Lecture Notes to Fuchsian Groups

Winter 2020, Technion

Lectures by Chen Meiri

Typed by Elad Tzorani

October 27, 2020

Contents

Chapter 1

Preliminaries

1.1 The Hyperbolic Plane

1.1.1 The Riemann Sphere

Definition 1.1.1 (The Riemann Sphere). The *Riemann sphere* is a one-dimensional complex manifold, denoted $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the charts of which are the following.

$$\begin{aligned} U_1 &= (\mathbb{C}, f_1) \\ U_2 &= \left(\hat{\mathbb{C}} \setminus \{0\}, f_2\right) \end{aligned}$$

where

$$\begin{aligned} f_1 \colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z \end{aligned}$$

and

$$\begin{aligned} f_2 \colon \mathbb{C} &\to \hat{\mathbb{C}} \setminus \{0\} \\ z &\mapsto \frac{1}{z}. \end{aligned}$$

Definition 1.1.2 (Möbius Transformation). A map $T \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $ad - bc \neq 0$ is called a *Möbius transformation*.

Notation 1.1.3. 1. We denote the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ in $\mathrm{PGL}_2(\mathbb{C})$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2. For every $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$, we denote by T_g the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$.

Lemma 1.1.4. *The set of Möbius transformations is a group under composition, and the map $g \mapsto T_g$ is an isomorphism between $\mathrm{PGL}_2(\mathbb{C})$ and the group of Möbius transformation.*

Proof. It holds that

$$\begin{aligned} T_{g_1} \circ T_{g_2}(z) &= \frac{a_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+1}{c_1\left(\frac{a_2z+b_2}{c_2z+d_2}\right)+d_1} \\ &= \frac{(a_1a_2+b_1c_2)z+(a_1b_2+b_1d_2)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)} \\ &= T_{g_1g_2}(z). \end{aligned}$$

In particular, $T_{g^{-1}}$ is the inverse of T_g.

The rest of the proof is clear. ■

Definition 1.1.5 (Generalised Circle). A generalised circle in $\mathbb{C}$ is either an Euclidean circle or an Euclidean straight line.

Lemma 1.1.6. *Let $T\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Then*

1. *T is an endomorphism of $\hat{\mathbb{C}}$.*

2. *T is conformal.*

3. *T sends generalised circles to generalised circles.*

1.1.2 Models of the Hyperbolic Plane

Definition 1.1.7 (The Upper Half Plane Model for the Hyperbolic Plane). 1. As a set, define $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path given by $\gamma(t) = x(t) + iy(t)$ for real functions $x(t), y(t)$. The *hyperbolic length* of γ is given by

$$h(\gamma) := \int_0^1 \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{y(t)}\,\mathrm{d}t = \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{y(t)}\,\mathrm{d}t.$$

3. The *hyperbolic distance* $\rho(z,w)$ between two points $z, w \in \mathbb{H}$ is defined as $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths γ from z to w.

Remark 1.1.8. $\mathbb{H}$ is a Riemann surface where for every $z \in \mathbb{H}$, the inner product of $T_z H$ is given by

$$((x_1, y_1), (x_2, y_2)) = \frac{x_1 x_2 + y_1 y_2}{(\Im z)^2}.$$

In particular, Euclidean angles are equal to hyperbolic angles.

Definition 1.1.9 (The Disc Model for the Hyperbolic Plane).

1. As a set, define $\mathbb{U} \coloneqq \{z \in \mathbb{C} \mid |z| < 1\}$.

2. Let $\gamma \colon [0,1] \to \mathbb{U}$ be a piecewise continuously differentiable path. The *hyperbolic length* of γ is given by

$$h_u(\gamma) \coloneqq \int_0^1 \frac{2\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{1 - |\gamma(t)|^2} \,\mathrm{d}t.$$

3. The *hyperbolic distance* $\rho_u(z, w)$ between $z, w \in \mathbb{U}$ is defined to be $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise differentiable paths from z to w.

Remark 1.1.10. It is clear that hyperbolic circles around 0 are exactly Euclidean circles around it (with a generally different radius).

Remark 1.1.11. Rotations around 0 are isometries in the disc model.

Lemma 1.1.12. *Let π be the Möbius transformation defined by*

$$\pi(z) = \frac{iz + 1}{z + i}.$$

Then

1. *π is a bijection from $\mathbb{H}$ to $\mathbb{U}$.*

2. *For every piecewise differentiable path $\gamma \colon [0,1] \to \mathbb{H}$,it holds that $h_u(\pi(\gamma)) = h(\gamma)$. In particular, π is an isometry.*

Proof.

1. It holds that

$$\begin{aligned} \pi(-1) &= -1 \\ \pi(0) &= -i \\ \pi(1) &= 1. \end{aligned}$$

Since Möbius transformations send generalised circles to generalised circles we get that π sends $\mathbb{R}$ to the unit circle. Since $\pi(i) = 0$ and π is a homeomorphism of the Riemann sphere, we get the result.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path. Denote $\psi = \pi^{-1}$ and $\delta = \pi(\gamma)$. Then

$$\psi(z) = \frac{iz-1}{-z+i} = \frac{(iz-1)-\bar{z}-i}{(-z+i)-\bar{z}-i} = \frac{(z+\bar{z})+i\left(1-|z|^2\right)}{|-z+i|^2}.$$

So,

$$\Im(\psi(z)) = \frac{1-|z|^2}{|-z+i|^2}.$$

Since

$$\frac{\mathrm{d}\psi}{\mathrm{d}z} = \frac{-2}{(-z+i)^2},$$

■

we get that

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\Im(\gamma(t))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi(\delta)}{\mathrm{d}t}\right|}{\Im(\psi(\delta(t)))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi}{\mathrm{d}z}(\delta(t))\frac{\mathrm{d}\delta}{\partial t}\right|}{\Im(\psi(\delta(t)))} \\ &= \int_0^1 \frac{2\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{1-|\delta(t)|^2}\,\mathrm{d}t \\ &= h_u(\delta). \end{aligned}$$