

Lecture Notes to Fuchsian Groups

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Chapter 1

Preliminaries

1.1 The Hyperbolic Plane

1.1.1 The Riemann Sphere

Definition 1.1.1 (The Riemann Sphere). The *Riemann sphere* is a one-dimensional complex manifold, denoted $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the charts of which are the following.

$$\begin{aligned}U_1 &= (\mathbb{C}, f_1) \\U_2 &= (\hat{\mathbb{C}} \setminus \{0\}, f_2)\end{aligned}$$

where

$$\begin{aligned}f_1: \mathbb{C} &\rightarrow \mathbb{C} \\z &\mapsto z\end{aligned}$$

and

$$\begin{aligned}f_2: \mathbb{C} &\rightarrow \hat{\mathbb{C}} \setminus \{0\} \\z &\mapsto \frac{1}{z}.\end{aligned}$$

Definition 1.1.2 (Möbius Transformation). A map $T: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ of the form

$$z \mapsto \frac{az + b}{cz + d}$$

where $ad - bc \neq 0$ is called a *Möbius transformation*.

Notation 1.1.3. 1. We denote the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ in $\mathrm{PGL}_2(\mathbb{C})$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2. For every $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PGL}_2(\mathbb{C})$, we denote by T_g the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$.

Lemma 1.1.4. *The set of Möbius transformations is a group under composition, and the map $g \mapsto T_g$ is an isomorphism between $\text{PGL}_2(\mathbb{C})$ and the group of Möbius transformation.*

Proof. It holds that

$$\begin{aligned} T_{g_1} \circ T_{g_2}(z) &= \frac{a_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + 1}{c_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)} \\ &= T_{g_1 g_2}(z). \end{aligned}$$

In particular, $T_{g^{-1}}$ is the inverse of T_g .

The rest of the proof is clear. ■

Definition 1.1.5 (Generalised Circle). A generalised circle in \mathbb{C} is either an Euclidean circle or an Euclidean straight line.

Lemma 1.1.6. *Let $T: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a Möbius transformation. Then*

1. *T is an endomorphism of $\hat{\mathbb{C}}$.*
2. *T is conformal.*
3. *T sends generalised circles to generalised circles.*

1.1.2 Models of the Hyperbolic Plane

Definition 1.1.7 (The Upper Half Plane Model for the Hyperbolic Plane). 1.

As a set, define $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

2. Let $\gamma: [0, 1] \rightarrow \mathbb{H}$ be a piecewise continuously differentiable path given by $\gamma(t) = x(t) + iy(t)$ for real functions $x(t), y(t)$. The *hyperbolic length* of γ is given by

$$h(\gamma) := \int_0^1 \frac{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}{y(t)} dt = \int_0^1 \frac{\left|\frac{d\gamma}{dt}\right|}{y(t)} dt.$$

3. The *hyperbolic distance* $\rho(z, w)$ between two points $z, w \in \mathbb{H}$ is defined as $\inf_{\gamma} h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths γ from z to w .

Remark 1.1.8. \mathbb{H} is a Riemann surface where for every $z \in \mathbb{H}$, the inner product of $T_z H$ is given by

$$((x_1, y_1), (x_2, y_2)) = \frac{x_1 x_2 + y_1 y_2}{(\Im z)^2}.$$

In particular, Euclidean angles are equal to hyperbolic angles.

Definition 1.1.9 (The Disc Model for the Hyperbolic Plane). 1. As a set, define $\mathbb{U} := \{z \in \mathbb{C} \mid |z| < 1\}$.

2. Let $\gamma: [0, 1] \rightarrow \mathbb{U}$ be a piecewise continuously differentiable path. The *hyperbolic length* of γ is given by

$$h_u(\gamma) := \int_0^1 \frac{2 \left| \frac{d\gamma}{dt} \right|}{1 - |\gamma(t)|^2} dt.$$

3. The *hyperbolic distance* $\rho_u(z, w)$ between $z, w \in \mathbb{U}$ is defined to be $\inf_{\gamma} h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths from z to w .

Remark 1.1.10. It is clear that hyperbolic circles around 0 are exactly Euclidean circles around it (with a generally different radius).

Remark 1.1.11. Rotations around 0 are isometries in the disc model.

Lemma 1.1.12. Let π be the Möbius transformation defined by

$$\pi(z) = \frac{iz + 1}{z + i}.$$

Then

1. π is a bijection from \mathbb{H} to \mathbb{U} .
2. For every piecewise continuously differentiable path $\gamma: [0, 1] \rightarrow \mathbb{H}$, it holds that $h_u(\pi(\gamma)) = h(\gamma)$. In particular, π is an isometry.

Proof. 1. It holds that

$$\begin{aligned} \pi(-1) &= -1 \\ \pi(0) &= -i \\ \pi(1) &= 1. \end{aligned}$$

Since Möbius transformations send generalised circles to generalised circles we get that π sends \mathbb{R} to the unit circle. Since $\pi(i) = 0$ and π is a homeomorphism of the Riemann sphere, we get the result.

2. Let $\gamma: [0, 1] \rightarrow \mathbb{H}$ be a piecewise continuously differentiable path. Denote $\psi = \pi^{-1}$ and $\delta = \pi(\gamma)$. Then

$$\psi(z) = \frac{iz - 1}{-z + i} = \frac{(iz - 1) - \bar{z} - i}{(-z + i) - \bar{z} - i} = \frac{(z + \bar{z}) + i(1 - |z|^2)}{|-z + i|^2}.$$

So,

$$\Im(\psi(z)) = \frac{1 - |z|^2}{|-z + i|^2}.$$

Since

$$\frac{d\psi}{dz} = \frac{-2}{(-z + i)^2},$$

we get that

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\left| \frac{d\gamma}{dt} \right|}{\Im(\gamma(t))} dt \\ &= \int_0^1 \frac{\left| \frac{d\psi(\delta)}{dt} \right|}{\Im(\psi(\delta(t)))} dt \\ &= \int_0^1 \frac{\left| \frac{d\psi}{dz}(\delta(t)) \frac{d\delta}{dt} \right|}{\Im(\psi(\delta(t)))} dt \\ &= \int_0^1 \frac{2 \left| \frac{d\delta}{dt} \right|}{1 - |\delta(t)|^2} dt \\ &= h_u(\delta). \end{aligned}$$

■

1.1.3 Isometries of the Hyperbolic Plane

Lemma 1.1.13. For every $g \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PSL}_2(\mathbb{R})$ it holds that

$$T_g(\mathbb{H}) \subseteq \mathbb{H}.$$

Proof. It's enough to show the inclusion $T_g(\mathbb{H}) \subseteq \mathbb{H}$ since then

$$T_{g^{-1}}(\mathbb{H}) = (T_g)^{-1}(\mathbb{H}) \subseteq \mathbb{H}$$

which implies $T_g(\mathbb{H}) \supseteq \mathbb{H}$ by applying T_g .

Now, we have

$$\begin{aligned} T_g(z) &= \frac{az + b}{cz + d} \\ &= \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} \\ &= \frac{ac|z|^2 + adz + bc\bar{z} + bg}{|cz + d|^2}. \end{aligned}$$

Thus,

$$\begin{aligned}
 \Im(T_g(z)) &= \frac{T_g(z) - \overline{T_g(z)}}{2i} \\
 &= \frac{(ad-bc)z - (ad-bc)\bar{z}}{2i|cz+d|^2} \\
 &\stackrel{ad-bc=1}{=} \frac{\Im(z)}{|cz+d|^2}.
 \end{aligned}$$

■

This lemma allows us to identify $\mathrm{PSL}_2(\mathbb{R})$ as a subgroup of $\mathrm{Sym}(\mathbb{H})$. The next lemma shows that even more is true.

Lemma 1.1.14. $\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{Isom}(\mathbb{H})$.

Proof. It's enough to show that for every $g \in \mathrm{PSL}_2(\mathbb{R})$ and every piecewise continuously differentiable path γ it holds that $h(\gamma) = h(T_g(\gamma))$. Denote $T = T_g$ and $\delta = T(\gamma)$. Then

$$\begin{aligned}
 h(\delta) &= \int_0^1 \frac{\left|\frac{d\delta}{dt}\right|}{\Im(\delta(t))} dt \\
 &= \int_0^1 \frac{\left|\frac{dT}{dz}(\gamma(t)) \frac{d\gamma}{dt}\right|}{\Im(\delta(t))} dt \\
 &= \int_0^1 \frac{\left|\frac{d\gamma}{dt}\right|}{\gamma(t)} dt \\
 &\stackrel{\star}{=} \int_0^1 \frac{\left|\frac{d\gamma}{dt}\right|}{\gamma(t)} dt \\
 &= h(\gamma)
 \end{aligned}$$

where \star follows from

$$\begin{aligned}
 \Im(T_g(z)) &= \frac{\Im(z)}{|cz+d|^2} \oplus \frac{dT}{dz} \\
 &= \frac{a(cz+d) - c(az+b)}{(cz+d)^2} \\
 &= \frac{1}{(cz+d)^2}.
 \end{aligned}$$

■

Corollary 1.1.15. $\mathrm{Isom}(\mathbb{H})$ acts transitively on \mathbb{H} .

Proof. It's enough to show that for every $z \in \mathbb{H}$ there's $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z) = i$.

If $z = x + yi$, take $g = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$, then

$$T_g(z) = \frac{1}{y}(x + yi) - \frac{x}{y} = i.$$

■

Lemma 1.1.16. Let $\pi: \mathbb{H} \rightarrow \mathbb{U}$ be the isometry $z \mapsto \frac{iz+1}{z+i}$ which we defined previously. Then

$$\{\pi T_g \pi^{-1} \mid g \in \mathrm{PSL}_2(\mathbb{R})\} = \left\{ \begin{pmatrix} r & s \\ \bar{r} & \bar{s} \end{pmatrix} \mid \begin{matrix} r, s \in \mathbb{C} \\ |r|^2 - |s|^2 = 1 \end{matrix} \right\}.$$

In particular, by taking $r = e^{i\theta}$ and $s = 0$ we see that the action of $\mathrm{PSL}_2(\mathbb{R})$ on \mathbb{U} contains all the rotations around 0.

Proof. It holds that

$$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (a+d) + i(b-c) & b+c + i(a-d) \\ (b+c) - i(a+d) & (a+d) - i(b-c) \end{pmatrix}.$$

Now, $(a+d, a-d, b+c, b-c)$ can be any 4-tuple. Specifically, for every $r, s \in \mathbb{C}$ we have $a, b, c, d \in \mathbb{R}$ such that $\pi T_g \pi^{-1} = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}$, and by the equality from the determinants we get that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. ■

Corollary 1.1.17. Let $z_1, z_2, w_1, w_2 \in \mathbb{H}$ be such that $\rho(z_1, w_1) = \rho(z_2, w_2)$, then there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z_1) = z_2$ and $T_g(w_1) = w_2$.

Proof. Since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on \mathbb{H} we can assume that $z_1 = z_2$ and show that $\mathrm{Stab}(z_1)$ acts transitively on $\{w \in \mathbb{H} \mid \rho(z_1, w) = \rho(z_1, w_1)\}$. We already showed this in the disc model, in the case $z_1 = i$. ■

Definition 1.1.18. Let (X, d) be a metric space.

1. Let $x, y \in X$. A path $\gamma: [a, b] \rightarrow X$ which joins x and y is called a *geodesic segment* if for every $a \leq t_1 \leq t_2 \leq b$ it holds that $|t_2 - t_1| = d(\gamma(t_1), \gamma(t_2))$.
2. A path $\gamma: \mathbb{R} \rightarrow X$ is called a *geodesic line* if for every $a < b$ it holds that $\gamma|_{[a, b]}$ is a geodesic segment.

Remark 1.1.19. Let γ be a geodesic segment or line. Then γ is determined by the image of γ up to a composition with an isometry of \mathbb{R} . Thus, we can identify geodesic segments and lines with their image up to orientation.