Lecture Notes to Fuchsian Groups

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Chapter 1

Preliminaries

1.1 The Hyperbolic Plane

1.1.1 The Riemann Sphere

Definition 1.1.1 (The Riemann Sphere). The *Riemann sphere* is a one-dimensional complex manifold, denoted $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the charts of which are the following.

$$\begin{aligned} U_1 &= (\mathbb{C}, f_1) \\ U_2 &= \left(\hat{\mathbb{C}} \setminus \{0\}, f_2\right) \end{aligned}$$

where

$$\begin{aligned} f_1 \colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z \end{aligned}$$

and

$$\begin{aligned} f_2 \colon \mathbb{C} &\to \hat{\mathbb{C}} \setminus \{0\} \\ z &\mapsto \frac{1}{z}. \end{aligned}$$

Definition 1.1.2 (Möbius Transformation). A map $T \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $ad - bc \neq 0$ is called a *Möbius transformation*.

Notation 1.1.3.

1. We denote the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ in $\mathrm{PGL}_2(\mathbb{C})$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2. For every $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$, we denote by T_g the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$.

Lemma 1.1.4. *The set of Möbius transformations is a group under composition, and the map $g \mapsto T_g$ is an isomorphism between $\mathrm{PGL}_2(\mathbb{C})$ and the group of Möbius transformation.*

Proof. It holds that

$$\begin{aligned} T_{g_1} \circ T_{g_2}(z) &= \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + 1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)} \\ &= T_{g_1 g_2}(z). \end{aligned}$$

In particular, $T_{g^{-1}}$ is the inverse of T_g.

The rest of the proof is clear. ■

Definition 1.1.5 (Generalised Circle). A generalised circle in $\mathbb{C}$ is either an Euclidean circle or an Euclidean straight line.

Lemma 1.1.6. *Let $T\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Then*

1. *T is an endomorphism of $\hat{\mathbb{C}}$.*

2. *T is conformal.*

3. *T sends generalised circles to generalised circles.*

1.1.2 Models of the Hyperbolic Plane

Definition 1.1.7 (The Upper Half Plane Model for the Hyperbolic Plane). 1.
As a set, define $\mathbb{H} \coloneqq \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path given by $\gamma(t) = x(t) + iy(t)$ for real functions $x(t), y(t)$. The *hyperbolic length* of γ is given by

$$h(\gamma) \coloneqq \int_0^1 \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{y(t)} \,\mathrm{d}t = \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{y(t)} \,\mathrm{d}t.$$

3. The *hyperbolic distance* $\rho(z,w)$ between two points $z, w \in \mathbb{H}$ is defined as $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths γ from z to w.

Remark 1.1.8. $\mathbb{H}$ is a Riemann surface where for every $z \in \mathbb{H}$, the inner product of T_zH is given by

$$((x_1, y_1), (x_2, y_2)) = \frac{x_1x_2 + y_1y_2}{(\Im z)^2}.$$

In particular, Euclidean angles are equal to hyperbolic angles.

Definition 1.1.9 (The Disc Model for the Hyperbolic Plane).

1. As a set, define $\mathbb{U} := \{z \in \mathbb{C} \mid |z| < 1\}$.
2. Let $\gamma\colon [0,1] \to \mathbb{U}$ be a piecewise continuously differentiable path. The *hyperbolic length* of γ is given by

$$h_u(\gamma) := \int_0^1 \frac{2\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{1 - |\gamma(t)|^2}\,\mathrm{d}t.$$

3. The *hyperbolic distance* $\rho_u(z, w)$ between $z, w \in \mathbb{U}$ is defined to be $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths from z to w.

Remark 1.1.10. It is clear that hyperbolic circles around 0 are exactly Euclidean circles around it (with a generally different radius).

Remark 1.1.11. Rotations around 0 are isometries in the disc model.

Lemma 1.1.12. *Let π be the Möbius transformation defined by*

$$\pi(z) = \frac{iz+1}{z+i}.$$

Then

1. *π is a bijection from $\mathbb{H}$ to $\mathbb{U}$.*
2. *For every piecewise continuously differentiable path $\gamma\colon [0,1] \to \mathbb{H}$, it holds that $h_u(\pi(\gamma)) = h(\gamma)$. In particular, π is an isometry.*

Proof.

1. It holds that

$$\begin{aligned} \pi(-1) &= -1 \\ \pi(0) &= -i \\ \pi(1) &= 1. \end{aligned}$$

Since Möbius transformations send generalised circles to generalised circles we get that π sends $\mathbb{R}$ to the unit circle. Since $\pi(i) = 0$ and π is a homeomorphism of the Riemann sphere, we get the result.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path. Denote $\psi = \pi^{-1}$ and $\delta = \pi(\gamma)$. Then

$$\psi(z) = \frac{iz-1}{-z+i} = \frac{(iz-1)\,-\bar{z}-i}{(-z+i)\,-\bar{z}-i} = \frac{(z+\bar{z}) + i\left(1-|z|^2\right)}{|-z+i|^2}.$$

So,

$$\Im(\psi(z)) = \frac{1-|z|^2}{|-z+i|^2}.$$

Since

$$\frac{\mathrm{d}\psi}{\mathrm{d}z} = \frac{-2}{(-z+i)^2},$$

we get that

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\Im(\gamma(t))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi(\delta)}{\mathrm{d}t}\right|}{\Im(\psi(\delta(t)))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi}{\mathrm{d}z}(\delta(t))\,\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{\Im(\psi(\delta(t)))} \\ &= \int_0^1 \frac{2\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{1-|\delta(t)|^2}\,\mathrm{d}t \\ &= h_u(\delta). \end{aligned}$$

∎

1.1.3 Isometries of the Hyperbolic Plane

Lemma 1.1.13. *For every* $g \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ *it holds that*

$$T_g(\mathbb{H}) \subseteq \mathbb{H}.$$

Proof. It's enough to show the inclusion $T_g(\mathbb{H}) \subseteq \mathbb{H}$ since then

$$T_{g^{-1}}(\mathbb{H}) = (T_g)^{-1}(\mathbb{H}) \subseteq \mathbb{H}$$

which implies $T_g(\mathbb{H}) \supseteq \mathbb{H}$ by applying T_g.

Now, we have

$$\begin{aligned} T_g(z) &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{ac\,|z|^2 + adz + bc\bar{z} + bg}{|cz+d|^2}. \end{aligned}$$

Thus,

$$\begin{aligned}\Im\left(T_g\left(z\right)\right) &= \frac{T_g\left(z\right)-\overline{T_g\left(z\right)}}{2i}\\ &= \frac{\left(ad-bc\right)z-\left(ad-bc\right)\bar{z}}{2i\left|cz+d\right|^2}\\ &\underset{ad-bc=1}{=} \frac{\Im\left(z\right)}{\left|cz+d\right|^2}.\end{aligned}$$ ■

This lemma allows us to identify $\mathrm{PSL}_2\left(\mathbb{R}\right)$ as a subgroup of $\mathrm{Sym}\left(\mathbb{H}\right)$. The next lemma shows that even more is true.

Lemma 1.1.14. $\mathrm{PSL}_2\left(\mathbb{R}\right) \subseteq \mathrm{Isom}\left(\mathbb{H}\right)$.

Proof. It's enough to show that for every $g \in \mathrm{PSL}_2\left(\mathbb{R}\right)$ and every piecewise continuously differentiable path γ it holds that $h\left(\gamma\right) = h\left(T_g\left(\gamma\right)\right)$. Denote $T = T_g$ and $\delta = T\left(\gamma\right)$. Then

$$\begin{aligned}h\left(\delta\right) &= \int_0^1 \frac{\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{\Im\left(\delta\left(t\right)\right)}\,\mathrm{d}t\\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}T}{\mathrm{d}z}\left(\gamma\left(t\right)\right)\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\Im\left(\delta\left(t\right)\right)}\,\mathrm{d}t\\ &\underset{\star}{=} \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\gamma\left(t\right)}\,\mathrm{d}t\\ &= h\left(\gamma\right)\end{aligned}$$

where $\star$ follows from

$$\begin{aligned}\Im\left(T_g\left(z\right)\right) &= \frac{\Im\left(z\right)}{\left|cz+d\right|^2} \oplus \frac{\mathrm{d}T}{\mathrm{d}z}\\ &= \frac{a\left(cz+d\right)-c\left(az+b\right)}{\left(cz+d\right)^2}\\ &= \frac{1}{\left(cz+d\right)^2}.\end{aligned}$$ ■

Corollary 1.1.15. $\mathrm{Isom}\left(\mathbb{H}\right)$ *acts transitively on* $\mathbb{H}$.

Proof. It's enough to show that for every $z \in \mathbb{H}$ there's $g \in \mathrm{PSL}_2\left(\mathbb{R}\right)$ such that $T_g\left(z\right) = i$.

If $z = x + yi$, take $g = \begin{pmatrix}\frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}}\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}$, then

$$T_g\left(z\right) = \frac{1}{y}\left(x+yi\right) - \frac{x}{y} = i.$$ ■

Lemma 1.1.16. *Let $\pi\colon \mathbb{H} \to \mathbb{U}$ be the isometry $z \mapsto \frac{iz+1}{z+i}$ which we defined previously. Then*

$$\left\{\pi T_g \pi^{-1} \mid g \in \mathrm{PSL}_2(\mathbb{R})\right\} = \left\{ \begin{pmatrix} r & s \\ \bar{r} & \bar{s} \end{pmatrix} \;\middle|\; {\scriptstyle r,s\in\mathbb{C} \atop |r|^2-|s|^2=1} \right\}.$$

In particular, by taking $r = e^{i\theta}$ and $s = 0$ we see that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{U}$ contains all the rotations around 0.

Proof. It holds that

$$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (a+d)+i(b-c) & b+c+i(a-d) \\ (b+c)-i(a+d) & (a+d)-i(b-c) \end{pmatrix}.$$

Now, $(a+d, a-d, b+c, b-c)$ can be any 4-tuple. Specifically, for every $r, s \in \mathbb{C}$ we have $a, b, c, d \in \mathbb{R}$ such that $\pi T_g \pi^{-1} = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}$, and by the equality from the determinants we get that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. ■

Corollary 1.1.17. *Let $z_1, z_2, w_1, w_2 \in \mathbb{H}$ be such that $\rho(z_1, w_1) = \rho(z_2, w_2)$, then there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z_1) = z_2$ and $T_g(w_1) = w_2$.*

Proof. Since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ we can assume that $z_1 = z_2$ and show that $\mathrm{Stab}(z_1)$ acts transitively on $\{w \in \mathbb{H} \mid \rho(z_1, w) = \rho(z_1, w_1)\}$. We already showed this in the disc model, in the case $z_1 = i$. ■

Definition 1.1.18. Let (X, d) be a metric space.

1. Let $x, y \in X$. A path $\gamma\colon [a,b] \to X$ which joins x and y is called a *geodesic segment* if for every $a \le t_1 \le t_2 \le b$ it holds that $|t_2 - t_1| = d(\gamma(t_1), \gamma(t_2))$.

2. A path $\gamma\colon \mathbb{R} \to X$ is called a *geodesic line* if for every $a < b$ it holds that $\gamma|_{[a,b]}$ is a geodesic segment.

Remark 1.1.19. Let γ be a geodesic segment or line. Then γ is determined by the image of γ up to a composition with an isometry of R. Thus, we can identify geodesic segments and lines with their image up to orientation.

Lemma 1.1.20. *Let $b > a > 0$ be real numbers. Then $\{iy \mid a \le y \le b\}$ is the unique geodesic segment between ia and ib and $\{iy \mid y > 0\}$ is the unique geodesic line through ia and ib.*

Proof. We begin with the first part of the lemma. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path joining ia and ib. For $t \in [0,1]$ denote

$\gamma(t) = x(t) + iy(t)$ where $x(t), y(t) \in \mathbb{R}$. Then

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{y(t)}\,\mathrm{d}t \\ &\underset{\star}{\geq} \int_0^1 \frac{\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|}{y(t)}\,\mathrm{d}t \\ &\geq \int_0^1 \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{y(t)}\,\mathrm{d}t \\ &= \ln\left(\frac{b}{a}\right). \end{aligned}$$

Thus, $\rho(ia, ib) \geq \ln\left(\frac{b}{a}\right)$. If $y(t) = i((b-a)t + a)$, the above inequalities are equalities so $\rho(ia, ib) = \ln\left(\frac{b}{a}\right)$. The inequality $\star$ is an equality if and only if $x(t) = 0$ for all $t \in [a, b]$. It follows that the unique geodesic segment between a and b is $\{iy \mid a \leq y \leq b\}$.

Now, it is clear that $\{iy \mid y > 0\}$ is a geodesic line which passes through ia and ib. We want to show it's unique.

Assume towards a contradiction that there exists a geodesic line ℓ between ia and ib which isn't the positive part of the y-axis. Then there's $z = x + iy \in \ell$ for which $x \neq 0$ and $\rho(z, ia) > \rho(z, ib)$. By the previous lemma, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(ia) = ia$ and $T_g(z) \in i\mathbb{R}$. Since T_g sends generalised circles to generalised circles, $T_g(ib) \notin i\mathbb{R}$. Indeed, otherwise the image of the segment between ia and ib would belong to $i\mathbb{R}$, and since T_g sends generalised circles to generalised circles, it would send $i\mathbb{R}$ to itself.

We get that there exists a geodesic between ia and $T_g(z) = ic$ which is not contained in $i\mathbb{R}$, and this is impossible. ■

Theorem 1.1.21. *1. Every distinct points $z, w \in \mathbb{H}$ are contained in a unique geodesic segment and a unique geodesic line.*

2. The geodesics in $\mathbb{H}$ are semicircles and lines orthogonal to the real axis.

Proof. 1. For every $g \in \mathrm{PSL}_2(\mathbb{R})$ it holds that $T_g(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$. If $z, w \in \mathbb{H}$, by a previous lemma there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z) = ia$ and $T_g(w) = ib$ for some $a, b \in \mathbb{R}_+$. Thus, $T_g^{-1}([ia, ib])$ is the unique geodesic segment between z and w.

2. This follows from the fact that Möbius circles are conformal, send generalised circles to generalised circles, and sends $\mathbb{R} \cup \{\infty\}$ to itself. ■

Corollary 1.1.22. *The geodesic segment in $\mathbb{U}$ are segments of straight lines through zero or arcs of circles which are orthogonal to the unit circles.*

Theorem 1.1.23. *Let $z, w \in \mathbb{H}$. Then*

$$\sinh\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|}{2\left(\Im(z)\Im(w)\right)^{\frac{1}{2}}}.$$

Proof. Since $\mathrm{PSL}_2(\mathbb{R}) \subseteq \mathrm{Isom}(\mathbb{H})$, the left side of the equation is invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$. We first show that the right side is also invariant.

It's clear that the right side is invariant under maps of the form $z \mapsto az+b$ for $a, b \in \mathbb{R}$. Since $\mathrm{PSL}_2(\mathbb{R})$ (viewed as a group of Möbius transformations) is generated by maps of the forms

$$\begin{aligned} &z \mapsto az+b,\ a,b \in \mathbb{R} \\ &z \mapsto -\frac{1}{z} \end{aligned}$$

it's enough to show that the right side is invariant under these maps.

The right side is indeed invariant under $\frac{1}{z}$ since

$$\begin{aligned} \frac{\left|\frac{1}{z}-\frac{1}{w}\right|}{2\left(\Im\left(\frac{1}{z}\right)\Im\left(\frac{1}{w}\right)\right)^{\frac{1}{2}}} &= \frac{\left|\frac{z-w}{zw}\right|}{2\left(\Im\left(\frac{z}{|z|^2}\right)\Im\left(\frac{w}{|w|^2}\right)\right)^{\frac{1}{2}}} \\ &= \frac{|z-w|}{2\left(\Im(z)\Im(w)\right)^{\frac{1}{2}}}. \end{aligned}$$

Since both sides of the equation are invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$, it's enough to prove the equality for $z = i$ and $w = ir$ for some $r \in \mathbb{R}_+$. Indeed,

$$\begin{aligned} \sinh\left(\frac{1}{2}\rho(i,ir)\right) &= \sinh\left(\frac{1}{2}|\ln r|\right) \\ &= \frac{\left|\sqrt{r}-\frac{1}{\sqrt{r}}\right|}{2} \\ &= \frac{|r-1|}{2\sqrt{r}} \\ &= \frac{|i-ir|}{2\left(\Im(i)\Im(ir)\right)^{\frac{1}{2}}}. \end{aligned}$$

■

Corollary 1.1.24.

1. *The hyperbolic topology is equal to the Euclidean topology.*
2. *$\mathbb{H}$ is a complete metric space.*

Proof. 1. Let $z \in \mathbb{H}$. If $|\Im(z) - \Im(w)| < \frac{1}{2}\Im(z)$ then

$$\frac{|z-w|}{\sqrt{6}\Im(z)} \leq \sinh\left(\frac{1}{2}\rho(z,w)\right) \leq \frac{|z-w|}{\sqrt{2}\Im(z)}.$$

2. We show that $\mathbb{U}$ is a complete metric space, which implies the result since there's an isometry between $\mathbb{U}$ and $\mathbb{H}$. Let $z, w \in \mathbb{U}$, we have

$$\sinh^2\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|^2}{\left(1-|z|^2\right)\left(z-|w|^2\right)}. \qquad \blacksquare$$

   Let $(z_n)_{n\in\mathbb{N}}$ be a hyperbolic Cauchy sequence. Then it's bounded in the hyperbolic metric, and (2) implies that it does not have a limit point on the unit circle and so is contained in a compact subset of the unit circle.

   The result follows since (2) implies that on such a subset the hyperbolic and Euclidean metric are Lipschitz equivalent.

Exercise 1. Prove that if $z, w \in \mathbb{U}$ then

$$\sinh^2\left(\frac{1}{2}\rho(z,w)\right) = \frac{|z-w|^2}{\left(1-|z|^2\right)\left(1-|w|^2\right)}.$$

Theorem 1.1.25. *Let* $\tau\colon \mathbb{H}\to\mathbb{H}$ *be* $\tau(z) = -\bar{z}$*. Then* $\operatorname{Isom}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R}) \rtimes \langle\tau\rangle$*. In particular,* $\mathrm{PSL}_2(\mathbb{R})$ *is a normal index two subgroup of* $\operatorname{Isom}(\mathbb{H})$*.*

Proof. Clearly, τ is of order two. Since every index two subgroup is normal, it is enough to prove that for every hyperbolic isometry $s \in \operatorname{Isom}(\mathbb{H})$ there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that sT_g is either the identity or τ.

There's $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(i) = s^{-1}(i)$ and $T_g(2i) = s^{-2}(2i)$ (since $\mathrm{PSL}_2(\mathbb{R})$ is 2-transitive). Then $sT_g(i) = i$ and $sT_g(2i) = 2i$. Since isometries send geodesics to geodesics, for every $t > 0$ it holds that $sT_g(ti) = ti$.

Denote

$$\begin{aligned} U_+ &\coloneqq \{z \in \mathbb{H} \mid \Re(z) > 0\} \\ U_- &\coloneqq \{z \in \mathbb{H} \mid \Re(z) < 0\}. \end{aligned}$$

Since sT_g is continuous it follows that $sT_g(U_+) \subseteq U_+$ or $sT_g(U_+) \subseteq U_-$. In the first case denote $R \coloneqq sT_g$, and in the second case denote $R \coloneqq \tau sT_g$. In either case, $R(U_+) \subseteq U_+$.

In order to finish the proof, we want to show that $R = \mathrm{id}$. For every $t > 0$ we have

$$\begin{aligned} \frac{|it-w|}{2\left(t\Im(w)\right)^{\frac{1}{2}}} &= \sinh\left(\frac{1}{2}\rho(it,w)\right) \\ &= \sinh\left(\frac{1}{2}\rho(R(it),R(w))\right) \\ &= \sinh\left(\frac{1}{2}\rho(it,R(w))\right) \\ &= \frac{|it-R(w)|}{2\left(t\Im(R(w))\right)^{\frac{1}{2}}}. \end{aligned}$$

So,

$$|it - w|^2 \Im(R(w)) = |it - R(w)|^2 \Im(w).$$

This holds for every t, which implies together with

$$\Im(R(w)) = \lim_{t\to\infty} \frac{|it - w|^2 \Im(R(w))}{t^2}$$

that

$$\Im(R(w)) = \lim_{t\to\infty} \frac{|it - R(w)|^2 \cdot \Im(w)}{t^2} = \Im(w).$$

Now, for every $t > 0$ we get

$$|it - w| = |it - R(w)|$$

which implies $w = R(w)$ or $w = -\overline{R(w)}$. The latter case is impossible since $R(U_+) \subseteq U_+$. ■

Corollary 1.1.26. *Every element of* $\operatorname{Isom}(\mathbb{H})$ *is either conformal or anti-conformal.*

An element of $\operatorname{Isom}(\mathbb{H})$ *is conformal if and only if it belongs to* $\mathrm{PSL}_2(\mathbb{R})$.

Definition 1.1.27. Let $\hat{\mathbb{C}}$ be the Riemann sphere. The cross ratio of distinct points $z_1, z_2, z_3, z_4 \in \hat{\mathbb{C}}$ is

$$(z_1, z_2 : z_3, z_3) \coloneqq \frac{(z_1 - z_2)\, z_3 - z_4}{(z_2 - z_3)(z_4 - z_1)}.$$

Lemma 1.1.28. *Möbius transformations preserve the cross ratio.*

Proof. We prove this when $z_1, z_2, z_3, z_4 \in \mathbb{C} \setminus \{0\}$. The other cases are left as exercise.

It is clear that maps of the form $z \mapsto az + b$, with $a \neq 0$, preserve the cross-ratio. Thus it's enough to prove that the map $z \mapsto -\frac{1}{z}$ preserves the cross-ratio. Indeed,

$$\begin{aligned}
(z_1, z_2; z_3, z_4) &= \frac{(z_1 - z_2)\, z_3 - z_4}{(z_2 - z_3)(z_4 - z_1)} \\
&= \frac{\left(\frac{z_1 - z_2}{z_1 z_2}\right) \frac{z_3 - z_4}{z_3 z_4}}{\left(\frac{z_2 - z_3}{z_2 z_3}\right)\left(\frac{z_4 - z_1}{z_1 z_4}\right)} \\
&= \frac{\left(\frac{1}{z_1} - \frac{1}{z_2}\right)\left(\frac{1}{z_3} - \frac{1}{z_4}\right)}{\left(\frac{1}{z_2} - \frac{1}{z_3}\right)\left(\frac{1}{z_4} - \frac{1}{z_1}\right)} \\
&= \left(\frac{1}{z_1}, \frac{1}{z_2}; \frac{1}{z_3}, \frac{1}{z_4}\right).
\end{aligned}$$

■

Theorem 1.1.29. *Let $z, w \in \mathbb{H}$ and let the geodesic joining z, w have have end points z^* and w^* in $\mathbb{R} \cup \{\infty\}$, chosen in a way that z lies between z^* and w. Then*

$$\rho(z, w) = \ln\left((w, z^*; z, w^*)\right).$$

Proof. Since both sides are invariant to the action of $\mathrm{PSL}_2(\mathbb{R})$ we can assume that $z = i$ and $w = ri$ with $r > 1$. Then $z^* = 0$ and $w^* = \infty$, so $r = (w, z^*, z, w^*)$ and $\rho(i, ir) = \ln(r)$. ∎

1.2 The Gauss-Bonnet Formula

Definition 1.2.1 (Hyperbolic Measure). We define a measure μ on subsets of $\mathbb{H}$ by

$$\mu(A) = \int_A \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

for which this exists.

Theorem 1.2.2. *The hyperbolic area is invariant under* $\mathrm{PSL}_2(\mathbb{R})$.

Proof. Let $f \colon \mathbb{C} \to \mathbb{C}$ given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

where $u, v \colon \mathbb{C} \to \mathbb{R}$.

By Cauchy-Riemann

$$\begin{aligned}\frac{\partial(u, v)}{\partial(x, y)} &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \\ &= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 \\ &= \dots\end{aligned}$$

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. Recall that

$$\begin{aligned}\left|\frac{\mathrm{d}T_g}{\mathrm{d}z}\right| &= \frac{1}{|cz + d|^2} \\ \Im(T_g(z)) &= \frac{\Im(z)}{|cz + d|^2}.\end{aligned}$$

Then

$$T_g(x + iy) = u(x, y) + iv(x, y)$$

so

$$\begin{aligned}
\mu\left(T_g\left(A\right)\right) &= \int_{T_g(A)} \frac{\mathrm{d}u\,\mathrm{d}v}{v^2} \\
&= \int_A \frac{\partial\left(u,v\right)}{\partial\left(x,y\right)} \frac{\mathrm{d}x\,\mathrm{d}y}{v^2} \\
&= \int_A \frac{1}{\left|cz+d\right|^4} \cdot \frac{\left|cz+d\right|^4}{y\,\mathrm{d}x\,\mathrm{d}y} \\
&= \mu\left(A\right).
\end{aligned}$$

■

Definition 1.2.3 ($\tilde{\mathbb{H}}$).

1. Define $\tilde{\mathbb{H}} = \mathbb{H} \cup \mathbb{R} \cup \{\infty\}$.
2. A hyperbolic n-sided polygon is a closed subset of $\tilde{\mathbb{H}}$ bounded by the closure of n hyperbolic geodesic segments or rays.
3. A side of a polygon is the closure of a geodesic segment or ray which bounds to polygon.
4. A point $z \in \tilde{\mathbb{H}}$ is called a vertex if it is the intersection of two distinct sides.

Example 1.2.4. There rare four types of hyperbolic triangles, which depend on the number of vertices on the boundary.

Theorem 1.2.5 (Gauss-Bonnet). *Let Δ be a hyperbolic triangle with angles α, β, γ. Then*

$$\mu\left(\Gamma\right) = \pi - \alpha - \beta - \gamma.$$

Proof. First assume that Δ has a vertex on the boundary. Since $\mathrm{PSL}_2\left(\mathbb{R}\right)$ preserves area, we may assume that this vertex is ∞. Thus, two sides are given by equations $x = a$ and $x = b$ (and assume $a < b$). By applying a transformation of the form

$$z \mapsto \lambda z + k$$

where $\lambda > 0$ and $k \in \mathbb{R}$, we can assume that the third side of Γ is an arc on the geodesic $\left|z\right|^2 = 1$.

Pass segments from 0 to the vertices of the triangle and call the angles between these and the real axis α and β. Then

$$\begin{aligned}
\mu\left(\Delta\right) &= \int_\Delta \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \\
&= \int_a^b \mathrm{d}x \int_{\sqrt{1-x^2}}^\infty \frac{\mathrm{d}y}{y^2} \\
&= \int_a^b \frac{\mathrm{d}x}{\sqrt{1-x^2}} \\
&\underset{x=\cos\theta}{=} \int_{\pi-\alpha}^{\beta} -\frac{\sin\theta}{\sin\theta}\,\mathrm{d}\theta = \pi - \alpha - \beta - \overset{0}{\cancel{\gamma}}.
\end{aligned}$$

In the other case, consider a triangle $\Delta = \mathrm{ABC}$ with respective angles α, β, γ. Continue the geodesic segment AB to get an intersection D with the boundary. Let $\Delta' = \mathrm{CBD}$ and $\Delta'' = \mathrm{ABD}$.

Now, Δ' and Δ'' have a vertex at infinity, so

$$\begin{aligned}\mu(\Delta) &= \mu(\Delta'') - \mu(\Delta') \\ &= \pi - (\alpha + \gamma + \theta) - (\pi - \theta - (\pi - \beta)) \\ &= \pi - \alpha - \beta - \gamma.\end{aligned}$$

■

1.3 Hyperbolic Geometry

Theorem 1.3.1. *Let Δ be a hyperbolic triangle with sides of hyperbolic lengths a, b, c and opposite angles α, β, γ. Assume that $\alpha, \beta, \gamma > 0$ (so there is no vertex at the boundary).*

The following holds.

The Sine Rule:

$$\frac{\sinh(a)}{\sin\alpha} = \frac{\sinh(b)}{\sin\beta} = \frac{\sinh(c)}{\sin\gamma}$$

The First Cosine Rule:

$$\cosh(c) = \cosh(a)\cosh(b) - \cos\gamma\sinh(a)\sinh(b)$$

The Second Cosine Rule:

$$\cosh(c) = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}.$$

Proof. **The First Cosine Rule:** We use the disc model to prove the rule. Let Δ be a triangle in $\mathbb{U}$ with sides a, b, c and let v_a, v_b, v_c be the vertices opposite to the respective sides.

We can assume $v_c = 0$ and $v_a = r \in (0, 1)$, and denote $v_b = z \in \mathbb{U}$. We have

$$\sinh^2\left(\frac{1}{2}\rho_u(z, w)\right) = \frac{|z - w|}{\left(1 - |z|^2\right)\left(1 - |w|^2\right)},$$

but

$$\sinh^2(\alpha) = \frac{1}{2}\cosh(2\alpha) - \frac{1}{2}\alpha$$

because

$$\begin{aligned}\left(\frac{e^\alpha - e^{-\alpha}}{2}\right)^\alpha &= \frac{e^{2\alpha} - 2 + e^{-2\alpha}}{4} \\ &= \frac{1}{2}\cdot\frac{e^{2\alpha} + e^{-2\alpha}}{2} - \frac{1}{2}.\end{aligned}$$

Hence

$$\cosh\left(\rho_u\left(z,w\right)\right)=\frac{2\left|z-w\right|}{\left(1-\left|z\right|^2\right)\left(1-\left|w\right|^2\right)}+1.$$

Then

$$\begin{aligned}
\cosh\left(a\right)&=\frac{1+\left|z\right|^2}{1-\left|z\right|^2}\\
\cosh\left(b\right)&=\frac{1+r^2}{1-r^2}\\
\cosh\left(c\right)&=\frac{2\left|z-r\right|^2}{\left(1-\left|z\right|^2\right)\left(1-r^2\right)}+1.
\end{aligned}$$

Using

$$\begin{aligned}
\sinh\left(a\right)&=\sqrt{\cosh^2\left(\left|z\right|\right)-1}=\frac{2\left|z\right|}{1-\left|z\right|^2}\\
\sinh\left(b\right)&=\sqrt{\cosh^2\left(r\right)-1}=\frac{2r}{1-r^2}
\end{aligned}$$

and the Euclidean cosine rule

$$\cos\gamma=\frac{r^2+\left|z\right|^2-\left|2-r\right|^2}{2r\left|z\right|}$$

we get

$$\begin{aligned}
\cosh\left(a\right)\cosh\left(b\right)-\sinh\left(a\right)\sinh\left(b\right)\cos\left(\gamma\right)&=\left(\frac{1+\left|z\right|^2}{1-\left|z\right|^2}\right)\left(\frac{1+r^2}{1-r^2}\right)-\frac{4r\left|z\right|}{\left(1-r^2\right)\left(1-\left|z\right|^2\right)}\cdot\frac{r^2+\left|z\right|^2}{2r}\\
&=\frac{\left(1+r^2\right)\left(1+\left|z\right|^2\right)-2r^2-2\left|z\right|^2+2\left|z-r\right|^2}{\left(1-r^2\right)\left(1-\left|z\right|^2\right)}\\
&=1+\frac{2\left|z-r\right|^2}{\left(1-r\right)^2\left(1-\left|z\right|^2\right)}\\
&=\cosh\left(c\right).
\end{aligned}$$

The Sine Rule: It holds by the first cosine rule that that

$$\begin{aligned}
\left(\frac{\sinh c}{\sin\gamma}\right)^2&=\frac{\sinh^2 c}{1-\left(\frac{\cosh a\cosh b-\cosh c}{\sinh(a)\sinh(b)}\right)^2}\\
&=\left(\cosh^2\left(a\right)-1\right)\left(\cosh^2\left(b\right)-1\right)-\left(\cosh\left(a\right)\cosh\left(b\right)-\cosh\left(c\right)\right)^2\\
&=1+2\cosh\left(a\right)\cosh\left(b\right)\cosh\left(c\right)-\cosh^2\left(a\right)-\cosh^2\left(b\right)-\cosh^2\left(c\right)
\end{aligned}$$

where the last term is symmetric in a,b,c. ■

Chapter 2

Fuchsian Groups

2.1 Fuchsian Groups

2.1.1 Definitions

Definition 2.1.1 ($\mathbf{SL_2}(\mathbb{R})$). Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2×2 real matrices with determinant 1, with the topology from $\mathbb{R}^4$ via $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$.

Throughout the course, we endow $\mathrm{PSL}_2(\mathbb{R})$ with the quotient topology from $\mathrm{GL}_2(\mathbb{R})$.

We endow $\mathrm{Isom}(\mathbb{H})$ with the following topology. Let $\tau \in \mathrm{Isom}(\mathbb{H}) \backslash \mathrm{PSL}_2(\mathbb{R})$ and. $U \subseteq \mathrm{Isom}(\mathbb{H})$ is open if and only if $U \cap \mathrm{PSL}_2(\mathbb{R})$ and $\tau U \cap \mathrm{PSL}_2(\mathbb{R})$ are open.

Exercise 2. 1. $\mathrm{SL}_2(\mathbb{R}), \mathrm{PSL}_2(\mathbb{R}), \mathrm{Isom}(\mathbb{H})$ are topological groups.

2. The actions of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ and $\mathbb{R} \cup \{\infty\}$ are continuous.

Definition 2.1.2. Let

$$S\mathbb{H} \coloneqq \{(z, \alpha) \mid z \in \mathbb{H}, \alpha \in \mathbb{C}, |\alpha| = \Im(z)\}$$

be the unit tangent bundle of $\mathbb{H}$, which is homeomorphic to $\mathbb{H} \times S^1$.

Definition 2.1.3. For every $g \in \mathrm{PSL}_2(\mathbb{R})$ and $(W, \alpha) \in S\mathbb{H}$, denote $T_g \cdot (w, \alpha) = (T_g(w), D(T_g)(w))$.

Definition 2.1.4 (Sharply Transitive Action). A group action is called *sharply transitive* if its transitive and the stabiliser of every element is trivial.

Lemma 2.1.5. *1. The map* $\mathrm{PSL}_2(\mathbb{R}) \times S\mathbb{H} \to S\mathbb{H}$ *is a group action.*

2. $\mathrm{PSL}_2(\mathbb{R})$ *acts sharply transitive on* $S\mathbb{H}$.

3. *The map* $g \mapsto T_g((i,i))$ *is a homeomorphism of* $\mathrm{PSL}_2(\mathbb{R})$ *and* $S\mathbb{H}$.

Proof. 1. Let $(w,\alpha) \in S\mathbb{H}$ and $g,h \in \mathrm{PSL}_2(\mathbb{R})$. We first show that $g\cdot(w,\alpha) \in S\mathbb{H}$. It holds that

$$\Im\left(T_g(w)\right) = \left|\frac{\mathrm{d}T_g}{\mathrm{d}z}(w)\right| \cdot \Im(w),$$

so

$$\left|DT_g|_{w(\alpha)}\right| = \left|\Im\left(T_g(w)\right)\right|.$$

We now have to check that this is an action. It holds that

$$\begin{aligned}(gh)\cdot(w,\alpha) &= \left(T_{gh}(\alpha), DT_{gh}(w)\alpha\right)\\ &= \left(T_g(T_h(w)), DT_g(T_h(w))\alpha\right)\\ &= g\cdot(h\cdot(w,\alpha)).\end{aligned}$$

2. Let $(w,\alpha) \in S\mathbb{H}$. It is enough to show that there exists a unique $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $g\cdot(i,i) = (w,\alpha)$. Recall that geodesic lines in $\mathbb{H}$ are oriented generalised semicircles orthogonal to the real axis. Hence there exists a unique geodesic $\ell\colon \mathbb{R} \to \mathbb{H}$ which passes through w and whose derivative at w is α. Let $\gamma\colon \mathbb{R} \to \mathbb{H}$ be the geodesic given by $\gamma(t) = ie^t$. Since T_g sends geodesics to geodesics, it must send i to w, send γ to ℓ, and respect the orientation of γ and ℓ. There exists a unique such g.

3. Prove this as an exercise. ■

We remind that $\mathrm{PSL}_2(\mathbb{R})$ is a topological group homeomorphic to $\mathbb{H}\times S^1$.

Definition 2.1.6 (Fuchsian Group). A subgroup of $\mathrm{PSL}_2(\mathbb{R})$ is called a *Fuchsian group* if it is discrete.

Example 2.1.7. $\mathrm{PSL}_2(\mathbb{Z})$ is a Fuchsian group.

Definition 2.1.8. Let X be a metric space and let $G \leq \mathrm{Isom}(X)$.

1. A *multiset* M of subsets of X is called locally finite if for every compact subset $K \subseteq X$, the multiset $[K\cap A | A \in M]$ is finite.

2. We say that G *acts properly discontinuously* on X if for every $x \in X$ the multiset $[\{gx\} | g \in G]$ is locally finite.

Exercise 3. Let G be a group which acts on a metric space X by isometries. Prove that the following conditions are equivalent.

1. $G \curvearrowright X$ is properly discontinuous.

2. Every G-orbit of X is discrete, and the stabiliser of each point is finite.

3. For every sequence $(g_n)_{n\in\mathbb{N}} \subseteq G$ of distinct elements of G and every $x \in X$ it holds that $\lim_{n\to\infty} g_n x \neq x$.

4. For every $x \in X$ there exists an open neighbourhood V of x such that the set $\{g \in G \mid gV \cap V \neq \varnothing\}$ is finite.

Example 2.1.9. $\mathrm{PSL}_2(\mathbb{Z})$ is discrete and acts continuously on $\partial\mathbb{H} = \mathbb{R}\cup\{\infty\}$. The orbit of 0 under $\mathrm{PSL}_2(\mathbb{Z})$ is $\mathbb{Q}$ which is not a discrete subset. Hence the action is not properly discontinuous.

Lemma 2.1.10. *For every $z \in \mathbb{H}$ the stabiliser $\mathrm{stab}_{\mathrm{PSL}_2(\mathbb{R})}(z)$ is compact.*

Proof. Since the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$ is continuous and transitive, it's enough to check the claim for a single point, say $z = i$.

Let $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. Assume

$$T_g(i) = \frac{a_i + b}{c_i + d} = i.$$

Then $ai + b = -c + di$ implies $a = d$ and $b = -c$. Then

$$\mathrm{stab}_{\mathrm{PSL}_2(\mathbb{R})}(i) = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \,\middle|\, {\scriptstyle a^2+b^2=1 \atop a,b\in\mathbb{R}} \right\}.$$

∎

Lemma 2.1.11. *Let $w \in \mathbb{H}$ and let $K \subseteq \mathbb{H}$ be compact. Then*

$$\{g \in \mathrm{PSL}_2(\mathbb{R}) \mid T_g(w) \in K\}$$

is compact.

Proof. The hyperbolic and Euclidean topologies on $\mathbb{H}$ are equal. Define a map

$$\begin{aligned} \rho \colon K &\to \mathrm{PSL}_2(\mathbb{R}) \\ z &\mapsto g_z \coloneqq \begin{bmatrix} a_z & b_z \\ 0 & a_z^{-1} \end{bmatrix} \end{aligned}$$

with

$$\begin{aligned} a_z &\coloneqq \sqrt{\frac{\Im(z)}{\Im(w)}} \\ b_z &\coloneqq -a_z^{-1}\Re(z) - a_z\Re(w). \end{aligned}$$

For every $z \in K$

$$\begin{aligned} T_{g_z}(w) &= a_z^2 w + b_z a_z w \\ &= \frac{\Im(z)}{\Im(w)} w + \Re(z) - \frac{\Im(z)}{\Im(w)}\Re(w) \\ &= z. \end{aligned}$$

It's clear from definition that ρ is continuous, so $M \coloneqq \mathrm{Im}(\rho)$ is compact. We get that

$$\{g \in \mathrm{PSL}_2(\mathbb{R}) \mid gw \in K\} = M \cdot \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(w)$$

where the last expression is the product of two compact subsets of $\mathrm{PSL}_2(\mathbb{R})$. A product of compact subsets of a topological group is compact, hence the result. ■

Theorem 2.1.12. *A subgroup $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ is discrete if and only if it acts properly discontinuously on $\mathbb{H}$.*

Proof.

- Assume first that Γ is discrete. If $K \subseteq \mathbb{H}$ is compact and $w \in \mathbb{H}$, the previous lemma tells us that $\{g \in \Gamma \mid gw \in K\}$ is the intersection of a discrete subset with a compact subset. Such an intersection is finite.
- Assume that Γ is not discrete. Then there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \Gamma$ of distinct elements, which converges to the identity. Let $z \in \mathbb{H}$. Then $\lim_{n\to\infty} g_n z = z$ so the Γ-action is not properly discontinuous. ■

Corollary 2.1.13. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$. Then Γ is discrete if and only if every Γ-orbit is discrete.*

Proof. The only if part is clear. For the other direction it is enough to show that if every orbit is discrete then the stabiliser of every element is finite.

Let $z \in \mathbb{H}$, we know $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})}(z)$ is compact, so if $\mathrm{Stab}_\Gamma(z)$ is not finite, it is not discrete. Thus there exists a sequence $(g_n)_{n\in\mathbb{N}} \subseteq \mathrm{Stab}_\Gamma(z)$ of distinct elements, which converges to the identity. Every element of $\mathrm{PSL}_2(\mathbb{R})$ stabilises at most one element of $\mathbb{H}$. So, there exists $w \in \mathbb{H}$ which is not fixed by any g_n. Since $g_n \xrightarrow{n\to\infty} \mathrm{id}$ it holds that $g_n w \xrightarrow{n\to\infty} w$ so the orbit of w under Γ is not discrete, which contradicts the assumption. ■

Definition 2.1.14. Let $g \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$.

1. *The trace of* g is $\mathrm{tr}(g) = |a + d|$.
2. g is called *elliptic* if $\mathrm{tr}(g) < 2$.
3. g is called *parabolic* if $\mathrm{tr}(g) = 2$.
4. g is called *hyperbolic* if $\mathrm{tr}(g) > 2$.

Lemma 2.1.15.

1. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ is elliptic, it is conjugate to $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ for some $\alpha \in \mathbb{R}$.*
2. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ is parabolic, it is conjugate to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ or to $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1}$.*
3. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic, it is conjugate to $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for some $\lambda > 0$.*

Remark 2.1.16. If we have a discrete group Γ with an elliptic/parabolic/hyperbolic element, we can assume that the element is of the form in 2.1.15 by conjugating Γ by the appropriate elements of $\mathrm{PSL}_2(\mathbb{R})$.

Notation 2.1.17. Let $g \in \mathrm{PSL}_2(\mathbb{R})$. We denote

$$\mathrm{Fix}(g) := \left\{ z \in \tilde{\mathbb{H}} \,\middle|\, T_g(z) = z \right\}$$

where $\tilde{\mathbb{H}} := \mathbb{H} \cup \partial\mathbb{H}$.

Lemma 2.1.18. *1. If* $g = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$ *for* $\alpha \in (0, \pi)$ *then* $\mathrm{Fix}(g) = i$.

2. *If* $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *then* $\mathrm{Fix}(g) = \{\infty\}$.
3. *If* $g = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ *for* $\lambda > 1$ *then* $\mathrm{Fix}(g) = \{0, \infty\}$. *Also, g preserves the geodesic* $\{iy \mid y > 0\}$.

Corollary 2.1.19. *1. If $g \in \mathrm{PSL}_2(\mathbb{R})$ is elliptic, $\mathrm{Fix}(g) = \{z\}$ is a unique point $z \in \mathbb{H}$.*

2. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ is parabolic, $\mathrm{Fix}(g) = \{z\}$ is a unique point $z \in \partial\mathbb{H}$.*
3. *If $g \in \mathrm{PSL}_2(\mathbb{R})$ is hyperbolic, then $\mathrm{Fix}(g) = \{z, w\}$ where $z \neq w$ and $z, w \in \partial\mathbb{H}$. Moreover, g preserves the unique geodesic through z and w. This geodesic is called* the axis of g.

Exercise 4. If $g \in \mathrm{PSL}_2(\mathbb{R})$ and $\langle g \rangle$ has an orbit of size 2 in $\tilde{\mathbb{H}}$, then g is elliptic of order 2.

Solution. Orbits under $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ are orbits under $z \mapsto z + 1$ which are infinite. So are orbits under $z \mapsto \lambda^2 z$. Orbits under $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ can be of order 2 if and only if $\alpha = \pm\pi$, from which g is of order 2.

Lemma 2.1.20. *Let $G \in$ **Grp** and let $X \in G-$**Set**. If $g, h \in G$ commute then*

$$g(\mathrm{Fix}(h)) = \mathrm{Fix}(h).$$

Proof. Let $x \in \mathrm{Fix}(h)$. Then $h(gx) = (gh)x = gx$ so $gx \in \mathrm{Fix}(h)$. Hence

$$g\mathrm{Fix}(h) \subseteq \mathrm{Fix}(h).$$

Similarly

$$g^{-1}\mathrm{Fix}(h) \subseteq \mathrm{Fix}(h)$$

by looking at g^{-1}, then

$$g(\mathrm{Fix}(h)) = \mathrm{Fix}(h).$$

■

Lemma 2.1.21. *The following hold, where isomorphisms are those of topological groups.*

1. *For every $\alpha \in (0, \pi)$,*
$$\mathrm{Cent}_{\mathrm{PSL}_2(\mathbb{R})}\left(\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}\right) = \left\{\begin{bmatrix}\cos\beta & \sin\beta\\ -\sin\beta & \cos\beta\end{bmatrix} \;\middle|\; \beta \in [0, \pi]\right\} \cong S^1.$$

2. *It holds that*
$$\mathrm{Cent}_{\mathrm{PSL}_2(\mathbb{R})}\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\right) = \left\{\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix} \;\middle|\; a \in \mathbb{R}\right\} \cong (\mathbb{R}, +)\,.$$

3. *For every $\lambda > 0$ different than 1 it holds that*
$$\mathrm{Cent}_{\mathrm{PSL}_2(\mathbb{R})}\left(\begin{bmatrix}\lambda & \\ & \lambda^{-1}\end{bmatrix}\right) \eta > 0 \cong (\mathbb{R}, +)\,,$$
where the isomorphism is given by $\ln$.

Remark 2.1.22. 1. In particular it follows from the lemma that two non-identity elements g, g' in $\mathrm{PSL}_2(\mathbb{R})$ commute if and only if $\mathrm{Fix}(g) = \mathrm{Fix}(g')$.

2. If Γ is a Fuchsian group, then the centraliser of every element $g \in \Gamma$ is cyclic and it is finite if and only if g is elliptic.

3. An abelian Fuchsian group is cyclic.

Example 2.1.23. A Fuchsian group does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

Lemma 2.1.24. *If $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ is non-abelian then Γ contains a non-elliptic element different from the identity.*

Proof. We use the disc model $\mathbb{U}$. Recall that we have an isomorphism $\rho\colon \mathrm{Isom}(\mathbb{H}) \to \mathrm{Isom}(\mathbb{U})$ which is given by conjugation, so it preserves traces.

Let $g, h \in \Gamma$ be non-commuting elements. We show that if g is elliptic then $[g, h]$ is not elliptic. We can assume that
$$\rho(g) \coloneqq \begin{bmatrix}a & 0\\ 0 & \bar{a}\end{bmatrix} \in \Gamma$$
and
$$\rho(h) \coloneqq \begin{bmatrix}b & c\\ \bar{c} & \bar{b}\end{bmatrix} \in \Gamma$$
where $\Im(a) \neq 0$ and $c \neq 0$ since g, h don't commute. Then
$$\rho([g, h]) = \begin{bmatrix}|b|^2 - a^2|c|^2 & *\\ * & |b|^2 - \bar{a}^2|c|^2\end{bmatrix}.$$

So

$$\begin{aligned}
\operatorname{tr}([g,h]) &= \operatorname{tr}(\rho\,[g,h]) \\
&= 2\,|b|^2 - \left(a^2+\bar{a}^2\right)|c|^2 \\
&= 2\,|b|^2 - \left(\left(a-a^{-1}\right)^2+2\right)|c|^2 \\
&= 2\cancelto{1}{\left(|b|^2-|c|^2\right)} - \left(a-\frac{1}{a}\right)|c|^2 \\
&= 2+4\Im(a)^2\,|c|^2 \\
&> 2.
\end{aligned}$$

■

Theorem 2.1.25. *Let Γ be a non-abelian Fuchsian groups. Then*

$$N \coloneqq N_{\mathrm{PSL}_2(\mathbb{R})}(\Gamma) \coloneqq \left\{ g \in \mathrm{PSL}_2(\mathbb{R}) \;\middle|\; g\Gamma g^{-1} = \Gamma \right\}$$

is a Fuchsian group.

Proof. Assume otherwise and let $(h_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements of N which converges to the identity. Let $g \in \Gamma$. For every $n \in \mathbb{N}$ it holds that $h_n g h_n^{-1} \in \Gamma$, and it holds that $\lim_{n\to\infty} h_n g h_n^{-1} = g$. Since Γ is discrete, there exists M_g such that for every $n > M_g$ it holds that $h_n g h_n^{-1} = g$. Then h_n, g commute so $\operatorname{Fix}(g) = \operatorname{Fix}(h_n)$. Since Γ is not abelian, there exist $g_1, g_2 \in \Gamma \setminus \{\mathrm{id}\}$ which do not commute, so $\operatorname{Fix}(g_1) \neq \operatorname{Fix}(g_2)$. On the other hand, for large enough $n \in \mathbb{N}$ it holds that

$$\operatorname{Fix}(g_1) = \operatorname{Fix}(h_n) = \operatorname{Fix}(g_2),$$

a contradiction. ■

2.1.2 Elementary Subgroups

Definition 2.1.26 (Elementary Subgroup). A subgroup $\Gamma \le \mathrm{PSL}_2(\mathbb{R})$ is called *elementary* if it has a finite orbit in $\tilde{\mathbb{H}}$.

Lemma 2.1.27. *A Fuchsian group is elementary if and only if it has a cyclic subgroup of index* 2.

Proof.

- Let $\Lambda \le \Gamma$ of finite index. Then Λ is elementary if and only if Γ is. The if part is clear since abelian subgroups of $\mathrm{PSL}_2(\mathbb{R})$ have a fixed point.

  Let Γ be an elementary Fuchsian group. It is enough to prove that Γ contains an index 2 abelian subroup since abelian Fuchsian groups are cyclic. If all non-identity elements are elliptic, then Γ is abelian. Assume Γ contains a hyperbolic element g. Assume by conjugation that $\operatorname{Fix}(g) = \{0,\infty\}$. Then $\{0,\infty\}$ are the only finite orbits of $\langle g\rangle$ in $\tilde{\mathbb{H}}$ so Γ preserves $\{0,\infty\}$.

Let $h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so that $T_h(z) = -\frac{1}{z}$. Recall that $k \in \mathrm{PSL}_2(\mathbb{R})$ commutes with g if and only if $\mathrm{Fix}(k) = \mathrm{Fix}(g)$. Thus

$$\Gamma \leq \mathrm{Cent}_{\mathrm{PSL}_2(\mathbb{R})}(g) \rtimes \langle h \rangle .$$

Indeed, h is of order 2 and conjugation by h sends g to g^{-1}.

It follows that $\Gamma \cap \mathrm{Cent}_{\mathrm{PSL}_2(\mathbb{R})}(g)$ is an abelian subgroup of Γ of index at most 2.

- Assume now that Γ contains a parabolic element g but no hyperbolic elements. $\langle g \rangle$ has a unique finite orbit in $\tilde{\mathbb{H}}$ and it is $\mathrm{Fix}(g)$. Thus every element of Γ fixes $\mathrm{Fix}(g) \subseteq \partial\mathbb{H}$, so every element is either either parabolic with the same fixed point, hyperbolic, or elliptic. It cannot be hyperbolic by assumption, and cannot be elliptic since it does not preserve a point in $\mathbb{H}$.

  We get that for non-identity elements $k \in \Gamma$ it holds that $\mathrm{Fix}(h) = \mathrm{Fix}(g)$ so Γ is abelian. ∎

Exercise 5. Show that if $g, h \in \mathrm{PSL}_2(\mathbb{R})$, g is hyperbolic, $[g, h] \neq \mathrm{id}$ and $\langle g, h \rangle$ is a Fuchsian elementary group then $\langle g, h \rangle \cong \mathbb{Z} \rtimes \mathbb{Z}_2$ and $h(\mathrm{Fix}(g)) = \mathrm{Fix}(g)$.

Lemma 2.1.28. *Any non-elementary group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ contains a hyperbolic element.*

Proof. Since Γ is not abelian, it contains a non-elliptic element $g \in \Gamma \setminus \{\mathrm{id}\}$. We can assume by conjugation that $g = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Since Γ is not abelian, it contains an element $h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ which does not commute with g. Hence $c \neq 0$ or $b \neq 0$. But,

$$\mathrm{tr}(g^n h) = \mathrm{tr} \begin{bmatrix} a + cn & b + dn \\ c & d \end{bmatrix} = |a + cn + d|$$
$$\mathrm{tr}(h g^n) = \mathrm{tr} \begin{bmatrix} a & b + an \\ c & d + bn \end{bmatrix} = |a + d + bn|$$

where both terms go to ∞ as $n \to \infty$, so at some point $g^n h$ and $h g^n$ are hyperbolic. ∎

Exercise 6. Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be non-elementary and let $X \subseteq \partial\mathbb{H}$ be finite. Then there's a hyperbolic $g \in \Gamma$ such that $\mathrm{Fix}(g) \cap \Gamma = \varnothing$.

Theorem 2.1.29. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be non-elementary and assume that Γ does not contain an elliptic element. Then Γ is discrete.*

Proof. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of elements of Γ which converges to id. Denote $g_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}$. We have to show that $g_n = \mathrm{id}$ for all n large enough. By the previous exercise, it is enough to show that for every hyperbolic $h \in \Gamma$, if n is large enough (depending on h) it holds that h, g_n have a common fixed point.

Let $h \in \Gamma$ be a hyperbolic element. We can assume that $h = \begin{bmatrix} u & \\ & u^{-1} \end{bmatrix}$ for some $u > 1$. If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$$[h, g] = \begin{bmatrix} ad - bcu^2 & ab\left(u^2 - 1\right) \\ cd\left(u^{-2} - 1\right) & ad - bcu^{-2} \end{bmatrix}.$$

So,

$$\mathrm{tr}\left([h, g]\right) = 2\left(ad - bc\right) - bc\left(u - u^{-1}\right)^2 = 2 - bc\left(u - u^{-1}\right)^2.$$

Also

$$\begin{aligned} \mathrm{tr}\left([h, [h, g]]\right) &= 2 - abcd\left(u^2 - 1\right)\left(u^{-2} - 1\right)\left(u - u^{-1}\right) \\ &= 2 + abcd\left|\left(u^2 - 1\right)\left(u^{-2} - 1\right)\left(u - u^{-1}\right)^2\right|. \end{aligned}$$

Adding these two equations gives that if $\mathrm{tr}\left([h, g]\right) \geq 2$ and $\mathrm{tr}\left([h, [h, g]]\right) \geq 2$ then $bc \leq 0$ and either $bc = 0$ or $ad \leq 0$. Applying this to the sequence $(g_n)_{n\in\mathbb{N}}$ and noting that $\lim_{n\to\infty} a_n d_n = 1$ for large enough n we get $b_n = c_n = 0$. If $b_n = 0$ then 0 is a fixed point of g_n and if $c_n = 0$ then ∞ is a fixed point of g_n. In either case, g_n and h have a common fixed point. ■

Theorem 2.1.30 (Jorgensen Inequality). *Let $g, h \in \mathrm{PSL}_2(\mathbb{R})$ and assume that $\langle g, h\rangle$ is a non-elementary discrete group. Then*

$$\left|\mathrm{tr}\left(g\right)^2 - 4\right| + \left|\mathrm{tr}\left[g, h\right] - 2\right| \geq 1.$$

Theorem 2.1.31. *A non-elementary group $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ is discrete if and only if for every $g, h \in \Gamma$ the group $\langle g, h\rangle$ is discrete.*

Theorem 2.1.32. *The only if part is clear. Assume therefore that $\langle g, h\rangle$ is discrete for every $g, h \in \Gamma$ and assume towards a contradiction that there's $(g_n)_{n\in\mathbb{N}} \subseteq \Gamma \setminus \{\mathrm{id}\}$ such that $g_n \xrightarrow{n\to\infty} \mathrm{id}$.*

We proved that Γ contains hyperbolic elements h_1, h_2 such that $\mathrm{Fix}\left(h_1\right) \cap \mathrm{Fix}\left(h_2\right) = \varnothing$. The only element of $\mathrm{PSL}_2(\mathbb{R})$ which fixes 4 points is the identity. Thus it is enough to prove that for every hyperbolic element h there exists M_h such that for every $n \geq M_h$ it holds that $\mathrm{Fix}\left(h\right) = \mathrm{Fix}\left(g_n\right)$.

Let h be a hyperbolic element. Choose M_h large enough such that for every $n \geq M_h$ it holds that

$$\left|\mathrm{tr}\left(g_n\right)^2 - 4\right| + \left|\mathrm{tr}\left[h, g_n\right] - 2\right| < 1$$

and the order of g_n is not 2. By Jorgensen inequality, $\langle g, h\rangle$ is elementary. Since the only finite orbits of $\langle h\rangle$ are contained in $\mathrm{Fix}\,(h)$ *we get that* $\mathrm{Fix}\,(g) = g_n\,(\mathrm{Fix}\,(h))$*. Since* $|\mathrm{Fix}\,(h)| = 2$*, either* $\mathrm{Fix}\,(g_n) = \mathrm{Fix}\,(h)$ *or g_n switches the two elements in* $\mathrm{Fix}\,(h)$*. The latter is impossible since the order of g_n is not 2, and thus there are no $\langle g_n\rangle$-orbits of size 2.*

Lemma 2.1.33. *Let $g, h \in \mathrm{PSL}_2\,(\mathbb{R})$. Define $g_1 = g$ and for every $n \geq 1$ define $g_N \coloneqq g_{n-1}hg_{n-1}^{-1}$. If for some $n \geq 0$ it holds that $gh = h$, then $\langle g, h\rangle$ is elementary and $g_2 = h$.*

Proof. The claim is clear if $g_0 = h$ so assume that $g_n = h$ for some $n \geq 1$. We claim that for every $k \in [n]$ it holds that $\mathrm{Fix}\,(h) = \mathrm{Fix}\,(g_k)$. Indeed, assume that $k \in [n]$ and $\mathrm{Fix}\,(h) = \mathrm{Fix}\,(g_k)$. The claim follows if $|\mathrm{Fix}\,(h)| = 1$ since $|\mathrm{Fix}\,(g_{k-1})| = |\mathrm{Fix}\,(h)|$ (since g_{k-1} and h are conjugate).

If $|\mathrm{Fix}\,(h)| = 2$ then h is hyperbolic so $g_{k-1} = g_{k-2}hg_{k-2}$ is hyperbolic and thus cannot switch the two points in $\mathrm{Fix}\,(h)$. We deduce that $\mathrm{Fix}\,(h) = \mathrm{Fix}\,(g_{k-1})$ also in this case.

It follows that h and g_1 have the same fixed points so they commute and $g_2 = g_1hg_1^{-1} = h$.

Finally,

$$\mathrm{Fix}\,(h) = \mathrm{Fix}\,(g_1) = g_0\,(\mathrm{Fix}\,(h)) = g\,(\mathrm{Fix}\,(h))$$

so $\mathrm{Fix}\,(h)$ contains a $\langle h, g\rangle$-orbit, so $\langle g, h\rangle$ is elementary. ■

Lemma 2.1.34. *Let $g, h \in \mathrm{PSL}_2\,(\mathbb{R})\backslash\{\mathrm{id}\}$. Assume that $\left|\mathrm{tr}\,(g)^2 - 4\right| + |\mathrm{tr}\,[g, h] - 2| < 1$. Define $g_0 \coloneqq g$, and for every $n \geq 1$ define $g_n \coloneqq g_{n-1}hg_{n-1}^{-1}$. Then*

1. *If h is parabolic, $\lim_{n\to\infty} g_n = h$.*
2. *If h is hyperbolic $\lim_{n\to\infty} h^n g_{2n} h^{-n} = h$.*
3. *If h is elliptic, $\lim_{n\to\infty} g_n = h$.*
4. *If h is*

Proof. Denote

$$g_n = \begin{bmatrix} a_n & b_n \\ c_n & d_n \end{bmatrix}.$$

1. We can assume that $h = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $\mathrm{tr}\,([h, g]) = 2 + c_0^2$ and

   $$g_{n+1} = g_nhg_n^{-1} = \begin{bmatrix} 1 - a_nc_n & a_n^2 \\ -c_n^2 & 1 + a_nc_n \end{bmatrix}.$$

   It follows that

   (a) $|c_0| < 1$ since $\left|c_0^2\right| = |\mathrm{tr}\,[h, g] - 2| < 1$. We know $c_n = -\left(c_0\right)^{2^n}$ and $|a_{n+1}| \leq 1 + |a_n|$. So $|a_n| \leq n + |a_0|$.

(b) Since $|c_0| < 1$ and $c_n = -(c_0)^{2^n}$ it holds that $c_n \to 0$. By the bound on a_n we get also $a_n c_n \to 0$. By the formula for a_{n+1} we get $a_n \to 1$. Then $g_n \to h$.

2. We can assume that
$$h = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$$
with $u > 1$. Then
$$\mu := \left|\operatorname{tr}(g)^2 - 4\right| + |\operatorname{tr}[h,g] - 2| = (1 + |bc|)\left(u - u^{-1}\right)^2 < 1,$$
and
$$g_{n+1} = g_n h g_n^{-1} = \begin{bmatrix} a_n d_n u - b_n c_n u^{-1} & a_n b_n \left(u^{-1} - u\right) \\ c_n d_N \left(u - u^{-1}\right) & a_n d_n u^{-1} - b_n c_n u \end{bmatrix}.$$
We deduce the following.

(a)
$$b_{n+1} c_{n+1} = a_n b_n c_n d_n \left(u - u^{-1}\right)\left(u^{-1} - u\right) = -b_n c_n \left(1 + b_n c_n\right)\left(u - \frac{1}{u}\right)^2,$$
so
$$|b_n - c_n| \leq \mu^n |b_0 c_0| \xrightarrow{n\to\infty} 0.$$

(b)
$$a_n d_n = 1 + b_n c_n \xrightarrow{n\to\infty} 1$$
so $a_n \to u$ and $d_n \to u^{-1}$.

(c)
$$\left|\frac{b_{n+1}}{b_n}\right| = \left|a_n \left(u - u^{-1}\right)\right| \xrightarrow{n\to\infty} \left|u\left(u - u^{-1}\right)\right| \leq \sqrt{\mu}\cdot|u|.$$
Hence
$$\left|\frac{b_{n+1}}{u^{n+1}}\right| \leq \sqrt{\mu}\left|\frac{b_n}{u^n}\right|$$
so $\frac{b_n}{\mu^n} \xrightarrow{n\to\infty} 0$.

(d) Similarly,
$$\left|\frac{c_{n+1}}{c_n}\right| = \left|d_n\left(u - u^{-1}\right)\right| \xrightarrow{n\to\infty} \left|u^{-1}\left(u - u^{-1}\right)\right| \leq \sqrt{\mu}u^{-1} \leq \left|c_{n+1}u^{n+1}\right| \leq \sqrt{\mu}\left|c_n u^n\right|,$$
so $c_n u_N \xrightarrow{n\to\infty} 0$.

(e) We get by the above parts that
$$h^n g_{2n} h^{-n} = \begin{bmatrix} a_{2n} & b_{2n} u^{-2} \\ c_{2n} u^{2n} & d_n \end{bmatrix} \xrightarrow{n\to\infty} h.$$

3. We use the disc model and regard $g, h \in \mathrm{Isom}(\mathbb{U})$. We can assume that $h = \begin{bmatrix} u & 0 \\ 0 & u^{-1} \end{bmatrix}$ where $u \in \mathbb{C}$ is such that $|u| = 1$.

   Let

$$\mu := \left|\mathrm{tr}\,(g)^2 - 4\right| + |\mathrm{tr}\,[g,h] - 2| = (1 + |bc|)\left(u - u^{-1}\right)^2 < 1$$

   and let

$$g_{n+1} = g_n h g_n^{-1} = \begin{bmatrix} a_n d_n u - b_n c_n u^{-1} & a_n b_n \left(u^{-1} - u\right) \\ c_n d_N \left(u - u^{-1}\right) & a_n d_n u^{-1} - b_n c_n u \end{bmatrix}.$$

   as above. We deduce that following.

   (a)

$$b_{n+1} c_{n+1} = -b_n c_n \left(1 + b_n c_n\right) \left(u - \frac{1}{u}\right)^2$$

   so $|b_n c_n| \le \mu^n |b_0 c_0|$ and $b_n c_n \xrightarrow{n \to \infty} 0$, as before.

   (b) $a_n d_N = 1 + b_n c_n \xrightarrow{n \to \infty} 1$, so $a_n \xrightarrow{n \to \infty} u$ and $d_n \xrightarrow{n \to \infty} u^{-1}$.

   (c)

$$\left|\frac{b_{n+1}}{b_n}\right| = \left|a_n \left(u - u^{-1}\right)\right| \xrightarrow{n \to \infty} \left|u \left(u - u^{-1}\right)\right| \le \sqrt{\mu} \cdot |u| = \sqrt{\mu} < 1,$$

   so $b_n \xrightarrow{n \to \infty} 0$.

   (d)

$$\left|\frac{c_{n+1}}{c_n}\right| = \left|d_n \left(u - u^{-1}\right)\right| \xrightarrow{n \to \infty} \left|u^{-1} \left(u - u^{-1}\right)\right| \le \sqrt{\mu} \cdot |u|^{-1} = \sqrt{\mu}$$

   so $c_n \xrightarrow{n \to \infty} 0$. ■

Proof (2.1.30). Assume

$$\left|\mathrm{tr}\,(g)^2 - 4\right| + |\mathrm{tr}\,[h,g] - 2| < 1.$$

Define $g_0 = g$ and for every $n \ge 1$ assume $g_n = g_{n-1} h g_{n-1}^{-1}$. We claim that there exists n such that $g_n = h$. If this is true, 2.1.33 implies that $\langle g, h \rangle$ is elementary.

We prove our claim. If h is parabolic or elliptic, this follows from 2.1.34 and discreteness. If g is hyperbolic, 2.1.34 and discreteness imply that that for large enough n it holds that $h^n g_{2n} h^{-n} = h$ so $g_{2n} = h$. ■

2.1.3 Fundamental Domains

Definition 2.1.35 (Fundamental Set). Let G be a group and X a G-set. A representative set for the G-orbits is called a *fundamental set*.

Definition 2.1.36 (Fundamental Domain). Let Γ be a Fuchsian group. A subset $D \subseteq \mathbb{H}$ is called a fundamental domain for Γ if the following holds.

1. D is a domain (i.e. connected & open).
2. There is a fundamental set F such that $D \subseteq F \subseteq \bar{D}$.
3. The hyperbolic area of ∂D is zero.

Lemma 2.1.37. *Let D be a fundamental domain. If $z_1, z_2 \in \bar{D}$ are in the same Γ-orbit, than $z_1, z_2 \in \partial D$.*

Proof. Part 2 of the definition implies that at least one of z_1, z_2 belongs to ∂D. Assume $z_1 \in D$. There exists a sequence $(w_n)_{n\in\mathbb{N}} \subseteq D$ which converges to z_2. If $gz_1 = z_2$ then $\left(g^{-1}w_n\right)_{n\in\mathbb{N}}$ converges to z_1. Since D is open, for large enough n, $g^{-1}w_n \in D$. But, since

$$z_1 = \lim_{n\to\infty} g^{-1}w_n \neq \lim_{n\to\infty} w_n = z_2$$

we get that $g^{-1}w_n \neq w_n$ for large enough n. This contradicts part 2 of the definition. ∎

Theorem 2.1.38. *Let F_1, F_2 be measurable fundamental sets for a Fuchsian group Γ. Then $\mathrm{h}-\mathrm{Area}\,(F_1) = \mathrm{h}-\mathrm{Area}\,(F_2)$.*

Proof. Let μ denote the hyperbolic area. We have

$$\begin{aligned}
\mu(F_1) &= \mu(F_1 \cap \mathbb{H}) \\
&= \mu\left(F_1 \cap \left[\bigcup_{g\in\Gamma} gF_2\right]\right) \\
&= \sum_{g\in\Gamma} \mu(F_1 \cap gF_2) \\
&= \sum_{g\in\Gamma} \mu\left(g^{-1}F_1 \cap F_2\right) \\
&= \sum_{g\in\Gamma} \mu(gF_1 \cap F_2) \\
&= \mu(F_2).
\end{aligned}$$

∎

Theorem 2.1.39. *Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group and let Λ be a subgroup of Γ of index m. If Γ has a measurable fundamental set F, Λ has a measurable fundamental set of measure $\mu(F) \cdot m$.*

Proof. Assume $(g_i)_{i\in[m]}$ are representatives to right cosets. Then

$$\Gamma = \bigcup_{i\in[m]} \Lambda g_i.$$

If $w, z \in F$, $h \in \Lambda$, $i, j \in [n]$ and $hg_1 z = g_2 w$ then

$$a_2^{-1} h g_1 z = w.$$

Since F is a fundamental set for Γ we get $z = w$ and $z \in \operatorname{Fix}\left(g_2^{-1} h g_1\right)$.

Clearly, $\bigcup_{i\in[m]} F g_i$ contains a fundamental set for Γ. Thus, there exists a fundamental set E of Γ such that

$$\bigsqcup_{i\in[m]} g_i \left(F \setminus \bigcup_{g\in\Gamma\setminus\{1\}} \operatorname{Fix}(g) \right) \subseteq E \subseteq \bigcup_{i\in[m]} g_i F.$$

The sets $\operatorname{Fix}(g)$ for $g \in \Gamma \setminus \{1\}$ are countable, hence E is measurable with $\mu(E) = m \cdot \mu(F)$. ■

Lemma 2.1.40. *Let D be a fundamental domain for a Fuchsian group Γ. Denote by D/Γ the quotient space after identifying points in the same orbit. We get the following commutative diagram.*

$$\begin{array}{ccc} \bar{D} & \xrightarrow{\tau} & \mathbb{H} \\ \downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\pi} \\ \bar{D}/\Gamma & \xrightarrow{\theta} & \mathbb{H}/\Gamma \end{array}$$

Then

1. *θ, τ are injective.*
2. *$\pi, \tilde{\pi}, \theta$ are surjective. Then, θ is bijective.*
3. *$\pi, \tilde{\pi}, \tau$ are continuous.*
4. *π is open.*

Proof. Every part of the proof is clear except maybe that θ is continuous and that π is open. We now have the following.

- Recall that $V \subseteq \mathbb{H}/\Gamma$ is open if and only if $\pi^{-1}(V)$ is open. If $U \subseteq \mathbb{H}$ is open,

  $$\pi^{-1}(\pi(U)) = \bigcup_{g\in\Gamma} gU$$

  is open as a union of open sets.

  Hence π is open.

- Let $V \subseteq \mathbb{H}/\Gamma$ be open. Then
$$\tilde{\pi}\left(\left(\tau^{-1}\left(\pi^{-1}\left(V\right)\right)\right) \cap \bar{D}\right).$$
Denoting $U \coloneqq \left(\tau^{-1}\left(\pi^{-1}\left(V\right)\right)\right) \cap \bar{D}$, this is open and
$$\tilde{\pi}^{-1}\left(\tilde{\pi}\left(U\right)\right) = U$$
is open, so $\tilde{\pi}\left(U\right)$ is open. ■

2.1.4 Examples

Example 2.1.41. Let $X = \mathbb{C} \setminus \{0\}$ and $g\colon X \to X$ given by $g\left(z\right) = 2z$. Then a fundamental domain for example is $\{z \in \mathbb{Z} \mid |z| \in (1,2)\}$. Then $\bar{D}/\Gamma \cong \mathbb{T}^2$ is compact.

A different fundamental domain would be the same set with the part where $x, y \geq 1$ replaced by the domain bounded by $y = e^{-x}$ and $y = \frac{1}{2}e^{-x}$. In this case $\bar{D}/\Gamma$ is non-compact.

Example 2.1.42. Let $g, h \in \mathrm{PSL}_2\left(\mathbb{Z}\right)$ be $g\left(z\right) = 2z$ and $h\left(z\right) = \frac{3z+4}{2z+3}$. Consider the domains in the figure Then
$$g\left(\mathbb{H} \setminus \bar{B}\right) = E$$
$$g^{-1}\left(\mathbb{H} \setminus \bar{E}\right) = B$$
and
$$h\left(\mathbb{H} \setminus A\right) = C$$
$$h^{-1}\left(\mathbb{H} \setminus \bar{C}\right) = A.$$
For every $k \in \Gamma \setminus \{\mathrm{id}\}$ where $\Gamma = \langle g, h\rangle$. We have $kw \notin D$. In particular, Γw is discrete, so Γ is discrete.

D is a fundamental domain for Γ.

- We showed that every orbit intersects D in at most one point.
- Assume $z \notin \bar{D}$ so that $z \in A \cup B \cup C \cup E$.
  - If $z \in B$, we have $\rho\left(z, w\right) > \rho\left(gz, w\right)$.
  - If $z \in C$, we have $\rho\left(z, w\right) > \rho\left(h^{-1}z, w\right)$.
  - If $z \in A$, we have $\rho\left(z, w\right) > \rho\left(hz, w\right)$.
  - If $z \in E$, we have $\rho\left(z, w\right) > \rho\left(g^{-1}z, w\right)$.

  By taking $z' \in \Gamma z$ that minimises the distance to w (which exists since Γ acts properly discontinuously), so we get $z' \in \bar{D}$.

  We have that $\bar{D}/\Gamma$ is a punctured torus. In particular, this has no boundary. We later construct a different fundamental domain which has a boundary.

Proposition 2.1.43. *Let ℓ be a geodesic line and let $x \in \mathbb{H}$ such that $x \notin \ell$. There exists a unique $y \in \ell$ such that $d_{\mathbb{H}}(x,y) = d_{\mathbb{H}}(x,\ell)$, and the geodesic segment $[x,y]$ is orthogonal to ℓ.*

Proof. Assume $z \in \ell$ doesn't satisfy $[x,z] \perp \ell$ and let y be such that $[x,y] \perp \ell$. By the Pythagorean theorem

$$\cosh(d_{\mathbb{H}}(x,z)) = \cosh(d_{\mathbb{H}}(x,y))\cosh(d_{\mathbb{H}})(y,z)$$

where $\cosh(d_{\mathbb{H}})(y,z) > 1$. ∎

Proposition 2.1.44. *Let ℓ, ℓ' be ultra-prallel geodeisc lines. There exists a unique $p \in \ell$ and $p' \in \ell'$ such that $d_{\mathbb{H}}(p,p') = d_{\mathbb{H}}(\ell,\ell')$.*

Proof. Existence follows from the fact that the hyperbolic distance increases near infinity, so we can restrict to compact subsets.

For uniqueness, assume there are points $q \in \ell$ and $q' \in \ell'$ not on the orthogonal line to $\mathbb{R}$ through the centre of the circle through ℓ. Let y be the centre of the circle in which the geodesic s through q, q' passes through, and let r be its Euclidean radius. Then $y^2 = r^2 + t^2 = r^2 + s^2$ and $s > t - x$ where t is the radius of the circle bounded by ℓ. ∎

Proposition 2.1.45. *Let $g \in \mathrm{PSL}_2(\mathbb{R})$ be hyperbolic and let ℓ, ℓ' be ultra-parallel geodesics such that $g(\ell) = \ell$, such that $g(z) = kz$ for $k > 1$ and that*

$$\begin{aligned} \ell &= \{z \in \mathbb{H} \mid |z| = r\} \\ \ell' &= \{z \in \mathbb{H} \mid |z| = kr\}. \end{aligned}$$

Let p, p' be the points minimising the distance between ℓ, ℓ' and let w be the mid-point of $[p,p']$, then $w = ir\sqrt{k}$. Let

$$\begin{aligned} A &:= \{z \in \mathbb{H} \mid |z| < r\} \\ B &:= \{z \in \mathbb{H} \mid |z| \in (r, kr)\} \\ C &:= \{z \in \mathbb{H} \mid |z| > kr\}. \end{aligned}$$

Then for every $z \in A$ it holds that

$$d_{\mathbb{H}}(z,w) > d(gz,w).$$

Proof. First consider a few cases.

- If $z = iy$ and $ky \le r\sqrt{k}$, the claim is clear.
- If $z = iy$ and $ky > r\sqrt{k}$, then

$$d_{\mathbb{H}}(z,w) = \ln\left(\frac{r\sqrt{k}}{y}\right) > \ln\sqrt{k} > \ln\left(\frac{ky}{r\sqrt{k}}\right) = \rho(gz,w).$$

Let $\gamma = [z, z']$, then $g\gamma = [gz, gz']$ because g sends geodesics to geodesics. Then by the Pythagorean theorem

$$\begin{aligned}\cosh d_{\mathbb{H}}(z,w) &= \cosh d_{\mathbb{H}}(z,z') \cosh d_{\mathbb{H}}(z',w) \\ &> \cosh d_{\mathbb{H}}(gz,gz') \cdot \cosh d_{\mathbb{H}}(gz',w) \\ &= \cosh d_{\mathbb{H}}(gz,w). \end{aligned}$$

■

Lemma 2.1.46 (Ping-Pong Lemma). *Let G be a group which acts on a set X. Let $A+, A^-, B^+, B^-$ be disjoint subsets of X and let $g, h \in G$ such that the following hold.*

1. $C \coloneqq X \setminus A^+ \cup A^- \cup B^+ \cup B^- \neq \varnothing$.
2. *For all $n \in \mathbb{N}_+$ it holds that*
$$\begin{aligned} g^n\left(X \setminus A^-\right) &\subseteq A^+ \\ g^{-n}\left(X \setminus A^+\right) &\subseteq A^-. \end{aligned}$$
3. *For all $n \in \mathbb{N}_+$ it holds that*
$$\begin{aligned} h^n\left(X \setminus B^-\right) &\subseteq B^+ \\ h^{-n}\left(X \setminus B^+\right) &\subseteq B^-. \end{aligned}$$

Then $\langle g, h\rangle$ is a non-abelian free group. Moreover, if $f \in \langle g, h\rangle$ is a non-identity element and $c \in C$ then

1. *If f ends with g as a reduced word then $fc \in A^+$.*
2. *If f ends with g^{-1} as a reduced word then $fc \in A^-$.*
3. *If f ends with h as a reduced word then $fc \in B^+$.*
4. *If f ends with h^{-1} as a reduced word then $fc \in B^-$.*

Proof. Let $w(x, y)$ be a reduced word different than 1. We will prove by induction on the length of w that $w(g, h) = f$ satisfies the properties.

In the base case $w \in \left\{x, x^{-1}, y, y^{-1}\right\}$ and the claim follows from the assumption.

Assume now that $w(x, y) = zw'$ where $z \in \left\{x, x^{-1}, y, y^{-1}\right\}$. Assume WLOG that $z = x$. Since w is reduced, w' does not end with x^{-1}. By induction we have $w'(g, h)c \in A^+ \cup B^+ \cup B^{-1}$, which is disjoint from A^{-1}, so by assumption $w(g, h)c = g(w'(g, h)c) \in A^+$. ■

Example 2.1.47. Let $g, h \in \mathrm{PSL}_2(\mathbb{Z})$, let $g(z) = 2z$ and let $h(z) = \frac{3z+4}{2z+3}$. Consider figure with A^-, A^+, B^-, B^+ closed. g, h satisfy the assumptions of the previous lemma, hence $\langle g, h\rangle$ is a free group and every orbit intersects C in at most one point.

We claim C is in fact a fundamental domain. Let $p \in \ell$, $p' \in \ell'$, $q \in m$ and $q' \in m'$ such that $d_{\mathbb{H}}(\ell, \ell') = d_{\mathbb{H}}(p, p')$ and $d_{\mathbb{H}}(m, m') = d_{\mathbb{H}}(q, q')$. Denote $w = i\sqrt{2}$. Then w is the mid-point of both $[p, p'], [q, q']$.

By the lemma if $x \in \bar{C}$ then x is in the interior of $A \cup A^{-} \cup B \cup B^{-}$. Hence there exists $f \in \{g, g^{-1}, h, h^{-1}\}$ such that $d(x, m) > d(fx, m)$. Since the action is properly discontinuous, there exists $y \in Gx$ such that $d(m, y) = d(Gx, m)$ so y must belong to $\bar{C}$.

Let $t \in \ell$. Then $t, 2t$ are two points on the boundary of C which are in the same G-orbit. We claim that $G \cdot t \cap \bar{C} = \{t, 2t\}$. Otherwise, there's $f \in G \setminus \{g\}$ such that $f(\bar{C}) \ni 2t$ so $fC \cap C \neq \varnothing$, a contradiction.

Definition 2.1.48 (Locally Finite Fundamental Domain). Let $\Gamma \leq \mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian group. A fundamental domain D for Γ is called locally-finite if every compact subset of $\mathbb{H}$ meets only finitely-many of $\{g\bar{D} \mid g \in \Gamma\}$.

Lemma 2.1.49. *D is locally finite if and only if for every sequence $(z_n)_{n\in\mathbb{N}}$ of elements of D and every sequence $(g_n)_{n\in\mathbb{N}}$ of distinct elements of Γ, the sequence $(g_n z_n)_{n\in\mathbb{N}}$ does not converge.*

Proof. Assume there are sequences $(z_n)_{n\ in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ as above such that $\lim_{n\to\infty} g_n z_N = w$. Then any compact neighbourhood of w meets infinitely many $g_n D$ so D isn't locally finite.

Assume now that D is not locally finite. Then there exist a compact subset $K \subseteq D$ and a sequence $(g_n)_{n\in\mathbb{N}}$ of distinct elements of Γ such that for every n, $g_n\bar{D} \cap K \neq \varnothing$. For every $n \in \mathbb{N}$ pick $z_n \in \bar{D} \cap g_n K$, so $g_n z_n \in K$. By passing to a subsequence we can assume that $(g_n z_n)_{n\in\mathbb{N}}$ converges to a point $w \in K$. Since every z_n belongs to $\bar{D}$ we can replace each z_n with $z_n' \in D$ such that $\lim_{n\to\infty} g_n z_n = w$. ■

Theorem 2.1.50. *Let Γ be a Fuchsian group and let D be a fundamental domain for Γ. Consider the following diagram.*

$$\begin{tikzcd} \bar{D} \arrow[r, "\tau"] \arrow[d, "\tilde{\pi}"'] & \mathbb{H} \arrow[d, "\pi"] \\ \bar{D}/\Gamma \arrow[r, "\theta"] & \mathbb{H}/\Gamma \end{tikzcd}$$

Here θ is a homeomorphism iff D is locally finite.

Proof. We know that θ is bijective and continuous. To show θ is a homeomorphism it is therefore enough to show that if $A \subseteq \bar{D}/\Gamma$ is closed then $\theta(A)$ is closed. Since $\mathbb{H}/\Gamma$ is endowed with the quotient topology, a set $C \subseteq \mathbb{H}/\Gamma$ is closed if and only if its inverse image is closed. In our case, this is true if and only if for every sequence $(g_n)_{n\in\mathbb{N}} \subseteq \Gamma$ and for every $(z_n)_{n\in\mathbb{N}} \subseteq \pi^{-1}(C)$, if $(g_n z_n)_{n\in\mathbb{N}}$ converge then $\lim_{n\to\infty} g_n z_n \in \pi^{-1}(c)$. By the lemma, this is equivalent to D being locally-finite.

Let $A \subseteq \bar{D}\big/\Gamma$ and denote $B = \tilde{\pi}^{-1}(A)$ and $C = \theta(A)$. Then C is closed if and only if for every sequence $(g_n)_{n\in\mathbb{N}}$ of elements of Γ and every sequence $(z_n)_{n\in\mathbb{N}}$ of elements of B such that $(g_n z_n)_{n\in\mathbb{N}}$ converges to an element $w \in B = \pi^{-1}(C) \cap \bar{D}$.

Assume that D is locally finite and $A \leq \bar{D}\big/\Gamma$ is closed. Then $B = \tilde{\pi}^{-1}(A)$ is closed in $\bar{D}$ and thus in $\mathbb{H}$. Let $(z_n)_{n\in\mathbb{N}}$ be a sequence of elements of B and $(g_n)_{n\in\mathbb{N}}$ a sequence of elements of Γ such that $\lim_{n\to\infty} g_n z_n = w \in \bar{D}$. Since D is locally finite, by passing to a subsequence we may assume that there's $g \in \Gamma$ such that $g_n = g$ for every $n \in \mathbb{N}$. Then $\lim_{n\to\infty} g z_n = w$ so $\lim_{n\to\infty} z_n = g^{-1}w \in B$ where this belongs to B since B is closed. Because B is invariant under Γ we get $w = g\left(g^{-1}w\right) \in B$.

Assume now that D is not locally finite. Let $(g_n)_{n\in\mathbb{N}}$ be a sequence of distinct elements of Γ and $(z_n)_{n\in\ltimes}$ a sequence of elements of D such that $\lim_{n\to\infty} g_n z_N = w \in \bar{D}$. Since the action is properly discontinuous. By passing to a subsequence we may assume that $(z_n)_{n\in\mathbb{N}}$ are distinct and that $w \notin (z_n)_{n\in\mathbb{N}}$. We claim that the set $B \coloneqq \{z_n \mid n \in \mathbb{N}\}$ is closed. Otherwise, by passing to a subsequence we may assume that $\lim_{n\to\infty} z_n = z$. Then $\lim_{n\to\infty} g_n z_n = w$ so $\lim_{n\to\infty} g_n^{-1} w = z$, which is impossible since the Γ-action is properly-discontinuous. Denote $A = \tilde{\pi}(B)$. Since $B \subseteq D$, we have $\tilde{\pi}^{-1}(A) = B$, so A is closed. But, $w \in \bar{D} \setminus B$ so $w \notin \pi^{-1}(\theta(A))$, so $\theta(A)$ is not closed. ■