Lecture Notes to Fuchsian Groups
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Lectures by Chen Meiri
Typed by Elad Tzorani

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Chapter 1

Preliminaries

1.1 The Hyperbolic Plane

1.1.1 The Riemann Sphere

Definition 1.1.1 (The Riemann Sphere). The *Riemann sphere* is a one-dimensional complex manifold, denoted $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the charts of which are the following.

$$\begin{aligned} U_1 &= (\mathbb{C}, f_1) \\ U_2 &= \left(\hat{\mathbb{C}} \setminus \{0\}, f_2\right) \end{aligned}$$

where

$$\begin{aligned} f_1 \colon \mathbb{C} &\to \mathbb{C} \\ z &\mapsto z \end{aligned}$$

and

$$\begin{aligned} f_2 \colon \mathbb{C} &\to \hat{\mathbb{C}} \setminus \{0\} \\ z &\mapsto \frac{1}{z}. \end{aligned}$$

Definition 1.1.2 (Möbius Transformation). A map $T \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $ad - bc \neq 0$ is called a *Möbius transformation.*

Notation 1.1.3.

1. We denote the image of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ in $\mathrm{PGL}_2(\mathbb{C})$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

2. For every $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$, we denote by T_g the Möbius transformation $z \mapsto \frac{az+b}{cz+d}$.

Lemma 1.1.4. *The set of Möbius transformations is a group under composition, and the map $g \mapsto T_g$ is an isomorphism between $\mathrm{PGL}_2(\mathbb{C})$ and the group of Möbius transformation.*

Proof. It holds that

$$\begin{aligned} T_{g_1} \circ T_{g_2}(z) &= \frac{a_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + 1}{c_1\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right) + d_1} \\ &= \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)} \\ &= T_{g_1 g_2}(z). \end{aligned}$$

In particular, $T_{g^{-1}}$ is the inverse of T_g.

The rest of the proof is clear. ■

Definition 1.1.5 (Generalised Circle). A generalised circle in $\mathbb{C}$ is either an Euclidean circle or an Euclidean straight line.

Lemma 1.1.6. *Let $T \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a Möbius transformation. Then*

1. *T is an endomorphism of $\hat{\mathbb{C}}$.*
2. *T is conformal.*
3. *T sends generalised circles to generalised circles.*

1.1.2 Models of the Hyperbolic Plane

Definition 1.1.7 (The Upper Half Plane Model for the Hyperbolic Plane). 1.

As a set, define $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path given by $\gamma(t) = x(t) + iy(t)$ for real functions $x(t), y(t)$. The *hyperbolic length* of γ is given by

$$h(\gamma) := \int_0^1 \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{y(t)} \mathrm{d}t = \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{y(t)} \mathrm{d}t.$$

3. The *hyperbolic distance* $\rho(z, w)$ between two points $z, w \in \mathbb{H}$ is defined as $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths γ from z to w.

Remark 1.1.8. $\mathbb{H}$ is a Riemann surface where for every $z \in \mathbb{H}$, the inner product of $T_z H$ is given by

$$((x_1, y_1), (x_2, y_2)) = \frac{x_1 x_2 + y_1 y_2}{(\Im z)^2}.$$

In particular, Euclidean angles are equal to hyperbolic angles.

Definition 1.1.9 (The Disc Model for the Hyperbolic Plane).

1. As a set, define $\mathbb{U} \coloneqq \{z \in \mathbb{C} \mid |z| < 1\}$.

2. Let $\gamma\colon [0,1] \to \mathbb{U}$ be a piecewise continuously differentiable path. The *hyperbolic length* of γ is given by

$$h_u(\gamma) \coloneqq \int_0^1 \frac{2\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{1 - |\gamma(t)|^2}\,\mathrm{d}t.$$

3. The *hyperbolic distance* $\rho_u(z, w)$ between $z, w \in \mathbb{U}$ is defined to be $\inf_\gamma h(\gamma)$ where the infimum is taken over all piecewise continuously differentiable paths from z to w.

Remark 1.1.10. It is clear that hyperbolic circles around 0 are exactly Euclidean circles around it (with a generally different radius).

Remark 1.1.11. Rotations around 0 are isometries in the disc model.

Lemma 1.1.12. *Let π be the Möbius transformation defined by*

$$\pi(z) = \frac{iz + 1}{z + i}.$$

Then

1. *π is a bijection from $\mathbb{H}$ to $\mathbb{U}$.*

2. *For every piecewise continuously differentiable path $\gamma\colon [0,1] \to \mathbb{H}$, it holds that $h_u(\pi(\gamma)) = h(\gamma)$. In particular, π is an isometry.*

Proof. 1. It holds that

$$\begin{aligned} \pi(-1) &= -1 \\ \pi(0) &= -i \\ \pi(1) &= 1. \end{aligned}$$

Since Möbius transformations send generalised circles to generalised circles we get that π sends $\mathbb{R}$ to the unit circle. Since $\pi(i) = 0$ and π is a homeomorphism of the Riemann sphere, we get the result.

2. Let $\gamma\colon [0,1] \to \mathbb{H}$ be a piecewise continuously differentiable path. Denote $\psi = \pi^{-1}$ and $\delta = \pi(\gamma)$. Then

$$\psi(z) = \frac{iz-1}{-z+i} = \frac{(iz-1)\,\overline{-z}-i}{(-z+i)\,\overline{-z}-i} = \frac{(z+\bar{z}) + i\left(1-|z|^2\right)}{|-z+i|^2}.$$

So,

$$\Im(\psi(z)) = \frac{1-|z|^2}{|-z+i|^2}.$$

Since

$$\frac{\mathrm{d}\psi}{\mathrm{d}z} = \frac{-2}{(-z+i)^2},$$

we get that

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\Im(\gamma(t))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi(\delta)}{\mathrm{d}t}\right|}{\Im(\psi(\delta(t)))}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}\psi}{\mathrm{d}z}(\delta(t))\,\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{\Im(\psi(\delta(t)))} \\ &= \int_0^1 \frac{2\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{1-|\delta(t)|^2}\,\mathrm{d}t \\ &= h_u(\delta). \end{aligned}$$

■

1.1.3 Isometries of the Hyperbolic Plane

Lemma 1.1.13. *For every* $g \in \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$ *it holds that*

$$T_g(\mathbb{H}) \subseteq \mathbb{H}.$$

Proof. It's enough to show the inclusion $T_g(\mathbb{H}) \subseteq \mathbb{H}$ since then

$$T_{g^{-1}}(\mathbb{H}) = (T_g)^{-1}(\mathbb{H}) \subseteq \mathbb{H}$$

which implies $T_g(\mathbb{H}) \supseteq \mathbb{H}$ by applying T_g.

Now, we have

$$\begin{aligned} T_g(z) &= \frac{az+b}{cz+d} \\ &= \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} \\ &= \frac{ac|z|^2 + adz + bc\bar{z} + bg}{|cz+d|^2}. \end{aligned}$$

Thus,

$$\begin{aligned}\Im\left(T_g\left(z\right)\right) &= \frac{T_g\left(z\right) - \overline{T_g\left(z\right)}}{2i} \\ &= \frac{\left(ad-bc\right)z - \left(ad-bc\right)\bar{z}}{2i\left|cz+d\right|^2} \\ &\underset{ad-bc=1}{=} \frac{\Im\left(z\right)}{\left|cz+d\right|^2}.\end{aligned}$$

■

This lemma allows us to identify $\mathrm{PSL}_2\left(\mathbb{R}\right)$ as a subgroup of $\mathrm{Sym}\left(\mathbb{H}\right)$. The next lemma shows that even more is true.

Lemma 1.1.14. $\mathrm{PSL}_2\left(\mathbb{R}\right) \subseteq \mathrm{Isom}\left(\mathbb{H}\right)$.

Proof. It's enough to show that for every $g \in \mathrm{PSL}_2\left(\mathbb{R}\right)$ and every piecewise continuously differentiable path γ it holds that $h\left(\gamma\right) = h\left(T_g\left(\gamma\right)\right)$. Denote $T = T_g$ and $\delta = T\left(\gamma\right)$. Then

$$\begin{aligned}h\left(\delta\right) &= \int_0^1 \frac{\left|\frac{\mathrm{d}\delta}{\mathrm{d}t}\right|}{\Im\left(\delta\left(t\right)\right)}\,\mathrm{d}t \\ &= \int_0^1 \frac{\left|\frac{\mathrm{d}T}{\mathrm{d}z}\left(\gamma\left(t\right)\right)\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\Im\left(\delta\left(t\right)\right)}\,\mathrm{d}t \\ &\underset{\star}{=} \int_0^1 \frac{\left|\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right|}{\gamma\left(t\right)}\,\mathrm{d}t \\ &= h\left(\gamma\right)\end{aligned}$$

where $\star$ follows from

$$\begin{aligned}\Im\left(T_g\left(z\right)\right) = \frac{\Im\left(z\right)}{\left|cz+d\right|^2} \oplus \frac{\mathrm{d}T}{\mathrm{d}z} &= \frac{a\left(cz+d\right) - c\left(az+b\right)}{\left(cz+d\right)^2} \\ &= \frac{1}{\left(cz+d\right)^2}.\end{aligned}$$

■

Corollary 1.1.15. $\mathrm{Isom}\left(\mathbb{H}\right)$ *acts transitively on* $\mathbb{H}$.

Proof. It's enough to show that for every $z \in \mathbb{H}$ there's $g \in \mathrm{PSL}_2\left(\mathbb{R}\right)$ such that $T_g\left(z\right) = i$.

If $z = x + yi$, take $g = \begin{pmatrix} \frac{1}{\sqrt{y}} & -\frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$, then

$$T_g\left(z\right) = \frac{1}{y}\left(x+yi\right) - \frac{x}{y} = i.$$

■

Lemma 1.1.16. *Let $\pi\colon \mathbb{H} \to \mathbb{U}$ be the isometry $z \mapsto \frac{iz+1}{z+i}$ which we defined previously. Then*

$$\left\{\pi T_g \pi^{-1} \;\middle|\; g \in \mathrm{PSL}_2(\mathbb{R})\right\} = \left\{\begin{pmatrix} r & s \\ \bar{r} & \bar{s} \end{pmatrix} \;\middle|\; {\scriptstyle r,s\in\mathbb{C} \atop |r|^2-|s|^2=1}\right\}.$$

In particular, by taking $r = e^{i\theta}$ and $s = 0$ we see that the action of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{U}$ contains all the rotations around 0.

Proof. It holds that

$$\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} i & -1 \\ -1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} (a+d)+i(b-c) & b+c+i(a-d) \\ (b+c)-i(a+d) & (a+d)-i(b-c) \end{pmatrix}.$$

Now, $(a+d, a-d, b+c, b-c)$ can be any 4-tuple. Specifically, for every $r, s \in \mathbb{C}$ we have $a, b, c, d \in \mathbb{R}$ such that $\pi T_g \pi^{-1} = \begin{pmatrix} r & s \\ \bar{s} & \bar{r} \end{pmatrix}$, and by the equality from the determinants we get that $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{PSL}_2(\mathbb{R})$. ■

Corollary 1.1.17. *Let $z_1, z_2, w_1, w_2 \in \mathbb{H}$ be such that $\rho(z_1, w_1) = \rho(z_2, w_2)$, then there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z_1) = z_2$ and $T_g(w_1) = w_2$.*

Proof. Since $\mathrm{PSL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ we can assume that $z_1 = z_2$ and show that $\mathrm{Stab}(z_1)$ acts transitively on $\{w \in \mathbb{H} \mid \rho(z_1, w) = \rho(z_1, w_1)\}$. We already showed this in the disc model, in the case $z_1 = i$. ■

Definition 1.1.18. Let (X, d) be a metric space.

1. Let $x, y \in X$. A path $\gamma\colon [a, b] \to X$ which joins x and y is called a *geodesic segment* if for every $a \le t_1 \le t_2 \le b$ it holds that $|t_2 - t_1| = d(\gamma(t_1), \gamma(t_2))$.

2. A path $\gamma\colon \mathbb{R} \to X$ is called a *geodesic line* if for every $a < b$ it holds that $\gamma|_{[a,b]}$ is a geodesic segment.

Remark 1.1.19. Let γ be a geodesic segment or line. Then γ is determined by the image of γ up to a composition with an isometry of R. Thus, we can identify geodesic segments and lines with their image up to orientation.

Lemma 1.1.20. *Let $b > a > 0$ be real numbers. Then $\{iy \mid a \le y \le b\}$ is the unique geodesic segment between ia and ib and $\{iy \mid y > 0\}$ is the unique geodesic line through ia and ib.*

Proof. We begin with the first part of the lemma. Let $\gamma\colon [0, 1] \to \mathbb{H}$ be a piecewise continuously differentiable path joining ia and ib. For $t \in [0, 1]$ denote

$\gamma(t) = x(t) + iy(t)$ where $x(t), y(t) \in \mathbb{R}$. Then

$$\begin{aligned} h(\gamma) &= \int_0^1 \frac{\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}}{y(t)} \mathrm{d}t \\ &\underset{\star}{\geq} \int_0^1 \frac{\left|\frac{\mathrm{d}y}{\mathrm{d}t}\right|}{y(t)} \mathrm{d}t \\ &\geq \int_0^1 \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{y(t)} \mathrm{d}t \\ &= \ln\left(\frac{b}{a}\right). \end{aligned}$$

Thus, $\rho(ia, ib) \geq \ln\left(\frac{b}{a}\right)$. If $y(t) = i((b-a)t + a)$, the above inequalities are equalities so $\rho(ia, ib) = \ln\left(\frac{b}{a}\right)$. The inequality $\star$ is an equality if and only if $x(t) = 0$ for all $t \in [a, b]$. It follows that the unique geodesic segment between a and b is $\{iy \mid a \leq y \leq b\}$.

Now, it is clear that $\{iy \mid y > 0\}$ is a geodesic line which passes through ia and ib. We want to show it's unique.

Assume towards a contradiction that there exists a geodesic line ℓ between ia and ib which isn't the positive part of the y-axis. Then there's $z = x + iy \in \ell$ for which $x \neq 0$ and $\rho(z, ia) > \rho(z, ib)$. By the previous lemma, there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(ia) = ia$ and $T_g(z) \in i\mathbb{R}$. Since T_g sends generalised circles to generalised circles, $T_g(ib) \notin i\mathbb{R}$. Indeed, otherwise the image of the segment between ia and ib would belong to $i\mathbb{R}$, and since T_g sends generalised circles to generalised circles, it would send $\mathbb{R}$ to itself.

We get that there exists a geodesic between ia and $T_g(z) = ic$ which is contained in $i\mathbb{R}$, and this is impossible. ■

Theorem 1.1.21. *1. Every distinct points $z, w \in \mathbb{H}$ are contained in a unique geodesic segment and a unique geodesic line.*

2. The geodesics in $\mathbb{H}$ are semicircles and lines orthogonal to the real axis.

Proof. 1. For every $g \in \mathrm{PSL}_2(\mathbb{R})$ it holds that $T_g(\mathbb{R} \cup \{\infty\}) = \mathbb{R} \cup \{\infty\}$. If $z, w \in \mathbb{H}$, by a previous lemma there exists $g \in \mathrm{PSL}_2(\mathbb{R})$ such that $T_g(z) = ia$ and $T_g(w) = ib$ for some $a, b \in \mathbb{R}_+$. Thus, $T_g^{-1}([ia, ib])$ is the unique geodesic segment between z and w.

2. This follows from the fact that Möbius circles are conformal, send generalised circles to generalised circles, and sends $\mathbb{R} \cup \{\infty\}$ to itself. ■

Corollary 1.1.22. *The geodesic segment in $\mathbb{U}$ are segments of straight lines through zero or arcs of circles which are orthogonal to the unit circles.*

Theorem 1.1.23. *Let $z, w \in \mathbb{H}$. Then*

$$\sinh\left(\frac{1}{2}\rho\left(z,w\right)\right) = \frac{|z-w|}{2\left(\Im\left(z\right)\Im\left(w\right)\right)^{\frac{1}{2}}}.$$

Proof. Since $\mathrm{PSL}_2\left(\mathbb{R}\right) \subseteq \mathrm{Isom}\left(\mathbb{H}\right)$, the left side of the equation is invariant under the action of $\mathrm{PSL}_2\left(\mathbb{R}\right)$. We first show that the right side is also invariant.

It's clear that the right side is invariant under maps of the form $z \mapsto az+b$ for $a, b \in \mathbb{R}$. Since $\mathrm{PSL}_2\left(\mathbb{R}\right)$ (viewed as a group of Möbius transformations) is generated by maps of the forms

$$\begin{aligned} &z \mapsto az+b,\ a,b \in \mathbb{R} \\ &z \mapsto -\frac{1}{z} \end{aligned}$$

it's enough to show that the right side is invariant under these maps.

The right side is indeed invariant under $\frac{1}{z}$ since

$$\begin{aligned} \frac{\left|\frac{1}{z}-\frac{1}{w}\right|}{2\left(\Im\left(\frac{1}{z}\right)\Im\left(\frac{1}{w}\right)\right)^{\frac{1}{2}}} &= \frac{\left|\frac{z-w}{zw}\right|}{2\left(\Im\left(\frac{z}{|z|^2}\right)\Im\left(\frac{w}{|w|^2}\right)\right)^{\frac{1}{2}}} \\ &= \frac{|z-w|}{2\left(\Im\left(z\right)\Im\left(w\right)\right)^{\frac{1}{2}}}. \end{aligned}$$

Since both sides of the equation are invariant under the action of $\mathrm{PSL}_2\left(\mathbb{R}\right)$, it's enough to prove the equality for $z = i$ and $w = ir$ for some $r \in \mathbb{R}_+$. Indeed,

$$\begin{aligned} \sinh\left(\frac{1}{2}\rho\left(i,ir\right)\right) &= \sinh\left(\frac{1}{2}\left|\ln r\right|\right) \\ &= \frac{\left|\sqrt{r}-\frac{1}{\sqrt{r}}\right|}{2} \\ &= \frac{|r-1|}{2\sqrt{r}} \\ &= \frac{|i-ir|}{2\left(\Im\left(i\right)\Im\left(ir\right)\right)^{\frac{1}{2}}}. \end{aligned}$$

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