Lecture Notes to Functional Analysis

Winter 2020, Technion

Lectures by Liran Rotem

Typed by Elad Tzorani

December 9, 2020

Contents

Chapter 1

Preliminaries

1.1 Banach Spaces & Examples

Definition 1.1.1 (Norm). Let X be a vector space over $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ (and assume from now until mentioned otherwise these are the only fields). A *norm* on X is a function

$$\|\cdot\| : X \to [0, \infty)$$

such that the following hold.

1. $\|x\| = 0$ iff $x = 0$.

2. $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in \mathbb{F}$.

3. $\|x + y\| \leq \|x\| + \|y\|$.

Remark 1.1.2. A norm $\|\cdot\|$ defines a metric by $d(x, y) = \|x - y\|$. This defines a topology generated by the open balls in the metric.

Definition 1.1.3 (Banach Space). A normed space $(X, \|\cdot\|)$ is a *Banach space* if it's complete (i.e. if every Cauchy sequence in it converges).

Definition 1.1.4 (Inner Product). Let X be a vector space over $\mathbb{F}$. An *inner product on X* is a function

$$\langle \cdot, \cdot \rangle : X \times X \to \mathbb{F}$$

such that the following hold.

1. $\langle \lambda x_1 + x_2, y \rangle = \lambda \langle x_1, y \rangle + \langle x_2, y \rangle$.

2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

3. $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$.

Remark 1.1.5. An inner product $\langle \cdot, \cdot \rangle$ defines a norm $\|x\| = \sqrt{\langle x, x\rangle}$. This can be proven by Cauchy-Schwarz $|\langle x, y\rangle| \leq \sqrt{\|x\| \, \|y\|}$.

Definition 1.1.6 (Hilbert Space). A *Hilbert space* is a complete inner product space.

Example 1.1.7 ($\boldsymbol{\ell_p}$ Spaces). Fix $p \in [1, \infty)$ and $(a_n)_{n=1}^{\infty}$ a sequence in $\mathbb{F}$. Define

$$\|(a_n)_{n=1}^{\infty}\|_p \coloneqq \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{\frac{1}{p}}$$
$$\|(a_n)_{n=1}^{\infty}\|_{\infty} \coloneqq \sup_{n \in \mathbb{N}_+} |a_n|$$

and

$$\ell_p \coloneqq \left\{(a_n)_{n=1}^{\infty} \;\middle|\; \|(a_n)_{n=1}^{\infty}\|_p < \infty\right\}$$
$$\ell_{\infty} \coloneqq \left\{(a_n)_{n=1}^{\infty} \;\middle|\; \|(a_n)_{n=1}^{\infty}\|_{\infty} < \infty\right\}$$

Example 1.1.8. Define

$$c \coloneqq \left\{(a_n)_{n \in \mathbb{N}_+} \in \ell_{\infty} \;\middle|\; \exists \lim_{n \to \infty} a_n \in \mathbb{C}\right\}.$$

Consider this with the ∞-norm, as a closed subspace of ℓ_{∞}.

Similarly, define

$$c_0 \coloneqq \left\{(a_n)_{n \in \mathbb{N}_+} \in \ell_{\infty} \;\middle|\; \lim_{n \to \infty} a_n = 0\right\}.$$

Example 1.1.9 (Continuous Functions over a Compact Space). Let X be a compact topological space. Define

$$\mathscr{C}(X) \coloneqq \{f \colon X \to \mathbb{F} \mid f \text{ is continuous}\}.$$

This is a Banach space with $\|f\| = \max_{x \in X} |f(x)|$.

Definition 1.1.10 (Support of a Function). Let $f \colon X \to V$ be a function of sets from a topological space into a vector space. Define *the support of f* to be

$$\sup(f) \coloneqq \overline{\{x \mid f(x) \neq 0\}}.$$

Example 1.1.11 (Continuous Functions over Locally Compact Hausdorff Spaces). Let X be a locally compact Hausdorff topological space (e.g. $X = \mathbb{R}^n$). Define

$$\mathscr{C}_c(X) \coloneqq \{f \colon X \to \mathbb{F} \mid f \text{ is compactly supported and continuous}\}.$$

This is a normed space which is *not* complete.

Define $f(x) = e^{-x^2}$. This can be approximated by continuous functions f_n which agree with it on $[-n, n]$ and go linearly to zero outside $[-n, n]$ until the boundary of $[-n-1, n+1]$. $(f_n)_{n=1}^{\infty} \subseteq \mathscr{C}_c(X)$ is Cauchy but not convergent because $f \notin \mathscr{C}_c(X)$.

Definition 1.1.12 (Completion of a N Space). Let Y be a normed space. There is a Banach space $\hat{Y}$ such that $Y \leq \hat{Y}$ and $\bar{Y} = \hat{Y}$. This is called *the completion of* Y.

Example 1.1.13.

$$\widehat{\mathscr{C}_c(X)} = \mathscr{C}_0(X) \coloneqq \left\{ f\colon X \to \mathbb{F} \;\middle|\; \substack{f \text{ is continuous} \\ \lim_{|x|\to\infty} f(x)=0} \right\}.$$

Example 1.1.14. Let (X, Σ, μ) be a measure space and let $p \in [1, \infty)$. We define

$$L_p(X, \Sigma, \mu) \coloneqq \left\{ f\colon X \to \mathbb{F} \;\middle|\; \substack{f \text{ is Borel measurable} \\ \|f\|_p < \infty} \right\}$$

where

$$\|f\|_p \coloneqq \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}$$

and where we take the Borel measure on $\mathbb{F}$.

Remark 1.1.15. If $X = \mathbb{N}_+$ and μ is the counting measure $\mu(a) = |A|$ then $L_p(X, \Sigma, \mu) = \ell_p$.

Remark 1.1.16. $\|\cdot\|_p$ which we defined isn't *exactly* a norm, since there are measurable functions $f\colon X \to \mathbb{F}$ which are 0 almost everywhere, but not everywhere.
We therefore look at L_p as the space of equivalence classes of functions up to equivalence almost-everywhere of functions.

On (X, Σ, μ), $\|\cdot\|_p$ is a norm. This follows from the following inequality.

Theorem 1.1.17 (Hölder Inequality). *If $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q}$ then for all $f, g\colon X \to \mathbb{F}$ it holds that*

$$\left| \int_X fg \, \mathrm{d}\mu \right| \leq \left(\int_X |f|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}} \left(\int_X |g|^q \, \mathrm{d}\mu \right)^{\frac{1}{q}}.$$

Corollary 1.1.18.

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

Corollary 1.1.19. *On (X, Σ, μ), $\|\cdot\|_p$ is a norm.*

Theorem 1.1.20. *$L_p(X, \Sigma, \mu)$ is a Banach space.*

Proof. It's enough to prove that if

$$\sum_{n=1}^{\infty} \|f_n\|_p < \infty$$

then $\sum_{n=1}^{\infty} f_n$ converges.

Assume first $f_n \geq 0$. Define $g \coloneqq \sum_{n=1}^{\infty} f_n \in [0,\infty]$ and $g_N \coloneqq \sum_{n=1}^{N} f_n$. We know by the triangle inequality that

$$\|g_N\|_p \leq \sum_{n=1}^{N} \|f_n\|_p \leq \sum_{n=1}^{\infty} \|f_n\|_p \eqqcolon S.$$

Now

$$\|g\|_p^p = \int_X |g|^p \, \mathrm{d}\mu = \int_X \lim_{N\to\infty} |g_N|^p \, \mathrm{d}\mu \underset{\text{Monotone Convergence}}{=} \lim_{N\to\infty} \int_X |g_N|^p \, \mathrm{d}\mu \leq S^p$$

hence $g \in L_p$. In particular, $g < \infty$ almost-everywhere.

Finally,

$$\|g - g_N\|_p = \left\| \sum_{n=N+1}^{\infty} f_n \right\|_p \leq \sum_{n=N+1}^{\infty} \|f_n\|_p \xrightarrow{N\to\infty} 0.$$

In general, we can write $f_n = f_n + -f_n^-$ where f_n^+, f_n^- are positive. We have $|f_n^{\pm}| \leq |f_n|$ so $\|f_n^{\pm}\|_p \leq \|f_n\|_p$. So $\sum_{n=1}^{\infty} \|f_n^{\pm}\|_p < \infty$ so $\sum_{n=1}^{\infty} f_n^{\pm}$ converge and so does $\sum_{n=1}^{\infty} (f_n^+ - f_n^-)$. ■

Definition 1.1.21. Let $f\colon X \to \mathbb{F}$. Define

$$\operatorname{essup}(f) \coloneqq \inf\{M > 0 \mid \mu(\{x \mid |f(x)| \geq M\}) = 0\}.$$

Definition 1.1.22 (L_∞). Define $\|f\|_\infty = \operatorname{essup}(f)$ and

$$L_\infty(X, \Sigma, \mu) = \{f\colon X \to \mathbb{F} \mid \|f\|_\infty < \infty\}.$$

Exercise 1. Assume $\mu(X) < \infty$.

1. If $1 \leq p < q \leq \infty$ then $L_q \subseteq L_p$.
2. If $f \in L_\infty(X, \Sigma, \mu)$ then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$.

Definition 1.1.23 (The Operator Norm). Let X, Y be normed spaces and let $T\colon X \to Y$ be linear. Define the *operator norm* as

$$\|T\| = \sup_{x \in X\setminus\{0\}} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\substack{x\in X \\ 0<\|x\|\leq 1}} \|Tx\|.$$

Definition 1.1.24 (Bounded Linear Map). A linear map $T\colon X \to Y$ between normed spaces is *bounded* if $\|T\| < \infty$.

Fact 1.1.25. *Let $T\colon X \to Y$ be a linear map between normed spaces. The following are equivalent.*

1. *T is bounded.*

2. *T is continuous.*

3. *T is continuous at $x = 0$.*

4. *T is Lipschitz.*

Notation 1.1.26. Let X, Y be normed spaces. We denote the class of bounded linear functions $X \to Y$ by $\mathscr{L}(X, Y)$.

Theorem 1.1.27. *If Y is a Banach space, so is $\mathscr{L}(X, Y)$.*

Definition 1.1.28 (The Dual Space). The *dual space* of a normed space X is $X^* := \mathscr{L}(X, \mathbb{F})$.

Remark 1.1.29. If X is a Banach space, so is X^*.

Example 1.1.30. Let H be a Hilbert space. By the Riesz representation theorem we know that every $f \in H^*$ is of the form

$$f(x) = f_y(x) := \langle x, y \rangle$$

for some $y \in H$.

The maps $y \mapsto f_y$ is a bijection, it holds that $\|f_y\| = \|y\|$, and

$$f_{\alpha y}(x) = \langle x, \alpha y \rangle = \bar{\alpha} \langle x, y \rangle = \bar{\alpha} f_y(x).$$

Example 1.1.31. Let $1 \leq p < \infty$. It holds that $(\ell_p)^* = \ell_q$ where $\frac{1}{p} + \frac{1}{q} = 1$.

For $c := (c_n)_{n \in \mathbb{N}_+}$ let

$$f_c(a_n) = \sum_{n \in \mathbb{N}_+} a_n c_n.$$

Then $c \mapsto f_c$ is an isometry and $\ell_q \cong (\ell_p)^*$.

Example 1.1.32. The same construction gives and embedding $\ell_1 \hookrightarrow (\ell_\infty)^*$, which is linear, norm-preserving and 1-1, but *not* surjective.

For $p < \infty$ it holds that

$$f\left((a_n)_{n \in \mathbb{N}_+}\right) = f\left(\sum_{n \in \mathbb{N}_+} a_n e_n\right) = \sum_{n \in \mathbb{N}_+} a_n f(a_n)$$

where e_n is the n^{th} basis vector. But, in ℓ^∞ it doesn't hold that $(a_n)_{n \in \mathbb{N}_+} \neq \sum_{n \in \mathbb{N}_+} a_n e_n$. For example,

$$\left\| (1, 1, 1, \dots) - \sum_{n \in [N]} e_n \right\|_\infty = \|(0, 0, \dots, 0, 1, 1, 1, \dots)\| = 1 \not\to 0.$$

But, we do have $c_0^* \cong \ell_1$ with the same proof as for ℓ_p.

Example 1.1.33. Let X be a compact metric space. We want to understand $\mathcal{C}(X)^*$. If μ is a finite Borel measure on X, we can define

$$\begin{aligned}\Phi_\mu \colon \mathcal{C}(X) &\to \mathbb{C}\\ f &\mapsto \int_X f \,\mathrm{d}\mu.\end{aligned}$$

This is a linear functional. It holds that

$$|\Phi_\mu(f)| = \left|\int_X f \,\mathrm{d}\mu\right| \leq \int_X |f| \,\mathrm{d}\mu \leq \|f\|_\infty \mu(X).$$

So, $\Phi_\mu \in \mathcal{C}(X)^*$ and $\|\Phi_\mu\|^* \leq \mu(X)$. Taking $f \equiv 1$ we see actually that $\|\Phi_\mu\|^* = \mu(X)$.

If μ, ν are two measure on X, let

$$\Phi(f) = \int_X f \,\mathrm{d}\mu - i \int_X f \,\mathrm{d}\nu \in \mathcal{C}(X)^* = \int_X f \,\mathrm{d}(\mu - i\nu).$$

Then $\Phi \in \mathcal{C}(X)^*$.

Definition 1.1.34 (Complex Measure). Let (X, Σ) be a measurable space. A *complex measure* μ on (X, Σ) is a map $\mu \colon \Sigma \to \mathbb{C}$ such that if $(A_n)_{n \in \mathbb{N}_+} \subseteq \Sigma$ are pairwise disjoint it holds that

$$\mu\left(\bigsqcup_{n \in \mathbb{N}_+ A_n}\right) = \sum_{n \in \mathbb{N}_+} \mu(A_n).$$

Fact 1.1.35.

1. *If μ is a complex measure, $\Re\mu, \Im\mu$ are real signed measures.*
2. *For every signed measure on a space X, there is a decomposition $X = P \sqcup N$ such that $\mu(A) \geq 0$ for every $A \in \Sigma$ such that $A \subseteq P$, and $\mu(A) \leq 0$ for every $A \in \Sigma$ such that $A \subseteq N$.*
   This is called the Hahn decomposition of a signed measure.
3. *If μ is a complex measure on a space X, we can write*

   $$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$$

   where $(\mu_i)_{i \in [4]}$ are (real, non-negative) measures on X. Then

   $$\int_X f \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu_1 - \int_X f \,\mathrm{d}\mu_2 + i\left(\int_X f \,\mathrm{d}\mu_3 - \int_X f \,\mathrm{d}\mu_4\right).$$

Definition 1.1.36 (Norm of a Measure). Given a complex norm μ, we define

$$\|\mu\| \coloneqq \sup\left\{\sum_{i \in \mathbb{N}_+} |\mu(A_i)| \;\middle|\; X = \bigsqcup_{i \in [n]} A_i\right\}.$$

Theorem 1.1.37. *Every $\Phi \in \mathcal{C}(X)^*$ is of the form $\Phi(f) = \int_X f \,\mathrm{d}\mu$ for some complex measure μ. Also $\|\Phi\| = \|\mu\|$.*

1.1.1 Applications of Duality & Some Measure Theory

Theorem 1.1.38 (Radon Nikodym). *Let (X, Σ, μ) be a finite measure space. Let ν be be another measure on (X, Σ) such that $\nu \ll \mu$. Then $\exists g \geq 0$ such that*

$$\nu(A) = \int_A g \, \mathrm{d}\mu$$

for all $A \in \Sigma$.

Proof. Let $H := L_2(X, \mu + \nu)$ and let

$$\begin{aligned} \ell \colon H &\to \mathbb{R} \\ f &\mapsto \int_X f \, \mathrm{d}\mu. \end{aligned}$$

Then

$$\begin{aligned} |\ell(f)| &= \left| \int_X f \, \mathrm{d}\mu \right| \\ &\leq \left(\int_X f^2 \, \mathrm{d}\mu \right)^{\frac{1}{2}} \cdot \left(\int_X 1^2 \, \mathrm{d}\mu \right)^{\frac{1}{2}} \\ &\leq \mu(X)^{\frac{1}{2}} \cdot \left(\int_X f^2 \, \mathrm{d}(\mu + \nu) \right)^{\frac{1}{2}} \\ &= \mu(X) \, \|f\|_{L_2(\mu+\nu)}. \end{aligned}$$

By Riesz, there's $h \in H$ such that

$$\int f \, \mathrm{d}\mu = \ell(f) = \langle f, h \rangle = \int_X f h \, \mathrm{d}(\mu + \nu).$$

Then

$$\int_X f(-h) \, \mathrm{d}\mu = \int f h \, \mathrm{d}\nu.$$

One can check $0 \leq h \leq 1$. Define $g = \frac{1-h}{h}$, then

$$\int_X f g \, \mathrm{d}\mu = \int_X \frac{f}{h}(1-h) \, \mathrm{d}\mu = \int_X \frac{f}{h} \cdot h \, \mathrm{d}\nu = \int_X f \, \mathrm{d}\nu.$$

Take $f = \chi_A$, from which

$$\int_A g \, \mathrm{d}\mu = \int_A 1 \, \mathrm{d}\nu = \nu(A).$$

∎

Remark 1.1.39. The theorem is also true if ν is a complex measure, with $g \colon X \to \mathbb{C}$ in $L_1(\mu)$. We get this by writing $\nu = (\nu_1 - \nu_2) + i(\nu_3 - \nu_4)$.

Theorem 1.1.40. *Let (X, Σ, μ) a finite measure space. Fix $1 \leq p < \infty$. Then*

$$(L_p(X, \Sigma, \mu))^* = L_q(X, \Sigma, \mu)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. We work over $\mathbb{C}$, and assume $p > 1$. For every $g \in L_q(X, \Sigma, \mu)$ define $\Phi_g \colon L_p \to \mathbb{C}$ by

$$\Phi_g(f) = \int_X fg \, \mathrm{d}\mu.$$

Obviously, Φ_g is linear. By Hölder we know

$$|\Phi_g(f)| \leq \|f\|_p \|g\|_q,$$

so $\Phi_g \in L_p^*$ and $\|\Phi_g\| \leq \|g\|_q$. Notice also that the map $g \mapsto \Phi_g$ is linear.

Fix $\Phi \in L_p(\mu)^*$, we should prove $\Phi = \Phi_g$ for some $g \in L_q$. Define a new measure ν on X by

$$\nu(A) = \Phi(\chi_A).$$

Here, $\chi_A \in L_p(\mu)$ since μ is a finite measure. We prove ν is indeed a (complex) measure. Fix $(A_n)_{n=1}^{\infty}$ such that $A = \sqcup_{n=1}^{\infty} A_n$. Note that $\sum_{n=1}^{\infty} \chi_{A_n} = \chi_A$ in L_p. Indeed,

$$\begin{aligned}
\left\| \chi_A - \sum_{n=1}^{N} \chi_{A_n} \right\|_p &= \left\| \chi_{\bigsqcup_{n=N+1}^{\infty} A_n} \right\|_p \\
&= \mu\left(\bigsqcup_{n=N+1}^{\infty} A_n \right)^{\frac{1}{p}} \\
&= \left(\sum_{n=N+1}^{\infty} \mu(A_n) \right)^{\frac{1}{p}} \\
&\xrightarrow{N \to \infty} 0.
\end{aligned}$$

Now, ν is indeed a measure since

$$\begin{aligned}
\nu(A) &= \Phi(\chi_A) \\
&= \Phi\left(\sum_{n=1}^{\infty} \chi_{A_n} \right) \\
&= \sum_{n=1}^{\infty} \Phi(\chi_{A_n}) \\
&= \sum_{n=1}^{\infty} \nu(A_n).
\end{aligned}$$

Now, if $\mu(A) = 0$ then $\chi_A = 0$ in $L_p(\mu)$. So

$$\nu(A) < \Phi(\chi_A) = \Phi(0) = 0,$$

so by Radon-Nikodym there exists $g \in L_1(\mu)$ such that

$$\Phi(\chi_A) = \nu(A) = \int_A g \, \mathrm{d}\mu = \int \chi_A g \, \mathrm{d}\mu = \Phi_g(\chi_A).$$

We got Φ_g such that $\Phi = \Phi_g$ on indicators. If $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ is a simple function, we have

$$\begin{aligned}\Phi(f) &= \sum_{i=1}^n \alpha_i \Phi(\chi_{A_i}) \\ &= \sum_{i=1}^n \alpha_i \Phi_g(\chi_{A_i}) \\ &= \Phi_g(f).\end{aligned}$$

Assume now that $f \in L_p$ is bounded with $|f| \le M$. Choose simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \to f$ almost-everywhere and that $|f_n| \le M$. Now

$$\begin{aligned}\Phi(f) &\underset{\text{DCT}}{=} \lim_{n\to\infty} \Phi(f_n) \\ &= \lim_{n\to\infty} \int f_n g \, \mathrm{d}\mu \\ &\underset{\text{DCT}}{=} \int \lim_{n\to\infty} f_n g \, \mathrm{d}\mu \\ &= \int f g \, \mathrm{d}\mu \\ &= \Phi_g(f).\end{aligned}$$

Define

$$A_n := \{x \mid |g(x)| \le n\}$$

and set

$$\rho(z) := \begin{cases} \frac{\bar{z}}{|z|} & z \ne 0 \\ 0 & z = 0 \end{cases}$$

so that $z\rho(z) = |z|$. Define $f_n := \chi_{A_n} \rho(g) |g|^{q-1}$ which is bounded.

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}
\|\Phi\| \left(\int_{A_n} |g|^q \, \mathrm{d}\mu \right)^{\frac{1}{p}} &= \|\Phi\| \left(\int_{A_n} |g|^{p(q-1)} \, \mathrm{d}\mu \right)^{\frac{1}{p}} \\
&= \|\Phi\| \, \|f_n\|_p \\
&\geq \Phi \left(f_n \right) \\
&= \Phi_g \left(f_n \right) \\
&= \int g f_n \, \mathrm{d}\mu \\
&= \int_{A_n} g \rho \left(g \right) |g|^{q-1} \, \mathrm{d}\mu \\
&= \int_{A_n} |g|^q \, \mathrm{d}\mu .
\end{aligned}$$

Hence

$$\left(\int_{A_n} |g|^q \, \mathrm{d}\mu \right)^{\frac{1}{q}} \leq \|\Phi\| .$$

Then by MCT we have

$$\|g\|_q \leq \|\Phi\| < \infty.$$

Hence $g \in L_q(\mu)$ so $\|\Phi_g\| \leq \|g\|_q < \infty$ so Φ_g, Φ are continuous. Since they're equal on simple functions, they're equal. Also $\|\Phi_g\| = \|\Phi\| \geq \|g\|_q$ where we've just shown the inequality. Hence $\|\Phi_g\| = \|g\|_q$. ■

Exercise 2. Extend the above theorem to σ-finite spaces.

Chapter 2

Linear Functionals

2.1 Bounded Linear Functionals

2.1.1 The Hahn-Banach Theorem

We said that $(\ell_p)^* \cong \ell_q$ for $1 \leq p < \infty$, but *not* for ℓ_∞. To show this, we need to find some linear functional $f \in (\ell_\infty)^*$ not of the form $f(a_n) = \sum_{n=1}^{\infty} a_n b_n$.

Definition 2.1.1 (Sublinear Functions). Let $X \in \mathbf{Vect}_{\mathbb{R}}$. A function $p\colon X \to \mathbb{R}$ is called *sublinear* if the following holds for all $x, y, \in X$ and $\lambda \in \mathbb{R}_{\geq 0}$.

1. $p(x+y) \leq p(x) + p(y)$.
2. $p(\lambda x) = \lambda p(x)$.

Theorem 2.1.2 (Hahn-Banach, #1). *Let $X \in \mathbf{Vect}_{\mathbb{R}}$ and $p\colon X \to \mathbb{R}$ sublinear. Assume $Y \subseteq X$ is a subspace, $\ell\colon Y \to \mathbb{R}$ is linear and $\ell \leq p$ on Y. Then $\exists \tilde{\ell}\colon X \to \mathbb{R}$ linear such that $\tilde{\ell} \leq p$ and $\tilde{\ell}\Big|_Y = \ell$.*

Lemma 2.1.3. *Theorem 2.1.2 holds if* $\operatorname{codim} Y = 1$.

Proof. Every $x \in X$ is of the form $x = y + \lambda x_0$. We want to define

$$\tilde{\ell}(x) = \tilde{\ell}(y) + \lambda \tilde{\ell}(x_0) = \ell(y) + \lambda \tilde{\ell}(x_0).$$

We only get to choose $a = \tilde{\ell}(x_0)$. The goal is to choose a to have

$$\forall y \in Y \forall \lambda \in \mathbb{R}\colon \ell(y) + \lambda a = \tilde{\ell}(y + \lambda x_0) \leq p(y + \lambda x_0).$$

For $\lambda > 0$ we get the requirement

$$a \leq \frac{1}{\lambda}\left(p(y + \lambda x_0) - \ell(y)\right) = p\left(\frac{y}{\lambda} + x_0\right) - \ell\left(\frac{y}{\lambda}\right).$$

For $\lambda < 0$, write $\lambda = -\mu$ with $\mu > 0$. Denote also $z \coloneqq y$. We get the requirement

$$\ell(z) - \mu a \leq p(z - \mu x_0)$$

so

$$a \geq \frac{1}{\mu}\left(\ell\left(z\right) - p\left(z - \mu x_0\right)\right) = \ell\left(\frac{z}{\mu}\right) - p\left(\frac{z}{\mu} - x_0\right).$$

Choosing such a is possible iff

$$\sup_{\substack{Z\in Y\\ \mu>0}}\left[\ell\left(\frac{z}{\mu}\right) - p\left(\frac{z}{\mu} - x_0\right)\right] \leq \inf_{\substack{y\in Y\\ \lambda\geq 0}}\left[p\left(\frac{y}{\lambda} + x_0\right) - \ell\left(\frac{y}{\lambda}\right)\right].$$

We prove this is the case.

Let $y, z \in Y$ and $\lambda, \mu \in \mathbb{R}_+$. We indeed have

$$\begin{aligned}\ell\left(\frac{z}{\mu}\right) + \ell\left(\frac{y}{\lambda}\right) &= \lambda\left(\frac{z}{\mu} + \frac{y}{\lambda}\right)\\ &\leq p\left(\frac{z}{\mu} - x_0 + \frac{y}{\lambda} + x_0\right)\\ &\leq p\left(\frac{z}{\mu} - x_0\right) + p\left(\frac{y}{\lambda} + x_0\right).\end{aligned}$$

■

Proof (2.1.2). Let

$$\mathcal{F} \coloneqq \left\{(Z, f) \;\middle|\; \substack{Y\subseteq Z\subseteq X\\ f\colon Z\to\mathbb{R}\\ f|_Y=\ell\\ f\leq p}\right\}.$$

We have $(Y, \ell) \in S$ so $S \neq \varnothing$. Say $(Z_1, f_1) \leq (Z_2, f_2)$ if $Z_1 \subseteq Z_2$ and $f_2|_{Z_1} = f_1$.

Take a chain $\{(Z_i, f_i)\}_{i\in I}$ in $\mathcal{F}$. We define $Z_\infty = \bigcup_{i\in I} Z_i$ and $f_\infty(x) = f_i(x)$ if $x \in Z_i$. Check that $(Z_\infty, f_\infty) \in S$ and this is an upper bound to the chain. By Zorn's lemma, there's a maximal element (Z_0, f_0) of $\mathcal{F}$. By 2.1.3 we get $Z_0 = X$, hence we're done. ■

Theorem 2.1.4 (Hahn Banach #2). *Let $(X, \|\cdot\|)$ be a normed space over $\mathbb{F}$, let $Y \subseteq X$ be a subspace and let $f \in Y^*$. Then $\exists \tilde{f} \in X^*$ such that $\tilde{f}\Big|_Y = f$ and $\left\|\tilde{f}\right\|_{X^*} = \|f\|_{Y^*}$.*

Proof.

- Assume first $\mathbb{F} = \mathbb{R}$. Just take $\ell = f$ and $p(x) = \|f\|_{Y^*}\|x\|$. Then for all $y \in Y$ we have

  $$f(y) = \ell(y) \leq p(y) = \|f\|\|y\|.$$

  Hence by 2.1.2 there's $\tilde{f}$ such that $\tilde{f}\Big|_Y = f$ and

  $$\forall x \in X\colon \tilde{f}(x) \leq \|f\|_{Y^*}\|x\|.$$

  Taking $-x$ instead of x we get

  $$-\tilde{f}(x) \leq \|f\|_{Y^*}\|x\|.$$

  Hence $\left\|\tilde{f}\right\| X^* \leq \|f\|_{Y^*}$.

- Assume now that $\mathbb{F} = \mathbb{C}$. We can think of $(X, \|\cdot\|)$ also as a space over $\mathbb{R}$. Write
$$f(y) = g(y) + ih(y)$$
where $g, h \colon Y \to \mathbb{R}$ are $\mathbb{R}$-linear. We have
$$g(iy) + ih(iy) = f(iy) = if(y) = -h(y) + ig(y)$$
so
$$h(y) - g(iy).$$
Hence
$$f(y) = g(y) - ig(iy).$$
Obviously,
$$|g(y)| \le |f(y)|,$$
so
$$\|g\|_{Y^*} \le \|f\|_{Y^*}.$$
By the real case, there's $\tilde{g} \colon X \to \mathbb{R}$ which is $\mathbb{R}$-linear such that $\tilde{g}|_{=}g$ and
$$\|\tilde{g}\|_{X^*} \le \|g\|_{Y^*} \le \|f\|_{Y^*}.$$
Define
$$\tilde{f}(x) = \tilde{g}(x) - i\tilde{g}(ix).$$
We get that $\tilde{f}(x)$ is $\mathbb{R}$-linear, but also
$$i\tilde{f}(x) = i\tilde{g}(x) + \tilde{g}(ix) = \tilde{f}(ix)$$
so $\tilde{f}$ is $\mathbb{C}$-linear. Also $\tilde{f}\Big|_Y = f$, so we're left to check the norm of $\tilde{f}$. Fix $x \in X$ and write $\tilde{f}(x) = re^{i\theta}$. Now
$$\begin{aligned}\left|\tilde{f}(x)\right| &= r \\ &= e^{-i\theta}\tilde{f}(x) \\ &= \tilde{f}\left(e^{-i\theta x}\right) \\ &= \tilde{g}\left(e^{-i\theta x}\right) \\ &\le \|\tilde{g}\| \left\|e^{-i\theta}x\right\| \\ &= \|\tilde{g}\| \|x\|.\end{aligned}$$
So,
$$\left\|\tilde{f}\right\|_{X^*} \le \|\tilde{g}\|_{X^*} \le \|f\|_{Y^*}.$$
■

Corollary 2.1.5. *For every $x_0 \in X \setminus \{0\}$ there's $f \in X^*$ such that $\|f\| = 1$ and $f(x_0) = \|x_0\|$.*

Proof. Let $Y = \text{Span}\{x_0\}$ and

$$\begin{aligned} f\colon Y &\to \mathbb{F} \\ \lambda x_0 &\mapsto \lambda \|x_0\| . \end{aligned}$$

Then $\|f\|_{Y^*} = 1$ and $f(x_0) = \|x_0\|$. Now extend f to all of X. ■

Corollary 2.1.6. *If $f(x_1) = f(x_2)$ for all $f \in X^*$, then $x_1 = x_2$.*

Proof. If $x_1 \neq x_2$, by 2.1.5 there's $f \in X^*$ with $\|f\| = 1$ such that $f(x_1 - x_2) = \|x_1 - x_2\| \neq 0$. Then $f(x_1) \neq f(x_2)$. ■

Corollary 2.1.7. *Let $E \leq X$ and let x_0 such that*

$$d \coloneqq d(x_0, E) > 0.$$

There's $f \in X^*$ with $\|f\| = 1$ such that $f(E) = 0$ and $f(x_0) = d$.

Proof. Take $Y = \text{Span}\{E, x_0\}$ and let

$$\begin{aligned} f\colon Y &\to \mathbb{F} \\ e + \lambda x_0 &\mapsto \lambda \cdot d. \end{aligned}$$

Then

$$\begin{aligned} \|e + \lambda x_0\| &= |\lambda| \left\| x_0 - \left(-\frac{1}{\lambda}\right) \right\| \\ &\geq |\lambda| \, d \\ &= |f(e + \lambda x_0)| . \end{aligned}$$

So $\|f\| \leq 1$, and one can check that actually $\|f\| = 1$.

By 2.1.4 we can extend f to X, which finishes the proof. ■

Corollary 2.1.8. *Let $A \subseteq X$ and $x \in X$. The following are equivalent.*

1. *$x \in \overline{\text{Span}(A)}$.*
2. *For every $f \in X^*$ such that $f(A) = 0$ it holds that $f(x) = 0$.*

Proof. $1 \implies 2$ is obvious.
For the other direction, let $E \coloneqq \overline{\text{Span}(A)}$. If $x \notin E$ then $d(x, E) = d > 0$. From 2.1.7 there's $f \in X^*$ with $\|f\| = 1$ such that $f(E) = 0$ and $f(x) = d \neq 0$. ■

2.1.2 Application of Hahn-Banach - Banach Limits

Definition 2.1.9 (Banach Limit). A *Banach limit* is a linear functional $f\colon \ell_\infty \to \mathbb{F}$ such that the following holds.

1. If $(a_n)_{n \in \mathbb{N}_+}$ is such that $a_n \geq 0$ for all $n \in \mathbb{N}_+$, then $f\left((a_n)_{n \in \mathbb{N}_+}\right) \geq 0$.

2. $f\left(\left(a_{n+1}\right)_{n\in\mathbb{N}_+}\right)=f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)$.
3. If $a_n \xrightarrow{n\to\infty} L$ then $f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)=L$.

Remark 2.1.10. It holds that

$$f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)\leq f\left(\left(\left\|\left(a_n\right)_{n\in\mathbb{N}_+}\right\|_\infty,\left\|\left(a_n\right)_{n\in\mathbb{N}_+}\right\|_\infty,\left\|\left(a_n\right)_{n\in\mathbb{N}_+}\right\|_\infty,\ldots\right)\right)=\left\|\left(a_n\right)_{n\in\mathbb{N}_+}\right\|_\infty.$$

Similarly

$$-f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)\leq\left\|\left(a_n\right)_{n\in\mathbb{N}_+}\right\|_\infty.$$

From this it follows that $\|f\|\leq 1$, and by looking at a constant sequence we actually get $\|f\|=1$.

Remark 2.1.11. A Banach limit f is in $(\ell_\infty)^*$ and this is *not* of the form $f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)=\sum_{n\in\mathbb{N}_+}a_nb_n$. We show the latter part.

Assume $f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)=\sum_{n=1}^\infty$. Then

$$\begin{aligned}0&=f\left(e_k\right)\\&=\sum_{n\in\mathbb{N}_+}\left(e_k\right)_n b_n\\&=\sum_{n\in\mathbb{N}_+}\delta_{k,n}b_n\\&=b_k\end{aligned}$$

so $f=0$, which is a contradiction to $f\left(1,1,1,\ldots\right)=1$.

Theorem 2.1.12. *There exists a Banach limit.*

Remark 2.1.13. Without parts (1,2) of the definition, this is obvious from 2.1.4. One can extend $f\left(\left(a_n\right)_{n\in\mathbb{N}_+}\right)=\lim_{n\to\infty}a_n$ from c to ℓ_∞.

Proof. Let

$$E\coloneqq\left\{\left(a_{n+1}-a_n\right)_{n\in\mathbb{N}_+}\;\middle|\;\left(a_n\right)_{n\in\mathbb{N}_+}\in\ell_\infty\right\}$$

and $x_0=(1,1,1,\ldots)$ the constant sequence 1. We claim $d\left(x_0,E\right)=1$. Let $b\coloneqq\left(b_n\right)_{n\in\mathbb{N}_+}\in E$ and let $d\coloneqq\|x_0-b\|$. Then

$$\begin{aligned}nd&\geq\sum_{k\in[n]}\left|\left(x_0\right)_k-b_k\right|\\&\geq\sum_{k\in[n]}\left(1-b_k\right)\\&=n-\sum_{k\in[n]}b_k.\end{aligned}$$

We claim the last sum is bounded. Indeed, if $b_k = a_{k+1} - a_k$ then

$$\sum_{k\in[n]} b_k = a_{n+1} - a_1.$$

Hence, dividing by n we get

$$d \geq 1 - \frac{1}{n}\sum_{k\in[n]} a_k \xrightarrow{n\to\infty} 1.$$

Hence $d(x_0, E) \geq 1$, but obviously $d(x_0, E) \leq \|x_0\| = 1$.

By 2.1.7 there's $f \in (\ell_\infty)^*$ such that $f(E) = 0$, $f(x_0) = 1$ and $\|f\| = 1$. Hence we have property 2 of the Banach limit. To get property 1 fix a sequence $a = (a_n)_{n\in\mathbb{N}_+}$ such that $a_n \geq 0$ and assume without loss of generality that $\|a\|_\infty = 1$. Then

$$\begin{aligned} f(a) &= f(x_0 - (x_0 - a)) \\ &= f(x_0) - f(x_0 - a) \geq 1 - \|x_0 - a\| \\ &\geq 1 - 1 \\ &= 0 \end{aligned}$$

where in the last inequality we use $a_n \in [0,1]$ for all $n \in \mathbb{N}_+$. This proves property 1.

Note that

$$\forall a \in \ell_\infty \forall k \inf_{mbbN} f\left((a_n)_{n\in\mathbb{N}_+}\right) = f\left((a_{n+k})_{n\in\mathbb{N}_+}\right) \leq \sup_{n\geq k} a_n.$$

Hence

$$f\left((a_n)_{n\in\mathbb{N}_+}\right) \leq \inf_{k\in\mathbb{N}} \sup_{n\geq k} a_n = \limsup_{n\to\infty} a_n.$$

Taking $-a$ instead of a we get

$$-f(a) \leq \limsup_{n\to\infty}(-a) = -\liminf_{n\to\infty} a_n$$

so

$$f(a) \geq \liminf_{n\to\infty} a_n.$$

Hence if $a_n \to L$ also $f(a) = L$. ■

2.1.3 Application of Hahn-Banach - Reflexive Spaces

Definition 2.1.14. For every $x \in X$ define $\mathrm{ev}_x \in X^{**} = (X^*)^*$ by

$$\forall f \in X^* \colon \mathrm{ev}_x(f) = f(x).$$

Proposition 2.1.15. *The map $x \mapsto \mathrm{ev}_x$ is a linear norm-preserving map $X \to X^{**}$.*

Proof. Linearity is easy. Note that

$$|\mathrm{ev}_x(f)| = |f(x)| \leq \|f\|\,\|x\| = \|x\|\,\|f\|,$$

so by definition $\|\mathrm{ev}_x\| \leq \|x\|$. By Hahn-Banach there's f with $\|f\| = 1$ and $f(x) = \|x\|$. Then

$$\|x\| = |f(x)| = |\mathrm{ev}_x(f)| \leq \|\mathrm{ev}_x\|\,\|f\| = \|\mathrm{ev}_x\|.$$

Hence $x \mapsto \mathrm{ev}_x$ is norm-preserving. ∎

Definition 2.1.16. A normed space X is called *reflexive* if X^{**} is of the form ev_x for some $x \in X$.

Example 2.1.17. If $p \in (1, \infty)$, the spaces ℓ_p, L_p are reflexive since for $\frac{1}{p} + \frac{1}{q} = 1$ we know $(L_p)^* \cong L_q$ and $(L_q)^* \cong L_p$, and similarly for ℓ_p.

However, ℓ_1 is not reflexive since $(\ell_1)^* \cong \ell_\infty$ and $(\ell_\infty)^* \ncong \ell_1$. Similarly, c_0 is not reflexive since $c_0^* \cong \ell_1$ but $\ell_1^* \cong \ell_\infty \ncong c_0$.

2.1.4 Application of Hahn-Banach - Approximation Theory

Example 2.1.18. The space $\mathrm{Span}\left\{x^k \;\middle|\; k \in \mathbb{Z}_{\geq 0}\right\}$ is dense in $\mathcal{C}([0,1])$ by the Weistrass approximation theorem.

Example 2.1.19. Consider

$$\mathcal{C}(\mathbb{T}) \coloneqq \{f \colon R \to \mathbb{C} \mid f \text{ is continuous and } 2\pi\text{-periodic}\}.$$

Then $\mathrm{Span}\left\{e^{2piin} \;\middle|\; n \in \mathbb{Z}_{\geq 0}\right\}$ is dense in $\mathcal{C}(\mathbb{T})$ by the second Weierstrass approximation theorem / Fejer's theorem.

Theorem 2.1.20. *Fix a weight function $w \colon \mathbb{R} \to \mathbb{R}$ such that $0 < w(t) < Ae^{-B|t|}$. Then*

$$E \coloneqq \mathrm{Span}\{t^n w \mid n \in \mathbb{Z}_{\geq 0}\}$$

is dense in $c_0(\mathbb{R})$.

Proof. **Step 1:** We show that $c^{i\alpha t} w \in \bar{E}$ for every $\alpha \in \mathbb{R}$. Let $f \in c_0(\mathbb{R})^*$ such that $f(E) = 0$. For $z \in \mathbb{C}$ write $\rho_z(t) \coloneqq e^{izt} w$. It holds that

$$\begin{aligned}
|\rho_z(t)| &= \left|e^{izt}\right| w \\
&= e^{\Re(izt)} w \\
&\leq e^{-(\Im z)t} A^{-B|t|} \\
&\leq Ae^{(|\Im z| - B)|t|}.
\end{aligned}$$

If $z \in S \coloneqq \{z \mid |\Im z| < B\}$ then $\rho_z \in c_0(\mathbb{R})$. Define $\varphi \colon S \to \mathbb{C}$ by $\varphi(t) = f(\rho_t)$.

Note that

$$\begin{aligned}\left(\frac{\rho_{z+h}-\rho_z}{h}\right)(t) &= \frac{e^{i(z+h)t}w(t)-e^{izt}w(t)}{h}\\ &= w(t)\,e^{izt}\frac{e^{iht}-1}{h}\\ &= w(t)\,e^{izt}it\cdot\frac{e^{iht}-1}{ith}\end{aligned}$$

where the last expression converges uniformly to e^{izt} as $h\to 0$ since $w(t)\,e^{izt}it$ is bounded and since $\frac{e^{iht}-1}{ith}$ converges uniformly to 1.

Hence

$$\begin{aligned}\varphi'(z) &= \lim_{h\to 0}\frac{\varphi(z+h)-\varphi(z)}{h}\\ &= \lim_{h\to 0} f\left(\frac{\rho_{z+h}-\rho_z}{h}\right)\\ &= f\left(\lim_{h\to 0}\frac{\rho_{z+h}-\rho_h}{h}\right)\\ &= f\left(ite^{izt}w(t)\right).\end{aligned}$$

In particular, φ is analytic and $\varphi'(0)=f(f(itw(t)))=0$ where the latter equality is by the choice of f.

Similarly

$$\varphi^{(n)}(t)=f\left((it)^n e^{izt}w(t)\right)$$

so

$$\varphi^{(n)}(0)=f(i^nt^nw(t))=0.$$

Hence by analyticity and complex analysis, $\varphi\equiv 0$. Hence $f(\rho_t)=0$ for every $z\in S$ and every $f\in c_0(\mathbb{R})^*$ for which $f(E)=0$. Hence $\rho_z\in\bar{E}$ for every $z\in S$.

Step 2: We'll show that $\mathcal{C}_c(\mathbb{R})\subseteq\bar{E}$ which finishes the proof because $\overline{\mathcal{C}_c(\mathbb{R})}=c_0(\mathbb{R})$.

Let $u\in\mathcal{C}_c(\mathbb{R})$ and let $(-M,M)$ contain $\operatorname{supp}(u)$. Define

$$u(t):=\frac{u\left(\frac{M}{\pi}t\right)}{w\left(\frac{M}{\pi}t\right)}.$$

This is supported in $(-\pi,\pi)$. We can then extend v to $\mathcal{C}(\mathbb{T})$. By Weirstrass' approximation there are trigonometric polynomials $(p_n)_{n\in\mathbb{N}}$ such that $p_n\xrightarrow{n\to\infty}v$ uniformly. Then

$$p_n\left(\frac{\pi}{M}t\right)w(t)\xrightarrow{n\to\infty}v\left(\frac{\pi}{M}t\right)w(t)=u(t)$$

uniformly since $w(t)$ is bounded. By the first step, we have $p_n\left(\frac{\pi}{M}t\right)w(t)\in\bar{E}$ hence also $u\in\bar{E}$, as required. ■

2.1.5 Application of Hahn-Banach - Convex Separation

Definition 2.1.21 (The Minkowski Functional). Let $X \in \mathbf{Vect}_{\mathbb{R}}$ and let $K \subseteq X$ be convex such that $0 \in K$. Define the *Minkowski function on X*

$$\begin{aligned} p_K \colon X &\to [0, \infty] \\ x &\mapsto \inf \left\{ \lambda > 0 \,\middle|\, \frac{x}{\lambda} \in K \right\}. \end{aligned}$$

Definition 2.1.22 (Internal Point). Let $X \in \mathbf{Vect}_{\mathbb{R}}$ and $A \subseteq X$. A point $a \in A$ is *internal* if

$$\forall y \in X \exists t_0 \in \mathbb{R} \forall t \leq t_0 \colon a + ty \in A.$$

Remark 2.1.23. If X is normed and $A \subseteq X$, every interior point of A is internal but the converse is generally false. E.g. take the union of two tangent circles and the line tangent to both of them. The tangential point is an internal point but not an interior point.

Fact 2.1.24. *Let $X \in \mathbf{Vect}_{\mathbb{R}}$ and let $K \subseteq X$ be convex such that $0 \in K$ is internal. Then $p_K \colon X \to [0, \infty)$ is sub-linear.*

Proof. One checks that $p_K(\lambda x) = \lambda p_K(x)$ for $\lambda > 0$.

We check the triangle inequality. Let $x, y \in X$. There are $(\lambda_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$ such that $\frac{x}{\lambda_n}, \frac{y}{\mu_n} \in K$ for all $n \in \mathbb{N}$ and that

$$\begin{aligned} \lambda_n &\xrightarrow{n \to \infty} p_k(x) \\ \mu_n &\xrightarrow{n \to \infty} p_k(y). \end{aligned}$$

We have

$$\frac{x+y}{\lambda_n + \mu_n} = \frac{\lambda_n}{\lambda_n + \mu_n} \frac{x}{\lambda_n} + \frac{\mu_n}{\lambda_n + \mu_n} \frac{y}{\mu_n} \in K$$

by convexity of K. So $p_K(x+y)$. So

$$p_K(x+y) \leq \lambda_n + \mu_n \xrightarrow{n \to \infty} p_K(x) + p_K(y). \qquad \blacksquare$$

Remark 2.1.25. We cannot in general reconstruct K from p_K. We know

$$p_K^{-1}([0,1)) \subseteq K \subseteq p_K^{-1}([0,1]),$$

but don't know if a point $p \in X$ for which $p_K(p) = 1$ are in K or not.

Definition 2.1.26 (Separating Function). Let X be a real Banach space. Let $A, B \subseteq X$ be disjoint. A linear functional $f \colon X \to \mathbb{R}$ separates A and B if $f \neq 0$ and

$$\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b).$$

Theorem 2.1.27. *Let X be a real vector space and $A, B \subseteq X$ be convex and disjoint such that A has an internal point. Then there exists $f\colon X \to \mathbb{R}$ which separates A, B.*

Proof. **Step 1:** Assume first that $B = \{x_0\}$ and that $0 \in A$ is internal. Let $Y = \operatorname{Span}\{x_0\}$ and define

$$\begin{aligned} \ell\colon Y &\to \mathbb{R} \\ \lambda x_0 &\mapsto \lambda p_A(x_0). \end{aligned}$$

Then

$$\begin{aligned} \forall \lambda > 0\colon \ell(\lambda x_0) &= \lambda p_A(x_0) = p_A(\lambda x_0) \\ \forall \lambda \leq 0\colon \ell(\lambda x_0) &= \lambda p_A(x_0) \leq 0 \leq p_A(\lambda x_0) \end{aligned}$$

so $\ell \leq p_A$.

We can extend ℓ to a function $f\colon X \to \mathbb{R}$ such that $f|_Y = \ell$ and $f \leq p_A$. But

$$\sup_{a \in A} f(a) \leq \sup_{a \in A} p_A(a) \leq 1 \leq p_A(x_0) = f(x_0).$$

Step 2: In general, let $a_0 \in A$ be internal and let $b_0 \in B$. Let

$$C \coloneqq A - B + b_0 - a_0 = \{a - b + b_0 - a_0 \mid a \in A, b \in B\}.$$

This is convex by writing down the definition, and $0 = a_0 - b_0 + b_0 - a_0 \in C$ is an internal point because a_0 is internal in A.

But, $b_0 - a_0 \notin C$ for otherwise $a = b$ which implies $A \cap B \neq \varnothing$. Hence, there's $f\colon X \to \mathbb{R}$ such that

$$\sup_{\substack{a \in A \\ b \in B}} \left(f(a) - f(b) + f(b_0) - f(a_0)\right) = \sup_{c \in C} f(c) \leq f(b_0 - a_0) = f(b_0) - f(a_0).$$

So

$$\sup_{\substack{a \in A \\ b \in B}} \left(f(a) - f(b)\right) \leq 0$$

which implies

$$\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b).$$

■

Example 2.1.28. Let $X \coloneqq \mathbb{R}[x]$ and let A be the subspace of polynomials with positive leading coefficient. This is convex. Take $B = \{0\}$, so that $A \cap B = \varnothing$.

Assume there's a separating $f\colon X \to \mathbb{R}$ for A, B. I.e.

$$\sup_{a \in A} f(a) \leq f(a) = 0.$$

Let $p \in X$ and $M > \deg p$. Then

$$\forall \delta > 0 \colon \delta x^m + p \in A.$$

Then

$$\forall \delta > 0 \colon \delta f(x^m) + f(p) = f(\delta x^m + p) \leq 0.$$

By sending $\delta \to 0$ we get $f(p) \leq 0$. By considering $f(-p)$ we get $f(p) \geq 0$ so $f \equiv 0$, a contradiction.

Note that A doesn't have an internal point, for otherwise the theorem would imply there's a separation for A, B.

We've so far discussed separation when X isn't normed, but there might be a norm on it.

Proposition 2.1.29. *Let X be a real Banach space and let $f \colon X \to \mathbb{R}$ be linear. Let $\lambda \in \mathbb{R}$ and $H \coloneqq f^{-1}(\lambda)$. Then*

1. *If f is bounded, H is closed.*
2. *If f is not bounded, H is dense.*

Proof. 1. This is immediate from topology.

2. It holds that

$$\sup_{x \in X} \frac{|f(x)|}{\|x\|} = \infty.$$

Take $(x_n)_{n \in \mathbb{N}}$ such that $\frac{|f(x_n)|}{\|x_n\|} \xrightarrow{n \to \infty} \infty$ and $y_n \coloneqq \frac{x_n}{f(x_n)}$ such that $f(y_n) = 1$ and $\|y_n\| \to \infty$. Then

$$\forall x \in X \colon x = \lim_{n \to \infty} (x - f(x) y_n + \lambda y_n).$$

But,

$$f(x - f(x) y_n + \lambda y_n) = f(x) - f(x) \cdot 1 + \lambda \cdot 1 = \lambda$$

so

$$x - f(x) y_n + \lambda y_n \in H,$$

so x is the limit of elements of H, which implies $x \in \bar{H}$. ■

Corollary 2.1.30. *Let X be a real Banach space and let $A, B \subseteq X$ be convex disjoint sets such that A has an interior point y_0. There's $f \in X^*$ separating A, B.*

Proof. We know there's some linear functional $f \colon X \to \mathbb{R}$, which we don't know is continuous. We have

$$\sup_{a \in B(y_0, r)} f(a) \leq \sup_{a \in A} f(a) \leq \lambda = \inf_{b \in B} f(b).$$

Let

$$H \coloneqq f^{-1}(\lambda + 1).$$

Then $H \cap B(y_0, r) = \varnothing$ so H is not dense, and by the proposition this implies f is bounded, i.e. continuous. ■

2.2 Banach Spaces

2.2.1 Baire Category Theorem & Uniform Boundedness

Definition 2.2.1. Let X be a topological space and let $A \subseteq X$. We say A is *nowhere dense* if $\int \bar{A} = \varnothing$. We say A is *meagre* (altenatively, a set of the set cateogry) if $A = \bigcup_{n \in \mathbb{N}} A_n$ where each A_n is nowhere dense. We say A is *comeagre* if A^C is meagre.

Theorem 2.2.2 (Baire). *A complete metric space is not meagre.*

Theorem 2.2.3 (Uniform Boundedness Principle). *Let Y be a Banach space and Y a normed space. Let $\mathcal{F} \subseteq \mathscr{L}(X, Y)$. Assume*

$$\forall x \in X \colon \sup_{T \in \mathcal{F}} \|Tx\| < \infty.$$

Then

$$\sup_{T \in \mathcal{F}} \|T\| < \infty.$$

This follows immediately from the following theorem and Baire's category theorem.

Theorem 2.2.4. *Let X be a Banach space, Y a normed space and $\mathcal{F} \subseteq \mathscr{L}(X, Y)$. Assume $\{x \mid \sup_{T \in \mathcal{F}} \|Tx < \infty\|\}$ is not meagre. Then $\sup_{T \in \mathcal{F}} < \infty$.*

Proof. Let

$$A_n := \left\{ x \;\middle|\; \sup_{T \in \mathcal{F}} \|Tx\| \le n \right\}.$$

Then A_n are closed because if $\varphi_T(x) = \|Tx\|$ we get

$$A_n = \bigcap_{T \in \mathcal{F}} \varphi_T^{-1}([0, n]).$$

Moreover, $\bigcup_{n \in \mathbb{N}_+} A_n = A$ so there's $n \in \mathbb{N}$ such that $\int \overline{A_n} \neq \varnothing$. Then there's $y_0 \in X$ and $r > 0$ such that $B(y_0, r) \subseteq A_n$. If $|z| < r$ we can write

$$\|Tz\| = \left\| T\left(\frac{y_0 + Z}{z}\right) - T\left(\frac{y_0 - z}{2}\right) \right\| \le \frac{1}{2}\left(\|T(y_0 + Z)\| + \|T(y_0 - z)\|\right) \le \frac{1}{2}(n + n) = n.$$

Now

$$\forall T \in \mathcal{F} \forall x \in X \colon \|Tx\| = \left\| T\left(\frac{rx}{2\|x\|}\right) \right\| \cdot \frac{2\|x\|}{r} \le \frac{2n}{r}\|x\|.$$

Hence

$$\sup_{T \in \mathcal{F}} \|T\| \le \frac{2n}{r} < \infty.$$

∎

Corollary 2.2.5. *Let X be a Banach space and $A \subseteq X$. Then A is bounded iff $f(A)$ is bounded for all $f \in X^*$.*

Proof. • Assume A is bounded and $f \in X^*$. We get

$$|f(a)| \leq \|f\| \|a\| \leq \left(\sup_{a \in A} \|a\|\right) \|f\|$$

so $f(A)$ is bounded.

• Assume $f(A)$ is bounded for all $f \in X^*$. Define

$$\mathcal{F} := \{\mathrm{ev}_x \mid x \in A\} \subseteq X^{**} \subseteq \mathscr{L}(X^*, \mathbb{F}).$$

Then

$$\forall f \in X^* \colon \sup_{T \in \mathcal{F}} \|Tf\| = \sup_{x \in A} |\mathrm{ev}_x(f)| = \sup_{x \in A} |f(x)| < \infty.$$

By the uniform boundedness theorem this implies

$$\sup_{T \in \mathcal{F}} \|T\| = \sup_{x \in A} \|\mathrm{ev}_x\| = \sup_{x \in A} \|x\| < \infty.$$

■

Proposition 2.2.6. *Let H be a Hilbert space and $T \colon H \to H$ be linear and self-adjoint. Then T is continuous.*

Proof. Let

$$A := \{Tx \mid \|x\| \leq 1\}.$$

We know that every $f \in H^*$ is of the form $f(x) = \langle x, y \rangle$. Hence

$$\begin{aligned} f(A) &= \{f(a) \mid a \in A\} \\ &= \{f(Tx) \mid \|x\| \leq 1\} \\ &= \{\langle Tx, y \rangle \mid \|x\| \leq 1\} \\ &= \{\langle x, Ty \rangle \mid \|x\| \leq 1\}. \end{aligned}$$

By Cauchy-Schwarz,

$$|\langle x, Ty \rangle| \leq \|x\| \|Ty\| \leq \|Ty\|$$

so $f(A)$ is bounded.
By the uniform boundedness theorem it follows that A is bounded, hence T is continuous. ■

2.2.2 Applications to Harmonic Analysis

Definition 2.2.7. Given $f \in \mathcal{C}(\mathbb{T})$ and define

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} \, \mathrm{d}x.$$

The Fourier series of f is

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx}.$$

Theorem 2.2.8 (Dirichlet). *Let $f \in \mathcal{C}prs\mathbb{T}$, it holds that*

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx} = f(x)$$

and the partial sums $S_N f = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$ converge uniformly to f.

Theorem 2.2.9. *Let*

$$A \coloneqq \{f \in \mathcal{C}(\mathbb{T}) \mid \forall q \in \mathbb{Q}\colon (S_N f(q)) \text{ isn't bounded}\}.$$

A is comeagre in $\mathcal{C}(\mathbb{T})$.

Proof. For $q \in \mathbb{Q}$ define

$$A_q \coloneqq \left\{f \in \mathcal{C}(\mathbb{T}) \;\middle|\; (S_N f(q))_{N\in\mathbb{N}} \text{ isn't bounded}\right\}.$$

Then $A = \bigcap_{q\in\mathbb{Q}} A_q$, so it's enough to show that each A_q is comeagre. WLOG assume $q = 0$. Define

$$\begin{aligned} T_n\colon \mathcal{C}(\mathbb{T}) &\to \mathbb{C} \\ f &\mapsto S_N f(0). \end{aligned}$$

We claim $T_n \in \mathcal{C}(\mathbb{T})^*$ but $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$. By uniform boundedness this would imply

$$A_0^C = \{f \mid T_n f \text{ is bounded}\}$$

is meagre.

Indeed,

$$\begin{aligned} T_n f &= \sum_{n=-N}^{N} \hat{f}(n) \\ &= \sum_{n=-N}^{N} \left(\frac{1}{2\pi}\int_0^{2\pi} f(x) e^{\pm inx}\,\mathrm{d}x\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x) \cdot \underbrace{\left(\sum_{n=-N}^{N} e^{inx}\right)}_{D_N(x)} \mathrm{d}x \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x) D_N(x)\,\mathrm{d}x \end{aligned}$$

from which

$$\begin{aligned} |T_n f| &\le \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,|D_N(x)|\,\mathrm{d}x \\ &\le \|f\| \cdot \underbrace{\frac{1}{2\pi}\int_0^{2\pi} |D_n(x)|\,\mathrm{d}x}_{I_n} \end{aligned}$$

so $\|T_n\| \leq I_n < \infty$.

Actually $\|T_n\| = I_n$ by picking $f = \operatorname{sgn} D_N$ and approximating it by continuous functions. But

$$I_N \coloneqq \frac{1}{2\pi} \int_0^{2\pi} \frac{\sin\left[\left(N + \frac{1}{2}\right) x\right]}{\sin\left(\frac{x}{2}\right)} \, \mathrm{d}x \xrightarrow{n\to\infty} \infty,$$

so $\sup_{n\in\mathbb{N}} \|T_n\| = \infty$ as required by the above reduction. ∎

2.2.3 The Open Mapping Theorem

Definition 2.2.10 (Open Map). Let X, Y be topological spaces. A map $\varphi\colon X \to Y$ is called *open* if $\varphi(U)$ is open for every open $U \subseteq X$.

Theorem 2.2.11 (The Open Mapping Theorem). *Let X, Y be Banach spaces and let $T \in \mathscr{L}(X, Y)$. If T is onto, it's open.*

Proof. We prove that for any open ball $B(x, r)$ the set $T(B(x, r))$ contains a ball around Tx. Note that

$$T(B(x,r)) = T(rB(0,1) + x) \qquad = r \cdot T(B(0,1)) + Tx.$$

Hence it's enough to show that $T(B(0,1))$ contains a ball around 0. Note that

$$Y = T(X) = \bigcup_{n\in\mathbb{N}_+} T(B_X(0,1)).$$

By Baire's theorem

$$\exists n \in \mathbb{N}_+ : \operatorname{int}\overline{T(B(0,1))} \neq \varnothing$$

or in other words

$$\overline{T(B(0,1))} \supseteq B(y,r).$$

We have to fix three things. We want a ball around zero, we want the image of a ball of radius 1, and we want the actual image to contain it, rather than the closure of the image.

Ball Around Zero & Image of 1-Ball: Write

$$\begin{aligned} B(0,r) &= B(y,r) - y \\ &\subseteq \overline{T(B(0,n)) - Tx} \\ &= \overline{T(B(0,n) - x)} \\ &\subseteq \overline{T(B(0, n + \|x\|))}. \end{aligned}$$

Dividing by $n + \|x\|$ we get

$$\overline{T(B(0,1))} \supseteq B\left(0, \frac{r}{n + \|x\|}\right) =: B(0, \varepsilon).$$

Closure: Note that for every $a > 0$ it holds that

$$\overline{T\left(B\left(0,a\right)\right)} \supseteq B\left(0,a\varepsilon\right). \tag{2.1}$$

We show that $T\left(B\left(0,1\right)\right) \supseteq B\left(0,\frac{\varepsilon}{2}\right)$. Let $y \in B\left(0,\frac{\varepsilon}{2}\right)$. By (2.1) with $a = \frac{1}{2}$ there's $x_1 \in B\left(\frac{1}{2}\right)$ such that

$$\left\|y - Tx_1\right\| < \frac{\varepsilon}{4}.$$

By (2.1) with $\frac{1}{2^n}$ there are $x_n \in B\left(0,\frac{1}{2^n}\right)$ such that

$$\left\|y - T\left(\sum_{i\in[n]} x_i\right)\right\| < \frac{\varepsilon}{2^{n+1}}. \tag{2.2}$$

Since

$$\sum_{n\in\mathbb{N}_+} \left\|x_n\right\| < \sum_{n\in\mathbb{N}_+} \frac{1}{2^n} = 1 < \infty$$

and X is complete, we get that $\sum_{n\in\mathbb{N}_+} x_n$ converges to sum x. We have

$$\left\|x\right\| \leq \sum_{n\in\mathbb{N}_+} \left\|x_n\right\| < 1$$

so $x \in B\left(0,1\right)$.

Let $n \to \infty$ in (2.2), we get $\left\|y - Tx\right\| \leq 0$ so $y = Tx$. ∎

Corollary 2.2.12. *Let X, Y be Banach spaces and $T \in \mathscr{L}\left(X,Y\right)$. If T is a bijection, T^{-1} is bounded and there's $c > 0$ such that $\left\|Tx\right\|_Y \geq c\left\|x\right\|_X$.*

Proof. We know by the open mapping theorem that $T\left(B\left(0,1\right)\right) \geq T\left(0,\varepsilon\right)$ for some $\varepsilon > 0$, or in other words $B\left(0,1\right) \supseteq T^{-1}\left(B\left(0,\varepsilon\right)\right)$. Then

$$\left\|T^{-1}x\right\| = \left\|T^{-1}\left(\frac{\varepsilon}{2}\cdot\frac{x}{\left\|x\right\|}\right)\right\| \cdot \frac{2\left\|x\right\|}{\varepsilon} \leq \frac{2}{\varepsilon}\left\|x\right\|$$

where in the last inequality we use $\frac{\varepsilon}{2}\frac{x}{\left\|x\right\|} \in B\left(0,\varepsilon\right)$. Hence $\left\|T^{-1}\right\| \leq \frac{2}{\varepsilon}$. In fact one can check $\left\|T^{-1}\right\| \leq \frac{1}{\varepsilon}$.

For every $x \in X$ we now have

$$\left\|x\right\| = \left\|T^{-1}Tx\right\| \leq \left\|T^{-1}\right\|\left\|Tx\right\| \leq \frac{2}{\varepsilon}\left\|Tx\right\|.$$

Hence $\left\|Tx\right\| \geq \frac{\varepsilon}{2}\left\|x\right\|$. ∎

Remark 2.2.13. The fact that T^{-1} is bounded can be shown more directly. Since T is surjective, it's open so T^{-1} is continuous and therefore bounded.

Corollary 2.2.14. *Let X be a complete Banach space with respect to two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. If*

$$\exists C > 0 \forall x \in X\colon\ \|x\|_1 \leq C\|x\|_2$$

then

$$\exists \tilde{C} > 0 \forall x\, \|x\|_2 \leq \tilde{C}\|x\|_1,$$

so the norms are equivalent.

Proof. Apply 2.2.12 to $i\colon (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$. ∎

2.2.4 Application of the Open Mapping Theorem to Harmonic Analysis

Definition 2.2.15 (Fourier Coefficients for Functions on the Circle). Write

$$L_1(\mathbb{T}) \coloneqq \left\{ f\colon \mathbb{R} \to \mathbb{C} \;\middle|\; \substack{f \text{ is } 2\pi\text{-periodic} \\ \int_0^{2\pi} |f|\,\mathrm{d}x < \infty} \right\} \cong L_1([0, 2\pi])$$

with the norm

$$\|f\| = \frac{1}{2\pi}\int_0^{2\pi} |f(x)|\,\mathrm{d}x.$$

This is a Banach space. For every $f \in L_1(\mathbb{T})$ and $z \in \mathbb{Z}$ define

$$\hat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\,\mathrm{d}x.$$

Fact 2.2.16.

1. *If $\hat{f}(n) = 0$ for every $n \in \mathbb{Z}$ then $f = 0$ in L_1.*
2. Riemann-Lebesgue*:* $\hat{f}(n) \xrightarrow{n\to\pm\infty} 0$.

Given $(a_n)_{n\in\mathbb{Z}} \subseteq \mathbb{C}$ such that $a_n \xrightarrow{n\to\pm\infty} 0$ we want to ask if there's $f \in L_1(\mathbb{T})$ such that $\hat{f}(n) = a_n$ for every $n \in \mathbb{Z}$. It turns out that the answer is no, which we prove using the open mapping theorem.

Definition 2.2.17. Define

$$c_0(\mathbb{Z}) \coloneqq \left\{ (a_n)_{n\in\mathbb{Z}} \;\middle|\; a_n \xrightarrow{n\to\pm\infty} 0 \right\}$$

with the supremum norm.

Definition 2.2.18. Define

$$\begin{aligned} \mathcal{F}\colon L_1(\mathbb{T}) &\to c_0(\mathbb{Z}) \\ f &\mapsto \hat{f}. \end{aligned}$$

Remark 2.2.19. $\mathcal{F}$ is linear. It's bounded, because

$$\left|\hat{f}(n)\right| \leq \frac{1}{2\pi}\int_0^{2\pi}\left|f(x)e^{-inx}\right|\mathrm{d}x = \|f\|.$$

It's injective by the first part of 2.2.16. If $\mathcal{F}$ is onto, then by the corollary $\|\mathcal{F}(f)\|_\infty \geq c\|f\|_1$. But, take $f = D_N = \sum_{n\in[-N,N]} e^{inx}$. Then

$$\|\mathcal{F}(f)\| = \|(0,0,\ldots,0,1,1,\ldots,1,1,0,0,0,\ldots)\| = 1.$$

But, $\|D_N\|_1 \xrightarrow{N\to\infty} \infty$. This is a contradiction, hence $\mathcal{F}$ is not onto $c_0(\mathbb{Z})$.

2.2.5 The Closed Graph Theorem

Definition 2.2.20 (Graph of a Map). Let X, Y be Banach spaces and let $E \leq X$. Let

$$T\colon E \to Y$$

be linear. The *graph of* T is

$$\Gamma(T) \coloneqq \{(x, Tx) \mid x \in E\} \subseteq X \times Y.$$

Remark 2.2.21. We can define a norm on $X \times Y$ by

$$\|(x,y)\| = \|x\| + \|y\|.$$

$X \times Y$ with this norm is denote $X \oplus_1 Y$ or sometimes $X \oplus Y$. This is a Banach space.

Definition 2.2.22. T is called closed if $\Gamma(T) \subseteq X \oplus Y$ is a closed set.

Proposition 2.2.23. *T is closed iff for every $(x_n)_{n\in\mathbb{N}_+} \subseteq E$ such that $x_n \to x$ implies $Tx_n \to y$, it holds that $x \in E$ and $y = Tx$.*

Proof.
- Assume T is closed. If $x_n \to x$ and $Tx_n \to y$ then $(x_n, Tx_n) \to (x,y)$ so $(x,y) \in \Gamma(T)$, so $x \in E$ and $y \in Tx$.
- The other direction is left as an exercise. ■

Example 2.2.24. Let $X = Y = \mathcal{C}[0,1]$ and let $E = \mathcal{C}^1[0,1]$. Define

$$\begin{aligned} T\colon E &\to Y \\ f &\mapsto f'. \end{aligned}$$

Then T is closed. Assume $(f_n)_{n\in\mathbb{N}_+} \subseteq E$ is such that $f_n \to f$ and $f_n' \to g$. Then

$$f_n(x) - f_n(0) = \int_0^x f_n'(t)\,\mathrm{d}t \to \int_0^x g(t)\,\mathrm{d}t$$

where the first expression converges also to $f(x) - f(0)$. Hence

$$f(x) = f(0) + \int_0^x g(t)\,\mathrm{d}t.$$

Hence $f \in E$ and $f' = g$.

Note that E is note closed (it's in fact dense) and that T is not bounded.

Theorem 2.2.25. *Let X, Y be Banach spaces. A closed map $T\colon X \to Y$ is continuous.*

Proof. T is closed, hence $\Gamma(T)$ is closed in $X \oplus Y$ and is therefore a Banach space. Let π_X, π_Y be the projections from Γ to X and to Y. π_X is a continuous bijection so π_X^{-1} is continuous by 2.2.12. Then $T = \pi_Y \circ \pi_X^{-1}$ is continuous as a composition of continuous maps. ■

2.2.6 Projections and Quotient Spaces

Definition 2.2.26 (Projection). A *projection* is a linear map $P\colon X \to X$ such that $P^2 = P$.

Proposition 2.2.27. *Given a projection $P\colon X \to X$ we have $X = \operatorname{Im} P \oplus \ker P$.*

Proof. Let $x \in X$, we can write $x = (x - Px) + Px$ where $Px \in \operatorname{Im}(P)$ and $x - P_x \in \ker(P)$. If $x \in \operatorname{Im}(P) \cap \ker(P)$ there's $y \in X$ such that $x = Py$ then

$$0 = Px = P^2 y = Py = x$$

so $x = 0$, so the sum is direct. ■

Remark 2.2.28. If $x = e + f$ where $e \in \operatorname{Im} P$ and $f \in \ker P$ we get

$$Px = Pe + Pf = Pe = e.$$

Definition 2.2.29 (Projection onto a Subspace). Let $P\colon X \to X$ be a projection, let $E = \operatorname{Im} P$ and $F = \ker P$. We say P is *the projection onto E parallel to F*.

Definition 2.2.30 (Complemented Subspace). A closed subspace $E \leq X$ of a Banach space is called *complemented* if there exists $F \leq X$ closed such that $X = E \oplus F$.

Theorem 2.2.31. *Let X be a Banach space. For a closed $E \leq X$ the following are equivalent.*

1. *E is complemented.*
2. *There is a continuous projection $P\colon X \to E$.*

Proof. **2** $\implies$ **1:** Take $F = \ker P$ which is closed, and $X = E \oplus F$ since $E = \operatorname{Im} P$ and $F = \ker P$.

1 $\implies$ **2:** Assume $X = E \oplus F$ where E, F are closed subspaces. Take P to be the projection onto E parallel to F.

We show that P is closed. Assume $(x_n)_{n\in\mathbb{N}_+}$ is such that $x_n \to x$ and $Px_n \to y$. Since E is closed, $y \in E$. But,

$$x - y = \lim_{n\to\infty} (x_n - Px_n) \in F$$

since F is closed. Hence

$$x = y + (x - y)$$

where $y \in E$ and $(x - y) \in F$. So, by definition, $Px = y$. Hence P is closed.

By the closed graph theorem, P is then continuous. ■

Fact 2.2.32. *1. c_0 is not complemented in ℓ_∞.*

2. Lindenstaruss-Tzafriri*: Every closed subspace of X is complemented if and only if X is isomorphic (in the sense that there's a bijection $T\colon X \to H$ such that $c\,\|x\| \le \|Tx\| \le C\,\|x\|$ for constants $c, C > 0$) to a Hilbert space.*

2.2.7 Quotient Spaces

Definition 2.2.33 (Quotient Space). Let X be a Banach space and $E \le X$ a closed subspace. We define *the quotient space*

$$X/E = \{x + E \mid x \in X\} = \{[x] \mid x \in X\}$$

where we identify $x \sim y$ iff $x - y \in E$. This is a Banach space with

$$\|x + E\| \coloneqq \inf_{y\sim x} \|y\| = \inf_{e\in E} \|x - e\| = d\,(x, E)\,.$$

Fact 2.2.34. *If X is a Banach space and E is closed, then X/E is a Banach spaces.*

Proof. One should check that X/E is a vector space and that $\|\cdot\|$ defines a norm on it. We prove completeness.

Let $(x_n)_{n\in\mathbb{N}_+} \subseteq X$ such that $\sum_{n\in\mathbb{N}_+} \|[x_n]\| < \infty$. It suffices to show that $(x_n)_{n\in\mathbb{N}_+}$ is convergent. Pick $y_n \sim x_n$ such that $\|y_n\| \le \|[x_n]\| + \frac{1}{2^n}$. Then

$$\sum_{n\in\mathbb{N}_+} \|y_n\| \le \sum_{n\in\mathbb{N}_+} \|[x_n]\| + \sum_{n\in\mathbb{N}_+} \frac{1}{2^n} < \infty.$$

Since X is complete, we get $\sum_{n\in\mathbb{N}_+} y_n = y$ for some $y \in X$. Then

$$\begin{aligned}\left\|\sum_{n\in[N]}[x_n]-[y]\right\| &= \left\|\left[\sum_{n\in[N]}x_n-y\right]\right\| \\ &= \left\|\left[\sum_{n\in[N]}y_n-y\right]\right\| \\ &\le \left\|\sum_{n\in[N]}y_n-y\right\| \\ &\xrightarrow{N\to\infty} 0.\end{aligned}$$

Hence X/E is complete. ■

Theorem 2.2.35. *Let X, Y be Banach spaces and let $T \in \mathcal{L}(X, Y)$ be surjective. There is an isomorphism*

$$X/\ker T \cong Y.$$

Proof. Let $E := \ker T$ and define

$$\begin{aligned} S\colon X/E &\to Y \\ [x] &\mapsto Tx. \end{aligned}$$

Note that

$$[x]=[y] \iff x-y\in E=\ker T \iff T(x-y)=0 \iff Tx=Ty \iff S([x])=S([y])$$

so S is well-defined and injective. S is surjective since T is surjective

Let $[x] \in X/E$. For every $\varepsilon > 0$ there's $y \in X/E$ such that $[y] = [x]$ and $\|y\| \le \|[x]\| + \varepsilon$. Then

$$\begin{aligned}\|S([x])\| &= \|S([y])\| \\ &= \|Ty\| \\ &\le \|T\|\,\|y\| \\ &\le \|T\|\left(\|[x]\|+\varepsilon\right).\end{aligned}$$

Letting $\varepsilon > 0$ we get $\|S\| \le \|T\|$ (in fact there's equality $\|S\| = \|T\|$). Hence S is continuous.

By the open mapping theorem a continuous bijection is an isomorphism, hence so is S. ■

Corollary 2.2.36. *Let X be a Banach space and let $E, F \le X$ be closed subspaces such that $X = E \oplus F$. It holds that*

$$X/E \cong F.$$

Proof. Let $P\colon X \to F$ be the projection onto F parallel to E. This is a surjective and continuous (which follows from the open graph theorem), hence 2.2.35 gives

$$X/_E = X/_{\ker P} \cong F. \qquad \blacksquare$$

2.2.8 Shauder Bases

Definition 2.2.37 (Shauder Base). A sequence $(e_n)_{n\in\mathbb{N}_+}$ is a Banach space X is a *(Shauder) base* if for every $x \in X$ there are unique $(\alpha_n)_{n\in\mathbb{N}_+} \subseteq \mathbb{F}$ such that

$$x = \sum_{n\in\mathbb{N}_+} \alpha_n e_n.$$

Example 2.2.38. In a Hilbert space, an orthonormal base is a (Shauder) base, and $\alpha_n = \langle x, e_n\rangle$.

Example 2.2.39. In ℓ_p for $p \in [1,\infty)$, the vectors e_n defined by $(e_n)_i = \delta_{n,i}$, form a base. Indeed $a = (a_1, a_2, a_3, \ldots)$ can be written uniquely as $\sum_{n\in\mathbb{N}_+} a_n e_n$.

Example 2.2.40. In $\mathscr{C}[0,1]$, take $e_n(t) = t^n$ for every $n \in \mathbb{N}$. We know by the Weierstrass approximation theorem that $\overline{\operatorname{Span}\{e_n \mid n \in \mathbb{N}\}} = \mathscr{C}[0,1]$.

However, $(e_n)_{n\in\mathbb{N}}$ is not a basis. If $f = \sum_{n=0}^{\infty} a_n t^n$ then $f \in \mathscr{C}^\infty$. Hence we cannot write any $f \in \mathscr{C}[0,1] \setminus \mathscr{C}^\infty[0,1]$ as $\sum_{n\in\mathbb{N}} a_n t^n$.

Definition 2.2.41. Given a basis $(e_n)_{n\in\mathbb{N}_+}$, define $\alpha_n\colon X \to \mathbb{F}$ by

$$x = \sum_{n\in\mathbb{N}_+} \alpha_n(x)\, e_n$$

and

$$\begin{aligned} P_N\colon X &\to X \\ X &\mapsto \sum_{n\in[N]} \alpha_n(x)\, e_n. \end{aligned}$$

Remark 2.2.42. In Hilbert spaces with an orthonormal basis it holds that $\|\alpha_n\| = 1$ and $\|P_N\| = 1$.

Remark 2.2.43. If X has a countable base, it's separable. One might ask if any separable space has a base. Enlfó showed in 1973 that the answer is no.

Theorem 2.2.44. *If $(e_n)_{n\in\mathbb{N}_+}$ is a base for a Banach space X. There exists $C > 0$ such that $\|P_k\| \leq C$, and $\alpha_n \in X^*$.*

Proof. Define $\|\cdot\|_1$ on X by

$$\|x\|_1 \coloneqq \sup_{k\in\mathbb{N}_+} \|P_k x\|.$$

This is finite since $P_k x \xrightarrow{n\to\infty} x$ and it is easily checked that this is a norm. We have

$$\forall x \in X\colon\ \|x\|_1 = \lim_{k\to\infty} \|P_k x\| = \|x\|.$$

- We show $(X, \|\cdot\|_1)$ is complete. Let $(x_n)_{n\in\mathbb{N}_+}$ be $\|\cdot\|_1$-Cauchy. By our bound, it's also $\|\cdot\|$-Cauchy. Hence there's

  $$x \coloneqq \|\cdot\|\text{-}\lim_{n\to\infty} x_n.$$

  For every $k \in \mathbb{N}_+$, $(P_k x_n)_{n\in\mathbb{N}_+}$ is also $\|\cdot\|$-Cauchy, so $P_k x_n \xrightarrow{n\to\infty} y_k$ for some $y_k \in X$. We want to show

  $$\lim_{n\to\infty}\lim_{k\to\infty} P_k x_n = \lim_{k\to\infty}\lim_{n\to\infty} P_k x_n$$

  which then implies

  $$\begin{aligned} x &= \lim_{n\to\infty} P_k x \\ &= \lim_{n\to\infty}\lim_{k\to\infty} P_k x_n \\ &= \lim_{k\to\infty}\lim_{n\to\infty} P_k x_n \\ &= \lim_{k\to\infty} y_k \end{aligned}$$

  Fix $\varepsilon > 0$. Because $(x_n)_{n\in\mathbb{N}_+}$ is Cauchy, there's $n_0 \in \mathbb{N}_+$ such that for all $m \geq n \geq n_0$ we have $\|x_m - x_n\|_1 < \varepsilon$. Hence

  1. For every $m \geq n \geq n_0$ it holds that $\|x_n - x_m\| < \varepsilon$. Letting $m \to \infty$ we get $\|x_n - x\| < \varepsilon$.
  2. For every $k \in \mathbb{N}_+$ and every $m \geq n \geq n_0$ we know $\|P_k x_m - P_k x_n\| < \varepsilon$ so by letting $m \to \infty$ we get $\|y_k - P_k x_n\| < \varepsilon$.

  Choose $k_0 \in \mathbb{N}_+$ such that $\|x_{n_0} - P_k x_{n_1}\| < \varepsilon$ for all $k \geq k_0$. Then

  $$\|y_k - x\| \leq \|y_k - P_k x_{n_0}\| + \|P_k x_{n_0} - x_{n_0}\| + \|x_{n_0} - x\| < 3\varepsilon.$$

  Then indeed $y_k \xrightarrow{k\to\infty} x$ in $\|\cdot\|$.

  Fix $1 \leq j \leq k$. Then α_j is bounded on $E_k \coloneqq \operatorname{Span}(e_i)_{i\in[k]}$. We have

  $$\lim_{n\to\infty} = \lim_{n\to\infty} \alpha_j(P_k x_n) = \alpha_j(y_k).$$

Hence $\alpha_j(y_k)$ doesn't depend on k. Denote $c_j \coloneqq \alpha_j(y_k)$. Hence

$$y_k = \sum_{j\in[k]} c_j e_j.$$

Hence

$$x = \sum_{j\in\mathbb{N}_+} c_j e_j.$$

We now show $x_n \xrightarrow{n\to\infty} x$ in $\|\cdot\|_1$. Fix $\varepsilon > 0$ as before. Then

$$\exists n_0 \in \mathbb{N}_+ \forall k \in \mathbb{N}_+ \forall n \geq n_0 \colon \|y_k - P_k x_n\| < \varepsilon.$$

Hence

$$\begin{aligned}\|x_n - x\|_1 &= \sup_{k\in\mathbb{N}_+} \|P_k x_n P_k x\| \\ &= \sup_{k\in\mathbb{N}_+} \|P_k x_n - y_k\| \\ &< \varepsilon.\end{aligned}$$

Hence $(X, \|\cdot\|_1)$ is complete.

- By the corollary we have $\|x\|_1 \leq C\|x\|$. This implies

  $$\forall k \in \mathbb{N}_+ \colon \|P_k x\| \leq C\|x\|$$

  so $\|P_k\| \leq C$.

  For the part $\alpha_n \in X^*$, we have

  $$\begin{aligned}|\alpha_n(x)|\,\|e_n\| &= \|\alpha_n(x)\,e_n\| \\ &= \|P_n x - P_{n-1}x\| \\ &\leq 2C\|x\|\end{aligned}$$

  so $\|\alpha_n\| \leq \frac{2C}{\|e_n\|}$. ■

2.3 Finite-Dimensional Spaces

2.3.1 Definitions

Definition 2.3.1 (Isomorphic Normed Spaces). Let X, Y be normed spaces. We say X, Y are *isomorphic* if there's a bijection $T \colon X \to Y$ and there are $c, C \in \mathbb{R}_+$ such that

$$\forall x \in X \colon c\|x\|_X \leq \|Tx\|_Y \leq C\|x\|_X .$$

Theorem 2.3.2. *Let X, Y be normed spaces with $\dim X = \dim Y = n \in \mathbb{N}$. Then X, Y are isomorphic.*

Corollary 2.3.3. *Every finite-dimensional normed space is a Banach space.*

Corollary 2.3.4. *If X is a finite-dimensional normed space, any linear $T\colon X \to Y$ is bounded.*

Definition 2.3.5 (The Banach-Mazur distance). Let X, Y be isomorphic Banach spaces. The *Banach-Mazur distance* is

$$\begin{aligned} d_{\mathrm{BM}}(X, Y) &:= \inf\left\{\|T\| \left\|T^{-1}\right\| \;\middle|\; T\colon X \to Y \text{ is a linear bijection}\right\} \\ &= \inf\left\{\frac{b}{a} \;\middle|\; \substack{\text{There's a linear bijection } T\colon X \to Y \text{ such that} \\ \forall x \in X\colon a\|x\| \le \|Tx\| \le b\|x\|}\right\}. \end{aligned}$$

Proposition 2.3.6.
1. *$d_{\mathrm{BM}}(X, Y) \ge 1$ and $d_{\mathrm{BM}}(X, Y) = 1$ if and only if X, Y are isometric.*
2. *$d_{\mathrm{BM}}(X, Y) = d_{\mathrm{BM}}(Y, X)$.*
3. *$d_{\mathrm{BM}}(X, Z) \le d_{\mathrm{BM}}(X, Y)\, d_{\mathrm{BM}}(Y, Z)$.*

Corollary 2.3.7. $\log d_{\mathrm{BM}}$ *is a metric on the space of isomorphism classes of finite-dimensional normed spaces quotiented by the isometric relation.*

Proposition 2.3.8. $d_{\mathrm{BM}}(\ell_1^n, \ell_2^n) = \sqrt{n}$.

Proof.
- We first show that $d_{\mathrm{BM}}(\ell_1^n, \ell_2^n) \le \sqrt{n}$.

  Recall that

$$\|x\|_1 = \sum_{i \in [n]} |x_i| \cdot 1 \le \left(\sum_{i \in [n]} |x_i|^2\right)^{\frac{1}{2}} \cdot \left(\sum_{i \in [n]} 1^2\right)^{\frac{1}{2}} = \sqrt{n}\, \|x\|_2 .$$

  For the other inequality, assume $\|x\|_1 = 1$. The $\forall i \in [n]\colon |x_i| \le 1$. Then $\forall i \in [n]\colon |x_i|^2 \le |x_i|$. Hence

$$\|x\|_2 = \left(\sum_{i \in [n]} |x_i|^2\right)^{\frac{1}{2}} \le \left(\sum_{i \in [n]} |x_i|\right)^{\frac{1}{2}} = 1.$$

  Hence by homogeneity

$$\forall x \in X\colon \|x\|_2 = \left\|\frac{x}{\|x\|_1}\right\|_2 \cdot \|x\|_1 \le 1 \cdot \|x\|_1 = \|x\|_1 .$$

  Hence

$$\|x\|_2 \le \|x\|_1 \le \sqrt{n}\, \|x\|_2 .$$

  Hence

$$d_{\mathrm{BM}}(\ell_1^n, \ell_2^n) \le \frac{\sqrt{n}}{1} = \sqrt{n}.$$

- Let $T\colon \ell_1^n \to \ell_2^n$ be a bijection such that

  $$\forall x \in X\colon a\left\|x\right\|_1 \leq \left\|Tx\right\|_2 \leq b\left\|x\right\|_1.$$

  By the parallelogram law

  $$\begin{aligned}\sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}\left\|\sum_{i\in[n]}\theta_i Te_i\right\|_2^2 &= \sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}\left\langle\sum_{i\in[n]}\theta_i Te_i, \sum_{j\in[n]}\theta_j Te_j\right\rangle\\ &= \sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}\sum_{i,j\in[n]}\theta_i\theta_j\left\langle Te_i, Te_j\right\rangle\\ &= \sum_{i,j\in[n]}^{n}\left(\sum_{(\theta_i)_{i\in[n]}\theta_i\theta_j}\right)\left\langle Te_i, Te_j\right\rangle\\ &= 2^n\sum_{i\in[n]}\left\langle Te_i, Te_i\right\rangle\\ &= 2^n\sum_{i\in[n]}\left\|Te_i\right\|_2^2\\ &\leq 2^n\sum_{i\in[n]}\left(b\left\|e_i\right\|_1\right)^2\\ &= 2^n\cdot n\cdot b^2.\end{aligned}$$

  On the other hand,

  $$\begin{aligned}\sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}\left\|\sum_{i\in[n]}\theta_i Te_i\right\|_2^2 &\geq \sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}\left(a\left\|\sum_{i\in[n]}\theta_i e_i\right\|\right)^2\\ &= \sum_{(\theta_i)_{i\in[n]}\subseteq\{\pm1\}}(an)^2\\ &= 2^n n^2 a^2.\end{aligned}$$

  Combining the inequalities we get $2^n n^2 a^2 \leq 2^n n b^2$, from which we get $\frac{b}{a} \geq \sqrt{n}$. ■

Theorem 2.3.9 (Auerbach). *Let X be a real normed space of dimension $n \in \mathbb{N}$. There exists a basis $(e_i)_{i\in[n]}$ such that for every $x = \sum_{i\in[n]}\alpha_i e_i$ in X we have $|\alpha_i| \leq \|x\|$ and $\|e_i\| = 1$.*

Equivalently, if we write $x = \sum_{i\in[n]}\alpha_i(x)e_i$, we have $\|\alpha_i\| = 1$ (where the norm on α_i is in X^*) and $\|e_i\| = 1$.

We call such a basis an Auerbach basis.

Proof. Without loss of generality, write $X = (\mathbb{R}^n, \|\cdot\|)$ for some norm $\|\cdot\|$. For $(y_i)_{i\in[n]} \subseteq \mathbb{R}^n$ define

$$D(y_1, \dots, y_n) := \det \begin{pmatrix} | & & | \\ y_1 & \cdots & y_n \\ | & & | \end{pmatrix}.$$

Choose $(e_i)_{i\in[n]}$ to maximise $D(e_1, \dots, e_n)$ such that $\|e_i\| = 1$ for all $i \in [n]$. This exists by compactness. Since $D(e_1, \dots, e_n) > 0$ we get that $(e_i)_{i\in[n]}$ is a base. Let

$$f_i(x) := \frac{D(e_1, \dots, e_{i-1}x, e_{i+1}, \dots, e_n)}{D(e_1, \dots, e_{i-1}, e_i, e_{i+1}, \dots, e_n)}.$$

Then f_i is linear, it holds that $f_i(e_j) = \delta_{i,j}$. Hence $f_i = \alpha_i$. Hence if $\|x\| = 1$ we have

$$|\alpha_i(x)| = |f_i(x)| \le 1$$

where inequality is by maximality of $D(e_1, \dots, e_n)$. Hence $\|\alpha_i\| \le 1$. ■

Corollary 2.3.10. *If* $\dim X = n \in \mathbb{N}$ *it holds that* $d_{\mathrm{BM}}(X, \ell^n) \le n$.

Proof. Let $(\tilde{e}_i)_{i\in[n]}$ be an Auerbach base of X and let

$$\begin{aligned} T\colon X &\to \ell_1^n \\ \tilde{e}_i &\mapsto e_i. \end{aligned}$$

Then

$$\begin{aligned} \|Tx\| &= \left\| T\left(\sum_{i\in[n]} \alpha_i(x)\tilde{e}_i\right)\right\| \\ &\le \sum_{i\in[n]} \alpha_i(x) \|T\tilde{e}_i\| \\ &= \sum_{i\in[n]} \alpha_i(x) \\ &\le n\|x\|. \end{aligned}$$

Conversely, for $a = (a_1, \dots, a_n) \in \ell_1$ we have

$$\begin{aligned} \left\|T^{-1}a\right\| &= \left\|T^{-1}\sum_{i\in[n]} a_i e_i\right\| \\ &= \left\|\sum_{i\in[N]} a_i\tilde{e}_i\right\| \\ &\le \sum_{i\in[n]} |a_i| \underbrace{\|\tilde{e}_i\|}_{1} = \|a\|_1. \end{aligned}$$

Hence

$$d_{\mathrm{BM}}(X, \ell_1^n) \le \|T\| \left\|T^{-1}\right\| \le n \cdot 1 = n.$$

■

Corollary 2.3.11. *If* $\dim X = \dim Y = n$, *then* $d_{\mathrm{BM}}(X, Y) \leq n^2$.

Fact 2.3.12 (F. John). *If* $\dim X = n$ *it holds that* $d_{\mathrm{BM}}(X, \ell_2^n) \leq \sqrt{n}$. *So, for every* X, Y *of dimension* n *it holds that* $d_{\mathrm{BM}}(X, Y) \leq n$.

Fact 2.3.13 (Gluskin). *There are* X, Y *of dimension* n *such that* $d_{\mathrm{BM}}(X, Y) \geq \frac{n}{1000000}$.

2.3.2 Geometric Interpretation

Let $X = (\mathbb{R}^n, \|\cdot\|)$ and let

$$K = \{x \mid \|x\| \leq 1\}$$

which is convex.

Aurbach's theorem says that we can apply a linear transformation to a convex body such that the image contains $\pm e_i$ for every base element e_i, and such that the 1-norm of points in the image is less than 1.

Theorem 2.3.14. *Let* X *be a Banach space. Then* $\dim X < \infty$ *if and only if* $\bar{B}_X := \bar{B}(0, 1)$ *is compact.*

Lemma 2.3.15 (Riesz Lemma). *Let* X *be a normed space and let* $E \subsetneq X$ *be a closed subspace. Then for every* $\varepsilon > 0$ *there's* $x_\varepsilon \in X$ *such that* $\|x_\varepsilon\| = 1$ *and* $d(x_\varepsilon, E) \geq 1 - \varepsilon$.

Proof. Fix $x \in X \setminus E$ and write $d := d(x, E) > 0$. Choose $e \in E$ such that $\|x - e\| \leq d(1 + \varepsilon)$ and define $x_\varepsilon = \frac{x-e}{\|x-e\|}$. Then $\|x_\varepsilon\| = 1$ and for all $y \in E$ it holds that

$$\|x_\varepsilon - y\| = \frac{\|x - e - y\|x - e\|\|}{\|x - e\|} \geq \frac{d}{d(1 + \varepsilon)} = \frac{1}{1 + \varepsilon} \geq 1 - \varepsilon. \qquad \blacksquare$$

Proof (2.3.14). If X is finite-dimensional, it's isomorphic to $\mathbb{R}^n$, so the closed unit ball is compact.

Assume $\dim X = \infty$, we want to show that $\bar{B}_X := \bar{B}(0, 1)$ is not compact. Let $x_1 \in X$ with $\|x_1\| = 1$, and let $E_1 = \operatorname{Span}\{x_1\} \neq X$. This is closed and by Riesz's lemma there's $x_2 \in X \setminus E_1$ such that $\|x_2\| = 1$ and $d(x_2, E_1) \geq 0.9$. Set $E_2 = \operatorname{Span}\{x_1, x_2\} \neq X$ which is closed. By Riesz lemma there's again $x_3 \in X \setminus E_2$ of norm 1 such that $d(x_3, E_2) \geq 0.9$, and so on. Then $(x_n)_{n \in \mathbb{N}_+} \subseteq \bar{B}_X$ but

$$\|x_n - x_m\| \geq d(x_n, E_n) \geq d(x_n, E_{n-1}) \geq 0.9$$

so there is no convergent subsequence, so $\bar{B}_X$ is not compact. $\blacksquare$

Corollary 2.3.16. *If* $\dim X = \infty$ *and* $A \subseteq X$ *is compact, it holds that* $\int A = \varnothing$.

Proof. Otherwise, $\bar{B}\left(X_0, \frac{r}{2}\right) \subseteq B\left(x_0, r\right) \subseteq A$ so $\bar{B}\left(x_0, \frac{r}{2}\right) = \frac{r}{2}\bar{B}_X + x_0$ is compact, so $\bar{B}_X$ is compact, a contradiction. ■

The above corollary raises a problem. Compact sets are comfortable to work with, and one wants to have compact sets other than those with empty interior. To fix this we later define new topologies on Banach spaces.

Chapter 3

Weak Topologies

3.1 Weak Topologies

3.1.1 Definitions

Definition 3.1.1 (Weak Convergence). Let X be a Banach space and let $(x_n)_{n\in\mathbb{N}} \subseteq X$ and $x \in X$. We say $x_n \xrightarrow{n\to\infty} x$ weakly if for every $f \in X^*$ it holds that $f(x_n) \xrightarrow{n\to\infty} f(x)$.

Notation 3.1.2. We denote weak convergence by $x_n \rightharpoonup x$, by $\underset{n\to\infty}{\text{w-lim}} x_n = x$ or by $x_n \xrightarrow{w} x$.

Example 3.1.3. Let H be a Hilbert space and let $(e_n)_{n\in\mathbb{N}} \subseteq H$ be orthonormal. Every $f \in H^*$ is of the form $f(x) = \langle x, y\rangle$ and by Bessel it holds that

$$\sum_{n\in\mathbb{N}} |\langle y, e_n\rangle|^2 \leq \|y\|^2 < \infty.$$

Then $\langle y, e_n\rangle \to 0$ so $f(e_n) = \langle e_n, y\rangle \xrightarrow{n\to\infty} 0 = f(0)$. Then $e_n \overset{n\to\infty}{\rightharpoonup} 0$. But, of course $\|e_n - e_m\| = \sqrt{2}$ for every n, m different, then $(e_n)_{n\in\mathbb{N}}$ doesn't converge (strongly).

Proposition 3.1.4. *A weak limit is unique.*

Proof. Assume that for every $f \in X^*$ it holds that $f(x_n) \xrightarrow{n\to\infty} f(x), f(y)$. Then $f(x) = f(y)$ for every $f \in X^*$, and by Hahn-Banach it follows that $x = y$. ■

Proposition 3.1.5.

1. *If $x_n \overset{n\to\infty}{\rightharpoonup} x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded.*
2. *If $x_n \overset{n\to\infty}{\rightharpoonup} x$ then $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.*

Proof.

1. For every f we have $f(x_n) \xrightarrow{n\to\infty} f(x)$. Hence $(f(x_n))_{n\in\mathbb{N}}$ is bounded. By uniform boundedness, $(x_n)_{n\in\mathbb{N}}$ is bounded.

2. By Hahn-Banach there's $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$. Now

$$\begin{aligned}\|x\| &< |f(x)| \\ &= \lim_{n\to\infty} |f(x_n)| \\ &\le \liminf_{n\to\infty} \|f\| \|x_n\| \\ &= \liminf_{n\to\infty} \|x_n\| . \end{aligned}$$ ■

Proposition 3.1.6. *Let S be a compact metric space and let $X = \mathcal{C}(S)$. Let $(f_n)_{n\in\mathbb{N}}, f \in X$. Then the following are equivalent.*

1. *$f_n \overset{n\to\infty}{\rightharpoonup} f$.*
2. *$f_n(s) \xrightarrow{n\to\infty} f(s)$ for all s, and $\sup_{n\in\mathbb{N}} \sup_{s\in S} |f_n(s)| < \infty$.*

Proof. 1. We prove 2 assuming 1. Pick $\delta_s \in \mathcal{C}(S)^*$ which is $\delta_s(f) = f(s)$. We know $\delta_s(f_n) \xrightarrow{n\to\infty} \delta_s(f)$. Also, $\sup_{n\in\mathbb{N}} \sup_{s\in S} |f_n(s)| = \sup_{n\in\mathbb{N}} \|f_n\| < \infty$ by the proposition.

We prove the other direction. By measure theory, $\varphi \in \mathcal{C}(S)^*$ is of the form $\varphi(f) = \int f \, \mathrm{d}\mu$. Now $\varphi(f_n) \xrightarrow{n\to\infty} \varphi(f)$ by dominated convergence. ■

Example 3.1.7 (Schur's Theorem). In ℓ_1, weak convergence implies strong convergence.

Definition 3.1.8 (The Weak Topology). The weak topology on a Banach space X is that with the sub-base $\left\{V_{f,a,\delta} \;\middle|\; \substack{f\in X^* \\ a\in\mathbb{F} \\ \delta>0}\right\}$ where

$$V_{f,a,\delta} = \{x \in X \mid |f(x) - a| < \delta\} .$$

Proposition 3.1.9. 1. *A local base for every $z \in X$ is given by sets of the form*

$$U_{f_i,z,\delta} := \{x \in X \mid \forall i \in [m] \colon \ |f_i(x) - f_i(z)|\}$$

for $\delta > 0$ and $f_1, \ldots, f_m \in X^*$.

2. *The weak topology is the weakest topology on X such that every $f \in X^*$ is continuous.*
3. *$x_n \xrightarrow{n\to\infty} x$ in the weak topology if and only if $x_n \overset{n\to\infty}{\rightharpoonup} x$.*
4. *The weak topology is weaker then the norm topology. It is strictly weaker if and only if $\dim X = \infty$.*
5. *The weak topology is Hausdorff.*
6. *$+ \colon X \times X \to X$ and $\cdot \colon \mathbb{F} \times X \to X$ are continuous in the weak topology.*

Proof. 1. Let U is weakly open and $z \in U$. Then

$$z \in \bigcap_{i\in[m]} V_{f_i,a_i,\delta_i} \subseteq U.$$

Then $f_i(z) \in B_{\mathbb{F}}(a_i, \delta_i)$ so we can choose $\delta > 0$ such that $B_{\mathbb{F}}(f_i(z), \delta) < B_{\mathbb{F}}(a_i, \delta_i)$ for all $i \in [m]$. Then $z \in U_{f_i,z,\delta} \subseteq U$.

2. Fix $f \in X^*$. For every $a \in \mathbb{F}$ and $\delta > 0$ we have

$$V_{f,a,\delta} = f^{-1}(B_{\mathbb{F}}(a, \delta))$$

is weakly-open. Hence f is weakly-continuous. If every $f \in X^*$ is continuous with respect to some topology τ, then $V_{f,a,\delta} \in \tau$ by the same argument, so the weak topology w is contained in τ.

3. Assume convergence in the weak topology. Since every $f \in X^*$ is continuous in the weak topology, we have weak convergence.

For the other direction, assume $x_n \overset{n\to\infty}{\rightharpoonup} x$. Fix $U_{f_i,x,\delta}$. Then

$$\forall i \in [m]\colon f_i(x_n) \xrightarrow{n\to\infty} f_i(x),$$

so

$$\exists n_0 \in \mathbb{N} \forall n \geq n_0\colon |f_i(x_n) - f_i(x)| < \delta.$$

Then

$$\forall n \geq n_0\colon x_n \in U_{f_i,x,\delta}.$$

4. We prove that if $\dim X = \infty$ then every weakly-open set U is unbounded. In particular $B(0,1)$ is not weakly open.

It's enough to prove the statement in the case $U = U_{f,x,\delta}$. Note that by linear algebra there's $y \neq 0$ such that $f_i(0)$ for all $i \in [m]$. Then $x + ty \in U_{f_i,x,\delta}$ for all $t \in \mathbb{F}$, so $U_{f_i,x,\delta}$ is unbounded.

The case where $\dim X < \infty$ is an exercise.

5. Fix $x, y \in X$ different. There's $f \in X^*$ such that $f(x) \neq f(y)$. Write $f = |f(x) - f(y)|$. Then

$$U_{f,x,\frac{\delta}{2}} \cap U_{f,y,\frac{\delta}{2}} = \varnothing$$

where x is in the first set and y in the second.

6. We prove that $+$ is continuous. It's enough to note

$$U_{f_i,x,\frac{\delta}{2}} + U_{f_i,y,\frac{\delta}{2}} \subseteq U_{f_i,x+y,\delta}$$

which follows from the triangle inequality in $\mathbb{F}$. ■

Proposition 3.1.10. *If* $\dim X = \infty$ *then* $\exists A \subseteq X$ *such that* $0 \in \bar{A}^{\mathrm{w}}$ *but there is no* $(x_n)_{n\in\mathbb{N}} \subseteq A$ *such that* $x_n \overset{n\to\infty}{\rightharpoonup} 0$.

Corollary 3.1.11. *Let* X *be a Banach space such that* $\dim X = \infty$ *and let* w *be the weak topology on it.* (X, w) *is not metrisable.*

Proof. Choose subspaces $E_1 \subseteq E_2 \subseteq \ldots$ such that $\dim E_n = n$. Let

$$A = \bigcup_{n\in\mathbb{N}_+} \{x \in E_n \mid \|x\| = n\}.$$

Fix an open set U around zero and assume WLOG that $U = U_{f_i,0,\delta}$. There's $y \in E_{m+1}$ such that $f_i(y) = 0$ for every $i \in [m]$. Then

$$(m+1)\frac{y}{\|y\|} \in U_{f_i,0,\delta} \cap A \neq \varnothing$$

so $0 \in \bar{A}^{\mathrm{w}}$.

Assume $(x_n)_{n\in\mathbb{N}} \subseteq A$ converges weakly to zero. Then $\|x_n\| \leq m$ for some m, so $(x_n)_{n\in\mathbb{N}} \subseteq E_m$. Since $\dim E_m < \infty$ we have $x_n \xrightarrow{n\to\infty} 0$ in norm, which is impossible since $\|x\| \geq 1$. ■

Proposition 3.1.12. *Let* $K \subseteq X$ *be convex. Then* K *is closed iff it's weakly-closed.*

Proof. If K is weakly-closed it's closed because the weak topology is weaker.

Assume K is norm-closed. Fix $x_0 \notin K$ Choose $d > 0$ such that $B(x_0, d) \cap K = \varnothing$. By Convex separation there exists $f \in X^*$ such that $\sup_{a\in K} f(a) \leq \inf_{b\in B(x_0,\delta)} f(b)$. Pick any $y \in X$ such that $f(y) > 0$ and $\|y\| = 1$. Then

$$f(x_0) = f\left(x_0 - \frac{d}{2}y\right) + f\left(\frac{d}{2}y\right)$$

where $x_0 - \frac{d}{2}y \in B(x_0, d)$ so $f(x_0) > c + 0 = c$. But now

$$U = \{x \mid f(x) > c\}$$

is weakly-open, $x_0 \in U$ and $U \cap K = \varnothing$. Hence K is weakly-closed. ■

Corollary 3.1.13.

1. $\bar{B}(0,1)$ *is weakly-closed.*
2. *If* $(x_n)_{n\in\mathbb{N}} \subseteq K$ *where* K *is convex and closed, and* $x_n \overset{n\to\infty}{\rightharpoonup} x$, *then* $x \in K$.

Corollary 3.1.14 (Mazur). *If* $x_n \overset{n\to\infty}{\rightharpoonup} x$ *then* $\exists (y_n)_{n\in\mathbb{N}}$ *such that*

1. $y_k \in \mathrm{Conv}(x_n)_{n\in\mathbb{N}}$ *for every* $k \in \mathbb{N}$.
2. $y_n \xrightarrow{n\to\infty} x$.

Proof. Take $K = \overline{\operatorname{Conv}(x_n)_{n\in\mathbb{N}}}$. This is convex and closed so it's w-closed. Then $K = \overline{\operatorname{Conv}(x_n)_{n\in\mathbb{N}}}^{\mathrm{w}}$. Therefore $x \in \overline{(x_n)_{n\in\mathbb{N}}}^{\mathrm{w}} \subseteq K$. Hence there's $(y_n)_{n\in\mathbb{N}} \subseteq \operatorname{Conv}(x_n)_{n\in\mathbb{N}}$ such that $y_n \xrightarrow{n\to\infty} x$. ■

Theorem 3.1.15 (Banach-Saks, 1). *Let H be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ weakly convergent to x. Then there's a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that*

$$\frac{1}{k}\sum_{i\in[k]} x_{n_i} \xrightarrow{k\to\infty} x.$$

Proof. Assume without loss of generality that $x = 0$. We have

$$\begin{aligned} I \coloneqq \left\| \frac{1}{k}\sum_{i\in[k]} x_{n_i} \right\|^2 &= \left\langle \frac{1}{k}\sum_{i\in[k]} x_{n_i}, \frac{1}{k}\sum_{j\in[k]} x_{n_j} \right\rangle \\ &\leq \frac{1}{k^2}\sum_{i\in[k]} \|x_{n_i}\|^2 + \frac{2}{k^2}\sum_{i,j\in[k]} \left|\langle x_{n_i}, x_{n_j}\rangle\right|. \end{aligned}$$

Choose $x_{n_1} = x_1$. Assume we already choose x_{n_i} for $i \in [k-1]$. Since $x_n \overset{n\to\infty}{\rightharpoonup} 0$ we have $\langle x_m, x_{n_i}\rangle \xrightarrow{m\to\infty} 0$ for all $i \in [k-1]$. Hence we can choose x_{n_k} such that $|\langle x_{n_k}, x_{n_i}\rangle| \leq \frac{1}{2^k}$ for all $i \in [k-1]$. Since $x_n \overset{n\to\infty}{\rightharpoonup} 0$, we have also $\|x_n\| \leq$ for some $C > 0$. Then

$$\begin{aligned} I &\leq \frac{1}{k^2}Ck + \frac{2}{k^2}\sum_{i,j\in[k]} \frac{1}{2^j} \\ &= \frac{C}{K} + \frac{2}{k^2}\sum_{j\in[k]} \frac{j}{2^j} \\ &\xrightarrow{k\to\infty} 0. \end{aligned}$$

■

3.1.2 Weak-* Topologies

Definition 3.1.16. Let X be a Banach space and let $F \leq X^*$. *The* $\sigma(X, F)$ *topology on* X is the weakest topology such that every $f \in F$ is continuous.

Proposition 3.1.17.

1. *A local base at X for the $\sigma(X, F)$ topology is given by sets*
$$U_{f_i,x,\delta} \coloneqq \{y \in X \mid \forall i \in [m]\colon\ |f_i(y) - f_i(x)| < \delta\}$$
for $f_1, \ldots, f_m \in F$ and $\delta > 0$.
2. *$\sigma(X, F) \subseteq w$.*
3. *$+, \cdot$ are continuous with respect to the $\sigma(X, F)$ topology.*
4. *$(X, \sigma(X, F))$ is Hausdorff if F separates points.*

Proposition 3.1.18. *If $f \in X^*$ then $f\colon (X, \sigma(X, F)) \to \mathbb{F}$ is continuous if and only if $f \in F$.*

Lemma 3.1.19. *Let V be a vector space over $\mathbb{F}$ and let $f_1, \ldots, f_m, g\colon V \to \mathbb{F}$ be linear functionals. Assume that for every x, $f_1(x) = \ldots = f_m(x) = 0$ implies $g(x) = 0$. Then $g \in \operatorname{Span}\{f_i \mid i \in [m]\}$.*

Proof. Define

$$\begin{aligned} T\colon V &\to \mathbb{F}^m \\ x &\mapsto (f_i(x))_{i \in [m]} \end{aligned}$$

and

$$\begin{aligned} \varphi\colon \operatorname{Im} T &\to \mathbb{F} \\ Tx &\mapsto g(x). \end{aligned}$$

φ is well-defined because $Tx = Ty$ implies $\forall i \in [m]\colon f_i(x) = f_i(y)$ which implies $f_i(x - y) = 0$ so $g(x - y) = 0$, so $g(x) = g(y)$. We know $\varphi(a_1, \ldots, a_m) = \sum_{i \in [m]} \lambda_i a_i$ so

$$g(x) = \varphi(Tx) = \varphi(f_1(x), \ldots, f_m(x)) = \sum_{i \in [m]} \lambda_i f_i(x).$$

■

Proposition 3.1.20. *If $f \in X^*$ then $f\colon (X, \sigma(X, F)) \to \mathbb{F}$ is continuous if and only if $f \in F$.*

Proof. Assume f is continuous. It holds that $f(0) = 0$ so continuity implies

$$f^{-1}(B(0,1)) \supseteq U_{f_i, 0, \delta}$$

for $f_1, \ldots, f_m \in F$. So, if $|f_i(z)| < \delta$ for all $i \in [m]$ then $abs f(z) < 1$. If $f_i(z) = 0$ for all $i \in [m]$ then for all $\lambda > 0$ it holds that $|f_i(\lambda z)| < \delta$, so $|\lambda| |f(z)| = |f(\lambda z)| < 1$, so $f(z) = 0$. By the lemma, $f \in \operatorname{Span}\{f_1, \ldots, f_m\} \subseteq F$, as required. ■

Example 3.1.21. $\sigma(X, X^*) = w$.

Example 3.1.22. Fix X a Banach space. The topology $\sigma(X^*, X)$ on X^* is called the weak-* topology on X^*. This is denoted w^* and is the weakest topology on X^* such that $f \mapsto f(x)$ are all continuous.

Example 3.1.23. Let $X = \mathcal{C}([-1,1])$. Then $X^* = M([-1,1])$ is the space of measures on $[-1,1]$. Choose $\mu_n \in X^*$ by $\mu_n(f) = \int f \, \mathrm{d}\mu_n = \frac{n}{2} \int_{[-\frac{1}{n}, \frac{1}{n}]} f \, \mathrm{d}t$. The for every $f \in X$ it holds that $\mu_n(f) \xrightarrow{n \to \infty} f(0) = \delta_0(f)$, so $\mu_n \xrightarrow{w^*} \delta_0$.

But, $\mu_n \not\xrightarrow{w} \delta_0$. Indeed, choose $\varphi \in X^{**}$ by $\varphi(\mu) = \mu(\{0\})$, it holds that

$$\varphi(\mu_n) = \mu(\{0\}) = 0 \not\to 1 = \delta_0(\{0\}) = \varphi(\delta_0).$$

Proposition 3.1.24. *$\sigma(X^*, X^{**}) = \sigma(X^*, X)$ if and only if X is reflexive.*

Proof. If X is reflexive, clearly $\sigma(X^*, X^{**}) = \sigma(X^*, X)$.

On the other hand, assume X isn't reflexive and fix $\varphi \in X^{**} \setminus X$. Then φ is continuous on (X^*, w) but not on (X^*, w^*), so $w \neq w^*$. ■

Theorem 3.1.25 (Banach Alaoglu). *For every Banach space X, $\left(\bar{B}_{X^*}, w^*\right)$ is compact.*

Proof. Let
$$S = \prod_{x \in X} \bar{B}_{\mathbb{F}}(0, \|x\|)$$
with the product topology. We identify this with
$$\{g \colon X \to \mathbb{F} \mid \forall x \in X \colon\ |g(x)| \leq \|x\|\}.$$
Then
$$\bar{B}_{X^*} \subseteq S$$
and in fact
$$\bar{B}_{X^*} = \{g \in S \mid g \text{ is linear}\}.$$
In fact,
$$\left(\bar{B}_{X^*}, w^*\right) \subseteq (S, \text{product})$$
by definition of the product topology. By Tychonoff, S is compact, so we just need to show that $\bar{B}_{X^*}$ is closed in it. For every $x, y \in X$ and $\lambda \in \mathbb{F}$ define
$$\begin{aligned} \varphi_{x,y,\lambda} \colon S &\to \mathbb{F} \\ g &\mapsto g(\lambda x + y) - (\lambda g(x) + g(y)). \end{aligned}$$
Then $\varphi_{x,y,\lambda}$ is continuous by definition of the product topology. Then
$$\bar{B}_{X^*} = \{g \in S \mid g \text{ is linear}\} = \bigcap_{x,y,\lambda} \ker \varphi_{x,y,\lambda}$$
is closed, so by the above it's compact. ■

Corollary 3.1.26. *$A \subseteq X^*$ is w^* compact if and only if it's w^*-closed and bounded.*

Proof. Assume A is w^*-closed bounded, so there's $r > 0$ such that $A \subseteq r \cdot \bar{B}_X$. $r\bar{B}_X$ is compact and A is w^*, hence A is w^*-compact.

For the other direction, assume A is w^*-compact. For every $x \in X$ define
$$\text{ev}_x(A) = \{f(x) \mid f \in A\} \subseteq \mathbb{F},$$
which is compact as the continuous image of a compact set. Hence this is bounded in $\mathbb{F}$. By the uniform boundedness principle, A is then bounded. ■

Corollary 3.1.27. *If X is reflexive, $A \subseteq X$ is w-compact if and only if it's w-closed and bounded.*

Proof. This is true because when X is reflexive, $(X, w) = \left((X^*)^*, w^*\right)$. ■

Example 3.1.28. Let $X = \ell_\infty$ and define $\varphi \in X^*$ by

$$f_n (a_i)_{i\in\mathbb{N}} = a_n.$$

Then $|f_n(a)| \leq \|a\|$ and specifically $\|f_n\| \leq 1$ so $(f_n)_{n\in\mathbb{N}} \subseteq \bar{B}_{X^*}$.

Assume $f_{n_k} \xrightarrow{w^*} f$ for some subsequence. Define $(a_m)_{m\in\mathbb{N}}$ by

$$a_m = \begin{cases} (-1)^k & m = n_k \\ 17 & \forall k \in \mathbb{N}\colon m \neq n_k \end{cases}.$$

Then $(a_m)_{m\in\mathbb{N}} \in \ell_\infty$. Note that

$$f_{n_k}(a) = (-1)^k \underset{k\to\infty}{\not\longrightarrow} f(a),$$

which is a contradiction. Hence no subsequence of $(f_n)_{n\in\mathbb{N}}$ is convergent. This isn't a contradiction to Banach-Alaoglu since in (X^*, w^*) compactness doesn't imply sequential compactness.

In separable spaces however, compactness and sequential compactness are equivalent, which is more comfortable.

Theorem 3.1.29. *Assume X is separable. Then $\left(\bar{B}_{X^*}, w^*\right)$ is metrisable.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a countable dense set in X. Define a metric on X^* by

$$d(f, g) = \sum_{n\in\mathbb{N}} \frac{\min(|f(x_n) - g(x_n)|, 1)}{2^n}.$$

We claim d is indeed a metric, which is a standard proof. We do show that d separates points. Assume $d(f, g) = 0$. Then $|f(x_n) - g(x_n)| = 0$ for all $n \in \mathbb{N}$, so f, g agree on a dense set and hence $f = g$.

We claim that $i\colon \left(\bar{B}_{X^*}, w^*\right) \to \left(\bar{B}_{X^*}, d\right)$ is continuous. Let $g_0 \in \bar{B}_{X^*}$ and $\varepsilon > 0$. Choose N such that $\frac{1}{2^n} < \frac{\varepsilon}{2}$. Define

$$U \coloneqq \left\{ f \in \bar{B}_{X^*} \;\middle|\; \forall i \in [N]\colon\; |f(x_i) - g_0(x_i)| < \frac{\varepsilon}{2} \right\} \in w^*.$$

We claim $U \subseteq B_d(q_0, \varepsilon)$, which implies continuity. Indeed, for every $f \in U$ we have

$$\forall f \in U\colon d(f, g) = \sum_{n\in\mathbb{N}} \frac{\min(|f(x_n) - g_0(x_n)|, 1)}{2^n} \leq \sum_{n\in[N]} \frac{|f(x_n) - g_0(x_n)|}{2^n} + \sum_{n=N+!}^{\infty} \frac{1}{2^n} \leq \sum_{n\in[N]} \frac{\varepsilon}{2^{n+1}} + \frac{1}{2^n}$$

Hence i is continuous. But by Banach-Alaoglu $\left(\bar{B}_{X^*}, w^*\right)$ is compact, and $\left(\bar{B}_{X^*}, d\right)$ is Hausdorff, hence i is in fact a homeomorphism. Hence $\left(\bar{B}_{X^*}, w^*\right)$ is metrisable. ■

Corollary 3.1.30. *If X is separable, $\bar{B}_{X*}$ is sequentially compact, i.e. every bounded sequence $(f_n) \subseteq x^*$ has a w^*-convergent subsequence.*

Exercise 3. If $\dim X = \infty$, (X^*, w^*) is not metrisable.

Example 3.1.31. Let S be a compact metric space and let $X = \mathcal{C}(S)$. Then X is separable, which is clear e.g. for $S = [0,1]$, but is true in general as we see later. $\left(\bar{B}_{X^*}, w^*\right)$ is metrisable and compact. We identify this with the space of Borel measures on S. Let $\mathrm{P}(S)$ denote the space of probability measures on S. We have by definition of the norm that

$$\mathrm{P}(S) \subseteq \bar{B}_{M(s)}$$

where $M(S)$ is the space of measures on S. In fact,

$$\mathrm{P}(S) = \left\{\mu \in \bar{B}_{M(S)} \,\middle|\, \forall f \in \mathcal{S} \colon f \geq 0 \implies \int f \,\mathrm{d}\mu \geq 0\right\},$$

so $\mathrm{P}(S)$ is closed in $\bar{B}_{M(S)}$, so $(\mathrm{P}(S), w^*)$ is compact and metrisable.

Hence, every sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on a compact set S has a w^*-convergent subsequence.

Theorem 3.1.32 (Eberlein-Šmulian). *For every Banach space X, $A \subseteq X$ is w-compact if and only if A is w-sequentially-compact.*

Proof. If $A \subseteq X$ is w-compact, it follows similarly to the proof of the last theorem that A is w-sequentially-compact.

The other direction is much harder but also much less useful. ∎

Corollary 3.1.33 (Banach-Saks, strong form). *Let H be a Hilbert space and let $(x_n)_{n\in\mathbb{N}} \subseteq H$ be bounded. There exists a subsequence $\imath k\,(x_{n_k})_{k\in\mathbb{N}}$ converge (in norm).*

Proof. Since $(x_n)_{n\in\mathbb{N}}$ is bounded, it's contained in some ball $r \cdot \bar{B}_H$ which is w^*-compact by Banach-Alaoglu and then w-compact since a Hilbert space is reflexive. Then $r\bar{B}_H$ is r-sequentially-compact, so there's a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent to some x. By the first form of Banach-Saks there's a subsequence $\left(x_{n_{k_\ell}}\right)_{\ell\in\mathbb{N}}$ such that

$$\frac{1}{\ell}\sum_{i\in[\ell]} x_{n_{k_i}} \xrightarrow{\ell\to\infty} x$$

in norm. ∎

3.1.3 Application: Haar Measures

Let (X, d) be a compact metric space and let G be a group that acts isometrically on X.

Theorem 3.1.34. *There exists a Borel probability measure μ on X such that $\mu(A) = \mu(gA)$ for all $g \in G$ and $A \subseteq X$ borel. μ is called the Haar measure.*

Example 3.1.35.
1. $\mathbb{R}^n$ acts on $\mathbb{R}^n / \mathbb{Z}^n$ by translations. The Haar measure here is the Lebesgue measure.
2. $\mathcal{O}(n)$, the group of orthogonal $n \times n$ matrices, acts on $S^{n-1} \subseteq \mathbb{R}^n$. The Haar measure is the uniform measure on S^{n-1}.

Theorem 3.1.36 (Hall's Marriage Theorem). *Let H be a bipartite graph with vertex sets V and W. Assume that for every $A \subseteq V$ it holds that*

$$|n(A)| = |\{w \in W \mid \exists v \in A \colon w \sim v\}| \geq |A| .$$

Then there's $\varphi \colon V \to W$ injective such that $v \sim \varphi(v)$ for all $v \in V$.

Lemma 3.1.37. *Recall that $N_\varepsilon \subseteq X$ is called an ε-net if for every $x \in X$ there's $y \in N_\varepsilon$ such that $d(x, y) \leq \varepsilon$. Then for every $\varepsilon > 0$ there exists a finite ε-net. We say that N_ε is a minimal ε-net if $|N_\varepsilon|$ is minimal. We claim that if $N_\varepsilon, \tilde{N}_\varepsilon$ are minimal ε-nets, there's a bijection $\varphi \colon N_\varepsilon \to \tilde{N}_\varepsilon$ such that $d(y, \varphi(y)) \leq 2\varepsilon$.*

Proof. Take a graph H with vertices $N_\varepsilon \sqcup \tilde{N}_\varepsilon$ and edges $y \sim z \iff \bar{B}(y, \varepsilon) \cap \bar{B}(z, \varepsilon) \neq \varnothing$. For every $\Gamma \subseteq N_\varepsilon$ the set $(N_\varepsilon \setminus A) \cup n(A)$ is also an ε-net. Indeed, $y \notin A$ implies $y \in (N_\varepsilon \setminus A) \cup n(A)$, and $y \in A$ implies $z \in (N_\varepsilon \setminus A) \cup n(A)$ since $z \sim y$.

Then $|(N_\varepsilon \setminus A) \cup n(A)| \geq |N_\varepsilon|$ so $|n(A)| \geq |A|$. By Hall's theorem there's then $\varphi \colon N_\varepsilon \to \tilde{N}_\varepsilon$ such that

$$\forall y \in N_\varepsilon \colon y \sim \varphi(y) \implies d(y, \varphi(y)) \leq 2\varepsilon. \qquad \blacksquare$$

Proof (3.1.34). For N_ε a minimal ε-net, and $\varepsilon > 0$, define

$$\mu_\varepsilon = \frac{1}{|N_\varepsilon|} \sum_{y \in N_\varepsilon} \delta_y$$

which is the uniform probability measure on N_ε. By Alaoglu / Prokhorov, there's a sequence $\varepsilon_i \to 0$ such that $\mu_{\varepsilon_i} \xrightarrow{w^*} \mu$. We claim μ is a Haar measure. We show that

$$\forall f \in \mathcal{C}(X) \, \forall g \in G \colon \int g(gx) \, \mathrm{d}\mu(x) = \int f(x) \, \mathrm{d}\mu(x) ,$$

which gives the result.

Note that weak convergence and the definition of μ_ε imply

$$\int f(gx)\,\mathrm{d}\mu(x) = \lim_{i\to\infty}\int f(gx)\,\mathrm{d}\mu_{\varepsilon_i}(x)$$
$$= \lim_{i\to\infty}\frac{1}{|N_{\varepsilon_i}|}\sum_{y\in N_{\varepsilon_i}} f(gy)$$
$$\lim_{i\to\infty}\frac{1}{|N_{\varepsilon_i}|}\sum_{y\in gN_{\varepsilon_i}} f(y).$$

Similarly,

$$int(x)\,\mathrm{d}\mu(x) = \lim_{i\to\infty}\frac{1}{|N_{\varepsilon_i}|}\sum_{y\in N_{\varepsilon_i}} f(y).$$

Because g acts by an isometry, gN_{ε_i} is a minimal ε-net.

f is continuous on a compact set X, so it's uniformly continuous on it. Fix $\eta > 0$, there's $\varepsilon > 0$ such that $d(x,y) < \varepsilon$ implies $|f(x) - f(y)| < \eta$. For large enough i we have $\varepsilon_i < \frac{\varepsilon}{2}$. Then, for every $y \in N_{\varepsilon_i}$ we have

$$d(y,\varphi(y)) \leq 2\varepsilon_i < \varepsilon$$

where $\varphi\colon N_{\varepsilon_i} \to gN_{\varepsilon_i}$ is the map from the lemma. Then

$$|f(y) - f(\varphi(y))| < \eta$$

so

$$\left|\frac{1}{|N_{\varepsilon_i}|}\sum_{y\in g(N_{\varepsilon_i})} f(y) - \frac{1}{|N_{\varepsilon_i}|}\sum_{y\in N_{\varepsilon_i}} f(y)\right| \leq \frac{1}{|N_{\varepsilon_i}|}\sum_{y\in N_{\varepsilon_i}} |f(\varphi(y)) - f(y)| < \eta.$$

Then

$$\forall\eta > 0\colon \left|\int f(gx)\,\mathrm{d}\mu - \int f(x)\,\mathrm{d}\mu\right| < \eta$$

so

$$\int f(gx)\,\mathrm{d}\mu = \int f(x)\,\mathrm{d}\mu$$

so μ is a Haar measure. ■

3.2 Locally-Convex Spaces

Definition 3.2.1 (Locally Convex Space (LCS)). A *locally-convex space (LCS)* X is a vector space over $\mathbb{R}$ with a topology τ such that

1. (X,τ) is Hausdorff.
2. $+\colon X\times X \to X$ and $\cdot\colon \mathbb{R}\times X \to X$ are continuous.

3. 0 has a local base of convex sets.

Remark 3.2.2. For every $y \in X$, the map $x \mapsto x + y$ is a homeomorphism, so having a local base of convex sets at 0 implies there's a local base of convex sets at every point.

Example 3.2.3. Let $(X, \|\cdot\|)$ be a Banach space with the norm topology. It obviously satisfies the first two conditions to be an LCS. Balls $\{x \mid \|X\| < \varepsilon\}$ are a convex local base at 0, so X is an LCS.

Example 3.2.4. Let X be a Banach space, let $F \subseteq X^*$ which separates points and consider $(X, \sigma(X, F))$. This is an LCS. It's Hausdorff since F separates points. We've seen addition and multiplication by scalars are continuous. We've seen that a local base at 0 is given by sets

$$U_{f_i,0,\varepsilon} = \{x \in X \mid \forall i \in [m] \colon \ |f_i(x)| < \varepsilon\}.$$

These are convex sets.

Example 3.2.5. Recall $X = \mathcal{C}^1([0,1])$ with the norm

$$\|f\| = \max|f| + \max|f'|$$

is a Banach space, where $f_n \xrightarrow{\|\cdot\|} f$ iff $f_n \to f$ and $f_n' \to f'$ uniformly.

We can generalise this and look at $X = \mathcal{C}^k([0,1])$ and $\|f\| = \sum_{i=0}^{k} \max\left|f^{(i)}\right|$. Here convergence in norm is equivalent to uniform convergence of all the derivatives.

Example 3.2.6. Let $X = \mathcal{C}^\infty([0,1])$. Set

$$\|f\|_n = \max_{t \in [0,1]} \left|f^{(n)}(t)\right|$$

and take the weakest topology on X such that all $\|\cdot\|_n$ are continuous. A local base at g is given by

$$U_{g,N,\varepsilon} = \{f \in X \mid \forall n \in \{0, \ldots, N\} : \ \|f - g\|_n < \infty\}.$$

X is metrisable with metric

$$d(f,g) = \sum_{n=0}^{\infty} \frac{\min(\|f-g\|_n, 1)}{2^n}.$$

This is the subspace topology from embedding $\mathcal{C}([0,1])$ in $\mathcal{C}([0,1])^{\mathbb{N}}$ by taking a function h to its derivatives.

We have $f_n \to f$ in X iff $f_n^{(i)} \to f^{(i)}$ uniformly for all i. This is an LCS because the $U_{g,N,\varepsilon}$ define a convex local base.

Remark 3.2.7. For any sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of semi-norms we can do exactly the same as in the above example, with the assumption that if $\|x\|_n = 0$ for all n then $x = 0$. Such an LCS is called a Fréchet space.

Example 3.2.8. Let $D \coloneqq \left\{x \in \mathbb{R}^d \;\middle|\; \|x\|_2 < 1\right\} \subseteq \mathbb{R}^d$. Take $X = \mathcal{C}(D)$. This is a Fréchet space with $\|f\|_n = \max_{\|t\|_2 < 1-\frac{1}{n}} |f(t)|$. If $\|f\|_n = 0$ for all n indeed $f = 0$. We have $f_n \to f$ iff $f_n \to f$ uniformly on each $\left\{\|x\|_2 < 1 - \frac{1}{n}\right\}$ iff $f_n \to f$ uniformly on all compact subsets $K \subseteq D$.

Theorem 3.2.9. *Let X be an LCS, $K \subseteq X$ convex and closed and $a \notin K$. There exists $f\colon X \to \mathbb{R}$ continuous and linear such that*

$$\sup_{x\in K} f(x) < f(a).$$

Remark 3.2.10. We can also separate open sets, we can separate a close set from a compact set, etc. as in the case of Banach spaces.

Proof. Without loss of generality assume $a = 0$ and let W be an open neighbourhood of 0 disjoint from K. Without loss of generality, by local convexity, we may assume W is convex. Let $U = W \cap (-W)$ which is a symmetric open neighbourhood of 0 disjoint from K. U is also convex, as the intersection of two convex sets.

We show that $0 \in U$ is internal. For $y \in X$, the map $t \mapsto t \cdot y$ from $\mathbb{R}$ to X is continuous. So, $V_y \coloneqq \{t \in \mathbb{R} \mid ty \in U\}$ is open in $\mathbb{R}$. We have $0 \in V_y$ and V_y is open, so there's $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subseteq V_y$. Hence $ty \in U$ for all $|t| < \varepsilon$.

We now have two convex disjoint sets U, K where U has an internal point. Hence there's $f\colon X \to \mathbb{R}$ linear and non-zero such that

$$\sup_{x\in K} f(x) \le \inf_{u\in U} f(u).$$

Fix $y \in X$ such that $f(y) < 0$ and $t > 0$ such that $ty \in U$. Then

$$\sup_{x\in K} f(x) \le \inf_{u\in U} f(u) \le f(ty) = tf(y) < 0 = f(0).$$

We're left to show f is continuous. Fix $(\alpha, \beta) \subseteq \mathbb{R}$ and $x \in f^{-1}((\alpha, \beta))$. Choose $0 < r < \min(\beta - f(x), f(x) - \alpha)$. Denote $-m = \sup_{x\in K} f(x)$, where $m > 0$. For every $y \in x + \frac{r}{m}U$ we have

$$\begin{aligned}
f(y) &= f\left(x + \frac{r}{m}U\right) \\
&= f(x) + \frac{r}{m} f(u) \\
&\ge f(x) - \frac{r}{m}(-m) \\
&= f(x) - r \\
&> f(x) - (f(x) - \alpha) \\
&= \alpha.
\end{aligned}$$

Similarly $f(y) < \beta$: Since U is symmetric we have $-u \in U$ so

$$\begin{aligned} f(y) &= f(x) - \frac{r}{m} f(-u) \\ &\leq f(x) - \frac{r}{m}(-m) \\ &= f(x) + r \\ &< f(x) + (\beta - f(x)) \\ &= \beta. \end{aligned}$$

Then

$$x + \frac{r}{m} U \subseteq f^{-1}((\alpha, \beta))$$

so

$$f^{-1}((\alpha, \beta))$$

is open, hence f is continuous. ■

Remark 3.2.11. We actually showed that i $f\colon X \to \mathbb{R}$ is linear and bounded from below on an open set, it's continuous.

Example 3.2.12. Let $p \in (0,1)$ and let $X = L^p([0,1])$. Define $\|f\| = \|f\|_p^p = \int_0^1 |f|^p$. This is *not* a norm since $\|\lambda f\| \neq |\lambda| \|f\|$. But, $\|f+g\| \leq \|f\| + \|g\|$ since $(a+b)^p < a^p + b^p$ for $p \in (0,1)$. Then $d(f,g) = \|f-g\|$ Then (X, d) is a complete metric space, but it's not locally convex since $X^* = \{0\}$ as we now show.

Assume towards a contradiction that $0 \neq \Phi \in X^*$. Choose $f \in X$ such that $a := \Phi(f) > 0$, and write $b := \|f\|$. By the mean value theorem, there's t_0 such that $\int_0^{t_0} |f|^p = \frac{b}{2}$.

Take $g = 2f\chi_{[0,t_0]}$ and $h = 2f\chi_{[t_0,1]}$. Then

$$a = \Phi(f) = \Phi\left(\frac{g+h}{2}\right) = \frac{\Phi(g) + \Phi(h)}{2}.$$

Then $\Phi(g) \geq a$ or $\Phi(h) \geq a$. Also,

$$\|g\| = \int_0^1 |g|^p = \int_0^{t_0} |2f|^p = 2^p \cdot \frac{b}{2} = \frac{b}{2^{1-p}}$$

and similarly $\|h\| = \frac{b}{2^{1-p}} = \|g\|$. Hence we can choose f_1 to be either $gorh$ (take g if $\Phi(g) \geq a$ and otherwise take h) such that $\Phi(f_1) \geq a$ and $\|f_1\| = \frac{b}{2^{1-p}}$.

Repeating this, we get a sequence $(f_n)_{n \in \mathbb{N}_+} \subseteq X$ such that $\Phi(f_n) \geq a$ and $\|f_n\| = \frac{b}{(2^{1-p})^n} \xrightarrow{n \to \infty} 0$. Since $\Phi(f_n) \not\to 0$ we get that Φ is not continuous.

Definition 3.2.13 (Extreme Point). Let X be a vector space and let $K \subseteq X$ be convex. A point $a \in K$ is *an extreme point of* K if

$$\forall \lambda \in (0,1)\, \forall y, z \in K\colon a = (1-\lambda) y + \lambda z \implies y = z = a.$$

Example 3.2.14. Let H be a Hilbert space and let $K = \bar{B}_H$. Then $\operatorname{Ext}(K)$, the set of extreme points of K, is the set $\{x \in H \mid \|x\| = 1\}$.

Assume $\|x\| = 1$, and write $x = (1-\lambda)y + \lambda z$ where $\lambda \in (0,1)$ and $y, z \in K$. Then

$$1 = \|x\| = \|(1-\lambda)y + \lambda z\| \le (1-\lambda)\|y\| + \lambda\|z\| \le (1-\lambda) + \lambda = 1$$

so there are equalities, so there's equality in the triangle inequality, which implies x, y, z are colinear, but $\|y\| = \|z\|$ since there's equality in the above inequality, hence $y = z = x$.

For the other direction, assume $\|x\| < 1$ and fix $y \perp x$ non-zero. Write

$$x = \frac{(x+ty) + (x-ty)}{2},$$

and $x \pm ty \in K$ for t small enough.

Example 3.2.15. Let $X = c_0$ and $K = \bar{B}_{c_0}$. Assume $a \in K$. Then $a_n \xrightarrow{n\to\infty} 0$ so there's $m \in \mathbb{N}$ such that $|a_m| < \frac{1}{2}$. Now write

$$a = \frac{\left(a + \frac{1}{2}e_m\right) + \left(a - \frac{1}{2}e_m\right)}{2}$$

where $a \pm \frac{1}{2}e_m \in K$ by assumption on a_m. Hence a is not an extreme point, so $\operatorname{Ext}(k) = \varnothing$.

Corollary 3.2.16. *c, c_0 are not isometric.*

Proof. If $T\colon c \to c_0$ is an isometry then $T\left(\bar{B}_c\right) = \bar{B}_{c_0}$ and so $T\left(\operatorname{Ext}\left(\bar{B}_c\right)\right) = \operatorname{Ext}\left(\bar{B}_{c_0}\right)$. But, $\operatorname{Ext}\left(\bar{B}_{c_0}\right) = \varnothing$ and $(1)_{n\in\mathbb{N}} \in \operatorname{Ext}\left(\bar{B}_c\right)$. ∎

Remark 3.2.17. c, c_0 are isomorphic. Define

$$\begin{aligned} T\colon c_0 &\to c \\ (a_i)_{i\in\mathbb{N}} &\mapsto (a_0 + a_{i+1})_{i\in\mathbb{N}}. \end{aligned}$$

This is an isomorphism.

Theorem 3.2.18 (Krein-Milman). *If X is an LCS and $K \subseteq X$ is compact and convex, then $K = \overline{\operatorname{conv}(\operatorname{Ext}(K))}$.*

Corollary 3.2.19. *There is no space X such that $X^* = c_0$ (isometrically).*

Proof. If $X^* = c_0$, we can look at $\bar{B}_{c_0} \subseteq (c_0, w^*)$ which is convex and compact (by Banach-Alaoglu). By Krein-Milman, $\bar{B}_{c_0} = \overline{\operatorname{conv}\left(\operatorname{Ext}\left(\bar{B}_{c_0}\right)\right)} = \varnothing$, a contradiction. ∎