Lecture Notes to a course in Algebraic Topology

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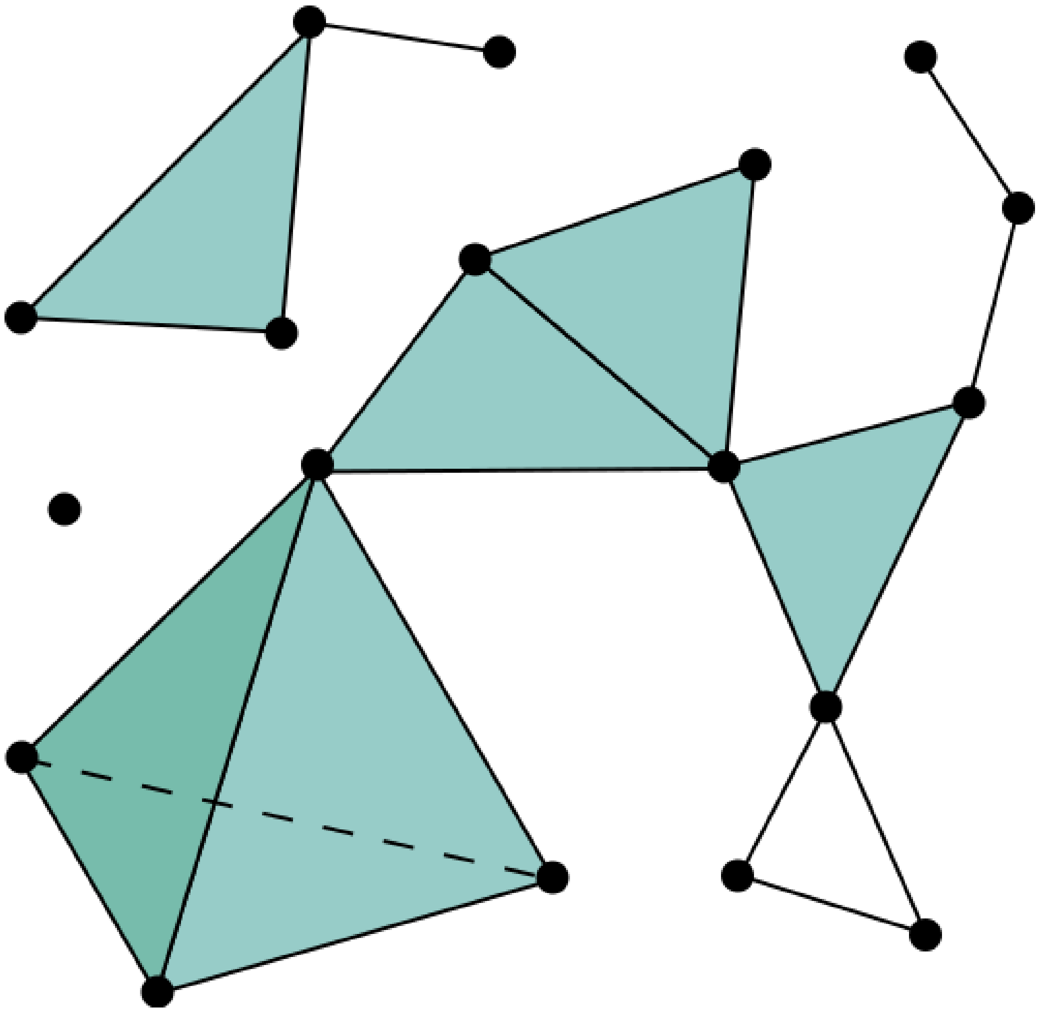

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.
In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.
Elad Tzorani.

Course Literature

The recommended course literature is as follows.

Hatcher: Algebraic Topology

Munkres: Elements of Algebraic Topology

Grade

The grade will be given depending on home-work assignments, an an oral examination, possibly together with group presentations, depending on the number of people in the course by then.

Chapter 1

Motivation

1.1 What is Algebraic Topology?

1.1.1 Homotopy groups

We'd want to study topological spaces, but that is generally a difficult task. For that reason we associate algebraic objects to topological spaces, through which we can study topology algebraically. Some reasons for associating algebraic objects to topological spaces are as follows.

1. Distinguishing spaces.
2. Studying properties of spaces.

Example (application of Algebraic Topology: Brouwer's fixed point theorem). Let $\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$. Every continuous map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point. I.e. there's $x \in \mathbb{D}^n$ such that $f(x) = x$.

Definition 1.1.1. Let X be a topological space and let $A \subseteq X$. A ***retraction*** is a continuous map $r\colon X \to A$ such that $r|_A = \mathrm{id}_A$.

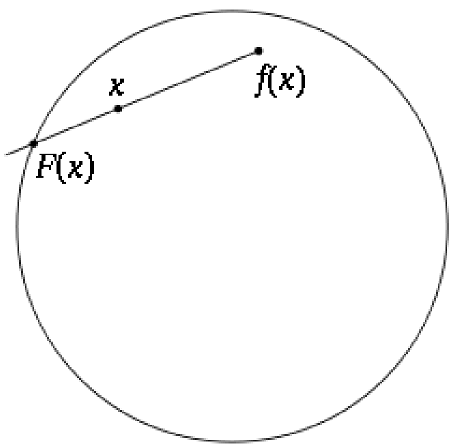
$$\begin{tikzcd} X \arrow[r, "r"] & A \\ A \arrow[u, hook, "i"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

Lemma 1.1.2. *There is no retraction* $\mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1} = \{x \in \mathbb{D}^n \mid \|x\| = 1\}$.

We show that the lemma implies Brouwer's fixed point theorem.

Proof. Assume $\forall x \in \mathbb{D}^n\colon f(x) \neq x$. Define $F(x)$ to be the point on $\mathbb{S}^{n-1}$ intersecting the ray from $f(x)$ to x. See figure 1.1. This is continuous (**check this!**), and hence a retraction, contradicting the lemma.

Figure 1.1: Retraction from the disk to the sphere.



∎

We shall now prove the lemma.

Proof (of the lemma). $n=1$**:** Define $\pi_0(X) \coloneqq \{\text{path connected componenets of } X\}$. A map $f\colon X \to Y$ which is continuous defines a map $\pi_0(f) = f_*\colon \pi_0(X) \to \pi_0(Y)$ by $[x] \mapsto [f(x)]$ (this is well defined). We observe that $\mathrm{id}_* = \mathrm{id}_{\pi_0(X)}$. Also, if $f\colon X \to Y$ and $g\colon Y \to Z$ then $(g \circ f)_* = g_* f_*$. Assume $r\colon \mathbb{D}^1 \to \mathbb{D}^1 = \mathbb{S}^0$ is a retraction. We apply π_0 to

$$\begin{tikzcd} \mathbb{D}^1 \arrow[r, "r"] & \mathbb{S}^0 \\ \mathbb{S}^0 \arrow[u, hook, "i"] \arrow[ur, "\mathrm{id}"'] & \end{tikzcd}$$

and get the following diagram.

$$\begin{tikzcd} \pi_0\left(\mathbb{D}^1\right) \arrow[r, "r_*"] & \pi_0\left(\mathbb{S}^0\right) \\ \pi_0\left(\mathbb{S}^0\right) \arrow[u, "i_*"] \arrow[ur, "\mathrm{id}_*=\mathrm{id}"'] & \end{tikzcd}$$

Now $\pi_0\left(\mathbb{D}^1\right) = \text{singleton}$ and $\pi_0\left(\mathbb{S}^0\right) = 2$ elements contradicting the diagram.

$n=2$**:** Let $\pi_1(X, x_0)$ be the fundamental group. That is

$$\pi_1(X, x_0) = \pi_0\,(\text{loops in } X \text{ that start and end in } x_0)\,.$$

Let $f\colon (X, x_0) \to (Y, y_0)$ be continuous (with $f(x_0) = y_0$). This defines $\pi_1(f) = f_*\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$. It can be checked (and is showed in another topology course) that $\pi_1\left(\mathbb{D}^2, x\right) \cong 1$, and that $\pi_1\left(\mathbb{S}^1, x\right) \cong \mathrm{Z}$. We get the following diagram, which gives a contradiction.

$$\begin{tikzcd} \pi_1\left(\mathbb{D}^2\right) \cong 1 \arrow[r, "r_*"] & \pi_1\left(\mathbb{S}^1\right) \cong \mathbb{Z} \\ \pi_1\left(\mathbb{S}^1\right) \cong \mathbb{Z} \arrow[u, "i_*"] \arrow[ur, "\mathrm{id}_*=\mathrm{id}"'] & \end{tikzcd}$$

■

We'd want to iterate such a construction by looking at loop spaces of loop spaces.

Definition 1.1.3.

$$\pi_n = \pi_0\,(\text{loop of}\,(\text{loop of} \ldots (X, x_0)\,\tilde{x}_0, \ldots))$$

For $n \geq 1$, π_n is a group. π_1 is a group with the operation of concatenation, and we can view π_n as π_1 of some space, if $n \geq 1$.
We don't really like this inductive definition of π_n, so we'd like to give another definition. We can view π_1 as homotopy classes of $\left(\mathbb{S}^1, *\right) \to (X, x_0)$. We'd like to generalise upon that idea.

Definition 1.1.4. π_n is the group of homotopy classes of maps $(\mathbb{S}^n, *) \to (X, x_0)$. We can view $\mathbb{S}^n$ as $I^n/\partial I^n$, and the group action of π_n is given by gluing the spheres at the identified boundary of I^n. It can be checked that π_n is abelian for $n \geq 2$.

1.1.2 Homology and cohomology of topological spaces

Homotopy groups are relatively difficult to compute. We'd want to introduce another algebraic object associate to topological spaces, $\tilde{H}_n$, which we shall see satisfies $\tilde{H}_n\left(\mathbb{S}^k\right) = \begin{cases} 0 & n \neq k \\ \mathrm{Z} & n = k \end{cases}$. We'll use this structure to prove Brouwer's theorem.
We define $H_0(X) = \bigoplus_{\pi_0} \mathrm{Z}$ ***the zero'th homology group***, and similarly $H_1(X) = \pi_1(X, x_0)^{\mathrm{ab}}$ ***the first homology group***.

Chapter 2

Simplicial Homology

2.1 Δ-complexes

Definition 2.1.1. Let X be a topological space and $\sim$ be an equivalence relation on X. The ***quotient space*** is $X/\sim$. Let $\pi\colon X \to X/\sim$ be the projection, we say that $U \subseteq X/\sim$ is open in the quotient if and only if $\pi^{-1}(U)$ is open in X.

Example. We can construct the torus $\mathbb{T}^2$ by gluing opposite sides of a square. We write $X = [0,1]^2$ and $\sim$ is the equivalence relation generated by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,y)$.

Example. We can construct the Klein bottle $\mathbb{K}^2$ by gluing opposite sides of a square, with one gluing in the opposite direction. We write $X = [0,1]^2$ and $\sim$ is the equivalence relation generated by $(x,0) \sim (x,1)$ and $(0,y) \sim (1,1-y)$.

Example. We can construct the ***real projective space*** $\mathbb{R}\mathrm{P}^n$ by

$$\mathbb{R}^{n+1} \setminus \{0\} \Big/ \{x \sim \lambda x \mid \lambda \in \mathbb{R} \setminus \{0\}\}.$$

Definition 2.1.2. Let $v_0, \ldots, v_n \in \mathbb{R}^m$. We say $\{v_i\}_{i=1}^n$ are ***geometrically independent*** if $\{v_i - v_0\}_{i=1}^n$ are linear-independent.

Example. **$n = 0$:** Any point in $\mathbb{R}^m$ is a 0-simplex.

$n = 1$: Any two *distinct* points in $\mathbb{R}^m$ are geometrically independent (we require $v_1 - v_0 \neq 0$). The convex hull is the segment connecting the points.

$n = 2$: Geometrical independence means the three points aren't collinear.

$n = 3$: Geometrical independence means the three points aren't coplanar, or equivalently that they span a full non-degenerate tetrahedron.

Definition 2.1.3. Given vectors $v_0, \ldots, v_n$, the ***convex hull*** of the vectors is defined by the following.

$$[v_0, \ldots, v_n] = \left\{ \sum_{i=0}^{n} t_i v_i \;\middle|\; \begin{array}{c} t_i \geq 0 \\ \sum t_i = 1 \end{array} \right\}$$

Definition 2.1.4. An ***n-simplex*** is the convex hull of $n+1$ geometrically independent points in $\mathbb{R}^m$.

Remark 2.1.5. A simplex comes with an ordering of its vertices.

Definition 2.1.6. Let $e_0, \ldots, e_n \in \mathbb{R}^{n+1}$ be basis vectors. $[e_0, \ldots, e_n]$ is the ***standard n-simplex***.

Definition 2.1.7. The ***i^{th} face*** of an n-simplex $[v_0, \ldots, v_n]$ is defined by $[v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$. We denote this by the following[1].

$$[v_0, \ldots, \hat{v}_i, \ldots, v_n]$$

[1] The hat means we remove the vector v_n

Definition 2.1.8. Let $v \in [v_0, \ldots, v_n]$. It can be written as $\sum_{i=0}^{n} t_i v_i$ for $t_i \geq 0$ with $\sum t_i = 1$, *uniquely*. $(t_0, \ldots, t_n)$ are the ***barycentric coordinates***. The inclusion map of the i^{th} face of $[v_0, \ldots, v_n]$ in barycentric coordinates is

$$\iota_i \colon (t_0, \ldots, t_{n-1}) \mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1})$$

Definition 2.1.9. We define the interior as

$$\mathring{\Delta}^n = \Delta^n \setminus \text{faces of } \Delta^n.$$

Remark 2.1.10. $\mathring{\Delta}^0 = \Delta^0$, because there are no faces.

Definition 2.1.11. A Δ***-complex*** structure on a topological space X is a collection Σ partitioned as

$$\Sigma_n = \{\text{set of } \sigma \colon \Delta^n \to X \text{ continuous}\}$$

such that the following properties hold.

1. For all $x \in X$ there's a unique σ such that there's a unique $y \in \mathring{\Delta}^n$ with $x = \sigma(y)$.[2]
2. For all $\sigma \in \Sigma_n$ and for all $0 \leq i \leq n$ there's $\tau \in \Sigma_{n-1}$ where $\sigma \circ \iota_i = \tau$.[3]
3. $U \subseteq X$ is open if and only if $\sigma^{-1}(U)$ is open for all $\sigma \in \Sigma$. This is called the ***CW topology***.

Example. See We have the following.

$$\Sigma = \begin{Bmatrix} \Sigma_0 & v & \text{0-simplex} \\ \Sigma_1 & e_1, e_2, e_3 & \text{1-simplex} \\ \Sigma_2 & \sigma_1, \sigma_2 & \text{2-simplex} \end{Bmatrix}$$

Example. See

Example. See this cannot be turned into a Δ-simplex by addition of the diagonal. However, if we give a different direction to the edges as in we can get a Δ-simplex.

Example. The figure in is not a Δ-simplex, because we cannot number the vertices in such a way that the direction of the arrows matches. However, we can turn it into a simplex as in

Example. The Dunce hat in figure is a Δ-complex.

Definition 2.1.12. Let X have a Δ-complex structure. The ***simplicial homology*** on X is the following.

$$C_i(X) = \bigoplus_{\sigma \in \Sigma_i} \mathbb{Z}\sigma = \left\{ \sum_{j=1}^{k} n_j \sigma_j \;\middle|\; \begin{array}{c} k \in \mathbb{Z} \\ n_j \in \mathbb{Z} \\ \sigma_j \in \Sigma_i \end{array} \right\}$$

where the sums are formal. The elements of $C_i(X)$ are called i***-chains***.

Definition 2.1.13. The ***boundary map*** on X with a Δ-complex structure is defined on $\sigma \in \Sigma_n$ by

$$\begin{aligned} \partial_n \colon C_n(X) &\to C_{n-1}(X) \\ \sigma &\to \sum_{i=0}^{n} (-1)^i \sigma \circ \iota_i \end{aligned}$$

and extended linearly to all maps.

Note 2.1.14. We notice that $\partial_0 = 0$.

[2] In words, every x is the image of exactly one point in the interior of a face.

[3] In words, this means that restricting to a face, we keep the same ordering of the vertices. This defines an orientation on the complexes.

Figure 2.1: Circle Δ complex.

Example. Let us calculate some boundary maps.

Lemma 2.1.15. $\partial_{n-1} \circ \partial_n = 0$.

Proof.

$$\begin{aligned}
\partial_{n-1}\partial_n\sigma &= \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) \\
&= \sum_{i=0}^{n} (-1)^i \left(\partial_{n-1}\, \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j<i} (-1)^j \, \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]} + \sum_{i<j} (-1)^{j-1} \, \sigma|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}\right)
\end{aligned}$$

The coefficient of $\sigma|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}$ is $(-1)^i (-1)^{j-1} + (-1)^j (-1)^i = 0$. ∎

Definition 2.1.16. A sequence of abelian groups

$$\dots \to C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \dots$$

with $\partial_{n-1} \circ \partial_n = 0$ is called a ***chain complex***.

Exercise. Calculate $\partial_0\partial_1$ of a line and $\partial_1\partial_2$ of a triangle.

Remark 2.1.17. Because the n in ∂_n is clear from the domain and codomain, we sometimes write ∂ instead of ∂_n.

Remark 2.1.18. $\partial_n \circ \partial_{n+1} = 0$ implies $\operatorname{Im}\partial_{n+1} \subseteq \ker \partial_n$.

Definition 2.1.19. ***The n^{th} homology*** is

$$H_n(X) := {\ker \partial_n}\big/{\operatorname{Im}\partial_{n+1}}.$$

Similarly we can write $H_n(X, A)$ where $C_n(X)$ is an A-span of Σ_n, A being any abelian group instead of $\mathbb{Z}$.

Definition 2.1.20. The elements of $\ker(\partial_n) =: Z_n$ are called ***cycles***.
The images of $\operatorname{Im}(\partial_{n+1}) =: B_n$ are called ***boundaries***.
The elements of $H_n(X)$ are called ***homoloy classes***.
If $x, y \in Z_n$ and $[x] = [y] \in H_n(X)$, then x and y are called ***homologous***.

Example. Take $X = \mathbb{S}^1$ with the Δ-complex structure in figure 2.1. We have $\partial(e) = v$ and therefore the following chain.

$$\dots \longrightarrow C_3 \longrightarrow C_2(\mathbb{S}^1) \longrightarrow C_1(\mathbb{S}^1) \longrightarrow C_0(\mathbb{S}^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

$$\dots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}e \xrightarrow{0} \mathbb{Z}v \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Computing the homologies, we have

$$\begin{aligned}
H_0(\mathbb{S}^1)\, {\mathbb{Z}r}/{0} &\cong \mathbb{Z} \\
H_1(\mathbb{S}^1) = {\mathbb{Z}e}/{0} &\cong \mathbb{Z} \\
\forall k \geq 1\colon H_k(\mathbb{S}^1) &= 0
\end{aligned}$$

Example. Take X to be as in We $\partial(e_i) = w - v$, and the following chain.

$$\dots \longrightarrow C_3 \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

$$\dots \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \xrightarrow{\partial} \mathbb{Z}v \oplus \mathbb{Z}w \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$$

Computing H_0 we have

$$H_0(X) = \mathbb{Z}v \oplus \mathbb{Z}w \big/ \langle w - v\rangle \cong \mathbb{Z}[v]$$

because in the quotient $w = v$.
Computing H_1 we have

$$H(X) = \ker\partial \big/ \cancel{\mathrm{Im}} \cong \ker\partial = \langle e_1 - e_2, e_2 - e_3\rangle .$$

Then $\ker\partial = \{a_1e_1 + a_2e_2 + a_3e_3 \mid a_1 + a_2 + a_3 = 0\}$. We have that for all $k \geq 2$, $H_k(X) = 0$.

Remark 2.1.21. H_0 "counts the number of connected components", H_1 "counts the number of holes", et cetera.

Example. Take $\mathbb{T}^2$ as in We have the following chain

$$\dots \longrightarrow C_3 \longrightarrow C_2(X) \longrightarrow C_1(X) \longrightarrow C_0(X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

$$\dots \xrightarrow{0} 0 \xrightarrow{0} \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \xrightarrow{\partial_2} \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \xrightarrow{0} \mathbb{Z}v \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \dots$$

with $\partial_2(\sigma_1) = e_2 + e_1 - e_3$ and $\partial_2(\sigma_2) = e_1 + e_2 - e_3$. We have

$$\begin{aligned}
&H_0(X) \cong \mathbb{Z}v \\
&H_1(X) \cong \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \big/ \langle e_1 + e_2 - e_3\rangle = \langle e_1, e_2, e_3\rangle \big/ e_3 = e_1 + e_2 \cong \langle e_1, e_2\rangle \cong \mathbb{Z}^2 \\
&H_2(X) \cong \ker\partial_2 = \langle \sigma_1 - \sigma_2\rangle \cong \mathbb{Z}\langle \sigma_1 - \sigma_2\rangle .
\end{aligned}$$

Example. Take $\mathbb{R}\mathrm{P}^2$ as in We have the chain

$$\langle \sigma_1, \sigma_2\rangle \to \langle e_1, e_2, e_3\rangle \to \mathbb{Z}r \oplus \mathbb{Z}w \xrightarrow{0} 0$$

with maps

$$\begin{aligned}
e_1 &\mapsto w - v \\
e_2 &\mapsto w - v \\
e_3 &\mapsto 0
\end{aligned}$$

and

$$\begin{aligned}
\sigma_1 &\mapsto e_3 + e_1 - e_2 \\
\sigma_2 &\mapsto e_3 + e_2 - e_1 .
\end{aligned}$$

Computing homologies we get that following.

$$\begin{aligned}
&H_0 \cong \mathbb{Z}[v] \\
&H_1 = \langle e_3, e_1 - e_2\rangle \big/ \langle \beta - \alpha, \beta + \alpha\rangle \cong \mathbb{Z}\alpha \big/ 2\alpha \cong \mathbb{Z}\big/ 2\mathbb{Z} \qquad \alpha := e_1 - e_2, \beta = e_3 \\
&H_2 = \ker\partial_2 = a_1\sigma_1 + a_2\sigma_2 = 0.
\end{aligned}$$

Example. Take $X = \mathbb{S}^n$. We can write the sphere as a result of gluing two discs of lower dimension.

$$\mathbb{S}^n = \mathbb{D}^n \coprod_{\partial\mathbb{D}^n \equiv \partial\mathbb{D}^n} \mathbb{D}^n$$

Therefore we write

$$\mathbb{S}^n = \overbrace{\Delta^n}^{\sigma_1} \coprod_{\partial\Delta^n \equiv \partial\Delta^n} \overbrace{\Delta^n}^{\sigma_2} .$$

The n$^{\text{th}}$ homology is $H_n(\mathbb{S}^n) = \ker\partial_n \big/ \cancel{\mathrm{Im}\,\partial_{n+1}}$. We have $C_{n+1} = 0$, $C_n = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2$ and the relation $\partial_n\sigma_1 = \partial_n\sigma_2$. Therefore

$$\ker\partial_n = \langle \sigma_1 - \sigma_2\rangle \cong \mathbb{Z}$$

which brings $H_n(\mathbb{S}^n) \cong \mathbb{Z}$.

Remark 2.1.22. The homology we have so far defined is called ***simplicial homology***. We sometimes write $H_n^{\Delta}(X)$ to note that. We shall define also ***singular homology***, and later we show that these are equivalent.

2.2 Singular Homology

Definition 2.2.1. Let X be a topological space, and let

$$\Sigma_n^{\text{sing}} = \{\sigma \colon \Delta^n \to X \mid \sigma \text{ is continuous}\}.$$

$\sigma \in \Sigma_n^{\text{sing}}$ is called a ***singular simplex***. $C_i^{\text{sing}} := \bigoplus_{\Sigma_i} \mathbb{Z}$ is an ***i-chain***. $\partial_n \colon C_n^{\text{sing}} \to C_{n-1}^{\text{sing}}$ are defined as before and satisfy $\partial_{n-1} \circ \partial_n = 0$. We obtain a chain

$$\ldots \to C_n^{\text{sing}}(X) \to C_{n-1}^{\text{sing}}(X) \to \ldots \to C_0^{\text{sing}}(X) \to 0$$

and define the ***singular homology*** $H_n^{\text{sing}}(X) = {\ker \partial_n}\big/{\operatorname{Im} \partial_{n+1}}$.

Remark 2.2.2. We may not write sing, but understand that the following computations are done in singular homology.

Lemma 2.2.3. *Let $X = \{x\}$. Then $H_n(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$.*

Proof. Note that $\Sigma_n = \{\sigma_{\text{const}}^n\}$ for $n \geq 0$. We have $C_i(X) = \mathbb{Z}\sigma_{\text{const}}^i$ and the following chain.

$$\ldots \to \mathbb{Z}\sigma_{\text{const}}^3 \to \mathbb{Z}\sigma_{\text{const}}^2 \to \mathbb{Z}\sigma_{\text{const}}^1 \to \mathbb{Z}\sigma_{\text{const}}^0 \to 0$$

Now

$$\partial_n\left(\sigma_{\text{const}}^n\right) = \sum_{i=0}^{n} (-1)^i \sigma_{\text{const}}^i$$

so ∂_n is the zero map where n is odd, and an isomorphism if n is even. Looking at the kernel and image gives $H_0 = \mathbb{Z}$ and $H_i = 0$ for all $i \geq 1$. ■

Remark 2.2.4. A map $f \colon X \to Y$ induces a map $\Sigma_n(X) \to \Sigma_n(Y)$ by $\sigma \mapsto f \circ \sigma$. This extends linearly to a map

$$f_\# \colon C_n(X) \to C_n(Y).$$

Definition 2.2.5. If we look at g as a map between $(C_n(X))_{n\in\mathbb{N}}$ and $(C_n(Y))_{n\in\mathbb{N}}$, and the following commutes, and we call g a ***chain map***.

$$\begin{tikzcd}
\ldots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d, "g"] & C_n(X) \arrow[r, "\partial"] \arrow[d, "g"] & C_{n-1}(X) \arrow[r, "\partial"] \arrow[d, "g"] & \ldots \\
\ldots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial"] & C_n(Y) \arrow[r, "\partial"] & C_{n-1}(Y) \arrow[r, "\partial"] & \ldots
\end{tikzcd}$$

Claim 2.2.6. $f_\#$ is a chain map.

Proof.

$$\begin{aligned}
f_\#(\partial\sigma) &= f_\#\left(\sum_{i=1}^{n} (-1)^i \sigma|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}\right) \\
&= \sum_{i=0}^{n} (-1)^i (f \circ \sigma)|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]} \\
&= \partial\left(f_\#(\sigma)\right)
\end{aligned}$$

■

Corollary 2.2.7. *$f_\#$ is a chain map, therefore it induces a map $f_* \colon H_n(X) \to H_n(Y)$.*

Claim 2.2.8. f_* is well defined.

Proof. $H_n(X) = {Z_n(X)}\big/{B_n(X)}$. If $z \in Z_n(X)$, then $f_\#(z) \in Z_n(Y)$ and

$$\partial f_\# z = f_\# \partial z \underset{z \in Z_n(X)}{=} f_\# 0 = 0.$$

So, have $f_\# \colon Z_n(X) \to Z_n(Y)$ by $[z] \mapsto [f_\#(z)]$.
If $b \in B_n(X)$, then $f_\# b \in B_n(Y)$ and there's $p \in C_n(X)$ such that $b = \partial p$. Now

$$\partial f_\#(p) = f_\#(\partial p) = f_\#(b).$$

■

Remark 2.2.9. 1. $(f\circ g)_* = f_*\circ g_*$. This is true because it's true for $(f\circ g)_\#$, and by definition of $(f\circ g)_*$.

2. $(\mathrm{id}_X)_* = \mathrm{id}_{H(X)}$.

Corollary 2.2.10. *$X\cong Y$ implies $H_i(X)\cong H_i(X)$ for all i.*

Definition 2.2.11. Let $f,g\colon X\to Y$ be continuous between topological spaces. We call f,g ***homotopic*** if there's a map

$$h\colon X\times[0,1]\to Y$$

such that $h|_{X\times\{0\}}=f$ and $h|_{X\times\{1\}}=g$. We write $f\approx x$ or $f\cong g$.

Definition 2.2.12. Let X,Y be topological. They're called ***homotopy equivalent*** if there's $f\colon X\to Y$ and $g\colon Y\to X$ such that $g\circ f\approx \mathrm{id}_X$ and $f\circ g\approx\mathrm{id}_Y$.

Proposition 2.2.13. *If $f,g\colon X\to Y$ and $f\approx g$, then $f_*=g_*$.*

Corollary 2.2.14. *If $X\approx Y$ are homotopy equivalent, then $H_i(X)\cong H_i(Y)$ for all i.*

Proof. The following commutes in homotopy.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}"'] & Y \arrow[r, "g"] & X \end{tikzcd}$$

So, applying $()_*$ we get

$$g_*\circ f_* = (g\circ f)_* = \mathrm{id}_* = \mathrm{id}\,.$$ ■

Definition 2.2.15. X is ***contractible*** if $X\approx *$.

Corollary 2.2.16. $H_n(X)=\begin{cases}\mathbb{Z} & n=0\\ 0 & n>0\end{cases}$ *for X contractible.*

Proof (of the proposition). We want to construct maps $P\colon C_n(X)\to C_{n+1}(Y)$ such that $\partial P = g_\# - f_\# - P\partial$. Then the following commutes.

$$\begin{tikzcd} \cdots \arrow[r] & C_n(X) \arrow[r] \arrow[dl, "P_n"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & C_{n-1}(X) \arrow[r] \arrow[dl, "P_{n-1}"'] \arrow[d, shift right, "g_\#"'] \arrow[d, shift left, "f_\#"] & \cdots \\ \cdots \arrow[r] & C_n(Y) \arrow[r] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

We obtain that for $z\in Z_n(X)$

$$g_\#(z)-f_\#(z)=\partial P(z)+P\underbrace{\partial(z)}_{=0}\in B_n(Y)$$

then $g_*-f_*=0\in H_n(Y)$ and then $g_*=f_*$.
Define

$$P(\sigma)=\sum_{i=0}^{n}(-1)^i\,h\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,v_i,w_i,w_n]}.$$

See figures 2.2 and 2.3.

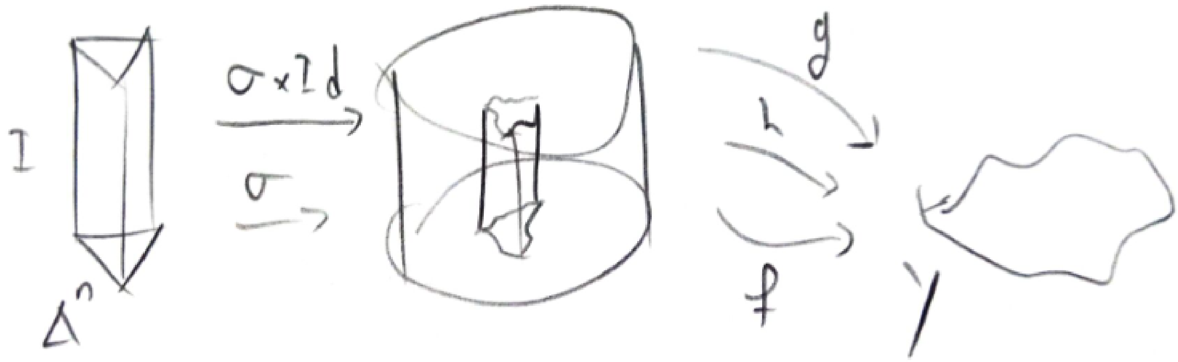
We have the following

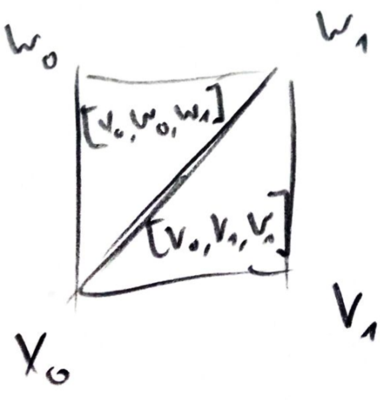
$$\begin{aligned}\partial P(\sigma) = &\sum_{0\le j\le i\le n}(-1)^i(-1)^j\,h\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,\hat v_j,\dots,v_i,w_i,\dots,w_n]}\\ &+\sum_{0\le i\le j\le n}(-1)^i(-1)^{j+1}\,h\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,v_i,w_i,\dots,\hat w_j,\dots,w_n]}\end{aligned}$$

Looking at $i=j=k$ we have in the first sum

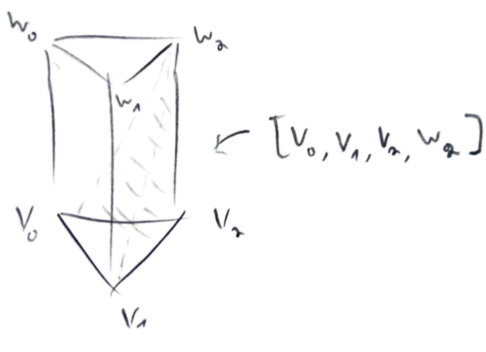
$$\cancel{(-1)^i(-1)^j}h\circ(\sigma\times\mathrm{id})|_{[v_0,\dots,v_{k-1},w_k,\dots,w_n]}.$$

Figure 2.2: Prism map.





(a) Prism one.



(b) Prism two.

Figure 2.3: Prism examples.

Looking at the second sum where $i = j = k-1$ we get

$$\cancel{(-1)^{k-1}}\cancel{(-1)^{k}}h \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_{k-1},w_k,\dots,w_n]}.$$

Therefore these cancel out. Among $i = j$ summands we are left with

$$\overbrace{(-1)^0 (-1)^0 h \circ (\sigma \times \mathrm{id})|_{[w_0,\dots,w_n]}}^{g_\# \sigma}$$
$$+ \overbrace{(-1)^n (-1)^{n+1} h \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_n]}}^{f_\# \sigma}.$$

Computing $P\partial(\sigma)$ we get the following.

$$\begin{aligned} P\partial(\sigma) &= P\left(\sum_{i=1}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) \\ &= \sum_{j<i} (-1)^i (-1)^j h \circ h \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,v_j,w_j,\dots,\hat{w}_i,\dots,w_n]} \\ &+ \sum_{i<j} (-1)^i (-1)^{j-1} h \circ (\sigma \times \mathrm{id})|_{[v_0,\dots,\hat{v}_i,\dots,v_j,w_j,\dots,w_n]} \\ &= \partial P - (g_\# - f_\#) \end{aligned}$$

So, P satisfies the required property. ■

Definition 2.2.17. A map $H\colon X \times [0,1] \to X$ is an ***isotopy*** if $H(\cdot, t)$ are all homeomorphisms.

Proposition 2.2.18. *If $X \neq \varnothing$ is path-connected, then $H_0(X) \cong \mathbb{Z}$.*

Proof. Examine the chain complex

$$C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \to 0.$$

Let $\varepsilon\colon C_0(X) \to \mathbb{Z}$ be the map defined by $v \mapsto 1$ for $v \in \Sigma_0$. Then

$$\sum \alpha_v v \mapsto \sum \alpha_v.$$

We want to show that $\ker \varepsilon = \operatorname{Im} \partial_1$.

$\supseteq$: Let $e \in \Sigma_1$ be viewed as $e\colon [v_0, v_1] \to X$. So,

$$\varepsilon(\partial e) = \varepsilon\left(e|_{v_1} - e|_{v_0}\right) = 1 - 1 = 0.$$

Remark 2.2.19.

$$\dots \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a chain complex. The homology of this chain complex is called the ***reduced (singular) homology of*** X and is denoted $\tilde{H}_i(X)$. Note that $\tilde{H}_i(X) = H_i(X)$ for all $i \geq 1$.

$\subseteq$: Let $\sum \alpha_v v \in \ker \varepsilon$. I.e. $\sum \alpha_v = 0$.
Let $v_0 \in \Sigma_0$. Let e_v be a path (i.e. a 1-simplex) from v_0 to v. Let $\lambda = \sum \alpha_v e_v \in C_1(X)$. Now

$$\partial \lambda = \sum \alpha_v (v - v_0) = \sum \alpha_v v - \overbrace{\left(\sum \alpha_v\right)}^{=0} v = \sum \alpha_v v.$$ ■

Proposition 2.2.20 (home-work). *If $\{C_i\}_{i \in A}$ are the path-connected components of X then $H_n(X) = \bigoplus_{i \in A} H_n(C_i)$.*

Corollary 2.2.21. *$H_0(X) = \oplus_{i \in A} \mathbb{Z}$.*

Figure 2.4: Simplex pairing.

Figure 2.5: More simplex pairing.

Figure 2.6: Manifold with boundary

Examine $\sum \alpha_e e \in Z_1$. We can form a space by pairing to get $\sum \alpha'_e e \in Z$ with $\alpha'_e \in \{\pm 1\}$ where the sum is taken with repetitions. Now $\partial \left(\sum \alpha_e e\right) = 0$. Take a disjoint union of such intervals and glue their boundaries according to the pairing. For example in (2.4) we get two circles and in (2.5). we get one. By gluing in such a way we obtain a one-dimensional topological manifold.

Definition 2.2.22. A ***topological manifold*** is a space X (Hausdorff, second countable) which is locally homeomorphic to $\mathbb{R}^n$.

Remark 2.2.23. 1-manifolds are $\mathbb{R}$ and $\mathbb{S}^1$, so as the above construction gives a compact space, we obtain a disjoint union of circles.

We can similarly repeat such a construction for Z_2.

Remark 2.2.24. The important property of the construction is that *every face (*codim $= 1$*) appears in exactly* 2 *pairs of n-simplices and faces.*

Remark 2.2.25. Take $\lambda \in C_2(X)$, we can view it as a 2-manifold with a surface. Then, taking a complex on the surface, the boundary map is trivial on inner edges, and is non-trivial on the boundary. See figure (2.6).

Our current *goal* is to relate homologies of X and $A \subseteq X$ to the homology of ${}^{X}/_{A}$.

2.3 Exact sequences

Definition 2.3.1. The sequence

$$\ldots \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\cdots}$$

is said to be ***exact at*** B if $\ker \beta = \operatorname{Im} \alpha$. The sequence is ***exact*** if it's exact at all objects.

Example. $0 \to A \xrightarrow{\alpha} B$ is exact if and only if α is injective.

Example. $B \xrightarrow{\beta} C \to 0$ is exact if and only if β is surjective.

Example. $0 \to A \xrightarrow{\alpha} B \to 0$ is exact if and only if α is surjective.

Definition 2.3.2. A ***short exact sequence*** is of the form

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$

Remark 2.3.3. From the isomorphism theorem, such a short exact sequence, β induces an isomorphism

$$C \underset{\beta}{\cong} {}^{B}\big/_{\alpha(A)}.$$

We have $A \overset{\alpha}{\hookrightarrow} B$ so we may write $C \cong {}^{B}/_{A}$.

Definition 2.3.4. Let $A \subseteq X$ be topological spaces. X ***deformation retracts to*** A if there's $h\colon X \times [0,1] \to X$ such that the following hold.

$$\begin{aligned} h|_{X\times\{0\}} &= \mathrm{id}_X \\ h(X \times \{1\}) &\subseteq A \\ \forall a \in A\colon h(a,t) &= a \end{aligned}$$

Example. $\mathbb{S}^{n-1}$ is a deformation retract of $\mathbb{R}^n \setminus \{0\}$. Take

$$h(v,t) = \left(1 - t + \frac{t}{\|v\|}\right) v.$$

Theorem 2.3.5. *Let X be a topological space and $A \subseteq X$ non-empty, closed, and a deformation retract of an open neighbourhood $V \supseteq A$. Then there is a long exact sequence*

$$\tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n\left({}^{X}/_{A}\right) \xrightarrow{\partial} \tilde{H}n-1(A) \to \ldots \to \tilde{H}_0\left({}^{X}/_{A}\right) \to 0.$$

This ∂ isn't the boundary map on chains, but a different boundary map we shall construct. $i\colon A \to X$ is the inclusion and $q\colon X \to {}^{X}/_{A}$ is the quotient map.

Corollary 2.3.6 (homologies of $\mathbb{S}^n$).

$$\tilde{H}_n(\mathbb{S}^n) = \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$$

Proof. By induction on n.

Basis, $n = 0$: We have $\mathbb{S}^0 = \{p_1, p_2\}$. Then

$$\tilde{H}_n\left(\mathbb{S}^0\right) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k \neq 0 \end{cases}$$

because

$$H_k\left(\mathbb{S}^0\right) = \begin{cases} \mathbb{Z}^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

and

$$H_k = \begin{cases} \tilde{H}_k & k > 0 \\ \tilde{H}_k \oplus \mathbb{Z} & k > 0 \end{cases}.$$

Step, $n > 0$: Note that $\mathbb{S}^n = {}^{\mathbb{D}^n}/_{\partial\mathbb{D}^n}$ and $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Also, $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ and $\mathbb{D}^n$ satisfy the assumption of the theorem. We obtain the following.

$$\ldots \to \tilde{H}_k(\partial\mathbb{D}^n) \to \tilde{H}_k(\mathbb{D}^n) \to \tilde{H}_k\left({}^{\mathbb{D}^n}/_{\partial\mathbb{D}^n}\right) \to \tilde{H}_{k-1}(\partial\mathbb{D}^n) \to \tilde{H}_{k-1}(\partial\mathbb{D}^n) \to \ldots$$

From homework, $\mathbb{D}^n$ is contractible so $\tilde{\mathrm{H}}_k(\mathbb{D}^n) = 0$ for all k. Hence we get the following

$$\ldots \to \tilde{H}_k(\partial\mathbb{D}^n) \to \cancelto{0}{\tilde{H}_k(\mathbb{D}^n)} \to \tilde{H}_k\left({}^{\mathbb{D}^n}/_{\partial\mathbb{D}^n}\right) \to \tilde{H}_{k-1}(\partial\mathbb{D}^n) \to \cancelto{0}{\tilde{H}_{k-1}(\partial\mathbb{D}^n)} \to \ldots$$

so

$$\tilde{H}_k\left({}^{\mathbb{D}^n}/_{\partial\mathbb{D}^n}\right) \cong \tilde{H}_{k-1}(\partial\mathbb{D}^n) \underset{\text{induction}}{\cong} \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}.$$

∎

Definition 2.3.7. Let $A \subseteq X$ be topological spaces. We call (X, A) a ***pair of spaces***. A map $f\colon (X, A) \to (Y, B)$ is ***a (continuous) map between pairs*** if $f\colon X \to Y$ is continuous such that $f(A) \subseteq B$.

Remark 2.3.8. $A \overset{i}{\hookrightarrow} X$ induces a map $C_n(A) \overset{i_\#}{\hookrightarrow} C_n(X)$. We denote $C_n(A) \subseteq C_n(X)$.

Definition 2.3.9. The ***relative chains*** are

$$C_n(X, A) := {C_n(X)}\Big/{C_n(A)}.$$

Remark 2.3.10. Note that $\partial C_n(A) \subseteq C_{n-1}(A)$ where ∂ is the boundary map of X.
∂ induces a map

$$\partial\colon C_n(X, A) \to C_{n-1}(X, A).$$

Remark 2.3.11. $\partial^2 = 0$ for $C_n(X)$ therefore $\partial^2 = 0$ for $C_n(X, A)$ (for we take a quotient of zero).

Definition 2.3.12.

$$C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \to \dots$$

is the ***relative chain complex***.

Definition 2.3.13. $H_n(X, A)$, the homologies of the relative chain complex, are the ***relative homologies of*** (X, A).

Example. A relative cycle happens to match a manifold in X that may have a boundary, only in A. I.e. after identifying the points of A, we the manifold wouldn't have a boundary.

Example. A relative boundary is a boundary of a manifold after identifying the points on A.

We'd like to phrase $\tilde{H}\left({X}/{A}\right)$ algebraically. We have a short exact sequence

$$0 \to C_n(A) \to C_n(X) \to C_n(X, A) \to 0.$$

The maps are chain maps (i.e. commute with ∂). We call such a thing a ***short exact sequence of chain complexes***. Are goal is to find a sequence

$$\dots \to H_n(A) \to H_n(X) \to H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \dots.$$

The map ∂ is really "the map ∂ of X."

Lemma 2.3.14 (The Snake lemma). *Let $\mathcal{A} = (A_\bullet, \partial)$, $\mathcal{B} = (B_\bullet, \partial)$ and $\mathcal{C} = (C_\bullet, \partial)$, and assume*

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{q} \mathcal{C} \to 0$$

is a short exact sequence of chain complexes. Then there is a long exact sequence in homology

$$\dots \to H_n(\mathcal{A}) \xrightarrow{i_*} H_n(\mathcal{B}) \xrightarrow{q_*} H_n(\mathcal{C}) \xrightarrow{\partial} H_n(\mathcal{A}) \to \dots.$$

Proof. We use diagram chasing on the following. Define the map $\partial\colon H_n(C) \to H_{n-1}(A)$ as follows. For $c_n \in Z_n(\mathcal{C})$ pick $b_n \in B_n$ such that $qb_n = c_n$ (q is surjective). Let $b_{n-1} := \partial b_n$. Note

$$qb_{n-1} = q\partial b \overset{q\partial = \partial q}{=} \partial qb = \partial c \overset{c \in Z_n(\mathcal{C})}{=} 0.$$

By exactness at B_{n-1} there's a_{n-1} such that $i(a_{n-1}) = b_{n-1}$.
We have

$$i\partial a_{n-1} = \partial i a_{n-1} = \partial b_{n-1} = \partial\partial b_n = 0$$

therefore $\partial a_{n-1} = 0$.
We therefore can define $\partial[c_n] = [a_{n-1}]$.

- The choice of a_{n-1} is fine because it is unique due to injectivity of i
- The choice of b_n from c_n is fine: If there are b_n, b'_n, we have $q(b_n - b'_n) = 0$. The difference has a source $\bar{a}_n$, and an image $b_{n-1} - b'_{n-1}$ with source $a_n - a'_{n-1}$. From commutativity of the diagram, $a_{n-1} - a'_{n-1} = \partial\left|a\right|_n$, so a_{n-1} and a'_{n-1} are homologous.

- The choice of c_n is similarly fine.

Claim 2.3.15. The sequence is exact.

We have to show the following.

1. $\operatorname{Im} i_* \subseteq \ker q_*$
2. $\operatorname{Im} q_* \subseteq \ker \partial$
3. $\operatorname{Im} \partial \subseteq \ker i_*$
4. $\supseteq$
5. $\supseteq$
6. $\supseteq$

Indeed

1. $\operatorname{Im} i_* = (qi)_* = 0$
2. Fill in as an exercise.
3. The image under ∂ is trivial in homology.

Exercise. Check the other inclusions through diagram chasing.

Example. Take the following exact sequence

$$0 \to C_n(A,B) \to C_n(X,B) \to C_n(X,A) \to 0$$

where $B \subseteq A \subseteq X$. That's exact from the isomorphism theorem. From this we obtain via the Snake lemma a long exact sequence.

Definition 2.3.16. The ***long exact sequence of reduced homologies*** is the one obtained by taking

$$0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0$$

for $n \geq 0$ and

$$0 \to \mathbb{Z} \to \mathbb{Z} \to 0 \to 0$$

for $n = -1$, with maps $\varepsilon\colon C_0(A) \to \mathbb{Z}$, $\varepsilon\colon C_0(X) \to \mathbb{Z}$ and $0\colon C_0(X,A) \to 0$. We obtain from this the long exact sequence

$$\ldots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \to \tilde{H}_n(A) \to \ldots.$$

Lemma 2.3.17. *Let $X \neq \varnothing$ and $x_0 \in X$. Then $H_n(X,x_0) \cong \tilde{H}_n(X)$.*

Proof. By the long exact sequence of reduced homologies:

$$\cancelto{0}{\tilde{H}_n(x_0)} \to \tilde{H}_n(X) \xrightarrow{q_*} H_n(X,x_0) \to \cancelto{0}{\tilde{H}_n(x_0)}$$

so q_* is an iso. ∎

Theorem 2.3.18 (Exision). *Let $Z \subseteq A \subseteq X$ such that $\bar{Z} \subseteq \operatorname{int}(A)$. Then*

$$H_n(X,A) \xleftarrow[i_*]{\sim} H_n(X \setminus Z, A \setminus Z).$$

Remark 2.3.19. Equivalently, the theorem states that for $A, B \subseteq X$ where $\operatorname{int}(A) \cup \operatorname{int}(B)$ them

$$H_n(X,A) \cong H_n(B, B \cap A).$$

Definition 2.3.20. Let $X \neq \varnothing$ be a topological space and let $\mathcal{U}$ be an open of X. Define $C_n^{\mathcal{U}}(X)$ to be the $\mathbb{Z}$-span of singular n-simplices whose image is in some $U \in \mathcal{U}$.

Remark 2.3.21. Clearly $\partial\colon C_n^{\mathcal{U}}(X) \to C_{n-1}^{\mathcal{U}}(X)$ and $\partial^2 = 0$. We obtain an homology $H_n^{\mathcal{U}}(X)$.

Theorem 2.3.22. *The embedding*

$$\iota\colon C_n^{\mathcal{U}}(X) \to C_n(X)$$

induces an isomorphism

$$H_n^{\mathcal{U}}(X) \xrightarrow[\iota_*]{\sim} H_n(X).$$

The idea of the proof is dividing the complex into complexes in the different sets. The actual proof uses the existence of Lebesgue numbers to show that the divisions stop, and homotopy of chains.

2.3.1 Barycentric subdivision (for Euclidean simplices)

Definition 2.3.23. If $v_0, \ldots, v_n$ is a Euclidean simplex, the ***barycenter*** (center of mass) of $[v_0, \ldots, v_n]$ is $b = \frac{1}{n+1} \sum v_i$.

Definition 2.3.24 (barrycentric subdivision). The n-simplices of the barycentric subdivision are spanned (geometrically) by barycenters of a decreasing choice of faces (of length $n+1$).

Remark 2.3.25. If $[w_1, \ldots, w_n]$ is a $(n-1)$-simplex of a [barycentric subdivision of a] face of $[v_0, \ldots, v_n]$, then $[b, w_1, \ldots, w_n]$ is an n-simplex in [...] of $[v_0, \ldots, v_n]$.

Lemma 2.3.26.

$$\operatorname{diam}[w_0, \ldots, w_n] \leq \frac{n}{n+1} \operatorname{diam}[v_0, \ldots, v_n]$$

where $[w_0, \ldots, w_n]$ *is a simplex in the barycentric subdivision of* $[v_0, \ldots, v_n]$.

Proof. To observe that $\operatorname{diam}[v_0, \ldots, v_n] = \max_{i,j} |v_i - v_j|$, take $u \in [v_0, \ldots, v_n]$ and $w \in \mathbb{R}^k$. Denote $u = \sum t_i v_i$ with $\sum t_i = 1$. Now

$$\begin{aligned} |w - u| &= \left| w - \sum t_i v_i \right| \\ &= \left| \sum t_i w - \sum t_i v_i \right| \\ &= \left| \sum t_i (w - v_i) \right| \\ &\leq \cancelto{1}{\sum t_i} \max_i |w - v_i| . \end{aligned}$$

Apply again for $w = v_j$ and obtain the result.
We continue to prove the lemma. Assume by induction that $|w_i - w_j| \leq \frac{n}{n+1} \operatorname{diam}[v_0, \ldots, v_n]$ for $w_i, w_j \neq b$.

Basis:

Step: By the observation we have $|b - w_i| \leq \max |b - v_i|$. Denote by w the barycenter of $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$. Then

$$b = \frac{n}{n+1} w + \frac{1}{n+1} v_i$$

and hence

$$|b - v_i| \leq \frac{n}{n+1} |w - v_i| \leq \frac{n}{n+1} \operatorname{diam}[v_0, \ldots, v_n]$$

■

We remind our goal theorem on excision.

Theorem 2.3.27 (excision). *Let* $X = \operatorname{int} A \cup \operatorname{int} B$. *Then*

$$H_n(X, A) \xleftarrow[i_*]{\sim} H_n(B, A \cap B) .$$

Proof. Take $\mathscr{U} = \{A, B\}$.

Claim 2.3.28. $C_n^{\mathscr{U}}(X, A) \cong C_n(B, A \cap B)$.

Proof (of claim).

$$C_n^{\mathscr{U}}(X) = \overbrace{C_n(A, \text{not contained in } B)}^{I} \oplus \overbrace{C_n(B, \text{not contained in } A)}^{II} \oplus \overbrace{\overbrace{C_n(A \cap B)}}^{III}$$

then

$$\begin{gathered} C_n^{\mathscr{U}}(X, A) = \frac{\cancel{I} \oplus II \oplus \cancel{III}}{\cancel{I} \oplus \cancel{III}} \\ C_n(B) = II \oplus III \\ C_n(B, A \cap B) = \frac{II \oplus \cancel{III}}{\cancel{III}} \end{gathered}$$

■

By the previous theorem (its proof)

$$H_n^{\mathscr{U}}(X,A) \cong H_n(X,A). \qquad \blacksquare$$

Definition 2.3.29. Let X be a topological space and $A \subseteq X$. (X,A) is called a ***good pair*** if $A \neq \varnothing$, is closed, and has an open neighbourhood that deformation-retracts to A.

Theorem 2.3.30. *Let (X,A) be a good pair. Then $H_n(X,A) \xrightarrow[q_*]{\sim} H_n\left({}^{X}/_{A}, {}^{A}/_{A}\right) \cong \tilde{H}_n\left({}^{X}/_{A}\right)$ where q is the quotient map $X \to {}^{X}/_{A}$.*

Proof. We already proved the isomorphism on the right.
Let $V \supseteq V$ be an open neighbourhood that deformation-retracts to A.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "j_*"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] \\
H_n\left({}^{X}/_{A}, {}^{A}/_{A}\right) \arrow[r] & H_n\left({}^{X}/_{A}, {}^{V}/_{A}\right)
\end{tikzcd}$$

We claim j_* is an iso. We have $A \subseteq V \subseteq X$, and we can use the long exact sequence of triples.[4]

$$\ldots \to H_n(V,A) \to H_n(X,A) \to H_n(X,V) \to H_{n-1}(V,A) \to \ldots$$

Claim 2.3.31. $H_n(V,A) = 0$.

Proof. Use the long exact sequence for (V,A)

$$H_n(A) \xrightarrow{\sim} H_n(V) \to H_n(V,A) \to H_{n-1}(A) \xrightarrow{\sim} H_n(V)$$

and from exactness, the inner maps are zero. Again from exactness $H_n(V,A) = 0$ (because it's trapped between zeroes). $\blacksquare$

We now have

$$\cancelto{0}{H_n(V,A)} \to H_n(X,A) \xrightarrow{j_*} H_n(X,V) \to \cancelto{0}{H_{n-1}(V,A)}$$

hence j_* is an iso. From excision we get a larger diagram.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "j_*"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X \setminus A, V \setminus A) \arrow[l, "\text{excision}"', "\sim"'] \arrow[d, "q_*"] \\
H_n\left({}^{X}/_{A}, {}^{A}/_{A}\right) \arrow[r] & H_n\left({}^{X}/_{A}, {}^{V}/_{A}\right) & H_n\left({}^{X}/_{A} \setminus {}^{A}/_{A}, {}^{V}/_{A} \setminus {}^{A}/_{A}\right) \arrow[l, "\text{excision}"', "\sim"']
\end{tikzcd}$$

Claim 2.3.32.

$$q_* \colon H_n(X \setminus A, V \setminus A) \to H_n\left({}^{X}/_{A} \setminus {}^{A}/_{A}, {}^{V}/_{A} \setminus {}^{A}/_{A}\right)$$

is an iso.

Proof.

$$q|_{X \setminus A} \colon X \setminus A \to {}^{X}/_{A} \setminus {}^{A}/_{A} \qquad \blacksquare$$

is a homeomorphism. Hence q_* is an iso. $\blacksquare$

---

[4] This comes from the short exact sequence.

$$0 \to C_n(V,A) \to C_n(X,A) \to C_n(X,V) \to 0$$

Remark 2.3.33. Let $\varnothing \neq A \subseteq X$. Form the mapping cone of (X, A)

$$\operatorname{MCone}(X, A) = X \amalg A \times [0,1] \Big/ {}_{A\times\{0\}\sim A}^{A\times\{1\}}$$

and similarly the mapping cylinder

$$\operatorname{MCyl}(X, A) = X \amalg A \times [0,1] \Big/ A \times \{0\} \sim A.$$

Then we get the long exact sequence

$$H_n(A) \to H_n(X) \to H_n(\operatorname{MCone}(X, A)) \to H_{n-1}(A) \to \dots.$$

We saw that $\tilde{H}_n\left(\mathbb{S}^k\right) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$, but we didn't find a generator for $\mathbb{Z}$. Let's solve a similar problem, finding a generator for $H_n\left(\mathbb{D}^k, \partial\mathbb{D}^k\right) \cong \begin{cases} \mathbb{Z} & n = k \\ 0 & n \neq k \end{cases}$. We "use the Δ-structure of $\mathbb{D}^n$ which has 1 n-simplex". We know $\mathbb{D}^n \cong \Delta^n$.

Claim 2.3.34. $i_n \colon \Delta^n \to \Delta^n$ (identity) is the generator of $H_N(\mathbb{D}^n, \partial\mathbb{D}^n)$.

Proof. By induction. $n = 0$ as an exercise. Let Λ be a union of all faces of Δ^n but one. Δ^n deformation-retracts to Λ so $H_n(\Delta^n, \Lambda) = 0$. We have the following sequence,

$$0 = H_n(\Delta^n, \Lambda) \to H_n(\Delta^n, \partial\Delta^n) \xrightarrow[\sim]{\partial} H_{n-1}(\partial\Delta^n, \Lambda) \to 0$$

hence ∂ is an iso. Now,

$$H_{n-1}(\partial\Delta^n, \Lambda) \cong H_{n-1}\left(\partial\Delta^{n-1}, \partial\Delta^{n-1}\right)$$

and by induction i_{n-1} is the generator for the homology on the right. So we get an identity on each of the faces, hence an identity. ∎

Example. We've seen $\tilde{H}_n\mathbb{S}^n \cong \mathbb{Z}$. We want to find the explicit generator of the homology group. Write $\mathbb{S}^n = \underset{\sigma}{\Delta^n} \amalg \underset{\tau}{\Delta^n} \Big/ \partial\sigma \sim \partial\tau$. By long exact sequences,

$$\tilde{H}_n(\mathbb{S}^n) = H_n(\mathbb{S}^n, \tau) \cong H_n(\sigma, \sigma \cap \tau) \cong H_n(\Delta^n, \partial\Delta^n).$$

Following the isomorphisms we get

$$[\sigma - \tau] \to [\sigma] \to [\sigma] \to [\sigma] = [i_n].$$

So, we know $\tilde{H}_n(\mathbb{S}^n) = \langle \sigma - \tau \rangle$.

Definition 2.3.35. Let $\{X_\alpha\}_{\alpha\in\mathcal{A}}$ be topological spaces and for each $\alpha \in \mathcal{A}$ let $x_\alpha \in X_\alpha$ such that (X_α, x_α) is a good pair. Define

$$\bigwedge_{\alpha\in A} (X_\alpha, x_\alpha) = \coprod_{\alpha\in A} X_\alpha \Big/ \{x_\alpha\}_{\alpha\in\mathcal{A}}$$

Theorem 2.3.36.

$$\tilde{H}_n\left(\bigwedge X_\alpha\right) = \bigoplus \tilde{H}_n(X_\alpha)$$

Proof. Look at the following long exact sequence.

$$0 = \tilde{H}_n\left(\{x_\alpha\}_{\alpha\in\mathcal{A}}\right) \to \overbrace{\tilde{H}_n\left(\coprod_{\alpha\in A} X_\alpha\right)}^{\bigoplus \tilde{H}_n(X_\alpha)} \to \tilde{H}_n\left(\bigwedge X_\alpha\right) \to \tilde{H}_{n-1}\left(\{x_\alpha\}_{\alpha\in\mathcal{A}}\right) = 0$$

∎

Example. $\tilde{H}_n$ of the 2-bouquet is $\mathbb{Z} \oplus \mathbb{Z} \cong \mathbb{Z}^2$ for $n = 1$ and 0 otherwise.

Theorem 2.3.37. $\mathbb{R}^n \cong \mathbb{R}^m$ *implies* $n = m$.

Remark 2.3.38. From the theorem follows that if M^n and L^m are n and m-manifolds respectively, then $M^n \cong L^m$ implies $n = m$.

Proof. Let $x \in X$. Then
$$H_n\left(X, X \setminus \{x\}\right)$$
is the ***local homology of** X **at** x*. This is local because for a neighbourhood $\mathcal{U}$ of x we get
$$H_n\left(X, X \setminus \{x\}\right) \cong H_n\left(\mathcal{U}, \mathcal{U} \setminus \{x\}\right).$$
It is enough to show that
$$H_k\left(\mathbb{R}^n, \mathbb{R}^n \setminus 0\right) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}.$$
Write the long exact sequence
$$\tilde{H}_k\left(\mathbb{R}^n \setminus 0\right) \to \cancelto{0}{\tilde{H}_k\left(\mathbb{R}^n\right)} \to \tilde{H}_k\left(\mathbb{R}^n, \mathbb{R}^n \setminus 0\right) \to \tilde{H}_{k-1}\left(\mathbb{R}^n \setminus 0\right) \to \cancelto{0}{\tilde{H}_{k-1}\left(\mathbb{R}^n\right)}.$$
Hence the here is an iso
$$\tilde{H}_k\left(\mathbb{R}, \mathbb{R}^n \setminus 0\right) \cong \tilde{H}_{k-1}\left(\mathbb{R}^k \setminus 0\right) \cong \tilde{H}_{k-1}\left(\mathbb{S}^{n-1}\right) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$
with the second iso being from homotopy equivalence. ∎

Chapter 3

Mayer-Vietories sequences

Theorem 3.0.1. *Let X be a topological space and $A, B \subseteq X$ such that $\mathring{A} \cup \mathring{B} = X$. Then there is a long exact sequence*

$$\ldots \to H_n\left(A \cap B\right) \to H_n\left(A\right) \oplus H_n\left(B\right) \to H_n\left(X\right) \to H_{n-1}\left(A \cap B\right) \to \ldots.$$

Proof. Consider the following short exact sequence where $\mathcal{U} = \{A, B\}$.

$$0 \to C_n\left(A \cap B\right) \xrightarrow[(\iota,-\iota)]{i} C_n\left(A\right) \oplus C_n\left(B\right) \xrightarrow[\iota_A \oplus \iota_B]{j} C_n^{\mathcal{U}}\left(X\right) \to 0$$

Let $(\alpha, \beta) \in \ker j$. Then $\alpha + \beta = 0$ so $\alpha = -\beta$ in $C_n^{\mathcal{U}}\left(X\right)$. Then $\alpha \in C_n\left(A \cap B\right)$. So $(\alpha, \beta) = (\alpha - \alpha) = i\left(\alpha\right)$. We get that this is indeed exact.
By use of the snake lemma we get the required long exact sequence with $H_n^{\mathcal{U}}\left(X\right)$ instead of $H_n\left(X\right)$. But these are isomorphic, so we get the required ■

3.1 Geometric interpretation

We want an intuition for the map $H_n\left(X\right) \to H_{n-1}\left(A \cap B\right)$ (as the other maps are already intuitive). Take $[c] \in H_n\left(X\right)$ a cycle. We break it into chains $[a] + [b]$ where $[a], [b]$ are chains in A and B respectively. Then the map is $[c] \to [\partial a]$.

Example (Homologies of $\mathbb{S}^n$ using MV[1]). Write $\mathbb{S}^n = \left(\mathbb{S}^n \setminus \{N\}\right) \cup \left(\mathbb{S}^n \setminus \{S\}\right)$ with $N, S \in \mathbb{S}^n$ being two different points. Via MV we get the following.

$$\cancelto{0}{\tilde{H}_k\left(\mathbb{S}^n \setminus \{N\}\right)} \oplus \cancelto{0}{\tilde{H}_k\left(\mathbb{S}^n \setminus \{S\}\right)} \to \tilde{H}_k\left(\mathbb{S}^n\right) \to \tilde{H}_{k-1}\left(\mathbb{S}^n \setminus \{N, S\}\right) \to \cancelto{0}{\tilde{H}_{k-1}\left(\mathbb{S}^n \setminus \{N\}\right)} \oplus \cancelto{0}{\tilde{H}_{k-1}\left(\mathbb{S}^n \setminus \{S\}\right)}$$

Hence

$$\tilde{H}_k\left(\mathbb{S}^n\right) \cong \tilde{H}_{k-1}\left(\mathbb{S}^n \setminus \{N, S\}\right) \cong \tilde{H}_{k-1}\left(\mathbb{S}^{n-1}\right)$$

where the last isomorphism is via deformation retract.

Example (Homology of the Klein bottle.). Write the Klein bottle as a gluing of two Möbius bands across the boundaries. $\mathbb{K}^2 = M_1 \coprod_{\partial M_1 \sim \partial M_2} M_2$. Via MV[2] we obtain the following.

$$0 \to \tilde{H}_2\left(\mathbb{K}^2\right) \to \tilde{H}_1\left(M_1 \cap M_2\right) \to \tilde{H}_1\left(M_1\right) \oplus \tilde{H}_1\left(M_2\right) \to \tilde{H}_1\left(\mathbb{K}^2\right) \to 0$$

We know $M_1 \cap M_2, M_1, M_2 \simeq \mathbb{S}^1$. Hence write

$$0 \to \tilde{H}_2\left(\mathbb{K}^2\right) \to \mathbb{Z}\lambda \to \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2 \to \tilde{H}_1\left(\mathbb{K}^2\right) \to 0$$

where λ is a cycle along $\partial M_1, \partial M_2$ and $\lambda_{1,2}$ are cycles across $\mathbb{S}^1$ in $M_{1,2}$ respectively. Then one can see that $[\lambda] = [2\lambda_i]$ in $\tilde{H}_1\left(M_i\right)$. So $[\lambda] \mapsto \left(2\left[\lambda_1\right], -2\left[\lambda_2\right]\right)$. By long exact sequence we get $\tilde{H}_2\left(\mathbb{K}\right) = 0$.
Similarly we get

$$\tilde{H}_1\left(K\right) = \mathbb{Z} \oplus \mathbb{Z} \big/ \{(2n, -2n) \mid n \in \mathbb{Z}\} \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

[2] M_1, M_2 don't have covering interiors, but we could take neighbourhoods of both, which would deformation-retract to them

3.2 Naturality

Take $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be chain complexes and examine the following diagram.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{A} \arrow[r, "i"] \arrow[d, "\alpha"] & \mathcal{B} \arrow[r, "j"] \arrow[d, "\beta"] & \mathcal{C} \arrow[r] & 0 \\ 0 \arrow[r] & \mathcal{A}' \arrow[r, "i'"] & \mathcal{B}' \arrow[r, "j'"] & \mathcal{C}' \arrow[r] & 0 \end{tikzcd}$$

We want to claim that the following commutes.

$$\begin{tikzcd} H_n(\mathcal{A}) \arrow[r, "i_*"] \arrow[d, "\alpha_*"] & H_n(\mathrm{B}) \arrow[r, "j_*"] \arrow[d, "\beta_*"] & H_n(\mathcal{C}) \arrow[r, "\partial"] \arrow[d, "\gamma_*"] & H_{n-1}(\mathbb{A}) \arrow[r] \arrow[d, "\alpha_*"] & \dots \\ H_n(\mathcal{A}') \arrow[r, "i_*"] & H_n(\mathcal{B}') \arrow[r, "j_*"] & H_n(\mathcal{C}') \arrow[r, "\partial"] & H_{n-1}(\mathcal{A}') \arrow[r] & \dots \end{tikzcd}$$

Commutativity of the first two squares should be apparent. For the first we have $\beta i = i'\alpha$ so $b_* i_* = i'_* \alpha_*$. We want to show commutativity of the third square. I.e. $\alpha_* \partial = \partial \gamma_*$.

Definition 3.2.1 (Homology theory). An homology theory matches a (nice) topological space a group $h_n(X)$ such that the following hold.

- $X \xrightarrow{f} Y$ induces a map $h_n(X) \xrightarrow{f_*} h_n(Y)$.
- $X \underset{g}{\xrightarrow{f}} Y$ with $f \simeq g$ implies f_*, g_*.
- *Excision:* Let (X, A) be a good pair. There is a a long exact sequence as follows.

$$h_n(A) \xrightarrow{i_*} h_n(X) \xrightarrow{q_*} h_n(X, A) \xrightarrow{\partial} h_{n-1}(A) \to \dots$$

- Naturality of ∂ in the above.
- $X = \bigwedge_\alpha X$ implies

$$h_n(X) \cong \bigoplus h_n(X_\alpha)$$

- If $h_n(\{\mathrm{pt}\}) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \neq 0 \end{cases}$ then $h_n(X) = H_n(X)$.

3.3 Equivalence of simplicial and singular homologies

Let X be a Δ-complex. We have two structures on X, with a map $C_n^{\triangle}(X) \xrightarrow{\iota} C_n(X)$. ι is a chain map.

Theorem 3.3.1. *$\iota_* \colon H_n^{\triangle}(X) \xrightarrow{\cong} H_n(X)$ is an iso.*

Proof.

- We first prove the theorem when X is finite dimensional. Let us denote by $X(k)$ ***the k-skeleton of*** X, the union of all simplices (of the Δ structure) of X up to dimension k. We can define $H_n^{\Delta}(X, A)$ where A is a sub-complex of X, via $C_n^{\Delta} = {C_n^{\Delta}(X)}\big/{C_n^{\Delta}(A)}$.
  X is finite dimensional, we prove that

$$H_n^{\Delta}\left(X^{(k)}\right) \xrightarrow{\cong} H_n\left(X^{(k)}\right),$$

  by induction of k.

  $k = 0$**:** ✓

  $k > 0$**:** Consider the good pair $\left(X^{(k)}, X^{(k-1)}\right)$, we get the following long exact sequences, with maps ι_* in between.

$$\begin{tikzcd} H_{n+1}^{\Delta}\left(X^{(k)}, X^{(k-1)}\right) \arrow[r] \arrow[d, "\iota_*"] & H_n^{\Delta}\left(X^{(k-1)}\right) \arrow[r] \arrow[d, "\iota_*"] & H_n^{\Delta}\left(X^{(k)}\right) \arrow[r] \arrow[d, "\iota_*"] & H_n^{\Delta}\left(X^{(k), X^{(k-1)}}\right) \arrow[r] \arrow[d, "\iota_*"] & H_{n-1}^{\Delta}\left(X^{(k-1)}\right) \arrow[d, "\iota_*"] \\ H_{n+1}^{\Delta}\left(X^{(k)}, X^{(k-1)}\right) \arrow[r] & H_n\left(X^{(k-1)}\right) \arrow[r] & H_n^{\Delta}\left(X^{(k)}\right) \arrow[r] & H_n\left(X^{(k), X^{(k-1)}}\right) \arrow[r] & H_{n-1}\left(X^{(k-1)}\right) \end{tikzcd}$$

From induction, we know that the 2^{nd} and 5^{th} from left ι_* are isomorphisms. We want to know what $H_n^\Delta\left(X^{(k)}, X^{(k-1)}\right)$ is. Write the following.

$$\overbrace{C_{k+1}^\Delta\left(X^{(k)},X^{(k-1)}\right)}^{=0} \xrightarrow{0} C_k^\Delta\left(X^{(k)},X^{(k-1)}\right) \xrightarrow{0} \overbrace{C_{k-1}^\Delta\left(X^{(k)}, X^{(k-1)}\right)}^{=0} \to \dots$$

So, $H_n^\Delta\left(X^{(k)}, X^{(k-1)}\right) \cong 0$ for $n \neq k$ and

$$H_k^\Delta\left(X^{(k)}, X^{(k-1)}\right) \cong C_k^\Delta\left(X^{(k)}, X^{(k-1)}\right) \cong \mathbb{Z} \text{ span of } k\text{-simplices of } X.$$

What is $H_n\left(X^{(k)},X^{(k-1)}\right)$? This is isomorphic to $\tilde{H}_n\left(X^{(k)}\middle/X^{(k-1)}\right)$, but this space is

$$\bigwedge_{\Sigma_k=\text{all } k\text{-simplices of X}} \mathbb{S}^k$$

so by a proposition,

$$\tilde{H}_n\left(X^{(k)}\middle/X^{(k-1)}\right) \cong \bigoplus_{\Sigma_k} \tilde{H}_n\left(\mathbb{S}^k\right) = \begin{cases} 0 & n \neq k \\ \oplus_{\sigma\in\Sigma_k}\mathbb{Z}\sigma & n = k \end{cases}.$$

Moreover, from the computation of the generators of $H_n^\Delta\left(X^{(k)},X^{(k-1)}\right)$ we see that

$$i_*\colon H_n^\Delta\left(X^{(k)},X^{(k-1)}\right) \to H_n\left(X^{(k)}, X^{(k-1)}\right), H_n\left(X^{(k)}, X^{(k-1)}\right)$$

are isomorphisms.

Lemma 3.3.2. *If $\alpha, \beta, \delta, \varepsilon$ are isomorphisms in the following commutative diagram, and $A \to B \to C \to D \to E$ and $A' \to B' \to C' \to D' \to E'$ are exact, then γ is an iso.*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Proof. By diagram chasing. ■

This proves the required i_* are the same, hence we know that simplicial and singular homologies are the same.

- ***Lemma 3.3.3.*** *Let X be a Δ-complex and let C be a compact subset of X. Then C intersects the interior of finitely many simplices of X.*

  Proof. By contradiction. Assume $\sigma_1, \sigma_2, \dots$ are simplices of X such that $\mathring{\sigma}_i \cap C \neq \varnothing$. LEt $x_i \in \mathring{\sigma}_i \cap C$. Let $U_i \coloneqq X \setminus \{x_j\}_{j\neq i}$. U_i is open since for every simplex σ, $\sigma \cap U_i = \sigma \setminus$ finitely many points. C is covered by the U_i, but doesn't have a finite sub-cover. ■

  Me move to prove $H_n^\Delta(X) \xrightarrow{\sim} H_n(X)$ for generalised Δ-complex X.

  Surjectivity: Let $[c] \in H_n(X)$ where c is a cycle. Write $c = \sum_{i=1}^M \alpha_i\sigma_i$ where $\alpha_i \in \mathbb{Z}$ and σ_i are singular simplices. The image of all σ_i is compact, hence their image is in some $X^{(k)}$. By $H_n^\Delta\left(X^{(k)}\right) \xrightarrow{\sim} H_n\left(X^{(k)}\right)$, there is $[c'] \in H_n^\Delta$ that maps to $[c]$.

  Injectivity: If $[c] \in H_n^\Delta(X)$ maps to $0 \in H_n(X)$, there is $d \in C_n(X)$ such that $\partial d = \iota c$. d is compact, and continue as for surjectivity. ■

Chapter 4

Cellular homology

4.1 CW complexes

Definition 4.1.1. A ***CW complex*** X is built inductively by the following.

- $X^{(0)}$ is a disjoint collection of points.
- build $X^{(n+1)}$ from $X^{(n)}$ by attaching disks $\mathbb{D}^{n+1}$ by gluing their boundaries to $X^{(n)}$.

$$X^{(n)} \amalg \coprod_{\alpha \in A} D_\alpha \Big/ x \sim \phi\alpha\left(x\right) \forall x \in \partial D_\alpha$$

  where $\phi\alpha \colon \partial D \to X^{(n)}$ is continuous.

- Take $X = \bigcup_n X^{(n)}$ with the weak CW topology. I.e. $U \subseteq X$ is open if and only if for all D_α, $U \cap D_\alpha$ is open.

Definition 4.1.2. $D_\alpha \sim \mathbb{D}^{n+1}$ are the ***cells*** of the CW complex.

Examples. 1. On $\mathbb{S}^n$ we can take one 0-cell v and one n-cell $\mathbb{D}^n$.

2. On $\mathbb{T}^2$ we can take one 0-cell v, two 1-cells e_1, e_2 and one 2-cell σ.
3. On $\mathbb{T}^3$ we can take one 0-cell, three 1-cells, three 2-cells and one 3-cell. We can write $\mathbb{T}^3 = \mathbb{S}^1 \times \ldots \times \mathbb{S}^1$, and it would be easier to compute using this.

Definition 4.1.3. The ***i-chain*** is the $\mathbb{Z}$-span of the i-cells.

Remark 4.1.4. We can see that $H_n\left(X^{(n)}, X^{(n-1)}\right)$ is the $\mathbb{Z}$-span of the n-cells.

Notation 4.1.5. Write the inclusion maps $e_\alpha \colon \mathbb{D}_\alpha \to X$, and the gluing maps $\phi_\alpha \colon \partial\mathbb{D}^n_\alpha \to X^{(n-1)}$.

Fact 4.1.6. *Let $A \subseteq X$ be a sub-complex. Then (X, A) is a good pair.*

We observe the following, within singular homology.

Proposition 4.1.7. *1.*

$$H_k\left(X^{(n)}, X^{(n-1)}\right) = \begin{cases} 0 & n \neq k \\ \bigoplus_\alpha \mathbb{Z}\left[e_\alpha\right] & n = k \end{cases}$$

2. $H_k\left(X^{(n)}\right) = 0$ for $k > n$.

3. $X^{(n)} \hookrightarrow X$ defines an iso $H_k\left(X^{(n)}\right) \xrightarrow{\sim} H_k\left(X\right)$ for $k < n$.

The first statement should be obvious.
To see the other two, we look at the long exact sequence of $\left(X^{(n)}, X^{(n-1)}\right)$.

$$H_{k+1}\left(X^{(n)}, X^{(n-1)}\right) \to H_k\left(X^{(n-1)}\right) \to H_k\left(X^{(n)}\right) \to H_k\left(X^{(n)}, X^{(n-1)}\right)$$

then $H_k\left(X^{(n-1)}\right) \xrightarrow{\sim} H_k\left(X^{(n)}\right)$ for $k \notin \{n, n-1\}$.[1]
If $k > n$ we get from the above iso the following

$$H_k\left(X^{(n)}\right) \cong H_k\left(X^{(n-1)}\right) \cong \ldots \cong H_k\left(X^{(0)}\right)$$

and $H_k\left(X^{(0)}\right) = 0$, so $H_k\left(X^{(n)}\right) = 0$.
If $k < n, m$ then $H_k\left(X^{(n)}\right) \cong H_k\left(X^{(m)}\right)$, and by the same compactness argument as in the end of the previous chapter, we get $H_k\left(X^{(n)}\right) = H_k(X)$.

4.2 Cellular homology

From long exact sequences, we have the following.

Definition 4.2.1. $\bar{\partial} = q_* \circ \partial$.

Remark 4.2.2. $\partial \circ q_* = 0$ hence $\bar{\partial}\bar{\partial} = 0$.

Proposition 4.2.3. *$H_n(X) = H_n^{\mathrm{CW}}(X)$ where the second is the homology of the cellular chain complex.*

Proof.

$$H_{n-1}(X) \cong H_{n-1}\left(X^{(n)}\right) \cong H_{n-1}\left(X^{(n-1)}\right) \big/ \partial H_n\left(X^{(n)}, X^{(n-1)}\right)$$

q_* is injective, hence

$$\ker \bar{\partial} = \ker \partial = q_* H_{n-1}\left(X^{(n-1)}\right).$$

q_* is an iso from

$$H_{n-1}\left(X^{(n-1)}\right) \xrightarrow{q_* \circ \partial} \ker \bar{\partial}.$$

Hence

$$H_{n-1}^{\mathrm{CW}}(X) = \ker \bar{\partial} \big/ \operatorname{Im} \bar{\partial} \cong H_{n-1}\left(X^{(n-1)}\right) \big/ \partial H_n\left(X^{(n)}, X^{(n-1)}\right) \cong H_{n-1}(X)$$

∎

Definition 4.2.4. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$, this defines a map $f_*\colon H_n(\mathbb{S}^n) \to H_n(\mathbb{S}^n)$. The ***degree of*** f is $\deg(f) = m$ such that $c \mapsto mc$ where $H_n(\mathbb{S}^n) \cong \mathbb{Z} \cong \langle c \rangle$.

Remark 4.2.5.

$\deg \mathbb{1} = 1$

$f \simeq g$ implies $\deg f = \deg g$.

If f isn't surjective, then $\deg f = 0$. If $x_0 \notin \operatorname{Im} f$ then

$$\mathbb{S}^n \xrightarrow{f} \mathbb{S}^n \setminus \{x_0\} \hookrightarrow \mathbb{S}^n$$

and $H_n(\mathbb{S}^n \setminus \{x_0\}) = 0$ hence $f_* = 0$ hence $\deg f = 0$.

$(\cdot)_*$ is functorial, hence $\deg(f \circ g) = \deg f \cdot \deg g$.

If f is a reflection of $\mathbb{S}^n$, then $\deg f = -1$. We can write $\mathbb{S}^n = \sigma_1 \underset{\partial\sigma_1 \sim \partial\sigma_2}{\cup} \sigma_2$ with $\sigma_i \cong \Delta^n$. Then

$$H_n(\mathbb{S}^n) = \langle [\sigma_1 - \sigma_2] \rangle$$

Take f a reflection along $\mathbb{S}^{n-1} = \partial\sigma_1$, this gives $f_*[\sigma_1] = \sigma_2$ and $f_*[\sigma_2] = \sigma_1$ hence $f_*(\sigma_1 - \sigma_2) = -(\sigma_1 - \sigma_2)$.

Corollary 4.2.6. *The reflection of the sphere isn't homotopic to the identity.*

[1] For $k \notin \{n, n-1\}$ the homologies on the sides are zero, then the map in between is an iso.

Take the antipodal map $-\mathbb{1}(x) = -x$ on $\mathbb{S}^n$, which has degree $(-1)^{n+1}$. That's true because $-\mathbb{1} = f_{n+1} \circ \ldots \circ f_1$ where f_i is the reflection along $\mathbb{R} \times \ldots \times \mathbb{R} \times \underbrace{\{0\}}_{i} \times \mathbb{R} \times \ldots \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$.

Corollary 4.2.7. *If f has no fixed points, then* $\deg f = (-1)^{n+1}$.

Proof. If $\forall x \in \mathbb{S}^n : f(x) \neq x$, then the line[2] connecting $f(x)$ and $-x$ avoids 0.
We get an homotopy $H_t(x) = \frac{-tx+(1-t)f(x)}{\|-tx+(1-t)f(x)\|}$ between f and $-\mathbb{1}$. ■

Theorem 4.2.8 (Hopf). $\deg f$ *determines f up to homotopy.*

[2] in $\mathbb{R}^{n+1}$

Theorem 4.2.9 (The hairy ball theorem). *There is a non-vanishing (tangent) vector field on $\mathbb{S}^n$ if and only if n is odd.*

Proof.

- Assume v is a non-vanishing vector field on $\mathbb{S}^n$. WLOG $\|v(x)\| = 1$ for all $x \in \mathbb{S}^n$. Define the homotopy
$$H_t(x) = \cos(\pi t)\, x + \sin(\pi t)\, v(x).$$
Then $H_t \colon \mathbb{S}^n \to \mathbb{S}^n$ and $H_0 = \mathbb{1}$, $H_1 = -\mathbb{1}$. So, taking degrees, $(-1)^{n+1} = 1$ hence n is odd.
- Take $\mathbb{S}^{2n-1} \subseteq \mathbb{R}^{2n}$. Then $v(x_1, \ldots, x_{2n}) = (x_1, -x_2, x_3, -x_4, \ldots, x_{2n-1}, -x_{2n})$. ■