Lecture Notes to a course on Lie Algebras
Winter 2018, Technion IIT

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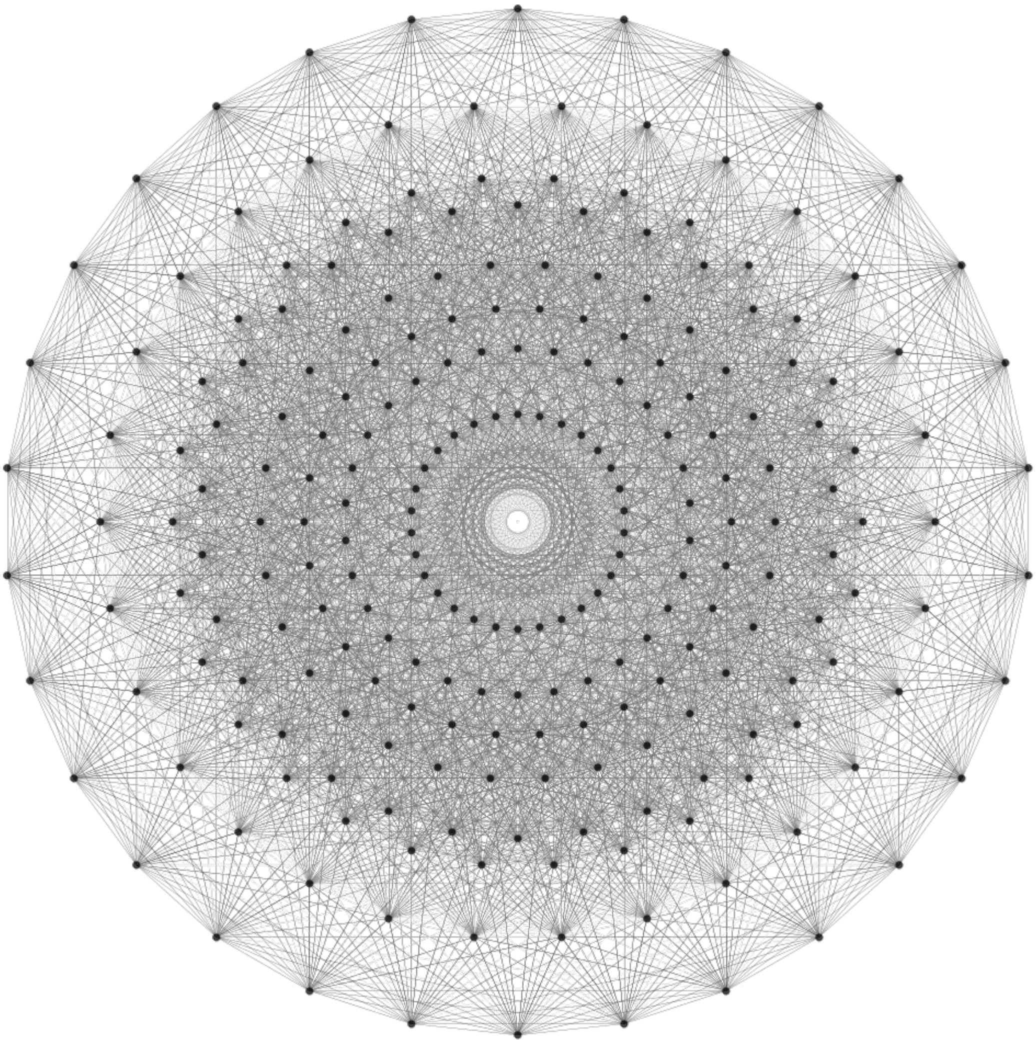

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.
In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.
Elad Tzorani.

Course Literature

The recommended course literature is as follows.

Humphreys, James E.: Introduction to Lie algebras and representation theory.

Jacobson, Nathan: Lie algebras. New York, 1962.

Part I

Lie Algebras

Chapter 1

Preliminaries

The course will be entirely algebraic, with possibly few examples from analysis. This will allow us to discuss issues regarding the algebraic properties of Lie algebras. We might be interested in infinite-dimensional Lie algebras, but in this course we discuss only finite-dimensional algebras. In this course one of our main goals is a classification theorem for simple Lie algebras. We assume knowledge in linear algebras and specifically bilinear forms.

1.1 Basic definitions

Let $\mathbb{F}$ be a field, and V a finite-dimensional vector-space over $\mathbb{F}$.

Definition 1.1.1. V is a ***generalised*** $\mathbb{F}$***-algebra*** if it comes with a map $m\colon V \times V \to V$ which is bilinear.

$$\begin{aligned} m(v_1 + v_2, w) &= m(v_1, w) + m(v_2, w) \\ m(v, w_1 + w_2) &= m(v, w_1) + m(v, w_2) \\ m(av, bw) &= abm(v, w) \end{aligned}$$

Example. Let V be an associative algebra. Here m is an associative operation which is left and right distributive on addition in v. Equivalently: If we denote $m(v, w) = v \odot w$ then

$$\begin{aligned} (v \odot w) \odot u &= v \odot (w \odot u) \\ v \odot (u + w) &= v \odot u + v \odot w \\ (u + w) \odot u &= u \odot v + w \odot v \end{aligned}$$

Remark 1.1.2. Here associativity means the following.

$$m(v, m(w, u)) = m(m(v, w), u)$$

Examples.

1. Every field k is an $\mathbb{F}$–algebra over any subfield $\mathbb{F}$.
2. $M_n(k)$ is an $\mathbb{F}$–algebra.
3. P_n, polynomials over k of degree smaller or equal to n, is an $\mathbb{F}$-algebra.

Definition 1.1.3. A Lie algebra L over $\mathbb{F}$ is an $\mathbb{F}$-algebra, so $\exists m\colon L \times L \to L$, which generally need not be associative, but instead satisfies the following ***Jacobi identity***,

$$m(X, m(Y, Z)) + m(Z, m(X, Y)) + m(Y, m(Z, X)) = 0$$

and additionally, antisymmetry of the multiplication

$$m(X, Y) = -m(Y, X).$$

If $\mathrm{char}\mathbb{F} = 2$ we require $m(X, X) = 0$.

Notation 1.1.4. The "multiplication" in L is called ***bracket***, and denoted $m(X, Y) = [X, Y]$ (X bracket Y).

Remark 1.1.5. In these terms we write the Jacobi identity as follows.

$$[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$$

Definition 1.1.6. A ***Lie algebra*** L is a vector space over $\mathbb{F}$ with a bilinear map $[,]:L\times L\to L$, which is anti-symmetric and satisfies the Jacobi identity.

Definition 1.1.7. Given a Lie algebra L, a vector subspace $L_0\subseteq L$ is called a sub-Lie-algebra if it is closed under brackets. I.e.

$$X,Y\in L_0 \implies [X,Y]\in L_0.$$

Examples. 1. *Abelian Lie algebras:* The bracket is the zero form.

$$\forall X,Y\in L\colon\ [X,Y]=0$$

Example. $\mathbb{F}$ is itself a Lie algebra as well as any $\mathbb{F}$-vector space V under the bracket

$$\forall u,v\in V\colon\ [u,v]=0.$$

Example. Let A be any associative $\mathbb{F}$-algebra, and define on A *another* bilinear operation, namely

$$[a,b]=ab-ba.$$

This is called ***the commutator of*** a ***and*** b. Then $[,]:A\times A\to A$.

Exercise. This bracket satisfies the Jacobi identity, and is anti-symmetric.

Given a solution to this exercise, $(A,[,])$ is a Lie algebra.
In particular, $M_n(k)$ is a Lie algebra under the bracket $[A,B]=AB-BA$. This algebra is *very important* and is denoted $\mathcal{GL}_n(k)$.

Exercise. Consider the subspace

$$\{A\in\mathcal{GL}_n(k)\mid \mathrm{tr}A=0\}\subseteq\mathcal{GL}_n(k).$$

Is the subspace a Lie algebra? Yes! Since for any $A,B\in\mathcal{GL}_n(k)$ we have that $\mathrm{tr}(AB)=\mathrm{tr}(BA)$, we get that $\mathrm{tr}[A,B]=0$. The sub-Lie-algebra of zero-trace matrices is denoted $\mathcal{SL}_n(k)$.

Exercise (Lie algebras associated with bilinear forms). Let V be a vector space over $\mathbb{F}$, and $B\colon V\times V\to\mathbb{F}$ be a bilinear form. Assume B is anti-symmetric. Define

$$L_B=\{X\in\mathrm{End}(V)\mid B(Xv,w)=-B(v,Xw)\}.$$

Check that L_B is a vector subspace of $\mathrm{End}(V)$. Consider the bracket operation on $\mathrm{End}(V)$, defined by $[T,S]=TS-ST$. Is L_B closed under brackets?

Solution. We compute as follows.

$$\begin{aligned}B([X,Y]v,w)&=B((XY-YX)v,w)\\&=B(XYv,w)-B(YXv,w)\\&=-B(Yv,Xw)+B(Xv,Yw)\\&=B(v,YXw)-B(v,XYw)\\&=B(v,(YX-XY)w)\\&=-B(v,[X,Y]w)\end{aligned}$$

In conclusion, L_B is a sub-Lie-algebra of $\mathrm{End}(V)$, the Lie algebra associated with the form B.

Exercise. Let S be a symmetric bilinear form, and let

$$L_S=\{X\in\mathrm{End}(V)\mid S(Xv,w)=-S(v,Xw)\}.$$

Then again, L_S is a Lie sub-algebra.

Examples (Sub-algebras of $\mathcal{GL}_n(\mathbb{F})$). 1.

$$\mathfrak{T}(n,\mathbb{F}) = \left\{ \begin{pmatrix} a_{11} & & a_{i,j} \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} \middle| \; a_{ij} \in \mathbb{F} \right\}$$

is closed under the bracket operation, for if $A, B \in \mathfrak{T}(n,\mathbb{F})$ then $AB \in \mathfrak{T}(n,\mathbb{F})$ and so $AB - BA \in \mathfrak{T}(n,\mathbb{F})$.

2.

$$\mathfrak{N}(n,\mathbb{F}) = \left\{ \begin{pmatrix} 0 & & a_{i,j} \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \middle| \; a_{ij} \in \mathbb{F} \right\}$$

is a Lie sub-algebra of $\mathfrak{T}(n,\mathbb{F})$.

3.

$$\mathfrak{D}(n,\mathbb{F}) = \left\{ \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \middle| \; a_i \in \mathbb{F} \right\}$$

an abelian sub-algebra.

1.2 Structure constants

Let L be a Lie algebra and let $X_1, \ldots, X_n$ be a basis of L, Then the bracket operation is completely determined by the structure constants with respect to the basis.

$$[X_i, X_j] = \sum_{k=1}^{n} c_k^{i,j} X_k$$

The ***structure constants*** $c_k^{i,j}$ contain full information on the bracket operation of course. These satisfy two properties associated with anti-symmetry and the Jacobi identity of the brackets. The property associated to anti-symmetry is $c_k^{i,j} = -c_k^{j,i}$. The other property (associated to the Jacobi identity) is left as an **Exercise**.

Example.

$$\mathcal{GL}_n(\mathbb{F}) = \operatorname{span}\{E_{i,j} \mid 1 \leq i,j \leq n\}$$

In the basis E_{ij} the structure constants are very simple. We have the following.

$$[E_{i,j}, E_{k,l}] = \delta_{j,k} E_{i,l} - \delta_{l,i} E_{k,j}$$

Hence all the structures constants are 1 or -1.

Definition 1.2.1. Let L_1, L_2 be Lie algebras. A ***Lie algebra homomorphism*** between L_1 and L_2 is a linear map $T \colon L_1 \to L_2$ satisfying

$$T[X,Y] = [TX, TY].$$

Definition 1.2.2. Let L be a Lie algebra. A sub-space $I \subseteq L$ is called a ***Lie-ideal*** of L if for all $X \in L$ and $Y \in I$, we have that $[X,Y] \in I$. This is written also by

$$[L, I] = \operatorname{span}\{[X,Y] \mid X \in L, Y \in I\} \subseteq I.$$

Definition 1.2.3. Let L be a Lie algebra and $L_0 \subseteq L$ be a sub-space. The ***Lie normaliser*** of L_0 is

$$N(L_0) = \{X \in L \mid [X, L_0] \subseteq L_0\}.$$

The ***Lie centraliser*** of L_0 is

$$Z(L_0) = \{X \in L \mid [X, L_0] = 0\}.$$

Definition 1.2.4. Let L be a Lie algebra. If $[X,Y]=0$ one says that X and Y commute. We sometimes refer to the bracket as the commutator.

Example. Two sub-spaces $L_1, L_2 \subseteq L$ of a Lie algebra commute if their commutators are zero. I.e.

$$[L_1, L_2] = 0.$$

Remark 1.2.5. Although we have linearity of the bracket, we do need to take the span in the above example. If we take $X, X' \in L_1$ and $Y, Y' \in L_2$ we can't always express $[X,Y]+[X',Y']$ as a bracket of two elements, although it certainly is in the span.

1.3 Linear representations

Definition 1.3.1. A ***linear representation*** of a Lie algebra L over $\mathbb{F}$ is a Lie-algebra homomorphism $T\colon L \to \mathrm{End}(V) \cong \mathcal{GL}_n(\mathbb{F})$ where V is an n-dimensional vector space over $\mathbb{F}$.

Remark 1.3.2. The bracket operation on $\mathrm{End}(V)$ is the usual one, namely $[A,B]=AB-BA$.

Let us define another large collection of Lie algebras. First, let A be a generalised $\mathbb{F}$–algebra, and denote $m(a,b)=a\odot b$.

Definition 1.3.3. A ***derivation*** of the generalised algebra A is a linear map $\delta\colon A\to A$ satisfying the following property.

$$\delta(a\odot b)=\delta(\alpha)\odot b + a\odot\delta(b)$$

Definition 1.3.4.

$$\mathrm{Der}(A) \coloneqq \{\delta\in\mathrm{End}(A) \mid \delta \text{ is a derivation.}\}$$

Remark 1.3.5. $\mathrm{Der}(A)$ is clearly a linear sub-space of $\mathrm{End}(A)$. Now, if δ_1 and δ_2 are derivations, $\delta_1\circ\delta_2$ is *not* a derivation, usually. But, $[\delta_1,\delta_2]=\delta_1\circ\delta_2-\delta_2\circ\delta_1$ *is* in fact a derivation.

Conclusion. $\mathrm{Der}(A)$, with the bracket inherited from $\mathrm{End}(A)$ is a Lie algebra.

Proof. We compute the following.

$$\begin{aligned}
[\delta_1,\delta_2](a\odot b) &= (\delta_1\circ\delta_2-\delta_2\circ\delta_1)(a\odot b)\\
&= \delta_1\circ\delta_2(a\odot b)-\delta_2\circ\delta_1(a\odot b)\\
&= \delta_1(\delta_2(a)\odot b + a\odot\delta_2(b)) - \delta_2(\delta_1(a)\odot b + a\odot\delta_1(b))\\
&= \delta_1\delta_2(a)\odot b+\delta_2(a)\odot\delta_1(b)+\delta_1(a)\odot\delta_2(b)+a\odot\delta_1\delta_2(b)\\
&- (\delta_2\delta_1(a)\odot b+\delta_1(a)\odot\delta_2(b)+\delta_2(a)\odot\delta_1(b)+a\odot\delta_2\delta_1(b))\\
&= (\delta_1\delta_2-\delta_2\delta_1)(a)\odot b + a\odot(\delta_1\delta_2-\delta_2\delta_1)(b)
\end{aligned}$$

∎

Example.
1. If A is an associative algebra, then $\mathrm{Der}(A)$ is a Lie algebra, $\mathrm{Der}(A)\subseteq\mathrm{End}(A)$. $\mathrm{Der}(A)$ is a sub-Lie-algebra of $\mathrm{End}(A)$ under bracket of linear transformations.
2. A Lie algebra is a generalised algebra and so $\mathrm{Der}(L)$ is another Lie algebra.

Fact 1.3.6 (important). *There is a very natural collection of derivations of any Lie algebras. For each $x\in L$, let us define a linear transformation denoted* $\mathrm{ad}(x)\colon L\to L$ *via* $\mathrm{ad}(x)(y)=[x,y]$. *(This is linear from the bi-linearity of the bracket) In fact,* $\mathrm{ad}(x)$ *is a derivation of L. Namely,*

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)y, z] + [y, \mathrm{ad}(x)(z)].$$

Indeed,

$$\begin{aligned}
\mathrm{ad}(x)([y,z]) &= [x,[y,z]]\\
&= [\mathrm{ad}(x)y,z]+[y,\mathrm{ad}(x)z]\\
&= [[x,y],z]+[y,[x,z]]
\end{aligned}$$

which is an identity as a consequence of the Jacobi identity.

Conclusion. The set $\{\mathrm{ad}(x) \mid x \in L\} \subseteq \mathrm{Der}(L)$ is a sub-algebra. We have the map $x \mapsto \mathrm{ad}(x)$ which is obviously linear (from bi-linearity of the bracket). So, $\mathrm{ad}(L) \coloneqq \{\mathrm{ad}(x) \mid x \in L\}$ is a linear sub-space. In fact it is a Lie sub-algebra of $\mathrm{Der}(L)$.

Proof. We have to show that $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x)$ is in the space $\mathrm{ad}(L)$. But, actually $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}\,[x, y]$, as the following proposition states.

Proposition 1.3.7. $\mathrm{ad}\colon L \to \mathrm{Der}(L)$ *is a Lie algebra homomorphism.*

Proof. Let us compute.

$$\begin{aligned}
[\mathrm{ad}(x), \mathrm{ad}(y)]\,(z) &= \mathrm{ad}\,(x)\,\mathrm{ad}\,(y)\,(z) - \mathrm{ad}\,(y)\,\mathrm{ad}\,(x)\,(z) \\
&= [x, [y, z]] - [y, [x, z]] \\
&\overset{\star}{=} [[x, y]\,, z] \\
&= \mathrm{ad}\,[x, y]\,(z)
\end{aligned}$$

where the $\star$ is given from the Jacobi identity. ■

In conclusion, $\mathrm{Der}(L)$ is a Lie sub-algebra of $\mathrm{End}(L)$ under bracket, and $\mathrm{ad}\colon L \to \mathrm{Der}(L) \subseteq \mathrm{End}(L)$ is a linear representation of the Lie algebra L with the image being $\mathrm{ad}(L) = \{\mathrm{ad}(x) \mid x \in L\}$. ■

Example. Given $L_0 \subseteq L$ a sub-space. Then $N\,(L_0) = \{x \mid [x, L_0] \subseteq L_0\}$ is the set of elements x such that the linear transformation $\mathrm{ad}\,(x)$ leaves the subspace L_0 invariant.

Example. The condition $[X, Y] = 0$ means $Y \in \ker\,(\mathrm{ad}(x))$ or equivalently $x \in \ker\,(\mathrm{ad}(y))$. Therefore

$$\begin{aligned}
Z\,(L_0) &= \{x \in L \mid [x, L_0] = 0\} \\
&= \{x \in L \mid L_0 \subseteq \ker\,(\mathrm{ad}(x))\}\,.
\end{aligned}$$

Remark 1.3.8. If $L_0 \subseteq L$ is a Lie sub-algebra, the $N\,(L_0)$ is a sub-algebra and is the largest sub-algebra such that L_0 is is an ideal in it.

$$\mathrm{End}\,(A, B)$$