1) Show that equivalence of categories is an equivalence relation on the collection of categories.

[Hint: Use the fact that an isomorphism $\alpha \in \mathrm{Hom}(X,Y)$ provides bijections

$$\mathrm{Hom}(Y,Z) \xrightarrow{\sim} \mathrm{Hom}(X,Z) \quad f \mapsto f\circ\alpha .$$
$$\mathrm{Hom}(Z,X) \xrightarrow{\sim} \mathrm{Hom}(Z,Y) \quad g \mapsto \alpha\circ g$$
]

2) Adjoint functors:

Let $\mathcal{C}$, $\mathcal{D}$ be given categories.

A pair of functors (L,R)

$L: \mathcal{C} \to \mathcal{D}$, $R: \mathcal{D} \to \mathcal{C}$ is

called adjoint (L is left-adjoint to R. R is right-adjoint to L.),

if for any $X \in \mathrm{obj}(\mathcal{C})$, $Y \in \mathrm{obj}(\mathcal{D})$

there is a bijection

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \xrightarrow[\Phi_{X,Y}]{\sim} \mathrm{Hom}_{\mathcal{C}}(X, R(Y)),$$

s.t.

$$
\begin{cases}
\Phi_{X_1,Y_1}\big(h \circ L(f)\big) = \Phi_{X_2,Y_1}(h) \circ f \\
\Phi_{X_2,Y_2}(g \circ h) = R(g) \circ \Phi_{X_2,Y_1}(h)
\end{cases}
$$

holds, for all

$f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$, $h \in \mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_1)$

$g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2)$, .

Show that an equivalence of categories
$F : \mathcal{C} \to \mathcal{D}$ always has
a right-adjoint and a left-adjoint
functor.

3) $\mathcal{C}$ = category of Lie algebras.
$\mathcal{D}$ = category of (all) Lie groups.

Show that the pair of functors

$(\tilde{\Gamma}, \mathrm{Lie})$ we constructed is an adjoint pair.

(Can assume facts that were proven in the matrix group case.)

6. Construct a double covering $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{C})$.

7. (a) Construct a double covering $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}_o(2,1)$.

   (b) Construct a double covering $\mathrm{SU}(1,1) \to \mathrm{SO}_o(2,1)$.

   (c) Find an isomorphism $\mathrm{SL}(2,\mathbb{R}) \approx \mathrm{SU}(1,1)$ compatible with (a) and (b).

   [Suggestion: for (c), consider conjugation by suitable 2×2 matrix.]

8. *The double covering* $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(4,\mathbb{C})$. Construct this covering. [Suggestion: $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ acts on $M_2(\mathbb{C}) \approx \mathbb{C}^4$ by $X \to aXb^{-1}$ preserving the quadratic form $\varphi(X,X) = \det(X)$.]

2. Show that $\mathrm{Aut}(\mathbb{T}^n) \approx \mathrm{GL}(n,\mathbb{Z}) = \{a \in M_n(\mathbb{Z}) \mid a^{-1} \in M_n(\mathbb{Z})\} = \{a \in M_n(\mathbb{Z}) \mid \det a = \pm 1\}$.

For 6,7 imitate what we did for $SU(2) \to SO(3)$.

$U(1,1) = \left\{ g \in GL_2(\mathbb{C}) : g \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overline{g^t} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

$$SU(1,1) = U(1,1) \cap SL_2(\mathbb{C}) .$$