הראו שלכל אופרטור ליניארי

$X \in End(V)$, (V מרחב וקטורי סוף-ממדי ממשי או מרוכב)

האופרטור $\exp(X) \in GL(V)$ מוגדר

היטב. כלומר, לכל בסיס B של V,

$\exp(X)$ מקיים

$$\exp([X]_B) = [\exp(X)]_B ,$$

כאשר $[A]_B$ היא המטריצה המייצגת של $A \in End(V)$ בבסיס B.

5. Show:

$$\exp \begin{bmatrix} \lambda & 1 & 0 \cdots & 0 \\ 0 & \lambda & 1 \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & \lambda \end{bmatrix} = \begin{bmatrix} e^\lambda & e^\lambda & e^\lambda/2! \cdots & e^\lambda/(n-1)! \\ 0 & e^\lambda & e^\lambda \cdots & e^\lambda/(n-2)! \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & e^\lambda \end{bmatrix}.$$

11. Two matrices P, Q are said to satisfy *Heisenberg's Commutation Relation* if

$$PQ - QP = k1$$

for some scalar k. Show that this is the case, if and only if

$$\exp \sigma P \exp \tau Q = e^{\sigma\tau k} \exp \tau Q \exp \sigma P$$

for all real σ, τ.

1. A matrix $X \in M$ is called *nilpotent* if $X^k = 0$ for some k (equivalently: all eigenvalues of X are equal to 0); $a \in M$ is called *unipotent* if $(1-a)^k = 0$ for some k (equivalently: all eigenvalues of a are equal to 1). Show:

   $X \to \exp X$ maps the nilpotent matrices bijectively onto the unipotent matrices with inverse $a \to \log a$.

   [Proposition 1 does not apply directly, nor does the Substitution Principle as stated; a minor adjustment will do.]

2. A matrix a is called *semisimple* if it is diagonalizable over $\mathbb{C}$. Show:

   (a) $X \to \exp X$ maps semisimple matrices to semisimple matrices.

   (b) If a is an invertible semisimple matrix, then there is a semisimple matrix X so that $a = \exp X$ and no two distinct eigenvalues of X differ by a multiple of $2\pi i$.

   (c) Assume X and X' are both semisimple and no two distinct eigenvalues of X differ by a multiple of $2\pi i$. Show that $\exp X = \exp X'$ if and only if X and X' are *simultaneously* diagonalizable with diagonal entries differing by multiples of $2\pi i$.

   Any matrix X can be *uniquely* written as $X = Y + Z$ where Y is semisimple, Z is nilpotent, and Y and Z *commute*. Furthermore, Y and Z are linear combinations of powers of X. $X = Y + Z$ is called the *Jordan decomposition* of X. [See Hoffman–Kunze (1961) Theorem 8, page 217, for example.]

8. (a) Prove the *Jacobi Identity*

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

Deduce that
(b) $(\operatorname{ad} Z)[X,Y] = [(\operatorname{ad} Z)X, Y] + [X, (\operatorname{ad} Z)Y]$,
(c) $\operatorname{ad}([X,Y]) = [\operatorname{ad} X, \operatorname{ad} Y]$.
(The bracket on the right side of (c) is that of linear transformations of the matrix space M.)

9. Show that for all $X \in M$,

$$\exp X = \lim_{k\to\infty} \left(1 + \frac{1}{k}X\right)^k.$$

[The formula has a 'physical' interpretation: subdivide the time interval $0 \le \tau \le 1$ into a large number of subintervals k; the fluid particle travelling on the trajectory $p(\tau) = \exp(\tau X)p_0$, with velocity $Xp(\tau)$ at $p(\tau)$, will move from p_0 to approximately $p_0 + (1/k)Xp_0 = (1 + (1/k)X)p_0$ in the first time interval, on to $(1 + (1/kX))^2 p_0$ in the second, etc., until at $\tau = 1$ it reaches approximately $(1 + 1/kX)^k p_0$, which must therefore approximate $\exp(X)p_0$].