1)

a) Consider the form

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = x_1 x_2 - y_1 y_2 \quad \text{on } \mathbb{R}^2.$$

Note that $O(1,1) = O(\langle , \rangle)$.

For a real number a, write

$$T_a = \{ v \in \mathbb{R}^2 : \langle v, v \rangle = a \}.$$

Show that for $v \in \mathbb{R}^2$ with $\langle v, v \rangle \neq 0$, $SO(1,1)$ acts transitively on $T_{\langle v, v \rangle}$.

Show that for $\langle v, v \rangle < 0$,

$$SO(1,1)^0 \cdot v = T_{\langle v, v \rangle} \cap \{ w \in \mathbb{R}^2 : \langle v, w \rangle < 0 \}.$$

Imagine the picture on the plane.

b)

Consider $G \subseteq SO(2,1)$ to be the subset of matrices whose 3,3 entry is positive.

Show that
$$G = \{ g \in SO(2,1) : g(P) \subseteq P \},$$
where $P = \{ v \in \mathbb{R}^3 : \langle v,v \rangle < 0, \ \langle v, e_3 \rangle < 0 \}$.

Here,
$$\left\langle \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\rangle = x_1 x_2 + y_1 y_2 - z_1 z_2$$
a form on $\mathbb{R}^3$, and $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^3$.

(Picture the cone P in the space and use its connectivity.)

c) Show that $G < SO(2,1)$ is an open subgroup.

d) Show that
$$\{ g \in SO(2,1) : g e_3 = e_3 \} \simeq SO(2).$$

e) For a given $g \in G$, with $g e_3 \neq e_3$, find $T_g \in O(2,1)$, s.t. $T_g \cdot e_3 = e_3$

and $T_g(e_2) = \alpha(ge_3 + \langle ge_3, e_3 \rangle e_3)$,
for a scalar α.

f) For $g \in G$ as before,
use a) to show that there is
$A \in SO(1,1)^\circ$, s.t. $h \cdot e_3 = T_g^{-1} \cdot g \cdot e_3$,
where $h = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$.

g) Show that every $g \in G$ can
be written as

$$g = T h T^{-1} h', \text{ where}$$

$$h \in \begin{pmatrix} 1 & 0 \\ 0 & SO(1,1)^\circ \end{pmatrix}, \quad h' \in \begin{pmatrix} SO(2) & 0 \\ 0 & 1 \end{pmatrix}.$$

h) Deduce that
$G = SO(2,1)^\circ$.