1) $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : 0 \neq a, b \in \mathbb{R} \right\} < GL_2(\mathbb{R})$

Consider the obvious coordinate chart $\varphi : G \longrightarrow \mathbb{R}^2$ on the Lie group G.

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} a \\ b \end{pmatrix}$$

Describe the space of left-invariant vector fields $\{ L_X : X \in Lie(G) \}$ on G in terms of φ-co-ordinates.

2) G - Lie group.

Consider $M = G \times Lie(G)$ as a smooth manifold, and the map

$\Phi : \mathbb{R} \times M \to M$ given by

$$\Phi(t, g, X) = (g \cdot \exp(tX), X).$$

Show that Φ is a flow defined by a vector field on M, hence, Φ is a smooth map.

Conclude that $\exp : \mathrm{Lie}(G) \to G$ is smooth.

3) M analytic manifold. X a vector field on M.

Consider $X^k = X \circ \cdots \circ X : C^\infty(M) \to C^\infty(M)$ as an operator on the space of smooth functions.

Suppose that $\exp(tX)\cdot p$ is defined for $t \in (-\varepsilon, \varepsilon)$ and a fixed $p \in M$.

For $\varphi \in C^\infty(M)$, set $f_\varphi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ as

$$f_\varphi(t) = \varphi(\exp(tX)p).$$

k-th derivative

Show that
$$f_\varphi^{(k)} = f_{X^k(\varphi)}.$$

Show that
$$f_\varphi(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \cdot X^k(f)(p)$$
holds pointwise.

4) Consider the charts given by exp to show that a Lie group G has an open nbd $e \in U \subset G$, s.t. no (abstract) subgroup $H < G$ is contained in U.

"no small subgroups" property.

5) G Lie group. $H < G$ abstract subgroup. Define

$$\mathrm{Lie}(H) = \left\{ \gamma'(0) \in \mathrm{Lie}(G) : \begin{array}{l} \gamma : (-\varepsilon, \varepsilon) \to G \\ \text{smooth}, \\ \gamma(0) = e,\ \gamma(t) \in H \\ \forall t \in (-\varepsilon, \varepsilon) \end{array} \right\}$$

i) Verify that $\mathrm{Lie}(H)$ is a subspace of $\mathrm{Lie}(G)$ closed under the bracket operation. (same as in the matrix case.)

[We would like to extend the proof that $\Gamma(\mathrm{Lie}(H)) \subset H$ from the matrix case. Please consult the notes of that proof for this exercise.]

ii) Find a chart

$$\mathrm{Lie}(G) \supset U' \xrightarrow[\varphi]{\sim} U \subset G, \text{ s.t.}$$

$$\varphi(0) = e, \quad \varphi(U' \cap \mathrm{Lie}(H)) \subset H, \quad d_0\varphi = \mathrm{Id}$$

iii) For $X \in \mathrm{Lie}(G)$, we identify

$$T_X(\mathrm{Lie}(G)) \simeq \mathrm{Lie}(G).$$

For $X \in U' \cap \mathrm{Lie}(H)$, show that the map

$$A_X = d_X(\ell_{\varphi(X)^{-1}} \circ \varphi) : T_X(\mathrm{Lie}(G)) \to T_e(G)$$

satisfies $A_X(\mathrm{Lie}(H)) \subset \mathrm{Lie}(H)$.

iv) Show that for all $Y \in Lie(H)$,
and small enough $X \in Lie(H)$,

$$L_Y(\varphi(X)) \in d_X(\varphi)(Lie(H))$$

$(L_Y(g) = g \cdot Y$ - left-invariant vector field$)$

v) Show that for all $Y \in Lie(H)$,

$$\exp(Y) \in H.$$