Problems for §2.2

1. Describe the Lie algebras of the following linear groups. (a) The group of invertible *block-triangular* matrices

$$\begin{bmatrix} a_1 & * & * & \cdots & * \\ 0 & a_2 & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & a_m \end{bmatrix}$$

($a_k \in M_{n_k}$, $\det a_k \neq 0$).

(b) The group of *unipotent block-triangular* matrices (as above with $a_k = 1_k \in M_{n_k}$).

2. (a) Find a basis for $\mathsf{so}(n)$ and show that $\dim \mathsf{so}(n) = \frac{1}{2}n(n-1)$. (b) Find a basis for $\mathsf{su}(n)$ and show that $\dim \mathsf{su}(n) = n^2 - 1$.

3. Let $f \in M$ be a non-singular matrix. Let $G = \{a \in G \mid f^{-1}a^t fa = 1\}$.
(a) Show that G is a group and find its Lie algebra.
(b) Take for f the $2m \times 2m$ matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(blocks of size $m \times m$). Describe G and its Lie algebra explicitly (as block-matrices).

4. (a) Describe all Lie subalgebras of $\mathsf{so}(3)$. (b) Describe all *complex* Lie subalgebras of $\mathsf{sl}(2, \mathbb{C})$.
[Suggestion for (b): classify the subalgebras up to conjugacy. Use the fact that every non-zero matrix of $\mathsf{sl}(2, C)$ is conjugate to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find first all complex subspaces of $\mathsf{sl}(2, C)$ which are stable under bracketing (i.e. under ad) with one of these matrices. You may want to use the following basis for $\mathrm{SL}(2, C)$:

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

These satisfy the famous relations

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H,$$

which you may use.]

5. (a) Describe all Lie subalgebras of $\mathsf{sl}(2, \mathbb{R})$. (b) Determine which of these are isomorphic. (c) Determine which of these are conjugate by an element of $\mathrm{SL}(2, \mathbb{R})$.

6. Let A be a finite-dimensional associative algebra over $\mathbb{R}$ or $\mathbb{C}$, $A^\times$ the group of invertible elements of A. Consider an element $a \in A$ as linear transformation $A \to A, x \to ax$, so that $A^\times$ becomes a linear group. Show that its Lie algebra is A with bracket $[x, y] = xy - yx$.

7. Let A be a finite-dimensional, not necessarily associative, algebra over $\mathbb{R}$ or $\mathbb{C}$, i.e. a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ with a bilinear operation $A \times A \to A$, written $(x, y) \to x \cdot y$. Let $\mathrm{Aut}(A)$ be the automorphism group of A, i.e. the group of invertible linear transformations a of A satisfying $a(x \cdot y) = (ax) \cdot (ay)$. Show that the Lie algebra of $\mathrm{Aut}(A)$ consists of all derivations of A, i.e. all linear maps $D : A \to A$ satisfying $D(x \cdot y) = (Dx) \cdot y + x \cdot (Dy)$.

8. Fix $c \in \mathrm{GL}(E)$. Let $G = \{a \in \mathrm{GL}(E) \mid ac = ca\}$.
(a) Check that G is a group and describe its Lie algebra $\mathfrak{g}$.
(b) Find an explicit matrix representation for the elements of G and of $\mathfrak{g}$ in case c is diagonal (with respect to a suitable basis for E).

9. Let F be a subspace of E. Let $G = \{a \in \mathrm{GL}(E) \mid aF \subset F\}$.
(a) Check that G is a group and describe its Lie algebra $\mathfrak{g}$.
(b) Find an explicit matrix representation for the elements of G and of $\mathfrak{g}$ (with respect to a suitable basis for E).

2.3 Coordinates on a linear group

The exponential map $\exp \colon \mathfrak{g} \to G$ of a linear group has many uses. First of all it can be used to introduce a *topology*[1] on G, which means specifying neighborhoods (or open sets). This is done much like for $\mathbb{R}^n$, as follows.

A *neighborhood of* $a \in G$ in G is any subset of G which contains $\{a \exp X \mid X \in \mathfrak{g}, \|X\| < \epsilon\}$ for some $\epsilon > 0$. (One gets the same notion of neighborhood using $\{(\exp X)a \mid X \in \mathfrak{g}, \|X\| < \epsilon\}$, although these sets themselves are generally not same.) A subset of G is *open in* G if it contains a neighborhood in G of each of its points, and *closed* in G if its complement is open in G. A *neighborhood of a subset* S of G is any set containing a neighborhood of each point of S. Convention: in the future we omit the qualification 'in G', with the understanding that *all topological notions pertaining to G (i.e. notions depending on notion of 'neighborhood' or 'open set') are understood in the sense just explained,* unless explicitly stated otherwise. This is of some importance, because the *group-topology* on G need not coincide with its *relative topology* in

[1] It is assumed this notion is familiar, at least from $\mathbb{R}^n$.

3. (a) Show the domain $a_oU = \{a_o \exp X \mid X \in \mathfrak{g}, \|X\| < R\}$ of the exponential coordinates around $a_o \in G$ is open in G.

(b) Show that the coordinate transformation relating *any* two coordinate systems on G is analytic.

(c) By definition, a function on a linear group G is said to be of class $C^k(0 \le k \le \infty, \omega)$ if it becomes such a function when elements of G are locally expressed in terms of coordinates. Show that this definition is independent of the coordinates chosen.

4. Let f be a real-valued function on a linear group. Show that if f extends to a C^k-function of the matrix entries in a neighborhood of each point in its domain, then f is a C^k-function on G. Give an example to show that the converse need not hold. (Any fixed k, $0 \le k \le \infty, \omega$.)

5. Show that every element of SU(2) can be written in the form

$$a = a_3(\theta)a_2(\phi)a_3(\psi)$$

where now

$$a_3(\theta) = \begin{bmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{bmatrix}, \quad a_2(\phi) = \begin{bmatrix} \cos\phi/2 & -\sin\phi/2 \\ \sin\phi/2 & \cos\phi/2 \end{bmatrix},$$

and $0 \le \theta,\ \psi < 2\pi, 0 \le \phi < \pi$. Show that θ, ϕ, ψ form a coordinate system on the domain where the inequalities are all strict. [See Example 2 of §2.1, which also explains the analogy with Example 4(b).]

6. Show that every element of $\mathrm{SL}(2,\mathbb{R})$ can be written in the form

$$a = k(\theta)n(\sigma)a(\tau)$$

where

$$k(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad n(\sigma) = \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix}, \quad a(\tau) = \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix},$$

and $0 \le \theta < 2\pi$, $\sigma, \tau \in \mathbb{R}$. Show that θ, σ, τ form a coordinate system on the domain where the first inequality is also strict. [Compare the decomposition $\mathrm{SL}(2,\mathbb{R}) = KB$ of Example 3 of §2.1.]

7. Show that every element of $\mathrm{SL}(2,\mathbb{R})$ can be written in the form

$$a = k(\theta)a(\tau)k(\phi)$$

with $k(\theta), a(\tau)$ as in the previous problem and $0 \le \theta < \pi$, $0 \le \phi < 2\pi$, $\tau \ge 0$. Show that θ, σ, τ form a coordinate system on the domain where all inequalities are strict. [Suggestion: to prove existence of the decomposition $a = k(\theta)a(\tau)k(\phi)$ consider the adjoint-action of $\mathrm{SL}(2,\mathbb{R})$ on one sheet of the hyperboloid (adjoint orbit) in $\mathfrak{sl}(2,\mathbb{R})$ given by

$$\det X = 1;$$

see Example 3, §2.1. Argue as in Example 4(b), with the help of a sketch. To determine the range of the variables (θ, τ, ϕ) note that $k(\theta+\pi)a(\tau)k(\phi+\pi) = k(\theta)a(\tau)k(\phi)$.]

8. Let N be a linear group consisting of unipotent matrices, $\mathfrak{n}$ its Lie algebra. Show that $\exp\mathfrak{n}$ is open in N and that $a = \exp X \to X$ defines a coordinate system on all of $\exp\mathfrak{n}$. [Suggestion: problem 1, §1.2.]

9. Let $f : G \to H$ be a differentiable homomorphism of linear groups. Show that for all $X \in \mathfrak{g}$ (the Lie algebra of G),

$$f(\exp X) = \exp\varphi(X),$$

where

$$\varphi(X) = \frac{d}{d\tau} f(\exp\tau X)\Big|_{\tau=0}.$$

[Suggestion: show that $f(\exp\tau X))$ satisfies the differential equation of exp.]

10. Let G be a linear group. Suppose there are arbitrarily small neighborhoods U of 1 in the matrix space M so that $G \cap U$ is C^1-connected, i.e. every element of $G \cap U$ can be joined to 1 by a C^1-curve in $G \cap U$. Show:

(a) There is a neighborhood U of 1 in M and a bi-analytic map $f : U \to V$ of U onto V an open ball in $\mathbb{R}^N$ $(N = \dim M)$ which carries $G \cap U$ onto the open ball in the m-dimensional subspace $\mathbb{R}^m$ of $\mathbb{R}^N$. ($m = \dim G$; $\mathbb{R}^m$ is considered as the subspace of $\mathbb{R}^N$ where the last $N-m$ coordinates are 0.) [Suggestion: review the proof of Theorem 3, §2.2.]

(b) Every element $a \in G$ has a neighborhood in G of the form $G \cap U$ where U is a neighborhood of a in M. The open sets in G are exactly the intersections with G of open sets in M. [This means that the group-topology on G is the relative topology. Neighborhoods in G are – as always – understood in the sense defined in the text.]

(c) A function on G (defined on an open subset with values in $\mathbb{R}$ or in a linear group) is of class $C^k(0 \le k \le \infty, \omega)$ if and only if it extends to a function of class C^k in a neighborhood in the matrix space M of each point $a_o \in G$ in its domain.

11. Show that the following groups satisfy the hypothesis of problem 10.

(a) $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$,

(b) $\mathrm{SU}(n)$, $\mathrm{SO}(n)$,

(c) Block-triangular and block-diagonal groups (Example 6, §2.1).

Give an example of a linear group G which does not satisfy the hypothesis of problem 10 and specify explicitly a continuous function on G which does not extend to a continuous function on M, even locally as in problem 10 (c).

The *normalizer* (*in* $\mathfrak{g}$) of a subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ is:

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{a}) = \{Y \in \mathfrak{g} \mid [Y, \mathfrak{a}] \subset \mathfrak{a}\}.$$

Proposition 10. *Let G be a linear group, $\mathfrak{g}$ its Lie algebra, A a connected subgroup of G, and $\mathfrak{a}$ its Lie algebra. Then $N_G(A) = N_G(\mathfrak{a})$ and has Lie algebra $\mathfrak{n}_{\mathfrak{g}}(A) = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{a})$.* (QED)

Again we omit the proof.

Problems for §2.5

1. (a) Prove Proposition 8, (b) prove Proposition 10.

2. Let $G \subset \mathrm{GL}(E)$ be a connected linear group, F a subspace of E. Show: F is G-stable [i.e. $aF \subset F$ for all $a \in G$] if and only if F is $\mathfrak{g}$-stable [i.e. $XF \subset F$ for all $X \in \mathfrak{g}$].

3. *Lie algebra cocycles.* Let $\mathfrak{g}$ be a real Lie algebra. A *cocycle* on $\mathfrak{g}$ is a bilinear function ω: $\mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ that satisfies

$$\omega(X, Y) = -\omega(Y, X),$$
$$\omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0.$$

A *coboundary* is a cocycle of the form $\omega(X, Y) = \varphi([X, Y])$ where $\varphi \in \mathfrak{g}^*$ is a linear functional on $\mathfrak{g}$.

Let G be a linear group, $\mathfrak{g}$ its Lie algebra, ω a cocycle on $\mathfrak{g}$. For any $a \in G$ define another cocycle ω^a on $\mathfrak{g}$ by the formula

$$\omega^a(X, Y) = \omega(\mathrm{Ad}(a)X, \mathrm{Ad}(a)Y).$$

Show that for a *connected* linear group G, ω^a differs from ω by a coboundary for any $a \in G$. [Suggestion: consider a C^1 path $a = a(\tau)$ and differentiate $\omega^a(X, Y)$ with respect to τ; write $a'(\tau)$ as $a'(\tau) = a(\tau)Z(\tau)$ with $Z(\tau) \in \mathfrak{g}$.]

4. If a linear group of dimension ≥ 2 has a dense one-parameter subgroup, then it is isomorphic with a torus $\mathbb{T}^n$.

Complex linear groups. A linear group $G \subset \mathrm{GL}(n, \mathbb{C})$ is complex if the Lie algebra $\mathfrak{g}$ of G is a complex subspace of $\mathfrak{gl}(n, \mathbb{C})$.

5. List *all* complex groups among the groups mentioned in §2.1. [Make sure your list is complete.]

6. Show that a complex abelian linear group which is a compact (i.e. closed and bounded) subset of the matrix space is finite. [Suggestion: use Liouville's Theorem from complex analysis.]

Comment. This is one of the few places where it is essential that one deals with matrix groups. For example, $\mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$ is compact and Abelian, but not finite.

Semidirect products. Let G be a group, M and N subgroups of G with N normal. We say G is the semidirect product of M and N if every $a \in G$ can be uniquely written in the form $a = mn$ with $m \in M$ and $n \in N$. (Equivalently: $G = MN$, $M \cap N = \{1\}$.) We then write $G = MN$ (semidirect), or $G = M \rtimes N$. One could also write $G = NM$ (semidirect) or $G = N \rtimes M$). If not clear from the context, it must be specified which of the two groups M, N is normal. If they are both normal, the product is direct. When G is a linear group we also require that the map $G = MN \to M \times N$, $mn \to (m, n)$ be analytic. This requirement is superfluous if M and N have countably many connected components, see problem 10.

7. Suppose the linear group G is a semidirect product $G = MN$.

(a) Show that its Lie algebra is the direct sum $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{n}$ of the Lie algebras $\mathfrak{m}$ and $\mathfrak{n}$, $\mathfrak{m}$ being a subalgebra, $\mathfrak{n}$ an ideal of $\mathfrak{g}$. [One says that the Lie algebra $\mathfrak{g}$ is the *semidirect product* of $\mathfrak{m}$ and $\mathfrak{n}$.]

(b) Show that the exponential map $\exp\colon \mathfrak{g} = \mathfrak{m} + \mathfrak{n} \to G = MN$ of G takes the form

$$\exp(X + Y) = \exp X \exp A(X)Y,$$

where $A(X)$ is a (generally non-linear) transformation of $\mathfrak{n}$ depending on $X \in \mathfrak{m}$. [It is understood that $X \in \mathfrak{m}$, $Y \in \mathfrak{n}$, both sufficiently close to 0.]

(c) Show that when N is abelian, $A(X) : \mathfrak{n} \to \mathfrak{n}$ is given by

$$A(X) = \frac{\exp(-\operatorname{ad} X) - 1}{\operatorname{ad} X}.$$

(d) Show that the group of affine transformations $x \to ax + b$, $a \in \mathrm{GL}(E)$, $b \in E$, is the semidirect product of the subgroup $\mathrm{GL}(E)$ of linear transformations $x \to ax$ and the subgroup E of translations $x \to x + b$. [See Example 6, §2.1.]

8. Let G be the group of block-triangular matrices of the form

$$\begin{bmatrix} a_1 & * & * & \cdots & * \\ 0 & a_2 & * & \cdots & * \\ & & \ddots & & \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & a_m \end{bmatrix}.$$

Each a_k is an invertible block of some fixed size. Let M be the subgroup of block-diagonal matrices

$$\begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ & & \ddots & & \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & a_m \end{bmatrix},$$

N the subgroup of unipotent block-triangular matrices

$$\begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & 1 & * & \cdots & * \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Show that G is the semidirect product $G = MN$.

9. Show that the analyticity requirement in the definition of 'semidirect product' is superfluous if M and N have countably many connected components. [Suggestion: use Lemma 2 to show that $\mathfrak{g} = \mathfrak{m} + \mathfrak{n}$.]

10. Let G be a linear group, A, B two subgroups of G, and $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ their algebras. Show:

   (a) Assume $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, G is connected, and AB is closed in G. Then $G = AB$.

   (b) Give an example with $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$, G connected, but $G \neq AB$ (even with A and B closed).

   (c) Assume $G = AB$ and A, B have countably many components. Then $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$. [Suggestion for (b). Take $G = \mathrm{SL}(2, R)$, $A = \{$upper triangular$\}$, and find a suitable B. Suggestion for (c): write $\mathfrak{g} = \mathfrak{a}' \oplus \mathfrak{b} \oplus \mathfrak{c}$ with $\mathfrak{a}' \subset \mathfrak{a}$ and consider $\mathfrak{a}' \times \mathfrak{b} \times \mathfrak{c} \to \exp \mathfrak{a}' \exp \mathfrak{b} \exp \mathfrak{c} \subset G$. Use Baire's Lemma to show that $\exp \mathfrak{a}'_\epsilon \exp \mathfrak{b}_\epsilon$ contains an open set for any ϵ–balls in $\mathfrak{a}', \mathfrak{b}$. Argue that $c = 0$.]

11. Let G be a connected linear group with the property that $G^* = G$. (a^* is the adjoint of a with respect to a positive definite form.) Fix a self-adjoint element $Z \in \mathfrak{g}$: $Z^* = Z$. For $\lambda \in \mathbb{R}$, let $\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid \mathrm{ad}(Z)X = \lambda X\}$. Show:

   (a) $\mathfrak{g} = \sum_\lambda \mathfrak{g}_\lambda$ (direct sum).

   (b) $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

   (c) $(\mathfrak{g}_\lambda)^* = \mathfrak{g}_{-\lambda}$.

$N_G(\mathfrak{q}) = \{g \in G : \mathrm{Ad}(G)\mathfrak{q} \subseteq \mathfrak{q}\}$

   (d) Let $\mathfrak{k} = \{X \in \mathfrak{g} \mid X^* = -X\}$, $\mathfrak{q} = \sum_{\lambda \geq 0} \mathfrak{g}_\lambda$. Then $\mathfrak{k}$ and $\mathfrak{q}$ are subalgebras of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$.

   (e) Let $K = \{k \in G \mid k^*k = 1\}$, $Q = N_G(\mathfrak{q})$. Then $L(K) = \mathfrak{k}$, $L(Q) = \mathfrak{q}$, and $G = KQ$. [Suggestion. For (c) and (d) use $(\mathrm{ad}\, Z)^* = \mathrm{ad}(Z^*)$. For $G = KQ$ in (e) use the previous problem. To show that KQ is closed, use the compactness of K to show that the limit of a convergent sequence $k_j q_j$ belongs to KQ.]

12. In problem 11 take $G = \mathrm{SL}(2, \mathbb{R})$,

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Find K and Q.

13. In problem 11, take $G = \mathrm{GL}(n, \mathbb{R})_o$ and $Z = (1, 0, \ldots, 0)$ (diagonal matrix). Find K and Q.

14. In problem 11, take $G = \mathrm{GL}(n, \mathbb{R})_o$ and $Z = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ (diagonal matrix). Find K and Q. [Compare your answer with exercise 9, §2.1. Suggestion: to find the $\mathfrak{g}_\lambda$, review the proof of Lemma 8, §1.2.]

15. Let J be the $n \times n$ Jordan block

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   (a) Show that the centralizer of J in $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$ consists of all matrices of the form

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & a_1 \end{bmatrix}.$$

   (b) Describe the one-parameter group generated by J and its centralizer in $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$.