Lie Groups — Exercise Page #5

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Exercise 1. Show that equivalence of categories is an equivalence relation on the collection of categories.

Solution. **Reflexivity:** Let $\mathcal{C}$ be a category, there's a functor $\mathrm{id}_{\mathcal{C}}$ acting as the identity on all objects and morphisms. Specifically, $\mathrm{id}_{\mathcal{C}}$ is bijective on Hom-sets and is essentially surjective, since for all $x \in \mathrm{Ob}(\mathcal{C})$ we have $X \cong X = \mathrm{id}_{\mathcal{C}}(X)$.

Transitivity: Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be equivalences of categories.

Let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow[F_{X,Y}]{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \xrightarrow[G_{F(X),F(Y)}]{\sim} \mathrm{Hom}_{\mathcal{E}}(GF(X), GF(Y))$$

so $(GF)_{X,Y} = G_{F(X),F(Y)} \circ F_{X,Y}$ is a bijection as a composition of bijections since F, G are equivalences of categories. Hence $(GF)_{X,Y}$ is a bijection so GF is fully-faithful.

Let $Z \in \mathcal{E}$. G is an equivalence of categories and is therefore essentially-surjective. Hence there's $Y \in \mathcal{D}$ such that $G(Y) \cong Z$. F is an equivalence of categories hence there's $X \in \mathcal{C}$ such that $F(X) \cong Y$.

Lemma 1.1. *A functor sends isomorphisms to isomorphisms.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be categories, let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be an isomorphism, and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We have

$$\mathrm{id}_{F(Y)} = F(\mathrm{id}_Y) = F\left(\varphi\varphi^{-1}\right) = F(\varphi) \circ F\left(\varphi^{-1}\right)$$

and similarly $F\left(\varphi^{-1}\right) \circ F(\varphi) = \mathrm{id}_{F(X)}$, so $F\left(\varphi^{-1}\right) = F(\varphi)^{-1}$, so $F(\varphi)$ is invertible and thus an isomorphism in $\mathcal{D}$. ■

We conclude from the lemma that $G(F(X)) \cong G(Y)$ so $(G \circ F)(X) = G(F(X)) \cong Z$, which means $G \circ F$ is essentially surjective.

Symmetry: Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. We construct a functor $G\colon \mathcal{D} \to \mathcal{C}$ by taking $Y \in \mathrm{Ob}(\mathcal{D})$ to any $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \cong Y$ (which we can do thanks to the axiom of choice). Let $f \in \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ and let $X = G(Y), X' = G(Y')$. Since F is fully-faithful we have $\mathrm{Hom}_{\mathcal{C}}(X, X') \cong \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ so there's $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $f = F\left(\tilde{f}\right)$. Define $G(f) = \tilde{f}$.

We have to show G is functorial and an equivalence of categories.

Functorial: Let $Y \in \mathrm{Ob}(\mathcal{D})$ and let $X = G(Y)$. We have $\mathrm{id}_Y = F(\mathrm{id}_X)$ hence by definition $G(\mathrm{id}_Y) = \mathrm{id}_X = \mathrm{id}_{G(Y)}$.

Let $f \in \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y', Y'')$, we want to show $G(g \circ f) = G(g) \circ G(f)$. By definition, $G(g \circ f)$ is the unique morphism such that $F(G(g \circ f)) = g \circ f$. However,

$$F(G(g) \circ G(f)) = F(G(g)) \circ F(G(f)) = g \circ f$$

by the same property and by functoriality of F. Hence

$$F(G(g) \circ G(f)) = F(G(g \circ f)),$$

but since F is faithful this implies $G(g) \circ G(f) = G(g \circ f)$.

Fully-Faithful: For $Y, Y' \in \mathrm{Ob}(\mathcal{D})$ and $X = G(Y), X' = G(Y')$ we have by definition $G_{Y,Y'} = F^{-1}_{X,X'}$, hence $G_{Y,Y'}\colon \mathrm{Hom}_{\mathcal{D}}(Y, Y') \to \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y'))$ is bijective.

Essentially-Surjective: Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $Y = F(X)$ and let $X' = G(Y)$. We want to show $X \cong X'$. We have a bijection $F_{X,X'}\colon \mathrm{Hom}_{\mathcal{C}}(X, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Y, Y)$ hence there's $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $F(f) = \mathrm{id}_Y$. Similarly, swapping roles between X, X' there's $g \in \mathrm{Hom}_{\mathcal{C}}(X', X)$ such that $F(g) = \mathrm{id}_Y$. We get

$$F(f \circ g) = F(f) \circ F(g) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y = F(\mathrm{id}_{X'})$$
$$F(g \circ f) = F(g) \circ F(f) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y = F(\mathrm{id}_X)$$

and since F is faithful this implies $f \circ g = \mathrm{id}_{X'}$ and $g \circ f = \mathrm{id}_X$, which together implies $g = f^{-1}$, so $f\colon X \xrightarrow{\sim} X'$ is an isomorphism, as required.

Exercise 2 (Adjoint Functors). Let $\mathcal{C}, \mathcal{D}$ be categories. A pair (L, R) with $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ is called *adjoint* (where L is called *left-adjoint to* R and R *right-adjoint to* L) if for any $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$ there is a bijection

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \underset{\Phi_{X,Y}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

such that

$$\Phi_{X_1,Y_1}(h \circ L(f)) = \Phi_{X_2,Y_1}(h) \circ f \Phi_{X_2,Y_2}(g \circ h) = R(g) \circ \Phi_{X_2,Y_1}(h)$$

for all

$$\begin{aligned}
&f \in \operatorname{Hom}_{\mathcal{C}}\left(X_1, X_2\right), \\
&g \in \operatorname{Hom}_{\mathcal{D}}\left(Y_1, Y_2\right), \\
&h \in \operatorname{Hom}_{\mathcal{D}}\left(F\left(X_2\right), Y_1\right).
\end{aligned}$$

Show that an equivalence of categories $F\colon \mathcal{C} \to \mathcal{D}$ always has a right-adjoint and a left-adjoint functor.

Solution. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. By the previous exercise there's an equivalence of categories $G\colon \mathcal{D} \to \mathcal{C}$, which we show is a left-adjoint and a right-adjoint to F.

Right-Adjoint: We have to construct a bijection

$$\Phi_{X,Y}\colon \operatorname{Hom}_{\mathcal{D}}\left(F\left(X\right), Y\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}\left(X, G\left(Y\right)\right).$$

By definition of G we have $F \circ G = \mathrm{id}_{\mathcal{D}}$, so Define

$$\Phi_{X,Y} = G_{F(X),Y}$$

which is a bijection because G is an equivalence of categories.

For

$$\begin{aligned}
&f \in \operatorname{Hom}_{\mathcal{C}}\left(X_1, X_2\right), \\
&g \in \operatorname{Hom}_{\mathcal{D}}\left(Y_1, Y_2\right), \\
&h \in \operatorname{Hom}_{\mathcal{D}}\left(F\left(X_2\right), Y_1\right)
\end{aligned}$$

we get

$$\begin{aligned}
\Phi_{X_1,Y_1}\left(h \circ F\left(f\right)\right) = G\left(h \circ F\left(f\right)\right) = G\left(h\right) \circ GF\left(f\right) = G\left(h\right) \circ f = \Phi_{X_2,Y_1}\left(h\right) \circ f \\
\Phi_{X_2,Y_2}\left(g \circ h\right) = G\left(g \circ h\right) = G\left(g\right) \circ G\left(h\right) = G\left(g\right) \circ \Phi_{X_2,Y_1}\left(h\right),
\end{aligned}$$

hence $\Phi_{X,Y}$ satisfies the required properties, so G is right-adjoint to F.

Left-Adjoint: Define $\Phi_{X,Y} = F_{G(X),Y}$ which is a bijection since F is an equivalence of categories.

Let

$$\begin{aligned}
&f \in \operatorname{Hom}_{\mathcal{D}}\left(X_1, X_2\right), \\
&g \in \operatorname{Hom}_{\mathcal{C}}\left(Y_1, Y_2\right), \\
&h \in \operatorname{Hom}_{\mathcal{C}}\left(F\left(X_2\right), Y_1\right).
\end{aligned}$$

We have

$$\begin{aligned}
\Phi_{X_1,Y_1}\left(h \circ G\left(f\right)\right) = F\left(h \circ G\left(f\right)\right) = F\left(h\right) \circ FG\left(f\right) = F\left(h\right) \circ f = \Phi_{X_2,Y_1}\left(h\right) \circ f \\
\Phi_{X_2,Y_2}\left(g \circ h\right) = F\left(g \circ h\right) = F\left(g\right) \circ F\left(h\right) = F\left(g\right) \circ \Phi_{X_2,Y_1}\left(h\right),
\end{aligned}$$

so Φ satisfies the required properties, so G is a left-adjoint to F.

Exercise 3. Show that the pair of functors $\left(\tilde{\Gamma}, \mathrm{Lie}\right)$ is an adjoint pair between the categories of Lie groups and of Lie algebras. You may assume facts that were proven in the case of matrix groups.

Solution. We use in the proof some tools from category theory which we illustrate below.

Definition 1.2 (Natural Transformation). Let $F, G \colon \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\alpha \colon F \to G$ is the data $\alpha_X \colon F(X) \to G(X)$ for all $X \in \mathrm{Ob}(\mathcal{C})$ and under the condition that the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

commutes for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Definition 1.3 (Natural Isomorphism). A natural transformation α is a *a natural isomorphism* if α_X is a bijection for all $X \in \mathrm{Ob}(\mathcal{C})$.

Proof. α, β are natural isomorphisms, so α_X, β_X are bijections for all $X \in \mathrm{Ob}(\mathcal{C})$, hence so are $\beta_X \circ \alpha_X$. $\beta \circ \alpha$ is a natural transformation by (**??**), hence this implies $\beta \circ \alpha$ is a natural isomorphism. ∎

Definition 1.4 (Product Category). Let $\mathcal{C}, \mathcal{D}$ be categories. We define $\mathcal{C} \times \mathcal{D}$ with $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) = \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and with

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}\left((X, Y), (X', Y')\right) = \mathrm{Hom}_{\mathcal{C}}(X, X') \times \mathrm{Hom}_{\mathcal{D}}(Y, Y')$$

and composition $(f_2, g_2) \circ (f_1, g_1) = (f_2 \circ f_1, g_2 \circ g_1)$.

Definition 1.5 (Hom-Functor). Let $\mathcal{C}$ be a category. We define

$$\mathrm{Hom}_{\mathcal{C}} \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

where **Set** is the category of sets. On objects, let $\mathrm{Hom}_{\mathcal{C}}$ be as defined in class. On morphisms, for $f^{\mathrm{op}} \colon X \to Y$ in $\mathcal{C}^{\mathrm{op}}$ and $g \colon X' \to Y'$ in $\mathcal{C}$, define

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(f^{\mathrm{op}}, g) : \ \mathrm{Hom}(X, X') &\to \mathrm{Hom}(Y, Y') \\ h &\mapsto g \circ h \circ f, \end{aligned}$$

where f^{op} is f when viewed as a morphism in the opposite category (so that $f^{\mathrm{op}} \colon X \to Y$ means $f \colon Y \to X$).

Lemma 1.6. $\mathrm{Hom}_{\mathcal{C}}$ *is a functor.*

Proof. Let $(f_1^{\mathrm{op}}, g_1): (X, X') \to (Y, Y')$ and $(f_2^{\mathrm{op}}, g_2): (Y, Y') \to (Z, Z')$ be morphisms in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Let $h \in \mathrm{Hom}(X, X')$, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}((f_2^{\mathrm{op}}, g_2) \circ (f_1^{\mathrm{op}}, g_1))(h) &= \mathrm{Hom}_{\mathcal{C}}(f_2^{\mathrm{op}} \circ f_1^{\mathrm{op}}, g_2 \circ g_1)(h) \\
&= \mathrm{Hom}_{\mathcal{C}}((f_1 \circ f_2)^{\mathrm{op}}, g_2 \circ g_1)(h) \\
&= g_2 \circ g_1 \circ h \circ f_1 \circ f_2 \\
&= g_2 \circ (g_1 \circ h \circ f_1) \circ f_2 \\
&= \mathrm{Hom}_{\mathcal{C}}(f_2^{\mathrm{op}}, g_2)(g_1 \circ h \circ f_1) \\
&= \mathrm{Hom}_{\mathcal{C}}(f_2^{\mathrm{op}}, g_2)(\mathrm{Hom}_{\mathcal{C}}(f_1^{\mathrm{op}}, g_1)(h)) \\
&= (\mathrm{Hom}_{\mathcal{C}}(f_2^{\mathrm{op}}, g_2) \circ \mathrm{Hom}_{\mathcal{C}}(f_1^{\mathrm{op}}, g_1))(h). \quad \blacksquare
\end{aligned}$$

Lemma 1.7. *Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{C}$ be functors. Then so are* $\mathrm{Hom}_{\mathcal{D}}(F(-), (-))$ *and* $\mathrm{Hom}_{\mathcal{C}}(-, G(-))$.

Proof.
- Let $f_1^{\mathrm{op}}, f_2^{\mathrm{op}}$ be morphisms in $\mathcal{D}$ such that $f_2^{\mathrm{op}} \circ f_1^{\mathrm{op}}$ is defined and let g_1, g_2 be morphisms in $\mathcal{C}$ such that $g_2 \circ g_1$ is defined.

  We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(F(f_1^{\mathrm{op}} \circ f_2^{\mathrm{op}}), g_2 \circ g_1) &= \mathrm{Hom}_{\mathcal{D}}(F(f_2 \circ f_1)^{\mathrm{op}}, g_2 \circ g_1) \\
&= \mathrm{Hom}_{\mathcal{D}}(F(f_2 \circ f_1)^{\mathrm{op}}, g_2 \circ g_1) \\
&= \mathrm{Hom}_{\mathcal{D}}((F(f_2) \circ F(f_1))^{\mathrm{op}}, g_2 \circ g_1)(h) \\
&= \mathrm{Hom}_{\mathcal{D}}(F(f_2)^{\mathrm{op}}, g_2) \circ \mathrm{Hom}_{\mathcal{D}}(F(f_1)^{\mathrm{op}}, g_2) \\
&= \mathrm{Hom}_{\mathcal{D}}(F(f_2^{\mathrm{op}}), g_2) \circ \mathrm{Hom}_{\mathcal{D}}(F(f_1^{\mathrm{op}}), g_2)
\end{aligned}$$

  where we write F also for the induced map $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$, and where the second-to-last equality is due to (1.6).

- Let $f_1^{\mathrm{op}}, f_2^{\mathrm{op}}$ be morphisms in $\mathcal{C}$ such that $f_2^{\mathrm{op}} \circ f_1^{\mathrm{op}}$ is defined and let g_1, g_2 be morphisms in $\mathcal{D}$ such that $g_2 \circ g_1$ is defined.

  We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(f_1^{\mathrm{op}} \circ f_2^{\mathrm{op}}, G(g_2 \circ g_1)) &= \mathrm{Hom}_{\mathcal{C}}(f_2 \circ f_1{}^{\mathrm{op}}, G(g_2 \circ g_1)) \\
&= \mathrm{Hom}_{\mathcal{C}}(f_2 \circ f_1{}^{\mathrm{op}}, G(g_2 \circ g_1)) \\
&= \mathrm{Hom}_{\mathcal{C}}(((f_2) \circ f_1)^{\mathrm{op}}, G(g_2 \circ g_1))(h) \\
&= \mathrm{Hom}_{\mathcal{C}}(f_2{}^{\mathrm{op}}, G(g_2)) \circ \mathrm{Hom}_{\mathcal{C}}(f_1{}^{\mathrm{op}}, G(g_2))
\end{aligned}$$

  where the last equality is due to (1.6). $\blacksquare$

Lemma 1.8. *Let $L: \mathcal{C} \to \mathcal{D}$ and $R: \mathcal{D} \to \mathcal{C}$ be functors. Then L is left-adjoint to R if and only if there's a natural isomorphism*

$$\alpha: \mathrm{Hom}_{\mathcal{D}}(L(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(-, R(-)).$$

Proof.

- Assume that L is left-adjoint to R, and let

$$\Phi_{X,Y} \colon \operatorname{Hom}_{\mathcal{D}}\left(L\left(X\right), Y\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}\left(X, R\left(Y\right)\right)$$

be bijections satisfying the conditions in the exercise. Let $\alpha \colon \operatorname{Hom}_{\mathcal{D}}\left(L\left(-\right), -\right) \to \operatorname{Hom}_{\mathcal{C}}\left(-, R\left(-\right)\right)$, given by $\alpha_{X,Y} \coloneqq \Phi_{X,Y}$ as above, which we show is a natural transformation.

To show α is natural we have to show that

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_1\right), Y_1\right) & \xrightarrow{\Phi_{X_1,Y_1}} & \operatorname{Hom}_{\mathcal{C}}\left(X_1, R\left(Y_1\right)\right) \\ {\scriptstyle \operatorname{Hom}_{\mathcal{D}}(L(f^{\mathrm{op}}),g)}\Big\downarrow & & \Big\downarrow{\scriptstyle \operatorname{Hom}_{\mathcal{C}}(f^{\mathrm{op}},R(g))} \\ \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_2\right), Y_2\right) & \xrightarrow{\Phi_{X_2,Y_2}} & \operatorname{Hom}_{\mathcal{C}}\left(X_2, R\left(Y_2\right)\right) \end{array}$$

commutes for any $X, X_2 \in \mathcal{C}$, $Y, Y_2 \in \mathcal{D}$ and $\left(f^{\mathrm{op}}, g\right) \colon \left(X_1, Y_1\right) \to \left(X_2, Y_2\right)$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$. Opening the definition of $\operatorname{Hom}_{\mathcal{D}}, \operatorname{Hom}_{\mathcal{C}}$ we need to show commutativity of the following.

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_1\right), Y_1\right) & \xrightarrow{\Phi_{X_1,Y_1}} & \operatorname{Hom}_{\mathcal{C}}\left(X_1, R\left(Y_1\right)\right) \\ {\scriptstyle g\circ(-)\circ L(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle R(g)\circ(-)\circ f} \\ \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_2\right), Y_2\right) & \xrightarrow{\Phi_{X_2,Y_2}} & \operatorname{Hom}_{\mathcal{C}}\left(X_2, R\left(Y_2\right)\right) \end{array}$$

We can decompose each vertical map so that this is equivalent to commutativity of the large square in the following.

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_1\right), Y_1\right) & \xrightarrow{\Phi_{X_1,Y_1}} & \operatorname{Hom}_{\mathcal{C}}\left(X_1, R\left(Y_1\right)\right) \\ {\scriptstyle g\circ(-)}\Big\downarrow & & \Big\downarrow{\scriptstyle R(g)\circ(-)} \\ \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_1\right), Y_2\right) & \xrightarrow{\Phi_{X_1,Y_2}} & \operatorname{Hom}_{\mathcal{C}}\left(X_1, R\left(Y_2\right)\right) \\ {\scriptstyle (-)\circ L(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle (-)\circ f} \\ \operatorname{Hom}_{\mathcal{D}}\left(L\left(X_2\right), Y_2\right) & \xrightarrow{\Phi_{X_2,Y_2}} & \operatorname{Hom}_{\mathcal{C}}\left(X_2, R\left(Y_2\right)\right) \end{array} \tag{1}$$

The smaller squares are both commutative, the bottom one by the first condition on $\Phi_{X,Y}$ and the top by the second condition. Hence the bigger square is commutative, so α is a natural transformation, hence thus a natural isomorphism.

- Assume There's a natural isomorphism

$$\alpha \colon \operatorname{Hom}_{\mathcal{D}}\left(L\left(-\right), Y\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}\left(-, R\left(-\right)\right)$$

and let $\Phi_{X,Y} = \alpha_{X,Y}$. Since α is natural, we have by the above equivalence that the large square in (1) commutes for all f, g. Taking $Y = Y_1 = Y_2$ and $g = \mathrm{id}_Y$ the large square is exactly the bottom one which therefore commutes. Taking $X = X_1 = X_2$ and $f = \mathrm{id}_X$, the large square is the same as the top one which therefore commutes. Commutativity of these squares is equivalent to the equation conditions on $\Phi_{X,Y}$, and $\Phi_{X,Y} = \alpha_{X,Y}$ is a bijection for all X, Y since α is an isomorphism.

Hence $\Phi_{X,Y}$ satisfies all the conditions, so L is left-adjoint to R. ■

Lemma 1.9. *Let L_i be left adjoints to R_i in the following.*

$$\mathcal{C} \underset{R_1}{\overset{L_1}{\leftrightarrows}} \mathcal{D} \underset{R_2}{\overset{L_2}{\leftrightarrows}} \mathcal{E}$$

Then $L_1 \circ L_2$ is left-adjoint to $R_2 \circ R_1$.

Proof.

Denote **LieGrp**, **LieAlg** the respective categories of Lie groups and algebras, with Lie group homomorphisms and Lie algebra homomorphisms. Denote by $\mathbf{LieGrp}_{\mathrm{sc}}$ the subcategory of simply-connected Lie groups within **LieGrp**.

The essential image of $\tilde{\Gamma}$ is contained in $\mathbf{LieGrp}_{\mathrm{sc}}$, hence it factors as $\iota \circ \hat{\Gamma}$ where $\iota\colon \mathbf{LieGrp}_{\mathrm{sc}} \hookrightarrow \mathbf{LieGrp}$ is the embedding, and we've seen that $\hat{\Gamma}$ is an equivalence of categories. Let $\mathrm{Lie}_0\colon \mathbf{LieGrp}_{\mathrm{sc}} \to \mathbf{LieAlg}$ be the restriction of Lie to $\mathbf{LieGrp}_{\mathrm{sc}}$.

Let $U\colon \mathbf{LieGrp} \to \mathbf{LieGrp}_{\mathrm{sc}}$ take a Lie group to its universal cover. For a group homomorphism $\varphi\colon G \to H$ let $U(\varphi)$ be obtained as follows.

Remark 1.10. We know $\hat{\Gamma}, \mathrm{Lie}_0$ form an equivalence between **LieAlg** and $\mathbf{LieGrp}_{\mathrm{sc}}$, hence by the solution to exercise 2 $\hat{\Gamma}$ is left-adjoint to Lie_0.

Lemma 1.11. *ι is left-adjoint to U.*

Proof.

We clearly have $\tilde{\Gamma} = \iota \circ \hat{\Gamma}$ and we have $\mathrm{Lie} = \mathrm{Lie}_0 \circ U$ since $G, \tilde{G}$ have the same Lie algebra (the covering map $p\colon \tilde{G} \to G$ induces an isomorphism $\mathrm{d}p\colon \mathrm{Lie}\left(\tilde{G}\right) \xrightarrow{\sim} \mathrm{Lie}(G)$). By (1.9), (1.10) and (1.11) we get that $\tilde{\Gamma}$ is left-adjoint to Lie, as required.