1) Let us write, for integers $p, q \geq 0$,

$$I_{p,q} = \begin{pmatrix} I_{p\times p} & 0 \\ 0 & -I_{q,q} \end{pmatrix} \in M_{p+q}.$$

Consider the groups

$$O(p,q) = \{ A \in GL_{p+q}(\mathbb{R}) : A I_{p,q} A^t = I_{p,q} \}.$$

Now, let V be a real vector space and $B : V \times V \to \mathbb{R}$ a symmetric bi-linear form on it.

Set

$$O(B) = \{ T \in GL(V) : B(Tx, Ty) = B(x,y), \forall x,y \in V \}$$

i) Assume that B is non-degenerate, that is, for every $0 \neq x \in V$, there is $y \in V$, s.t. $B(x,y) \neq 0$.

Prove that as a group $O(B) \simeq O(p,q)$, for some $p+q = n$.

(Recall Sylvester's intertia theorem.)

bonus: Try to describe $O(B)$ without the non-degeneracy assumption.

ii) Identify $O(B)$ with matrices and compute $Lie(O(B))$.

iii) Prove that $O(p,q) \simeq O(q,p)$.

and writing out the first few terms of the inner exp, or by using (a) twice:

$$\exp(X)\exp(Y)\exp(-X) = \exp(X + Y + \tfrac{1}{2}[X,Y] + \cdots)\exp(-X) \text{ [once]}$$
$$= \exp(Y + [X,Y] + \cdots)\text{[twice]}$$

(c) Same method. QED

Work on marked ex's.

Problems for §1.3

1. Prove part (c) of Proposition 2.

2. Use Dynkin's formula (4) to show that

$$C(X,Y) = X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}[X,[X,Y]] + \tfrac{1}{12}[Y,[Y,X]] + \cdots .$$

Check that this agrees with what one obtains by writing out the terms up to order three of the series (4).

3. Prove that the series C(X,Y) can also be written in the following form:

$$C(X,Y) = \sum \frac{(-1)^k}{k+1}\frac{1}{i_1 + \cdots + i_k + 1}\frac{[X^{(i_1)}Y^{(j_1)}\cdots X^{(i_k)}Y^{(j_k)}X]}{i_1!j_1!\cdots i_k!j_k!}.$$

[Suggestion: Start with $Z = Z(\tau)$ defined by $\exp(Z) = \exp(\tau X)\exp(Y)$ instead of (6); imitate the proof. Comment: this formula might seem slightly simpler than (4), but is equally unmanageable and less symmetric. If one reverses the roles of X and Y in this procedure one obtains a formula reflecting the relation $C(-Y,-X) = -C(X,Y)$, which is evident form the definition of $C(X,Y)$].

4. Write $\exp(Z) = \exp(\tau X)\exp(\tau Y)$ as in (6). Let

$$Z = \sum_k \tau^k C_k,$$

be the expansion of Z in powers of τ. Derive the recursion formula

$$(k+1)C_{k+1} = -[C_k, X] + \sum \gamma_j [C_{k_1} \cdots [C_{k_j}, X + Y] \cdots],$$

where the γ_j are defined as the coefficients of the series

$$\frac{x}{1 - e^{-x}} = \sum_j \gamma_j x^j.$$

(Compare with the *Bernoulli numbers* β_j defined by

$$\frac{x}{e^x - 1} = \sum_j \beta_j \frac{x^j}{j!},$$

i.e. $\gamma_j = (-1)^j \beta_j / j!$).

[Suggestion: Show first that

$$\frac{dZ}{d\tau} = -\operatorname{ad}(Z)X + \frac{\operatorname{ad} Z}{1 - \exp(-\operatorname{ad} Z)}(X + Y).$$

Then substitute power series.]

A *linear Lie algebra* is a space $\mathfrak{n} \subset M$ of linear transformations that is closed under the bracket operation:

$$X, Y \in \mathfrak{n} \quad \text{implies} \quad [X,Y] \in \mathfrak{n}.$$

5. Let $\mathfrak{n}$ be a linear Lie algebra consisting of *nilpotent* matrices. Let $N = \{\exp \mathfrak{n} = \exp X | X \in \mathfrak{n}\}$. Show that N is a *group* under matrix multiplication, i.e.

$$a \in N \quad \text{implies} \quad a^{-1} \in N,$$
$$a, b \in N \quad \text{implies} \quad ab \in N.$$

[Suggestion: for $a = \exp X$ and $b = \exp Y$, consider $Z(\tau)$ defined by $\exp(\tau X)\exp(\tau Y) = \exp Z(\tau)$ as a power series in τ with matrix coefficients. All nilpotent $n \times n$ matrices A satisfy $A^n = 0$.]

6. Let $\mathfrak{n}_1$ and $\mathfrak{n}_2$ be two linear Lie algebras consisting of nilpotent matrices as in problem 5, N_1 and N_2 be the corresponding groups. Let $\varphi : \mathfrak{n}_1 \to \mathfrak{n}_2$ be a linear map. Show that the rule $f(\exp X) = \exp \varphi(X)$ defines a *group homomorphism* $f : N_1 \to N_2$ (i.e. a well-defined map satisfying $f(ab) = f(a)f(b)$ for all $a, b \in N_1$) if and only if $\varphi([X,Y]) = [\varphi X, \varphi Y]$ for all $X, Y \in \mathfrak{n}_1$.

Problems 7 and 8 are meant to illustrate problems 5 and 6. Assume known the results of those problems.

7. (a) Describe all subspaces $\mathfrak{n}$ consisting of nilpotent upper triangular matrices real 3×3 matrices

$$\begin{bmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{bmatrix}$$

which satisfy $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$. Describe the corresponding groups N.

(b) Give an example of a subspace $\mathfrak{n}$ of M with $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{n}$ for which $N = \exp \mathfrak{n}$ is *not* a group. [Suggestion: Consider Example 9 of §1.2.]

8. Let $\mathfrak{n}$ be the space of all nilpotent upper triangular real $n \times n$ matrices

$$\begin{bmatrix} 0 & * & * \cdots & * \\ 0 & 0 & * \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & 0 \end{bmatrix}. \qquad (1)$$

infinite groups, such as the groups $\mathrm{GL}(n,\mathbb{Z})$ or $\mathrm{GL}(n,\mathbb{Q})$ of integral or rational matrices with inverses of the same kind.

Even though we shall be exclusively concerned with linear groups, it is sometimes appropriate to think in terms of *abstract groups*, when it is irrelevant that the elements of the groups in question are matrices. Such is the case, for example, when one defines subgroups or homomorphisms (although for linear groups homomorphisms will be required to be differentiable in a sense to be explained later). We assume known the rudiments of abstract group theory, such as can be found in the first few sections of any introduction to that subject (for example in Herstein (1964)). It should be remarked at this point that while subgroups and direct products of linear groups are again linear groups, such is not the case (in a general or natural way) for quotient groups. The *direct product* $G \times H$ of two linear groups G and H is in this context realized as the group of block-diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad a \in G,\ b \in H.$$

Problems for §2.1

1. Prove Lemma 1B.
2. Check (7)
3. Check (8).
4. Check (10).
5. (a) As in Example 2, identify complex numbers with quaternions of the form $\lambda + i\mu$. Show that, the map

$$\alpha + j\beta \to \begin{bmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{bmatrix}$$

sets up a one-to-one correspondence between $\mathbb{H}$ and complex 2×2 matrices of the form indicated which turns multiplication of quaternions into matrix multiplication. Verify that the conjugate of a quaternion corresponds to the Hermitian adjoint of the matrix. Deduce that the group of norm-one quaternions, $\mathrm{Sp}(1) = \{\alpha \in \mathbb{H} \mid |\alpha| = 1\}$, gets mapped isomorphically onto $\mathrm{SU}(2) = \{a \in M_2(\mathbb{C}) \mid aa^* = 1, \det a = 1\}$.

(b) Show that any $\gamma \in \mathbb{H}$ satisfying $\bar\gamma = -\gamma$ can be written in the form $\gamma = \bar\alpha j \alpha$ for some $\alpha \in \mathbb{H}$.

6. Let $H(3,\mathbb{R})$ be the group of a real 3×3 matrices of the form

$$\begin{bmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix}$$

(called the three-dimensional *Heisenberg group*).

(a) Describe the set $\mathsf{h}(3,\mathbb{R})$ of all matrices X for which $\exp(\tau X) \in H(3,\mathbb{R})$ for all $\tau \in \mathbb{R}$. Verify that $h(3,\mathbb{R})$ is a three-dimensional vector space satisfying $[X,Y] \in h(3,\mathbb{R})$ for all $X,Y \in h(3,\mathbb{R})$.

(b) Prove that $\exp\colon h(3,\mathbb{R}) \to H(3,\mathbb{R})$ is bijective.

(c) Describe the subsets of $h(3,\mathbb{R})$ which correspond to the conjugacy classes in $H(3,\mathbb{R})$ under exp. Sketch.

7. Define the group of Euclidean motions in space, $\mathbb{R}^3 \rtimes \mathrm{SO}(3)$ in analogy with Example 5.

(a) Define an exponential map $\exp\colon \mathbb{R}^3 \times \mathsf{so}(3) \to \mathbb{R}^3 \rtimes \mathrm{SO}(3)$ and show that it is surjective.

(b) Describe the subsets of $\mathbb{R}^3 \times \mathsf{so}(3)$ which get mapped onto the conjugacy classes in $\mathbb{R}^3 \rtimes \mathrm{SO}(3)$ in analogy with Example 5.

8. Let $\mathrm{SO}(n) = \{a \in M_n(\mathbb{R}) \mid aa^* = 1 \text{ and } \det a = +1\}$. Define an exponential map $\exp\colon \mathsf{so}(n) \to \mathrm{SO}(n)$ and show that it is surjective. [Suggestion: For the second part, show first that every element of $\mathrm{SO}(n)$ is conjugate to a block-diagonal matrix with 2×2 blocks of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

together with a single 1×1 block $[1]$ when n is odd.]

9. (a) Show that $\mathrm{GL}(n,\mathbb{R}) = \mathrm{O}(n)B$, where B is the group upper triangular matrices with strictly positive diagonal entries. [Suggestion: write the Gram–Schmidt process

$$\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{(v_2,w_1)}{(w_1,w_1)} w_1 \\
w_3 &= v_3 - \frac{(v_3,w_1)}{(w_1,w_1)} w_1 - \frac{(v_3,w_2)}{(w_2,w_2)} w_2, \\
&\cdots \\
u_1 &= \frac{w_1}{\|w_1\|}, \quad u_2 = \frac{w_2}{\|w_2\|}, \quad u_3 = \frac{w_3}{\|w_3\|}, \cdots
\end{aligned}$$

as a matrix equation

$$[v_1, v_2, v_3, \dots] = [u_1, u_2, u_3, \dots] b$$

with b upper triangular.]

(b) Show that $\mathrm{GL}(n,\mathbb{R})_+ = \{a \in \mathrm{GL}(n,\mathbb{R}) | \det(a) > 0\}$ and $\mathrm{SL}(n,\mathbb{R})$ are *connected* in the sense that any two of its elements can be joined by a continuous (even analytic) path.

2) Show that $\exp : \mathfrak{sl}_2(\mathbb{R}) \to SL_2(\mathbb{R})$ is not surjective.

(Consider the eigenvalues of $\exp(X)$ and of $\exp(\frac{1}{2}X)$.)