Lie Groups — Exercise Page #5

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Exercise 1. Show that equivalence of categories is an equivalence relation on the collection of categories.

Solution. **Reflexivity:** Let $\mathcal{C}$ be a category, there's a functor $\mathrm{id}_{\mathcal{C}}$ acting as the identity on all objects and morphisms. Specifically, $\mathrm{id}_{\mathcal{C}}$ is bijective on Hom-sets and is essentially surjective, since for all $x \in \mathrm{Ob}(\mathcal{C})$ we have $X \cong X = \mathrm{id}_{\mathcal{C}}(X)$.

Transitivity: Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{E}$ be equivalences of categories.

Let $X, Y \in \mathrm{Ob}(\mathcal{C})$. We have

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \underset{F_{X,Y}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)) \underset{G_{F(X),F(Y)}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{E}}(GF(X), GF(Y))$$

so $(GF)_{X,Y} = G_{F(X),F(Y)} \circ F_{X,Y}$ is a bijection as a composition of bijections since F, G are equivalences of categories. Hence $(GF)_{X,Y}$ is a bijection so GF is fully-faithful.

Let $Z \in \mathcal{E}$. G is an equivalence of categories and is therefore essentially-surjective. Hence there's $Y \in \mathcal{D}$ such that $G(Y) \cong Z$. F is an equivalence of categories hence there's $X \in \mathcal{C}$ such that $F(X) \cong Y$.

Lemma 1.1. *A functor sends isomorphisms to isomorphisms.*

Proof. Let $\mathcal{C}, \mathcal{D}$ be categories, let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ be an isomorphism, and let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We have

$$\mathrm{id}_{F(Y)} = F(\mathrm{id}_Y) = F\left(\varphi\varphi^{-1}\right) = F(\varphi) \circ F\left(\varphi^{-1}\right)$$

and similarly $F\left(\varphi^{-1}\right) \circ F(\varphi) = \mathrm{id}_{F(X)}$, so $F\left(\varphi^{-1}\right) = F(\varphi)^{-1}$, so $F(\varphi)$ is invertible and thus an isomorphism in $\mathcal{D}$. ■

We conclude from the lemma that $G(F(X)) \cong G(Y)$ so $(G \circ F)(X) = G(F(X)) \cong Z$, which means $G \circ F$ is essentially surjective.

Symmetry: Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. We construct a functor $G\colon \mathcal{D} \to \mathcal{C}$ by taking $Y \in \mathrm{Ob}(\mathcal{D})$ to any $X \in \mathrm{Ob}(\mathcal{C})$ such that $F(X) \cong Y$ (which we can do thanks to the axiom of choice). Let $f \in \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ and let $X = G(Y), X' = G(Y')$. Since F is fully-faithful we have $\mathrm{Hom}_{\mathcal{C}}(X, X') \cong \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ so there's $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $f = F\left(\tilde{f}\right)$. Define $G(f) = \tilde{f}$.

We have to show G is functorial and an equivalence of categories.

Functorial: Let $Y \in \mathrm{Ob}(\mathcal{D})$ and let $X = G(Y)$. We have $\mathrm{id}_Y = F(\mathrm{id}_X)$ hence by definition $G(\mathrm{id}_Y) = \mathrm{id}_X = \mathrm{id}_{G(Y)}$.

Let $f \in \mathrm{Hom}_{\mathcal{D}}(Y, Y')$ and $g \in \mathrm{Hom}_{\mathcal{D}}(Y', Y'')$, we want to show $G(g \circ f) = G(g) \circ G(f)$. By definition, $G(g \circ f)$ is the unique morphism such that $F(G(g \circ f)) = g \circ f$. However,

$$F(G(g) \circ G(f)) = F(G(g)) \circ F(G(f)) = g \circ f$$

by the same property and by functoriality of F. Hence

$$F(G(g) \circ G(f)) = F(G(g \circ f)),$$

but since F is faithful this implies $G(g) \circ G(f) = G(g \circ f)$.

Fully-Faithful: For $Y, Y' \in \mathrm{Ob}(\mathcal{D})$ and $X = G(Y), X' = G(Y')$ we have by definition $G_{Y,Y'} = F^{-1}_{X,X'}$, hence $G_{Y,Y'}\colon \mathrm{Hom}_{\mathcal{D}}(Y, Y') \to \mathrm{Hom}_{\mathcal{C}}(G(Y), G(Y'))$ is bijective.

Essentially-Surjective: Let $X \in \mathrm{Ob}(\mathcal{C})$. Let $Y = F(X)$ and let $X' = G(Y)$. We want to show $X \cong X'$. We have a bijection $F_{X,X'}\colon \mathrm{Hom}_{\mathcal{C}}(X, X') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Y, Y)$ hence there's $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that $F(f) = \mathrm{id}_Y$. Similarly, swapping roles between X, X' there's $g \in \mathrm{Hom}_{\mathcal{C}}(X', X)$ such that $F(g) = \mathrm{id}_Y$. We get

$$\begin{aligned} F(f \circ g) &= F(f) \circ F(g) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y = F(\mathrm{id}_{X'}) \\ F(g \circ f) &= F(g) \circ F(f) = \mathrm{id}_Y \circ \mathrm{id}_Y = \mathrm{id}_Y = F(\mathrm{id}_X) \end{aligned}$$

and since F is faithful this implies $f \circ g = \mathrm{id}_{X'}$ and $g \circ f = \mathrm{id}_X$, which together implies $g = f^{-1}$, so $f\colon X \xrightarrow{\sim} X'$ is an isomorphism, as required.

Exercise 2 (Adjoint Functors). Let $\mathcal{C}, \mathcal{D}$ be categories. A pair (L, R) with $L\colon \mathcal{C} \to \mathcal{D}$ and $R\colon \mathcal{D} \to \mathcal{C}$ is called *adjoint* (where L is called *left-adjoint to* R and R *right-adjoint to* L) if for any $X \in \mathrm{Ob}(\mathcal{C}), Y \in \mathrm{Ob}(\mathcal{D})$ there is a bijection

$$\mathrm{Hom}_{\mathcal{D}}(L(X), Y) \underset{\Phi_{X,Y}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{C}}(X, R(Y))$$

such that

$$\Phi_{X_1,Y_1}(h \circ L(f)) = \Phi_{X_2,Y_1}(h) \circ f \Phi_{X_2,Y_2}(g \circ h) = R(g) \circ \Phi_{X_2,Y_1}(h)$$

for all

$$f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2),$$
$$g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2),$$
$$h \in \mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_1).$$

Show that an equivalence of categories $F\colon \mathcal{C} \to \mathcal{D}$ always has a right-adjoint and a left-adjoint functor.

Solution. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. By the previous exercise there's an equivalence of categories $G\colon \mathcal{D} \to \mathcal{C}$, which we show is a left-adjoint and a right-adjoint to F.

Right-Adjoint: We have to construct a bijection

$$\Phi_{X,Y}\colon \mathrm{Hom}_{\mathcal{D}}(F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

By definition of G we have $F \circ G = \mathrm{id}_{\mathcal{D}}$, so Define

$$\Phi_{X,Y} = G_{F(X),Y}$$

which is a bijection because G is an equivalence of categories.

For

$$f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2),$$
$$g \in \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2),$$
$$h \in \mathrm{Hom}_{\mathcal{D}}(F(X_2), Y_1)$$

we get

$$\Phi_{X_1,Y_1}(h \circ F(f)) = G(h \circ F(f)) = G(h) \circ GF(f) = G(h) \circ f = \Phi_{X_2,Y_1}(h) \circ f$$
$$\Phi_{X_2,Y_2}(g \circ h) = G(g \circ h) = G(g) \circ G(h) = G(g) \circ \Phi_{X_2,Y_1}(h),$$

hence $\Phi_{X,Y}$ satisfies the required properties, so G is right-adjoint to F.

Left-Adjoint: Define $\Phi_{X,Y} = F_{G(X),Y}$ which is a bijection since F is an equivalence of categories.

Let

$$f \in \mathrm{Hom}_{\mathcal{D}}(X_1, X_2),$$
$$g \in \mathrm{Hom}_{\mathcal{C}}(Y_1, Y_2),$$
$$h \in \mathrm{Hom}_{\mathcal{C}}(F(X_2), Y_1).$$

We have

$$\Phi_{X_1,Y_1}(h \circ G(f)) = F(h \circ G(f)) = F(h) \circ FG(f) = F(h) \circ f = \Phi_{X_2,Y_1}(h) \circ f$$
$$\Phi_{X_2,Y_2}(g \circ h) = F(g \circ h) = F(g) \circ F(h) = F(g) \circ \Phi_{X_2,Y_1}(h),$$

so Φ satisfies the required properties, so G is a left-adjoint to F.

Exercise 3. Denote **LieGrp**, **LieAlg** the respective categories of Lie groups and algebras, with Lie group homomorphisms and Lie algebra homomorphisms.

We find a natural isomorphism

$$\Phi\colon \operatorname{Hom}_{\mathbf{LieGrp}}\left(\tilde{\Gamma}\left(-\right), -\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{LieAlg}}\left(-, \operatorname{Lie}\left(-\right)\right).$$

I.e. for any $\mathfrak{g} \in \mathbf{LieAlg}$ and $H \in \mathbf{LieGrp}$ we construct a bijection

$$\Phi_{\mathfrak{g},H}\colon \operatorname{Hom}_{\mathbf{LieGrp}}\left(\tilde{\Gamma}\left(\mathfrak{g}\right), H\right) \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{LieAlg}}\left(\mathfrak{g}, \operatorname{Lie}\left(H\right)\right)$$

such that the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathbf{LieGrp}}\left(\tilde{\Gamma}\left(\mathfrak{g}_2\right), G_1\right) \arrow[r, "\Phi_{\mathfrak{g}_2,G_1}"] \arrow[d, "g\circ(-)\circ\tilde{\Gamma}(f)"'] & \operatorname{Hom}\left(\mathfrak{g}_2, \operatorname{Lie}\left(G_1\right)\right) \arrow[d, "\operatorname{Lie}(g)\circ(-)\circ f"] \\
\operatorname{Hom}_{\mathbf{LieGrp}}\left(\tilde{\Gamma}\left(\mathfrak{g}_1\right), G_2\right) \arrow[r, "\Phi_{\mathfrak{g}_1,G_2}"] & \operatorname{Hom}\left(\mathfrak{g}_1, \operatorname{Lie}\left(G_2\right)\right)
\end{tikzcd}$$

commutes for any $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ in **LieAlg** and any $g\colon G_1 \to G_2$ in **LieGrp**. The latter description matches our definition of an adjunction since one can take $f = \mathrm{id}_{X_2}$ or $g = \mathrm{id}_{Y_1}$ to get the desired equations.