Lecture Notes to Linear Algebraic Groups

Winter 2021-2022, Technion IIT

Typed by Elad Tzorani

A cat.

Last updated January 9, 2022

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.
In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.
Elad Tzorani.

Grade

The course grade will consist of the following.

- 60% for homework
- 40% for giving lectures on more advanced topics at the end of the semester

Chapter 1

Linear Algebraic Groups

1.1 Preliminaries

1.1.1 Motivation & Historical Background

Linear Algebraic Groups From Differential Equations

Algebraic groups developed from the study of Lie groups. The latter were studied by Sophus Lie around 1870 in the context of differential equations. Lie groups can describe symmetries of solutions of differential equations; e.g. solutions of $\nabla y = 0$ are *harmonic functions* and one is interested in linear isomorphisms $g \colon \mathbb{R}^n \to \mathbb{R}^n$ such that $\Delta(y) = 0$ implies $\Delta(y \circ g) = 0$. Lie noticed that such g form a group $\mathcal{O}_n(\mathbb{R}) := \left\{ g \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; g^T g = I_n \right\}$. Such groups for operators different than Δ are smooth manifolds with smooth group actions, called *Lie groups*. One of Lie's motivation was to have Galois theory for differentiable equations. It had already been known that in order to find roots of polynomials one uses the symmetries of field extensions.
Around 1880, Picard looked at differentiable equations of the form

$$\frac{(\mathrm{d}y)^{n\,n}}{\mathrm{d}x} + p_1(x) \frac{\mathrm{d}y^{n-1}}{(\mathrm{d}x)^{n-1}} + \ldots + p_n(x)\, y = 0$$

for p_i rational functions. The solution space for such an equation is the n-dimensional space

$$\mathrm{Span}\left\{ y_1(x), \ldots, y_n(x) \right\}.$$

Picard looked at a subgroup $G \leq \mathrm{GL}_n(\mathbb{R})$ which preserves the algebraic dependencies of the y_i (i.e. preserves polynomials $p \in \mathbb{R}_n[x]$ for which $p(y_1(x), \ldots, y_n(x)) = 0$). These were the first treatments of algebraic groups.
Around 1870-1900, Mauren took homogeneous rational functions $f \colon \mathbb{C}^n \to \mathbb{C}$ (such as $f(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2$ for which $G_f = \mathcal{O}_n(\mathbb{C})$) and studied the structure of

$$G_f := \left\{ g \in \mathrm{GL}_n(\mathbb{C}) \mid f \circ g = f \right\}.$$

One can take f to be any quadratic form, e.g.

$$\mathrm{Sp}_{2n}(\mathbb{C}) = \left\{ g \in \mathrm{GL}_{2n}(\mathbb{C}) \;\middle|\; g^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

is such group. Such groups are called *classical groups*.
Mauren looked at the tangent space of such group. His motivation was his interest in Hilbert's 14^{th} problem: Given $g \in G \leq \mathrm{GL}_n(\mathbb{C})$ we can consider g as a map $\mathbb{C}^n \to \mathbb{C}^n$. Considering the action $G \curvearrowright \mathbb{C}[x_1, \ldots, x_n]$ on the coefficient, the problem is understanding the invariant space of this action. E.g. the invariant space of S_n are *symmetric polynomials*.

Later Developments

The field of Lie groups gave great success. Semisimple Lie groups have complete combinatorical classification due to Cartan and Killing. This is considered one of the greatest achievements in mathematics.

Chevalley found out the every semisimple Lie group is defined by polynomials in integer coefficients, circa 1940. This led to the definition of algebraic groups and a new goal: to algebrize Lie theory and develop tools to study smooth symmetric "without analysis" and over more general fields. This should form a bridge between continuous groups and finite groups. Chevalley used in his studies of the subject the formal expression

$$\exp(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

and required $\operatorname{char}(\mathbb{F}) = 0$. Later Kolchin returned to Picard's ideas and developed a differential Galois theory over a general field.

Modern Developments

From 1950 onwards, many mathematicians developed the study of algebraic groups, which was possible thanks to advances in algebraic geometry. Some of the advances of the field are the following.

1. The classification of finite simple groups. Most of these groups are of "Lie type", which are of the form $\mathrm{Sp}_{2n}(\mathbb{F}_q)$.
2. Results on p-adic groups. For example, Bruhat-Tits buildings are homogeneous spaces with p-adic group actions and which are "non-archimedean" analogues to classical symmetric spaces.
3. Results in number theory.

The Langlands Program

The Laglands program, circa 1960, tries to study properties in number theory through the study of groups. There are analogues to Riemann's zeta function which one hopes all arise from group actions in the following way. Taking an algebraic group G, one looks at *automorphic spaces* V with $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$ actions which commute with each other, for some groups $G(\mathbb{R}), G(\mathbb{Q}_p)$ over the respective fields.

1.1.2 Definitions & Course Goals

What are Algebraic Groups?

Write $\mathbb{F}$ for a field, and write $M_n(\mathbb{F}) \cong \mathbb{F}^{n^2}$ for the space of $n \times n$ matrices over $\mathbb{F}$.

Definition 1.1.1 (Affine Algebraic Group). A subset $G \leq M_n(\mathbb{F})$ closed under multiplication and inverse is called an *affine algebraic group* over $\mathbb{F}$ if there are $f_1, \ldots, f_k \in \mathbb{F}\left[\{x_{i,j}\}_{i,j\in[n]}\right]$ such that

$$G = \{A \in M_n(\mathbb{F}) \mid f_1(A) = \ldots = f_k(A) = 0\}.$$

Example 1. $\mathrm{SL}_n(\mathbb{F})$ is an affine algebraic group. The determinant,

$$\det(X) = \sum_{\sigma\in S_n} (-1)^{\operatorname{sgn}(\sigma)} x_{1,\sigma(1)} \cdot \ldots \cdot x_{n,\sigma(n)},$$

is a polynomial and $\mathrm{SL}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \det(A) - 1 = 0\}$.

Example 2. Let $Q \in M_n(\mathbb{F})$ and denote

$$\mathcal{O}_Q(\mathbb{F}) := \left\{A \in \mathrm{GL}_n(\mathbb{F}) \;\middle|\; A^t Q A = Q\right\}.$$

Taking $Q = I_n$ one gets $\mathcal{O}_Q(\mathbb{R}) = \mathcal{O}_n(\mathbb{R})$. More generally, matrix multiplication is polynomial and one can write $A^t Q A - Q = 0$ as a polynomial equation in the coefficients of A. We explain the condition $A \in \mathrm{GL}_n(\mathbb{F})$ later.

Example 3. Let $N \subseteq M_n(\mathbb{F})$ be the subset of upper-triangular matrices with 1 on the diagonal. This is an algebraic group with polynomial conditions $x_{i,j} = 0$ for $i > j$ and $x_{i,i} = 1$ for all $i \in [n]$. One has $N \cong \mathbb{F}^{\frac{n(n-1)}{2}}$ as vector space, but this doesn't remember the group structure.

Example 4. The vector space $\mathbb{F}^n$ with addition is an algebraic group. We have

$$V := \left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 & * \\ 0 & & & & * \\ \vdots & & \ddots & & \vdots \\ & & & & * \\ 0 & & \cdots & & 1 \end{pmatrix} \in M_{n+1}(\mathbb{F}) \right\} \cong \mathbb{F}^n.$$

One denotes $G_a(\mathbb{F}) := (\mathbb{F}, +)$ and calls this *the additive group over* $\mathbb{F}$.

Remark 1.1.2. One has

$$\begin{aligned} \mathrm{GL}_n(\mathbb{F}) &= \{A \in M_n(\mathbb{F}) \mid \det(A) \neq 0\} \\ &\cong \left\{ \begin{pmatrix} A & 0 \\ 0 & a \end{pmatrix} \in M_{n+1}(\mathbb{F}) \;\middle|\; \det(A) \cdot a = 1 \right\} \end{aligned}$$

and a bijection $A \leftrightarrow \begin{pmatrix} A & 0 \\ 0 & \det(A)^{-1} \end{pmatrix}$, but this looks weird. We then want the definition to be more general and capture groups that are isomorphic to what we defined as affine algebraic groups. We do that later in the course.

Example 5. Every finite group G is an algebraic group. One has an inclusion $G \hookrightarrow S_n$, and S_n is an algebraic group where σ is considered as $(x_{i,j})_{i,j\in[n]}$ with $x_{i,\sigma(i)} = 1$ and $x_{i,j} = 0$ for any other $i, j \in [n]$.

Exercise 1. Every finite subset of $M_n(\mathbb{F})$ is an algebraic set, in the sense that it's defined by the vanishing of polynomials.

To study properties of algebraic groups, one needs to use tools from algebraic geometry. Here there are two possible difficulties:

1. One needs to ask what generality is to be worked with. With our current definition it is difficult to use strong algebro-geometric tools, but with "too general" definitions it is more difficult to look at simple examples.

2. One should decide how much they want to rely on geometric results as facts and how much is to be proved.

Our answer to the latter question is proving things at the beginning of the course and later on taking more things as facts. For the first difficulty...you'll see as we go.

A Course Overview

During the course we plan to go over the following.

- Basic algebraic geometry.

- General structure properties of algebraic groups. For example, a generalization of Jordan's decomposition to GL_n.

- Generalization of the notion of an algebraic group.

- Study of algebraic groups by looking at algebraic groups over the Galois closure and via Galois theory.

- The classification of reductive groups over algebraically closed fields. An algebraic version of the Cartan-Killing classification.

1.1.3 Preliminary Algebraic Geometry

Embedded $\mathbb{F}$-Affine Varieties

Notation 1.1.3. Denote $A_n \coloneqq \mathbb{F}[x_1, \ldots, x_n]$.

Definition 1.1.4. For $C \subseteq A_n$ define

$$V(C) \coloneqq \{p \in \mathbb{F}^n \mid \forall f \in C \colon f(p) = 0\} \subseteq \mathbb{F}^n.$$

A set of this form is called an *embedded* $\mathbb{F}$*-Affine Variety.*

Definition 1.1.5. For $S \subseteq \mathbb{F}^n$ define

$$I(S) \coloneqq \{f \in A_n \mid \forall p \in S \colon f(p) = 0\}.$$

Exercise 2. For $S \subseteq \mathbb{F}^n$ one has $I(S) \trianglelefteq A_n$.

Example 6. One has $I(\varnothing) = A_n$ and whenever $\mathbb{F}$ is infinite one has $I(\mathbb{F}^n) = \{0\}$.

Example 7. For $I = \left\{x_1^2 - x_2, x_1^3 - x_3\right\}$ one has $V(I) = \left\{\left(x, x^2, x^3\right) \mid x \in \mathbb{F}\right\}$ which one calls the *twisted cubic* over $\mathbb{F}$

Proposition 1.1.6. *One notices that for $S \subseteq \mathbb{F}^n$ and $C \subseteq A_n$ we have*

$$S \subseteq V(I(S))$$
$$C \subseteq I(V(C)).$$

Definition 1.1.7 (The Zariski Topology). The *Zariski topology* on $\mathbb{F}^n$ is the topology given by taking sets of the form $V(C)$ for $C \subseteq A_n$ as the closed subsets.

Exercise 3. Check that the above definition gives a well-defined topology.

Exercise 4. For $S \subseteq \mathbb{F}^n$ one has $\overline{S} = V(I(S))$.

Example 8. Consider the case $n = 1$. Then closed subsets of $\mathbb{F}$ are sets of the form $V(C)$ for $C \subseteq A_n$. If C contains a nonzero polynomial, $V(C)$ is finite, and otherwise $V(C) = \mathbb{F}$. We get that the nontrivial closed sets are exactly the finite subsets of $\mathbb{F}^n$.

Remark 1.1.8. $\mathbb{F}^n$ with the Zariski topology is always *quasi-compact*, meaning it's compact but not Hausdorff.

Theorem 1.1.9 (Hilbert's Basis Theorem). *Every ideal $I \trianglelefteq A_n$ is finitely-generated.*

Proof. We prove the statement by induction on $n \in \mathbb{N}$. The case $n = 0$ is trivial since $\mathbb{F}$ is a field. Assume the statement is true for $n - 1$, we show it for n. Write $A_n \cong A_{n-1}[x_n]$ and assume $I \trianglelefteq A_n$ is nonzero. Choose $f_1 \in I \setminus \{0\}$ of minimal degree and write $d_1 \coloneqq \deg_{A_{n-1}} f_1$. If $(f_1) \neq I$, choose $f_2 \in I \setminus (f_1)$ of minimal degree $d_2 \coloneqq \deg_{A_{n_1}}(f_2)$. Continue this way to get f_i with $d_i \coloneqq \deg_{A_{n-1}}(f_i)$ and $d_1 \leq d_2 \leq d_3 \leq \ldots$.
Assume that this doesn't end a finite point (for otherwise we're done). Denote by $a_i \in A_{n-1}$ the leading coefficient of f_i. By assumption, $I' \coloneqq (a_1, \ldots, a_i, \ldots) \trianglelefteq A_{n-1}$ is finitely-generated. We can then write $I' = (a_1, \ldots, a_h)$ for some $h \in \mathbb{N}$. Then

$$a_{h+1} = x_1 a_1 + \ldots + x_h a_h$$

for some $x_1, \ldots, x_h \in A_{n-1}$. Let

$$g \coloneqq f_{h+1} - \sum_{i \in [h]} x_i \cdot f_i \cdot x^{d_{h+1} - d_i} \in I.$$

The coefficient of $x^{d_{h+1}}$ in g vanishes so $\deg(g) < d_{h+1}$ and $g \in (f_1, \ ldots, f_h)$. Then also $f_{h+1} \in (f_1, \ldots, f_h)$, in contradiction. ∎

Definition 1.1.10 (Noetherian Topological Space). A topological space X is called *Noetherian* if every decreasing sequence of closed subsets stabilises.

Corollary 1.1.11. *$\mathbb{F}^n$ with the Zariski topology is Noetherian.*

Proof. Let $(X_i)_{i\in\mathbb{N}_+} \subseteq \mathbb{F}^n$ be a decreasing sequence of closed subsets, and for every i denote $I_i \coloneqq I(X_i)$. Then $X_i = V(I_i)$. We get $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$. Denote $I = \bigcup_{i\in\mathbb{N}_+} I_i$. By Theorem 1.1.9 we have $I = (f_1, \ldots, f_k)$ for some $k \in \mathbb{N}_+$. Then $f_1, \ldots, f_k \in I_m$ for some $m \in \mathbb{N}_+$. We get $I = I_m = I_{m+1} = \ldots$ and $X_m = X_{m+1} = X_{m+2} = \ldots$, as required. ■

Exercise 5. Every closed subspace $X \subseteq \mathbb{F}^n$ is quasi-compact.

Example 9. Consider $xy \in \mathbb{F}[x, y]$. $V(x, y)$ is connected, but we would like to say it has two components. E.g. if $\mathbb{F} = \mathbb{R}$, the set $V(x, y)$ is the union of two perpendicular axes. This leads to the following definition.

Definition 1.1.12 (Irreducible Topological Space). A topological space X is *irreducible* if there aren't strict closed subsets $X_1, X_2 \subsetneq X$ such that $X = X_1 \cup X_2$.

Exercise 6. An irreducible Hausdorff topological space is a point.

Exercise 7. In a Noetherian space X there are finitely many maximal irreducible subsets $X_1, \ldots, X_k$, and $X = \bigcup_{i\in[k]} X_i$.

Proposition 1.1.13. *An algebraic variety $V \subseteq \mathbb{F}^n$ is irreducible if and only if $I(V)$ is prime.*

Proof. Assume V is irreducible. Let $f_1, f_2 \in A_n$, such that $f_1 f_2 \in I(V)$, we want to show $f_1 \in I(V)$ or $f_2 \in I(V)$. We have $V \subseteq V(f_1 f_2) = V(f_1) \cup V(f_2)$. Now $V = (V \cap V(f_1)) \cup (V \cap V(f_2))$ and by irreducibility $V = V \cap V(f_i)$ for $i \in [2]$, in which case $V \subseteq V(f_i)$ and therefore $f_i \in I(V)$.
The other direction is left as an exercise. ■

Example 10. Consider $G \coloneqq \mathrm{GL}_1(\mathbb{F}) \cong \mathbb{F}^\times \subseteq \mathbb{F}$. We have $V(x-1) = \{1\}$ and similarly $V\left((x-1)^2\right) = \{1\}$. If $\mathbb{F} = \mathbb{R}$, one has $V\left(x^5 - 1\right) = \{1\}$ and one gets $V\left(x^5 - 1\right) = V(x-1)$. However, over $\mathbb{F} = \mathbb{C}$ the group $V\left(x^5 - 1\right)$ is the roots of unity of order 5.

Example 11. Let $V \subseteq \mathbb{F}^n$ be closed, and for $f \in A_n$ define

$$V_f \coloneqq \{x \in V \mid f(x) \neq 0\} = V \setminus V(f).$$

This is open in V and such a set is called a *principal open set*. Every open set U is a finite union of such sets, so the principal open sets form a basis for the Zariski topology:
If U is open in V let $W \coloneqq V \setminus U$ so that there are $(f_i)_{i\in[k]} \subseteq A_n$ for which

$$V \setminus U = V \cap W = V(f_1, \ldots, f_k)$$

so

$$U = \bigcup_{i\in[k]} V_{f_i}.$$

We sometimes want to think of V_f as closed sets. This can be done by considering

$$\tilde{V}_f \coloneqq \left\{ (v, y) \;\middle|\; \substack{v\in\mathbb{F}^n \\ y\in\mathbb{F} \\ f(v)\cdot y=1} \right\} \subseteq \mathbb{F}^{n+1}.$$

There's a clear bijection $V_f \xrightarrow{\sim} \tilde{V}_f$.

Regular Maps

Definition 1.1.14 (Regular Map). For embedded algebraic varieties $V \subseteq \mathbb{F}^n$ and $W \subseteq \mathbb{F}^m$, a *morphism* $\varphi\colon V \to W$, called also a *regular map* is a map of the form

$$\varphi(x) = (f_1(x), \ldots, f_m(x))$$

for $(f_i)_{i\in[m]} \subseteq A_n$.

Example 12. The map

$$\begin{aligned}\varphi\colon \mathbb{F}^2 &\to \mathbb{F}^2\\ (x,y) &\mapsto (xy, y)\end{aligned}$$

is a regular map.

Exercise 8. A regular map is continuous in the Zariski topology.

Definition 1.1.15. A *regular function* on an embedded algebraic variety V over $\mathbb{F}$ is a regular map $V \to \mathbb{F}$.

Remark 1.1.16. Regular functions on V are of the form $f|_V$ for $f \in A_n$. We can think of these as elements of $\mathbb{F}[V] \coloneqq A_n/I(V)$.

Definition 1.1.17. A regular map $\varphi\colon V \to W$ gives an $\mathbb{F}$-algebra homomorphism

$$\begin{aligned}\varphi^*\colon \mathbb{F}[W] &\to \mathbb{F}[V]\\ f &\mapsto f \circ \varphi.\end{aligned}$$

Remark 1.1.18. Sending every V to $\mathbb{F}[V]$ and every φ to φ^* is a contravariant functor from the category of embedded algebraic varieties to that of finite-dimensional $\mathbb{F}$-algebras.

Exercise 9.

1. Consider regular maps

$$\varphi_1, \varphi_2\colon V \to W$$

   such that $\varphi_1^* = \varphi_2^*$. Show that $\varphi_1 = \varphi_2$.

2. Show that if $\varphi\colon V \to W$ is a regular map such that φ^* is an isomorphism, φ is also an isomorphism.

Exercise 10. Let $\varphi\colon V \to W$ be a regular map, and assume V is irreducible. Show that $\overline{\varphi(V)}$ is irreducible.

Revising Affine $\mathbb{F}$-Varieties

For a set X one can consider the algebra of functions $M \coloneqq \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{F})$. Every $x \in X$ defines a homomorphism $\mathrm{ev}_x\colon M \to \mathbb{F}$ given by $\mathrm{ev}_x(f) = f(x)$.

Definition 1.1.19 (Affine $\mathbb{F}$-Variety). An *affine* $\mathbb{F}$*-variety* is a pair (X, A) where X is any set and $A \subseteq \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{F})$ such that the following conditions hold.

1. A is finite-generated.
2. The map

$$\begin{aligned}X &\to \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(A, \mathbb{F})\\ x &\mapsto \mathrm{ev}_x|_A\end{aligned}$$

   is a bijection.

Remark 1.1.20. This new definition of an affine $\mathbb{F}$-variety is coordinate-free and gives good results even for non-algebraically-closed fields.

Remark 1.1.21. We could take any finitely-generated $\mathbb{F}$-algebra A and define $X = \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(A, \mathbb{F})$. This would given an homomorphism $A \to \mathrm{Hom}_{\mathbf{Set}}(X, \mathbb{F})$ taking $f \in A$ to the map $x \mapsto x(f)$. If we require that this map is injective, we get an equivalent definition to that of an affine $\mathbb{F}$-variety.

Definition 1.1.22 (Regular Maps). Let (X, A) and (Y, B) be affine $\mathbb{F}$-varieties. A *regular map* $\varphi\colon (X, A) \to (Y, B)$ is a map of sets $\varphi\colon X \to Y$ such that for every $f \in B$ it holds that $f \circ \varphi \in A$.

Definition 1.1.23 (Regular Isomorphism). A regular map $\varphi\colon (X, A) \to (Y, B)$ is an *isomorphism* if there's $\psi\colon (Y, B) \to (X, A)$ such that $\varphi \circ \psi = \mathrm{Id}$ and $\psi \circ \varphi = \mathrm{Id}$.

Remark 1.1.24. Every homomorphism $\alpha\colon B \to A$ gives a regular map $\varphi\colon X \to Y$ by sending x to the $y \in Y$ for which $\mathrm{ev}_x \circ \alpha = \mathrm{ev}_Y$.

Proposition 1.1.25. *Embedded $\mathbb{F}$-affine varieties are $\mathbb{F}$-affine varieties.*

Proof. Let $V \subseteq \mathbb{F}^n$ be an embedded $\mathbb{F}$-affine variety. We claim $(V, \mathbb{F}[V])$ is an affine $\mathbb{F}$-variety. $\mathbb{F}[V] = A_n / I(V)$ is finitely-generated, so we have to show that all maps $\mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(\mathbb{F}[V], \mathbb{F})$ are of the form ev_x (since $\mathbb{F}[V]$ separate points). Let $\varepsilon\colon \mathbb{F}[V] \to \mathbb{F}$, we find $x \in V$ such that $\varepsilon = \mathrm{ev}_x$. Consider the quotient map $\pi\colon A_n \to \mathbb{F}[V]$. Then $\varepsilon \circ \pi\colon A_n \to \mathbb{F}$. Let $x = (\varepsilon \circ \pi(x_i))_{i \in [n]} \in \mathbb{F}^n$. Now $\ker(\varepsilon \circ \pi)$ is a maximal ideal contained in $I(x)$ so

$$I(V) = \ker(\pi) \subseteq \ker(\varepsilon \circ \pi) = I(x)$$

so $x \in V$. ■

Remark 1.1.26. Let (X, A) be an affine $\mathbb{F}$-variety. Write $A \cong A_n / I$ and $\pi\colon A_n \xrightarrow{\sim} A_n / I$ the quotient map, and define a map

$$\begin{aligned} \varphi\colon X &\to \mathbb{F}^n \\ x &\mapsto (\mathrm{ev}_x \circ \pi(x_i))_{i \in [n]}. \end{aligned}$$

Now $I = \ker(\pi) \subseteq \ker(\mathrm{ev}_p)$ for all $p \in \varphi(X)$, so $I \subseteq I(\varphi(X))$. In the other direction, if $f \in I(\varphi(X))$ then $\mathrm{ev}_x(\pi(f))$ for all $x \in X$. Therefore $\pi(f) = 0$ so $I(\varphi(X)) \subseteq \ker(\pi) = I$. Then $A \cong A_n / I(\varphi(X)) = \mathbb{F}[\varphi(X)]$. Then $(X, A) \cong (\varphi(X), \mathbb{F}[\varphi(X)])$.
We want $\varphi(X)$ to be closed, which we explain later.

Exercise 11. Check that φ in the above remark is injective.

Definition 1.1.27 (Zariski Topology on an Affine $\mathbb{F}$-Variety). Let (X, A) be an affine $\mathbb{F}$-variety. Define the *Zariski topology* on (X, A) by choosing the closed sets to be sets of the form

$$V(C) \coloneqq \{x \in X \mid \forall f \in C\colon f(x) = 0\}$$

for $C \subseteq A$.

Definition 1.1.28. Let (X, A) be an affine $\mathbb{F}$-variety. For $S \subseteq X$ we define

$$I(S) \coloneqq \{f \in A \mid \forall x \in X\colon f(x) = 0\} \trianglelefteq A.$$

Remark 1.1.29. As before, if $Y \subseteq X$ is closed, we have $Y = V(I(Y))$. Then Y is itself an affine $\mathbb{F}$-variety as $(Y, A/I(Y))$ where $A/I(Y)$ is considered as embedded in $\mathrm{Hom}_{\mathbf{Set}}(Y, \mathbb{F})$ by considering the restriction to Y. Elements of $\mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(A/I(Y), \mathbb{F})$ are elements of $\mathrm{Hom}(A, \mathbb{F}) \cong X$ that vanish on $I(Y)$. These are exactly $V(I(Y)) \cong Y$.

Definition 1.1.30 (Closed Embedding). A regular map $\varphi\colon (X, A) \to (Y, B)$ is called a *closed embedding* if $\mathrm{Im}\,\varphi$ is closed and $\varphi|_X$ is a regular isomorphism.

Remark 1.1.31. Requiring that φ is injective would not suffice. Consider

$$\begin{aligned} \varphi\colon \mathbb{F}_p &\to \mathbb{F}_p \\ x &\mapsto x^p. \end{aligned}$$

We have

$$\begin{aligned} \varphi^*\colon \mathbb{F}_p[x] &\to \mathbb{F}_p[x] \\ x &\mapsto x^p, \end{aligned}$$

which isn't surjective. Hence φ isn't an isomorphism.

Proposition 1.1.32. *Let* $\varphi\colon (X,A) \to (Y,B)$*.* φ *is a closed embedding if and only if* $\varphi^*\colon \mathbb{F}[Y] \to \mathbb{F}[X]$ *is surjective.*

Proof.

- Assume φ is a closed embedding. Consider the inclusion $i\colon \varphi(X) \to Y$ and write $\varphi = i \circ \varphi_0$. Then $\varphi^* = \varphi_0^* \circ i^*$. φ_0^* is an isomorphism, hence φ_0 is an isomorphism. We're left to show that i^* is surjective. $\varphi(X) \subseteq Y$ is closed and i^* is the quotient map from $\mathbb{F}[Y]$ to $\mathbb{F}[\varphi(X)]$, hence we get the result.
- Assume φ^* is surjective. We show that $\varphi(X) = V(\ker(\varphi^*))$ and $\ker(\varphi^*) = I(\varphi(X))$.

  $\varphi(X) = V(\ker(\varphi^*))$: Let $f \in \ker(\varphi^*) \subseteq \mathbb{F}[Y]$. For every $x \in X$ we have

  $$\mathrm{ev}_{\varphi(x)}(f) = \mathrm{ev}_x(\varphi^*(f)) = \mathrm{ev}_x(0) = 0.$$

  Hence $f \in I(\varphi(X))$. Hence $\ker(\varphi^*) \subseteq I(\varphi(X))$ so $\varphi(X) \subseteq V(\ker(\varphi^*))$.
  In the other direction, let $y \in V(\ker(\varphi^*))$. Then $\mathrm{ev}_y|_{\ker(\varphi^*)} = 0$. Then $\mathrm{ev}_y\colon \mathbb{F}[Y] \to \mathbb{F}$ factors through $\lambda\colon \mathbb{F}[X] \to \mathbb{F}$ where $\lambda = \mathrm{ev}_y \circ \varphi^*$. Hence there's $x \in X$ such that $\lambda = \mathrm{ev}_x$. Hence $y = \varphi(x) \in \varphi(X)$ so $V(\ker(\varphi^*)) \subseteq \varphi(X)$.

  $\ker(\varphi^*) = I(\varphi(X))$: We saw one inclusion in the previous part. Let $f \in I(\varphi(X))$, we have to show $f \in \ker(\varphi^*)$. Indeed, $\varphi^*(f) = f \circ \varphi = 0$. ■

1.2 Algebraic Groups

1.2.1 Definitions

In order to define affine algebraic groups, we want the product and inverse maps to be regular. For that, we need to define the product variety. This structure comes from tensor products of $\mathbb{F}$-algebras.
Let $(X,A),(Y,B)$ be affine $\mathbb{F}$-varieties. We have an embedding

$$\begin{aligned} \iota\colon A \otimes B &\to \mathrm{Hom}_{\mathbf{Set}}(X \times Y, F) \\ f \otimes g &\mapsto ((x,y) \mapsto f(x) \cdot g(y)). \end{aligned}$$

We show this is injective. Let $f_1,\ldots,f_k \in A$ linearly independent and $g_1,\ldots,g_\ell \in B$ linearly independent. Using properties of tensor products, we show that for $F = \sum_{\substack{i\in[k]\\ j\in[\ell]}} a_{i,j} f_i \otimes g_j$ such that $\iota(F) = 0$ it holds that $F = 0$. Indeed, for $y \in Y$ define

$$b_i(y) = \sum_{j\in[\ell]} a_{i,j} g_j(y).$$

Then

$$0 = \iota(F)(x,y) = \sum_{i\in[k]} b_i(y) f_i(x)$$

for all $x \in X$. Hence

$$\sum_{i\in[k]} b_i(y) f_i = 0$$

so for every $i \in [k]$ we have $b_i(y) = 0$. Hence

$$0 = b_i = \sum_{j\in[\ell]} a_{i,j} g_j \in B$$

so $a_{i,j} = 0$ for every $i \in [k]$ and $j \in [\ell]$.

Definition 1.2.1 ($\mathbb{F}$-Algebraic Group). An $\mathbb{F}$-*algebraic group* G is an $\mathbb{F}$-affine variety such that the product and inverse maps are regular.

Exercise 12. Let (X,A) be an affine $\mathbb{F}$-variety. Let $f \in A$, and define the *principle open set*

$$X_f := \{v \in X \mid f(v) \neq 0\} \subseteq X.$$

Show that $\left(X_f, A\left[\frac{1}{f}\right]\right)$ is an affine $\mathbb{F}$-variety.

Remark 1.2.2. 1. Consider $\{0\} \subseteq A_n$ and $X = V(\{0\}) = \mathbb{F}^n$. Then $I(X) = (0)$ if and only if $\mathbb{F}$ is infinite. This is equivalent to $\mathbb{F}[X] \cong A_n$. In particular, when $\mathbb{F}$ is finite, $(\mathbb{F}, \mathbb{F}[x])$ isn't an affine $\mathbb{F}$-variety in our sense.

2. Consider $\mathbb{F} = \overline{\mathbb{F}_p}$ and take $X = \mathbb{F}$. We have

$$\mathbb{F}[X] = \mathbb{F}[X]$$

and the map

$$\begin{aligned} \varphi \colon \mathbb{F} &\to \mathbb{F} \\ v &\mapsto v^p \end{aligned}$$

is bijective (check this). The map

$$\begin{aligned} \varphi^* \colon \mathbb{F}[x] &\to \mathbb{F}[x] \\ x &\mapsto x^p \end{aligned}$$

isn't surjective, so φ isn't an isomorphism!

Our current goals are the following.

1. Every embedded $\mathbb{F}$-group is an $\mathbb{F}$ algebraic group.
2. Every $\mathbb{F}$ algebraic group has an embedding into $\mathbb{F}^n$ for some $n \in \mathbb{N}_+$.

Proposition 1.2.3. *Every embedded $\mathbb{F}$-group is an algebraic group.*

Proof. Firstly we notice that a restriction of a regular map to a closed subset is regular. Hence, if G is an algebraic group and $H \leq G$ is closed, H is algebraic. It therefore suffices to show that $\mathrm{GL}_n(\mathbb{F})$ is an $\mathbb{F}$-algebraic group.
Now,

$$\mathrm{GL}_n(\mathbb{F}) = \{A \in M_n(\mathbb{F}) \mid \det(A) \neq 0\} = \left(\mathbb{F}^{n^2}\right)_{\det}$$

is a principle open set in $\mathbb{F}^{n^2}$. We have

$$\mathbb{F}[\mathrm{GL}_n(\mathbb{F})] = \mathbb{F}\left[\bigcup_{i,j \in [n]} \left\{X_{i,j}, \frac{1}{\det(X_{i,j})}\right\}\right].$$

Let $G := \mathrm{GL}_n(\mathbb{F})$ and $m \colon G \times G \to G$ be the multiplication map. Then

$$m^*(X_{i,j}) = \sum_{k \in [n]} X_{i,k} \otimes X_{k,j} \in \mathbb{F}[G \times G] \cong \mathbb{F}[G] \otimes \mathbb{F}[G].$$

From multiplicativity of det we get

$$m^*\left(\frac{1}{\det(X_{i,j})}\right) = \frac{1}{\det(X_{i,j})} \otimes \frac{1}{\det(X_{i,j})}.$$

Let $i \colon G \to G$ be the inverse map. Cramer's rule described the inverse of a matrix as a polynomial by the coefficients, hence the inverse map is also regular. ∎

Definition 1.2.4. The [Multiplicative Group] Define *the multiplicative group* over $\mathbb{F}$ as $\mathbb{G}_m(\mathbb{F}) := \mathrm{GL}_1(\mathbb{F})$.

Exercise 13. The product of irreducible algebraic varieties is irreducible.

Proposition 1.2.5. *Let G be an $\mathbb{F}$ algebraic group. Let $G^\circ \subseteq G$ be the irreducible component of G containing the identity element e of G. Then*

1. *$G^\circ \trianglelefteq G$ is a normal subgroup of G of finite index.*
2. *Every closed subgroup $H \leq G$ of finite index contains G°.*
3. *$G = G^\circ$ if and only if G is connected.*

Proof. 1. By exercise 13, $G^\circ \times G^\circ \subseteq G \times G$ is irreducible. Hence $e = e \cdot e \in \overline{G^\circ \cdot G^\circ}$. Since $G^\circ \subseteq \overline{G^\circ \cdot G^\circ}$ we get $G^\circ = \overline{G^\circ \cdot G^\circ}$. Hence G° is closed to multiplication. Now, $g \mapsto g^{-1}$ is a homeomorphism so $(G^\circ)^{-1}$ is irreducible. Now $\overline{G_0 \cdot (G_0)^{-1}}$ is irreducible containing e so one gets $G_0 = \overline{G_0 \cdot (G_0)^{-1}} = (G_0)^{-1}$.

For every $x \in G$, the map $\lambda_x \colon G \to G$ given by $x \mapsto xy$ is a homeomorphism. Every coset xG° is an irreducible component of G and $\bigsqcup_{x\in G} xG^\circ$. In an affine variety, there are finitely many irreducible components, hence G° has finitely many cosets in G. This proves part 3 as well.

We're left to show that G° is normal. Indeed, conjugation $y \mapsto xyx^{-1}$ is a homeomorphism. For every $x \in G$, $xG^\circ x^{-1}$ is an irreducible component. Hence $e \in G^\circ x^{-1}$ implies $G_0 = xG_0x^{-1}$.

2. If $H \leq G$ is a closed subgroup of finite index, we can write $G = \bigcup_{x\in G} xH$ which is a finite disjoint union of the different cosets. Then $G^\circ = \bigcup_{x\in G} (xH \cap G^\circ)$. From irreducibility there's $x \in G$ such that $G^\circ = xH \cap G^\circ$. Then $e \in G^\circ \subseteq xH = H$. ∎

Example 13. $\mathrm{GL}_1(\mathbb{R})$ isn't connected as a Lie group (with the Euclidean topology from $\mathbb{R}$, but *is* connected as an algebraic group.

Example 14. Finite non-trivial groups aren't connected, since $G^\circ = \{1\}$.

Example 15. Consider the group

$$\mathrm{O}_n(\mathbb{F}) \coloneqq \left\{ M_n(\mathbb{F}) \mid g^t Q g = Q \right\}$$

where $Q = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$. For $g \in O_n(\mathbb{F})$ we get

$$\begin{aligned} \det(g)^2 \det(Q) &= \det\left(g^t Q g\right) \\ &= \det(Q) \end{aligned}$$

so $\det(g) \in \{\pm 1\}$. $\det \colon \mathrm{O}_n(\mathbb{F}) \to \{\pm 1\}$ is a homomorphism. Now, $\mathrm{SO}_n(\mathbb{F}) \coloneqq \mathrm{O}_n(\mathbb{F}) \cap \mathrm{SL}_n(\mathbb{F})$ is a subgroup of index 2 (or 1). For this, it suffices to show that there's $g \in \mathrm{O}_n(\mathbb{F})$ such that $\det(g) = -1$, which is the cast when $1 + 1 \neq 0$ (otherwise, the index is 1). Usually, $\mathrm{SO}_n(\mathbb{F}) = \mathrm{O}_n(\mathbb{F})^\circ$. When $n = 2$ and $\mathbb{F}$ is algebraically-closed we get

$$\mathrm{GL}_1(\mathbb{F}) \cong \mathrm{SO}_2(\mathbb{F}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \middle| \, a \in \mathbb{F} \setminus \{0\} \right\}$$

which is connected. Then there's

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{O}_2(\mathbb{F}) \setminus \mathrm{SO}_2(\mathbb{F}).$$

1.2.2 Embedding Algebraic Groups

Group Actions

We want to show that every $\mathbb{F}$-algebraic group is isomorphic to a closed subgroup of $\mathrm{GL}_n(\mathbb{F})$. To every G, we later find a rational representation $\varphi \colon G \to \mathrm{GL}_n(\mathbb{F})$ (i.e. with φ regular) such that φ is a closed embedding. For this, we look at actions of algebraic groups.

Definition 1.2.6 ($\mathbb{F}$-Affine G-Space). Let (X, A) be an affine $\mathbb{F}$-variety. We say X is an $\mathbb{F}$-*affine G-space* if it has a regular map $G \times X \to X$ for an $\mathbb{F}$-affine algebraic group G.

Example 16. Take $X = G$. Then G acts on X by left multiplication $x \mapsto gx$, by right multiplication $x \mapsto xg^{-1}$ or by conjugation $x \mapsto gxg^{-1}$.

For X an $\mathbb{F}$-affine G-space, we get an action of G on $\mathbb{F}[X]$. Let $f \in \mathbb{F}[X]$ and $g \in G$. We have an action

$$(gf)(x) = f\left(g^{-1} \cdot x\right).$$

The map

$$\begin{aligned} \mathbb{F}[X] &\to \mathbb{F}[x] \\ f &\mapsto gf \end{aligned}$$

is $\mathbb{F}$-linear. We get a map

$$G \to \mathrm{GL}(\mathbb{F}[X]).$$

Writing the G-action as

$$a \colon G \times X \to X$$

we get

$$a^* \colon \mathbb{F}[X] \to \mathbb{F}[G] \otimes \mathbb{F}[X].$$

Now,

$$g \circ f = \left(\mathrm{ev}_{g^{-1}} \otimes \mathrm{Id}\right)(a^*(f)).$$

Proposition 1.2.7. *1. If $V \subseteq \mathbb{F}[X]$ is a finite-dimensional G-invariant subspace, then*

$$a^*(V) \subseteq \mathbb{F}[G] \otimes V,$$

and the resulting homomorphism $\varphi \colon G \to \mathrm{GL}(V)$ is regular (given any choice of basis for V).

2. For every finite-dimensional $W \subseteq \mathbb{F}[X]$ there's $V \subseteq \mathbb{F}[X]$ which contains W and is G-invariant.

Proof. 1. Let $V \subseteq \mathbb{F}[X]$ be finite-dimensional and G-invariant. Choose $\{r_1, r_2, \ldots\} \subseteq \mathbb{F}[X]$ linearly independent such that $\mathbb{F}[X] = V \oplus \mathrm{Span}\{r_1, r_2, \ldots\}$. For $f \in V$ we can write

$$a^*(f) = s + \sum_{i \in [k]} u_i \otimes r_{j_i}$$

for $s \in \mathbb{F}[G] \otimes V$ and $u_i \in \mathbb{F}[G]$. Since $g^{-1} \cdot f \in V$ we get

$$(\mathrm{ev}_v \otimes \mathrm{Id})(a^*(f)) \in V$$

so

$$\sum_{i \in [k]} u_i(g)\, r_{j_i} = 0 \in \mathbb{F}[X]$$

for all $g \in G$. By linear independency we get $u_i(g) = 0$ for all i, so $u_i \equiv 0$. Hence $a^*(f) \in \mathbb{F}[G] \otimes V$.

Choose a basis $(f_1, \ldots, f_n)$ for V. We get $a^*(f_i) = \sum_{j \in [n]} m_{j,i} f_j$ for $m_{i,j} \in \mathbb{F}[G] \otimes V$. If we look at the action of G on V and write the map $\varphi \colon G \to \mathrm{GL}_n(\mathbb{F})$ given by choosing the basis $(f_1, \ldots, f_n)$, we get

$$\varphi(g) = \left(m_{j,i}\left(g^{-1}\right)\right)$$

and

$$g \cdot f_i = \sum_{j \in [k]} m_{j,i}\left(g^{-1}\right) f_j.$$

Then, the matrix coefficients of $\varphi(g)$ are given by regular functions.

2. Take a basis $(h_1, \ldots, h_k)$ of W and examine the finite sum $a^*(h_i) = \sum_j u_j \otimes f_{i_j}$. We have

$$g \cdot h_i = \sum_j u_j \left(g^{-1}\right) f_{i,j} \in \mathbb{F}[X].$$

Take $V' = \operatorname{Span}\{f_{i,j}\}_{i,j} \subseteq \mathbb{F}[X]$ which is finite-dimensional. Then

$$V = \operatorname{Span}\left\{g \cdot f \;\middle|\; \substack{g \in G \\ f \in \mathbb{F}[X]}\right\} \subseteq V'$$

is a G-invariant subspace. ■

Proposition 1.2.8. *Let G be an $\mathbb{F}$-algebraic group. There's a regular homomorphism $\varphi\colon G \to \mathrm{GL}_n(\mathbb{F})$ which is a closed embedding.*

Proof. Consider the right action of G on itself. Take generators $f_1, \ldots, f_k \in \mathbb{F}[G]$ of the $\mathbb{F}$-algebra $\mathbb{F}[G]$. From the previous proposition, there's a finite-dimensional G-invariant $V \subseteq \mathbb{F}[G]$ under the induced action on $\mathbb{F}[G]$, which contains each f_i.
Choose a basis $(e_1, \ldots, e_n)$ of V. We get a homomorphism $\varphi\colon G \to \mathrm{GL}_n(\mathbb{F}) \cong \mathrm{GL}(V)$ as in the previous proposition. We want to show that

$$\varphi^*\colon \mathbb{F}[\mathrm{GL}_n(\mathbb{F})] \to \mathbb{F}[G]$$

is surjective. We may assume $\varphi(g) = \left(m_{j,i}\left(g^{-1}\right)\right)$ for $m_{i,j} \in \mathbb{F}[G]$ for which

$$\forall x, g \in G\colon e_i(xg) = \sum_j m_{j,i}\left(g^{-1}\right) e_j(x).$$

Taking $x = e$ we get

$$e_i(g) = \sum_j m_{j,i} e_j(e)\, m_{j,i}\left(g^{-1}\right).$$

Write $\tilde{m}_{i,j}(g) = m_{j,i}\left(g^{-1}\right)$. Then

$$e_i = \sum_j e_j(e)\, \tilde{m}_{j,i} \in \mathbb{F}[G]$$

but $\tilde{m}_{i,j} = \varphi^*(T_{i,j})$. Hence $e_i \in \operatorname{Im}(\varphi)$. Hence $f_j \in \operatorname{Im}(\varphi^*)$ and these generate $\mathbb{F}[G]$ so $\operatorname{Im}(\varphi^*) = \mathbb{F}[G]$. ■

Consider a homomorphism $\varphi\colon G \to H$ of $\mathbb{F}$-algebraic groups. The kernel $\ker(\varphi) = \varphi^{-1}(e) \leq G$ is a closed subgroup of G. A question we would like to ask whether or not $\operatorname{Im}(\varphi)$ is closed. This happens to be true when $\mathbb{F}$ is algebraically-closed, but requires use of an intricate result in algebraic geometry.

Remark 1.2.9. In Lie groups, the situation is different than that mentioned above. Consider the map

$$\begin{aligned}\varphi\colon \mathbb{R} &\to \mathrm{GL}_4(\mathbb{R})\\ \theta &\mapsto \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos(\alpha\theta) & \sin(\alpha\theta) \\ 0 & 0 & -\sin(\alpha\theta) & \cos(\alpha\theta) \end{pmatrix}.\end{aligned}$$

If $\alpha/2\pi$ is irrational, this is an embedding of $\mathbb{R}$ which isn't closed. Moreover, $\operatorname{Im}\varphi$ has an empty interior.

We later show that if $\varphi\colon X \to Y$ is regular between $\mathbb{F}$-affine varieties and $\mathbb{F}$ is algebraically-closed, there's an open $U \subseteq \overline{\varphi(X)}$ such that $U \subseteq \varphi(X)$.

Example 17. Consider $X := \left\{(x,y) \in \mathbb{F}^2 \,\middle|\, xy = 1\right\}$ and the map

$$\begin{aligned}\varphi\colon X &\to \mathbb{F}\\ (x,y) &\mapsto x.\end{aligned}$$

The image $\operatorname{Im}(\varphi) = \mathbb{F} \setminus \{0\}$ is dense, non-closed and open.

Example 18. Consider

$$\begin{aligned}\varphi\colon \mathrm{GL}_1(\mathbb{R}) &\to \mathrm{GL}_1(\mathbb{R})\\ x &\mapsto x^2.\end{aligned}$$

We have $\operatorname{Im}(\varphi) = \{y \in \mathbb{R} \mid y > 0\}$ which is (Zariski) dense in $\mathrm{GL}_1(\mathbb{R})$ but doesn't contain an open set.

Definition 1.2.10 (Dominant Regular Map). Let $\varphi\colon X \to Y$ be a regular map between $\mathbb{F}$-affine varieties. We say φ is *dominant* if $\overline{\varphi(X)} = Y$.

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Proposition 1.2.11. *$\varphi\colon X \to Y$ is dominant if and only if $\varphi^*\colon \mathbb{F}[Y] \to \mathbb{F}[X]$ is injective.*

Proof. Assume φ isn't dominant. We have $Z := \overline{\varphi(X)} \subsetneq Y$. Then there's $f \in \mathbb{F}[Y] \setminus 0$ such that $f|_Z = 0$. Then $\varphi^*(f) = 0$ so $\ker(\varphi^*) \neq (0)$ so φ^* isn't injective.
Assume φ^* isn't injective. Then $\ker(\varphi^*) \neq (0)$. Then there's $f \in \mathbb{F}[Y] \setminus 0$ such that $\varphi^*(f) = 0$. Then $\varphi(X) \subseteq V(f) \subsetneq Y$ so $\varphi(X)$ isn't dense, a contradiction. ■

Theorem 1.2.12. *Let $\varphi\colon X \to Y$ be a dominant regular map between $\mathbb{F}$-affine varieties, for $\mathbb{F}$ algebraically-closed. Then $\operatorname{Im}(\varphi)$ has non-empty interior.*

An equivalent statement is the following.

Theorem 1.2.13. *Let $\varphi\colon X \to Y$ be a dominant regular map between $\mathbb{F}$-affine varieties, for $\mathbb{F}$ algebraically-closed. There's $f \in \mathbb{F}[Y] \setminus \{0\}$ such that $\mathrm{ev}_y(f) \neq 0$ implies there's $x \in X$ for which $\mathrm{ev}_x \circ \varphi^* = \mathrm{ev}_y$.*

Example 19. $\varphi\colon \mathrm{GL}_1(\mathbb{F}) \to \mathbb{F}$ sending $x \mapsto x$ gives a map $\varphi^*\colon \mathbb{F}[x] \to \mathbb{F}\left[x, x^{-1}\right]$. For $y \in \mathbb{F} \setminus \{0\}$ we can lift ev_y to a homomorphism

$$\mathbb{F}\left[x, x^{-1}\right] \to f \qquad\qquad \mapsto f(y).$$

If we take $y = 0$ we get

$$\begin{aligned}\mathrm{ev}_0\colon \mathbb{F}[x] &\to \mathbb{F}\\ f &\mapsto f(0),\end{aligned}$$

which cannot be lifted to $\mathbb{F}\left[x, x^{-1}\right]$. The polynomial $x \in \mathbb{F}[x]$ satisfies our required property. We get that we can lift every ev_y with $\mathrm{ev}_y(x) \neq 0$.

Definition 1.2.14 (Reduced Ring). A ring is *reduced* if it has no non-zero nilpotent elements.

Lemma 1. *Let $\mathbb{F}$ be an algebraically-closed field. Let $A \hookrightarrow B$ be a subring of a reduced ring B (which is unital and commutative) such that B is generated by A and $t \in B$. Write $B \cong A[t]/I$. Assume there are a homomorphism $\varepsilon\colon A \to \mathbb{F}$ and an element $f := f_0 + f_1 t + \ldots + f_n t^n \in I$ such that $\varepsilon(f_n) \neq 0$. Then there's $\varepsilon'\colon B \to \mathbb{F}$ such that $\varepsilon'|_A = \varepsilon$.*

Proof. Assume that m is minimal for which $\varepsilon(f_m) \neq 0$. We prove the statement by induction on m. Consider

$$\begin{aligned}\tilde{\varepsilon}\colon A[t] &\to \mathbb{F}[t]\\ \sum_{i=0}^{k} a_i t^i &\mapsto \sum_{i=0}^{k} \varepsilon(a_i)\, t^i.\end{aligned}$$

If $\langle \tilde{\varepsilon}(I) \rangle \neq \mathbb{F}[t]$, then $(\tilde{\varepsilon}(I)) = (p)$ for $p \in \mathbb{F}[t]$. Since $\mathbb{F}$ is algebraically-closed, there's a root $\alpha \in \mathbb{F}$ of p. Consider $\varepsilon' = \mathrm{ev}_\alpha \circ \tilde{\varepsilon}$, we have $I \subseteq \ker(\varepsilon)$.
We prove that indeed $\langle \tilde{\varepsilon}(I) \rangle \neq \mathbb{F}[t]$ via contradiction. Otherwise, there is a polynomial $g = \sum_{i=0}^{n} g_i t^i \in I$ for which $\deg \tilde{\varepsilon}(g) = 0$. (check this!) Then $\varepsilon(g_0) \neq 0$ and $\varepsilon(g_i) = 0$ for all $i > 0$. We have $A \leq B$ so $A \cap I = \{0\}$. Hence $n \geq 1$. We may assume $n < m$ by the following argument: We "divide" g by f. It can be shown that there are $q, r \in A[t]$ for which $f_m^d g = qf + r$ for some $d \geq 1$. Applying $\tilde{\varepsilon}$ on the equation we get

$$0 \neq \varepsilon(f_m)^d \cdot \varepsilon(g_0) = \tilde{\varepsilon}(q)\,\tilde{\varepsilon}(f) + \tilde{\varepsilon}(r).$$

Since the left-hand-side is of degree 0, the right-hand side is a constant polynomial. But, $\deg \tilde{\varepsilon}(m) = 0$, so $\tilde{\varepsilon}(q) = 0$. Also $r = f_m^d \cdot g - q \cdot f \in I$. We therefore may consider r instead of g as it satisfies the same conditions. This shows that case $m = 1$.
We now show the induction step. Assume $m > 1$. For $h = \sum_{i=0}^{s} h_i t^i \in A[t]$ we define

$$\tilde{h} = \sum_{i=0}^{s} h_{s-i} t^i.$$

As a function $\tilde{h}(t) = (t^s) \circ h \circ \left(t^{-1}\right)$ which is an involution reminding of those in complex function theory. Consider the ideal

$$\tilde{I} \coloneqq \left\langle \left\{ \tilde{h} \,\middle|\, h \in I \right\} \right\rangle \trianglelefteq A[t].$$

We have $\tilde{g} \in \tilde{I}$ and $\varepsilon(g_0) \neq 0$. Define

$$\bar{A} = A/\left(A \cap \tilde{I}\right)$$

and

$$\tilde{B} = A[t]/\tilde{I},$$

so that there's an inclusion $\bar{A} \hookrightarrow \bar{B}$. For $h \in I$ with $\tilde{h} \in \tilde{I} \cap A$ we have $h = a \cdot t^s$ for some $a \in$. Then $(at)^s = a^{s-1}(at^s) \in I$. Since B has no nilpotent elements we get that $at \in I$. Since $m > 1$ we get $\varepsilon(a) = 0$. Now, $\tilde{h} = a$ so $\varepsilon\left(\tilde{h}\right) = 0$, so $\varepsilon\left(A \cap \tilde{I}\right) = 0$. We get that ε factors via $\bar{\varepsilon} \colon \bar{A} \to \mathbb{F}$. Since $n < m$, and by the induction step (check that $\bar{B}$ is reduced), there's $\bar{\varepsilon}' \colon \bar{B} \to \mathbb{F}$ such that $\bar{\varepsilon}'|_{\bar{A}} = \bar{\varepsilon}$. We have

$$\bar{\varepsilon}'(\tilde{g}) = \varepsilon(g_0) \cdot \bar{\varepsilon}(t)^n,$$

but since $\tilde{g} \in \tilde{I}$ we have $\bar{\varepsilon}'(\tilde{g}) = 0$. So, $\bar{\varepsilon}'(t)^n$. Hence

$$\begin{aligned} 0 &= \bar{\varepsilon}'\left(\tilde{f}\right) \\ &= \sum_{i=0}^{m} \varepsilon(f_{m-i})\,\bar{\varepsilon}'(t)^i \\ &= \varepsilon(f_m) \neq 0, \end{aligned}$$

a contradiction. ∎

Example 20. In the previous example, take $A \coloneqq \mathbb{F}[x]$ and $B \coloneqq \mathbb{F}\left[x, x^{-1}\right]$. Then $B \cong A[t]/(xt-1)$. Then $f(t) = -1 + xt$ is the element in the lemma.

Corollary 1.2.15. *Let $\varphi \colon G \to H$ be a homomorphism of algebraic groups over an algebraically-closed field $\mathbb{F}$. Then $\mathrm{Im}(\varphi)$ is closed.*

A generalization of the lemma is the following.

Proposition 1.2.16. *Let $\mathbb{F}$ be an algebraically-closed field. Let B be a finitely-generated $\mathbb{F}$-algebra, which is an integral domain. Let $A \leq B$. Then for every $b \in B \setminus \{0\}$ there's $a \in A \setminus \{0\}$ such that for every homomorphism $\varepsilon \colon A \to \mathbb{F}$ with $\varepsilon(a) \neq 0$ there is a homomorphism $\varepsilon' \colon B \to \mathbb{F}$ with $\varepsilon'|_A = \varepsilon$ and $\varepsilon(b) \neq 0$.*

Proof. Since B is finitely-generated, we can write

$$A = A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_k = B$$

with $A_i \cong A_{i-t}[t]/I_i$. By induction, it suffices to prove the case $k = 1$. Write $B \cong A[t]/I$ and pick $b \in A[t]/I$ non-zero. Let $h \in A[t]$ with quotient image $\bar{h} = b$.

- If $I = (0)$ we have $B = A[t]$, in which case we define

$$\tilde{\varepsilon} \colon A[t] \to \mathbb{F}[t]$$
$$\sum_{i=0}^{n} a_i t^i \mapsto \sum_{i=0}^{n} \varepsilon(a_i) t^i.$$

  Now, $b \neq 0$ so $h \coloneqq \sum_{i=0}^{n} h_i t^i \neq 0$. Then there's i such that $h_i \neq 0$. Take $a = h_i$. If $\varepsilon(a) \neq 0$ we have $\tilde{\varepsilon} \in \mathbb{F}[t] \setminus 0$. Then there's $\alpha \in \mathbb{F}$ such that $\tilde{\varepsilon}(h)(\alpha) \neq 0$. Define $\varepsilon' = \mathrm{ev}_\alpha \circ \tilde{\varepsilon}$. Then $\varepsilon'(b) = \varepsilon'(h) \neq 0$.

- Take $f = \sum_{i=0}^{m} f_i t^i \in I$ of minimal degree. B is an integral domain, so I is prime so f is irreducible. Since $b \neq 0$ we have $h \notin I$. Hence $f \nmid h$. By working over the field of fractions and then multiplying by a common denominator, there are $u, v \in A[t]$ such that $uf + vh = a'$ for some $a' \in A$. Here we have $vh \equiv a' \pmod{I}$. Take $a \coloneqq f_M \cdot a' \in A \setminus \{0\}$. If $\varepsilon \colon A \to \mathbb{F}$ is homomorphism such that $\varepsilon(a) \neq 0$, we have $\varepsilon(a'), \varepsilon(f_m) \neq 0$. From the lemma it follows that there's $\varepsilon' \colon B \to \mathbb{F}$ such that $\varepsilon'|_A = \varepsilon$. Now,

$$0 \neq \varepsilon(a') = \tilde{\varepsilon}(v)\,\tilde{\varepsilon}(h) = \tilde{\varepsilon}(v) \cdot \varepsilon'(b)$$

  where $\tilde{\varepsilon} = \varepsilon' \circ \pi$. Hence $\varepsilon(b) \neq 0$. ■

Proof (1.2.12).

- If X is irreducible, so is Y and we get $\varphi^* \colon \mathbb{F}[X] \hookrightarrow \mathbb{F}[X]$. X is irreducible, hence $\mathbb{F}[X]$ is an integral domain. From 1.2.16 it follows that there's $f \in \mathbb{F}[Y] \setminus \{0\}$ such that for $\varepsilon \colon \mathbb{F}[Y] \to \mathbb{F}$ satisfying $\varepsilon(f) \neq 0$ there is a lift $\varepsilon' \colon \mathbb{F}[x] \to \mathbb{F}$.

  For $y \in Y$ with $\mathrm{ev}_y(f) = f(y) \neq 0$ we can find $\varepsilon' \colon \mathbb{F}[X] \to \mathbb{F}$ lifting ev_y. But, $\varepsilon' = \mathrm{ev}_x$ for some $x \in X$. We get that $\mathrm{ev}_y = \mathrm{ev}_x \circ \varphi^*$, which is equivalent to $\varphi(x) = y$. Then $V_f \subseteq \varphi(X)$.

- Assume $X = \bigcup_{i \in [s]} X_i$ is a decomposition of X to irreducible components. It follows from the irreducible case that there are open sets $U_1, \dots, U_s \subseteq Y$ such that $\overline{\varphi(X_i)} \cap U_i \subseteq \varphi(X_i)$ for all $i \in [s]$. Write $U \coloneqq \bigcap_{i \in [s]} U_i'$ for $U_i' = U_i \cup \left(Y \setminus \overline{\varphi(X_i)}\right)$. For every $i \in [s]$ we get

$$\overline{\varphi(X_i)} \cap U \subseteq \overline{\varphi(X_i)} \cap U_i' \subseteq \varphi(X_i).$$

  Also, $U \subseteq \bigcup_{i \in [s]} \varphi(X_i) = \varphi(X)$. Check that $U \neq \varnothing$ by using the irreducibility of one of the X_i. ■

Back to Algebraic Groups

Proposition 1.2.17. *Let G be an $\mathbb{F}$-algebraic group and let $H \leq G$ be an abstract subgroup. Then*

1. *$\bar{H} \leq G$ is a subgroup.*
2. *If H contains a non-empty open subset of $\bar{H}$ then $H = \bar{H}$.*

Proof.

1. For $x \in H$ we have

$$H = x^{-1} H \subseteq x^{-1} \bar{H}.$$

   Hence $xH \subseteq \bar{H}$ for all x. Since $g \mapsto x^{-1} g$ is a homeomorphism we get $x\bar{H} \subseteq \bar{H}$. Hence $H \cdot \bar{H} \subseteq \bar{H}$. For $y \in \bar{H}$ we have $Hy \subseteq \bar{H}$ s $H \subseteq \bar{H} y^{-1}$. Hence $\bar{H} \subseteq \bar{H} y^{-1}$ hence $\bar{H} y \subseteq \bar{H}$, so $\bar{H} \cdot \bar{H} \subseteq \bar{H}$.

   Since $g \mapsto g^{-1}$ is a homeomorphism we also get $\bar{H}^{-1} = \overline{H^{-1}} = \bar{H}$.

2. Assume $U \subseteq H$ is open in $\bar{H}$. Then $H = \bigcup_{x \in H} xU$ is open in $\bar{H}$ as a union of open sets. For every $y \in \bar{H}$ we get $yH \cap H$ is an intersection of dense open subsets of $\bar{H}$, which is therefore non-empty. Hence there are $x_1, x_@ \in H$ such that $yx_1 = x_2$. Then $y = x_2 x_1^{-1} \in H$. ■

Proposition 1.2.18. *If $\varphi \colon G \to H$ is a regular homomorphism of algebraic groups, $\varphi(G)$ is a closed subgroup of H.*

Proof. $\varphi(G) \leq H$ is a subgroup, and from 1.2.12, $\varphi(G)$ contains an open subset of $\overline{\varphi(G)}$. From the previous proposition we get that $\varphi(G)$ is closed. ■

1.2.3 Jordan Decomposition

Let $\mathbb{F}$ be an algebraically-closed field. It follows from Jordan's theorem that for $A \in M_n(\mathbb{F})$ one can write

$$A = P^{-1}DP + PNP$$

for D diagonal and N nilpotent. Denote $A_s := P^{-1}DP$ which is then semi-simple, and $A_n := P^{-1}NP$ which is nilpotent. We have $A_sA_n = A_nA_s$. We call $A = A_n + A_s$ the *Jordan-Chevalley decomposition* for A.

Definition 1.2.19. We call $g \in \mathrm{GL}(V)$ *unipotent* if $g - I$ is nilpotent.

Remark 1.2.20. g is unipotent iff its only eigenvalue is 1.

For $g \in \mathrm{GL}(V)$, write $g = g_s + g_n$ for its Jordan-Chevalley decomposition. Let $g_u := I + g_s^{-1}g_n$. We have

$$g = g_sg_u = g_ug_s.$$

We want to show that for an algebraic group $G \leq \mathrm{GL}(V)$ and $g \in G$ we have $g_sg_u \in G$ and that g_s, g_u are independent of the embedding of G.

Remark 1.2.21. In fact, if $\mathbb{F}$ is a perfect field, one can consider $A \subseteq M_n(\mathbb{F}) \subseteq M_n\left(\overline{\mathbb{F}}\right)$. One gets that $A_s, A_n \in M_n(\mathbb{F})$ which requires a proof. The rest of our statements in this section will work for perfect fields.

Proposition 1.2.22. *Let $A \in M_n(\mathbb{F})$. There is $p \in \mathbb{F}[t]$ for which $A_s = p(A)$ and $A_n = (1-p)(A)$.*

Proof. Write $p_A(t) = \prod_{i\in[k]}(t-\lambda_i)^{r_a(\lambda_i)}$ for the characteristic polynomial of A where $(\lambda_i)_{i\in[k]}$ are the different eigenvalues of A. It suffices to find a polynomial $p \in \mathbb{F}[t]$ such that for every $i \in [k]$ there's $q_i \in \mathbb{F}[t]$ for which

$$p(t) = q_i(t)(t-\lambda_i)^{r_a(\lambda_i)} + \lambda_i.$$

This is equivalent to

$$L_{p(A)}\big|_{\ker((L_A - \lambda_i \operatorname{Id})^n)} = \lambda_i I = L_{A_s}\big|_{\ker((L_A-\lambda_i \operatorname{Id})^n)}$$

where

$$\begin{aligned} L_A \colon \mathbb{F}^n &\to \mathbb{F}^n \\ v &\mapsto Av. \end{aligned}$$

Since $(t-\lambda_i)^{r_a(\lambda_i)}$ are coprime, it follows from the Chinese Remainder Theorem that such p exists. ∎

Proposition 1.2.23. *Let $a \in \mathrm{End}(V)$. There are polynomials $p_s, p_u \in \mathbb{F}[x]$ such that $a_s = p_s(a)$ and $a_n = p_n(a)$.*

Corollary 1.2.24. *Let $a \in \mathrm{End}(V)$ (or $g \in \mathrm{GL}(V)$). The decomposition $a = a_s + a_n$ with a_s semi-simple and a_n nilpotent such that $a_sa_n = a_na_s$ (or $g = g_sg_u$ such that g_s is semi-simple and g_u is unipotent such that $g_sg_u = g_ug_s$), is unique.*

Proof. Write $a = a_s + a_n = b_s + b_n$ where a_s, b_s are semi-simple, a_n, b_n are nilpotent and each of the pairs a_s, a_n and b_s, b_n commutes. Write $a_s = p_s(a)$ for a polynomial $p_s \in \mathbb{F}[x]$. Then

$$b_sa_s = a_sb_s$$

since b_s commutes with a and $a_s = p_s(a)$. Similarly, $b_Sa_n = a_nb_s$ and $b_na_n = a_nb_n$. Then $a_s - b_s = b_-a_n$ is semi-simple and nilpotent (a sum of commuting semi-simple/nilpotent matrices is semi-simple/nilpotent), hence zero. ∎

Lemma 2. *Let $a \in \mathrm{End}(V)$, $b \in \mathrm{End}(W)$, and $\varphi \in \mathrm{Hom}(V, W)$ such that $b \circ \varphi = \varphi \circ a$. Then $b_s \circ \varphi = \varphi \circ a_s$ and $b_n \circ \varphi = \varphi \circ a_n$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} V \arrow[rr, "\varphi"] \arrow[dr, "\iota"'] & & W \\ & V \oplus W \arrow[ur, "p"'] & \end{tikzcd}$$

with $i(v) = (v, \varphi(v))$ and $p(v, w) = w$. By considering $a \oplus b \in \operatorname{End}(V \oplus W)$ we see that we may assume φ is either injective or surjective.
In other word, for $T \in \operatorname{End}(V)$ and $Z \leq V$ which is T-invariant, we want to show that $prsT|_{Zs} = (T_s)|_Z$ (for when φ is injective) and if $\tilde{T} \in \operatorname{End}(V/Z)$ is the induced map on the quotient, we have to show $\tilde{T}_s = \widetilde{(T_s)}$.
Now, $(T_s)|_Z$ is semi-simple as a restriction of such, and $T_n|_Z$ is nilpotent, and these commute. By uniqueness, we get the case for the restriction. Similarly one gets the case for quotients. ■

Jordan-Chevalley Decomposition and Groups

Let $G \leq \mathrm{GL}(V)$ be a closed subgroup. Choose $v \in V$ and $\varphi \in V^*$, and write $f_{v,\varphi} \in \mathbb{F}[G]$ for $f_{v,\varphi}(g) = \varphi(g \cdot v)$. This is an analogue for a representing matrix of a linear transformation given two bases.
For every $\varphi \in V^*$ let

$$\begin{aligned} r^{\varphi} \colon V &\to \mathbb{F}[G] \\ v &\mapsto f_{v,\varphi}. \end{aligned}$$

This commutes with the group action.
Consider $G \curvearrowright G$ with the right-action $g * h = hg^{-1}$. This gives a linear action $G \curvearrowright \mathbb{F}[G]$ given by

$$\rho(g_0)(f)(g) = f(gg_0)$$

for $g_0 \in G$ and $f \in \mathbb{F}[G]$. We call this right-translation. For every $g \in G$ we get

$$\rho(g) \circ r^{\varphi} = r^{\varphi} \circ g.$$

We would like to say the following things.

1. That
$$\rho(g)_s \circ r^{\varphi} = r^{\varphi} \circ g_s,$$
but $\rho(g)$ is an operator on an infinite-dimensional space.
2. That $\rho(g)_s = \rho(\hat{g})$ for some $\hat{g} \in G$.
3. That $g_s = \hat{g} \in G$.

Notation 1.2.25. For a group G denote by λ, ρ its action on itself from the left and right respectively. Denote the actions of an element g by λ_g, ρ_g.

Remark 1.2.26. For an algebraic group G we get maps

$$\begin{aligned} &\lambda(g) : \mathbb{F}[G] \to \mathbb{F}[G] \\ &\lambda(g)(f)(h) = f\left(\lambda_{g^{-1}}(h)\right) = f\left(g^{-1}h\right) \end{aligned}$$

and

$$\begin{aligned} &\rho(g) : \mathbb{F}[G] \to \mathbb{F}[G] \\ &\rho(g)(f)(h) = f\left(\rho_{g^{-1}}(h)\right) = f(hg). \end{aligned}$$

We would like to write each $\rho(g)$ as $\rho(g)_s + \rho(g)_n$ for semi-simple and nilpotent parts, but we don't have such a decomposition yet since $\mathbb{F}[G]$ is infinite-dimension.

Definition 1.2.27. Let V be a vector space and let $a \in \operatorname{End}(V)$ such that for every $v \in V$ there's $W_v \leq V$ finite-dimensional and a-invariant such that $v \in W_v$. Define

$$\begin{aligned} a_s \colon V &\to V \\ v &\mapsto \left(a|_W\right)_s(v) \end{aligned}$$

for W_v as above.

Exercise 14. The above definition is well-defined. This follows from the fact that restriction and taking the semi-simple part commute.

Theorem 1.2.28. *Let $\mathbb{F}$ be algebraically-closed and let $G \leq \mathrm{GL}(V)$ be an embedded $\mathbb{F}$-algebraic group. For every $g \in G$ it holds that $g_s, g_u \in G$.*

Exercise 15. It holds that $(a \otimes b)_s = a_s \otimes b_s$.

Proof (1.2.28). Consider the map

$$\begin{aligned} m \colon \mathbb{F}[G] \otimes \mathbb{F}[G] &\to \mathbb{F}[G] \\ f_1 \otimes f_2 &\mapsto f_1 \cdot f_2. \end{aligned}$$

Since $\rho(g)$ is a homomorphism for every $g \in G$, it holds that

$$m(\rho(g) \otimes \rho(g)) = \rho(g) \circ m.$$

Hence

$$m\left(\rho(g)_s \otimes \rho(g)_s\right) = \rho(g)_s \circ m$$

since this is true locally (on finite-dimensional subspaces). Hence $\rho(g)_s : \mathbb{F}[G] \to \mathbb{F}[G]$ is a homomorphism of $\mathbb{F}$-algebras.
We get a regular map $\psi \colon G \to G$ such that $\psi^* = \rho(g)_s$. Let $\hat{g} := \psi(e) \in G$. For $h \in G$ we have

$$\rho(g) \circ \rho(h) = \lambda(h) \circ \rho(g)$$

so

$$\rho(g)_s \circ \rho(h) = \lambda(h) \circ \rho(g)_s.$$

Hence $\lambda(h) = \lambda^*_{h^{-1}}$. Then

$$\psi^* \circ \lambda^*_h = \lambda^*_h \circ \psi^*$$

so

$$\psi \circ \lambda_h = \lambda_h \circ \psi$$

and then

$$\psi(h) = \psi(\lambda_h(e)) = \lambda_h(\psi(e)) = h\hat{g}.$$

Hence $\psi = \rho_{\hat{g}^{-1}}$ so $\rho(g)_s = \rho(\hat{g})$.
For $\varphi \in V^*$ remind that

$$\begin{aligned} r^\varphi \colon V &\to \mathbb{F}[G] \\ \varphi &\mapsto f_{v,\varphi} \end{aligned}$$

and $f_{v,\varphi}(g) = \varphi(g \cdot v)$. It holds that $\rho(g) \circ r^\varphi = r^\varphi \circ g$ for all $g \in G$. Then

$$r^\varphi \circ g_s = \rho(g)_s \circ r^\varphi = \rho(\hat{g}) \circ r^\varphi = r^\varphi \circ \hat{g}$$

for every $\varphi \in V^*$. Hence $g_s(v) = \hat{g}(v)$ for all $v \in V$, so $g_s = \hat{g} \in G$. Then also $g_u = g g_s^{-1} \in G$. ∎

Theorem 1.2.29. *Let $\mathbb{F}$ be an algebraically-closed field.*

1. *Let G be an $\mathbb{F}$-algebraic group. For every $g \in G$, the elements $g_s, g_u \in G$ are independent of the embedding of G.*

2. *For a homomorphism $\varphi \colon G_1 \to G_2$ of $\mathbb{F}$-algebraic groups it holds that*

$$\begin{aligned} \varphi(g_s) &= \varphi(g)_s \\ \varphi(g_u) &= \varphi(g)_u \end{aligned}$$

for all $g \in G$.

Proof. 1. In the previous proof, we saw that for any embedding, g_s is determined by $\rho(g)_s$.

2. For $g, h \in G$ it holds that
$$\varphi(hg) = \varphi(h)\varphi(g)$$
so
$$\rho_{G_1}(g) \circ \varphi^* = \varphi^* \circ \rho_{G_2}(\varphi(g)).$$
Indeed, for $f \in \mathbb{F}[G_2]$ we have
$$\rho_{G_1}(g)\varphi^*(f)(g') = f(\varphi(g'g))$$
and
$$\varphi^*(\rho_{G_2}(\varphi(g))(f)(g')) = f(\varphi(g')\varphi(g))$$
so these are equal. We get
$$\rho_{G_1}(g_s) \circ \varphi^* = \varphi^* \circ \rho_{G_2}(\varphi(g)_s).$$
Then
$$\begin{aligned} \mathrm{ev}_{g_s} \circ \varphi^* &= \mathrm{ev}_{e_{G_2}} \circ \rho_{G_2}(g_s) \circ \varphi^* \\ &= \mathrm{ev}_{e_{G_2}} \circ \varphi^* \circ \rho_{G_2}(\varphi(g)_s) \\ &= \mathrm{ev}_{e_{G_1}} \circ \rho(\varphi(g)_s) \\ &= \mathrm{ev}_{\varphi(g)_s} \end{aligned}$$
so $\rho(g_s) = \rho(g)_s$. ■

Definition 1.2.30. Let G be an $\mathbb{F}$-algebraic group. Call an element $g \in G$ *semi-simple* if $g = g_s$ and *unipotent* if $g = g_u$.

1.2.4 Unipotent and Reductive Groups

Definition 1.2.31 (Unipotent Group). An $\mathbb{F}$-algebraic group G is called *unipotent* if every $g \in G$ is unipotent.

Notation 1.2.32. Denote $U_n(\mathbb{F}) \leq M_n(\mathbb{F})$ for the group of upper-triangular matrices with 1 on the diagonal.

Some Representation Theory

Every closed subgroup of $U_n(\mathbb{F})$ is unipotent. We want to show these are all the unipotent subgroups.

Definition 1.2.33 (Representation). Let G be an $\mathbb{F}$-algebraic group. A *representation* of G is an $\mathbb{F}$-vector space V together with a regular homomorphism $\varphi\colon G \to \mathrm{GL}(V)$.

Definition 1.2.34 (Irreducible Representation). A representation (φ, V) of G is *irreducible* if there's no non-trivial $W \leq V$ which is $\varphi(G)$-invariant.

Definition 1.2.35 (Subrepresentation). If $W \leq V$ is $\varphi(G)$-invariant, the restrictions $\varphi|_W(g) := \varphi(g)|_W \in \mathrm{GL}(W)$ form a representation $(\varphi|_W, W)$ which we call a *subrepresentation* of (φ, V).

Definition 1.2.36 (Quotient Representation). If $W \leq V$ is $\varphi(G)$-invariant, the quotient maps $\overline{\varphi(g)} \in \mathrm{GL}(V/W)$ form a representation $(\bar{\varphi}, V/W)$ called the *quotient representation.*

Lemma 3 (Schur). *Let $\mathbb{F}$ be algebraically-closed and let V be a finite-dimensional vector space over $\mathbb{F}$. Let $S \subseteq \mathrm{End}(V)$ be such that there's no non-trivial $W \leq V$ which is S-invariant (usually one takes $S = \varphi(G)$). If $T \in \mathrm{End}(V)$ is such that $T \circ a = a \circ T$ for all $a \in S$, then $T = \lambda\,\mathrm{Id}_V$ for some $\lambda \in \mathbb{F}$.*

Proof. Since $\mathbb{F}$ is algebraically-closed, T has an eigenvalue $\lambda \in \mathbb{F}$. For every $a \in S$ and $v \in \ker(T - \lambda \operatorname{Id}_V)$ it holds that

$$\begin{aligned} \mathrm{T}(a(v)) &= a(T(v)) \\ &= a(\lambda v) \\ &= \lambda a(v). \end{aligned}$$

Since $a(v) \in \ker(T - \lambda \operatorname{Id}_V)$ we get that $\ker(T - \lambda I)$ is S-invariant. From the assumption, it follows that $\ker(T - \lambda \operatorname{Id}_V) = V$ so $T = \lambda \operatorname{Id}_V$. ∎

Theorem 1.2.37 (Burnside, Wedderburn; Density Theorem). *Let $\mathbb{F}$ be an algebraically-closed field, let V be a finite-dimensional $\mathbb{F}$-vector space and let $S \subseteq \operatorname{End}(V)$ such that there's no non-trivial S-invariant $W \leq V$. Then*

$$\operatorname{Span}_{\mathbb{F}}(S) = \operatorname{End}(V).$$

Notation 1.2.38. For $A \in M_{r,n}(\mathbb{F})$ let

$$\begin{aligned} \tau_A \colon V^{\oplus n} &\to V^{\oplus n} \\ (v_1, \dots, v_r) &\mapsto (v_1, \dots, v_r) \cdot A. \end{aligned}$$

(If we identify $V^{\oplus n} \cong \mathbb{F}^n \otimes_{\mathbb{F}} V$ we get $\tau_A = L_A \otimes \operatorname{Id}_V$.) For every $a \in \operatorname{End}(V)$ one gets

$$\tau_A \circ \varphi_r(a) = \varphi_n(a) \circ \tau_A.$$

Denote also

$$\begin{aligned} \varphi_n \colon \operatorname{End}(V) &\to \operatorname{End}\left(V^{\oplus n}\right) \\ a &\mapsto a^{\oplus n}. \end{aligned}$$

Lemma 4. *Let $\mathbb{F}$ be an algebraically-closed field, let V be a finite-dimensional $\mathbb{F}$-vector space and let $S \subseteq \operatorname{End}(V)$ such that there's no non-trivial S-invariant $W \leq V$. Let $W \lneq V^{\oplus n}$ which is $\varphi_n(S)$-invariant. Then there's $A \in M_{n,r}(\mathbb{F})$ for $r < n$ and such that $W = \operatorname{Im}(\tau_A)$.*

Proof.

- Assume first that $W \neq \{0\}$ is irreducible (i.e. has no non-trivial $\varphi_n(S)$-invariant subspace). Let $p_i \colon V^{\oplus n} \to V$ be the projection on the i^{th} component. Since W is non-zero, there's $i_0 \in [n]$ such that $p_{i_0}|_W \neq 0$. Then $\psi := p_{i_0}|_W$ is an isomorphism since its domain and range are irreducible (one gets $\ker(\psi) = \{0\}$ and $\operatorname{Im}(\psi) = V$).

  Let $t_i := p_i \circ \psi^{-1} \in \operatorname{End}(V)$. This commutes with every $a \in S$. By Schur's lemma, $t_i(v) = \lambda_i v$ for some $\lambda_i \in \mathbb{F}$. Now

  $$\begin{aligned} W &= \{(\lambda_1 v, \dots, \lambda_n v) \mid v \in V\} \\ &= \operatorname{Im}\left(\tau_{\left(\lambda_1, \dots, \lambda_n\right)}\right). \end{aligned}$$

- Now let $\{0\} \neq W \subseteq V^{\oplus n}$ be general. We use induction on n. There's $\{0\} \neq W_0 \subseteq W$ irreducible and $\varphi_n(S)$-invariant. By the previous case, we can write $W_0 = \operatorname{Im} \tau_{(q_1, \dots, q_n)}$. There's $g \in \operatorname{GL}_n(\mathbb{F})$ such that

  $$(q_1, \dots, q_n) \cdot g = (1, 0, \dots, 0).$$

  Then

  $$\begin{aligned} \tau_g(W) = \operatorname{Im} \tau_{(1,0,\dots,0)} &= \{(v, 0, \dots, 0) \mid v \in V\} \\ &= \tau_g(W_0) \oplus (\tau_g(W) \ker p_1) \end{aligned}$$

  where p_1 is the projection the the first component. Denote $W' := (\tau_g(W) \ker p_1)$. By induction we can write $W' = \operatorname{Im} \tau_{A'}$ for $A \in M_{r',n-1}(\mathbb{F})$ with some $r' < n-1$. Then $\tau_g(W) = \operatorname{Im} \tau_{A''}$ for $A'' := \begin{pmatrix} 1 & 0 \\ 0 & A' \end{pmatrix}$. Then $W = \operatorname{Im}\left(\tau_{A'' g^{-1}}\right) \in M_{r'+1,n}(\mathbb{F})$. ∎

Proof (1.2.37). We assume that S is multiplicatively closed, which might be necessary.
Let $(e_1, \dots, e_n)$ be a basis for V and let $\underline{e} = (e_1, \dots, e_n) \in V^{\oplus n}$. Let $W \coloneqq \operatorname{Span}\{\varphi_n(S)(\underline{e})\} \leq V^{\oplus n}$ which is $\varphi_n(S)$-invariant. If $W \lneq V^{\oplus n}$, it follows from the lemma that there's $A \in M_{r,n}(\mathbb{F})$ with $r < n$ and $W = \operatorname{Im}(\tau_A)$. In particular, in that case $\underline{e} \in \operatorname{Im}(\tau_A)$, so $\dim(\operatorname{Span}\{e_1, \dots, e_n\}) \leq r$, a contradiction. Hence $W = V^{\oplus n}$.
Take $T \in \operatorname{End}(V)$. Then $(T(e_1), \dots, T(e_n)) \in W$, so there are $(s_i)_{i\in[k]} \subseteq S$ and $(\alpha_i)_{i\in[k]} \subseteq \mathbb{F}$ such that $U \coloneqq \sum_{i\in[k]} \alpha_i s_i$ gives $T(e_i) = U(e_i)$ for all $i \in [k]$, so $T = U \in \operatorname{Span}(S)$. ■

Theorem 1.2.39. *Let G be a unipotent group over an algebraically-closed field $\mathbb{F}$. Let $\varphi\colon G \to \mathrm{GL}(V)$ be an irreducible rational representation. Then ϕ is the trivial representation.*

Proof. From properties we've shown, elements of $\tilde{G} \coloneqq \varphi(G)$ are unipotent transformations. For $g \in \tilde{G}$ we have $\operatorname{tr}(g) = n$ and for every $g, h \in \tilde{G}$ we have

$$\operatorname{tr}((\mathrm{Id} - g)h) = \operatorname{tr}(h) - \operatorname{tr}(gh) = n - n = 0.$$

By 1.2.37 we get that

$$\operatorname{tr}((\mathrm{Id} - g)A) = 0$$

for all $A \in \operatorname{End}(V)$.
Now,

$$\langle Y, X\rangle \coloneqq \operatorname{tr}\left(Y^t X\right)$$

pulls back to a non-degenerate form on $\operatorname{End}(V)$, so we get that $\mathrm{Id} - g = 0$ for all $g \in \tilde{G}$. Then $\tilde{G} = \{I\}$ and by irreducibility of V we get $\dim(V) = 1$. ■

Corollary 1.2.40. *Every unipotent group G over an algebraically-closed field $\mathbb{F}$ is isomorphic to a closed subgroup of* $\mathrm{U}_n \coloneqq \left\{\begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right\}$.
In other words, for any rational representation $\varphi\colon G \to \mathrm{GL}(V)$ there's a basis B of V such that $\{[T]_B \mid T \in \varphi(G)\} \subseteq U_n$.

Proof. Let $\varphi\colon G \to \mathrm{GL}(V)$ be a rational representation. There's a minimal irreducible subrepresentation $\{0\} \neq W \leq V$ and by the theorem W is 1-dimensional. Write $W = \operatorname{Span}\{e_1\}$ for some e_1. Then $\varphi(G)e_1 = e_1$, and we look at the $\varphi(G)$-action on V/W. By induction, there's a basis $f_2, \dots, f_n$ of the quotient for which the action of $\varphi(G)$ is represented by matrices in U_{n-1}. Lift $f_2, \dots, f_n$ to elements $e_2, \dots, e_n \in V$. We get a basis $(e_1, \dots, e_n)$ of V for which elements of $\varphi(G)$ are of the given form. ■

Proposition 1.2.41. *Let G be a unipotent group acting algebraically on an affine $\mathbb{F}$-variety X. Then every G-orbit is closed.*

Proof. Let $x_0 \in X$ and

$$\begin{aligned} \varphi\colon G &\to X \\ g &\mapsto g \cdot x_0. \end{aligned}$$

Let $Y \coloneqq \overline{\varphi(G)} \leq X$, we want to show $Y = \varphi(G)$.
There's $U \leq Y$ open and non-empty such that $U \subseteq \varphi(G)$. It holds that $\varphi(G) = \bigcup_{g\in G} g \cdot U$ so $\varphi(G) \leq Y$ is open. Indeed, taking $u_0 \in U$ there's $g_0 \in G$ such that $u_0 = g_0 \cdot x_0$, so for all $g \in G$ we have

$$g \cdot x_0 = g g_0^{-1} u_0 \in g g_0^{-1} U.$$

Now, $Z \coloneqq Y \setminus \varphi(G)$ is closed, and assume towards a contradiction that it's non-empty. We get $(0) \neq I(Z) \trianglelefteq \mathbb{F}[Y]$. Y is G-invariant as the closure of an orbit. (Check this. It's similar to showing that $H \leq G$ implies $\bar{H} \leq G$.) We have a linear action $G \curvearrowright \mathbb{F}[Y]$ via $g \cdot f(x) = f\left(g^{-1}x\right)$ and Z is G-invariant, so $I(Z)$ is G-invariant. There's a finite-dimensional G-invariant $V \leq I(Z)$. From the theorem on unipotent groups, there's $f \in I(Z)$ non-zero such that $G \cdot f = f$. Then $f \in \mathbb{F}[Y]$ is constant on the G-orbits. $\varphi(G)$ is dense in Y so $f|_{\varphi(G)} \equiv 0$ (since $f|_Z \equiv 0$), so $f = 0$, a contradiction. ■

Remark 1.2.42. The above isn't true without the unipotent requirement. For example, consider $G := \mathbb{F}^\times$ acting on $\mathbb{F}$ via multiplication. The orbits are $\{0\}$ and $\mathbb{F}^\times$, where the latter isn't closed unless $\mathbb{F}$ is finite.

Example 21. Consider the action of $G := \mathrm{GL}_2(\mathbb{F})$ on $M_2(\mathbb{F})$ by conjugation. We have

$$\left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \middle| a \in \mathbb{F}^\times \right\} \subseteq G \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

but $0 \notin G \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and this is clearly in the closure.

Example 22. Examine

$$(\mathbb{F}, +) \cong U_2 := \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{F} \right\}$$

with the usual action $U_2 \curvearrowright \mathbb{F}^2$. For $a \in \mathbb{F} \setminus \mathbb{F}$ let

$$Y_a := \left\{ \begin{pmatrix} x \\ a \end{pmatrix} \middle| x \in \mathbb{F} \right\}.$$

These are closed orbits, but the the rest of the orbits are points $\left\{ \begin{pmatrix} b \\ 0 \end{pmatrix} \right\}$ with $b \in \mathbb{F}$. These orbits are all closed.

1.3 Tangent Spaces

1.3.1 Definitions

Definition 1.3.1 (Tangent Space). Let X be an embedded affine $\mathbb{F}$-variety and let $p \in X$. Define *the tangent space of X at p* to be

$$T_pX := \left\{ v \in \mathbb{F}^n \middle| \frac{\partial f_i}{\partial v}(p) = 0 \right\}$$

where

$$\frac{\partial f}{\partial v}(p) = \sum_{i \in [n]} \frac{\mathrm{d}f}{\mathrm{d}x_i} c\dot{v}_i.$$

Notation 1.3.2. Denote $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n} \right)$.

Example 23. Let $X = V\left(y - x^2, z - x^3\right) \subseteq \mathbb{F}^3$ where $f_1 := y - x_2$ and $f_2 := z - x^3$. Then $X = \left\{ \left(x, x^2, x^3\right) \mid x \in \mathbb{F} \right\}$ and

$$\begin{aligned} \nabla f_1 &= (-2x, 1, 0) \\ \nabla f_2 &= \left(-3x^2, 0, 1\right). \end{aligned}$$

So,

$$T_pX = \left\{ (v_1, v_2, v_3) \middle| \begin{array}{r} -2xv_1 + v_2 = 0 \\ -3x^2 v_1 + v_3 = 0 \end{array} \right\}.$$

Example 24. Let $X := V(xy) \subseteq \mathbb{F}^2$ and $f = xy$ so that $\nabla f = (y, x)$. For $p = (p_1, p_2) \in X$ we have

$$T_pX = \begin{cases} \mathrm{Span}\{(0,1)\} & p_1 = 0 \wedge p_2 \neq 0 \\ \mathrm{Span}\{(1,0)\} & p_1 \neq 0 \wedge p_2 = 0 \\ \mathbb{F}^2 & p = (0,0) \end{cases}.$$

We later call a point such as p where the tangent space is of *different dimension*, a *singular point.*

Example 25. Given $X \coloneqq V\left(y^2 - x^3\right)$ we have $T_{(0,0)}X = \mathbb{F}^2$.

Examine the Jacobian

$$J\left(f_1, \ldots, f_k\right) \coloneqq \begin{pmatrix} \nabla f_1 \\ \vdots \\ \nabla f_k \end{pmatrix} \in M_{k,n}\left(A_n\right)$$

where $A_n \coloneqq \mathbb{F}\left[x_1, \ldots, x_n\right]$. At every point p we can evaluate

$$J\left(f_1, \ldots, f_k\right)(p) \in M_n\left(\mathbb{F}\right)$$

and

$$T_pX = \operatorname{null}\left(J\left(f_1, \ldots, f_k\right)\right) \leq \mathbb{F}^n.$$

Assume now that X is irreducible, so that $\mathbb{F}\left[X\right]$ is an integral domain. Let K be the fraction field of $\mathbb{F}\left[X\right]$, and consider $\Omega \coloneqq \overline{J\left(f_1, \ldots, f_k\right)} \in M_n\left(K\right)$ where $\mathbb{F}\left[X\right] \cong \mathbb{F}\left[x_1, \ldots, x_n\right] / \left(f_1, \ldots, f_k\right)$ and $f_i \in \mathbb{F}\left[x_1, \ldots, x_n\right]$. Let $d \coloneqq \dim\left(\operatorname{null}\left(\Omega\right)\right)$. Note that Ω is supposedly dependent on the choice of generators, but it is in fact not.

Proposition 1.3.3. *Let X be an irreducible $\mathbb{F}$-variety and let Ω and d be as above. For every $p \in X$ it holds that $\dim T_pX \geq d$ and for every non-empty $U \subseteq X$ with $p \in U$ it holds that $\dim T_pX = d$.*

Proof. We have $\operatorname{rank}\left(\Omega\right) = n - d$. Examine the matrix's minors of size $n - d$ which we denote $g_1, \ldots, g_t \in \mathbb{F}\left[X\right]$. Now $U \coloneqq X \setminus V\left(g_1, \ldots, g_t\right) \neq \varnothing$ since the g_i aren't all zero. For $p \in U$ there's $i \in [t]$ for which $g_i\left(p\right) \neq 0$, hence

$$\operatorname{rank}\left(\Omega p\right) \geq n - d$$

so

$$\operatorname{rank}\left(\Omega p\right) \leq \operatorname{rank}\left(\Omega\right) = n - d$$

and for $p \in X$ we have $\dim T_pX \geq d$. ∎

1.3.2 Definitions Independent of the Embedding

Definition 1.3.4 (Module Over an Algebra). Let A be an $\mathbb{F}$-algebra. An A-module M is a vector space over $\mathbb{F}$ together with an $\mathbb{F}$-algebra homomorphism

$$\varphi\colon A \to \operatorname{End}_{\mathbb{F}}\left(M\right).$$

Notation 1.3.5. We usually write $a \cdot m$ instead of $\varphi\left(a\right)\left(m\right)$, and don't write φ.

Example 26. An ideal $I \trianglelefteq A$ is an A-module.

Example 27. A/I is an A-module.

For X an affine $\mathbb{F}$-variety and $p \in X$, examine $\mathfrak{m}_p \coloneqq \mathbb{F}\left[X\right] / \ker\left(\operatorname{ev}_p\right) \cong \mathbb{F}$. This is an $\mathbb{F}\left[X\right]$-module where $f \in \mathbb{F}\left[X\right]$ acts on $t \in \mathbb{F}$ via $f \cdot t = f\left(p\right) \cdot t$ under the above isomorphism.

Definition 1.3.6 (Derivation Space). Let M be an A-module. We define the *derivation space*

$$\operatorname{Der}_{\mathbb{F}}\left(A, M\right) \coloneqq \left\{D \in \operatorname{Hom}_{\mathbb{F}}\left(A, M\right) \mid \forall a, b \in A\colon D\left(ab\right) = a \cdot D\left(b\right) + b \cdot D\left(a\right)\right\}.$$

Example 28. Let $v \in \mathbb{F}^n$. We have $\frac{\partial}{\partial v} \in \operatorname{Der}_{\mathbb{F}}\left(A_n, A_n\right)$ since (regular) derivation satisfies the Leibniz rule.

Definition 1.3.7. Let A be a finitely-generated $\mathbb{F}$-algebra with generators $x_1, \ldots, x_n \in A$. Write $A \cong A_n/I$ and $I = \left(f_1, \ldots, f_k\right)$. Let $\Omega \coloneqq J\left(f_1, \ldots, f_k\right) \in M_{k,n}\left(A\right)$. Taking an A-module M, we get that Ω acts on the left of M^n via matrix multiplication. We define

$$\mathcal{I}_{A,M} = \left\{v \in M^n \mid \Omega v = 0\right\}.$$

Notation 1.3.8. For $M \in \mathbb{F}\left[X\right] - \mathbf{Mod}$, denote

$$\mathcal{I}_{\mathbb{F}[X],M} \coloneqq \left\{v \in M^n \mid \Omega v = 0\right\} \leq M^n.$$

Denote also $\mathbb{F}_p \coloneqq \mathbb{F}\left[X\right] / \left(p\right)$.

Remark 1.3.9. It holds that
$$T_pX = \mathcal{I}_{\mathbb{F}[X],\mathbb{F}_p}$$
and we've seen that
$$\dim_k \mathcal{I}_{\mathbb{F}[X],K} \leq \dim_{\mathbb{F}} T_pX$$
where there's equality on each p in an open subset of X.

Definition 1.3.10 (Derivation Space). For $M \in A-\mathbf{Mod}$ let
$$\mathrm{Der}_{\mathbb{F}}(A, M) := \{D \in \mathrm{Hom}_{\mathbb{F}}(A, M) \mid \forall a, b \in A \colon D(ab) = a \cdot D(b) + b \cdot D(A)\}.$$

Definition 1.3.11. For $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \in \mathcal{I}_{A,M}$ consider the quotient map $\pi \colon \mathbb{F}[x_1, \ldots, x_n] \to \mathbb{F}[x_1, \ldots, x_n] / (f_1, \ldots, f_k)$, and define
$$\begin{aligned} D_v \colon A &\to M \\ \pi(f) &\mapsto \sum_{i\in[n]} \pi\left(\frac{\mathrm{d}f}{\mathrm{d}x_i}\right) \cdot v_i. \end{aligned}$$

Proposition 1.3.12. *1. $D_v \in \mathrm{Der}_{\mathbb{F}}(A, M)$, and in particular this is well-defined.*

2. The map
$$\begin{aligned} \eta \colon \mathcal{I}_{A,M} &\to \mathrm{Der}_{\mathbb{F}}(A, M) \\ v &\mapsto D_v \end{aligned}$$
is an A-module isomorphism.

Proof. 1. Let $v \in \mathcal{I}_{A,M}$. We want to show that $D_v(\ker(\pi)) = \{0\}$ so that D_v is well-defined. For $j \in [n]$ it holds that
$$\sum_{i\in[n]} \pi\left(\frac{\mathrm{d}f_j}{\mathrm{d}x_i}\right) \cdot v_i = 0.$$
For a general $f \in \ker(\pi)$ we can write $f = \sum_{i\in[k]} h_i f_i$, so
$$\pi\left(\frac{\mathrm{d}f}{\mathrm{d}x_i}\right) = \sum_{j\in[k]} \pi(h_j)\,\pi\left(\frac{\mathrm{d}f_j}{\mathrm{d}x_i}\right) + \pi\left(\frac{\mathrm{d}h_j}{\mathrm{d}x_i}\right) \cancelto{0}{\pi(f_j)} = 0$$
where the last equality follows from the previous computation. Hence D_v is well-defined. It easily seen that the Leibniz law holds, so D_v is a derivation.

2. If $D_v = 0$ we have
$$0 = D_v(\pi(x_j)) = v_j$$
for all $j \in [k]$, so $v = 0$. Hence η is injective.
Assume $D \in \mathrm{Der}_{\mathbb{F}}(A, M)$ and let $\tilde{D} := D \circ \pi \in \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[x_1, \ldots, x_n], M)$. Let
$$v := \begin{pmatrix} \tilde{D}(x_1) \\ \vdots \\ \tilde{D}(x_n) \end{pmatrix} \in M^n.$$
For $j \in [n]$ we have
$$\tilde{D}(x_j) = \sum_{i\in[n]} \frac{\mathrm{d}x_j}{\mathrm{d}x_i} \cdot v_i = \tilde{D}_v(x_j).$$
As before, derivations are determined by generators and we get that
$$\tilde{D} = \tilde{D}_v.$$
If we show that $v \in \mathcal{I}_{A,M}$ we get that η is surjective. This is true since for every $j \in [k]$ it holds that
$$\tilde{D}_v(f_j) = \tilde{D}(f_j) = 0.$$
■

Corollary 1.3.13. *For every $p \in X$ it holds that*

$$T_o X \cong \mathrm{der}_{\mathbb{F}}\left(\mathbb{F}\left[X\right], \mathbb{F}_p\right).$$

Proposition 1.3.14. *For $p \in X$ denote $\mathfrak{m}_p \coloneqq \ker\left(\mathrm{ev}_p\right) \trianglelefteq \mathbb{F}\left[X\right]$, which is maximal. There is a natural isomorphism*

$$T_p X \cong \left(\mathfrak{m}/\mathfrak{m}^2\right)^*.$$

Proof. Let $\ell \in \mathrm{Hom}_{\mathbb{F}-\mathbf{Vect}}\left(\mathfrak{m}, \mathfrak{F}\right)$ such that $\ell|_{\mathfrak{m}^2} \equiv 0$. Define

$$\begin{aligned}\mu \colon \mathbb{F}\left[X\right] &\to \mathbb{F}\\ f &\mapsto \ell\left(f - f\left(p\right) \cdot 1\right).\end{aligned}$$

One can check that $\mu \in \mathrm{Der}_{\mathbb{F}}\left(\mathbb{F}\left[X\right], \mathbb{F}\left(p\right)\right)$.
For the other direction, given μ in this space, we can take $\ell = \mu|_{\mathfrak{m}}$. ■

Let A, B be $\mathbb{F}$-algebras, N a B-module and $\psi \colon A \to B$ an $\mathbb{F}$-module homomorphism. There is an induced A-module structure on N by $a \cdot n \coloneqq \psi\left(a\right) n$. In particular, there is an induced map

$$\begin{aligned}\tilde{\psi} \colon \mathrm{Der}_{\mathbb{F}}\left(B, N\right) &\to \mathrm{Der}_{\mathbb{F}}\left(A, N\right)\\ D &\mapsto D \circ \psi.\end{aligned}$$

If X, Y are affine $\mathbb{F}$-varieties and $\varphi \colon X \to Y$ is regular, we have a map $\varphi^* \colon \mathbb{F}\left[Y\right] \to \mathbb{F}\left[X\right]$. For $p \in X$ the module $\mathbb{F}\left(p\right) \coloneqq \mathbb{F}\left[X\right] / \ker\left(\mathrm{ev}_p\right)$ has the structure of an $\mathbb{F}\left[Y\right]$-module and is isomorphic to $\mathbb{F}\left(\varphi\left(p\right)\right)$ with this structure.

Definition 1.3.15. Denote

$$\mathrm{d}_p \varphi \coloneqq \tilde{\varphi}^* \colon T_p\left(X\right) \to T_{\varphi(p)}\left(Y\right)$$

and call this the *differential* of φ at p.

Exercise 16. If one identifies $T_p\left(X\right) \leq \mathbb{F}^n$ and $T_{\varphi(p)}\left(Y\right) \leq \mathbb{F}^m$, show that φ is a restriction of a polynomial $\mathbb{F}^n \to \mathbb{F}^m$ and that $\mathrm{d}_p \varphi$ is identified with the matrix $M_{n,m}\left(\mathbb{F}\right)$ which is the differential of φ (as over $\mathbb{R}$).

1.3.3 Dimension

In analogy to differentiable manifolds, we define algebraic varieties as a space which locally looks like $\mathbb{F}^n$.

Definition 1.3.16. Let X be an irreducible affine $\mathbb{F}$-variety, and let $K \coloneqq \mathrm{Frac}\left(\mathbb{F}\left[X\right]\right)$. Define $\dim X \coloneqq \mathrm{trdeg}_{\mathbb{F}}\left(K\right)$.

Fact 1.3.17. *Let $\mathbb{F}$ be a perfect field and X be an irreducible affine $\mathbb{F}$-variety. Then*

$$\dim X = \dim_K \mathrm{Der}_{\mathbb{F}}\left(\mathbb{F}\left[X\right], K\right).$$

Corollary 1.3.18. *Let X be an irreducible affine $\mathbb{F}$-variety over a perfect field $\mathbb{F}$. There is $U \subseteq X$ open and non-empty such that for every $p \in U$ it holds that $\dim T_p X = \dim X$.*

Definition 1.3.19. For X as above, a point $p \in X$ is called *simple* or *smooth* if $\dim T_p X = \dim X$, and *singular* otherwise (in which case $\dim T_p X > \dim X$).

Definition 1.3.20 (Smooth Variety). For X as above, we say X is *smooth* if it has no singular points.

Proposition 1.3.21. *Let G be an $\mathbb{F}$-algebraic group for $\mathbb{F}$ a perfect field. Let X be an $\mathbb{F}$-affine G-space on which G acts transitively. Then, X is smooth.*

Proof. Let $x_0 \in X$ be a smooth point (which exists by the corollary). For $x \in X$ there's $g \in G$ such that $x = g \cdot x_0$.

$$\begin{aligned} \lambda_g \colon X &\to X \\ x &\mapsto g \cdot x \end{aligned}$$

is an isomorphism, so

$$\mathrm{d}_{x_0}(\lambda_g) : T_{x_0}(X) \to T_x(X)$$

is an isomorphism. Hence x is smooth. ∎

Proposition 1.3.22. *Let X be an irreducible affine $\mathbb{F}$-variety and let $Y \lneq X$ be closed and irreducible. Then* $\dim Y < \dim X$.

Proof. We can write

$$\mathbb{F}[Y] \cong \mathbb{F}[X]/I$$

for some non-zero ideal $I \trianglelefteq \mathbb{F}[X]$, which is prime since Y is irreducible. Let $f_1, \ldots, f_k \in \mathbb{F}[Y]$ be algebraically-independent where $k \coloneqq \dim Y$. Let $g_1, \ldots, g_k \in \mathbb{F}[X]$ be such that $\bar{g}_i = f_i$. Let $g_0 \in I \setminus \{0\}$, we want to show that $g_0, \ldots, g_k$ are algebraically-independent.
Assume towards a contradiction that there's a non-zero $p \in \mathbb{F}[y_0, \ldots, y_k]$ such that $p(g_0, \ldots, g_k) = 0$. We may assume p is irreducible. Write

$$p(y_0, \ldots, y_k) = y_0 p_1(y_0, \ldots, y_k) + p_2(y_1, \ldots, y_k)$$

for polynomials p_1, p_2. We have

$$\begin{aligned} \bar{0} &= \overline{p(g_0, \ldots, g_k)} \\ &= \overline{g_0 p_1(g_0, \ldots, g_k) + p_2(g_1, \ldots, g_k)} \\ &= \overline{\cancelto{0}{g_0} p_1(g_0, \ldots, g_k)} + \overline{p_2(g_1, \ldots, g_k)} \\ &= \overline{p_2(g_1, \ldots, g_k)} \\ &= p_2(f_1, \ldots, f_k). \end{aligned}$$

Hence $p_2 \equiv 0$ so $p(y_0, \ldots, y_k) = y_0 p_1(y_0, \ldots, y_k)$, so p is reducible, a contradiction. ∎

Lemma 5. *Let $X \subseteq \mathbb{F}^n$ be an embedded affine irreducible variety. For $p \in X$ identify $T_p X \leq \mathbb{F}^n$. For $p_0 \in X$ simple there's $f \in \mathbb{F}[X]$ such that $p_0 \in X_f$ and there are regular maps*

$$\psi_i \colon X_f \to \mathbb{F}^n$$

for $i \in d \coloneqq \dim X$ such that $(\psi_1(x), \ldots, \psi_d(x))$ is a basis for $T_x X$ for all $x \in X_f$.

Proof. Identify

$$\mathbb{F}[X] = \mathbb{F}[x_1, \ldots, x_n] / (f_1, \ldots, f_k).$$

There's a non-vanishing minor of size $n - d$ in $J(f_1, \ldots, f_k)(p_0)$. We have

$$T_{p_0}(X) = \mathrm{null}(J(f_1, \ldots, f_{n-d})(p_0))$$

for $n - d \leq k$. Then we can write

$$J(f_1, \ldots, f_{n-d}) = \begin{pmatrix} A & B \end{pmatrix} \in M_{n-d,n}(A_n)$$

for $A \in M_{n-d}(A_n)$ and $B \in M_{n-d,d}(A_n)$. Let $f = \det(A)$. Then up to reordering the columns we have $\det A(p_0) \neq 0$. X_f consists of simple points of X. For $x \in X_f$ we have

$$T_x(X) = \mathrm{null}(J(f_1, \ldots, f_{n-d})(x))$$

so

$$\psi(x) = (\psi_1(x), \ldots, \psi_d(x)) = \begin{pmatrix} -A^{-1}B \\ I_{d \times d} \end{pmatrix}(x) \in M_{n,d}(\mathbb{F})$$

so

$$J(f_1, \ldots, f_{n-d})(x) \cdot \psi(x) = 0.$$

∎

Proposition 1.3.23 (Differential Criterion for Dominance). *Let $\mathbb{F}$ be perfect and infinite. Let X, Y be irreducible affine $\mathbb{F}$-varieties and let $\varphi\colon X \to Y$. Assume there's $p \in X$ such that $\mathrm{d}_p(\varphi)$ is surjective onto $T_{\varphi(p)}(Y)$ and that $\varphi(p) \in Y$ is simple (the second condition might be unnecessary). Then $Y = \overline{\varphi(X)}$.*

Proof. Using 5 with respect to $p \in X$, we get $X_f \subseteq X$ such that for all $x \in X_f$ one has

$$T_x X = \operatorname{Span}(\psi_1(x), \dots, \psi_d(x))$$

where

$$\psi(x) = (\psi_1(x), \dots, \psi_d(x)) \in M_{n,d}(A_n).$$

Now

$$p \in U_1 \coloneqq \{x \in X_f \mid \operatorname{rank}(\mathrm{d}_x(\varphi)) \geq \dim(Y)\} \subseteq X$$

is open since the condition is equivalent to non-vanishing of minors of the matrix $\mathrm{d}\varphi(x) \cdot \psi(x)$. Denote $Z \coloneqq \overline{\varphi(X)}$ and assume towards a contradiction that $Z \subsetneq Y$. Then $\dim Z \leq \dim Y$. Let

$$U_2 \coloneqq \{x \in X \mid \varphi(x) \text{ is simple in } Z\} \subseteq X.$$

Then U_2 is the inverse image of an open set under φ, and is therefore open. Hence $\varphi(X)$ contains an open subset of Z. Using irreducibility we get that $\varphi(X)$ contains simple points of Z, so $U_1 \cap U_2 \neq \varnothing$. Let $x_0 \in U_1 \cap U_2$. We get

$$\dim Z = \dim T_{\varphi(x_0)}(Z) \geq \operatorname{rank} \mathrm{d}_{x_0}(\varphi) \geq \dim(Y),$$

a contradiction. ∎

1.4 Lie Algebras

1.4.1 Definitions

Definition 1.4.1 (Lie Algebra). A *Lie algebra* $\mathfrak{g}$ over $\mathbb{F}$ is a nonassociative algebra over $\mathbb{F}$ whose (bilinear) product, denoted $[x, y]$, and called the *Lie bracket*, satisfying the following for all $x, y, z \in \mathfrak{g}$.

Skew-symmetry: $[x, y] = -[y, x]$

Jacobi's Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 1.4.2 (The Primordial Lie Algebra). Let C be an $\mathbb{F}$-algebra. We define on C the *commutator*

$$[a, b] = ab - ba.$$

This turns C into a Lie algebra.

Example 29. Let A be an $\mathbb{F}$-algebra. One has $\operatorname{Der}_{\mathbb{F}-\mathbf{Vec}}(A, A) \leq \operatorname{End}_{\mathbb{F}}(A, A)$. In particular, for $X, Y \in \operatorname{Der}_{\mathbb{F}}(A, A)$ we have $X \circ Y\colon A \to A$. The composition $X \circ Y$ isn't a derivation, but $[X, Y]$ is by direct computation. We get a Lie algebra structure on $\operatorname{Der}_{\mathbb{F}}(A, A)$.

Example 30. Given an $\mathbb{F}$-affine variety X one gets a "large" Lie algebra $\operatorname{Der}_{\mathbb{F}}(\mathbb{F}[X], \mathbb{F}[X])$ which is infinite-dimensional over $\mathbb{F}$.
If $X = G$ is an algebraic group over $\mathbb{F}$ there is a more interesting construction. Let $\lambda\colon G \to \operatorname{GL}(\mathbb{F}[G])$ be the map with $\lambda(g)(f)(x) = f(g^{-1}x)$ for $g \in G, f \in \mathbb{F}[X], x \in G$.
For $D \in \operatorname{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}(G))$ and $g \in G$ we have

$$\lambda(g) \circ D \circ \lambda(g)^{-1} \in \operatorname{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}(G)).$$

Geometrically, this moves the vector field matching D by the g action.

Definition 1.4.3. For G as above, define

$$\operatorname{Lie}(G) \coloneqq \{D \in \operatorname{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}[G]) \mid \forall g \in G\colon \lambda(g) \circ D = D \circ \lambda(g)\}.$$

Remark 1.4.4. It is easily checked that for $D_1, D_2 \in \mathrm{Lie}(G)$ one has $[D_1, D_2] \in \mathrm{Lie}(G)$. We then call $\mathrm{Lie}(G)$ the *Lie algebra of G*.

Notation 1.4.5. For $D \in \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}[G])$ and $g \in G$ denote

$$D_g \coloneqq \mathrm{ev}_g \circ D.$$

Remark 1.4.6. It holds that $D_g \in \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}[g]) = T_g(G)$. We have

$$\mathrm{d}(\lambda_g)_h : T_h(G) \to T_{gh}(G)$$

and

$$\lambda(g) \circ D_h \circ \lambda(g)^{-1} = \mathrm{d}(\lambda_g)_{g^{-1}h}\left(D_{g^{-1}h}\right).$$

(Check this.) We get that $D \in \mathrm{Lie}(G)$ if and only if

$$\mathrm{d}(\lambda_g)_e(D_e) = D_g.$$

Example 31. Let $G = (\mathbb{F}^n, +)$. Then $A_n \coloneqq \mathbb{F}[G] = \mathbb{F}[x_1, \ldots, x_n]$. We have

$$\mathrm{Der}_{\mathbb{F}}(A_n, A_n) = \left\{ \sum_{i\in[n]} f_i \frac{\mathrm{d}}{\mathrm{d}x_i} \;\middle|\; (f_i)_{i\in[n]} \subseteq A_n \right\}.$$

These correspond to global vector fields on G. Then $D = \sum_{i\in[n]} f_i \frac{\mathrm{d}}{\mathrm{d}x_i}$ is in $\mathrm{Lie}(G)$ if all the f_i satisfy $f_i(x+y) = f_i(x)$ for all $y \in \mathbb{F}^n$. This is true exactly when $f_i \equiv a_i$ are constant, so $D = \sum_{i\in[n]} a_i \frac{\mathrm{d}}{\mathrm{d}x_i}$ corresponds to a constant vector field.

Example 32. Let $G \coloneqq \mathrm{GL}_1(\mathbb{F})$, so that $\mathbb{F}[G] \equiv \mathbb{F}[t,u]/(tu-1)$. Then

$$\mathrm{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}[G]) = \left\{ f\frac{\mathrm{d}}{\mathrm{d}t} \;\middle|\; f \in \mathbb{F}[G] \right\}.$$

Now $f\frac{\mathrm{d}}{\mathrm{d}t} \in \mathrm{Lie}(G)$ if and only if for all $a \in \mathbb{F}^\times$ and $\rho \in \mathbb{F}[G] \cong \mathbb{F}\left[t, t^{-1}\right]$ we have

$$f(t)\frac{\mathrm{d}}{\mathrm{d}t}(\rho(at)) = f(at)\frac{\mathrm{d}\rho}{\mathrm{d}t}(at).$$

Then

$$af(t)\frac{\mathrm{d}\rho}{\mathrm{d}t}(at) = f(at)\frac{\mathrm{d}\rho}{\mathrm{d}t}(at)$$

so

$$af(t) = f(at)$$

so $f(t) = \alpha t$ for some $\alpha \in \mathbb{F}$. We get that

$$\mathrm{Lie}(G) = \mathrm{Span}\left\{ t\frac{\mathrm{d}}{\mathrm{d}t} \right\}.$$

Proposition 1.4.7. *The map*

$$\begin{aligned} \eta\colon \mathrm{Lie}(G) &\to T_e(G) \\ D &\mapsto D_e \end{aligned}$$

is a vector-space isomorphism.

Proof. **Injectivity:** Assume D is such that $D_e = 0$. Then for all $g \in G$ we have

$$D_g = \mathrm{d}(\lambda_g)_e(D_e) = 0.$$

For $f \in \mathbb{F}[G]$ we then have

$$\mathrm{ev}_g(D(f)) = 0$$

so $D(f) = 0$.

Surjectivity: Let $X \in T_e(G)$. Define

$$D_X \colon \mathbb{F}[G] \to \mathbb{F}[G]$$

via

$$D_X(f)(g) = X\left(\lambda\left(g^{-1}\right)(f)\right)$$

for all $g \in G$. We have to show that $D_X(f) \in \mathbb{F}[G]$, and then check that D_X is a derivation. Let $m \colon G \times G \to G$ be the group product. Then for $f \in \mathbb{F}[G]$ with

$$f(xy) = \sum_i u_i(x) v_i(y)$$

we have

$$\lambda(g)(f) = \sum_i u_i\left(g^{-1}\right) v_i$$

and

$$D_X(f)(g) = \sum_i u_i\left(g^{-1}\right) X(v_i) \in \mathbb{F}[X].$$

Now, $D_X \in \mathrm{Lie}(G)$ because for $h \in G$ we have

$$\begin{aligned} D_X(\lambda(h)(f))(g) &= X\left(\lambda\left(\left(h^{-1}g\right)^{-1}\right)(f)\right) \\ &= \lambda(h) D_X(f)(g) \end{aligned}$$

so

$$D_X(\lambda(h) f) = \lambda(h) D_X(f).$$

■

Knowing the above proposition, we sometimes think of $\mathrm{Lie}(G)$ as the tangent space $T_e(G)$. In fact, we could define a Lie algebra structure on $T_e(G)$ directly by

$$[X, Y] \coloneqq (X \otimes Y - Y \otimes X) \circ m^* \tag{1.1}$$

for every $X, Y \in T_e(G) = \mathrm{Der}_{\mathbb{F}}(\mathbb{F}[G], \mathbb{F}(G))$ and where $m \colon G \times G \to G$ is the group-multiplication. But, this might lack motivation.
From the above proposition, we can identify $T_e(G) \cong \mathrm{Lie}(G)$, and, pulling back this Lie-algebra structure we can define the bracket $[X, Y] \in T_e G$ for any $X, Y \in T_e(G)$.
For an embedded algebraic group $G \leq \mathrm{GL}_n(\mathbb{F})$ we get an embedding $\mathrm{Lie}(G) \cong T_e(G) \leq M_n(\mathbb{F})$. Write $\mathbb{F}[G] = \mathbb{F}\left[\{T_{i,j}\}_{i,j \in [n]}\right]/I$. Any $X \in T_e(G)$ is determined by its action on generators. Write $x_{i,j} \coloneqq X(T_{i,j})$. We have

$$\begin{aligned} m^* \colon \mathbb{F}[G] &\to \mathbb{F}[G] \times \mathbb{F}[G] \\ T_{i,j} &\mapsto \sum_{k \in [n]} T_{i,k} \otimes T_{k,j}. \end{aligned}$$

Then

$$D_X(T_{i,j}) = \sum_{k \in [n]} T_{i,k} x_{k,j}.$$

Hence $D_X(T_{i,j}) \in M_n(\mathbb{F}[G])$ and

$$(D_X(T_{i,j}))_{i,j} = (T_{i,j}) \cdot X$$

under the identification of X as the matrix $(x_{i,j})_{i,j \in [n]}$ in $M_n(\mathbb{F}[G])$.
Now,

$$\begin{aligned} [X, Y](f) &= (D_X \circ D_Y - D_Y \circ D_X)(f)(e) \\ &= X(D_Y(f)) - Y(D_X(f)). \end{aligned}$$

Under the embedding we have

$$Y(D_X(T_{i,j})) = (YX)_{i,j}$$

so

$$[X, Y](T_{i,j}) = (XY - YX)_{i,j}.$$

We then get the under the identification with matrices, $[X, Y] = XY - YX$. I.e. for every embedding $G \hookrightarrow \mathrm{GL}_n(\mathbb{F})$ we get an isomorphism from $\mathrm{Lie}(G)$ to a Lie subalgebra of $M_n(\mathbb{F})$.

Exercise 17. Check that the definition of $[\cdot,\cdot]$ on $T_e(G)$ is the same as that of (1.1).

Proposition 1.4.8. *Let $\varphi\colon G \to H$ be a homomorphism of algebraic groups. Then*

$$\mathrm{d}\varphi = (\mathrm{d}\varphi)_e : \operatorname{Lie}(G) \to \operatorname{Lie}(H)$$

satisfies

$$[\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)] = \mathrm{d}\varphi[X,Y].$$

In other words, $\mathrm{d}\varphi\colon \operatorname{Lie}(G) \to \operatorname{Lie}(H)$ is a Lie-algebra homomorphism.

Proof. φ is a homomorphism, so $\varphi \circ m_G = m_H \circ (\varphi \times \varphi)$, so

$$(\varphi^* \otimes \varphi^*) \circ m_H^* = m_G^* \circ \varphi^*.$$

Then

$$\begin{aligned}
[\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)] &= ((X \circ \varphi^*) \otimes (Y \circ \varphi^*) - (Y \circ \varphi^*) \otimes (X \circ \varphi^*)) \circ m_H^* \\
&= (X \otimes Y - Y \otimes X) \circ m_G^* \circ \varphi^* \\
&= [X,Y] \circ \varphi^* \\
&= \mathrm{d}\varphi([X,Y]).
\end{aligned}$$

■

Let G be an affine $\mathbb{F}$-algebraic group. G acts on itself by conjugation. For $g \in G$, denote

$$\begin{aligned}
\operatorname{In}(g) : G &\to G \\
h &\mapsto ghg^{-1}.
\end{aligned}$$

Then

$$\operatorname{Ad}(g) \coloneqq \mathrm{d}(\operatorname{In}(g))_e : \operatorname{Lie}(G) \to \operatorname{Lie}(G)$$

gives a regular map

$$\operatorname{Ad}\colon G \to \operatorname{GL}(\operatorname{Lie}(G))$$

which is a representation of G called *the adjoint representation of G*. We get a map

$$\operatorname{ad} \coloneqq \mathrm{d}\operatorname{Ad}\colon \operatorname{Lie}(G) \to \operatorname{End}(\operatorname{Lie}(G))$$

such that

$$\operatorname{ad}(X)(Y) = [X,Y].$$

Exercise 18. We could define $[X,Y]$ as $\operatorname{ad}(X)(Y)$.

Remark 1.4.9. From the above proposition, one has

$$[\operatorname{ad}(X), \operatorname{ad}(Y)] = \operatorname{ad}[X,Y],$$

which is equivalent to Jacobi's identity.

Exercise 19. In finite dimensional it holds that $T_e(G)^\circ = T_e(G)$.

Corollary 1.4.10. *One has* $\dim \operatorname{Lie}(G) = \dim T_e(G) = \dim(G^\circ)$.

1.5 Categorical Approach to Algebraic Groups

1.5.1 Definitions

An affine algebraic variety is defined by polynomial equations, $X = V(f_1, \ldots, f_k) \leq \mathbb{F}^n$ for $(f_i)_{i\in[k]}\, \mathbb{F}[x_1, \ldots, x_n]$. One can treat the equations $f_i = 0$ over any (commutative) $\mathbb{F}$-algebra R.

Definition 1.5.1 (Functor of Points). Let X be an affine algebraic variety over $\mathbb{F}$, and let R be an $\mathbb{F}$-algebra. Define

$$X(R) = \operatorname{Hom}_{\mathbb{F}-\mathbf{Alg}}(\mathbb{F}[X], R)$$

and call the resulting functor $R \mapsto X(R)$ the *functor of points of X*.

Definition 1.5.2 (Category). A *(small) category* $\mathcal{C}$ is a *collection* (the exact type of which depends on the axiomatic approach) $\mathrm{Ob}(\mathcal{C})$ together with a *set* of morphism $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ for every $X,Y\in\mathcal{C}$ and a *composition map*

$$\begin{aligned}\mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)&\to\mathrm{Hom}(X,Z)\\(f,g)&\mapsto g\circ f\end{aligned}$$

satisfying the following.

1. For all $X\in\mathrm{Ob}(\mathcal{C})$ there's $\mathrm{Id}_X\in\mathrm{Hom}_{\mathcal{C}}(X,X)$ such that $\mathrm{Id}_X\circ f=f$ and $g\circ\mathrm{Id}_X$ for all f,g (which make sense).
2. Composition is associative.

Example 33. 1. Sets with functions between them form a category.

2. Groups with group homomorphisms.
3. $\mathbb{F}$-algebras with their homomorphisms.

Definition 1.5.3 (Functor). A *(covariant) functor* $F\colon\mathcal{C}\to\mathcal{D}$ between categories is a "function" $F\colon\ \mathrm{Ob}(\mathcal{C})\to\mathrm{Ob}(\mathcal{D})$ and for each $X,Y\in\mathrm{Ob}(\mathcal{C})$ a function

$$F\colon\ \mathrm{Hom}_{\mathcal{C}}(X,Y)\to\mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

such that F respects composition and identities.

Definition 1.5.4. If instead of respecting composition, F satisfies $F(g\circ f)=F(f)\circ F(g)$, we call F a *contravariant functor.*

Example 34. The correspondence $X\mapsto\mathbb{F}[X]$ and $\varphi\mapsto\varphi^*$ forms a functor from affine $\mathbb{F}$-varieties to commutative $\mathbb{F}$-algebras.

Example 35. There is a functor $F\colon\mathbf{Vec}_{\mathbb{F}}\to\mathbf{Vec}_{\mathbb{F}}$ from the category of $\mathbb{F}$-vector spaces to itself, sending $V\mapsto V^*$ and $T\mapsto T^*$.

Example 36. We have a functor $G\mapsto\mathrm{Lie}(G)$ sending an affine algebraic group over $\mathbb{F}$ to an vector space over $\mathbb{F}$.

Definition 1.5.5. We call a functor which "forgets" some of the structure a *forgetful functor.*

Definition 1.5.6. Let $\mathcal{C}$ be a category, and let $X\in\mathrm{Ob}(\mathcal{C})$. Define a (covariant) functor

$$\begin{aligned}h^X:\mathcal{C}&\to\mathbf{Set}\\Y&\mapsto\mathrm{Hom}_{\mathcal{C}}(X,Y)\\\varphi&\mapsto(f\mapsto\varphi\circ f)\,.\end{aligned}$$

Corollary 1.5.7. *Let X be an affine variety over $\mathbb{F}$. One can see X as a functor $X:\mathbb{F}{-}\mathbf{Alg}\to\mathbf{Set}$ sending R to $X(R)$. This is the functor $X=h^{\mathbb{F}[X]}$.*

Definition 1.5.8 (Natural Transformation). Let $F,G\colon\mathcal{C}\to\mathcal{D}$ be functors between categories. A *natural transformation* $\eta\colon F\to G$ is the data $\eta_X\in\mathrm{Hom}_{\mathcal{D}}(F(X),G(X))$ for all $X\in\mathrm{Ob}(\mathcal{C})$, such that the following diagram commutes for every $X,Y\in\mathrm{Ob}(\mathcal{C})$.

$$\begin{tikzcd}F(X) \arrow[r, "\eta_X"] \arrow[d, "F(\varphi)"] & G(X) \arrow[d, "G(\varphi)"] \\ F(Y) \arrow[r, "\eta_Y"] & G(Y)\end{tikzcd}$$

Corollary 1.5.9. *The collection $\mathbf{Fun}(\mathcal{C},\mathcal{D})$ of functors $\mathcal{C}\to\mathcal{D}$ is a category together with natural transformations as morphisms.*

Example 37. Consider the functors $\mathrm{Id}_{\mathbb{F}-\mathbf{Vec}}$ and $\left((-)^*\right)^*$ on the category of $\mathbb{F}$-vector spaces. The canonical identification $V \cong (V^*)^*$ is a natural transformation $\mathrm{Id}_{\mathbb{F}-\mathbf{Vec}} \to \left((-)^*\right)^*$.

Theorem 1.5.10 (Yoneda's Lemma). *Let $\mathcal{C}$ be a category and let $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Consider the induced natural transformation $T(\varphi) \in \mathrm{Hom}\left(h^Y, h^X\right)$ with*

$$\begin{aligned} T(\varphi)_Z \colon \mathrm{Hom}(Y, Z) to \mathrm{Hom}(X, Z) \\ f \mapsto f \circ \varphi. \end{aligned}$$

Then

$$T \colon \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathbf{Fun}(\mathcal{C}, \mathbf{Set})}\left(h^Y, h^X\right)$$

is a bijection.

Corollary 1.5.11. *Any $\mathcal{C}$ is embedded as a subcategory of* $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$*.*

Definition 1.5.12 (Yoneda Embedding). Define the *Yoneda embedding* to be the contravariant functor

$$\begin{aligned} \mathcal{C} &\to \mathbf{Fun}(\mathcal{C}, \mathbf{Set}) \\ X &\mapsto h^X. \end{aligned}$$

This is an *embedding* due to Yoneda's lemma.

Remark 1.5.13. We saw that $X(R)$ can be thought of as the solution of solutions defined in $\mathbb{F}[X]$ over R. This is a way to construct geometric objects from an $\mathbb{F}$-algebra R.

Definition 1.5.14 (Corepresentable Functor). A functor $F \colon \mathcal{C} \to \mathbf{Set}$ is *corepresentable* if there's $X \in \mathrm{Ob}(\mathcal{C})$ such that $F \cong h^X$.

Definition 1.5.15 ($\mathbb{F}$-Affine Scheme). An $\mathbb{F}$-*affine scheme* is a corepresentable functor $X : \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$. If X is corepresentable by a finitely-generated object, we say X is an *algebraic scheme*.

Remark 1.5.16. Remembering our definition of affine varieties over $\mathbb{F}$, the algebra of regular functions has no non-trivial nilpotent elements. So, the case of $\mathbb{F}$-schemes is more general, but one has a similar duality.

Definition 1.5.17. For an $\mathbb{F}$-scheme $X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$, let

$$A_X \coloneqq \mathrm{Hom}_{\mathbb{F}}\left(X, \mathbb{A}^1\right)$$

where $\mathbb{A}^1 \coloneqq h^{\mathbb{F}[t]}$.

Any $f \in \mathrm{Hom}\left(X, \mathbb{A}^1\right)$ (a natural transformation between the functors) is composed of functions $f_R \colon X(R) \to \mathbb{A}^1(R)$, such that for a morphism $t \colon R \to S$ the following diagram commutes.

$$\begin{array}{ccc} X(R) & \xrightarrow{f_R} & \mathbb{A}^1(R) \\ \downarrow{\scriptstyle X(t)} & & \downarrow{\scriptstyle t} \\ X(S) & \xrightarrow{f_S} & S \end{array}$$

In particular, A_X is an $\mathbb{F}$-algebra. We get a natural transformation

$$\alpha^X \in \mathrm{Hom}\left(X, h^{A_X}\right)$$

with

$$\begin{aligned} \alpha^X_R \colon X(R) &\to \mathrm{Hom}(A_X, R) \\ g &\mapsto (f_R \mapsto f_R(g)). \end{aligned}$$

This is analogous to ev_g.

Proposition 1.5.18. *$X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ is corepresentable (i.e. a scheme) if and only if α^X is an isomorphism.*

Proof. If α^X is an isomorphism, $X \cong h^{A_X}$ which is a scheme by definition.
If $X \cong h^B$ then

$$A_X = \mathrm{Hom}\left(h^B, h^{\mathbb{F}[t]}\right) \underset{\text{Yoneda}}{\cong} \mathrm{Hom}(\mathbb{F}[t], B) \cong B.$$

Hence $X \cong h^{A_X}$. Check that then $\alpha^X \in \mathrm{Hom}\left(h^{A_X}, h^{A_X}\right)$ is a natural isomorphism. ∎

1.5.2 Interlude: General (Non-Affine) Schemes

We defined affine schemes, but what are non-affine schemes? Similarly to our definition of an $\mathbb{F}$-affine variety (X, A) where $X = \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(A, \mathbb{F})$, an affine scheme is a functor

$$X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$$

where $X(R) = \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}(A, R)$, for some $\mathbb{F}$-algebra A. But, there are in general non-affine spaces of algebraic interest which aren't affine. For example, there's projective space.

Example 38. We define the n-dimensional projective space $\mathbb{P}^n(\mathbb{F})$ as $\left(\mathbb{F}^{n+1} \setminus \{0\}\right)/\mathbb{F}^\times$ under the $\mathbb{F}^\times \curvearrowright \mathbb{F}^{n+1} \setminus \{0\}$ action given by multiplication by the scalar.
While projective space is *not* affine, it is *locally affine*. For example, removing $p_0 := [(\alpha, 0)]$ from $\mathbb{P}^1(\mathbb{F})$, we get an algebraic variety isomorphic to $\mathbb{F}$ via

$$(\alpha, \beta) \leftrightarrow \frac{\alpha}{\beta}.$$

This fits the intuition of the Riemann sphere where $\mathbb{P}^1(\mathbb{C}) \cong \mathbb{C} \cup \{\infty\}$. We can get such identification by removing any other point, so one has $\mathbb{P}^1(\mathbb{F}) = A_1 \cup A_2$ where the A_i are open and isomorphic to $\mathbb{F}$.
Zariski closed subsets of $\mathbb{P}^n$ (under the correct definition) are called *projective varieties* and are locally isomorphic to affine varieties. This leads us to the general definition of an $\mathbb{F}$-scheme.

Definition 1.5.19 (Presheaf). A functor in $\mathbf{Fun}(\mathcal{C}, \mathbf{Set})$ is called a *presheaf.*

Definition 1.5.20. A functor $X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ is an $\mathbb{F}$-scheme if as a presheaf

We define a functor $X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ to be an algebraic $\mathbb{F}$-scheme if the presheaf is "locally corepresentable". The Zariski to topology on affine $\mathbb{F}$-varieties then gives a "topology" on $\mathbb{F}-\mathbf{Alg}$, called the *Grothendieck topology.* With respect to this topology, one can say that $X \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ is "locally corepresentable" if there are $\mathbb{F}$-affine algebraic schemes $U_1, \ldots, U_k \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ such that $U_i \hookrightarrow X$ are open embeddings (w.r.t. to the Grothendieck topology) and $X = \bigcup_{i \in [k]} U_i$.

1.5.3 A Categorical Approach to Algebraic Groups

Definitions

Definition 1.5.21 (Algebraic $\mathbb{F}$-Affine Group-Scheme). An *algebraic $\mathbb{F}$-affine group scheme* is a functor $G \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Group}$ such that $U \circ G$ is an algebraic $\mathbb{F}$-affine scheme, where $U \colon \mathbf{Group} \to \mathbf{Set}$ is the forgetful functor.

Remark 1.5.22. An $\mathbb{F}$-affine scheme is defined by an $\mathbb{F}$-algebra A, being isomorphic to h^A. If $U \circ G \cong h^A$ we want to deduce something on A. In other words, we ask what $\mathbb{F}$-algebras corepresent algebraic $\mathbb{F}$-affine group-schemes. Such $\mathbb{F}$-algebras are called *Hopf algebras.*

Definition 1.5.23. For $\mathbb{F}$-affine varieties $X_1, X_2 \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ we define their product by

$$\begin{aligned} X_1 \times X_2 \colon \mathbb{F}-\mathbf{Alg} &\to \mathbf{Set} \\ R &\mapsto X_1(R) \times X_2(R). \end{aligned}$$

Remark 1.5.24. In the above definition one should show that $X_1 \times X_2$ is corepresentable. If X_i is corepresented by A_i one has $h^{A_1 \otimes A_2} \cong X_1 \times X_2$ since we've shown

$$\mathrm{Hom}(A_1 \otimes A_2, R) \cong \mathrm{Hom}(A_1, R) \times \mathrm{Hom}(A_2, R).$$

Let F be a group-scheme, and $t \colon R \to S$ an $\mathbb{F}$-algebra homomorphism. Then $G(R), G(S)$ are groups, and the following diagram commutes since $G \colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Group}$ is a functor.

$$\begin{tikzcd} G(R) \times (S) \arrow[r, "m_R"] \arrow[d, "(G\times G)(t)"] & G(R) \arrow[d, "G(t)"] \\ G(S) \times G(S) \arrow[r, "m_S"] & G(S) \end{tikzcd}$$

On the other hand, the collection $(m_R)_{R\in \mathrm{Ob}(\mathbb{F}-\mathbf{Alg})}$ defines a natural transformation $m\colon G\times G \to G$, and

$$m \in \mathrm{Hom}\,(G\times G, G) \cong \mathrm{Hom}\left(h^{A\otimes A}, h^A\right) \cong \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}\,(A, A\otimes A)\,.$$

The image $\Delta := m^*$ of m in $\mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}\,(A, A\otimes A)$ is called the comultiplication on A. If $A = \mathbb{F}\,[G]$ we have

$$\begin{aligned}\Delta = m^*\colon \mathbb{F}\,[G] &\to \mathbb{F}\,[G\times G]\\ f &\mapsto ((g_1, g_2) \mapsto f\,(g_1 g_2))\,.\end{aligned}$$

We can write the comultiplication Δ in another way. We have

$$G\,(R)\times G\,(R) = \mathrm{Hom}\,(A,R)\times \mathrm{Hom}\,(A,R) \cong \mathrm{Hom}\,(A\otimes A, R) \xrightarrow{\mathrm{ev}\,\circ\Delta} \mathrm{Hom}\,(A,R) = G\,(R)$$

and via these morphisms Δ defines the multiplication structure on $G\,(R)$. Associativity of the multiplication on $G\,(R)$ for all R gives the following equation.

$$(\mathrm{Id}\otimes\Delta)\,\Delta = (\Delta\otimes \mathrm{Id})\,\Delta.$$

Similarly, we would like to describe all the properties defining a group strucutre on $G\,(R)$ as properties of A. We have an identity element

$$G_\varepsilon\colon \mathbb{F}-\mathbf{Alg} \to \mathbf{Group}$$

with $G_\varepsilon = h^{\mathbb{F}}$, so that

$$G_\varepsilon = \mathrm{Hom}_{\mathbb{F}-\mathbf{Alg}}\,(\mathbb{F}, R) = \{e\}\,.$$

We have maps

$$G_\varepsilon \xrightarrow{\varepsilon_R} G\,(R) \xrightarrow{\eta_R} G_\varepsilon\,(R)$$

where

$$\varepsilon_R\,(g) = e.$$

One can check that these maps ε_R, η_R coalesce into natural transformations

$$G_\eta \xrightarrow{\varepsilon} G \xrightarrow{\eta} G_\varepsilon.$$

Now $\varepsilon \in \mathrm{Hom}\left(h^F, h^A\right) \cong \mathrm{Hom}\,(A, \mathbb{F})$ corresponds to evaluation at $e\in G$, i.e. $\varepsilon = \mathrm{ev}_e$. (Check this!) The property $g = g\cdot e = e\cdot g$ for all $g\in G$ is then expressed as

$$\begin{aligned}\mathrm{Id} &= (\mathrm{Id}\otimes\varepsilon)\,\Delta\\ \mathrm{Id} &= (\varepsilon\otimes \mathrm{Id})\,\Delta.\end{aligned}$$

Now, (A, Δ, ε) is called a *bi-algebra*: it has a comultiplication with counit, and also an algebra structure.
We have the inverse map $i\colon g\mapsto g^{-1}$ which gives an homomorphism $S = i^*\colon A\to A$ called the *antipode*. The properties $gg^{-1} = g^{-1}g = e$ can be expressed by the commutativity of the following diagram.

$$\begin{tikzcd}
 & G \arrow[r, "g\mapsto(g,g)"] \arrow[dl] & G\times G \arrow[dd, "(g,h)\mapsto\left(g,h^{-1}\right)"] \\
\{e\} \arrow[dr] & & \\
 & G & G\times G \arrow[l, "gh\mapsfrom(g,h)"']
\end{tikzcd}$$

In the algebras we get

$$m\,(\mathrm{Id}\otimes S)\,\Delta\,(f) = m\,(S\otimes \mathrm{Id})\,\Delta\,(f) = \varepsilon\,(f)\cdot 1. \tag{1.2}$$

Definition 1.5.25. An $\mathbb{F}$-bialgebra with an antipode $S\colon A\to A$ satisfying condition (1.2) is called a *Hopf* $\mathbb{F}$*-algebra*.

One can check that a Hopf-algebra structure on A gives a functor $\mathbb{F}-\mathbf{Alg} \to \mathbf{Group}$ structure on h^A.

Remark 1.5.26. Given the above descriptions, we can think of $\mathbb{F}$-algebraic groups in the following ways.

- An $\mathbb{F}$-affine variety with regular multiplication and inverse maps.
- A corepresentable functor $\mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$ together with a lift to a functor $\mathbb{F}-\mathbf{Alg} \to \mathbf{Set}$.
- An Hopf $\mathbb{F}$-algebra A.

Concrete Application of the Categorical Approach

We can use the categorical approach to think of Lie algebras as a natural object. Consider the $\mathbb{F}$-algebra $R := \mathbb{F}[t] / (t^2) =: \mathbb{F}[\varepsilon]$ with $\varepsilon^2 = 0$. Then R has a unique maximal ideal (0). There's an homomorphism

$$\begin{aligned} \varphi\colon \mathbb{F}[\varepsilon] &\to \mathbb{F} \\ \varepsilon &\mapsto 0 \end{aligned}$$

which induces a homomorphism

$$\alpha\colon G(\mathbb{F}[\varepsilon]) \to G(\mathbb{F}).$$

For $p \in G(F)$, if $v \in G(\mathbb{F}[\varepsilon])$ is such that $\alpha(v) = p$, we have

$$\begin{aligned} v\colon \mathbb{F}[G] &\to \mathbb{F}[\varepsilon] \\ f &\mapsto v_0(f) \cdot 1 + v_1(f) \cdot \varepsilon \end{aligned}$$

and since $\alpha(v) = p$ we get $v_0 = \mathrm{ev}_p$. One can also check that

$$v_1\colon \mathbb{F}[G] \to \mathbb{F}$$

is a derivation, hence in $T_p(G)$. Then $\ker(\alpha) = T_e(G) = \mathrm{Lie}(G)$ is a Lie algebra.

Extension and Restriction of Scalars

Let E/F be a field extension. Any E-algebra is also naturally an F algebra. If $G\colon F-\mathbf{Alg} \to \mathbf{Group}$ is an F-algebraic group, we get a restriction

$$G_E := G|_{E-\mathbf{Alg}}\colon E-\mathbf{Alg} \to \mathbf{Group}.$$

Example 39. We can consider $\mathrm{SL}_n(\mathbb{C})$ as an algebraic group over $\mathbb{R}$.

We have to show that G_E is actually an algebraic group. I.e. that forgetting the group structure gives a corepresentable functor. For every E-algebra R we get

$$\begin{aligned} G_E(R) &= \mathrm{Hom}_{F-\mathbf{Alg}}(F[G], R) \\ &\cong \mathrm{Hom}_{E-\mathbf{Alg}}(E \otimes_F F[G], R). \end{aligned}$$

We describe the rightmost isomorphism. We send $\varphi\colon F[G] \to R$ to

$$\begin{aligned} \varphi_E\colon E \otimes_F F[G] &\to R \\ e \otimes f &\mapsto e \cdot \varphi(f) \end{aligned}$$

where $e \in E$ and $f \in F[G]$.

Example 40. The algebraic group $(F, +)$ is denoted $\mathbb{G}_a$ and is called *the additive group*. This is the functor $G_a(R) = R$. We have $\mathbb{G}_a = h^{F[t]}$.

Example 41. The group $(F^\times, \cdot)$ is denoted GL_1 or $\mathbb{G}_m$ and is called *the multiplicative group*. We have $\mathbb{G}_m(R) = R^\times$ and $\mathbb{G}_m = h^{F[t,t^{-1}]}$.
Extending scalars we get $\mathrm{GL}_1(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{GL}_1(\mathbb{R})$. We have

$$\mathbb{G}_m\colon \mathbb{Q}-\mathbf{Alg} \to \mathbf{Group}$$

and the extension of scalars is

$$(\mathbb{G}_m)_{\mathbb{R}} = h^{\mathbb{R} \otimes \mathbb{Q}[t,t^{-1}]}.$$

Here

$$\mathbb{R} \otimes \mathbb{Q}[t, t^{-1}] \cong \mathbb{R}[t, t^{-1}]$$

are Laurent polynomials with coefficients in $\mathbb{R}$.

Example 42 (A Typical Application). Let G be an $\mathbb{F}$-algebraic group. It is sometimes easier to understand the extension of scalars $G_{\bar{\mathbb{F}}}$ to the algebraic closure.
Conversely, given an $\bar{\mathbb{F}}$-algebraic group G we want to understand those G' which are $\mathbb{F}$-algebraic such that $G \cong (G')_{\bar{\mathbb{F}}}$. Such G' is called an $\mathbb{F}$-form of G.

Example 43 (Orthogonal Groups). Any symmetric $Q \in M_n(\mathbb{R})$ defines a bilinear form

$$\langle x, y\rangle_Q = x^t Q y.$$

We get an $\mathbb{R}$-algebraic group

$$\mathcal{O}^Q(\mathbb{R}) := \left\{g \in \mathrm{GL}_n(\mathbb{R}) \mid gQg^t = Q\right\} = \left\{g \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; \forall x, y \in \mathbb{R}^n\colon\ \langle gx, gy\rangle_Q = \langle x, y\rangle_Q\right\}$$

and similarly a $\mathbb{C}$-algebraic group

$$O^Q(\mathbb{C}) := \left\{g \in \mathrm{GL}_n(\mathbb{C}) \mid gQg^t = Q\right\}.$$

If $\det Q \neq 0$, the form $\langle,\rangle_Q$ is non-degenerate.
In $M_n(\mathbb{C})$, for any such Q_1, Q_2 there's $g_0 \in \mathrm{GL}_n(\mathbb{C})$ such that

$$g_0 Q_1 g_0^t = Q_2.$$

Then

$$g_0 O^{Q_1}(\mathbb{C})\, g_0^{-1} = O^{Q_2}(\mathbb{C})$$

so one denotes

$$\mathrm{O}_n := \mathrm{O}^I$$

which is equal to any O^Q.
Going back to $\mathbb{R}$, we get that for any symmetric Q one has

$$\left(\mathrm{O}^Q\right)_{\mathbb{C}} = (\mathrm{O}_n)_{\mathbb{C}}.$$

Over $\mathbb{R}$ itself there can be non-equivalent forms, for example $Q_1 = I_n$ and $Q_2 = \begin{pmatrix} I_k & \\ & -I_\ell \end{pmatrix}$ where $k + \ell = n$. Check that in this case $O^{Q_1}(\mathbb{R}) \cong O^{Q_2}(\mathbb{R})$ as abstract groups.

Example 44. Let

$$U_n := \left\{g \in \mathrm{GL}_n(\mathbb{C}) \mid g^t \cdot \bar{g} = I_n\right\} \subseteq M_n(\mathbb{C})$$

be the unitary group. If $z \in \mathbb{C}$, the conjugate $\bar{z}$ is not a polynomial, but if we treat this over $\mathbb{R}$, it is. Hence U_n is an $\mathbb{R}$-algebraic group. We can embed $M_n(\mathbb{C}) \subseteq M_{2n}(\mathbb{R}) \subseteq M_{2n}(\mathbb{C})$, and then ask what is $(U_n)_{\mathbb{C}}$.
We embed

$$s\colon M_n(\mathbb{C}) \hookrightarrow M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

with $s(A) = \left(A, \bar{A}\right)$. This is $\mathbb{R}$-linear and multiplicative, hence an embedding of $\mathbb{R}$-algebras. In fact,

$$M_n(\mathbb{C}) \oplus M_n(\mathbb{C}) = s(M_n(\mathbb{C})) \oplus is(M_n(\mathbb{C})).$$

For example,

$$\mathbb{C} \oplus \mathbb{C} = \{(z, \bar{z}) \mid z \in \mathbb{C}\} \oplus \{(z, -\bar{z}) \mid z \in \mathbb{C}\}.$$

Now,

$$s(U_n) = s(M_n(\mathbb{C})) \cap \left\{(A, B) \mid A^t B = I\right\} = s(M_n(\mathbb{C})) \cap \left\{\left(A, \left(A^t\right)^{-1}\right) \;\middle|\; A \in \mathrm{GL}_n(\mathbb{C})\right\}.$$

We get an embedding

$$\begin{aligned} \iota\colon \mathrm{GL}_n(\mathbb{C}) &\hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \\ g &\mapsto \left(g, \left(g^t\right)^{-1}\right) \end{aligned}$$

such that

$$s(U_n) = s(M_n(\mathbb{C})) \cap \iota(\mathrm{GL}_n(\mathbb{C}))$$

and $(U_n)_{\mathbb{C}} \cong \mathrm{GL}_n(\mathbb{C})$.
We could also embed

$$\begin{aligned} \ell\colon \mathrm{GL}_n(\mathbb{C}) &\hookrightarrow \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \\ g &\mapsto (g, g) \end{aligned}$$

in which case we get

$$\mathrm{GL}_n(\mathbb{R}) \cong s(M_n(\mathbb{R})) \cap \ell(\mathrm{GL}_n(\mathbb{C}))$$

and $(\mathrm{GL}_n)_{\mathbb{C}} \cong \mathrm{GL}_n$.