Proof of 1000-digit Fibonacci Number

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We want to invert the explicit formula for the Fibonacci numbers as to get an $\mathrm{O}(1)$ solution. Since F_n is approximately $\frac{\phi^n}{\sqrt{5}}$, the index n of F_n will be approximated by $\log_\phi\left(\sqrt{5}F_n\right)$.

Theorem 0.1. *Let F_n be the n^{th} Fibonacci number, for $n > 1$. Then if F is a Fibonacci number, its index n such that $F = F_n$ is given by $n(F) = \left[\log_\varphi\left(\sqrt{5}F\right)\right]$, where $[\cdot]$ is rounding to the nearest integer.*

Proof. It is known that $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and where $\psi = \frac{1-\sqrt{5}}{2}$. We get

$$\begin{aligned}\left|F_n - \frac{\psi^n}{\sqrt{5}}\right| &= \left|\frac{\psi^n}{\sqrt{5}}\right| \\ &= \frac{1}{\sqrt{5}}\left|\psi^n\right|\end{aligned}$$

where $|\psi^n| < \frac{1}{2}$ for $n > 1$ since

$$|\psi| = \frac{\sqrt{5}-1}{2} < \frac{3-1}{2} = 1 < \sqrt{2}.$$

Hence

$$\left|F_n - \frac{\psi^n}{\sqrt{5}}\right| \leq \frac{1}{2\sqrt{5}}.$$

Taking $N(F) = \log_\varphi\left(\sqrt{5}F\right)$, since φ^x is convex in x we get that

$$\begin{aligned}\left|N\left(F_n\right) - n\right| &\leq \left|\varphi^{N(F_n)} - \varphi^n\right| \\ &= \left|\varphi^{\log_\varphi\left(\sqrt{5}F\right)} - \varphi^n\right| \\ &= \left|\sqrt{5}F_n - \varphi^n\right| \\ &= \sqrt{5}\left|F_n - \frac{\varphi^n}{\sqrt{5}}\right| \\ &\leq \frac{\sqrt{5}}{2\sqrt{5}} = \frac{1}{2}.\end{aligned}$$

Hence

$$n = [N(F)] = \left[\log_{\varphi}\left(\sqrt{5}F\right)\right],$$

as required. ◻

Corollary 0.2. *The minimal* $n \in \mathbb{N}$ *such that* F_n *has at least* k *digits is one of the following*

$$\left[\log_{\varphi}\left(\sqrt{5}\right) + (k-1)\log_{\varphi}(10)\right],$$
$$\left[\log_{\varphi}\left(\sqrt{5}\right) + (k-1)\log_{\varphi}(10)\right] + 1.$$

Proof. We need to find n such that $F_n \geq 10^{k-1}$ and $F_{n-1} \leq 10^{k-1}$. We have

$$n = \left[\log_{\varphi}\left(\sqrt{5}F_n\right)\right] \geq \left[\log_{\varphi}\left(\sqrt{5}\cdot 10^{k-1}\right)\right]$$

and

$$n-1 = \left[\log_{\varphi}\left(\sqrt{5}F_{n-1}\right)\right] \leq \left[\log_{\varphi}\left(\sqrt{5}\cdot 10^{k-1}\right)\right].$$

Therefore, n is either $\left[\log_{\varphi}\left(\sqrt{5}\cdot 10^{k-1}\right)\right]$ or $\left[\log_{\varphi}\left(\sqrt{5}\cdot 10^{k-1}\right)\right]+1$. Using basic properties of the logarithm, we get the result. ◻