Proof of Correctedness for the Sum Square Difference

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May 16, 2024

Lemma 0.1. *Let* $n \in \left[6, e^{95}\right]$ *be an integer and let* p_n *be the* n^{th} *prime number. We have*

$$p_n \leq n\left(\log n + \log\log n\right).$$

Proof. See [Ros41]. □

Theorem 0.2. *Let* $n \in \left[6, e^{95}\right]$ *be an integer, let* $C = n\left(\log n + \log\log n\right)$*, such that* $p_n \leq C$ *by Theorem 0.1. The* n^{th} *prime number can be computed as follows.*

1. *Let* $S_1 = [C] \setminus \{1\}$ *be a set of candidates, and let* $i = 1$*.*
2. *Let* $m = \min(S)$ *be the minimum of* S*.*
3. *If* $i = n$*, take* $p_n = m$*. Otherwise, take* S_{i+1} *to be the set* S_i *minus all the multiples of* m*, and then increase* i *by* 1*.*
4. *Repeat steps 2-3.*

Proof. We prove inductively on i that $\min(S_i) = p_i$, which shows the result.

Base: We have $S_1 = [C] \setminus \{1\}$ so $\min(S_1) = 2$, which is the first prime.

Step: Assume the minimum of S_k is the k^{th} prime for all $k < i$. Then $\min(S_i)$ is not divisible by these primes $p_1, \ldots, p_{i-1}$, by definition of S_i. Hence $\min(S_i) \geq p_i$. However, $p_i \in S_i$ since $p_i \in S_1$ and since at each step we only removed multiples of one of $p_1, \ldots, p_{i-1}$. Hence $\min(S_i) = p_i$.

□

References

[Ros41] Barkley Rosser. "Explicit Bounds for Some Functions of Prime Numbers". In: *American Journal of Mathematics* 63.1 (1941), pp. 211–232. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2371291 (visited on 05/16/2024).