Proof of Correctedness for Highly Divisible Triangular Numbers

Alan Sorani

May 26, 2024

Lemma 0.1. *Let $m \in \mathbb{N}_+$ and let*

$$m = \prod_{i \in [k]} p_i^{\alpha_i}$$

be its prime factorization. Then

$$\omega(m) = \prod_{i \in [k]} (\alpha_i + 1),$$

where $\omega(m)$ denotes the number of divisors of m.

Proof. The divisors of m are the numbers of the form $\prod_{i \in [k]} p_i^{\beta_i}$ for $\beta_i \in \{0, \ldots, \alpha_i\}$, and since one can choose each β_i separately and each choice of $(\beta_i)_{i \in [k]}$ gives a different divisor, we get the result. $\square$

Corollary 0.2. *For coprime $m, n \in \mathbb{N}_+$ we have*

$$\omega(mn) = \omega(m)\,\omega(n).$$

Corollary 0.3. *Let T_n denote the n^{th} triangular number, i.e.*

$$T_n = \sum_{k \in [n]} k = \frac{n(n+1)}{2}.$$

We have

$$\omega(T_n) = \begin{cases} \omega\left(\frac{n}{2}\right)\omega(n+1) & n \in 2\mathbb{N}_+ \\ \omega(n)\,\omega\left(\frac{n+1}{2}\right) & n \notin 2\mathbb{N}_+ \end{cases}$$