Lecture Notes to Fundamental Concepts in Representation Theory

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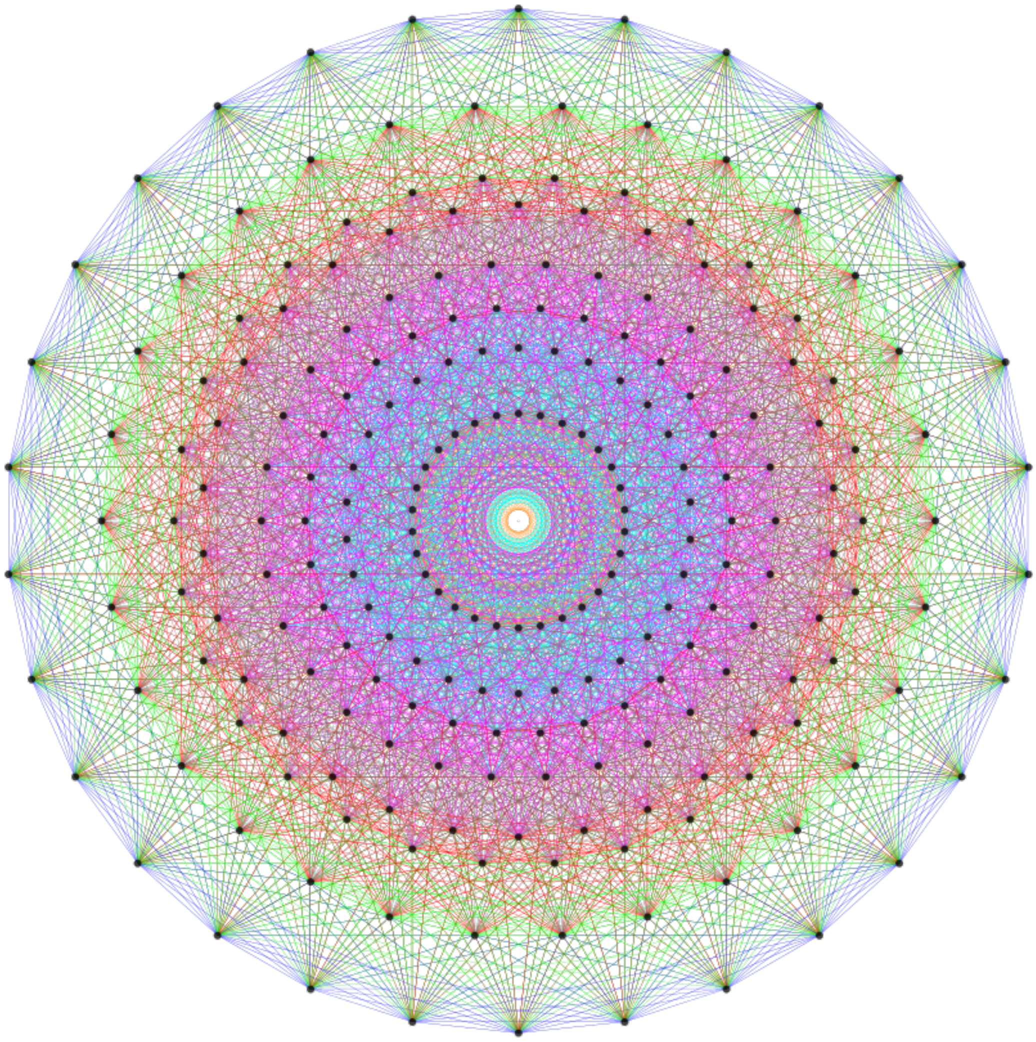

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Preface

Technicalities

These aren't formal notes related to the course and henceforward there is *absolutely no guarantee* that the recorded material is in correspondence with the course expectations, or that these notes lack any mistakes.
In fact, there probably are mistakes in the notes! I would highly appreciate if any comments or corrections were sent to me via email at tzorani.elad@gmail.com.
Elad Tzorani.

Overview

This course will first discuss representation theory of finite groups and semisimple modules and rings. These are somehow related as described in the notes.

We later discuss character theory, induction and related topics.

If time permits, follows an introduction to representation theory of compact groups.

Chapter 1

G-Groups and G-Sets

Notation 1.0.1. Let $X \in \mathbf{Set}$, we denote by S_X the group of set automorphisms of X.

Definition 1.0.2 (G-Set). Let $G \in \mathbf{Grp}$. A ***G-set*** is a set X equipped with a group homomorphism $\rho\colon G \to S_X$.

Remark 1.0.3. Equivalently a G-set is a set X with a map $a\colon G \times X \to X$ such that

$$\begin{aligned} a(1,x) &= x \\ a(g_1 g_2, x) &= a(g_1, a(g_2, x)). \end{aligned}$$

This is given by

$$\rho(g)(x) = a(g,x).$$

Notation 1.0.4. We don't usually write ρ, a explicitly. We instead write $gx \coloneqq \rho(g)(x)$.

Example 1. Let $V \in \mathbf{Vect}_{\mathbb{R}}^{\mathrm{fin}}$ equipped with an inner product.
Let $\mathrm{O}(V) \subseteq \mathrm{GL}(V)$ be the orthogonal group and let

$$S(V) \coloneqq \{v \in V \mid \|v\| = 1\}.$$

Then $\mathrm{O}(V)$ acts on $S(V)$, which we denote $O(V) \curvearrowright S(V)$.

Example 2 (Action on Quotient). Let $G \in \mathbf{Grp}$ and $H \leq G$ a subgroup. Then $G \curvearrowright {}^{G}/_{H}$ by

$$g \cdot \tilde{g}H = g\tilde{g}H.$$

Definition 1.0.5 (Morphism of G-Sets). A ***morphism*** of G-Sets X, Y is a map $f\colon X \to Y$ such that

$$\forall g \in G \forall x \in X\colon f(gx) = gf(x).$$

Definition 1.0.6 (Isomorphism of G-Sets). An ***isomorphism*** of G-sets is a morphism which admits an inverse morphism.

Exercise 1. A bijective morphism of G-sets is an isomorphism.

Notation 1.0.7. We denote the set of morphisms from G-sets X to Y by $\mathrm{Maps}_G(X,Y)$.

Definition 1.0.8 (Transitive G-Set). A G-set X is ***transitive*** if it's nonempty and

$$\forall x_0, x_1 \in X \exists g \in G\colon gx_0 = x_1.$$

Notation 1.0.9. Let $G \in \mathbf{Grp}$, we denote the category of G-sets with morphisms of G-sets by G-$\mathbf{Set}$.

Definition 1.0.10. Let $X \in G$-**Set** and let $x_0 \in X$. Denote the ***stabiliser***

$$\mathrm{Stab}_G(x_0) = \{g \in G \mid gx_0 = x_0\} \subseteq G$$

which is a subgroup of G.

Remark 1.0.11.

$$G \big/ \mathrm{Stab}_G(x_0) \cong X.$$

Example 3. Let $e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in V$ and let $W = \mathrm{Span}(e)^{\perp}$. Then

$$\mathrm{Stab}_{\mathrm{O}(V)}(e) \cong \mathrm{O}(W).$$

Example 4. Let $X \in$ **Set**, there's a natural action $S_X \curvearrowright X$.

Example 5. $S_2 \coloneqq S_{\{1,2\}} \curvearrowright \{1,2\}$ is indecomposable in some sense. We cannot decompose the permutation (12).

We have however $(12)^2 = \mathrm{id}$. In the case of basis permutation in linear algebra, in such cases we'd like to decompose the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to its restriction to eigenspaces with eigenvalues ± 1.

Notation 1.0.12. Let henceforth $k \in$ **Field**, $G \in$ **Grp**.

Definition 1.0.13 (Group Representation). A ***representation of*** G ***over*** k is $V \in \mathbf{Vect}_k$ together with a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V).$$

Notation 1.0.14. We denote as above $gv \coloneqq \rho(g)(v)$.

Definition 1.0.15 (Morphism of G-Representations). A ***morphism of*** G***-representations from*** V ***to*** W is a linear map $T\colon V \to W$ such that

$$\forall v \in V \forall g \in G\colon T(gv) = gT(v).$$

Notation 1.0.16. Denote by $\mathrm{Hom}_G(V,W)$ the set of G-representation-morphisms from V to W.

Remark 1.0.17. $\mathrm{Hom}_G(V,W) \subseteq \mathrm{Hom}(V,W)$ is a k-sub-vector-space.

Definition 1.0.18 (Isomorphism of G-Representations). A ***isomorphism of*** G***-representations from*** V ***to*** W is a morphism which admits an inverse.

Exercise 2. An bijective morphism of G-representations is an isomorphism.

Notation 1.0.19. Denote $S_n \coloneqq S_{\{1,\dots,n\}}$.

Notation 1.0.20. Let $X \in$ **Set**, we denote by $\mathrm{Fun}_k(X)$ the vector space of functions $X \to k$.

Example 6. $S_n \curvearrowright k^n$ via

$$\sigma\left(x_1 \vdots x_n\right) = \begin{pmatrix} x_{\sigma^{-1}(1)} \\ \vdots \\ x_{\sigma^{-1}(n)} \end{pmatrix}.$$

More generally $\mathrm{Fun}_k(X)$ has an action $S_X \curvearrowright \mathrm{Fun}_k(X)$ via

$$(\sigma f)(x) \coloneqq f\left(\sigma^{-1}(x)\right).$$

Notation 1.0.21. We denote by $k[X]$ the vector space spanned by the symbols $\{\delta_x \mid x \in X\}$.

Remark 1.0.22. If X is finite, there's a natural isomorphism $\mathrm{Fun}_k(X) \cong k[X]$.

Remark 1.0.23. There's an action $S_X \curvearrowright k[X]$ via

$$\forall \sigma \in S_X \forall x \in X\colon \sigma\delta_x = \delta_{\sigma(x)}.$$

Remark 1.0.24. Given a map $X \to Y$ we have induced maps

$$k[X] \to k[Y]$$
$$\mathrm{Fun}_k(Y) \to \mathrm{Fun}_k(X).$$

Example 7 (Trivial Representations). Let $c \in k$, there's an action $G \curvearrowright k$ via $g \cdot c = c$.

Definition 1.0.25 (Character). A ***character*** is a group homomorphism $G \to k^\times$.

Notation 1.0.26. Given a character $\chi\colon G \to k^\times$ we consider the G-representation k_χ, which as a vector space is k, and the homomorphism being $g \cdot c := \chi(g) \cdot c$.

Exercise 3. Show that

$$\chi \mapsto k_\chi$$

givens a bijection from the set of characters $G \to k^\times$ to the set of isomorphism classes of 1-dimensional G-representations over k.

Definition 1.0.27 (G-Subrepresentation). Let V a G-representation and $W \leq V$ a subspace. We say W is a ***G-subrepresentation of V*** if

$$\forall g \in G \forall w \in W\colon gw \in W.$$

Example 8. Let $S_n \curvearrowright k^n$, we can consider the subrepresentation

$$W := \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \, x_1 + \ldots + x_n = 0 \right\} \subseteq k^n$$

where the condition $x_1 + \ldots + x_n = 0$ is indeed invariant under permutations.

Definition 1.0.28 (Quotient Representation). Let V a G representation and $W \subseteq V$ a G-subrepresentation. We define ***the quotient representation*** V/W via

$$g(v + W) = gv + W.$$

Definition 1.0.29 (Irreducible (alt. Simple) Representation). We say that a G-representation V is ***irreducible*** if $V \neq 0$ and there are no subresentations of V except 0 and V.

Definition 1.0.30 (Indecomposable Representation). We say that a G-representation V is ***indecomposable*** if given $W_1, W_1 \leq V$ such that $V = W_1 \oplus W_2$ we have

$$W_1 = 0 \vee W_2 = 0.$$

Example 9. Let $S_2 \curvearrowright k^2$ as before. Assume that $\mathrm{char}(k) = 2$. Then k^2 is not irreducible. It has the subrepresentation

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \, x_1 + x_2 = 0 \right\}.$$

However, k^2 is indecomposable. Informally, if $k^2 = L_1 \oplus L_2$ is nontrivial, L_1, L_2 are one-dimensional so are isomorphic to the characters Then

$$(\chi(12))^2 = 1 \implies \chi(12) = 1$$

so $L_1, L_2 \cong k$ so (12) should act as the identity on k^2, a contradiction.

Notation 1.0.31. We sometimes say G-morphism to mean a morphism of G-representations.

Definition 1.0.32 (Kernel). Let $T\colon V \to W$ a G-morphism, we define the ***kernel of*** T to be

$$\ker(T) = \{v \in V \mid T(v) = 0\}.$$

Definition 1.0.33 (Direct Sum). Let V, W be G-representations. We define $V \oplus W$ to be $V \times W$ as a set, with the action

$$g(v, w) = (gv, gw).$$

Definition 1.0.34 (Semisimple Representation). Let V a G-representation, we say V is ***semisimple*** if for any subrepresentation $W \leq V$ there exists a subrepresentation $U \leq V$ such that

$$V = W \oplus U.$$

Remark 1.0.35. A representation which is semisimple and indecomposable is irreducible.

Theorem 1.0.36 (Maschke). *Let* $G \in \mathbf{Grp}^{\mathrm{fin}}$ *and assume* $\operatorname{char}(k) \nmid |G|$.
Then every finite-dimensional G*-representation over* k *is semisimple.*

Proof. Let V be a finite-dimensional G-representation over k and let $\rho\colon G \to \mathrm{GL}(V)$ be the corresponding homomorphism. Let $W \leq V$ a subrepresentation.
Let $P\colon V \to V$ be a projection operator onto W. I.e.

$$P|_W = \mathrm{id}_W, \quad \mathrm{Im}(P) = W.$$

Let

$$Q \coloneqq \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ P \circ \rho(g)^{-1} \in \mathrm{End}(V).$$

We want to show Q is a projection on W. Indeed

- We have $\rho(g)(W) \subseteq W$ for every g, hence $\mathrm{Im}\, Q \subseteq W$.
- If $w \in W$ we have
  $$\rho(g) P \rho(g)^{-1} w = \rho(g) \rho(g)^{-1}(w) = w$$
  where in the first equality we use the fact that $P|_W = \mathrm{id}_W$. Hence $Qw = w$ so $Q|_W = \mathrm{id}_W$.

We want to show Q is a G-morphism. I.e. that for every $h \in G$ we have $Q \circ \rho(h) = \rho(h) \circ Q$. Indeed

$$\begin{aligned}
Q \circ \rho(h) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ P \circ \rho(g)^{-1} \circ \rho(h) \\
&= \frac{1}{|G|} \sum_{g \in G} \rho(g) \circ P \circ \rho\left(\left(h^{-1} g\right)^{-1}\right) \\
&\underset{g \mapsto hg}{=} \frac{1}{|G|} \sum_{g \in G} \rho(hg) \circ P \circ \rho\left(g^{-1}\right) \\
&= \rho(h) \circ \left(\frac{1}{|G|} \sum_{g \in G} \rho(g) \circ P \circ \rho(g)^{-1}\right) \\
&= \rho(h) \circ Q.
\end{aligned}$$

So, Q is a G morphisms so $V = W \oplus \ker Q$, as required. ∎

Definition 1.0.37. Let V, W be G-representations. We define $G \curvearrowright \mathrm{Hom}(V, W)$ via

$$\forall g \in G \forall T \in \mathrm{Hom}(V, W) \forall v \in V\colon\ (gT)(v) = gT\left(g^{-1} v\right).$$

Definition 1.0.38 (Subspace of Invariants). Let V be a G-representation. We define the ***subspace of invariants***

$$V^G := \{v \in V \mid \forall g \in G \colon gv = v\} \subseteq V.$$

Definition 1.0.39 (Averaging Operator). Let V be a G-representation and assume $G \in \mathbf{Grp}^{\mathrm{fin}}$. We define the ***averaging operator***

$$\mathrm{Av}_V^G(v) := \frac{1}{|G|} \sum_{g \in G} gv.$$

Remark 1.0.40. Av_V^G is a G-morphism and a projection operator with image V^G.

Assume throughout that G is finite, V, W are G-representations and $\mathrm{char}(k) \nmid |G|$.

Exercise 4. Prove that

$$\mathrm{Hom}_G(V, W) = \mathrm{Hom}(V, W)^G$$

where the right-hand side is the set of fixed points under the G-action.

Proof (2^{nd} proof for 1.0.36 where $k \in \{\mathbb{R}, \mathbb{C}\}$). Assume V is finite-dimensional and $W \leq V$. Let $\langle \cdot, \cdot \rangle_0$ be an inner product on V.

Denote

$$\langle v_1, v_2 \rangle := \frac{1}{|G|} \sum_{g \in G} \langle gv_1, gv_2 \rangle_0 .$$

This is then an inner product and is g-invariant in the sense that $\langle gv_1, gv_2 \rangle = \langle v_1, v_2 \rangle$.

Let U be the orthogonal complement of W with respect to $\langle \cdot, \cdot \rangle$. Then $V = W \oplus U$ and we're left to check that U is G-invariant (then a G-representation).

Indeed

$$\langle w, gu \rangle = \left\langle g^{-1}w, g^{-1}gu \right\rangle = \left\langle g^{-1}w, u \right\rangle = 0. \qquad \blacksquare$$

Proof (3^{rd} proof for 1.0.36). Consider the projection

$$P \colon V \twoheadrightarrow {}^{V}\!/_{W}.$$

We want to find a G-morphism $i \colon {}^{V}\!/_{W} \to V$ such that $P \circ i = \mathrm{id}$. Then $\mathrm{Im}(i) \oplus W = V$ by the splitting lemma.

More generally, given G-representations X, Y, Z and a G-morphism $\varphi \colon Y \to Z$ we want to see that

$$\mathrm{Hom}_G(X, Y) \xrightarrow{\varphi \circ -} \mathrm{Hom}_G(X, Z)$$

is surjective. In our case $Z = {}^{V}\!/_{W}$ and $X := {}^{V}\!/_{W}$ so $\mathrm{id} \in \mathrm{Hom}_G\left({}^{V}\!/_{W}, {}^{V}\!/_{W}\right)$ has a preimage in $\mathrm{Hom}_G\left({}^{V}\!/_{W}, V\right)$ under $P \circ -$.

Consider

$$\mathrm{Hom}(X, Y) \xrightarrow{P \circ -} \mathrm{Hom}(X, Z).$$

This is a G-morphism and is surjective by linear algebra.

Reformulating the problem, given G-representations V, Z and a surjective morphism $p \colon V \twoheadrightarrow Z$ the resulting map

$$V^G \to Z^G$$

is also surjective.

Indeed, let $z \in Z^G$. Let $v \in V$ be such that $p(v) = z$. Then

$$p\left(\mathrm{Av}_V^G(v)\right) = \mathrm{Av}_Z^G(p(v)) = \mathrm{Av}_Z^G(z) = z. \qquad \blacksquare$$

Chapter 2

Decomposition to Irreducible Subspaces

Lemma 2.0.1. *Assume V is finite-dimensional, there exist G-representations $E_1, \dots, E_n$ such that*

$$V \cong \bigoplus_{i \in [n]} E_i$$

as G-representations.

Proof. If $V = 0$, it's the empty sum. Then argue by induction on $\dim V$ using 1.0.36. ■

We want to understand in which sense the above decomposition is unique.

Lemma 2.0.2 (Schur). *Let E, F be irreducible G-representations. Let $T \colon E \to F$ a G-morphism. Then either $T = 0$ or T is an isomorphism.*

In particular, if E, F are not isomorphic then $\operatorname{Hom}_G(E, F) = 0$.

Proof. Let $T \colon E \to F$ be a non-zero G-morphism.

Consider $\ker(T) \leq E$ which is a G-subrepresentation. Therefore since E is irreducible $\ker(T) = 0$ or $\ker(T) = E$, the latter of which implies $T = 0$ in contradiction. Hence $\ker(T) = 0$ so T is injective.

Consider $\operatorname{Im}(T) \leq F$, by irreducibility $\operatorname{Im}(T) = 0$ or $\operatorname{Im}(T) = F$ the first of which is a contradiction. Hence $\operatorname{Im}(T) = F$ so T is surjective.

T is then bijective, hence an isomorphism. ■

Claim 2.0.3. Assume V is finite-dimensional with decompositions as sums of irreducible G-representations.

$$V \cong \bigoplus_{i \in [r]} E_i$$
$$V \cong \bigoplus_{j \in [s]} F_j.$$

Then for an irreducible G-representation E, the number of $i \in [r]$ for which E_i is isomorphic to E i equal to the number of $j \in [s]$ for which F_j is isomorphic to E, both of which are equal to

$$\frac{\dim \operatorname{Hom}_G(V, E)}{\dim \operatorname{Hom}_G(E, E)}.$$

Definition 2.0.4 (G-endomorphism). Denote $\operatorname{End}_G(E) = \operatorname{Hom}_G(E, E)$ and call an element of $\operatorname{End}_G(E)$ a ***G-endomorphism***.

Proof. Denote $d \coloneqq \dim \operatorname{End}_G(E) \in \mathbb{Z}_{\geq 1}$. For an irreducible G-representation F we have $\dim \operatorname{Hom}_G(F, E)$ is zero if F is not isomorphic to E by 2.0.2, and is d is F is isomorphic to E. In the latter case there are isomorphisms $F \cong E$ which induce isomorphisms

$$\operatorname{Hom}_G(F, E) \cong \operatorname{Hom}_G(E, E)$$

defined via pre-composition.

Now

$$\begin{aligned}\dim \mathrm{Hom}_G(V,E) &= \dim \mathrm{Hom}_G\left(\bigoplus_{i\in[r]} E_i, E\right)\\ &= \dim\left(\bigoplus_{i\in[r]} \mathrm{Hom}_G(E_i,E)\right)\\ &= \dim \sum_{i\in[r]} \mathrm{Hom}_G(E_i,E)\\ &= d\cdot|\{i\in[r]\mid E_i\cong E\}|\end{aligned}$$

where all equalities before the last are due to isomorphisms of the vector spaces. So

$$|\{i\in[r]\mid E_i\cong E\}| = \frac{\dim \mathrm{Hom}_G(V,E)}{d},$$

as required. ■

Definition 2.0.5. Assume V is finite-dimensional and let E an irreducible G-representation. The ***multiplicity of*** E ***in*** V is

$$[V:E] := |\{i\in[r]\mid E_i\cong E\}|$$

where $V\cong\bigoplus_{i\in[r]} E_i$ is a decomposition to irreducible representations and where $[V:E]$ is well-defined by the previous claim.

Lemma 2.0.6 (Schur (2)). *Assume $k=\bar{k}$ and let E an irreducible irreducible G-representation. Then* $\mathrm{End}_G(E) = k\cdot\mathrm{id}_E$.

In particular $\dim \mathrm{End}_G(E) = 1$.

Exercise 5. Assume $H\in\mathbf{Grp}^{\mathrm{fin}}$. An irreducible H-representation is finite-dimensional.

Proof. Let $T\in\mathrm{End}_G(E)$. Since E is finite-dimensional and $k=\bar{k}$ there's $\lambda\in k$ such that $T-\lambda\cdot\mathrm{id}_E$ is not invertible. Then by 2.0.2 $T-\lambda\cdot\mathrm{id}_E=0$, so $T=\lambda\cdot\mathrm{id}_E$, as required. ■

Corollary 2.0.7. *By 2.0.6 and the previous claim, if $k=\bar{k}$ and E one has*

$$[V:E] = \dim \mathrm{Hom}_G(V,E).$$

Definition 2.0.8 (The Regular *G*-Set). Let G with the left G-action by multiplication be ***the regular*** G***-set***.

I.e. the action is

$$a(g,\tilde{g}) = g\tilde{g}.$$

Definition 2.0.9 (The Regular *G*-Representation). ***The regular*** G***-representation*** is the corresponding $k[G]$ of the regular G-set.

Remark 2.0.10. The regular G-representation has a basis $\{\delta_g\}_{g\in G}$ with $g\cdot\delta_{\tilde{g}} = \delta_{g\tilde{g}}$.

Remark 2.0.11. From linear algebra we have a natural isomorphism of vector spaces

$$\mathrm{Hom}(k[X],V)\cong\mathrm{Maps}(X,V)$$

where $\mathrm{Maps}(X,V)$ are arbitrary maps. This is given by

$$T\mapsto(x\mapsto T(\delta_x)).$$

If X is a G-set the isomorphism induces an isomorphism of vector spaces

$$\mathrm{Hom}_G(k[X],V)\cong\mathrm{Maps}_G(X,V).$$

Fact 2.0.12 (For those who know category theory). *$k[-]$ is in fact left-adjoint to the functor forgetting the linear structure of a G-representation.*

Proposition 2.0.13. *Let E be an irreducible G-representation. Then*

$$[k[G]:E] = \frac{\dim E}{\dim \mathrm{End}_G(E)}.$$

Proof. Using the observation in the last remark we get

$$\begin{aligned}[k[G]:E] &= \frac{\dim \mathrm{Hom}_G(k[G],E)}{\dim \mathrm{End}_G(E)} \\ &= \frac{\dim \mathrm{Maps}_G(G,E)}{\dim \mathrm{End}_G(E)} \\ &= \frac{\dim E}{\dim \mathrm{End}_G(E)}.\end{aligned}$$

■

Corollary 2.0.14. *There are finitely many irreducible G-representations up to isomorphism. If $k[G] = \bigoplus_{i\in[r]} E_i$ and E is irreducible, it is isomorphic to one of the E_i as by the above proposition $[k[G]:E] \neq 0$.*

Corollary 2.0.15. *Assume $k = \bar{k}$. Let $(E_i)_{i\in[r]}$ be a section of the irreducible G-representation up to isomorphism. Then*

$$|G| = \sum_{i\in[r]} (\dim E_i)^2.$$

Proof. By direct computation

$$\begin{aligned}|G| &= \dim k[G] \\ &= \sum_{i\in[r]} [k[G]:E_i]\cdot \dim E_i \\ &= \sum_{i\in[r]} (\dim E_i)^2\end{aligned}$$

where in the last equality we use 2.0.13 and 2.0.6. ■

Example 10. Let $G = S_3$ and assume $\mathrm{char}(k) \notin \{2,3\}$ and $k = \bar{k}$.

There are two characters $S_3 \to k^\times$, the trivial one and sgn. Corresponding to them are two one-dimensional irreducible representations $k_{\mathrm{triv}}, k_{\mathrm{sgn}}$.

Using 2.0.15 we get that $G = 1^2 + 1^2 + 2^2$ is the only solution, so there's one other representation which is of dimension 2. We've seen another representation of S_3,

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \; x_1 + x_2 + x_3 = 0 \right\},$$

which is then the required 2-dimensional representation.

Chapter 3

Preliminary Algebraic Structures

Definition 3.0.1 (k-Algebra). A k***-algebra*** $A \in$ **Ring** which is also a k-vector space, such that the multiplication map $A \times A \to A$ is k-bilinear.

Remark 3.0.2. Given a k-algebra A we have a ring homomorphism $k \to Z(A)$ given by $c \mapsto c \cdot 1$. It is injective, so we can think of k as sitting inside A.

One can therefore think of a k-algebra as a ring A equipped with a ring homomorphism $k \to Z(A)$.

Definition 3.0.3 (Module). Let $R \in$ **Ring**. A ***(left)*** R***-module*** is $M \in$ **Ab** equipped with a biadditive map

$$\begin{aligned} R \times M &\to M \\ (r, m) &\mapsto rm \end{aligned}$$

such that

$$\begin{gathered} \forall m \in M \colon 1 \cdot m = m \\ \forall r_1, r_2 \in R \forall m \in M \colon \ (r_1 r_2) \cdot m = r_1 (r_2 m) . \end{gathered}$$

Notation 3.0.4. Denote by R-**Mod** the category of left R-modules.

Definition 3.0.5 (Morphism of Modules). Let $R \in$ **Ring** and $f \colon M \to N$ a map of sets such that $M, N \in$ R-**Mod**.

f is ***a morphism of*** R***-modules*** if it's additive such that

$$\forall r \in R \forall m \in M \colon \varphi(rm) = r\varphi(m) .$$

Notation 3.0.6. Denote the category of k-algebras, with morphisms between them being morphisms of the respective modules, by $\mathbf{Alg}_k$.

Remark 3.0.7. Let $A \in \mathbf{Alg}_k$. Any $M \in A$-**Mod** has a structure of a k-vector-space by

$$k \hookrightarrow Z(A) \subseteq A \curvearrowright M .$$

Morphisms of A-modules will be k-linear.

Often do we start with a k-vector space M and give it the structure of an A-module. We then assume implicitly that the k-vector-space structure on M arising from this module structure is the same as the original one.

Definition 3.0.8 (Group Algebra). The ***group algebra of*** G ***over*** k, denoted $k[G]$, is, as a k-vector space $k[G]$ per definition 2.0.9. The ring structure of $k[G]$ is characterised as the unique $k - bilinear$

$$k[G] \times k[G] \to k[G]$$

sending

$$(\delta_g, \delta_h) \mapsto \delta_{gH} .$$

Notation 3.0.9. Let $A \in \mathbf{Ring}$, we denote by $A^{\times}$ the multiplicative group of invertible elements in A.

Exercise 6. Let $A \in \mathbf{Alg}_k$, we have a natural isomorphism

$$\alpha\colon \mathrm{Hom}_{\mathbf{Alg}_k}(k[G], A) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}\left(G, A^{\times}\right).$$

Remark 3.0.10. In categorical language, $(-)^{\times}$ is a right-adjoint functor to $k[-]$.
This isomorphism is given by

$$T \mapsto (g \mapsto T(\delta_g)).$$

Remark 3.0.11 (Naturality). By naturality of α we mean naturality in $\mathbf{Alg}_k$ and in $\mathbf{Grp}$ in the sense that if $f\colon A_1 \to A_2$ in $\mathbf{Alg}_k$ the diagram

$$\begin{array}{ccc} \mathrm{Hom}(k[G], A_1) & \xrightarrow{\alpha_{G,A_1}} & \mathrm{Hom}\left(G, A_1^{\times}\right) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(k[G], A_2) & \xrightarrow{\alpha_{G,A_2}} & \mathrm{Hom}\left(G, A_2^{\times}\right) \end{array}$$

Corollary 3.0.12. *Let $V \in \mathbf{Vect}_k$. We have a natural bijection between structures of a G-representation on V and of a $k[G]$-modules on V.*

Proof. Consider $A \coloneqq \mathrm{End}(V)$. By 6 we get

$$\mathrm{Hom}(k[G], \mathrm{End}(V)) \cong \mathrm{Hom}\left(G, \mathrm{End}(V)^{\times}\right) \cong \mathrm{Hom}(G, \mathrm{GL}(V))$$

where elements in the rightmost term define are G-representation on V. ■

Exercise 7. Let V, W be G-representations and so also $k[G]$-modules. Then $T\colon V \to W$ is a G-morphism iff it's a $k[G]$-morphism.

Example 11. We have

$$k[S_3] \cong k_1 \oplus k_{\mathrm{sgn}} \oplus W \oplus W.$$

The components k_1, k_{sgn} are called ***the isotopic components*** and we have

$$W \oplus \cong M_{2\times 2}(k).$$

Definition 3.0.13 (Division Ring). A ***division ring*** R is a ring such that every element of $R \setminus \{0\}$ is invertible.

Definition 3.0.14 (Division Algebra). A k-algebra is a ***division k-algebra*** if it's a division ring as a ring.

Example 12 (Hamilton's Quaternion Algebra).

Fields are division rings.

The Hamilton quaternion algebra is a 4-dimensional algebra over $\mathbb{R}$ with is a division algebra.

Lemma 3.0.15 (Schur). *Remind our assumptions G is finite and $\mathrm{char}(k) \nmid |G|$.*
Then $\mathrm{End}_G(E)$ is a division k-algebra under composition.

Proof. This is a reformulation of 2.0.2. ■

Claim 3.0.16. Assume $k = \bar{k}$ and let A be a finite-dimensional division k-algebra. Then $A = k$.

Remark 3.0.17. 2.0.6 follows easily from the above claim.

Proof. ***Lemma 3.0.18.*** *Let $\mathbb{F} \in$ **Field**. Let A be a finite-dimensional division $\mathbb{F}$-algebra and let $B \leq A$ a subalgebra. Then B is also a division algebra.*

Proof. Let $b \in B \setminus \{0\}$. Consider the $\mathbb{F}$-linear map

$$\begin{aligned} m_b \colon A &\to A \\ a &\mapsto ba \end{aligned} \quad .$$

Then m_b preserves B and is invertible with $m_b^{-1} = m_{b^{-1}}$. Then $m_b|_B \colon B \to B$ is invertible. Hence there's $x \in B$ such that

$$bx = m_b(x) = 1 \in B.$$

Hence b is right-invertible in B. Similarly b is left-invertible in B, and hence b is invertible in B. ∎

Let $a \in A$ and consider the subalgebra of A generated by a,

$$B := \mathrm{Span}\{a^n \mid n \in \mathbb{N}\}.$$

By the lemma, B is a division k-algebra. On the other hand, B is commutative. So B is a finite extension field over k and since $k = \bar{k}$ we get $a \in k$.

Hence $A = k$. ∎

Remark 3.0.19 (Basis of a Module). Any D-module V admits a ***basis*** $\{v_i\}_{i \in I} \subseteq V$ such that

$$V = \bigoplus_{i \in I} D \cdot v_i.$$

Definition 3.0.20 (Finitely Generated Module). An R-module is ***finitely generated*** if it has a finite generating set.

Definition 3.0.21. The ***dimension*** of a D-module V is the cardinality of a basis of V.

Proposition 3.0.22. *Let $D \in \mathbf{Alg}_k$ and $V \in D\text{-}\mathbf{Mod}$. Then*

$$\dim_k V = \dim_D V \cdot \dim_k D.$$

Definition 3.0.23. Let $R \in \mathbf{Ring}$, we define R^{op} to be the same abelian group with the product

$$x \cdot_{R^{\mathrm{op}}} y = yx.$$

Definition 3.0.24 (Right R-Module). A right R-module is a left R^{op}-module.

Let $R \in \mathbf{Ring}$, $M \in R\text{-}\mathbf{Mod}$ and $S := \mathrm{End}_R(M)$.

Then S has the structure of a ring and M of an S-module.

Given an element in R we get an endomorphism in $\mathrm{End}(M)$ via the action of R. This endomorphism commutes with the endomorphisms of S by definition of S. We get a ring homomorphism

$$R \to \mathrm{End}_S(M).$$

Notation 3.0.25. Let E be an irreducible G-representation, we denote

$$D_E := \mathrm{End}_G(E).$$

Remark 3.0.26. By 3.0.15 D_E is a finite-dimensional division k-algebra.

Remark 3.0.27. We get a map

$$\mathcal{F}_E \colon k[G] \to \mathrm{End}_{D_E}(E).$$

Proposition 3.0.28 (Special Case of Artin-Wederburn). *Let $(E_i)_{i \in [r]}$ be a section of the classes of irreducible representations of G.*

We obtain a morphism

$$\prod_{i \in [r]} \mathbb{F}_{E_i} = \mathcal{F} \colon k[G] \to \prod_{i \in [r]} \mathrm{End}_{D_{E_i}}(E_i).$$

This is an isomorphism.

Proof. **Injectivity:** Let $d \in k[G]$ and assume $F(d) = 0$. d acts by zero on all G-representations by the definition of $\mathcal{F}$.

Since every finite-dimensional G-representation is isomorphic to a direct sum of irreducible G-representations, d acts by zero on any finite-dimensional G-representation. In particular d acts by zero on the regular representation $k[G]$. In particular $d = d \cdot 1 = 0$.

Surjectivity: It's enough to see that the dimensions over k match.

On one hand we have $\dim_k k[G] = |G|$.

Denote $e_i \coloneqq \dim_k E_i$ and $d_i \coloneqq \dim_k D_{E_i}$. Then

$$e_i = \dim_k E_i = \dim_{D_{E_i}} E_i \cdot \underbrace{\dim_k D_{E_i}}_{d_i}$$

so

$$\dim_{D_{E_i}} E_i = \frac{e_i}{d_i}.$$

For every $i \in [r]$ choose a basis of E_i over D_{E_i}. We can identify $\mathrm{End}_{D_{E_i}}(E_i)$ with the k-algebra of matrices over $D_{E_i}^{\mathrm{op}}$ of order $\dim_{D_{E_i}} E_i$. Then

$$\dim_k \mathrm{End}_{D_{E_i}}(E_i) = \dim_k D_{E_i}^{\mathrm{op}} \cdot \dim_{D_{E_i}^{\mathrm{op}}} M_{\frac{e_i}{d_i} \times \frac{e_i}{d_i}}\left(D_{E_i}^{\mathrm{op}}\right) = d_i \cdot \left(\frac{e_i}{d_i}\right)^2 = \frac{e_i^2}{d_i}$$

so

$$\dim(\mathrm{codomain}(\mathcal{F})) = \sum_{i \in [r]} \frac{e_i^2}{d_i}.$$

Then

$$k[G] \cong \bigoplus_{i \in [r]} E_i^{[k[G]:E_i]} \cong \bigoplus_{i \in [r]} E_i^{\oplus \frac{e_i}{d_i}}$$

so

$$|G| = \dim_k k[G] = \sum_{i \in [r]} \frac{e_i}{d_i} \cdot e_i = \sum_{i \in [r]} \frac{e_i^2}{d_i}.$$

∎

Definition 3.0.29 (Center). Let $G \in \mathbf{Grp}$, we call

$$Z(G) \coloneqq \{g \in G \mid \forall a \in G \colon ga = ag\}$$

the center of G.

Remark 3.0.30. There is an isomorphism

$$\mathrm{Fun}_k(G) \xrightarrow{\sim} k[G]$$
$$f \mapsto \sum_{g \in G} f(g) \cdot \delta_g.$$

Definition 3.0.31 (Class Functions). Let

$$\mathrm{Fun}_k(G)^{\mathrm{cl}} \coloneqq \left\{f \in \mathrm{Fun}_k(G) \;\middle|\; \forall g, h \in G \colon f\left(ghg^{-1} = f(h)\right)\right\}$$

be the space of ***class functions on*** G.

Remark 3.0.32. Equivalently

$$\mathrm{Fun}_k(G)^{\mathrm{cl}} = \{f \in \mathrm{Fun}_k(G) \mid \forall g, h \in G \colon f(gh) = f(hg)\}.$$

Exercise 8.

$$Z(k[G]) \cong \mathrm{Fun}_k(G)^{\mathrm{cl}}.$$

I.e. if $d = \sum_{g \in G} c_g \delta_g \in k[G]$ we get

$$\in Z(k[G]) \iff \forall g, h \in G \colon c_{ghg^{-1}} = c_h.$$

Remark 3.0.33. Class functions are functions which are constant on conjugacy classes.
Then
$$\dim Z\left(k\left[G\right]\right) = \dim \mathrm{Fun}_k\left(G\right)^{\mathrm{cl}}$$
is the number of conjugacy classes in G.

Exercise 9.
$$Z\left(\mathrm{End}_{D_{E_i}}\left(E_i\right)\right) = Z\left(M_{\frac{e_i}{d_i}\times\frac{e_i}{d_i}}\left(D_{E_i}^{\mathrm{op}}\right)\right) = Z\left(D_{E_i}^{\mathrm{op}}\cdot\mathrm{id}\right) = Z\left(D_{E_i}^{\mathrm{op}}\right) = Z\left(D_{E_i}\right).$$
Deduce
$$\dim Z\left(\prod_{i\in[r]}\mathrm{End}_{D_{E_i}}\left(E_i\right)\right) = \sum_{i\in[r]}\dim_k Z\left(D_{E_i}\right).$$

Corollary 3.0.34. *Assume $k=\bar{k}$. Then $D_{E_i}=k$ by 2.0.6. The number of conjugacy classes in G is then*
$$\dim_k Z\left(k\left[G\right]\right) = \dim_k Z\left(\prod_{i\in[r]}\mathrm{End}_{D_{E_i}}\left(E_i\right)\right) = \sum_{i\in[r]}1 = r$$
is the number of G-representation classes.

Definition 3.0.35. For $s\in 2\mathbb{Z}$ let
$$\delta_G\left(s\right) \coloneqq \sum_{i\in[r]}\left(\dim_k E_i\right)^{-s}.$$

Corollary 3.0.36. *$xi_G\left(0\right)$ is the number of conjugacy classes of G, which by the above is the number of irreducible G-representation classes.*
Also
$$\xi_G\left(-2\right) = \left|G\right| = \sum_{i\in[r]}\left(\dim_k E_i\right)^2.$$

Lemma 3.0.37. *Assume G is commutative and $k=\bar{k}$. Let E be an irreducible G-representation. Then $\dim_k E = 1$.*

Proof. Check first that every $g\in G$ acts by scalar on E. Let λ be an eigenvalue of g, we get that $E_{g,\lambda}$ is a subrepresentation so $E_{g,\lambda}=E$. ∎

Notation 3.0.38. Denote by
$$\mathrm{Ch}_k\left(G\right) \coloneqq \mathrm{Hom}_{\mathbf{Grp}}\left(G,k^\times\right)$$
the set of characters of G to k.

Corollary 3.0.39. *The family*
$$\left(k_\chi\right)_{\chi\in\mathrm{Ch}_k(g)}$$
contains a section of the classes of irreducible G-representations.

Remark 3.0.40. Let
$$\mu_3 = \left\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\right\}.$$
Then $\mu_3 \curvearrowright \mathbb{C}$ as an $\mathbb{R}$-vector space and this is an irreducible representation on μ_3.

Fact 3.0.41. *If $V\in\mathbf{Vect}_k$ is one-dimensional, $\mathrm{End}_k\left(V\right)\cong k$.*

Definition 3.0.42 (Fourier Transform). Let ***the Fourier Transform of** G*
$$\mathcal{F}\colon k\left[G\right] \xrightarrow{\sim} \prod_{\chi\in\mathrm{Ch}_k(G)} k \xrightarrow{\sim} \mathrm{Fun}_k\left(\mathrm{Ch}_k\left(G\right)\right)$$
by
$$\mathcal{F}\left(\sum_{g\in G}c_g\cdot\delta_g\right) = \left(\chi\mapsto\sum_{g\in G}c_g\cdot\chi\left(g\right)\right).$$

Let $\chi \in \mathrm{Ch}_k(G)$, we want to find $x \in k[G]$ such that $\sigma_\chi \in \mathrm{Fun}_k(\mathrm{Ch}_k(G))$ such that $\mathcal{F}(x) = \delta_\chi$.
We want $\sum_{g\in G} c_g \cdot \delta_g$ such that for $\theta \in \mathrm{Ch}_k(G)$ we have

$$\sum_{g\in G} c_g \theta(g) = \begin{cases} 1 & \theta = \chi \\ 0 & \theta \neq \chi \end{cases}.$$

Take

$$e_\chi \coloneqq \frac{1}{|G|} \sum_{g\in G} \chi(g)^{-1} \cdot \delta_g \in k[G].$$

Then

$$\mathcal{F}(e_\chi) = \delta_\chi.$$

For this we want to check the "orthogonality relations"

$$\frac{1}{|G|} \sum_{g\in G} \chi(g)^{-1} \theta(g) = \delta_\chi.$$

The LHS is

$$\frac{1}{|G|} \sum_{g\in G} \left(\chi^{-1}\theta\right)(g)$$

so this reduces to

$$\frac{1}{|G|} \sum_{g\in G} \chi(g) = \delta_\chi.$$

This is clear when $\chi = 1$, and otherwise requires to prove $\sum_{g\in G} \chi(g) = 0$. Pick $g_0 \in G$ such that $\chi(g_0) \neq 1$, we get

$$\sum_{g\in G} \chi(g) = \sum_{g\in G} \chi(gg_0) = \left(\sum_{g\in G} \chi(g)\right) \chi(g_0).$$

Hence

$$(1 - \chi(g_0)) \sum_{g\in G} \chi(0) = 0$$

so

$$\sum_{g\in G} \chi(g) = 0.$$

For $g \in G$, δ_g acts on χ by

$$\mathcal{F}(\delta_g) = (\chi \mapsto \chi(g)) = \sum_{\chi\in \mathrm{Ch}_k(G)} \chi(g)\, \delta_\chi.$$

Applying $\mathcal{F}^{-1}$ we get

$$\delta_g = \sum_{\chi\in \mathrm{Ch}_k(G)} \chi(g) \cdot e_\chi.$$

Remark 3.0.43. We have two bases for $k[G]$, the ***geometric basis*** $(\delta_g)_{g\in G}$ and the ***spectral basis*** $(e_\chi)_{\chi\in \mathrm{Ch}_k(G)}$.
We have

$$e_\chi = \sum_{g\in G} \frac{1}{|G|} \chi(g)^{-1} \cdot \delta_g$$

where

$$\delta_g = \sum_{\chi\in \mathrm{Ch}_k(G)} \chi(g) \cdot e_\chi.$$

Definition 3.0.44 (Collapse Operators). Let $g_0 \in G$, we define

$$\mathrm{Collapse}_{g_0} \colon k[G] \to k[G]$$
$$\delta_g \mapsto \begin{cases} \delta_g & g = g_0 \\ 0 & g \neq g_0 \end{cases}.$$

Definition 3.0.45 (Shift Operators). Let $g_0 \in G$, we define

$$\begin{aligned} \mathrm{Shift}_{g_0} \colon k\left[G\right] &\to k\left[G\right] \\ \delta_g &\mapsto \delta_{gg_0}. \end{aligned}$$

Fact 3.0.46. *The geometric bases of $k\left[G\right]$ is the eigenbasis for the collapse operators, and the spectral basis is the eigenbasis for the shift operators.*

Theorem 3.0.47 (Dirichlet). *Let $d \in \mathbb{N}_+, a \in \mathbb{Z}$ relatively prime, there are infinitely many primes congruent to a mod d.*

Proof. Let $f \in \mathrm{Fun}_{\mathbb{C}}\left(\left(\mathbb{Z}/d\mathbb{Z}\right)^\times\right)$ and extend this by 0 to $\mathbb{Z}/d\mathbb{Z}$. Let

$$M_f\left(s\right) \coloneqq \sum_{p \text{ prime}} \frac{f\left(p \left(\bmod d\right)\right)}{p^s}.$$

We want $f = \delta_{a (\bmod d)}$.

We have

$$f\left(g\right) = \frac{1}{\varphi\left(d\right)} \sum_{\chi \in \mathrm{Ch}_{\mathbb{C}}\left(\left(\mathbb{Z}/d\mathbb{Z}\right)^\times\right)} \chi\left(a\right)^{-1} \chi$$

where χ are “multiplicative” from which we obtain a sum easier to work with. ■