Homological Algebra — Exercise Page #1

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Exercise 1 (2.9). Let $T=(V,E)$ be a tree. Prove that if $T_i=(V_i,E_i)$ are m subtrees of T such that $V_i\cap V_j\neq\varnothing$ for all i,j then $\bigcap_{i\in[m]}V_i\neq\varnothing$.

Solution. Assume towards a contradiction this isn't the case and let $n\in[m]$ be minimal such that there are $i_1,\ldots,i_n\in[m]$ for which $\bigcap_{j\in[n]}V_{i_j}=\varnothing$.

For every $j\in[n]$ let

$$x_j\in\bigcap_{\substack{j'\in[n]\\ j'\neq j}}V_{i_j},$$

and consider the minimal subtree $T'=(V',E')$ of T that contains each x_j. Notice that T' is finite. Any tree has two leaves so in particular there's $v\in V'$ of degree 1. By minimality of T' there's $j_1\in[n]$ for which $v=x_{j_1}$. WLOS assume $j_1=1$.

Lemma 0.1. *There's $u_1\in V'$ closest to x_1 and such that* $\deg(u_1)\geq 3$.

Proof. Otherwise, T' is linear so we can embed $\iota\colon T'\hookrightarrow\mathbb{R}^1$. Numbering the x_j such that $d(x_1,x_i)>d(x_1,x_j)$ for $i>j$ we may choose $\iota(x_i)=i$, and send each path $x_i\to x_{i+1}$ to $[i,i+1]$ with constant speed.

Denote by $p_{j,k}$ the shortest path from x_j to x_k in T'. Consider

$$p_{1,2},p_{2,3},\ldots,p_{n-1,n},p_{n,1}.$$

Each two such paths intersect, since $p_{j,k},p_{k,\ell}$ both contain x_k. Hence each two $\iota(p_{j,j+1})$ (where the indexes are considered $(\mathrm{mod}\ n)$) intersect. By Helly's theorem, all of them intersect. Hence all the $p_{j,j+1}$ intersect. For $j,k\in[n]$ take $\ell\in[n]\setminus\{j,k\}$ (which we can since $n\geq 3$). We have $x_j,x_k\in V_{i_\ell}$ hence $p_{j,k}\subseteq T_{i_\ell}$ since T_{i_ℓ} is connected. In particular, $p_{j,j+1}\subseteq T_{i_{j-1}}$. We get

$$\varnothing\neq\bigcap_{j\in[n]}p_{j,j+1}\subseteq\bigcap_{j\in[n]}T_{i_j}$$

hence all the V_{i_j} intersect, a contradiction. ∎

Now, let $u_1\in V'$ closest to x_1 and such that $\deg(u_1)\geq 3$. Assume first that there's x_j on the path p from x_1 to u_1, for $j>1$. Then $x_1\in V_{i_j}$ and $x_k\in V_{i_j}$

for all $k \in [n] \setminus \{j\}$. Since x_j is on the path from x_1 to x_k for any such k, we get $x_j \in V_{i_j}$, a contradiction. We get that the path p from x_1 to u_1 doesn't contain any x_j for $j \neq 1$.

Assume now that $u_1 \notin V_{i_1}$. Since $\deg(u_1) \geq 3$ there are different edges $(u_1, f), (u_1, g)$ in T'. By minimality, there are $j, k \in [n]$ such that u_1 lies on the path from x_j to x_k. Then, $x_j, x_k \in V_{i_1}$ so by connectedness $u_1 \in V_{i_1}$, a contradiction.

We got that $u_1 \in V_{i_1}$, and we want to show

$$u_1 \in \bigcap_{j \in [n]} V_{i_j},$$

which would give a contradiction. Let $j \in [n] \setminus \{1\}$. Since $n \geq 3$ there's $k \in [n] \setminus \{1, j\}$. Then $x_1, x_k \in V_{i_j}$. By choice of u_1 and by connectedness of T' we get $u_1 \in V_{i_j}$.