Special topics: Complex Systems.

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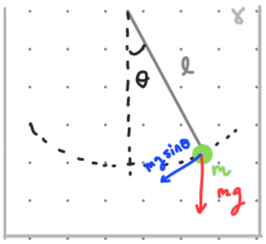
1. A plane pendulum of mass m and length l moves in a medium with a coefficient of friction $\gamma > 0$ that produces a friction force proportional to its velocity. In addition, the pendulum is subject to a force $F = A\cos(\nu t)$, where A and ν are the amplitude and the frequency of the force, respectively.
a) Find the equation of motion for this system. [1]
c) Is this system autonomous? Is it conservative or dissipative? [1]

a) The classical equation for a simple pendulum is $m\frac{d^2s}{dt} + mg\sin\theta$, then adding the dumping for $-bv$ where b is a constant $(b>0)$. Finally we need to constrain to a external for $F = A\cos(\nu t)$:



$$m\frac{d^2s}{dt^2} + mg\sin\theta + bv = A\cos(\nu t)$$

$$\Rightarrow ml\frac{d^2\theta}{dt^2} + mg\sin\theta + bl\frac{d\theta}{dt} = A\cos(\nu t)$$

$$\boxed{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta + \frac{b}{m}\frac{d\theta}{dt} = \frac{A}{ml}\cos(\nu t)}$$

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b) Is this system autonomous?

$\boxed{\text{It is not autonomous}}$ since the driven function F depends on time:

$$F = F(t)$$

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Therefore is a forced system.

c)

$$\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{b}{m}\dot{\theta} = \frac{A}{ml}\cos(\nu t).$$

Given $x_1 = \theta$ and $x_2 = \dot{\theta}$, then $\dot{x}_1 = x_2$.

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 + \frac{A}{ml}\cos(\nu t)$$

In this way we can write the vector vield as follows:

$$\vec{f} = \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = -\dfrac{g}{l}\sin x_1 - \dfrac{b}{m}x_2 + \dfrac{A}{ml}\cos(\nu t) \end{cases}$$

$$\vec{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 + \frac{A}{ml}\cos(\nu t) \end{bmatrix}$$

Now, using Liouville's theorem.

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial x_2}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\frac{g}{l}\sin x_1 - \frac{b}{m}x_2 - \frac{A}{ml}\cos(\nu t)\right)$$

$$\vec{\nabla}\cdot\vec{F} = 0 - \frac{b}{m} = -\frac{b}{m}$$

Since $b>0$ then we $\vec{\nabla}\cdot\vec{F}<0$ and the system is dissipative.

$$\boxed{\vec{\nabla}\cdot\vec{F}<0 \Rightarrow \text{The system is dissipative}}$$

2. The competition dynamics between two populations of sizes x (rabbits) and y (sheep) can be described by the equations

$$\begin{aligned}\dot{x} &= x(3-x-2y)\\ \dot{y} &= y(2-x-y),\end{aligned}$$

a) Find the fixed points on the phase space of the system. [1]
b) Classify the fixed points according to their stability. [1]
c) Can both species survive? [1]

a) Given our system:

$$\vec{F} = \begin{cases} f_1 = \frac{dx}{dt} = x(3-x-2y) \\ f_2 = \frac{dy}{dt} = y(2-x-y) \end{cases}$$

Then a fixed point $\vec{x}^*$ is such that $\vec{f}(\vec{x}^*)=0$, then:

$$\begin{cases} 0 = x^*(3-x^*-2y^*) \\ 0 = y^*(2-x^*-y^*) \end{cases}$$

i) Trivial solution:

$$\vec{x}_1^* = (x_1^*, y_1^*) = (0,0)$$

ii) when $x^*\neq 0$, and $y^*\neq 0$

$$\begin{cases} 0 = 3-x^*-2y \\ 0 = 2-x^*-y^* \end{cases} \Rightarrow \begin{cases} 0 = 3-x^*-2y^* \\ 0 = -2+x^*+y^* \end{cases}$$

$$0 = 1-y^*$$

$$y^* = 1 \quad \text{and} \quad 0 = 3-x^*-2, \quad x^* = 1$$

$$\vec{x}_2 = (x_2^*, y_2^*) = (1,1)$$

iii) When $x=0$, then

$$0 = y^*(2-y^*) \Rightarrow y^* = 2$$

$$\vec{x}_3^* = (0,2)$$

iv) When $y=0$, then:

$$0 = x^*(3-x^*) \Rightarrow x^* = 3$$

$$\vec{x}_4 = (3,0)$$

Fixed points $\rightarrow \begin{cases} \vec{x}_1^* = (0,0) \\ \vec{x}_2^* = (1,1) \\ \vec{x}_3^* = (0,2) \\ \vec{x}_4^* = (3,0) \end{cases}$

b) Given an inestability $\Delta\vec{x}$ we saw in class that it can be expressed as the linear combination:

$$\Delta\vec{x} = c_1 e^{w_1 t}\vec{u}_1 + c_2 e^{w_2 t}\vec{u}_2 ,$$

where $\vec{u}_i$ and w_i are the eigenvectors and eigenvalues of the $J(\vec{x}^*)$ matrix, respectively.

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix} = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

i) $\vec{x}_1^* = \vec{0}$

$$\det\left(J(\vec{x}_1^*) - w I_2\right) = \begin{vmatrix} 3-w & 0 \\ 0 & 2-w \end{vmatrix} = (3-w)(2-w)$$

$$\Rightarrow w_1 = 3 \quad \wedge \quad w_2 = 2$$

Since $w_{1,2} > 0$ the fixed point $\vec{x}_1 = \vec{0}$ is an unstable node.

ii) $\vec{x}_2 = (1,1)$

$$\det\left(J(\vec{x}_2^*) - w I_2\right) = \begin{vmatrix} -1-w & -2 \\ -1 & -1-w \end{vmatrix} = (-1-w)(-1-w) - 2 = w^2+2w+1-2$$

$$= w^2+2w-1$$

$$w_{1,2} = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$w_1 \approx 0.414 \quad \wedge \quad w_2 \approx -2.414$$

Since at least one of the eigenvalues is more than zero ($w_2 > 0$) the fixed point $\vec{x}_2^* = (1,1)$ is unstable (saddle point).

iii) $\vec{x}_3^* = (0,2)$

$$\det\left(J(\vec{x}_3^*) - w I\right) = \begin{vmatrix} -1-w & 0 \\ -2 & -2-w \end{vmatrix} = (1+w)(2+w) = 0$$

$$\Rightarrow w_1 = -1 \quad \wedge \quad w_2 = -2$$

As both w_1 and w_2 are less than zero $\vec{x}_3^*$ is a stable node.

iv) $\vec{x}_4^* = (3,0)$

$$\det\left(J(\vec{x}_4^*) - w I\right) = \begin{vmatrix} -3-w & -6 \\ 0 & -1-w \end{vmatrix} = (3+w)(1+w) = 0$$

$$w_1 = -1 \quad \wedge \quad w_2 = -3$$

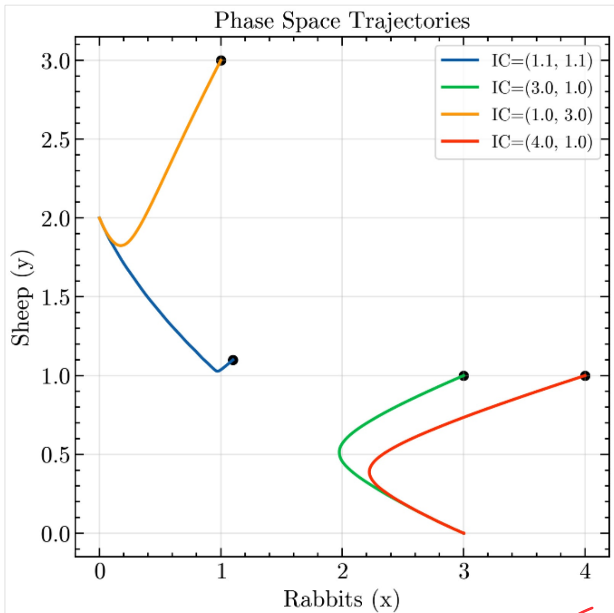
As well as the previous fixed point $\vec{x}_4^*$ is stable since $w_1, w_2 < 0$.

Fixed points $\rightarrow$ $\begin{cases} \vec{x}_1^* = (0,0) \rightarrow \text{unstable} \\ \vec{x}_2^* = (1,1) \rightarrow \text{unstable} \\ \vec{x}_3^* = (0,2) \rightarrow \text{stable} \\ \vec{x}_4^* = (3,0) \rightarrow \text{stable} \end{cases}$

c) The fixed point at which both species survive is $\vec{x}_2^* = (1,1)$ but it is not stable so any perturbation will make the system reach a stable point ($\vec{x}_3^*$ or $\vec{x}_4^*$), however in neither case both species survive. Therefore both species cannot survive.

The next figure shows the phase space system with different initial conditions. As it is seen all of them end in the stable fixed points:

$$\vec{x}_3^* = (0,2) \quad \text{and} \quad \vec{x}_4^* = (3,0)$$



Asymptotically ($t \to \infty$) both species can not survive at the same time.

3. Consider the following dynamical system with parameters $a, b \in \mathbb{R}$,

$$\dot{x} = y - ax$$
$$\dot{y} = \frac{x^2}{1+x^2} - by,$$

a) Find the fixed points for this system. [1]
b) Classify the fixed points according to their stability. [1]
c) Can this system be chaotic for some range of parameters a and b? [1]

a) $\vec{x}^*$ is a fixed point when $f(\vec{x}^*)=0$, then:

$$\vec{f}(\vec{x}^*) = \begin{bmatrix} f_1(x^*, y^*) \\ f_2(x^*, y^*) \end{bmatrix} = \begin{bmatrix} y - ax \\ \frac{x^2}{1+x^2} - by \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} y - ax = 0 & (1) \\ \frac{x^2}{1+x^2} - by = 0 & (2) \end{cases}$$

Remplace (1) in (2):

$$\frac{x^2}{1+x^2} - b(ax) = 0 \Rightarrow \frac{x^2 - bax(1+x^2)}{1+x^2} = 0$$

$$1 + x^2 \neq 0 \Rightarrow x(x - ab(1+x^2)) = 0$$

$$x = 0 \wedge x - ab - abx^2 = 0$$
$$-abx^2 + x - ab = 0$$

$$x_1^* = 0 \wedge x_{2,3}^* = \frac{-1 \pm \sqrt{1-4(-ab)(-ab)}}{-2ab} = \frac{-1 \pm \sqrt{1-4a^2b^2}}{-2ab} = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$$

Now, let's find y:

For $x_1^* = 0$,

$$y_1^* = 0$$

$$\vec{x}_1^* = (0, 0)$$

For $x_2^* = \frac{1 + \sqrt{1-4a^2b^2}}{2ab}$

$$y_2^* = a\left(\frac{1 + \sqrt{1-4a^2b^2}}{2ab}\right) = \frac{1 + \sqrt{1-4a^2b^2}}{2b}$$

$$\vec{x}_2^* = \left(\frac{1 + \sqrt{1-4a^2b^2}}{2ab}, \frac{1 + \sqrt{1-4a^2b^2}}{2b}\right)$$

For $x_3^* = \frac{1 - \sqrt{1-4a^2b^2}}{2ab}$

$$\vec{x}_3^* = \left(\frac{1 - \sqrt{1-4a^2b^2}}{2ab}, \frac{1 - \sqrt{1-4a^2b^2}}{2b}\right)$$

b) Given an inestability $\Delta\vec{x}$ we saw in class that it can be expressed as the linear combination:

$$\Delta\vec{x} = c_1 e^{w_1 t}\vec{u}_1 + c_2 e^{w_2 t}\vec{u}_2 ,$$

where $\vec{u}_i$ and w_i are the eigenvectors and eigenvalues of the $J(\vec{x}^*)$ matrix, respectively.

$$J = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$

$$\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{1+x^2}\right) = 2x\left(\frac{1}{1+x^2}\right) - \frac{x^2}{(1+x^2)^2}(2x) = \frac{2x}{1+x^2} - \frac{2x^3}{(1+x^2)^2}$$
$$= \frac{2x(1+x^2)-2x^3}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$$

$$\Rightarrow J = \begin{bmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{bmatrix}$$

i) For $\vec{x}_1^* = (0,0)$:

$$\det(J(\vec{x}_1^*) - wI) = \begin{vmatrix} -a-w & 1 \\ 0 & -b-w \end{vmatrix} = (a+w)(b+w) = 0$$

$$\Rightarrow w_1 = -a, \quad w_2 = -b$$

In this case we have three options.

1) If both $b>0$ and $a>0$, $\vec{x}_1^* = (0,0)$ is stable.
2) If both $b<0$ and $a<0$, $\vec{x}_1 = (0,0)$ is unstable.
3) If they are opposite sign, $\vec{x}_1^* = (0,0)$ is unstable (saddle point).

ii) For $\vec{x}_{2,3} = \left(\frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}, \frac{1 \pm \sqrt{1-4a^2b^2}}{2b}\right)$

$$J_{21}(x^*) = \left.\frac{2x}{(1+x^2)^2}\right|_{x=x^*_{2,3}} = \frac{\cancel{2}}{\left[1+\left(\frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}\right)^2\right]^2} \cdot \left(\frac{1 \pm \sqrt{1-4a^2b^2}}{\cancel{2}ab}\right)$$

$$= \frac{1 \pm \sqrt{1-4a^2b^2}}{ab\left[1 + \frac{1 \pm 2\sqrt{1-4a^2b^2} + (1-4a^2b^2)}{4a^2b^2}\right]^2} = \frac{1 \pm \sqrt{1-4a^2b^2}}{ab\left[\frac{4a^2b^2 + 1 \pm 2\sqrt{1-4ab^2} + 1 - 4a^2b^2}{4a^2b^2}\right]^2}$$

$$= \frac{1 \pm \sqrt{1-4a^2b^2}}{ab\,\frac{(2 \pm 2\sqrt{1-4a^2b^2})^2}{16a^4b^4}} = \frac{16a^3b^3(1 \pm \sqrt{1-4a^2b^2})}{4(1 \pm \sqrt{1-4a^2b^2})^2}$$

$$= \frac{4a^3b^3}{1 \pm \sqrt{1-4a^2b^2}}$$

$$\det\left(J(\vec{x}^*_{2,3}) - w\mathbb{I}_2\right) = \begin{vmatrix} -a-w & 1 \\ \frac{4a^3b^3}{1 \pm \sqrt{1-4a^2b^2}} & -b-w \end{vmatrix} = (a+w)(b+w) - \frac{4a^3b^3}{1 \pm \sqrt{1-4a^2b^2}}$$

$$= w^2 + (a+b)w + ab - \frac{4a^3b^3}{1 \pm \sqrt{1-4a^2b^2}} = 0$$

$$= w^2 - T(J)w + \Delta \quad , \quad \text{where } T(J) = -(a+b) \text{ and } \Delta = ab - \frac{4a^3b^3}{1 \pm \sqrt{1-4a^2b^2}}$$

$$\Rightarrow \quad w_{1,2} = \frac{T \pm \sqrt{T^2 - 4\Delta}}{2}$$

1) When $T^2 - 4\Delta < 0 \Rightarrow \mathrm{Im}(w_{1,2}) \neq 0$.

1.1) If $T = -(a+b) < 0$ and $\Delta < 0 \Rightarrow (a+b) > 0$

Then, $\vec{x}^*_{2,3}$ are stable focus.

1.2) If $T = -(a+b) > 0 \Rightarrow (a+b) < 0$

Then, $\vec{x}^*_{2,3}$ are unstable focus

1.3) Finally in $T = 0 \Rightarrow (a+b) = 0$

Then, there is a closed orbit around $\vec{x}^*_{2,3}$.

2) When $T^2 - 4\Delta > 0 \Rightarrow \mathrm{Im}(w_{2,3}) = 0$ and $w_{1,2} \in \mathbb{R}$

2.1) $T > 0$ and $\Delta > 0$
at least one $w_i > 0 \Rightarrow \vec{x}^*_{2,3}$ is unstable

2.2) $T < 0$ and $\Delta > 0$
so $\mathrm{Re}(w_{1,2}) < 0 \Rightarrow \vec{x}^*_{2,3}$ is stable

c)

According to the Poncaire-Bendixson theorem in a two dimentional phase space the only asymtotic solution for $\vec{x}(t)$ when $t \to \infty$ are:

i) fixed points
ii) limit cycle (closed orbit, trayectory).

In this sense, we will not find "strange" atractors that could exhibit chaotic behaviour.

4. The evolution of a system is described by the equation

$$\dddot{x} + r\ddot{x} + \dot{x} - |x| + 1 = 0,$$

where $r > 0$ is a real parameter.
a) Find the fixed points of the system. [1]
b) Plot the asymptotic trajectory of the system on its phase space for $r = 0.6$. [1]

a)

Let's change this system to a 3-dim phase space (3 first order ODEs).

Given $\vec{x} = (x_1, x_2, x_3)$, change the variable x as follows:

$$x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = \ddot{x}.$$

This implies that $\dot{x}_1 = x_2 = \dot{x}$ and $\dot{x}_2 = x_3 = \ddot{x}$. Therefore our system is:

$$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dfrac{dx_3}{dt} = -rx_3 - x_2 + |x_1| - 1 \end{cases} \Rightarrow \frac{d\vec{F}}{dt} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r & -1 & \end{bmatrix}$$

Now, the fixed points $\vec{x}_i^*$ are those which $\dfrac{d\vec{x}_i}{dt} = 0$, then

$$\begin{cases} x_2^* = 0 \\ x_3^* = 0 \\ -rx_3^* - x_2^* + |x_1^*| - 1 = 0 \end{cases}$$

Then, the system reduces to:

$$|x_1^*| = 1 \Rightarrow \begin{cases} x_1^* = -1 \\ x_1^* = 1 \end{cases}$$

The fixed points are:

$$\begin{cases} \vec{x}_1^* = (1, 0, 0) \\ \vec{x}_2^* = (-1, 0, 0) \end{cases}$$

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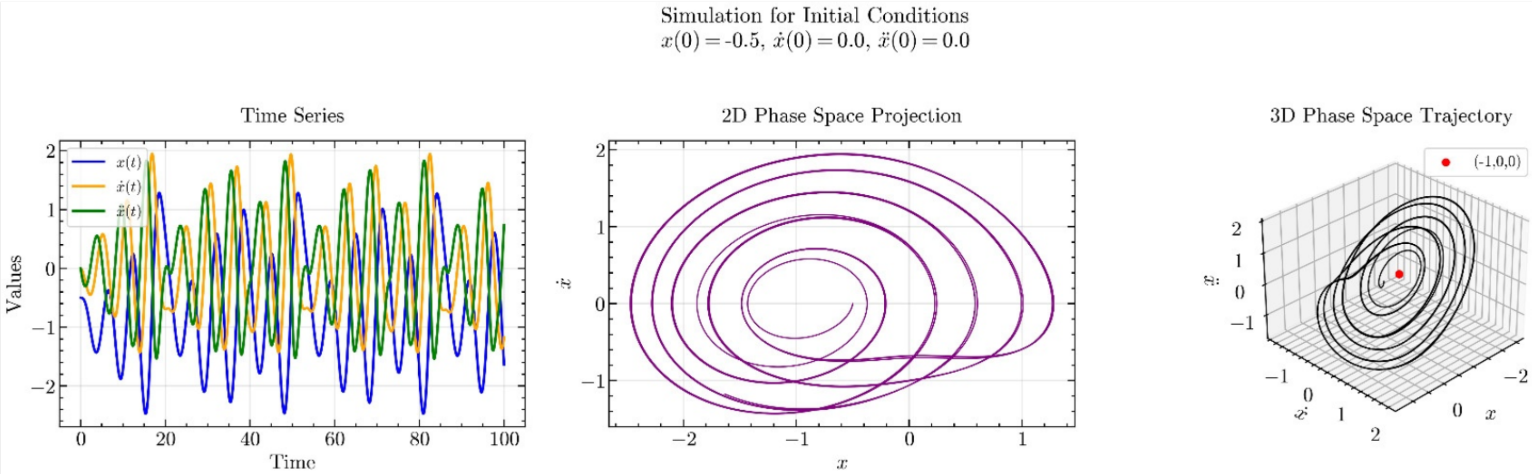
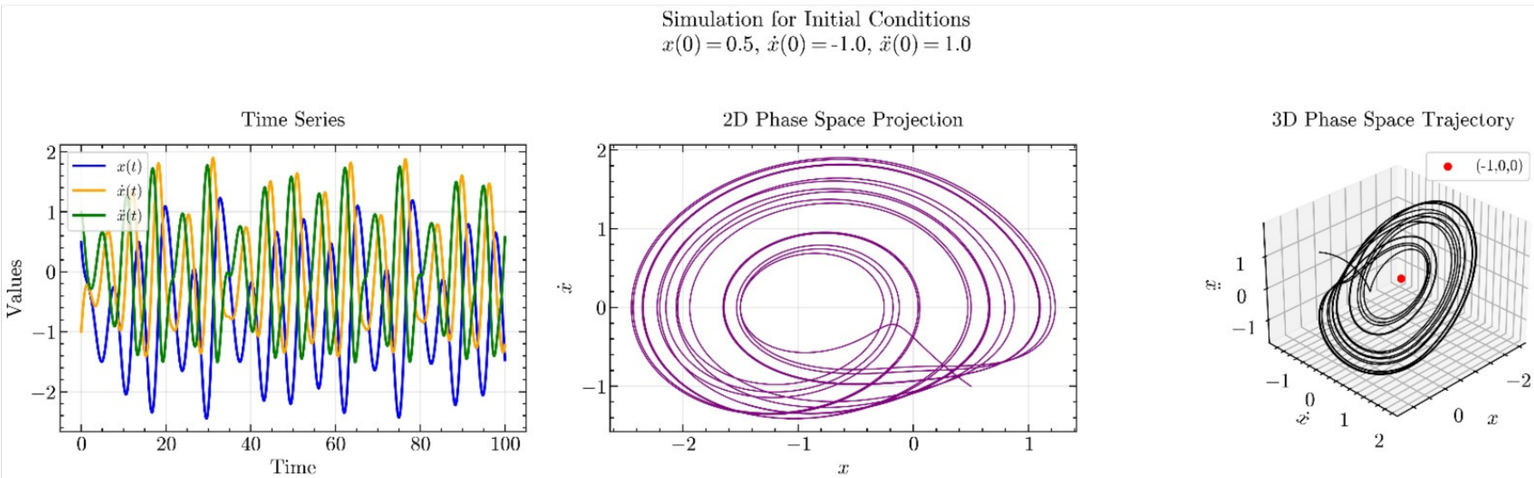
c)

Here there is some figures representing the phase space of this system for some initial conditions around the fixed points.

- It is seen that the fixed point $\vec{x}_2^* = (-1, 0, 0)$ is an unstable focus since the spiral is going out. This shows us that the eigenvalues of this specific case are complex and $\mathrm{Re}(w_i) > 0$.
- When we set our intial conditions near to the another fixed point $\vec{x}_1 = (1, 0, 0)$. Specifically when $x_1 > 1$ the phase space is a right line in 3D. This also tell us that this point is unstable as well.





Nice!

This is a strange attractor, characteristic of chaotic systems.