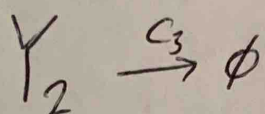
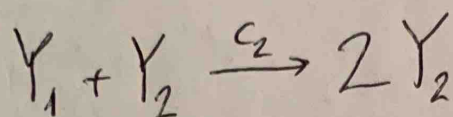
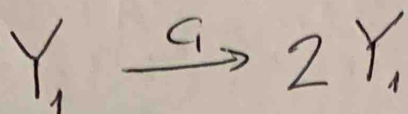


(1)

Stochastic
Lotka - Volterra

The corresponding ODE model is:

$$\dot{y}_1 = \alpha y_1 - \beta y_1 y_2$$

$$\dot{y}_2 = \delta y_1 y_2 - \gamma y_2$$

where $y_1 \leftrightarrow Y_1$ (prey)
 $y_2 \leftrightarrow Y_2$ (predator)

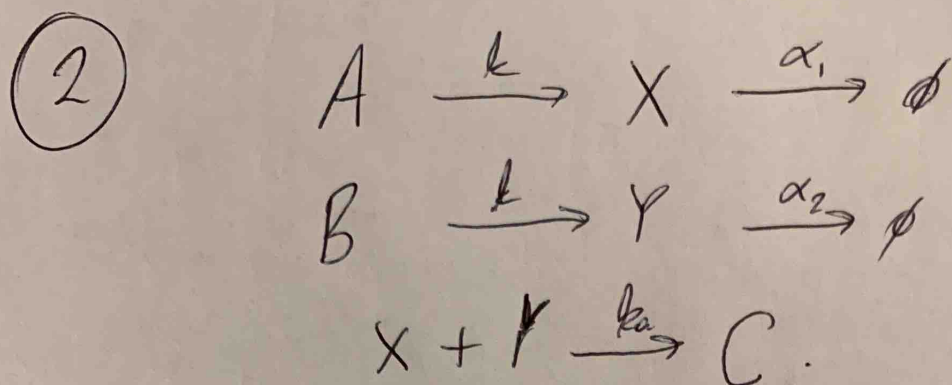
- The stochastic Lotka-Volterra is a predator-prey model because Y_1 acts as a prey that reproduces with rate c_1 and is eaten by Y_2 with rate c_2 .
(predator)
 Y_2 then dies with rate c_3 .

In this ~~version~~ formulation of the model, Y_1 can exist in isolation, however the Y_1 population would ~~be~~ reproduce uncontrollably and the number of Y_1 converges to ∞ .

① (cont.)

On the other hand, Y_2 cannot exist in isolation; because in the absence of X_1 , Y_2 is not able to reproduce and all Y_2 agent would eventually die.

- Please see Python Jupyter Notebook for simulations and answers to the last two points.
-



$$\frac{d[X]}{dt} = k - \alpha_1[X] - k_a[X][Y]$$

$$\frac{d[Y]}{dt} = k - \alpha_2[Y] - k_a[X][Y].$$

• Fixed points occur when $\frac{d[X]}{dt} = 0$ and $\frac{d[Y]}{dt} = 0$.

$$\frac{dX}{dt} = 0 \Rightarrow [X]_* = \frac{k}{\alpha_1 + k_a[Y]_*}$$

$$\frac{dY}{dt} = 0 \Rightarrow [Y]_* = \frac{k}{\alpha_2 + k_a[X]_*}$$

Solve simultaneously.

(2) (cont.)

By substitution, $X = \frac{k}{\alpha_1 + k\alpha \left(\frac{k}{\alpha_2 + k\alpha X} \right)}$

$$\Rightarrow \alpha_1 X + k\alpha \cdot \frac{k}{\alpha_2 + k\alpha X} X = k$$

$$\Rightarrow k\alpha \cdot k \cdot X = (\cancel{k} - \alpha_1 X)(\alpha_2 + k\alpha X)$$

$$\Rightarrow \cancel{k\alpha k X} = k\alpha_2 + \cancel{k\alpha k X} - \alpha_1 \alpha_2 X - \alpha_1 k\alpha X^2$$

$$\Rightarrow 0 = \alpha_1 k\alpha X^2 + \alpha_1 \alpha_2 X - k\alpha_2$$

by quadratic formula, $X = \frac{-\alpha_1 \alpha_2 \pm \sqrt{(\alpha_1 \alpha_2)^2 - 4\alpha_1 k\alpha(-k\alpha_2)}}{2\alpha_1 k\alpha}$

Since concentrations must be positive,

fixed point:

$$X_* = \frac{-\alpha_1 \alpha_2 + \sqrt{(\alpha_1 \alpha_2)^2 + 4\alpha_1 \alpha_2 k\alpha}}{2\alpha_1 k\alpha}$$

$$Y_* = \frac{k}{\alpha_2 + k\alpha \left(\frac{-\alpha_1 \alpha_2 + \sqrt{(\alpha_1 \alpha_2)^2 + 4\alpha_1 \alpha_2 k\alpha}}{2\alpha_1 k\alpha} \right)}$$

(2)

Check that the f.p. is the same for:

$$(*) \quad k = 10, \quad \alpha_1 = 10^{-6}, \quad \alpha_2 = 10^{-5}, \quad k_a = 10^{-5}$$

and $(**) \quad k = 10^3, \quad \alpha_1 = 10^{-4}, \quad \alpha_2 = 10^{-3}, \quad k_a = 10^{-3}.$

For $(*)$, the ^{nulclines} ~~fixed points~~ are

$$X_* = \left(\frac{10}{10^{-6} + 10^{-5} Y_*} \right) \cdot \frac{10^5}{10^5} = \frac{10^6}{10^{-1} + Y_*}$$

$$Y_* = \frac{10}{10^{-5} + 10^{-5} X_*} \left(\frac{10^5}{10^5} \right) = \frac{10^6}{1 + X_*}$$

for $(**)$, we have:

$$X_{**} = \frac{10^3}{10^{-4} + 10^{-3} Y_{**}} \left(\frac{10^3}{10^3} \right) = \frac{10^6}{10^{-1} + Y_{**}}$$

and $Y_{**} = \frac{10^3}{10^{-3} + 10^{-3} X_{**}} \left(\frac{10^3}{10^3} \right) = \frac{10^6}{1 + X_{**}}$

$$(X_*, Y_*) = (X_{**}, Y_{**}).$$

Since the nulclines are the same, the fixed pts will be the same.

• Please see code for answers to ~~the~~ parts (b), (c).

③

$$\frac{dr}{dt} = k_L + \phi(p) - \gamma_r r$$

$$\frac{dp}{dt} = r \cdot k_p - \gamma_p \cdot p$$

(a) The transition matrix is given by the following probabilities, assuming h is small enough:

$$P(r \rightarrow r+1, p \rightarrow p) = (k_L + \phi(p)) \cdot h + o(h)$$

$$P(r \rightarrow r-1, p \rightarrow p) = (\gamma_r \cdot r) \cdot h + o(h)$$

$$P(r \rightarrow r, p \rightarrow p+1) = (k_p \cdot r) \cdot h + o(h)$$

$$P(r \rightarrow r, p \rightarrow p-1) = (\gamma_p \cdot p) h + o(h)$$

$$P(r \rightarrow r, p \rightarrow p) = 1 - (k_L + \phi(p) + \gamma_r r + k_p r + \gamma_p p) \cdot h + o(h)$$

(b) Nullclines: $\frac{dr}{dt} = 0 \Rightarrow r = \frac{k_L + \phi(p)}{\gamma_r}$, $\frac{dp}{dt} = 0 \Rightarrow p = \frac{k_p}{\gamma_p} r$

We found the fixed points using the nullclines and we determined their stability using phase plane analysis. See code for answers to this and part (c).