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Homework #3

① (a) Derive (8.47) from (8.37) in Gerstner et al.

Given u' at time t, the probability density of finding a potential u at time $t+\Delta t$ is:

$$(8.37)\quad P^{trans}(u, t+\Delta t \mid u'; t) = \left[1 - \Delta t \sum_k \nu_k(t)\right] \delta\left(u - u' e^{-\Delta t/\tau_m}\right) + \Delta t \sum_k \nu_k(t)\, \delta\left(u - u' e^{-\Delta t/\tau_m} - w_k\right)$$

Since the process is memoryless,

$$p(u, t+\Delta t) = \int P^{trans}(u, t+\Delta t \mid u', t) \cdot p(u'; t)\, du' \quad - (8.38)$$

Insert (8.37) into (8.38):

$$p(u, t+\Delta t) = \int \left[\left[1 - \Delta t \sum_k \nu_k\right] \delta\left(u - u' e^{-\Delta t/\tau_m}\right) + \Delta t \sum_k \nu_k\, \delta\left(u - u' e^{-\Delta t/\tau_m} - w_k\right)\right] p(u'; t)\, du'$$

It follows that

$$p(u, t+\Delta t) = \left[1 - \Delta t \sum_k \nu_k\right] \int \delta\left(u - u' e^{-\Delta t/\tau_m}\right) p(u'; t)\, du'$$
$$+ \Delta t \sum_k \nu_k \int \delta\left(u - u' e^{-\Delta t/\tau_m} - w_k\right) \cdot p(u'; t)\, du'$$

① Since $\delta(au) = a^{-1}\delta(u)$ for any $a \in \mathbb{R}$,

$$p(u,t+\Delta t) = \left[1-\Delta t \sum_k \nu_k\right] e^{\Delta t/\tau_m} \int \delta(e^{\Delta t/\tau_m} u - u')\, p(u';t)\, du'$$

$$+ \Delta t \sum_k \nu_k e^{\Delta t/\tau_m} \int \delta(e^{\Delta t/\tau_m}(u-w_k) - u')\, p(u';t)\, du'.$$

The integrals are really convolutions of delta functions.

Since generally $\int \delta(\tau)\, g(t-\tau)\, d\tau = g(t)$, we obtain:

$$p(u,t+\Delta t) = \left[1-\Delta t \sum_k \nu_k\right] e^{\Delta t/\tau_m} \cdot p(e^{\Delta t/\tau_m} \cdot u, t)$$

$$+ \Delta t \sum_k \nu_k e^{\Delta t/\tau_m} p(e^{\Delta t/\tau_m}(u-w_k), t) \qquad -(8.39)$$

We now expand around $\Delta t = 0$ up to first order in Δt:

$$p(u,t+\Delta t) = \left[1-\Delta t \sum_k \nu_k\right] \cdot (1+\Delta t/\tau_m) \cdot p(e^{\Delta t/\tau_m} \cdot u, t)$$

$$+ \Delta t \sum_k \nu_k (1+\Delta t/\tau_m)\, p(e^{\Delta t/\tau_m}(u-w_k), t)$$

where we used $e^{+\Delta t/\tau_m} \simeq 1 + \Delta t/\tau_m$ (order 1 term of Taylor exp).

(1) We also use the expansion $\rho(e^{\Delta t/\tau_m}\cdot u, t) \approx \rho(u,t) + \frac{u\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}$ to arrive at the expression:

$$\rho(u,t+\Delta t) = \left[1-\Delta t\sum_k \nu_k\right]\left(1+\Delta t/\tau_m\right)\cdot\left[\rho(u,t)+\frac{u\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}\right]$$

$$+\Delta t\sum_k \nu_k \left(1+\Delta t/\tau_m\right)\left[\rho(u-w_k,t)+\frac{(u-w_k)\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}\right].$$

Rearranging,

$$\rho(u,t+\Delta t) = \left(\rho(u,t)+\frac{u\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}\right)\left(1+\Delta t/\tau_m\right)$$

$$+\Delta t\sum_k \nu_k \left(1+\frac{\Delta t}{\tau_m}\right)\left[-\rho(u,t)-\underbrace{\frac{u\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}}_{\text{will have } O((\Delta t)^2)}+\rho(u-w_k,t)+\underbrace{\frac{(u-w_k)\Delta t}{\tau_m}\frac{\partial \rho(u,t)}{\partial u}}_{\text{Also order } (\Delta t)^2}\right]$$

Expand and ignore terms of order $(\Delta t)^2$ or higher.

$$\rho(u,t+\Delta t) \simeq \rho(u,t) + \frac{\Delta t}{\tau_m}u\frac{\partial \rho(u,t)}{\partial u} + \frac{\Delta t}{\tau_m}\rho(u,t)$$

$$+\Delta t\sum_k \nu_k\left[-\rho(u,t)+\rho(u-w_k,t)\right]$$

1 Rearrange once again to obtain:

$$\frac{p(u,t+\Delta t)-p(u,t)}{\Delta t} \approx \frac{1}{\tau_m} p(u,t) + \frac{1}{\tau_m}\cdot u \cdot \frac{\partial p(u,t)}{\partial u} + \sum_k \nu_k \left[p(u-w_k,t) - p(u,t) \right]. \qquad (8.40)$$

Let's now expand on $p(u-w_k,t)$.

$$\frac{p(u-t+\Delta t)-p(u,t)}{\Delta t} \approx \frac{1}{\tau_m} p(u,t) + \frac{1}{\tau_m} u \frac{\partial p(u,t)}{\partial u}$$

$$+ \sum_k \nu_k \left[-\cancel{p(u,t)} + \cancel{p(u,t)} - w_k \frac{\partial p(u,t)}{\partial u} + \frac{w_k^2}{2!} \frac{\partial^2 p(u,t)}{\partial u^2} \right]$$

$$\approx \frac{1}{\tau_m} p(u,t) + \left[\frac{1}{\tau_m}\cdot u - \sum_k \nu_k w_k \right] \frac{\partial p(u,t)}{\partial u} + \frac{1}{2} \sum_k \nu_k w_k^2 \frac{\partial^2 p(u,t)}{\partial u^2}$$

Lastly, take the limit as $\Delta t \to 0$,

$$\tau_m \frac{\partial p(u,t)}{\partial t} = p(u,t) + \left[u - \tau_m \sum_k \nu_k w_k \right] \frac{\partial p(u,t)}{\partial u} + \frac{\tau_m}{2} \sum_k \nu_k w_k^2 \frac{\partial^2 p(u,t)}{\partial u^2}$$

Rearrange the first two terms:

$$\tau_m \frac{\partial p(u,t)}{\partial t} = -\frac{\partial}{\partial u}\left[-u + \tau_m \sum_k \nu_k w_k \right] p(u,t) + \frac{1}{2}\left[\tau_m \sum_k \nu_k w_k^2 \right] \frac{\partial^2}{\partial u^2} p(u,t) \qquad (8.41)$$

① (c) Please See Python Notebook for simulations.

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② Please also ~~see~~ see Python Notebook for the answer.
& Mathematica Notebook

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③ Let isi ~ $\Gamma(a,b)$. Find CV(isi).

$$CV(isi) = \frac{\sqrt{Var[isi]}}{E[isi]} = \frac{\sqrt{\alpha/\beta^2}}{\alpha/\beta} = \frac{1}{\sqrt{\alpha}}.$$

Erlang dist is sum of k iid exponentials w/ parameter λ.

Let isi ~ Erlang(k, λ).

Then $$CV(isi) = \frac{\sqrt{Var[isi]}}{E[isi]} = \frac{\sqrt{k/\lambda^2}}{k/\lambda} = 1/\sqrt{k}.$$

- CV does not depend on rate λ.
- As $k \to \infty$, $CV \to 0$, so interevents become more regular with increasing k.

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(4) Set $x = n/N$ and Taylor expand the master equation keeping only terms up to order $1/N^2$ to derive the Fokker-Planck eqn.

For Moran w/o selection, we have:

$$T(n+1|n) = (1-u)\cdot x(1-x) + v(1-x)^2$$

$$T(n|n-1) = (1-v)(1-x)(x) + u\cdot x^2$$

$$T(n|n+1) = (1-v)(1-x-1/N)(x+1/N) + u(x+1/N)^2$$

$$T(n|n-1) = (1-u)(x-1/N)(1-x+1/N) + v(1-x+1/N)^2$$

Note $p(n,t) = f(x,t)$.

Then

$$\frac{\partial}{\partial t} f(x,t) = -f(x,t)\cdot\left[x(1-x)(2-u-v) + u x^2 + v(1-x)^2\right]$$

$$+ f(x+1/N, t)\cdot\left[(1-v)(1-x-1/N)(x+1/N) + u(x+1/N)^2\right]$$

$$+ f(x-1/N, t)\cdot\left[(1-u)(x-1/N)(1-x+1/N) + v(1-x+1/N)^2\right]$$

4 Let's expand up to order $1/N^2$:

$$\frac{\partial}{\partial t} f(x,t) = -f(x,t)\cdot\left[x(1-x)(2-u-v)+ux^2+v(1-x)^2\right]$$

$$+\left(f(x,t)+\frac{1}{N}\frac{\partial}{\partial x}f(x,t)+\frac{1}{2!}\cdot\frac{1}{N^2}\frac{\partial^2}{\partial x^2}f(x,t)\right)\left[(1-v)(1-x-1/N)(x+1/N)+u(x+1/N)^2\right]$$

$$+\left(f(x,t)-\frac{1}{N}\frac{\partial}{\partial x}f(x,t)+\frac{1}{2}\frac{1}{N^2}\frac{\partial^2}{\partial x^2}f(x,t)\right)\left[(1-u)(x-1/N)(1-x+1/N)+v(1-x+1/N)^2\right]$$

$$= f(x,t)\Big(-\cancel{2x}+\cancel{ux}+\cancel{vx}-2x^2-\cancel{ux^2}-\cancel{vx^2}-\cancel{ux^2}-\cancel{v}+vx^2$$
$$+\cancel{x}-x^2-\cancel{\frac{2x}{N}}-\cancel{vx}+\cancel{vx^2}+\cancel{\frac{2vx}{N}}+\cancel{\frac{1}{N}}-\cancel{\frac{v}{N}}+\cancel{\frac{v}{N^2}}+\cancel{ux^2}+\cancel{\frac{2ux}{N}}$$
$$+\cancel{x}-\cancel{\frac{1}{N}}-\cancel{ux}+\frac{u}{N}-x^2+\cancel{\frac{x}{N}}+\cancel{ux^2}-\cancel{\frac{2ux}{N}}+\cancel{\frac{x}{N}}-\frac{1}{N^2}+\frac{u}{N^2}+\cancel{v}-2vx$$
$$+\frac{v}{N}+vx^2-\cancel{\frac{ux}{N}}+\cancel{\frac{u}{N}}-\cancel{\frac{xv}{N}}+\frac{v}{N^2}\Big)$$

$$+\frac{1}{N}\frac{\partial}{\partial x}f(x,t)\cdot\Big(\cancel{x}-\cancel{x^2}-\frac{2x}{N}+vx+\cancel{vx^2}+\frac{2vx}{N}+\frac{1}{N}-\frac{v}{N}+\cancel{\frac{v}{N^2}}+\cancel{ux^2}+\cancel{\frac{2ux}{N}}$$
$$-\Big(+\cancel{x}-1/N-ux+u/N-\cancel{x^2}+x/N+\cancel{ux^2}-\cancel{\frac{2ux}{N}}+x/N-1/N^2+\frac{u}{N^2}+v-\cancel{2vx}$$
$$+v/N+\cancel{vx^2}-vx/N+v/N-xv/N+\cancel{\frac{v}{N^2}}\Big)$$

$$\cancel{+\frac{1}{N^2}\frac{\partial^2}{\partial x^2}f(x,t)\frac{1}{2}\left(-\frac{ux}{N}+vx+\frac{uvx}{N}+2/N-3v/N+ux-u/N-\frac{2x}{N}+\frac{1}{N^2}-\frac{u}{N^2}-v\right)}$$

Sorry!

$$+\frac{1}{2}\frac{1}{N^2}\frac{\partial^2}{\partial x^2} f(x,t)\Big(x - x^2 - \cancel{\cdots} + vx + vx^2 + \cancel{\cdots} + \cancel{\cdots} - \cancel{\cdots} + \cancel{\cdots} + ux^2 + \frac{2ux}{N}$$

$$+ x - \cancel{\cdots} - ux + u/N - x^2 + \cancel{\cdots} + ux^2 + \frac{2ux}{N} + \cancel{\cdots} - \cancel{\cdots}$$

$$+ u/N^2 + v - 2vx + \cancel{\cdots} + vx^2 - \cancel{\cdots} + v/N - \cancel{\cdots} + v/N^2\Big)$$

$$\frac{\partial}{\partial t} f(x,t) = f(x,t)\cdot\Big(-4x^2 + 2vx^2 + \overbrace{\frac{2v}{N^2}}^{\sigma(1/N^3)} + \frac{u}{N} - \frac{1}{N^2} + \underbrace{\frac{u}{N^2}}_{\sigma(1/N^3)} - 2vx + \frac{v}{N}\Big)$$

$$+ \frac{1}{N}\frac{\partial}{\partial x} f(x,t)\Big(-4x + vx + \overbrace{\frac{4vx}{N}}^{\sigma(1/N^3)} + 2/N \underbrace{-\frac{3v}{N}}_{\sigma(1/N^3)} + ux - \overbrace{u/N}^{\sigma(1/N^3)} - \frac{2x}{N} + 1/N^2 - \underbrace{\frac{u}{N^2}}_{\sigma(1/N^4)} - v\Big)$$

$$+ \frac{1}{2}\frac{1}{N^2}\frac{\partial^2}{\partial x^2} f(x,t)\cdot\Big(2x - 2x^2 - \underbrace{vx}_{\sigma(1/N^3)} + \overbrace{2vx^2}^{\sigma(1/N^3)} + \underbrace{2ux^2}_{\sigma(1/N^3)} + \overbrace{\frac{4ux}{N}}^{\sigma(1/N^4)} - \underbrace{ux}_{\sigma(1/N^3)} + \overbrace{\frac{u}{N}}^{\sigma(1/N^4)} + \underbrace{\frac{u}{N^2}}_{\sigma(1/N^5)} + \overbrace{v + \frac{v}{N} + \frac{v}{N^2}}^{\text{higher order}}\Big)$$

We can get rid of terms order $1/N^3$ or higher:

$$\frac{\partial}{\partial t} f(x,t) = f(x,t)\Big(-4x^2 + 2vx^2 + \frac{u}{N} - 1/N^2 - 2vx + v/N\Big)$$

$$+ \frac{1}{N}\frac{\partial}{\partial x} f(x,t)\cdot\Big(-4x + vx + 2/N + ux - \frac{2x}{N} - v\Big)$$

$$+ \cancel{\tfrac{1}{2}}\frac{1}{N^2}\frac{\partial^2}{\partial x^2} f(x,t)\cdot\Big(\cancel{2}x - \cancel{2}x^2\Big) + \mathcal{O}(1/N^3)$$

(4) Regroup terms:

$$\frac{\partial}{\partial t} f(x,t) = f(x,t)\cdot\left(-4x^2 + 2vx^2 - 2vx\right)$$

$$+ \frac{1}{N}\left[f(x,t)\cdot(u+v) + \frac{\partial}{\partial x} f(x,t)\cdot\left(-4x + vx + ux - v\right)\right]$$

$$+ \frac{1}{N^2}\left[f(x,t)\cdot(-1) + \frac{\partial}{\partial x} f(x,t)\cdot(2-2x) + \frac{\partial^2}{\partial x^2} f(x,t)\cdot\left(x - x^2\right)\right]$$

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(5) Please see Python Notebook for results.
