Math 6397: Homework #1 Alan Akil

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① Claim: In a pure birth process,

$$\sum_{n=0}^{\infty} P_n(t) = 1 \quad \forall t \quad \text{if and only if} \quad \sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \infty.$$

Proof:

($\Rightarrow$) Proof by contrapositive. Let $S_n \sim Exp(\lambda_n)$ be the waiting times. Assume $\sum_{n=0}^{\infty} 1/\lambda_n < \infty$.

Then $E\left[\sum_{n=0}^{\infty} S_n\right] = \sum_{n=0}^{\infty} E[S_n] = \sum_{n=0}^{\infty} 1/\lambda_n < \infty.$

Hence the expected time for the population size to reach ∞ is finite.

Thus $P\left(\sum_{n=0}^{\infty} S_n < \infty\right) = 1$ which implies

that $\sum_{n=0}^{\infty} P_n(t) < 1.$

(1) Pf (continued):

($\Leftarrow$) Assume $\sum_{n=0}^{\infty} \lambda_n^{-1} = \infty$.

Consider the moment generating function of the sum of waiting times.

mgf of exponentially distributed random variable.

$$\mathbb{E}\left[e^{-\sum_{n=0}^{\infty} S_n}\right] = \lim_{N\to\infty} \prod_{n=0}^{N} \mathbb{E}\left[e^{-S_n}\right] = \lim_{N\to\infty} \prod_{n=0}^{N} \frac{1}{1+\lambda_n^{-1}}$$

$$= \lim_{N\to\infty} \exp\left[-\sum_{n=0}^{N} \ln\left(1+\lambda_n^{-1}\right)\right]$$

$$= \exp\left(-\sum_{n=0}^{\infty} \ln(1+\lambda_n^{-1})\right) = 0 \,, \quad \text{because } \sum_{n=0}^{\infty} \lambda_n^{-1} = \infty.$$

Thus, $\sum_{n=0}^{\infty} S_n = \infty$, i.e. the population size reaches ∞ in infinite time.

This implies $\sum_{n=0}^{\infty} P_n(t) = 1$.

$\square$

(1) (cont.)

Derive an explosive process that explodes in finite time.

Consider the pure birth process with waiting times $S_n \sim Exp(\lambda_n)$ $\forall n \in \mathbb{N} \cup \{0\}$

with $\lambda_n = c \cdot n^2$, c - constant.

Please see Python Notebook for results on this.

$\square$

(2) <u>Claim:</u> The birth and death process with immigration given by $\lambda_n = \lambda n$, $\mu_n = \mu \cdot n$, ν - rate of immigration has mean that follows $\frac{dn}{dt} = (\lambda - \mu) n + \nu$.

<u>Proof:</u> For this process, we have:

$$(\star)\ \begin{cases} P_0'(t) = -a\, P_0(t) + \mu\, P_1(t), & n = 0 \\ P_n'(t) = \big(\lambda(n-1) + \nu\big) P_{n-1}(t) - \big((\lambda+\mu) n + \nu\big) P_n(t) + \mu(n+1) P_{n+1}(t) & n \geq 1. \end{cases}$$

We will use the probability generating function (pgf) to obtain the evolution first moment of ~~the~~ n in the following way:

$$\frac{\partial}{\partial z} \sum_{n=0}^{\infty} P_n'(t) \cdot z^n \Big|_{z=1} = \sum_{n=0}^{\infty} n \cdot P_n'(t) \cdot z^{n-1} \Big|_{z=1} = \sum_{n=0}^{\infty} n \cdot P_n'(t)$$

$$= \frac{d}{dt} \sum_{n=0}^{\infty} n \cdot P_n(t) = \frac{d}{dt} \mathbb{E}[n].$$

② (cont.)

Using (*) we know that:

$$R := \sum_{n=0}^{\infty} P_n'(t)\, z^{n+1}$$

$$= \sum_{n=0}^{\infty} \left[(\lambda(n-1)+\nu) P_{n-1}(t) z^{n+1} \right] - \sum_{n=0}^{\infty} \left[((\lambda+\mu)n+\nu) P_n(t) \right] z^{n+1}$$

$$+ \sum_{n=0}^{\infty} \mu (n+1) P_{n+1}(t)\, z^{n+1}$$

$$= z^2 \lambda \sum_{n=0}^{\infty} (n-1) P_{n-1}(t) z^{n-1} + z^2 \nu \sum_{n=0}^{\infty} P_{n-1}(t) z^{n-1}$$

$$- z(\lambda+\mu) \sum_{n=0}^{\infty} n \cdot P_n(t) z^n - z \cdot \nu \sum_{n=0}^{\infty} P_n(t) z^n$$

$$+ \mu \sum_{n=0}^{\infty} (n+1) P_{n+1}(t) z^{n+1}$$

So, $R = \left(z^2\lambda - z(\lambda+\mu)+\mu\right) \cdot \sum_{n=0}^{\infty} n P_n(t) z^n + \left(z^2\nu - z\nu\right) \sum_{n=0}^{\infty} P_n(t) \cdot z^n$

and $\frac{\partial}{\partial z} R = \left(2z\lambda - (\lambda+\mu)\right) \cdot \sum_{n=0}^{\infty} n P_n(t) z^n + \left(\cancel{z}\lambda - z(\lambda+\mu)+\mu\right) \sum_{n=0}^{\infty} n^2 P_n(t) z^{n-1}$

$$+ \nu(2z-1) \sum_{n=0}^{\infty} P_n(t) \cdot z^n + \nu(z^2 - z) \cdot \sum_{n=0}^{\infty} n^2 P_n(t) z^{n-1}.$$

→

② (cont.)

Evaluate at $z=1$:

$$\frac{d}{dt}\mathbb{E}[n] = \frac{\partial}{\partial z}R\Big|_{z=1} = (2\lambda-\lambda-\mu)\sum_{n=0}^{\infty} n P_n(t)$$
$$+\underbrace{(\lambda-\cancel{\lambda}-\cancel{\mu}+\mu)}_{0}\sum_{n=0}^{\infty} n^2 P_n(t)\, z^{n-1}$$
$$+\nu(2-1)\underbrace{\sum_{n=1}^{\infty} P_n(t)}_{=1} + \nu\underbrace{\cancel{(1-1)}}_{0}\sum n^2 P_n z$$

$$\frac{d}{dt}\mathbb{E}[n] = (\lambda-\mu)\cdot\mathbb{E}[n]+\nu.$$

□

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② Please see jupyter notebook for the numerical simulation regarding this question.

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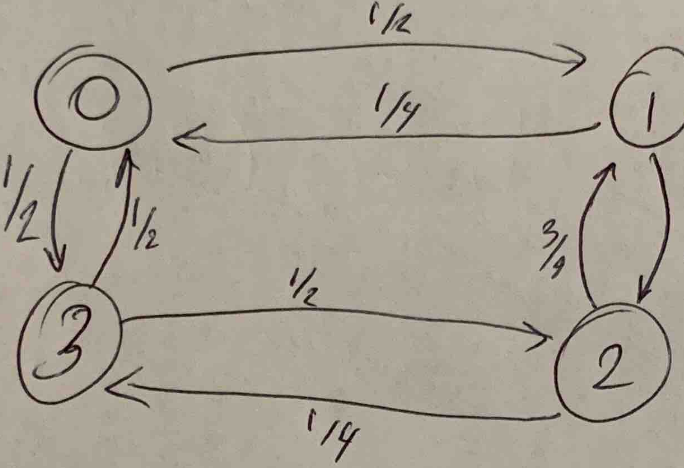
(3)

$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 3/4 & 0 & 1/4 \\ 1/2 & 0 & 1/2 & 0 \end{pmatrix}$$

(a)



(b) Irreducibility.

Consider $P^7 = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix}$ and $P^8 = \begin{pmatrix} 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \end{pmatrix}$

Looking at P^7, P^8, we see that for all states i, j, there exists $t, t+1$ such that:

$$P(X(k+t) = j \mid X(k) = i) > 0$$

or $$P(X(k+t+1) = j \mid X(k) = i) > 0$$

Therefore this Markov Chain (MC) is irreducible. $\square$

③ (b) (cont).

Since the ~~[illegible]~~ MC is irreducible and has finite state space, it must also be positive recurrent. $\square$

Show periodicity.

The MC is periodic, because as shown early the probability of returning to a state is 1.

Also given P^7, P^8 earlier, we see that for any state the smallest number of steps possible to return is 2.

So the period is 2. $\square$

③ (c) Let π be the unique stationary distribution.

Then $\pi = \pi_0 \cdot P^n$ for n large enough

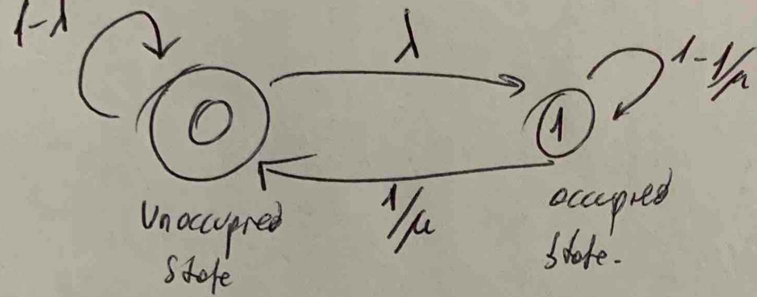
π_0: arbitrary initial distribution.

(in this case $n=8$ is large enough)

then (solving in Python), $\pi = [1/6, 1/3, 1/3, 1/6]$.

$\square$

(4) We can write this system as a Markov Chain (MC) given by:



with transition matrix $P = \begin{pmatrix} 1-\lambda & \lambda \\ 1/\mu & 1-1/\mu \end{pmatrix}$.

The long run fraction of time that the promoter is unoccupied is the fraction of time spent in state 0 which is given by the stationary distribution,

$\pi = [\pi_0, \pi_1]$, where we want π_0.

π is given by $\pi P = \pi$ and $\pi_0 + \pi_1 = 1$.

$$\Rightarrow \pi_0 (1-\lambda) + \pi_1 (1/\mu) = \pi_0$$
$$\pi_0 (\lambda) + \pi_1 (1-1/\mu) = \pi_1$$

Thus, $\pi_1 = \lambda\mu \cdot \pi_0$

So $\pi_0 + \lambda\mu\pi_0 = 1 \Rightarrow \pi_0 = \frac{1}{1+\lambda\mu}$. $\square$

(5)

Assume that a spike train can be described by a Poisson process. Then for small Δt, the probability of firing k action potentials in Δt is given by:

$$P(X(t+\Delta t)-X(t)=k) = \frac{(\lambda\Delta t)^k e^{-\lambda \Delta t}}{k!}.$$

For a fixed time t, let $T(t)$ be the time to the nearest AP in time.

Consider two cases: (1) spike at time $t+s$, $s \in \mathbb{R}$
(2) spike at time $t-u$, $u \in \mathbb{R}$.

Case 1: $$P(T(t) \leq s) = P\begin{pmatrix}\text{no spike between} \\ t \text{ and } t+s\end{pmatrix} = 1 - \frac{(\lambda s)^0}{0!}e^{-\lambda s} = 1-e^{-\lambda s}$$

Case 2: $$P(T(t) \leq u) = P\begin{pmatrix}\text{no spike between} \\ t-u \text{ and } t\end{pmatrix} = 1 - \frac{(\lambda u)^0}{0!}e^{-\lambda u} = 1-e^{-\lambda u}.$$

In any case, we see that the CDF of $T(t)$ corresponds to the exponential distribution with:

PDF: $f_T(s) = \lambda e^{-\lambda s}$

and mean: $\mathbb{E}[T(t)] = 1/\lambda$. $\square$