Math 6397: Hwk #4

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Problem 1

If $S_1 + S_2 \leq 10^6$, then the sons get S_1 and S_2.
If $S_1 + S_2 > 10^6$, then the sons get \$0.

Assume the two men do not talk to each other. Then the Nash equilibrium is any S_1, S_2 such that $S_1 + S_2 = 1{,}000{,}000$, because if one brother asks for more, then he gets \$0 instead. And if he asks for less, he would obviously get something less than S_1. Therefore any S_1, S_2 that satisfy $S_1 + S_2 = 1{,}000{,}000$ is a Nash equilibrium.

Note: if S_1, S_2 are such that $S_1 + S_2 < 10^6$, then they are not a Nash equilibrium because any brother can ask for more and get a higher payoff.

$\square$

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Problem 2

Single shot Prisoner's Dilemma:

payoff matrix.

		Player 2 C	Player 2 D
Player 1	C	R	S
	D	T	P

provided $T > R > P > S$.

(a) Compute payoff matrix after m rounds of GRIM & ALLD.

GRIM – GRIM → both players always cooperate so in m rounds, the payoff is $m \cdot R$.

GRIM – ALLD → Player 1 cooperates in round 1, but player 2 defects. After that, both defect $(m-1)$ times.

Therefore payoff to player 1 is $S + (m-1) \cdot P$

ALLD – GRIM → Same as above, but now player 1 gets $T + (m-1)P$.

ALLD – ALLD → Both players always defect so payoff is mP.

Payoff matrix:

	GRIM	ALLD
GRIM	mR	$S+(m-1)P$
ALLD	$T+(m-1)P$	mP

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Problem 2

(b) GRIM is stable under ALLD invasion if

$$\text{Payoff}_{\text{GRIM GRIM}} > \text{Payoff}_{\text{GRIM ALLD}}$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$mR > T + (m-1)P$$

$$mR > T + mP - P$$

$$mR - mP > T - P$$

$$m > \frac{T-P}{R-P}.$$

$\square$

(c)

GRIM* - GRIM* $\longrightarrow$ Both player cooperate in $(m-1)$ rounds and both defect on the last one.

Payoff is then: $(m-1)R + P$.

GRIM* - GRIM $\rightarrow$ Both cooperate in $(m-1)$ rounds

Player 1 defects on the last one and player 2 cooperates.

Payoffs are $(m-1)R+T$ & $(m-1)R+S$

GRIM-GRIM $\rightarrow$ mR (see part (b)).

Payoff matrix:

	GRIM*	GRIM
GRIM*	$(m-1)R+P$	$(m-1)R+T$
GRIM	$(m-1)R+S$	mR

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Problem 2

(c) GRIM* dominates GRIM

because

$$(m-1)R + P > (m-1)R + S,$$

because $P > S$ by assumption.

(d) Denote by GRIM** the strategy where the player cooperates for $(m-2)$ rounds and defects on the last two rounds.

The payoff matrix for GRIM** vs GRIM* is:

	GRIM**	GRIM*
GRIM**	$(m-2)R + 2\cdot P$	$(m-2)R + T + P$
GRIM*	$(m-2)R + S + P$	$(m-1)R + P$

GRIM** dominates GRIM*

because:

$$(m-2)R + 2P > (m-2)R + P + S$$

since $P > S$.

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Problem 2

(e) If we continue to defect on more and more rounds, then eventually ~~thus~~ we will always defect and the strategy becomes ALLD.

Problem 3

(a) Assume the game continues with probability δ. Then the probability of playing the 1^{st} round is 1. And the probability of playing 2 rounds is δ.

$\vdots$

Continuing in this manner, the prob. of playing k rounds is δ^{k-1}.

Therefore, the expected number of rounds is:

$$\langle m \rangle = \lim_{n \to \infty}\left(1 + \delta + \delta^2 + \delta^3 + \cdots + \delta^n\right)$$

$$= \sum_{k=0}^{\infty} \delta^k = \frac{1}{1-\delta} = \frac{\delta + 1 - \delta}{1-\delta} = \frac{\delta}{1-\delta} + 1.$$

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Problem 3

(b) Matchup between GRIM & ALLD.

GRIM - GRIM → Always cooperate → $\underbrace{R \cdot \sum_{k=0}^{\infty} \delta^k}_{\text{expected \# of rounds.}} = \frac{R}{1-\delta}$.

GRIM - ALLD → Cooperate once and defects forever after.

$$S + \sum_{k=1}^{\infty} \delta^k P = S + \frac{\delta}{-\delta+1} P$$

Other player that always defects gets:

$$T + \sum_{k=1}^{\infty} \delta^k \cdot P = T + \frac{\delta}{-\delta+1} P.$$

ALLD - ALLD → Both always defect: $P \sum_{k=0}^{\infty} \delta^k = P \cdot \frac{1}{1-\delta}$.

Payoff matrix:

	GRIM	ALLD
GRIM	$\frac{1}{1-\delta} R$	$S + \frac{\delta}{1-\delta} P$
ALLD	$T + \frac{\delta}{1-\delta} P$	$\frac{1}{1-\delta} P$

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Problem 3

(c) GRIM is stable against ALLD if:

$$\frac{1}{1-\delta} R > T + \frac{\delta}{1-\delta} P$$

$$R > (1-\delta) T + \delta P$$

$$R > T - \delta T + \delta P$$

$$\delta T - \delta P > T - R$$

$$\delta (T - P) > T - R$$

$$\delta > \frac{T-R}{T-P}.$$

$\square$

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Problem 4

(i) TFT $\rightarrow \{\overset{P_{cc}}{1-\epsilon}, \overset{P_{cd}}{\epsilon}, \overset{P_{dc}}{1-\epsilon}, \overset{P_{dd}}{\epsilon}\}$

(ii) GRIM $\rightarrow \{1-\epsilon, \epsilon, \epsilon, \epsilon\}$

(iii) ALLC $\rightarrow \{1-\epsilon, 1-\epsilon, 1-\epsilon, 1-\epsilon\}$

Possible combinations: TFT-TFT, TFT-GRIM, TFT-ALLC, GRIM-GRIM, GRIM-ALLC, ALLC-ALLC

(a)

• TFT-TFT.

$$M = \begin{pmatrix} (1-\epsilon)^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & \epsilon^2 \\ \epsilon(1-\epsilon) & \epsilon^2 & (1-\epsilon)^2 & \epsilon(1-\epsilon) \\ \epsilon(1-\epsilon) & (1-\epsilon)^2 & \epsilon^2 & \epsilon(1-\epsilon) \\ \epsilon^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & (1-\epsilon)^2 \end{pmatrix}$$

• TFT-GRIM.

$$M = \begin{pmatrix} (1-\epsilon)^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & \epsilon^2 \\ \epsilon^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & (1-\epsilon)^2 \\ \epsilon(1-\epsilon) & (1-\epsilon)^2 & \epsilon^2 & \epsilon(1-\epsilon) \\ \epsilon^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & (1-\epsilon)^2 \end{pmatrix}$$

• TFT-ALLC:

$$M = \begin{pmatrix} (1-\epsilon)^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & \epsilon^2 \\ \epsilon(1-\epsilon) & \epsilon^2 & (1-\epsilon)^2 & \epsilon(1-\epsilon) \\ (1-\epsilon)^2 & \epsilon(1-\epsilon) & \epsilon(1-\epsilon) & \epsilon^2 \\ \epsilon(1-\epsilon) & \epsilon^2 & (1-\epsilon)^2 & \epsilon(1-\epsilon) \end{pmatrix}$$

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Problem 4

(a)

- GRIM-GRIM

$$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \\ \varepsilon^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & (1-\varepsilon)^2 \end{pmatrix}$$

- GRIM-ALLC :

$$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \\ \varepsilon(1-\varepsilon) & \varepsilon^2 & (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) \end{pmatrix}$$

- ALLC-ALLC.

$$M = \begin{pmatrix} (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \\ (1-\varepsilon)^2 & \varepsilon(1-\varepsilon) & \varepsilon(1-\varepsilon) & \varepsilon^2 \end{pmatrix}$$

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Problem 4

(b) Collaborated with Youssef Menacer in this question.

To obtain the stationary distribution μ,
we need to solve the system:

$$\mu = \mu M \quad \text{with} \quad \sum_{i=1}^{4} \mu_i = 1.$$

However, we can use the fact that $\mu_1 = 1 - \mu_2 - \mu_3 - \mu_4$
and solve the simpler system instead:

$$A \cdot \begin{pmatrix} \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} = b.$$

where $A = \begin{pmatrix} 1+m_{12}-m_{22}, & m_{12}-m_{32}, & m_{12}-m_{42} \\ m_{13}-m_{23}, & 1+m_{13}-m_{33}, & m_{13}-m_{43} \\ m_{14}-m_{24}, & m_{14}-m_{34}, & 1+m_{24}-m_{44} \end{pmatrix}$

and $b = \begin{pmatrix} m_{12} \\ m_{13} \\ m_{14} \end{pmatrix}$

and $m = M$ — transition matrix.

and $\mu_1 = 1 - \mu_2 - \mu_3 - \mu_4$.

We solved this system on Matlab to
obtain the distribution:

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Problem 4

(b)

- **TFT - TFT:**

$$\mu = \left[\frac{1}{2(2\varepsilon-3)} + \frac{1}{2} \,,\; \frac{1}{4} \,,\; \frac{1}{4} \,,\; \frac{-1}{2(2\varepsilon-3)} \right]$$

$$\mu_{\varepsilon\to 0} = \left[\frac{1}{3} \,,\; \frac{1}{4} \,,\; \frac{1}{4} \,,\; \frac{1}{6} \right].$$

- **TFT-GRIM:**

$$\mu = \begin{bmatrix} \frac{2\varepsilon(\varepsilon-1)}{2\varepsilon+1} - \frac{2(2\varepsilon^3-2\varepsilon^2+\varepsilon)}{2\varepsilon+1} - \frac{\varepsilon}{2\varepsilon+1} + 1 \\ -2\varepsilon(\varepsilon-1)/(2\varepsilon+1) \\ \varepsilon/(2\varepsilon+1) \\ \frac{2(2\varepsilon^3-2\varepsilon^2+\varepsilon)}{2\varepsilon+1} \end{bmatrix}^T$$

$$\mu_{\varepsilon\to 0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

- **TFT-ALLC.**

$$\mu = \begin{bmatrix} \frac{\varepsilon(-2\varepsilon^3+\varepsilon^2+\varepsilon-1)}{4\varepsilon^4+4\varepsilon^3-\varepsilon^2+\varepsilon+1} + \frac{2\varepsilon^2(\varepsilon-1)}{4\varepsilon^4-4\varepsilon^3-\varepsilon^2+\varepsilon+1} + \frac{2\varepsilon(\varepsilon-1)(2\varepsilon^3-3\varepsilon^2+1)}{4\varepsilon^4-4\varepsilon^3-\varepsilon^2+\varepsilon+1} + 1 \\ \frac{-\varepsilon(-2\varepsilon^3+\varepsilon^2+\varepsilon-1)}{4\varepsilon^4+4\varepsilon^3-\varepsilon^2+\varepsilon+1} \\ \frac{-2\varepsilon(\varepsilon-1)(2\varepsilon^3-3\varepsilon^2+1)}{4\varepsilon^4+4\varepsilon^3-\varepsilon^2+\varepsilon+1} \\ \frac{-2\varepsilon^2(\varepsilon-1)}{4\varepsilon^4+4\varepsilon^3-\varepsilon^2+\varepsilon+1} \end{bmatrix}^T$$

$$\mu_{\varepsilon\to 0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

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Problem 4

(b) • **GRIM-GRIM:**

$$\mu = \begin{bmatrix} 2\varepsilon(\varepsilon-1) - 2\varepsilon + 5\varepsilon^2 - 4\varepsilon^3 + 1 \\ -\varepsilon(\varepsilon-1) \\ -\varepsilon(\varepsilon-1) \\ 4\varepsilon^3 - 5\varepsilon^2 + 2\varepsilon \end{bmatrix}^T$$

$$\mu_{\varepsilon \to 0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

• **GRIM-ALLC:**

$$\mu = \begin{bmatrix} \dfrac{2\varepsilon(\varepsilon-1) - 4\varepsilon^4 + 6\varepsilon^3 - 2\varepsilon^2 - \varepsilon + 2(2\varepsilon+1)(\varepsilon-1)(\varepsilon^2-2\varepsilon+1)}{8\varepsilon^3 - 12\varepsilon^2 + 2\varepsilon + 3} + 1 \\ \dfrac{4\varepsilon^4 - 6\varepsilon^3 + 2\varepsilon^2 + \varepsilon}{8\varepsilon^3 - 12\varepsilon^2 + 2\varepsilon + 3} \\ \dfrac{-2(2\varepsilon+1)(\varepsilon-1)(\varepsilon^2-2\varepsilon+1)}{8\varepsilon^3 - 12\varepsilon^2 + 2\varepsilon + 3} \\ \dfrac{-2\varepsilon(\varepsilon-1)}{8\varepsilon^3 - 12\varepsilon^2 + 2\varepsilon + 3} \end{bmatrix}^T$$

$$\mu_{\varepsilon \to 0} = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \\ 0 \end{bmatrix}^T$$

• **ALLC-ALLC:**

$$\mu = \begin{bmatrix} 2\varepsilon(\varepsilon-1) - \varepsilon^2 + 1 \\ -\varepsilon(\varepsilon-1) \\ -\varepsilon(\varepsilon-1) \\ \varepsilon^2 \end{bmatrix}$$

$$\mu_{\varepsilon \to 0} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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Problem 4

(c)

- TFT-TFT payoff: $\frac{1}{3}R + \frac{1}{4}S + \frac{1}{4}T + \frac{1}{6}P$.
- TFT-GRIM payoff: $1 \cdot R + 0 \cdot S + 0 \cdot T + 0 \cdot P = R$.
- TFT-ALLC payoff: $1R + 0 \cdot S + 0 \cdot T + 0 \cdot P = R$
- GRIM-GRIM payoff: $1R + 0 \cdot S + 0 \cdot T + 0 \cdot P = R$
- GRIM ALLC payoff: $\frac{1}{3}R + \frac{1}{3}T$.
- ALLC-ALLC payoff: $1R + 0 \cdot S + 0 \cdot T + 0 \cdot P = R$.

(d)

	TFT	GRIM
TFT	$\frac{1}{3}R+\frac{1}{4}S+\frac{1}{4}T+\frac{1}{6}P$	R
GRIM	R	R

→ Nash eq.

	TFT	ALLC
TFT	$\frac{1}{3}R+\frac{1}{4}S+\frac{1}{4}T+\frac{1}{6}P$	R
ALLC	R	R

→ b/c $R \geq R$

	ALLC	GRIM
ALLC	R	$\frac{1}{3}R+\frac{2}{3}T$
GRIM	$\frac{1}{3}R+\frac{2}{3}T$	R

→ Nash eq. b/c $\frac{1}{3}R+\frac{2}{3}T \geq R$.