Math 6397: Homework #2

(1) $Y_1 \xrightarrow{c_1} 2Y_1$

$Y_1 + Y_2 \xrightarrow{c_2} 2Y_2$

$Y_2 \xrightarrow{c_3} \phi$

Stochastic Lotka - Volterra

The correspony ODE model is:

$$\dot{y}_1 = \alpha y_1 - \beta y_1 y_2$$

$$\dot{y}_2 = \delta \cdot y_1 \cdot y_2 - \gamma y_2 .$$

where $y_1 \leftrightarrow Y_1$ (prey)
$y_2 \leftrightarrow Y_2$ (predator)

- The stochastic Lotka-Volterra is a predator-prey model because Y_1 acts as a prey that reproduces with rate c_1 and is eaten by Y_2 (predator) with rate c_2. Y_2 then dies with rate c_3.

In this ~~[illegible]~~ formulation of the model, Y_1 can exist in isolation, however the Y_1 population would reproduce uncontrollably and the number of Y_1 diverges to ∞.

1 (cont.)

On the other hand, Y_2 cannot exist in isolation; because in the absence of Y_1, Y_2 is not able to reproduce and all Y_2 agent would eventually die.

- Please see Python Jupyter Notebook for simulations and answers to the last two points.

(2) $A \xrightarrow{k} X \xrightarrow{\alpha_1} \phi$

$B \xrightarrow{k} Y \xrightarrow{\alpha_2} \phi$

$X + Y \xrightarrow{k_a} C.$

$$\frac{d[X]}{dt} = k - \alpha_1[X] - k_a \cdot [X][Y]$$

$$\frac{d[Y]}{dt} = k - \alpha_2[Y] - k_a[X][Y].$$

- Fixed points occur when $\frac{d[X]}{dt} = 0$ and $\frac{d[Y]}{dt} = 0$.

$$\frac{dX}{dt} = 0 \Rightarrow [X]_* = \frac{k}{\alpha_1 + k_a[Y]_*}$$

$$\frac{dY}{dt} = 0 \Rightarrow [Y]_* = \frac{k}{a_2 + k_a[X]_*}$$

Solve simultaneously.

2 (cont.)

By substitution, $X = \dfrac{k}{\alpha_1 + k_a\left(\dfrac{k}{\alpha_2 + k_a X}\right)}$.

$$\Rightarrow \alpha_1 X + k_a \cdot \frac{k}{\alpha_2 + k_a X} X = k$$

$$\Rightarrow k_a \cdot k \cdot X = (k - \alpha_1 X)(\alpha_2 + k_a X)$$

$$\Rightarrow \cancel{k_a k X} = k\alpha_2 + \cancel{k_a k X} - \alpha_1 \alpha_2 X - \alpha_1 k_a X^2$$

$$\Rightarrow 0 = \alpha_1 k_a X^2 + \alpha_1 \alpha_2 X - k\alpha_2 .$$

By quadratic formula, $X = \dfrac{-\alpha_1\alpha_2 \pm \sqrt{(\alpha_1\alpha_2)^2 - 4\alpha_1 k_a(-k\alpha_2)}}{2\alpha_1 k_a}$

Since concentrations must be positive,

fixed point:

$$\boxed{\begin{aligned} X_* &= \frac{-\alpha_1\alpha_2 + \sqrt{(\alpha_1\alpha_2)^2 + 4\alpha_1\alpha_2 k k_a}}{2\alpha_1 k_a} \\ Y_* &= \frac{k}{\alpha_2 + k_a\left(\dfrac{-\alpha_1\alpha_2 + \sqrt{(\alpha_1\alpha_2)^2 + 4\alpha_1\alpha_2 k \cdot k_a}}{2\alpha_1 k_a}\right)} \end{aligned}}$$

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(2)

Check that the f.p. is the same for:

$$(*)\ k = 10,\quad \alpha_1 = 10^{-6},\quad \alpha_2 = 10^{-5},\quad k_a = 10^{-5}$$

and

$$(**)\ k = 10^3,\quad \alpha_1 = 10^{-4},\quad \alpha_2 = 10^{-3},\quad k_a = 10^{-3}.$$

For $(*)$, the ~~fixed points~~ nullclines are

$$X_* = \left(\frac{10}{10^{-6}+10^{-5}Y_*}\right)\cdot\frac{10^5}{10^5} = \frac{10^6}{10^{-1}+Y_*}$$

$$Y_* = \frac{10}{10^{-5}+10^{-5}X_*}\left(\frac{10^5}{10^5}\right) = \frac{10^6}{1+X_*}$$

For $(**)$, we have:

$$X_{**} = \frac{10^3}{10^{-4}+10^{-3}Y_{**}}\left(\frac{10^3}{10^3}\right) = \frac{10^6}{10^{-1}+Y_{**}}$$

and

$$Y_{**} = \frac{10^3}{10^{-3}+10^{-3}X_{**}}\cdot\left(\frac{10^3}{10^3}\right) = \frac{10^6}{1+X_{**}}.$$

Since the nullclines are the same, fixed pts will be the same:

$$(X_*, Y_*) = (X_{**}, Y_{**}).$$

- Please see code for answers to parts (b), (c).

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(3)

$$\frac{dr}{dt} = k_\ell + \phi(p) - \gamma_r r$$

$$\frac{dp}{dt} = r \cdot k_p - \gamma_p \cdot p.$$

(a) The transition matrix is given by the following probabilities, assuming h is small enough:

$$P(r \to r+1, p \to p) = (k_\ell + \phi(p)) \cdot h + o(h)$$

$$P(r \to r-1, p \to p) = (\gamma_r \cdot r) \cdot h + o(h)$$

$$P(r \to r, p \to p+1) = (k_p \cdot r) \cdot h + o(h)$$

$$P(r \to r, p \to p-1) = (\gamma_p \cdot p)\, h + o(h)$$

$$P(r \to r, p \to p) = 1 - (k_\ell + \phi(p) + \gamma_r r + k_p r + \gamma_p p) \cdot h + o(h)$$

(b) Nullclines: $\frac{dr}{dt} = 0 \Rightarrow r = \frac{k_\ell + \phi(p)}{\gamma_r}$, $\frac{dp}{dt} = 0 \Rightarrow p = \frac{k_p}{\gamma_p} r$

We found the fixed points using the nullclines and we determined their stability using phase plane analysis. See code for answers to this and part (c).