Dynamic Demand Estimation for Storable Goods

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Abstract

This paper develops a dynamic discrete-continuous demand model for storable goods - a class of goods that account for a large fraction of grocery expenditures. To estimate and solve the dynamic demand model, we use techniques from: (i) Approximate Dynamic Programming, (ii) large scale dynamic programming in economics, (iii) machine learning, and (iv) statistical computing. The benefits of this approach are three-fold. First, the dynamic demand model is compatible high-dimensional choice sets. Second, it can capture rich inter- and intra-temporal substitution patterns. Third, the dimension reduction strategies do not rely on the idiosyncratic features of the industry being studied. As a result, it can be estimated for any storable good industry. In this paper we apply the model to the UK laundry detergent sector using household level purchase data.

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1 Introduction

To date much of the demand estimation literature has focussed on static demand models.[1] Underlying all static demand estimation models is the assumption that goods are consumed within the same period in which they are purchased. Moreover, future expectations over changes to the market environment are assumed to have no impact on current demand. However, these assumptions are not appropriate for storable goods - a class of fast moving consumer goods that account for a large portion of household grocery expenditures.[2]

There are two key features of storable good demand. First, the demand for storable goods is inherently dynamic. Their durability enables them be stored in inventory and consumed at a later date. As a result, households need only make infrequent purchases to meet consumption needs. The process of building up stocks with purchases and depleting them through consumption creates inter-temporal links in the demand model.

Second, storable goods are typically sold at a constant price for sustained periods of time interspersed with short, temporary sales periods.[3] As such, prices exhibit persistence. Consequently, household's are likely to form expectations over future prices and factor them into the current purchase decisions.

Due to the presence of dynamic links through inventories and price expectations, storable good demand models must solve a dynamic optimisation problem while estimating the structural parameters. This problem is further complicated because households typically have the option to purchase, store and consume many different variants of the good. Therefore, in addition to being inherently dynamic, the demand model is high-dimensional.

Moreover, inventories are unobserved in the data. As a result, they must be integrated out of the demand model. This further complicates the computational challenge of mitigating the curse of dimensionality.

Faced with these issues, the existing literature has sought to trade-off the flexibility of the dynamic demand model with the computational resources needed to estimate it. In general, there have been two approaches.

One is to select applications that have low dimensional choice sets (i.e. few brands and few size choices) and exploit other idiosyncratic features of those industry to produce a tractable dynamic demand model (Erdem et al. (2003)). The other is to assume that the consumer's decision can be split into a static brand choice and a dynamic size choice (Hendel and Nevo (2006a); Wang (2012, 2013); Osborne (2016)). Further, price expectations are assumed to be captured by ex-ante expected utility of consuming a particular size (see Melnikov (2013)). Both approaches require further restrictions on the functional form of utility from consumption and require that the products are available in only a handful of sizes.

[1] See Deaton and Muellbauer (1980); Train (2009); Chintagunta and Nair (2011).

[2] For example, tinned products, frozen products, soft drinks, alcohol, personal hygiene products, table sauces, and pet food are all examples of storable goods.

[3] Erdem et al. (2003); Hendel and Nevo (2006a,b); Nevo and Hendel (2012); Osborne (2013); Wang (2012, 2013) all report that prices typically exhibit these features.

If the number of choices entering the dynamic choice set are large, these novel approaches no longer yield computationally tractable demand models. Indeed, in many industries, products are available in many different varieties and sizes and the choice sets are high-dimensional.

One such example it the UK laundry detergent industry - the application used in this paper. In this industry, there are around 100 products in the choice set in each week. Moreover, over 20 pack sizes are needed to cover 95 percent of sales. Therefore, even if we adopt the static-brand dynamic-size choice split used by Hendel and Nevo (2006a), the dynamic choice set is still large.

This paper proposes an alternative approach to alleviate the curse of dimensionality in storable good demand estimation. In turn, it can allow for high-dimensional dynamic choice sets. Moreover, the dimension reduction strategies do not rely on idiosyncratic features of the industry studied. As a result this model can be applied to a wide range of storable good industries.

With fewer restrictions needed for computational tractability, a flexible dynamic demand model can be estimated. As as result, the model presented in this paper can incorporate and extend desirable features of existing models. As in Erdem et al. (2003) this allows for unobserved heterogeneity in the continuation value function in *both* brand and size of the purchased product. As a result, it can capture rich substitution patterns both over time and within a time period. Like Hendel and Nevo (2006a) and related models, consumption is modelled as an endogenous continuous choice. In addition, we allow consumption to be vector-valued because product differentiation enters utility at the point of consumption, not purchase.

To estimate and solve such a flexible, high-dimensional dynamic demand model, we use techniques from: (i) Approximate Dynamic Programming (ADP), (ii) large scale dynamic programming in economics, (iii) machine learning, and (iv) statistical computing.

There exists a nascent literature where ADP methods have been used to estimate dynamic models in economics. Hendel and Nevo (2006a), Sweeting (2013) and Fowlie et al. (2016) use parametric policy function iteration described by Benitez-Silva et al. (2000) - an early ADP algorithm. Arcidiacono et al. (2012) show how to use sieve value function iteration to estimate and approximate the solution to dynamic single agent models with large-state spaces - an approach closely related to the ADP methods used in this paper. Other ADP techniques have also been used to approximate solutions to large scale dynamic games (see Farias et al. (2012)).

In this paper, a low-dimensional representation of the inventory state space is used to approximate the value function associated with a particular configuration of inventory. Since the resulting state space is still moderately sized, the value function is approximated using a Smolyak polynomial to further mitigate the curse of dimensionality (Judd et al. (2014)). We solve for its coefficients by combining a forward simulation algorithm from the ADP literature called λ-policy iteration (λ-PI) (Bertsekas (2015)) with the Envelope Condition Method (Maliar and Maliar (2013)). Further, to mitigate numerical instability issues that can arise when fitting models using simulated data from dynamic models, we draw on the insights of Judd et al. (2011).[4]

Dimension reduction techniques from the machine learning and statistics literature are employed to reduce the size of the price state space used by household's to form price expectations.[5] In particular, we assume household's use an Interactive Fixed Effects model (Bai (2009)) to forecast prices. This produces a low-rank approximation to the evolution of a high-dimensional set of prices. The basis of the low-rank approximation serve state variables and its dynamics govern their transition.

The dynamic demand model for the UK laundry detergent industry is estimated using the simulated method of moments. To fit the structural parameters we use recently developed derivative free optimisation methods from statistical computing (Łącki and Miasojedow (2015); Baragatti et al. (2013)). As noted by Imai et al. (2009) and Norets (2009), solving the dynamic demand model at every parameter guess is very costly for such methods. In line with their suggested approaches, we alternate between fitting the structural parameters and solving the dynamic demand model.[6]

The remainder of the paper is structured as follows. In section 2 we provide an overview of the UK laundry detergent industry. Section 3 describes the dynamic demand model for the UK laundry detergent industry and details the dimension reduction strategies used from machine learning and ADP. Sections 4 and 5 discuss the identification and estimation of the model, respectively. Section 6 concludes.

2 UK Laundry Detergent Market

In this section we provide a brief overview of the UK detergent industry. We highlight features of the industry that make application of existing storable good demand models challenging. Next, we present evidence that both price and inventory dynamics are present and impact on demand. Finally, we show evidence that suggests consumer taste heterogeneity is likely to be an important factor in effectively modelling and successfully predicting consumer demand. Before doing so, we briefly provide an overview of the data.

2.1 Data

The analysis of the UK laundry detergent industry uses household level purchase diary data from Kantar. The data spans the period from 1st January 2009 until 31st December 2011. It focusses on households who make the vast majority of their purchases at one store - Tesco, the UK's largest grocery retailer.

[4]In addition to the use of orthogonal basis functions in the Smolyak polynomial, we fit coefficient using PCA regression.

[5]We also use tools from machine learning to reduce the number of structural parameters to be estimated. In particular, high dimensional positive definite matrices of structural demand parameters are flexibly, yet parsimoniously specified using a Mercer Kernel of Radial Basis Functions. See Murphy (2012), Chapter 14

[6]This technique is used by Osborne (2016).

The same dataset is used for demand estimation and contains further modifications as a result. A description of the construction of the sample used can be found in section 5. The sample contains 620 households. The 10th and 90th percentile of grocery spend per equivalent adult are £30 and £70, respectively. Households in the sample contain 1.7 equivalent adults on average and are observed for an average of 135 weeks. The average household spends £48 per equivalent adult on weekly groceries of which £2.31 is spend on laundry detergent.

2.2 Overview of UK Laundry Detergent

Laundry detergent is purchased in discrete bundles of washes called Stock Keeping Units (SKUs). A SKU is defined by the type of detergent it contains and the number of washes it provides.[7] The laundry detergent a SKU contains is defined by two characteristics: format and brand.[8]

Formats Detergent is available in one of five formats: liquid capsules, gel, liquid, powder and tablets. Each format differs in how it is used, its efficacy, the amount of physical storage it occupies, and its ease of storage.

At the point of consumption, households can choose how much liquid, powder or gel to use in a wash. In contrast, capsules and tablets are sold in pre-measured, discrete dosages. Further, the format may also impact on the type of laundry it is being used for. In particular, the presence of bleach in powder makes it especially suitable for removing deep stains, whereas liquid and gel might be better for delicate garments.

Formats also differ in the the physical amount of material needed for one wash. In particular, the physical amount of liquid based detergents for a single wash is less than solid detergent. Consequently, powder and tablets are likely to take up more physical storage space than other formats per wash.

These differences also impact on the price per wash for each detergent. Figure 1 uses box plots to summarise the observed distribution of price per wash for each format in the data.

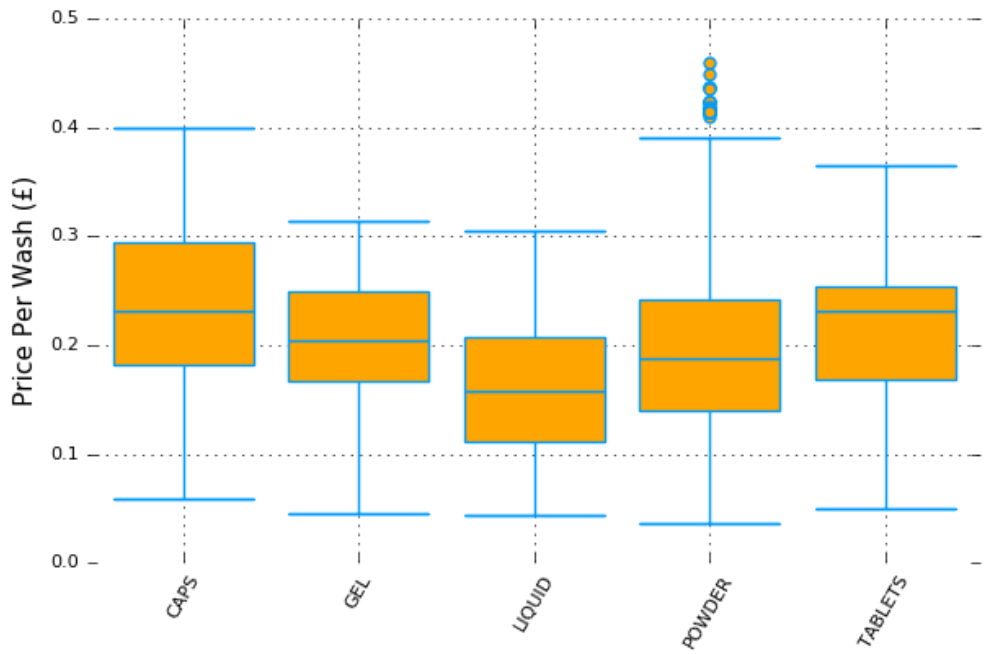
Notably the convenience and storage flexibility of capsules and tablets appears to command a price premium. While performance of gel is broadly similar to liquid, its novelty, combined with its low storage costs and ease of use commands a premium relative to liquid.

Given these differences, we might expect (dis)utility of consumption and storage costs to vary by format. As such, allowing household's utility from consumption and cost of

[7] In addition to the price of the detergent, the configuration of the set of SKUs the manufacturers make available to retailers is a competitive variable - albeit over a longer time horizon.

[8] Detergents are also differentiated in other ways. In particular, a detergent can be classified as biological or non-biological depending on the enzymes used. Further, they may be specialised to protect against colour fading. Other aspects of differentiation include scent, effective temperature range, additional stain removal capacity, etc. However, many of these aspects of the product are imperfectly observed in the data and cannot be accurately measured. As such, the model focuses on the two key defining characteristics of detergents: brand and format.

Figure 1: Price per wash by format



storage to differ by detergent format would be desirable in any demand model of the UK laundry industry.

Brands In the UK there are only two firms that sell branded detergent: Proctor and Gamble (P&G) and Unilever.[9] There are six major brands: four are owned by P&G (Ariel, Bold, Daz and Fairy) and two are owned by Unilever (Persil and Surf). In addition, Tesco sell Private Label (PL) detergent, as well as several other smaller niche brands.

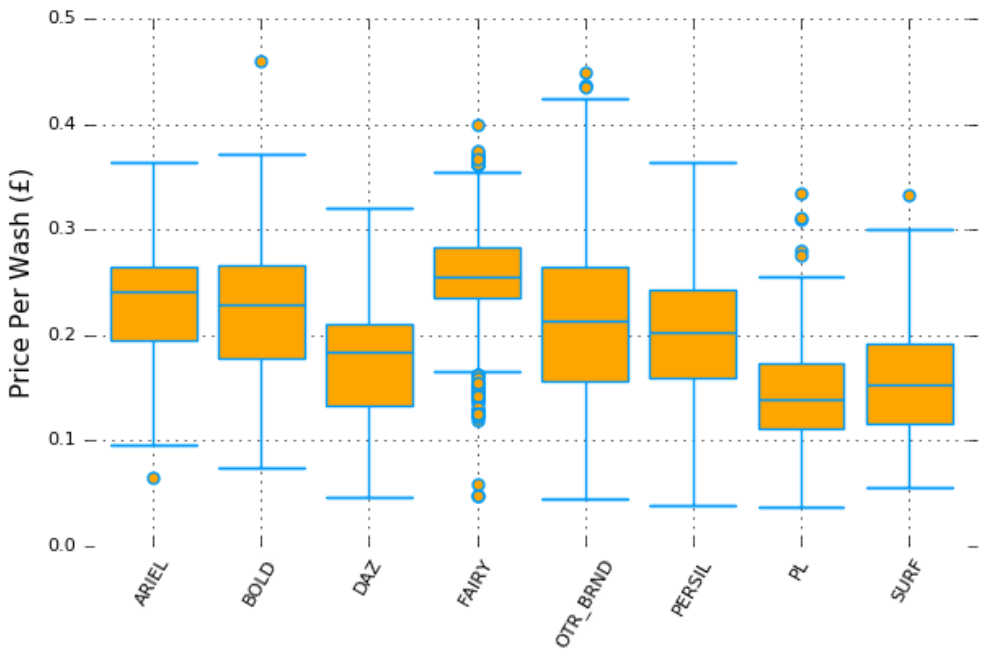
Not all brands are available in all formats. In particular, gel detergent is an innovation of P&G introduced in 2008. In the sample, Ariel, Bold and Fairy are the only major brands sold in a gel format. Tesco's PL is also available in a gel format - highlighting that these major brands are key suppliers of PL products for supermarkets.

The distribution of the price per wash for each brand is summarised by box plots in Figure 2. The figure suggests that Fairy and, to a lesser extent, Ariel are premium brands. Further, it would appear Bold and Persil can be classified as mid-range brands. Budget brands, Daz and Surf, are competitively priced against Tesco's PL.

Figure 3 displays the market shares based on the number of washes purchased for each detergent over the sample period. The most purchased brands are Tesco's PL (34%) and Persil (21%). In terms of formats, powder detergents are the most popular (38%), followed by liquid.

[9]There is only 1 other major manufacturer in Europe - Henkel. They do not sell any brands in the UK.

Figure 2: Price per wash by brand



Turning to specific detergents, Tesco's PL sales are equally dispersed over liquid, powder and tablets. Unilever's products, Persil and Surf, are most heavily focussed on liquid and powder. Of P&G's products, Ariel gel detergent and Daz powder are also quite popular, each with 5% market share.

SKU sizes The distribution of SKU sizes available in the data by format is shown in Figure 4. In contrast to dynamic demand applications in the existing literature, it shows that laundry detergent in the UK can be purchased a wide variety of sizes. The 5 most popular SKU sizes only account for 53 percent of washes purchased. To cover 95 percent of all washes purchased, in excess of 20 different SKU sizes are needed.

This feature of the UK laundry detergent industry is especially important in the context of applying existing dynamic storable good demand models. This is because having a low dimensional set of size choices is crucial to achieving computational tractability with existing models.

2.3 Demand Dynamics

Underlying a static demand model is the assumption that all purchases are intended for current consumption. As such, there does not exist a link between current demand and inventories or price expectations. However, in a dynamic inventory demand model, household purchases are stockpiled to be used for both current and future consumption. In turn, price expectations and existing inventories can impact on current demand.

Figure 3: Market Shares by brand and format of washes purchased

	Formats					
Brands	Caps	Gel	Liquid	Powder	Tablets	Total
Ariel	2.87	5.25	0.18	2.03	1.24	11.59
Bold	2.13	3.31	0.21	3.64	1.22	10.52
Daz	0.13		0.70	5.51	0.25	6.58
Fairy	0.78	1.14	0.74	3.44	1.13	7.24
Persil	1.91		8.36	7.22	3.04	20.54
Surf	0.32		3.64	5.20	0.17	9.33
Private Label	2.88	1.89	8.82	10.84	9.29	33.72
Other Brands	0.01		0.22	0.08	0.17	0.49
Total	11.03	11.59	22.89	37.96	16.53	100.00

Source: Kantar

Figure 4: Sales are not concentrated in a handful of SKU sizes

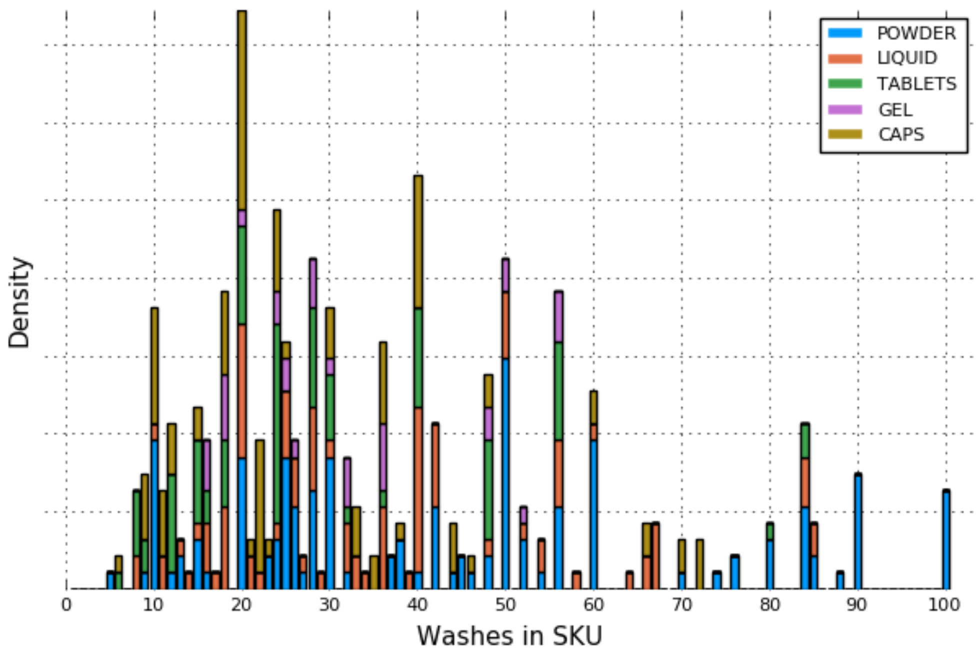
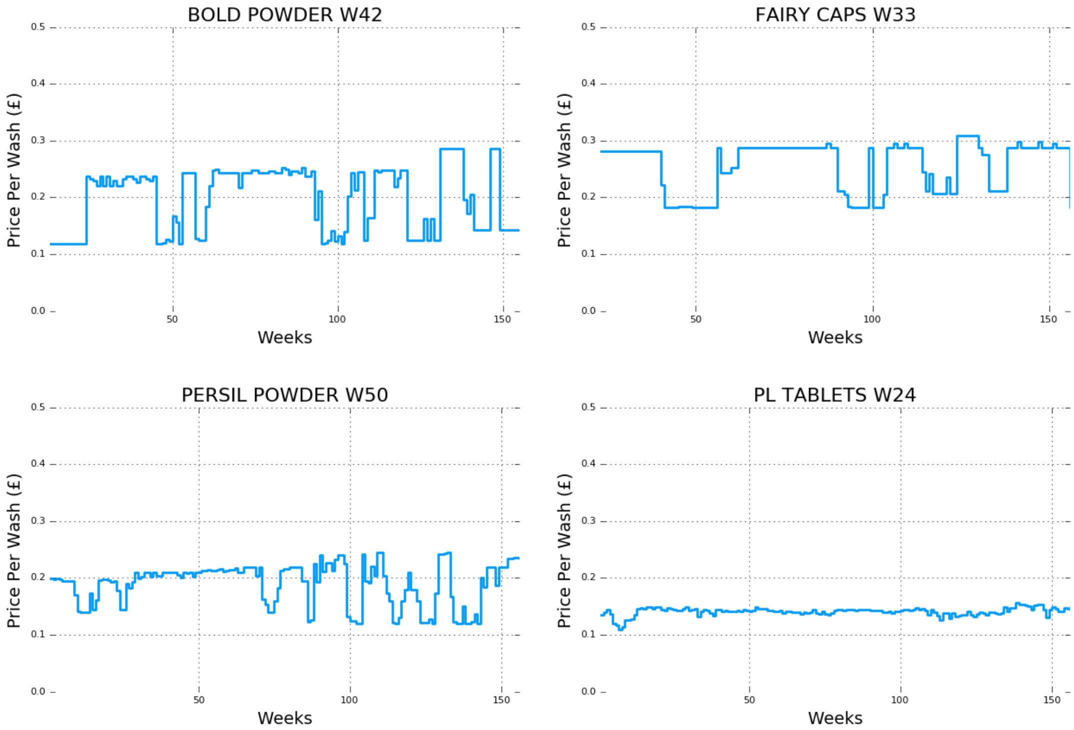


Figure 5: Selected price per wash time series



Like other storable good demand studies, UK laundry detergent is often sold on promotion. Figure 5 shows examples of time series of price per wash for four SKUs. These SKUs differ in number of washes, brand and format but exhibit typical price patterns for other SKUs in the data.

The price series in the figure highlights that the length, depth, and frequency of promotions vary with SKU size, brand and format. As such, any model of household price forecasting must be sufficiently flexible to capture differences in pricing strategies across SKUs.

If households' price expectations affect current demand, we would expect them to accelerate purchases to take advantage of promotional prices that they believe to be short lived. As such, we expect to observe positive conditional correlation between inter-purchase duration and current price per wash.[10]

Figure 6 shows the result of a regression of the logarithm of weeks since purchase on the current and previous logarithm of the price per wash, brand and format together with fixed effects for households. The current price per wash is positively correlated with inter-purchase duration; this suggests that price expectations impact on the timing of purchases and are an important aspect of households' purchasing behaviour.

In addition, the figure shows that past prices are negatively conditionally correlated with inter-purchase duration. This provides indirect evidence that households take ad-

[10]See Boizot et al. (2001)

Figure 6: Conditional correlation of current and previous price per wash with inter-purchase duration

Variable	Estimate	Std. Err.
Current purchase: ln(Price Per Wash)	0.085	0.027
Last purchase: ln(Price Per Wash)	-0.380	0.027
Num. Obs.	11,592	

Note: Includes HH fixed effects and controls for current and previous brand and format purchased
Source: Kantar

Figure 7: Quantity purchased, inventory, prices and sales

Variable	Estimate	Std. Err.
Inventory/100	-1.847	0.509
Price Per Wash	-99.18	3.582
Promotion	11.24	0.891
Price Per Wash x Promotion	-64.88	5.320
Num. Obs.	5,438	

Note: Includes fixed effects for brand, format, fascia and household
Source: Kantar

vantage of previous sales to stock up on detergent. As a result, they can wait longer before making their next purchase.

To further explore whether current demand is a function of inventories, and explore whether households take advantage of promotions, we regress washes purchased on a proxy for inventory, the price per wash and a dummy indicating that the a purchase was made on promotion.[11]

The results, shown in Figure 7, are also consistent with stockpiling behaviour. Namely, higher inventory tends to reduce current demand, promotional activity increases likelihood of purchasing more washes (i.e. purchase acceleration), and households' demand is inversely linked to price per wash.

Overall, the reduced form analysis of household purchasing behaviour presented in this section supports the view that a static demand model would be mis-specified for the UK laundry detergent industry.

[11]See Annex E for a description of the construction of the inventory proxy.

2.4 Persistent taste heterogeneity

Incorporating persistent taste heterogeneity into static demand models is often necessary to produce rich substitution patterns. This may be especially important in dynamic demand models because persistent taste heterogeneity may have an impact on the timing of households' purchases.

For example, households might accelerate their purchase to take advantage of a promotion of their favoured detergent that they perceive to be short-lived. Alternatively, they may choose to purchase small, cheap "stop-gap" detergents. In turn, enabling them to continue to smooth their consumption without unduly increasing the cost of purchasing their favoured detergent on sale in the near future.

Without persistent taste heterogeneity, the distribution of format, brand and size should be of independent if the previous purchase.

Figure 8 shows the format, brand, and size per equivalent adult of consecutive household purchases. The rows in the two tables record the previous purchase and the columns record the most recent purchase. The figures show the share of purchases in the current period conditional on the previous purchase. The bottom figure shows the contours of a kernel density estimate of the joint distribution of last and current quantity per equivalent adult purchased.

This analysis shows that households tend to purchase the same brand, format and size. In turn, suggesting that persistent taste heterogeneity may be important in explaining households purchase decisions.

3 Dynamic Demand Model

In the model time is discrete, households are infinitely lived and discount the future at a rate $\delta \in (0, 1)$. In each period, a household faces a discrete-continuous decision: (i) they choose whether or not to purchase laundry detergent, and (ii) how much laundry detergent to consume in the current period.

When deciding which, if any, SKU to purchase, households account for price dynamics. From a household's viewpoint, the timing and length of these sales and the depth of the discounts are uncertain. Households are assumed to use prices in previous periods to model their beliefs over price evolution.[12]

Once the purchase decision is made, the newly purchased detergent is added to inventories. Households then choose how much to consume of each of the laundry detergents held in inventory.

Households can store unused laundry detergent. Since households have limited storage space they incur convex costs of storage in the amount of inventory held. As such, they may elect to infrequently make purchases to restock laundry detergent.

In the remainder of this section, the household's choice problem is then described in detail.

[12]The strategic interactions on the supply-side that generate price dynamic is assumed to be too complex for households to comprehend.

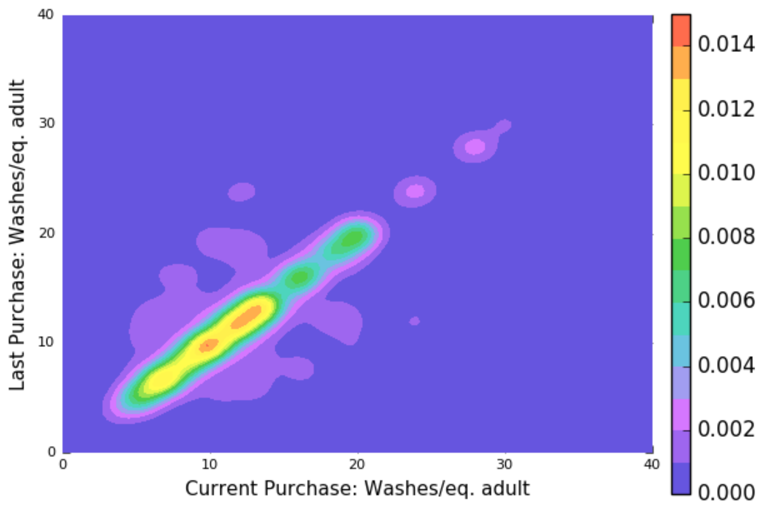
Figure 8: Conditional purchase shares: Format, Brand and Size purchased

	Current Format				
Last Format	Caps	Gel	Liquid	Powder	Tablets
Caps	82.0	3.1	5.1	3.2	6.6
Gel	3.1	78.8	13.5	3.5	1.0
Liquid	3.0	8.1	81.4	5.5	2.0
Powder	1.3	1.3	3.5	91.8	2.0
Tablets	6.1	1.6	3.2	4.3	84.7

Source: Kantar

	Current Brand							
Last Brand	Ariel	Bold	Daz	Fairy	Others	Persil	PL	Surf
Ariel	72.8	6.2	2.1	1.9	0.2	8.7	5.3	2.9
Bold	7.0	67.5	3.2	1.1	0.1	7.0	6.1	8.2
Daz	4.2	6.5	62.3	0.6	0.1	6.7	7.5	12.0
Fairy	3.5	1.7	0.7	79.2	0.6	8.5	4.7	1.0
Others	5.3	2.1		3.2	37.2	17.0	28.7	6.4
Persil	5.3	4.8	2.1	4.2	0.8	67.5	9.9	5.3
PL	1.6	2.4	1.6	0.8	0.6	4.6	86.0	2.3
Surf	4.2	9.2	8.7	0.9	0.6	11.4	10.3	54.8

Source: Kantar



3.1 Household's choice problem

There are $j = 1, \ldots, J$ laundry detergents that are defined by their brand and format. Detergents are purchase in $M - 1$ distinct SKUs indexed by $m = 1, \ldots, M - 1$.

SKU m provides $q_{j,m}$ washes of a detergent j and costs

$$p_m = ppw_m \times q_{j,m} \tag{1}$$

where p_m is the SKU price and ppw_m is the price per wash of SKU.[13] As noted in the description of the UK laundry detergent industry, laundry detergent is available in a variety of pack sizes (i.e. $M > J$) .[14]

At the start of each time period households receive their income, Y, observe existing inventories, $I = [I_1, \ldots, I_J]^\top$, and observe the current SKU prices, p. As was highlighted in section 2, laundry detergent is often available for purchase on promotion. From the perspective of the household, the depth of the price discount, the timing, and the length of the promotion are uncertain.[15]

The possibility of a change in the price in the near future may impact on the current purchase decision.[16] Therefore, before making purchases, households forecast likely future prices. To forecast SKU prices, households are assumed use a statistical model based on previous prices. In particular, households believe that price dynamics evolve as an exogenous $\tau + 1$-Order Markov Process,

$$p^{'} \sim G_{p'|P}$$

where $P = [p, p_{-1}, \ldots, p_{-\tau}]$ is the $M - 1 \times (\tau + 1)$ matrix of past prices used to forecast next period's SKU prices, $p^{'}$.

Other SKU related factors observed by the household, but not by the econometrician are captured by an additively separable i.i.d. random utility shocks for each SKU, $\varepsilon \overset{iid}{\sim}$ Type I Extreme Value.

A household's decision to buy SKU m is recorded by the $M-$vector

$$d = [d_1, \ldots, d_M]^\top \tag{2}$$

where

[13] Each SKU contains only one type of laundry detergent, j. In other storable goods industry, SKUs might contain discrete quantities of several different storable products. For example, cereals, fruit, drinks, wine, meat, shower gel, etc. can all be purchased in tied and/or mixed bundles. The framework outlined here can be adapted to allow for bundles of storable goods.

[14] In the data, the number of SKUs on sale in Tesco in each week is 100 and there are 37 types of detergent available for purchase over the sample period.

[15] Erdem et al. (2003); Hendel and Nevo (2006a,b); Nevo and Hendel (2012); Osborne (2013); Wang (2012, 2013) all report that prices typically exhibit these features.

[16] For example, suppose that a SKU is available on a deep discount that the household believes to be short lived. Even though they may have enough inventory to service consumption needs for the near future, they may elect to accelerate the SKU purchase to take advantage of the relatively low purchase price.

Table 1: Consumption Options

	$C < I + Qd$	$C = I + Qd$
$C > 0$	Interior Solution	Exhaust Inventory
$C = 0$	Save Inventory	No Inventory

$$d_m = \begin{cases} 1 & \text{if purchase SKU } m \\ 0 & \text{otherwise} \end{cases}$$

and $m = M$ represents the decision not to purchase any SKU. Households are restricted to a single purchase, $d^\top d = 1$.

After purchases, the remaining income is

$$\bar{Y} = Y - d^\top p \tag{3}$$

and newly purchased inventories are added to existing inventories. The post-purchase inventory available for consumption is

$$\bar{I} = I + Qd \tag{4}$$

where Q is a $J \times M$ matrix that maps the number of washes provided in the $M - 1$ SKUs into J inventory holdings

$$\underset{(J\times M)}{Q} = \begin{bmatrix} q_{1,1} & q_{1,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q_{2,3} & q_{2,4} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{3,6} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{J,M-2} & q_{J,M-1} & 0 \end{bmatrix}$$

Then, households choose how much of the inventory held in stock, if any, to consume. The are four possibilities listed in Table 1

where $C = [C_1, \ldots, C_J]^\top$ denotes consumption. In addition to consumption of laundry detergent, households purchase and consume an essential composite good, C_0 whose price is normalised to 1.

In each period households receive utility from consumption

$$U(C) := \psi^\top C - \frac{1}{2} C^\top \Psi C + \psi_Y C_0 \tag{5}$$

where ψ, is a J-vector of utility weights, and Ψ, is a $J \times J$ symmetric positive semi-definite matrix of utility coefficients. The diagonal components of the matrix, Ψ, impact on the satiation of consumption of product j. The off-diagonal elements of the matrix allow marginal utility of consumption to depend on the consumption of other detergents. Household utility from C_0 is assumed to be additive and linear.[17]

Any unused inventory is stored ready for consumption in the next period,

$$I' = I + Qd - C \tag{6}$$

However, the remaining inventory occupies valuable storage space and the household incurs an inventory cost,

$$IC\left(I'\right) := \gamma^{\top} I' + \frac{1}{2} I'^{,\top} \Gamma I' \tag{7}$$

where γ, is a J-vector of inventory costs, and Γ, is a $J \times J$ symmetric positive semi-definite matrix of inventory cost coefficients. This cost is paid out of remaining income, $\bar{Y}$.

Combining the budget constraint with SKU utility shocks, the households per-period flow utility function of consuming laundry detergent bundle C is

$$U(C) + \psi_Y \left(Y - d^{\top} p - IC\left(I + Qd - C\right)\right) + d^{\top} \varepsilon \tag{8}$$

Bringing together the elements of the model described above, Theorem 1 describes the household's choice problem as a two stage discrete-continuous Markov decision problem expressed in recursive form.

Theorem 1 In the first stage, households choose which, if any, SKU to purchase

$$\begin{aligned} V\left(I, P, Y, \varepsilon\right) &= \max_{d} d^{\top} \left[V_1\left(I, P, Y\right) + \varepsilon_1, \ldots, V_M\left(I, P, Y\right) + \varepsilon_M\right]^{\top} \\ s.t. & \\ & d^{\top} d = 1 \\ & \varepsilon \overset{iid}{\sim} \text{Type I EV} \end{aligned}$$

where $V_m\left(I, P, Y\right)$ is the post-purchase indirect utility function or choice specific value function (CSVF) corresponding to the purchase SKU $m = 1, \ldots, M-1$, or the no purchase option, $m = M$.

In the second stage, households choose how much to consume from the inventory held in stock by solving

[17] This rules out that the composite good contains complementary products to storage. Further, it imposes that the marginal utility from consuming the composite good, ψ_Y, is constant. Implicitly this assumes that utility from consumption of all other goods has a lower rate of satiation than laundry detergent. In particular, with a linear specification, the marginal utility from consuming the composite good is constant and exhibits non-satiation. Further, the composite good is assumed to be a (weak) substitute to laundry detergent.

$$V_m(I,P,Y) = \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^\top p - IC(I+Qd-C)\right)$$
$$+\delta\int \ln\sum_{m'=1}^{M}\exp\left\{V_{m'}\left(I+Qd-C,P',Y'\right)\right\} dG_{p'|P}\ \forall\, m = 1,\ldots,M \quad (9)$$

where $P' = [p', \ldots, p_{-\tau+1}]$. ■

3.2 Taste heterogeneity

For an individual household, the parameter space is $\theta = [\psi, \Psi, \psi_Y, \gamma, \Gamma]$ with $J^2 + J + 1$ parameters.

To allow for multiple households with persistent taste heterogeneity will require even more parameters. To ensure that the resulting model is estimable, a more parsimonious parameterisation of the model is needed.

The proposed approach is to two-fold. First, project J dimensional parameter vectors, ψ and γ, onto a K-vector of detergent characteristics. Second, combine Radial Basis Function kernels with projections on to characteristic space. This provides a flexible, low-dimensional parameterisation of the positive semi-definite matrices, Ψ and Γ.

3.2.1 Utility parameters

The linear utility weights are specified as non-negative functions of detergent and household characteristics. In particular, for household i and detergent j

$$\psi_{i,j} = \exp(X_j^\top \beta^\psi Z_i) \quad (10)$$

where X_j is a K-vector of detergent j's brand and characteristics, Z_i is a L-vector of household i's characteristics and β^ψ is a $K \times L$ matrix of parameters.

The elements of Ψ are specified as a Radial Basis Functions (RBF) kernels whose inputs are q1detergent and household characteristics.[18] That is,

$$\Psi_i\left[j,j'\right] = \alpha_i^\Psi \exp\left\{-\frac{1}{2}\Delta X_{j,j'}^\top \Sigma^\Psi \Delta X_{j,j'}\right\} \quad (11)$$

where$\Delta X_{j,j'}$ vector of differences between pairs of detergents, $j, j' = 1, \ldots, J$. Also, $\alpha_i^\Psi = \exp(Z_i^\top \beta^\Psi)$ where β^Ψ is a L-vector of parameters and Σ^Ψ is a $K \times K$ diagonal matrix whose elements are non-negative. Using this specification, $\Psi_i\left[j,j'\right] \to \alpha_i^\Psi$ when products are 'close' in product space and $\Psi_i\left[j,j'\right] \to 0$ when they are distant.

[18] The resulting matrices are Mercer Kernels whose elements can be shown to be approximations to an infinite order cross-product polynomial. They are guaranteed to be symmetric and positive definite. See Murphy (2012), Chapter 14.

3.2.2 Marginal utility of income

Marginal utility of income is a non-negative function of household characteristics

$$\psi_{i,Y} = \exp\left(Z_i^\top \beta^Y\right) \tag{12}$$

where β^Y is an L-vector of parameters.

3.2.3 Inventory cost

For any household, i, the cost of storing inventory only depends on the format stocked, we can further reduce the number of parameters.

$$IC_i\left(I^{'}\right) = \alpha_i^{IC}\left[\gamma^\top I' + \frac{1}{1} I^{',\top}\Gamma I^{'}\right] \tag{13}$$

where $\alpha_i^{IC} = \exp(Z_i^\top \beta^{IC})$ captures heterogeneity in storage costs across households. Then, for all $j, j^{'} = 1, \ldots, J$

$$\gamma_j = \exp\left(X_{fmt,j}^\top \beta^\gamma\right) \tag{14}$$

$$\Gamma\left[j, j^{'}\right] = \exp\left\{\left(X_{fmt,j} + X_{fmt,j'}\right)^\top \beta^\Gamma\right\} \tag{15}$$

where $X_{fmt,j}$ denotes the format of detergent j. The parameter vectors β^γ and β^Γ are the linear and quadratic inventory cost weights associated with different formats.

3.3 The curse of dimensionality

As highlighted in section 2, there are many different laundry detergents a household can store and they can purchase. Consequently, both household inventory and the past prices used in forecasting future prices are high-dimensional continuous state spaces. This acutely exacerbates the curse of dimensionality associated with solving the household's choice problem.

To see how the curse of dimensionality arises through a high dimensional SKU choice set and inventory space consider a 'small' laundry detergent industry and with a first order Markov Price Process. For example, let $J = 4$, $M = 10$ and $\tau = 0$. Further suppose we allow households to choose from $\{0, 1, 2, 3\}$ washes per week of each detergent in stock (if inventories permit it). The resulting action space contains $MJ^4 = 10 \times 4^4 \approx 2500$ possibilities.

Suppose that up to 20 washes of each detergent can be held in inventory and that the corresponding state space takes integer values. Discretise prices into 5 levels.

Despite the resulting discretisation of the state space being very coarse, it is still of the order of $4^{20} \times 10^5 \approx 110$ Quadrillion state space nodes. At each node the household evaluates 2500 actions. Solving this problem is infeasible with existing computational resources. This simple example shows how even a 'small' industry with a coarse state space is hindered by the curse of dimensionality.

To make the model amenable to estimation, it is therefore desirable to reduce the computational burden associated with the state space of inventory and prices.

A popular strategy to circumvent the curse of dimensionality to exploit idiosyncratic features of the industry being studied to simplify the household's problem. Unfortunately, the novel approaches that yield a computationally tractable demand models in one industry may not do so in others.

This paper proposes an alternative approach to alleviate the curse of dimensionality in storable good demand estimation that can, in principle, be applied to other storable good industries. Central to the dimension reduction strategies is to address the different ways in which the curse of dimensionality arises from the size of the choice set, the information needed to forecast future prices and the number of different detergents that a household can store.

The proposed approaches to alleviate the curse of dimensionality that arise from each source are discussed in turn.

3.3.1 Size of the choice set

To evaluate which, if any, SKU is purchased, we must solve a bellman equation for optimal consumption conditional on the purchase of SKU $m = 1, \ldots, M$. Since each of these bellman equations is high dimensional, solving M of them is likely to be computationally demanding - most likely infeasible. To remedy this issue, we can re-cast the household's choice problem onto the post-purchase state space.[19]

Theorem 2 Recall from section $\bar{I} := I + Qd$ is the inventory held immediately after purchases and $\bar{Y} := Y - d^\top p$ is the income remaining after having paid for SKU m. The household's choice problem is as two-stage discrete continuous choice defined on the post-purchase state space, $\left[\bar{I}, P, \bar{Y}\right]$. In the first stage, households make purchasing decisions by solving

$$\begin{aligned} V\left(I, P, Y, \varepsilon\right) \quad &= \quad \max_{d} W\left(\bar{I}, P, \bar{Y}\right) + d^\top \varepsilon \\ s.t. \quad & \quad d^\top d = 1 \\ & \quad \varepsilon \overset{iid}{\sim} \text{Type I EV} \end{aligned} \tag{16}$$

where $W\left(\bar{I}, P, \bar{Y}\right)$ is a function that evaluates the value arising from purchases of SKU $m = 1, \ldots, M$ through the impact they have on the value inventory ready for consumption.

In the second stage, households choose how much to consume from the inventory held in stock by solving

[19] Powell (2011); Bertsekas (2011) describe how redefining a problem on the post-decision states can reduce computational costs associated with solving high-dimensional dynamic programs.

$$W\left(\bar{I},P,\bar{Y}\right) := \max_{0\leq C\leq\bar{I}} U(C)+\psi_Y\left(\bar{Y}-IC\left(\bar{I}-C\right)\right) \\ +\delta\int \ln\sum_{d'}\exp\left\{W\left(\bar{I}',P',\bar{Y}'\right)\right\}dG_{p'|P} \tag{17}$$

where $P' = [p', \ldots, p_{-\tau+1}]$. ■

Theorem 2 shows that by redefining the household's problem on the post-purchase state vector, the household's consumption problem collapses to a single bellman equation.

In particular, by allowing purchases to change utility by altering the state space, we do not require M different CSVF to reflect the differences in household's utility from purchasing different SKUs. In turn, M CSVFs are replaced by an indirect utility function, $W(\cdot)$, whose arguments reflect changes in inventory holdings following purchases.

Reflecting this dependence on a specific configuration of inventories held, rather than SKUs purchased, hereafter we refer to $W(\cdot)$ as the inventory specific value function (ISVF).

3.3.2 Forecasting SKU Prices

Even with the post-purchase state space, the dimension of the choice set has another avenue through which it contributes to the curse of dimensionality. Namely, through the dimension of the price state space needed to forecast future prices.

Where the dimension of the price forecasting problem is prohibitively high, the existing storable good dynamic demand literature sought to reduce it one of two ways. One approach is to embed the dimension reduction of the price state space into the choice problem.[20] Another is to use a price forecasting model exploiting the specific features of the industry being studied.[21] A drawback of these approaches is that they are not generally applicable to all industries.

The approach taken here is to use dimension reduction techniques from the machine learning and statistics literature before estimating the dynamic model.[22] In particular, households use a factor model as a low-rank approximation of the high dimensional price

[20]Examples of this approach include Hendel and Nevo (2006a); Wang (2012, 2013). In this approach, the household's choice problem is a dynamic choice of SKU pack size and a static choice of SKU given the chosen pack size. Because the choice of SKU conditional on pack size is assumed to be static, it can be estimated outside the household's choice problem. Households believe that all SKUs of the same size have the same price process and their continuation values are identical. Therefore, only the choice of pack size is dynamic. With only a few pack sizes, a first-order Markov Process of expected utility of purchasing a given pack can be estimated.

[21]Erdem et al. (2003) use a model of price evolution that relies on the idiosyncratic features of the US Ketchup industry. In particular, they use the presence of a common pack size across brands that accounts for the majority of sales in the US Ketchup industry. Moreover, there are a small number of pack sizes and few brands (i.e. $J = 3$ in their application).

[22]Households are price takers and prices are exogenous state variables. Therefore, the household's model of beliefs over price evolution can be estimated before solving the household's choice problem.

forecasting problem, $p^{'} = G_{p'|P}$.[23] For $t = 1, \ldots, T$ periods observed in the data, the household's forecasting model for SKU prices is an Interactive Fixed Effects (IFE) model (Bai (2009))

$$z\left(p_{mt}\right) = \lambda_m^\top F_t + X_m^\top \alpha + \epsilon_{mt} \tag{18}$$

where X_m is a vector of SKU m's characteristics, $z(\cdot)$ is a transformation of SKU prices, and $\mathbb{E}\left[\epsilon_{mt}|F_t, X_m\right] = 0$.[24] There are R-factors in the price model. In particular, define $F_t := \left[f_{1,t}, \ldots, f_{R,t}\right]^\top$ where $f_{r,t}$ are elements of the T-vector f_r for $r = 1, \ldots, R$. The factor loading matrix is

$$\underset{M\times R}{\Lambda} = \begin{bmatrix} \lambda_1^\top \\ \lambda_2^\top \\ \vdots \\ \lambda_M^\top \end{bmatrix}$$

where λ_m is a R-vector of factor loadings for SKU m. The following scale and rotation normalisations restrictions are imposed

$$f_r^\top f_s = \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

$$\Lambda^\top \Lambda \text{ is diagonal} \tag{20}$$

The expected SKU price movements are governed by the dynamics of price factors, F_t. Factors are assumed to follow a stationary, $(\tau + 1)$-order exogenous Markov process,

$$F_t = A_0 + \sum_{s=1}^{\tau} A_s F_{t-s} + u_t \tag{21}$$

where $u_t \perp \epsilon_{mt}|F_t, F_{t-1}, \ldots F_{t-\tau}, X_m$ and $u_t \overset{i.i.d.}{\sim} N(0, \Sigma_u)$.

Using this model, households need only keep track of a coarser partition of the price state space containing $\tau + 1$ R-vectors, $F = [F_t, F_{t-1}, \ldots F_{t-\tau}]$, to forecast SKU prices. Incorporating the household's price forecasting model into the choice problem, the ISVF is $W\left(\bar{I}, F, \bar{Y}\right)$.[25]

The resulting dimension reduction of the state vector is $(\tau + 1)\left(M - 1 - R\right)$. When prices are correlated over time, the price factor state space, F, will likely contain only a

[23]Formally, the factor models can be shown to be equivalent to a truncated SVD decomposition of the price matrix. See Murphy (2012), Chapter 12 for more details.

[24]In this application $z\left(\cdot\right)$ is a scaled logistic transformation. This prevents the prediction of negative prices and provides a sensible upper bound motivated by observed data.

[25]To avoid an unwieldy proliferation of function notation, with a small abuse of notation we redefine $W\left(\cdot\right)$ on the coarser partition of the price state space.

few dimensions. The resulting reduction in the size of the state space can be substantial and the curse of dimensionality is mitigated.

A further benefit is that this approach can, in principle, be applied to any industry and does not require modification of the choice problem nor does it rely on idiosyncratic features of the industry being studied.

3.3.3 Inventories

Since inventory is endogenously determined by the model, any dimension reduction strategy must be embedded into the solution of household's choice problem. This creates some additional challenges when attempting to address this curse of dimensionality arising from inventories.

Where the inventory dimension contains more than a handful of products, the existing literature has sought to reduce the dimension of inventory by simplifying the household's problem. However, rather than modify the problem itself, we follow the Approximate Dynamic Programming (ADP) approach and find a low-dimensional approximate solution to the household's choice problem.[26]

The key is then how to approximate the ISVF.[27] In ADP, there are two inter-linked components to the low-dimensional approximation to the value function: the approximation architecture and the selection of a set of 'features' of the high-dimensional problem that characterise the dimension reduction. We discuss each in turn.

Approximation architecture Adopting Powell (2011) terminology, there are three broad categories of value function approximation architectures: a lookup table approximation (i.e. valuations at pre-defined state space vectors), a linear-in-parameter approximation (i.e. polynomials, B-splines, etc), or, a non-linear approximation (i.e. neural nets). In this paper, we focus on linear-in-parameter approximations

$$W\left(\bar{I}, F, \bar{Y}\right) \approx \phi\left(\bar{I}, F, \bar{Y}\right)^{\top} r \tag{22}$$

where $\phi(\cdot)$ is the N-vector of basis functions and r is a N-vector of parameters.

The reasons for choosing a linear-in-parameters approximation are two-fold. First, lookup tables tend to suffer from the curse of dimensionality for even moderately sized problems. In this case, even with an aggressive dimension reduction strategy, it is likely that the model could only be solved on a very coarse discretisation of the state space.[28]

[26]ADP is a field that combines tools from statistics and machine learning to approximate the solution to computationally intractable dynamic programming problems. ADP encompasses a wide variety of techniques from a collection of disparate fields that have developed specific approaches to approximate solutions to complex dynamic programs they encounter. Indeed, some of the techniques used are closely linked to a Generalised Stochastic Search Algorithms (GSSA) that have been used to solve large-scale dynamic programs economics (see Maliar and Maliar (2014); Judd et al. (2011) for a detailed overview).

[27]Since the state space contains many continuous state variables, any solution is necessarily an approximation.

[28]Nevertheless, other more sophisticated approximations that work with discretised high-dimensional

Second, linear-in-parameters approximations can exploit their structure to ensure updating is relatively straightforward and can be easily modified to address any numerically stability issue that might arise. Whereas, non-linear architectures can be more complex to update and are likely to be less numerically stable. As such there are fewer convergence guarantees and they often require *sui generis* modifications.[29]

To approximate ISVF we use a class of flexible polynomials suitable for large-scale dynamic programming problems called Anisotrophic Smolyak Polynomials.[30] Beneficially, the number of basis functions used by this class of models grows polynomially, rather than exponentially as the state space expands. Moreover, the accuracy of the approximation can be flexed in each dimension.

Taken together, these features of the approximation architecture may allow us include a richer set of low-dimensional features without incurring too high a computational penalty. In turn, this adds a degree of flexibility that may be useful when selecting which 'features' to include in our approximation to the ISVF.

Feature selection In ADP, the low dimensional approximation to the value function typically involves some transformation and/or aggregation of the high dimensional state space that exploits some special structure in the dynamic program.

In storable good industries, like laundry detergent, households are unlikely to stock more than a handful of products at once. As a result, their inventory vector is likely to be sparse. It is this feature of the problem we propose to exploit to reduce the dimensionality of the inventory vector. Specifically, the inventory state space entering the approximate ISVF is restricted to include only those detergents that are held in stock after purchases. Further we assume household's never stock more than 3 products at once. As a result, the dimension of the approximate ISVF is greatly reduced.

To formally describe this approximation strategy we begin by describing how to approximate the ISVF for an individual household at a given point in time. Suppose the household holds a small number of detergents, $\bar{J}$, in inventory after making purchases (i.e. $\bar{J} \leq 3$ from $J = 37$). Let $\bar{I}_+$ denote the resulting $\bar{J}$-vector of post-purchase inventories.[31] Using the assumptions outlined above, the household's ISVF is defined on $\bar{I}_+$ and is approximated by

$$W\left(\bar{I}, F, \bar{Y}\right) \approx \phi\left(\bar{I}_+, F, \bar{Y}\right)^\top r_+ \tag{23}$$

where $\phi\left(\bar{I}_+, F, \bar{Y}\right)$ is a $N_{\bar{J}}$-vector of basis functions and r_+ is a $N_{\bar{J}}$-vector of parameters.

state spaces, such as hierarchical approximations (see Powell (2011); Bertsekas (2011))) and adaptive grid methods (Brumm and Scheidegger (2015)) may prove to be fruitful.

[29]See Powell (2011) for a discussion of updating nonlinear approximation architectures. Note that Neural Nets can be easily updated in some instances. As highlighted by Bertsekas (2011); Judd et al. (2011), an additional benefit of using a linear architecture is that other projections can be considered (i.e. regularisation can be added).

[30]See Judd et al. (2014) for an overview of this class of polynomials and a detailed discussion of how to efficiently implement them.

[31]Formally, let $\bar{I}_+ := \omega \bar{I}$ be the $\bar{J}$-vector of inventories held in stock and ω is a $\bar{J} \times J$ matrix whose rows are basis vectors whose j-th element is 1 if $\bar{I}_j > 0$.

To illustrate the benefits of this approximation strategy, note that the number of basis functions grows rapidly in the dimension of the state space. As such, by reducing the number of inventory state variables in the approximate ISVF, the number of basis functions and parameters is greatly reduced (i.e. $N_{\bar{J}} \ll N$).[32] Since the approximate ISVF is calculated very many times in the solution to model, it is very desirable to limit the computational costs incurred during its formation.

So far we have discussed the approximate ISVF in the context of a particular combination detergents held inventory, $\bar{I}_+$. Of course, we expect different inventory holdings over time and across households. For example, even assuming household's never stock more than 3 of the 37 detergents sold, there are over 8000 different configurations of detergents that could be held in inventory at once.

To accommodate the different combinations of $1, 2, \cdots, \bar{J}$ detergents that can be chosen from J alternatives, we therefore need to construct the approximate ISVF for any inventory configuration. (see equation 22). To this end, we apply an inventory-specific selection matrix, Ω. The role of Ω is to select the relevant subset of basis functions and corresponding parameters needed to construct the approximate ISVF for any combination of $1, 2, \ldots, \bar{J} \ll J$ detergents of inventory held. That is,

$$\phi\left(\bar{I}_+, F, \bar{Y}\right) = \Omega\phi\left(\bar{I}, F, \bar{Y}\right) \tag{24}$$

$$r_+ = \Omega r \tag{25}$$

where Ω is a $N_{\bar{J}} \times N$ matrix of N-dimensional basis vectors. Therefore, by adjusting Ω to reflect the post-purchase inventory, $\bar{I}_+$, we can efficiently construct the low-dimensional approximate ISVF.[33]

$$W\left(\bar{I}, F, \bar{Y}\right) \approx \phi\left(\bar{I}, F, \bar{Y}\right)^\top \Omega^\top \Omega r = \phi\left(\bar{I}_+, F, \bar{Y}\right)^\top r_+$$

Therefore, even though the dimension of the approximate ISVF for a particular inventory configuration is lowered, this dimension reduction strategy for inventory does not further reduce the number parameters, N.[34] That is, we need to solve for the N-parameters in r.

While this may seem to be an obstacle, compared to the cost of forming the approximate ISVF defined on even moderately sized state spaces, it is relatively cheap to fit r.

[32] The dimension of the state vector entering the approximate ISVF has $\bar{J} + (\tau + 1) R + 1$ dimensions. Assuming that the household's forecasting problem is a low-order Markov process with only a few price factors, and noting that $\bar{J}$ is small, then the dimension of the state vector in the approximate ISVF is moderately sized.

[33] To illustrate how Ω is constructed, partition it into 2 components, Ω^1 and Ω^2. The first component, Ω^1, includes all N-dimensional basis vectors that select basis functions that do not depend on the subset of detergents held. These are included in all approximate ISVF. The subset of N-dimensional basis vector in Ω^2 depend on the specific configuration of the post-purchase inventory. To construct Ω^2, we only include basis vectors whose l-th element is 1 if $\phi_l(\cdot)$ is a function of one of the $\bar{J}$ detergents held in inventory after purchases. The resulting matrix $\Omega = \left[\Omega^1, \Omega^2\right]^\top$ defines the set of basis vectors that select the relevant subset of basis functions and corresponding parameters.

[34] The use of the post-purchase state space already reduces the number of parameters by a factor of M.

Indeed, as we discuss in the next section, to solve for the parameters of the approximate ISVF we use an iterative procedure. Like many ADP methods it proceeds by forward simulating the model and iteratively updating r using least squares - a relatively cheap computational procedure.[35] Since we must construct the approximate ISVF very many times during simulation of the model, reducing the computational cost of evaluating it is the focus of the dimension reduction strategy.

3.4 Approximate Household's Choice Problem

Bringing together the household's price forecasting model with the value function approximation strategy for inventories we can describe the household's approximated dynamic program on the post-purchase state space, $\left[\bar{I}_{+}, F, \bar{Y}\right]$ as searching for an N-vector of parameters, r, that solves for the fixed point of the projected Bellman Equation (PBE) that approximates the solution to the household's consumption problem,

$$\phi^{\top} r = \Pi T_{\mu^{BE}}\left(\phi^{\top} r\right) \tag{26}$$

where Π is the projection with respect to the weighted Euclidean norm $\|\cdot\|_{\nu}$ with $\nu > 0 \; s.t \sum \nu = 1$.

The Bellman Optimality operator is

$$T_{\mu^{BE}}\left(\phi^{\top} r\right) = \max_{d} \max_{0 \leq C \leq \bar{I}} U(C) + \psi_{Y}\left(\bar{Y} - IC(\bar{I} - C)\right) \tag{27}$$

$$+\delta \int \ln \sum_{d'} \exp\left(\phi^{',\top} r'\right) dG_{F'|F} + d^{\top}\varepsilon \tag{28}$$

where μ^{BE} denotes the resulting policy. Price forecasts are made using (18) and (21). To solve the PBE, we combine the Envelope Condition Method (ECM) (Maliar and Maliar (2013)) with an ADP algorithm called λ-Policy Iteration (Bertsekas (2015)).[36]

4 Identification

Like other dynamic demand models of storable goods, formal identification of the model is complex. As such, and in line with this existing literature, we provide an informal discussion of identification of model parameters.

In section 2, data on inter-purchase durations, current and past prices, quantities purchased, and the sequences SKUs chosen was used to demonstrate that price expectations, inventory holdings, and taste heterogeneity are important features of demand for laundry detergent in the UK. This data also identifies the parameters of the model.

[35] However, in order to make the fitting procedure robust to additional difficulties that arise in the context of fitting parameters using data simulated from the model, computational costs can increase (see Judd et al. (2011)).

[36] Like other ADP algorithms, these algorithms are related to Generalised Stochastic Search Algorithms (GSSA) used to solve high-dimensional dynamic programming problems (see Judd et al. (2011); Maliar and Maliar (2014)). See Annex D for a detailed description of the algorithm used.

The marginal utility of income is identified by standard arguments using variation in prices over markets.

Inventory costs are identified by comparing inter-purchase durations of households with the same consumption rate. To illustrate, consider two households that always purchase detergent in one particular format. Over the same time period, they purchase the same number of washes. However, because one household has higher inventory costs than the other, we observe that they purchase smaller SKUs more frequently. As such, comparing inter-purchase duration for households with the same consumption rate identifies storage costs. To identify inventory costs differences between formats, this analysis can be conditioned on format purchased.

For a particular detergent j, the ratio ψ_j/Ψ_{jj} is an important determinant of the rate of consumption. Holding fixed ψ_j, Ψ_{jj} can be identified by comparing inter-purchase durations of households conditional on purchasing the same quantity of detergent j. To illustrate these ideas, consider two households who face the same prices and always buy the same number of washes. However, one household consistently purchases less frequently than the other. Therefore, over the same period the household that purchases more frequently will consume more washes. That is, it consumes detergent washes at a higher rate. Therefore, Ψ_{jj} can be identified by comparing inter-purchase durations of households conditional on purchasing the same quantity of detergent j. Conditioning this analysis on consecutive purchases for each detergent identifies all diagonal elements of Ψ.

In the above discussion, we held ψ_j fixed. This is because, in addition to impacting the rate of consumption, ψ_j directly impacts on the level of utility from consumption of detergent j. In turn, the linear utility weights are important parameters for matching market shares. As such, observed market shares will aid identification of ψ over and above consumption rates.

Interaction between quantities purchased, duration between purchases and identities of consecutive purchases of detergent help identify off diagonal terms in Ψ. This is because households that tend to use different detergents for different types of washing are likely to maintain more than one type of inventory. Since detergent is costly to store, all else equal, households with two sets of inventory will tend to purchase smaller SKUs of different inventories at close intervals. Therefore, by comparing the joint distribution of quantities purchased and inter-purchase duration of households whose consecutive purchases are of different detergents to those who purchase the same detergent helps identify off-diagonal terms in Ψ.

Finally, by conditioning the above analysis on household type, we can identify sources of taste heterogeneity on observed household attributes.

5 Estimation

The model is estimated in two steps:

1. Estimate the household's price forecasting model (see section 3.3.2)

2. Estimate and solve the dynamic demand model

The model is estimated using household's purchase diary data that predominantly shopped in Tesco - the UK's largest supermarket - between 2009 and 2011.[37] Details of the data used and estimation procedure are discussed in turn.

5.1 Step 1: Price Forecast Model

To estimate the the price forecasting model, we use all purchases recorded in Tesco between 2009 and 2011. The underlying assumption is that all households in the sample observed the same set of prices. For the UK this is actually the case. Following a ruling by the UK Competition Commission in 2000, all supermarkets must charge the same price for a given SKU.[38]

Because the data is only recorded when household's make purchases, the price series for each SKU is only partially observed. While, the Interactive Fixed Effects model can be used to impute missing prices, the panel is too unbalanced for this to work in practice. As an alternative, we group SKUs of a similar size together. This is because SKUs of a similar size are likely to exhibit similar promotional activity.[39] For each of the 37 types of detergent we assign SKUs to one of four groups: less that 14 washes, between 15 and 24 washes, 25 to 40 washes and more than 40 washes.[40]

Finally, to estimate the price forecast model we need to specify the number of factors and lags. The model is estimated for up to 5 factors and 4 lags. The number of factors used and lags selected is informed by a combination of within sample fit and information criteria.[41]

5.2 Step 2: Dynamic Demand model

To ensure that household's purchase records are likely to be informative for the estimation of the dynamic demand model, we impose some additional sample selection criteria.

First, to make sure that we observe a long enough sequence of purchases to draw inference about demand dynamics, household's are required to makes at least 10 purchases. Second, to make sure that each household's purchase record covers an unbroken time period, household must purchase detergent at least every 24 weeks.[42] Third, as in the model, households are permitted to make no more than one purchase per week. Finally,

[37] A household is classified as a Tesco customer if at least 75 percent of their purchases are made in a Tesco store.

[38] Some price variation is permitted for 'small' supermarkets that are similar in size to convenience stores (less than 280 sq m).

[39] For example, very large SKUs are offered at quantity discounts and are therefore not promoted very regularly.

[40] These thresholds correspond to the SKU size quartiles in the data.

[41] See Stock and Watson (2002); Bai and Ng (2006, 2008).

[42] This problem arises because some households periodically drop in and out of the sample. The 24 weeks cut off is designed to strike a balance between mistakenly omitting households that purchase infrequently and households who records are likely to be 'missing' purchases when they were out of the sample. An inter-purchase duration 24 weeks corresponds to the 95th percentile.

to avoid unduly complicating demand dynamics, households are required to have stable family dynamics. The resulting data set is described in detail in section 2.

The structural parameters, θ, are fitted using Simulated Methods of Moments. Let $\hat{h}$ define the vector of moments observed in the data and $h(\theta, r)$ define the simulated moments at the current estimate of θ and r. Define the distance between the observed data moments and the simulated counterparts as $g(\theta, r) \coloneqq \hat{h} - h(\theta, r)$.

To fit the structural parameters we use a Nested Fixed Point Estimator (NXFP). It solves the following optimisation problem

$$\begin{aligned} \theta^\star, r^\star &= \arg\min_{\theta, r} g(\theta, r)^\top \Sigma^{-1} g(\theta, r) \\ s.t. \quad & \phi^\top r = \Pi T_{\mu^{BE}} \left(\phi^\top r\right) \end{aligned}$$

where Σ is a positive definite symmetric weighting matrix and Π is the projection with respect to the weighted Euclidean norm $\|\cdot\|_\nu$ with $\nu > 0 \; s.t \sum \nu = 1$.

To implement the estimation we use ABC-MCMC derivative free optimisation.[43] As noted by Imai et al. (2009) and Norets (2009) solving the dynamic demand model at every parameter guess is very costly. Their suggested remedy is to alternate between iterations of fitting the structural parameters and doing dampened updates of the value function.[44] In line with this approach, we alternate between fitting the structural parameters, θ, using ABC-MCMC and then solving the dynamic demand model using λ-PI.[45]

6 Conclusion

This paper develops a dynamic discrete-continuous demand model for storable goods - a class of fast moving consumer goods that account for a large fraction of grocery expenditures. It is applied to the UK laundry detergent industry using household level purchase data.

To estimate and solve the dynamic demand model, we use techniques from: (i) Approximate Dynamic Programming (ADP), (ii) large scale dynamic programming in economics, (iii) machine learning, and (iv) statistical computing. The benefits of this approach are three-fold.

First, the dynamic demand model is compatible high-dimensional dynamic choice sets. In turn, making dynamic demand estimation possible for storable good industries with many sizes - the UK laundry detergent industry is an example.

Second, the model can combine the most desirable features of existing models. In particular, it allows for persistent taste heterogeneity to interact with product varieties

[43] For details of algorithm used see Baragatti et al. (2013); Łącki and Miasojedow (2015)

[44] Intuitively this avoids the costly step of fitting the solution to a dynamic program whose parameters may be far away from the true parameters. Moreover, they show that the alternating procedure converges to the same posterior distribution.

[45] In practice, it is more computationally efficient to conduct several parameter guesses per λ-PI iteration.

in flow utility and continuation values. Furthermore, utility from product differentiation accrues the point of consumption, not purchase. Together these features enable the model to capture rich inter- and intra temporal substitution patterns.

Finally, these dimension reduction techniques do not hinge idiosyncratic features of the industry being studied nor do they impose restrictive assumptions on purchase decisions and/or consumption. This means this dynamic demand can be applied to any storable good industry with only minor modifications.

This model is likely to be of both policy and commercial interest. In a policy setting, understanding how consumers react to short and long run price dynamics may be important for effective design of taxation policy. In addition, consistent estimation of short and medium to long run elasticities is a key input into antitrust analysis of merger analysis, assessment of cartel damages, etc. Finally, these storable demand can be used to construct new cost-of-living indices to reflect differences in the prices recorded in baskets of goods and purchases prices (Osborne (2016)).

From a commercial perspective, this structural dynamic demand model can be applied to consumer level purchase data and estimated using ever increasing computational resources. The resulting demand model enables firms to better understand demand dynamics - a key input into optimisation of promotional price strategies and demand forecasting. Moreover, it provides a new way to explore counterfactual market outcomes when new products are introduced or old products are withdrawn.

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Annex A: Equivalence of the two stage problem

In recursive form, the household's choice problem is

$$
\begin{aligned}
V(I,P,Y,\varepsilon) &= \max_{d} \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^T p - IC(I+Qd-C)\right) + d^\top \varepsilon \\
&\quad +\delta \int\int V\left(I+Qd-C, P', Y, \varepsilon'\right) dG_{\varepsilon'} dG_{p'|P} \qquad (29)
\end{aligned}
$$

where $\varepsilon \overset{iid}{\sim}$ Type I EV and $\tau = 0$ so $p' \sim G_{p'|P}$.
Integrating the value function over SKU specific shocks, ε,

$$
\begin{aligned}
\bar{V}(I,P,Y) &:= \int V(I,P,Y,\varepsilon)\, dG_\varepsilon \qquad (30) \\
&= \ln \sum_d \exp\{ \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^\top p - IC(I+Qd-C)\right) \\
&\quad +\delta \int\int V\left(I+Qd-C, P', Y, \varepsilon'\right) dG_{\varepsilon'} dG_{p'|P} \} \qquad (31)
\end{aligned}
$$

Define $V_m(I,P,Y)$ as the indirect utility function conditional on purchasing SKU m

$$
\begin{aligned}
V_m(I,P,Y) &:= \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^\top p - IC(I+Qd-C)\right) \\
&\quad +\delta \int \bar{V}\left(I+Qd-C, P', Y'\right) dG_{p'|P} \qquad (32)
\end{aligned}
$$

Then substituting eq (32) into eq (31)

$$
\bar{V}(I,P,Y) = \ln \sum_{m=1}^{M} \exp\{V_m(I,P,Y)\} \qquad (33)
$$

Rolling forward eq (33) next period's integrated value function

$$
\bar{V}\left(I', P', Y'\right) = \ln \sum_{m'=1}^{M} \exp\left\{V_{m'}\left(I', P', Y'\right)\right\} \qquad (34)
$$

Substituting eq (34) into eq (32) yields a Bellman Equation,

$$
\begin{aligned}
V_m(I,P,Y) &= \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^\top p - IC(I+Qd-C)\right) \\
&\quad +\delta \int \ln \sum_{m'=1}^{M} \exp\left\{V_{m'}\left(I+Qd-C, P', Y'\right)\right\} dG_{p'|P} \qquad (35)
\end{aligned}
$$

Finally, we show that the household's discrete choice problem is to choose the largest indirect utility function once SKU specifics are realised

$$
\begin{aligned}
V(I,P,Y,\varepsilon) &= \max_{d} \max_{0\leq C\leq I+Qd} U(C)+\psi_Y\left(Y-d^\top p-IC\left(I+Qd-C\right)\right)+d^\top\varepsilon \\
&\quad +\delta\int\int V\left(I+Qd-C,P^{'},Y,\varepsilon^{'}\right)dG_{\varepsilon'}dG_{p'|P} \qquad (36)\\
&= \max_{d} \max_{0\leq C\leq I+Qd} U(C)+\psi_Y\left(Y-d^\top p-IC\left(I+Qd-C\right)\right)+d^\top\varepsilon \\
&\quad +\delta\int \bar{V}\left(I+Qd-C,P^{'},Y^{'}\right)dG_{p'|P} \qquad (37)\\
&= \max_{d} d^\top\left[V_1(I,P,Y)+\varepsilon_1,\ldots,V_M(I,P,Y)+\varepsilon_m\right]^\top \qquad (38)
\end{aligned}
$$

where the first line is eq (29). Eq (34) is substituted into the second line. Finally, we can substitute in eq (35) and represent ε element-wise. This gives the desired expression.

Annex B: Households DP

Proof of Theorem 2. Define the post-purchase state space, $s = \left[\bar{I}, F, \bar{Y}\right]$ where $\bar{I} = I + Qd$ is the inventory available for consumption and $\bar{Y} = Y - d^{\top}p$ is net income after purchases. Consider the CSVF,

$$\begin{aligned} V_m\left(I, P\left(F\right), Y\right) &= \max_{0\le C\le I+Qd} U(C) + \psi_Y\left(Y - d^{\top}p - IC\left(I + Qd - C\right)\right) \\ &\quad + \delta\int \ln\sum_{m'=1}^{M}\exp\left\{V_{m'}\left(I + Qd - C, P^{'}(F^{'}), Y^{'}\right)\right\} dG_{F'|F} \quad (39) \\ &= \max_{0\le C\le \bar{I}} U(C) + \psi_Y\left(\bar{Y} - IC\left(\bar{I} - C\right)\right) \\ &\quad + \delta\int \ln\sum_{m'=1}^{M}\exp\left\{V_{m'}\left(\bar{I} - C, P^{'}(F^{'}), \bar{Y} + p^{'}_{m'}\right)\right\} dG_{F'|F} \quad (40) \\ &:= W\left(\bar{I}, F, \bar{Y}\right) \quad (41) \end{aligned}$$

In equation (39) the household's price forecasting model is incorporated into the CSVF. Equation (40) re-writes the household's choice problem in terms of the post-purchase state space. This defines the CSVF on the post-purchase state space, $W\left(\bar{I}, F, \bar{Y}\right)$.

Rolling forward the CSVF of choosing SKU $m^{'}$ in the next period we have

$$\begin{aligned} V_{m'}\left(\bar{I} - C, P^{'}(F^{'}), \bar{Y} + p_m\right) &= \max_{0\le C'\le \bar{I}'} U(C^{'}) + \psi_Y\left(\bar{Y}^{'} - IC\left(\bar{I}^{'} - C^{'}\right)\right) \\ &\quad + \delta\int \ln\sum_{m''=1}^{M}\exp\left\{V_{m''}\left(\bar{I}^{'} - C^{'}, P^{''}(F^{''}), \bar{Y}^{'} + d^{'',\top}p^{''}\right)\right\} dG_{F''|F'} \quad (42) \\ &= W\left(\bar{I}^{'}, F^{'}, \bar{Y}^{'}\right) \quad (43) \end{aligned}$$

Substituting (43) into (41) gives the second stage of the household's problem

$$\begin{aligned} W\left(\bar{I}, F, \bar{Y}\right) &= \max_{0\le C\le \bar{I}} U(C) + \psi_Y\left(\bar{Y} - IC\left(\bar{I} - C\right)\right) \\ &\quad + \delta\int \ln\sum_{d'}\exp\left\{W\left(\bar{I} - C + Qd^{'}, F^{'}, Y - d^{',\top}p^{'}\left(F^{'}\right)\right)\right\} dG_{F'|F} \quad (44) \end{aligned}$$

and substituting in eq (41) gives the first stage of the household's choice problem as required.

$$\begin{aligned} V\left(I, P, Y, \varepsilon\right) &= \max_{d} d^{\top}\left[V_1\left(I, P, Y\right) + \varepsilon_1, \ldots, V_M\left(I, P, Y\right) + \varepsilon_M\right]^{\top} \\ &= \max_{d} W\left(\bar{I}, F, \bar{Y}\right) + d^{\top}\varepsilon \end{aligned}$$

Annex C: Derivation of KKT conditions

Suppose that product m is chosen in the current period and for ease of exposition set $\tau = 0$. With knowledge of the available inventories, household h finds the optimal consumption profile

$$\begin{aligned} W\left(\bar{I}, P, \bar{Y}\right) &= \max_{0 \leq C \leq \bar{I}} U(C) + \psi_Y\left(\bar{Y} - IC\left(\bar{I}, C\right)\right) \\ &\quad + \delta \int \ln \sum_{d'} \exp\left\{W\left(\bar{I} - C + Qd', P', Y - d'^{,\top} p'\right)\right\} dG_{p'|P} \end{aligned}$$

Setting up the Lagrangian,

$$\begin{aligned} L &= U(C) + \psi_Y\left(\bar{Y} - IC\left(\bar{I} - C\right)\right) \\ &\quad + \delta \int \ln \sum_{d'} \exp\left\{W\left(\bar{I} - C + Qd', P', Y - d'^{,\top} p'\right)\right\} dG_{p'|P} \\ &\quad - \lambda_{I'}^{\top}\left[C - \bar{I}\right] + \lambda_0^{\top} C \end{aligned}$$

The first-order conditions

$$\frac{\partial L}{\partial C^{\top}} = \nabla U + \psi_Y \nabla IC - \delta \int \sum_{d'} \Pr\left(d'\right) \nabla W' dG_{p'|P} - \lambda_{I'} + \lambda_0 \quad (45)$$

$$\frac{\partial L}{\partial \lambda_0^{\top}} = C \quad (46)$$

$$\frac{\partial L}{\partial \lambda_{I'}^{\top}} = \bar{I} - C \quad (47)$$

with Envelope Condition

$$\nabla_{\bar{I}} W = -\psi_Y \nabla IC + \delta \int \sum_{d'} \Pr\left(d'\right) \nabla_{\bar{I}'} W' + \lambda_{I'} \quad (48)$$

where

$$\Pr\left(d'\right) := \frac{\exp\left\{W\left(\bar{I} - C + Qd', P', Y - d'^{,\top} p'\right)\right\}}{\sum_{\tilde{d}} \exp\left\{W\left(\bar{I} - C + Q\tilde{d}, P', Y - \tilde{d}^{\top} p'\right)\right\}} \quad (49)$$

is the probability that the household buys SKU m in the next period and W' is the shorthand notation for $W\left(\bar{I} - C + Qd', P', Y - d'^{,\top} p'\right)$.

Karush-Kuhn-Tucker (KKT) In summary, the Karush-Kuhn-Tucker (KKT) conditions satisfied by the optimal consumption choice for laundry detergents are,

$$
\begin{aligned}
\nabla U + \psi_Y \nabla IC - \lambda_{I'} + \lambda_0 &= \delta \int \sum_{d'} \Pr\left(d'\right) \nabla_{\bar{I}'} W' dG_{p'|P} && (50)\\
\nabla_{\bar{I}} W + \psi_Y \nabla IC - \lambda_{I'} &= \delta \int \sum_{d'} \Pr\left(d'\right) \nabla_{\bar{I}'} W' dG_{p'|P} && (51)\\
C \geq 0 \quad &\perp \quad \lambda_0 \geq 0 && (52)\\
\bar{I} \geq C \quad &\perp \quad \lambda_{I'} \geq 0 && (53)
\end{aligned}
$$

Note that with approximations for W, equations (50), (52) and (53) define the system of equations that define $C_{\mu^{BE}}$ from the ADP.

Equating the RHS of eq (50) and (51) we have

$$
\begin{aligned}
\nabla U + \psi_Y \nabla IC - \lambda_{I'} + \lambda_0 &= \nabla_{\bar{I}} W + \psi_Y \nabla IC - \lambda_{I'} && (54)\\
\implies \nabla U + \lambda_0 &= \nabla_{\bar{I}} W && (55)
\end{aligned}
$$

Euler Equation Rolling forward eq. (55) and substituting it into RHS of eq. (50) yields the Euler Equation

$$
\nabla U + \psi_Y \nabla IC - \lambda_{I'} + \lambda_0 = \delta \int \sum_{d'} \Pr\left(d'\right) \left(\nabla U' + \lambda_0'\right) dG_{p'|P} \qquad (56)
$$

Annex D: ADP: Solving the Household's PBE

This annex describes the Approximate Dynamic Programming techniques used to solve the PBE corresponding to the approximation of the household's consumption problem. We will describe how the value function approximation is combined with λ-Policy Iteration to solve the PBE.[46] In addition, we will describe how incorporating the Envelope Condition Method (ECM) (Maliar and Maliar (2013)) can considerably simplify the computational burden in implementing the ADP. We begin by introducing some additional notation.

D.1 Notation

To succinctly describe λ-PI it is useful to express the approximate ISVF evaluated at $s_i = \left[\bar{I}_{+,i}, F_i, \bar{Y}_i\right]$ for $i = 1, \ldots, N$ state space points. Let

$$\begin{bmatrix} W(s_1) \\ \vdots \\ W(s_N) \end{bmatrix} \approx \Phi r \text{ where } \underset{(N\times L)}{\Phi} = \begin{bmatrix} \phi(s_1)^\top \Omega_1^\top \Omega_1 \\ \vdots \\ \phi(s_N)^\top \Omega_N^\top \Omega_N \end{bmatrix} \tag{57}$$

where Ω_i is the inventory-specific selection matrix corresponding the post-purchase inventory holdings, $\bar{I}_{+,i}$. Similarly, let

$$\begin{bmatrix} W\left(s_1^{'}\right) \\ \vdots \\ W\left(s_N^{'}\right) \end{bmatrix} \approx \Phi^{'} r \text{ where } \underset{(N\times L)}{\Phi^{'}} = \begin{bmatrix} \phi\left(s_{1,d'}^{'}\right)^\top \Omega_1^{',\top} \Omega_1^{'} \\ \vdots \\ \phi\left(s_{N,d'}^{'}\right)^\top \Omega_N^{',\top} \Omega_N^{'} \end{bmatrix} \tag{58}$$

D.2 ADP and λ-Policy Iteration (λ-PI)

To explore how λ-policy iteration is used to solve the PBE, we first introduce a closely linked exact DP method called optimistic policy iteration. In turn, we describe how the addition of value function approximation links optimistic policy iteration to an important family of algorithms in ADP called Approximate Policy Iteration (API).

Our discussion of API exhibits a key feature of ADP methods - namely, they use a current guess at value and/or policy functions and use them to step forward through time using the model in time. As such, we show how API, like many ADP methods, uses iterative procedures based on forward simulation to approximate the solution to PBE. In particular, we will highlight the importance of the policy evaluation step.

We are then ready to show how, by changing the mapping used to complete this important policy evaluation step of API, λ-PI is linked to API. Finally, we will discuss how we adapt λ-PI to use a policy improvement step from called the Envelope Condition Method (ECM).

[46] See Bertsekas and Ioffe (1996).

D.2.1 Optimistic Policy Iteration

Optimistic policy iteration is a hybrid of value iteration (VI) and policy iteration (PI). Like λ-PI, optimistic PI has two steps: policy evaluation and policy improvement. The policy evaluation step estimates the value of policy μ in a given states. Then, using the values obtained in the policy evaluation step, the policy is improved and updated. Often this update step uses the Bellman Optimality Operator to generate a new policy function.[47]

The key idea behind optimistic PI is that we do not need to exactly evaluate the policy to get a good policy update. That is, the algorithm is inherently 'optimistic' about the quality of the evaluation of the policy given a target number of applications of $T_\mu\,()$. As such, optimistic PI can be characterised by the number of times that a policy, μ, is evaluated before the policy improvement step. If a policy is evaluated by just one application of $T_\mu\,(\cdot)$, optimistic PI corresponds to VI. On the other hand, if $T_\mu\,(\cdot)$ is applied an infinite number of times, optimistic PI corresponds to exact PI. Often, however, Bertsekas (2015) reports that an intermediate number of applications of $T_\mu\,(\cdot)$ are used. When the computational burden of exact PI is substantial, optimistic PI is widely thought to be more computationally efficient.

D.2.2 Approximate Policy Iteration (API)

When combined with value function approximation, optimistic PI is referred to as Approximate Policy Iteration (API).[48]

To illustrate how a typical API algorithm uses forward simulation to solve the PBE, suppose we are at iteration k of a simulation and have a current guess of the parameter vector, r_k. The next step of which is to evaluate the new policy, μ_{k+1}, by applying the Bellman Operator m_k times to a batch of N state vectors. We take the following steps:

1. **Policy Evaluation**:
   a) Given a current guess of the approximation to the ISVF, evaluate a new policy, μ_{k+1}. That is,
   $$\hat{w} := T^{(m_k)}_{\mu_{k+1}}\left(\Phi r_k\right) \tag{59}$$
   b) Then, complete the policy evaluation step using a least squares fitting procedure to solve for the parameter vector that best fits the batch of simulated values, r_{k+1}. For example, we could use OLS
   $$r_{k+1} = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top \hat{w} \tag{60}$$
   The combination of steps 1 and 2 is referred to as Least Squares Policy Evaluation (LSPE).

[47]Though, as we shall discuss below, other methods can be employed.

[48]API includes the special case of Approximate Value Iteration (AVI) when a single iteration of the Bellman Operator is used to evaluate the policy in question.

2. **Policy Improvement**:

   a) From $\{\mu_{k+1}, r_{k+1}\}$ create a new, improved policy, $\bar{\mu}$. For example, apply the optimal Bellman Operator to define

$$T_{\bar{\mu}}\left(\Phi r_{k+1}\right) := T_{\mu^{BE}}\left(\Phi r_{k+1}\right) \tag{61}$$

   b) Then increment the simulation counter k by 1 and set $\mu_{k+1} = \bar{\mu}$.

In the API simulation, these steps are repeated K times (or until convergence) starting from an initial parameter guess r_0. This algorithm generates a sequence of policy-parameter tuples $\{\mu_k, r_k\}$ for $k = 1, \ldots, K$. As such, the approximate solution to the PBE is defined by the tuple $\{\mu_K, r_K\}$.[49]

D.2.3 λ-Policy Iteration (λ-PI)

Next we explain how changes to the policy evaluation step in the API algorithm lead to a λ-PI algorithm. Then, we introduce a different policy improvement step using the Envelope Condition Method (ECM). Finally, we bring these two steps together to provide a stylised overview of the key mechanisms of the λ-PI algorithm used in this paper.

Policy Evaluation: LSPE(λ) λ-PI is closely related to API but differs in the application of the policy evaluation step. Rather than repeatedly apply the Bellman Operator under policy μ, λ-PI uses a geometrically weighted **multi-step Bellman Operator** for policy evaluation

$$T_{\mu}^{(\lambda)}\left(\Phi r\right) := (1-\lambda)\sum_{l=0}^{\infty}\lambda^{l}T_{\mu}^{l+1}\left(\Phi r\right)$$

where $\lambda \in [0,1)$ and has the same fixed points as T_{μ}. When $T_{\mu}^{(\lambda)}\left(\Phi r\right)$ is combined with a least squares fitting procedure, this particular method of policy evaluation is called LSPE(λ).

Like API, λ-PI is an optimistic algorithm that can be seen as a hybrid of VI and PI with value and/or policy function approximations. When $\lambda = 0$ we have Approximate VI (i.e. $m_k = 1$), as $\lambda \to 1$ we have API when $m_k \to \infty$.

[49] Note that the API algorithm above describes a typical implementation in the context of the model above. Its main goal is to illustrate the role of the approximation, the way forward simulation is used by ADP algorithms to evaluate policies, the policy update step and the inherently optimistic nature of the algorithm. In practice, the details of API implementation can differ widely depending on the nature of the ADP at hand. For example, sometimes the dynamic system is observed but the model is unknown. In others, the Bellman Operator is linear in parameters. Alternatively, the ADP might have a finite, rather than infinite horizon. Or the state space might be discrete and/or the value function approximation might be discrete. For an exposition of the many algorithmic possibilities and how they might be suited to different ADP see Bertsekas (2011) and Powell (2011).

Policy Improvement: ECM As described above, we focus on API algorithms where new policies are based on value function approximations.[50] In particular, our new policies are formulated by applying the Bellman Optimality Operator to our newly evaluated parameter vector r_{k+1}.

Households' consumption choice is a continuous vector-valued variable. This is calculated by solving the non-linear system of Euler Equations]and feasibility constraints at each state vector. As is often noted in implementations of Euler Equations methods, root-finding in such a system of equations can be computationally intensive. This is because they involve costly computational tasks such as the calculation of multi-dimensional integrals to evaluate expectations and the need to call a non-linear equation solver.

Even if the computational costs are moderate, this can be especially problematic for ADP algorithms. This is because they make heavy use of simulation and μ^{BE} is applied after every transition in simulated trajectories. As such, the system of Euler Equations would have to be solved very many times.

To mitigate this issue, I follow Maliar and Maliar (2013) and use the Envelope Condition Method (ECM). ECM solves the following system of equations evaluated at the current guess of the parameters of the approximated ISVF, r, for vector of consumption, $C_{\mu^{ECM}}$

$$\psi - \Psi C_{\mu^{ECM}} + \lambda_0 = \nabla_{\bar{I}} \Phi r \tag{62}$$

$$0 \leq C_{\mu^{ECM}} \leq \bar{I} \tag{63}$$

where $\nabla_{\bar{I}} \Phi r := \frac{\partial \Phi(s)}{\partial \bar{I}^{\top}}$.[51] The associated Bellman Operator mapping is

$$T_{\mu^{ECM}} (\phi r) := U \left(C_{\mu^{ECM}}\right) + \psi_Y \left(\bar{Y} - IC\left(\bar{I} - C\right)\right) + \delta \int \ln \sum_{m=1}^{M} \exp \left(\Phi'_m r\right) dG_{F'|F} \tag{64}$$

and has the same fixed point as $T_{\mu^{BE}} (\cdot)$.

The benefit of using the ECM is three-fold. First, the continuous choice vector can be obtained using the gradient of the current period's ISVF in place of the gradient of the derivatives continuation value. As such, there is no need to compute expectations. Second, we only have to solve a linear system of equations to find $C_{\mu^{ECM}}$. As such, we avoid the additional computational burden associated with calling a non-linear equation solver. Third, ECM works with the estimate of the current ISVF, Φr_k, rather than estimate the continuation value function in the next period.[52] As such, we can

[50] In practice, several other options are available to specify policies. For example, we could look to approximate policy outcomes directly as a functions of states. See Powell (2011) chapter 6 for a detailed overview of policy types.

[51] See Annex C for a derivation of this system of equations.

[52] Traditional DP methods like VI, PI and Endogenous Grid Method (EGM) necessarily guess the next period's value function and its derivatives. This is particularly useful when stepping backwards through time to solve the model.

easily combine ECM with forward simulation techniques used by λ-PI and other ADP algorithms.

As such, the computational burden likely to be associated with $C_{\mu^{BE}}$ does not arise for $C_{\mu^{ECM}}$. Indeed, Maliar and Maliar (2013) and Arellano et al. (2014) find that ECM achieves similar speed-ups in computation time over VI as the Endogenous Grid Method.[53] In the context of an algorithm that heavily utilises simulation, such as λ-PI, this is likely to be an important practical consideration.

There are also drawbacks of using the ECM method. Most notably, $T_{\mu^{ECM}}(\cdot)$, is not necessarily a contraction mapping. However, Arellano et al. (2014) show that there exists a damping parameter, $\xi \in (0,1)$, that can be used to exponentially smooth updates to the value function such that $T_{\mu^{ECM}}(\cdot)$ is a contraction mapping.[54]

λ-PI with ECM By bringing together LSPE(λ) with policy improvement using the ECM we outline how the λ-PI version of the API algorithm described above is altered at iteration k :

1. Policy Evaluation, LSPE(λ):
    a) Given a current guess of an approximation to the ISVF, evaluate a new policy, μ_{k+1}. That is, apply the multi-step Bellman Operator under policy μ_{k+1} to a batch of N state space points to give a vector of 'optimistic' valuations, $\hat{w}$. That is
    $$\hat{w} := T^{(\lambda)}_{\mu_{k+1}}(\Phi r_k) \tag{65}$$
    b) Use a least squares fitting procedure to solve for the parameter vector that best fits the batch of simulated values, $\hat{r}_{k+1}$. For example, we could use OLS
    $$\hat{r}_{k+1} = \left(\Phi^\top \Phi\right)^{-1} \Phi^\top \hat{w} \tag{66}$$
    c) Complete the policy evaluation step by updating the parameter vector of the approximate value function using $\hat{r}_{k+1}$
    $$r_{k+1} = (1-\xi) r_k + \xi \hat{r}_{k+1} \tag{67}$$
    where $\xi \in (0,1)$.
2. Policy Improvement using $T_{\mu^{ECM}}(\cdot)$:

[53] See Fella (2014) and Iskhakov et al. (2015) for a detailed discussion of the difficulties that can arise for EGM when there are occasionally binding constraints, discrete choices, and vector valued continuous choices.

[54] See Arellano et al. (2014) for a detailed exposition. Intuitively, this occurs because ECM does not impose the first order conditions at every step in the iteration, only in the limit. Likewise, VI does not impose the Envelope Condition during the contraction, only in the limit. However, imposing the first order condition is necessary to guarantee that VI and other backward iteration methods, like Endogenous Grid Method, have the contraction mapping property.

a) From $\{\mu_{k+1}, r_{k+1}\}$ create a new, improved policy, $\bar{\mu}$. For example, apply the Bellman Operator under the ECM policy μ^{ECM}

$$T_{\bar{\mu}}\left(\Phi r_{k+1}\right) := T_{\mu^{ECM}}\left(\Phi r_{k+1}\right) \tag{68}$$

b) Next, increment the simulation counter k by 1 and set $\mu_{k+1} = \bar{\mu}$.

D.3 Choosing λ

To implement λ-PI we need to choose a value $\lambda \in [0, 1)$. There is bias and variance trade-off that influences our choice of λ.

The bias is the difference between the approximated value function evaluated at the parameter vector that solves the PBE, r_μ, and the fixed point of $W = T_\mu^{(\lambda)} W$, W_μ; that is, $W_\mu - \Phi r_\mu$. In general, the bias depends on λ. In particular, as $\lambda \to 1$ the bias is reduced.

On the other hand, setting λ close to 1 also results in more periods being evaluated. In turn, the simulation variance increases. Bertsekas (2011) reports thats the quality of the approximation can quickly degrade with larger simulation variance. Consequently there is a bias-variance trade-off when choosing λ.

Bertsekas (2011) states that while there is no obvious rule of thumb, it might be desirable to set λ close to 1 if computing many transitions is computationally feasible. This is because, in addition to the bias being reduced, any convergence concerns over the Bellman Operator mapping $T_\mu^{(\lambda)}(\cdot)$ can be mitigated.[55]

[55]This is especially useful in practice because it enables us to make desirable modification to ADP algorithms that may mean the mapping $\Pi T_\mu^{(\lambda)}(\cdot)$ is no longer a contraction (i.e. using off-policy transitions can lead to 'better policies' being chosen in the policy improvement step. This is especially useful in the finite state case).

Annex E: Inventory Proxy

Construct proxy for inventory for household h whose first period in the data is T_0 and last is T.

1. Calculate average consumption, $\bar{C}$, over the sample

$$\bar{C} = \frac{\sum_{t=T_0}^{T} Q_t}{T - T_0 + 1}$$

   where Q_t is the number of washes purchased in period t

2. Set inventory of 0 immediately prior to first purchase,

$$I_{T_{-1}} = 0$$

3. Then for $t = T_0, \ldots, T$ calculate inventory before purchase are made

$$I_t = \max\left\{0, I_{t-1} + Q_{t-1} - \bar{C}\right\}$$

4. The first 10 periods are omitted to reduce dependence on initial inventory assumption