Dynamic Demand Estimation for Storable Goods

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This paper develops a dynamic discrete-continuous demand model for storable goods - a class of goods that account for a large fraction of grocery expenditures. To estimate and solve the dynamic demand model, we use techniques from: (i) Approximate Dynamic Programming, (ii) large scale dynamic programming in economics, (iii) machine learning, and (iv) statistical computing. The benefits of this approach are three-fold. First, the dynamic demand model relaxes assumptions of existing models while retaining computational feasibility. Second, it can capture rich inter- and intra-temporal substitution patterns. Third, it can incorporate high-dimensional choice sets. As a result, the model is easily adapted to other storable good industries - widening the applicability of this class of demand models. In this paper we apply the model to the UK laundry detergent sector using household level purchase data.

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1 Introduction

Storable goods are a class of fast moving consumer goods that account for a large portion of household grocery expenditures. Amongst others, this class of goods includes alcohol, tinned products, frozen products, table sauces, condiments, personal hygiene products, household cleaning products, and pet food. In the UK, the storable goods market is worth in excess of £1 trillion.

There are two key features of storable goods. First, many retail storable goods are typically sold at a constant price for sustained periods of time interspersed with frequent, uncertain, temporary sales.[1] As such, prices exhibit persistence. Through repeated shopping trips, households form new expectations over future prices and factor them into their current purchase decisions.

Second, by definition, the durability of storable goods enables them be held in inventory and consumed at a later date. As a result, households need only make infrequent purchases to meet current and future consumption needs. The process of building up stocks with purchases and depleting them through consumption creates inter-temporal links in demand behaviour.

Since storage space is limited, inventories are a key determinant of these demand dynamics. When inventories are high, storage space for new purchases is costly. Moreover, households can service current consumption with existing inventory. The combination of these factors results in low demand for new purchases. At the other extreme, when a household's stocks are depleted, demand for new stock is high because they need to make purchases to consume. At interim levels of inventory, households can choose to make purchases before their stocks run out. In particular, they may accelerate purchases in response to price promotions.

Problematically, inventories are unobserved by the researcher.[2] Because current prices and inventories are a function of past prices, the omission of inventories leads to price endogeneity. In turn, elasticities and welfare analysis based on a static demand estimation are biased. This problem could be resolved if instruments that are correlated with current prices but uncorrelated with past prices were available.[3] However, since observed prices are serially correlated, finding such instruments would seem to be challenging - if not impossible.

An alternative is to integrate out over inventories during estimation. To implement this computationally intensive solution, consumption is calculated using the demand model and combined with a sequence of observed purchases to calculate inventory in each period.

This approach is further complicated by the fact that households have the option to

[1]Erdem et al. (2003); Hendel and Nevo (2006a,b); Nevo and Hendel (2012); Osborne (2013); Wang (2012, 2013) all report that prices typically exhibit these features.

[2]Even though detailed household level purchase diary data records price, quantity purchased, product characteristics and date of all purchases for each household in the survey, inventories cannot be constructed without data on initial inventories and consumption .

[3]Armed with these instruments, marginal utility of income could be consistently estimated using a micro-BLP procedure.

purchase, store and consume many different variants of the good. For example, in the UK laundry detergent industry, the households' choice set contains approximately 100 purchase options with over 35 different types of detergent in any given week .

Even without the unobserved inventories, the curse of dimensionality is severe when solving and estimating a high-dimensional dynamic choice model. The need to integrate out over the high dimensional inventory state space further exacerbates this sizeable computational challenge.

Faced with these issues, existing dynamic demand models for storable goods have sought to trade-off the flexibility for the computational resources needed to estimate it. In general, there have been two approaches.

One approach is to select applications that have low dimensional choice sets (i.e. few brands or size choices) and exploit idiosyncratic features of the industry to produce a tractable dynamic demand model (Erdem et al. (2003)). The other is to impose restrictions of convenience on the model that allow the consumer's decision can be split into a static brand choice and a dynamic size choice (Hendel and Nevo (2006a); Wang (2012, 2013); **?**). Further, price expectations are assumed to be captured by ex-ante expected utility of consuming a particular size (see Melnikov (2013)). Both approaches adopt significant restrictions on the functional form of utility from consumption and require that the products are available in only a handful of sizes.

If the number of choices entering the dynamic choice set is large, these approaches no longer yield computationally tractable demand models. While successful, the range of applications is highly limited. Indeed, in many industries, products are available in many different varieties and sizes and the choice sets are high-dimensional.

The UK laundry detergent industry is one such industry and is the subject of this paper. This industry is chosen because it highlights many challenges that arise when estimating dynamic demand models of storable goods. First, as noted above, the choice set is high-dimensional - there are around 100 products in the choice set in each week. [4] Second, differentiation of products in the choice set gives rise to differences the way that products are consumed and storage costs incurred. Third, it exhibits the promotional price patterns that are often observed in many storable good industries. Therefore, the dynamic demand model for UK laundry detergent provides a framework that can be readily adapted and applied to many other storable good industries.

The key innovation presented in this paper is to take an alternative approach to alleviate the curse of dimensionality in storable good demand estimation. First, household's are assumed to use an interactive fixed effects model (Bai (2009)) to forecast prices. This produces a low-rank approximation to the a high-dimensional time series process. The resulting price factors from the interactive fixed effect models are the state variables whose dimension is determined by statistical criteria (i.e. see Bai and Ng (2008)). A vector autoregression of the time series process of the low-dimensional price factors is the kernel that household's use to forecast future prices.

[4] There are eight major brands, available in five different formats sold in many different pack sizes (i.e. over 20 pack sizes are needed to cover 95 percent of sales). Therefore, even if the static-brand dynamic-size choice restriction used in Hendel and Nevo (2006a) is adopted, the dynamic choice set will be large.

Second, the tendency of households to stock no more than a few products at once means that the household's inventory is sparse. Only the detergents stocked and ready for consumption are included as arguments in an approximation to the value function that solves the household's choice problem. This provides a low-dimensional representation of the inventory state space with minimal loss of information.

This dimension reduction strategy imposes fewer restrictions than existing models. As such, it can build on the seminal work of Erdem et al. (2003) and Hendel and Nevo (2006a) and make significant advances. As in Erdem et al. (2003) the model can allow for unobserved heterogeneity in the continuation value function in *both* brand-format and size of the purchased product. The model can therefore capture rich substitution patterns both over time and within a time period.

Like Hendel and Nevo (2006a) and related models, consumption is modelled as an endogenous continuous choice. This is especially important because the evolution of households' inventories are a critical component of demand. By allowing consumption to be vector-valued and to impact on utility at the point of consumption we build on Hendel and Nevo (2006a). Like static demand models, this model compatible with high-dimensional dynamic choice sets and can incorporate persistent unobserved heterogeneity. As a result it can be applied to a wide range of dynamic choice problems with only minor modifications.

To estimate and solve this flexible, high-dimensional dynamic demand model, techniques from: (i) approximate dynamic programming (ADP), (ii) large scale dynamic programming in macroeconomics, (iii) machine learning, and (iv) statistical computing are used. There exists a nascent literature where ADP methods have been used to estimate dynamic models in economics. Hendel and Nevo (2006a), Sweeting (2013) and Fowlie et al. (2016) use parametric policy function iteration described by Benitez-Silva et al. (2000) - an early ADP algorithm. Arcidiacono et al. (2012) show how to use sieve value function iteration to estimate and approximate the solution to dynamic single agent models with large-state spaces - an approach closely related to the ADP methods used in this paper. Other ADP techniques have also been used to approximate solutions to large scale dynamic games (see Farias et al. (2012)).

Since the resulting state space is still moderately sized, the value function is approximated using a Smolyak polynomial to further mitigate the curse of dimensionality (Judd et al. (2014)). To solve for its coefficients a forward simulation algorithm from the ADP literature called λ-policy iteration (Bertsekas (2015)) with the envelope condition method (Maliar and Maliar (2013)) are combined. In implementation, stochastic projected gradient descent methods are used in place of penalised least squares methods to address numerical stability issues that arise from the high-dimensional, sparse nature of the household's choice problem.

The dynamic demand model is estimated using the simulated method of moments. To fit the structural parameters a derivative free optimisation methods from statistical computing is used (Łącki and Miasojedow (2015); Baragatti et al. (2013)). As noted by Imai et al. (2009) and Norets (2009), solving the dynamic demand model at every parameter guess is costly for these methods. In line with their suggested approaches, the model is estimated by alternating between fitting the structural parameters and solving

the dynamic demand model.[5]
The model is then applied to the UK laundry detergent industry using household level purchase data from Kantar. The data spans the period from 1st January 2009 until 31st December 2011. It focusses on households who make the vast majority of their purchases at one store - Tesco, the UK's largest grocery retailer. We show that the model matches the distribution of brand shares across the household's with different levels of income and reproduces key price and inventory demand dynamics.
The remainder of the paper is structured as follows. In section 2 we provide an overview of the UK laundry detergent industry and highlight the key economic issues. Section 3 describes the dynamic demand model for the UK laundry detergent industry and details the dimension reduction strategies used. Section 4 discusses the identification and estimation of the model, respectively. The results of the empirical application to the UK laundry detergent industry are presented in section 5. Section 6 concludes.

2 UK Laundry Detergent Market

The nature of products sold in the UK laundry detergent industry make the application of existing storable good demand models to it challenging.
In addition to the price and inventory dynamics that are a feature of storable good industries, products are highly differentiated. In the UK, there is a range of brands, formats, and pack sizes of laundry detergent. They vary in their efficacy, ease of use and costs of storage. In turn, consumers tend to use and store them differently. Finally, there are no dominant pack sizes - preventing straightforward application of existing methods.
Using micro-data on individual households laundry detergent purchases, the remainder of this section discusses these issues in more depth. The brands, formats and different pack size available are documented. Further, reduced form evidence of price and inventory demand dynamics is presented. We begin with brief description of the micro data used.

2.1 Data

We analyse the UK laundry detergent industry using household level purchase data from Kantar. The data spans the period from 1st January 2009 until 31st December 2011. It focusses on households who make the vast majority of their purchases at one store - Tesco, the UK's largest grocery retailer.
Additional filters are added to the sample of Tesco customers to make them suitable for use in estimation of a dynamic demand model. To ensure households purchase records are likely to be informative for an analysis demand dynamics, they are required to make at least 10 purchases with at most one purchase per week. To guard against including households who temporarily drop out of the sample, the maximum gap between any two purchases is 24 weeks.

[5] This technique is used by Osborne (2017).

In addition, to only include households whose purchases are for personal consumption, any households that purchases more than 100 washes or buy more than 2 packs of detergent in a single shopping trip are omitted. Overall, the filtered sample contains 620 households.

The 10th and 90th percentile of grocery spending per equivalent adult are £30 and £70, respectively.[6] Households in the sample contain 1.7 equivalent adults on average and are observed for an average of 135 weeks. A household spends £48 per equivalent adult on average on weekly groceries, of which £2.31 is spent on laundry detergent.

2.2 Overview of UK Laundry Detergent

There are two key aspects of the UK laundry detergent industry: (i) how physical and quality differences in laundry detergents impact on utility, storage costs and prices, and (ii) the multitude of pack sizes that laundry detergent is bought in. In the remainder of this section we highlight the implications these features have on the design of dynamic demand models, and the ramifications they have on the applicability of existing approaches.§

2.2.1 Brands and formats

Laundry detergent is differentiated in many ways. For example, a detergent can be classified as biological or non-biological depending on the enzymes used. Further, they may be specialised to protect against colour fading. Other aspects of differentiation include scent, effective temperature range, additional stain removal capacity, etc. However, many of these aspects of the product are imperfectly observed in the data and cannot be accurately measured. Therefore, the model focuses on the two key defining characteristics of detergents: brand and format.

Brands There are six major brands: four are owned by P&G (Ariel, Bold, Daz and Fairy) and two are owned by Unilever (Persil and Surf). In addition, Tesco sell Private Label (PL) detergent, as well as several other smaller niche brands.

The distribution of the price per wash for each brand is summarised by box plots in Figure 1. The average price per wash is £0.20 - the same as Persil. Fairy, Bold and Ariel are the premium brands and are priced 24%, 14%, and 10% higher than Persil, respectively. The cheapest is Tesco's Private Label (PL) and is around 30% cheaper that Persil on average. Budget brands Surf and Daz are priced close to Tesco's PL; their average price per wash is 15% and 22% cheaper than Persil respectively.

In terms of market shares, the bottom panel of Figure 1 shows that the most purchased brand is Tesco's private label with 34% of all washes purchase - suggesting many consumers are price sensitive and elect to purchase the cheapest product. The budget brands are less popular; Daz and Surf have 7% and 9% respectively. Persil, the mid-range brand, is the second most popular with 21% market share. Suggesting that while

[6]The modified equivalent adults scale is used. The first the adult counts 1, then for anyone else over the age of fourteen add 0.5. For those under the age of 14 add 0.3.

some consumers focus on price they also like to consume branded products. This is further supported by the fact that the higher quality P&G brands - Fairy, Ariel and Bold - together account for around 30% of the market.
This highlights that the demand model must also be able to capture consumer heterogeneity over the quality of detergent purchased and allow it to impact on the amount consumed.

Formats Detergent is available in one of five formats: liquid capsules, gel, liquid, powder and tablets. Although not all brands are available in all formats. In particular, gel detergent is an innovation of P&G introduced in 2008. In the sample, only P&G's premium brands are sold in a gel format. Tesco's PL is also available in a gel format - highlighting that these major brands are key suppliers of PL products for supermarkets. Each format differs in how it is used, its efficacy, the amount of physical storage it occupies, and its ease of storage.[7] As a result the format being consumed materially affects both consumer utility and storage costs.
These differences impact on the price per wash for each detergent. Figure 1 uses box plots to summarise the observed distribution of price per wash for each format in the data. Notably the convenience and storage flexibility of capsules and tablets appears to command a 10% price premium relative to powder. While performance of gel is broadly similar to liquid, its novelty, combined with its low storage costs and ease of use commands a 25% price per wash premium relative to liquid.
In terms of volume of washes sold, the bottom panel of Figure 1 shows that powder detergents are the most popular (38%), followed by liquid (21%). Tablets account for 17% and capsules and gel account for around 11% of washes sold.

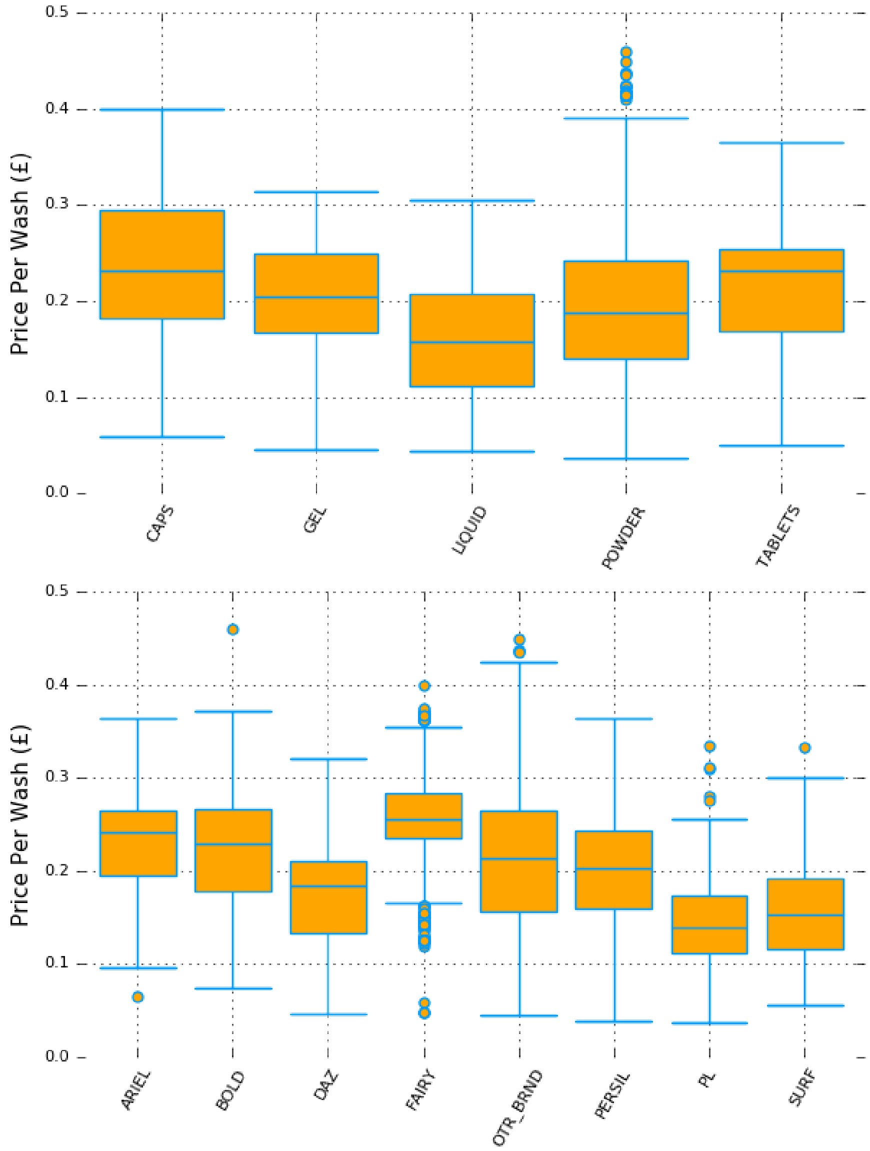
Given these differences, we might expect utility of consumption and storage costs to vary by format. Therefore, it is desirable to allow household utility from consumption and cost of storage to differ by detergent format.

2.2.2 Pack sizes

Laundry detergent is purchased in discrete bundles of washes called Stock Keeping Units (SKUs). A SKU is defined by the type of detergent and the number of washes it contains. The number of washes added to inventory is a key determinant of the number of weeks of consumption that can be serviced without making new purchases. The number of washes of each type of detergent purchased interacts with existing stocks, household size, and price expectations in households' expected utility from purchases.

[7] At the point of consumption, households can choose how much liquid, powder or gel to use in a wash. In contrast, capsules and tablets are sold in pre-measured, discrete dosages. Further, the format may also impact on the type of laundry it is being used for. In particular, the presence of bleach in powder makes it especially suitable for removing deep stains, whereas liquid and gel might be better for delicate garments. Formats also differ in the physical amount of material needed for one wash. In particular, the physical amount of liquid based detergents for a single wash is less than solid detergent. Consequently, powder and tablets are likely to take up more physical storage space than other formats per wash.

Figure 1: Price per wash and market share: by format and brand



Market Shares by number of purchases (%)

	Formats					
Brands	Caps	Gel	Liquid	Powder	Tablets	Total
Ariel	2.87	5.25	0.18	2.03	1.24	11.59
Bold	2.13	3.31	0.21	3.64	1.22	10.52
Daz	0.13		0.70	5.51	0.25	6.58
Fairy	0.78	1.14	0.74	3.44	1.13	7.24
Persil	1.91		8.36	7.22	3.04	20.54
Surf	0.32		3.64	5.20	0.17	9.33
Private Label	2.88	1.89	8.82	10.84	9.29	33.72
Other Brands	0.01		0.22	0.08	0.17	0.49
Total	11.03	11.59	22.89	37.96	16.53	100.00

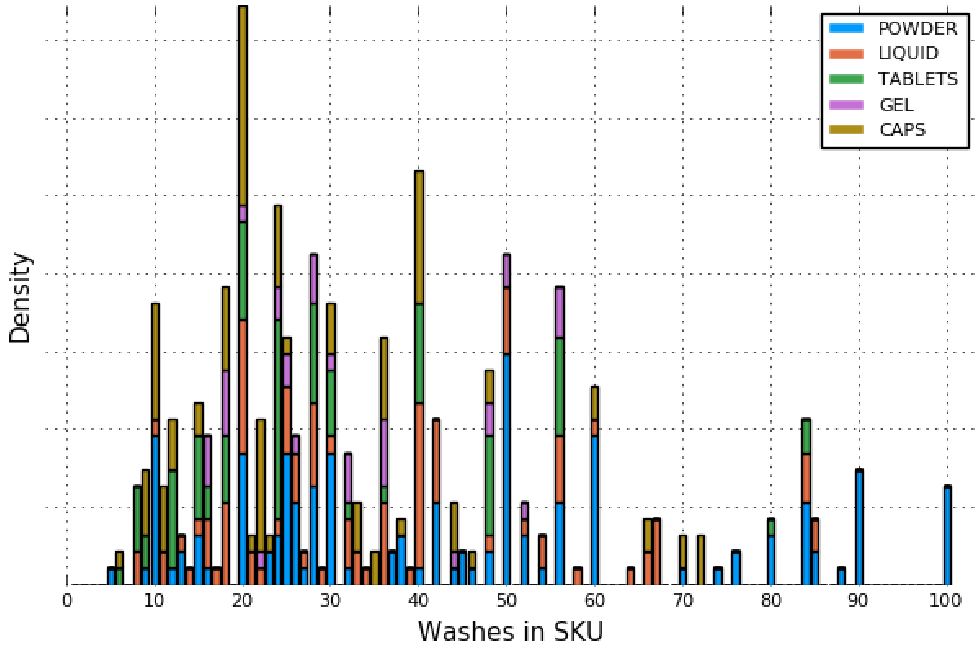
Source: Kantar

In existing approaches to estimating dynamic demand models of storable goods a dominant SKU size (Erdem et al. (2003)) or a low-dimensional number of SKU sizes (e.g. Hendel and Nevo (2006a)) is used to reduce the dimension of the price state space to help address the curse of dimensionality. For example, Hendel and Nevo (2006a) impose restrictions on the dynamic demand model to split the household's purchase decision into a static brand choice that is conditional on the size of the SKU purchased and a dynamic discrete choice over which size to purchase. Even with these restrictions, the price state space is high dimensional and gives rise to the curse of dimensionality.
Using the approach pioneered by Melnikov (2013) and with additional restrictions, the high-dimensional price state space can be replaced by an 'inclusive value' state space with dimension equal to the number of SKU sizes.[8] For example, Hendel and Nevo (2006a) analyse the US liquid detergent market segment between June 1991 to June 1993. In their demand analysis of this market segment there are around ten brands and four sizes. Therefore, these additional restrictions replace a 40 dimensional price state space with an inclusive value state space with four dimensions - one for each size.
In addition to the restrictive assumptions, one drawback of this approach is that it relies on there only being a handful of different SKU sizes. When there are many different SKU sizes, the inclusive value state state is also high dimensional. In this case, this approach does not yield a computationally tractable model. Indeed, there are many markets where the products are sold in many different SKU sizes. The UK laundry detergent is one such industry.
The distribution of SKU sizes available in the UK laundry detergent industry is shown in Figure 2. It shows that laundry detergent in the UK can be purchased in a wide variety of sizes. The 5 most popular SKU sizes only account for 55 percent of purchases purchased. To cover 95 percent of all purchases, in excess of 13 different SKU sizes are needed. As such, the model must be able to incorporate a choice set with a large number of different size choices.
In the spirit of static nested logit models, one might construct groups of ranges of SKU sizes. Then estimate the demand model using a static-brand dynamic-size group split. Problematically, conditioning on the size group purchased, only the upper and lower bound of the quantity purchased is known and the inventory available for consumption can take on a range of values. Therefore, consumption, utility, inventory costs, and the next inventory in the next period are set valued. In this case, the dynamic program conditional on purchasing from a group of different size SKUs is not well defined and cannot be solved using standard dynamic programming methods. Hendel and Nevo's (2006) estimation cannot be applied.[9]
It may be tempting to rectify this by using the expected SKU size purchase or another summary statistic of the size group as a size proxy in the state transition function. However, this will necessarily result in a mis-measured inventory and incorrect specification

[8] 'Inclusive value' refers to the ex-ante expected utility for a household using a random utility model to choose the utility maximising option from a set of alternatives.

[9] **?** estimates a nested logit model dynamic demand for storable goods by adding assumptions on the ratios of pack sizes within size groups. Under these additional assumptions, the inventory states are single valued.

Figure 2: Sales are not concentrated in a handful of SKU sizes



of the transition function of the model.[10]
Notably this approach runs counter to best practice in the approximate dynamic programming literature developed across a wide range of applications. Powell (2011) states

> "It has been our repeated experience in many industrial applications that it is far more important to capture a high degree of realism in the transition function than it is to produce truly optimal decisions".

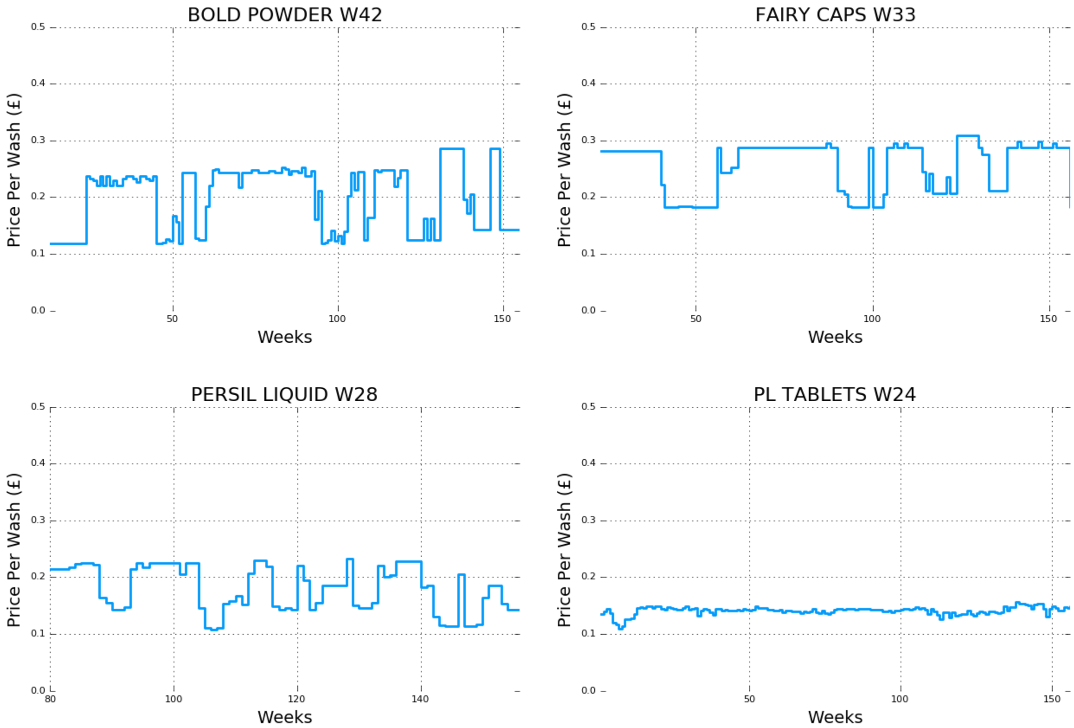
This statement reflects the approach taken in this paper. Rather than artificially restrict the households' dynamic choice problem by adding assumptions of convenience, this paper aims to accurately model the high-dimensional price and inventory dynamics. The computational tractability of the model is achieved by combining statistical models of price dynamics and approximates the solution to the model.

2.3 Demand Dynamics

In a dynamic demand for storable goods there are two avenues through which demand behaviour is inter-temporally linked. One is the presence of household beliefs of future movements of promotional pricing activity. The other is the household's ability to use costly storage technology to store past purchases ready for future consumption.

[10]These errors might be especially large for size groups with a wide range of sizes.

Figure 3: Selected price per wash time series



Using observed price and household purchase data, this section assesses whether there is any evidence that both households' beliefs over price expectations and existing inventories create inter-temporal links in demand behaviour.
Like other storable good markets in the UK, laundry detergent is often sold on promotion. Figure 3 shows examples of time series of the price per wash for four SKUs that differ in number of washes, brand and format. These price series highlight that the length, depth, and frequency of promotions vary with SKU size, brand and format. Other SKUs exhibit similar price patterns.
The upper left panel of Figure 3 shows that Bold powder with 42 washes is sold on relatively infrequent deep discounts of around 50% that can last for several weeks. In comparison, the 33 wash Fairy capsules SKU (top-right) discounts are less pronounced and less frequent but typically last longer. Persil liquid with 28 washes (bottom-left) is often sold on promotion with around 1/3 off the price - though the discounts vary between 10% to 50%. In contrast, Tesco's private label 24 wash tablets SKU is never promoted. This reflects and everyday low price strategy typical of Tesco's private label products.

Through these repeated shopping trips households are likely to be able to anticipate price movements, and factor them into their purchase decisions. For example, household's may respond to their belief that a promotional price may be short lived by accelerating purchases. In turn, building up inventories that delay the need to purchase again in

Figure 4: Conditional correlation of current and previous price per wash with inter-purchase duration

Variable	Estimate	Std. Err.
Current purchase: ln(Price Per Wash)	0.085	0.027
Last purchase: ln(Price Per Wash)	-0.380	0.027
Num. Obs.	11,592	

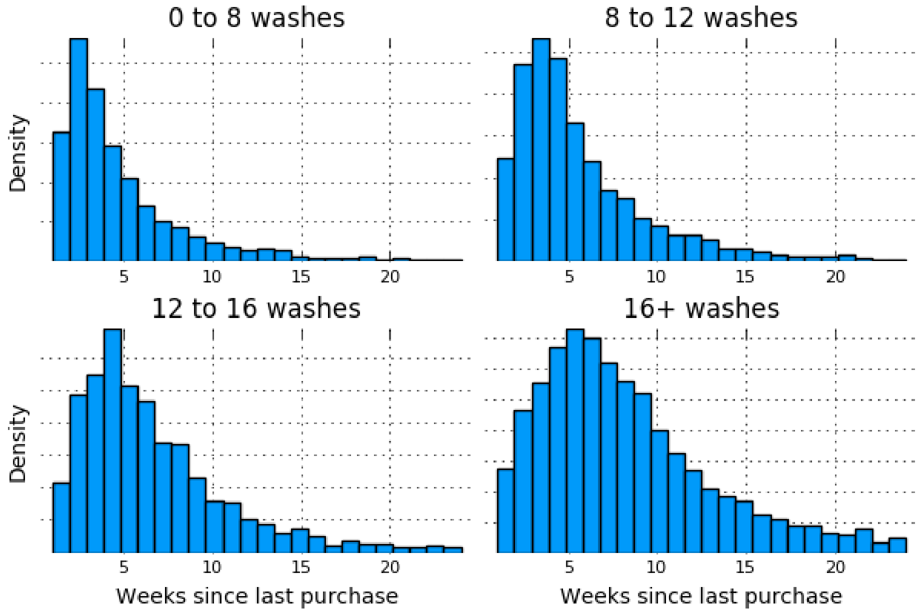
Note: Includes HH fixed effects and controls for current and previous brand and format purchased
Source: Kantar

the near future. To test for such behaviour, we evaluate the impact of current and past prices per wash on the time between purchases - a measure of purchase acceleration. As discussed by Boizot et al. (2001), if price dynamics impact on current demand, past and current purchase price per wash have opposing impacts on inter-purchase duration. That is, conditional on past and current purchase prices, the expected inter-purchase duration is increasing in the current purchase price and decreasing in the price of the previous purchase.
Figure 4 shows the result of a regression of the logarithm of weeks since purchase on the current and previous logarithm of the price per wash, brand and format together with fixed effects for households. The current price per wash is positively correlated with inter-purchase duration; this suggests that price expectations impact on the timing of purchases and are an important aspect of households' purchasing behaviour.

To explore whether the inventories built up as a result of purchase acceleration impact on current demand we plot the time between purchases conditional on the amount of washes purchased per equivalent adult (W/eq. ad.). Figure 5 shows the distribution of inter-purchase duration conditional on the washes per equivalent adult most recently purchased. It partitions the sample into four groups: households that purchase 0 to 8 W/eq. ad. (top left), 8-12 W/eq.ad. (top right), 12-16 W/eq. ad. (bottom left) and those who buy more than W/eq. ad. (bottom-right). ed on the amount.
Figure 5 shows that the distribution of time between purchases shifts rightwards when more washes per equivalent adult are purchased. This is consistent with household's preferring to run down inventories before purchasing again. That is, higher inventories reduce current demand. Indeed, in this industry household's choose not to purchase detergent in 85 percent of weeks.[11]
Overall, the reduced form analysis of household purchasing behaviour presented in this section supports the view that a static demand model would be mis-specified for the UK laundry detergent industry.

[11] An alternative, but unlikely explanation is that household's consumption per equivalent adult is markedly decreasing in the amount purchased.

Figure 5: Inter-purchase duration given washes purchase per equivalent adult



3 Dynamic Demand Model

In the dynamic demand model households visit the supermarket in each week. On each visit, a household decides whether or not to purchase laundry detergent. If so, a household chooses the format, brand and pack size they wish to purchase from around 100 alternatives.

When making purchases households take into account their preferences, the number of washes of laundry detergents held in stock from past purchases, current prices and their beliefs of the likely evolution of future prices.

From a household's viewpoint, the timing, length and depth of the discounts of price promotions are uncertain. In the face of price uncertainty, households use a statistical model to forecast prices.[12] Because they frequently shop at the same supermarket, they observe a long history of prices they can use to build such a forecasting model. Recognising that tracking and forecasting 100 or so prices is cognitively challenging, households use a low rank approximation of the underlying price movements. The forecasting model is estimated using observed prices and its dimensions and parameters are chosen using statistical criteria.

New purchases are added to stocks held at home. After purchases, households choose how much to consume of each of the laundry detergents available at home. For example, the household may choose to use high quality liquid detergent for delicate clothing and lower quality powder detergent for washing other clothes.

Any detergent left over from purchases and consumption is added to inventories. House-

[12]The strategic interactions on the supply-side that generate price dynamics is assumed to be too complex for households to comprehend.

holds use valuable space in their homes to store to laundry detergent.[13] Because detergent competes with many other products for limited storage space, the marginal cost of storage another wash is (weakly) increasing. Moreover, the ease of storing different formats of detergent is reflected in marginal inventory costs.

With convex storage costs the increment in inventory costs for new purchases is minimised when stocks are low. As a result, household may elect to purchase infrequently to restock laundry detergent - a feature of observed purchase sequences in the data.

3.1 Household utility

Households are infinitely lived and discount the future at a rate $\delta \in (0, 1)$. The utility from consumption of laundry detergent is assumed to be quadratic. In each time period, the utility a household receives from consumption is,

$$U(C_1, \dots, C_J, C_0) = \psi^\top C - \frac{1}{2} C^\top \Psi C + \psi_0 C_0 \tag{1}$$

where $C = [C_1, \dots, C_J]^\top$ and C_j is the amount consumed of detergent j by a household. C_0 is a composite good of all other products purchased. To simplify the model and focus on the demand dynamics of the laundry detergent market in the UK, the composite good is assumed to be essential to the household and is not storable.[14]

There are three sets of parameters: ψ, Ψ and ψ_0. First, the J-vector of utility weights for a household, $\psi = [\psi_1, \dots, \psi_J]^\top$, allow a household's utility from consumption of a detergent to depend on its quality and ease of use.

Second, to ensure the utility from consuming detergent is weakly concave in consumption, Ψ is a $J \times J$ symmetric positive semi-definite matrix of utility coefficients. The j-th diagonal element of the matrix impacts on the rate of consumption of detergent j.[15] The off-diagonal elements of the matrix allow marginal utility of consumption to depend on the consumption of other detergents. If all elements of Ψ are zero, then the utility function collapses to the linear specification in Erdem et al. (2003).[16]

Finally, ψ_0 is the marginal utility of consuming the composite good. Normalising the price of the composite good to 1, ψ_0 is also the marginal utility of income.

[13] In 2016 in the UK the average cost per square foot is around £200. Using the average rent per month-to-house price ratio, this translates into around £1.20 per square foot per month.

[14] That is, it is always purchased in positive quantities and entirely consumed in the period in which it is bought.

[15] To see this consider a static model with this utility function and a single good, the interior optimal consumption is given by $C_{ij} = \psi_{ij}/\Psi_{i,(j,j)}$.

[16] This specification also nests the key assumptions of the utility function from consumption used in Hendel and Nevo (2006a). Namely, all detergents are perfect substitutes at the point of consumption and the utility function is strictly concave. This occurs when all detergents have the same linear utility weight, the same non-zero value of the diagonal component of Ψ_i, and the off-diagonal terms of Ψ_i are constant and identical. The sum of these off diagonal terms should be less that the diagonal element.

3.2 Purchases and consumption

Prior to choosing how much to consume, a household decides which, if any, detergent SKU to buy. Detergents are purchased in M distinct SKUs indexed by $m = 1, \ldots, M$ and costs p_m. SKU m provides $q_{j,m}$ washes of a detergent j. As noted in the description of the UK laundry detergent industry in section 2, the number of SKUs on sale in Tesco in each week is around 100 and there are 37 types of detergent available for purchase over the sample period (i.e. $M > J$).

A household's purchase decision to buy SKU m is recorded by the $(M + 1)$ −vector

$$d = [d_0, d_1, \ldots, d_M]^\top \tag{2}$$

where

$$d_m = \begin{cases} 1 & \text{if the household purchases SKU } m \\ 0 & \text{otherwise} \end{cases}$$

Households are restricted to a single purchase, $d^\top d \in \{0, 1\}$. Adopting the convention that $m = 0$ indexes a household's decision not to purchase, the no purchase option is denoted by $d_0 = 1$.[17]

Utility from other SKU related factors observed by the household, but not by the econometrician are captured by an additively separable identically and independently distributed random utility shocks for each SKU, $\varepsilon_m \overset{iid}{\sim}$ Type I Extreme Value with scale parameter σ_ε for $m = 0, 1, \ldots, M$.

When making purchases, the household also incurs purchase costs. The fixed costs of purchase are captured by τ_0. These include costs of going to the store and search costs. However, search costs may also be linked to SKU size. For example, the most popular SKUs are often placed in convenient locations - especially when on promotion. If the best located SKUs tend to be medium sized (i.e. around 20 to 30 washes) then searching for them might be less costly relative to small and large SKUs. Other purchase costs, like physical carriage costs, are likely to be strictly increasing in SKU size.

To flexibly capture the fixed purchase costs and SKU size related search and carriage costs, purchase costs are

$$PC(Qd) = \rho_0 1 \left[Qd \geq 0 \right] + \rho_1 Qd + \rho_2 \left(Qd \right)^2 \tag{3}$$

where Q is a $J \times (M + 1)$ matrix that maps the number of washes provided in the M SKUs into J inventory holdings

$$Q = \begin{bmatrix} q_{1,1} & q_{1,2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & q_{2,3} & q_{2,4} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{3,6} & \cdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & q_{J,M-1} & q_{J,M} & 0 \end{bmatrix}$$

[17]Using this notation, $d_{imt} = 0 \, \forall \, m = 1, \ldots, M$ when household's decide not to purchase any SKU.

Fixed costs are restricted to be non-negative (i.e. $\rho_0 \geq 0$) and are a convex function of SKU size (i.e. $\rho_2 \geq 0$). To allow for the possibility that search costs are lower for the best located medium size products the sign of ρ_1 is unrestricted.
Newly purchased detergent is added to existing inventories, $I = [I_1, \ldots, I_J]^\top$. The post-purchase inventory available for consumption contain K different detergents and is

$$\bar{I} = I + Qd \tag{4}$$

Only after purchases, households choose how much of the inventory held in stock, if any, to consume. The are four possibilities listed in Table 1. First, they can choose to consume some, but not all of the detergent. Second, the remaining stock of the detergent can be consumed (i.e. only a small amount is left). Third, they can elect not to consume any, even if it is held in stock. The final case is that there is no detergent consumed or held in stock.

Table 1: Consumption Options

	$C < I + Qd$	$C = I + Qd$
$C > 0$	Interior Solution	Exhaust Inventory
$C = 0$	Save Inventory	No Inventory

Any unused inventory, $\bar{I} - C$, is stored ready for consumption in the next period. The household incurs an inventory cost,[18]

$$IC\left(\bar{I} - C\right) = \gamma^\top\left(\bar{I} - C\right) + \frac{1}{2}\left(\bar{I} - C\right)^\top \Gamma \left(\bar{I} - C\right) + \kappa_1 D + \kappa_2 D^2 \tag{5}$$

where γ and Γ are a J-vector and a $J \times J$ symmetric matrix of inventory cost parameters, respectively.
Since detergent competes with many other products for limited storage space, adding new purchases of detergent to existing inventories is costly. As such, the marginal costs of storing new purchases are positive and increasing. As such, the cost of stocking D types of detergent is convex, the elements of γ are non-negative, and Γ is positive definite.
Inventory costs can also impact on purchase patterns. With convex inventory costs, the incremental storage costs of adding new purchases to inventory are at their lowest when existing stocks are low. All else equal, households prefer to run down detergent stocks before repurchasing. Provided that purchased quantities are large relative to weekly consumption, a household would choose to infrequently restock laundry detergent - a feature of the observed purchase sequences in the data.
However, the timing of purchases is also affected by expected price movements. As was highlighted in section 2, laundry detergent is often available for purchase on promotion.

[18]In 2016 in the UK the average cost per square foot is around £200. Using the average rent per month-to-house price ratio, this translates into around £1.20 per square foot per month.

From the perspective of the household, the depth of the price discount, the timing, and the length of the promotion are uncertain.[19]

The possibility of a change in the price in the near future will affect the current purchase decision. For example, suppose that a SKU is available on a deep discount that the household believes to be short lived. Even though they may have enough inventory to service consumption needs for the near future, they may elect to accelerate the SKU purchase to take advantage of the relatively low purchase price.

To forecast prices, households build a forecasting model using a long history of prices observed on repeatedly making shopping trips to the same supermarket. Since the strategic interactions that generate price dynamics are too complex for households to comprehend, households use a statistical model to forecast prices. In particular, household's use τ-order Markov Process to forecast M SKU prices in the next period, $p' = \left[0, p_1', \ldots, p_M'\right]^\top$,

$$p' \sim G_{p'|P} \tag{6}$$

where $P = [p_t, p_{t-1}, \ldots, p_{t+1-\tau}]$ is the matrix of past τ-period's prices.

Recognising that tracking and forecasting 100 or so prices is cognitively challenging, households use a low rank approximation of the underlying price movements. The forecasting model is estimated using observed prices and its dimensions and parameters are chosen using statistical criteria. The details of this are deferred until section 3.4.1.

Any purchases of detergent and inventory costs incurred are paid out of per-period income, Y, received at the beginning of the period. The household spends the remainder of the income on the composite good whose price is normalised to 1. The resulting budget constraint is

$$Y = C_0 + d^\top p + PC(Qd) + IC\left(\bar{I} - C\right) \tag{7}$$

3.3 Household choice problem

Adding the SKU utility shocks and substituting in the budget constraint, the households per-period flow utility function of consuming laundry detergent bundle C after purchasing is

$$U(C) + \psi_0\left(Y - d^\top p - PC(Qd) - IC\left(\bar{I} - C\right)\right) + d^\top \varepsilon \tag{8}$$

where $U(C) = \psi^\top C - \frac{1}{2} C^\top \Psi C$.

Bringing together the elements of the model described above, Theorem 1 describes the household's choice problem as a two stage discrete-continuous Markov decision problem expressed in recursive form.

In each period, a household faces a discrete-continuous decision: (i) they choose whether or not to purchase laundry detergent, and (ii) how much laundry detergent to consume in the current period.

Before making any purchase, the household observes inventories, prices, and utility shocks. Using this information, they make price forecasts and choose which, if any, of the SKUs to purchase.

[19] Erdem et al. (2003); Hendel and Nevo (2006a,b); Nevo and Hendel (2012); Osborne (2013); Wang (2012, 2013) all report that prices typically exhibit these features.

The amount of each detergent available for consumption after the purchase of SKU m is measured by the post-purchase inventories, $\bar{I}$. This is unique to SKU m. Further, the post-purchase income before inventory costs are incurred, $\bar{Y} := Y - d^{\top}p$, is also specific to the purchase of SKU m. These SKU m specific elements of the post-purchase state space, $s = [\bar{I}, P, \bar{Y}]$, give rise to different values of the conditional indirect utility function, $W(s)$.
Adding SKU specific utility shocks, the household maximises utility by choosing whether or not to purchase and if so, which SKU to purchase.
Then, in the second stage, the household solves the Bellman equation for optimal consumption conditional on the purchase of SKU m (or the no purchase option) in stage 1.

Theorem 1 Recall $\bar{I} := I + Qd$ is the inventory held immediately after purchases and $\bar{Y} := Y - d^{\top}p$ is the income remaining after having paid for SKU m. The household's choice problem is as two-stage discrete continuous choice defined on the post-purchase state space, $s = \left[\bar{I}, P, \bar{Y}\right]$.
In the first stage, households make purchasing decisions by solving

$$\begin{aligned} V(s,\varepsilon) &= \max_{d} W(s) + d^{\top}\varepsilon \\ s.t. &\quad d^{\top}d = 1 \\ &\quad \varepsilon \overset{iid}{\sim} \text{Type I EV} \end{aligned} \tag{9}$$

where $W(s)$ is the indirect utility function corresponding to the purchasing SKU $m = 0, 1, \ldots, M$.
In the second stage, households choose how much to consume from the inventory held in stock by solving

$$\begin{aligned} W(s) = \max_{0 \le C \le \bar{I}} & \; U(C) + \psi_0 \left(\bar{Y} - PC(Qd) + IC\left(\bar{I} - C\right)\right) \\ & + \delta \int \ln \sum_{d'} \exp\left\{W\left(s^{'}\right)\right\} dG_{p'|P} \end{aligned} \tag{10}$$

where $P = [p_t, \ldots, p_{t+1-\tau}]$. ■
See Annex A for the proof in Theorem 1.
As specified the model has a very large state space. It includes 37 inventory variables - one for each detergent and in excess of 100 price variables. The curse of dimensionality bites hard and renders the problem computationally intractable. The next section discusses this issue in more detail and outlines the approach taken in this paper that make the model computationally estimable.

3.4 The curse of dimensionality

As highlighted in section 2, there are many different laundry detergents a household can purchase and store. Consequently, the inventory and price components of the state space are high-dimensional continuous variables. This acutely exacerbates the curse of dimensionality.

To see how a high dimensional price and inventory state space can quickly suffer from the curse of dimensionality consider a 'small' laundry detergent industry and assume a first order Markov price process. For example, let $J = 4$, $M = 10$ and $\tau = 1$. Further suppose households choose from $\{0, 1, 2, 3\}$ washes per week of each detergent in stock (if inventories permit it).

Assume that up to 100 washes of each detergent can be held in inventory. Discretise inventory and prices into 20 and 5 points, respectively. Despite the resulting discretisation of the state space being very coarse, it is still of the order of $4^{20} \times 10^5 \approx 110$ quadrillion state space nodes. At each node the household evaluates 2500 actions. Solving this problem is infeasible with existing computational resources. This simple example shows how even a 'small' industry with a coarse state space is hindered by the curse of dimensionality.

This paper proposes an approach to alleviate the curse of dimensionality. Unlike existing approaches it can be easily adapted for application to other storable good industries. Central to the dimension reduction strategies is to address the different ways in which the curse of dimensionality arises from the information needed to forecast future prices and the number of different detergents that a household can store.

The proposed approaches to alleviate the curse of dimensionality that arise from price and inventories are discussed in turn. Subsequently, the household's choice problem is revisited.

3.4.1 Forecasting SKU Prices

Where the dimension of the price forecasting problem is prohibitively high, the existing storable good dynamic demand literature sought to reduce it in one of two ways.

One approach, introduced by Hendel and Nevo (2006a), is to impose restrictions on the dynamic demand model that split the household's purchase decision in two. First, households solve a dynamic choice problem for the optimal size of SKU. Then, conditional on the chosen SKU size, they choose a brand.

Since the brand choice is entirely static, forecasted price changes only affect current purchases through their impact on expected utility of future purchases for each SKU size. Therefore, to decide which size to purchase, households need only forecast changes in the expected utility for each size choice, not the underlying expected SKU-level price movements. As such, Hendel and Nevo (2006a) refer to these inclusive values for each SKU size as quality-adjusted price indices.[20]

To compute the optimal size choice, households therefore need only to forecast their expected inclusive values for each size, rather than all prices.[21] By itself, this simpli-

[20]In a discrete choice setting, the ex-ante expected utility of choosing an item from a choice set is often referred to as the 'inclusive value'. Hendel and Nevo (2006a) use these terms interchangeably.

[21]In Hendel and Nevo (2006a) each household has their own set of inclusive values. The reason is two-fold. First, households' choice sets contain only those brand they are observed to purchase in the sample. Second, linked to the first assumption, brand-size fixed effects are unique to each household.

fication does not lead to a smaller state space. This is because all of the past prices are potentially useful for predicting future expected quality-adjusted price indices. The state space is still very large.

To reduce the size of the state space, individual SKU prices are assumed to have no predictive power beyond their contribution to the quality-adjusted price indices. Under this additional assumption, households need only track the quality-adjusted price indices for each size. If the industry contains a handful of SKU sizes and many SKUs, the dimension of the price forecasting problem is greatly reduced.

Another approach is to use a price forecasting model exploiting the specific features of the industry being studied. Erdem et al. (2003) use a model of price evolution that relies on the idiosyncratic features of the US Ketchup industry. In particular, they use the presence of a common pack size across brands that accounts for the majority of sales in the US Ketchup industry. Moreover, there are a small number of pack sizes and few brands (i.e. $J = 3$ in their application). A drawback of both these approaches is that they are not generally applicable to all industries and rely on ad hoc assumptions.

The approach taken here is to use dimension reduction techniques from the machine learning and statistics literature before estimating the dynamic model.[22] In particular, households use a factor model as a low-rank approximation of the high dimensional price forecasting problem, $p^{'} = G_{p'|P_\tau}$.[23] For $t = 1, \ldots, T$ periods observed in the data, the household's forecasting model for SKU prices is an interactive fixed effects model (Bai (2009))

$$z\left(p_{mt}\right) \;=\; \lambda_m^\top F_t + X_m^\top \alpha + \epsilon_{mt} \tag{11}$$

where X_m is a vector of SKU m's characteristics, $z(\cdot)$ is a transformation of SKU prices, and $\mathbb{E}\left[\epsilon_{mt} | F_t, X_m\right] = 0$.[24] There are R-factors in the price model. In particular, define $F_t := \left[f_{1,t}, \ldots, f_{R,t}\right]^\top$ where $f_{r,t}$ are elements of the T-vector f_r for $r = 1, \ldots, R$. The factor loading matrix is

$$\underset{M \times R}{\Lambda} = \begin{bmatrix} \lambda_1^\top \\ \lambda_2^\top \\ \vdots \\ \lambda_M^\top \end{bmatrix}$$

where λ_m is an R-vector of factor loadings for SKU m. The following scale and rotation normalisations restrictions are imposed

These are estimated by including brand-size dummies for each household in the static brand choice model.

[22] Households are price takers and prices are exogenous state variables. Therefore, the household's model of beliefs over price evolution can be estimated before solving the household's choice problem.

[23] Formally, the factor models can be shown to be equivalent to a truncated SVD decomposition of the price matrix. See Murphy (2012), Chapter 12 for more details.

[24] In this application $z(\cdot)$ is a scaled logistic transformation. This prevents the prediction of negative prices and provides a sensible upper bound motivated by observed data.

$$f_r^\top f_s \quad = \quad \begin{cases} 1 & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

$$\Lambda^\top \Lambda \text{ is diagonal} \tag{13}$$

The expected SKU price movements are governed by the dynamics of price factors, F_t. Factors follow a stationary, τ-order exogenous Markov process,

$$F_t = A_0 + \sum_{s=1}^{\tau} A_s F_{t-s} + u_t \tag{14}$$

where $u_t \perp \epsilon_{mt}|F, X_m$, $F = [F_t, F_{t-1}, \ldots F_{t-\tau+1}]^\top$ and $u_t \overset{i.i.d.}{\sim} N(0, \Sigma_u)$. The number of lags included τ is informed by statistical criteria and relies on data.

The resulting dimension reduction of the state vector is $\tau\,(M - R)$. When prices are correlated over time, the price factor state space, F, will likely contain only a few dimensions. In many applications the resulting reduction in the size of the state space is substantial.

Beneficially, this approach does not rely on idiosyncratic features of the industry being studied and can be applied to any industry. Moreover, rather than restrictive ad hoc assumptions, the curse of dimensionality is substantially mitigated using a statistical criterion that relies on observed price data.

3.4.2 Inventories

Since inventory is endogenously determined by the model, any dimension reduction strategy must be embedded into the solution of the household's choice problem. This creates some additional challenges when attempting to address this curse of dimensionality arising from inventories.

Where the inventory dimension contains more than a handful of products, the existing literature has sought to reduce the dimension of inventory by imposing restrictions on the household's problem. However, rather than add assumptions, an approximate dynamic programming (ADP) approach is used.[25]

The defining feature of an ADP method is that it retains the high dimensional representation of the household's choice problem but seeks a high quality but computational feasible approximate solution.

There are two components to the approximation to $W\,(s)$: the approximation architecture and the selection of a set of 'features' of the high-dimensional problem that characterise the dimension reduction. Each is discussed in turn.

[25] ADP combines tools from statistics and machine learning to approximate the solution to computationally intractable dynamic programming problems. It encompasses a wide variety of techniques from a collection of disparate fields that have developed specific approaches to approximate solutions to complex dynamic programs they encounter. Indeed, some of the techniques used are closely linked to generalised stochastic search algorithms that have been used to solve large-scale dynamic programs economics. See Maliar and Maliar (2014); Judd et al. (2011) for a detailed overview.

Approximation architecture Adopting Powell's (2011) terminology, there are three broad categories of value function approximation architectures: a lookup table approximation (i.e. valuations at pre-defined state space vectors), a linear-in-parameter approximation (i.e. polynomials, B-splines, etc), and a non-linear approximation (i.e. neural nets). In this paper, we focus on linear-in-parameter approximations

$$W(s) \approx \phi(s)^{\top} r \tag{15}$$

where $\phi(s)$ is a L-vector of basis functions and r is a L-vector of parameters.

The reasons for choosing a linear-in-parameters approximation are two-fold. First, lookup tables tend to suffer from the curse of dimensionality for even moderately sized problems. In this case, even with an aggressive dimension reduction strategy, it is likely that the model could only be solved on a very coarse discretisation of the state space.[26]

Second, linear-in-parameters approximations can exploit the linear structure to ensure updating is relatively straightforward and are easily modified to address numerical stability issues. Whereas, non-linear architectures are more complex to update and less numerically stable. As a result there are no convergence guarantees and *sui generis* modifications to are needed improve performance.[27]

To approximate $W(s)$ a class of flexible polynomials suitable for large-scale dynamic programming problems called Smolyak polynomials is used.[28] Beneficially, the number of basis functions used by this class of models grows polynomially, rather than exponentially as the state space expands. Moreover, if an anisotrophic Smolyak polynomial is used, the accuracy of the approximation can be varied in each state space variable.[29]

Taken together, the linearity, smoothness and sparseness of the approximation architecture allow a richer set of low-dimensional features to be included without incurring too high a computational penalty. In turn, this adds a degree of flexibility that may be useful when selecting which 'features' to include in the approximation to $W(s)$.

Feature selection ADP methods make heavy use of forward simulation. As the model steps forward through the state space, $W(s)$ has to be evaluated many times to calculate expectations and solve for optimal consumption. Therefore, reducing the computational cost of evaluating $\phi(s)$ is the focus of the dimension reduction strategy.

When using a Smolyak polynomial to approximate $W(s)$, this is tantamount to reducing the number of arguments entering the basis functions. When the dimension of the state

[26]Nevertheless, other more sophisticated approximations that work with discretised high-dimensional state spaces, such as hierarchical approximations (see Powell (2011); Bertsekas (2011a))) and adaptive grid methods (Brumm and Scheidegger (2015)) may prove to be fruitful.

[27]See Powell (2011) for a discussion of updating nonlinear approximation architectures. Note that neural nets can be easily updated in some instances. As highlighted by Bertsekas (2011a); Judd et al. (2011), an additional benefit of using a linear architecture is that other projections can be considered (i.e. regularisation can be added).

[28]See Judd et al. (2014) for an overview of this class of polynomials and a detailed discussion of how to efficiently implement them.

[29]The polynomial growth of basis functions involving each variable is linked to the chosen level of accuracy.

vector is high, there are very many basis functions to compute. For example, doubling the state vector from 5 to 10 variables increases the number of basis functions of a level 2 precision isotrophic Smolyak polynomial from 61 to 221, an increase of over 360%. At level 3 precision, the the number of basis functions increases by over 650%.[30]
In ADP, a high quality, low dimensional approximation to the solution is created by identifying and extracting the salient features of the problem. Typically this involves some transformation and/or aggregation of the high dimensional state space and exploits the structure in the dynamic program.
In retail storable good industries, like laundry detergent, households are unlikely to stock more than a handful of products at once. As a result, their inventory vector is likely to be sparse. This sparsity is the structure used to reduce the dimensionality of the inventory vector in this class of models..
Specifically, the inventory state space entering the approximation is restricted to include only those detergents that are held in stock after purchases. The resulting state vector has $D + \tau R + 1$ dimensions where $D \in \left\{0, 1, \ldots, \bar{D}\right\}$ and $\bar{D}$ is the maximum number of detergent that can be held at once by any household. Assuming that the household's forecasting problem is a first-order Markov process with only a handful of price factors, the dimension of the state vector is, at worst, moderately sized.
By reducing the number of inventory state variables, the number of basis functions and parameters is much smaller (i.e. $L_D \ll L$). In turn, dramatically lowering the computational burden of approximating the solution to the household's choice problem.
To implement this approximation strategy define a $L_D \times L$ matrix Ω to select the subset of basis functions whose arguments contains the D non-zero elements of the post-purchase inventory and other non-inventory state variables. The lower-dimensional subset of basis functions and corresponding parameters with D different detergents held in inventory are calculated by pre-multiplying the high-dimensional counterparts by Ω,

$$\tilde{\phi}(s) := \Omega\phi(s) \tag{16}$$

$$\tilde{r} := \Omega r \tag{17}$$

The approximate conditional indirect utility function is

$$W(s) \approx \phi(s)^{\top} \Omega^{\top}\Omega r = \tilde{\phi}(s)^{\top} \tilde{r} \tag{18}$$

So far the value function approximation has been discussed in the context of a particular combination of detergents held inventory. Of course, different inventory holdings over time and across households are to be expected. For example, even assuming households never stock more than 3 of the 37 detergents sold, there are over 8000 different configurations of detergents that could be held in inventory at once.
To accommodate these different combinations, the components of Ω are adjusted to reflect the post-purchase inventory holdings. As such, a low-dimensional approximation

[30] A isotrophic Smolyak polynomial with precision level p can exactly interpolate an order monomial from x^0 to x^p. See Burkhardt (2012) for a more general discussion of sparse grids in the context of integration rules.

to $W(s)$ for any combination of detergents can be efficiently constructed.[31]
Even though the dimension of the approximation for a particular inventory configuration is lowered, this dimension reduction strategy for inventory does not reduce the number of parameters. That is, we need to solve for the L-parameters in r. While this may seem to be an obstacle, compared to the cost of forming the approximation to $W(s)$ defined on even moderately sized state spaces, it is relatively cheap to fit r.

3.5 Approximating the household's choice problem

Incorporating the household's price forecasting model into the choice problem, households need only keep track of a coarser partition of the price state space containing τ R-vectors to forecast SKU prices. The conditional indirect utility function is redefined on the coarser partition of the price state space, $W\left(\bar{I}, F, \bar{Y}\right)$.[32]
Implementing the dimension reduction strategy for inventories, the approximation to the solution of the household's choice problem is constrained to lie in the approximation subspace spanned by the basis functions, $\mathcal{S}^{\phi} := \left\{\phi(s)^{\top}\Omega^{\top}\Omega r | r \in \mathbb{R}^{L}\right\}$.
However, the conditional indirect utility function that solves the Bellman equation (eq. (10)) may not lie in the subspace of function $\mathcal{S}^{\phi}$. If so, there is no $r \in \mathbb{R}^{L}$ that solves the household's optimal consumption problem.
The goal then is to project $W(s)$ onto S^{ϕ} and find the value of r that minimises the distance between $W(s)$ and its approximation with respect to some norm.
One approach is to choose r by minimising the Bellman equation approximation errors with respect to a weighted Euclidean norm $\|\cdot\|_{\omega}$ with positive weights ω summing to one. The $r^{\star}$ that minimises the Bellman equation approximation errors is the fixed point the projected Bellman equation,

$$\Phi r^{\star} = \Pi T\left(\Phi r^{\star}\right) \tag{19}$$

where $\Phi = \left[\phi(s_1)^{\top}\Omega_1^{\top}\Omega_1, \ldots, \phi(s_N)^{\top}\Omega_N^{\top}\Omega_N\right]^{\top}$ is a sparse matrix of basis functions evaluated at $N \geq L$ state space points, Π is the projection with respect to the a weighted Euclidean norm $\|\cdot\|_{\omega}$ with $\omega > 0, \sum\omega = 1$, and $T(\Phi r)$ is the Bellman optimality operator evaluated at r.[33]
To solve for the fixed point the of projected Bellman equation we use an ADP algorithm

[31] To illustrate how Ω is constructed, partition it into 2 components, Ω^1 and Ω^2. The first component, Ω^1, includes all L-dimensional basis vectors that select basis functions that do not depend on the subset of detergents held. These are included in all approximation to $W(s)$. The subset of L-dimensional basis vectors in Ω^2 depends on the specific configuration of the post-purchase inventory. To construct Ω^2, we only include basis vectors whose l-th element is 1 if $\phi_l(\cdot)$ is a function of one of the D detergents held in inventory after purchases. The resulting matrix $\Omega = \left[\Omega^1, \Omega^2\right]^{\top}$ defines the set of basis vectors that select the relevant subset of basis functions and corresponding parameters.

[32] To avoid an unwieldy proliferation of function notation, with a small abuse of notation $W(\cdot)$ is redefined on the coarser partition of the price state space.

[33] $T(\Phi r) := \max_{0 \leq C \leq \bar{I}} U(C) + \psi_0\left(\bar{Y} - PC(Qd) - IC\left(\bar{I} - C\right)\right) + +\delta \int \ln \sum_{d'} \exp\{\Phi r\}\, dG_{F'|F}$

called λ-policy iteration (λ-PI).[34] Like policy iteration, λ-PI has two steps: policy evaluation and a policy improvement.
Starting from an initial parameter guess r_0, these steps are repeated K times (or until convergence) in an outer loop. The outer loop of this algorithm generates a sequence of policy-parameter tuples $\{\mu_k, r_k\}$ for $k = 1, \ldots, K$. The approximate solution to the projected Bellman is given by the final policy-parameter vector tuple $\{\mu_K, r_K\}$.
Next we discuss the two steps of λ-PI in more detail in the context of the household's choice problem. The section closes with a discussion on the factors affecting the choice of λ.

3.5.1 Policy Evaluation

At iteration k of the λ-PI algorithm, the policy evaluation step approximates the value of the following new consumption policy function, μ_{k+1}, at states visited along simulated trajectory of the model, s_i for $i = 1, \ldots, N$.[35] The approximate value is calculated by applying a geometrically weighted multi-step Bellman operator for policy μ evaluated at the current parameter guess r_k

$$T_\mu^{(\lambda)}(\Phi r_k) := (1-\lambda)\sum_{l=0}^{\infty} \lambda^l T_\mu^{l+1}(\Phi r_k) \tag{20}$$

where $\lambda \in [0,1)$ and $T_\mu^l(\cdot)$ indicates the that Bellman operator under policy μ is applied l times to the each of states in $\Phi(s)$.[36]
The quality of the approximation depends on the value of λ. At one extreme, each state is evaluated by a single application of the Bellman operator under policy μ - as is the case for value function iteration. At the other, as $\lambda \to 1$ the policy evaluation step converges to policy function iteration. As such, the λ-PI method can be viewed as being a convex combination of the two. The choice of λ is left for discussion at the end of this section.
The value at state s_i in the simulated trajectory of the model under policy μ_{k+1} is $W_{\mu_{k+1}}(s)$ and is approximated by,[37]

$$\hat{W}^{(\lambda)}_{\mu_{k+1}}(s_i) = T_\mu^{(\lambda)}\left(\tilde{\phi}(s_i)^\top \tilde{r}_k\right) \;\forall\, i = 1, \ldots, N \tag{21}$$

To evaluate the policy, the simulated valuations are projected on to the subspace S^ϕ. At this point, shape restrictions are added that ensure the value function is concave in each of detergents held in inventory.[38] The resulting optimisation problem solved in the policy evaluation step is

[34] λ-policy iteration was developed by Bertsekas and Ioffe (1996). The specific algorithm used is called λ-PI(1) and was proposed by Bertsekas (2015), where further details can be found.

[35] In the λ-PI(1) algorithm there are several batches of simulated model trajectory with randomly selected initial states terminated with probability $1-\lambda$ in each period.

[36] Formally $T_\mu(\Phi r) := U(C_\mu) + \psi_0\left(\bar{Y} - PC(Qd) - IC\left(\bar{I} - C_\mu\right)\right) + \delta \int \ln \sum_{d'} \exp\{\Phi r\}\, dG_{F'|F}$

[37] In practice, the transitions have an additional exploratory component to encourage exploration of all of the state space, not just those visited by stepping forward through the model under the policy being evaluated.

[38] In principle we could add the restriction that the Hessian matrix with respect to inventories is negative definite. However, this is computationally burdensome to ensure at all N state space

$$
\begin{aligned}
r_{k+1} \quad = \quad & \arg\min_{r\in\mathbb{R}^L} \sum_{i=1}^{N} \left(\hat{W}^{(\lambda)}_{\mu_{k+1}}(s_i) - \tilde{\phi}(s_i)^\top \tilde{r}\right)^2 \\
s.t. \quad & \nabla^2_{j,j}\tilde{\phi}(s_i)^\top \tilde{r} \leq 0 \,\forall\, j = 1,\ldots,J,\; i = 1,\ldots,N \\
& \tilde{\phi}(s_i) = \Omega_i \phi(s_i) \,\forall\, i = 1,\ldots,N \\
& \tilde{r} = \Omega_i r \,\forall\, i = 1,\ldots,N
\end{aligned}
$$

where $\nabla^2_{j,j}\tilde{\phi}(s_i)$ is the vector of second derivatives of the basis function with respect to the inventory of detergent j.

As noted above, the inventory and the vector of basis functions used in the approximation to $W(s)$ is high-dimensional and sparse. When simulating under a given policy, it is possible that not all areas of the state space are visited with high probability. Even with a very long simulated trajectory, the matrix of basis functions is very sparse. When stacked over the N state space points being evaluated it is likely to be rank deficient. Further noting the iterative nature of procedure and the the fact that policies are changing as the model is solved, the policy is evaluated using a proximal algorithm called stochastic projected gradient descent.[39]

3.5.2 Policy Improvement

From $\{\mu_{k+1}, r_{k+1}\}$ a new, improved policy is obtained. The exact method used depends on how the the consumption policy function is specified.[40] In this model, consumption is specified as a function of the state and the new policy evaluation parameters calculated in the previous step, r_{k+1}. This class of policies are referred to by Powell (2011) as value function approximation policies.

For example, the consumption policy function can be defined as the Bellman optimality operator evaluated at r_{k+1} (i.e. $C^\star = \arg T(\Phi r_{k+1})$). For vector-valued continuous controls, like household consumption, one way to implement this is to solve a system of Euler equations. As is often noted in implementations of Euler equations methods, root-finding in such a system of equations can be computationally intensive. This is because they involve costly computational tasks such as the calculation of multi-dimensional integrals to evaluate expectations and the need to call a non-linear equation solver. Even if the computational costs are moderate, this can be especially problematic for ADP algorithms. As noted above, this is because ADP's use of forward simulation and requires the system of Euler equations to be solved very many times.

Another approach is to use the envelope condition method (ECM) developed by Maliar and Maliar (2013). ECM solves for consumption using the envelope condition rather than the first order conditions.

points in the simulation. Therefore, the computationally burdensome necessary condition that the second derivatives for each detergent are negative is imposed.

[39] See Bertsekas (1999)

[40] Several other options are available. For example, we could look to approximate policy outcomes directly as a functions of states. See Powell (2011) chapter 6 for a detailed overview of policy types.

The benefit of using the ECM is three-fold. First, the continuous choice vector can be obtained using the gradient of the current period's approximation to $W(s)$ in place of the gradient of the continuation value. Unlike Euler equations, there is no need to compute expectations. Second, because the flow payoff is quadratic in consumption, this system of equations is linear in consumption. This avoids the additional computational burden associated with calling a non-linear equation solver and always has a unique solution. Third, ECM works with the estimate of the parameters, r_{k+1}, rather than estimate the continuation value function in the next period.[41] Therefore, ECM can easily be combined with forward simulation techniques used by λ-policy iteration and other ADP algorithms.

As such, the computational burden likely to be associated with Euler equations does not arise for ECM. Maliar and Maliar (2013) and Arellano et al. (2014) find that ECM achieves similar speed-ups in computation time over value iteration and the endogenous grid method.[42] In the context of an algorithm that heavily utilises simulation, such as λ-policy iteration, this is likely to be an important practical consideration. [43]

There are also drawbacks of using the ECM method. Most notably, $T^{(\lambda)}_{\mu^{ECM}}(\cdot)$, is not necessarily a contraction mapping. However, Arellano et al. (2014) show that there exists a damping parameter, $\xi \in (0, 1)$, that can be used to exponentially smooth updates to the value function such that $T^{(\lambda)}_{\mu^{ECM}}(\cdot)$ is a contraction mapping.[44]

Because a stochastic projected gradient descent algorithm used to find r_{k+1} in the policy evaluation step, partial updating is equivalent to damping.

3.5.3 Choice of λ

The convergence properties of the policy evaluation step and the approximation bias, $W(s) - \tilde{\phi}(s)^\top \tilde{r}$, depend on λ.

Despite $T(\cdot)$ and $T_\mu(\cdot)$ being contraction mappings on the sup-norm, there is no guarantee that the projected Bellman operator mappings are also contractions. Even if the set of weights satisfy $\omega > 0$ and $\sum \omega = 1$ of the Euclidean norm above, neither $\Pi T(\cdot)$ nor $\Pi T_\mu(\cdot)$ are, in general, contractions.[45]

If, however, the weights correspond to the invariant distribution of the Markov trans-

[41] Traditional DP methods like VI, PI and Endogenous Grid Method (EGM) necessarily guess the next period's value function and its derivatives. This is particularly useful when stepping backwards through time to solve the model.

[42] See Fella (2014) and Iskhakov et al. (2015) for a detailed discussion of the difficulties that can arise for the endogenous grid method when there are occasionally binding constraints, discrete choices, and vector valued continuous choices.

[43] Further, unlike the endogenous grid method it is much easier to work with occasionally binding constraints.

[44] See Arellano et al. (2014) for a detailed exposition. Intuitively, this occurs because ECM does not impose the first order conditions at every step in the iteration, only in the limit. Likewise, value iteration does not impose the envelope condition during the contraction, only in the limit. However, imposing the first order condition is necessary to guarantee that value iteration and other backward iteration methods, like endogenous grid method, have the contraction mapping property.

[45] See Bertsekas (2011b).

ition function of the household's choice problem then mappings ΠT and ΠT_μ are contractions.[46] This highlights the importance of simulating from the model under policy μ to produce states for policy evaluation when doing approximate value iterations (i.e. $\lambda = 0$).

The drawback of this is that evaluation of the policy function is heavily weighted to areas of the state space visited most often. Ideally, the whole of the state space would be explored to produce better policies in the improvement step.

However, Bertsekas (2011b) shows that for any weighted Euclidean projection norm, $\Pi T^{(\lambda)}$ is a contraction mapping if λ is sufficiently close to 1. Therefore, by setting λ close to 1 we can sample states from outside of the transition density and the policy evaluation step a contraction mapping. The state vector sampling process can therefore be augmented to promote exploration of the state space. In turn, leading to better policy improvements.

A beneficially corollary of setting λ closer to 1 is that the the approximation bias, $W(s) - \tilde{\phi}(s)^\top \tilde{r}$, is lower. There is, however, a bias-variance trade-off. If λ is too close to 1 then simulation noise can cause the quality of the approximation to degrade.

Bertsekas (2011a) states that while there is no obvious rule of thumb, it might be desirable to set λ close to 1 if computing many transitions is computationally feasible. After some experimentation, λ is set to 0.95.

4 Econometrics

4.1 Estimation

The structural parameters, θ, are fitted using simulated methods of moments. Let $\hat{h}$ define the vector of moments observed in the data and $h(\theta, r)$ define the simulated moments at the current estimate of θ and r. Define the distance between the observed data moments and the simulated counterparts as $g(\theta, r) := \hat{h} - h(\theta, r)$.

To fit the structural parameters we solve the following optimisation problem

$$\begin{aligned} \theta^\star, r^\star &= \arg\min_{\theta, r} g(\theta, r)^\top \Sigma^{-1} g(\theta, r) \\ s.t. \quad & \Phi r = \Pi T(\Phi r) \end{aligned}$$

where Σ is a positive definite symmetric weighting matrix and Π is the projection with respect to the weighted Euclidean norm $\|\cdot\|_\omega$ with $\omega > 0 \; s.t \sum \omega = 1$.

To implement the estimation we use adaptive Markov chain MC derivative free optimisation.[47] As noted by Imai et al. (2009) and Norets (2009) solving the dynamic demand model at every parameter guess is very costly. Their suggested remedy is to alternate between iterations of fitting the structural parameters and do dampened updates of the

[46] See Bertsekas (2011a). His proof is restricted to finite states for ergodic Markov Chains. Ma and Powell (2011) extend this result to continuous space ADP whose Markov transition functions are positive Harris Chains.

[47] For details of the adaptive MCMC algorithm used see Łącki and Miasojedow (2015).

value function.[48] In line with this approach, we alternate between fitting the structural parameters, θ, and solving the dynamic demand model by doing a single iteration of λ-PI described in section 3.5.

At the start of iteration k, the value of structural parameters is θ_k and the current state of the solution to the dynamic program is captured by the parameters r_k. The objective function at the beginning of the iteration is

$$Q(\theta_k, r_k) := g(\theta_k, r_k)^\top \Sigma^{-1} g(\theta_k, r_k)$$

From the proposal density for the structural parameters draw $\tilde{\theta}_k$ and compute a new objective function, $Q(\tilde{\theta}_k, r_k)$. Then, accept or reject the draw of the structural parameters by comparing $Q(\tilde{\theta}_k, r_k)$ to $Q(\theta_k, r_k)$. If the draw is accepted, set $\theta_{k+1} = \tilde{\theta}_k$. If not, leave the structural parameters unchanged.

Next we do a single iteration of the λ-PI algorithm at θ_{k+1}. Applying the policy evaluation and policy improvement steps described in sections 3.5.1 and 3.5.2 yields r_{k+1}. To prepare the algorithm for the next iteration, the objective function needs to be recalculated as $Q(\theta_{k+1}, r_{k+1})$. This is used in the accept-reject decision for the new structural parameter draws at r_{k+1}

Conducting three separate simulations at each iteration is likely to be computationally burdensome. Clearly the simulation used to evaluate the draw is unavoidable and updating the ADP at each iteration is highly advisable. However, as highlighted by Imai et al. (2009), even if the objective function is not re-evaluated after the λ-PI step, we may have a good enough approximation to the objective function for the purpose of deciding whether to accept or reject the next structural parameters draw in the next iteration.

Following the suggested approach in Imai et al. (2009) a third simulation is conducted if the existing objective function is likely to be a poor approximation to $Q(\theta_{k+1}, r_{k+1})$. If $\tilde{\theta}_k$ is accepted then there is no need to re-simulate. The value of the objective function used in the structural parameter update can be used as an approximation; $Q(\theta_{k+1}, r_{k+1}) \approx Q(\theta_{k+1}, r_k)$.

If the draw is rejected the quality of the approximation to $Q(\theta_{k+1}, r_{k+1})$ may depend on how many iterations have passed since it was last updated. If a draw has been accepted in last $i \leq \bar{n}$ iterations, the objective function is left unchanged. That is, $Q(\theta_{k+1}, r_{k+1}) \approx Q(\theta_k, r_k)$ for $1 \leq i < \bar{n}$. After $i > \bar{n}$ successive rejections, after the λ-PI step in iteration k, the objective function is re-evaluated at (θ_{k+1}, r_{k+1}). In the algorithm, if there is no update in the last five iterations, the objective function is re-evaluated.[49]

[48] Intuitively this avoids the costly step of fitting the solution to a dynamic program whose parameters may be far away from the true parameters. Moreover, they show that the alternating procedure converges to the same posterior distribution.

[49] Given that the adaptive Markov Chain MC targets an accepted draw every 234 per 1000 draws, a 'natural' update occurs approximately once every 5 draws.

4.2 Identification

Like other dynamic demand models of storable goods, formal identification of the model is complex. As such, and in line with this existing literature, we provide an informal discussion of identification of model parameters.

In section 2, data on inter-purchase durations, current and past prices, quantities purchased, and the sequences of SKUs chosen was used to demonstrate that price expectations, inventory holdings, and taste heterogeneity are important features of demand for laundry detergent in the UK. This data also identifies the parameters of the model.

The marginal utility of income is identified by standard arguments using variation in prices over markets.

Inventory costs are identified by comparing inter-purchase durations of households. To illustrate, consider two households that always purchase detergent in one particular format. Over the same time period, they purchase the same number of washes. However, because one household has higher inventory costs than the other, we observe that they purchase smaller SKUs more frequently. As such, comparing inter-purchase duration for households with the same consumption rate identifies storage costs. To identify inventory costs differences between formats, this analysis can be conditioned on format purchased.

For a particular detergent j, the ratio ψ_j/Ψ_{jj} is an important determinant of the rate of consumption. Holding fixed ψ_j, Ψ_{jj} can be identified by comparing inter-purchase durations of households conditional on purchasing the same quantity of detergent j. To illustrate these ideas, consider two households who face the same prices and always buy the same number of washes. However, one household consistently purchases less frequently than the other. Therefore, over the same period the household that purchases more frequently will consume more washes. That is, it consumes detergent washes at a higher rate. Therefore, Ψ_{jj} can be identified by comparing inter-purchase durations of households conditional on purchasing the same quantity of detergent j. Conditioning this analysis on consecutive purchases for each detergent identifies all diagonal elements of Ψ.

In the above discussion, we held ψ_j fixed. This is because, in addition to impacting the rate of consumption, ψ_j directly impacts on the level of utility from consumption of detergent j. In turn, the linear utility weights are important parameters for matching market shares. As such, observed market shares will aid identification of ψ over and above consumption rates.

Interaction between quantities purchased, duration between purchases and identities of consecutive purchases of detergent help identify off diagonal terms in Ψ. This is because households that tend to use different detergents for different types of washing are likely to maintain more than one type of inventory. Since detergent is costly to store, all else equal, households with two sets of inventory will tend to purchase smaller SKUs of different inventories at close intervals. Therefore, by comparing the joint distribution of quantities purchased and inter-purchase duration of households whose consecutive purchases are of different detergents to those who purchase the same detergent helps identify off-diagonal terms in Ψ.

5 Empirical Results

The model is estimated in two steps:

1. Estimate the household's price forecasting model
2. Estimate and solve the dynamic demand model

5.1 Price Model

All purchases recorded in Tesco by Kantar between 2009 and 2011 are used to estimate the price forecasting model. The underlying assumption is that all households in the sample observe the same set of prices. For the UK this is actually the case. Following a ruling by the UK Competition Commission in 2000, all large supermarkets must charge the same price for a given SKU.[50]

Because the data is only recorded when households make purchases, the price series for each SKU is only partially observed. While, the interactive fixed effects model can be used to impute missing prices, the panel is too unbalanced for this to work well in practice. Instead SKUs of a similar size are grouped together. This is because they are likely to exhibit similar promotional activity.[51] For each of the 37 types of detergent we assign SKUs to one of four groups: less that 18 washes, between 18 and 24 washes, 25 to 40 washes and more than 40 washes. This results in 112 groups of SKUs.

The dependent variable is a monotonic transformation of the price per wash that take the support of price per wash in to the real line; $z(\cdot) : (0, \frac{1}{2}) \rightarrow \mathbb{R}$.[52]

The interactive fixed effects model is estimated for up to three factors using non-linear least squares. In each case we estimate a vector autoregression with up to four lags.

To choose the optimal number of factors use the information criteria proposed by Bai and Ng (2002). These are shown in the top panel of 2. Two of of the information criteria are minimised by a price model with two factors, the other with three factors.

The bottom panel reports the results of the Schwarz-Bayes information criterion (SBIC) when up to four lags are included in the vector-autoregression for the factors from each of the interactive fixed effect models. When there is one factor, the SBIC is minimised by including two lags. Otherwise, the SBIC is minimised by choosing one lag.

[50]Some price variation is permitted for 'small' supermarkets that are similar in size to convenience stores (less than 280 sq m).

[51]For example, very large SKUs are offered at quantity discounts and are therefore not promoted very regularly.

[52]Specifically, $z(p_{mt}) = \text{logit}(\frac{2p_{m,t}}{q_{j,m}})$.

Table 2: Price Model: interactive fixed effects model

Number of Factors	R=1	R=2	R=3	$R^\star$
ln(Obj. Func.)+R×Pen. Fn.				
Penalty 1: $\frac{N+T}{NT} log(\frac{NT}{N+T})$	-2.064	-2.109	-2.102	2
Penalty 2: $\frac{N+T}{NT}\log(C)$	-2.052	-2.084	-2.065	2
Penalty 3: $\log(C)/C$	-2.094	-2.170	-2.193	3
Number of Lags, SBIC				
τ=1	-8.241	-16.620	-25.481	
τ=2	-8.242	-16.574	-25.321	
τ=3	-8.227	-16.492	-25.089	
τ=4	-8.190	-16.364	-24.848	
$\tau^\star$	2	1	1	

Note: $C = \min\{N, T\}$

Taken together, these criteria suggest that either a two or three factor model with one lag would could be used. Since there is an additional premium to increasing the state space we opt for the 2 factor model with 1 lag.[53] The results of this interactive fixed effect model and the $VAR(1)$ applied to the factors are shown in Tables 3 and 4, respectively. When translated back into price per wash, 90 percent of the fitted average price per wash for the 112 groups are within 2% of the average price per wash in the data. Therefore, the model fits average prices well over brands, formats and sizes.
The results of the vector-autoregression show that factors, like prices, are positively serially correlated.

[53]The results of coefficients of all three interactive fixed effect models are shown in Annex B.

Table 3: Factor Model: Interactive fixed effect model with 2 factors

		Estimate	Std. Error
Formats (excl. Caps):			
	Powder	-0.449	(0.119)
	Liquid	-0.709	(0.148)
	Tablets	-0.574	(0.147)
	Gel	-0.589	(0.119)
Brands (excl. Ariel)			
	Bold	0.068	(0.110)
	Daz	-0.539	(0.106)
	Fairy	0.277	(0.106)
	Persil	-0.688	(0.170)
	Surf	-1.032	(0.175)
	Tesco	-1.069	(0.112)
	Others	0.377	(0.243)
Other Chars (excl. Single SKU)			
	Multi-Pack	0.007	(0.118)
	Washes	-0.383	(1.708)
	Washes^2	-1.106	(2.439)
	Washes^3	0.808	(1.267)
Constant		0.689	(0.196)
R^2: Overall		0.710	
R^2: Within		0.132	
Num. Obs.		14,927	

Table 4: Factor Model: VAR(1) with 2 factors

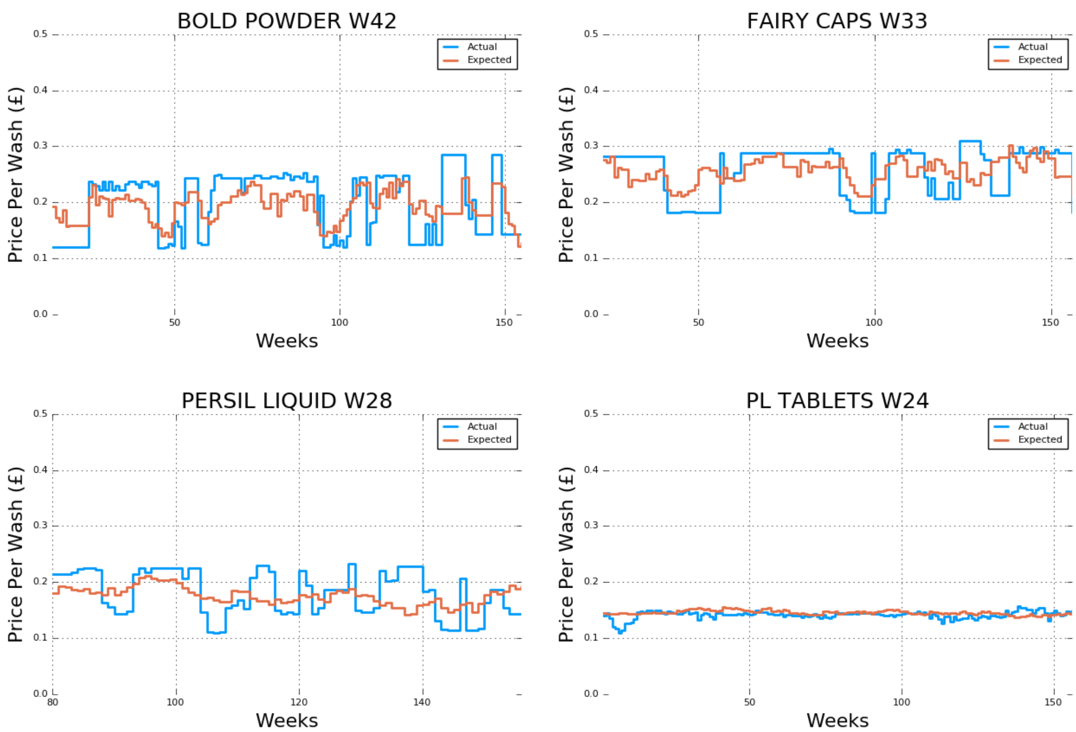
	$F_{1,t}$		$F_{2,t}$	
	Estimate	Std. Err	Estimate	Std Error
$F_{1,t-1}$	0.795	(0.067)	0.192	(0.163)
$F_{2,t-1}$	0.054	(0.025)	0.884	(0.061)
Constant	0.017	(0.006)	-0.018	(0.014)

To explore how the price model matches time series variation in the data we plot the expected price per wash of the four SKUs from Figure 3. The expected price per wash is calculated by taking the observed factor from the previous period and calculating the one-step ahead forecast. This is then plugged into the interactive fixed effect model to calculate the expected price per wash. The results shown in Figure 6 demonstrate that the price forecasting model can effectively capture a different inter-temporal pricing patterns.

The upper left panel of Figure 6 shows that the price per wash for Bold powder with 42

washes are predicted well in terms of both depth and timings of promotions. Similarly, the 33 wash Fairy capsules SKU (top-right) also predict observed prices quite well. Both of these SKUs are promoted relatively infrequently.
Persil liquid with 28 washes (bottom-left) is a frequently promoted SKU. Rather than closely track observed prices, the predicted prices tend to lie close to the average price. This is consistent with the household effectively randomising over the likelihood of a sale in the next period - perhaps unsurprising for a SKU promoted so often.
The predicted price for Tesco's private label 24 wash tablets SKU (bottom-right) is flat because it is almost never promoted. By setting its factor loadings to close to zero and matching the average price per wash well through the hedonic part of the model, the households can easily capture the everyday low price strategy typical of Tesco's private label products.

Figure 6: Selected price per wash time series



5.2 Dynamic Demand Model

The model is estimated using data household purchase diary data described in section 2. For an individual household, the parameter space is $\theta = [\psi, \Psi, \psi_0, \rho_0, \rho_1, \rho_2, \gamma, \Gamma, \kappa_1, \kappa_2]$ with $J^2 + J + 5$ parameters. With 37 detergents, there are 1,411 parameters. To ensure that the resulting model is estimable, a more parsimonious parameterisation of the model is needed.

5.2.1 Parameterisation

Household's flow utility is slightly modified for estimation. There are two main alterations to eq (8). First the marginal utility of income is included in the parameters of the purchase and inventory costs. Second, since income is constant across purchase alternatives it is omitted from the model. The resulting flow utility function is

$$U(C) - \psi_0 d^\top p - PC(Qd) - IC\left(\bar{I} - C\right) + d^\top \varepsilon \tag{22}$$

Turning to the parameters themselves, the detergent specific linear utility weights are the same across all households and are constrained to be strictly positive. The (j,k)-th entry for quadratic utility weights for all $j,k = 1,\ldots,J$ are

$$\ln \Psi_{jk} = \frac{1}{2}\ln\psi_j + \frac{1}{2}\ln\psi_k + \ln\upsilon_{jk} - \ln(\alpha_1 + \alpha_2 Z) \tag{23}$$

where Z is the number of equivalent adults and $\upsilon_{jj} = 1$ and $\upsilon_{jk} \in (0,1)$ for all $j \neq k$. The final term allows the consumption rate to vary with household size.

The price coefficient is allowed to vary across households and is function of income per equivalent adults

$$\psi_0 = f_Y\left(\mu - \sigma\frac{\tilde{Y}}{Z}\right) \tag{24}$$

where $\frac{\tilde{Y}}{Z}$ is a standardised income per equivalent adult, $\sigma > 0$, and $f_Y : \mathbb{R} \to \mathbb{R}^+$.

Fixed costs of purchase and the costs of stocking several detergent are identified, inter alia, by the frequency of purchase. As such, their effects on utility may be difficult to separate in practice. For this reason fix $\rho_0 = 0$ and estimate κ_1 and $\kappa_2 \geq 0$.

The increment to costs from the linear components of both purchase and inventory costs depend on the quantity of washes purchased. As such it is difficult to separately identify γ and ρ_1. For this reason we fix $\gamma = 0$. The quadratic parameters of purchase and inventory costs are separately identified. Incremental inventory costs depend on both existing inventory and quantity purchase, whereas purchase costs depend on the the latter.

The quadratic inventory parameters are a function of the detergent's format and are constrained to be strictly positive. The (j,k)-element of Γ is $\Gamma_{j,k} = \beta_j\beta_k$ where $\beta_j > 0$ is the cost of storing detergent $j = 1,\ldots,J$. If j and k have the same format, then $\beta_j = \beta_k$. Finally, the quadratic purchase cost parameter, ρ_2 is constrained to be strictly positive.

5.2.2 Results

The results of the estimation are show in Table 5.[54] For the purpose of this calibration the quadratic parameter on purchase costs are fixed at zero. Also, the parameters on the number of items held in stock, κ_1 and κ_2 are also fixed at zero.

[54]The results presented here are an interim output of the estimation used to calibrate the model. As a result there are no standard errors.

Table 5: Model results

	Linear Utility Parameters				
	Formats				
Brands	Caps	Gel	Liquid	Powder	Tablets
Ariel	7.653	2.301	9.627	3.923	8.938
	(-)	(-)	(-)	(-)	(-)
Bold	5.324	3.789	3.540	7.006	2.092
	(-)	(-)	(-)	(-)	(-)
Daz	5.758		2.422	2.311	6.048
	(-)		(-)	(-)	(-)
Fairy	8.506	0.552	4.855	6.136	0.162
	(-)	(-)	(-)	(-)	(-)
Persil	4.320		1.352	3.983	2.597
	(-)		(-)	(-)	(-)
Surf	3.343		7.326	5.821	1.190
	(-)		(-)	(-)	(-)
Private Label	0.184	1.722	1.772	4.411	8.277
	(-)	(-)	(-)	(-)	(-)
Other Brands	0.475	6.813	0.909	3.730	0.1254
	(-)	(-)	(-)	(-)	(-)
	Quadratic Utility Parameters				
υ	0.731				
	(-)				
α_1	0.861				
	(-)				
α_2	1.596				
	(-)				
	ψ_0 Parameters				
μ	9.405				
	(-)				
σ_0	1.978				
	(-)				
	Purchase Cost Parameters				
ρ_0	0				
	(-)				
ρ_1	6.665				
	(-)				
ρ_2	0				
	(-)				
	Inventory Cost Parameters				
	Caps	Gel	Liquid	Powder	Tablets
γ	0	0	0	0	0
	(-)	(-)	(-)	(-)	(-)
Γ:	6.541	4.325	4.954	1.678	2.104
	(-)				
κ_1	0				
	(-)				
κ_2	0				
	(-)				
	Other Parameters				
Scale of utility	0.495				
	(-)				
δ	0.99				
	(-)				

To interpret the fit of the model it is instructive to see how well it fits the data. There are three main sets of moments: the brand shares, format shares, and inter-purchase duration.
Figure 7 plots the share of each brand purchased in a simulation of the model against the the share purchased in the data. The red line is the 45 degree line. This analysis partitions the sample into income per equivalent adult quartiles.
The fit of shares is reasonably good for the wealthiest two quartiles of households (top row). Only Bold shares are materially overstated by the model at the expense of Persil. However, as noted in section 2, these brands are quite similar in terms of their perceived quality and price levels.
For the poorest quartiles (bottom-row), the model captures the fact that they purchase more Tesco private label that brands - especially the very poorest households. For these households the share of cheaper brands, like Daz, Surf and Persil are a little understated while the premium brands, Ariel, Bold and Fairy, tend to be slightly overstated.

Figure 7: Brand Shares by Income Quartile

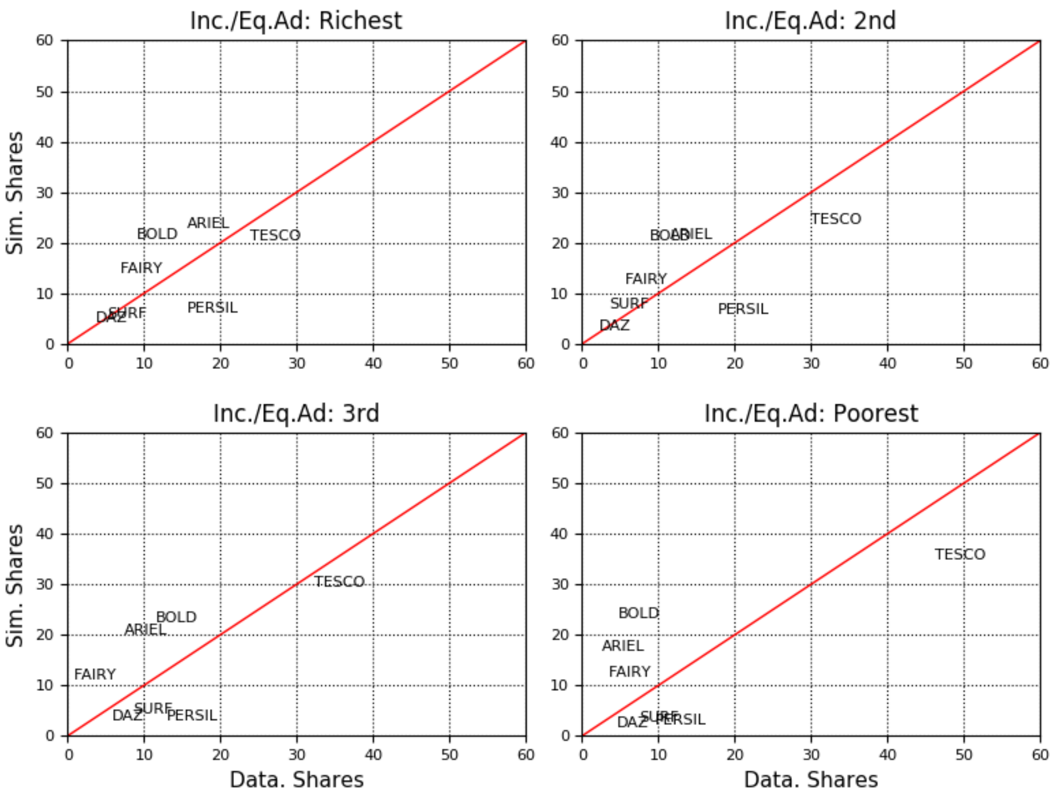


Figure 8 replicates Figure 7 but for the detergent formats. It shows that the share of tablets purchased is overstated in the model for all income groups. For the most wealthy households, the share of powder detergent purchased is overstated by the model, while for the poorest households the share of liquid detergent sales is understated.

Figure 8: Format Shares by Income Quartile

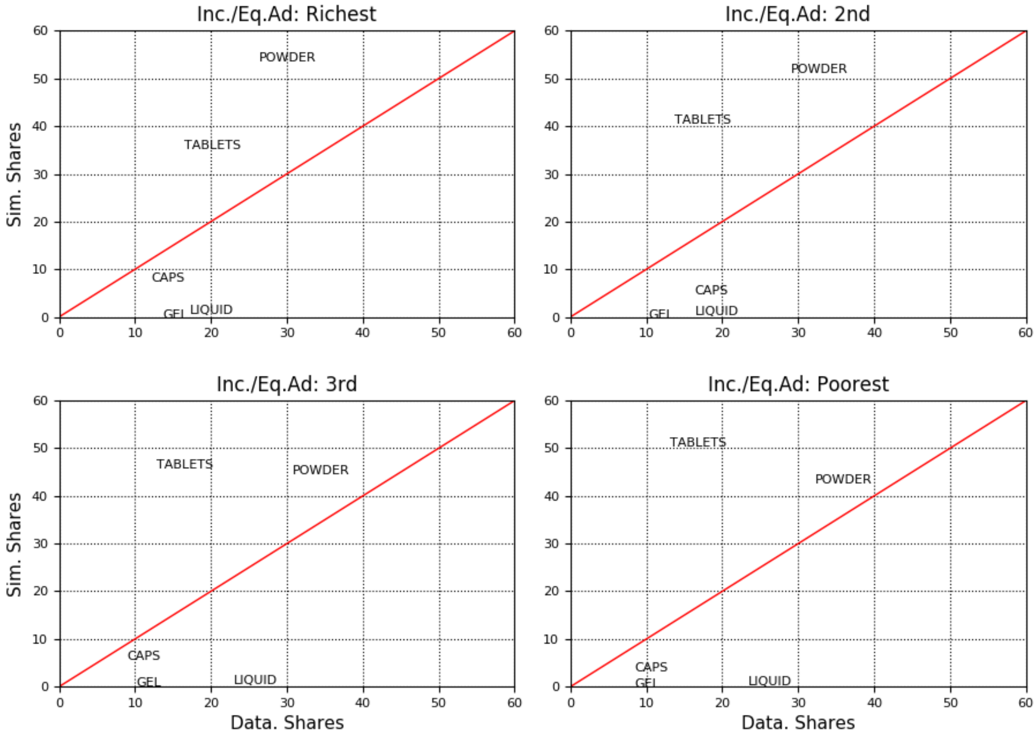
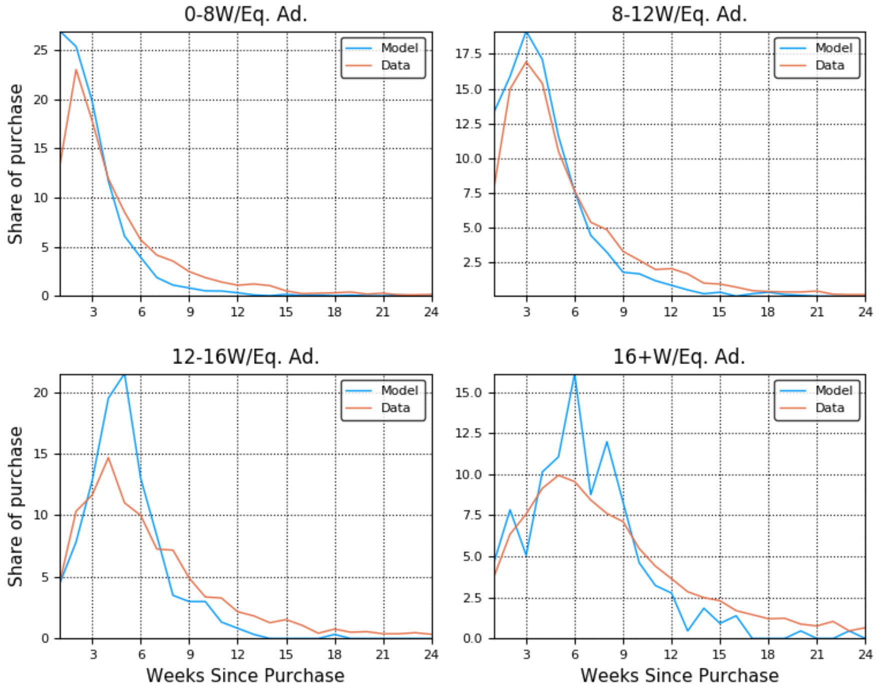


Figure 9 shows the distribution of inter-purchase duration condition on the amount of detergent per equivalent adult last purchased in the model and in the data. The red line corresponds to Figure 5 in section 2.

Overall, it is clear that the model does a good job reproducing these important demand dynamics. In each case, the simulated inter-purchase distribution shifts rightward as more washes per equivalent adult are purchased. In turn, indicating that having higher existing inventories reduces the propensity to purchase in the model.

Figure 9: Inter-purchase duration by income per equivalent adult



6 Conclusion

This paper develops a dynamic discrete-continuous demand model for storable goods - a class of fast moving consumer goods that account for a large fraction of grocery expenditures. It is applied to the UK laundry detergent industry using household level purchase data.

To estimate and solve the dynamic demand model, we use techniques from: (i) Approximate Dynamic Programming (ADP), (ii) large scale dynamic programming in economics, (iii) machine learning, and (iv) statistical computing. The benefits of this approach are three-fold.

First, the dynamic demand model is compatible high-dimensional dynamic choice sets. In turn, making dynamic demand estimation possible for storable good industries with many sizes - the UK laundry detergent industry is an example.

Second, the model can combine the most desirable features of existing models. In particular, it allows for persistent taste heterogeneity to interact with product varieties in flow utility and continuation values. Furthermore, utility from product differentiation accrues the point of consumption, not purchase. Together these features enable the model to capture rich inter- and intra temporal substitution patterns.

Finally, these dimension reduction techniques do not hinge idiosyncratic features of the industry being studied nor do they impose restrictive assumptions on purchase decisions and/or consumption. This means this dynamic demand can be applied to any storable good industry with only minor modifications.

This model is likely to be of both policy and commercial interest. In a policy setting, understanding how consumers react to short and long run price dynamics may be important for effective design of taxation policy. In addition, consistent estimation of short and medium to long run elasticities is a key input into antitrust analysis of merger analysis, assessment of cartel damages, etc. Finally, these storable demand can be used to construct new cost-of-living indices to reflect differences in the prices recorded in baskets of goods and purchases prices (Osborne (2017)).

From a commercial perspective, this structural dynamic demand model can be applied to consumer level purchase data and estimated using ever increasing computational resources. The resulting demand model enables firms to better understand demand dynamics - a key input into optimisation of promotional price strategies and demand forecasting. Moreover, it provides a new way to explore counterfactual market outcomes when new products are introduced or old products are withdrawn.

Annex A: Equivalence of the two stage problem

In recursive form, the household's choice problem is

$$\begin{aligned} V(I,P,Y,\varepsilon) &= \max_{d} \max_{0\le C\le I+Qd} U(C)+\psi_0\left(Y-d^{T}p-PC(Qd)-IC(I+Qd-C)\right) \\ &\quad +\delta\int\int V\left(I+Qd-C,P^{'},Y,\varepsilon^{'}\right)dG_{\varepsilon'}dG_{p'|P}+d^{\top}\varepsilon \end{aligned} \tag{25}$$

where $\varepsilon \overset{iid}{\sim}$ Type I EV and $\tau=1$ so $p'\sim G_{p'|P}$.
Integrating the value function over SKU specific shocks, ε,

$$\bar{V}(I,P,Y) := \int V(I,P,Y,\varepsilon)\,dG_{\varepsilon} \tag{26}$$

$$\begin{aligned} &= \ln\sum_{d}\exp\{\max_{0\le C\le I+Qd} U(C)+\psi_0\left(Y-d^{\top}p-PC(Qd)-IC(I+Qd-C)\right) \\ &\quad +\delta\int\int V\left(I+Qd-C,P^{'},Y,\varepsilon^{'}\right)dG_{\varepsilon'}dG_{p'|P}\} \end{aligned} \tag{27}$$

Define $W(I,P,Y)$ as the indirect utility function conditional on purchasing SKU m

$$\begin{aligned} W(I,P,Y) &:= \max_{0\le C\le I+Qd} U(C)+\psi_0\left(Y-d^{\top}p-PC(Qd)-IC(I+Qd-C)\right) \\ &\quad +\delta\int \bar{V}\left(I+Qd-C,P^{'},Y^{'}\right)dG_{p'|P} \end{aligned} \tag{28}$$

Then substituting eq (28) into eq (27)

$$\bar{V}(I,P,Y) = \ln\sum_{m=1}^{M}\exp\{W(I,P,Y)\} \tag{29}$$

Rolling forward eq (29) next period's integrated value function

$$\bar{V}\left(I^{'},P^{'},Y^{'}\right) = \ln\sum_{m'=1}^{M}\exp\left\{W\left(I^{'},P^{'},Y^{'}\right)\right\} \tag{30}$$

Substituting eq (30) into eq (28) yields a Bellman Equation,

$$\begin{aligned} W(I,P,Y) &= \max_{0\le C\le I+Qd} U(C)+\psi_0\left(Y-d^{\top}p-PC(Qd)-IC(I+Qd-C)\right) \\ &\quad +\delta\int \ln\sum_{m'=1}^{M}\exp\left\{W\left(I+Qd-C,P^{'},Y^{'}\right)\right\}dG_{p'|P} \end{aligned} \tag{31}$$

Finally, we show that the household's discrete choice problem is to choose the largest indirect utility function once SKU specifics are realised

$$
\begin{aligned}
V\left(I,P,Y,\varepsilon\right) &= \max_{d}\max_{0\leq C\leq I+Qd} U\left(C\right)+\psi_0\left(Y-d^{\top}p-PC(Qd)-IC\left(I+Qd-C\right)\right) \\
&\quad +\delta\int\int V\left(I+Qd-C,P^{'},Y,\varepsilon^{'}\right)dG_{\varepsilon'}dG_{p'|P}+d^{\top}\varepsilon \qquad (32)\\
&= \max_{d}\max_{0\leq C\leq I+Qd} U\left(C\right)+\psi_0\left(Y-d^{\top}p-PC(Qd)-IC\left(I+Qd-C\right)\right) \\
&\quad +\delta\int\bar{V}\left(I+Qd-C,P^{'},Y^{'}\right)dG_{p'|P}+d^{\top}\varepsilon \qquad (33)\\
&= \max_{d} W\left(\bar{I},P,\bar{Y}\right)+d^{\top}\varepsilon \qquad (34)
\end{aligned}
$$

where the first line is eq (25). Eq (30) is substituted into the second line. Finally, we can substitute in eq (31) and represent ε element-wise. This gives the desired expression.

Annex B: Empirical Results

B.1 Interactive fixed effects model

Table 6: Price Model: interactive fixed effects model

		R=1	R=2	R=3
Formats:	Powder	-0.219	-0.449	-1.812
		(0.057)	(0.119)	(2.027)
	Liquid	-0.234	-0.709	-0.027
		(0.107)	(0.148)	(0.832)
	Tablets	-0.137	-0.574	-1.265
		(0.097)	(0.147)	(1.433)
	Gel	-0.396	-0.589	-1.822
		(0.062)	(0.119)	(1.905)
Brands:	Bold	-0.143	0.068	0.673
		(0.073)	(0.110)	(1.190)
	Daz	-0.463	-0.539	0.388
		(0.081)	(0.106)	(1.268)
	Fairy	0.120	0.277	0.795
		(0.057)	(0.106)	(0.916)
	Persil	-0.312	-0.688	-1.693
		(0.093)	(0.170)	(1.855)
	Surf	-0.712	-1.032	-1.436
		(0.089)	(0.175)	(1.229)
	Tesco	-1.020	-1.069	-1.466
		(0.090)	(0.112)	(0.964)
	Others	-0.828	0.377	-8.487
		(0.264)	(0.243)	(8.971)
Other Chars.:	Multipack	0.18	0.007	0.024
		(0.049)	(0.118)	(0.156)
	Washes	-2.922	-0.383	-0.465
		(0.975)	(1.708)	(2.316)
	Washes^2	2.096	-1.106	-1.050
		(1.758)	(2.439)	(3.261)
	Washes^3	-0.684	0.808	0.815
		(1.016)	(1.267)	(1.666)
Constant		0.789	0.689	1.551
		(0.138)	(0.196)	(1.363)
R^2: Overall		0.674	0.710	0.728
R^2: Within		0.364	0.132	0.061
Num. Obs.		14,927	14,927	14,927

B.1 Demand Model

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