Estimating dynamic diversion ratios in storable good industries

Online Appendix

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A. Dynamic GUPPI

This appendix derives approximations to static and dynamic GUPPI using product margins.

A.1. Static demand and supply

In this section the GUPPI for an industry without demand or supply dynamics is derived. GUPPI measured over multiple period is derived from first order condition of the workhorse differentiated Bertrand price setting game. This makes clear the nature of the approximation of the GUPPI index typically used in practice when their are no demand or supply dynamics.

A.1.1. Single Period

Consider the first order conditions of two single product firms

$$q_{jt} + \Delta^{q}_{jjt} m_{jt} = 0 \tag{1}$$
$$q_{kt} + \Delta^{q}_{kkt} m_{kt} = 0 \tag{2}$$

The optimal markup for firm j is

$$m_{jt} = -\frac{q_{jt}}{\Delta^{q}_{jjt}} \tag{3}$$

The corresponding optimal margin is

$$\mu^{PRE}_{jt} := \frac{m_{jt}}{p_{jt}} = -\frac{1}{\varepsilon^{SR}_{jjt}} \tag{4}$$

Now merge j and k. First order effects for j internalising recaptured sales from k

$$
\begin{aligned}
q_{jt} + \Delta^q_{jjt} m_{jt} + \Delta^q_{jkt} m_{kt} = 0 \quad (5)\\
\implies \Delta^q_{jjt} m_{jt} = -q_{jt} - \Delta^q_{jkt} m_{kt} \quad (6)\\
\implies m_{jt} = -\frac{q_{jt}}{\Delta^q_{jjt}} - \frac{\Delta^q_{jkt}}{\Delta^q_{jjt}} m_{kt} \quad (7)
\end{aligned}
$$

In each period, GUPPI measures the strength of the competitive constraint internalised in the first rounds of effects

$$
\begin{aligned}
\mu^{POST}_{jt} := \frac{m_{jt}}{p_{jt}} &= -\frac{1}{\varepsilon^{SR}_{jjt}} - \frac{\Delta^q_{jkt}}{\Delta^q_{jjt}} \frac{m_{kt}}{p_{jt}} \quad (8)\\
&= \mu^{PRE}_{jt} + GUPPI_{jkt} \quad (9)
\end{aligned}
$$

A.1.2. Multiple Periods

In practice, the margins used to populate GUPPI in practice span multiple periods. To measure pre-merger market power accruing over multiple periods, sum profits over the T periods

$$
\sum_{t=1}^{T} q_{jt} m_{jt} = -\sum_{t=1}^{T} \frac{q^2_{jt}}{\Delta^q_{jjt}} \quad (10)
$$

Let $R_j := \sum_{t=1}^{T} q_{jt} p_{jt}$ be total revenues from sales of product j over T periods and define μ^{PRE}_j as the percentage margin earned on j's sales over T periods

$$
\begin{aligned}
\mu^{PRE}_j := \frac{\sum_{t=1}^{T} q_{jt} m_{jt}}{\sum_{t=1}^{T} q_{jt} p_{jt}} &= -\sum_{t=1}^{T} \frac{q^2_{jt}}{R_j} \frac{1}{\Delta^q_{jjt}} = -\sum_{t=1}^{T} \frac{p_{jt} q_{jt}}{R_j} \frac{1}{\varepsilon^{SR}_{jjt}} \quad (11)\\
&= -\sum_{t=1}^{T} \sigma_{jt} \frac{1}{\varepsilon^{SR}_{jjt}} \quad (12)
\end{aligned}
$$

where σ_{jt} is period t's share of product j's revenue over T periods.

Now merge j and k and again sum profits over the T periods

$$
\sum_{t=1}^{T} q_{jt} m_{jt} = -\sum_{t=1}^{T} \frac{q^2_{jt}}{\Delta^q_{jjt}} - \sum_{t=1}^{T} q_{jt} \frac{\Delta^q_{jkt}}{\Delta^q_{jjt}} m_{kt} \quad (13)
$$

Define μ^{POST}_j as the percentage margin earned on j's sales over T periods once the first round effects of merging with product k are taken into account

$$\mu_j^{POST} := \frac{\sum_{t=1}^T q_{jt}m_{jt}}{\sum_{t=1}^T q_{jt}p_{jt}} = -\sum_{t=1}^{T} \frac{q_{jt}^2}{R_j}\frac{1}{\Delta_{jjt}^q} - \sum_{t=1}^{T} \frac{q_{jt}}{R_j}\frac{\Delta_{jkt}^q}{\Delta_{jjt}^q} m_{kt} \tag{14}$$

$$= -\sum_{t=1}^{T} \frac{p_{jt}q_{jr}}{R_j}\frac{1}{\varepsilon_{jjt}^{SR}} - \sum_{t=1}^{T} \frac{p_{jt}q_{jt}}{R_j} DR_{jkt}\frac{m_{kt}}{p_{jt}} \tag{15}$$

$$= -\sum_{t=1}^{T} \sigma_{jt}\frac{1}{\varepsilon_{jjt}^{SR}} - \sum_{t=1}^{T} \sigma_{jt} DR_{jkt}\frac{m_{kt}}{p_{jt}} \tag{16}$$

$$= \mu_j^{PRE} + \sum_{t=1}^{T} \sigma_{jt} GUPPI_{jkt} \tag{17}$$

The long-run measure of the GUPPI is the revenue weighted sum of the GUPPI in each period

$$GUPPI_{jk}^{LR} = \sum_{t=1}^{T} \sigma_{jt} GUPPI_{(j,k),t} \tag{18}$$

To obtain long-run measures of pricing pressure, an analyst can, in principle, estimate a static demand model and back out marginal cost in each period given a model of supply in each period. Then, by combining prices, estimated marginal costs and estimates of diversion ratios from the demand model the analyst can compute the revenue-weighted sum of $GUPPI_{jkt}$ to measure $GUPPI_{jk}^{LR}$.

In practice, however, margins derived from internal accounts are used to compute price pressure indices.[1] Since accounts are costly to compile, financial statistics are produced periodically. Implicitly, therefore, these margins (approximately) measure market power over multiple products and across a period of time (i.e. brand-level annual gross margins). As such, the estimate of market power reflects a sequence of pricing decisions for multiple products spanning several time periods.

Assuming the analyst has obtained an estimate of the percentage margin earned on sales of product k over T periods, $\hat{\mu}_k$, they can approximate the $GUPPI_{jk}^{LR}$ using DR_{jk}^{π} is the diverted value ratio evaluated at (revenue) weighted average prices and market shares over T periods

$$GUPPI_{jk}^{LR} \approx DR_{jk}^{\pi}\mu_j \tag{19}$$

where $DR_{jk}^{\pi} = DR_{jk}\frac{\mu_k \bar{p}_k}{\mu_j \bar{p}_j}$.

A.2. Dynamic case

Next, the dynamic GUPPI is derived starting from the first order conditions in the dynamic demand and price setting model from section ??. As for the static case, dGUPPI is derived for a single period, then generalised to multi-period setting. Finally, analogous to the approximation implicitly used in the workhorse static demand and supply model, a practical approximation to a long-run measure of dGUPPI is proposed.

[1] In the simplest case these are based on gross-margins produced by internal accounts. The can also be estimated from accounting data using the methodology in [?].

A.2.1. Single Period

Consider the first order conditions assuming there are two single product storable good firms that face dynamic consumer demand and engage in dynamic price competition

$$q_{jt} + \Delta^q_{jjt}(1+\Psi_{jjt})m_{jt} + \Upsilon_{jjt} = 0 \tag{20}$$

$$q_{kt} + \Delta^q_{kkt}(1+\Psi_{kkt})m_{kt} + \Upsilon_{kkt} = 0 \tag{21}$$

Pre-merger firm j in period t

$$\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)m_{jt} = -q_{jt} - \Upsilon_{jjt} \tag{22}$$

$$\implies m_{jt} = -\frac{q_{jt} + \Upsilon_{jjt}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)} \tag{23}$$

Then the pre-merger margin for product j in period t is

$$\mu^{PRE}_{jt} := \frac{m_{jt}}{p_{jt}} = -\frac{1}{\varepsilon^{SR}_{jjt}\left(1+\Psi_{jjt}\right)} - \frac{\Upsilon_{jjt}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)p_{jt}} \tag{24}$$

$$= -\frac{1}{\varepsilon^{LR}_{jjt}} - \frac{\Upsilon_{jjt}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)p_{jt}} \tag{25}$$

where ε^{SR}_{jjt} is the short-run price elasticity of demand for product j in period t.
Now merge j and k. The first order condition for j internalising recaptured sales from k

$$q_{jt} + \Delta^q_{jjt}\left(1+\Psi_{jjt}\right)m_{jt} + \Upsilon_{jjt} + \Delta^q_{jkt}\left(1+\Psi_{jkt}\right)m_{kt} + \Upsilon_{jkt} = 0 \tag{26}$$

$$q_{jt} + \sum_{i\in\{j,k\}} \Delta^q_{jit}\left(1+\Psi_{jjt}\right)m_{it} + \sum_{i\in\{j,k\}} \Upsilon_{jit} = 0 \tag{27}$$

Rearranging, gives and expression for the mark-up for product j

$$m_{jt} = -\frac{q_{jt}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)} - \frac{\Delta^q_{jkt}\left(1+\Psi_{jkt}\right)m_{kt}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)} - \sum_{i\in\{j,k\}} \frac{\Upsilon_{jit}}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)} \tag{28}$$

To simplify notation, let Z_{jkt} := -$\frac{\Upsilon_{jkt}}{\Delta^q_{jjt}(1+\Psi_{jj})p_{jt}}$ be the effect on margins from the internalisation of new expected revenues accruing from changes in future prices to product k as a result of p_{jt} changing today. Now, using this notation, the post-merger margin for product j in period t is

$$\mu^{POST}_{jt} := \frac{m_{jt}}{p_{jt}} = -\frac{1}{\varepsilon^{SR}_{jjt}\left(1+\Psi_{jjt}\right)} - \frac{\Delta^q_{jkt}\left(1+\Psi_{jkt}\right)}{\Delta^q_{jjt}\left(1+\Psi_{jjt}\right)}\frac{m_{kt}}{p_{jt}} - \sum_{i\in\{j,k\}} Z_{jit} \tag{29}$$

$$= \mu^{PRE}_{jt} + \frac{1+\Psi_{jkt}}{1+\Psi_{jjt}} DR_{jkt}\frac{m_{kt}}{p_{jt}} + Z_{jkt} \tag{30}$$

$$= \mu^{PRE}_{jt} + \frac{1+\Psi_{jkt}}{1+\Psi_{jjt}} GUPPI_{jkt} + Z_{jkt} \tag{31}$$

$$= \mu^{PRE}_{jt} + dGUPPI_{jkt} + Z_{jkt} \tag{32}$$

This holds fixed pre-merger dynamic competition pricing strategies and is likely a lower bound on what a merging firm can expect to gain as a result of controlling competitors dynamic pricing strategies (i.e. promote less frequently, space out promotions from the most similar products).

A.2.2. Multiple Periods

The exercise above is repeated but with margins measured over T periods - as is observed in data. To measure pre-merger market power over multiple periods sum profits over T periods

$$\sum_{t=1}^{T} q_{jt} m_{jt} = \sum_{t=1}^{T} -\frac{q_{jt}^2}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} - \frac{\Upsilon_{jjt}}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} q_{jt} \tag{33}$$

Then the pre-merger markup over T periods

$$\mu_j^{PRE} := \frac{\sum_{t=1}^{T} q_{jt} m_{jt}}{\sum_{t=1}^{T} q_{jt} p_{jt}} = -\sum_{t=1}^{T} \frac{q_{jt}^2}{R_j} \frac{1}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} - \sum_{t=1}^{T} \frac{\Upsilon_{jjt}}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} \frac{q_{jt}}{R_j} \tag{34}$$

$$= -\sum_{t=1}^{T} \sigma_{jt} \frac{1}{\varepsilon_{jjt}^{LR}} + \sum_{t=1}^{T} \sigma_{jt} Z_{jjt} \tag{35}$$

To measure post-merger market power accruing over multiple periods, sum product j's profits over T periods

$$\begin{aligned} \sum_{t=1}^{T} q_{jt} m_{jt} = &-\sum_{t=1}^{T} \frac{q_{jt}^2}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} - \sum_{t=1}^{T} \frac{\Upsilon_{jjt}}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} q_{jt} \\ &-\sum_{t=1}^{T} \frac{\Delta_{jkt}^q \left(1 + \Psi_{jkt}\right) m_{kt}}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} q_{jt} - \sum_{t=1}^{T} \frac{\Upsilon_{jkt}}{\Delta_{jjt}^q \left(1 + \Psi_{jjt}\right)} q_{jt} \end{aligned} \tag{36}$$

Dividing by product j's revenues over the T periods

$$
\begin{aligned}
\mu_j^{POST} := \frac{\sum_{t=1}^{T} q_{jt} m_{jt}}{\sum_{t=1}^{T} q_{jt} p_{jt}} &= -\sum_{t=1}^{T} \sigma_{jt} \frac{1}{\varepsilon_{jjt}^{LR}} + \sum_{t=1}^{T} \sigma_{jt} Z_{jjt} \\
&\quad - \sum_{t=1}^{T} \sigma_{jt} \frac{\Delta_{jkt}^{q} \left(1 + \Psi_{jkt}\right) m_{kt}}{\Delta_{jjt}^{q} \left(1 + \Psi_{jjt}\right) p_{jt}} + \sum_{t=1}^{T} \sigma_{jt} Z_{jkt} && (37) \\
&= \mu_j^{PRE} - \sum_{t=1}^{T} \sigma_{jt} \frac{\left(1 + \Psi_{jkt}\right) \Delta_{jkt}^{q} m_{kt}}{\left(1 + \Psi_{jjt}\right) \Delta_{jjt}^{q} p_{jt}} + \sum_{t=1}^{T} \sigma_{jt} Z_{jkt} && (38) \\
&= \mu_j^{PRE} + \sum_{t=1}^{T} \sigma_{jt} \frac{1 + \Psi_{jkt}}{1 + \Psi_{jjt}} DR_{jkt} \frac{m_{kt}}{p_{jt}} + \sum_{t=1}^{T} \sigma_{jt} Z_{jkt} && (39) \\
&= \mu_j^{PRE} + \sum_{t=1}^{T} \sigma_{jt} \frac{1 + \Psi_{jkt}}{1 + \Psi_{jjt}} GUPPI_{jkt} + \sum_{t=1}^{T} \sigma_{jt} Z_{jkt} && (40) \\
&= \mu_j^{PRE} + \sum_{t=1}^{T} \sigma_{jt} dGUPPI_{jkt} + \sum_{t=1}^{T} \sigma_{jt} Z_{jkt} && (41)
\end{aligned}
$$

The long-run measure of the $dGUPPI$ is the revenue weighted sum of the $dGUPPI$ in each period

$$dGUPPI_{jk}^{LR} := \sum_{t=1}^{T} \sigma_{jt} dGUPPI_{jkt} \tag{42}$$

As in the static case, assuming the analyst has obtained an estimate of the percentage margin earned on sales of product k over T periods, $\hat{\mu}_k$, $dGUPPI_{jk}$ can be can approximate using DR_{jk} is the diversion ratio evaluated at (revenue) weighted average prices and market shares over T periods

$$dGUPPI_{jk} \approx \frac{1 + \Psi_{jkt}}{1 + \Psi_{jjt}} DR_{jk}^{\pi} \mu_j \tag{43}$$

A.2.3. dGUPPI Example

To illustrate how dGUPPI captures demand dynamics, consider a firm assessing the impact on profits of its promotional pricing decisions on two substitutable products j and k over a one month planning horizon. Suppose the firm producing product j is considering temporarily cutting its price from £15 to £10 for the current week. If it does so, it expects to generate an additional 100 sales as a result. However, the firm expects that 20 of those sales would occur at the regular higher price of £15 within the subsequent weeks over the next month without the price cut.

Assuming a marginal cost of £5 in all periods, 40% of short-run profits made on sales of good j immediately following the price cut are drawn from expected profits from future profit flows. Further suppose that the effect of the price cut of product j on product k is the loss of 10 sales today and a further 3 sales over the next month. Assuming its price and cost remain unchanged over the period and its sales yield a £10 margin, the firm anticipates that consumer demand dynamics will lead to an additional 30% of losses on top of short-run profits lost on product k over the next month.

In this simple example, only 20% of the firm's profits are the result of sales diverted away from product k in the period of the price change and the GUPPI is 0.1. Therefore, $DDR^{\pi}_{jk} = 0.43$ overstating its short-run counterpart by 54 percent. The dGUPPI that factors in anticipated demand dynamics is 0.217 - more than double the GUPPI calibrated with contemporaneous profit movements alone. This higher dynamic diversion ratio demonstrates that products j and k are much closer substitutes than implied by a static short-run analysis.

B. Price Forecasting

B.1. VAR(κ)

The analyst may consider linear-in-prices κ-order Markov process as a statistical model to approximate the price process

$$\boldsymbol{p}_t = \boldsymbol{A}_0 + \sum_{s=1}^{\kappa} \boldsymbol{A}_s \boldsymbol{p}_{t-s} + \boldsymbol{u}_t \tag{44}$$

where $\boldsymbol{A}_0$ is a J-vector, each $\boldsymbol{A}_s$ is a $J \times J$ matrix of price coefficients for $s = 1, \ldots, \kappa$ and $\boldsymbol{u}_t$ is a J-vector of price shocks.

One benefit of this approach is that the price impulse response functions are constants calculated from the coefficient matrices and estimate $\boldsymbol{\Delta}^p_{t+h}$ for $h = 1, \ldots, H$ in each time period, t. In this case computation of the term in the first order condition capturing the change expected present value of revenues is simple to calculate.

$$\Upsilon_{nt} := \mathbb{E} \sum_{h=t+1}^{H} \delta^{h-t} \boldsymbol{\Delta}^p_{nh} \boldsymbol{q}_h = \sum_{h=t+1}^{H} \sum_{s=1}^{\kappa} \delta^{h-t} \boldsymbol{A}_{ns} \bar{\boldsymbol{q}} \tag{45}$$

However, there are three key drawbacks. First, this model has $J(1+\ \kappa J)$ parameters and is ill-suited to high-dimensional applications. Even if J is only 100 and $\kappa = 1$ then there are over 10,000 parameters to estimate. With more than a handful of products in the choice set, regularisation methods and/or ad hoc restrictions on the coefficient matrices may be required (i.e. assume prices follow a univariate AR(κ) process).

Second, prices are often not observed for all products in all time periods. This might be due to a strategic removal from the stores' shelves, unexpected stock-outs, or a feature of the sampling process. Missing data complicates estimation linear-in-price forecast model and requires careful use of interpolation methods.

Finally, a linear-in-price model may be too smooth to adequately capture the effect of frequent, irregularly spaced promotions. In that case, the approximation of the underlying price dynamics may degrade.

B.2. Dynamic Factor Model

The state space representation of the dynamic factor model of the price process is

$$\boldsymbol{p}_t = \boldsymbol{\Lambda} \boldsymbol{F}_t + \boldsymbol{\epsilon}_t \tag{46}$$

$$\boldsymbol{F}_{t+1} = \boldsymbol{A} \boldsymbol{F}_t + \boldsymbol{U}_{t+1} \tag{47}$$

where $\boldsymbol{F}_t = [\boldsymbol{f}_t, \ldots, \boldsymbol{f}_{t-\kappa}]^\top$ is a κR vector comprising of a low dimensional set of R-vector, $\boldsymbol{f}_t$, capturing underlying price trends or 'factors'. Also, $\boldsymbol{\Lambda} = [\boldsymbol{L}, \boldsymbol{0}, \ldots, \boldsymbol{0}]$ is a $J \times \kappa R$ matrix and $\boldsymbol{L} = \left[\boldsymbol{\lambda}_1^\top, \ldots, \boldsymbol{\lambda}_J^\top\right]^\top$ be a $J \times R$ matrix of factor loadings of factor loadings. Standard normalisations are imposed: $\frac{\boldsymbol{f}\boldsymbol{f}^\top}{T} = \boldsymbol{I}_R$ and $\boldsymbol{L}^\top \boldsymbol{L}$ is diagonal where $\boldsymbol{f} := [\boldsymbol{f}_1, \ldots, \boldsymbol{f}_T]$ is the $R \times T$ vector of price factors.

In the factor VAR, the coefficient matrix is $\kappa R \times \kappa R$

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_1} & \boldsymbol{A_2} & \cdots & \boldsymbol{A_{\kappa-1}} & \boldsymbol{A_\kappa} \\ \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \boldsymbol{0} & \vdots \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0} \end{bmatrix} \tag{48}$$

Finally, $\boldsymbol{\epsilon}_t$ is J-vector of price shocks and $\boldsymbol{U}_{t+h} = [\boldsymbol{u}_{t+h}, 0 \ldots, 0]^\top$ is a κR vector containing innovations to $\boldsymbol{F}_{t+h}$ where

$$\begin{bmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{u}_{t+h} \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{pp} & 0 \\ 0 & \boldsymbol{\Sigma}_{ff} \end{bmatrix}\right) \tag{49}$$

where $\boldsymbol{\Sigma}_{pp}$ and $\boldsymbol{\Sigma}_{ff}$ are the covariance matrices for equations (46) and (47), respectively.

Define covariance matrix of current prices and the h-step ahead price factors conditional on current price factors in period t

$$cov\left(\begin{bmatrix} \boldsymbol{p}_t \\ \boldsymbol{F}_{t+h} \end{bmatrix} \mid \boldsymbol{F}_t\right) = \begin{bmatrix} \boldsymbol{V}_{pp} & \boldsymbol{V}^h_{pf} \\ \boldsymbol{V}^h_{fp} & \boldsymbol{V}^h_{ff} \end{bmatrix} \tag{50}$$

For brevity, conditioning variables are suppressed, then the components of the covariance matrix are

$$\underset{J\times J}{\boldsymbol{V}_{pp}} = \mathbb{E}\left[(\boldsymbol{\Lambda}\boldsymbol{F}_t + \boldsymbol{\epsilon}_t)(\boldsymbol{\Lambda}\boldsymbol{F}_t + \boldsymbol{\epsilon}_t)^\top\right] \tag{51}$$

$$= \boldsymbol{\Lambda}\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top + \boldsymbol{\Sigma}_{pp} \tag{52}$$

$$\underset{R\times R}{\boldsymbol{V}^h_{ff}} = \mathbb{E}\left[\left(\boldsymbol{A}^h\boldsymbol{F}_t + \sum_{r=0}^{h}\boldsymbol{A}^r\boldsymbol{U}_{t+r+1}\right)\left(\boldsymbol{A}^h\boldsymbol{F}_t + \sum_{r=0}^{h}\boldsymbol{A}^r\boldsymbol{U}_{t+r+1}\right)^\top\right] \tag{53}$$

$$= \boldsymbol{A}^h\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{A}^{h,\top} + \sum_{r=0}^{h-1}\boldsymbol{A}^r\boldsymbol{\Sigma}_{ff}\boldsymbol{A}^{r,\top} \tag{54}$$

$$\underset{J\times R}{\boldsymbol{V}^h_{pf}} = \mathbb{E}\left[(\boldsymbol{\Lambda}\boldsymbol{F}_t + \boldsymbol{\epsilon}_t)\left(\boldsymbol{A}^h\boldsymbol{F}_t + \sum_{r=0}^{h}\boldsymbol{A}^r\boldsymbol{U}_{t+r+1}\right)^\top\right] \tag{55}$$

$$= \boldsymbol{\Lambda}\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{A}^{h,\top} \tag{56}$$

$$\underset{R\times J}{\boldsymbol{V}_{fp}^{h}} = \mathbb{E}_t\left[\left(\boldsymbol{A}^h\boldsymbol{F}_t + \sum_{r=0}^{h}\boldsymbol{A}^r\boldsymbol{U}_{t+r+1}\right)(\boldsymbol{\Lambda}\boldsymbol{F}_t + \boldsymbol{\epsilon}_t)^\top\right] \tag{57}$$

$$= \boldsymbol{A}^h\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top \tag{58}$$

From the definition of the conditional Normal distribution

$$\boldsymbol{F}_{t+h} \mid \boldsymbol{p}_t, \boldsymbol{F}_t \sim N(\bar{\boldsymbol{\mu}}^h, \bar{\boldsymbol{\Sigma}}^h) \tag{59}$$

where

$$\bar{\boldsymbol{\mu}}^h = \boldsymbol{A}^h\boldsymbol{F}_t + \boldsymbol{V}_{fp}^h(\boldsymbol{V}_{pp})^{-1}(\boldsymbol{p}_t - \boldsymbol{\Lambda}\boldsymbol{F}_t) \tag{60}$$

$$\bar{\boldsymbol{\Sigma}}^h = \boldsymbol{V}_{ff}^h - \boldsymbol{V}_{fp}^h(\boldsymbol{V}_{pp})^{-1}\boldsymbol{V}_{pf}^h \tag{61}$$

So

$$\mathbb{E}\left[\boldsymbol{p}_{t+h} \mid \boldsymbol{p}_t\right] = \boldsymbol{\Lambda}\mathbb{E}\left[\boldsymbol{F}_{t+h} \mid \boldsymbol{p}_t\right] + \mathbb{E}\left[\boldsymbol{\epsilon}_{t+h} \mid \boldsymbol{p}_t\right] \tag{62}$$

$$= \boldsymbol{\Lambda}\boldsymbol{A}^h\boldsymbol{F}_t + \boldsymbol{\Lambda}\boldsymbol{\Sigma}_{fp}^h\left(\boldsymbol{\Sigma}_{pp}^h\right)^{-1}(\boldsymbol{p}_t - \boldsymbol{\Lambda}\boldsymbol{F}_t) \tag{63}$$

Then

$$\frac{d\mathbb{E}\left[\boldsymbol{p}_{t+h} \mid \boldsymbol{p}_t\right]}{d\boldsymbol{p}_t} = \boldsymbol{\Lambda}\boldsymbol{V}_{fp}^h(\boldsymbol{V}_{pp})^{-1} \tag{64}$$

$$\implies \mathbb{E}\boldsymbol{\Delta}_{t+h}^p = \boldsymbol{\Lambda}\boldsymbol{A}^h\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top\left[\boldsymbol{\Lambda}\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top + \boldsymbol{\Sigma}_{pp}\right]^{-1} \tag{65}$$

Then the dynamic pricing term in the first order condition is

$$\mathbb{E}\sum_{h=t+1}^{H}\delta^{h-t}\boldsymbol{\Delta}_{nh}^p\boldsymbol{q}_h = \sum_{h=t+1}^{H}\delta^{h-t}\mathbb{E}\left[\boldsymbol{\Delta}_{nh}^p\boldsymbol{q}_{nh}\right] \tag{66}$$

$$= \sum_{h=t+1}^{H}\delta^{h-t}\mathbb{E}\left[\boldsymbol{\Delta}_{nh}^p\right]\mathbb{E}\left[\boldsymbol{q}_{nh}\right] + cov\left(\boldsymbol{\Delta}_{nh}^p, \boldsymbol{q}_{nh}\right) \tag{67}$$

$$= \sum_{h=t+1}^{H}\delta^{h-t}\boldsymbol{\Gamma}_{nh}\bar{\boldsymbol{q}}_n \tag{68}$$

where assumption A1 is imposed in the final line, then

$$\boldsymbol{\Gamma}_{t+h} = \boldsymbol{\Lambda}\boldsymbol{A}^h\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top\left[\boldsymbol{\Lambda}\boldsymbol{F}_t\boldsymbol{F}_t^\top\boldsymbol{\Lambda}^\top + \boldsymbol{\Sigma}_{pp}\right]^{-1} \tag{69}$$