Voting in Corporations

Alan D. Miller[*†]

October 9, 2017

Abstract

I introduce a model of shareholder voting. I describe and provide two characterizations of a family of shareholder voting rules, the thresholds rules. One characterization relies on an axiom, merger, that requires consistency in voting outcomes following stock-for-stock mergers; the other relies on an axiom, reallocation invariance, that requires the shareholder voting rule to be immune to certain manipulative techniques used by shareholders to hide their ownership. The family of thresholds rules includes many common voting methods including the shareholder majority rule. *JEL classification: D71, G34, K22*

1 Introduction

I introduce a model of shareholder voting; that is, voting by individuals with ownership stakes in a corporation. I then use the model to define and characterize an important class of shareholder voting rules, *thresholds rules.* In doing so, I provide normative justifications for the "one share-one vote" principle, according to which each shareholder receives a number of votes proportional to the size of her holding.[1]

[*]Faculty of Law and Department of Economics, University of Haifa, Mount Carmel, Haifa, 31905, Israel. Email: admiller@econ.haifa.ac.il. Web: http://econ.haifa.ac.il/∼admiller/. Phone: +972.52.790.3793.

[†]I would like to thank Bernie Black, Chris Chambers, Kate Litvak, Herve Moulin, Murat Mungan, and participants at the 34[th] Annual Conference of the European Association of Law and Economics for their comments.

[1]The one share-vote principle may be understood as requiring that all shares have an equal vote (that is, that there is only one class of shares), or alternatively, that the voting strength of a small shareholder must be linear in her holdings. The Delaware General

The shareholder franchise is understood to be an essential element of corporate governance. Corporations are owned by shareholders, and shareholders exercise their power through voting. They vote to elect the board of directors, which runs the corporation directly. They vote on major corporate transactions such as mergers. They vote on shareholder resolutions.

The most common rule is that shareholders receive one vote per share owned, and that shareholder votes are decided according to the majority (or supermajority) of votes cast.[2] However, this is not the only possibility. In the early nineteenth century, the shareholders often received one vote regardless of the number of shares owned. Today, many corporations issue multiple classes of voting stock to allow the founders to sell their shares without losing control of the corporation or to provide extra voting power to long-term investors. Others use "voting rights ceilings" to limit the voting power of larger shareholders. Recently, Posner and Weyl (2014) have proposed "square-root voting," under which a shareholder receives a number of votes proportional to the square-root of her holdings.

The positive effects of the one share-one vote rule have been studied extensively, including by Grossman and Hart (1988) and Harris and Raviv (1988) who analyze conditions under which a single class of equity stock and majority voting are optimal, and Ritzberger (2005) who provides conditions under which pure-strategy equilibria exist.[3] However, despite the large literature in social choice theory devoted to voting and despite the economic importance of the rules of corporate governance, to my knowledge there has never been a model to evaluate the normative properties of shareholder voting rules.

This paper contributes to the literature on shareholder voting in two ways. First, it introduces a formal model of shareholder voting, through which different voting rules can be compared and evaluated in terms of their normative characteristics. Second, the paper describes an important class of shareholder voting rules and provides an axiomatic justification of the rules in this class.

In the model, there is a set of shareholders, each of whom has preferences

Corporation Law provides, as a default, that: "Unless otherwise provided in the certificate of incorporation and subject to §213 of this title, each stockholder shall be entitled to 1 vote for each share of capital stock held by such stockholder." 8 Del. C. 1953, §212(a).

[2]The default is generally majority rule. For example, when a firm does not specify otherwise in its certificate of incorporation or bylaws, the Delaware code provides: "In all matters other than the election of directors, the affirmative vote of the majority of shares present in person or represented by proxy at the meeting and entitled to vote on the subject matter shall be the act of the stockholders." 8 Del. C. 1953, §216(2).

[3]For a surveys of the theoretical and empirical literature see Burkart and Lee (2008) and Adams and Ferreira (2008), respectively. For more on shareholder voting generally, see Easterbrook and Fischel (1983) and Thompson and Edelman (2009).

on a shareholder resolution and each of whom owns a portion of the firms' common stock.[4] Individuals may favor or oppose the resolution, or they may be indifferent between these two alternatives. A shareholder voting rule takes into account the preferences of the individuals and their shareholdings, and then uses this information to determine whether the resolution passes or fails, or whether the vote results in a tie.

The results in this paper revolve around two key axioms, *merger* and *reallocation invariance*. Each of these two properties, combined with others described below, are used to characterize a family of shareholder voting rules called the "thresholds rules." These are the rules most closely connected with "one share-one vote," and in this sense the axioms provide normative justifications for this principle.

The merger axiom requires a certain type of consistency in connection with votes related to stock-for-stock mergers. Imagine that there are two firms (firm A and firm B), that wish to merge in a stock-for-stock transaction, and there is a common resolution in front of the shareholders of each firm. For example, that resolution may be to approve the merger, or it may relate to post-merger plans. The merger axiom requires that, if the outcome of the shareholder vote held in firm A is the same as the outcome of the shareholder vote held in firm B, then this must also be the outcome of a (hypothetical) shareholder vote in the combined firm (after the merger). One may assume that preferences are formed in anticipation of the merger, and thus do not change, and that no shares are sold in the interim. The merger axiom is formally related to the consistency axiom of Young (1974, 1975).

The *reallocation invariance* axiom is motivated by the idea that individuals may be able to manipulate the identity of their shares owners' to the extent that ownership is relevant as far as voting rights are concerned. For example, if large blocks of shares were to receive disproportionately strong voting rights, likeminded shareholders may be able to combine their shares into a holding company, which becomes the sole owner of the shares.[5] The shareholders would then receive stock in the holding company. The transaction could be structured so that these shareholders could leave the holding company, and take their stock with them, in case that they wish to sell it or wish to vote differently from their fellow holding company participants. On the other hand, if small blocks of shares were to receive disproportionately strong voting rights, then a larger shareholder could partition her shares into several

[4]While this assumption is somewhat limiting, it is possible to modify the model to study multiple classes of stock. This is discussed in the conclusion.

[5]Depending on its size, such a transaction may trigger S.E.C. reporting requirements.

holding corporations. These are but a few of a wide variety of techniques that can be used to disguise the true ownership of the shares; for more see Hu and Black (2005, 2006, 2007, 2008). Reallocation independence is formally related to the no advantageous reallocation axiom introduced by Moulin (1985, 1987) in the context of bargaining and cost-sharing problems.

These axioms are then combined with several others to prove several results. The *anonymity* axiom (May, 1952) requires the shareholder voting rule to treat each voter equally; it accomplishes this by requiring the result to be invariant to changes in the names of the individuals. The *unanimity* axiom (called "weak Pareto" in Arrow, 1963) requires the resolution to pass when all shareholders are in favor, and to fail when all are opposed. The *repurchase* axiom requires that the outcome of the vote not be affected by corporate repurchases of stock held by indifferent shareholders. The *strategyproofness* axiom requires that shareholders not be able to benefit by misrepresenting their preferences.

Combinations of these axioms are then used to provide characterizations of the "thresholds rules." Thresholds rules can be described as a two-step process. First, they ask whether any shareholders have an opinion on the resolution. If the answer to this question is no—that is, if all shareholders are indifferent between the passing or failure of the resolution—the rule decides the matter on the basis of that single piece of information alone. If the answer is yes (as we would generally expect), the rules then look at the ratio of the number of shares owned by supporters of the resolution to the total number of shares owned by non-indifferent shareholders. If this proportion is sufficiently high, above the "passing" threshold, the resolution passes; if it is sufficiently low, below the "failing" threshold, the resolution fails. If these thresholds are different and the proportion is in between the two, then it results in a tie. Thresholds rules also specify the result if the proportion coincides exactly with one of the thresholds.

Thresholds rules include the methods most commonly used to determine the outcome of votes on shareholder resolutions: the shareholder majority rule, under which a resolution is passed if the supporters own more shares than the opponents, and supermajority rules which require a higher ratio between the supporters' and opponents' shareholdings for the resolution to pass. These rules have the characteristic that the upper and lower thresholds are the same. I characterize the subset of thresholds rules with this features with an additional axiom, *share monotonicity*, a form of positive responsiveness (May, 1952) that applies to changes in shareholdings rather than changes in preferences.

Another interesting class of rules are those for which the thresholds are symmetric around fifty percent. For example, for the shareholder majority rule, both thresholds are set at fifty percent; one can similarly envision a form of unanimity rule in which the vote results in a tie unless there is unanimous agreement among the shareholders. I characterize the subset of thresholds rules with symmetric thresholds using a neutrality condition introduced by May (1952).

The model allows for the possibility of ties to represent the idea that a voting rule need not always provide an answer to every question. This follows Arrow (1963) and May (1952), each of whom worked in a setting of weak preference. However, one might question the relevance of ties in the corporate setting; arguably, shareholder resolutions must either pass or fail. A tie may be interpreted as a a third outcome, different from either the passing or complete failure of a resolution. For example, under rules promulgated by the Securities and Exchange Commission, it is easier to reintroduce a failed shareholder resolution if it receives a certain percentage of the vote.[6] The possibility of ties can easily be eliminated by imposing a "no-tie" axiom. Because the implications of this axiom is straightforward if imposed on the class of thresholds rules, it is not formally introduced in this paper.

2 The Model

Let $\mathbb{N}$ be the set of all possible shareholders, and let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$. Let $\mathcal{R} \equiv \{-1, 0, 1\}$ be a set of preferences, with preferences R_i. For a set $N \in \mathcal{N}$, let $\mathbf{x} \in \Delta(N)$ be a distribution of shares. For $N \in \mathcal{N}$, let $\mathcal{Q}_N \equiv \mathcal{R}^N \times \Delta(N)$. The class of problems is the set $\mathcal{Q} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{Q}_N$.

For $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $N' \subseteq N$ for which $\sum_{i \in N'} \mathbf{x}_i = 1$, let $(R, \mathbf{x})|_{N'}$ denote the restriction of $(R, \mathbf{x})$ to $\mathcal{Q}_{N'}$. A function $f : \mathcal{Q} \to \mathcal{R}$ is **invariant to non-shareholders** if for $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $\sum_{i \in N'} \mathbf{x}_i = 1$ for $N' \subseteq N$ implies that $f(R, \mathbf{x}) = f\left((R, \mathbf{x})|_{N'}\right)$. A **shareholder voting rule** is a function $f : \mathcal{Q} \to \mathcal{R}$ that is invariant to non-shareholders.

The main results rely on six axioms. The first axiom, merger, requires a certain type of consistency in merged firms, as described in the introduction. The parameter λ represents the portion of the new firm that will be owned by the shareholders of first firm, while $1 - \lambda$ represents the portion that will be owned by the shareholders of the second firm. Because the model allows for null shareholders, the axiom can be limited to the case where the sets of shareholders are the same.

[6]See SEC Rule 14a-8(i)(12). Under this particular interpretation, the strategyproofness axiom would not be appropriate.

Merger: For $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $\lambda \in (0,1)$, if $f(R, \mathbf{x}) = f(R, \mathbf{x}')$, then $f(R, \mathbf{x}) = f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}')$.

The second axiom, reallocation invariance, is motivated by the idea that individuals may be able to manipulate the identity of their shares owners' to the extent that ownership affects voting rights. Several techniques that can be used to accomplish this result are described in the introduction. Formally, if there is a group $S \subseteq N$ of likeminded individuals (so that $R_j = R_k$ for all $j, k \in S$), and there are no changes in the ownership of stock among individuals outside of this group (so that $\mathbf{x}'_\ell = \mathbf{x}_\ell$ for all $\ell \notin S$), then the outcome of the vote must not change.

Reallocation invariance: For $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$, if $R_j = R_k$ for all $j, k \in S$ and $\mathbf{x}'_\ell = \mathbf{x}_\ell$ for all $\ell \notin S$, then $f(R, \mathbf{x}') = f(R, \mathbf{x})$.

The third axiom, anonymity, requires that the result of the vote be independent of the names of the shareholders. Anonymity is implied by reallocation invariance. For $N \in \mathcal{N}$, let Π_N refer to the set of permutations of N. For $\pi \in \Pi_N$, define $\pi R \equiv \left(R_{\pi(1)}, \ldots, R_{\pi(n)}\right)$ and $\pi\mathbf{x} \equiv \left(\mathbf{x}_{\pi(1)}, \ldots, \mathbf{x}_{\pi(n)}\right)$.

Anonymity For every $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$, $f(R, \mathbf{x}) = f(\pi R, \pi\mathbf{x})$.

The fourth axiom, unanimity, requires that a resolution must pass when all shareholders support it, and must fail when it is opposed by all.

Unanimity: For every $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, if there exists $k \in \{1, -1\}$ such that $R_i = k$ for all $i \in N$, then $f(R, \mathbf{x}) = k$.

The fifth axiom, repurchase, requires the voting rule to be invariant to corporate repurchases of stock held by indifferent shareholders. When a firm repurchases stock and places it in the corporate treasury, the number of outstanding shares decreases. As a consequence, each remaining share has a greater claim on the assets of the firm. In the formal definition of the axiom, $\mathbf{x}$ refers to the allocation of shares before the repurchase, and $\mathbf{x}'$ refers to the allocation of shares after all indifferent shares have been purchased. The absolute value $|R_i|$ serves as an indicator function describing whether individual i is non-indifferent. Note that the axiom only applies in the case where $\sum_i |R_i|\mathbf{x}_i > 0$; that is, some shares are held by non-indifferent shareholders.

Repurchase: For every $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_i |R_i|\mathbf{x}_i > 0$, if for all $j \in N$, $|R_j|\mathbf{x}_j = |R_j|\mathbf{x}'_j \sum_i |R_i|\mathbf{x}_i$, then $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

The sixth axiom, strategyproofness, requires that an individual shareholder not be able to benefit from misrepresenting her preferences. For $i \in N$ and $R \in \mathcal{R}^N$, let $[R_{-i}, k] \equiv (R_1, ..., R_{i-1}, k, R_{i+1}, ..., R_n)$.

Strategyproofness: For $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$, neither $R_i \leqslant f([R_{-i}, k], \mathbf{x}) < f(R, \mathbf{x})$ nor $R_i \geqslant f([R_{-i}, k], \mathbf{x}) > f(R, \mathbf{x})$.

2.1 Thresholds rules

I introduce a family of shareholder voting rules, the *thresholds rules*. These rules determine the outcome of a corporate vote using two pieces of information. The first piece of information is whether any shares are held by non-indifferent shareholders; that is, whether $\sum_i |R_i|\mathbf{x}_i > 0$. If the answer to this question is yes (as will commonly be the case), the rules use a second piece of information: the proportion of shares held by non-indifferent shareholders that are voted in favor of the resolution. Thresholds rules are constrained in how they can use this latter piece of information; the mapping from this proportion to the outcome of the vote must be monotonic and must respect a basic unanimity condition.

Thresholds rules are defined by five parameters. The first parameter, p, is an element of $\mathcal{R}$; it describes the outcome of the vote when all shareholders are indifferent; that is, when $\sum_i |R_i|\mathbf{x}_i = 0$. All outcomes are possible—that is, the resolution may pass, fail, or tie—but a thresholds rule must treat all cases of complete indifference the same way, regardless of the identities of the shareholders or the distribution of the shares between them.

The other parameters determine the outcome of the voting rule when some shareholders are not indifferent; that is, when $\sum_i |R_i|\mathbf{x}_i > 0$. The second and third parameters, q and r, are elements of $[0, 1]$; these represent the lower and upper thresholds, respectively. If the proportion of non-indifferent shares cast in favor exceeds the upper threshold, the resolution passes; if it is below the lower threshold, the resolution fails. If the proportion is in between the two thresholds, the vote results in a tie. The fourth and fifth parameters, s and t, are elements of $\{-1, 0\}$ and $\{0, 1\}$ respectively; these determine the result when the proportion of non-indifferent shares cast in favor coincides exactly with the lower and upper thresholds. If the proportion is exactly at the lower threshold, the resolution fails when $s = -1$; if it is exactly at the upper threshold, the resolution passes when $t = 1$.

The thresholds rules are pictured in Figure 1. The top of the figure explains the result in the case when all shares are held by indifferent shareholders. In

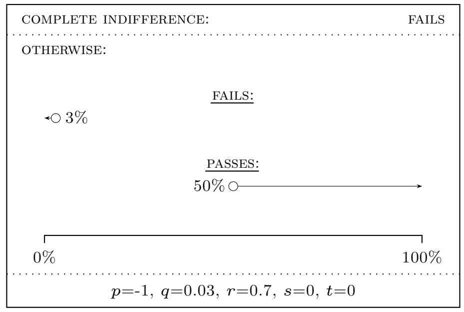


Figure 1: Thresholds rules

this example, p = -1, consequently, complete indifference leads to the failure of the resolution. In the middle section are two rays that represent the sets of votes that lead to the failure or passing of the resolution. In this representation, the right endpoint of the first ray is determined by q and the left endpoint of the second ray is determined by r. The circles representing the endpoints are empty if the relevant parameter (s for the first ray, t for the second) has the value of zero, and are filled in otherwise. Because $q = 0.03$, the right endpoint of the first ray is set at thirty percent, and because $s = 0$, the right endpoint of the first ray is left empty. Thus the resolution fails if the proportion of non-indifferent shares voted in favor of the resolution is less or equal to three percent. Similarly, because $r = 0.5$ the left endpoint of the second ray is set at seventy percent, and because $t = 0$, it is empty. Thus the resolution passes if the proportion of non-indifferent shares voted in favor of the resolution is greater than fifty percent. If the resolution neither fails nor passes (that is, if the proportion of non-indifferent shares voted in favor of the resolution is greater or equal to three percent but less or equal to fifty percent), the vote results in a tie. This example describes majority rule subject to SEC Rule 14a-8(i)(12), where a tie represents the outcome under which a resolution fails but is easier to reintroduce.

Not every combination of the five parameters leads to a valid rule. It cannot be the case that $q < r$ (as illustrated in Figure 2(a)), or that $q = r$ and st = -1 (as illustrated in Figure 2(b)). Each of these combinations of parameters could lead to the outcome where a resolution both passes and fails. (One can see that in both Figures 2(a) and 2(b), this result would occur if a shareholder resolution was supported by exactly seventy-five percent of the non-indifferent shares.) There is no such problem, however, if $q < r$ or $q = r$ and s or t equals zero, as is demonstrated in Figures 2(c) and 2(d).

Furthermore, it cannot be the case that $q = 0$ and $s = 0$, as shown in

Figure 2(e), or that $r = 1$ and $t = 0$, as shown in Figure 2(f). In the former case, a resolution will fail only when strictly less than zero percent of the non-indifferent shares are cast in favor the resolution; in the latter case a resolution will pass only when strictly greater than one hundred percent of the non-indifferent shares are cast in its favor. As both of these are impossible events; the former rule would mean that a resolution could never fail, while the latter rule would mean that a resolution could never pass. Both would violate the unanimity condition. If $q = 0$ then $s = -1$ and if $r = 1$ then $t = 1$, as shown in Figures 2(g) and 2(h).

I use two additional pieces of notation to define the thresholds rules. First, let $\sigma : \mathcal{Q} \to \Delta(\mathcal{R})$ such that for all $k \in \mathcal{R}$, $\sigma_k(R, \mathbf{x}) = \sum_{i:R_i=k} \mathbf{x}_i$. To simplify notation, I will make use of the fact that $\sum_i |R_i|\mathbf{x}_i = \sigma_1(R, \mathbf{x}) + \sigma_{-1}(R, \mathbf{x})$. Second, for $x \in \mathbb{R}$, denote the sign function τ as:

$$\tau(x) \equiv \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x = 0 \\ -1 & \text{if} \quad x < 0. \end{cases}$$

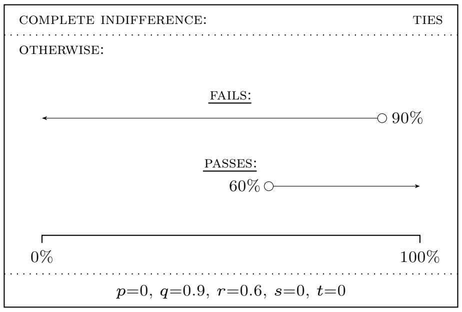
Thresholds rules: A shareholder voting rule f is a *thresholds rule* if there exist constants $p \in \mathcal{R}$, $q, r \in [0, 1]$, $s \in \{-1, 0\}$, and $t \in \{0, 1\}$, where $q \leqslant r$, $(q, s) \neq (0, 0)$, $(r, t) \neq (1, 0)$, and $q = r$ implies $st = 0$, such that for all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, if $\sum_i |R_i|\mathbf{x}_i = 0$ then $f(R, \mathbf{x}) = p$, and if $\sum_i |R_i|\mathbf{x}_i > 0$, then

$$f(R, \mathbf{x}) = \begin{cases} -1, & \text{if} \quad \tau\left(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q\right) < -s \\ 1, & \text{if} \quad \tau\left(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - r\right) > -t \\ 0, & \text{otherwise.} \end{cases}$$

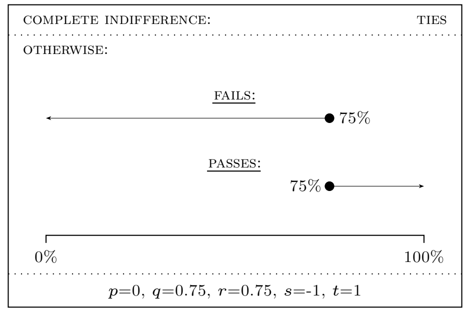
I provide two characterizations of the thresholds rules. The first theorem states that thresholds rules is the family of rules satisfying merger, anonymity, unanimity, and repurchase. The proof of this theorem is in the appendix.

Theorem 1. *A shareholder voting rule satisfies merger, anonymity, unanimity, and repurchase if and only if it is a thresholds rule. Furthermore, the four axioms are independent.*

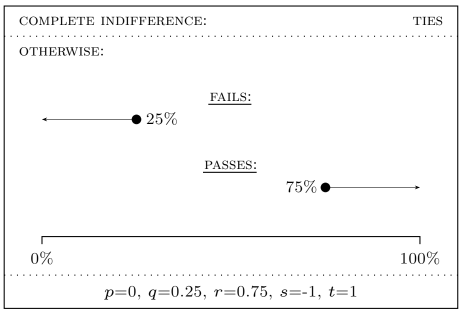
The second theorem states that the thresholds rules is the family of rules satisfying reallocation invariance, unanimity, repurchase, and strategyproofness. The proof of this theorem is also in the appendix.



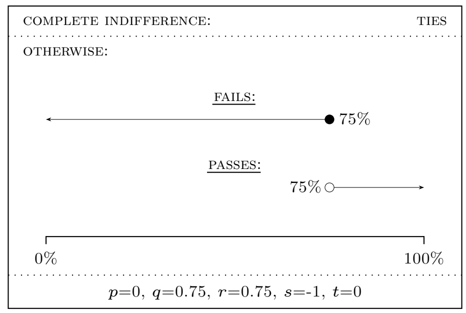
(a) Invalid rule: $q > r$



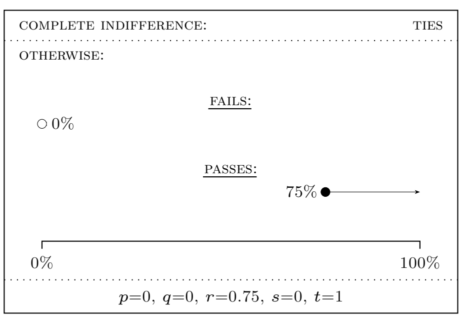
(b) Invalid rule: $q = r$ but $st = -1$



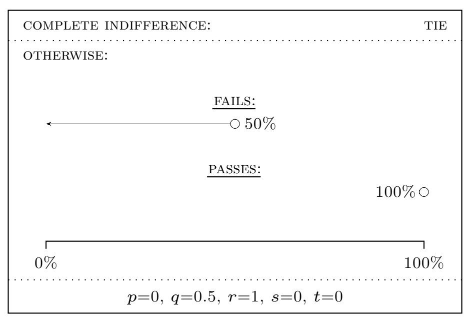
(c) Valid rule: $q \leqslant r$



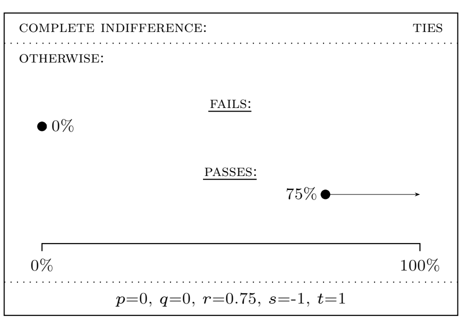
(d) Valid rule: $q = r$ and $st = 0$



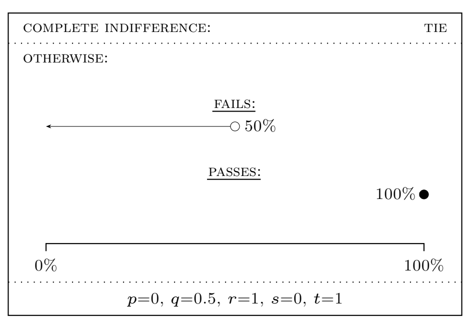
(e) Invalid rule: $(q, s) = (-1, 0)$



(f) Invalid rule: $(r, t) = (1, 0)$



(g) Valid rule: $(q, s) = (-1, -1)$



(h) Valid rule: $(r, t) = (1, 1)$

Figure 2: Invalid and valid rules

Theorem 2. *A shareholder voting rule satisfies reallocation invariance, unanimity, repurchase, and strategyproofness if and only if it is a thresholds rule. Furthermore, the four axioms are independent.*

Among the most prominent of the thresholds rules is the shareholder majority rule (Figure 3(a)), under which a resolution passes if it receives more than one half of the non-indifferent votes, fails if it receives less than one half, and leads to a tie in the event that the resolution receives exactly one half of the non-indifferent votes. The shareholder majority rule is the thresholds rule where $p = s = t = 0$ and $q = r = 0.5$. Note that $\sum_i R_i \mathbf{x}_i = \sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x})$.

Shareholder majority rule: For all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $f(R, \mathbf{x}) = \tau(\sum_i R_i \mathbf{x}_i)$.

The thresholds rules also contains two important subclasses. First, the *supermajority* rules are those rules for which there is a single threshold. The resolution passes if the number of votes in favor exceeds the threshold; it fails if the number of votes in favor falls short. The class of "supermajority rules" includes the shareholder majority rule as a special case, as well as "submajority" rules that are biased against the status quo.

Supermajority rules: A shareholder voting rule f is a *supermajority rule* if it is a thresholds rule for which $q = r$.

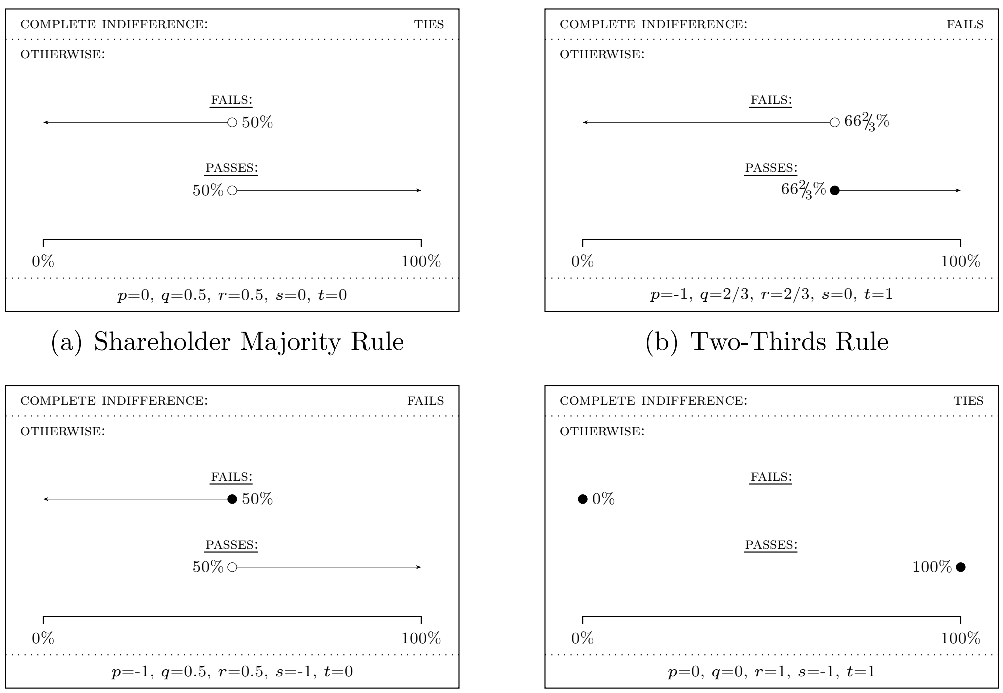
An example of a supermajority rule is the *two-thirds rule* (Figure 3(b)), where $p = -1$, $q = r = \frac{2}{3}$, $s = 0$, and $t = 1$. Under the two-thirds rule, a resolution passes if it gets at least two-thirds of the non-indifferent vote, and fails otherwise.

Another example of a supermajority rule is *majority rule without ties* (Figure 3(c)), where $p = -1$, $q = r = 0.5$, $t = 0$, and $s = -1$ (see Ritzberger, 2005). This is a form of majority rule in which a resolution passes if it receives greater than one half of the non-indifferent vote, and fails otherwise. Neither the two-thirds rule nor majority rule without indifference allow for the possibility of a tie. While some supermajority rules do allow for the possibility of ties, all thresholds rules which do not allow for the possibility of a tie are supermajority rules.

The second important subclass of thresholds rules is that of the balanced threshold rules, where the thresholds are symmetric around zero.

Balanced threshold rules: A shareholder voting rule f is a *balanced threshold rule* if it is a thresholds rule for which $p = 0$, $q = 1 - r$, and $s = -t$.

For example, consider the *unanimity rule* (Figure 3(d)) where $p = 0$, $q = 0$, $r = 1$, $s = -1$, and $t = 1$. In this rule, a resolution passes if it receives exactly one-hundred percent the non-indifferent shares, and it fails if it receives exactly none of those shares. If some but not all of the non-indifferent shares are in favor, then this rule leads to a tie. The shareholder majority rule is also a balanced threshold rule, as both of its thresholds are equal to zero. The unanimity rule is the thresholds rule that leads to the greatest likelihood of a tie. The shareholder majority rule is also a balanced threshold rule, as both of its thresholds are equal to zero.



(a) Shareholder Majority Rule

(b) Two-Thirds Rule

(c) Majority Rule without Ties

(d) Unanimity Rule

Figure 3: Other thresholds rules

These two classes, supermajority rules and balanced threshold rules, can be characterized with the imposition of additional axioms. One such axiom, share monotonicity, requires that, if a particular resolution does not fail (that is, either it passes or there is a tie), and then an individual who supports the resolution receives shares from an individual who does not support the resolution, the result is that the resolution is now chosen. This axiom requires, essentially, that having more shares helps one's vote. It is related to the positive responsiveness axiom of May (1952).

Share monotonicity: For every $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, and (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, if $f(R, \mathbf{x}) \neq -1$ then $f(R, \mathbf{x}') = 1$.

The other axiom, neutrality, was introduced by May (1952). it requires the shareholder voting rule not to favor the passing of the resolution over its failure. For $R \in \mathcal{R}^N$, define $-R = (-R_1, \ldots, -R_n)$.

Neutrality For every $(R, \mathbf{x}) \in \mathcal{Q}$, $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Supermajority rules are thresholds rules that satisfy share monotonicity.

Proposition 1. *A shareholder voting rule satisfies reallocation independence, anonymity, unanimity, and share monotonicity if and only if it is a supermajority rule. Furthermore, the four axioms are independent.*

Corollary 1. *A shareholder voting rule satisfies merger, anonymity, unanimity, repurchase, and share monotonicity if and only if it is a supermajority rule. Furthermore, the five axioms are independent.*

Balanced threshold rules are thresholds rules that satisfy neutrality.

Proposition 2. *A shareholder voting rule satisfies merger, anonymity, unanimity, repurchase, and neutrality if and only if it is a balanced threshold rule. Furthermore, the five axioms are independent.*

Corollary 2. *A shareholder voting rule satisfies reallocation independence, unanimity, repurchase, strategyproofness, and neutrality if and only if it is a balanced threshold rule. Furthermore, the five axioms are independent.*

As described above, the shareholder majority rule is both a supermajority rule and a balanced threshold rule. It is the only such rule.[7] Not all axioms are independent however; it is possible to characterize this rule with only three or four.

Theorem 3. *A shareholder voting rule satisfies merger, anonymity, share monotonicity and neutrality if and only if it is the shareholder majority rule. Furthermore, the four axioms are independent.*

[7] For a balanced threshold rule, $p = 0$, $q = 1 - r$, and $s = -t$. For a supermajority rule, $q = r$. Clearly, $q = r = 1 - r$ implies that $q = r = 0.5$, $q = r$ implies that $st = 0$, and $s = -t$ implies that $s = t = 0$. Therefore we have $p = s = t = 0$ and $q = r = 0.5$, the shareholder majority rule.

The reallocation invariance axiom, in conjunction with neutrality and share monotonicity, characterizes with the shareholder majority rule.

Theorem 4. *A shareholder voting rule satisfies reallocation invariance, share monotonicity and neutrality if and only if it is the shareholder majority rule. Furthermore, the three axioms are independent.*

The proofs of both propositions and theorems are in the appendix.

3 Other rules

In this section I describe several important shareholder voting rules that are worthy of further study. I do not provide full characterizations of these rules, but I do explain which of the above axioms are satisfied by them. These claims are then used to prove the independence of the axioms used in the characterization results above.

Polynomial majority rules are those for which majority rule is applied to the shareholdings transformed by an exponent. Three polynomial majority rules are of particular importance. The common law *one person-one vote rule* ($\alpha = 0$), where each shareholder receives an equal vote regardless of the size of the shareholding, was used during the nineteenth century until replaced by statute (see Dunlavy, 2006). More recently, Posner and Weyl (2014) have proposed a new method they call *square-root voting* ($\alpha = \frac{1}{2}$), where each shareholder receives a number of votes equal to the square-root of her holdings. The shareholder majority rule is the polynomial majority rule where $\alpha = 1$. Higher values of α give more voting power to larger shareholders.

Polynomial majority rules: A shareholder voting rule f is a *polynomial majority rule* if there is an $\alpha \in \mathbb{R}_+$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \tau\left(\sum_i R_i(\mathbf{x}_i)^\alpha\right)$.

The polynomial majority rules fail to satisfy merger and reallocation invariance, except in the case of the shareholder majority rule. They satisfy all of the other axioms, except for share monotonicity in the specific case of the one person-one vote rule.

Claim 1. *The polynomial majority rules satisfy anonymity, unanimity, repurchase, strategyproofness, and neutrality, but satisfy share monotonicity only for $\alpha > 0$ and reallocation invariance and merger only for $\alpha = 1$.*

Capped voting rules are those for which majority rule is applied to the shareholdings limited by a cap, so that there is a limit on each voter's voting strength regardless of the size of her shareholding. These rules are sometimes referred to as "voting rights ceilings." A study found that fifteen percent of the firms in the FTSEurofirst 300 Index use capped voting rules (see Dunlavy, 2006).

Capped majority rules: A shareholder voting rule f is a *capped majority rule* if there is an $c \in (0, 0.5)$ such that, for all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \tau\left(\sum_i R_i \min\{\mathbf{x}_i, c\}\right)$.

The capped majority rules do not satisfy merger, reallocation invariance, repurchase, and share monotonicity, but do satisfy the other axioms. However, it is important to note that for this class of rules, c is interpreted as a percentage cap on voting power. While this is a common restriction, it is also possible in practice to have a cap on the number of shares that can be voted. In that case, the repurchase axiom as formulated would not be applicable, because a repurchase of stock affects per-share voting power in percentage terms.

Claim 2. *The capped majority rules satisfy anonymity, unanimity, strategyproofness, and neutrality, but fail to satisfy merger, reallocation invariance, repurchase, and share monotonicity.*

I provide two examples of rules that fail anonymity, and therefore reallocation invariance. The first is the class of weighted majority rules, which assign a weight to each shareholder, by which the shareholdings are multiplied. The shareholder majority rule is a majority style rule where $\delta_i = \frac{1}{|N|}$ for all $i \in N$.

Weighted majority rules: A shareholder voting rule f is a *weighted majority rule* if there is a strictly positive set of weights $\delta \in \text{int}\{\Delta(N)\}$ for which $f(R, \mathbf{x}) = \tau\left(\sum_i R_i \delta_i \mathbf{x}_i\right)$.

The weighted majority rules satisfy all axioms except for reallocation invariance and anonymity.

Claim 3. *The weighted majority rules satisfy merger, unanimity, repurchase, strategyproofness, share monotonicity, and neutrality, but do not necessarily satisfy reallocation invariance or anonymity.*

The second example is the lexicographic dictator rule. According to this rule, there is a list of individuals, and the rule proceeds by choosing the opinion of the first shareholder on the list with a strict preference and a positive holding. If no such individual exists, then the rule leads to a tie.

For $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, let

$$d(R, \mathbf{x}) = \begin{cases} \min\{i : |R_i|\mathbf{x}_i > 0\}, & \text{if } \{i : |R_i|\mathbf{x}_i > 0\} \neq \varnothing \\ \min\{i : \mathbf{x}_i > 0\}, & \text{otherwise.} \end{cases}$$

Lexicographic dictator rule: $f(R, \mathbf{x}) = R_{d(R,\mathbf{x})}$.

The lexicographic dictator rule satisfies all axioms except for reallocation invariance and anonymity.

Claim 4. *The lexicographic dictator rule satisfies merger, unanimity, repurchase, strategyproofness, share monotonicity, and neutrality, but does not satisfy reallocation invariance and anonymity.*

The next rule is the constant rule, under which voting is irrelevant. The outcome under a constant rule is always the same, regardless of the preferences and the shareholdings.

Constant rules: There exists $k \in \mathcal{R}$ such that $f(R, \mathbf{x}) = k$.

All constant rules satisfy share monotonicity or neutrality, but not both. They fail unanimity, but satisfy the other axioms.

Claim 5. *The constant rules satisfy merger, reallocation invariance, anonymity, repurchase, and strategyproofness, satisfy share monotonicity if and only if $k \neq 0$, satisfy neutrality if and only if $k = 0$, and fail to satisfy unanimity for all $k \in \mathcal{R}$.*

Firms commonly require the existence of a quorum before a vote can be conducted. In this model, an indifferent shareholder is understood as one who arrives to the vote, but does not have a preference for or against the resolution to be decided on. Normally such shareholders would be counted within the quorum, but a shareholder voting rule could be defined so that a vote cannot take place if more than a certain percentage of shares are held by indifferent shareholders. As defined below, a quorum rule is one in which the outcome is determined by the shareholder majority rule, under the condition that not too many shares are held by indifferent shareholders. Otherwise, the result is a tie.

Quorum rules: There exists an $r \in (0, 1)$ such that $f(R, \mathbf{x}) = \tau(\sum_i R_i \mathbf{x}_i)$ if $\sigma_1(R, \mathbf{x}) + \sigma_{-1}(R, \mathbf{x}) > r$; otherwise $f(R, \mathbf{x}) = 0$.

The quorum rules satisfy all axioms except for merger, repurchase, strategyproofness, and share monotonicity. In conjunction with Claim 3, it establishes that merger and reallocation invariance are logically independent.

Claim 6. *The quorum rules satisfy reallocation invariance, anonymity, unanimity, and neutrality, but do not satisfy merger, repurchase, strategyproofness, and share monotonicity.*

Alternatively, a firm could use a form of an absolute majority rule, which ignores indifferent votes. The resolution passes if greater than one-half of all votes, including those that abstain, are in favor. It fails if greater than one-half of all votes are opposed. If neither of these events occur, the result is a tie.

Absolute majority rule: For $k \in \{-1, 1\}$, $f(R, \mathbf{x}) = k$ if $\sigma_k(R, \mathbf{x}) > \frac{1}{2}$.

The absolute majority rule satisfies all axioms except for repurchase and share monotonicity.

Claim 7. *The absolute majority rule satisfies merger, reallocation invariance, anonymity, unanimity, strategyproofness, and neutrality, but does not satisfy repurchase and share monotonicity.*

Phantom voter rules are similar to majority rule, but with a handicap; they operate as if there are "phantom" voters who have already voted their shares.[8] Here, t is the number of phantom votes, as a fraction of the total outstanding stock, that are in favor of the resolution. A handicap of $t < 0$ implies that these phantom votes are opposed. The shareholder majority rule is a phantom voter rule where $t = 0$.

Phantom voter rules: A shareholder voting rule f is a *phantom voter rule* if there is a $t \in (-1, 1)$ such that $f(R, \mathbf{x}) = \tau\left(t + \sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x})\right)$.

The phantom voter rules satisfy all axioms but repurchase and neutrality.

Claim 8. *The phantom voter rules satisfy merger, reallocation invariance, anonymity, unanimity, strategyproofness, and share monotonicity, but may fail to satisfy repurchase and neutrality.*

[8] These rules are clearly different from those in the "phantom voter" result of Moulin (1980), but are similar in that they operate as if shares have been voted.

The last three rules are provided to demonstrate logical independence of the voting rules, and not because they are particularly interesting in their own right. The next rule is the "alternating rule," in which the resolution passes it if is supported by more than one quarter and less than one half of the vote, or if it is supported by three quarters or more. The vote leads to a tie if it is supported by exactly fifty percent, and otherwise it fails.

Alternating rule: A shareholder voting rule f is the *alternating rule* if

$$f(R, \mathbf{x}) = \begin{cases} 1, & \text{if} \quad \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} \in (0.25, 0.5) \cup [0.75, 1] \\ -1, & \text{if} \quad \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} \in (0.5, 0.75) \cup [0, 0.25] \\ 0, & \text{otherwise.} \end{cases}$$

The alternating rule satisfies all axioms except for merger, strategyproofness, and share monotonicity.

Claim 9. *The alternating rule satisfies reallocation invariance, anonymity, unanimity, repurchase, and neutrality, but fails to satisfy merger, strategyproofness, and share monotonicity.*

In the proof of Theorem 2, it was shown that the reallocation invariance and strategyproofness axioms together imply that a rule satisfies merger on $\mathcal{Q}^2$. A natural question is whether these axioms together imply merger. The next rule is constructed to demonstrate that the answer to this question is no. A rule can satisfy reallocation invariance and strategyproofness, as well as unanimity and share monotonicity, without implying merger.

Rule X: A shareholder voting rule f is a *rule X* if

$$f(R, \mathbf{x}) = \begin{cases} 1, & \text{if} \quad \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} > 0.5 \\ 0, & \text{if} \quad \sigma_1(R, \mathbf{x}) = \sigma_{-1}(R, \mathbf{x}) = 0.5 \text{ or } \sigma_0(R, \mathbf{x}) = 1 \\ -1, & \text{otherwise.} \end{cases}$$

Rule X satisfies all axioms except for merger, repurchase, and neutrality.

Claim 10. *Rule X satisfies reallocation invariance, anonymity, unanimity, strategyproofness, and share monotonicity, but fails to satisfy merger, repurchase, and neutrality.*

The last rule demonstrates that strategyproofness does not imply share monotonicity. Together with Claim 1 it establishes that the two axioms are independent.

Indifference is two against rule: A shareholder voting rule f is the *indifference is two against rule* if $f(R, \mathbf{x}) = \tau\left(\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}) - 2\sigma_0(R, \mathbf{x})\right)$

The indifference is two against rule satisfies all axioms except for repurchase, strategyproofness, and neutrality.

Claim 11. *The indifference is two against rule satisfies merger, reallocation invariance, anonymity, unanimity, and share monotonicity, but fails to satisfy repurchase, strategyproofness, and neutrality.*

Conclusion

I have introduced a model of shareholder voting in which the preferences of shareholders are aggregated to decide on a shareholder resolution. I describe an important class of shareholder voting rules, the thresholds rules, and provide two characterizations of this class. One characterization relies on a merger axiom that requires consistency in corporate decisions following mergers; the other relies on a reallocation invariance axiom that requires the decision to be invariant to certain manipulative techniques used by shareholders to hide their ownership.

The thresholds rules are closely connected with the one share-one vote principle, in that each shareholder gets a number of votes that is linear in her shareholdings. In this sense the merger and the reallocation invariance axioms may be used as independent justifications of this principle. These axioms are easiest to defend in a setting in which mergers are a realistic possibility or in which incorporation can be done at low cost. While this clearly descries the present day, this assumption would have been more questionable in the early days of corporate law. The characterization theorems suggest a normative justification for the parallel growth of the one share-one vote principle and of the corporation as a vehicle for organizing economic activity in the latter part of the nineteenth century. However, there are other potential explanations, and establishing the existence of a causal link is a matter for economic historians, outside the scope of this paper.

As mentioned in the introduction, the possibility of ties can be easily eliminated by imposing a "no-tie" axiom that requires the outcome to not be a tie. By imposing such an axiom on the thresholds rules, we would be able to characterize the subset of the supermajority rules where $s + t \neq 0$. While the no-tie axiom is stronger than share monotonicity in the presence of the axioms used in Theorems 1 and 2, as a general matter the no-tie axiom neither implies nor is implied by any combination of the eight axioms described above. It is, however, inconsistent with the neutrality axiom.[9]

In this model shareholders are assumed to show up at the annual meeting. Indifference represents the preference of the shareholder who is not interested in the outcome. The strategyproofness axiom ensures that interested shareholders will not make the strategic choice to pretend to be indifferent. However, in practice, shareholder voting can be more complicated; shareholders may have a choice to avoid being counted in a quorum by avoiding the annual meeting. It is possible to model this choice explicitly to gain a better understanding of quorum requirements in shareholder voting.

In the model, preferences are defined on a single pair of alternatives. In practice, virtually all shareholder votes involve only a single pair of alternatives, where one of the alternatives is the status quo. However, it is theoretically possible that some corporate decisions, such as elections for the board of directors, could involve multiple alternatives. The model can be generalized to allow more complicated preferences simply by redefining $\mathcal{R}$ as appropriate.

Similarly, the model assumes a single class of stock, and that shares are infinitely divisible. While this is useful as a simplifying assumption, one may wish to allow the possibility for multiple classes of stock. One simple way to do this would be to define a finite set $\mathcal{S}$ of shares, where $\mathbf{x}$ is a partition of $\mathcal{S}$, and where $\mathbf{x}_i$ then represents the shares assigned to agent i. If the model was extended further to incorporate the date the specific shares were purchased, then it might be possible to study time-phased voting rules, where the voting power of the share is increasing in the amount of time that it has been held by the shareholder.

[9]To see this, note that for $N = \{1\}$ and $(R, \mathbf{x}) = (0, 1)$, $f(R, \mathbf{x}) = f(-R, \mathbf{x})$. Neutrality implies that $f(R, \mathbf{x}) = -f(-R, \mathbf{x})$ and therefore $f(R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$.

Appendix

Let $\mathscr{G}$ be the set of all functions $\mathbf{g} : \Delta(\mathcal{R}) \to \mathcal{R}$. For $p \in \mathcal{R}$, $q, r \in [0,1]$, $s \in \{-1, 0\}$, and $t \in \{0, 1\}$, define $g^{pqrst} \in \mathscr{G}$ such that:

$$g^{pqrst}(\mathbf{x}_1, \mathbf{x}_{-1}, \mathbf{x}_0) = \begin{cases} p, & \text{if} \quad \mathbf{x}_0 = 1 \\ -1, & \text{if} \quad \mathbf{x}_0 \neq 1 \text{ and } \tau(\frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_{-1}} - q) < -s \\ 1, & \text{if} \quad \mathbf{x}_0 \neq 1 \text{ and } \tau(\frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_{-1}} - r) > -t \\ 0, & \text{otherwise.} \end{cases}$$

Note that for a thresholds rule f defined by constants p, q, r, s, and t, $f(R, \mathbf{x}) = \mathbf{g}^{pqrst}(\sigma(R, \mathbf{x}))$. Let $\mathscr{G}^T \subseteq \mathscr{G}$ be the set of all functions g^{pqrst}. For a function $\mathbf{g} \in \mathscr{G}$ and a domain $\mathcal{Q}^* \subseteq \mathcal{Q}$, I define the following property:

g-Reducible on $\mathcal{Q}^*$: For all $(R, \mathbf{x}) \in \mathcal{Q}^*$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

Let $\mathcal{N}^3 \equiv \{N \in \mathcal{N} : |N| = 3\}$. Let $\mathcal{Q}^3 \subseteq \bigcup_{N \in \mathcal{N}^3} \mathcal{Q}_N$ be the set of problems for which, for all $N \in \mathcal{N}^3$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, $R_i \neq R_j$ for all $\{i, j\} \subseteq N$.

I begin by stating and proving three lemmas.

Lemma 1. *If f is anonymous then f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$ for some $\mathbf{g} \in \mathscr{G}$.*

Proof of Lemma 1. Let f satisfy anonymity. Note that for $N \in \mathcal{N}^3$, there exists $\mathbf{g} \in \mathscr{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3 \cap \mathcal{Q}_N$. Let $N, N' \in \mathcal{N}^3$. Then there exists $\mathbf{g}, \mathbf{g}' \in \mathscr{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}^3 \cap \mathcal{Q}_N$ and $f(R', \mathbf{x}') = \mathbf{g}'(\sigma(R', \mathbf{x}'))$ for all $(R', \mathbf{x}') \in \mathcal{Q}^3 \cap \mathcal{Q}_{N'}$.

Let $j, k, \ell \in N$ and $j', k', \ell' \in N'$. Let $R \in \mathcal{R}^N$ and $R' \in \mathcal{R}^{N'}$ such that $R_j = R'_j = 1$, $R_k = R'_k = -1$, and $R_\ell = R'_\ell = 0$. Let $\mathbf{x} \in \Delta(N)$ and $\mathbf{x}' \in \Delta(N')$ such that $\mathbf{x}_j = \mathbf{x}'_{j'}$, $\mathbf{x}_k = \mathbf{x}'_{k'}$, and $\mathbf{x}_\ell = \mathbf{x}'_{\ell'}$. Note that $\sigma(R, \mathbf{x}) = \sigma(R', \mathbf{x}')$. Thus to prove that $\mathbf{g} = \mathbf{g}'$, it is sufficient to show that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Let $N^* = \{j^*, k^*, \ell^*\} \in \mathcal{N}^3$ such that $N \cap N^* = N' \cap N^* = \varnothing$. Let $R^\circ \in \mathcal{R}^{N \cup N^*}$ such that $R^\circ_j = R^\circ_{j*} = 1$, $R^\circ_k = R^\circ_{k*} = -1$, and $R^\circ_\ell = R^\circ_{\ell*} = 0$. Let $\mathbf{x}^\circ, \mathbf{x}^{\circ\circ} \in \Delta(N \cup N^*)$ such that (a) $\mathbf{x}^\circ_j = \mathbf{x}_j$, $\mathbf{x}^\circ_k = \mathbf{x}_k$, $\mathbf{x}^\circ_\ell = \mathbf{x}_\ell$, and $\mathbf{x}^\circ_{j*} = \mathbf{x}^\circ_{k*} = \mathbf{x}^\circ_{\ell*} = 0$, and (b) $\mathbf{x}^{\circ\circ}_{j*} = \mathbf{x}_j$, $\mathbf{x}^{\circ\circ}_{k*} = \mathbf{x}_k$, $\mathbf{x}^{\circ\circ}_{\ell*} = \mathbf{x}_\ell$, and $\mathbf{x}^{\circ\circ}_j = \mathbf{x}^{\circ\circ}_k = \mathbf{x}^{\circ\circ}_\ell = 0$. Let $\pi \in \Pi_{N \cup N^*}$ such that $\pi(j) = j^*$, $\pi(k) = k^*$, $\pi(\ell) = \ell^*$, $\pi(j^*) = j$, $\pi(k^*) = k$ and $\pi(\ell^*) = \ell$. Then $\pi R^\circ = R^\circ$ and $\pi \mathbf{x}^\circ = \mathbf{x}^{\circ\circ}$. It follows from anonymity that $f(R^\circ, \mathbf{x}^\circ) = f(\pi R^\circ, \pi \mathbf{x}^\circ) = f(R^\circ, \mathbf{x}^{\circ\circ})$. Because $(R, \mathbf{x}) = (R^\circ, \mathbf{x}^\circ)|_N$, it follows from invariance to non-shareholders that $f(R, \mathbf{x}) = f(R^\circ, \mathbf{x}^{\circ\circ})$.

Let $R^\bullet \in \mathcal{R}^{N' \cup N^*}$ such that $R^\bullet_{j'} = R^\bullet_{j*} = 1$, $R^\bullet_{k'} = R^\bullet_{k*} = -1$, and $R^\bullet_{\ell'} = R^\bullet_{\ell*} = 0$. Let $\mathbf{x}^\bullet, \mathbf{x}^{\bullet\bullet} \in \Delta(N' \cup N^*)$ such that (a) $\mathbf{x}^\bullet_{j'} = \mathbf{x}_j$, $\mathbf{x}^\bullet_{k'} = \mathbf{x}_k$, $\mathbf{x}^\bullet_{\ell'} = \mathbf{x}_\ell$, and $\mathbf{x}^\bullet_{j*} = \mathbf{x}^\bullet_{k*} = \mathbf{x}^\bullet_{\ell*} = 0$, and (b) $\mathbf{x}^{\bullet\bullet}_{j*} = \mathbf{x}_j$, $\mathbf{x}^{\bullet\bullet}_{k*} = \mathbf{x}_k$, $\mathbf{x}^{\bullet\bullet}_{\ell*} = \mathbf{x}_\ell$, and

$\mathbf{x}^{\bullet\bullet}_{j'} = \mathbf{x}^{\bullet\bullet}_{k'} = \mathbf{x}^{\bullet\bullet}_{\ell'} = 0$. Let $\pi' \in \Pi_{N' \cup N^*}$ such that $\pi'(j') = j^*$, $\pi'(k') = k^*$, $\pi'(\ell') = \ell^*$, $\pi'(j^*) = j'$, $\pi'(k^*) = k'$ and $\pi'(\ell^*) = \ell'$. Then $\pi R^\bullet = R^\bullet$ and $\pi \mathbf{x}^\bullet = \mathbf{x}^{\bullet\bullet}$. It follows from anonymity that $f(R^\bullet, \mathbf{x}^\bullet) = f(\pi R^\bullet, \pi \mathbf{x}^\bullet) = f(R^\bullet, \mathbf{x}^{\bullet\bullet})$. Because $(R', \mathbf{x}') = (R^\bullet, \mathbf{x}^\bullet)|_N$, it follows from invariance to non-shareholders that $f(R', \mathbf{x}') = f(R^\bullet, \mathbf{x}^{\bullet\bullet})$. Because $(R^\circ, \mathbf{x}^{\circ\circ})|_{N^*} = (R^\bullet, \mathbf{x}^{\bullet\bullet})|_{N^*}$, it follows from invariance to non-shareholders that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$. $\square$

Let $\mathcal{N}^2 \equiv \{N \in \mathcal{N} : |N| = 2\}$. Let $\mathcal{Q}^2 \subseteq \bigcup_{N \in \mathcal{N}^2} \mathcal{Q}_N$ be the set of problems for which, for all $N \in \mathcal{N}^2$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, there exists $j, k \in N$ such that $R_j = 1$ and $R_k = -1$. The next lemma makes use of the following property:

Merger on $\mathcal{Q}^2$: For $N \in \mathcal{N}^2$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}^2 \cap \mathcal{Q}_N$, and $\lambda \in (0, 1)$, if $f(R, \mathbf{x}) = f(R, \mathbf{x}')$, then $f(R, \mathbf{x}) = f(R, \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}')$.

Lemma 2. *If f satisfies anonymity, unanimity, repurchase, and merger on $\mathcal{Q}^2$ then there is $\mathbf{g} \in \mathscr{G}^T$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$.*

Proof of Lemma 2. Let f satisfy anonymity, unanimity, repurchase, and merger on $\mathcal{Q}^2$. Let $N \in \mathcal{N}^3$. I show that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3 \cap \mathcal{Q}_N$ for some $\mathbf{g} \in \mathscr{G}^T$. Consequently, Lemma 1 implies (by anonymity) that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$.

Let $j, k, \ell \in N$. Let $R \in \mathcal{R}^N$ such that $R_j = 1$, $R_k = -1$, $R_\ell = 0$. For $z \in [0, 1]$, let $\mathbf{x}^z \in \Delta(N)$ such that $\mathbf{x}^z_j = z$, $\mathbf{x}^z_k = 1 - z$, and $\mathbf{x}^z_\ell = 0$. By unanimity and invariance to non-shareholders, $f(R, \mathbf{x}^1) = 1$ and $f(R, \mathbf{x}^0) = -1$.

Let $q = \sup\{z \in [0, 1] : f(R, \mathbf{x}^z) = -1\}$. Let $s = -1$ if $f(R, \mathbf{x}^q) = -1$, otherwise let $s = 0$. Let $r = \inf\{z \in [0, 1] : f(R, \mathbf{x}^z) = 1\}$. Let $t = 1$ if $f(R, \mathbf{x}^r) = 1$, otherwise let $t = 0$. Unanimity implies that $s = -1$ if $q = 0$ and $t = 1$ if $r = 1$. If $q = r$ then $st = 0$; otherwise $-1 = f(R, \mathbf{x}^q) = f(R, \mathbf{x}^r) = 1$, a contradiction. Let $\ddot{\mathbf{x}} \in \Delta(N)$ such that $\ddot{\mathbf{x}}_\ell = 1$. Let $p = f(R, \ddot{\mathbf{x}})$.

For $\mathbf{x} \in \Delta(N) \backslash \{\ddot{\mathbf{x}}\}$, let $z(\mathbf{x}) = \frac{\mathbf{x}_j}{\mathbf{x}_j + \mathbf{x}_k}$. By repurchase, $f(R, \mathbf{x}) = f(R, \mathbf{x}^{z(\mathbf{x})})$. For $\mathbf{x} \in \Delta(N) \backslash \{\ddot{\mathbf{x}}\}$, I show that $f(R, \mathbf{x}) = -1$ if and only if $\tau(\frac{\sigma_1(R, \mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - q) < -s$. A dual argument proves that $f(R, \mathbf{x}) = 1$ if and only if $\tau(\frac{\sigma_1(R, \mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - r) > -t$.

Let $\mathbf{x} \in \Delta(N) \backslash \{\ddot{\mathbf{x}}\}$ such that $\tau(\frac{\sigma_1(R, \mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - q) < -s$. Then $\tau(z(\mathbf{x}) - q) < -s$. It follows that $z(\mathbf{x}) \leqslant q$. If $z(\mathbf{x}) = q$, then $s = -1$; by construction of s, $f(R, \mathbf{x}^q) = -1$. If $z(\mathbf{x}) = 0$, then by unanimity, $f(R, \mathbf{x}^0) = -1$. If $z(\mathbf{x}) \in (0, q)$, then there must exist $z' \in (z(\mathbf{x}), q]$ such that $f(R, \mathbf{x}^{z'}) = -1$. Because $0 < z(\mathbf{x}) < z'$, it follows that there is $\lambda \in (0, 1)$ such that $\mathbf{x}^{z(\mathbf{x})} = \lambda \mathbf{x}^0 + (1 - \lambda)\mathbf{x}^{\mathbf{z}'}$. Because $\mathbf{x}^{z(\mathbf{x})}_\ell = \mathbf{x}^0_\ell = \mathbf{x}^{z'}_\ell = 0$, it follows from invariance to non-shareholders and merger on $\mathcal{Q}^2$ that $f(R, \mathbf{x}^{z'}) = f(R, \lambda \mathbf{x}^0 + (1 - \lambda)\mathbf{x}^{z'}) = f(R, \mathbf{x}^{z(\mathbf{x})}) = -1$. By repurchase, $f(R, \mathbf{x}) = -1$. Next, let $\mathbf{x} \in \Delta(N) \backslash \{\ddot{\mathbf{x}}\}$ such that $\tau(\frac{\sigma_1(R, \mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} -$

$q) \geqslant -s$. Then $\tau(z(\mathbf{x}) - q) \geqslant -s$. It follows that $z(\mathbf{x}) \geqslant q$. If $z(\mathbf{x}) = q$, then $s = 0$; by construction of s, $f(R, \mathbf{x}^q) \geqslant 0$. If $z(\mathbf{x}) > q$, then by construction of q, $f(R, \mathbf{x}^{z(\mathbf{x})}) \geqslant 0$. By repurchase, $f(R, \mathbf{x}) \geqslant 0$.

Lastly, I show that $q \leqslant r$. Suppose, contrariwise, that $q > r$. Let $z = \frac{1}{2}(q+r)$. Then $\frac{\sigma_1(R,\mathbf{x}^z)}{\sum_i |R_i|\mathbf{x}_i^z} - q = \frac{1}{2}(r-q) = -(\frac{\sigma_1(R,\mathbf{x}^z)}{\sum_i |R_i|\mathbf{x}_i^z} - r)$. Because $q > r$, it follows that $\tau(\frac{\sigma_1(R,\mathbf{x}^z)}{\sum_i |R_i|\mathbf{x}_i^z} - q) < -s$ and that $\tau(\frac{\sigma_1(R,\mathbf{x}^z)}{\sum_i |R_i|\mathbf{x}_i^z} - r) > -t$, a contradiction. □

Lemma 3. *If f satisfies merger and there is $\mathbf{g} \in \mathscr{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$, then f is $\mathbf{g}$-reducible on $\mathcal{Q}$.*

Proof of Lemma 3: Let f satisfy merger and let $\mathbf{g} \in \mathscr{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}^3$. Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\mathbf{x}_i > 0$ for all $i \in N$. I show that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

For $k \in \mathcal{R}$, define $S^k \equiv \{i : R_i = k\}$. For $k \in \mathcal{R}$ and $i \in S^k$, let $\omega_i = \mathbf{x}_i \left[\sigma_k(R, \mathbf{x})\right]^{-1}$ and let $\mathbf{z}^i \in [0,1]^N$ such that $\mathbf{z}^i_i = \sigma_k(R, \mathbf{x})$ and $\mathbf{z}^i_j = 0$ for $j \neq i$. For $k \in \mathcal{R}$, if $S^k \neq \varnothing$, then let $\mathcal{S}^k = S^k$. If $S^k = \varnothing$, let $\mathcal{S}^k = \{i^k\}$, where $\mathcal{S}^k \cap N = \varnothing$, where $\omega_{i^k} = 1$, and where $\mathbf{z}^{i^k}_j = 0$ for all $j \in N$. For $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, let $\mathbf{x}^{jk\ell} \in \Delta(N)$ such that $\mathbf{x}^{jk\ell} = \mathbf{z}^j + \mathbf{z}^k + \mathbf{z}^\ell$.

$$\text{Note that } \mathbf{x} = \sum_{j \in \mathcal{S}^1} \sum_{k \in \mathcal{S}^{-1}} \sum_{\ell \in \mathcal{S}^0} \omega_j \omega_k \omega_\ell \mathbf{x}^{jk\ell}.$$

For all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f((R, \mathbf{x}^{jk\ell})|_{\{j,k,\ell\} \cap N}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. By construction, the sets $\mathcal{S}^k$ are finite; thus, it follows from merger that for all $j \in \mathcal{S}^1$, $k \in \mathcal{S}^{-1}$, and $\ell \in \mathcal{S}^0$, $f(R, \mathbf{x}^{jk\ell}) = f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. □

Proof of Theorem 1. **Only if:** Let f satisfy merger, anonymity, unanimity, and repurchase. Because f satisfies merger it satisfies merger on $\mathcal{Q}^2$. Thus, by Lemma 2, there is $\mathbf{g} \in \mathscr{G}^T$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. By Lemma 3, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; that is, f is a thresholds rule.

If: Let f be a thresholds rule with constants p, q, r, s, and t. I show that it satisfies the four axioms.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Note that for all $k \in \mathcal{R}$, $\sigma_k(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = \lambda\sigma_k(R, \mathbf{x}) + (1-\lambda)\sigma_k(R, \mathbf{x}')$. If $\sum_i |R_i|\mathbf{x}_i = \sum_i |R_i|\mathbf{x}'_i = 0$, then clearly $\lambda \sum_i |R_i|\mathbf{x}_i + (1-\lambda)\sum_i |R_i|\mathbf{x}'_i = \sum_i |R_i|(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) = 0$, so merger is satisfied. If $\sum_i |R_i|\mathbf{x}_i = 0$ but $\sum_i |R_i|\mathbf{x}'_i > 0$, then for $k \in \{1, -1\}$, $\sigma_k(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = (1-\lambda)\sigma_k(R, \mathbf{x}')$. Furthermore, for all $i \in N$ such that $\mathbf{x}_i > 0$, $R_i = 0$. Thus $\sum_i |R_i|(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) = \sum_i |R_i|(1-\lambda)\mathbf{x}'_i = (1-\lambda)\sum_i |R_i|\mathbf{x}'_i$. Consequently, $\frac{\sigma_1(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')}{\sum_i |R_i|(\lambda\mathbf{x}_i+(1-\lambda)\mathbf{x}'_i)} = \frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i}$ and therefore $f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = f(R, \mathbf{x}')$. If $\sum_i |R_i|\mathbf{x}_i > 0$ and $\sum_i |R_i|\mathbf{x}'_i > 0$, note that for $q \in [0,1]$, $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q) =$

$\tau(\sigma_1(R,\mathbf{x}) - q\sum_i |R_i|\mathbf{x}_i)$. Because $\sigma_1(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') - q\sum_i |R_i|(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) = \lambda(\sigma_1(R,\mathbf{x}) - q\sum_i |R_i|\mathbf{x}_i) + (1-\lambda)(\sigma_1(R,\mathbf{x}') - q\sum_i |R_i|\mathbf{x}'_i)$, it follows that $\tau(\sigma_1(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') - q\sum_i |R_i|(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i)) \in (\tau(\sigma_1(R,\mathbf{x}) - q\sum_i |R_i|\mathbf{x}_i), \tau(\sigma_1(R,\mathbf{x}') - q\sum_i |R_i|\mathbf{x}'_i))$. This proves that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$.

Anonymity. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. For $k \in \mathcal{R}$, $\sum_{i:R_i=k} \mathbf{x}_i = \sum_{\pi(i):R_{\pi(i)}=k} \mathbf{x}_{\pi(i)} = \sum_{i:R_{\pi(i)}=k} \mathbf{x}_{\pi(i)}$. Thus $\sigma(R,\mathbf{x}) = \sigma(\pi R, \pi\mathbf{x})$. Consequently, $f(R,\mathbf{x}) = \mathbf{g}^{pqrst}(\sigma(R,\mathbf{x})) = \mathbf{g}^{pqrst}(\sigma(\pi R, \pi\mathbf{x})) = f(\pi R, \pi\mathbf{x})$.

Unanimity. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R,\mathbf{x}) = 1$. Consequently, $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R|\mathbf{x}_i} = \max\{k, 0\}$. This implies that $f(R,\mathbf{x}) = 1$ if $\tau(\max\{k,0\} - r) > -t$ and that $f(R,\mathbf{x}) = -1$ if $\tau(\max\{k,0\} - q) < -s$. If $k = 1$ then $\tau(\max\{k,0\} - r) > -t$, and $f(R,\mathbf{x}) = 1$. If $k = -1$ then $\tau(\max\{k,0\} - q) < -s$, and $f(R,\mathbf{x}) = -1$. Thus, $f(R,\mathbf{x}) = k$.

Repurchase. Let $N \in \mathcal{N}$ and $(R,\mathbf{x}), (R,\mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_j |R_j|\mathbf{x}_j > 0$ and, for all $i \in N$, $|R_i|\mathbf{x}_i = |R_i|\mathbf{x}'_i \sum_j |R_j|\mathbf{x}_j$. Then, for $k \in \{-1, 1\}$, $\sigma_k(R,\mathbf{x}) = \sum_j |R_j|\mathbf{x}_j \sigma_k(R,\mathbf{x}')$. Note that $\sum_i |R_i|\mathbf{x}_i = \sum_i |R_i|\mathbf{x}'_i \sum_j |R_j|\mathbf{x}_j = \sum_j |R_j|\mathbf{x}_j \sum_i |R_i|\mathbf{x}'_i$. It follows that $\frac{s_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} = \frac{(\sum_j |R_j|\mathbf{x}_j) s_1(R,\mathbf{x})}{(\sum_j |R_j|\mathbf{x}_j)\sum_i |R_i|\mathbf{x}'_i} = \frac{s_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i}$. Therefore $f(R,\mathbf{x}) = f(R,\mathbf{x}')$.

Independence of the Axioms: That the four axioms are independent follows from Claims 1, 3, 5, and 7. $\square$

The next lemma was described in the body of the paper.

Lemma 4. *A shareholder voting rule satisfies reallocation invariance only if it satisfies anonymity.*

Proof of Lemma 4. Let f be a shareholder voting rule that satisfies reallocation invariance. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Without loss of generality, let $N = \{1, ..., n\}$. Suppose, contrariwise, that $f(R,\mathbf{x}) \neq f(\pi R, \pi\mathbf{x})$.

Step one: I show that if there is a set $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, then for $(R',\mathbf{x}') \in \mathcal{Q}_{N'}$, if there is a one-to-one mapping $\omega : N' \to N$ such that $(R,\mathbf{x}) = (\omega R', \omega\mathbf{x}')$, then $f(R,\mathbf{x}) = f(R',\mathbf{x}')$.

Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$. Without loss of generality, let $N' = \{n+1, ..., 2n\}$. Let $R' \in \mathcal{R}^{N'}$ and $\mathbf{x}' \in \Delta(N')$ such that $R'_i = R_{i-n}$ and $\mathbf{x}'_i = \mathbf{x}_{i-n}$ for $i \in N'$. For $\omega(i) = n - i$, $(R,\mathbf{x}) = (\omega R', \omega\mathbf{x}')$.

Let $R^* \in \mathcal{R}^{N \cup N'}$ such that $R^*_i = R_i$ for $i \in N$ and $R^*_i = R'_i$ for $i \in N'$. Let $\mathbf{x}^{\circ\circ} \in \Delta(N \cup N')$ such that (a) $\mathbf{x}^{\circ\circ}_i = \mathbf{x}_i$ for $i \in N$ and (b) $\mathbf{x}^{\circ\circ}_i = 0$ for $i \in N'$. Note that $(R^*, \mathbf{x}^{\circ\circ})|_N = (R,\mathbf{x})$. For $k \in \mathcal{R}$, let $S^k \equiv \{i \in N \cup N' : R^*_i = k\}$.

Let $\mathbf{x}^{\circ\bullet} \in \Delta(N \cup N')$ such that (a) for $i \in N$, $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i = 1$ and $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}_i$ if $R^*_i \neq 1$, and (b) for $i \in N'$, $\mathbf{x}^{\circ\bullet}_i = \mathbf{x}'_i$ if $R^*_i = 1$ and $\mathbf{x}^{\circ\bullet}_i = 0$ if $R^*_i \neq 1$. For all $i \notin S^1$, $\mathbf{x}^{\circ\circ}_i = \mathbf{x}^{\circ\bullet}_i$. By reallocation invariance, $f(R^*, \mathbf{x}^{\circ\circ}) = f(R^*, \mathbf{x}^{\circ\bullet})$.

Let $\mathbf{x}^{\bullet\circ} \in \Delta(N \cup N')$ such that (a) for $i \in N$, $\mathbf{x}_i^{\bullet\circ} = 0$ if $R_i^* \neq 0$ and $\mathbf{x}_i^{\bullet\circ} = \mathbf{x}_i$ if $R_i^* = 0$, and (b) for $i \in N'$, $\mathbf{x}_i^{\bullet\circ} = \mathbf{x}'_i$ if $R_i^* \neq 0$ and $\mathbf{x}_i^{\bullet\circ} = 0$ if $R_i^* = 0$. For all $i \notin S^{-1}$, $\mathbf{x}_i^{\circ\bullet} = \mathbf{x}_i^{\bullet\circ}$. By reallocation invariance, $f(R^*, \mathbf{x}^{\circ\bullet}) = f(R^*, \mathbf{x}^{\bullet\circ})$.

Let $\mathbf{x}^{\bullet\bullet} \in \Delta(N \cup N')$ such that $\mathbf{x}_i^{\bullet\bullet} = 0$ for $i \in N$ and $\mathbf{x}_i^{\bullet\bullet} = \mathbf{x}'_i$ for $i \in N'$. For all $i \notin S^0$, $\mathbf{x}_i^{\bullet\circ} = \mathbf{x}_i^{\bullet\bullet}$. By reallocation invariance, $f(R^*, \mathbf{x}^{\bullet\circ}) = f(R^*, \mathbf{x}^{\bullet\bullet})$. It follows that $f(R, \mathbf{x}) = f\left((R^*, \mathbf{x}^{\bullet\bullet})|_{N'}\right) = f(R', \mathbf{x}')$.

Step two: Let $N' \in \mathcal{N}$ such that $|N'| = |N|$ and $N' \cap N = \varnothing$, and let ω be a one-to-one mapping from N' to N. Let $(R', \mathbf{x}') \in \mathcal{Q}_{N'}$ such that $(R, \mathbf{x}) = (\omega R', \omega \mathbf{x}')$. It follows from step one that $f(R, \mathbf{x}) = f(R', \mathbf{x}')$.

Let ω' be a one-to-one mapping from N' to N such that for all $i \in N'$, $\omega'(i) = \pi(\omega(i))$. Then $(\pi R, \pi \mathbf{x}) = (\omega' R', \omega' \mathbf{x}')$. It follows from step one that $f(\pi R, \pi \mathbf{x}) = f(R', \mathbf{x}')$. Therefore, $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$. $\square$

Lemma 5. *A shareholder voting rule f satisfies reallocation invariance if and only if there is $\mathbf{g} \in \mathscr{G}$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}$.*

Proof of Lemma 5: **Only if:** Let f satisfy reallocation invariance. By Lemma 4, f satisfies anonymity. Thus by Lemma 1, there is a $\mathbf{g} \in \mathscr{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}^3$. Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N$. Without loss of generality, assume that $\{1,2,3\} \cap N = \varnothing$. I show that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

Let $(R^*, \mathbf{x}^*) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R_1^* = 1$, $R_2^* = -1$, $R_3^* = 0$, $\mathbf{x}_1^* = \sigma_1(R, \mathbf{x})$, $\mathbf{x}_2^* = \sigma_{-1}(R, \mathbf{x})$, and $\mathbf{x}_3^* = \sigma_0(R, \mathbf{x})$. Define $N^+ \equiv \{1,2,3\} \cup N$.

Let $(R', \mathbf{x}'), (R', \mathbf{x}'') \in \mathcal{Q}_{N^+}$ such that (a) for $i \in \{1,2,3\}$, $R'_i = R^*_i$, $\mathbf{x}'_i = \mathbf{x}^*_i$, and $\mathbf{x}''_i = 0$, and (b) for $j \in N$, $R'_j = R_j$, $\mathbf{x}'_j = 0$, and $\mathbf{x}''_j = \mathbf{x}_j$.

For $k \in R$, define $S^k \equiv \{i \in N^+ : R'_i = k\}$. Let $\mathbf{x}^\circ \in \Delta(N^+)$ such that $\mathbf{x}_1^\circ = \mathbf{x}'_1$, $\mathbf{x}_i^\circ = 0$ for $i \in S^1 \backslash \{1\}$, and $\mathbf{x}_j^\circ = \mathbf{x}''_j$ for $j \notin S^1$. Let $\mathbf{x}^\bullet \in \Delta(N^+)$ such that $\mathbf{x}_2^\bullet = \mathbf{x}'_2$, $\mathbf{x}_i^\bullet = 0$ for $i \in S^{-1} \backslash \{2\}$, and $\mathbf{x}_j^\bullet = \mathbf{x}_j^\circ$ for $j \notin S^{-1}$.

Because $R'_i = R'_j$ for all $i, j \in S^1$ and because $\mathbf{x}_k^\circ = \mathbf{x}''_k$ for $k \notin S^1$, reallocation invariance implies that $f(R', \mathbf{x}^\circ) = f(R', \mathbf{x}'')$. Because $R'_i = R'_j$ for all $i, j \in S^{-1}$ and because $\mathbf{x}_k^\bullet = \mathbf{x}_k^\circ$ for $k \notin S^{-1}$, reallocation invariance implies that $f(R', \mathbf{x}^\bullet) = f(R', \mathbf{x}^\circ)$. Because $R'_i = R'_j$ for all $i, j \in S^0$ and because $\mathbf{x}'_k = \mathbf{x}_k^\bullet$ for $k \notin S^0$, reallocation invariance implies that $f(R', \mathbf{x}') = f(R', \mathbf{x}^\bullet)$. Hence, $f(R', \mathbf{x}') = f(R', \mathbf{x}'')$. By invariance to non-shareholders, it follows that $f(R, \mathbf{x}) = f((R', \mathbf{x}'')|_N) = f((R', \mathbf{x}')|_{\{1,2,3\}}) = f(R^*, \mathbf{x}^*)$. Because $(R^*, \mathbf{x}^*) \in \mathcal{Q}^3$, it follows that $f(R^*, \mathbf{x}^*) = \mathbf{g}(\sigma(R, \mathbf{x}))$. Therefore, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$.

If: Let f be a shareholder voting rule and let $\mathbf{g} \in \mathscr{G}$ such that $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$ for all $(R, \mathbf{x}) \in \mathcal{Q}$. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $S \subseteq N$ such that, for all $i, j \in S$, $R_i = R_j$ and for all $k \notin S$, $\mathbf{x}_k = \mathbf{x}'_k$. For all $k \in \mathcal{R}$, $\sum_{i:R_i=k} \mathbf{x}_i = \sum_{i:R_i=k} \mathbf{x}'_i$, therefore $\sigma_k(R, \mathbf{x}) = \sigma_k(R, \mathbf{x}')$. Consequently, $\mathbf{g}(\sigma(R, \mathbf{x})) = \mathbf{g}(\sigma(R, \mathbf{x}'))$. It follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. $\square$

Proof of Theorem 2. **Only if:** Let f satisfy reallocation invariance, unanimity, repurchase, and strategyproofness.

First, I show that f satisfies merger on $\mathcal{Q}^2$. Let $N \in \mathcal{N}^2$, $j, k \in N$, and $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Let $\mathbf{x}, \mathbf{x}', \mathbf{x}'' \in \Delta(N)$ such that $\mathbf{x}_j < \mathbf{x}'_j < \mathbf{x}''_j$ and $f(R, \mathbf{x}) = f(R, \mathbf{x}'')$. Note that $(R, \mathbf{x}), (R, \mathbf{x}'), (R, \mathbf{x}'') \in \mathcal{Q}^2 \cap \mathcal{Q}_N$.

Let $N' = N \cup \{\ell\}$. Let $R', R'' \in \mathcal{R}^{N'}$ such that $R'_j = R''_j = R''_\ell = 1$ and $R'_k = R''_k = R'_\ell = -1$. Note that $R'' = [R'_{-\ell}, R''_\ell]$. By strategyproofness, for $\dot{\mathbf{x}} \in \Delta(N')$, it cannot be that $R'_\ell \leqslant f(R'', \dot{\mathbf{x}}) < f(R', \dot{\mathbf{x}})$. Because $R'_\ell = -1$, it follows that $f(R', \dot{\mathbf{x}}) \leqslant f(R'', \dot{\mathbf{x}})$.

Let $\mathbf{x}^*, \mathbf{x}^{**}, \mathbf{y}, \mathbf{y}', \mathbf{y}'' \in \Delta(N')$ such that $\mathbf{x}^* = (\mathbf{x}_j, \mathbf{x}'_k, \mathbf{x}'_j - \mathbf{x}_j)$, $\mathbf{x}^{**} = (\mathbf{x}'_j, \mathbf{x}''_k, \mathbf{x}''_j - \mathbf{x}'_j)$, $\mathbf{y} = (\mathbf{x}_j, \mathbf{x}_k, 0)$, $\mathbf{y}' = (\mathbf{x}'_j, \mathbf{x}'_k, 0)$, and $\mathbf{y}'' = (\mathbf{x}''_j, \mathbf{x}''_k, 0)$.

Because $R'_k = R'_\ell$ and $\mathbf{y}_j = \mathbf{x}^*_j$, reallocation invariance implies that $f(R', \mathbf{y}) = f(R', \mathbf{x}^*)$. By invariance to non-shareholders, $f(R', \mathbf{y}) = f(R, \mathbf{x})$. Because $R''_j = R''_\ell$ and $\mathbf{y}'_k = \mathbf{x}^*_k$, reallocation invariance implies that $f(R'', \mathbf{y}') = f(R'', \mathbf{x}^*)$. By invariance to non-shareholders, $f(R'', \mathbf{y}') = f(R, \mathbf{x}')$. Because $f(R', \mathbf{x}^*) \leqslant f(R'', \mathbf{x}^*)$, it follows that $f(R, \mathbf{x}) \leqslant f(R, \mathbf{x}')$.

Because $R'_k = R'_\ell$ and $\mathbf{y}'_j = \mathbf{x}^{**}_j$, reallocation invariance implies that $f(R', \mathbf{y}') = f(R', \mathbf{x}^{**})$. By invariance to non-shareholders, $f(R', \mathbf{y}') = f(R, \mathbf{x}')$. Because $R''_j = R''_\ell$ and $\mathbf{y}''_k = \mathbf{x}^{**}_k$, reallocation invariance implies that $f(R'', \mathbf{y}'') = f(R'', \mathbf{x}^{**})$. By invariance to non-shareholders, $f(R'', \mathbf{y}'') = f(R, \mathbf{x}'')$. Because $f(R', \mathbf{x}^{**}) \leqslant f(R'', \mathbf{x}^{**})$, it follows that $f(R, \mathbf{x}') \leqslant f(R, \mathbf{x}'')$, and therefore, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = f(R, \mathbf{x}'')$. Because there is $\lambda \in (0, 1)$ such that $\mathbf{x}' = \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'$ it follows that f satisfies merger on $\mathcal{Q}^2$.

By Lemma 4, reallocation invariance implies anonymity. Because f satisfies anonymity, unanimity, repurchase, and merger on $\mathcal{Q}^2$, it follows from Lemma 2 that there is $\mathbf{g} \in \mathscr{G}^T$ such that f is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. Consequently, by Lemma 5, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$; that is, it is a thresholds rule.

If: Let f be a thresholds rule with constants p, q, r, s, and t. By Theorem 1, f satisfies unanimity and repurchase. By Lemma 5, it follows that f satisfies reallocation invariance. I show that f satisfies strategyproofness.

Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. I show that neither $R_i \leqslant f(R', \mathbf{x}) < f(R, \mathbf{x})$ nor $R_i \geqslant f(R', \mathbf{x}) > f(R, \mathbf{x})$. Note that (i) $\sigma_{R_i}(R, \mathbf{x}) = \sigma_{R_i}(R', \mathbf{x}) + \mathbf{x}_i$ for $k \neq R_i$ and $\sigma_{R_i}(R, \mathbf{x}) = \sigma_{R_i}(R', \mathbf{x})$ for $k = R_i$, and (ii) $\sum_j |R_j|\mathbf{x}_j = \sum_j |R'_j|\mathbf{x}_j + \mathbf{x}_i$ for $k = 0$ and $\sum_j |R_j|\mathbf{x}_j = \sum_j |R'_j|\mathbf{x}_j$ for $k \neq 0$.

If $\sum_j |R_j|\mathbf{x}_j = 0$ then $\mathbf{x}_i = 0$, consequently $\sum_j |R'_j|\mathbf{x}_j = 0$. Therefore $f(R, \mathbf{x}) = f(R', \mathbf{x})$. If $\sum_j |R_j|\mathbf{x}_j > 0$ and $\sum_j |R'_j|\mathbf{x}_j = 0$, then $R_j\mathbf{x}_j = 0$ for all $j \neq i$ and therefore $f(R, \mathbf{x}) = R_i$.

If $\sum_j |R_j|\mathbf{x}_j > 0$ and $\sum_j |R'_j|\mathbf{x}_j > 0$, then there are three cases. Case 1:

$R_i = k$. In this case $R = R'$ and thus $f(R,\mathbf{x}) = f(R',\mathbf{x})$. Case 2: $R_i = -k$. In this case, $\sum_j |R_j|\mathbf{x}_j = \sum_j |R'_j|\mathbf{x}_j$ and $\sigma_1(R,\mathbf{x}) = \sigma_1(R',\mathbf{x}) + R_i\mathbf{x}_i$. If $R_i = 1$ then $\frac{\sigma_1(R,\mathbf{x})}{\sum_j |R_j|\mathbf{x}_j} \geqslant \frac{\sigma_1(R',\mathbf{x})}{\sum_j |R'_j|\mathbf{x}_j}$, and therefore $R_i \geqslant f(R,\mathbf{x}) \geqslant f(R',\mathbf{x})$. If $R_i = -1$ then $\frac{\sigma_1(R,\mathbf{x})}{\sum_j |R_j|\mathbf{x}_j} \leqslant \frac{\sigma_1(R',\mathbf{x})}{\sum_j |R'_j|\mathbf{x}_j}$, and therefore $R_i \leqslant f(R,\mathbf{x}) \leqslant f(R',\mathbf{x})$. Case 3: $k = 0$. If $R_i = 1$ then, by (i) and (ii), $\frac{\sigma_1(R,\mathbf{x})}{\sum_j |R_j|\mathbf{x}_j} = \frac{\sigma_1(R',\mathbf{x})+\mathbf{x}_i}{\sum_j |R'_j|\mathbf{x}_j+\mathbf{x}_i} \geqslant \frac{\sigma_1(R',\mathbf{x})}{\sum_j |R'_j|\mathbf{x}_j}$, and therefore $R_i \geqslant f(R,\mathbf{x}) \geqslant f(R',\mathbf{x})$. If $R_i = -1$ then $\frac{\sigma_1(R,\mathbf{x})}{\sum_j |R_j|\mathbf{x}_j} = \frac{\sigma_1(R',\mathbf{x})}{\sum_j |R'_j|\mathbf{x}_j+\mathbf{x}_i} \leqslant \frac{\sigma_1(R',\mathbf{x})}{\sum_j |R'_j|\mathbf{x}_j}$, and therefore $R_i \leqslant f(R,\mathbf{x}) \leqslant f(R',\mathbf{x})$.

Independence of the Axioms: That the four axioms are independent follows from Claims 1, 5, 7, and 9. $\square$

Proof of Proposition 1. **Only if:** Let f satisfy reallocation invariance, unanimity, repurchase, and share monotonicity.

First I show that f satisfies merger on $\mathcal{Q}^2$. Let $N \in \mathcal{N}^2$, let $j,k \in N$, and let $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Let $\mathbf{x},\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}_j < \mathbf{x}'_j$ and such that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Then $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}^2 \cap \mathcal{Q}_N$. Note that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \lambda\mathbf{x}_j + (1-\lambda)\mathbf{x}'_j$ and (d) $\lambda\mathbf{x}_j + (1-\lambda)\mathbf{x}'_j < \mathbf{x}'_j$.

First, if $f(R,\lambda\mathbf{x} + (1-\lambda)\mathbf{x}') \geqslant 0$, then share monotonicity along with (a), (b), and (d) imply that that $f(R,\mathbf{x}') = 1$. Therefore, $f(R,\mathbf{x}) = 1$. Consequently, share monotonicity along with (a), (b), and (c) imply that $f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') = 1$. Second, if $f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') = -1$ but $f(R,\mathbf{x}) = f(R,\mathbf{x}') \geqslant 0$, then share monotonicity along with (a), (b), and (c) imply that $f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') = 1$, a contradiction. Thus $f(R,\mathbf{x}) = f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')$ and therefore f satisfies merger on $\mathcal{Q}^2$.

By Lemma 4, reallocation invariance implies anonymity. Because f satisfies anonymity, unanimity, repurchase, and merger on $\mathcal{Q}^2$, it follows from Lemma 2 that there is $\mathbf{g} \in \mathscr{G}^T$ such that$f$ is $\mathbf{g}$-reducible on $\mathcal{Q}^3$. Consequently, by Lemma 5, it follows that f is $\mathbf{g}$-reducible on $\mathcal{Q}$. Thus, there are constants p, q, r, s, and t such that f is a thresholds rule.

To show that f is a supermajority rule, suppose by means of contradiction that $p < q$. Let $N \in \mathcal{N}$ such that $|N| = 2$, let $j,k \in N$, and let $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Note that for all $\mathbf{x}$, $\sum_i |R_i|\mathbf{x}_i = 1$ and $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} = \mathbf{x}_j$.

Let $\mathbf{x},\mathbf{x}' \in \Delta(N)$ such that $\mathbf{x}_j = \frac{1}{3}(2q+r)$ and $\mathbf{x}'_j = \frac{1}{3}(q+2r)$. It follows that (a) $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q = \frac{1}{3}(r-q)$, (b) $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - r = \frac{2}{3}(q-r)$, (c) $\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} - q = \frac{2}{3}(r-q)$, and (d) $\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} - r = \frac{1}{3}(q-r)$. Because, by supposition, $q < r$, it follows that (a) and (c) are positive and that (b) and (d) are negative. Because (a) and (c) are positive, it follows that $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q) = \tau(\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} - q) = 1$ and thus $f(R,\mathbf{x}) \geqslant 0$ and $f(R,\mathbf{x}') \geqslant 0$. Because (b) and (d) are negative, it follows that

$\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - r) = \tau(\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} - r) = -1$ and therefore that $f(R,\mathbf{x}) = f(R,\mathbf{x}') = 0$. However, because $R_j = 1$, $R_k = -1$, $\mathbf{x}_j < \mathbf{x}'_j$, and $f(R,\mathbf{x}) = 0$; share monotonicity implies that $f(R,\mathbf{x}') = 1$, a contradiction.

If: Let f be a thresholds rule with constants p, q, r, s, and t such that $q = r$. By Theorem 2, f satisfies unanimity and repurchase and reallocation invariance. I show that f satisfies share monotonicity.

Let $N \in \mathcal{N}$, $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$, and $j,k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \neq j,k$, and (e) $f(R,\mathbf{x}) \in \{0,1\}$. I show that $f(R,\mathbf{x}') = 1$.

From (e) it follows that $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q) \geqslant -s$, which implies that $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} \geqslant q$. From (a-d) it follows that $\sigma_1(R,\mathbf{x}) < \sigma_1(R,\mathbf{x}')$. From (c-d) it follows that either (i) $\sum_i |R_i|\mathbf{x}'_i = \sum_i |R_i|\mathbf{x}_i$ or (ii) $\sum_i |R_i|\mathbf{x}'_i = \sum_i |R_i|\mathbf{x}_i + \sigma_1(R,\mathbf{x}') - \sigma_1(R,\mathbf{x})$. If (i) then $\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} > \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i}$. If (ii) then $\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} = \frac{\sigma_1(R,\mathbf{x}) + (\sigma_1(R,\mathbf{x}') - \sigma_1(R,\mathbf{x}))}{\sum_i |R_i|\mathbf{x}_i + (\sigma_1(R,\mathbf{x}') - \sigma_1(R,\mathbf{x}))} > \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i}$. Thus $\frac{\sigma_1(R,\mathbf{x}')}{\sum_i |R_i|\mathbf{x}'_i} > q = r$ which implies that $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - r) = 1 > -t$. Therefore $f(R,\mathbf{x}') = 1$.

Independence of the Axioms: A thresholds rule for which $q \neq r$ satisfies reallocation invariance, repurchase, and unanimity by Theorem 2, but fails share monotonicity because it is not a supermajority rule. This fact, in conjunction with Claims 1, 5, and 8, is sufficient to establish the independence of the axioms. □

Proof of Proposition 2. **Only if:** Let f be a thresholds rule with constants p, q, r, s, and t that satisfies neutrality but is not a balanced threshold rule. I will derive a contradiction. Let $N \in \mathcal{N}$ such that $|N| = 2$, let $j,k \in N$, and let $R \in \mathcal{R}^N$ such that $R_j = 1$ and $R_k = -1$. Note that for all $\mathbf{x}$, $\sum_i |R_i|\mathbf{x}_i = 1$, $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} = \mathbf{x}_j$, and $\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |-R_i|\mathbf{x}_i} = 1 - \mathbf{x}_j$. Let $\mathbf{x} \in \Delta(N)$ such that $\mathbf{x}_j = \frac{1}{2}(q - r + 1)$.

Then $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i|\mathbf{x}_i} - q = \frac{1}{2}(-q - r + 1)$. Consequently $f(R,\mathbf{x}) = -1$ if and only if (a) $\tau(1 - q - r) < -s$. Furthermore, $\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |-R_i|\mathbf{x}_i} - r = \frac{1}{2}(-q - r + 1)$. Consequently $f(-R,\mathbf{x}) = 1$ if and only if (b) $\tau(1 - q - r) > -t$.

Neutrality implies that $f(R,\mathbf{x}) = -f(-R,\mathbf{x})$. Therefore (a) holds if and only if (b) holds. It follows from neutrality that (a) $\tau(1 - q - r) < -s$ if and only if (b) $\tau(1 - q - r) > -t$. If $1 - q - r < 0$ then (a) holds (for all s) but not (b) (for any t). Thus $1 - q - r \geqslant 0$. If $1 - q - r > 0$ then (b) holds (for all t) but not (a) (for any s). Thus $1 - q - r = 0$. If $s = 0$ then (a) holds, which implies that (b) holds, which implies that $t = 0$. If $s = -1$ then (a) does not hold, which implies that (b) does not hold, which implies that $t = 1$.

Let $N' \in \mathcal{N}$ such that $|N'| = 1$, let $\ell \in N'$, let $R \in \mathcal{R}^{N'}$ such that $R_\ell = 0$, and let $\mathbf{x} \in \Delta(N')$ such that $\mathbf{x}_\ell = 1$. Because f is not a balanced threshold rule, it follows that $p \neq 0$. By the definition of the thresholds rule, $f(R, \mathbf{x}) = p$. By neutrality, $f(-R, \mathbf{x}) = -p$. Because $R = -R$, this implies that $p = -p \neq 0$, a contradiction.

If: Let f be a thresholds rule with constants p, q, r, s, and t such that $p = 0$, $q = 1 - r$, and $s = -t$. I show that for all $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$, that $f(R, \mathbf{x}) \geqslant 0$ implies that $f(-R, \mathbf{x}) \leqslant 0$ and that $f(R, \mathbf{x}) \leqslant 0$ implies that $f(-R, \mathbf{x}) \geqslant 0$.

Let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$. If $f(R, \mathbf{x}) \geqslant 0$ then $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - q) \geqslant -s$. This would imply that $\tau(-\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} + q) \leqslant s$, which would imply that $\tau(\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - 1 + q) \leqslant s$. Substituting $r = 1 - q$ and $s = -t$ we have that $\tau(\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - r) \leqslant -t$. Consequently, we know that (a) $f(R, \mathbf{x}) \geqslant 0$ implies that $f(-R, \mathbf{x}) \leqslant 0$. Because (a) must hold for all $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$, it follows that (b) $f(-R, \mathbf{x}) \geqslant 0$ implies that $f(R, \mathbf{x}) \leqslant 0$.

Next, if $f(R, \mathbf{x}) \leqslant 0$ then $\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - r) \leqslant -t$. This would imply that $-\tau(\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} + r) \geqslant t$, which would imply that $\tau(\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - 1 + r) \geqslant t$. Substituting $r = 1 - q$ and $s = -t$ we have that $\tau(\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} - q) \geqslant -s$. Consequently, we know that (c) $f(R, \mathbf{x}) \leqslant 0$ implies that $f(-R, \mathbf{x}) \geqslant 0$. Because (c) must hold for all $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$, it follows that (d) $f(-R, \mathbf{x}) \leqslant 0$ implies that $f(R, \mathbf{x}) \geqslant 0$. Statements (a), (b), (c), and (d) together imply that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$ for all $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i > 0$.

Finally, let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $\sum_i |R_i| \mathbf{x}_i = 0$. Then $f(R, \mathbf{x}) = p = 0$. Because $\sum_i |R_i| \mathbf{x}_i = 0 = \sum_i | - R_i| \mathbf{x}_i$, it follows that $f(-R, \mathbf{x}) = 0 = -f(R, \mathbf{x})$.

Independence of the Axioms: A thresholds rule for which $r \neq 1 - q$ satisfies merger, anonymity, unanimity, and repurchase by Theorem 1, but fails neutrality because it is not a balanced threshold rule. This fact, in conjunction with Claims 1, 3, 5, and 7, is sufficient to establish the independence of the axioms. $\square$

Proof of Corollary 1. **Only If:** Let f satisfy merger, anonymity, unanimity, repurchase, and share monotonicity. By Theorem 1 f must be a thresholds rule. By Theorem 2 it follows that f satisfies reallocation invariance. Consequently, by Proposition 1 it follows that f is a supermajority rule.

If: Let f be a supermajority rule. Then it is a thresholds rule and therefore satisfies merger, anonymity, unanimity, and repurchase by Theorem 1, By Proposition 1 it follows that f satisfies share monotonicity.

Independence of the Axioms: A thresholds rule for which $q \neq r$ satisfies merger, anonymity, unanimity, and repurchase by Theorem 1, but fails share monotonicity because it is not a supermajority rule. This fact, in conjunction with Claims 1, 3, 5, and 8, is sufficient to establish the independence of the axioms. $\square$

Proof of Corollary 2. **Only If:** Let f satisfy reallocation independence, unanimity, repurchase, strategyproofness, and neutrality. By Theorem 2 f must be a thresholds rule. By Theorem 1 it follows that f satisfies merger and anonymity. Consequently, by Proposition 2 it follows that f is a balanced threshold rule.

If: Let f be a balanced threshold rule. Then it is a thresholds rule and therefore satisfies reallocation invariance, unanimity, repurchase, and strategyproofness by Theorem 1, By Proposition 2 it follows that f satisfies neutrality.

Independence of the Axioms: A thresholds rule for which $r \neq 1 - q$ satisfies reallocation invariance, unanimity, repurchase, and strategyproofness by Theorem 1, but fails neutrality because it is not a balanced threshold rule. This fact, in conjunction with Claims 1, 5, 7, and 9, is sufficient to establish the independence of the axioms. $\square$

Let $\mathbf{g}^{sm} \in \mathscr{G}^T$ such that $\mathbf{g}^{sm}(\sigma(R, \mathbf{x}))$ is the shareholder majority rule.

Lemma 6. *A shareholder voting rule satisfies anonymity, neutrality, and share monotonicity only if it is* $\mathbf{g}^{sm}$*-reducible on* $\mathcal{Q}^3$.

Proof of Lemma 6. Let f satisfy anonymity, neutrality, and share monotonicity. Let $N \in \mathcal{N}^3$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N \cap \mathcal{Q}^3$. Let $j, k, \ell \in N$ such that $R_j = 1$, $R_k = -1$, and $R_\ell = 0$. Let $\pi \in \Pi_N$ such that $\pi(j) = k$ and $\pi(k) = j$. Note that $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = \mathbf{x}_j - \mathbf{x}_k$.

Step one: I show that $f(R, \mathbf{x}) = -f(R, \pi\mathbf{x})$. By anonymity, $f(R, \mathbf{x}) = f(\pi R, \pi\mathbf{x})$. Because $\pi R = -R$, it follows that $f(R, \mathbf{x}) = f(-R, \pi\mathbf{x})$. By neutrality, $f(-R, \pi\mathbf{x}) = -f(R, \pi\mathbf{x})$, and therefore $f(R, \mathbf{x}) = -f(R, \pi\mathbf{x})$.

Step two: I show that if $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = 0$ then $f(R, \mathbf{x}) = 0$. Let $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = 0$. Then $\mathbf{x}_j = \mathbf{x}_k$, which implies that $\mathbf{x} = \pi\mathbf{x}$. From step one it follows that $f(R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$.

Step three: I show that if $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = 1$ then $f(R, \mathbf{x}) = 1$. Let $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = 1$ and assume contrariwise that that $f(R, \mathbf{x}) \neq 1$. Then by step one, $f(R, \pi\mathbf{x}) \in (0, 1)$. Because $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = 1$, it follows that $\mathbf{x}_j > \mathbf{x}_k$. Because (a) $R_j = 1$, (b) $R_k = -1$, (c) $\pi\mathbf{x}_j < \mathbf{x}_j$, and (d) $\pi\mathbf{x}_\ell = \mathbf{x}_\ell$, it follows from share monotonicity that $f(R, \mathbf{x}) = 1$, a contradiction.

Step four: I show that if $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = -1$ then $f(R, \mathbf{x}) = -1$. Let $\mathbf{g}^{sm}(\sigma(R, \mathbf{x})) = -1$. Then $\mathbf{x}_j < \mathbf{x}_k$. By step three, $f(R, \pi\mathbf{x}) = 1$. By step one, $f(R, \mathbf{x}) = -1$. $\square$

Proof of Theorem 3. **Only if:** Let f satisfy merger, anonymity, share monotonicity, and neutrality. Thus, by Lemma 6, f is $\mathbf{g}^{sm}$-reducible on $\mathcal{Q}^3$. Consequently, by Lemma 3, it follows that f is $\mathbf{g}^{sm}$-reducible on $\mathcal{Q}$; that is, it is the shareholder majority rule.

If: The shareholder majority rule is both a polynomial majority rule and a weighted majority rule. Therefore, it satisfies anonymity and neutrality (by Claim 1) and merger and share monotonicity (by Claim 3).

Independence of the Axioms: That the four axioms are independent follows from Claims 1, 3, and 5. □

Proof of Theorem 4. **Only if:** Let f satisfy reallocation invariance, share monotonicity, and neutrality. By Lemma 4, because f satisfies reallocation invariance it satisfies anonymity. Therefore, by Lemma 6, f is $\mathbf{g}^{sm}$-reducible on $\mathcal{Q}^3$. By Lemma 5, because f is $\mathbf{g}^{sm}$-reducible on $\mathcal{Q}^3$, then f is $\mathbf{g}^{sm}$-reducible on $\mathcal{Q}$; that is, f is the shareholder majority rule.

If: The shareholder majority rule is a polynomial majority rule and a phantom voter rule. Therefore, it satisfies neutrality (by Claim 1) and reallocation invariance and share monotonicity (by Claim 8).

Independence of the Axioms: That the three axioms are independent follows from Claims 1 and 5. □

Proof of Claim 1. I show that all polynomial majority rules satisfy anonymity, unanimity, repurchase, strategyproofness and neutrality, that share monotonicity is satisfied if and only if $\alpha > 0$, and that merger and reallocation invariance are satisfied if and only if $\alpha = 1$.

Anonymity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, $\pi \in \Pi_N$, and $\alpha \in \mathbb{R}_+$. Note that $\sum_{i \in N} R_i(\mathbf{x}_i)^\alpha = \sum_{\pi(i) \in N} R_{\pi(i)}(\mathbf{x}_{\pi(i)})^\alpha = \sum_{i \in N} R_{\pi(i)}(\mathbf{x}_{\pi(i)})^\alpha$; thus $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

Unanimity. Let $k \in \{-1, 1\}$ and let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $\sum_i R_i(\mathbf{x}_i)^\alpha = k \sum_i (\mathbf{x}_i)^\alpha$, and therefore $\tau(\sum_i R_i(\mathbf{x}_i)^\alpha) = \tau(k) = k$.

Repurchase. Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that (a) $\sum_j |R_j| \mathbf{x}_j > 0$ and (b) $|R_i| \mathbf{x}_i = |R_i| \mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j$ for all $i \in N$. Then $\sum_i R_i(\mathbf{x}_i)^\alpha = \sum_i R_i(\mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j)^\alpha = (\sum_j |R_j| \mathbf{x}_j)^\alpha \sum_i R_i(\mathbf{x}'_i)^\alpha$. Consequently $\tau(\sum_i R_i(\mathbf{x}_i)^\alpha) = \tau(\sum_i R_i(\mathbf{x}'_i)^\alpha)$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}')$

Strategyproofness. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. Note that $\sum_j R_j(\mathbf{x}_j)^\alpha = R_i(\mathbf{x}_i)^\alpha + \sum_{j \neq i} R_j(\mathbf{x}_j)^\alpha$ and that $\sum_j R'_j(\mathbf{x}_j)^\alpha = k(\mathbf{x}_i)^\alpha + \sum_{j \neq i} R_j(\mathbf{x}_j)^\alpha$. Consequently, $\sum_j R_j(\mathbf{x}_j)^\alpha - \sum_j R'_j(\mathbf{x}_j)^\alpha = (R_i - k)(\mathbf{x}_i)^\alpha$. If $R_i = 1$ then $(R_i - k) \geqslant 0$ which implies that $R_i \geqslant f(R, \mathbf{x}) \geqslant f(R', \mathbf{x})$. If $R_i = -1$ then $(R_i - k) \leqslant 0$ which implies that $R_i \leqslant f(R, \mathbf{x}) \leqslant f(R', \mathbf{x})$.

NEUTRALITY. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Because $\sum_i -R_i(\mathbf{x}_i)^\alpha = -\sum_i R_i(\mathbf{x}_i)^\alpha$, it follows that $\tau(\sum_i -R_i(\mathbf{x}_i)^\alpha) = -\tau(\sum_i R_i(\mathbf{x}_i)^\alpha)$, and therefore that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

SHARE MONOTONICITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -1$. If $\alpha = 0$ and $f(R, \mathbf{x}) = 0$, then $f(R, \mathbf{x}') = 0$, a contradiction. Let $\alpha > 0$. From (e) it follows that $\sum_i R_i(\mathbf{x}_i)^\alpha \geqslant 0$. Therefore, $R_j(\mathbf{x}_j)^\alpha + R_k(\mathbf{x}_k)^\alpha \geqslant 0 - \sum_{\ell \in N \backslash \{j,k\}} R_\ell(\mathbf{x}_\ell)^\alpha$. From (c) it follows that $(\mathbf{x}_j)^\alpha < (\mathbf{x}'_j)^\alpha$. From (d) it follows that $\mathbf{x}_j + \mathbf{x}_k = \mathbf{x}'_j + \mathbf{x}'_k$ and thus from (c) that $(\mathbf{x}'_k)^\alpha < (\mathbf{x}_k)^\alpha$. Together this implies that $(\mathbf{x}'_j)^\alpha - (\mathbf{x}'_k)^\alpha > (\mathbf{x}_j)^\alpha - (\mathbf{x}_k)^\alpha$. If $R_k = -1$, then $(\mathbf{x}'_j)^\alpha - (\mathbf{x}'_k)^\alpha > (\mathbf{x}_j)^\alpha - (\mathbf{x}_k)^\alpha$ implies that $R_j(\mathbf{x}'_j)^\alpha + R_k(\mathbf{x}'_k)^\alpha > R_j(\mathbf{x}_j)^\alpha + R_k(\mathbf{x}_k)^\alpha$. If $R_k = 0$, then this is implied by the fact that $(\mathbf{x}_j)^\alpha < (\mathbf{x}'_j)^\alpha$. By (d) it follows that $\sum_{\ell \in N \backslash \{j,k\}} R_\ell(\mathbf{x}_\ell)^\alpha = \sum_{\ell \in N \backslash \{j,k\}} R_\ell(\mathbf{x}'_\ell)^\alpha$. Putting this together, we have that $R_j(\mathbf{x}'_j)^\alpha + R_k(\mathbf{x}'_k)^\alpha > 0 - \sum_{\ell \in N \backslash \{j,k\}} R_\ell(\mathbf{x}'_\ell)^\alpha$, and therefore that $\sum_i R_i(\mathbf{x}'_i)^\alpha > 0$. Therefore $f(R, \mathbf{x}') = 1$.

MERGER. If $\alpha = 1$ then f is the shareholder majority rule, which is a weighted majority rule, and therefore it satisfies merger by Claim 3. Let $\alpha \neq 1$. Let $N = \{1, 2, 3\}$, and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, -1)$, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, and $\mathbf{x}' = (\frac{1}{2}, 0, \frac{1}{2})$. In this case, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. However, $f(R, \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}') = 0$ if and only if $(\frac{1}{2})^\alpha = 2(\frac{1}{4})^\alpha$ which is false for $\alpha \neq 1$.

REALLOCATION INVARIANCE. If $\alpha = 1$ then f is the shareholder majority rule, which is a phantom voter rule, and therefore it satisfies reallocation invariance by Claim 8. Let $\alpha \neq 1$. Let $N = \{1, 2, 3\}$, and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, -1)$, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, and $\mathbf{x}' = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Note that for $S = \{2, 3\}$ the predicate of the axiom is satisfied. However, for $\sum_i R_i(\mathbf{x}_i)^\alpha = 0$ while $\sum_i R_i(\mathbf{x}'_i)^\alpha = (\frac{1}{2})^\alpha - 2(\frac{1}{4})^\alpha$. By reallocation invariance, $(\frac{1}{2})^\alpha - 2(\frac{1}{4})^\alpha = 0$, which is false for $\alpha \neq 1$. $\square$

Proof of Claim 2. I show that all capped majority rules satisfy anonymity, unanimity, strategyproofness, and neutrality, but fail merger, reallocation invariance, repurchase, and share monotonicity.

ANONYMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $\pi \in \Pi_N$. Then $\sum_{i \in N} R_i \min\{\mathbf{x}_i, c\} = \sum_{\pi(i) \in N} R_{\pi(i)} \min\{\mathbf{x}_{\pi(i)}, c\} = \sum_{i \in N} R_{\pi(i)} \min\{\mathbf{x}_{\pi(i)}, c\}$; thus $f(R, \mathbf{x}) = f(\pi R, \pi \mathbf{x})$.

UNANIMITY. Let $k \in \{-1, 1\}$, $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $\sum_i R_i \min\{\mathbf{x}_i, c\} = k \sum_i \min\{\mathbf{x}_i, c\}$, and therefore $\tau(\sum_i R_i \min\{\mathbf{x}_i, c\}) = \tau(k) = k$.

NEUTRALITY. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Because $\sum_i -R_i i \min\{\mathbf{x}_i, c\} = -\sum_i R_i i \min\{\mathbf{x}_i, c\}$, it follows that $\tau(\sum_i -R_i i \min\{\mathbf{x}_i, c\}) = -\tau(\sum_i R_i i \min\{\mathbf{x}_i, c\})$, and therefore that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Strategyproofness. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. Note that $\sum_j R_j \min\{\mathbf{x}_j, c\} = R_i \min\{\mathbf{x}_i, c\} + \sum_{j \neq i} R_j \min\{\mathbf{x}_j, c\}$ and that $\sum_j R'_j \min\{\mathbf{x}_j, c\} = k \min\{\mathbf{x}_i, c\} + \sum_{j \neq i} R_j \min\{\mathbf{x}_j, c\}$. Consequently, $\sum_j R_j \min\{\mathbf{x}_j, c\} - \sum_j R'_j \min\{\mathbf{x}_j, c\} = (R_i - k) \min\{\mathbf{x}_i, c\}$. If $R_i = 1$ then $(R_i - k) \geqslant 0$ which implies that $R_i \geqslant f(R, \mathbf{x}) \geqslant f(R', \mathbf{x})$. If $R_i = -1$ then $(R_i - k) \leqslant 0$ which implies that $R_i \leqslant f(R, \mathbf{x}) \leqslant f(R', \mathbf{x})$.

Repurchase. Let $N = \{1, 2, 3\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$, $R_2 = -1$, and $R_3 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (c, c\frac{1+2c}{3-2c}, \frac{3-6c}{3-2c})$ and $\mathbf{x}' = (\frac{3-2c}{4}, \frac{1+2c}{4}, 0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1(\mathbf{x}_1 + \mathbf{x}_2)$ and $\mathbf{x}_2 = \mathbf{x}'_2(\mathbf{x}_1 + \mathbf{x}_2)$, it follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\frac{3-2c}{4} > \frac{1+2c}{4} > c$ for all $c \in (0, 0.5)$, $\sum_i R_i \min\{\mathbf{x}'_i, c\} = 0$ and consequently, $f(R, \mathbf{x}') = 0$. However, because $\frac{1+2c}{3-2c} < 1$ for all $c \in (0, 0.5)$, $\sum_i R_i \min\{\mathbf{x}'_i, c\} = c - c\frac{1+2c}{3-2c} > 0$, and consequently $f(R, \mathbf{x}) = 1$, a contradiction.

Merger. Let $N = \{1, 2, 3, 4\}$, let $R \in \mathcal{R}^N$ such that $R_1 = R_2 = 1$ and $R_3 = R_4 = -1$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (\frac{c}{2}, \frac{c}{2}, 0, 1 - c)$ and $\mathbf{x}' = (\frac{c}{2}, \frac{c}{2}, 1 - c, 0)$. In this case, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. However, $\sum_i R_i \min\{\frac{1}{2}\mathbf{x}_i + \frac{1}{2}\mathbf{x}'_i, c\} = c - \min\{1 - c, 2c\} < 0$, which implies that $f(R, \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}') = -1$, a contradiction.

Share monotonicity. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = -1$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (\frac{1+2c}{4}, \frac{3-2c}{4})$ and $\mathbf{x}' = (\frac{3-2c}{4}, \frac{1+2c}{4})$. Then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, and $f(R, \mathbf{x}) = 0$, it follows that $f(R, \mathbf{x}') = 1$, a contradiction.

Reallocation invariance. Let $N = \{1, 2, 3\}$, and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, -1)$, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, and $\mathbf{x}' = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$. Note that for $S = \{2, 3\}$ the predicate of the axiom is satisfied. Because $\sum_i R_i \min\{\mathbf{x}_i, c\} = 0$ it follows that $f(R, \mathbf{x}) = 0$. But because $\sum_i R_i \min\{\mathbf{x}'_i, c\} < 0$ it follows that $f(R, \mathbf{x}') = -1$, a contradiction. $\square$

Proof of Claim 3. I show that weighted majority rules satisfy merger, unanimity, repurchase, strategyproofness, share monotonicity, and neutrality, but may fail to satisfy reallocation invariance and anonymity.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0, 1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\sum_i R_i \delta_i (\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i) = \lambda(\sum_i R_i \delta_i \mathbf{x}_i) + (1 - \lambda)(\sum_i R_i \delta_i \mathbf{x}'_i)$, it follows that

$$\max\{\sum_i R_i \delta_i \mathbf{x}_i, \sum_i R_i \delta_i \mathbf{x}'_i\} \geqslant \sum_i R_i \delta_i (\lambda \mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) \geqslant \min\{\sum_i R_i \delta_i \mathbf{x}_i, \sum_i R_i \delta_i \mathbf{x}'_i\},$$

and consequently that $\tau(\sum_i R_i \delta_i \mathbf{x}_i) = \tau(\sum_i R_i \delta_i (\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i))$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}') = f(R, \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}')$.

Unanimity. Let $k \in \{-1, 1\}$ and let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $\sum_i R_i \delta_i \mathbf{x}_i = k \sum_i \delta_i \mathbf{x}_i$, and therefore $\tau(\sum_i R_i \delta_i \mathbf{x}_i) = \tau(k) = k$.

Repurchase. Let $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_j |R_j| \mathbf{x}_j > 0$ and, for all $i \in N$, $|R_i| \mathbf{x}_i = |R_i| \mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j$. Then $\sum_i R_i \delta_i \mathbf{x}_i = \sum_i R_i \delta_i \mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j = (\sum_j |R_j| \mathbf{x}_j) \sum_i R_i \delta_i \mathbf{x}_i$, which implies that $\tau(\sum_i R_i \delta_i \mathbf{x}_i) = \tau(\sum_i R_i \delta_i \mathbf{x}'_i)$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}')$

Strategyproofness. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. Note that $\sum_j R_j \delta_j \mathbf{x}_j = R_i \delta_i \mathbf{x}_i + \sum_{j \neq i} R_j \delta_j \mathbf{x}_j$ and that $\sum_j R'_j \delta_j \mathbf{x}_j = k \delta_i \mathbf{x}_i + \sum_{j \neq i} R_j \delta_j \mathbf{x}_j$. Consequently, $\sum_j R_j \delta_j \mathbf{x}_j - \sum_j R'_j \delta_j \mathbf{x}_j = (R_i - k) \delta_i \mathbf{x}_i$. If $R_i = 1$ then $(R_i - k) \geqslant 0$ which implies that $R_i \geqslant f(R, \mathbf{x}) \geqslant f(R', \mathbf{x})$. If $R_i = -1$ then $(R_i - k) \leqslant 0$ which implies that $R_i \leqslant f(R, \mathbf{x}) \leqslant f(R', \mathbf{x})$.

Share monotonicity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -1$. From (e) it follows that $\sum_i R_i \delta_i \mathbf{x}_i \geqslant 0$. Therefore, $R_j \delta_j \mathbf{x}_j + R_k \delta_k \mathbf{x}_k \geqslant 0 - \sum_{\ell \in N \backslash \{j,k\}} R_\ell \delta_\ell \mathbf{x}_\ell$. From (d) it follows that $\mathbf{x}_j + \mathbf{x}_k = \mathbf{x}'_j + \mathbf{x}'_k$ and thus from (c) it follows that $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$. If $R_k = -1$, then $\mathbf{x}'_j - \mathbf{x}'_k > \mathbf{x}_j - \mathbf{x}_k$ implies that $R_j \delta_j \mathbf{x}'_j + R_k \delta_k \mathbf{x}'_k > R_j \delta_j \mathbf{x}_j + R_k \delta_k \mathbf{x}_k$. If $R_k = 0$, then this fact is implied by (c). By (d) it follows that $\sum_{\ell \in N \backslash \{j,k\}} R_\ell \delta_\ell \mathbf{x}_\ell = \sum_{\ell \in N \backslash \{j,k\}} R_\ell \delta_\ell \mathbf{x}'_\ell$. Putting this together, we have that $R_j \delta_j \mathbf{x}'_j + R_k \delta_k \mathbf{x}'_k > 0 - \sum_{\ell \in N \backslash \{j,k\}} R_\ell \delta_\ell \mathbf{x}'_\ell$, and therefore that $\sum_i R_i \delta_i \mathbf{x}'_i > 0$. Therefore $f(R, \mathbf{x}') = 1$.

Neutrality. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Because $\sum_i -R_i \delta_i \mathbf{x}_i = -\sum_i R_i \delta_i \mathbf{x}_i$, it follows that $\tau(\sum_i -R_i \delta_i \mathbf{x}_i) = -\tau(\sum_i R_i \delta_i \mathbf{x}_i)$, and therefore that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Reallocation invariance. Because weighted majority rules may fail to satisfy anonymity, they may fail to satisfy reallocation invariance, by Lemma 4.

Anonymity. Let $(R, \mathbf{x}) \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1, -1)$ and $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$, let $\delta = (\frac{2}{3}, \frac{1}{3}) \in \operatorname{int}\{\Delta(\{1,2\})\}$, and let $\pi \in \Pi_{\{1,2\}}$ such that $\pi(1) = 2$ and $\pi(2) = 1$. Then $f(R, \mathbf{x}) = \tau(\frac{1}{6}) = 1$ but $f(\pi R, \pi \mathbf{x}) = \tau(-\frac{1}{6}) = -1$. □

Proof of Claim 4. I show that the lexicographic dictator rule satisfies merger, unanimity, repurchase, strategyproofness, share monotonicity, and neutrality, but fails reallocation invariance and anonymity.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0, 1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $f(R, \mathbf{x}) = f(R, \mathbf{x}')$ it follows that $R_{d(R,\mathbf{x})} = R_{d(R,\mathbf{x}')}$. Because $\lambda \mathbf{x}_i + (1 - \lambda) \mathbf{x}'_i > 0$ if and only if $\max\{\mathbf{x}_i, \mathbf{x}'_i\} > 0$, it follows that $d(R, \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}') \in \{d(R, \mathbf{x}), d(R, \mathbf{x}')\}$ and hence $f(R, \mathbf{x}) = f(R, \mathbf{x}') = f(R, \lambda \mathbf{x} + (1 - \lambda) \mathbf{x}')$.

Unanimity. Let $k \in \{-1, 1\}$ and let $N \in \mathcal{N}$ and $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i = k$ for all $i \in N$. Then $R_{d(R,\mathbf{x})} = k$ and therefore $f(R, \mathbf{x}) = k$.

Repurchase. Let $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_j |R_j|\mathbf{x}_j > 0$ and, for all $i \in N$, $|R_i|\mathbf{x}_i = |R_i|\mathbf{x}'_i \sum_j |R_j|\mathbf{x}_j$. Because, for all $j \in N$ such that $R_j \neq 0$, $\mathbf{x}_j > 0$ if and only if $\mathbf{x}'_j > 0$, it follows that $d(R, \mathbf{x}) = d(R, \mathbf{x}')$. Consequently, $f(R, \mathbf{x}) = f(R, \mathbf{x}')$.

Strategyproofness. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. If $i > d(R, \mathbf{x})$ then $i > d(R', \mathbf{x})$, which would imply that $f(R, \mathbf{x}) = f([R_{-i}, k], \mathbf{x})$. If $i = d(R, \mathbf{x})$ then $f(R, \mathbf{x}) = R_i$.

Share monotonicity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -1$. From (e) there are two cases, $f(R, \mathbf{x}) = 1$ and $f(R, \mathbf{x}) = 0$.

Case 1: $f(R, \mathbf{x}) = 1$. From (b) we know that $d(R, \mathbf{x}) \neq k$. If $j \geqslant d(R, \mathbf{x})$, then $j \geqslant d(R, \mathbf{x}')$. So then $d(R, \mathbf{x}) = d(R, \mathbf{x}')$ and therefore $f(R, \mathbf{x}') = R_{d(R,\mathbf{x}')} = R_{d(R,\mathbf{x})} = 1$. If $j < d(R, \mathbf{x})$, then $d(R, \mathbf{x}') = j$, and therefore, by (a), $f(R, \mathbf{x}') = 1$.

Case 2: $f(R, \mathbf{x}) = 0$. In this case, by the definition of $d(R, \mathbf{x})$, there does not exist an individual $\ell \in N$ for which $x_\ell > 0$ and $R_\ell \neq 0$. Thus it must be the case that $R_k = \mathbf{x}_j = 0$. By (c) and (d) it follows that $d(R, \mathbf{x}') = j$, and therefore that $f(R, \mathbf{x}') = 1$.

Neutrality. Let $N \in \mathcal{N}$ and let $(R, \mathbf{x}) \in \mathcal{Q}_N$. By the lexicographic dictator rule, $f(-R, \mathbf{x}) = -R_{d(-R,\mathbf{x})}$ and $-f(R, \mathbf{x}) = -R_{d(R,\mathbf{x})}$. By the definition of $d(R, \mathbf{x})$, $d(R, \mathbf{x}) = d(-R, \mathbf{x})$, and consequently, $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Reallocation invariance. Because the lexicographic dictator rule fails to satisfy anonymity, it fails to satisfy reallocation invariance, by Lemma 4.

Anonymity. Let $(R, \mathbf{x}) \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1, -1)$ and $\mathbf{x} = (\frac{1}{2}, \frac{1}{2})$, and let $\pi \in \Pi$ such that $\pi(1) = 2$ and $\pi(2) = 1$. Note that $d(R, \mathbf{x}) = 1$, while $d(\pi R, \pi \mathbf{x}) = 2$. Then $f(R, \mathbf{x}) = R_1 = 1 \neq f(\pi R, \pi \mathbf{x}) = R_2 = -1$. □

Proof of Claim 5. The constant rules satisfy merger, reallocation invariance, anonymity, and repurchase because these axioms require invariance, and strategyproofness because it requires only that a weak inequality be satisfied. I show that the constant rules satisfy share monotonicity if and only if $k \neq 0$, neutrality if and only if $k = 0$, and fail to satisfy unanimity for all $k \in \mathcal{R}$.

Share monotonicity. Let $k \neq 0$. Then for all $\mathbf{x}, \mathbf{x}'$, $f(R, \mathbf{x}) \in \{0, 1\}$ implies that $f(R, \mathbf{x}') = 1$. Let $k = 0$. Let $N = \{1, 2\}$, let $R_1 = 1$, let $R_2 \neq 1$, and let $\mathbf{x}_1 < \mathbf{x}'_1$. Then $f(R, \mathbf{x}) = 0$ but $f(R, \mathbf{x}') = 0$, a contradiction.

Neutrality. Let $k = 0$. Then clearly $f(R, \mathbf{x}) = 0$ and $f(-R, \mathbf{x}) = -f(R, \mathbf{x}) = 0$. Let $k \neq 0$. Then $f(R, \mathbf{x}) = k$ and $f(-R, \mathbf{x}) = -f(R, \mathbf{x}) =$

$-k \neq k$, a contradiction.

Unanimity. If $k = 0$, let $R_i = 1$ for all $i \in N$. In this case, $f(R, \mathbf{x}) = 0 \neq 1$, a contradiction. If $k \neq 0$ let $R_i = -k$ for all $i \in N$. In this case, $f(R, \mathbf{x}) = k \neq -k$, a contradiction. □

Proof of Claim 6. Let $r \in (0, 1)$ and let f be the quorum rule defined by r. I show that f satisfies reallocation invariance, anonymity, unanimity, and neutrality, but fails to satisfy merger, repurchase, strategyproofness, and share monotonicity.

Reallocation invariance. Let $\mathbf{g}^r \in \mathscr{G}$ such that $\mathbf{g}^r(\mathbf{x}) = \tau(\mathbf{x}_1 - \mathbf{x}_{-1})$ if $\mathbf{x}_1 + \mathbf{x}_{-1} > r$ and $\mathbf{g}(\mathbf{x}) = 0$ otherwise. For all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}^r(\sigma(R, \mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

Anonymity. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

Unanimity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R, \mathbf{x}) = 1$. Consequently, $\sigma_1(R, \mathbf{x}) + \sigma_{-1}(R, \mathbf{x}) = 1$ and $\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}) = k$. Therefore, $f(R, \mathbf{x}) = \tau(k) = k$.

Neutrality. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Note that by neutrality, for $k \in \mathcal{R}$, $\sigma_k(-R, \mathbf{x}) = \sum_{i:-R_i=k} \mathbf{x}_i = \sum_{i:R_i=-k} \mathbf{x}_i = \sigma_{-k}(R, \mathbf{x})$. Consequently, $\sigma_1(-R, \mathbf{x}) + \sigma_{-1}(-R, \mathbf{x}) = \sigma_1(R, \mathbf{x}) + \sigma_{-1}(R, \mathbf{x})$ and $\sigma_1(-R, \mathbf{x}) - \sigma_{-1}(-R, \mathbf{x}) = \sigma_{-1}(R, \mathbf{x}) - \sigma_1(R, \mathbf{x})$. This implies that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Merger. Let $N = \{1, 2, 3\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$, $R_2 = -1$, and $R_3 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.5, 0.5, 0)$ and $\mathbf{x}' = (r, 0, 1-r)$. In this case, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. Then $\sum_i |R_i|(\lambda\mathbf{x}_i + (1-\lambda)\mathbf{x}'_i) = \lambda + (1-\lambda)r > r$, thus $f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = \tau(\sum_i R_i\lambda\mathbf{x} + (1-\lambda)\mathbf{x}')$. However, $\sum_i R_i(\lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = \lambda\frac{1}{2} + (1-\lambda)r - \lambda\frac{1}{2} = (1-\lambda)r > 0$, which implies that $f(R, \mathbf{x}) = 1$, a contradiction.

Repurchase. Let $N = \{1, 2, 3\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$, $R_2 = -1$, and $R_3 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (\frac{2r}{6}, \frac{r}{6}, 1 - \frac{r}{2})$ and $\mathbf{x}' = (\frac{2}{3}, \frac{1}{3}, 0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1(\mathbf{x}_1 + \mathbf{x}_2)$ and $\mathbf{x}_2 = \mathbf{x}'_2(\mathbf{x}_1 + \mathbf{x}_2)$, it follows that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\sum_i |R_i|\mathbf{x}'_i = 1$ and $\sum_i R_i\mathbf{x}'_i = \frac{1}{3}$, $f(R, \mathbf{x}') = 1$. However, because $\sum_i |R_i|\mathbf{x}_i = \frac{r}{2} < r$, $f(R, \mathbf{x}) = 0$, a contradiction.

Strategyproofness. Let $N = \{1, 2, 3\}$, let $R, R' \in \mathcal{R}^N$ such that $R = (1, -1, 0)$ and $R' = (0, -1, 0)$, and let $\mathbf{x} \in \Delta(N) = (r^2, r, 1 - r - r^2)$. Because $\sum_i |R_i|\mathbf{x}_i = r + r^2 > r$, $f(R, \mathbf{x}) = \tau(r^2 - r) = -1$. Because $\sum_i |R'_i|\mathbf{x}_i = r$, $f(R', \mathbf{x}) = 0$. However, $R' = [R_{-1}, 0]$, but $1 \geqslant f(R', \mathbf{x}) > f(R, \mathbf{x})$, a contradiction.

Share monotonicity. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0, 1)$ and $\mathbf{x}' = (r, 1-r)$.

Then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, and $f(R, \mathbf{x}) = 0$, it follows that $f(R, \mathbf{x}') = 1$, a contradiction. $\square$

Proof of Claim 7. Let f be the absolute majority rule. I show that f satisfies anonymity, unanimity, merger, reallocation invariance, strategyproofness, and neutrality, but fails to satisfy repurchase and share monotonicity.

Merger. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Note that for all $k \in \mathcal{R}$, $\sigma_k(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') = \lambda\sigma_k(R, \mathbf{x}) + (1-\lambda)\sigma_k(R, \mathbf{x}')$. Thus for all $k \in \mathcal{R}$, if $s_k(R, \mathbf{x}) > 0.5$ and $s_k(R, \mathbf{x}') > 0.5$ it follows that $\sigma_k(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') > 0.5$. If $s_k(R, \mathbf{x}) \leqslant 0.5$ and $s_k(R, \mathbf{x}') \leqslant 0.5$ it follows that $\sigma_k(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') \leqslant 0.5$. Therefore $f(R, \mathbf{x}) = f(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}')$.

Reallocation invariance. Let $\mathbf{g} \in \mathscr{G}$ such that $\mathbf{g}(\mathbf{x}) = k$ if $s_k(R, \mathbf{x}) > \frac{1}{2}$ and $\mathbf{g}(\mathbf{x}) = 0$ otherwise. For all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

Anonymity. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

Unanimity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R, \mathbf{x}) = 1 > 0.5$. Therefore, $f(R, \mathbf{x}) = k$.

Strategyproofness. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. Note that $\sigma_{R_i}(R, \mathbf{x}) \geqslant \sigma_{R_i}(R', \mathbf{x})$. If $f(R, \mathbf{x}) = R_i$ then we are done. If $f(R, \mathbf{x}) = 0$ then $0.5 \geqslant \sigma_{R_i}(R, \mathbf{x})$ which implies that $0.5 \geqslant \sigma_{R_i}(R', \mathbf{x})$ and therefore $f(R', x) \neq R_i$. If $f(R, \mathbf{x}) = -R_i$ then $\sigma_{-R_i}(R, \mathbf{x}) > 0.5$. Because $\sigma_{R_i}(-R, \mathbf{x}) \leqslant \sigma_{-R_i}(R', \mathbf{x})$ it follows that $\sigma_{-R_i}(R', \mathbf{x}) > 0.5$, and consequently, $f(R', \mathbf{x}) = -R_i$.

Neutrality. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Note that by neutrality, for $k \in \mathcal{R}$, $\sigma_k(-R, \mathbf{x}) = \sum_{i:-R_i=k} \mathbf{x}_i = \sum_{i:R_i=-k} \mathbf{x}_i = \sigma\_k(R, \mathbf{x})$. Consequently, $\sigma_k(-R, \mathbf{x}) > 0.5$ if and only if $\sigma_{-k}(R, \mathbf{x}) > 0.5$. This implies that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Repurchase. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = 0$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.4, 0.6)$ and $\mathbf{x}' = (1, 0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1\mathbf{x}_1$ it follows from repurchase that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Because $\sum_{i:R_i=1} \mathbf{x}_i = 0.4 \leqslant \frac{1}{2}$ it must be that $f(R, \mathbf{x}) = 0$. However, because $\sum_{i:R_i=1} \mathbf{x}'_i = 1 > \frac{1}{2}$, $f(R, \mathbf{x}') = 1$, a contradiction.

Share monotonicity. Let $N = \{1, 2, 3\}$, let $R = (1, -1, 0) \in \mathcal{R}^N$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.35, 0.45, 0.2)$ and $\mathbf{x}' = (0.45, 0.35, 0.2)$. In this case, $\sum_{i:R_i=1} \mathbf{x}_i = 0.35 \leqslant \frac{1}{2}$ and $\sum_{i:R_i=-1} \mathbf{x}_i = 0.45 \leqslant \frac{1}{2}$ so $f(R, \mathbf{x}) = 0$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, $\mathbf{x}_3 = \mathbf{x}'_3$, and $f(R, \mathbf{x}) = 0$, it follows that $f(R, \mathbf{x}') = 1$. But $\sum_{i:R_i=1} \mathbf{x}'_i = 0.45 \leqslant \frac{1}{2}$, a contradiction. $\square$

Proof of Claim 8. Let $t \in (-1, 1)$ and let f be the phantom voter rule defined by t. I show that f satisfies merger,reallocation invariance, anonymity, unanimity, strategyproofness, and share monotonicity, but fails to satisfy repurchase and neutrality.

MERGER. Let $N \in \mathcal{N}$, $\lambda \in (0, 1)$, and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. Let $t \in (-1, 1)$. Note that for all $k \in \mathcal{R}$, $\sigma_k(R, \lambda\mathbf{x} + (1 - \lambda)\mathbf{x}') = \lambda\sigma_k(R, \mathbf{x}) + (1-\lambda)\sigma_k(R, \mathbf{x}')$. This implies that $t + \sigma_1(R, \lambda\mathbf{x} + (1-\lambda)\mathbf{x}') - \sigma_{-1}(R, \lambda\mathbf{x} + (1 - \lambda)\mathbf{x}') \in (t + \sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}), t + \sigma_1(R, \mathbf{x}') - \sigma_{-1}(R, \mathbf{x}'))$ Therefore $f(R, \mathbf{x}) = f(R, \lambda\mathbf{x} + (1 - \lambda)\mathbf{x}')$.

REALLOCATION INVARIANCE. Let $\mathbf{g}^t \in \mathscr{G}$ such that $\mathbf{g}^t(\mathbf{x}) = \tau(t + \mathbf{x}_1 - \mathbf{x}_{-1})$. For all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}^t(\sigma(R, \mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

ANONYMITY. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

UNANIMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R, \mathbf{x}) = 1$. Consequently, $\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}) = k$. Therefore, $f(R, \mathbf{x}) = \tau(t + k)$. Because $t \in (-1, 1)$, $\tau(t + k) = \tau(k) = k$.

STRATEGYPROOFNESS. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' \equiv [R_{-i}, k]$. Note that $t + \sum_j R_j\mathbf{x}_j = R_i\mathbf{x}_i + t + \sum_{j \neq i} R_j\mathbf{x}_j$ and that $t + \sum_j R'_j\mathbf{x}_j = k\mathbf{x}_j + t + \sum_{j \neq i} R_j\mathbf{x}_j$. Consequently, $t + \sum_j R_j\mathbf{x}_j - (t + \sum_j R'_j\mathbf{x}_j) = (R_i - k)\mathbf{x}_i$. If $R_i = 1$ then $(R_i - k) \geqslant 0$ which implies that $R_i \geqslant f(R, \mathbf{x}) \geqslant f(R', \mathbf{x})$. If $R_i = -1$ then $(R_i - k) \leqslant 0$ which implies that $R_i \leqslant f(R, \mathbf{x}) \leqslant f(R', \mathbf{x})$.

SHARE MONOTONICITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -1$. From (e) it follows that $\sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x}) \geqslant -t$. From (a-d) it follows that $\sigma_1(R, \mathbf{x}) < \sigma_1(R, \mathbf{x}')$. From (c-d) it follows that $\sigma_{-1}(R, \mathbf{x}) \geqslant \sigma_{-1}(R, \mathbf{x}')$. Thus $\sigma_1(R, \mathbf{x}') - \sigma_{-1}(R, \mathbf{x}') > \sigma_1(R, \mathbf{x}) - \sigma_{-1}(R, \mathbf{x})$. Therefore, $\sigma_1(R, \mathbf{x}') - \sigma_{-1}(R, \mathbf{x}') > -t$ which implies that $f(R, \mathbf{x}') = 1$.

REPURCHASE. Let $t = -\frac{1}{2}$, and let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1, 0)$, $\mathbf{x} = (\frac{1}{3}, \frac{2}{3})$, and $\mathbf{x}' = (1, 0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1\mathbf{x}_1$ it follows from repurchase that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. However, $f(R, \mathbf{x}) = \tau(t + \frac{1}{3}) = -1$, but $f(R, \mathbf{x}') = \tau(t + 1) = 1$, a contradiction.

NEUTRALITY. Let $t = \frac{1}{2}$, and let $(R, \mathbf{x}) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, 0)$ and $\mathbf{x} = (\frac{2}{3}, \frac{1}{3}, 0)$. Then $f(-R, x) = \tau(\frac{1}{6}) \neq -\tau(\frac{5}{6}) = -f(R, x)$. □

Proof of Claim 9. Let f be the alternating rule. I show that f satisfies reallocation invariance, anonymity, unanimity, repurchase, and neutrality, but fails to satisfy merger, strategyproofness, and share monotonicity.

REALLOCATION INVARIANCE. Let $\mathbf{g} \in \mathscr{G}$ such that $\mathbf{g}(\mathbf{x}) = 1$ if $\frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_{-1}} \in$

$(0.25, 0.5) \cup [0.75, 1]$, $\mathbf{g}(\mathbf{x}) = -1$ if $\frac{\mathbf{x}_1(R,\mathbf{x})}{\mathbf{x}_1 + \mathbf{x}_{-1}} \in (0.5, 0.75) \cup [0, 0.25]$, and $\mathbf{g}(\mathbf{x}) = 0$ otherwise. For all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

Anonymity. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

Unanimity. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R, \mathbf{x}) = 1$. Consequently, $\frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} = \max\{k, 0\}$. Therefore, $f(R, \mathbf{x}) = k$.

Repurchase. Let $N \in \mathcal{N}$ and $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$ such that $\sum_j |R_j| \mathbf{x}_j > 0$ and, for all $i \in N$, $|R_i| \mathbf{x}_i = |R_i| \mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j$. Then for $k \in \{-1, 1\}$, $\sigma_k(R, \mathbf{x}) = (\sum_j |R_j| \mathbf{x}_j) \sigma_k(R, \mathbf{x}')$. Note that $\sum_i |R_i| \mathbf{x}_i = \sum_i |R_i| \mathbf{x}'_i \sum_j |R_j| \mathbf{x}_j = (\sum_j |R_j| \mathbf{x}_j) \sum_i |R_i| \mathbf{x}'_i$. It follows that $\frac{s_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} = \frac{(\sum_j |R_j| \mathbf{x}_j) s_1(R,\mathbf{x})}{(\sum_j |R_j| \mathbf{x}_j) \sum_i |R_i| \mathbf{x}'_i} = \frac{s_1(R,\mathbf{x}')}{\sum_i |R_i| \mathbf{x}'_i}$. Therefore $f(R, \mathbf{x}) = f(R, \mathbf{x}')$

Neutrality. Let $(R, \mathbf{x}) \in \mathcal{Q}$. Note that by neutrality, for $k \in \mathcal{R}$, $\sigma_k(-R, \mathbf{x}) = \sum_{i:-R_i=k} \mathbf{x}_i = \sum_{i:R_i=-k} \mathbf{x}_i = \sigma_-k(R, \mathbf{x})$. Consequently, $\frac{\sigma_1(-R,\mathbf{x})}{\sum_i |-R_i| \mathbf{x}_i} = \frac{\sigma_{-1}(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i} = 1 - \frac{\sigma_1(R,\mathbf{x})}{\sum_i |R_i| \mathbf{x}_i}$. This implies that $f(-R, \mathbf{x}) = -f(R, \mathbf{x})$.

Merger. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = -1$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.4, 0.6)$ and $\mathbf{x}' = (1, 0)$. In this case, $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 1$. However, $f(R, \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{x}') = -1$, a contradiction.

Strategyproofness. Let $N = \{1, 2, 3\}$, let $R, R' \in \mathcal{R}^N$ such that $R = (1, 1, -1)$ and $R' = (1, -1, -1)$, and let $\mathbf{x} \in \Delta(N) = (0.4, 0.2, 0.4)$. Then $\sigma_1(R, \mathbf{x}) = 0.6$ and $\sigma_1(R', \mathbf{x}) = 0.4$. In both cases $\sum_i |R_i| \mathbf{x}_i = \sum_i |R'_i| \mathbf{x}_i = 1$. Thus $f(R, \mathbf{x}) = -1$ and $f(R', \mathbf{x}) = 1$. However, $R' = [R_{-2}, -1]$ but $1 \geqslant f(R', \mathbf{x}) > f(R, \mathbf{x})$, a contradiction.

Share monotonicity. Let $N = \{1, 2\}$, let $R \in \mathcal{R}^N$ such that $R_1 = 1$ and $R_2 = -1$, and let $\mathbf{x}, \mathbf{x}' \in \Delta(N)$ such that $\mathbf{x} = (0.4, 0.6)$ and $\mathbf{x}' = (0.6, 0.4)$. Then $f(R, \mathbf{x}) = 1$ and $f(R, \mathbf{x}') = -1$. By share monotonicity, because $R_1 = 1$, $R_2 \neq 1$, $\mathbf{x}_1 < \mathbf{x}'_1$, and $f(R, \mathbf{x}) = 1$, it follows that $f(R, \mathbf{x}') = 1$, a contradiction. □

Proof of Claim 10. Let f be rule X. I show that f reallocation invariance, anonymity, unanimity, strategyproofness, and share monotonicity, but fails to satisfy merger, repurchase, and neutrality.

Reallocation invariance. Let $\mathbf{g} \in \mathscr{G}$ such that $\mathbf{g}(\mathbf{x}) = 1$ if $\frac{\mathbf{x}_1}{\mathbf{x}_1 + \mathbf{x}_{-1}} > 0.5$, $\mathbf{g}(\mathbf{x}) = 0$ if $\mathbf{x}_1 = \mathbf{x}_{-1} = 0.5$ or $\mathbf{x}_0 = 1$, and $\mathbf{g}(\mathbf{x}) = -1$ otherwise. For all $(R, \mathbf{x}) \in \mathcal{Q}$, $f(R, \mathbf{x}) = \mathbf{g}(\sigma(R, \mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

Anonymity. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

UNANIMITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1, 1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R, \mathbf{x}) = 1$. Consequently, $f(R, \mathbf{x}) = k$.

STRATEGYPROOFNESS. Let $N \in \mathcal{N}$, $i \in N$, $(R, \mathbf{x}) \in \mathcal{Q}_N$ such that $R_i \neq 0$, and $k \in \mathcal{R}$. Let $R' = [R_{-i}, k]$. Note that $\sigma_{R_i}(R, \mathbf{x}) \geqslant \sigma_{R_i}(R', \mathbf{x})$. **Case 1:** $R_i = 1$. If $f(R, \mathbf{x}) = 1$ then we are done. If $f(R, \mathbf{x}) = 0$ then $\sigma_1(R, \mathbf{x}) \leqslant \sigma_{-1}(R, \mathbf{x})$. Then $\sigma_1(R, \mathbf{x}) \geqslant \sigma_1(R', \mathbf{x})$ and $\sigma_{-1}(R', \mathbf{x}) \geqslant \sigma_{-1}(R, \mathbf{x})$ implies that $\sigma_1(R', \mathbf{x}) \leqslant \sigma_{-1}(R', \mathbf{x})$ and therefore $f(R', \mathbf{x}) \leqslant 0$. If $f(R, \mathbf{x}) = -1$ then either (a) $\sigma_1(R, \mathbf{x}) < \sigma_{-1}(R, \mathbf{x})$ or (b) $\sigma_1(R, \mathbf{x}) = \sigma_{-1}(R, \mathbf{x})$ and $\sigma_0(R, \mathbf{x}) \in (0, 1)$. If (a) then $\sigma_1(R, \mathbf{x}) \geqslant \sigma_1(R', \mathbf{x})$ and $\sigma_{-1}(R', \mathbf{x}) \geqslant \sigma_{-1}(R, \mathbf{x})$ implies that $\sigma_1(R', \mathbf{x}) < \sigma_{-1}(R', \mathbf{x})$ and therefore $f(R', \mathbf{x}) = -1$. If (b) then by a previous argument $f(R', \mathbf{x}) \leqslant 0$. Because $\sigma_0(R, \mathbf{x}) > 0$ it follows that $\sigma_0(R', \mathbf{x}) > 0$. Because $\sigma_{-1}(R, \mathbf{x}) > 0$ it follows that $\sigma_{-1}(R, \mathbf{x}) > 0$ and therefore that $\sigma_0(R', \mathbf{x}) < 1$. It follows that $f(R', \mathbf{x}) \neq 0$. **Case 2:** $R_i = -1$. If $f(R, \mathbf{x}) = -1$ then we are done. If $f(R, \mathbf{x}) = 0$ then either (a) $\sigma_1(R, \mathbf{x}) = \sigma_{-1}(R, \mathbf{x}) = 0.5$ or (b) $\sigma_0(R, \mathbf{x}) = 1$. If (a) then $\sigma_{-1}(R, \mathbf{x}) \geqslant \sigma_{-1}(R', \mathbf{x})$ which implies that $\sigma_{-1}(R', \mathbf{x}) \leqslant 0.5$ and $\sigma_1(R', \mathbf{x}) \geqslant \sigma_1(R, \mathbf{x}) = 0.5$ which implies that $f(R, \mathbf{x}) \geqslant 0$. If (b) then $\sigma_{-1}(R, \mathbf{x}) = 0$ which implies that $\sigma_{-1}(R', \mathbf{x}) = 0$ and therefore $f(R', \mathbf{x}) \geqslant 0$. If $f(R, \mathbf{x}) = 1$ then $\sigma_1(R, \mathbf{x}) > \sigma_{-1}(R, \mathbf{x})$. Because $\sigma_1(R, \mathbf{x}) \leqslant \sigma_1(R', \mathbf{x})$ and $\sigma_{-1}(R, \mathbf{x}) \geqslant \sigma_{-1}(R', \mathbf{x})$ it follows that $\sigma_1(R', \mathbf{x}) > \sigma_{-1}(R', \mathbf{x})$ and therefore that $f(R', \mathbf{x}) = 1$.

SHARE MONOTONICITY. Let $N \in \mathcal{N}$, $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_N$, and $j, k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N \backslash \{j, k\}$, and (e) $f(R, \mathbf{x}) \neq -1$. From (e) it follows that $f(R, \mathbf{x}) \geqslant 0$. If $f(R, \mathbf{x}) = 0$ then either (i) $\sigma(R, \mathbf{x}) = (0.5, 0.5, 0)$ or (ii) $\sigma(R, \mathbf{x}) = (0, 0, 1)$. If (i) then there is $a \in (0, 0.5]$ such that $\sigma(R, \mathbf{x}') = (0.5 + a, 0.5 - a, 0)$. If (ii) then there is $b \in (0, 1]$ such that $\sigma(R, \mathbf{x}') = (b, 0, 1 - b)$ Consequently, $f(R, \mathbf{x}') = 1$. If $f(R, \mathbf{x}) = 1$ then $\sigma_1(R, \mathbf{x}) > \sigma_{-1}(R, \mathbf{x})$. From (a-d) it follows that $\sigma_1(R, \mathbf{x}') > \sigma_1(R, \mathbf{x})$. From (c-d) it follows that $\sigma_{-1}(R, \mathbf{x}) \geqslant \sigma_{-1}(R, \mathbf{x}')$. Consequently, $\sigma_1(R, \mathbf{x}') > \sigma_{-1}(R, \mathbf{x}')$ and therefore $f(R, \mathbf{x}') = 1$.

MERGER. Let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, 0)$, $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0)$, $\mathbf{x}' = (0, 0, 1)$, and $\mathbf{x}'' = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Then $f(R, \mathbf{x}) = f(R, \mathbf{x}') = 0$ and $f(R, \mathbf{x}'') = -1$. However, for $\lambda = 0.5$, $\mathbf{x}'' = \lambda \mathbf{x} + (1 - \lambda)\mathbf{x}'$, a contradiction.

REPURCHASE. Let $(R, \mathbf{x}), (R, \mathbf{x}') \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, 0)$, $\mathbf{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and $\mathbf{x}' = (\frac{1}{2}, \frac{1}{2}, 0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1(\mathbf{x}_1 + \mathbf{x}_2)$ and $\mathbf{x}_2 = \mathbf{x}'_2(\mathbf{x}_1 + \mathbf{x}_2)$ it follows from repurchase that $f(R, \mathbf{x}) = f(R, \mathbf{x}')$. However, $f(R, \mathbf{x}) = -1$ and $f(R, \mathbf{x}') = 0$, a contradiction.

NEUTRALITY. Let $(R, \mathbf{x}) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1, -1, 0)$ and $\mathbf{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. Then $f(R, \mathbf{x}) = -1$ and $f(-R, \mathbf{x}) = -1$, a contradiction. □

Proof of Claim 11. Let f be the indifference is two against rule. I show that f satisfies anonymity, unanimity, merger, share monotonicity, and reallocation invariance, but fails to satisfy repurchase, strategyproofness, and neutrality.

MERGER. Let $N \in \mathcal{N}$, $\lambda \in (0,1)$, and $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$ such that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. Let $t \in (-1,1)$. Note that for all $k \in \mathcal{R}$, $\sigma_k(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') = \lambda\sigma_k(R,\mathbf{x})+(1-\lambda)\sigma_k(R,\mathbf{x}')$. This implies that $\sigma_1(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')-\sigma_{-1}(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')-2\sigma_0(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}') \in (\sigma_1(R,\mathbf{x})-\sigma_{-1}(R,\mathbf{x})-2\sigma_0(R,\mathbf{x}),\sigma_1(R,\mathbf{x}')-\sigma_{-1}(R,\mathbf{x}')-2\sigma_0(R,\mathbf{x}'))$ Therefore $f(R,\mathbf{x}) = f(R,\lambda\mathbf{x}+(1-\lambda)\mathbf{x}')$.

REALLOCATION INVARIANCE. Let $\mathbf{g} \in \mathscr{G}$ such that $\mathbf{g}(\mathbf{x}) = \tau(\mathbf{x}_1-\mathbf{x}_{-1}-2\mathbf{x}_0)$. For all $(R,\mathbf{x}) \in \mathcal{Q}$, $f(R,\mathbf{x}) = \mathbf{g}(\sigma(R,\mathbf{x}))$. Therefore, by Lemma 5, f satisfies reallocation invariance.

ANONYMITY. Because f satisfies reallocation invariance it satisfies anonymity by Lemma 4.

UNANIMITY. Let $N \in \mathcal{N}$, $(R,\mathbf{x}) \in \mathcal{Q}_N$, and $k \in \{-1,1\}$ such that $R_i = k$ for all $i \in N$. Then $\sigma_k(R,\mathbf{x}) = 1$. Consequently, $\sigma_1(R,\mathbf{x})-\sigma_{-1}(R,\mathbf{x})-2\sigma_0(R,\mathbf{x}) = k$. Therefore, $f(R,\mathbf{x}) = \tau(k) = k$.

SHARE MONOTONICITY. Let $N \in \mathcal{N}$, $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_N$, and $j,k \in N$ such that (a) $R_j = 1$, (b) $R_k \neq 1$, (c) $\mathbf{x}_j < \mathbf{x}'_j$, (d) $\mathbf{x}_\ell = \mathbf{x}'_\ell$ for all $\ell \in N\backslash\{j,k\}$, and (e) $f(R,\mathbf{x}) \neq -1$. From (e) it follows that $\sigma_1(R,\mathbf{x})-\sigma_{-1}(R,\mathbf{x})-2\sigma_0(R,\mathbf{x}) \geqslant 0$. From (a-d) it follows that $\sigma_1(R,\mathbf{x}) < \sigma_1(R,\mathbf{x}')$. From (c-d) it follows that $\sigma_{-1}(R,\mathbf{x}) \geqslant \sigma_{-1}(R,\mathbf{x}')$ and that $\sigma_0(R,\mathbf{x}) \geqslant \sigma_0(R,\mathbf{x}')$. Thus $\sigma_1(R,\mathbf{x}')-\sigma_{-1}(R,\mathbf{x}')-2\sigma(R,\mathbf{x}') > \sigma_1(R,\mathbf{x})-\sigma_{-1}(R,\mathbf{x})-2\sigma(R,\mathbf{x})$. Therefore, $\sigma_1(R,\mathbf{x}')-\sigma_{-1}(R,\mathbf{x}')-2\sigma(R,\mathbf{x}') > 0$ which implies that $f(R,\mathbf{x}') = 1$.

REPURCHASE. Let $(R,\mathbf{x}),(R,\mathbf{x}') \in \mathcal{Q}_{\{1,2\}}$ such that $R = (1,0)$, $\mathbf{x} = (\frac{1}{2},\frac{1}{2})$, and $\mathbf{x}' = (1,0)$. Because $\mathbf{x}_1 = \mathbf{x}'_1\mathbf{x}_1$ it follows from repurchase that $f(R,\mathbf{x}) = f(R,\mathbf{x}')$. However, $f(R,\mathbf{x}) = -1$ and $f(R,\mathbf{x}') = 1$, a contradiction.

STRATEGYPROOFNESS. Let $N = \{1,2\}$, let $R,R' \in \mathcal{R}^N$ such that $R = (1,-1)$ and $R' = (1,0)$, and let $\mathbf{x} \in \Delta(N) = (0.6,0.4)$. Then $f(R,\mathbf{x}) = 1$ and $f(R',\mathbf{x}) = -1$. However, $R' = [R_{-2},0]$ but $-1 \leqslant f(R',\mathbf{x}) < f(R,\mathbf{x})$, a contradiction.

NEUTRALITY. Let $(R,\mathbf{x}) \in \mathcal{Q}_{\{1,2,3\}}$ such that $R = (1,-1,0)$ and $\mathbf{x} = (\frac{1}{3},\frac{1}{3},\frac{1}{3})$. Then $f(R,\mathbf{x}) = -1$ and $f(-R,x) = -1$, a contradiction. □

References

ADAMS, R. AND D. FERREIRA (2008): “One share-one vote: The empirical evidence,” *Review of Finance*, 12, 51–91.

Arrow, K. J. (1963): *Social Choice and Individual Values*, Yale University Press, second ed.

Burkart, M. and S. Lee (2008): "One share-one vote: The theory," *Review of Finance*, 12, 1–49.

Dunlavy, C. A. (2006): "Social conceptions of the corporation: Insights from the history of shareholder voting rights," *Wash. & Lee L. Rev.*, 63, 1347.

Easterbrook, F. H. and D. R. Fischel (1983): "Voting in corporate law," *The Journal of Law and Economics*, 26, 395–427.

Grossman, S. J. and O. D. Hart (1988): "One Share-One Vote and the Market for Corporate Control," *Journal of Financial Economics*, 20, 175–202.

Harris, M. and A. Raviv (1988): "Corporate Governance: Voting Rights and Majority Rules," *Journal of Financial Economics*, 20, 203–235.

Hu, H. T. and B. Black (2005): "The new vote buying: Empty voting and hidden (morphable) ownership," *S. Cal. L. Rev.*, 79, 811.

——— (2006): "Empty voting and hidden (morphable) ownership: Taxonomy, implications, and reforms." *Business Lawyer*, 61, 1011–1070.

——— (2007): "Hedge funds, insiders, and the decoupling of economic and voting ownership: Empty voting and hidden (morphable) ownership," *Journal of Corporate Finance*, 13, 343–367.

——— (2008): "Equity and Debt Decoupling and Empty Voting II: Importance and Extensions," *University of Pennsylvania Law Review*, 625–739.

May, K. O. (1952): "A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision," *Econometrica*, 20, 680–684.

Moulin, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

——— (1985): "Egalitarianism and Utilitarianism in Quasi-Linear Bargaining," *Econometrica*, 53, 49–67.

——— (1987): "Equal or Proportional Division of a Surplus, and Other Methods," *International Journal of Game Theory*, 16, 161–186.

Posner, E. A. and E. G. Weyl (2014): "Quadratic Voting as Efficient Corporate Governance," *University of Chicago Law Review*, 81, 251–272.

Ritzberger, K. (2005): "Shareholder voting," *Economics Letters*, 86, 69–72.

Thompson, R. B. and P. H. Edelman (2009): "Corporate voting," *Vand. L. Rev.*, 62, 127.

Young, H. P. (1974): "An Axiomatization of Borda's Rule," *Journal of Economic Theory*, 9, 43–52.

——— (1975): "Social choice scoring functions," *SIAM Journal on Applied Mathematics*, 28, 824–838.