and therefore

$$R_{total} = R_{plastic} + R_{conv} = 0.76 + 0.18 = 0.94°C/W$$

Then the interface temperature can be determined from

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{total}} \longrightarrow T_1 = T_\infty + \dot{Q}R_{total}$$
$$= 30°C + (80 \text{ W})(0.94°C/\text{W}) = \mathbf{105°C}$$

Note that we did not involve the electrical wire directly in the thermal resistance network, since the wire involves heat generation.

To answer the second part of the question, we need to know the critical radius of insulation of the plastic cover. It is determined from Eq. 3–50 to be

$$r_{cr} = \frac{k}{h} = \frac{0.15 \text{ W/m·K}}{12 \text{ W/m}^2\text{·K}} = 0.0125 \text{ m} = 12.5 \text{ mm}$$

which is larger than the radius of the plastic cover. Therefore, increasing the thickness of the plastic cover will *enhance* heat transfer until the outer radius of the cover reaches 12.5 mm. As a result, the rate of heat transfer $\dot{Q}$ will *increase* when the interface temperature T_1 is held constant, or T_1 will *decrease* when $\dot{Q}$ is held constant, which is the case here.

Discussion It can be shown by repeating the calculations above for a 4-mm-thick plastic cover that the interface temperature drops to 90.6°C when the thickness of the plastic cover is doubled. It can also be shown in a similar manner that the interface reaches a minimum temperature of 83°C when the outer radius of the plastic cover equals the critical radius.

3–6 ▪ HEAT TRANSFER FROM FINNED SURFACES

The rate of heat transfer from a surface at a temperature T_s to the surrounding medium at T_∞ is given by Newton's law of cooling as

$$\dot{Q}_{conv} = hA_s(T_s - T_\infty)$$

where A_s is the heat transfer surface area and h is the convection heat transfer coefficient. When the temperatures T_s and T_∞ are fixed by design considerations, as is often the case, there are *two ways* to increase the rate of heat transfer: to increase the *convection heat transfer coefficient h* or to increase the *surface area* A_s. Increasing h may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate. The alternative is to increase the surface area by attaching to the surface *extended surfaces* called *fins* made of highly conductive materials such as aluminum. Finned surfaces are manufactured by extruding, welding, or wrapping a thin metal sheet on a surface. Fins enhance heat transfer from a surface by exposing a larger surface area to convection and radiation.

An interesting application of fins from about 150 million years ago, the Jurassic era, is shown in Fig. 3–33. The dinosaur stegosaurus lived during

FIGURE 3–33
Presumed cooling fins on dinosaur stegosaurus.
© Alamy RF

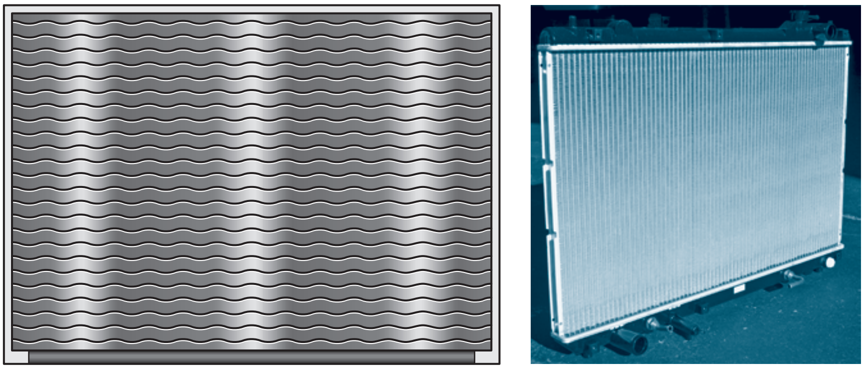


FIGURE 3–34
The thin plate fins of a car radiator greatly increase the rate of heat transfer to the air.
Left: © Yunus A. Çengel, photo by James Kleiser, right:© McGraw-Hill Education / Christopher Kerrigan

this era and it had two rows of big (and bizarre) bony plates down its back. For a long time, scientists thought that the plates were some kind of armor to protect the vegetarian from predators. We now know that a lot of blood flowed through the plates, and they may have acted like a car radiator. The heart pumped blood through the plates, and the plates acted like cooling fins to cool the blood down.

Finned surfaces are commonly used in practice to enhance heat transfer, and they often increase the rate of heat transfer from a surface severalfold. The car radiator shown in Fig. 3–34 is an example of a finned surface. The closely packed thin metal sheets attached to the hot-water tubes increase the surface area for convection and thus the rate of convection heat transfer from the tubes to the air many times. There are a variety of innovative fin designs available in the market, and they seem to be limited only by imagination (Fig. 3–35).

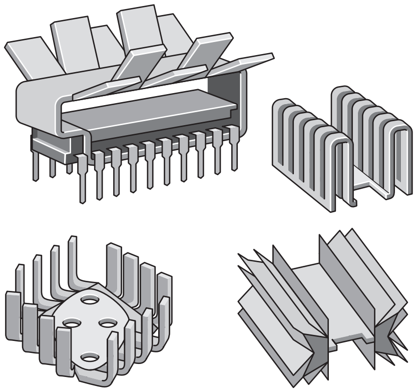


FIGURE 3–35
Some innovative fin designs.

In the analysis of fins, we consider *steady* operation with *no heat generation* in the fin, and we assume the thermal conductivity k of the material to remain constant. We also assume the convection heat transfer coefficient h to be *constant* and *uniform* over the entire surface of the fin for convenience in the analysis. We recognize that the convection heat transfer coefficient h, in general, varies along the fin as well as its circumference, and its value at a point is a strong function of the *fluid motion* at that point. The value of h is usually much lower at the *fin base* than it is at the *fin tip* because the fluid is surrounded by solid surfaces near the base, which seriously disrupt its motion to the point of "suffocating" it, while the fluid near the fin tip has little contact with a solid surface and thus encounters little resistance to flow. Therefore, adding too many fins on a surface may actually decrease the overall heat transfer when the decrease in h offsets any gain resulting from the increase in the surface area.

Fin Equation

Consider a volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and a perimeter of p, as shown in Fig. 3–36. Under steady conditions, the energy balance on this volume element can be expressed as

$$\begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ into} \\ \text{the element at } x\end{pmatrix} = \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{conduction} \text{ from the} \\ \text{element at } x + \Delta x\end{pmatrix} + \begin{pmatrix}\text{Rate of } \textit{heat} \\ \textit{convection} \text{ from} \\ \text{the element}\end{pmatrix}$$

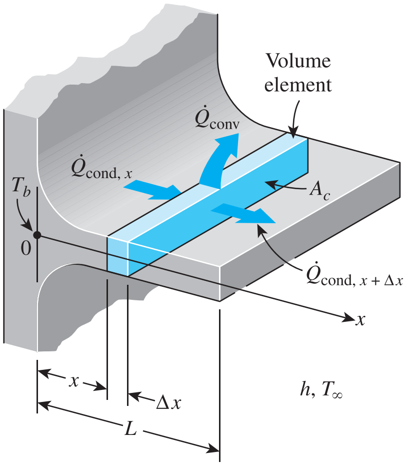


FIGURE 3–36
Volume element of a fin at location x having a length of Δx, cross-sectional area of A_c, and perimeter of p.

or

$$\dot{Q}_{\text{cond}, x} = \dot{Q}_{\text{cond}, x + \Delta x} + \dot{Q}_{\text{conv}}$$

where

$$\dot{Q}_{\text{conv}} = h(p\,\Delta x)(T - T_\infty)$$

Substituting and dividing by Δx, we obtain

$$\frac{\dot{Q}_{\text{cond}, x+\Delta x} - \dot{Q}_{\text{cond}, x}}{\Delta x} + hp(T - T_\infty) = 0 \qquad \textbf{(3–52)}$$

Taking the limit as $\Delta x \to 0$ gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_\infty) = 0 \qquad \textbf{(3–53)}$$

From Fourier's law of heat conduction we have

$$\dot{Q}_{\text{cond}} = -kA_c \frac{dT}{dx} \qquad \textbf{(3–54)}$$

where A_c is the cross-sectional area of the fin at location x. Substitution of this relation into Eq. 3–53 gives the differential equation governing heat transfer in fins,

$$\frac{d}{dx}\left(kA_c \frac{dT}{dx}\right) - hp(T - T_\infty) = 0 \qquad \textbf{(3–55)}$$

In general, the cross-sectional area A_c and the perimeter p of a fin vary with x, which makes this differential equation difficult to solve. In the special case of *constant cross section* and *constant thermal conductivity,* the differential equation Eq. 3–55 reduces to

$$\frac{d^2T}{dx^2} - \frac{hp}{kA_c}(T - T_\infty) = 0 \quad \text{or} \quad \frac{d^2\theta}{dx^2} - m^2\theta = 0 \qquad \textbf{(3–56)}$$

where

$$m^2 = \frac{hp}{kA_c} \qquad \textbf{(3–57)}$$

and $\theta = T - T_\infty$ is the *temperature excess.* At the fin base we have $\theta_b = T_b - T_\infty$.

Equation 3–56 is a linear, homogeneous, second-order differential equation with constant coefficients. A fundamental theory of differential equations states that such an equation has two linearly independent solution functions, and its general solution is the linear combination of those two solution functions. A careful examination of the differential equation reveals that subtracting a constant multiple of the solution function θ from its second derivative yields zero. Thus we conclude that the function θ and its second derivative must be *constant multiples* of each other. The only functions whose derivatives are constant multiples of the functions themselves are the *exponential functions* (or a linear combination of exponential functions such as sine and cosine hyperbolic functions). Therefore, the solution functions of the differential equation above are the exponential functions e^{-mx} or e^{mx} or constant multiples of them. This can be verified by direct substitution. For example, the second derivative of e^{-mx} is m^2e^{-mx}, and its substitution into Eq. 3–56 yields zero. Therefore, the general solution of the differential equation Eq. 3–56 is

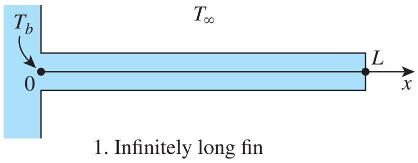
$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \tag{3–58}$$

where C_1 and C_2 are arbitrary constants whose values are to be determined from the boundary conditions at the base and at the tip of the fin. Note that we need only two conditions to determine C_1 and C_2 uniquely.

The temperature of the plate to which the fins are attached is normally known in advance. Therefore, at the fin base we have a *specified temperature* boundary condition, expressed as

Boundary condition at fin base: $$\theta(0) = \theta_b = T_b - T_\infty \tag{3–59}$$

At the fin tip we have several possibilities, including infinitely long fins, negligible heat loss (idealized as an adiabatic tip), specified temperature, and convection (Fig. 3–37). Next, we consider each case separately.



1. Infinitely long fin
2. Negligible heat loss (adiabatic tip)
3. Specified temperature
4. Convection

FIGURE 3–37
Boundary conditions at the fin base and the fin tip.

1 Infinitely Long Fin ($T_{\text{fin tip}} = T_\infty$)

For a sufficiently long fin of *uniform* cross section (A_c = constant), the temperature of the fin at the fin tip approaches the environment temperature T_∞ and thus θ approaches zero. That is,

Boundary condition at fin tip: $$\theta(L) = T(L) - T_\infty = 0 \quad \text{as} \quad L \to \infty$$

This condition is satisfied by the function e^{-mx}, but not by the other prospective solution function e^{mx} since it tends to infinity as x gets larger. Therefore, the general solution in this case will consist of a constant multiple of e^{-mx}. The value of the constant multiple is determined from the requirement that at the fin base where $x = 0$ the value of θ is θ_b. Noting that $e^{-mx} = e^0 = 1$, the proper value of the constant is θ_b, and the solution function we are looking for is $\theta(x) = \theta_b e^{-mx}$. This function satisfies the differential equation as well as the requirements that the solution reduce to θ_b at the fin base and approach zero at the fin tip for large x. Noting that $\theta = T - T_\infty$ and $m = \sqrt{hp/kA_c}$, the variation of temperature along the fin in this case can be expressed as

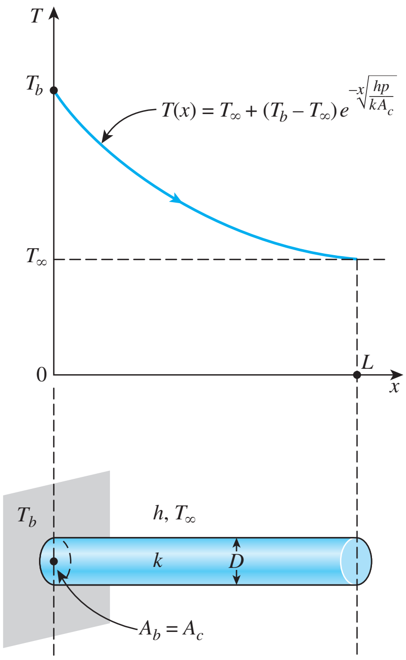
Very long fin: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-mx} = e^{-x\sqrt{hp/kA_c}} \tag{3–60}$$

Note that the temperature along the fin in this case decreases *exponentially* from T_b to T_∞, as shown in Fig. 3–38. The steady rate of *heat transfer* from the entire fin can be determined from Fourier's law of heat conduction

Very long fin: $$\dot{Q}_{\text{long fin}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty) \tag{3–61}$$

where p is the perimeter, A_c is the cross-sectional area of the fin, and x is the distance from the fin base. Alternatively, the rate of heat transfer from the fin could also be determined by considering heat transfer from a differential volume element of the fin and integrating it over the entire surface of the fin:

$$\dot{Q}_{\text{fin}} = \int_{A_{\text{fin}}} h[T(x) - T_\infty]\, dA_{\text{fin}} = \int_{A_{\text{fin}}} h\theta(x)\, dA_{\text{fin}} \tag{3–62}$$



($p = \pi D$, $A_c = \pi D^2/4$ for a cylindrical fin)

FIGURE 3–38
A long circular fin of uniform cross section and the variation of temperature along it.

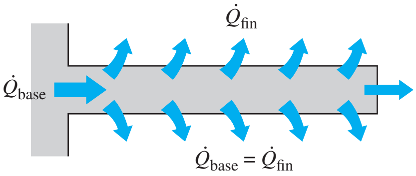


FIGURE 3–39
Under steady conditions, heat transfer from the exposed surfaces of the fin is equal to heat conduction to the fin at the base.

The two approaches described are equivalent and give the same result since, under steady conditions, the heat transfer from the exposed surfaces of the fin is equal to the heat transfer to the fin at the base (Fig. 3–39).

2 Negligible Heat Loss from the Fin Tip (Adiabatic fin tip, $\dot{Q}_{\text{fin tip}} = 0$)

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic situation is for heat transfer from the fin tip to be negligible since the heat transfer from the fin is proportional to its surface area, and the surface area of the fin tip is usually a negligible fraction of the total fin area. Then the fin tip can be assumed to be adiabatic, and the condition at the fin tip can be expressed as

Boundary condition at fin tip: $$\left.\frac{d\theta}{dx}\right|_{x=L} = 0 \qquad \textbf{(3–63)}$$

The condition at the fin base remains the same as expressed in Eq. 3–59. The application of the boundary conditions given by Eqs. 3–59 and 3–63 on the general solution (Eq. 3–58) requires that $\theta(0) = \theta_b = C_1 + C_2$ and $mC_1e^{mL} - mC_2e^{-mL} = 0$, respectively. Solving these two equations simultaneously for C_1 and C_2 yields $C_1 = \theta_b/(1+e^{2mL})$ and $C_2 = \theta_b/(1+e^{-2mL})$. Substituting the relations for C_1 and C_2 into Eq. 3–58 and using the definition of the hyperbolic cosine function $\cosh x = (e^x + e^{-x})/2$ gives the desired relation for the temperature distribution:

Adiabatic fin tip: $$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L - x)}{\cosh mL} \qquad \textbf{(3–64)}$$

The rate of heat transfer from the fin can be determined again from Fourier's law of heat conduction:

Adiabatic fin tip: $$\dot{Q}_{\text{adiabatic tip}} = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = \sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL \qquad \textbf{(3–65)}$$

where the equation for the hyperbolic tangent function is

$$\tanh x = \sinh x/\cosh x = (e^x - e^{-x})/(e^x + e^{-x}).$$

Note that the heat transfer relations for the very long fin and the fin with negligible heat loss at the tip differ by the factor $\tanh mL$, which approaches 1 as L becomes very large.

3 Specified Temperature ($T_{\text{fin, tip}} = T_L$)

In this case the temperature at the end of the fin (the fin tip) is fixed at a specified temperature T_L. This case could be considered as a generalization of the case of *Infinitely Long Fin* where the fin tip temperature was fixed at T_∞. The condition at the fin tip for this case is

Boundary condition at fin tip: $$\theta(L) = \theta_L = T_L - T_\infty \qquad \textbf{(3–66)}$$

The fin base boundary condition remains the same as given in Eq. 3–59. Applying the boundary conditions given by Eqs. 3–59 and 3–66 on the general solution (Eq. 3–58) gives, after some lengthy algebra and using the definition of the hyperbolic sine function, $\sinh x = (e^x - e^{-x})/2$, the desired temperature distribution:

Specified fin tip temperature:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{[(T_L - T_\infty)/(T_b - T_\infty)]\sinh mx + \sinh m(L - x)}{\sinh mL} \tag{3–67}$$

Using the Fourier's law of heat conduction, the *rate of heat transfer* from the fin is

Specified fin tip temperature:

$$\dot{Q}_{\text{specified temp.}} = -kA_c \frac{dT}{dx}\bigg|_{x=0}$$
$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\cosh mL - [(T_L - T_\infty)/(T_b - T_\infty)]}{\sinh mL} \tag{3–68}$$

Note that Eqs. 3–67 and 3–68 reduce to Eqs. 3–60 and 3–61 for the case of *infinitely long fin* ($L \to \infty$).

4 Convection from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that may also include the effects of radiation. Consider the case of convection only at the tip. The condition at the fin tip can be obtained from an energy balance at the fin tip ($\dot{Q}_{\text{cond}} = \dot{Q}_{\text{conv}}$) That is,

Boundary condition at fin tip:

$$-kA_c \frac{dT}{dx}\bigg|_{x=L} = hA_c[T(L) - T_\infty] \tag{3–69}$$

The boundary condition at the fin base is Eq. 3–59, which is the same as the three previous cases. Substituting the two boundary conditions given by Eqs. 3–59 and 3–69 in the general solution (Eq. 3–58), it may be shown, after some lengthy manipulation that the temperature distribution is

Convection from fin tip:

$$\frac{T(x) - T_\infty}{T_b - T_\infty} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL} \tag{3–70}$$

The *rate of heat transfer* from the fin can be found by substituting the temperature gradient at the base of the fin, obtained from Eq. 3–70, into the Fourier's law of heat conduction. The result is

Convection from fin tip:

$$\dot{Q}_{\text{convection}} = -kA_c \frac{dT}{dx}\bigg|_{x=0}$$
$$= \sqrt{hpkA_c}(T_b - T_\infty)\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL} \tag{3–71}$$

The solution to the general fin equation for the case of *convection from fin tip* is rather complex. An approximate, yet practical and accurate, way of accounting for the loss from the fin tip is to replace the *fin length L* in the relation for the *insulated tip* case by a **corrected fin length** defined as (Fig. 3–40)

Corrected fin length:

$$L_c = L + \frac{A_c}{p} \tag{3–72}$$

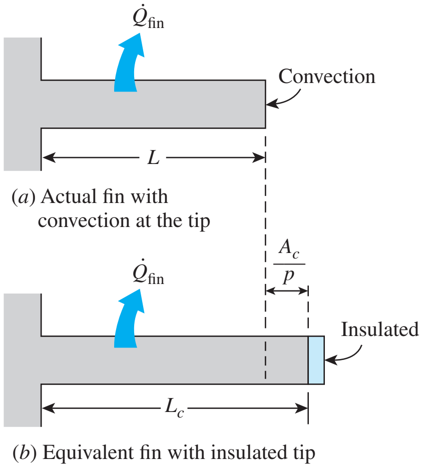
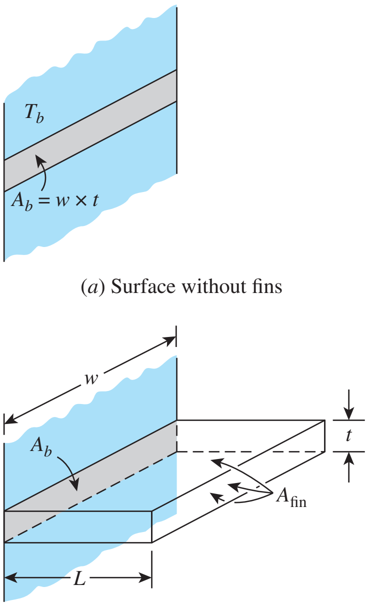


FIGURE 3–40
Corrected fin length L_c is defined such that heat transfer from a fin of length L_c with insulated tip is equal to heat transfer from the actual fin of length L with convection at the fin tip.

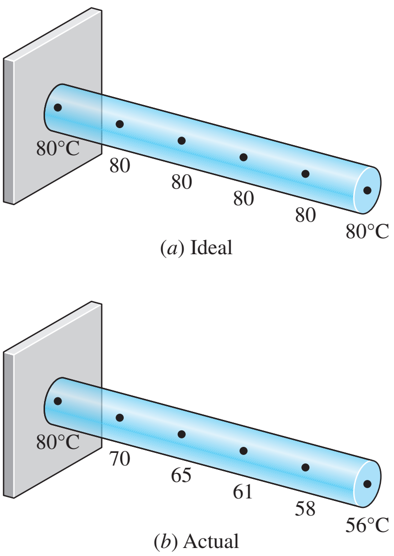


(*a*) Surface without fins

(*b*) Surface with a fin

$A_{\text{fin}} = 2 \times w \times L + w \times t$
$\cong 2 \times w \times L$

FIGURE 3–41
Fins enhance heat transfer from a surface by enhancing surface area.



(*a*) Ideal

(*b*) Actual

FIGURE 3–42
Ideal and actual temperature distribution along a fin.

where A_c is the cross-sectional area and p is the perimeter of the fin at the tip. Multiplying the relation above by the perimeter gives $A_{\text{corrected}} = A_{\text{fin (lateral)}} + A_{\text{tip}}$, which indicates that the fin area determined using the corrected length is equivalent to the sum of the lateral fin area plus the fin tip area.

The corrected length approximation gives very good results when the variation of temperature near the fin tip is small (which is the case when $mL \geq 1$) and the heat transfer coefficient at the fin tip is about the same as that at the lateral surface of the fin. Therefore, *fins subjected to convection at their tips can be treated as fins with insulated tips by replacing the actual fin length by the corrected length in Eqs. 3–64 and 3–65.*

Using the proper relations for A_c and p, the corrected lengths for rectangular and cylindrical fins are easily determined to be

$$L_{c,\text{ rectangular fin}} = L + \frac{t}{2} \quad \text{and} \quad L_{c,\text{ cylindrical fin}} = L + \frac{D}{4}$$

where t is the thickness of the rectangular fins and D is the diameter of the cylindrical fins.

Fin Efficiency

Consider the surface of a *plane wall* at temperature T_b exposed to a medium at temperature T_∞. Heat is lost from the surface to the surrounding medium by convection with a heat transfer coefficient of h. Disregarding radiation or accounting for its contribution in the convection coefficient h, heat transfer from a surface area A_s is expressed as $\dot{Q} = hA_s(T_s - T_\infty)$.

Now let us consider a fin of constant cross-sectional area $A_c = A_b$ and length L that is attached to the surface with a perfect contact (Fig. 3–41). This time heat is transferred from the surface to the fin *by conduction* and from the fin to the surrounding medium *by convection* with the same heat transfer coefficient h. The temperature of the fin is T_b at the fin base and gradually decreases toward the fin tip. Convection from the fin surface causes the temperature at any cross section to drop somewhat from the midsection toward the outer surfaces. However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross section can be considered to be uniform. Also, the fin tip can be assumed for convenience and simplicity to be adiabatic by using the corrected length for the fin instead of the actual length.

In the limiting case of *zero thermal resistance* or *infinite thermal conductivity* ($k \to \infty$), the temperature of the fin is uniform at the base value of T_b. The heat transfer from the fin is *maximum* in this case and can be expressed as

$$\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}\,(T_b - T_\infty) \tag{3–73}$$

In reality, however, the temperature of the fin drops along the fin, and thus the heat transfer from the fin is less because of the decreasing temperature difference $T(x) - T_\infty$ toward the fin tip, as shown in Fig. 3–42. To account for the effect of this decrease in temperature on heat transfer, we define a **fin efficiency** as

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\begin{array}{c}\text{Ideal heat transfer rate from the fin}\\ \text{if the entire fin were at base temperature}\end{array}} \tag{3–74}$$

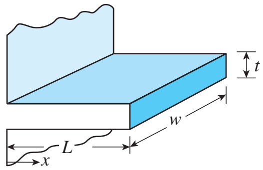
TABLE 3–3

Efficiency and surface areas of common fin configurations

Straight rectangular fins

$$m = \sqrt{2h/kt}$$
$$L_c = L + t/2$$
$$A_{\text{fin}} = 2wL_c$$

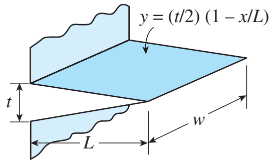
$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$



Straight triangular fins

$$m = \sqrt{2h/kt}$$
$$A_{\text{fin}} = 2w\sqrt{L^2 + (t/2)^2}$$

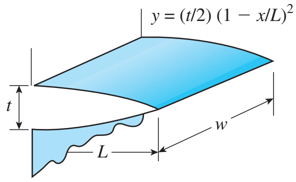
$$\eta_{\text{fin}} = \frac{1}{mL}\frac{I_1(2mL)}{I_1(2mL)}$$



Straight parabolic fins

$$m = \sqrt{2h/kt}$$
$$A_{\text{fin}} = wL[C_1 + (L/t)\ln(t/L + C_1)]$$
$$C_1 = \sqrt{1 + (t/L)^2}$$

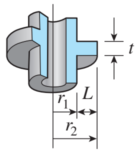
$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL)^2 + 1}}$$



Circular fins of rectangular profile

$$m = \sqrt{2h/kt}$$
$$r_{2c} = r_2 + t/2$$
$$A_{\text{fin}} = 2\pi(r_{2c}^2 - r_1^2)$$

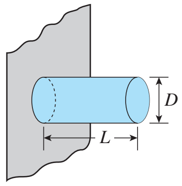
$$\eta_{\text{fin}} = C_2\frac{K_1(mr_1)I_1(mr_{2c}) - I_1(mr_1)K_1(mr_{2c})}{I_0(mr_1)K_1(mr_{2c}) + K_0(mr_1)I_1(mr_{2c})}$$
$$C_2 = \frac{2r_1/m}{r_{2c}^2 - r_1^2}$$



Pin fins of rectangular profile

$$m = \sqrt{4h/kD}$$
$$L_c = L + D/4$$
$$A_{\text{fin}} = \pi DL_c$$

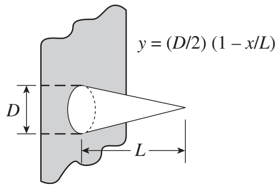
$$\eta_{\text{fin}} = \frac{\tanh mL_c}{mL_c}$$



Pin fins of triangular profile

$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi D}{2}\sqrt{L^2 + (D/2)^2}$$

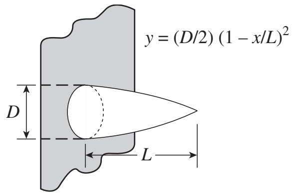
$$\eta_{\text{fin}} = \frac{2}{mL}\frac{I_0(2mL)}{I_1(2mL)}$$
$$I_2(x) = I_0(x) - (2/x)I_1(x) \text{ where } x = 2mL$$



Pin fins of parabolic profile

$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi L^3}{8D}[C_3C_4 - \frac{L}{2D}\ln(2DC_4/L + C_3)]$$
$$C_3 = 1 + 2(D/L)^2$$
$$C_4 = \sqrt{1 + (D/L)^2}$$

$$\eta_{\text{fin}} = \frac{2}{1 + \sqrt{(2mL/3)^2 + 1}}$$



Pin fins of parabolic profile (blunt tip)

$$m = \sqrt{4h/kD}$$
$$A_{\text{fin}} = \frac{\pi D^4}{96L^2}\left\{[16(L/D)^2 + 1]^{3/2} - 1\right\}$$

$$\eta_{\text{fin}} = \frac{3}{2mL}\frac{I_1(4mL/3)}{I_0(4mL/3)}$$

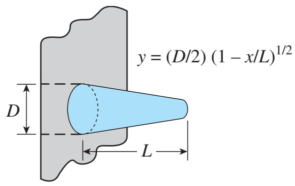


TABLE 3–4

Modified Bessel functions of the first and second kinds*

x	$e^{-x}I_0(x)$	$e^{-x}I_1(x)$	$e^{x}K_0(x)$	$e^{x}K_1(x)$
0.0	1.0000	0.0000	∞	∞
0.2	0.8269	0.0823	2.1408	5.8334
0.4	0.6974	0.1368	1.6627	3.2587
0.6	0.5993	0.1722	1.4167	2.3739
0.8	0.5241	0.1945	1.2582	1.9179
1.0	0.4658	0.2079	1.1445	1.6362
1.2	0.4198	0.2153	1.0575	1.4429
1.4	0.3831	0.2185	0.9881	1.3011
1.6	0.3533	0.2190	0.9309	1.1919
1.8	0.3289	0.2177	0.8828	1.1048
2.0	0.3085	0.2153	0.8416	1.0335
2.2	0.2913	0.2121	0.8057	0.9738
2.4	0.2766	0.2085	0.7740	0.9229
2.6	0.2639	0.2047	0.7459	0.8790
2.8	0.2528	0.2007	0.7206	0.8405
3.0	0.2430	0.1968	0.6978	0.8066
3.2	0.2343	0.1930	0.6770	0.7763
3.4	0.2264	0.1892	0.6580	0.7491
3.6	0.2193	0.1856	0.6405	0.7245
3.8	0.2129	0.1821	0.6243	0.7021
4.0	0.2070	0.1788	0.6093	0.6816
4.2	0.2016	0.1755	0.5953	0.6627
4.4	0.1966	0.1725	0.5823	0.6454
4.6	0.1919	0.1695	0.5701	0.6292
4.8	0.1876	0.1667	0.5586	0.6143
5.0	0.1835	0.1640	0.5478	0.6003
5.2	0.1797	0.1614	0.5376	0.5872
5.4	0.1762	0.1589	0.5280	0.5749
5.6	0.1728	0.1565	0.5188	0.5634
5.8	0.1697	0.1542	0.5101	0.5525
6.0	0.1667	0.1521	0.5019	0.5422
6.5	0.1598	0.1469	0.4828	0.5187
7.0	0.1537	0.1423	0.4658	0.4981
7.5	0.1483	0.1380	0.4505	0.4797
8.0	0.1434	0.1341	0.4366	0.4631
8.5	0.1390	0.1305	0.4239	0.4482
9.0	0.1350	0.1272	0.4123	0.4346
9.5	0.1313	0.1241	0.4016	0.4222
10.0	0.1278	0.1213	0.3916	0.4108

*Evaluated from EES using the mathematical functions Bessel_I(x) and Bessel_K(x)

or

$$\dot{Q}_{\text{fin}} = \eta_{\text{fin}}\,\dot{Q}_{\text{fin, max}} = \eta_{\text{fin}}\,hA_{\text{fin}}\,(T_b - T_\infty) \qquad \textbf{(3–75)}$$

where A_{fin} is the total surface area of the fin. This relation enables us to determine the heat transfer from a fin when its efficiency is known. For the cases of constant cross section of *very long fins* and *fins with adiabatic tips,* the fin efficiency can be expressed as

$$\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}(T_b - T_\infty)}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{1}{L}\sqrt{\frac{kA_c}{hp}} = \frac{1}{mL} \qquad \textbf{(3–76)}$$

and

$$\eta_{\text{adiabatic tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{hA_{\text{fin}}\,(T_b - T_\infty)} = \frac{\tanh mL}{mL} \qquad \textbf{(3–77)}$$

since $A_{\text{fin}} = pL$ for fins with constant cross section. Equation 3–77 can also be used for fins subjected to convection provided that the fin length L is replaced by the corrected length L_c.

Table 3–3 provides fin efficiency relations for fins with uniform and non-uniform cross section. For fins with non-uniform profile, Eq. 3–56 is no longer valid and the general form of the differential equation governing heat transfer in fins of arbitrary shape, Eq. 3–55, must be used. For these cases the solution is no longer in the form of simple exponential or hyperbolic functions. The mathematical functions I and K that appear in some of these relations are the *modified Bessel functions,* and their values are given in Table 3–4. Efficiencies are plotted in Fig. 3–43 for fins on a *plain surface* and in Fig. 3–44 for *circular fins* of constant thickness. For most fins of constant thickness encountered in practice, the fin thickness t is too small relative to the fin length L, and thus the fin tip area is negligible.

Note that fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles, and thus are more suitable for applications requiring minimum weight such as space applications.

An important consideration in the design of finned surfaces is the selection of the proper *fin length L.* Normally the *longer* the fin, the *larger* the heat transfer area and thus the *higher* the rate of heat transfer from the fin. But also the larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction. Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost. Also, the fin efficiency decreases with increasing fin length because of the decrease in fin temperature with length. Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically and should be avoided. The efficiency of most fins used in practice is above 90 percent.

Fin Effectiveness

Fins are used to *enhance* heat transfer, and the use of fins on a surface cannot be recommended unless the enhancement in heat transfer justifies the added cost and complexity associated with the fins. In fact, there is no assurance that

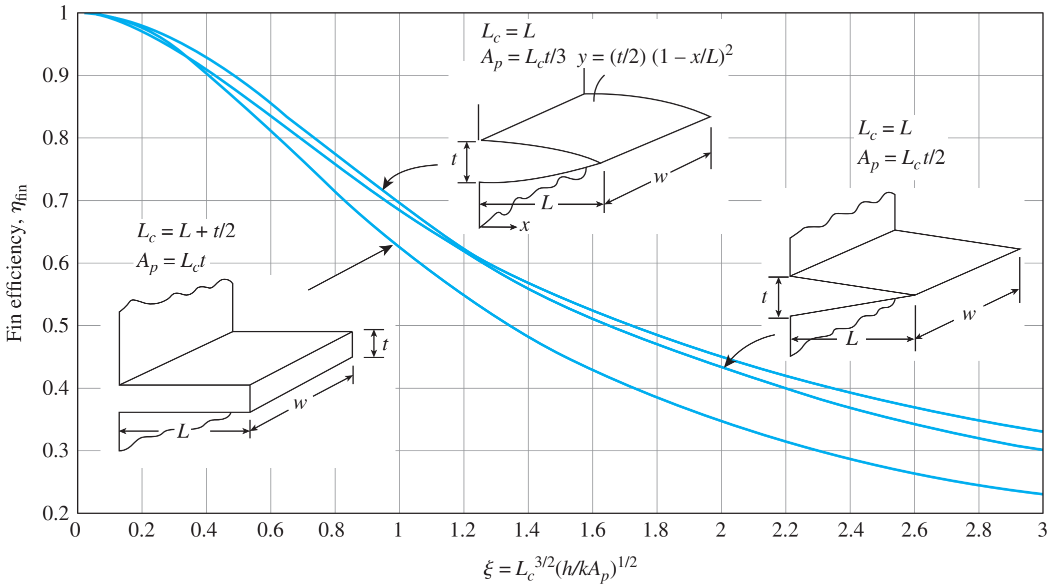


FIGURE 3–43
Efficiency of straight fins of rectangular, triangular, and parabolic profiles.

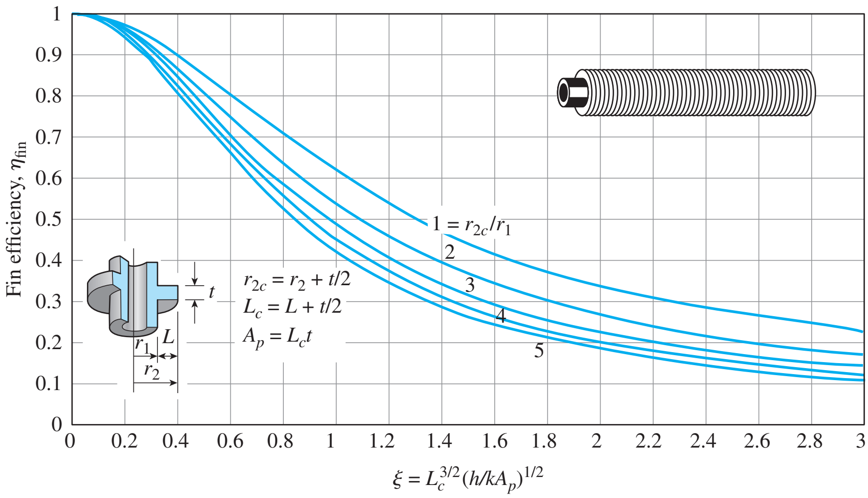


FIGURE 3–44
Efficiency of annular fins of constant thickness *t*.

adding fins on a surface will *enhance* heat transfer. The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case. The performance of fins is expressed in terms of the *fin effectiveness* ε_{fin} defined as (Fig. 3–45)

$$\varepsilon_{fin} = \frac{\dot{Q}_{fin}}{\dot{Q}_{no\ fin}} = \frac{\dot{Q}_{fin}}{hA_b(T_b - T_\infty)} = \frac{\text{Heat transfer rate from the fin of } \textit{base area } A_b}{\text{Heat transfer rate from the surface of } \textit{area } A_b} \quad (3\text{–}78)$$

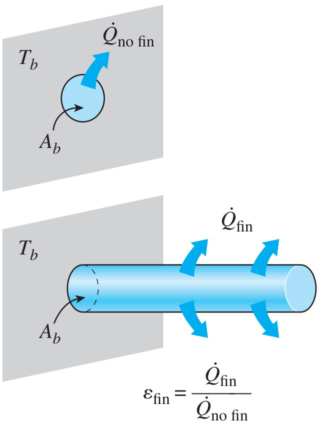


FIGURE 3–45
The effectiveness of a fin.

Hence, A_b is the cross-sectional area of the fin at the base and $\dot{Q}_{\text{no fin}}$ represents the rate of heat transfer from this area if no fins are attached to the surface. An effectiveness of $\varepsilon_{\text{fin}} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area A_b is equal to the heat transferred from the same area A_b to the surrounding medium. An effectiveness of $\varepsilon_{\text{fin}} < 1$ indicates that the fin actually acts as *insulation,* slowing down the heat transfer from the surface. This situation can occur when fins made of low thermal conductivity materials are used. An effectiveness of $\varepsilon_{\text{fin}} > 1$ indicates that fins are *enhancing* heat transfer from the surface, as they should. However, the use of fins cannot be justified unless ε_{fin} is sufficiently larger than 1. Finned surfaces are designed on the basis of *maximizing* effectiveness for a specified cost or *minimizing* cost for a desired effectiveness.

Note that both the fin efficiency and fin effectiveness are related to the performance of the fin, but they are different quantities. However, they are related to each other by

$$\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_\infty)} = \frac{\eta_{\text{fin}}\, hA_{\text{fin}}(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \frac{A_{\text{fin}}}{A_b}\eta_{\text{fin}} \tag{3–79}$$

Therefore, the fin effectiveness can be determined easily when the fin efficiency is known, or vice versa.

The rate of heat transfer from a sufficiently *long* fin of *uniform* cross section under steady conditions is given by Eq. 3–61. Substituting this relation into Eq. 3–78, the effectiveness of such a long fin is determined to be

$$\varepsilon_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)}{hA_b(T_b - T_\infty)} = \sqrt{\frac{kp}{hA_c}} \tag{3–80}$$

since $A_c = A_b$ in this case. We can draw several important conclusions from the fin effectiveness relation above for consideration in the design and selection of the fins:

- The *thermal conductivity* k of the fin material should be as high as possible. Thus it is no coincidence that fins are made from metals, with copper, aluminum, and iron being the most common ones. Perhaps the most widely used fins are made of aluminum because of its low cost and weight and its resistance to corrosion.
- The ratio of the *perimeter* to the *cross-sectional area* of the fin p/A_c should be as high as possible. This criterion is satisfied by *thin* plate fins and *slender* pin fins.
- The use of fins is *most effective* in applications involving a *low convection heat transfer coefficient.* Thus, the use of fins is more easily justified when the medium is a *gas* instead of a liquid and the heat transfer is by *natural convection* instead of by forced convection. Therefore, it is no coincidence that in liquid-to-gas heat exchangers such as the car radiator, fins are placed on the *gas* side.

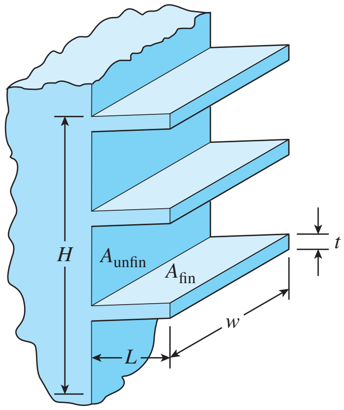
When determining the rate of heat transfer from a finned surface, we must consider the *unfinned portion* of the surface as well as the *fins.* Therefore, the rate of heat transfer for a surface containing n fins can be expressed as

$$\dot{Q}_{\text{total, fin}} = \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}}$$
$$= hA_{\text{unfin}}(T_b - T_\infty) + \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_\infty)$$
$$= h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty) \quad \text{(3–81)}$$

We can also define an **overall effectiveness** for a finned surface as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins,

$$\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_\infty)}{hA_{\text{no fin}}(T_b - T_\infty)} = \frac{A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}}}{A_{\text{no fin}}} \quad \text{(3–82)}$$

where $A_{\text{no fin}}$ is the area of the surface when there are no fins, A_{fin} is the total surface area of all the fins on the surface, and A_{unfin} is the area of the unfinned portion of the surface (Fig. 3–46). Note that the overall fin effectiveness depends on the fin density (number of fins per unit length) as well as the effectiveness of the individual fins. The overall effectiveness is a better measure of the performance of a finned surface than the effectiveness of the individual fins.



$A_{\text{no fin}} = w \times H$
$A_{\text{unfin}} = w \times H - 3 \times (t \times w)$
$A_{\text{fin}} = 2 \times L \times w + t \times w$
$\cong 2 \times L \times w$ (one fin)

FIGURE 3–46
Various surface areas associated with a rectangular surface with three fins.

Proper Length of a Fin

An important step in the design of a fin is the determination of the appropriate length of the fin once the fin material and the fin cross section are specified. You may be tempted to think that the longer the fin, the larger the surface area and thus the higher the rate of heat transfer. Therefore, for maximum heat transfer, the fin should be infinitely long. However, the temperature drops along the fin exponentially and reaches the environment temperature at some length. The part of the fin beyond this length does not contribute to heat transfer since it is at the temperature of the environment, as shown in Fig. 3–47. Therefore, designing such an "extra long" fin is out of the question since it results in material waste, excessive weight, and increased size and thus increased cost with no benefit in return (in fact, such a long fin will hurt performance since it will suppress fluid motion and thus reduce the convection heat transfer coefficient). Fins that are so long that the temperature approaches the environment temperature cannot be recommended either since the little increase in heat transfer at the tip region cannot justify the disproportionate increase in the weight and cost.

To get a sense of the proper length of a fin, we compare heat transfer from a fin of finite length to heat transfer from an infinitely long fin under the same conditions. The ratio of these two heat transfers is

Heat transfer ratio:
$$\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpkA_c}\,(T_b - T_\infty)\tanh mL}{\sqrt{hpkA_c}\,(T_b - T_\infty)} = \tanh mL \quad \text{(3–83)}$$

Using a hand calculator, the values of tanh mL are evaluated for some values of mL and the results are given in Table 3–5. We observe from the table that heat transfer from a fin increases with mL almost linearly at first, but the curve reaches a plateau later and reaches a value for the infinitely long fin at about $mL = 5$. Therefore, a fin whose length is $L = 5/m$ can be considered to be an infinitely long fin. We also observe that reducing the fin length by half in that case (from $mL = 5$ to $mL = 2.5$) causes a drop of just 1 percent in

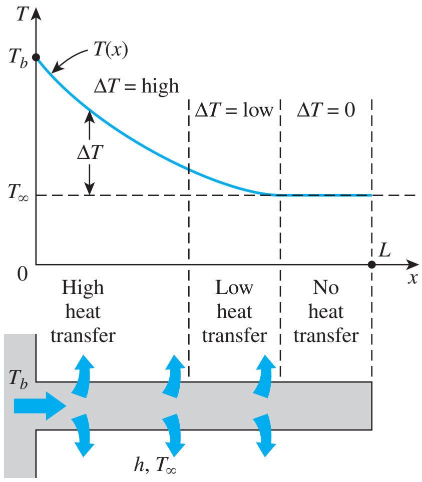


FIGURE 3–47
Because of the gradual temperature drop along the fin, the region near the fin tip makes little or no contribution to heat transfer.

TABLE 3–5

The variation of heat transfer from a fin relative to that from an infinitely long fin

mL	$\dfrac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \tanh mL$
0.1	0.100
0.2	0.197
0.5	0.462
1.0	0.762
1.5	0.905
2.0	0.964
2.5	0.987
3.0	0.995
4.0	0.999
5.0	1.000

heat transfer. We certainly would not hesitate sacrificing 1 percent in heat transfer performance in return for 50 percent reduction in the size and possibly the cost of the fin. In practice, a fin length that corresponds to about $mL = 1$ will transfer 76.2 percent of the heat that can be transferred by an infinitely long fin, and thus it should offer a good compromise between heat transfer performance and the fin size.

A common approximation used in the analysis of fins is to assume the fin temperature to vary in one direction only (along the fin length) and the temperature variation along other directions is negligible. Perhaps you are wondering if this one-dimensional approximation is a reasonable one. This is certainly the case for fins made of thin metal sheets such as the fins on a car radiator, but we wouldn't be so sure for fins made of thick materials. Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1 percent) when

$$\frac{h\delta}{k} < 0.2$$

where δ is the characteristic thickness of the fin, which is taken to be the plate thickness t for rectangular fins and the diameter D for cylindrical ones.

Specially designed finned surfaces called *heat sinks,* which are commonly used in the cooling of electronic equipment, involve one-of-a-kind complex geometries, as shown in Table 3–6. The heat transfer performance of heat sinks is usually expressed in terms of their *thermal resistances* R in °C/W, which is defined as

$$\dot{Q}_{\text{fin}} = \frac{T_b - T_\infty}{R} = hA_{\text{fin}}\,\eta_{\text{fin}}\,(T_b - T_\infty) \qquad \textbf{(3–84)}$$

A small value of thermal resistance indicates a small temperature drop across the heat sink, and thus a high fin efficiency.

EXAMPLE 3–10 **Maximum Power Dissipation of a Transistor**

Power transistors that are commonly used in electronic devices consume large amounts of electric power. The failure rate of electronic components increases almost exponentially with operating temperature. As a rule of thumb, the failure rate of electronic components is halved for each 10°C reduction in the junction operating temperature. Therefore, the operating temperature of electronic components is kept below a safe level to minimize the risk of failure.

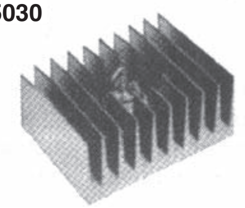
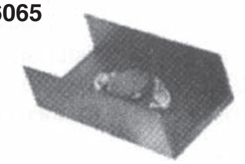
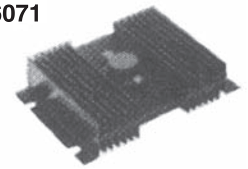
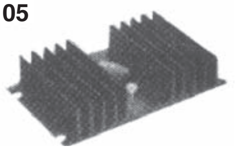
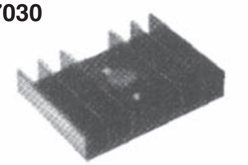
The sensitive electronic circuitry of a power transistor at the junction is protected by its case, which is a rigid metal enclosure. Heat transfer characteristics of a power transistor are usually specified by the manufacturer in terms of the case-to-ambient thermal resistance, which accounts for both the natural convection and radiation heat transfers.

The case-to-ambient thermal resistance of a power transistor that has a maximum power rating of 10 W is given to be 20°C/W. If the case temperature of

(*Continued on page 184*)

TABLE 3–6

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in) long.

HS 5030		$R = 0.9^\circ\text{C/W}$ (vertical) $R = 1.2^\circ\text{C/W}$ (horizontal) Dimensions: 76 mm $\times$ 105 mm $\times$ 44 mm Surface area: 677 cm^2
HS 6065		$R = 5^\circ\text{C/W}$ Dimensions: 76 mm $\times$ 38 mm $\times$ 24 mm Surface area: 387 cm^2
HS 6071		$R = 1.4^\circ\text{C/W}$ (vertical) $R = 1.8^\circ\text{C/W}$ (horizontal) Dimensions: 76 mm $\times$ 92 mm $\times$ 26 mm Surface area: 968 cm^2
HS 6105		$R = 1.8^\circ\text{C/W}$ (vertical) $R = 2.1^\circ\text{C/W}$ (horizontal) Dimensions: 76 mm $\times$ 127 mm $\times$ 91 mm Surface area: 677 cm^2
HS 6115		$R = 1.1^\circ\text{C/W}$ (vertical) $R = 1.3^\circ\text{C/W}$ (horizontal) Dimensions: 76 mm $\times$ 102 mm $\times$ 25 mm Surface area: 929 cm^2
HS 7030		$R = 2.9^\circ\text{C/W}$ (vertical) $R = 3.1^\circ\text{C/W}$ (horizontal) Dimensions: 76 mm $\times$ 97 mm $\times$ 19 mm Surface area: 290 cm^2