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# 1. Teoría de números

## 1.1. Funciones básicas

### 1.1.1. Función piso y techo

```
lli piso(lli a, lli b){
    if((a >= 0 && b > 0) || (a < 0 && b < 0)){
        return a / b;
    }else{
        if(a % b == 0) return a / b;
        else return a / b - 1;
    }
}
```

```
lli techo(lli a, lli b){
    if((a >= 0 && b > 0) || (a < 0 && b < 0)){
        if(a % b == 0) return a / b;
        else return a / b + 1;
    }else{
        return a / b;
    }
}
```

### 1.1.2. Exponenciación y multiplicación binaria

```
lli power(lli b, lli e){
    lli ans = 1;
    while(e){
        if(e & 1) ans *= b;
        e >>= 1;
        b *= b;
    }
    return ans;
}
```

```
lli multMod(lli a, lli b, lli n){
    lli ans = 0;
    a %= n, b %= n;
    if(abs(b) > abs(a)) swap(a, b);
    if(b < 0){
        a = -a, b = -b;
    }
```

```
    }
    while(b){
        if(b & 1) ans = (ans + a) % n;
        b >>= 1;
        a = (a + a) % n;
    }
    return ans;
}
```

```
uint64_t mul_mod(uint64_t a, uint64_t b, uint64_t m){
    if(a >= m) a %= m;
    if(b >= m) b %= m;
    uint64_t c = (long double)a * b / m;
    int64_t r = (int64_t)(a * b - c * m) % (int64_t)m;
    return r < 0 ? r + m : r;
}
```

### 1.1.3. Mínimo común múltiplo y máximo común divisor

```
lli gcd(lli a, lli b){
    lli r;
    while(b != 0) r = a % b, a = b, b = r;
    return a;
}
```

```
lli lcm(lli a, lli b){
    return b * (a / gcd(a, b));
}
```

```
lli gcd(const vector<lli>& nums){
    lli ans = 0;
    for(lli num : nums) ans = gcd(ans, num);
    return ans;
}
```

```
lli lcm(const vector<lli>& nums){
    lli ans = 1;
    for(lli num : nums) ans = lcm(ans, num);
    return ans;
}
```

### 1.1.4. Euclides extendido e inverso modular

```
tuple<lli, lli, lli> extendedGcd(lli a, lli b){
    if(b == 0){
        if(a > 0) return {a, 1, 0};
        else return {-a, -1, 0};
    }else{
        auto[d, x, y] = extendedGcd(b, a%b);
        return {d, y, x - y*(a/b)};
    }
}

lli modularInverse(lli a, lli m){
    auto[d, x, y] = extendedGcd(a, m);
    if(d != 1) return -1; // inverse doesn't exist
    if(x < 0) x += m;
    return x;
}
```

### 1.1.5. Todos los inversos módulo $p$

```
//find all inverses (from 1 to p-1) modulo p
vector<lli> allInverses(lli p){
    vector<lli> ans(p);
    ans[1] = 1;
    for(lli i = 2; i < p; ++i)
        ans[i] = p - (p / i) * ans[p % i] % p;
    return ans;
}
```

### 1.1.6. Exponenciación binaria modular

```
lli powerMod(lli b, lli e, lli m){
    lli ans = 1;
    b %= m;
    if(e < 0){
        b = modularInverse(b, m);
        e = -e;
    }
    while(e){
        if(e & 1) ans = ans * b % m;
        e >>= 1;
    }
```

```
        b = b * b % m;
    }
    return ans;
}
```

### 1.1.7. Teorema chino del residuo

```
//generalized chinese remainder theorem
//the modulus doesn't need to be pairwise coprime
pair<lli, lli> crt(const vector<lli>& a, const vector<lli>& m){
    lli x = a[0], mod = m[0];
    for(int i = 1; i < a.size(); ++i){
        auto[d, s, t] = extendedGcd(mod, -m[i]);
        if((a[i] - x) % d != 0) return {-1, -1};
        lli step = m[i] / d;
        lli k = s * ((a[i] - x) / d) % step % step;
        if(k < 0) k += step;
        x += mod*k;
        mod *= step;
    }
    return {x, mod};
}
```

### 1.1.8. Coeficiente binomial

```
lli ncr(lli n, lli r){
    if(r < 0 || r > n) return 0;
    r = min(r, n - r);
    lli ans = 1;
    for(lli den = 1, num = n; den <= r; den++, num--){
        ans = ans * num / den;
    }
    return ans;
}
```

### 1.1.9. Fibonacci

```
//very fast fibonacci
inline void modula(lli & n, lli mod){
    while(n >= mod) n -= mod;
}
```

```
lli fibo(lli n, lli mod){
    array<lli, 2> F = {1, 0};
    lli p = 1;
    for(lli v = n; v >= 1; p <= 1);
    array<lli, 4> C;
    do{
        int d = (n & p) != 0;
        C[0] = C[3] = 0;
        C[d] = F[0] * F[0] % mod;
        C[d+1] = (F[0] * F[1] << 1) % mod;
        C[d+2] = F[1] * F[1] % mod;
        F[0] = C[0] + C[2] + C[3];
        F[1] = C[1] + C[2] + (C[3] << 1);
        modula(F[0], mod), modula(F[1], mod);
    }while(p >= 1);
    return F[1];
}
```

## 1.2. Cribas

### 1.2.1. Criba de divisores

```
vector<lli> divsSum;
vector<vector<int>> divs;
void divisorsSieve(int n){
    divsSum.resize(n + 1, 0);
    divs.resize(n + 1);
    for(int i = 1; i <= n; ++i){
        for(int j = i; j <= n; j += i){
            divsSum[j] += i;
            divs[j].push_back(i);
        }
    }
}
```

### 1.2.2. Criba de primos

```
vector<int> primesSieve(int n){
    vector<bool> is(n+1, true);
    vector<int> primes = {2};
    is[0] = is[1] = false;
    for(int i = 4; i <= n; i += 2) is[i] = false;
```

```
    for(int i = 3; i <= n; i += 2){
        if(is[i]){
            primes.push_back(i);
            if((long long)i*i <= n)
                for(int j = i*i; j <= n; j += 2*i)
                    is[j] = false;
        }
    }
    return primes;
}
```

### 1.2.3. Criba de factor primo más pequeño

```
vector<int> lowestPrimeSieve(int n){
    vector<int> lp(n+1);
    iota(lp.begin(), lp.end(), 0);
    for(int i = 4; i <= n; i += 2) lp[i] = 2;
    for(int i = 3; i*i <= n; i += 2)
        if(lp[i] == i)
            for(int j = i*i; j <= n; j += 2*i)
                lp[j] = min(lp[j], i);
    return lp;
}
```

### 1.2.4. Criba de factor primo más grande

```
vector<int> greatestPrimeSieve(int n){
    vector<int> gp(n+1);
    iota(gp.begin(), gp.end(), 0);
    for(int i = 2; i <= n; i++)
        if(gp[i] == i)
            for(int j = 2*i; j <= n; j += i)
                gp[j] = i;
    return gp;
}
```

### 1.2.5. Criba de factores primos

```
vector<vector<int>> primeFactorsSieve(int n){
    vector<vector<int>> primeFactors(n+1);
    for(int p = 2; p <= n; ++p){
```



```

    if(primeFactors[p].empty())
        for(int j = p; j <= n; j += p)
            primeFactors[j].push_back(p);
}
return primeFactors;
}

```

### 1.2.6. Criba de la función $\varphi$ de Euler

```

vector<int> phiSieve(int n){
    vector<int> Phi(n+1);
    iota(Phi.begin(), Phi.end(), 0);
    for(int i = 2; i <= n; ++i)
        if(Phi[i] == i)
            for(int j = i; j <= n; j += i)
                Phi[j] -= Phi[j] / i;
    return Phi;
}

```

### 1.2.7. Criba de la función $\mu$

```

vector<int> muSieve(int n){
    vector<int> Mu(n+1, -1);
    Mu[0] = 0, Mu[1] = 1;
    for(int i = 2; i <= n; ++i)
        if(Mu[i])
            for(int j = 2*i; j <= n; j += i)
                Mu[j] -= Mu[i];
    return Mu;
}

```

### 1.2.8. Triángulo de Pascal

```

vector<vector<lli>> ncrSieve(int n){
    vector<vector<lli>> Ncr(n+1);
    Ncr[0] = {1};
    for(int i = 1; i <= n; ++i){
        Ncr[i].resize(i + 1);
        Ncr[i][0] = Ncr[i][i] = 1;
        for(int j = 1; j <= i / 2; j++)

```

```

        Ncr[i][i - j] = Ncr[i][j] = Ncr[i - 1][j - 1] + Ncr[i - 1][j];
    }
    return Ncr;
}

```

### 1.2.9. Segmented sieve

```

vector<int> segmented_sieve(int limit){
    const int L1D_CACHE_SIZE = 32768;
    int raiz = sqrt(limit);
    int segment_size = max(raiz, L1D_CACHE_SIZE);
    int s = 3, n = 3;
    vector<int> primes(1, 2), tmp, next;
    vector<char> sieve(segment_size);
    vector<bool> is_prime(raiz + 1, 1);
    for(int i = 2; i * i <= raiz; i++)
        if(is_prime[i])
            for(int j = i * i; j <= raiz; j += i)
                is_prime[j] = 0;
    for(int low = 0; low <= limit; low += segment_size){
        fill(sieve.begin(), sieve.end(), 1);
        int high = min(low + segment_size - 1, limit);
        for(; s * s <= high; s += 2){
            if(is_prime[s]){
                tmp.push_back(s);
                next.push_back(s * s - low);
            }
        }
        for(size_t i = 0; i < tmp.size(); i++){
            int j = next[i];
            for(int k = tmp[i] * 2; j < segment_size; j += k)
                sieve[j] = 0;
            next[i] = j - segment_size;
        }
        for(; n <= high; n += 2)
            if(sieve[n - low])
                primes.push_back(n);
    }
    return primes;
}

```

### 1.2.10. Criba de primos lineal

```
vector<int> linearPrimeSieve(int n){
    vector<int> primes;
    vector<bool> isPrime(n+1, true);
    for(int i = 2; i <= n; ++i){
        if(isPrime[i])
            primes.push_back(i);
        for(int p : primes){
            int d = i * p;
            if(d > n) break;
            isPrime[d] = false;
            if(i % p == 0) break;
        }
    }
    return primes;
}
```

### 1.2.11. Criba lineal para funciones multiplicativas

```
//suppose f(n) is a multiplicative function and
//we want to find f(1), f(2), ..., f(n)
//we have f(pq) = f(p)f(q) if gcd(p, q) = 1
//and f(p^a) = g(p, a), where p is prime and a>0
vector<int> generalSieve(int n, function<int(int, int)> g){
    vector<int> f(n+1, 1), cnt(n+1), acum(n+1), primes;
    vector<bool> isPrime(n+1, true);
    for(int i = 2; i <= n; ++i){
        if(isPrime[i]){ //case base: f(p)
            f[i] = g(i, 1);
            primes.push_back(i);
            cnt[i] = 1;
            acum[i] = i;
        }
        for(int p : primes){
            int d = i * p;
            if(d > n) break;
            isPrime[d] = false;
            if(i % p == 0){ //gcd(i, p) != 1
                f[d] = f[i / acum[i]] * g(p, cnt[i] + 1);
                cnt[d] = cnt[i] + 1;
                acum[d] = acum[i] * p;
                break;
            }
        }
    }
}
```

```
        }else{ //gcd(i, p) = 1
            f[d] = f[i] * g(p, 1);
            cnt[d] = 1;
            acum[d] = p;
        }
    }
    return f;
}
```

## 1.3. Factorización

### 1.3.1. Factorización de un número

```
vector<pair<lli, int>> factorize(lli n){
    vector<pair<lli, int>> f;
    for(lli p : primes){
        if(p * p > n) break;
        int pot = 0;
        while(n % p == 0){
            pot++;
            n /= p;
        }
        if(pot) f.emplace_back(p, pot);
    }
    if(n > 1) f.emplace_back(n, 1);
    return f;
}
```

### 1.3.2. Potencia de un primo que divide a un factorial

```
lli potInFactorial(lli n, lli p){
    lli ans = 0, div = n;
    while(div /= p) ans += div;
    return ans;
}
```

### 1.3.3. Factorización de un factorial

```
vector<pair<lli, lli>> factorizeFactorial(lli n){
    vector<pair<lli, lli>> f;
```

```

for(lli p : primes){
    if(p > n) break;
    f.emplace_back(p, potInFactorial(n, p));
}
return f;
}

```

### 1.3.4. Factorial módulo $p$

```

//Finds  $(n!/p^m) \bmod p^s$ , where  $m$  is the largest power of  $p$ 
//that divides  $n!$ ,  $p$  must be prime
lli factmod(lli n, lli p, int s){
    lli ans = 1;
    lli ps = power(p, s);
    while(n > 1){
        lli q = n / ps, r = n % ps;
        ans = ans * (q % 2 == 1 && !(p == 2 && s >= 3) ? ps-1 : 1) %
        ↪ ps;
        for(lli i = 2; i <= r; ++i){
            if(i % p == 0) continue;
            ans = ans * i % ps;
        }
        n /= p;
    }
    return ans;
}

```

### 1.3.5. Factorización usando Pollard-Rho

```

bool isPrimeMillerRabin(lli n){
    if(n < 2) return false;
    if(!(n & 1)) return n == 2;
    lli d = n - 1, s = 0;
    for(; !(d & 1); d >>= 1, ++s);
    for(int a : {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37}){
        if(n == a) return true;
        lli m = powerMod(a, d, n);
        if(m == 1 || m == n - 1) continue;
        int k = 0;
        for(; k < s; ++k){
            m = m * m % n;
            if(m == n - 1) break;

```

```

        }
        if(k == s) return false;
    }
    return true;
}

```

```

mt19937_64
↪ rng(chrono::steady_clock::now().time_since_epoch().count());
lli aleatorio(lli a, lli b){
    std::uniform_int_distribution<lli> dist(a, b);
    return dist(rng);
}
lli getFactor(lli n){
    lli a = aleatorio(1, n - 1), b = aleatorio(1, n - 1);
    lli x = 2, y = 2, d = 1;
    while(d == 1){
        x = x * ((x + b) % n) % n + a;
        y = y * ((y + b) % n) % n + a;
        y = y * ((y + b) % n) % n + a;
        d = gcd(abs(x - y), n);
    }
    return d;
}

map<lli, int> fact;
void factorizePollardRho(lli n, bool clean = true){
    if(clean) fact.clear();
    while(n > 1 && !isPrimeMillerRabin(n)){
        lli f = n;
        for(; f == n; f = getFactor(n));
        n /= f;
        factorizePollardRho(f, false);
        for(auto & it : fact){
            while(n % it.first == 0){
                n /= it.first;
                ++it.second;
            }
        }
    }
    if(n > 1) ++fact[n];
}

```

## 1.4. Funciones aritméticas famosas

### 1.4.1. Función $\sigma$

```
//divisor power sum of n
//if pot=0 we get the number of divisors
//if pot=1 we get the sum of divisors
lli sigma(lli n, lli pot){
    lli ans = 1;
    auto f = factorize(n);
    for(auto & factor : f){
        lli p = factor.first;
        int a = factor.second;
        if(pot){
            lli p_pot = power(p, pot);
            ans *= (power(p_pot, a + 1) - 1) / (p_pot - 1);
        }else{
            ans *= a + 1;
        }
    }
    return ans;
}
```

### 1.4.2. Función $\Omega$

```
//number of total primes with multiplicity dividing n
int Omega(lli n){
    int ans = 0;
    auto f = factorize(n);
    for(auto & factor : f)
        ans += factor.second;
    return ans;
}
```

### 1.4.3. Función $\omega$

```
//number of distinct primes dividing n
int omega(lli n){
    int ans = 0;
    auto f = factorize(n);
    for(auto & factor : f)
        ++ans;
}
```

```
    return ans;
}
```

### 1.4.4. Función $\varphi$ de Euler

```
//number of coprimes with n less than n
lli phi(lli n){
    lli ans = n;
    auto f = factorize(n);
    for(auto & factor : f)
        ans -= ans / factor.first;
    return ans;
}
```

### 1.4.5. Función $\mu$

```
//1 if n is square-free with an even number of prime factors
//-1 if n is square-free with an odd number of prime factors
//0 is n has a square prime factor
int mu(lli n){
    int ans = 1;
    auto f = factorize(n);
    for(auto & factor : f){
        if(factor.second > 1) return 0;
        ans *= -1;
    }
    return ans;
}
```

## 1.5. Orden multiplicativo, raíces primitivas y raíces de la unidad

### 1.5.1. Función $\lambda$ de Carmichael

```
//the smallest positive integer k such that for
//every coprime x with n, x^k=1 mod n
lli carmichaelLambda(lli n){
    lli ans = 1;
    auto f = factorize(n);
    for(auto & factor : f){
        lli p = factor.first;
```

```

    int a = factor.second;
    lli tmp = power(p, a);
    tmp -= tmp / p;
    if(a <= 2 || p >= 3) ans = lcm(ans, tmp);
    else ans = lcm(ans, tmp >> 1);
}
return ans;
}

```

### 1.5.2. Orden multiplicativo módulo $m$

```

// the smallest positive integer k such that  $x^k = 1 \pmod m$ 
lli multiplicativeOrder(lli x, lli m){
    if(gcd(x, m) != 1) return 0;
    lli order = phi(m);
    auto f = factorize(order);
    for(auto & factor : f){
        lli p = factor.first;
        int a = factor.second;
        order /= power(p, a);
        lli tmp = powerMod(x, order, m);
        while(tmp != 1){
            tmp = powerMod(tmp, p, m);
            order *= p;
        }
    }
    return order;
}

```

### 1.5.3. Número de raíces primitivas (generadores) módulo $m$

```

//number of generators modulo m
lli numberOfGenerators(lli m){
    lli phi_m = phi(m);
    lli lambda_m = carmichaellambda(m);
    if(phi_m == lambda_m) return phi(phi_m);
    else return 0;
}

```

### 1.5.4. Test individual de raíz primitiva módulo $m$

```

//test if  $\text{order}(x, m) = \phi(m)$ , i.e.,  $x$  is a generator for  $\mathbb{Z}/m\mathbb{Z}$ 
bool testPrimitiveRoot(lli x, lli m){
    if(gcd(x, m) != 1) return false;
    lli order = phi(m);
    auto f = factorize(order);
    for(auto & factor : f){
        lli p = factor.first;
        if(powerMod(x, order / p, m) == 1) return false;
    }
    return true;
}

```

### 1.5.5. Test individual de raíz $k$ -ésima de la unidad módulo $m$

```

//test if  $x^k = 1 \pmod m$  and  $k$  is the smallest for such  $x$ , i.e.,
 $\hookrightarrow x^{(k/p)} \neq 1$  for every prime divisor of  $k$ 
bool testPrimitiveKthRootUnity(lli x, lli k, lli m){
    if(powerMod(x, k, m) != 1) return false;
    auto f = factorize(k);
    for(auto & factor : f){
        lli p = factor.first;
        if(powerMod(x, k / p, m) == 1) return false;
    }
    return true;
}

```

### 1.5.6. Encontrar la primera raíz primitiva módulo $m$

```

lli findFirstGenerator(lli m){
    lli order = phi(m);
    if(order != carmichaellambda(m)) return -1; //just an
     $\hookrightarrow$  optimization, not required
    auto f = factorize(order);
    for(lli x = 1; x < m; x++){
        if(gcd(x, m) != 1) continue;
        bool test = true;
        for(auto & factor : f){
            lli p = factor.first;
            if(powerMod(x, order / p, m) == 1){
                test = false;
            }
        }
        if(test) return x;
    }
    return -1;
}

```

```

        break;
    }
}
if(test) return x;
}
return -1; //not found
}

```

### 1.5.7. Encontrar la primera raíz $k$ -ésima de la unidad módulo $m$

```

lli findFirstPrimitiveKthRootUnity(lli k, lli m){
    if(carmichaelLambda(m) % k != 0) return -1; //just an
    ↪ optimization, not required
    auto f = factorize(k);
    for(lli x = 1; x < m; x++){
        if(powerMod(x, k, m) != 1) continue;
        bool test = true;
        for(auto & factor : f){
            lli p = factor.first;
            if(powerMod(x, k / p, m) == 1){
                test = false;
                break;
            }
        }
        if(test) return x;
    }
    return -1; //not found
}

```

### 1.5.8. Logaritmo discreto

```

// Solves for x in the equation a^x = b mod m
pair<lli, lli> discreteLogarithm(lli a, lli b, lli m){
    lli m1 = m, pw = 1, div, nonRep = 0;
    for(; (div = gcd(a, m1)) > 1; ++nonRep, m1 /= div, pw = pw * a %
    ↪ m){
        if(pw == b) return {nonRep, 0}; //aperiodic solution found
    }
    auto[d, x, y] = extendedGcd(pw, m);
    if(b % d > 0) return {-1, 0}; //solution not found
    b = x * (b / d) % m;
}

```

```

if(b < 0) b += m;
lli order = multiplicativeOrder(a, m1);
lli n = sqrt(order) + 1;
lli a_n = powerMod(a, n, m1);
unordered_map<lli, lli> firstHalf;
pw = a_n;
for(lli p = 1; p <= n; ++p, pw = pw * a_n % m1){
    firstHalf[pw] = p;
}
pw = b % m1;
for(lli q = 0; q <= n; ++q, pw = pw * a % m1){
    if(firstHalf.count(pw)) return {nonRep + (n * firstHalf[pw] -
    ↪ q) % order, order}; //periodic solution found
}
return {-1, 0}; //solution not found
}

```

### 1.5.9. Raíz $k$ -ésima discreta

```

// x^k = b mod m, m has at least one generator
vector<lli> discreteRoot(lli k, lli b, lli m){
    if(b % m == 0) return {0};
    lli g = findFirstGenerator(m);
    lli power = powerMod(g, k, m);
    auto y0 = discreteLogarithm(power, b, m);
    if(y0.first == -1) return {};
    lli phi_m = phi(m);
    lli d = gcd(k, phi_m);
    vector<lli> x(d);
    x[0] = powerMod(g, y0.first, m);
    lli inc = powerMod(g, phi_m / d, m);
    for(lli i = 1; i < d; i++)
        x[i] = x[i - 1] * inc % m;
    sort(x.begin(), x.end());
    return x;
}

```

### 1.5.10. Algoritmo de Tonelli-Shanks para raíces cuadradas módulo $p$

```

//finds x such that x^2 = a mod p
lli sqrtMod(lli a, lli p){

```

```

a %= p;
if(a < 0) a += p;
if(a == 0) return 0;
assert(powerMod(a, (p - 1) / 2, p) == 1);
if(p % 4 == 3) return powerMod(a, (p + 1) / 4, p);
lli s = p - 1;
int r = 0;
while((s & 1) == 0) ++r, s >>= 1;
lli n = 2;
while(powerMod(n, (p - 1) / 2, p) != p - 1) ++n;
lli x = powerMod(a, (s + 1) / 2, p);
lli b = powerMod(a, s, p);
lli g = powerMod(n, s, p);
while(true){
    lli t = b;
    int m = 0;
    for(; m < r; ++m){
        if(t == 1) break;
        t = t * t % p;
    }
    if(m == 0) return x;
    lli gs = powerMod(g, 1 << (r - m - 1), p);
    g = gs * gs % p;
    x = x * gs % p;
    b = b * g % p;
    r = m;
}
}

```

## 1.6. Particiones

### 1.6.1. Función $P$ (particiones de un entero positivo)

```

lli mod = 1e9 + 7;

vector<lli> P;

//number of ways to write n as a sum of positive integers
lli partitionsP(int n){
    if(n < 0) return 0;
    if(P[n]) return P[n];
    int pos1 = 1, pos2 = 2, inc1 = 4, inc2 = 5;
    lli ans = 0;

```

```

    for(int k = 1; k <= n; k++){
        lli tmp = (n >= pos1 ? P[n - pos1] : 0) + (n >= pos2 ? P[n -
        ↪ pos2] : 0);
        if(k & 1) ans += tmp;
        else ans -= tmp;
        if(n < pos2) break;
        pos1 += inc1, pos2 += inc2;
        inc1 += 3, inc2 += 3;
    }
    ans %= mod;
    if(ans < 0) ans += mod;
    return ans;
}

void calculateFunctionP(int n){
    P.resize(n + 1);
    P[0] = 1;
    for(int i = 1; i <= n; i++)
        P[i] = partitionsP(i);
}

```

### 1.6.2. Función $Q$ (particiones de un entero positivo en distintos sumandos)

```

vector<lli> Q;

bool isPerfectSquare(int n){
    int r = sqrt(n);
    return r * r == n;
}

int s(int n){
    int r = 1 + 24 * n;
    if(isPerfectSquare(r)){
        int j;
        r = sqrt(r);
        if((r + 1) % 6 == 0) j = (r + 1) / 6;
        else j = (r - 1) / 6;
        if(j & 1) return -1;
        else return 1;
    }else{
        return 0;
    }
}

```

```

}

//number of ways to write n as a sum of distinct positive integers
//number of ways to write n as a sum of odd positive integers
lli partitionsQ(int n){
    if(n < 0) return 0;
    if(Q[n]) return Q[n];
    int pos = 1, inc = 3;
    lli ans = 0;
    int limit = sqrt(n);
    for(int k = 1; k <= limit; k++){
        if(k & 1) ans += Q[n - pos];
        else ans -= Q[n - pos];
        pos += inc;
        inc += 2;
    }
    ans <= 1;
    ans += s(n);
    ans %= mod;
    if(ans < 0) ans += mod;
    return ans;
}

void calculateFunctionQ(int n){
    Q.resize(n + 1);
    Q[0] = 1;
    for(int i = 1; i <= n; i++)
        Q[i] = partitionsQ(i);
}

```

### 1.6.3. Número de factorizaciones ordenadas

```

//number of ordered factorizations of n
lli orderedFactorizations(lli n){
    //skip the factorization if you already know the powers
    auto fact = factorize(n);
    int k = 0, q = 0;
    vector<int> powers(fact.size() + 1);
    for(auto & f : fact){
        powers[k + 1] = f.second;
        q += f.second;
        ++k;
    }
}

```

```

vector<lli> prod(q + 1, 1);
//we need Ncr until the max_power+Omega(n) row
//module if needed
for(int i = 0; i <= q; i++){
    for(int j = 1; j <= k; j++){
        prod[i] = prod[i] * ncr(powers[j] + i, powers[j]);
    }
}
lli ans = 0;
for(int j = 1; j <= q; j++){
    int alt = 1;
    for(int i = 0; i < j; i++){
        ans = ans + alt * ncr(j, i) * prod[j - i - 1];
        alt *= -1;
    }
}
return ans;
}

```

### 1.6.4. Número de factorizaciones no ordenadas

```

//Number of unordered factorizations of n with
//largest part at most m
//Call unorderedFactorizations(n, n) to get all of them
//Add this to the main to speed up the map:
//mem.reserve(1024); mem.max_load_factor(0.25);
struct HASH{
    size_t operator()(const pair<int, int>&x) const{
        return hash<long long>()(((long long)x.first)^(((long
        long)x.second)<<32));
    }
};
unordered_map<pair<int, int>, lli, HASH> mem;
lli unorderedFactorizations(int m, int n){
    if(m == 1 && n == 1) return 1;
    if(m == 1) return 0;
    if(n == 1) return 1;
    if(mem.count({m, n})) return mem[{m, n}];
    lli ans = 0;
    int l = sqrt(n);
    for(int i = 1; i <= l; ++i){
        if(n % i == 0){
            int a = i, b = n / i;

```



```

    if(a <= m) ans += unorderedFactorizations(a, b);
    if(a != b && b <= m) ans += unorderedFactorizations(b, a);
}
}
return mem[{m, n}] = ans;
}

```

## 1.7. Otros

### 1.7.1. Cambio de base

```

string decimalToBaseB(lli n, lli b){
    string ans = "";
    lli d;
    do{
        d = n % b;
        if(0 <= d && d <= 9) ans = (char)(48 + d) + ans;
        else if(10 <= d && d <= 35) ans = (char)(55 + d) + ans;
        n /= b;
    }while(n != 0);
    return ans;
}

lli baseBtoDecimal(const string & n, lli b){
    lli ans = 0;
    for(const char & d : n){
        if(48 <= d && d <= 57) ans = ans * b + (d - 48);
        else if(65 <= d && d <= 90) ans = ans * b + (d - 55);
        else if(97 <= d && d <= 122) ans = ans * b + (d - 87);
    }
    return ans;
}

```

### 1.7.2. Fracciones continuas

```

//continued fraction of (p+sqrt(n))/q, where p,n,q are positive
↪ integers
//returns a vector of terms and the length of the period,
//the periodic part is taken from the right of the array
pair<vector<lli>, int> ContinuedFraction(lli p, lli n, lli q){
    vector<lli> coef;
    lli r = sqrt(n);

```

```

//Skip this if you know that n is not a perfect square
if(r * r == n){
    lli num = p + r;
    lli den = q;
    lli residue;
    while(den){
        residue = num % den;
        coef.push_back(num / den);
        num = den;
        den = residue;
    }
    return {coef, 0};
}
if((n - p * p) % q != 0){
    n *= q * q;
    p *= q;
    q *= q;
    r = sqrt(n);
}
lli a = (r + p) / q;
coef.push_back(a);
int period = 0;
map<pair<lli, lli>, int> pairs;
while(true){
    p = a * q - p;
    q = (n - p * p) / q;
    a = (r + p) / q;
    //if p=0 and q=1, we can just ask if q==1 after inserting a
    if(pairs.count({p, q})){
        period -= pairs[{p, q}];
        break;
    }
    coef.push_back(a);
    pairs[{p, q}] = period++;
}
return {coef, period};
}

```

### 1.7.3. Ecuación de Pell

```

//first solution (x, y) to the equation x^2-ny^2=1, n IS NOT a
↪ perfect square
pair<lli, lli> PellEquation(lli n){

```

```

vector<lli> cf = ContinuedFraction(0, n, 1).first;
lli num = 0, den = 1;
int k = cf.size() - 1;
for(int i = ((k & 1) ? (2 * k - 1) : (k - 1)); i >= 0; i--){
    lli tmp = den;
    int pos = i % k;
    if(pos == 0 && i != 0) pos = k;
    den = num + cf[pos] * den;
    num = tmp;
}
return {den, num};
}

```

#### 1.7.4. Números de Bell

```

//number of ways to partition a set of n elements
//the nth bell number is at Bell[n][0]
vector<vector<int>> Bell;
void bellNumbers(int n){
    Bell.resize(n + 1);
    Bell[0] = {1};
    for(int i = 1; i <= n; ++i){
        Bell[i].resize(i + 1);
        Bell[i][0] = Bell[i - 1][i - 1];
        for(int j = 1; j <= i; ++j)
            Bell[i][j] = Bell[i][j - 1] + Bell[i - 1][j - 1];
    }
}

```

#### 1.7.5. Números de Stirling

```

//s(n, k) represents the number of permutations
//of n elements with k disjoint cycles
vector<vector<lli>> stirring1;
void stirringNumber1stKind(lli n){
    stirring1.resize(n+1, vector<lli>(n+1));
    stirring1[0][0] = 1;
    for(int i = 1; i <= n; ++i)
        for(int j = 1; j <= i; ++j)
            stirring1[i][j] = (i-1) * stirring1[i-1][j] +
                stirring1[i-1][j-1];
}

```

```

//S(n, k) represents the number of ways to
//partition a set of n object into k non-empty
//distinct subsets
vector<vector<lli>> stirring2;
void stirringNumber2ndKind(lli n){
    stirring2.resize(n+1, vector<lli>(n+1));
    stirring2[0][0] = 1;
    for(int i = 1; i <= n; ++i)
        for(int j = 1; j <= i; ++j)
            stirring2[i][j] = j * stirring2[i-1][j] +
                stirring2[i-1][j-1];
}

```

#### 1.7.6. Números de Euler

```

//euler(n, k) represents the number of permutations
//of 1,...,n with exactly k numbers greater than
//the previous number
vector<vector<lli>> euler;
void eulerianNumbers(lli n){
    euler.resize(n+1, vector<lli>(n+1));
    for(int i = 1; i <= n; ++i){
        euler[i][0] = 1;
        for(int j = 1; j < i; ++j)
            euler[i][j] = (i-j) * euler[i-1][j-1] + (j+1) *
                euler[i-1][j];
    }
}

```

#### 1.7.7. Prime counting function in sublinear time

```

#include <bits/stdc++.h>
using namespace std;
using lli = long long int;

const lli Mod = 1e9 + 7;

template<typename T>
struct SumPrimePi{
    int v, k;
    lli n;
}

```

```

vector<T> lo, hi;
vector<int> primes;

SumPrimePi(lli n, int k = 0): n(n), v(sqrt(n)), k(k){
    lo.resize(v+2), hi.resize(v+2);
}

T power(T a, lli b){
    T ans = 1;
    while(b){
        if(b & 1) ans *= a;
        b >>= 1;
        a *= a;
    }
    return ans;
}

T powerSum(T n, int k){
    if(k == 0) return n;
    if(k == 1) return n * (n + 1) / 2;
    return 0;
}

void build(){
    lli p, q, j, end, i, d;
    T temp;
    for(p = 1; p <= v; p++){
        lo[p] = powerSum(p, k) - 1;
        hi[p] = powerSum(n/p, k) - 1;
    }
    for(p = 2; p <= v; p++){
        T pk = power(p, k);
        if(lo[p] == lo[p-1]) continue;
        primes.push_back(p);
        temp = lo[p-1];
        q = p * p;
        end = (v <= n/q) ? v : n/q;
        for(i = 1; i <= end; ++i){
            d = i * p;
            if(d <= v)
                hi[i] -= (hi[d] - temp) * pk;
            else
                hi[i] -= (lo[n/d] - temp) * pk;
        }
    }
}

```

```

        for(i = v; i >= q; i--){
            lo[i] -= (lo[i/p] - temp) * pk;
        }
    }
}

T get(lli i) const{
    if(i <= v) return lo[i];
    else return hi[n/i];
}

};

template<typename T>
struct MultiplicativeSum{
    int v;
    lli n;
    vector<T> lo, hi, smallFP;
    vector<int> primes;

    MultiplicativeSum(lli n, const vector<int> & primes): n(n),
        ↪ v(sqrt(n)), primes(primes){
        lo.resize(v+2), hi.resize(v+2), smallFP.resize(v+2);
    }

    void add(T coef, const auto & pi){
        assert(pi.n == n);
        for(int i = 1; i <= v; ++i){
            smallFP[i] += coef * pi.get(i);
            hi[i] += coef * (pi.get(n/i) - pi.get(v));
        }
    }

    T getAdded(lli i, lli p){
        if(i <= v){
            return lo[i] + smallFP[max(i, p)] - smallFP[p];
        }else{
            return hi[n/i] + smallFP[v] - smallFP[p];
        }
    }

    void build(function<T(lli, int)> g){
        for(int i = 1; i <= v; ++i){
            lo[i] += 1;
            hi[i] += 1;
        }
    }
}

```

```

}
for(int r = (int)primes.size()-1; r >= 0; --r){
    lli p = primes[r];
    vector<lli> p_power(1, 1);
    vector<T> gs(1, T(1));
    lli p_pow = p;
    for(int e = 1; ; ++e){
        p_power.push_back(p_pow);
        gs.push_back(g(p, e));
        if(p_pow > n/p) break;
        p_pow *= p;
    }
    for(int i = 1; i <= v; ++i){
        lli next = n / i;
        if(next < p*p) break;
        for(int e = 1; e < p_power.size() && p_power[e] <= next;
            ↪ ++e){
            hi[i] += gs[e] * getAdded(next / p_power[e], p);
        }
        hi[i] -= gs[1];
    }
    for(int i = v; i >= 1; --i){
        if(i < p*p) break;
        for(int e = 1; e <= p_power.size() && p_power[e] <= i;
            ↪ ++e){
            lo[i] += gs[e] * getAdded(i / p_power[e], p);
        }
        lo[i] -= gs[1];
    }
}
for(int i = 1; i <= v; ++i){
    lo[i] += smallFP[i];
    hi[i] += smallFP[v];
}
}

T get(lli i) const{
    if(i <= v) return lo[i];
    else return hi[n/i];
}
};

template<typename T>

```

```

T f(const SumPrimePi<T> & pi, function<T(lli, int)> g, lli n, int
↪ idx = 0){
    // sum of g(p, 1) for primes p such that primes[idx] <= p <= n
    int lo = idx ? pi.primes[idx-1] : 0;
    T ans = pi.get(n) - pi.get(lo);
    if(idx == 0) ans++;
    for(int i = idx; i < pi.primes.size(); ++i){
        lli p = pi.primes[i];
        if(p * p > n) break;
        int e = 1;
        lli curr = n / p;
        while(curr >= p){
            ans += g(p, e) * f(pi, g, curr, i+1) + g(p, e+1);
            curr /= p;
            ++e;
        }
    }
    return ans;
}

int main(){
    lli n;
    int k;
    cin >> n >> k;
    clock_t start = clock();
    SumPrimePi<lli> pi(n, k);
    pi.build();
    lli ans = pi.get(n);
    clock_t end = clock();
    cout << ans << "\n" << (double)(end - start) /
    ↪ (double)CLOCKS_PER_SEC << "s\n";
    return 0;
}

```

### 1.7.8. Suma de la función piso

```

//finds sum(floor(p*i/q), 1<=i<=n)
lli floorsSum(lli p, lli q, lli n){
    lli t = gcd(p, q);
    p /= t, q /= t;
    lli s = 0, z = 1;
    while(q && n){
        t = p/q;

```

```

    s += z*t*n*(n+1)/2;
    p -= q*t;
    t = n/q;
    s += z*p*t*(n+1) - z*t*(p*q*t + p + q - 1)/2;
    n -= q*t;
    t = n*p/q;
    s += z*t*n;
    n = t;
    swap(p, q);
    z = -z;
}
return s;
}

```

### 1.7.9. Periodo de Pisano

```

lli pisano_prime(lli p){
    if(p == 2) return 3;
    if(p == 5) return 20;
    lli order = 0;
    if(p%10 == 1 || p%10 == 9) order = p - 1;
    else order = 2*p + 2;
    auto fact = factorize(order);
    for(auto par : fact){
        lli q; int a;
        tie(q, a) = par;
        order /= power(q, a);
        while(!(fibo(order, p) == 0 && fibo(order+1, p) == 1)){
            order *= q;
        }
    }
    return order;
}

lli pisano(lli mod){
    lli ans = 1;
    auto fact = factorize(mod);
    for(auto par : fact){
        lli p; int a;
        tie(p, a) = par;
        ans = lcm(ans, power(p, a-1) * pisano_prime(p));
    }
    return ans;
}

```

```

}

```

### 1.7.10. Suma en dos cuadrados de un primo

```

pair<lli, lli> sq2(lli p){
    assert(p >= 3 && p % 4 == 1);
    lli z;
    for(lli a = 2; a < p-1; ++a){
        if(powerMod(a, (p-1)/2, p) == p-1){
            z = powerMod(a, (p-1)/4, p);
            break;
        }
    }
    lli w0 = p, w1 = 0, z0 = z, z1 = 1;
    while(z0 || z1){
        lli n = z0*z0 + z1*z1;
        lli u0 = (w0*z0 + w1*z1) / n;
        lli u1 = (w1*z0 - w0*z1) / n;
        lli r0 = w0 - z0*u0 + z1*u1;
        lli r1 = w1 - z0*u1 - z1*u0;
        w0 = z0, w1 = z1, z0 = r0, z1 = r1;
    }
    return {abs(w0), abs(w1)};
}

```

### 1.7.11. Polinomio ciclotómico

```

vector<int> cyclotomic(int n){
    if(n == 1) return {-1, 1};
    int deg = phi(n);
    vector<int> a(deg+1);
    a[0] = 1;
    for(int d : divs[n]){
        if(mu(n/d) == 0) continue;
        if(mu(n/d) == 1){
            for(int i = deg; i >= d; --i){
                a[i] -= a[i-d];
            }
        }else{
            for(int i = d; i <= deg; ++i){
                a[i] += a[i-d];
            }
        }
    }
}

```

```

    }
}
return a;
}

```

## 2. Números racionales

### 2.1. Estructura fraccion

```

struct fraccion{
    ll num, den;
    fraccion(){
        num = 0, den = 1;
    }
    fraccion(ll x, ll y){
        if(y < 0)
            x *= -1, y *= -1;
        ll d = __gcd(abs(x), abs(y));
        num = x/d, den = y/d;
    }
    fraccion(ll v){
        num = v;
        den = 1;
    }
    fraccion operator+(const fraccion& f) const{
        ll d = __gcd(den, f.den);
        return fraccion(num*(f.den/d) + f.num*(den/d), den*(f.den/d));
    }
    fraccion operator-() const{
        return fraccion(-num, den);
    }
    fraccion operator-(const fraccion& f) const{
        return *this + (-f);
    }
    fraccion operator*(const fraccion& f) const{
        return fraccion(num*f.num, den*f.den);
    }
    fraccion operator/(const fraccion& f) const{
        return fraccion(num*f.den, den*f.num);
    }
    fraccion operator+=(const fraccion& f){
        *this = *this + f;
        return *this;
    }
    fraccion operator-=(const fraccion& f){
        *this = *this - f;
        return *this;
    }
}

```

```

fraccion operator++(int xd){
    *this = *this + 1;
    return *this;
}
fraccion operator--(int xd){
    *this = *this - 1;
    return *this;
}
fraccion operator*=(const fraccion& f){
    *this = *this * f;
    return *this;
}
fraccion operator/=(const fraccion& f){
    *this = *this / f;
    return *this;
}
bool operator==(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) == (den/d)*f.num);
}
bool operator!=(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) != (den/d)*f.num);
}
bool operator >(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) > (den/d)*f.num);
}
bool operator <(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) < (den/d)*f.num);
}
bool operator >=(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) >= (den/d)*f.num);
}
bool operator <=(const fraccion& f) const{
    ll d = __gcd(den, f.den);
    return (num*(f.den/d) <= (den/d)*f.num);
}
fraccion inverso() const{
    return fraccion(den, num);
}
fraccion fabs() const{

```

```

    fraccion nueva;
    nueva.num = abs(num);
    nueva.den = den;
    return nueva;
}
double value() const{
    return (double)num / (double)den;
}
string str() const{
    stringstream ss;
    ss << num;
    if(den != 1) ss << "/" << den;
    return ss.str();
}
};

ostream &operator<<(ostream &os, const fraccion & f) {
    return os << f.str();
}

istream &operator>>(istream &is, fraccion & f){
    ll num = 0, den = 1;
    string str;
    is >> str;
    size_t pos = str.find("/");
    if(pos == string::npos){
        istringstream(str) >> num;
    }else{
        istringstream(str.substr(0, pos)) >> num;
        istringstream(str.substr(pos + 1)) >> den;
    }
    f = fraccion(num, den);
    return is;
}

```

### 3. Álgebra lineal

#### 3.1. Estructura matrix

```
template <typename T>
struct matrix{
    vector<vector<T>> A;
    int m, n;

    matrix(int m, int n): m(m), n(n){
        A.resize(m, vector<T>(n, 0));
    }

    vector<T> & operator[] (int i){
        return A[i];
    }

    const vector<T> & operator[] (int i) const{
        return A[i];
    }

    static matrix identity(int n){
        matrix<T> id(n, n);
        for(int i = 0; i < n; i++){
            id[i][i] = 1;
        }
        return id;
    }

    matrix operator+(const matrix & B) const{
        assert(m == B.m && n == B.n); //same dimensions
        matrix<T> C(m, n);
        for(int i = 0; i < m; i++){
            for(int j = 0; j < n; j++){
                C[i][j] = A[i][j] + B[i][j];
            }
        }
        return C;
    }

    matrix operator+=(const matrix & M){
        *this = *this + M;
        return *this;
    }

    matrix operator-() const{
```

```
        matrix<T> C(m, n);
        for(int i = 0; i < m; i++){
            for(int j = 0; j < n; j++){
                C[i][j] = -A[i][j];
            }
        }
        return C;
    }

    matrix operator-(const matrix & B) const{
        return *this + (-B);
    }

    matrix operator-=(const matrix & M){
        *this = *this + (-M);
        return *this;
    }

    matrix operator*(const matrix & B) const{
        assert(n == B.m); //#columns of 1st matrix = #rows of 2nd
        ↪ matrix
        matrix<T> C(m, B.n);
        for(int i = 0; i < m; i++){
            for(int j = 0; j < B.n; j++){
                for(int k = 0; k < n; k++){
                    C[i][j] += A[i][k] * B[k][j];
                }
            }
        }
        return C;
    }

    matrix operator*(const T & c) const{
        matrix<T> C(m, n);
        for(int i = 0; i < m; i++){
            for(int j = 0; j < n; j++){
                C[i][j] = A[i][j] * c;
            }
        }
        return C;
    }

    matrix operator*=(const matrix & M){
        *this = *this * M;
        return *this;
    }

    matrix operator*=(const T & c){
        *this = *this * c;
        return *this;
    }
}
```



```

matrix operator^(lli b) const{
    matrix<T> ans = matrix<T>::identity(n);
    matrix<T> A = *this;
    while(b){
        if(b & 1) ans *= A;
        b >>= 1;
        if(b) A *= A;
    }
    return ans;
}

matrix operator^=(lli n){
    *this = *this ^ n;
    return *this;
}

bool operator==(const matrix & B) const{
    if(m != B.m || n != B.n) return false;
    for(int i = 0; i < m; i++)
        for(int j = 0; j < n; j++)
            if(A[i][j] != B[i][j]) return false;
    return true;
}

bool operator!=(const matrix & B) const{
    return !(*this == B);
}

void scaleRow(int k, T c){
    for(int j = 0; j < n; j++)
        A[k][j] *= c;
}

void swapRows(int k, int l){
    swap(A[k], A[l]);
}

void addRow(int k, int l, T c){
    for(int j = 0; j < n; j++)
        A[k][j] += c * A[l][j];
}

```

### 3.2. Transpuesta y traza

```

matrix<T> transpose(){
    matrix<T> tr(n, m);
    for(int i = 0; i < m; i++)
        for(int j = 0; j < n; j++)
            tr[j][i] = A[i][j];
    return tr;
}

T trace(){
    T sum = 0;
    for(int i = 0; i < min(m, n); i++)
        sum += A[i][i];
    return sum;
}

```

### 3.3. Gauss Jordan

```

//full: true: reduce above and below the diagonal, false: reduce
↪ only below
//makeOnes: true: make the elements in the diagonal ones, false:
↪ leave the diagonal unchanged
//For every elemental operation that we apply to the matrix,
//we will call to callback(operation, k, l, value).
//operation 1: multiply row "k" by "value"
//operation 2: swap rows "k" and "l"
//operation 3: add "value" times the row "l" to the row "k"
//It returns the rank of the matrix, and modifies it
int gauss_jordan(bool full = true, bool makeOnes = true,
↪ function<void(int, int, int, T)>callback = NULL){
    int i = 0, j = 0;
    while(i < m && j < n){
        if(A[i][j] == 0){
            for(int f = i + 1; f < m; f++){
                if(A[f][j] != 0){
                    swapRows(i, f);
                    if(callback) callback(2, i, f, 0);
                    break;
                }
            }
        }
        if(A[i][j] != 0){

```

```

    T inv_mult = A[i][j].inverso();
    if(makeOnes && A[i][j] != 1){
        scaleRow(i, inv_mult);
        if(callback) callback(1, i, 0, inv_mult);
    }
    for(int f = (full ? 0 : (i + 1)); f < m; f++){
        if(f != i && A[f][j] != 0){
            T inv_adit = -A[f][j];
            if(!makeOnes) inv_adit *= inv_mult;
            addRow(f, i, inv_adit);
            if(callback) callback(3, f, i, inv_adit);
        }
    }
    i++;
}
j++;
}
return i;
}

void gaussian_elimination(){
    gauss_jordan(false);
}

```

### 3.4. Matriz escalonada por filas y reducida por filas

```

matrix<T> reducedRowEchelonForm(){
    matrix<T> asoc = *this;
    asoc.gauss_jordan();
    return asoc;
}

matrix<T> rowEchelonForm(){
    matrix<T> asoc = *this;
    asoc.gaussian_elimination();
    return asoc;
}

```

### 3.5. Matriz inversa

```

bool invertible(){
    assert(m == n); //this is defined only for square matrices

```

```

    matrix<T> tmp = *this;
    return tmp.gauss_jordan(false) == n;
}

matrix<T> inverse(){
    assert(m == n); //this is defined only for square matrices
    matrix<T> tmp = *this;
    matrix<T> inv = matrix<T>::identity(n);
    auto callback = [&](int op, int a, int b, T e){
        if(op == 1){
            inv.scaleRow(a, e);
        }else if(op == 2){
            inv.swapRows(a, b);
        }else if(op == 3){
            inv.addRow(a, b, e);
        }
    };
    assert(tmp.gauss_jordan(true, true, callback) == n); //check
    ↪ non-invertible
    return inv;
}

```

### 3.6. Determinante

```

T determinant(){
    assert(m == n); //only square matrices have determinant
    matrix<T> tmp = *this;
    T det = 1;
    auto callback = [&](int op, int a, int b, T e){
        if(op == 1){
            det /= e;
        }else if(op == 2){
            det *= -1;
        }
    };
    if(tmp.gauss_jordan(false, true, callback) != n) det = 0;
    return det;
}

```

### 3.7. Matriz de cofactores y adjunta

```

matrix<T> minor(int x, int y){
    matrix<T> M(m-1, n-1);
    for(int i = 0; i < m-1; ++i)
        for(int j = 0; j < n-1; ++j)
            M[i][j] = A[i < x ? i : i+1][j < y ? j : j+1];
    return M;
}

T cofactor(int x, int y){
    T ans = minor(x, y).determinant();
    if((x + y) % 2 == 1) ans *= -1;
    return ans;
}

matrix<T> cofactorMatrix(){
    matrix<T> C(m, n);
    for(int i = 0; i < m; i++)
        for(int j = 0; j < n; j++)
            C[i][j] = cofactor(i, j);
    return C;
}

matrix<T> adjugate(){
    if(invertible()) return inverse() * determinant();
    return cofactorMatrix().transpose();
}

```

### 3.8. Factorización $PA = LU$

```

tuple<matrix<T>, matrix<T>, matrix<T>> PA_LU(){
    matrix<T> U = *this;
    matrix<T> L = matrix<T>::identity(n);
    matrix<T> P = matrix<T>::identity(n);
    auto callback = [&](int op, int a, int b, T e){
        if(op == 2){
            L.swapRows(a, b);
            P.swapRows(a, b);
            L[a][a] = L[b][b] = 1;
            L[a][a + 1] = L[b][b - 1] = 0;
        }else if(op == 3){
            L[a][b] = -e;
        }
    };
    U.gauss_jordan(false, false, callback);
    return {P, L, U};
}

```

```

    }
};
U.gauss_jordan(false, false, callback);
return {P, L, U};
}

```

### 3.9. Polinomio característico

```

vector<T> characteristicPolynomial(){
    matrix<T> M(n, n);
    vector<T> coef(n + 1);
    matrix<T> I = matrix<T>::identity(n);
    coef[n] = 1;
    for(int i = 1; i <= n; i++){
        M = (*this) * M + I * coef[n - i + 1];
        coef[n - i] = -((*this) * M).trace() / i;
    }
    return coef;
}

```

### 3.10. Gram-Schmidt

```

matrix<T> gram_schmidt(){
    //vectors are rows of the matrix (also in the answer)
    //the answer doesn't have the vectors normalized
    matrix<T> B = (*this) * (*this).transpose();
    matrix<T> ans = *this;
    auto callback = [&](int op, int a, int b, T e){
        if(op == 1){
            ans.scaleRow(a, e);
        }else if(op == 2){
            ans.swapRows(a, b);
        }else if(op == 3){
            ans.addRow(a, b, e);
        }
    };
    B.gauss_jordan(false, false, callback);
    return ans;
}

```

### 3.11. Recurrencias lineales

```
//Solves a linear homogeneous recurrence relation of degree "deg"
//of the form  $F(n) = a(d-1)*F(n-1) + a(d-2)*F(n-2) + \dots +$ 
 $\hookrightarrow a(1)*F(n-(d-1)) + a(0)*F(n-d)$ 
//with initial values  $F(0), F(1), \dots, F(d-1)$ 
//It finds the  $n$ th term of the recurrence,  $F(n)$ 
//The values of  $a[0, \dots, d]$  are in the array  $P[]$ 
lli solveRecurrence(const vector<lli> & P, const vector<lli> &
 $\hookrightarrow$  init, lli n){
    int deg = P.size();
    vector<lli> ans(deg), R(2*deg);
    ans[0] = 1;
    lli p = 1;
    for(lli v = n; v >>= 1; p <<= 1);
    do{
        int d = (n & p) != 0;
        fill(R.begin(), R.end(), 0);
        for(int i = 0; i < deg; i++)
            for(int j = 0; j < deg; j++)
                (R[i + j + d] += ans[i] * ans[j]) %= mod;
        for(int i = deg-1; i >= 0; i--)
            for(int j = 0; j < deg; j++)
                (R[i + j] += R[i + deg] * P[j]) %= mod;
        copy(R.begin(), R.begin() + deg, ans.begin());
    }while(p >>= 1);
    lli nValue = 0;
    for(int i = 0; i < deg; i++)
        (nValue += ans[i] * init[i]) %= mod;
    return nValue;
}
```

### 3.12. Berlekamp-Massey

```
//Finds the shortest linear recurrence relation for the
//given init values. Only works for prime modulo.
vector<lli> BerlekampMassey(const vector<lli> & init){
    vector<lli> cur, ls;
    lli ld;
    for(int i = 0, m; i < init.size(); ++i){
        lli eval = 0;
        for(int j = 0; j < cur.size(); ++j)
            eval = (eval + init[i-j-1] * cur[j]) % mod;
```

```
eval -= init[i];
if(eval < 0) eval += mod;
if(eval == 0) continue;
if(cur.empty()){
    cur.resize(i + 1);
    m = i;
    ld = eval;
}else{
    lli k = eval * inverse(ld, mod) % mod;
    vector<lli> c(i - m - 1);
    c.push_back(k);
    for(int j = 0; j < ls.size(); ++j)
        c.push_back((mod-ls[j]) * k % mod);
    if(c.size() < cur.size()) c.resize(cur.size());
    for(int j = 0; j < cur.size(); ++j){
        c[j] += cur[j];
        if(c[j] >= mod) c[j] -= mod;
    }
    if(i - m + ls.size() >= cur.size())
        ls = cur, m = i, ld = eval;
    cur = c;
}
}
if(cur.empty()) cur.push_back(0);
reverse(cur.begin(), cur.end());
return cur;
}
```

### 3.13. Simplex

```
/*
Parametric Self-Dual Simplex method
Solve a canonical LP:
    min or max.  $c x$ 
    s.t.  $A x \leq b$ 
         $x \geq 0$ 
*/
#include <bits/stdc++.h>
using namespace std;
const double eps = 1e-9, oo = numeric_limits<double>::infinity();

typedef vector<double> vec;
typedef vector<vec> mat;
```

```

pair<vec, double> simplexMethodPD(const mat &A, const vec &b,
↪ const vec &c, bool mini = true){
    int n = c.size(), m = b.size();
    mat T(m + 1, vec(n + m + 1));
    vector<int> base(n + m), row(m);

    for(int j = 0; j < m; ++j){
        for(int i = 0; i < n; ++i)
            T[j][i] = A[j][i];
        row[j] = n + j;
        T[j][n + j] = 1;
        base[n + j] = 1;
        T[j][n + m] = b[j];
    }

    for(int i = 0; i < n; ++i)
        T[m][i] = c[i] * (mini ? 1 : -1);

    while(true){
        int p = 0, q = 0;
        for(int i = 0; i < n + m; ++i)
            if(T[m][i] <= T[m][p])
                p = i;

        for(int j = 0; j < m; ++j)
            if(T[j][n + m] <= T[q][n + m])
                q = j;

        double t = min(T[m][p], T[q][n + m]);

        if(t >= -eps){
            vec x(n);
            for(int i = 0; i < m; ++i)
                if(row[i] < n) x[row[i]] = T[i][n + m];
            return {x, T[m][n + m] * (mini ? -1 : 1)}; // optimal
        }

        if(t < T[q][n + m]){
            // tight on c -> primal update
            for(int j = 0; j < m; ++j)
                if(T[j][p] >= eps)
                    if(T[j][p] * (T[q][n + m] - t) >= T[q][p] * (T[j][n + m]
↪ - t))

```

```

        q = j;

        if(T[q][p] <= eps)
            return {vec(n), oo * (mini ? 1 : -1)}; // primal
↪ infeasible
        }else{
            // tight on b -> dual update
            for(int i = 0; i < n + m + 1; ++i)
                T[q][i] = -T[q][i];

            for(int i = 0; i < n + m; ++i)
                if(T[q][i] >= eps)
                    if(T[q][i] * (T[m][p] - t) >= T[q][p] * (T[m][i] - t))
                        p = i;

            if(T[q][p] <= eps)
                return {vec(n), oo * (mini ? -1 : 1)}; // dual infeasible
        }

        for(int i = 0; i < m + n + 1; ++i)
            if(i != p) T[q][i] /= T[q][p];

        T[q][p] = 1; // pivot(q, p)
        base[p] = 1;
        base[row[q]] = 0;
        row[q] = p;

        for(int j = 0; j < m + 1; ++j){
            if(j != q){
                double alpha = T[j][p];
                for(int i = 0; i < n + m + 1; ++i)
                    T[j][i] -= T[q][i] * alpha;
            }
        }

        return {vec(n), oo};
    }

    int main(){
        int m, n;
        bool mini = true;
        cout << "Numero de restricciones: ";
        cin >> m;
    }

```

```

cout << "Numero de incognitas: ";
cin >> n;
mat A(m, vec(n));
vec b(m), c(n);
for(int i = 0; i < m; ++i){
    cout << "Restriccion #" << (i + 1) << ": ";
    for(int j = 0; j < n; ++j){
        cin >> A[i][j];
    }
    cin >> b[i];
}
cout << "[0]Max o [1]Min?: ";
cin >> mini;
cout << "Coeficientes de " << (mini ? "min" : "max") << " z: ";
for(int i = 0; i < n; ++i){
    cin >> c[i];
}
cout.precision(6);
auto ans = simplexMethodPD(A, b, c, mini);
cout << (mini ? "Min" : "Max") << " z = " << ans.second << ",
↪ cuando: \n";
for(int i = 0; i < ans.first.size(); ++i){
    cout << "x_" << (i + 1) << " = " << ans.first[i] << "\n";
}
return 0;
}

```

## 4. FFT

### 4.1. Declaraciones previas

```

using lli = long long int;
using comp = complex<double>;
const double PI = acos(-1.0);

```

```

int nearestPowerOfTwo(int n){
    int ans = 1;
    while(ans < n) ans <= 1;
    return ans;
}

```

### 4.2. FFT con raíces de la unidad complejas

```

void fft(vector<comp> & X, int inv){
    int n = X.size();
    for(int i = 1, j = 0; i < n - 1; ++i){
        for(int k = n >> 1; (j ^= k) < k; k >>= 1);
        if(i < j) swap(X[i], X[j]);
    }
    vector<comp> wp(n >> 1);
    for(int k = 1; k < n; k <= 1){
        for(int j = 0; j < k; ++j)
            wp[j] = polar(1.0, PI * j / k * inv); // best precision but
            ↪ slower
        for(int i = 0; i < n; i += k << 1){
            for(int j = 0; j < k; ++j){
                comp t = X[i + j + k] * wp[j];
                X[i + j + k] = X[i + j] - t;
                X[i + j] += t;
            }
        }
    }
    if(inv == -1)
        for(int i = 0; i < n; ++i)
            X[i] /= n;
}

```

### 4.3. FFT con raíces de la unidad en $\mathbb{Z}_p$ (NTT)

```
lli powerMod(lli b, lli e, lli m){
    lli ans = 1;
    e %= m-1;
    if(e < 0) e += m-1;
    while(e){
        if(e & 1) ans = ans * b % m;
        e >>= 1;
        b = b * b % m;
    }
    return ans;
}

template<int p, int g>
void ntt(vector<int> & X, int inv){
    int n = X.size();
    for(int i = 1, j = 0; i < n - 1; ++i){
        for(int k = n >> 1; (j ^ k) < k; k >>= 1);
        if(i < j) swap(X[i], X[j]);
    }
    vector<lli> wp(n>>1, 1);
    for(int k = 1; k < n; k <= 1){
        lli wk = powerMod(g, inv * (p - 1) / (k<<1), p);
        for(int j = 1; j < k; ++j)
            wp[j] = wp[j - 1] * wk % p;
        for(int i = 0; i < n; i += k << 1){
            for(int j = 0; j < k; ++j){
                int u = X[i + j], v = X[i + j + k] * wp[j] % p;
                X[i + j] = u + v < p ? u + v : u + v - p;
                X[i + j + k] = u - v < 0 ? u - v + p : u - v;
            }
        }
    }
    if(inv == -1){
        lli nrev = powerMod(n, p - 2, p);
        for(int i = 0; i < n; ++i)
            X[i] = X[i] * nrev % p;
    }
}
```

#### 4.3.1. Valores para escoger el generador y el módulo

Generador ( $g$ )	Tamaño máxi- mo del arreglo ( $n$ )	Módulo $p$
3	$2^{16}$	$1 \times 2^{16} + 1 = 65537$
10	$2^{18}$	$3 \times 2^{18} + 1 = 786433$
3	$2^{19}$	$11 \times 2^{19} + 1 = 5767169$
<b>3</b>	<b><math>2^{20}</math></b>	<b><math>7 \times 2^{20} + 1 = 7340033</math></b>
3	$2^{21}$	$11 \times 2^{21} + 1 = 23068673$
3	$2^{22}$	$25 \times 2^{22} + 1 = 104857601$
3	$2^{22}$	$235 \times 2^{22} + 1 = 985661441$
26	$2^{23}$	$105 \times 2^{23} + 1 = 880803841$
<b>3</b>	<b><math>2^{23}</math></b>	<b><math>119 \times 2^{23} + 1 = 998244353</math></b>
11	$2^{24}$	$45 \times 2^{24} + 1 = 754974721$
3	$2^{25}$	$5 \times 2^{25} + 1 = 167772161$
3	$2^{26}$	$7 \times 2^{26} + 1 = 469762049$
31	$2^{27}$	$15 \times 2^{27} + 1 = 2013265921$

### 4.4. Multiplicación de polinomios (convolución lineal)

```
vector<comp> convolution(vector<comp> A, vector<comp> B){
    int sz = A.size() + B.size() - 1;
    int size = nearestPowerOfTwo(sz);
    A.resize(size), B.resize(size);
    fft(A, 1), fft(B, 1);
    for(int i = 0; i < size; i++)
        A[i] *= B[i];
    fft(A, -1);
    A.resize(sz);
    return A;
}

template<int p, int g>
vector<int> convolution(vector<int> A, vector<int> B){
    int sz = A.size() + B.size() - 1;
    int size = nearestPowerOfTwo(sz);
    A.resize(size), B.resize(size);
    ntt<p, g>(A, 1), ntt<p, g>(B, 1);
    for(int i = 0; i < size; i++)
```

```

    A[i] = (lli)A[i] * B[i] % p;
    ntt<p, g>(A, -1);
    A.resize(sz);
    return A;
}

const int p = 7340033, g = 3; //default values for NTT

```

## 4.5. Aplicaciones

### 4.5.1. Multiplicación de números enteros grandes

```

string multiplyNumbers(const string & a, const string & b){
    int sgn = 1;
    int pos1 = 0, pos2 = 0;
    while(pos1 < a.size() && (a[pos1] < '1' || a[pos1] > '9')){
        if(a[pos1] == '-') sgn *= -1;
        ++pos1;
    }
    while(pos2 < b.size() && (b[pos2] < '1' || b[pos2] > '9')){
        if(b[pos2] == '-') sgn *= -1;
        ++pos2;
    }
    vector<int> X(a.size() - pos1), Y(b.size() - pos2);
    if(X.empty() || Y.empty()) return "0";
    for(int i = pos1, j = X.size() - 1; i < a.size(); ++i)
        X[j--] = a[i] - '0';
    for(int i = pos2, j = Y.size() - 1; i < b.size(); ++i)
        Y[j--] = b[i] - '0';
    X = convolution<p, g>(X, Y);
    stringstream ss;
    if(sgn == -1) ss << "-";
    int carry = 0;
    for(int i = 0; i < X.size(); ++i){
        X[i] += carry;
        carry = X[i] / 10;
        X[i] %= 10;
    }
    while(carry){
        X.push_back(carry % 10);
        carry /= 10;
    }
    for(int i = X.size() - 1; i >= 0; --i)

```

```

        ss << X[i];
    return ss.str();
}

```

### 4.5.2. Recíproco de un polinomio

```

vector<int> inversePolynomial(const vector<int> & A){
    vector<int> R(1, powerMod(A[0], p - 2, p));
    //R(x) = 2R(x) - A(x)R(x)^2
    while(R.size() < A.size()){
        size_t c = 2 * R.size();
        R.resize(c);
        vector<int> R2 = R;
        vector<int> a(min(c, A.size()));
        for(int i = 0; i < a.size(); ++i)
            a[i] = A[i];
        R2 = convolution<p, g>(R2, R2);
        R2.resize(c);
        R2 = convolution<p, g>(R2, a);
        for(int i = 0; i < c; ++i){
            R[i] = R[i] + R[i] - R2[i];
            if(R[i] < 0) R[i] += p;
            if(R[i] >= p) R[i] -= p;
        }
    }
    R.resize(A.size());
    return R;
}

```

### 4.5.3. Raíz cuadrada de un polinomio

```

const int inv2 = powerMod(2, p - 2, p);

vector<int> sqrtPolynomial(const vector<int> & A){
    int r0 = 1; //verify that r0^2 = A[0] mod p
    vector<int> R(1, r0);
    //R(x) = R(x)/2 + A(x)/(2R(x))
    while(R.size() < A.size()){
        size_t c = 2 * R.size();
        R.resize(c);
        vector<int> a(min(c, A.size()));
        for(int i = 0; i < a.size(); ++i)

```



```

    a[i] = A[i];
    a = convolution<p, g>(a, inversePolynomial(R));
    for(int i = 0; i < c; ++i){
        R[i] = R[i] + a[i];
        if(R[i] >= p) R[i] -= p;
        R[i] = (lli)R[i] * inv2 % p;
    }
}
R.resize(A.size());
return R;
}

```

#### 4.5.4. Logaritmo y exponencial de un polinomio

```

vector<int> derivative(vector<int> A){
    for(int i = 0; i < A.size(); ++i)
        A[i] = (lli)A[i] * i % p;
    if(!A.empty()) A.erase(A.begin());
    return A;
}

vector<int> integral(vector<int> A){
    for(int i = 0; i < A.size(); ++i)
        A[i] = (lli)A[i] * (powerMod(i+1, p-2, p)) % p;
    A.insert(A.begin(), 0);
    return A;
}

vector<int> logarithm(vector<int> A){
    assert(A[0] == 1);
    int n = A.size();
    A = convolution<p, g>(derivative(A), inversePolynomial(A));
    A.resize(n);
    A = integral(A);
    A.resize(n);
    return A;
}

vector<int> exponential(const vector<int> & A){
    assert(A[0] == 0);
    //E(x) = E(x)(1-ln(E(x))+A(x))
    vector<int> E(1, 1);
    while(E.size() < A.size()){

```

```

        size_t c = 2*E.size();
        E.resize(c);
        vector<int> S = logarithm(E);
        for(int i = 0; i < c && i < A.size(); ++i){
            S[i] = A[i] - S[i];
            if(S[i] < 0) S[i] += p;
        }
        S[0] = 1;
        E = convolution<p, g>(E, S);
        E.resize(c);
    }
    E.resize(A.size());
    return E;
}

```

#### 4.5.5. Cociente y residuo de dos polinomios

```

//returns Q(x), where A(x)=B(x)Q(x)+R(x)
vector<int> quotient(vector<int> A, vector<int> B){
    int n = A.size(), m = B.size();
    if(n < m) return vector<int>{0};
    reverse(A.begin(), A.end());
    reverse(B.begin(), B.end());
    A.resize(n-m+1), B.resize(n-m+1);
    A = convolution<p, g>(A, inversePolynomial(B));
    A.resize(n-m+1);
    reverse(A.begin(), A.end());
    return A;
}

//returns R(x), where A(x)=B(x)Q(x)+R(x)
vector<int> remainder(vector<int> A, const vector<int> & B){
    int n = A.size(), m = B.size();
    if(n >= m){
        vector<int> C = convolution<p, g>(quotient(A, B), B);
        A.resize(m-1);
        for(int i = 0; i < m-1; ++i){
            A[i] -= C[i];
            if(A[i] < 0) A[i] += p;
        }
    }
    return A;
}

```

#### 4.5.6. Multievaluación rápida

```
//evaluates all the points in P(x), both the size of P and points
↪ must be the same
vector<int> multiEvaluate(const vector<int> & P, const vector<int>
↪ & points){
    int n = points.size();
    vector<vector<int>> prod(2*n - 1);
    function<void(int, int, int)> pre = [&](int v, int l, int r){
        if(l == r) prod[v] = vector<int>{(p - points[l]) % p, 1};
        else{
            int y = (l + r) / 2;
            int z = v + (y - l + 1) * 2;
            pre(v + 1, l, y);
            pre(z, y + 1, r);
            prod[v] = convolution<p, g>(prod[v + 1], prod[z]);
        }
    };
    pre(0, 0, n - 1);

    function<int(const vector<int>&, int)> eval = [&](const
    ↪ vector<int> & poly, int x0){
        int ans = 0;
        for(int i = (int)poly.size()-1; i >= 0; --i){
            ans = (lli)ans * x0 % p + poly[i];
            if(ans >= p) ans -= p;
        }
        return ans;
    };

    vector<int> res(n);
    function<void(int, int, int, vector<int>)> evaluate = [&](int v,
    ↪ int l, int r, vector<int> poly){
        poly = remainder(poly, prod[v]);
        if(poly.size() < 400){
            for(int i = l; i <= r; ++i)
                res[i] = eval(poly, points[i]);
        }else{
            if(l == r)
                res[l] = poly[0];
            else{
                int y = (l + r) / 2;
                int z = v + (y - l + 1) * 2;
                evaluate(v + 1, l, y, poly);
```

```
                evaluate(z, y + 1, r, poly);
            }
        }
    };
    evaluate(0, 0, n - 1, P);
    return res;
}
```

#### 4.5.7. DFT con tamaño de vector arbitrario (algoritmo de Bluestein)

```
//it evaluates 1, w, w^2, ..., w^(n-1) on the polynomial a(x)
//in this example we do a DFT with arbitrary size
vector<comp> bluestein(vector<comp> A){
    int n = A.size(), m = nearestPowerOfTwo(2*n-1);
    comp w = polar(1.0, 2*PI/n), w1 = 1, w2 = 1;
    vector<comp> p(m), q(m), b(n);
    for(int k = 0; k < n; ++k, w2 *= w1, w1 *= w){
        b[k] = w2;
        p[n-1-k] = A[k] / b[k];
        q[k] = b[k];
        if((n&1) == 1 && k < n-1) q[k+n] = q[k];
        else if((n&1) == 0 && k < n-1) q[k+n] = -q[k]; // q[k]*w^(n/2)
    }
    fft(p, 1), fft(q, 1);
    for(int i = 0; i < m; i++){
        p[i] *= q[i];
    }
    fft(p, -1);
    for(int k = 0; k < n; ++k)
        A[k] = p[k+n-1] / b[k];
    return A;
}
```

#### 4.6. Convolución de dos vectores reales con solo dos FFT's

```
//A and B are real-valued vectors, just do 2 fft's instead of 3
vector<double> convolutionTrick(const vector<double> & A, const
↪ vector<double> & B){
    int sz = A.size() + B.size() - 1;
    int size = nearestPowerOfTwo(sz);
    vector<comp> C(size);
    comp I(0, 1);
```

```

for(int i = 0; i < A.size() || i < B.size(); ++i){
    if(i < A.size()) C[i] += A[i];
    if(i < B.size()) C[i] += I*B[i];
}
fft(C, 1);
vector<comp> D(size);
for(int i = 0, j = 0; i < size; ++i){
    j = (size-1) & (size-i);
    D[i] = (conj(C[j]*C[j]) - C[i]*C[i]) * 0.25 * I;
}
fft(D, -1);
vector<double> E;
for_each(D.begin(), D.begin() + sz, [&](comp
    ↪ x){E.push_back(x.real());});
return E;
}

```

#### 4.7. Convolución con módulo arbitrario

```

//convolution with arbitrary modulo using only 4 fft's
vector<int> convolutionMod(const vector<int> & A, const
    ↪ vector<int> & B, int mod){
    int s = sqrt(mod);
    int sz = A.size() + B.size() - 1;
    int size = nearestPowerOfTwo(sz);
    vector<comp> a(size), b(size);
    for(int i = 0; i < A.size(); ++i)
        a[i] = comp(A[i] % s, A[i] / s);
    for(int i = 0; i < B.size(); ++i)
        b[i] = comp(B[i] % s, B[i] / s);
    fft(a, 1), fft(b, 1);
    comp I(0, 1);
    vector<comp> c(size), d(size);
    for(int i = 0, j = 0; i < size; ++i){
        j = (size-1) & (size-i);
        comp e = (a[i] + conj(a[j])) * 0.5;
        comp f = (conj(a[j]) - a[i]) * 0.5 * I;
        comp g = (b[i] + conj(b[j])) * 0.5;
        comp h = (conj(b[j]) - b[i]) * 0.5 * I;
        c[i] = e * g + I * (e * h + f * g);
        d[i] = f * h;
    }
    fft(c, -1), fft(d, -1);
}

```

```

vector<int> D(sz);
for(int i = 0, j = 0; i < sz; ++i){
    j = (size-1) & (size-i);
    int p0 = (lli)round(real(c[i])) % mod;
    int p1 = (lli)round(imag(c[i])) % mod;
    int p2 = (lli)round(real(d[i])) % mod;
    D[i] = p0 + s*(p1 + (lli)p2*s % mod) % mod;
    if(D[i] >= mod) D[i] -= mod;
    if(D[i] < 0) D[i] += mod;
}
return D;
}

//convolution with arbitrary modulo using CRT
//slower but with no precision errors
const int a = 998244353, b = 985661441, c = 754974721;
const lli a_b = powerMod(a, b-2, b), a_c = powerMod(a, c-2, c),
    ↪ b_c = powerMod(b, c-2, c);
vector<int> convolutionModCRT(const vector<int> & A, const
    ↪ vector<int> & B, int mod){
    vector<int> P = convolution<a, 3>(A, B);
    vector<int> Q = convolution<b, 3>(A, B);
    vector<int> R = convolution<c, 11>(A, B);
    vector<int> D(P.size());
    for(int i = 0; i < D.size(); ++i){
        int x1 = P[i] % a;
        if(x1 < 0) x1 += a;
        int x2 = a_b * (Q[i] - x1) % b;
        if(x2 < 0) x2 += b;
        int x3 = (a_c * (R[i] - x1) % c - x2) * b_c % c;
        if(x3 < 0) x3 += c;
        D[i] = x1 % mod + a*(x2 + (lli)x3*b % mod) % mod;
        if(D[i] >= mod) D[i] -= mod;
        if(D[i] < 0) D[i] += mod;
    }
    return D;
}

```

#### 4.8. Transformada rápida de Walsh–Hadamard

```

//Fast Walsh-Hadamard transform, works with any modulo p
//op: 0(OR), 1(AND), 2(XOR), A.size() must be power of 2
void fwt(vector<int> & A, int op, int inv){

```

```

int n = A.size();
for(int k = 1; k < n; k <= 1)
  for(int i = 0; i < n; i += k < 1)
    for(int j = 0; j < k; ++j){
      int u = A[i + j], v = A[i + j + k];
      int sum = u + v < p ? u + v : u + v - p;
      int rest = u - v < 0 ? u - v + p : u - v;
      if(inv == -1){
        if(op == 0)
          A[i + j + k] = rest ? p - rest : 0;
        else if(op == 1)
          A[i + j] = rest;
        else if(op == 2)
          A[i + j] = sum, A[i + j + k] = rest;
      }else{
        if(op == 0)
          A[i + j + k] = sum;
        else if(op == 1)
          A[i + j] = sum;
        else if(op == 2)
          A[i + j] = sum, A[i + j + k] = rest;
      }
    }
  }
if(inv == -1 && op == 2){
  lli nrev = powerMod(n, p-2, p);
  for(int i = 0; i < n; ++i)
    A[i] = A[i] * nrev % p;
}
}

```

## 5. Geometría

### 5.1. Estructura point

```

using ld = long double;
const ld eps = 1e-9, inf = numeric_limits<ld>::max(), pi =
  ↪ acos(-1);
// For use with integers, just set eps=0 and everything remains
  ↪ the same
bool geq(ld a, ld b){return a-b >= -eps;} //a >= b
bool leq(ld a, ld b){return b-a >= -eps;} //a <= b
bool ge(ld a, ld b){return a-b > eps;} //a > b
bool le(ld a, ld b){return b-a > eps;} //a < b
bool eq(ld a, ld b){return abs(a-b) <= eps;} //a == b
bool neq(ld a, ld b){return abs(a-b) > eps;} //a != b

struct point{
  ld x, y;
  point(): x(0), y(0){}
  point(ld x, ld y): x(x), y(y){}

  point operator+(const point & p) const{return point(x + p.x, y +
    ↪ p.y);}
  point operator-(const point & p) const{return point(x - p.x, y -
    ↪ p.y);}
  point operator*(const ld & k) const{return point(x * k, y * k);}
  point operator/(const ld & k) const{return point(x / k, y / k);}

  point operator+=(const point & p){*this = *this + p; return
    ↪ *this;}
  point operator-=(const point & p){*this = *this - p; return
    ↪ *this;}
  point operator*=(const ld & p){*this = *this * p; return *this;}
  point operator/=(const ld & p){*this = *this / p; return *this;}

  point rotate(const ld & a) const{return point(x*cos(a) -
    ↪ y*sin(a), x*sin(a) + y*cos(a));}
  point perp() const{return point(-y, x);}
  ld ang() const{
    ld a = atan2l(y, x); a += le(a, 0) ? 2*pi : 0; return a;
  }
  ld dot(const point & p) const{return x * p.x + y * p.y;}
  ld cross(const point & p) const{return x * p.y - y * p.x;}
}

```

```

ld norm() const{return x * x + y * y;}
ld length() const{return sqrt1(x * x + y * y);}
point unit() const{return (*this) / length();}

bool operator==(const point & p) const{return eq(x, p.x) &&
↪ eq(y, p.y);}
bool operator!=(const point & p) const{return !(*this == p);}
bool operator<(const point & p) const{return le(x, p.x) ||
↪ (eq(x, p.x) && le(y, p.y));}
bool operator>(const point & p) const{return ge(x, p.x) ||
↪ (eq(x, p.x) && ge(y, p.y));}
bool half(const point & p) const{return le(p.cross(*this), 0) ||
↪ (eq(p.cross(*this), 0) && le(p.dot(*this), 0));}
};

istream &operator>>(istream &is, point & p){return is >> p.x >>
↪ p.y;}
ostream &operator<<(ostream &os, const point & p){return os << "("
↪ << p.x << ", " << p.y << ")";}

int sgn(ld x){
    if(ge(x, 0)) return 1;
    if(le(x, 0)) return -1;
    return 0;
}

void polarSort(vector<point> & P, const point & o, const point &
↪ v){
    //sort points in P around o, taking the direction of v as first
    ↪ angle
    sort(P.begin(), P.end(), [&](const point & a, const point & b){
        return point((a - o).half(v), 0) < point((b - o).half(v), (a -
↪ o).cross(b - o));
    });
}

```

## 5.2. Líneas y segmentos

### 5.2.1. Verificar si un punto pertenece a una línea o segmento

```

bool pointInLine(const point & a, const point & v, const point &
↪ p){
    //line a+tv, point p

```

```

    return eq((p - a).cross(v), 0);
}

```

```

bool pointInSegment(const point & a, const point & b, const point
↪ & p){
    //segment ab, point p
    return pointInLine(a, b - a, p) && leq((a - p).dot(b - p), 0);
}

```

### 5.2.2. Intersección de líneas

```

int intersectLinesInfo(const point & a1, const point & v1, const
↪ point & a2, const point & v2){
    //lines a1+tv1 and a2+tv2
    ld det = v1.cross(v2);
    if(eq(det, 0)){
        if(eq((a2 - a1).cross(v1), 0)){
            return -1; //infinity points
        }else{
            return 0; //no points
        }
    }else{
        return 1; //single point
    }
}

```

```

point intersectLines(const point & a1, const point & v1, const
↪ point & a2, const point & v2){
    //lines a1+tv1, a2+tv2
    //assuming that they intersect
    ld det = v1.cross(v2);
    return a1 + v1 * ((a2 - a1).cross(v2) / det);
}

```

### 5.2.3. Intersección línea-segmento

```

int intersectLineSegmentInfo(const point & a, const point & v,
↪ const point & c, const point & d){
    //line a+tv, segment cd
    point v2 = d - c;
    ld det = v.cross(v2);
    if(eq(det, 0)){

```

```

    if(eq((c - a).cross(v), 0)){
        return -1; //infinity points
    }else{
        return 0; //no point
    }
}
}
}

```

#### 5.2.4. Intersección de segmentos

```

int intersectSegmentsInfo(const point & a, const point & b, const
↪ point & c, const point & d){
    //segment ab, segment cd
    point v1 = b - a, v2 = d - c;
    int t = sgn(v1.cross(c - a)), u = sgn(v1.cross(d - a));
    if(t == u){
        if(t == 0){
            if(pointInSegment(a, b, c) || pointInSegment(a, b, d) ||
↪ pointInSegment(c, d, a) || pointInSegment(c, d, b)){
                return -1; //infinity points
            }else{
                return 0; //no point
            }
        }else{
            return 0; //no point
        }
    }else{
        return sgn(v2.cross(a - c)) != sgn(v2.cross(b - c)); //1:
↪ single point, 0: no point
    }
}

```

#### 5.2.5. Distancia punto-recta

```

ld distancePointLine(const point & a, const point & v, const point
↪ & p){
    //line: a + tv, point p
    return abs(v.cross(p - a)) / v.length();
}

```

### 5.3. Polígonos

#### 5.3.1. Perímetro y área de un polígono

```

ld perimeter(vector<point> & P){
    int n = P.size();
    ld ans = 0;
    for(int i = 0; i < n; i++){
        ans += (P[i] - P[(i + 1) % n]).length();
    }
    return ans;
}

```

```

ld area(vector<point> & P){
    int n = P.size();
    ld ans = 0;
    for(int i = 0; i < n; i++){
        ans += P[i].cross(P[(i + 1) % n]);
    }
    return abs(ans / 2);
}

```

#### 5.3.2. Envoltente convexa (convex hull) de un polígono

```

vector<point> convexHull(vector<point> P){
    sort(P.begin(), P.end());
    vector<point> L, U;
    for(int i = 0; i < P.size(); i++){
        while(L.size() >= 2 && leq((L[L.size() - 2] -
↪ P[i]).cross(L[L.size() - 1] - P[i]), 0)){
            L.pop_back();
        }
        L.push_back(P[i]);
    }
    for(int i = P.size() - 1; i >= 0; i--){
        while(U.size() >= 2 && leq((U[U.size() - 2] -
↪ P[i]).cross(U[U.size() - 1] - P[i]), 0)){
            U.pop_back();
        }
        U.push_back(P[i]);
    }
    L.pop_back();
}

```

```

    U.pop_back();
    L.insert(L.end(), U.begin(), U.end());
    return L;
}

```

### 5.3.3. Verificar si un punto está en el perímetro o dentro de un polígono

```

bool pointInPerimeter(const vector<point> & P, const point & p){
    int n = P.size();
    for(int i = 0; i < n; i++){
        if(pointInSegment(P[i], P[(i + 1) % n], p)){
            return true;
        }
    }
    return false;
}

```

```

bool crossesRay(const point & a, const point & b, const point &
    ↪ p){
    return (geq(b.y, p.y) - geq(a.y, p.y)) * sgn((a - p).cross(b -
    ↪ p)) > 0;
}

```

```

int pointInPolygon(const vector<point> & P, const point & p){
    if(pointInPerimeter(P, p)){
        return -1; //point in the perimeter
    }
    int n = P.size();
    int rays = 0;
    for(int i = 0; i < n; i++){
        rays += crossesRay(P[i], P[(i + 1) % n], p);
    }
    return rays & 1; //0: point outside, 1: point inside
}

```

### 5.3.4. Verificar si un punto pertenece a un polígono convexo $O(\log n)$

*//point in convex polygon in  $O(\log n)$*   
*//make sure that P is convex and in ccw*  
*//before the queries, do the preprocess on P:*

```

// rotate(P.begin(), min_element(P.begin(), P.end()), P.end());
// int right = max_element(P.begin(), P.end()) - P.begin();
//returns 0 if p is outside, 1 if p is inside, -1 if p is in the
↪ perimeter
int pointInConvexPolygon(const vector<point> & P, const point & p,
    ↪ int right){
    if(p < P[0] || P[right] < p) return 0;
    int orientation = sgn((P[right] - P[0]).cross(p - P[0]));
    if(orientation == 0){
        if(p == P[0] || p == P[right]) return -1;
        return (right == 1 || right + 1 == P.size()) ? -1 : 1;
    }else if(orientation < 0){
        auto r = lower_bound(P.begin() + 1, P.begin() + right, p);
        int det = sgn((p - r[-1]).cross(r[0] - r[-1])) - 1;
        if(det == -2) det = 1;
        return det;
    }else{
        auto l = upper_bound(P.rbegin(), P.rend() - right - 1, p);
        int det = sgn((p - l[0]).cross((l == P.rbegin() ? P[0] :
        ↪ l[-1]) - l[0])) - 1;
        if(det == -2) det = 1;
        return det;
    }
}

```

### 5.3.5. Cortar un polígono con una recta

```

vector<point> cutPolygon(const vector<point> & P, const point & a,
    ↪ const point & v){
    //returns the part of the convex polygon P on the left side of
    ↪ line a+tv
    int n = P.size();
    vector<point> lhs;
    for(int i = 0; i < n; ++i){
        if(geq(v.cross(P[i] - a), 0)){
            lhs.push_back(P[i]);
        }
        if(intersectLineSegmentInfo(a, v, P[i], P[(i+1)%n]) == 1){
            point p = intersectLines(a, v, P[i], P[(i+1)%n] - P[i]);
            if(p != P[i] && p != P[(i+1)%n]){
                lhs.push_back(p);
            }
        }
    }
}

```

```

    }
    return lhs;
}

```

### 5.3.6. Centroide de un polígono

```

point centroid(vector<point> & P){
    point num;
    ld den = 0;
    int n = P.size();
    for(int i = 0; i < n; ++i){
        ld cross = P[i].cross(P[(i + 1) % n]);
        num += (P[i] + P[(i + 1) % n]) * cross;
        den += cross;
    }
    return num / (3 * den);
}

```

### 5.3.7. Pares de puntos antipodales

```

vector<pair<int, int>> antipodalPairs(vector<point> & P){
    vector<pair<int, int>> ans;
    int n = P.size(), k = 1;
    auto f = [&](int u, int v, int w){return
        ↪ abs((P[v%n]-P[u%n]).cross(P[w%n]-P[u%n]));};
    while(ge(f(n-1, 0, k+1), f(n-1, 0, k))) ++k;
    for(int i = 0, j = k; i <= k && j < n; ++i){
        ans.emplace_back(i, j);
        while(j < n-1 && ge(f(i, i+1, j+1), f(i, i+1, j)))
            ans.emplace_back(i, ++j);
    }
    return ans;
}

```

### 5.3.8. Diámetro y ancho

```

pair<ld, ld> diameterAndWidth(vector<point> & P){
    int n = P.size(), k = 0;
    auto dot = [&](int a, int b){return
        ↪ (P[(a+1)%n]-P[a]).dot(P[(b+1)%n]-P[b]);};

```

```

    auto cross = [&](int a, int b){return
        ↪ (P[(a+1)%n]-P[a]).cross(P[(b+1)%n]-P[b]);};
    ld diameter = 0;
    ld width = inf;
    while(ge(dot(0, k), 0)) k = (k+1) % n;
    for(int i = 0; i < n; ++i){
        while(ge(cross(i, k), 0)) k = (k+1) % n;
        //pair: (i, k)
        diameter = max(diameter, (P[k] - P[i]).length());
        width = min(width, distancePointLine(P[i], P[(i+1)%n] - P[i],
            ↪ P[k]));
    }
    return {diameter, width};
}

```

### 5.3.9. Smallest enclosing rectangle

```

pair<ld, ld> smallestEnclosingRectangle(vector<point> & P){
    int n = P.size();
    auto dot = [&](int a, int b){return
        ↪ (P[(a+1)%n]-P[a]).dot(P[(b+1)%n]-P[b]);};
    auto cross = [&](int a, int b){return
        ↪ (P[(a+1)%n]-P[a]).cross(P[(b+1)%n]-P[b]);};
    ld perimeter = inf, area = inf;
    for(int i = 0, j = 0, k = 0, m = 0; i < n; ++i){
        while(ge(dot(i, j), 0)) j = (j+1) % n;
        if(!i) k = j;
        while(ge(cross(i, k), 0)) k = (k+1) % n;
        if(!i) m = k;
        while(le(dot(i, m), 0)) m = (m+1) % n;
        //pairs: (i, k) , (j, m)
        point v = P[(i+1)%n] - P[i];
        ld h = distancePointLine(P[i], v, P[k]);
        ld w = distancePointLine(P[j], v.perp(), P[m]);
        perimeter = min(perimeter, 2 * (h + w));
        area = min(area, h * w);
    }
    return {area, perimeter};
}

```



## 5.4. Círculos

### 5.4.1. Distancia punto-círculo

```
ld distancePointCircle(const point & c, ld r, const point & p){
    //point p, circle with center c and radius r
    return max((ld)0, (p - c).length() - r);
}
```

### 5.4.2. Proyección punto exterior a círculo

```
point projectionPointCircle(const point & c, ld r, const point &
↪ p){
    //point p (outside the circle), circle with center c and radius
    ↪ r
    return c + (p - c).unit() * r;
}
```

### 5.4.3. Puntos de tangencia desde punto exterior

```
pair<point, point> pointsOfTangency(const point & c, ld r, const
↪ point & p){
    //point p (outside the circle), circle with center c and radius
    ↪ r
    point v = (p - c).unit() * r;
    ld d2 = (p - c).norm(), d = sqrt(d2);
    point v1 = v * (r / d), v2 = v.perp() * (sqrt(d2 - r*r) / d);
    return {c + v1 - v2, c + v1 + v2};
}
```

### 5.4.4. Intersección línea-círculo y segmento-círculo

```
vector<point> intersectLineCircle(const point & a, const point &
↪ v, const point & c, ld r){
    //line a+tv, circle with center c and radius r
    ld h2 = r*r - v.cross(c - a) * v.cross(c - a) / v.norm();
    point p = a + v * v.dot(c - a) / v.norm();
    if(eq(h2, 0)) return {p}; //line tangent to circle
    else if(1e(h2, 0)) return {}; //no intersection
    else{
        point u = v.unit() * sqrt(h2);
```

```
        return {p - u, p + u}; //two points of intersection (chord)
    }
}
```

```
vector<point> intersectSegmentCircle(const point & a, const point
↪ & b, const point & c, ld r){
    //segment ab, circle with center c and radius r
    vector<point> P = intersectLineCircle(a, b - a, c, r), ans;
    for(const point & p : P){
        if(pointInSegment(a, b, p)) ans.push_back(p);
    }
    return ans;
}
```

### 5.4.5. Centro y radio a través de tres puntos

```
pair<point, ld> getCircle(const point & m, const point & n, const
↪ point & p){
    //find circle that passes through points p, q, r
    point c = intersectLines((n + m) / 2, (n - m).perp(), (p + n) /
    ↪ 2, (p - n).perp());
    ld r = (c - m).length();
    return {c, r};
}
```

### 5.4.6. Intersección de círculos

```
vector<point> intersectionCircles(const point & c1, ld r1, const
↪ point & c2, ld r2){
    //circle 1 with center c1 and radius r1
    //circle 2 with center c2 and radius r2
    point d = c2 - c1;
    ld d2 = d.norm();
    if(eq(d2, 0)) return {}; //concentric circles
    ld pd = (d2 + r1*r1 - r2*r2) / 2;
    ld h2 = r1*r1 - pd*pd/d2;
    point p = c1 + d*pd/d2;
    if(eq(h2, 0)) return {p}; //circles touch at one point
    else if(1e(h2, 0)) return {}; //circles don't intersect
    else{
        point u = d.perp() * sqrt(h2/d2);
        return {p - u, p + u};
```

```

}
}

```

#### 5.4.7. Contención de círculos

```

int circleInsideCircle(const point & c1, ld r1, const point & c2,
↳ ld r2){
    //test if circle 2 is inside circle 1
    //returns "-1" if 2 touches internally 1, "1" if 2 is inside 1,
    ↳ "0" if they overlap
    ld l = r1 - r2 - (c1 - c2).length();
    return (ge(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}

```

```

int circleOutsideCircle(const point & c1, ld r1, const point & c2,
↳ ld r2){
    //test if circle 2 is outside circle 1
    //returns "-1" if they touch externally, "1" if 2 is outside 1,
    ↳ "0" if they overlap
    ld l = (c1 - c2).length() - (r1 + r2);
    return (ge(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}

```

```

int pointInCircle(const point & c, ld r, const point & p){
    //test if point p is inside the circle with center c and radius
    ↳ r
    //returns "0" if it's outside, "-1" if it's in the perimeter,
    ↳ "1" if it's inside
    ld l = (p - c).length() - r;
    return (le(l, 0) ? 1 : (eq(l, 0) ? -1 : 0));
}

```

#### 5.4.8. Tangentes comunes externas e internas

```

vector<vector<point>> tangents(const point & c1, ld r1, const
↳ point & c2, ld r2, bool inner){
    //returns a vector of segments or a single point
    if(inner) r2 = -r2;
    point d = c2 - c1;
    ld dr = r1 - r2, d2 = d.norm(), h2 = d2 - dr*dr;
    if(eq(d2, 0) || le(h2, 0)) return {};
    point v = d*dr/d2;

```

```

    if(eq(h2, 0)) return {{c1 + v*r1}};
    else{
        point u = d.perp()*sqrt(h2)/d2;
        return {{c1 + (v - u)*r1, c2 + (v - u)*r2}, {c1 + (v + u)*r1,
        ↳ c2 + (v + u)*r2}};
    }
}

```

#### 5.4.9. Intersección polígono-círculo

```

ld signed_angle(const point & a, const point & b){
    return sgn(a.cross(b)) * acosl(a.dot(b) / (a.length() *
    ↳ b.length()));
}

ld intersectPolygonCircle(const vector<point> & P, const point &
↳ c, ld r){
    //Gets the area of the intersection of the polygon with the
    ↳ circle
    int n = P.size();
    ld ans = 0;
    for(int i = 0; i < n; ++i){
        point p = P[i], q = P[(i+1)%n];
        bool p_inside = (pointInCircle(c, r, p) != 0);
        bool q_inside = (pointInCircle(c, r, q) != 0);
        if(p_inside && q_inside){
            ans += (p - c).cross(q - c);
        }else if(p_inside && !q_inside){
            point s1 = intersectSegmentCircle(p, q, c, r)[0];
            point s2 = intersectSegmentCircle(c, q, c, r)[0];
            ans += (p - c).cross(s1 - c) + r*r * signed_angle(s1 - c, s2
            ↳ - c);
        }else if(!p_inside && q_inside){
            point s1 = intersectSegmentCircle(c, p, c, r)[0];
            point s2 = intersectSegmentCircle(p, q, c, r)[0];
            ans += (s2 - c).cross(q - c) + r*r * signed_angle(s1 - c, s2
            ↳ - c);
        }else{
            auto info = intersectSegmentCircle(p, q, c, r);
            if(info.size() <= 1){
                ans += r*r * signed_angle(p - c, q - c);
            }else{
                point s2 = info[0], s3 = info[1];

```

```

    point s1 = intersectSegmentCircle(c, p, c, r)[0];
    point s4 = intersectSegmentCircle(c, q, c, r)[0];
    ans += (s2 - c).cross(s3 - c) + r*r * (signed_angle(s1 -
    ↪ c, s2 - c) + signed_angle(s3 - c, s4 - c));
}
}
}
return abs(ans)/2;
}

```

#### 5.4.10. Smallest enclosing circle

```

pair<point, ld> mec2(vector<point> & S, const point & a, const
↪ point & b, int n){
    ld hi = inf, lo = -hi;
    for(int i = 0; i < n; ++i){
        ld si = (b - a).cross(S[i] - a);
        if(eq(si, 0)) continue;
        point m = getCircle(a, b, S[i]).first;
        ld cr = (b - a).cross(m - a);
        if(le(si, 0)) hi = min(hi, cr);
        else lo = max(lo, cr);
    }
    ld v = (ge(lo, 0) ? lo : le(hi, 0) ? hi : 0);
    point c = (a + b) / 2 + (b - a).perp() * v / (b - a).norm();
    return {c, (a - c).norm()};
}

pair<point, ld> mec(vector<point> & S, const point & a, int n){
    random_shuffle(S.begin(), S.begin() + n);
    point b = S[0], c = (a + b) / 2;
    ld r = (a - c).norm();
    for(int i = 1; i < n; ++i){
        if(ge((S[i] - c).norm(), r)){
            tie(c, r) = (n == S.size() ? mec(S, S[i], i) : mec2(S, a,
            ↪ S[i], i));
        }
    }
    return {c, r};
}

pair<point, ld> smallestEnclosingCircle(vector<point> S){
    assert(!S.empty());

```

```

    auto r = mec(S, S[0], S.size());
    return {r.first, sqrt(r.second)};
}

```

#### 5.4.11. Área de unión de círculos

```

struct circ{
    point c;
    ld r;
    circ() {}
    circ(const point & c, ld r): c(c), r(r) {}
    set<pair<ld, ld>> ranges;

    void disable(ld l, ld r){
        ranges.emplace(l, r);
    }

    auto getActive() const{
        vector<pair<ld, ld>> ans;
        ld maxi = 0;
        for(const auto & dis : ranges){
            ld l, r;
            tie(l, r) = dis;
            if(l > maxi){
                ans.emplace_back(maxi, l);
            }
            maxi = max(maxi, r);
        }
        if(!eq(maxi, 2*pi)){
            ans.emplace_back(maxi, 2*pi);
        }
        return ans;
    }
};

ld areaUnionCircles(const vector<circ> & circs){
    vector<circ> valid;
    for(const circ & curr : circs){
        if(eq(curr.r, 0)) continue;
        circ nuevo = curr;
        for(circ & prev : valid){
            if(circleInsideCircle(prev.c, prev.r, nuevo.c, nuevo.r)){
                nuevo.disable(0, 2*pi);
            }
        }
        valid.push_back(nuevo);
    }
    ld ans = 0;
    for(const circ & c : valid){
        ans += c.r*c.r*pi;
        for(const circ & d : valid){
            if(d == c) continue;
            ld l, r;
            tie(l, r) = c.ranges.intersection(d.ranges);
            if(l < r) ans += (r - l)*c.r*d.r;
        }
    }
    return ans;
}

```

```

}else if(circleInsideCircle(nuevo.c, nuevo.r, prev.c,
↪ prev.r)){
    prev.disable(0, 2*pi);
}else{
    auto cruce = intersectionCircles(prev.c, prev.r, nuevo.c,
↪ nuevo.r);
    if(cruce.size() == 2){
        ld a1 = (cruce[0] - prev.c).ang();
        ld a2 = (cruce[1] - prev.c).ang();
        ld b1 = (cruce[1] - nuevo.c).ang();
        ld b2 = (cruce[0] - nuevo.c).ang();
        if(a1 < a2){
            prev.disable(a1, a2);
        }else{
            prev.disable(a1, 2*pi);
            prev.disable(0, a2);
        }
        if(b1 < b2){
            nuevo.disable(b1, b2);
        }else{
            nuevo.disable(b1, 2*pi);
            nuevo.disable(0, b2);
        }
    }
}
}
valid.push_back(nuevo);
}
ld ans = 0;
for(const circ & curr : valid){
    for(const auto & range : curr.getActive()){
        ld l, r;
        tie(l, r) = range;
        ans += curr.r*(curr.c.x * (sin(r) - sin(l)) - curr.c.y *
↪ (cos(r) - cos(l))) + curr.r*curr.r*(r-l);
    }
}
return ans/2;
};

```

## 5.5. Par de puntos más cercanos

```

bool comp1(const point & a, const point & b){
    return le(a.y, b.y);
}
pair<point, point> closestPairOfPoints(vector<point> P){
    sort(P.begin(), P.end(), comp1);
    set<point> S;
    ld ans = inf;
    point p, q;
    int pos = 0;
    for(int i = 0; i < P.size(); ++i){
        while(pos < i && geq(P[i].y - P[pos].y, ans)){
            S.erase(P[pos++]);
        }
        auto lower = S.lower_bound({P[i].x - ans - eps, -inf});
        auto upper = S.upper_bound({P[i].x + ans + eps, -inf});
        for(auto it = lower; it != upper; ++it){
            ld d = (P[i] - *it).length();
            if(le(d, ans)){
                ans = d;
                p = P[i];
                q = *it;
            }
        }
        S.insert(P[i]);
    }
    return {p, q};
}

```

## 5.6. Vantage Point Tree (puntos más cercanos a cada punto)

```

struct vantage_point_tree{
    struct node
    {
        point p;
        ld th;
        node *l, *r;
    }*root;

    vector<pair<ld, point>> aux;

```

```

vantage_point_tree(vector<point> &ps){
    for(int i = 0; i < ps.size(); ++i)
        aux.push_back({ 0, ps[i] });
    root = build(0, ps.size());
}

node *build(int l, int r){
    if(l == r)
        return 0;
    swap(aux[l], aux[l + rand() % (r - l)]);
    point p = aux[l++].second;
    if(l == r)
        return new node({ p });
    for(int i = l; i < r; ++i)
        aux[i].first = (p - aux[i].second).dot(p - aux[i].second);
    int m = (l + r) / 2;
    nth_element(aux.begin() + l, aux.begin() + m, aux.begin() +
        ↪ r);
    return new node({ p, sqrt(aux[m].first), build(l, m), build(m,
        ↪ r) });
}

priority_queue<pair<ld, node*>> que;

void k_nn(node *t, point p, int k){
    if(!t)
        return;
    ld d = (p - t->p).length();
    if(que.size() < k)
        que.push({ d, t });
    else if(ge(que.top().first, d)){
        que.pop();
        que.push({ d, t });
    }
    if(!t->l && !t->r)
        return;
    if(le(d, t->th)){
        k_nn(t->l, p, k);
        if(leq(t->th - d, que.top().first))
            k_nn(t->r, p, k);
    }else{
        k_nn(t->r, p, k);
        if(leq(d - t->th, que.top().first))

```

```

        k_nn(t->l, p, k);
    }
}

vector<point> k_nn(point p, int k){
    k_nn(root, p, k);
    vector<point> ans;
    for(; !que.empty(); que.pop())
        ans.push_back(que.top().second->p);
    reverse(ans.begin(), ans.end());
    return ans;
}
};

```

## 5.7. Suma Minkowski

```

vector<point> minkowskiSum(vector<point> A, vector<point> B){
    int na = (int)A.size(), nb = (int)B.size();
    if(A.empty() || B.empty()) return {};

    rotate(A.begin(), min_element(A.begin(), A.end()), A.end());
    rotate(B.begin(), min_element(B.begin(), B.end()), B.end());

    int pa = 0, pb = 0;
    vector<point> M;

    while(pa < na && pb < nb){
        M.push_back(A[pa] + B[pb]);
        ld x = (A[(pa + 1) % na] - A[pa]).cross(B[(pb + 1) % nb] -
            ↪ B[pb]);
        if(leq(x, 0)) pb++;
        if(geq(x, 0)) pa++;
    }

    while(pa < na) M.push_back(A[pa++] + B[0]);
    while(pb < nb) M.push_back(B[pb++] + A[0]);

    return M;
}

```

## 5.8. Triangulación de Delaunay

*//Delaunay triangulation in  $O(n \log n)$*

```
const point inf_pt(inf, inf);
```

```
struct QuadEdge{
    point origin;
    QuadEdge* rot = nullptr;
    QuadEdge* onext = nullptr;
    bool used = false;
    QuadEdge* rev() const{return rot->rot;}
    QuadEdge* lnext() const{return rot->rev()->onext->rot;}
    QuadEdge* oprev() const{return rot->onext->rot;}
    point dest() const{return rev()->origin;}
};
```

```
QuadEdge* make_edge(const point & from, const point & to){
    QuadEdge* e1 = new QuadEdge;
    QuadEdge* e2 = new QuadEdge;
    QuadEdge* e3 = new QuadEdge;
    QuadEdge* e4 = new QuadEdge;
    e1->origin = from;
    e2->origin = to;
    e3->origin = e4->origin = inf_pt;
    e1->rot = e3;
    e2->rot = e4;
    e3->rot = e2;
    e4->rot = e1;
    e1->onext = e1;
    e2->onext = e2;
    e3->onext = e4;
    e4->onext = e3;
    return e1;
}
```

```
void splice(QuadEdge* a, QuadEdge* b){
    swap(a->onext->rot->onext, b->onext->rot->onext);
    swap(a->onext, b->onext);
}
```

```
void delete_edge(QuadEdge* e){
    splice(e, e->oprev());
    splice(e->rev(), e->rev()->oprev());
    delete e->rot;
```

```
    delete e->rev()->rot;
    delete e;
    delete e->rev();
}
```

```
QuadEdge* connect(QuadEdge* a, QuadEdge* b){
    QuadEdge* e = make_edge(a->dest(), b->origin);
    splice(e, a->lnext());
    splice(e->rev(), b);
    return e;
}
```

```
bool left_of(const point & p, QuadEdge* e){
    return ge((e->origin - p).cross(e->dest() - p), 0);
}
```

```
bool right_of(const point & p, QuadEdge* e){
    return le((e->origin - p).cross(e->dest() - p), 0);
}
```

```
ld det3(ld a1, ld a2, ld a3, ld b1, ld b2, ld b3, ld c1, ld c2, ld
↪ c3) {
    return a1 * (b2 * c3 - c2 * b3) - a2 * (b1 * c3 - c1 * b3) + a3
↪ * (b1 * c2 - c1 * b2);
}
```

```
bool in_circle(const point & a, const point & b, const point & c,
↪ const point & d) {
    ld det = -det3(b.x, b.y, b.norm(), c.x, c.y, c.norm(), d.x, d.y,
↪ d.norm());
    det += det3(a.x, a.y, a.norm(), c.x, c.y, c.norm(), d.x, d.y,
↪ d.norm());
    det -= det3(a.x, a.y, a.norm(), b.x, b.y, b.norm(), d.x, d.y,
↪ d.norm());
    det += det3(a.x, a.y, a.norm(), b.x, b.y, b.norm(), c.x, c.y,
↪ c.norm());
    return ge(det, 0);
}
```

```
pair<QuadEdge*, QuadEdge*> build_tr(int l, int r, vector<point> &
↪ P){
    if(r - l + 1 == 2){
        QuadEdge* res = make_edge(P[l], P[r]);
        return {res, res->rev()};
    }
```

```

}
if(r - l + 1 == 3){
    QuadEdge *a = make_edge(P[l], P[l + 1]), *b = make_edge(P[l +
    ↪ 1], P[r]);
    splice(a->rev(), b);
    int sg = sgn((P[l + 1] - P[l]).cross(P[r] - P[l]));
    if(sg == 0)
        return {a, b->rev()};
    QuadEdge* c = connect(b, a);
    if(sg == 1)
        return {a, b->rev()};
    else
        return {c->rev(), c};
}
int mid = (l + r) / 2;
QuadEdge *ldo, *ldi, *rdo, *rdi;
tie(ldo, ldi) = build_tr(l, mid, P);
tie(rdi, rdo) = build_tr(mid + 1, r, P);
while(true){
    if(left_of(rdi->origin, ldi)){
        ldi = ldi->lnext();
        continue;
    }
    if(right_of(ldi->origin, rdi)){
        rdi = rdi->rev()->onext;
        continue;
    }
    break;
}
QuadEdge* basel = connect(rdi->rev(), ldi);
auto valid = [&basel](QuadEdge* e){return right_of(e->dest(),
    ↪ basel);};
if(ldi->origin == ldo->origin)
    ldo = basel->rev();
if(rdi->origin == rdo->origin)
    rdo = basel;
while(true){
    QuadEdge* lcand = basel->rev()->onext;
    if(valid(lcand)){
        while(in_circle(basel->dest(), basel->origin, lcand->dest(),
    ↪ lcand->onext->dest())){
            QuadEdge* t = lcand->onext;
            delete_edge(lcand);
            lcand = t;

```

```

}
}
QuadEdge* rcand = basel->oprev();
if(valid(rcand)){
    while(in_circle(basel->dest(), basel->origin, rcand->dest(),
    ↪ rcand->oprev()->dest())){
        QuadEdge* t = rcand->oprev();
        delete_edge(rcand);
        rcand = t;
    }
}
if(!valid(lcand) && !valid(rcand))
    break;
if(!valid(lcand) || (valid(rcand) && in_circle(lcand->dest(),
    ↪ lcand->origin, rcand->origin, rcand->dest())))
    basel = connect(rcand, basel->rev());
else
    basel = connect(basel->rev(), lcand->rev());
}
return {ldo, rdo};
}

vector<tuple<point, point, point>> delaunay(vector<point> & P){
    sort(P.begin(), P.end());
    auto res = build_tr(0, (int)P.size() - 1, P);
    QuadEdge* e = res.first;
    vector<QuadEdge*> edges = {e};
    while(le((e->dest() - e->onext->dest()).cross(e->origin -
    ↪ e->onext->dest()), 0))
        e = e->onext;
    auto add = [&P, &e, &edges]() {
        QuadEdge* curr = e;
        do{
            curr->used = true;
            P.push_back(curr->origin);
            edges.push_back(curr->rev());
            curr = curr->lnext();
        }while(curr != e);
    };
    add();
    P.clear();
    int kek = 0;
    while(kek < (int)edges.size())
        if(!(e = edges[kek++])->used)

```

```

    add();
    vector<tuple<point, point, point>> ans;
    for(int i = 0; i < (int)P.size(); i += 3){
        ans.emplace_back(P[i], P[i + 1], P[i + 2]);
    }
    return ans;
}

```

## 6. Grafos

### 6.1. Disjoint Set

```

struct disjointSet{
    int N;
    vector<short int> rank;
    vi parent, count;

    disjointSet(int N): N(N), parent(N), count(N), rank(N){}

    void makeSet(int v){
        count[v] = 1;
        parent[v] = v;
    }

    int findSet(int v){
        if(v == parent[v]) return v;
        return parent[v] = findSet(parent[v]);
    }

    void unionSet(int a, int b){
        a = findSet(a), b = findSet(b);
        if(a == b) return;
        if(rank[a] < rank[b]){
            parent[a] = b;
            count[b] += count[a];
        }else{
            parent[b] = a;
            count[a] += count[b];
            if(rank[a] == rank[b]) ++rank[a];
        }
    }
};

```

### 6.2. Definiciones

```

struct edge{
    int source, dest, cost;

    edge(): source(0), dest(0), cost(0){}
}

```



```

edge(int dest, int cost): dest(dest), cost(cost){}

edge(int source, int dest, int cost): source(source),
    ↪ dest(dest), cost(cost){}

bool operator==(const edge & b) const{
    return source == b.source && dest == b.dest && cost == b.cost;
}
bool operator<(const edge & b) const{
    return cost < b.cost;
}
bool operator>(const edge & b) const{
    return cost > b.cost;
}
};

struct path{
    int cost = inf;
    deque<int> vertices;
    int size = 1;
    int prev = -1;
};

struct graph{
    vector<vector<edge>> adjList;
    vector<vb> adjMatrix;
    vector<vi> costMatrix;
    vector<edge> edges;
    int V = 0;
    bool dir = false;

    graph(int n, bool dir): V(n), dir(dir), adjList(n), edges(n),
    ↪ adjMatrix(n, vb(n)), costMatrix(n, vi(n)){
        for(int i = 0; i < n; ++i)
            for(int j = 0; j < n; ++j)
                costMatrix[i][j] = (i == j ? 0 : inf);
    }

    void add(int source, int dest, int cost){
        adjList[source].emplace_back(source, dest, cost);
        edges.emplace_back(source, dest, cost);
        adjMatrix[source][dest] = true;
        costMatrix[source][dest] = cost;
        if(!dir){

```

```

            adjList[dest].emplace_back(dest, source, cost);
            adjMatrix[dest][source] = true;
            costMatrix[dest][source] = cost;
        }
    }

    void buildPaths(vector<path> & paths){
        for(int i = 0; i < V; i++){
            int u = i;
            for(int j = 0; j < paths[i].size; j++){
                paths[i].vertices.push_front(u);
                u = paths[u].prev;
            }
        }
    }
}

```

### 6.3. DFS genérica

```

void dfs(int u, vi & status, vi & parent){
    status[u] = 1;
    for(edge & current : adjList[u]){
        int v = current.dest;
        if(status[v] == 0){ //not visited
            parent[v] = u;
            dfs(v, status, parent);
        }else if(status[v] == 1){ //explored
            if(v == parent[u]){
                //bidirectional node u<-->v
            }else{
                //back edge u-v
            }
        }else if(status[v] == 2){ //visited
            //forward edge u-v
        }
    }
    status[u] = 2;
}

```

### 6.4. Dijkstra

```

vector<path> dijkstra(int start){
    priority_queue<edge, vector<edge>, greater<edge>> cola;

```

```

vector<path> paths(V);
cola.emplace(start, 0);
paths[start].cost = 0;
while(!cola.empty()){
    int u = cola.top().dest; cola.pop();
    for(edge & current : adjList[u]){
        int v = current.dest;
        int nuevo = paths[u].cost + current.cost;
        if(nuevo == paths[v].cost && paths[u].size + 1 <
        ↪ paths[v].size){
            paths[v].prev = u;
            paths[v].size = paths[u].size + 1;
        }else if(nuevo < paths[v].cost){
            paths[v].prev = u;
            paths[v].size = paths[u].size + 1;
            cola.emplace(v, nuevo);
            paths[v].cost = nuevo;
        }
    }
}
buildPaths(paths);
return paths;
}

```

## 6.5. Bellman Ford

```

vector<path> bellmanFord(int start){
    vector<path> paths(V, path());
    vi processed(V);
    vb inQueue(V);
    queue<int> Q;
    paths[start].cost = 0;
    Q.push(start);
    while(!Q.empty()){
        int u = Q.front(); Q.pop(); inQueue[u] = false;
        if(paths[u].cost == inf) continue;
        ++processed[u];
        if(processed[u] == V){
            cout << "Negative cycle\n";
            return {};
        }
        for(edge & current : adjList[u]){
            int v = current.dest;

```

```

        int nuevo = paths[u].cost + current.cost;
        if(nuevo == paths[v].cost && paths[u].size + 1 <
        ↪ paths[v].size){
            paths[v].prev = u;
            paths[v].size = paths[u].size + 1;
        }else if(nuevo < paths[v].cost){
            if(!inQueue[v]){
                Q.push(v);
                inQueue[v] = true;
            }
            paths[v].prev = u;
            paths[v].size = paths[u].size + 1;
            paths[v].cost = nuevo;
        }
    }
}
buildPaths(paths);
return paths;
}

```

## 6.6. Floyd

```

vector<vi> floyd(){
    vector<vi> tmp = costMatrix;
    for(int k = 0; k < V; ++k)
        for(int i = 0; i < V; ++i)
            for(int j = 0; j < V; ++j)
                if(tmp[i][k] != inf && tmp[k][j] != inf)
                    tmp[i][j] = min(tmp[i][j], tmp[i][k] + tmp[k][j]);
    return tmp;
}

```

## 6.7. Cerradura transitiva $O(V^3)$

```

vector<vb> transitiveClosure(){
    vector<vb> tmp = adjMatrix;
    for(int k = 0; k < V; ++k)
        for(int i = 0; i < V; ++i)
            for(int j = 0; j < V; ++j)
                tmp[i][j] = tmp[i][j] || (tmp[i][k] && tmp[k][j]);
    return tmp;
}

```

## 6.8. Cerradura transitiva $O(V^2)$

```
vector<vb> transitiveClosureDFS(){
    vector<vb> tmp(V, vb(V));
    function<void(int, int)> dfs = [&](int start, int u){
        for(edge & current : adjList[u]){
            int v = current.dest;
            if(!tmp[start][v]){
                tmp[start][v] = true;
                dfs(start, v);
            }
        }
    };
    for(int u = 0; u < V; u++){
        dfs(u, u);
    }
    return tmp;
}
```

## 6.9. Verificar si el grafo es bipartito

```
bool isBipartite(){
    vi side(V, -1);
    queue<int> q;
    for (int st = 0; st < V; ++st){
        if(side[st] != -1) continue;
        q.push(st);
        side[st] = 0;
        while(!q.empty()){
            int u = q.front();
            q.pop();
            for (edge & current : adjList[u]){
                int v = current.dest;
                if(side[v] == -1) {
                    side[v] = side[u] ^ 1;
                    q.push(v);
                }else{
                    if(side[v] == side[u]) return false;
                }
            }
        }
    }
    return true;
}
```

## 6.10. Orden topológico

```
vi topologicalSort(){
    int visited = 0;
    vi order, indegree(V);
    for(auto & node : adjList){
        for(edge & current : node){
            int v = current.dest;
            ++indegree[v];
        }
    }
    queue<int> Q;
    for(int i = 0; i < V; ++i){
        if(indegree[i] == 0) Q.push(i);
    }
    while(!Q.empty()){
        int source = Q.front();
        Q.pop();
        order.push_back(source);
        ++visited;
        for(edge & current : adjList[source]){
            int v = current.dest;
            --indegree[v];
            if(indegree[v] == 0) Q.push(v);
        }
    }
    if(visited == V) return order;
    else return {};
}
```

## 6.11. Detectar ciclos

```
bool hasCycle(){
    vi color(V);
    function<bool(int, int)> dfs = [&](int u, int parent){
        color[u] = 1;
        bool ans = false;
        int ret = 0;
        for(edge & current : adjList[u]){
            int v = current.dest;
            if(color[v] == 0)
                ans |= dfs(v, u);
            else if(color[v] == 1 && (dir || v != parent || ret++))
                return true;
        }
        return ans;
    };
    return dfs(0, -1);
}
```

```

        ans = true;
    }
    color[u] = 2;
    return ans;
};
for(int u = 0; u < V; ++u)
    if(color[u] == 0 && dfs(u, -1))
        return true;
return false;
}

```

## 6.12. Puentes y puntos de articulación

```

pair<vb, vector<edge>> articulationBridges(){
    vi low(V), label(V);
    vb points(V);
    vector<edge> bridges;
    int time = 0;
    function<int(int, int)> dfs = [&](int u, int p){
        label[u] = low[u] = ++time;
        int hijos = 0, ret = 0;
        for(edge & current : adjList[u]){
            int v = current.dest;
            if(v == p && !ret++) continue;
            if(!label[v]){
                ++hijos;
                dfs(v, u);
                if(label[u] <= low[v])
                    points[u] = true;
                if(label[u] < low[v])
                    bridges.push_back(current);
                low[u] = min(low[u], low[v]);
            }
            low[u] = min(low[u], label[v]);
        }
        return hijos;
    };
    for(int u = 0; u < V; ++u)
        if(!label[u])
            points[u] = dfs(u, -1) > 1;
    return make_pair(points, bridges);
}

```

## 6.13. Componentes fuertemente conexas

```

vector<vi> scc(){
    vi low(V), label(V);
    int time = 0;
    vector<vi> ans;
    stack<int> S;
    function<void(int)> dfs = [&](int u){
        label[u] = low[u] = ++time;
        S.push(u);
        for(edge & current : adjList[u]){
            int v = current.dest;
            if(!label[v]) dfs(v);
            low[u] = min(low[u], low[v]);
        }
        if(label[u] == low[u]){
            vi comp;
            while(S.top() != u){
                comp.push_back(S.top());
                low[S.top()] = V + 1;
                S.pop();
            }
            comp.push_back(S.top());
            S.pop();
            ans.push_back(comp);
            low[u] = V + 1;
        }
    };
    for(int u = 0; u < V; ++u)
        if(!label[u]) dfs(u);
    return ans;
}

```

## 6.14. Árbol mínimo de expansión (Kruskal)

```

vector<edge> kruskal(){
    sort(edges.begin(), edges.end());
    vector<edge> MST;
    disjointSet DS(V);
    for(int u = 0; u < V; ++u)
        DS.makeSet(u);
    int i = 0;
}

```

```

while(i < edges.size() && MST.size() < V - 1){
    edge current = edges[i++];
    int u = current.source, v = current.dest;
    if(DS.findSet(u) != DS.findSet(v)){
        MST.push_back(current);
        DS.unionSet(u, v);
    }
}
return MST;
}

```

### 6.15. Máximo emparejamiento bipartito

```

bool tryKuhn(int u, vb & used, vi & left, vi & right){
    if(used[u]) return false;
    used[u] = true;
    for(edge & current : adjList[u]){
        int v = current.dest;
        if(right[v] == -1 || tryKuhn(right[v], used, left, right)){
            right[v] = u;
            left[u] = v;
            return true;
        }
    }
    return false;
}

```

```

bool augmentingPath(int u, vb & used, vi & left, vi & right){
    used[u] = true;
    for(edge & current : adjList[u]){
        int v = current.dest;
        if(right[v] == -1){
            right[v] = u;
            left[u] = v;
            return true;
        }
    }
    for(edge & current : adjList[u]){
        int v = current.dest;
        if(!used[right[v]] && augmentingPath(right[v], used, left,
        ↪ right)){
            right[v] = u;
            left[u] = v;
        }
    }
}

```

```

return true;
}
}
return false;
}

//vertices from the left side numbered from 0 to l-1
//vertices from the right side numbered from 0 to r-1
//graph[u] represents the left side
//graph[u][v] represents the right side
//we can use tryKuhn() or augmentingPath()
vector<pair<int, int>> maxMatching(int l, int r){
    vi left(l, -1), right(r, -1);
    vb used(l);
    for(int u = 0; u < l; ++u){
        tryKuhn(u, used, left, right);
        fill(used.begin(), used.end(), false);
    }
    vector<pair<int, int>> ans;
    for(int u = 0; u < r; ++u){
        if(right[u] != -1){
            ans.emplace_back(right[u], u);
        }
    }
    return ans;
}

```

### 6.16. Circuito euleriano

## 7. Árboles

### 7.1. Estructura tree

```

struct tree{
    vi parent, level, weight;
    vector<vi> dists, DP;
    int n, root;

    void dfs(int u, graph & G){
        for(edge & curr : G.adjList[u]){
            int v = curr.dest;
            int w = curr.cost;
            if(v != parent[u]){
                parent[v] = u;
                weight[v] = w;
                level[v] = level[u] + 1;
                dfs(v, G);
            }
        }
    }

    tree(int n, int root): n(n), root(root), parent(n), level(n),
        ↪ weight(n), dists(n, vi(20)), DP(n, vi(20)){
        parent[root] = root;
    }

    tree(graph & G, int root): n(G.V), root(root), parent(G.V),
        ↪ level(G.V), weight(G.V), dists(G.V, vi(20)), DP(G.V,
        ↪ vi(20)){
        parent[root] = root;
        dfs(root, G);
    }

    void pre(){
        for(int u = 0; u < n; u++){
            DP[u][0] = parent[u];
            dists[u][0] = weight[u];
        }
        for(int i = 1; (1 << i) <= n; ++i){
            for(int u = 0; u < n; ++u){
                DP[u][i] = DP[DP[u][i - 1]][i - 1];
            }
        }
    }
}

```

```

        dists[u][i] = dists[u][i - 1] + dists[DP[u][i - 1]][i -
        ↪ 1];
    }
}
}

```

### 7.2. $k$ -ésimo ancestro

```

int ancestor(int p, int k){
    int h = level[p] - k;
    if(h < 0) return -1;
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    for(int i = lg; i >= 0; --i){
        if(level[p] - (1 << i) >= h){
            p = DP[p][i];
        }
    }
    return p;
}

```

### 7.3. LCA

```

int lca(int p, int q){
    if(level[p] < level[q]) swap(p, q);
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    for(int i = lg; i >= 0; --i){
        if(level[p] - (1 << i) >= level[q]){
            p = DP[p][i];
        }
    }
    if(p == q) return p;

    for(int i = lg; i >= 0; --i){
        if(DP[p][i] != -1 && DP[p][i] != DP[q][i]){
            p = DP[p][i];
            q = DP[q][i];
        }
    }
}

```

```

    return parent[p];
}

```

## 7.4. Distancia entre dos nodos

```

int dist(int p, int q){
    if(level[p] < level[q]) swap(p, q);
    int lg;
    for(lg = 1; (1 << lg) <= level[p]; ++lg);
    lg--;
    int sum = 0;
    for(int i = lg; i >= 0; --i){
        if(level[p] - (1 << i) >= level[q]){
            sum += dists[p][i];
            p = DP[p][i];
        }
    }
    if(p == q) return sum;

    for(int i = lg; i >= 0; --i){
        if(DP[p][i] != -1 && DP[p][i] != DP[q][i]){
            sum += dists[p][i] + dists[q][i];
            p = DP[p][i];
            q = DP[q][i];
        }
    }
    sum += dists[p][0] + dists[q][0];
    return sum;
}

```

## 7.5. Link Cut

# 8. Flujos

## 8.1. Estructura flowEdge

```

template<typename T>
struct flowEdge{
    int dest;
    T flow, capacity, cost;
    flowEdge *res;

    flowEdge(): dest(0), flow(0), capacity(0), cost(0), res(NULL){}
    flowEdge(int dest, T flow, T capacity, T cost = 0): dest(dest),
        ↪ flow(flow), capacity(capacity), cost(cost), res(NULL){}

    void addFlow(T flow){
        this->flow += flow;
        this->res->flow -= flow;
    }
};

```

## 8.2. Estructura flowGraph

```

template<typename T>
struct flowGraph{
    T inf = numeric_limits<T>::max();
    vector<vector<flowEdge<T>*>> adjList;
    vector<int> dist, pos;
    int V;
    flowGraph(int V): V(V), adjList(V), dist(V), pos(V){}
    ~flowGraph(){
        for(int i = 0; i < V; ++i)
            for(int j = 0; j < adjList[i].size(); ++j)
                delete adjList[i][j];
    }
    void addEdge(int u, int v, T capacity, T cost = 0){
        flowEdge<T> *uv = new flowEdge<T>(v, 0, capacity, cost);
        flowEdge<T> *vu = new flowEdge<T>(u, capacity, capacity,
            ↪ -cost);
        uv->res = vu;
        vu->res = uv;
        adjList[u].push_back(uv);
        adjList[v].push_back(vu);
    }
};

```

```
}

```

### 8.3. Algoritmo de Edmonds-Karp $O(VE^2)$

```
//Maximun Flow using Edmonds-Karp Algorithm  $O(VE^2)$ 
T edmondsKarp(int s, int t){
    T maxFlow = 0;
    vector<flowEdge<T>*> parent(V);
    while(true){
        fill(parent.begin(), parent.end(), nullptr);
        queue<int> Q;
        Q.push(s);
        while(!Q.empty() && !parent[t]){
            int u = Q.front(); Q.pop();
            for(flowEdge<T> *v : adjList[u]){
                if(!parent[v->dest] && v->capacity > v->flow){
                    parent[v->dest] = v;
                    Q.push(v->dest);
                }
            }
        }
        if(!parent[t]) break;
        T f = inf;
        for(int u = t; u != s; u = parent[u]->res->dest)
            f = min(f, parent[u]->capacity - parent[u]->flow);
        for(int u = t; u != s; u = parent[u]->res->dest)
            parent[u]->addFlow(f);
        maxFlow += f;
    }
    return maxFlow;
}
```

### 8.4. Algoritmo de Dinic $O(V^2E)$

```
//Maximun Flow using Dinic Algorithm  $O(EV^2)$ 
T blockingFlow(int u, int t, T flow){
    if(u == t) return flow;
    for(int &i = pos[u]; i < adjList[u].size(); ++i){
        flowEdge<T> *v = adjList[u][i];
        if(v->capacity > v->flow && dist[u] + 1 == dist[v->dest]){
            T fv = blockingFlow(v->dest, t, min(flow, v->capacity -
                ↪ v->flow));

```

```
            if(fv > 0){
                v->addFlow(fv);
                return fv;
            }
        }
    }
    return 0;
}
T dinic(int s, int t){
    T maxFlow = 0;
    dist[t] = 0;
    while(dist[t] != -1){
        fill(dist.begin(), dist.end(), -1);
        queue<int> Q;
        Q.push(s);
        dist[s] = 0;
        while(!Q.empty()){
            int u = Q.front(); Q.pop();
            for(flowEdge<T> *v : adjList[u]){
                if(dist[v->dest] == -1 && v->flow != v->capacity){
                    dist[v->dest] = dist[u] + 1;
                    Q.push(v->dest);
                }
            }
        }
        if(dist[t] != -1){
            T f;
            fill(pos.begin(), pos.end(), 0);
            while(f = blockingFlow(s, t, inf))
                maxFlow += f;
        }
    }
    return maxFlow;
}
```

### 8.5. Flujo máximo de costo mínimo

```
//Max Flow Min Cost
pair<T, T> maxFlowMinCost(int s, int t){
    vector<bool> inQueue(V);
    vector<T> distance(V), cap(V);
    vector<flowEdge<T>*> parent(V);
    T maxFlow = 0, minCost = 0;

```



```

while(true){
    fill(distance.begin(), distance.end(), inf);
    fill(parent.begin(), parent.end(), nullptr);
    fill(cap.begin(), cap.end(), 0);
    distance[s] = 0;
    cap[s] = inf;
    queue<int> Q;
    Q.push(s);
    while(!Q.empty()){
        int u = Q.front(); Q.pop(); inQueue[u] = 0;
        for(flowEdge<T> *v : adjList[u]){
            if(v->capacity > v->flow && distance[v->dest] >
                distance[u] + v->cost){
                distance[v->dest] = distance[u] + v->cost;
                parent[v->dest] = v;
                cap[v->dest] = min(cap[u], v->capacity - v->flow);
                if(!inQueue[v->dest]){
                    Q.push(v->dest);
                    inQueue[v->dest] = true;
                }
            }
        }
    }
    if(!parent[t]) break;
    maxFlow += cap[t];
    minCost += cap[t] * distance[t];
    for(int u = t; u != s; u = parent[u]->res->dest)
        parent[u]->addFlow(cap[t]);
}
return {maxFlow, minCost};
}

```

## 8.6. Hungariano

*//Given a  $n \times m$  cost matrix ( $n \leq m$ ), it finds a minimum cost  
 $\hookrightarrow$  assignment.  
 //The actual assignment is in the vector returned.  
 //To find the maximum, negate the values and the answer.*

```

template<typename T>
pair<T, vector<int>>> hungarian(const vector<vector<T>> & a){
    int n = a.size(), m = a[0].size();
    assert(n <= m);
    vector<int> ans(n), pa(n+1, -1), pb(m+1, -1), way(m, -1);

```

```

    vector<T> minv(m), u(n+1), v(m+1);
    vector<bool> used(m+1);
    T inf = numeric_limits<T>::max();
    for(int i = 0; i < n; ++i){
        fill(minv.begin(), minv.end(), inf);
        fill(used.begin(), used.end(), false);
        pb[m] = i;
        pa[i] = m;
        int j0 = m;
        do{
            used[j0] = true;
            int i0 = pb[j0];
            T delta = inf;
            int j1 = -1;
            for(int j = 0; j < m; ++j){
                if(used[j]) continue;
                T cur = a[i0][j] - u[i0] - v[j];
                if(cur < minv[j]){
                    minv[j] = cur;
                    way[j] = j0;
                }
                if(minv[j] < delta){
                    delta = minv[j];
                    j1 = j;
                }
            }
            for(int j = 0; j <= m; ++j){
                if(used[j]){
                    u[pb[j]] += delta;
                    v[j] -= delta;
                }else{
                    minv[j] -= delta;
                }
            }
            j0 = j1;
        }while(pb[j0] != -1);
        do{
            int j1 = way[j0];
            pb[j0] = pb[j1];
            pa[pb[j0]] = j0;
            j0 = j1;
        }while(j0 != m);
    }
    for(int i = 0; i < n; ++i)

```

```

    ans[pb[i]] = i;
    return {-v[m], ans};
}

```

## 9. Estructuras de datos

### 9.1. Segment Tree

#### 9.1.1. Minimalistic: Point updates, range queries

```

template<typename T>
struct SegmentTree{
    int N;
    vector<T> ST;

    //build from an array in O(n)
    SegmentTree(int N, vector<T> & arr): N(N){
        ST.resize(N << 1);
        for(int i = 0; i < N; ++i)
            ST[N + i] = arr[i];
        for(int i = N - 1; i > 0; --i)
            ST[i] = ST[i << 1] + ST[i << 1 | 1];
    }

    //single element update in i
    void update(int i, T value){
        ST[i += N] = value; //update the element accordingly
        while(i >>= 1)
            ST[i] = ST[i << 1] + ST[i << 1 | 1];
    }

    //single element update in [l, r]
    void update(int l, int r, T value){
        l += N, r += N;
        for(int i = l; i <= r; ++i)
            ST[i] = value;
        l >>= 1, r >>= 1;
        while(l >= 1){
            for(int i = r; i >= l; --i)
                ST[i] = ST[i << 1] + ST[i << 1 | 1];
            l >>= 1, r >>= 1;
        }
    }

    //range query, [l, r]
    T query(int l, int r){
        T res = 0;

```

```

    for(l += N, r += N; l <= r; l >>= 1, r >>= 1){
        if(l & 1) res += ST[l++];
        if(!(r & 1)) res += ST[r--];
    }
    return res;
}
};

```

### 9.1.2. Dynamic: Range updates and range queries

```

template<typename T>
struct SegmentTreeDin{
    SegmentTreeDin *left, *right;
    int l, r;
    T sum, lazy;

    SegmentTreeDin(int start, int end, vector<T> & arr): left(NULL),
        ↪ right(NULL), l(start), r(end), sum(0), lazy(0){
        if(l == r) sum = arr[l];
        else{
            int half = l + ((r - l) >> 1);
            left = new SegmentTreeDin(l, half, arr);
            right = new SegmentTreeDin(half+1, r, arr);
            sum = left->sum + right->sum;
        }
    }

    void propagate(T dif){
        sum += (r - l + 1) * dif;
        if(l != r){
            left->lazy += dif;
            right->lazy += dif;
        }
    }

    T sum_query(int start, int end){
        if(lazy != 0){
            propagate(lazy);
            lazy = 0;
        }
        if(end < l || r < start) return 0;
        if(start <= l && r <= end) return sum;
    }
};

```

```

        else return left->sum_query(start, end) +
            ↪ right->sum_query(start, end);
    }

    void add_range(int start, int end, T dif){
        if(lazy != 0){
            propagate(lazy);
            lazy = 0;
        }
        if(end < l || r < start) return;
        if(start <= l && r <= end) propagate(dif);
        else{
            left->add_range(start, end, dif);
            right->add_range(start, end, dif);
            sum = left->sum + right->sum;
        }
    }

    void add_pos(int i, T sum){
        add_range(i, i, sum);
    }
};

```

### 9.1.3. Static: Range updates and range queries

```

template<typename T>
struct SegmentTreeEst{
    int size;
    vector<T> sum, lazy;

    void rec(int pos, int l, int r, vector<T> & arr){
        if(l == r) sum[pos] = arr[l];
        else{
            int half = l + ((r - l) >> 1);
            rec(2*pos+1, l, half, arr);
            rec(2*pos+2, half+1, r, arr);
            sum[pos] = sum[2*pos+1] + sum[2*pos+2];
        }
    }

    SegmentTreeEst(int n, vector<T> & arr): size(n){
        int h = ceil(log2(n));
        sum.resize((1 << (h + 1)) - 1);
    }
};

```

```

    lazy.resize((1 << (h + 1)) - 1);
    rec(0, 0, n - 1, arr);
}

void propagate(int pos, int l, int r, T dif){
    sum[pos] += (r - l + 1) * dif;
    if(l != r){
        lazy[2*pos+1] += dif;
        lazy[2*pos+2] += dif;
    }
}

T sum_query_rec(int start, int end, int pos, int l, int r){
    if(lazy[pos] != 0){
        propagate(pos, l, r, lazy[pos]);
        lazy[pos] = 0;
    }
    if(end < l || r < start) return 0;
    if(start <= l && r <= end) return sum[pos];
    else{
        int half = l + ((r - l) >> 1);
        return sum_query_rec(start, end, 2*pos+1, l, half) +
            sum_query_rec(start, end, 2*pos+2, half+1, r);
    }
}

T sum_query(int start, int end){
    return sum_query_rec(start, end, 0, 0, size - 1);
}

void add_range_rec(int start, int end, int pos, int l, int r, T
    ↪ dif){
    if(lazy[pos] != 0){
        propagate(pos, l, r, lazy[pos]);
        lazy[pos] = 0;
    }
    if(end < l || r < start) return;
    if(start <= l && r <= end) propagate(pos, l, r, dif);
    else{
        int half = l + ((r - l) >> 1);
        add_range_rec(start, end, 2*pos+1, l, half, dif);
        add_range_rec(start, end, 2*pos+2, half+1, r, dif);
        sum[pos] = sum[2*pos+1] + sum[2*pos+2];
    }
}

```

```

}

void add_range(int start, int end, T dif){
    add_range_rec(start, end, 0, 0, size - 1, dif);
}

void add_pos(int i, T sum){
    add_range(i, i, sum);
}

};

```

#### 9.1.4. Persistent: Point updates, range queries

```

template<typename T>
struct StPer{
    StPer *left, *right;
    int l, r;
    T sum;

    StPer(int start, int end): left(NULL), right(NULL), l(start),
        ↪ r(end), sum(0){
        if(l != r){
            int half = l + ((r - l) >> 1);
            left = new StPer(l, half);
            right = new StPer(half+1, r);
        }
    }

    StPer(int start, int end, T val): left(NULL), right(NULL),
        ↪ l(start), r(end), sum(val){}

    StPer(int start, int end, StPer* left, StPer* right):
        ↪ left(left), right(right), l(start), r(end){
        sum = left->sum + right->sum;
    }

    T sum_query(int start, int end){
        if(end < l || r < start) return 0;
        if(start <= l && r <= end) return sum;
        else return left->sum_query(start, end) +
            ↪ right->sum_query(start, end);
    }

    StPer* update(int pos, T val){
        if(l == r) return new StPer(l, r, sum + val);
    }
}

```

```

    int half = 1 + ((r - 1) >> 1);
    if(pos <= half) return new StPer(l, r, left->update(pos, val),
        ↪ right);
    return new StPer(l, r, left, right->update(pos, val));
}
};

```

## 9.2. Fenwick Tree

```

template<typename T>
struct FenwickTree{
    int N;
    vector<T> bit;

    //build from array in O(n), indexed in 0
    FenwickTree(int N, vector<T> & arr): N(N){
        bit.resize(N);
        for(int i = 0; i < N; ++i){
            bit[i] += arr[i];
            if((i | (i + 1)) < N)
                bit[i | (i + 1)] += bit[i];
        }
    }

    //single element increment
    void update(int pos, T value){
        while(pos < N){
            bit[pos] += value;
            pos |= pos + 1;
        }
    }

    //range query, [0, r]
    T query(int r){
        T res = 0;
        while(r >= 0){
            res += bit[r];
            r = (r & (r + 1)) - 1;
        }
        return res;
    }

    //range query, [l, r]

```

```

    T query(int l, int r){
        return query(r) - query(l - 1);
    }
};

```

## 9.3. SQRT Decomposition

```

struct MQuery{
    int l, r, index, S;
    bool operator<(const MQuery & q) const{
        int c_o = l / S, c_q = q.l / S;
        if(c_o == c_q)
            return r < q.r;
        return c_o < c_q;
    }
};

template<typename T>
struct SQRT{
    int N, S;
    vector<T> A, B;

    SQRT(int N): N(N){
        this->S = sqrt(N + .0) + 1;
        A.assign(N, 0);
        B.assign(S, 0);
    }

    void build(vector<T> & arr){
        A = vector<int>(arr.begin(), arr.end());
        for(int i = 0; i < N; ++i) B[i / S] += A[i];
    }

    //single element update
    void update(int pos, T value){
        int k = pos / S;
        A[pos] = value;
        T res = 0;
        for(int i = k * S, end = min(N, (k + 1) * S) - 1; i <= end;
            ↪ ++i) res += A[i];
        B[k] = res;
    }
}

```

```

//range query, [l, r]
T query(int l, int r){
    T res = 0;
    int c_l = l / S, c_r = r / S;
    if(c_l == c_r){
        for(int i = l; i <= r; ++i) res += A[i];
    }else{
        for(int i = l, end = (c_l + 1) * S - 1; i <= end; ++i) res
            += A[i];
        for(int i = c_l * S + 1; i <= c_r * S - 1; ++i) res += B[i];
        for(int i = c_r * S; i <= r; ++i) res += A[i];
    }
    return res;
}

//range queries offline using MO's algorithm
vector<T> MO(vector<MOquery> & queries){
    vector<T> ans(queries.size());
    sort(queries.begin(), queries.end());
    T current = 0;
    int prevL = 0, prevR = -1;
    int i, j;
    for(const MOquery & q : queries){
        for(i = prevL, j = min(prevR, q.l - 1); i <= j; ++i){
            //remove from the left
            current -= A[i];
        }
        for(i = prevL - 1; i >= q.l; --i){
            //add to the left
            current += A[i];
        }
        for(i = max(prevR + 1, q.l); i <= q.r; ++i){
            //add to the right
            current += A[i];
        }
        for(i = prevR; i >= q.r + 1; --i){
            //remove from the right
            current -= A[i];
        }
        prevL = q.l, prevR = q.r;
        ans[q.index] = current;
    }
    return ans;
}

```

```
};
```

## 9.4. AVL Tree

```

template<typename T>
struct AVLNode{
    AVLNode<T> *left, *right;
    short int height;
    int size;
    T value;

    AVLNode(T value = 0): left(NULL), right(NULL), value(value),
        height(1), size(1){}

    inline short int balance(){
        return (right ? right->height : 0) - (left ? left->height :
            0);
    }

    AVLNode *maxLeftChild(){
        AVLNode *ret = this;
        while(ret->left) ret = ret->left;
        return ret;
    }
};

template<typename T>
struct AVLTree{
    AVLNode<T> *root;

    AVLTree(): root(NULL){}

    inline int nodeSize(AVLNode<T> *& pos){return pos ? pos->size :
        0;}

    inline int nodeHeight(AVLNode<T> *& pos){return pos ?
        pos->height : 0;}

    inline void update(AVLNode<T> *& pos){
        if(!pos) return;
        pos->height = 1 + max(nodeHeight(pos->left),
            nodeHeight(pos->right));
        pos->size = 1 + nodeSize(pos->left) + nodeSize(pos->right);
    }
};

```

```

}

int size(){return nodeSize(root);}

void leftRotate(AVLNode<T> *& x){
    AVLNode<T> *y = x->right, *t = y->left;
    y->left = x, x->right = t;
    update(x), update(y);
    x = y;
}

void rightRotate(AVLNode<T> *& y){
    AVLNode<T> *x = y->left, *t = x->right;
    x->right = y, y->left = t;
    update(y), update(x);
    y = x;
}

void updateBalance(AVLNode<T> *& pos){
    if(!pos) return;
    short int bal = pos->balance();
    if(bal > 1){
        if(pos->right->balance() < 0) rightRotate(pos->right);
        leftRotate(pos);
    }else if(bal < -1){
        if(pos->left->balance() > 0) leftRotate(pos->left);
        rightRotate(pos);
    }
}

void insert(AVLNode<T> *&pos, T & value){
    if(pos){
        value < pos->value ? insert(pos->left, value) :
        ↪ insert(pos->right, value);
        update(pos), updateBalance(pos);
    }else{
        pos = new AVLNode<T>(value);
    }
}

AVLNode<T> *search(T & value){
    AVLNode<T> *pos = root;
    while(pos){
        if(value == pos->value) break;

```

```

        pos = (value < pos->value ? pos->left : pos->right);
    }
    return pos;
}

void erase(AVLNode<T> *&pos, T & value){
    if(!pos) return;
    if(value < pos->value) erase(pos->left, value);
    else if(value > pos->value) erase(pos->right, value);
    else{
        if(!pos->left) pos = pos->right;
        else if(!pos->right) pos = pos->left;
        else{
            pos->value = pos->right->maxLeftChild()->value;
            erase(pos->right, pos->value);
        }
    }
    update(pos), updateBalance(pos);
}

void insert(T value){insert(root, value);}

void erase(T value){erase(root, value);}

void updateVal(T old, T New){
    if(search(old))
        erase(old), insert(New);
}

T kth(int i){
    assert(0 <= i && i < nodeSize(root));
    AVLNode<T> *pos = root;
    while(i != nodeSize(pos->left)){
        if(i < nodeSize(pos->left)){
            pos = pos->left;
        }else{
            i -= nodeSize(pos->left) + 1;
            pos = pos->right;
        }
    }
    return pos->value;
}

int lessThan(T & x){

```

```

int ans = 0;
AVLNode<T> *pos = root;
while(pos){
    if(x > pos->value){
        ans += nodeSize(pos->left) + 1;
        pos = pos->right;
    }else{
        pos = pos->left;
    }
}
return ans;
}

int lessThanOrEqual(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
    while(pos){
        if(x < pos->value){
            pos = pos->left;
        }else{
            ans += nodeSize(pos->left) + 1;
            pos = pos->right;
        }
    }
    return ans;
}

int greaterThan(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;
    while(pos){
        if(x < pos->value){
            ans += nodeSize(pos->right) + 1;
            pos = pos->left;
        }else{
            pos = pos->right;
        }
    }
    return ans;
}

int greaterThanOrEqual(T & x){
    int ans = 0;
    AVLNode<T> *pos = root;

```

```

    while(pos){
        if(x > pos->value){
            pos = pos->right;
        }else{
            ans += nodeSize(pos->right) + 1;
            pos = pos->left;
        }
    }
    return ans;
}

int equalTo(T & x){
    return lessThanOrEqual(x) - lessThan(x);
}

void build(AVLNode<T> *& pos, vector<T> & arr, int i, int j){
    if(i > j) return;
    int m = i + ((j - i) >> 1);
    pos = new AVLNode<T>(arr[m]);
    build(pos->left, arr, i, m - 1);
    build(pos->right, arr, m + 1, j);
    update(pos);
}

void build(vector<T> & arr){
    build(root, arr, 0, (int)arr.size() - 1);
}

void output(AVLNode<T> *pos, vector<T> & arr, int & i){
    if(pos){
        output(pos->left, arr, i);
        arr[++i] = pos->value;
        output(pos->right, arr, i);
    }
}

void output(vector<T> & arr){
    int i = -1;
    output(root, arr, i);
}
};

```



## 9.5. Treap

```

template<typename T>
struct TreapNode{
    TreapNode<T> *left, *right;
    T value;
    int key, size;

    //fields for queries
    bool rev;
    T sum, add;

    TreapNode(T value = 0): value(value), key(rand()), size(1),
        ↪ left(NULL), right(NULL), sum(value), add(0), rev(false){}
};

template<typename T>
struct Treap{
    TreapNode<T> *root;

    Treap(): root(NULL) {}

    inline int nodeSize(TreapNode<T>* t){return t ? t->size: 0;}

    inline T nodeSum(TreapNode<T>* t){return t ? t->sum : 0;}

    inline void update(TreapNode<T>* &t){
        if(!t) return;
        t->size = 1 + nodeSize(t->left) + nodeSize(t->right);
        t->sum = t->value; //reset node fields
        push(t->left), push(t->right); //push changes to child nodes
        t->sum = t->value + nodeSum(t->left) + nodeSum(t->right);
        ↪ //combine(left,t,t), combine(t,right,t)
    }

    int size(){return nodeSize(root);}

    void merge(TreapNode<T>* &t, TreapNode<T>* t1, TreapNode<T>*
        ↪ t2){
        if(!t1) t = t2;
        else if(!t2) t = t1;
        else if(t1->key > t2->key)
            merge(t1->right, t1->right, t2), t = t1;
        else

```

```

            merge(t2->left, t1, t2->left), t = t2;
        update(t);
    }

    void split(TreapNode<T>* t, T & x, TreapNode<T>* &t1,
        ↪ TreapNode<T>* &t2){
        if(!t)
            return void(t1 = t2 = NULL);
        if(x < t->value)
            split(t->left, x, t1, t->left), t2 = t;
        else
            split(t->right, x, t->right, t2), t1 = t;
        update(t);
    }

    void insert(TreapNode<T>* &t, TreapNode<T>* x){
        if(!t) t = x;
        else if(x->key > t->key)
            split(t, x->value, x->left, x->right), t = x;
        else
            insert(x->value < t->value ? t->left : t->right, x);
        update(t);
    }

    TreapNode<T>* search(T & x){
        TreapNode<T> *t = root;
        while(t){
            if(x == t->value) break;
            t = (x < t->value ? t->left : t->right);
        }
        return t;
    }

    void erase(TreapNode<T>* &t, T & x){
        if(!t) return;
        if(t->value == x)
            merge(t, t->left, t->right);
        else
            erase(x < t->value ? t->left : t->right, x);
        update(t);
    }

    void insert(T & x){insert(root, new TreapNode<T>(x));}

```

```

void erase(T & x){erase(root, x);}

void updateVal(T & old, T & New){
    if(search(old))
        erase(old), insert(New);
}

T kth(int i){
    assert(0 <= i && i < nodeSize(root));
    TreapNode<T> *t = root;
    while(i != nodeSize(t->left)){
        if(i < nodeSize(t->left)){
            t = t->left;
        }else{
            i -= nodeSize(t->left) + 1;
            t = t->right;
        }
    }
    return t->value;
}

int lessThan(T & x){
    int ans = 0;
    TreapNode<T> *t = root;
    while(t){
        if(x > t->value){
            ans += nodeSize(t->left) + 1;
            t = t->right;
        }else{
            t = t->left;
        }
    }
    return ans;
}

//OPERATIONS FOR IMPLICIT TREAP
inline void push(TreapNode<T>* t){
    if(!t) return;
    //add in range example
    if(t->add){
        t->value += t->add;
        t->sum += t->add * nodeSize(t);
        if(t->left) t->left->add += t->add;
        if(t->right) t->right->add += t->add;

        t->add = 0;
    }
    //reverse range example
    if(t->rev){
        swap(t->left, t->right);
        if(t->left) t->left->rev ^= true;
        if(t->right) t->right->rev ^= true;
        t->rev = false;
    }
}

void split2(TreapNode<T>* t, int i, TreapNode<T>* &t1,
    ↪ TreapNode<T>* &t2){
    if(!t)
        return void(t1 = t2 = NULL);
    push(t);
    int curr = nodeSize(t->left);
    if(i <= curr)
        split2(t->left, i, t1, t->left), t2 = t;
    else
        split2(t->right, i - curr - 1, t->right, t2), t1 = t;
    update(t);
}

inline int aleatorio(){
    return (rand() << 15) + rand();
}

void merge2(TreapNode<T>* &t, TreapNode<T>* t1, TreapNode<T>*
    ↪ t2){
    push(t1), push(t2);
    if(!t1) t = t2;
    else if(!t2) t = t1;
    else if(aleatorio() % (nodeSize(t1) + nodeSize(t2)) <
        ↪ nodeSize(t1))
        merge2(t1->right, t1->right, t2), t = t1;
    else
        merge2(t2->left, t1, t2->left), t = t2;
    update(t);
}

//insert the element "x" at position "i"
void insert_at(T & x, int i){
    if(i > nodeSize(root)) return;

```

```

    TreapNode<T> *t1 = NULL, *t2 = NULL;
    split2(root, i, t1, t2);
    merge2(root, t1, new TreapNode<T>(x));
    merge2(root, root, t2);
}

//delete element at position "i"
void erase_at(int i){
    if(i >= nodeSize(root)) return;
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, i, t1, t2);
    split2(t2, 1, t2, t3);
    merge2(root, t1, t3);
}

void update_at(TreapNode<T>* t, T & x, int i){
    push(t);
    assert(0 <= i && i < nodeSize(t));
    int curr = nodeSize(t->left);
    if(i == curr)
        t->value = x;
    else if(i < curr)
        update_at(t->left, x, i);
    else
        update_at(t->right, x, i - curr - 1);
    update(t);
}

T nth(TreapNode<T>* t, int i){
    push(t);
    assert(0 <= i && i < nodeSize(t));
    int curr = nodeSize(t->left);
    if(i == curr)
        return t->value;
    else if(i < curr)
        return nth(t->left, i);
    else
        return nth(t->right, i - curr - 1);
}

//update value of element at position "i" with "x"
void update_at(T & x, int i){update_at(root, x, i);}

//ith element

```

```

T nth(int i){return nth(root, i);}

//add "val" in [l, r]
void add_update(T & val, int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    t2->add += val;
    merge2(root, t1, t2);
    merge2(root, root, t3);
}

//reverse [l, r]
void reverse_update(int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    t2->rev ^= true;
    merge2(root, t1, t2);
    merge2(root, root, t3);
}

//rotate [l, r] k times to the right
void rotate_update(int k, int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL, *t4 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    k %= nodeSize(t2);
    split2(t2, nodeSize(t2) - k, t2, t4);
    merge2(root, t1, t4);
    merge2(root, root, t2);
    merge2(root, root, t3);
}

//sum query in [l, r]
T sum_query(int l, int r){
    TreapNode<T> *t1 = NULL, *t2 = NULL, *t3 = NULL;
    split2(root, l, t1, t2);
    split2(t2, r - l + 1, t2, t3);
    T ans = nodeSum(t2);
    merge2(root, t1, t2);
    merge2(root, root, t3);
    return ans;
}

```

```

void inorder(TreapNode<T>* t){
    if(!t) return;
    push(t);
    inorder(t->left);
    cout << t->value << " ";
    inorder(t->right);
}

void inorder(){inorder(root);}
};

```

## 9.6. Sparse table

### 9.6.1. Normal

```

template<typename T>
struct SparseTable{
    vector<vector<T>> ST;
    vector<int> logs;
    int K, N;

    SparseTable(vector<T> & arr){
        N = arr.size();
        K = log2(N) + 2;
        ST.assign(K + 1, vector<T>(N));
        logs.assign(N + 1, 0);
        for(int i = 2; i <= N; ++i)
            logs[i] = logs[i >> 1] + 1;
        for(int i = 0; i < N; ++i)
            ST[0][i] = arr[i];
        for(int j = 1; j <= K; ++j)
            for(int i = 0; i + (1 << j) <= N; ++i)
                ST[j][i] = min(ST[j - 1][i], ST[j - 1][i + (1 << (j - 1))]); //put the function accordingly
    }

    T sum(int l, int r){ //non-idempotent functions
        T ans = 0;
        for(int j = K; j >= 0; --j){
            if((1 << j) <= r - l + 1){
                ans += ST[j][l];
                l += 1 << j;
            }
        }
        return ans;
    }
};

```

```

    }
}

return ans;
}

T minimal(int l, int r){ //idempotent functions
    int j = logs[r - l + 1];
    return min(ST[j][l], ST[j][r - (1 << j) + 1]);
}
};

```

### 9.6.2. Disjoint

```

//build on O(n log n), queries in O(1) for any operation
template<typename T>
struct DisjointSparseTable{
    vector<vector<T>> left, right;
    int K, N;

    DisjointSparseTable(vector<T> & arr){
        N = arr.size();
        K = log2(N) + 2;
        left.assign(K + 1, vector<T>(N));
        right.assign(K + 1, vector<T>(N));
        for(int j = 0; (1 << j) <= N; ++j){
            int mask = (1 << j) - 1;
            T acum = 0; //neutral element of your operation
            for(int i = 0; i < N; ++i){
                acum += arr[i]; //your operation
                left[j][i] = acum;
                if((i & mask) == mask) acum = 0; //neutral element of your
                ↪ operation
            }
            acum = 0; //neutral element of your operation
            for(int i = N-1; i >= 0; --i){
                acum += arr[i]; //your operation
                right[j][i] = acum;
                if((i & mask) == 0) acum = 0; //neutral element of your
                ↪ operation
            }
        }
    }
};

```

```

T query(int l, int r){
    if(l == r) return left[0][l];
    int i = 31 - __builtin_clz(l^r);
    return left[i][r] + right[i][l]; //your operation
}
};

```

## 9.7. Wavelet Tree

```

struct WaveletTree{
    int lo, hi;
    WaveletTree *left, *right;
    vector<int> freq;
    vector<int> pref; //just use this if you want sums

    //queries indexed in base 1, complexity for all queries:
    ↪ O(log(max_element))
    //build from [from, to) with non-negative values in range [x, y]
    //you can use vector iterators or array pointers
    WaveletTree(vector<int>::iterator from, vector<int>::iterator
    ↪ to, int x, int y): lo(x), hi(y){
        if(from >= to) return;
        int m = (lo + hi) / 2;
        auto f = [m](int x){return x <= m;};
        freq.reserve(to - from + 1);
        freq.push_back(0);
        pref.reserve(to - from + 1);
        pref.push_back(0);
        for(auto it = from; it != to; ++it){
            freq.push_back(freq.back() + f(*it));
            pref.push_back(pref.back() + *it);
        }
        if(hi != lo){
            auto pivot = stable_partition(from, to, f);
            left = new WaveletTree(from, pivot, lo, m);
            right = new WaveletTree(pivot, to, m + 1, hi);
        }
    }

    //kth element in [l, r]
    int kth(int l, int r, int k){
        if(l > r) return 0;
        if(lo == hi) return lo;

```

```

        int lb = freq[l - 1], rb = freq[r];
        int inLeft = rb - lb;
        if(k <= inLeft) return left->kth(lb + 1, rb, k);
        else return right->kth(l - lb, r - rb, k - inLeft);
    }

    //number of elements less than or equal to k in [l, r]
    int lessThanOrEqual(int l, int r, int k){
        if(l > r || k < lo) return 0;
        if(hi <= k) return r - l + 1;
        int lb = freq[l - 1], rb = freq[r];
        return left->lessThanOrEqual(lb + 1, rb, k) +
            right->lessThanOrEqual(l - lb, r - rb, k);
    }

    //number of elements equal to k in [l, r]
    int equalTo(int l, int r, int k){
        if(l > r || k < lo || k > hi) return 0;
        if(lo == hi) return r - l + 1;
        int lb = freq[l - 1], rb = freq[r];
        int m = (lo + hi) / 2;
        if(k <= m) return left->equalTo(lb + 1, rb, k);
        else return right->equalTo(l - lb, r - rb, k);
    }

    //sum of elements less than or equal to k in [l, r]
    int sum(int l, int r, int k){
        if(l > r || k < lo) return 0;
        if(hi <= k) return pref[r] - pref[l - 1];
        int lb = freq[l - 1], rb = freq[r];
        return left->sum(lb + 1, rb, k) + right->sum(l - lb, r - rb,
            ↪ k);
    }
};

```

## 9.8. Ordered Set C++

```

#include <ext/pb_ds/assoc_container.hpp>
#include <ext/pb_ds/tree_policy.hpp>
using namespace __gnu_pbds;

template<typename T>

```

```

using ordered_set = tree<T, null_type, less<T>, rb_tree_tag,
↪ tree_order_statistics_node_update>;

int main(){
    int t, n, m;
    ordered_set<int> conj;
    while(cin >> t && t != -1){
        cin >> n;
        if(t == 0){ //insert
            conj.insert(n);
        }else if(t == 1){ //search
            if(conj.find(n) != conj.end()) cout << "Found\n";
            else cout << "Not found\n";
        }else if(t == 2){ //delete
            conj.erase(n);
        }else if(t == 3){ //update
            cin >> m;
            if(conj.find(n) != conj.end()){
                conj.erase(n);
                conj.insert(m);
            }
        }else if(t == 4){ //lower bound
            cout << conj.order_of_key(n) << "\n";
        }else if(t == 5){ //get nth element
            auto pos = conj.find_by_order(n);
            if(pos != conj.end()) cout << *pos << "\n";
            else cout << "-1\n";
        }
    }
    return 0;
}

```

## 9.9. HLD

```

struct HeavyLight{
    int n;
    vector<vector<int>> adj;
    vector<int> parent, level, size, heavy, head, pos, ipos;
    int cur_pos;
    SegmentTree<int> * st;

    HeavyLight(int n, SegmentTree<int> * st): n(n), st(st){
        adj.resize(n), ipos.resize(n);
    }
}

```

```

parent.resize(n), level.resize(n), size.resize(n);
heavy.resize(n, -1), head.resize(n), pos.resize(n);
}

void dfs(int u){
    size[u] = 1;
    int mx = 0;
    for(int v : adj[u]){
        if(v != parent[u]){
            parent[v] = u;
            level[v] = level[u] + 1;
            dfs(v);
            if(size[v] > mx){
                mx = size[v];
                heavy[u] = v;
            }
            size[u] += size[v];
        }
    }
}

void build(int u, int h){
    head[u] = h;
    pos[u] = cur_pos;
    ipos[cur_pos++] = u;
    if(heavy[u] != -1) build(heavy[u], h);
    for(int v : adj[u])
        if(v != parent[u] && v != heavy[u])
            build(v, v);
}

void init(int root = 0){
    cur_pos = 0;
    dfs(root);
    build(root, root);
}

int query(int a, int b){
    int mx = 0;
    while(head[a] != head[b]){
        if(level[head[a]] > level[head[b]]) swap(a, b);
        mx = max(mx, st->query(pos[head[b]], pos[b]));
        b = parent[head[b]];
    }
}

```

```

if(level[a] > level[b]) swap(a, b);
// if(pos[a] + 1 <= pos[b]) for values in edges
mx = max(mx, st->query(pos[a], pos[b]));
//LCA at a
return mx;
}

int kth_ancestor(int u, int k){
    while(pos[u] - pos[head[u]] < k){
        k -= pos[u] - pos[head[u]] + 1;
        u = parent[head[u]];
    }
    return ipos[pos[u] - k];
}

```

## 9.10. Splay Tree

## 9.11. Red Black Tree

# 10. Cadenas

## 10.1. Trie

```

struct Node{
    bool isWord = false;
    map<char, Node*> letters;
};

struct Trie{
    Node* root;

    Trie(){
        root = new Node();
    }

    inline bool exists(Node * actual, const char & c){
        return actual->letters.find(c) != actual->letters.end();
    }

    void InsertWord(const string& word){
        Node* current = root;
        for(auto & c : word){
            if(!exists(current, c))
                current->letters[c] = new Node();
            current = current->letters[c];
        }
        current->isWord = true;
    }

    bool FindWord(const string& word){
        Node* current = root;
        for(auto & c : word){
            if(!exists(current, c))
                return false;
            current = current->letters[c];
        }
        return current->isWord;
    }

    void printRec(Node * actual, string acum){
        if(actual->isWord){
            cout << acum << "\n";

```

```

    }
    for(auto & next : actual->letters)
        printRec(next.second, acum + next.first);
}

void printWords(const string & prefix){
    Node * actual = root;
    for(auto & c : prefix){
        if(!exists(actual, c)) return;
        actual = actual->letters[c];
    }
    printRec(actual, prefix);
}
};

```

## 10.2. KMP

```

struct kmp{
    vector<int> aux;
    string pattern;

    kmp(string pattern){
        this->pattern = pattern;
        aux.resize(pattern.size());
        int i = 1, j = 0;
        while(i < pattern.size()){
            if(pattern[i] == pattern[j])
                aux[i++] = ++j;
            else{
                if(j == 0) aux[i++] = 0;
                else j = aux[j - 1];
            }
        }
    }

    vector<int> search(string & text){
        vector<int> ans;
        int i = 0, j = 0;
        while(i < text.size() && j < pattern.size()){
            if(text[i] == pattern[j]){
                ++i, ++j;
                if(j == pattern.size()){
                    ans.push_back(i - j);

```

```

                j = aux[j - 1];
            }
        }else{
            if(j == 0) ++i;
            else j = aux[j - 1];
        }
    }
    return ans;
}
};

```

## 10.3. Aho-Corasick

```

const int M = 26;
struct node{
    vector<int> child;
    int p = -1;
    char c = 0;
    int suffixLink = -1, endLink = -1;
    int id = -1;

    node(int p = -1, char c = 0) : p(p), c(c){
        child.resize(M, -1);
    }
};

struct AhoCorasick{
    vector<node> t;
    vector<int> lenghts;
    int wordCount = 0;

    AhoCorasick(){
        t.emplace_back();
    }

    void add(const string & s){
        int u = 0;
        for(char c : s){
            if(t[u].child[c-'a'] == -1){
                t[u].child[c-'a'] = t.size();
                t.emplace_back(u, c);
            }
            u = t[u].child[c-'a'];

```



```

    }
    t[u].id = wordCount++;
    lengths.push_back(s.size());
}

void link(int u){
    if(u == 0){
        t[u].suffixLink = 0;
        t[u].endLink = 0;
        return;
    }
    if(t[u].p == 0){
        t[u].suffixLink = 0;
        if(t[u].id != -1) t[u].endLink = u;
        else t[u].endLink = t[t[u].suffixLink].endLink;
        return;
    }
    int v = t[t[u].p].suffixLink;
    char c = t[u].c;
    while(true){
        if(t[v].child[c-'a'] != -1){
            t[u].suffixLink = t[v].child[c-'a'];
            break;
        }
        if(v == 0){
            t[u].suffixLink = 0;
            break;
        }
        v = t[v].suffixLink;
    }
    if(t[u].id != -1) t[u].endLink = u;
    else t[u].endLink = t[t[u].suffixLink].endLink;
}

void build(){
    queue<int> Q;
    Q.push(0);
    while(!Q.empty()){
        int u = Q.front(); Q.pop();
        link(u);
        for(int v = 0; v < M; ++v)
            if(t[u].child[v] != -1)
                Q.push(t[u].child[v]);
    }
}

```

```

    }

int match(const string & text){
    int u = 0;
    int ans = 0;
    for(int j = 0; j < text.size(); ++j){
        int i = text[j] - 'a';
        while(true){
            if(t[u].child[i] != -1){
                u = t[u].child[i];
                break;
            }
            if(u == 0) break;
            u = t[u].suffixLink;
        }
        int v = u;
        while(true){
            v = t[v].endLink;
            if(v == 0) break;
            ++ans;
            int idx = j + 1 - lengths[t[v].id];
            cout << "Found word #" << t[v].id << " at position " <<
                << idx << "\n";
            v = t[v].suffixLink;
        }
    }
    return ans;
}
};

```

## 10.4. Suffix Automaton

```

struct state{
    int len, link;
    vector<int> child;
    state(int len = 0, int link = -1): len(len), link(link),
        << child(M, -1){}
    state(int len, int link, const vector<int> & child): len(len),
        << link(link), child(child){}
};

struct SuffixAutomaton{
    vector<state> st;

```

```

int last = 0;

SuffixAutomaton(){
    st.emplace_back();
}

void extend(char c){
    int curr = st.size();
    st.emplace_back(st[last].len + 1);
    int p = last;
    while(p != -1 && st[p].child[c-'A'] == -1){
        st[p].child[c-'A'] = curr;
        p = st[p].link;
    }
    if(p == -1){
        st[curr].link = 0;
    }else{
        int q = st[p].child[c-'A'];
        if(st[p].len + 1 == st[q].len){
            st[curr].link = q;
        }else{
            int clone = st.size();
            st.emplace_back(st[p].len + 1, st[q].link, st[q].child);
            while(p != -1 && st[p].child[c-'A'] == q){
                st[p].child[c-'A'] = clone;
                p = st[p].link;
            }
            st[q].link = st[curr].link = clone;
        }
    }
    last = curr;
}
};

        ++z[i];
        if(i + z[i] - 1 > r)
            l = i, r = i + z[i] - 1;
    }
    return z;
}

```

## 10.5. Función Z

```

vector<int> z_function(const string & s){
    int n = s.size();
    vector<int> z(n);
    for(int i = 1, l = 0, r = 0; i < n; ++i){
        if(i <= r)
            z[i] = min(r - i + 1, z[i - l]);
        while(i + z[i] < n && s[z[i]] == s[i + z[i]])

```

## 11. Varios

### 11.1. Lectura y escritura de \_\_int128

```
//cout for __int128
ostream &operator<<(ostream &os, const __int128 & value){
    char buffer[64];
    char *pos = end(buffer) - 1;
    *pos = '\\0';
    __int128 tmp = value < 0 ? -value : value;
    do{
        --pos;
        *pos = tmp % 10 + '0';
        tmp /= 10;
    }while(tmp != 0);
    if(value < 0){
        --pos;
        *pos = '-';
    }
    return os << pos;
}

//cin for __int128
istream &operator>>(istream &is, __int128 & value){
    char buffer[64];
    is >> buffer;
    char *pos = begin(buffer);
    int sgn = 1;
    value = 0;
    if(*pos == '-'){
        sgn = -1;
        ++pos;
    }else if(*pos == '+'){
        ++pos;
    }
    while(*pos != '\\0'){
        value = (value << 3) + (value << 1) + (*pos - '0');
        ++pos;
    }
    value *= sgn;
    return is;
}
```

### 11.2. Longest Common Subsequence (LCS)

```
int lcs(string & a, string & b){
    int m = a.size(), n = b.size();
    vector<vector<int>> aux(m + 1, vector<int>(n + 1));
    for(int i = 1; i <= m; ++i){
        for(int j = 1; j <= n; ++j){
            if(a[i - 1] == b[j - 1])
                aux[i][j] = 1 + aux[i - 1][j - 1];
            else
                aux[i][j] = max(aux[i - 1][j], aux[i][j - 1]);
        }
    }
    return aux[m][n];
}
```

### 11.3. Longest Increasing Subsequence (LIS)

```
int lis(vector<int> & arr){
    if(arr.size() == 0) return 0;
    vector<int> aux(arr.size());
    int ans = 1;
    aux[0] = arr[0];
    for(int i = 1; i < arr.size(); ++i){
        if(arr[i] < aux[0])
            aux[0] = arr[i];
        else if(arr[i] > aux[ans - 1])
            aux[ans++] = arr[i];
        else
            aux[lower_bound(aux.begin(), aux.begin() + ans, arr[i]) -
                aux.begin()] = arr[i];
    }
    return ans;
}
```

### 11.4. Levenshtein Distance

```
int LevenshteinDistance(string & a, string & b){
    int m = a.size(), n = b.size();
    vector<vector<int>> aux(m + 1, vector<int>(n + 1));
    for(int i = 1; i <= m; ++i)
        aux[i][0] = i;
```

```

for(int j = 1; j <= n; ++j)
    aux[0][j] = j;
for(int j = 1; j <= n; ++j)
    for(int i = 1; i <= m; ++i)
        aux[i][j] = min({aux[i-1][j] + 1, aux[i][j-1] + 1,
        ↪ aux[i-1][j-1] + (a[i-1] != b[j-1])});
return aux[m][n];
}

```

## 11.5. Día de la semana

```

//0:saturday, 1:sunday, ..., 6:friday
int dayOfWeek(int d, int m, lli y){
    if(m == 1 || m == 2){
        m += 12;
        --y;
    }
    int k = y % 100;
    lli j = y / 100;
    return (d + 13*(m+1)/5 + k + k/4 + j/4 + 5*j) % 7;
}

```

## 11.6. 2SAT

```

struct satisfiability_twosat{
    int n;
    vector<vector<int>> imp;

    satisfiability_twosat(int n) : n(n), imp(2 * n) {}

    void add_edge(int u, int v){imp[u].push_back(v);}

    int neg(int u){return (n << 1) - u - 1;}

    void implication(int u, int v){
        add_edge(u, v);
        add_edge(neg(v), neg(u));
    }

    vector<bool> solve(){
        int size = 2 * n;
        vector<int> S, B, I(size);
    }
}

```

```

function<void(int)> dfs = [&](int u){
    B.push_back(I[u] = S.size());
    S.push_back(u);

    for(int v : imp[u])
        if(!I[v]) dfs(v);
        else while (I[v] < B.back()) B.pop_back();

    if(I[u] == B.back())
        for(B.pop_back(), ++size; I[u] < S.size(); S.pop_back())
            I[S.back()] = size;
};

for(int u = 0; u < 2 * n; ++u)
    if(!I[u]) dfs(u);

vector<bool> values(n);

for(int u = 0; u < n; ++u)
    if(I[u] == I[neg(u)]) return {};
    else values[u] = I[u] < I[neg(u)];

return values;
}
};

```

## 11.7. Código Gray

```

//gray code
int gray(int n){
    return n ^ (n >> 1);
}

//inverse gray code
int inv_gray(int g){
    int n = 0;
    while(g){
        n ^= g;
        g >>= 1;
    }
    return n;
}

```

## 11.8. Contar número de unos en binario en un rango

```
//count the number of 1's in the i-th bit of all
//representations in binary of numbers in [1,n]
lli count(lli n, int i){
    if(n <= 0) return 0ll;
    lli ans = ((n + 1) >> (i + 1)) << i;
    ans += max(((n + 1) & ((1ll << (i + 1)) - 1)) - (1ll << i),
        ↪ 0ll);
    return ans;
}
```

## 11.9. Números aleatorios en C++11

```
//Random number generation in C++11
mt19937_64
↪ rng(chrono::steady_clock::now().time_since_epoch().count());

//Generate a random integer in [a, b], you can also use long long
↪ int
int aleatorio_int(int a, int b){
    uniform_int_distribution<int> dist(a, b);
    return dist(rng);
}

//Generate a random double in [a, b], you can also use long double
double aleatorio_double(double a, double b){
    uniform_real_distribution<double> dist(a, b);
    return dist(rng);
}
```

## 11.10. Lower and upper bound

```
//Let S be an ordered vector. Returns the cardinality of the set
↪ {x∈S : a <= x <= b}
template<typename T>
int count(const vector<T> & S, T a, T b){
    return upper_bound(S.begin(), S.end(), b) -
        ↪ lower_bound(S.begin(), S.end(), a);
}
```

## 12. Fórmulas y notas

### 12.1. Números de Stirling del primer tipo

$\begin{bmatrix} n \\ k \end{bmatrix}$  representa el número de permutaciones de  $n$  elementos en exactamente  $k$  ciclos disjuntos.

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} &= 1 \\ \begin{bmatrix} 0 \\ n \end{bmatrix} &= \begin{bmatrix} n \\ 0 \end{bmatrix} = 0, & \quad n > 0 \\ \begin{bmatrix} n \\ k \end{bmatrix} &= (n-1) \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, & \quad k > 0 \\ \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} &= n! \\ \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix} x^k &= \prod_{k=0}^{n-1} (x+k) \end{aligned}$$

### 12.2. Números de Stirling del segundo tipo

$\{n \atop k\}$  representa el número de formas de particionar un conjunto de  $n$  objetos distinguibles en  $k$  subconjuntos no vacíos.

$$\begin{aligned} \{0 \atop 0\} &= 1 \\ \{0 \atop n\} &= \{n \atop 0\} = 0, & \quad n > 0 \\ \{n \atop k\} &= k \{n-1 \atop k\} + \{n-1 \atop k-1\}, & \quad k > 0 \\ &= \sum_{j=0}^k \frac{j^n}{j!} \cdot \frac{(-1)^{k-j}}{(k-j)!} \end{aligned}$$

### 12.3. Números de Euler

$\langle n \atop k \rangle$  representa el número de permutaciones de 1 a  $n$  en donde exactamente  $k$  números son mayores que el número anterior, es decir, cuántas

permutaciones tienen  $k$  “ascensos”.

$$\begin{aligned}\langle 1 \rangle_0 &= 1 \\ \langle n \rangle_k &= (n-k) \langle n-1 \rangle_{k-1} + (k+1) \langle n-1 \rangle_k, \quad n \geq 2 \\ &= \sum_{j=0}^k (-1)^j \binom{n+1}{j} (k+1-j)^n \\ \sum_{k=0}^{n-1} \langle n \rangle_k &= n!\end{aligned}$$

#### 12.4. Números de Catalan

$$\begin{aligned}C_0 &= 1 \\ C_n &= \frac{1}{n+1} \binom{2n}{n} = \sum_{j=0}^{n-1} C_j C_{n-1-j} \\ \sum_{n=0}^{\infty} C_n x^n &= \frac{1 - \sqrt{1-4x}}{2x}\end{aligned}$$

#### 12.5. Números de Bell

$B_n$  representa el número de formas de particionar un conjunto de  $n$  elementos.

$$\begin{aligned}B_n &= \sum_{k=0}^n \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k \\ \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n &= e^{e^x - 1}\end{aligned}$$

#### 12.6. Números de Bernoulli

$$\begin{aligned}B_0^+ &= 1 \\ B_n^+ &= 1 - \sum_{k=0}^{n-1} \binom{n}{k} \frac{B_k^+}{n-k+1} \\ \sum_{n=0}^{\infty} \frac{B_n^+ x^n}{n!} &= \frac{x}{1-e^{-x}} = \frac{1}{\frac{1}{1!} - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \dots}\end{aligned}$$

#### 12.7. Fórmula de Faulhaber

$$S_p(n) = \sum_{k=1}^n k^p = \frac{1}{p+1} \sum_{k=0}^p \binom{p+1}{k} B_k^+ n^{p+1-k}$$

#### 12.8. Función Beta

$$\begin{aligned}B(x, y) &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2 \int_0^{\pi/2} \sin^{2x-1}(\theta) \cos^{2y-1}(\theta) d\theta \\ &= \int_0^1 t^{x-1} (1-t)^{y-1} dt = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt\end{aligned}$$

#### 12.9. Función zeta de Riemann

La siguiente fórmula converge rápido para valores pequeños de  $n$  ( $n \approx 20$ ):

$$\begin{aligned}\zeta(s) &\approx \frac{1}{d_0(1-2^{1-s})} \sum_{k=1}^n \frac{(-1)^{k-1} d_k}{k^s} \\ d_k &= \sum_{j=k}^n \frac{4^j}{n+j} \binom{n+j}{2j}\end{aligned}$$

### 12.10. Funciones generadoras

$$\sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k \right) x^n = \frac{1}{1-x} \sum_{n=0}^{\infty} a_n x^n$$

$$\sum_{n=0}^{\infty} \binom{n+k-1}{k-1} x^n = \frac{1}{(1-x)^k}$$

$$\sum_{n=0}^{\infty} p_n x^n = \frac{1}{\prod_{k=1}^{\infty} (1-x^k)} = \frac{1}{\sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{1}{2}n(3n+1)}}$$

$$\sum_{p=0}^{\infty} \frac{S_p(n)}{p!} x^p = \frac{e^{x(n+1)} - e^x}{e^x - 1}$$

$$\sum_{n=0}^{\infty} n^k x^n = \frac{\sum_{i=0}^{k-1} \langle k \rangle_i x^{i+1}}{(1-x)^{k+1}}, \quad k \geq 1$$

Sean  $a_1, a_2, \dots, a_n$  números complejos. Sean  $p_m = \sum_{i=1}^n a_i^m$  y  $s_m$  el  $m$ -ésimo polinomio elemental simétrico de  $a_1, a_2, \dots, a_n$ . Entonces se cumple que  $xS'(x) + P(x)S(x) = 0$ , donde  $P(x) = \sum_{m=1}^{\infty} p_m x^m$  y  $S(x) = \prod_{i=1}^n (1 - a_i x) = \sum_{m=0}^n (-1)^m s_m x^m$ .

### 12.11. Números armónicos

$$H_n = \sum_{k=1}^n \frac{1}{k} \approx \ln(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2}$$

$$\gamma \approx 0.577215664901532860606512$$

### 12.12. Aproximación de Stirling

$$\ln(n!) \approx n \ln(n) - n + \frac{1}{2} \ln(2\pi n)$$

$$\# \text{ de dígitos de } n! = 1 + \left\lfloor n \log \left( \frac{n}{e} \right) + \frac{1}{2} \log(2\pi n) \right\rfloor \quad (n \geq 30)$$

### 12.13. Ternas pitagóricas

- Una terna de enteros positivos  $(a, b, c)$  es pitagórica si  $a^2 + b^2 = c^2$ . Además es primitiva si  $\gcd(a, b, c) = 1$ .
- Generador de ternas primitivas:

$$a = m^2 - n^2$$

$$b = 2mn$$

$$c = m^2 + n^2$$

donde  $n \geq 1$ ,  $m > n$ ,  $\gcd(m, n) = 1$  y  $m, n$  tienen distinta paridad.

- Árbol de ternas pitagóricas primitivas: al multiplicar cualquiera de estas matrices:

$$\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}, \quad \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

por una terna primitiva  $\mathbf{v}^T$ , obtenemos otra terna primitiva diferente. En particular, si empezamos con  $\mathbf{v} = (3, 4, 5)$ , podremos generar todas las ternas primitivas.

### 12.14. Árbol de Stern–Brocot

Todos los racionales positivos se pueden representar como un árbol binario de búsqueda completo infinito con raíz  $\frac{1}{1}$ .

- Dado un racional  $q = [a_0; a_1, a_2, \dots, a_k]$  donde  $a_k \neq 1$ , sus hijos serán  $[a_0; a_1, a_2, \dots, a_k + 1]$  y  $[a_0; a_1, a_2, \dots, a_k - 1, 2]$ , y su padre será  $[a_0; a_1, a_2, \dots, a_k - 1]$ .

- Para hallar el camino de la raíz  $\frac{1}{1}$  a un racional  $q$ , se usa búsqueda binaria iniciando con  $L = \frac{0}{1}$  y  $R = \frac{1}{0}$ . Para hallar  $M$  se supone que  $L = \frac{a}{b}$  y  $R = \frac{c}{d}$ , entonces  $M = \frac{a+c}{b+d}$ .

### 12.15. Combinatoria

- Principio de las casillas: al colocar  $n$  objetos en  $k$  lugares hay al menos  $\lceil \frac{n}{k} \rceil$  objetos en un mismo lugar.
- Número de funciones: sean  $A$  y  $B$  conjuntos con  $m = |A|$  y  $n = |B|$ . Sea  $f : A \rightarrow B$ :
  - Si  $m \leq n$ , entonces hay  $m! \binom{n}{m}$  funciones inyectivas  $f$ .
  - Si  $m = n$ , entonces hay  $n!$  funciones biyectivas  $f$ .
  - Si  $m \geq n$ , entonces hay  $n! \left\{ \begin{smallmatrix} m \\ n \end{smallmatrix} \right\}$  funciones suprayectivas  $f$ .
- Barras y estrellas: ¿cuántas soluciones en los enteros no negativos tiene la ecuación  $\sum_{i=1}^k x_i = n$ ? Tiene  $\binom{n+k-1}{k-1}$  soluciones.
- ¿Cuántas soluciones en los enteros positivos tiene la ecuación  $\sum_{i=1}^k x_i = n$ ? Tiene  $\binom{n-1}{k-1}$  soluciones.
- Desordenamientos:  $a_0 = 1$ ,  $a_1 = 0$ ,  $a_n = (n-1)(a_{n-1} + a_{n-2}) = na_{n-1} + (-1)^n$ .
- Sea  $f(x)$  una función. Sea  $g_n(x) = xg'_{n-1}(x)$  con  $g_0(x) = f(x)$ . Entonces  $g_n(x) = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} x^k f^{(k)}(x)$ .
- Supongamos que tenemos  $m+1$  puntos:  $(0, y_0), (1, y_1), \dots, (m, y_m)$ . Entonces el polinomio  $P(x)$  de grado  $m$  que pasa por todos ellos es:

$$P(x) = \left[ \prod_{i=0}^m (x-i) \right] (-1)^m \sum_{i=0}^m \frac{y_i (-1)^i}{(x-i)i!(m-i)!}$$

- Sea  $a_0, a_1, \dots$  una recurrencia lineal homogénea de grado  $d$  dada por  $a_n = \sum_{i=1}^d b_i a_{n-i}$  para  $n \geq d$  con términos iniciales  $a_0, a_1, \dots, a_{d-1}$ . Sean  $A(x)$  y  $B(x)$  las funciones generadoras de las sucesiones  $a_n$  y  $b_n$  respectivamente, entonces se cumple que  $A(x) = \frac{A_0(x)}{1-B(x)}$ , donde  $A_0(x) = \sum_{i=0}^{d-1} \left[ a_i - \sum_{j=0}^{i-1} a_j b_{i-j} \right] x^i$ .
- Si queremos obtener otra recurrencia  $c_n$  tal que  $c_n = a_{kn}$ , las raíces del polinomio característico de  $c_n$  se obtienen al elevar todas las raíces del polinomio característico de  $a_n$  a la  $k$ -ésima potencia; y sus términos iniciales serán  $a_0, a_k, \dots, a_{k(d-1)}$ .

### 12.16. Grafos

- Sea  $d_n$  el número de grafos con  $n$  vértices etiquetados:  $d_n = 2^{\binom{n}{2}}$ .
- Sea  $c_n$  el número de grafos conexos con  $n$  vértices etiquetados. Tenemos la recurrencia:  $c_1 = 1$  y  $d_n = \sum_{k=1}^n \binom{n-1}{k-1} c_k d_{n-k}$ . También se cumple, usando funciones generadoras exponenciales, que  $C(x) = 1 + \ln(D(x))$ .
- Sea  $t_n$  el número de torneos fuertemente conexos en  $n$  nodos etiquetados. Tenemos la recurrencia  $t_1 = 1$  y  $d_n = \sum_{k=1}^n \binom{n}{k} t_k d_{n-k}$ . Usando funciones generadoras exponenciales, tenemos que  $T(x) = 1 - \frac{1}{D(x)}$ .
- Número de spanning trees en un grafo completo con  $n$  vértices etiquetados:  $n^{n-2}$ .
- Número de bosques etiquetados con  $n$  vértices y  $k$  componentes conexas:  $kn^{n-k-1}$ .
- Para un grafo no dirigido simple  $G$  con  $n$  vértices etiquetados de 1 a  $n$ , sea  $Q = D - A$ , donde  $D$  es la matriz diagonal de los grados de



cada nodo de  $G$  y  $A$  es la matriz de adyacencia de  $G$ . Entonces el número de spanning trees de  $G$  es igual a cualquier cofactor de  $Q$ .

- Sea  $G$  un grafo. Se define al polinomio  $P_G(x)$  como el polinomio cromático de  $G$ , en donde  $P_G(k)$  nos dice cuántas  $k$ -coloraciones de los vértices admite  $G$ . Ejemplos comunes:
  - Grafo completo de  $n$  nodos:  $P(x) = x(x-1)(x-2)\dots(x-(n-1))$
  - Grafo vacío de  $n$  nodos:  $P(x) = x^n$
  - Árbol de  $n$  nodos:  $P(x) = x(x-1)^{n-1}$
  - Ciclo de  $n$  nodos:  $P(x) = (x-1)^n + (-1)^n(x-1)$

## 12.17. Teoría de números

$$(f * e)(n) = f(n)$$

$$(\varphi * \mathbf{1})(n) = n$$

$$(\mu * \mathbf{1})(n) = e(n)$$

$$\varphi(n^k) = n^{k-1}\varphi(n)$$

$$\sum_{\substack{k=1 \\ \gcd(k,n)=1}}^n k = \frac{n\varphi(n)}{2}, \quad n \geq 2$$

$$\sum_{k=1}^n \text{lcm}(k, n) = \frac{n}{2} + \frac{n}{2} \sum_{d|n} d\varphi(d) = \frac{n}{2} + \frac{n}{2} \prod_{p^a|n} \frac{p^{2a+1} + 1}{p + 1}$$

$$\sum_{k=1}^n \gcd(k, n) = \sum_{d|n} d\varphi\left(\frac{n}{d}\right) = \prod_{p^a|n} p^{a-1}(1 + (a+1)(p-1))$$

- Lifting the exponent: sea  $p$  un primo,  $x, y$  enteros y  $n$  un entero positivo tal que  $p \mid x - y$  pero  $p \nmid x$  ni  $p \nmid y$ . Entonces:
  - Si  $p$  es impar:  $v_p(x^n - y^n) = v_p(x - y) + v_p(n)$
  - Si  $p = 2$  y  $n$  es par:  $v_p(x^n - y^n) = v_p(x - y) + v_p(n) + v_p(x + y) - 1$
 donde  $v_p(n)$  es el exponente de  $p$  en la factorización en primos de  $n$ .

- Suma de dos cuadrados: sea  $\chi_4(n)$  una función multiplicativa igual a 1 si  $n \equiv 1 \pmod{4}$ ,  $-1$  si  $n \equiv 3 \pmod{4}$  y cero en otro caso. Entonces, el número de soluciones enteras  $(a, b)$  de la ecuación  $a^2 + b^2 = n$  es  $4(\chi_4 * \mathbf{1})(n) = 4 \sum_{d|n} \chi_4(d)$ .

- Teorema de Lucas:

$$\binom{m}{n} \equiv \prod_{i=0}^k \binom{m_i}{k_i} \pmod{p}$$

$$m = \sum_{i=0}^k m_i p^i, \quad n = \sum_{i=0}^k n_i p^i$$

$$0 \leq m_i, n_i < p$$

- Sean  $a, b, c \in \mathbb{Z}$  con  $a \neq 0$  y  $b \neq 0$ . La ecuación  $ax + by = c$  tiene como soluciones:

$$x = \frac{x_0 c - b k}{d}$$

$$y = \frac{y_0 c + a k}{d}$$

para toda  $k \in \mathbb{Z}$  si y solo si  $d|c$ , donde  $ax_0 + by_0 = \gcd(a, b) = d$  (Euclides extendido). Si  $a$  y  $b$  tienen el mismo signo, hay exactamente  $\max\left(\left\lfloor \frac{x_0 c}{|b|} \right\rfloor + \left\lfloor \frac{y_0 c}{|a|} \right\rfloor + 1, 0\right)$  soluciones no negativas. Si tienen el signo distinto, hay infinitas soluciones no negativas.

- Dada una función aritmética  $f$  con  $f(1) \neq 0$ , existe otra función aritmética  $g$  tal que  $(f * g)(n) = e(n)$ , dada por:

$$g(1) = \frac{1}{f(1)}$$

$$g(n) = -\frac{1}{f(1)} \sum_{d|n, d < n} f\left(\frac{n}{d}\right) g(d), \quad n > 1$$

- Sean  $h(n) = \sum_{k=1}^n f\left(\left\lfloor \frac{n}{k} \right\rfloor\right) g(k)$ ,  $G(n) = \sum_{k=1}^n g(k)$  y  $m = \lfloor \sqrt{n} \rfloor$ , entonces:

$$h(n) = \sum_{k=1}^{\lfloor n/m \rfloor} f\left(\left\lfloor \frac{n}{k} \right\rfloor\right) g(k) + \sum_{k=1}^{m-1} \left( G\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right) \right) f(k)$$

- Sean  $F(n) = \sum_{k=1}^n f(k)$ ,  $G(n) = \sum_{k=1}^n g(k)$ ,  $h(n) = (f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$  y  $H(n) = \sum_{k=1}^n h(k)$ , entonces:

$$H(n) = \sum_{k=1}^n f(k)G\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$$

- Sean  $\Phi_p(n) = \sum_{k=1}^n k^p \varphi(k)$  y  $M_p(n) = \sum_{k=1}^n k^p \mu(k)$ . Aplicando lo anterior, podemos calcular  $\Phi_p(n)$  y  $M_p(n)$  con complejidad  $O(n^{2/3})$  si precalculamos con fuerza bruta los primeros  $\lfloor n^{2/3} \rfloor$  valores, y para los demás, usamos las siguientes recurrencias (DP con `map`):

$$\Phi_p(n) = S_{p+1}(n) - \sum_{k=2}^{\lfloor n/m \rfloor} k^p \Phi_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left( S_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - S_p\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right) \right) \Phi_p(k)$$

$$M_p(n) = 1 - \sum_{k=2}^{\lfloor n/m \rfloor} k^p M_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left( S_p\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - S_p\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right) \right) M_p(k)$$

- En general, si queremos hallar  $F(n)$  y existe una función mágica  $g(n)$  tal que  $G(n)$  y  $H(n)$  se puedan calcular en  $O(1)$ , entonces:

$$F(n) = \frac{1}{g(1)} \left[ H(n) - \sum_{k=2}^{\lfloor n/m \rfloor} g(k)F\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - \sum_{k=1}^{m-1} \left( G\left(\left\lfloor \frac{n}{k} \right\rfloor\right) - G\left(\left\lfloor \frac{n}{k+1} \right\rfloor\right) \right) F(k) \right]$$

## 12.19. Números primos de Mersenne

Números primos de la forma  $M_p = 2^p - 1$  con  $p$  primo. Todos los números perfectos pares son de la forma  $2^{p-1}M_p$  y viceversa.

Los primeros 47 valores de  $p$  son: 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497, 86243, 110503, 132049, 216091, 756839, 859433, 1257787, 1398269, 2976221, 3021377, 6972593, 13466917, 20996011, 24036583, 25964951, 30402457, 32582657, 37156667, 42643801, 43112609.

## 12.20. Números primos de Fermat

Números primos de la forma  $F_p = 2^{2^p} + 1$ , solo se conocen cinco: 3, 5, 17, 257, 65537. Un polígono de  $n$  lados es construible si y solo si  $n$  es el producto de algunas potencias de dos y distintos primos de Fermat.

## 12.18. Primos

$10^2 + 1$ ,  $10^3 + 9$ ,  $10^4 + 7$ ,  $10^5 + 3$ ,  $10^6 + 3$ ,  $10^7 + 19$ ,  $10^8 + 7$ ,  $10^9 + 7$ ,  $10^{10} + 19$ ,  $10^{11} + 3$ ,  $10^{12} + 39$ ,  $10^{13} + 37$ ,  $10^{14} + 31$ ,  $10^{15} + 37$ ,  $10^{16} + 61$ ,  $10^{17} + 3$ ,  $10^{18} + 3$ .

$10^2 - 3$ ,  $10^3 - 3$ ,  $10^4 - 27$ ,  $10^5 - 9$ ,  $10^6 - 17$ ,  $10^7 - 9$ ,  $10^8 - 11$ ,  $10^9 - 63$ ,  $10^{10} - 33$ ,  $10^{11} - 23$ ,  $10^{12} - 11$ ,  $10^{13} - 29$ ,  $10^{14} - 27$ ,  $10^{15} - 11$ ,  $10^{16} - 63$ ,  $10^{17} - 3$ ,  $10^{18} - 11$ .