HW 6

1. a. Reflexive: $A \sim A$ is a bijection
$B \sim B$ is a bijection
since $id_A : A \to A$ $x \mapsto x$ is a bijection
and similarly $id_B : B \to B$ is a bijection
Symmetric: $B \sim A$ is a bijection
because $f^{-1} : B \to A$ is bijective
Transitive: $A \sim B$ and $B \sim C \Rightarrow A \sim C$
A bijective function composed of another bijective function is bijective

b. $f(x) = x - 1$, $X = \{x \in X : x \in \mathbb{N}\}$
is a bijection where $\mathbb{N} \sim \mathbb{Z}_{\geq 0}$
since $\mathbb{N} \sim \mathbb{N}$, $\mathbb{Z}_{\geq 0} \sim \mathbb{Z}_{\geq 0}$
$\mathbb{Z}_{\geq 0} \sim \mathbb{N}$ since $f(x) = x + 1$ is a bijection
The relation is transitive because $\sim$ is an equivalence relation

c. $[\{1, 2, 3\}]$ the set of naturals that are $(\text{mod } 3) + 1$.

2. $f : X \to A$ $x, y \in X$ $x \sim y$ means $f(x) = f(y)$

a. $=$ operator is an equivalence relation

b. $[x]$ contains all elements related to x which is $\{y \in X : f(x) = f(y)\}$
$I_f(\{a\}) := \{y \in X : f(y) = f(x)\}$ so by definition,
$[x] = I_f(\{a\})$

c. $A = \{0, 1, 2, 3, 4\}$

$x \sim y$ if $x \equiv y \pmod 5$

3. a. $f \sim g \Leftrightarrow f \circ \sigma = g$

Reflexive: $f \sim f$ $\quad f \circ \sigma = f$ since σ is bijective

Symmetric: $g \sim f \Leftrightarrow g \circ \sigma = f$

g and f are both from X so if $f \sim g$, then $g \sim f$

Transitive: $f \sim g$ and $g \sim h \Rightarrow f \sim h$

$f \circ \sigma = g$ and $g \circ \sigma = h$, $f \circ \sigma = f$, $g \circ \sigma = g$

$g = h$ so $f \circ \sigma = h$

b. $f \sim g \Leftrightarrow \mathrm{Im}(f) = \mathrm{Im}(g)$

[$\Rightarrow$] $f \circ \sigma = g$ $\quad (\mathrm{Im}(f) \supseteq \mathrm{Im}(g))$

$y \in \mathrm{Im}(g) \to \exists x : g(x) = y$

$\to \exists x : f(\sigma(x)) = y$

$y = f(\sigma(x))$ so $y \in \mathrm{Im}(f)$

Do same for $\mathrm{Im}(g) \subseteq \mathrm{Im}(f)$

[$\Leftarrow$] $\mathrm{Im}(f) = \mathrm{Im}(g)$ since it's $\{1, \ldots, 5\}$

so $f(\sigma(x)) = g(y)$

c. $|\{1, \ldots, 5\}| = 5$, $|A| = 5$

so there can be a one-to-one correspondence between $X/\sim$ and Y.

7.1.9. R is a relation on set A. $S = R \cup R^{-1}$, S symmetric
$T = \{B \subseteq A \times A : B$ symmetric and $R \subseteq B\}$
then $S = \bigcap_{B \in T} B$
$R \cup R^{-1}$ makes S symmetric.
If a symmetric relation on A contains R, then the intersection of all these relations is also symmetric and also contains R, but this intersection produces the smallest relation in A which is the symmetric closure.

7.3.10. a. Reflexive: $a \sim a$ since $\frac{a}{a} = 1$ is in A when $m = 0$

Symmetric: $b \sim a$ since if $\frac{a}{b} = 2^m$
$\frac{b}{a} = 2^{-m}$.
Transitive: $a \sim b$ and $b \sim c \Rightarrow a \sim c$
$\frac{a}{b} \in A$ and $\frac{b}{c} \in A$
$2^m = \frac{a}{b}$, $2^n = \frac{b}{c}$ $\quad \frac{a}{b} \cdot \frac{b}{c} = \frac{a}{c} = 2^{m+n}$
which is in A

b. $[3]$ is the set of elements which don't belong to A, since there is no m such that $2^m = 3$.

7.3.11. $A = \{a + b\sqrt{2} : a, b \in \mathbb{Q}, a + b\sqrt{2} \neq 0\}$

$x \sim y \Leftrightarrow \frac{x}{y} \in \mathbb{Q}$

Reflexive : $x \sim x$, $\frac{x}{x} = 1 \in \mathbb{Q}$

Symmetric : $x \sim y \Rightarrow y \sim x$ since $\frac{y}{x} \in \mathbb{Q}$
and $x \neq 0$

Transitive : $x \sim y$ and $y \sim z \Rightarrow x \sim z$
$\frac{x}{y}$ and $\frac{y}{z}$ are rational numbers
$\frac{x}{y} \cdot \frac{y}{z} = \frac{x}{z}$ is also rational

Every distinct a, b is its own equivalence class since $a + b\sqrt{2} \neq c + d\sqrt{2}$ unless $a = c$ and $b = d$

7.3.12. Reflexive closure : $\{(1,1), (2,2), (3,3)\} \cup R$
Symmetric closure : $\{(2,1), (3,2)\} \cup R$
Transitive closure : $\{(1,3)\} \cup R$