HW 8

1. $F(Z) := \{f \mid f: Z \to \{0,1\}\}$
   $P(Z) = \{A \mid A \subset Z\}$
   $$\Psi(A)(z) := \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A \end{cases}$$
   $(\Psi(A): P(Z) \to F(Z)) \to (\Psi(A): Z \to \{0,1\})$
   $F(Z)$ is an infinite sequence of 0's and 1's and $P(Z)$ is a set of subsets of Z
   $\Psi(A): P(Z) \to F(Z)$ is injective and $\Psi(A): Z \to \{0,1\}$ is surjective, so the whole equation bijective.

2. Let $f: \mathbb{N} \to \{0,1\}$ be an arbitrary function not in the image of G.
   $f: \mathbb{N} \to \{0,1\}$ is an infinite sequence of 0's and 1's.
   $F(\mathbb{N})$ is composed of an infinite number of functions $f_1, f_2 \ldots$ Mapping $\mathbb{N}$ to $F(\mathbb{N})$ will not be possible because every element in F has cardinality $|\mathbb{N}|$, so $|\mathbb{N}| < |F(\mathbb{N})|$.

3. a. $$\Psi(X_\phi)(n) = \begin{cases} 1 & \text{if } n \in X_\phi \\ 0 & \text{if } n \notin X_\phi \end{cases}$$
   $\phi: \mathbb{N} \to P(\mathbb{N})$

$X_\phi \in P(\mathbb{N})$, $X_\phi := \{n \in \mathbb{N} : n \notin \phi(n)\}$

$$\Psi(X_\phi)(n) = \begin{cases} 1 & \text{if } n \notin \phi \\ 0 & \text{if } n \in \phi \end{cases}$$

b. $G = \Psi \circ \phi : \mathbb{N} \to F(\mathbb{N})$

$f : \mathbb{N} \to \{0, 1\}$

$$\Psi(\phi(z))(a) = \begin{cases} 1 & \text{if } a \in \phi(z) \\ 0 & \text{if } a \notin \phi(z) \end{cases}$$

$$= \Psi(X_\phi)(a)$$

4. $d : P(\mathbb{N}) \to [0, 1]$ $\quad d(X) = \sum_{n=1}^{\infty} \frac{X_n}{2^n}$

$X_n = 1$ if $n \in X$ and $X_n = 0$ if $n \notin X$

a. 1 is not mapped to anything so it's not injective

$0 \leq \frac{X_n}{2^n} \leq \frac{1}{2}$ which is in the domain of $P(\mathbb{N})$

b. $A \subset P(\mathbb{N})$ such that $d ; P(\mathbb{N}) \setminus A \to [0, 1]$ is bijective

If $A = (\frac{1}{2}, 1]$, then d is $[0, \frac{1}{2}]$, which is bijective to $[0, \frac{1}{2}]$

c. $f : P(\mathbb{N}) \to \mathbb{R}$

$|P(\mathbb{N})| \leq |\mathbb{R}| \leq |P(\mathbb{N})|$ so $|P(\mathbb{N})| = |\mathbb{R}|$

8.1.13. If a set is infinite, taking away elements from it would still leave it infinite since adding anything to infinity is still infinity.

8.2.7. a. $|(0,1)| \leq |\mathbb{R} \setminus \mathbb{N}| \leq |\mathbb{R}|$

$(0,1) \subseteq \mathbb{R} \setminus \mathbb{N}$

$\mathbb{R} \setminus \mathbb{N} \subseteq \mathbb{R}$

b. $f: (0,1) \to (-\frac{\pi}{2}, \frac{\pi}{2})$

$f(x) = \pi x - \frac{\pi}{2}$

c. $g: (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R} : x \to \tan x$

All points are touched once and only once

The domain of $\tan x$ is $(-\frac{\pi}{2}, \frac{\pi}{2})$ and the range is $\mathbb{R}$

d. $|\mathbb{R} \setminus \mathbb{N}| = |\mathbb{R}| = |[0,1]|$

$|\mathbb{R}|$ is uncountable and $\mathbb{N}$ is countable so $\mathbb{R} \setminus \mathbb{N}$ is uncountable. $[0,1]$ is uncountable as well since it is a subset of $\mathbb{R}$, so $\mathbb{R} \subseteq \mathbb{R} \setminus \mathbb{N}$ and $\mathbb{R} \setminus \mathbb{N} \subseteq [0,1]$

From part a, use the CSB Theorem to conclude that $|\mathbb{R} \setminus \mathbb{N}| = |\mathbb{R}| = |[0,1]|$