Homework 4: Math 347

Instructions: You may use any results from the book or lecture in your proofs. You are welcome and encouraged to work together with other students on these problems, but the final write up should be done alone. If people submit solutions that look highly similar, then they will get automatic zeros on the problem.

Problem 1. Let $f : X \to Y$ be a function.

(a) Prove that there is a set $Y' \subset Y$ so that $f : X \to Y'$ is surjective. (This proves every function can be made surjective by shrinking the codomain.)

(b) Prove there is a set $X' \subset X$ so that $f : X' \to Y$ is injective and $f(X') = f(X)$. (This proves every function can be made injective by shrinking the domain.)

Hint: If you are stuck, then work out some small examples and try to generalize.

Problem 2. Let $f : X \to Y$ be a function and let A and B be subsets of Y. Recall the preimage of A under f is defined as $I_f(A) := \{x \in X : f(x) \in A\}$, i.e. the elements in X that get mapped to A.

(a) Prove $I_f(A \cup B) = I_f(A) \cup I_f(B)$

(b) Prove $I_f(A \cap B) = I_f(A) \cap I_f(B)$

(c) Prove $I_f(A \setminus B) = I_f(A) \setminus I_f(B)$

Problem 3. For $f : X \to Y$ and $g : Y \to Z$ functions, prove items $2), 3), 4)$ from page 12.3 of lecture notes:

(a) f and g surjective implies that $g \circ f$ is surjective.

(b) f and g bijective implies that $g \circ f$ is bijective.

(c) $g \circ f$ injective implies that f is injective.

Problem 4. Prove the result on page 12.4 of the lecture notes: For a function $f : A \to B$

(a) f is injective if and only if there is a $\ell : B \to A$ so that $\ell \circ f = id_A$

(b) f is surjective if and only if there is a $h : B \to A$ so that $f \circ h = id_B$

(c) f is bijective if and only if there is a $g : B \to A$ so that $g \circ f = id_A$ and $f \circ g = id_B$

Hint: For the $\Rightarrow$ implications in (a) and (b) you need to build the functions. For both parts you can use Problem 1 to turn f into a bijective function $\tilde{f}$ and then use $\tilde{f}^{-1}$ to define the required function.

Problem 5. Two fake proofs are given on page 12.6 of the lecture notes, with the same major error made in both proofs.

(a) Correct the proof of $2) \Rightarrow 1)$ by judicious use of 'without loss of generality' and explain why it is indeed without loss of generality. Your proof should be different than the proof I gave.

(b) Correct the proof of $3) \Rightarrow 2)$ by judicious use of 'without loss of generality' and explain why it is indeed without loss of generality. Your proof should be different than the proof I gave.

From Donaldson and Pantanto: 4.4: 12