Midterm 2

1. $n = \min(c, d)$

Base case: $(c,d) = (1,1)$ holds because player A places the piece on $(1,1)$ and wins

$(c,d) = (2,2)$: Player A places the piece on $(2,2)$. Player B moves to $(1,2)$ or $(2,1)$ and player A moves to $(1,1)$.

Inductive hypothesis: Player A can win by choosing the top right corner to be (x,y), where $x+y = 2n$ and x is the right-most or y is the top most coordinate, $(1 \leq x \leq c$ and $1 \leq y \leq d)$

Inductive step: Suppose that the top right of the board is $(c+1, d+1)$. Player A can choose to start at $(c+1, j)$ or $(i, d+1)$ where $c+1+j = i+d+1 = 2(n+1)$. After two moves, the piece will be on (x,y) where $x+y = 2n$. Then, player A wins using the inductive hypothesis.

2. $P_n = \{ x^n + a_n x^{n-1} + \cdots a_2 x + a_1 : a_j \in \mathbb{Z} \text{ for all } j \}$

$A_n = \{ z \in \mathbb{R} : p(z) = 0 \text{ for some } p(x) \in P_n \}$

a. $n \in \mathbb{N}_0$ and $a_j \in \mathbb{Z}$

To get all combinations of P_n, we need to get the Cartesian product of $\mathbb{Z}$ n times. The Cartesian product of denumerable sets is denumerable so $\mathbb{Z} \times \cdots \times \mathbb{Z}$ is denumerable which means that $|P_n| = |\mathbb{N}|$.

b. Each expression P_n has n roots at most, so the cardinality of A_n is at most $|P_n| \times |P_n|$ which is denumerable since P_n is denumerable. Since A_n is denumerable, $|A_n| = |\mathbb{N}|$.

c. We proved in part b that A_n is denumerable. We know that the union of denumerable sets is denumerable. Therefore, $|A| = |\mathbb{N}|$. $A \neq \mathbb{R}$ since $|\mathbb{N}| \neq |\mathbb{R}|$.

3. If every equivalence class has exactly k elements, this means that $\sim$ has been partitioned into n classes, which means k divides n.

4. $f: \mathbb{Z}/a \to \mathbb{Z}/b$ by $f([x]_a) = [5x^2+3]_b$ is well-defined $\Leftrightarrow$ b divides $5a^2$ and $10a$

[$\Rightarrow$] Assume $f: \mathbb{Z}/a \to \mathbb{Z}/b$ by $f([x]_a) = [5x^2+3]_b$ is well-defined. Then, we have $[5a^2+3]_b = f([a]_a)$ $= f([0]_a) = [3]_b$. b does not divide $5a^2$

[$\Leftarrow$] Assume b divides $5a^2$ and $10a$ so

$5a^2 = b \cdot n_1$ and $10a = b \cdot n_2$.

Let $[x]_a = [y]_a$, so $x = y + ak$ for some $k \in \mathbb{Z}$

$x = y + \frac{b \cdot n_2}{10} \cdot k = y + b\left(\frac{n_2 k}{10}\right)$

$x = y \pm \sqrt{\frac{b \cdot n_1}{5}}\, k$ which means that f is not well-defined.

EC. Base case: Only one student has a red card. They see that the other students all have black cards so they know they're the only one with the red card and wins on round 1.

Inductive hypothesis: Suppose that a player will declare victory at the end of the $(n-1)$th round if there are exactly $n-1$ students with red cards $(n \geq 2)$

Inductive step: Suppose there are n students with red cards.

A student can't declare victory on the nth round because there's at least one other student with a red card. After one round, it will be the $(n-1)$th round and using the inductive hypothesis, someone will declare victory in $n-1$ rounds.

Therefore, a student will declare victory on the nth round if there are n red cards.