Homework 6: Math 347

Instructions: You may use any results from the book or lecture in your proofs. You are welcome and encouraged to work together with other students on these problems, but the final write up should be done alone. If people submit solutions that look highly similar, then they will get automatic zeros on the problem.

Problem 1. Let $\mathcal{U}$ be a set of sets. For sets $A, B \in \mathcal{U}$ define $A \sim B$ to mean there is a bijection $f : A \to B$.

(a) Prove $\sim$ is an equivalence relation on $\mathcal{U}$.

(b) Prove that $\mathbb{N} \sim \mathbb{Z}_{\geq 0}$ (assuming both sets are in $\mathcal{U}$).

(c) Give a description of $[\{1, 2, 3\}]$ without referencing the equivalence relation or bijections (assuming $\{1, 2, 3\}$ is in $\mathcal{U}$).

Problem 2. Fix a function $f : X \to A$. For $x, y \in X$ define $x \sim y$ to mean $f(x) = f(y)$.

(a) Prove $\sim$ is an equivalence relation on X.

(b) Recall the preimage construction from Lecture 12. If $x \in X$ and $f(x) = a$, prove that

$$[x] = I_f(\{a\})$$

(c) Find a set A and a function $f : \mathbb{Z} \to A$ so the resulting equivalence relation $x \sim y$ means $x \equiv y \mod 5$.

Problem 3. Let $\mathcal{X}$ be the set of injective functions $\{1, \dots, 5\} \to \mathbb{R}$. For $f, g \in \mathcal{X}$, define $f \sim g$ to mean $f \circ \sigma = g$ for some permutation (i.e. bijection) $\sigma : \{1, \dots, 5\} \to \{1, \dots, 5\}$.

(a) Prove $\sim$ is an equivalence relation on $\mathcal{X}$.

(b) Prove that $f \sim g$ if and only if f and g have the same image.

(c) Prove there is a bijection between $\mathcal{X}/_\sim$ and $\mathcal{Y} = \{A \subset \mathbb{R} : |A| = 5\}$.

From Donaldson and Pantanto:
7.1: 9
7.3: 10, 11, 12 (find the equivalence relation closure as well)