Homework 7: Math 347

Instructions: You may use any results from the book or lecture in your proofs. You are welcome and encouraged to work together with other students on these problems, but the final write up should be done alone. If people submit solutions that look highly similar, then they will get automatic zeros on the problem.

Problem 0. Do a nonacademic activity away from a computer to take care of yourself, e.g. get some exercise, take a walk with a friend outside, read a book for fun, etc.

Problem 1. Use Problem 4 on Homework 4 to give a very short proof of the following:

$$\exists\, q : X \to Y \text{ that is injective} \quad \text{if and only if} \quad \exists\, w : Y \to X \text{ that is surjective}$$

Problem 2. Let $f : A \to A'$ and $g : B \to B'$ be bijections.

(a) Assume $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$, prove there is a bijection between $A \cup B$ and $A' \cup B'$.

(b) Give a counterexample to (a) if the conditions $A \cap B = \emptyset$ and $A' \cap B' = \emptyset$ are removed. Identify where your proof in (a) used these assumptions.

Problem 3. Do not use Theorem 8.6 in the book in this problem.

(a) If $A \subset \mathbb{N}$ is infinite, prove that $|A| = |\mathbb{N}|$. (Your proof should use some property of $\mathbb{N}$, since the statement is not true if $\mathbb{N}$ is replaced by $\mathbb{R}$.)

(b) If B is infinite and $|B| \leq |\mathbb{N}|$, prove that $|B| = |\mathbb{N}|$. (Hint: There is a bijection between B and a subset of $\mathbb{N}$, now apply part (a) to that subset...)

Problem 4. Let X and Y be denumerable sets.

(a) If $X \cap Y = \emptyset$, then prove $X \cup Y$ is denumerable. (Do not just cite Lecture 23/24.)

(b) Use part (a) and Problem 3(b) to prove $X \cup Y$ is denumerable even if $X \cap Y \neq \emptyset$. (In Lecture 24, I show how to replace X and Y with sets that are disjoint.)

From Donaldson and Pantanto:
7.4: 4
7.5: 4
7.6: 6