Homework 8: Math 347

Instructions: You may use any results from the book or lecture in your proofs. You are welcome and encouraged to work together with other students on these problems, but the final write up should be done alone. If people submit solutions that look highly similar, then they will get automatic zeros on the problem.

Definition For a set Z define

$$F(Z) := \{f \mid f : Z \to \{0,1\}\}$$

to be the *set of functions from Z to $\{0,1\}$* and let $\mathcal{P}(Z) = \{A \mid A \subset Z\}$ be the *power set* of Z.

Problem 1. Prove that the function $\Psi : \mathcal{P}(Z) \to F(Z)$ where $A \in \mathcal{P}(Z)$, i.e. $A \subset Z$, maps to the function $\Psi(A) : Z \to \{0,1\}$ defined by

$$\Psi(A)(z) := \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{if } z \notin A \end{cases}$$

is a bijection between $\mathcal{P}(Z)$ and $F(Z)$.

Note about notation: Here $A \in \mathcal{P}(Z)$, i.e. $A \subset Z$, so $\Psi(A) : Z \to \{0,1\}$ is the function in $F(Z)$ where Ψ maps A and $\Psi(A)(z)$ is the result of plugging $z \in Z$ into $\Psi(A)$.

Problem 2. Adapt the 'diagonal argument' we used to prove $[0,1)$ was uncountable to prove that $|\mathbb{N}| < |F(\mathbb{N})|$. Namely for an arbitrary function $G : \mathbb{N} \to F(\mathbb{N})$, construct a function $f : \mathbb{N} \to \{0,1\}$ not in the image of G.

Hint: To setup the 'diagonal argument', observe that a function $h : \mathbb{N} \to \{0,1\}$ is equivalent to an infinite sequence of 0's and 1's, namely the sequence $\langle h(1), h(2), h(3), \ldots \rangle$.

Problem 3. Let $\phi : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ be a function and let $X_\phi \in \mathcal{P}(\mathbb{N})$ be defined as

$$X_\phi := \{n \in \mathbb{N} : n \notin \phi(n)\}.$$

This is the set from the proof of Cantor's theorem (Thm 8.11 in book) and in this problem you will show Cantor's argument is a generalization of the 'diagonal argument'.

(a) For the bijection Ψ from Problem 1, what function is $\Psi(X_\phi)$?

(b) Verify that if you pick G in Problem 2 to be $\Psi \circ \phi : \mathbb{N} \to F(\mathbb{N})$, the resulting function f from the 'diagonal argument' is the function $\Psi(X_\phi)$.

Problem 4. Consider the base 2 decimal function

$$d : \mathcal{P}(\mathbb{N}) \to [0,1] \quad \text{defined by} \quad d(X) = \sum_{n=1}^{\infty} \frac{x_n}{2^n}$$

where $x_n = 1$ if $n \in X$ and $x_n = 0$ if $n \notin X$.

(a) Prove that $d : \mathcal{P}(\mathbb{N}) \to [0,1]$ is surjective but not injective.

(b) Prove there is a denumerable set $\mathcal{A} \subset \mathcal{P}(\mathbb{N})$ such that $d : \mathcal{P}(\mathbb{N}) \setminus \mathcal{A} \to [0,1]$ is bijective.

(c) Define an injective map $f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ and use the CSB Theorem to deduce $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

From Donaldson and Pantanto:
8.1: 13
8.2: 7 Note: $\mathfrak{c}$ is the book's notation for $\big|[0,1]\big|$