ABSTRACT ALGEBRA I: HOMEWORK 2 SOLUTIONS

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Problem (Pg. 59-61, 1(b,d))**.** In each of the following parts, determine whether the given function is one-to-one and whether it is onto.

(b) $f\colon \mathbb{C} \to \mathbb{C}$; $f(x) = x^2 + 2x + 1$
(d) $f\colon \mathbb{R}^+ \to \mathbb{R}$; $f(x) = \ln x$

Proof. (b) Note that $f(x) = x^2 + 2x + 1 = (x+1)^2$. Since $f(0) = f(-2) = 1$, we know that f is not one-to-one. Now take an arbitrary $z \in \mathbb{C}$. Let $w = \sqrt{z} - 1 \in \mathbb{C}$. Then $f(w) = (w+1)^2 = (\sqrt{z})^2 = z$. Since for every $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that $f(w) = z$, f is onto.

(d) Let $\ln x = \ln y$. Then $e^{\ln x} = e^{\ln y}$, hence $x = y$. This shows that f is one-to-one. Now let $y \in \mathbb{R}$. Take $x = e^y \in \mathbb{R}^+$. Then $\ln x = \ln e^y = y$. Since for any $y \in \mathbb{R}$ there exists $x \in \mathbb{R}^+$ such that $\ln x = y$, f is onto.

□

Problem (Pg. 59-61, 3(b,d))**.** For each one-to-one and onto function in Exercise 1, find the inverse of the function. *Hint*: It might not hurt to review the section on inverse functions in your calculus book.

Proof. (b) Since the function is not both one-to-one and onto, it has no inverse.

(d) Let $g(x) = e^x$. Then $g(f(x)) = g(\ln x) = e^{\ln x} = x$, and $f(g(x)) = f(e^x) = \ln e^x = x$, so $g = f^{-1}$.

□

Problem (Pg. 59-61, 5(a))**.** In each of the following parts, determine whether the given function is one-to-one and whether it is onto. If the function is both one-to-one and onto, find the inverse of the function.

(a) $f\colon \mathbb{R}^2 \to \mathbb{R}^2$; $f(x,y) = (x+y, y)$

Proof. Suppose $f(a,b) = f(c,d)$. Then $(a+b, b) = (c+d, d)$, hence $b = d$. We also have $a + b = c + d$, in which we can substitute b for d on the right hand side to get $a + b = c + b$. Subtracting b from both sides, we have $a = c$, hence $(a,b) = (c,d)$. This shows that f is one-to-one.

Now take $(a,b) \in \mathbb{R}^2$ to be an arbitrary point. Let $x = a - b$ and $y = b$. Then $f(x,y) = f(a-b, b) = (a-b+b, b) = (a,b)$. Since for any point in $(a,b) \in \mathbb{R}^2$ there exists a point $(x,y) \in \mathbb{R}^2$ such that $f(x,y) = (a,b)$, f is onto.

Define $g(x,y) = (x-y, y)$. Then $g = f^{-1}$, since

$$g(f(x,y)) = g(x+y, y) = (x+y-y, y) = (x,y), \text{ and}$$
$$f(g(x,y)) = f(x-y, y) = (x-y+y, y) = (x,y).$$

□

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Problem (Pg. 59-61, 9(c))**.** Show that each of the following formulas yields a well-defined function.

(c) $h\colon \mathbb{Z}_{12} \to \mathbb{Z}_4$ defined by $h([x]_{12}) = [x]_4$

Proof. To check the function is well-defined, we need to check that two equivalent elements of $\mathbb{Z}_{12}$ get mapped to the same place. Let $[x]_{12} = [y]_{12}$. Then there is some $k \in \mathbb{Z}$ such that $x = y + 12k = y + 4(3k)$. It follows that $[x]_4 = [y]_4$, so

$$h([x]_{12}) = [x]_4 = [y]_4 = f([y]_{12}),$$

and the map is well-defined. □

Problem (Pg. 59-61, 10(a))**.** In each of the following cases, give an example to show that the formula does not define a function.

(a) $f\colon \mathbb{Z}_8 \to \mathbb{Z}_{10}$ defined by $f([x]_8) = [6x]_{10}$

Proof. We must find x and y such that $[x]_8 = [y]_8$, but $f([x]_8) \neq f([y]_8)$. Let $x = 1$ and $y = 9$. We have $[x]_8 = [y]_8 = 1$, but

$$f([x]_8) = [6]_{10} \neq [4]_{10} = [6 \cdot 9]_{10} = f([y]_8).$$

□

Problem (Pg. 59-61, 11)**.** Let k and n be positive integers. For a fixed $m \in \mathbb{Z}$, define the formula $f\colon \mathbb{Z}_n \to \mathbb{Z}_k$ by $f([x]_n) = [mx]_k$, for $x \in \mathbb{Z}$. Show that f defines a function if and only if $k|mn$.

Proof. Suppose $k|mn$. Let $[x]_n = [y]_n$, i.e. $n|(x-y)$. Then $nm|m(x-y)$. Since $k|mn$, by transitivity we have $k|m(x-y)$, hence $[mx]_k = [my]_k$, and the map is well-defined.

Conversely, suppose the map is well-defined. Note that $[n]_n = [0]_n$, and since the map is well-defined we have

$$[mn]_k = f([n]_n) = f([0]_n) = [m \cdot 0]_k = [0]_k.$$

Since $mn \equiv 0 \pmod{k}$, we have $k|mn$. □

Bad Proof. Many people had the following incorrect proof (or variations of it) of the claim: if f is well-defined, then $k|mn$:

Suppose f is a function. Let $[x]_n = [y]_n$ (so $n|(x-y)$). Then $[mx]_k = [my]_k$, so $k|m(x-y)$. Write $x - y = nq$ for some $q \in \mathbb{Z}$. Then we have

$$k|m(x-y) \Rightarrow k|m(nq) \overset{?}{\Rightarrow} k|mn. \qquad \square\,?$$

The problem with the proof, as indicated above, is that the last implication doesn't hold. There is the possibility that $k|q$, or that $k = ab$, where $a|mn$ and $b|q$. In either of these cases, we could have $k|mnq$, but not $k|mn$.

Problem (Pg. 59-61, 15)**.** Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove that if $g \circ f$ is one-to-one, then f is one-to-one, and if $g \circ f$ is onto, then g is onto.

Proof. Suppose $g \circ f$ is one-to-one. Let $f(x) = f(y)$. Since g is a well-defined function, we have $g(f(x)) = g(f(y))$. Since $g \circ f$ is one-to-one, this implies $x = y$. Since $f(x) = f(y)$ implies $x = y$, this shows f is one-to-one.

Now suppose that $g \circ f$ is onto. Then for any $z \in C$ there is some $x \in A$ such that $(g \circ f)(x) = g(f(x)) = c$. Take $y \in B$ to be $f(x)$. Then $g(y) = z$. Since for any $z \in C$ there exists $y \in B$ such that $g(y) = z$, we have that g is onto. □