Midterm II: Math 347

Instructions: Write your proofs in complete sentences.

Problem 1. For natural numbers $c, d \in \mathbb{N}$ consider the following two person game:

(i) The board is the grid $\{1, \ldots, c\} \times \{1, \ldots, d\} \subset \mathbb{R}^2$ (so $(1,1)$ is the bottom left corner and (c,d) is the top right corner) and there is a single game piece.

(ii) Player A starts the game by placing the piece anywhere on the right most column or top most row.

(iii) Starting with player B, the two players now alternate turns moving the piece according to the following rule: If the piece is at (i,j), then they can move it down its current column to (i,j') with $j' < j$ or left on its current row (i',j) with $i' < i$.

(iv) The winner is the player that moves the piece to $(1,1)$.

Use strong induction on $n = \min(c,d)$ to prove there is a strategy so that player A always wins.

Problem 2. Consider the set of degree n monic polynomials with integer coefficients

$$\mathcal{P}_n = \{x^n + a_n x^{n-1} + \cdots a_2 x + a_1 : a_j \in \mathbb{Z} \text{ for all } j\}$$

and consider the set real numbers that are the root of some such a polynomial

$$\mathcal{A}_n = \{z \in \mathbb{R} : p(z) = 0 \text{ for some } p(x) \in \mathcal{P}_n\}.$$

(a) Prove that $|\mathcal{P}_n| = |\mathbb{N}|$.

(b) Prove that $|\mathcal{A}_n| = |\mathbb{N}|$.

(c) Let $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Prove $|\mathcal{A}| = |\mathbb{N}|$ and deduce $\mathcal{A} \neq \mathbb{R}$.

Problem 3. Fix natural numbers n and k. Let X be a finite set with n elements and let $\sim$ be an equivalence relation on X. Prove that if every equivalence class for $\sim$ has exactly k elements, then k divides n.

Problem 4. For $a, b \in \mathbb{N}$ prove that

$$f : \mathbb{Z}/a \to \mathbb{Z}/b \quad \text{by} \quad f([x]_a) = [5x^2 + 3]_b$$

is well-defined if and only if b divides both $5a^2$ and $10a$. (Recall that $\mathbb{Z}/n$ is the equivalence classes of the integers mod n.)

Extra Credit (Worth 10%) Ten math majors are sitting around a table and playing a game at a bar (that is designed not to end) with the following rules:

(i) Using a standard deck of cards each player is a dealt a single card face down, which they then place on their forehead with the card face being displayed to the table. (Each player knows everyone else's card but does not know their own.)

(ii) The game is played in 60 second rounds. At the end of a round each player must either take a sip of their drink (to pass) or stand to declare victory.

(iii) A player may only stand to declare victory if they can prove they have a red card.

(iv) During the game players are not allowed to make any type of communication about the cards or do something that would allow a player to see their own card.

One night a clueless math professor wanders by the students' table and remarks that at least one of the students has a red card, which is a fact that all the students now use in the game. Prove via induction that a player declares victory at the end of the n-th round if there are exactly n students with red cards.