Math 347 HW 3

3.1.10. From the hint: $345 = 3\cdot 10^2 + 4\cdot 10 + 5\cdot 10^0$

Thm 3.8.2: $a \equiv b$ and $c \equiv d \Rightarrow a \pm c \equiv b \pm d$

Assume a number $n \cdot 10^k \pmod 9 \equiv n \pmod 9$

$n \cdot 10^k - n = n(10^k - 1) = 9l$

$9 \mid 10^k - 1$ so $n \equiv l \pmod 9$

From the theorem, summing up the digits of a number n won't affect its congruence to n (mod 9). ∎

3.1.11. $\forall (p \in \text{primes} \geq 3)$, $p \equiv 1 \pmod 3 \Rightarrow p \equiv 1 \pmod 6$

Since p is always odd, p will be of the form $2n+1$ for an integer n. For $2n+1$ to be congruent to $1 \pmod 3$, n must be a multiple of 3. 2n will be a multiple of 6, so $2n+1 \equiv 1 \pmod 6$.

∎

3.1.12. $7x \equiv 28 \pmod{42}$, $x \equiv 4 \pmod 6$

a. Thm 3.6: $a \equiv b \pmod n \Leftrightarrow n \mid b - a$

$42 \mid 28 - 7x \Leftrightarrow 28 - 7x = 42k$

$= 7(4-x) = 7(6k) = 4 - x = 6k$

b. Yes, x can just be 4.

c. $7x \equiv 28 \pmod{42} \Leftrightarrow 4 - x = 6k$

$x \equiv 4 \pmod{42} \Leftrightarrow 4 - x = 42k_2$

No, it doesn't work if $x = 52$.

d. Thm 3.9: $ka \equiv kb \pmod{kn} \Rightarrow a \equiv b \pmod n$

$kb - ka = kn \cdot n_2$

$k(b-a) = k(n \cdot n_2)$

$b - a = n \cdot n_2$

$n \mid (b-a)$

$a \equiv b \pmod{n}$ $\blacksquare$

3.2.7. $239 = 5a + 7b$, $a, b \geq 0$

$239 - 5a = 7b$

$7 \mid 239 - 5a$

$5 \mid 239 - 7b$

No, there are at two solutions. (also see Thm. 3.13)

$a = 3$, $b = 32$ and $a = 38$, $b = 7$

3.2.10. Prove: $\gcd(m,n) = 1 \Rightarrow \exists x, y \in \mathbb{Z}$, $mx + ny = 1$

Apply Bezout's Identity

$\exists x, y \in \mathbb{Z}$, $mx + ny = 1 \Rightarrow \gcd(m,n) = 1$

Assume $\gcd(m,n) \neq 1$. Then, there is some common factor a that divides both m and n, where $a > 1$.

m, n can be rewritten as am_1, an_1.

$am_1 x + an_1 y = a(m_1 x + n_1 y) = ak$

where $k = m_1 x + n_1 y$. If $ak = 1$, k must be $\frac{1}{a}$.

k must be an integer so $ak \neq 1$. $\blacksquare$

3.2.11. $a, b, c \in \mathbb{Z}$; $(x_0, y_0), (x_1, y_1)$ solutions to $ax + by = c$

Thm 3.13: $ax + by = c$ has solution iff $d \mid c$, $d = \gcd(a,b)$

$x = x_0 + \frac{b}{d} t$, $y = y_0 - \frac{a}{d} t$

a. $ax_0 + by_0 = c$

$ax_1 + by_1 = c$

$a(x_0 - x_1) + b(y_0 - y_1)$

$= ax_0 - ax_1 + by_0 - by_1 = c - c = 0$

b. $\gcd(a,b) = d$

$d = ax + by$

$1 = \frac{a}{d}x + \frac{b}{d}y$

$\gcd(\frac{a}{d}, \frac{b}{d}) = 1$

c. $ax + by = 0$

$x = -\frac{b}{a}y$

$y = -\frac{a}{b}x$

$\gcd(a, b) = 0$

$(x_0, y_0) = (0,0)$

d. $\frac{a}{d}x + \frac{b}{d}y = 1$

$ax_0 + by_0 = c$

$a = \frac{c - by_0}{x_0} = \left(1 - \frac{b}{d}y\right)\frac{x}{d}$

(Not sure)

3.2.13

n	1	2	3	4	5	6
gcd(2n, n+1)	2	1	2	1	2	1

$\gcd(2n, n+1) = 2$ if n is odd, else 1

If n is odd, $2n = 2(2a+1)$ for some $a \in \mathbb{Z}$

$n+1 = (2a+1) + 1 = 2a+2$

$\gcd(4a+2, 2a+2) = \gcd(2a+2, 2a) = 2$

If n is even, $2n = 2(2a)$ for some $a \in \mathbb{Z}$

$n+1 = 2a+1$

$\gcd(4a, 2a+1) = 1$ because no common

factors.

3.2.14. a. $2b \equiv 1 \pmod 6$?

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

$2 \cdot 0 \pmod 6 = 0$ $\quad$ $2 \cdot 3 \pmod 6 = 0$

$2 \cdot 1 \pmod 6 = 2$ $\quad$ $2 \cdot 4 \pmod 6 = 2$

$2 \cdot 2 \pmod 6 = 4$ $\quad$ $2 \cdot 5 \pmod 6 = 4$

b. $n = n_1 n_2 \Rightarrow ab \not\equiv 1 \pmod{n_1 n_2}$

$(n_1, n_2 \geq 2)$

When $a = n_1$, $n_1 b \not\equiv 1 \pmod{n_1 n_2}$

c. If n is prime, n can't be the product of two numbers n_1, n_2 where $n_1, n_2 \geq 2$.

$ab \equiv 1 \pmod n$ if n is prime

$\gcd(a, n) = 1 \Rightarrow \exists\, x, y \in \mathbb{Z}$ such that

$ax + ny = 1$