Final

1. $L = [(\neg A \Leftrightarrow B) \vee C]$ $\quad R = [(A \wedge B) \Rightarrow C]$

a. $L \Rightarrow R$

Assume $\neg R$, which is $(A \wedge B) \wedge \neg C$, so
A and B are True and C is False.
L is False after plugging in (False $\Leftrightarrow$ True) $\vee$ False
which means that $\neg L$ is True.
Since $\neg R \Rightarrow \neg L$, $L \Rightarrow R$ is True.

b. $R \Rightarrow L$

Assume R.
Case 1: A, B, C True
Then, L is True since C is True.
Case 2: A or B is False
If A is False, B is False, and C is False,
then $R \Rightarrow L$ is False since R is True but
L is False.
Therefore, $R \Rightarrow L$ is False.

2. Base case: $n=1$

According to (i), $\forall(x \in S)\exists(y \in S): P(x,y)$

Since there's only one element, $x=y$ so $P(x,x)$.

However, that's a contradiction to (iii), so $|S| \geq 1$.

Base case: $n=2$

From (i), there exists a $P(x,y)$ and $P(y,x)$.

This means from (ii), $P(x,x)$ which contradicts (iii), so $|S| \geq 2$.

Inductive hypothesis: Suppose $|S| \geq n$ for $n = 1 \ldots k$, and that the three statements hold.

Inductive step: Suppose there is a set with $k+1$ elements. Let the elements of the set be from x_1 to x_{k+1} in any order. According to the pigeonhole principle, there will be an x mapped to a y where that y is already mapped to another element. However, this will cause (iii) to eventually be contradicted.

Therefore, S can't be a set with finite elements since then we can use the pigeonhole principle as a contradiction. ∎

4. Let $n_1, n_2 \in \mathbb{N}$

a. $\gcd(n_1, n_2) = 1$, $l \in \mathbb{Z}$

Prove $n_1 | l$ and $n_2 | l \Rightarrow n_1 n_2 | l$

$n_1 | l \iff n_1 k_1 = l$

$n_2 | l \iff n_2 k_2 = l \quad (k_1, k_2 \in \mathbb{Z})$

$$n_1 \cdot n_2 = \frac{l^2}{k_1 k_2}$$

Since $\gcd(n_1, n_2) = 1$, k_1 is not a multiple of k_2 and vice versa.

Therefore, $k_1 \cdot k_2 \geq l$ and $k_1 \cdot k_2$ is a multiple of l so $k_1 \cdot k_2 = cl$, $c \in \mathbb{Z}$.

$$n_1 \cdot n_2 = \frac{l^2}{cl} = \frac{l}{c}$$

$c n_1 \cdot n_2 = l$, so $n_1 n_2 | l$. ∎

b. f is bijective $\Leftrightarrow \gcd(n_1, n_2) = 1$

$[\Rightarrow]$ Assume f is bijective $\gcd(n_1, n_2) \neq 1$,

$f([x]_8) = ([x]_2, [x]_4) = ([x]_1, [x]_8)$,

which means that f is not bijective.

This is a contradiction so therefore

f is bijective $\Rightarrow \gcd(n_1, n_2) = 1$.

$[\Leftarrow]$ Injective: $\forall a, b \in \mathbb{Z}/N$, $f([x]_a) = f([x]_b)$

$\Rightarrow [x_a] = [x_b]$

$f([x]_a) = f([x]_b) \Leftrightarrow ([x]_{an_1}, [x]_{an_2}) = ([x]_{bn_1}, [x]_{bn_2})$

$\Leftrightarrow [x]_{an_1} = [x]_{bn_1}$ and $[x]_{an_2} = [x]_{bn_2}$

$\Rightarrow [x]_a = [x]_b$

Surjective: $\forall$ elements in $\mathbb{Z}/n_1 \times \mathbb{Z}/n_2$,

$\exists q \in \mathbb{Z}/N$

Since $N = n_1 \cdot n_2$ from the problem, f is surjective from part a.

5. Let X be a countably infinite set.
Let the elements of X start from
$a_1, a_2, a_3, \ldots$
Take every other element of X to create a new set Y so that Y contains $a_2, a_4, a_6, \ldots$
$|X| = |X \setminus Y|$ as shown by the function $f(n) = 2n$.
Now let X be an uncountable set.
Create a set Y such that $|Y| = |\mathbb{N}|$.
From any uncountable set, you can remove a countable set and it will still remain uncountable so $|X| = |X \setminus Y|$.

Therefore, we have shown that from any infinite set X, there is a countably infinite subset $Y \subset X$ so that $|X| = |X \setminus Y|$.

8. If there exists a K such that $a_n \geq 0$ for all $n \geq K$, then we know that $\langle a_n \rangle$ is not alternating. Since Σa_n converges, the series is Cauchy and for every $\varepsilon > 0$, there exists an $M \in \mathbb{N}$ such that for every $n \geq M$ and every $k > n$ we have

$$\left| \sum_{j=n+1}^{k} x_j \right| < \varepsilon.$$

If we square all the terms in the sequence, the series will still converge because the terms must get arbitrarily smaller where $a_n^2 < a_n$, and there are only finite terms before it gets there.

b. $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{n}}$ converges due to the alternating series test

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because of p-series test

9. $\frac{b_{n+1}}{b_n} \leq \frac{a_{n+1}}{a_n}$ for all n sufficiently large

$$\sum a_n \text{ converges} \Rightarrow \sum b_n \text{ converges}$$

Since $\frac{b_{n+1}}{b_n} \leq \frac{a_{n+1}}{a_n}$, either $\langle b_n \rangle$ is an alternating sequence so that $\frac{b_{n+1}}{b_n}$ is negative, or its terms are getting closer than the terms in $\langle a_n \rangle$. If $\sum b_n$ is an alternating series, it converges. Else, $\sum b_n$ also converges since the terms before n are finite and so after n, the partial sums for $\langle b_n \rangle$ will be less than the partial sums for $\langle a_n \rangle$.

b. Rearrange the sequence for the alternating harmonic sequence to be
$1, \frac{1}{3}, -\frac{1}{2}, -\frac{1}{4}, \frac{1}{5}, \frac{1}{7}, -\frac{1}{6}, -\frac{1}{8}, \ldots$ so it alternates between positive terms and negative terms every two terms. The limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist because it can either be 1 or -1.

Proof of convergence:

Group this sequence into subsequences of four terms each. The sums of these subsequences are positive but the partials are Increasing at a decreasing rate, which means for any ε, the difference will eventually be smaller.