HW 4

1. $f: X \to Y$

   a. For every element y in Y there are two possibilities; either it is not mapped from an element in X or it is. If it is, add it to a new set Y'. Then, $Y' \subset Y$ and Y' is surjective.

   b. Let x_1 and x_2 be two unique elements in X. If $f(x_1) = f(x_2)$, set x_2 as an alias to x_1 so $x_1 = x_2$. This new domain X' makes $f: X' \to Y$ injective.

2. $f: X \to Y \quad I_f(A) := (x \in X : f(x) \in A)$

   a. Prove $I_f(A \cup B) = I_f(A) \cup I_f(B)$

   $I_f(B) = (x \in X ; f(x) \in B)$

   Let $C = A \cup B$

   $$I_f(C) = (x \in X ; f(x) \in C) = (x \in X : f(x) \in (A \cup B))$$
   $$= (x \in X : f(x) \in A \cup f(x) \in B)$$
   $$= (x \in X : f(x) \in A) \cup (x \in X ; f(x) \in B)$$
   $$= I_f(A) \cup I_f(B) \quad \blacksquare$$

   b. Prove $I_f(A \cap B) = I_f(A) \cap I_f(B)$

   Let $C = A \cap B$

   $$I_f(C) = (x \in X ; f(x) \in C) = (x \in X : f(x) \in (A \cap B))$$
   $$= (x \in X : f(x) \in A \cap f(x) \in B)$$
   $$= (x \in X : f(x) \in A) \cap (x \in X ; f(x) \in B)$$
   $$= I_f(A) \cap I_f(B) \quad \blacksquare$$

c. Prove $I_f(A \setminus B) = I_f(A) \setminus I_f(B)$

Let $C = A \setminus B$

$I_f(C) = (x \in X ; f(x) \in C) = (x \in X : f(x) \in (A \setminus B))$

$= (x \in X : f(x) \in A \setminus f(x) \in B)$

$= (x \in X : f(x) \in A) \setminus (x \in X ; f(x) \in B)$

$= I_f(A) \setminus I_f(B)$ ▨

3. a. f and g surjective $\Rightarrow g \circ f$ surjective

Assume f and g surjective.

Suppose $f : X \to Y$ and $g : Y \to Z$.

Since f and g are surjective, $\forall y \in Y, \exists x \in X$ such that $f(x) = y$ and $\forall z \in Z, \exists y \in Y$ such that $g(y) = z$.

Then, it follows that a function from $X \to Z$ is surjective. ▨

b. f and g bijective $\Rightarrow g \circ f$ bijective

Assume f and g bijective.

We need to show that if f and g are injective, then $g \circ f$ is injective.

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \Rightarrow g(f(x_1)) \neq g(f(x_2))$.

In part a we already proved the case of $g \circ f$ surjective. ▨

c. $g \circ f$ injective $\Rightarrow f$ injective

Assume f is not injective. Then, there exists x_1 and x_2 where $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. $g(f(x_1)) = g(f(x_2))$ so $g \circ f$ is not injective. ▨

4. a. f is injective iff $\exists\, l: B \to A$ so that $l \circ f = id_A$

Assume there does not exist $l: B \to A$ so that $l \circ f = id_A$. Then, since f does not have an inverse, it can't be injective.

$\Leftarrow$ proved in lecture

b. f is surjective iff $\exists\, h: B \to A$ so that $f \circ h = id_B$

Assume there does not exist $h: B \to A$ so that $f \circ h = id_B$. Then, the inverse of f does not exist so f is not surjective.

$\Leftarrow$ already proved in lecture.

c. f is bijective iff $\exists\, g: B \to A$ so that $g \circ f = id_A$ and $f \circ g = id_B$

Assume there does not exist $g: B \to A$ so that $g \circ f = id_A$ and $f \circ g = id_B$

Then, the inverse of f does not exist so f is not bijective.

$\Leftarrow$ proved in lecture.

5. a. Must include the fact that distinct values of X map to distinct values of Y.

b. We must declare the same thing as from part a or else f doesn't have to be injective.

4.4.12 a.

x	0	1	2	3	4	5	6	7	8	9
$1\cdot x \pmod{10}$	0	1	2	3	4	5	6	7	8	9

Range: 0 to 9 Cardinality: 10

x	0	1	2	3	4	5	6	7	8	9
$2x \pmod{10}$	0	2	4	6	8	0	2	4	6	8

Range: $\{0,2,4,6,8\}$ Cardinality: 5

x	0	1	2	3	4	5	6	7	8	9
$3x \pmod{10}$	0	3	6	9	2	5	8	1	4	7

Range: 0 to 10 Cardinality: 10

x	0	1	2	3	4	5	6	7	8	9
$4x \pmod{10}$	0	4	8	2	6	0	4	8	2	6

Range: $\{0,2,4,6,8\}$ Cardinality: 5

k is odd

no: Range: $\{0,2,4,6,8\}$ Cardinality: 5

yes: Range: $\{0 \ldots 10\}$ Cardinality: 10

b. k is odd

no: Range: $\{x \in A : x \text{ even}\}$ Cardinality: $\lfloor n/2 \rfloor$

yes: Range: $\{0 \ldots n-1\}$ Cardinality: n