Math 347 HW 2

1.a. $\quad L \qquad = \qquad R$

$\forall(x\in S):[P(x)\vee Q(x)] \Rightarrow [\forall(x\in S):P(x)]\vee[\forall(y\in S):Q(y)]$

Assume that L is True, so either $P(x)$ or $Q(x)$ is True.
If $P(x)$ is True, then both L and R are True.
If $Q(x)$ is True, $Q(y)$ must also be True because
$\forall(x\in S):Q(x) \Leftrightarrow \forall(y\in S):Q(y)$. $\blacksquare$

$R \qquad \Rightarrow \qquad L$

$[\forall(x\in S):P(x)]\vee[\forall(y\in S):Q(y)] \Rightarrow [\forall(x\in S):P(x)\vee Q(x)]$

Assume that R is True, so $P(x)$ or $Q(y)$ is True $\forall(x,y\in S)$
R can be rewritten as $\forall(x,y\in S): P(x)\vee Q(y)$.
Because x,y can be any element of the same set, R can be simplified to one variable, x. $\blacksquare$

b. $\quad L \qquad \Rightarrow \qquad R$

$\forall(x\in S):[P(x)\wedge Q(x)] \Rightarrow [\forall(x\in S):P(x)]\wedge[\forall(y\in S):Q(y)]$

Assume L is True so $P(x)$ and $Q(x)$ are True for all $x\in S$.
Then $Q(y)$ must be True for all $y\in S$, so R is True. $\blacksquare$

$R \qquad \Rightarrow \qquad L$

$[\forall(x\in S):P(x)]\wedge[\forall(y\in S):Q(y)] \Rightarrow \forall(x\in S):[P(x)\wedge Q(x)]$

Assume R is True so $P(x)$ and $Q(y)$ are True for all elements x,y in S. R can be simplified to L. $\blacksquare$

2.a. $\quad L \qquad \Rightarrow \qquad R$

$\forall(x\in S)\,\exists(y\in S): C(x,y) \Rightarrow \exists(y\in S)\,\forall(x\in S): C(x,y)$

Assume L to be True. There exists a y for every x such that $C(x,y)$ is True; y depends on what x is. R will be false because y doesn't depend on the value of x and L does not guarantee there exists a value y to make $C(x,y)$ always True. $\blacksquare$

b. $\quad R \qquad \Rightarrow \qquad L$

$\exists(y\in S)\,\forall(x\in S): C(x,y) \Rightarrow \forall(x\in S)\,\exists(y\in S): C(x,y)$

Assume R to be True, so there exists a value of y such that for all x, $C(x,y)$ is True. This means L is True there will always be a y to make $C(x,y)$ True. $\blacksquare$

3.a. $\forall(\varepsilon>0)\,\exists(\delta>0)\,\forall(x\in\mathbb{R}): |x|<\delta \Rightarrow |f(x)|<\varepsilon$

$\forall x\in\mathbb{R}: |x|<\delta \Rightarrow |f(x)|<\varepsilon \Leftrightarrow \forall x\in(-\delta,\delta) \Rightarrow |f(x)|<\varepsilon$

From lecture, $x=0 \Leftrightarrow \forall(\varepsilon>0): |x|<\varepsilon$

The exists $\delta>0$, $|f|$ is bounded by ε on the interval $(-\delta,\delta)$

For all $\varepsilon>0$, there exits $\delta>0$, $|f|$ bounded by ε on interval $(-\delta,\delta)$

This means $f(x)=0$ at some point x.

A valid function would be $f(x)=x$.

An invalid function would be $f(x)=x^2+1$ because $f(x)$ is always greater than 0.

b. $\exists(\varepsilon>0)\,\forall(\delta>0)\,\forall(x\in\mathbb{R}): |x|<\delta \Rightarrow |f(x)|<\varepsilon$

$x=0 \Leftrightarrow \forall(\varepsilon>0): |x|<\varepsilon$

$\exists(\varepsilon>0)\,\forall(\delta>0)\,\forall x\in(-\delta,\delta): |f(x)|<\varepsilon$

This means that $f(x)=0$ for any δ so $f(x)$ is

always 0.
A valid function would be $f(x)=0$.
An invalid function would be $f(x)=x$ because $f(x)$ is not always 0.

2.3.7. $\forall(x,y\in\mathbb{R})\,\exists(c\in\mathbb{R}): x<y \Rightarrow x<c<y$
Let $c=(x+y)/2$. c is the average and will always be between x and y. ∎

13. a. $\forall x\in\mathbb{R}, \exists y\in\mathbb{R}$ such that $y^4=4x$
Let $x=-1$. y won't be a real number so this statement is False ∎

b. $\exists y\in\mathbb{R}$ such that $\forall x\in\mathbb{R}$ we have $y^4=4x$
Negation of this: $\forall y\in\mathbb{R}\,\exists x\in\mathbb{R}: y^4\neq 4x$
Let $y=1$ and $x=1$. $1\neq 4$ so this statement is False. ∎

c. $\forall y\in\mathbb{R}, \exists x\in\mathbb{R}$ such that $y^4=4x$
Negation: $\exists y\in\mathbb{R}\,\forall x\in\mathbb{R}: y^4\neq 4x$
Let $y=2$ and $x=4$. $y^4=4x$ so this statement is True. ∎

d. $\exists x\in\mathbb{R}$ such that $\forall y\in\mathbb{R}$ we have $y^4=4x$
Negation: $\forall x\in\mathbb{R}\,\exists y\in\mathbb{R}: y^4\neq 4x$
Let $x=0$ and $y=1$. $0\neq 1$ so this statement is False. ∎

17. $\forall m,n\in\mathbb{R},\ m>n \Rightarrow m^2>n^2$
a. Negation: $\exists m,n\in\mathbb{R}: m>n$ and $m^2\leq n^2$
b. Let $m=-3$ and $n=-4$. $m>n$ and $m^2\leq n^2$.
c. $\forall m,n\in A,\ m>n \Rightarrow m^2>n^2$
A has to be real numbers greater than zero, because squares of negative numbers increase as the numbers decrease.

18. a. $\forall(x,y\in\mathbb{R})\,\exists n\in\mathbb{Z}^+: x<y \Rightarrow n(y-x)>1$
$y-x>0$ since $y>x$ so $\lceil y-x\rceil\geq 1$.
$n(y-x)>\lceil y-x\rceil\geq 1$. Let $n=\lceil \lceil y-x\rceil+1/y-x\rceil$
Since $(\lceil y-x\rceil/(y-x))\ y-x=\lceil y-x\rceil\geq 1$,
$\lceil\lceil y-x\rceil+1/(y-x)\rceil>\lceil y-x\rceil$. ∎

b. $\forall x,y\in\mathbb{R}$ with $x<y$, $\exists m,n\in\mathbb{Z}$ for which $nx<m<ny$
rewritten: $\forall(x,y\in\mathbb{R}), \exists(m,n\in\mathbb{Z}): x<y \Rightarrow nx<m<ny$
Assume $x<y$. There are two cases: $y-x\leq 1$ and $y-x>$
If $y-x>1$, n can be 1 and m can be $\lceil x\rceil+1$.

If $y-x\leq 1$, n can be a power of ten for which $n(y-x)>1$. Then, m can be $\lceil nx\rceil+1$. ∎

c. From b, $nx<m<ny$ $(n,m\in\mathbb{Z})$
Then, $x<m/n<y$ ∎

d. Yes, since irrational numbers are also real numbers,
$x + \pi < m/n < y + \pi$ is still True.
Therefore, $x < m/n - \pi < y$.