HW 10

2.2.4. $X_1 = \frac{1}{2}$ $\quad X_{n+1} = X_n^2$

$X_n > 0$ for all n

Base case: $x_1 = \frac{1}{2} > 0$

Suppose $X_n > 0$; $X_{n+1} = X_n^2$

$X_n = \frac{1}{2^n}$, $\lim X_n = 0$

2.2.8. $\lim\limits_{n\to\infty} \frac{n^2}{2^n} = 0$

$$\lim_{n\to\infty} \frac{|(n+1)^2/2^{n+1}|}{|n^2/2^n|} = \lim_{n\to\infty} \left| \frac{(n^2+2n+1)}{2^{n+1}} \cdot \frac{2^n}{n^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^2+2n+1}{2n^2} \right| = \lim_{n\to\infty} \left| \frac{1}{2} + \frac{2}{n} + \frac{1}{n^2} \right| = \frac{1}{2}$$

2.2.12. a. $a_n \leq k$ for all k, $\lim b_n = 0$

$a_n b_n < k\varepsilon$ so $\lim a_n b_n = 0$

b. $\{a_n\} = \{n^2\}$, $\{b_n\} = \{\frac{1}{n}\}$

$\{a_n b_n\} = \{n\}$ not convergent

c. $\{a_n\} = \{\sin(n)\}$ $\{b_n\} = \{1\}$

$\{a_n b_n\} = \{\sin(n)\}$

2.2.16. $\{(n!)^{1/n}\}$

$$\lim_{n\to\infty} \frac{|C^{n+1}/(n+1)!|}{|C^n/n!|} = \lim_{n\to\infty} \left|\frac{C}{n+1}\right| = 0$$

so $\lim_{n\to\infty} \frac{C^n}{n!} = 0$ which means $C^n << n!$

so $\{(n!)^{1/n}\}$ is unbounded

2.3.1. $\{x_n\}$ is bounded

$\{a_n\} := \sup\{x_k : k \geq n\}$

$\{b_n\} := \inf\{x_k : k \geq n\}$

a_n and b_n are the bounds

2.3.2. $\{x_n\}$ is bounded

$\{b_n\}$ is $\inf\{x_k : k \geq n\}$

b_n is the least lower bound for $\{x_k : k \geq n\}$ so it is also the lower bound for $\{x_k : k \geq n+1\}$. Therefore, b_{n+1} has to be greater or equal to b_n.

2.3.9. If S is infinite, then S contains a countably infinite subset, and there is a subsequence S_k that converges according to the Bolzano-Weierstrauss Theorem.

Since S_k converges there has to be at least one cluster point at the infimum or supremum because the points ~~keep~~ getting closer to a point x.

2.3.11. a. $\{x_n\}$ is bounded by x since every subsequence has a subsequence that converges to x.

b. Since every subsequence of x_{n_i} converges, let there be a subsequence starting at x_2. $\{x_{n+1}\}$ converges to x so $\{x_n\}$ also converges to x.