# Midterm I: Math 347

**Instructions:** Write your proofs in complete sentences.

**Problem 1.** Let $A$, $B$, $C$, and $D$ be logical statements. For the following logical statements $L$ and $R$, prove without a truth table (or give a counterexample to) the implications $L \Rightarrow R$ and $R \Rightarrow L$:

$$(C \Rightarrow A) \Rightarrow (A \wedge B \wedge \neg C \wedge D) \quad \text{and} \quad B \wedge D \wedge ((A \Rightarrow C) \Rightarrow \neg(C \Rightarrow A))$$

**Problem 2.** Let $P(x)$ mean '$x$ has property $P$'. For a set $S$, prove the following two statements are equivalent:

L: There is an unique element in $S$ with property $P$

R: $\exists(x \in S) : \Big(P(x) \wedge \forall(y \in S) : (P(y) \Rightarrow x = y)\Big)$

**Problem 3.** Name a function $f : \mathbb{R} \to \mathbb{R}$ that does not satisfy the following property and prove your answer:

$$\forall(B \in \mathbb{R})\exists(C \in \mathbb{R})\forall(x \geq C) : f(x) \geq B$$

**Problem 4.** Let $\mathbb{Z}_n = \{0, 1, \dots n-1\}$ be equipped with addition and multiplication mod $n$ (like in problem 3.2.14 in the book). For $b \in \mathbb{Z}_n$ define the function

$$f_b : \mathbb{Z}_n \to \mathbb{Z}_n \quad \text{by} \quad f_b(x) = bx + 1$$

(where $bx + 1$ is done in $\mathbb{Z}_n$ i.e. mod $n$) and prove the following:

$$f_b \text{ is bijective if and only if } \gcd(b, n) = 1$$

**Extra Credit** (Worth 10%) Consider a kingfisher that lives in the Euclidian plane $\mathbb{R}^2$ that can hop a unit distance, but only in the North/South and East/West directions, i.e. from $(x, y)$ it can hop to one of the points $(x \pm 1, y)$ or $(x, y \pm 1)$. If the kingfisher starts out at the origin $(0, 0)$, prove it can move to $(m, n) \in \mathbb{Z}^2$ in exactly $\ell$ hops if and only if $\ell \geq |m| + |n|$ and $\ell \equiv m + n \mod 2$.

(Hint: The hard part is establishing the mod 2 condition when proving the $\Rightarrow$ implication. If you proceed by contradiction you can construct a loop with an odd number of hops ...)