Math 347 HW5

5.3.4. a. They didn't say they were using $P(n)$ to get $P(n+1)$ in the induction step.

b. Let $P(n)$ be $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$

Base Case: $\sum_{k=0}^{0} r^0 = 1 = \frac{1-r^1}{1-r} = 1$

Induction Step:

Assume $P(n)$ is true for some fixed $n \in \mathbb{N}_0$.

By assumption in our inductive hypothesis,

$$P(n+1) = \sum_{k=0}^{n+1} r^k = \sum_{k=0}^{n} r^k + r^{n+1}$$

$$= \frac{1-r^{n+1}}{1-r} + r^{n+1} = \frac{1-r^{n+1}}{1-r} + \frac{(1-r)r^{n+1}}{1-r}$$

$$= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} = \frac{1-r^{(n+1)+1}}{1-r}$$

which says that $P(n+1)$ is true.

By mathematical induction, we conclude that $P(n)$ is true for all $n \in \mathbb{N}_0$.

5.3.9. a. $q = \frac{k-m}{n}$ Let $k-m$ be multiples of n so $k-m = an$ for any $a \in \mathbb{Z}$. Then, $k = m + an$. $k \geq 0$ when $m \leq an$. Therefore, S is non-empty.

b. $m \pmod n$ gives an element from 0 to $n-1$.
r is the result of this

c. r_1 and r_2 are both the smallest element in S, which means they are equal

5.4.4. Since p is prime, $p \mid ab \Rightarrow p \mid a$ or $p \mid b$ $\forall a,b \in \mathbb{N}$. Therefore, a_i is either a or b.

5.4.5. Base case: $f_1 = 1 = \dfrac{\left(\frac{1+\sqrt5}{2}\right)^1 - \left(\frac{1-\sqrt5}{2}\right)^1}{\sqrt5} = \dfrac{\frac{1+\sqrt5-1+\sqrt5}{2}}{\sqrt5} = 1$

$$f_2 = 1 = \frac{\left(\frac{1+\sqrt5}{2}\right)^2 - \left(\frac{1-\sqrt5}{2}\right)^2}{\sqrt5} = \frac{\frac{1+2\sqrt5+5}{4} - \left(\frac{1-2\sqrt5+5}{4}\right)}{\sqrt5} = 1$$

Induction step: Fix $n \geq 2$ and suppose that $P(1), \ldots, P(n)$ are true.

$$f_{n+1} = f_n + f_{n-1} = \frac{\phi^n - \hat\phi^n}{\sqrt5} + \frac{\phi^{n-1} - \hat\phi^{n-1}}{\sqrt5}$$

$$= \frac{\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n}{\sqrt5} + \frac{\left(\frac{1+\sqrt5}{2}\right)^{n-1} - \left(\frac{1-\sqrt5}{2}\right)^{n-1}}{\sqrt5}$$

$$= \frac{\left(\frac{1+\sqrt5}{2}+1\right)\left(\frac{1+\sqrt5}{2}\right)^{n-1} - \left(\frac{1-\sqrt5}{2}+1\right)\left(\frac{1-\sqrt5}{2}\right)^{n-1}}{\sqrt5}$$

$$= \frac{\left(\frac{1+\sqrt5}{2}\right)^2\left(\frac{1+\sqrt5}{2}\right)^{n-1} - \left(\frac{1-\sqrt5}{2}\right)^2\left(\frac{1-\sqrt5}{2}\right)^{n-1}}{\sqrt5}$$

$$= \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} = \frac{\phi^{n+1} - \hat{\phi}^{n+1}}{\sqrt{5}}$$

Hence $P(n)$ is true for all $n \geq 1$

6.1.9. Prove $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$

Base case: $n=1$ $|A_1| = |A_1|$

Induction step: Assume $P(n)$ is true where $P(n)$ is $|A_1 \times \cdots \times A_n| = |A_1| \cdots |A_n|$, $n \geq 1$

Using $P(n)$, $P(n+1) = |A_1 \times \cdots \times A_{n+1}|$

$= P(n) \cdot |A_{n+1}| = |A_1| \cdots |A_{n+1}|$

Hence, $P(n)$ is true for all $n \geq 1$

6.1.10. a. $(1,1) \in E$

Rule 1: $(1,2) \in E$ Rule 1: $(1,3) \in E$

Rule 1: $(1,4) \in E$ Rule 2: $(4,1) \in E$

Rule 1: $(4,2) \in E$ Rule 1: $(4,3) \in E$

b. Base case: $(1,1) \in E$

Induction step: Assume $(1,k)$ in E for $k \leq n$

Using Rule 1, $(1, n+1) \in E$. Therefore, the induction hypothesis is true.

c. This is false since in part a, $(4,2) \in E$ and $\gcd(4,2)$ is 2.

6.2.7. Either f contains the element b or it doesn't. If it doesn't, then $f(X,1)$ is just X. If it does, $f(X,2) = X \cup \{b\}$.

$$|P(A)| = 2^n \text{ since } |P(B)| = 2^{n+1}$$

$$|P(A) \times \{1,2\}| = |P(A)| \cdot |\{1,2\}| = 2^{n+1}$$

which is the cardinality of $|P(B)|$

6.2.8. a. $\binom{n+1}{r} = \dfrac{(n+1)!}{r!\,((n+1)-r)!} = \dfrac{(n+1)\,n!}{r!\,(n+1-r)(n-r)!}$

$$= \frac{n!}{r!\,(n-r)!} \cdot \frac{n+1}{n+1-r} = \binom{n}{r}\left(\frac{(n-r+1)+r}{n-r+1}\right)$$

$$= \binom{n}{r}\left(1 + \frac{r}{n-r+1}\right) = \binom{n}{r} + \frac{n!\,r}{r!\,(n-r)!\,(n-r+1)}$$

$$= \frac{n!}{r!\,(n-r)!} + \frac{n!\,r}{r!\,(n-r+1)\,(n-r)!}$$

$$= \binom{n}{r} + \frac{n!}{(r-1)!\,(n-r+1)!} = \binom{n}{r} + \frac{n!}{(r-1)!\,(n-(r-1))!}$$

$$= \binom{n}{r} + \binom{n}{r-1} \quad P(n) \text{ true for } n \geq 1$$

b. $\forall n \in \mathbb{N}_0, \ \sum_{r=0}^{n} \binom{n}{r} = 2^n$

Base case: $n=1$

$$\binom{1}{0} + \binom{1}{1} = 1+1 = 2$$

Inductive step: Assume the statement holds for all $n \in \mathbb{N}_0$. Using the equation,

$$\sum_{r=0}^{n+1} \binom{n+1}{r} = \sum_{r=0}^{n} \binom{n+1}{r} + \binom{n+1}{n+1}$$

$$= \sum_{r=0}^{n} \left(\binom{n}{r} + \binom{n}{r-1} \right) + 1$$

$$= 2^n + \sum_{r=0}^{n} \binom{n}{r-1} + 1$$

$$= 2^n + 2^n - 1 + 1 = 2^{n+1}$$

Therefore, $P(n)$ is true for $n \geq 1$

c. $|\mathcal{P}(A)| = 2^{|A|}$

In part b, we are just choosing how many sets can be in A so n is how many elements A has.

6.3.4. a. $A_n = \{ x \in [1, 2 + \frac{1}{n}) : n \in \mathbb{N} \}$

$$\cup A_n = \{[1,3)\} \quad \cap A_n = \{[1,2]\}$$

b. $A_n = \left\{ x \in \left(\frac{-a}{b}, a+1\right) : \begin{array}{l} a \pmod 2 = 1 \\ a, b \in \mathbb{N} \end{array} \right\}$

$\bigcup A_n = (-\infty, \infty) \quad \bigcap A_n = (-1, 2)$

c. $A_n = \left\{ x \in \left(\frac{1}{2^n}, \frac{1}{2^{n-2}}\right) : n \in \mathbb{N}_{\geq 2} \right\}$

$\bigcup A_n = (0, 1) \quad \bigcap A_n = \emptyset$

6.3.8. $f(x) = x$ has only one zero, so

$\bigcap A_f = \{0\}$

$f(x) = x(x-a)$ where a is between 0 and 1 has zeroes at all points between 0 and 1 so $\bigcup A_f = [0,1]$

6.3.11. a. $A_1 = \emptyset \qquad A_2 = \{\frac{1}{2}\}$

$A_3 = \{\frac{1}{3}, \frac{2}{3}\} \quad A_4 = \{\frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$

b. $A_m = \emptyset$ since $A_m = \{\frac{m}{m} : 0 < m < m\}$

$\emptyset \subseteq A_{pm}$

c. $\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{Q} \cap (0,1)$

Any element in $A_n < 1$ since $m < n$ and greater than one since $m, n \in \mathbb{N}$

$$\mathbb{Q} \cap (0,1) \subseteq \bigcup_{n \in \mathbb{N}} A_n$$

All elements in A_n are rational numbers.

d. From b, $A_n \subseteq A_{2n}$ so $\bigcup_{n \in \mathbb{N}} A_{2n} = \mathbb{Q} \cap (0,1)$

e. From b, $A_n \subseteq A_{pn}$, $\mathbb{Q} \cap (0,1) \subseteq A_{pn}$

so $A_{pn} = \mathbb{Q} \cap (0,1)$