

A Functional Approach to (parallelised) Monte Carlo Simulation

Advanced R for Econometricians

Final Project

Submitted to the Faculty of
Business Administration and Economics
at the
University of Duisburg-Essen

from:

Alexander Langnau, Öcal Kaptan, Sunyoung Ji

Matriculation Number:	232907, 230914, 229979
Study Path:	M.Sc. Econometrics
Reviewer:	Prof. Dr. Christoph Hanck
Secondary Reviewer:	M.Sc. Martin C. Arnold, M.Sc. Jens Klenke
Semester:	1 st Semester
Graduation (est.):	Summer Term 2022
Deadline:	09. 09. 2022

Contents

1	Introduction	1
2	Pre-process: Creating Helper functions	1
2.1	Helper function 1: <code>create_grid()</code>	1
2.2	Helper function 2: <code>data_generation()</code>	3
2.3	Helper function 3: <code>summary_function()</code>	6
2.4	Helper function 4: <code>create_array_function()</code>	8
2.5	Helper function 5: <code>average_function()</code>	11
2.6	Helper function 6: <code>output_function()</code>	12
3	Monte Carlo simulation function: <code>main_function()</code>	14
3.1	Test 1 : Summary performance	15
3.2	Test 2 : Visualization performance	16
4	Examples	19
4.1	Comparing the execution times of OLS and GLS simulations	19
4.2	Visualisation	22
5	Conclusion	25
6	Contributions	26
7	References	27
8	Appendix	28
8.1	Appendix A	28
8.2	Appendix B	47

List of Tables

List of Figures

1 Introduction

The Monte Carlo method is a simulation method for calculating the probabilistic value of the desired function using random numbers. A repeated pseudo-random number generator estimates sample statistics and returns a probability distribution of the sample, representing the parameter (Barbu_2020). Monte Carlo methods are combined with programming in modern research and contribute to various studies in statistics, economics and many other scientific fields. This paper progresses on developing a collection of different wrapper functions to partially automatize the process of running a Monte Carlo simulation. The main function, that combines all the other helper functions, provides a convenient interface for Monte Carlo simulations and allows the user to create a parameter grid and iterate homogeneous function calls over the parameter grid. It also offers informative summary statistics, including visualization with `ggplot2`-methods and an option to use a parallelisation plan utilizing the `furrr` package. The paper proceeds as follows: Chapter 2 describes pre-processes to establish the Monte Carlo simulation function. The pre-process contains functions to create a parameter grid and the respective data points drawn from a user-defined distribution, along with functions that provide summary statistics. Chapter 3 details the main Monte Carlo simulation function, which consists of the helper functions introduced in chapter 2. Chapter 4 presents specific examples for the simulation process. Finally, chapter 5 summarizes the work presented in this paper. Since the output of the simulation in chapter 4 is quite lengthy, it was moved to the Appendix in order not to overload the structure of the actual paper.

2 Pre-process: Creating Helper functions

The pre-process comprises 5 helper functions: `create_grid()`, `data_generation()`, `summary_function()`, `create_array_function()` and `output_function()`. The functions are named in a way that the underlying purpose is directly clear. These functions are the building blocks of the `main_function()`, which takes the user input and runs the Monte Carlo simulation by itself.

2.1 Helper function 1: `create_grid()`

The first helper function introduced is the `create_grid()`-function, which automatically creates a hyper-parameter grid over all permutations specified by the user.

```
create_grid <- function(parameters, nrep){  
  input <- parameters  
  storage <- list()  
  name_vec <- c()  
  
  for(i in 1:length(input)){ #1:3  
    a <- as.numeric(input[[i]][[2]])
```

```

    b <- as.numeric(input[[i]][[3]])
    c <- as.numeric(input[[i]][[4]])
    output <- seq(from=a, to=b, by=c)
    storage[[i]] <- output
    name_vec[i] <- input[[i]][[1]]
  }

  grid <- expand_grid(unlist(storage[1])
                    , unlist(storage[2])
                    , unlist(storage[3])
                    , unlist(storage[4])
                    , unlist(storage[5])
                    , c(1:nrep))

  names(grid) <- c(name_vec, "rep")

  return(grid)
}

```

Input for the parameter list has to follow a specific format as shown below:

```

parameter_list <- list(c("variable name 1", from, to, by)
                      ,c("variable name 2", from, to, by)
                      ,c("variable name 3", from, to, by)
                      ,c("variable name 4", from, to, by))

```

`create_grid()` works with a minimum of 1 and a maximum of 4 variables. The structure of arguments in `parameter_list` is similar to `seq()` in R: Each vector contained in the list needs four arguments specified, that is, the function name, the start of the sequence, the end of the sequence and the steps, by which the interval gets divided. It would be easy to adapt this helper function for more parameters, but it is assumed that a grid with up to 4 parameters offers enough complexity for the simulation. The function basically takes the information of the input parameter list and creates a grid with `tidyr::expand_grid()`. The argument `nrep` specifies how many repetitions per parameter constellation are created, where a separate row in the parameter grid gets created for each repetition.

Example to create_grid():

```

#four parameters
param_list0 <- list(c("n", 10, 20, 10)
                  ,c("mu", 0, 0.5, 0.25)
                  ,c("sd", 0, 0.3, 0.1)
                  ,c("gender", 0, 1, 1))

```

```
head(create_grid(param_list0, nrep=3), n=10)
```

```
## # A tibble: 10 x 5
##       n      mu    sd gender  rep
##   <dbl> <dbl> <dbl>   <dbl> <int>
## 1     10      0    0         0     1
## 2     10      0    0         0     2
## 3     10      0    0         0     3
## 4     10      0    0         1     1
## 5     10      0    0         1     2
## 6     10      0    0         1     3
## 7     10      0  0.1         0     1
## 8     10      0  0.1         0     2
## 9     10      0  0.1         0     3
## 10    10      0  0.1         1     1
```

2.2 Helper function 2: data_generation()

The second helper function called `data_generation()` takes the arguments `grid` and `simulation` as inputs. `simulation` is the argument for the user-defined function for the data generation process, while `grid` is the parameter grid previously created. The user can choose between a variety of probability distributions by entering the name of the function, for example, `rnorm` for the normal distribution or `runif` for the uniform distribution.

The n data points created for each set of parameters are stored as separate elements in a list, since this format is a very flexible way of storing data.

`data_generation()` chooses relative mapping function base on specification shown as on the table below:

Mapping	Mapping with parallelisation	Number of parameters
<code>purrr::map()</code>	<code>furrr::future_map()</code>	$n = 1$
<code>purrr::map2()</code>	<code>furrr::future_map2()</code>	$n = 2$
<code>purrr::pmap()</code>	<code>furrr::future_pmap()</code>	$n \geq 3$

`options = furrr_options(seed = TRUE)` is for reproducible random number generation (RNG) processes. This argument takes control of the RNG process for parallelisation and always generates the same numbers according to the given seed. More details can be found by running the command `?furrr_options` in RStudio.

```
data_generation <- function(simulation, grid){
```

```

if(ncol(grid)==2){
  var1 <- c(unlist(grid[,1]))
  if(cores>1){
    data <- future_map(var1, simulation,
                        .options = furrr_options(seed = TRUE))
  }else{
    data <- map(var1, simulation)
  }
}

if(ncol(grid)==3){
  var1 <- c(unlist(grid[,1]))
  var2 <- c(unlist(grid[,2]))
  if(cores>1){
    data <- future_map2(var1, var2, simulation,
                        .options = furrr_options(seed = TRUE))
  } else{
    data <- map2(var1, var2, simulation)
  }
}

if(ncol(grid)==4){
  var1 <- c(unlist(grid[,1]))
  var2 <- c(unlist(grid[,2]))
  var3 <- c(unlist(grid[,3]))
  list1 <- list(var1,var2,var3)
  if(cores>1){
    data <- future_pmap(list1, .f=simulation,
                        .options = furrr_options(seed = TRUE))
  }else{
    data <- pmap(list1, .f=simulation)
  }
}

return(data)
}

```

Monte Carlo simulations can become very demanding in terms of computing time. In that case, parallel processing may be used to reduce the time of the simulation. In parallel processes, each process is executed simultaneously but independently. Interconnections are proceeded through communication channel (**Czech_2017**).

`data_generation()` automatically used the proper function from the `furrrrr`-package, if the

user specified more than one core. Otherwise, the function will stick to the respective mapping function from the `purrr`-package.

Example to `data_generation()`: The example below demonstrates the data generation process using a Poisson distribution without parallelisation. The advantage in computation time with parallelisation will be discussed in Chapter 3.

```
cores <- 1

param_list2 <- list(c("n", 10, 20, 10)
                    ,c("lambda", 0.5, 1, 0.5))

grid2 <- create_grid(param_list2, nrep=1)
sim1 <- data_generation(simulation=rpois, grid=grid2)

names(sim1) <- c("n=10, lamda = 0.5", "n=10, lamda = 1",
                 "n=20, lamda = 0.5", "n=20, lamda = 1")
grid2

## # A tibble: 4 x 3
##       n lambda  rep
##   <dbl> <dbl> <int>
## 1    10   0.5     1
## 2    10     1     1
## 3    20   0.5     1
## 4    20     1     1

sim1

## $`n=10, lamda = 0.5`
## [1] 0 1 0 1 2 0 0 1 0 0
##
## $`n=10, lamda = 1`
## [1] 3 1 1 1 0 2 0 0 0 3
##
## $`n=20, lamda = 0.5`
## [1] 1 1 1 3 1 1 0 0 0 0 2 1 1 1 0 0 1 0 0 0
##
## $`n=20, lamda = 1`
## [1] 0 1 1 1 0 0 0 1 0 2 0 1 2 0 1 0 0 2 2 1
```

`sim1` contains the generated data by `data_generation()` with `grid2` and Poisson distribution as input. Below, the second example is demonstrated to showcase the capabilities using a uniform distribution. The user could even define his own function for generating the data and use it for the simulation process. The information stored in the parameter list is used by position, not by variable name. That's why it's important to specify any variables in the correct order, that shall be used for the simulation.

```
# Application to uniform distribution
param_list_runif <- list(c("n", 10, 30, 10)
                        ,c("min", 0, 0, 0)
                        ,c("max", 2, 2, 0))

grid_unif <- create_grid(param_list_runif, nrep=3)
head(data_generation(simulation=runif, grid=grid_unif),4)

## $n1
## [1] 1.3302304 0.1896813 0.7679393 0.5487673 1.6292801 0.8970327 1.6201287
## [8] 1.6247790 1.5886846 0.8796634
##
## $n2
## [1] 1.508950317 1.258442263 1.420364803 0.001249547 0.950633148 0.440237770
## [7] 0.759633075 1.225542007 0.703595818 0.222270849
##
## $n3
## [1] 0.4872389 1.3361112 0.8352936 1.5763917 0.2057293 0.8697855 1.9699140
## [8] 1.7861022 1.7729381 0.3501053
##
## $n4
## [1] 0.2613914 1.3062039 0.6870329 1.3135163 0.6407465 0.3753822 1.5645886
## [8] 0.1871900 0.9335581 1.0230109 1.1999779 0.6656471 0.9772261 1.9089477
## [15] 0.9658048 1.7807004 1.8288764 1.2174700 0.8213796 0.2941894
```

As explained before, the data is stored in a list, where data points for each set of parameters are stored under a different variable. The first variable (saved under `$n1`) is based on 10 draws from a uniform distribution with $\mathcal{U}_{[0,2]}$. Each variable in the list relates to a row in the parameter grid.

2.3 Helper function 3: `summary_function()`

So far, the paper explains how to create *raw* data with the functions `create_grid()` and `data_generation()` based on a chosen probability distribution. This chapter introduces `summary_function()` which applies a user-defined function to calculate a summary statistic on the raw data generated before. The function uses a `sapply`-loop to apply the user-defined

summary function (`sum_fun`) on each element of the list containing the raw data. Any summary may be applied, that returns a singular output value. Results are stored in a (`nrow(grid)` X 1)-dimensional matrix, which is combined with the parameter grid in the next step.

```
summary_function <- function(sum_fun, data_input){

  count <- length(data_input)
  summary_matrix <- matrix(nrow=count, ncol=1)

  for(i in 1:count){
    input <- list(data_input[[i]])
    output <- sapply(sum_fun, do.call, input)
    summary_matrix[i] <- output
  }

  colnames(summary_matrix) <- sum_fun
  return(summary_matrix)
}
```

Example to `summary_function()`:

For this example, data is generated using the normal distribution, with the respective parameters n , μ and σ specified in the parameter list. The list containing the raw data created by `data_generation()` is stored in the variable `test_data`, which is the data input for `summary_function()`. The arithmetic mean, using base R's `mean()`-function, is supplied as the second input to calculate the average of all the data points created under a unique set of parameters. The output relates exactly to each row of the used parameter grid. Later, these summary statistics are merged with the parameter grid into a data frame.

```
param_list3 <- list(c("n", 10, 20, 10)
                   ,c("mu", 1, 2, 0.25)
                   ,c("sd", 0.5, 1, 0.1))

grid_test <- create_grid(param_list3, nrep=3)
test_data <- data_generation(simulation=rnorm, grid=grid_test)
summary_data <- summary_function(sum_fun=list("mean"),
                                data_input=test_data)

nrow(summary_data)
```

```
## [1] 180
```

```
head(summary_data)
```

```
##           mean
## [1,] 0.9447809
## [2,] 1.0168699
## [3,] 0.7629786
## [4,] 1.0764221
## [5,] 1.1742643
## [6,] 1.0260035
```

2.4 Helper function 4: `create_array_function()`

Even though the example above applies a relatively small parameter grid, the simulation still returns 180 summarize data points. Like explained before, in `main_function()` the results from the previous step get merged with the parameter grid into one data frame. This way of storing the data allows the user to apply further data wrangling processes, but is not the most overseeable way to display the data. A multidimensional array is more suitable for printing the output in a tidy and clear way. `create_array_function()` takes all relevant data from the previous steps and transforms it into an array with the correct dimensions. The function works with up to three variables supplied in the parameter list. Using the number of repetitions as an additional dimension, `create_array_function()` can create up to 4-dimensional arrays. Due to the same reasoning regarding the size of the parameter grid, no further capabilities were implemented beyond this point, since five or more dimensional arrays would be extremely hard to overlook, even though it would be easy to expand the use for more variables.

```
create_array_function <- function(comb, parameters, nrep){
  storage <- list()
  name_vec <- c()

  for(i in 1:length(parameters)){
    a <- as.numeric(parameters[[i]][[2]])
    b <- as.numeric(parameters[[i]][[3]])
    c <- as.numeric(parameters[[i]][[4]])
    output <- seq(from=a, to=b, by=c)
    storage[[i]] <- output
    name_vec[i] <- parameters[[i]][[1]]
  }

  matrix.numeration <- paste("rep", "=", 1:nrep, sep = "")

  if(length(parameters)==1){ #for 1 variable
```

```

comb_ordered <- comb %>% arrange(comb[,2])
seq1 <- c(unlist(storage[1]))

row.names <- paste(name_vec[1], "=", seq1, sep = "")

dimension_array <- c(length(seq1), nrep)
dim_names_list <- list(row.names, matrix.numeration)
}

if(length(parameters)==2){ #for 2 variables
  comb_ordered <- comb %>% arrange(comb[,2]) %>% arrange(comb[,3])
  seq1 <- c(unlist(storage[1]))
  seq2 <- c(unlist(storage[2]))

  row.names <- paste(name_vec[1], "=", seq1, sep = "")
  column.names <- paste(name_vec[2], "=", seq2, sep = "")

  dimension_array <- c(length(seq1), length(seq2), nrep)
  dim_names_list <- list(row.names, column.names, matrix.numeration)
}

if(length(parameters)==3){ #for 3 variable
  comb_ordered <- comb %>% arrange(comb[,2]) %>%
    arrange(comb[,3]) %>% arrange(comb[,4])
  seq1 <- c(unlist(storage[1]))
  seq2 <- c(unlist(storage[2]))
  seq3 <- c(unlist(storage[3]))

  row.names <- paste(name_vec[1], "=", seq1, sep = "")
  column.names <- paste(name_vec[2], "=", seq2, sep = "")
  matrix.names1 <- paste(name_vec[3], "=", seq3, sep = "")

  dimension_array <- c(length(seq1), length(seq2), length(seq3), nrep)
  dim_names_list <- list(row.names, column.names,
                        matrix.names1, matrix.numeration)
}

array1 <- array(comb_ordered[,ncol(comb)]
               , dim = dimension_array
               , dim_names_list)

```

```

    return(array1)
}

```

In order to test `create_array_function()`, we need to set up an altered version of `main_function()`. The `main_function()` will be discussed in the next chapter. Also, a slightly modified version of the example using `rnorm` is used, where the parameter grid spans over a larger sequence, to demonstrate, that `create_array_function()` also works for a complex parameter grid.

Example to `create_array_function()` :

```

# PREP TEST `create_array_function`
main_function_array_test <- function(parameters #list of parameters
                                     , nrep #number of repetitions
                                     , simulation #data generation
                                     , sum_fun){ #summary statistics

  grid <- create_grid(parameters, nrep) #Step 1: create grid
  raw_data <- data_generation(simulation, grid) #Step 2: simlate data
  summary <- summary_function(sum_fun, data_input=raw_data) #Step 3: Summary statistics
  comb <- cbind(grid, summary) #Step 4: Combine resuluts with parameters

  return(comb)
}

param_list3x <- list(c("n", 10, 30, 10)
                    ,c("mu", 0, 6, 1)
                    ,c("sd", 1, 2, 1))

comb1 <- main_function_array_test(parameters=param_list3x
                                  , nrep = 2
                                  , simulation = rnorm
                                  , sum_fun="mean")

array_test <- create_array_function(comb=comb1,
                                    parameters=param_list3x, nrep=2)

array_test

```

```
## , , sd=1, rep=1
```

```

##
##          mu=0      mu=1      mu=2      mu=3      mu=4      mu=5      mu=6
## n=10  0.2614310  1.0561405  1.783523  3.547164  4.400955  4.649202  5.979642
## n=20 -0.1674596  1.0302339  1.654583  3.156313  3.810663  4.906353  6.260126
## n=30 -0.1080901  0.7546153  2.106712  2.879157  3.762768  5.146553  5.924848
##
## , , sd=2, rep=1
##
##          mu=0      mu=1      mu=2      mu=3      mu=4      mu=5      mu=6
## n=10 -0.5766303  1.081483  1.540717  3.001723  4.244725  4.861292  6.250286
## n=20 -0.2090269  1.149640  2.185703  2.899424  4.206897  4.987184  5.791887
## n=30 -0.4031820  1.498366  2.251058  3.012474  3.784264  4.631249  5.324699
##
## , , sd=1, rep=2
##
##          mu=0      mu=1      mu=2      mu=3      mu=4      mu=5      mu=6
## n=10  0.002933627  0.8484961  1.893848  3.459722  4.035440  4.569448  6.111295
## n=20 -0.371251268  1.3979844  1.944946  3.084847  4.367529  4.832309  5.953284
## n=30  0.076642145  0.9738097  2.346561  2.793762  4.110612  4.962612  6.259721
##
## , , sd=2, rep=2
##
##          mu=0      mu=1      mu=2      mu=3      mu=4      mu=5      mu=6
## n=10  1.2275878  0.9011456  0.9410799  4.187987  4.570954  4.951318  5.424411
## n=20 -0.3172296  1.2432931  1.9439902  2.993698  3.495244  5.694815  6.022008
## n=30 -0.5051576  0.9019707  1.8288620  3.064289  3.894587  5.358014  5.693080

```

Under these specifications, the data was transformed into a 4-dimensional array. Sample size n and the mean μ are the variables at the side of each row and column. Also, it was specified that two repetitions are done for each set of parameters, while a range of two values ($\sigma = \{1, 2\}$) was given for the standard deviation. The output contains the correct amount of tables, namely two repetitions for $\sigma = 1$ and two repetitions for $\sigma = 2$. This output style allows the user to easily oversee a wide variety of different parameter constellations.

2.5 Helper function 5: `average_function()`

The example above used only two repetitions for each parameter constellation to keep the output simple. However, in practice the user would probably repeat the simulations over a higher number of repetitions, which would also drastically increase the size of the array. It may be of interest to see the average over all repetitions, in order to get a more compact output. `average_function()` calculates the average over all repetitions and stores the results in an array with a dimension, that is reduced by one, since the dimension for repetitions is not relevant anymore. At this point no further example is demonstrated for this function, because the output would look similar to

the array before. But the final simulation will include the averaged simulation results, that are created the help of this function.

```
average_function <- function(grid_for_avg, summary, nrep){
  grid_for_avg <- grid_for_avg[-ncol(grid_for_avg)] #remove column for reps
  n_rows <- nrow(grid_for_avg)
  n_col <- ncol(grid_for_avg)

  for(i in 1:n_rows){
    start <- 1 + (i-1)*nrep
    end <- i*nrep
    grid_for_avg[i, n_col+1] <- mean(summary[start:end, ])
  }

  grid_plus_mc <- data.frame(grid_for_avg)

  colnames(grid_plus_mc)[n_col+1] <- "avg"

  return(grid_plus_mc)
}
```

2.6 Helper function 6: output_function()

`output_function()` is the last part of the pre-process. This function takes results and parameters of the Monte Carlo simulation as inputs and converts them into an tidy output format. Thus, the user can obtain a tidy form of a simulation summary.

`array_1`, `average_over_reps`, `parameters`, `cores`, `simulation`, `nrep` and `cpt` are used as input parameters in the `output_function()`. The variable `cpt` is used to store the execution time of the simulation. The next chapter will discuss this in more detail.

Regarding the structure of `output_function()`, (1) `out` is an empty list for storing simulation results, averaged results and summary. (2) `Eco` is the class name of `out`. The class should be assigned to visualize simulation results with `ggplot2` methods. (3) Results from `array_1` and `average_over_reps` are assigned as `out$results` and `out$average` respectively. (4) These results have classes with their own names: `out$results` and `out$average`, thereby preventing future error when visualizing simulation results with `ggplot2`-methods. `ggplot2`-methods require specific classes, such as `data.frame`. (5) `cat` is useful for producing output in user-defined functions. It converts its arguments to character vectors, concatenates them to a single character vector, appends the given `sep = string(s)` to each element and then outputs them (`rdocumentation`). (6) The neat output is returned at the end.


```

output_function <- function(array_1,average_over_reps,parameters,cores,simulation,
                             nrep,cpt){

  # (1)
  out <- list()
  # (2)
  class(out) <- "Eco"
  # (3)
  out$results <- array_1
  out$aaverage <- average_over_reps
  # (4)
  class(out$aaverage) <- c("Eco",class(out$aaverage))
  class(out$results) <- c("Eco",class(out$results))
  # (5)
  if(cores>1){
    parallel = "Multisession"
  } else {
    parallel = "Sequential"
  }
  text <- cat("\n",
              "Repetition(nrep)      : ",nrep,"\n\n",
              "Parallelization Type   : ",parallel,"\n\n",
              "Number of Cores Used in Parallelization : ",cores,
              " out of",detectCores(),"\n\n",
              "Input Parameters : ",paste(parameters),"\n\n",
              "Simulation Length : ",length(array_1),"\n",
              "Minumum : ",min(array_1),"\n",
              "Maximum : ",max(array_1),"\n",
              "Mean      : ", mean(array_1),"\n",
              "Median   : ",median(array_1),"\n\n",
              "Execution Time of Monte Carlo Simulation",
              as.numeric(cpt),"secs \n\n",
              "Name of The Class : ",class(out))
  # (6)
  return(out)
}

```

`output_function()` combines all information relevant for the simulation and stores it into a tidy output format, which gets returned by the `main_function()`, that combines all steps needed for running the simulation.

3 Monte Carlo simulation function: `main_function()`

The `main_function()` is built-up by the helper functions introduced in chapter 2 and includes additional arguments to combine different steps of the helper functions. (1) `parameters` is a parameter list, `nrep` is the number of repetitions, `simulation` is a data generation process, `sum_fun` is summary function that the user defines, `seed set` is for reproducibility of the simulation and `cores` is the number of cores, that exist in the CPU. (2) `if` and `else` commands check how many cores are used in the `main_function()` and whether the user defined variable `cores` exceeds the maximum number of cores contained in the CPU. By using `detectCores()` from the `parallel` package, the maximum number of cores in the CPU is saved under the variable `max.cores`. (3) Setting the seed is an important step to obtain reproducibility. `main_function()` either takes the seed that the user inputs or creates a seed randomly using `sample.int()`, when no seed is provided. (4) The function `Sys.time()` measures the execution time of the simulation. `startTime` and `endTime` save the simulation's starting time and ending time, respectively. The difference between them is the execution time stored in `cpt`, which is also included in the summary statistics. (5) `plan()` is used to parallelise the simulation. `plan()` has two arguments, "sequential" or "multisession". "Sequential" runs non-parallel processing with one core. On the other hand, user can define the number of cores with the "multisession" argument. Lastly, (6) `main_function()` produces tidy summary statistics with `output_function()` in chapter 2.

```
#(1)
main_function <- function(parameters
                           , nrep
                           , simulation
                           , sum_fun
                           , seed = NULL
                           , cores=NULL){

  #(2)
  max.cores <- detectCores()
  if(cores>max.cores){
    stop("Number of Cores cannot be bigger than total number of cores")
  }

  #(3)
  if(!is.null(seed)) {
    set.seed(seed)}
  else {
    warning("No seed provided!", call. = FALSE)
    seed <- sample.int(10000, 1)
    set.seed(seed)
    message("Random seed = ", seed, "\n")}
```

```

#(4) and (5)
startTime <- Sys.time()#Starting time

grid <- create_grid(parameters, nrep)

if(cores > 1){
  plan(multisession,workers = cores)
} else{
  plan(sequential)
}
suppressMessages(raw_data <- data_generation(simulation, grid))

summary <- summary_function(sum_fun, data_input=raw_data)

average_over_reps <- average_function(grid_for_avg=create_grid(parameters, 1),
                                     summary, nrep)

comb <- cbind(grid, summary)

array_1 <- create_array_function(comb, parameters, nrep)

endTime <- Sys.time()

cpt <- endTime - startTime

#(6)
summary_1 <- output_function(array_1,
                             average_over_reps,
                             parameters,cores,
                             simulation,nrep,cpt)

return(summary_1)
}

```

The tests below present the performance of `main_function()` with the normal distribution.

3.1 Test 1 : Summary performance

```

param_list3x <- list(c("n", 10, 100, 10)
                    ,c("mu", 0, 10, 1)
                    ,c("sd", 0, 5, 1))

test_me <- main_function(parameters=param_list3x
                        , nrep = 5
                        , simulation = rnorm
                        , sum_fun="mean"
                        , seed=123
                        , cores=1)

```

```

##
## Repetition(nrep)      : 5
##
## Parallelization Type : Sequential
##
## Number of Cores Used in Parallelization : 1 out of 8
##
## Input Parameters : c("n", "10", "100", "10") c("mu", "0", "10", "1") c("sd", "0", "5",
##
## Simulation Length : 3300
## Minumum : -1.844634
## Maximum : 13.43665
## Mean : 5.001471
## Median : 5
##
## Execution Time of Monte Carlo Simulation 0.2251358 secs
##
## Name of The Class : Eco

```

All summary components, results, average result and simulation summary are clear.

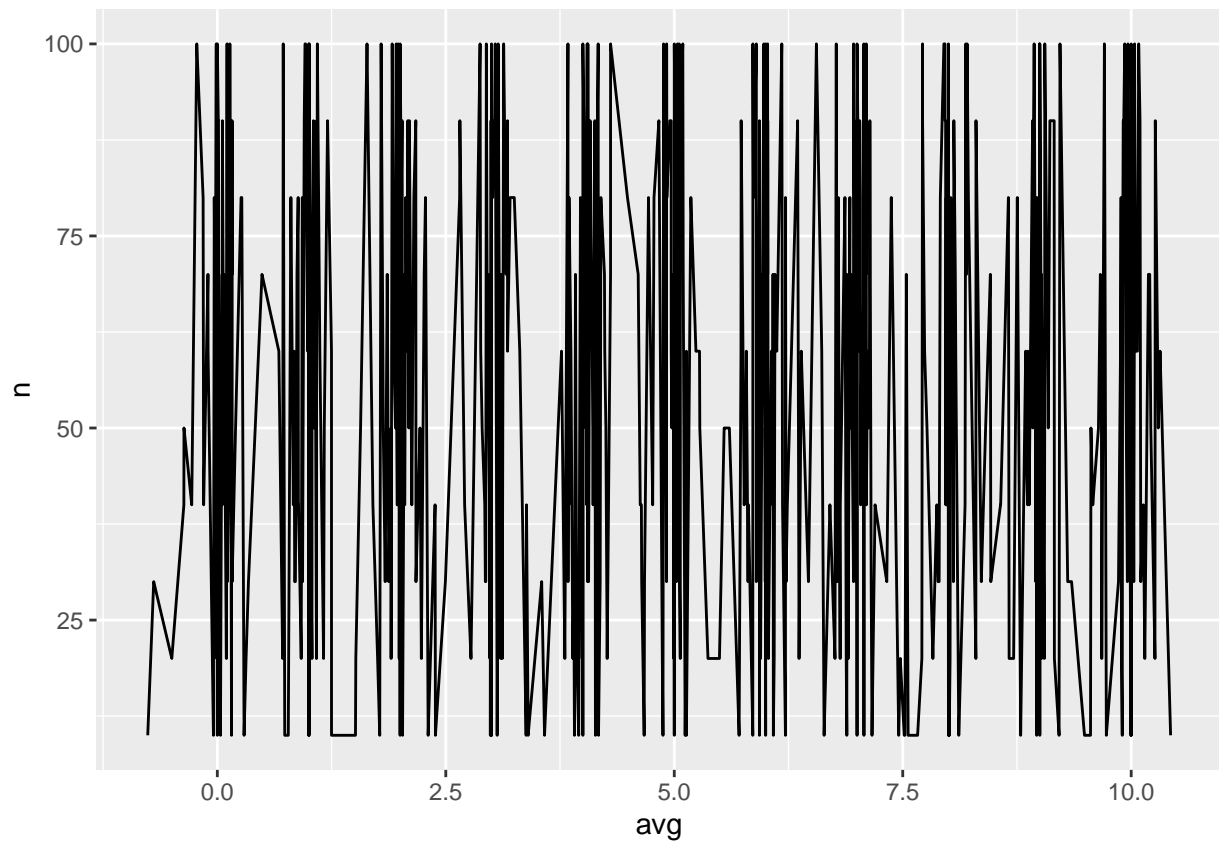
3.2 Test 2 : Visualization performance

- Visualizing average

```

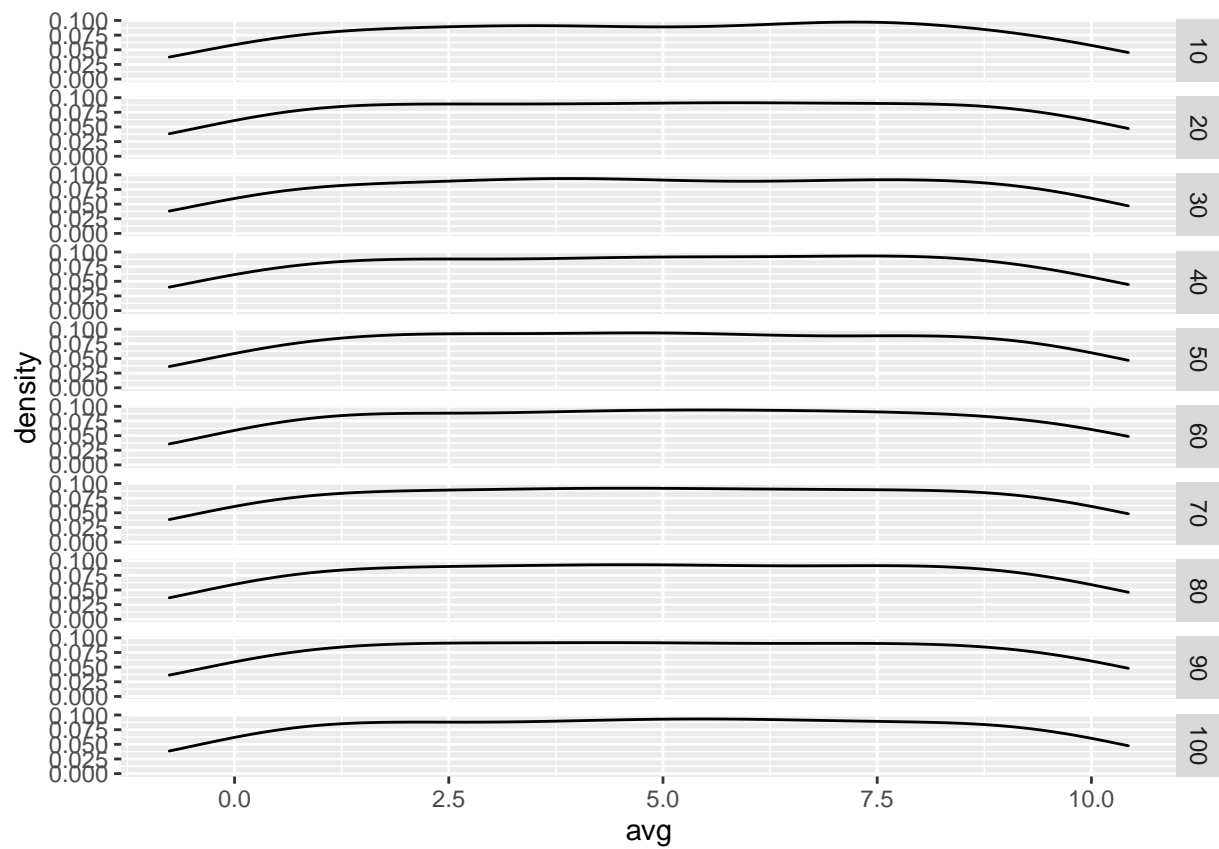
ggplot(test_me$average,aes(x=avg,y=n))+geom_line()

```



- Visualizing average by using `facet_grid()`

```
ggplot(test_me$average,aes(x=avg))+facet_grid(n~.)+
  geom_density()
```



As given graphs above, the simulation result works well with **ggplot2**-methods.

4 Examples

4.1 Comparing the execution times of OLS and GLS simulations

As an example of Monte Carlo simulation, OLS and GLS coefficients β are estimated with and without parallelisation to compare the execution time.

```
ols_f <- function(n,mu,sd){
  e <- rnorm(n,mu,sd)
  x <- runif(n)
  y <- 0.5*x + e
  ols.hat <- t(x) %*% y / t(x)%*%x
  return("ols"=ols.hat)}

gls_f <- function(n,mu,sd){
  e <- rnorm(n,mu,sd)
  x <- runif(n)
  y <- 0.5*x + e
  v.inv <- diag(1/(1:n))
  c <- chol(v.inv)
  cy <- c %*% y
  cx <- c %*% x
  gls_hat <- t(cx) %*% cy / t(cx)%*%cx
  return("gls"=gls_hat)

  param_list <- list(c("n",100,1000,100),c("mu",0,1,0.25),c("sd",1,2,.5))
}
```

As shown above, simple OLS and GLS functions are defined to find β coefficients. However, the execution time of the GLS would be much longer than OLS since the Cholesky Decomposition (`chol()`) is applied to the GLS function.

OLS simulation without parallel processing:

```
param_list <- list(c("n",100,1000,100),c("mu",0,1,0.25),c("sd",1,2,.5))
ols <- main_function(parameters = param_list,
                     nrep=5,
                     simulation = ols_f,
                     sum_fun="mean",
                     seed=123,
                     cores=1)
```

```
##
```

```
## Repetition(nrep)      : 5
```

```
##
## Parallelization Type : Sequential
##
## Number of Cores Used in Parallelization : 1 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : 0.08611764
## Maximum : 2.598461
## Mean : 1.251827
## Median : 1.246948
##
## Execution Time of Monte Carlo Simulation 8.46029388 secs
##
## Name of The Class : Eco
```

The total execution time of OLS simulation is 8.46 seconds when only one core is used.

OLS simulation with parallel processing:

```
ols <- main_function(parameters = param_list,
                      nrep=5,
                      simulation = ols_f,
                      sum_fun="mean",
                      seed=123,
                      cores=4)
```

```
##
## Repetition(nrep) : 5
##
## Parallelization Type : Multisession
##
## Number of Cores Used in Parallelization : 4 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : 0.08611764
## Maximum : 2.598461
## Mean : 1.251827
## Median : 1.246948
##
```



```
## Execution Time of Monte Carlo Simulation 6.048335 secs
##
## Name of The Class : Eco
```

The total execution time of OLS simulation is 6 seconds when four cores are used.

GLS with parallel processing:

```
gls <- main_function(parameters = param_list,
                      nrep=5,
                      simulation = gls_f,
                      sum_fun="mean",
                      seed=123,
                      cores=4)
```

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type : Multisession
##
## Number of Cores Used in Parallelization : 4 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : -0.9971
## Maximum : 3.383637
## Mean : 1.260196
## Median : 1.284057
##
## Execution Time of Monte Carlo Simulation 29.79384 secs
##
## Name of The Class : Eco
```

GLS without parallel processing:

```
gls <- main_function(parameters = param_list,
                      nrep=5,
                      simulation = gls_f,
                      sum_fun="mean",
                      seed=123,
                      cores=1)
```

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type  : Sequential
##
## Number of Cores Used in Parallelization : 1 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : -0.9971
## Maximum : 3.383637
## Mean : 1.260196
## Median : 1.284057
##
## Execution Time of Monte Carlo Simulation 39.71279 secs
##
## Name of The Class : Eco
```

As seen in the summary part, the total execution time of the simulation took 36.35 seconds which also proves that the parallel process works well. Therefore, the execution times of the simulation might differ on other computers.

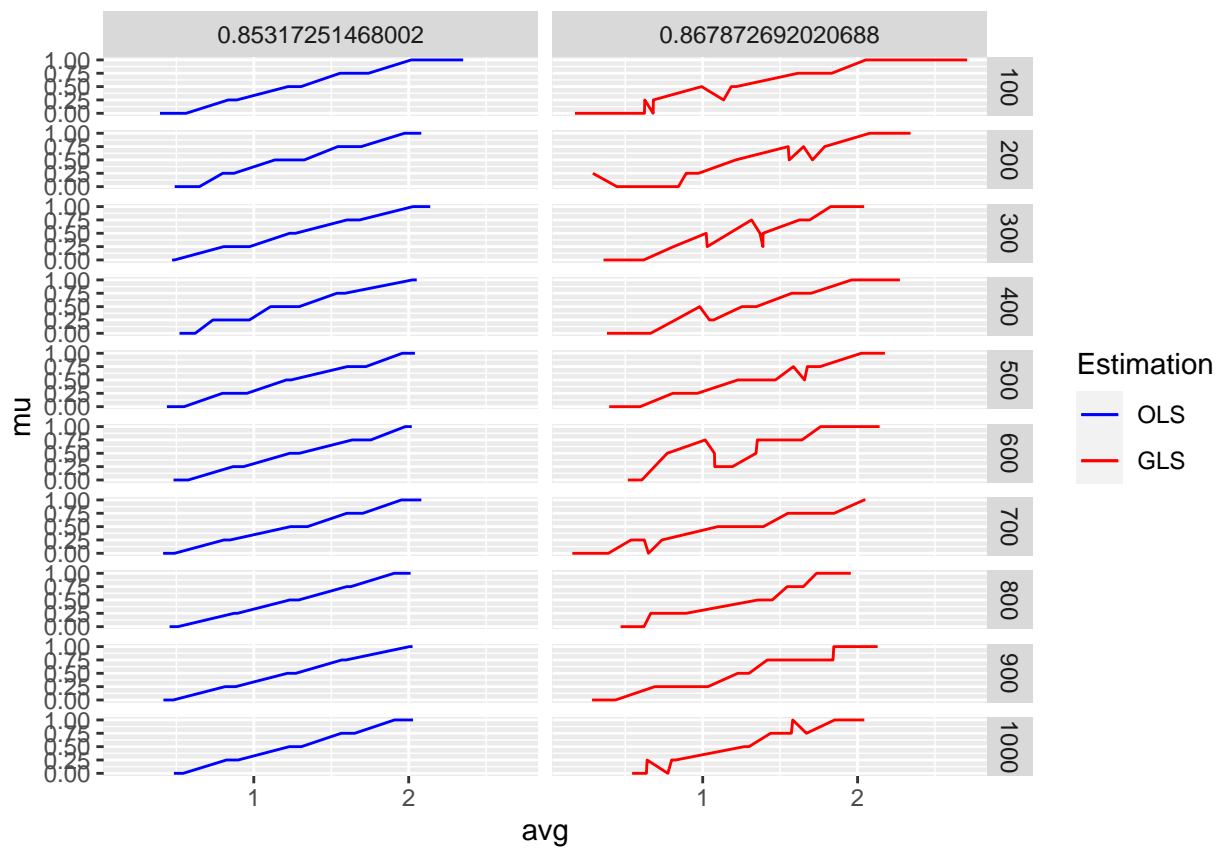
4.2 Visualisation

Visualizing MSE(Mean Square Error) of OLS and GLS simulations:

MSE is calculated by `out$average$mse` for each simulation.

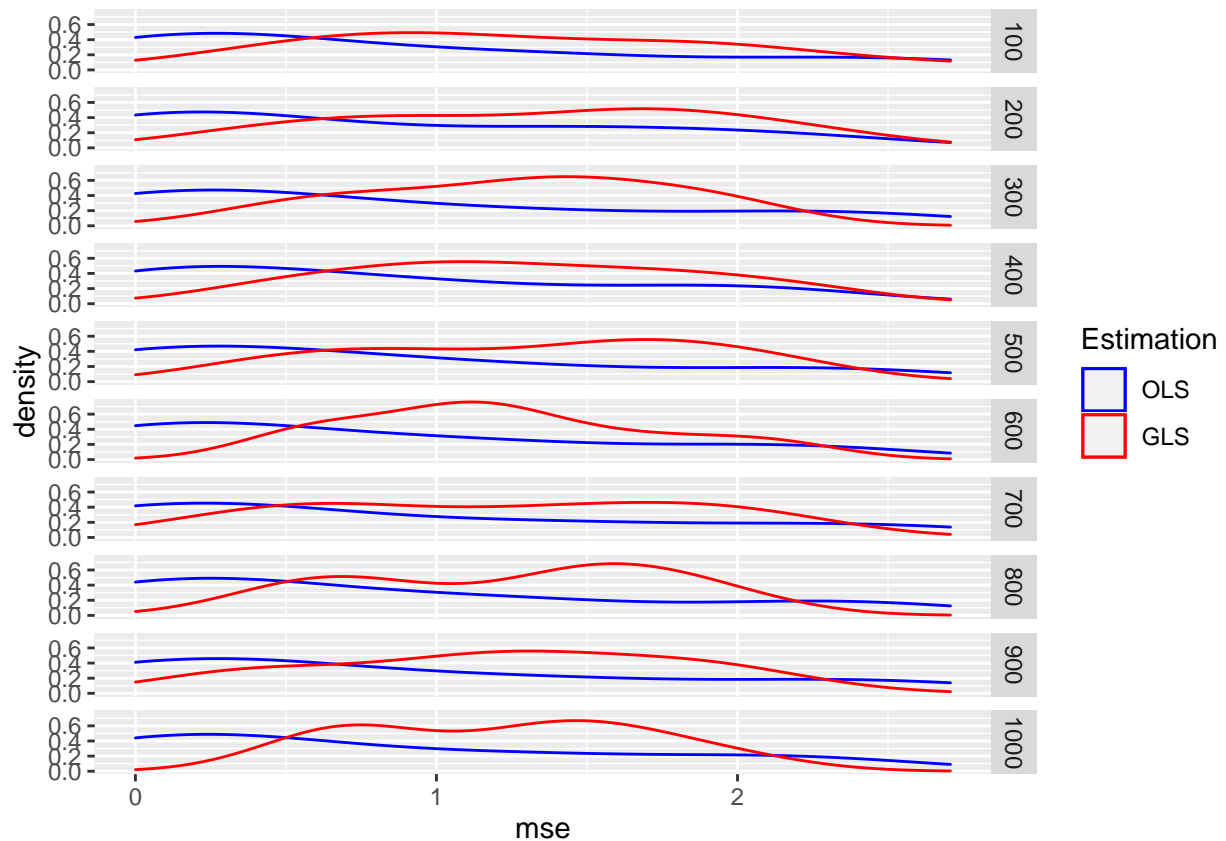
```
gls$average <- gls$average %>% mutate(mse =(2-avg)^2 )
ols$average <- ols$average %>% mutate(mse =(2-avg)^2 )

ols$average %>% ggplot(aes(x=avg,y=mu,col="OLS"))+
  facet_grid(n~mean(mse))+geom_line()+
  geom_line(data=gls$average,aes(x=avg,y=mu,col="GLS"))+
  scale_color_manual(name = "Estimation",
                     values = c("OLS" = "blue", "GLS" = "red"))
```



Density graph of MSE of β in OLS and GLS simulations

```
ggplot(ols$average,aes(x=mse,col="OLS"))+facet_grid(n~.)+
  geom_density()+
  geom_density(data=gl$average,aes(x=avg,col="GLS"))+
  scale_color_manual(name = "Estimation",
                    values = c("OLS" = "blue", "GLS" = "red"))
```



5 Conclusion

The above section illustrates the power of the implemented model and provides a fairly easy-to-use tool that still allows for various specifications in terms of used parameters, data generation processes and summary functions. Researchers, who use Monte Carlo studies regularly, may save a lot of time using a tool like this in the long run. It also lowers the threshold to run a Monte Carlo simulation for users, who don't have the ability or time to code a complete Monte Carlo simulation, like it was done for this paper.

It was shown, that the code works for many different situations, but there are some limitations, f.e. regarding the number of variables used in the parameter grid or the fact, that only a singular output can be returned by the summary statistic. The functional programming approach applied in this work would easily allow for adjustments, in case the purpose of the program needs to be altered or expanded.

6 Contributions

	Alexander Langnau	Öcal Kaptan	Sunyoung Ji
Planning	X	X	X
create_grid()	X	X	0
data_generation()	X	X	0
summary_function()	X	0	0
create_array_function()	X	0	0
average_function()	X	0	0
output_function()	0	X	0
main_function()	X	X	0
ggplot2 part	0	X	0
formatting	X	X	X
writing the report	X	X	X
literature research	X	X	X
proof-reading	X	X	X

7 References

Adrian G. Barbu, Song Chun Zhu, “Monte Carlo Methods”, Springer Singapore, 2020, pp.1-4,
[doi:https://doi.org/10.1007/978-981-13-2971-5](https://doi.org/10.1007/978-981-13-2971-5)

Czech, Z. “In Introduction to Parallel Computing”, Cambridge University Press, 2017, pp. 1-34,
[doi:10.1017/9781316795835.002](https://doi.org/10.1017/9781316795835.002)

rdocumentation.org, DataCamp, “cat: Concatenate and Print”, url:<https://www.rdocumentation.org/packages/base/versions/3.6.2/topics/cat>

8 Appendix

8.1 Appendix A

Output of the OLS simulation

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type  : Sequential
##
## Number of Cores Used in Parallelization : 1 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : 0.08611764
## Maximum : 2.598461
## Mean : 1.251827
## Median : 1.246948
##
## Execution Time of Monte Carlo Simulation 0.07933807 secs
##
## Name of The Class : Eco

## $results
## , , sd=1, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.6824267 0.7726517 1.390428 1.352546 2.033425
## n=200  0.4786798 0.7268500 1.551036 1.469126 2.016927
## n=300  0.5430408 0.6925565 1.412873 1.516396 2.020209
## n=400  0.4916201 0.9335676 1.240309 1.588818 1.939011
## n=500  0.4665197 0.8658038 1.302298 1.621465 2.071480
## n=600  0.4135505 1.0087766 1.263624 1.564343 1.873183
## n=700  0.6062303 0.8060657 1.423163 1.677571 1.915217
## n=800  0.5215150 0.8373901 1.384251 1.536015 1.845518
## n=900  0.4377088 0.8522425 1.323030 1.625229 2.113246
## n=1000 0.4272311 0.9598533 1.171133 1.607383 2.007037
##
## , , sd=1.5, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
```



```

## n=100 0.2325414 0.8709612 1.007916 1.187153 1.646406
## n=200 0.2684302 0.5956376 1.066756 1.572480 1.780621
## n=300 0.3463280 0.5074475 1.249930 1.526170 2.281022
## n=400 0.5532419 0.6727324 1.069168 1.587438 2.040513
## n=500 0.5088940 0.9917867 1.254739 1.687302 2.191762
## n=600 0.5817868 1.0516531 1.388582 1.787617 2.039833
## n=700 0.4085830 0.9266206 1.177307 1.618533 1.795094
## n=800 0.4788177 0.7297284 1.261184 1.635941 1.990439
## n=900 0.4293894 0.7521720 1.281461 1.550926 1.969204
## n=1000 0.4379182 0.7061176 1.281116 1.604609 1.875328
##
## , , sd=2, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.6237415 0.8008305 1.639701 1.897970 2.513486
## n=200 0.6110261 1.2155626 1.815507 1.725145 2.033192
## n=300 0.4066040 0.8631502 1.239882 1.684178 2.106188
## n=400 0.4235146 0.6718906 1.072217 1.373997 2.000179
## n=500 0.4979046 1.0646659 1.248795 1.624880 2.036934
## n=600 0.4167380 0.9594940 1.222973 1.538267 1.972503
## n=700 0.4474413 0.7043636 1.318657 1.298269 2.086695
## n=800 0.3558261 0.9032952 1.340988 1.480559 1.852471
## n=900 0.4184591 0.8440885 1.312351 1.481778 1.959888
## n=1000 0.5482327 0.8486597 1.325121 1.557654 2.174067
##
## , , sd=1, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5699594 0.6347600 1.240574 1.409578 2.102247
## n=200 0.3411974 0.9859774 1.127496 1.654986 2.005527
## n=300 0.5568138 0.7710960 1.309760 1.640136 1.868297
## n=400 0.5892899 0.8630584 1.420656 1.488524 2.106760
## n=500 0.4565660 0.9875469 1.169863 1.642037 2.125870
## n=600 0.6074920 0.8311391 1.144387 1.561809 2.127146
## n=700 0.5311162 0.8793250 1.213227 1.758464 2.019087
## n=800 0.4907978 0.8137087 1.243065 1.544169 1.890569
## n=900 0.5006696 0.9126527 1.185916 1.551420 2.011440
## n=1000 0.5132049 0.8425093 1.323331 1.665086 2.067430
##
## , , sd=1.5, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1

```

```

## n=100 0.4100172 1.0050342 1.3586338 1.594119 2.030257
## n=200 0.7621120 1.1129076 1.2050693 1.766159 2.052134
## n=300 0.4495498 0.9131922 1.3998834 1.844595 2.048539
## n=400 0.5189146 0.9392205 0.9085613 1.502305 2.159537
## n=500 0.7089984 0.6966299 1.1168839 1.472302 1.942223
## n=600 0.3991638 0.7265354 1.1572410 1.748560 1.735354
## n=700 0.4141086 0.7800295 1.3853238 1.765437 1.860513
## n=800 0.3797112 0.8637205 1.1967238 1.669677 1.904555
## n=900 0.4946693 0.9968749 1.3024325 1.747231 2.093768
## n=1000 0.6315810 0.8282354 1.1035843 1.541674 1.946093
##
## , , sd=2, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.2670268 1.0417428 0.9765324 1.779177 2.139938
## n=200 0.2064741 0.8966665 1.2011619 1.562213 1.929578
## n=300 0.2953656 0.8280923 1.4380720 1.301385 2.089569
## n=400 0.7044943 0.8050854 1.4651276 1.320472 2.051137
## n=500 0.4125427 0.8432624 0.9844522 1.838044 2.144216
## n=600 0.7200719 0.9085732 1.0035755 1.809966 2.205790
## n=700 0.5352158 0.8738860 1.2227186 1.564884 2.114450
## n=800 0.3849922 0.7982686 1.2963208 1.625716 1.980291
## n=900 0.3505717 0.7606954 1.0653161 1.494651 1.986403
## n=1000 0.5160520 0.8734081 1.3842792 1.766722 2.085573
##
## , , sd=1, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5968253 1.0425781 1.207359 1.494982 1.755818
## n=200 0.6013728 0.8169956 1.089477 1.759701 2.050444
## n=300 0.4185383 0.8722548 1.202220 1.632546 2.054984
## n=400 0.5761413 0.9439121 1.215469 1.611440 2.019500
## n=500 0.6625234 0.8265606 1.301594 1.551010 1.973081
## n=600 0.6338517 0.9970815 1.257844 1.716586 2.047527
## n=700 0.4954136 0.8538731 1.202218 1.656197 1.921142
## n=800 0.4847987 0.8606272 1.280399 1.681161 1.939804
## n=900 0.4581839 0.8963640 1.363804 1.640401 2.005699
## n=1000 0.4964781 0.9341992 1.243164 1.696117 1.985360
##
## , , sd=1.5, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1

```

```

## n=100 0.5659700 0.7903221 1.254210 1.886166 1.790293
## n=200 0.5821258 0.6511703 1.343681 1.782181 1.894019
## n=300 0.6550444 0.9367739 1.380756 1.705998 2.028275
## n=400 0.7048194 0.9633662 1.120887 1.688024 1.811344
## n=500 0.3923555 0.7626086 1.104116 1.537444 2.047353
## n=600 0.4717144 0.8688271 1.175261 1.826726 1.983990
## n=700 0.3871651 0.6201482 1.305263 1.669981 2.042779
## n=800 0.4834246 0.9357010 1.305622 1.523736 2.081715
## n=900 0.4433156 0.7916531 1.359010 1.580634 1.940405
## n=1000 0.4946876 0.8067392 1.480151 1.486763 1.900808
##
## , , sd=2, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.4237920 0.5144245 0.4718755 2.004104 2.186295
## n=200 0.5777614 0.7299343 1.6298909 1.499530 1.929751
## n=300 0.6276662 0.8600749 0.9626478 2.011661 2.175176
## n=400 0.5785860 0.9158230 1.1544399 1.389809 2.169336
## n=500 0.2516702 0.9907678 1.0602298 1.532465 1.682624
## n=600 0.5165529 0.8616768 1.5189127 1.652623 1.941119
## n=700 0.4674128 0.7479103 1.2133761 1.738886 1.996975
## n=800 0.5090721 0.8062749 1.3959268 1.542798 1.994769
## n=900 0.4197560 0.8280591 1.2129929 1.504396 1.947569
## n=1000 0.6527630 0.8873246 1.2662086 1.656169 2.115128
##
## , , sd=1, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.7359835 1.0087000 1.1157127 1.851686 2.286974
## n=200 0.5466285 0.8691083 0.9809655 1.677466 1.914859
## n=300 0.4092951 0.8421824 1.1834134 1.463671 2.105769
## n=400 0.4870693 0.8787365 1.3442915 1.641625 2.038680
## n=500 0.6247941 0.9737552 1.1814126 1.583607 1.981457
## n=600 0.5543342 0.9385104 1.2433161 1.676785 1.916412
## n=700 0.4178732 0.8808745 1.1952781 1.530920 1.960542
## n=800 0.5618498 0.8943310 1.2475019 1.686068 1.946561
## n=900 0.5052058 0.8785736 1.2679218 1.571624 2.009535
## n=1000 0.4550461 0.8622264 1.2189931 1.640547 2.027112
##
## , , sd=1.5, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1

```

```

## n=100 0.4449720 0.8435500 1.5231357 1.528876 2.461235
## n=200 0.3352203 0.8038153 0.9593336 1.823112 2.149320
## n=300 0.4930487 0.6791231 1.0272955 1.574880 2.304307
## n=400 0.3806721 1.1714969 1.4056411 1.543264 2.258722
## n=500 0.4589277 0.7928819 1.5044037 1.632672 1.829121
## n=600 0.4884702 0.6958911 1.2883971 1.606977 2.109358
## n=700 0.5038252 0.8396545 1.4939591 1.598702 2.198315
## n=800 0.5515391 0.8256058 1.2838778 1.644795 2.020077
## n=900 0.4102779 0.7769137 1.2511958 1.567744 1.925168
## n=1000 0.6885496 0.9734467 1.2688095 1.525439 1.946295
##
## , , sd=2, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5717011 0.8796835 2.1004676 1.351107 2.598461
## n=200 1.1810891 0.4725781 0.9070217 1.490537 2.444452
## n=300 0.5900949 1.0569764 1.3134750 1.766744 2.089249
## n=400 0.7380901 0.4503391 1.0841655 1.704982 1.998472
## n=500 0.6158445 1.0384364 1.6255235 1.911106 1.868147
## n=600 0.6300586 1.1140343 1.1895053 1.822727 1.933571
## n=700 0.3625215 0.9113498 1.2522415 1.654996 1.981563
## n=800 0.4427674 0.8570775 1.1367413 1.796147 2.103874
## n=900 0.3816606 0.8423425 1.3168243 1.712630 2.102760
## n=1000 0.4265370 0.9495205 1.3021528 1.603875 1.875588
##
## , , sd=1, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.2288956 0.9259850 1.345124 1.767008 1.904891
## n=200 0.4782098 0.9611524 1.326272 1.689215 2.050280
## n=300 0.4520819 0.9920557 1.238380 1.753581 2.087970
## n=400 0.4573970 1.1576259 1.246390 1.497974 2.019858
## n=500 0.5434649 0.9485599 1.191573 1.631702 2.049623
## n=600 0.4632066 0.8962950 1.266618 1.657375 1.924914
## n=700 0.3907963 0.8164093 1.155234 1.615976 1.949318
## n=800 0.4930138 0.9572131 1.295102 1.684709 1.936302
## n=900 0.4979275 0.8736681 1.214887 1.584737 1.985072
## n=1000 0.5301261 0.8977935 1.313722 1.630559 1.935747
##
## , , sd=1.5, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1

```

```

## n=100  0.4688968 0.9435859 1.380401 1.586464 2.192555
## n=200  0.7831954 0.8507725 1.104056 1.524999 1.987862
## n=300  0.4957244 0.9953593 1.100958 1.596888 2.032147
## n=400  0.6377009 1.1108968 1.039524 1.621890 1.989716
## n=500  0.2757090 0.7425858 1.221020 1.687816 1.782811
## n=600  0.4674703 0.9888671 1.462157 1.805296 2.070940
## n=700  0.3602414 0.8718152 1.369915 1.868538 1.981321
## n=800  0.3886475 1.1178513 1.246394 1.612441 2.065348
## n=900  0.4814983 0.8016443 1.148795 1.443979 2.099358
## n=1000 0.4683020 0.8041379 1.013566 1.662726 1.873865
##
## , , sd=2, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.08611764 0.9398614 0.9146867 1.669708 2.319204
## n=200  0.67138321 0.6832920 1.0754579 1.433456 2.068870
## n=300  0.44312742 1.2584762 1.3301563 1.650963 2.100398
## n=400  0.66455181 0.8364694 1.2167647 1.894041 1.989744
## n=500  0.41708387 0.8419375 1.1330271 1.713047 2.102631
## n=600  0.60004510 0.7267214 1.2282300 1.533603 2.044685
## n=700  0.42035536 0.8611750 1.3460957 1.744825 2.228779
## n=800  0.59792089 1.0213057 0.9896550 1.553520 1.600474
## n=900  0.51242473 0.7970967 1.1751429 1.646252 2.039864
## n=1000 0.56705287 0.8053208 1.2585926 1.673913 1.889833
##
## attr("class")
## [1] "Eco"    "array"
##
## $average
##      n  mu  sd      avg
## 1   100 0.00 1.0 0.5628181
## 2   100 0.00 1.5 0.4244795
## 3   100 0.00 2.0 0.3944758
## 4   100 0.25 1.0 0.8769350
## 5   100 0.25 1.5 0.8906907
## 6   100 0.25 2.0 0.8353085
## 7   100 0.50 1.0 1.2598397
## 8   100 0.50 1.5 1.3048593
## 9   100 0.50 2.0 1.2206527
## 10  100 0.75 1.0 1.5751601
## 11  100 0.75 1.5 1.5565558
## 12  100 0.75 2.0 1.7404132

```

## 13	100	1.00	1.0	2.0166709
## 14	100	1.00	1.5	2.0241493
## 15	100	1.00	2.0	2.3514768
## 16	200	0.00	1.0	0.4892177
## 17	200	0.00	1.5	0.5462168
## 18	200	0.00	2.0	0.6495468
## 19	200	0.25	1.0	0.8720167
## 20	200	0.25	1.5	0.8028606
## 21	200	0.25	2.0	0.7996067
## 22	200	0.50	1.0	1.2150492
## 23	200	0.50	1.5	1.1357791
## 24	200	0.50	2.0	1.3258080
## 25	200	0.75	1.0	1.6500989
## 26	200	0.75	1.5	1.6937863
## 27	200	0.75	2.0	1.5421762
## 28	200	1.00	1.0	2.0076074
## 29	200	1.00	1.5	1.9727911
## 30	200	1.00	2.0	2.0811684
## 31	300	0.00	1.0	0.4759540
## 32	300	0.00	1.5	0.4879391
## 33	300	0.00	2.0	0.4725716
## 34	300	0.25	1.0	0.8340291
## 35	300	0.25	1.5	0.8063792
## 36	300	0.25	2.0	0.9733540
## 37	300	0.50	1.0	1.2693292
## 38	300	0.50	1.5	1.2317647
## 39	300	0.50	2.0	1.2568465
## 40	300	0.75	1.0	1.6012659
## 41	300	0.75	1.5	1.6497061
## 42	300	0.75	2.0	1.6829863
## 43	300	1.00	1.0	2.0274458
## 44	300	1.00	1.5	2.1388580
## 45	300	1.00	2.0	2.1121160
## 46	400	0.00	1.0	0.5203035
## 47	400	0.00	1.5	0.5590698
## 48	400	0.00	2.0	0.6218474
## 49	400	0.25	1.0	0.9553801
## 50	400	0.25	1.5	0.9715426
## 51	400	0.25	2.0	0.7359215
## 52	400	0.50	1.0	1.2934233
## 53	400	0.50	1.5	1.1087563
## 54	400	0.50	2.0	1.1985429

## 55	400	0.75	1.0	1.5656764
## 56	400	0.75	1.5	1.5885842
## 57	400	0.75	2.0	1.5366603
## 58	400	1.00	1.0	2.0247618
## 59	400	1.00	1.5	2.0519664
## 60	400	1.00	2.0	2.0417733
## 61	500	0.00	1.0	0.5507736
## 62	500	0.00	1.5	0.4689769
## 63	500	0.00	2.0	0.4390092
## 64	500	0.25	1.0	0.9204453
## 65	500	0.25	1.5	0.7972986
## 66	500	0.25	2.0	0.9558140
## 67	500	0.50	1.0	1.2293479
## 68	500	0.50	1.5	1.2402327
## 69	500	0.50	2.0	1.2104054
## 70	500	0.75	1.0	1.6059639
## 71	500	0.75	1.5	1.6035074
## 72	500	0.75	2.0	1.7239084
## 73	500	1.00	1.0	2.0403021
## 74	500	1.00	1.5	1.9586539
## 75	500	1.00	2.0	1.9669104
## 76	600	0.00	1.0	0.5344870
## 77	600	0.00	1.5	0.4817211
## 78	600	0.00	2.0	0.5766933
## 79	600	0.25	1.0	0.9343605
## 80	600	0.25	1.5	0.8663548
## 81	600	0.25	2.0	0.9141000
## 82	600	0.50	1.0	1.2351578
## 83	600	0.50	1.5	1.2943275
## 84	600	0.50	2.0	1.2326393
## 85	600	0.75	1.0	1.6353797
## 86	600	0.75	1.5	1.7550350
## 87	600	0.75	2.0	1.6714372
## 88	600	1.00	1.0	1.9778364
## 89	600	1.00	1.5	1.9878951
## 90	600	1.00	2.0	2.0195340
## 91	700	0.00	1.0	0.4882859
## 92	700	0.00	1.5	0.4147847
## 93	700	0.00	2.0	0.4465893
## 94	700	0.25	1.0	0.8473095
## 95	700	0.25	1.5	0.8076536
## 96	700	0.25	2.0	0.8197369

## 97	700	0.50	1.0	1.2378238
## 98	700	0.50	1.5	1.3463535
## 99	700	0.50	2.0	1.2706179
## 100	700	0.75	1.0	1.6478257
## 101	700	0.75	1.5	1.7042384
## 102	700	0.75	2.0	1.6003720
## 103	700	1.00	1.0	1.9530612
## 104	700	1.00	1.5	1.9756043
## 105	700	1.00	2.0	2.0816926
## 106	800	0.00	1.0	0.5103950
## 107	800	0.00	1.5	0.4564280
## 108	800	0.00	2.0	0.4581157
## 109	800	0.25	1.0	0.8726540
## 110	800	0.25	1.5	0.8945214
## 111	800	0.25	2.0	0.8772444
## 112	800	0.50	1.0	1.2900639
## 113	800	0.50	1.5	1.2587603
## 114	800	0.50	2.0	1.2319263
## 115	800	0.75	1.0	1.6264243
## 116	800	0.75	1.5	1.6173179
## 117	800	0.75	2.0	1.5997479
## 118	800	1.00	1.0	1.9117509
## 119	800	1.00	1.5	2.0124269
## 120	800	1.00	2.0	1.9063757
## 121	900	0.00	1.0	0.4799391
## 122	900	0.00	1.5	0.4518301
## 123	900	0.00	2.0	0.4165744
## 124	900	0.25	1.0	0.8827002
## 125	900	0.25	1.5	0.8238516
## 126	900	0.25	2.0	0.8144564
## 127	900	0.50	1.0	1.2711118
## 128	900	0.50	1.5	1.2685788
## 129	900	0.50	2.0	1.2165253
## 130	900	0.75	1.0	1.5946823
## 131	900	0.75	1.5	1.5781026
## 132	900	0.75	2.0	1.5679413
## 133	900	1.00	1.0	2.0249983
## 134	900	1.00	1.5	2.0055806
## 135	900	1.00	2.0	2.0072967
## 136	1000	0.00	1.0	0.4844173
## 137	1000	0.00	1.5	0.5442077
## 138	1000	0.00	2.0	0.5421275


```
## 139 1000 0.25 1.0 0.8993163
## 140 1000 0.25 1.5 0.8237354
## 141 1000 0.25 2.0 0.8728468
## 142 1000 0.50 1.0 1.2540685
## 143 1000 0.50 1.5 1.2294453
## 144 1000 0.50 2.0 1.3072707
## 145 1000 0.75 1.0 1.6479383
## 146 1000 0.75 1.5 1.5642421
## 147 1000 0.75 2.0 1.6516665
## 148 1000 1.00 1.0 2.0045373
## 149 1000 1.00 1.5 1.9084779
## 150 1000 1.00 2.0 2.0280377
##
## attr(,"class")
## [1] "Eco"
```

Output of the OLS simulation with parallelisation

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type  : Multisession
##
## Number of Cores Used in Parallelization : 4 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : 0.08611764
## Maximum : 2.598461
## Mean : 1.251827
## Median : 1.246948
##
## Execution Time of Monte Carlo Simulation 0.712235 secs
##
## Name of The Class : Eco

## $results
## , , sd=1, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.6824267 0.7726517 1.390428 1.352546 2.033425
## n=200  0.4786798 0.7268500 1.551036 1.469126 2.016927
## n=300  0.5430408 0.6925565 1.412873 1.516396 2.020209
## n=400  0.4916201 0.9335676 1.240309 1.588818 1.939011
## n=500  0.4665197 0.8658038 1.302298 1.621465 2.071480
## n=600  0.4135505 1.0087766 1.263624 1.564343 1.873183
## n=700  0.6062303 0.8060657 1.423163 1.677571 1.915217
## n=800  0.5215150 0.8373901 1.384251 1.536015 1.845518
## n=900  0.4377088 0.8522425 1.323030 1.625229 2.113246
## n=1000 0.4272311 0.9598533 1.171133 1.607383 2.007037
##
## , , sd=1.5, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.2325414 0.8709612 1.007916 1.187153 1.646406
## n=200  0.2684302 0.5956376 1.066756 1.572480 1.780621
## n=300  0.3463280 0.5074475 1.249930 1.526170 2.281022
## n=400  0.5532419 0.6727324 1.069168 1.587438 2.040513
```

```

## n=500 0.5088940 0.9917867 1.254739 1.687302 2.191762
## n=600 0.5817868 1.0516531 1.388582 1.787617 2.039833
## n=700 0.4085830 0.9266206 1.177307 1.618533 1.795094
## n=800 0.4788177 0.7297284 1.261184 1.635941 1.990439
## n=900 0.4293894 0.7521720 1.281461 1.550926 1.969204
## n=1000 0.4379182 0.7061176 1.281116 1.604609 1.875328
##
## , , sd=2, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.6237415 0.8008305 1.639701 1.897970 2.513486
## n=200 0.6110261 1.2155626 1.815507 1.725145 2.033192
## n=300 0.4066040 0.8631502 1.239882 1.684178 2.106188
## n=400 0.4235146 0.6718906 1.072217 1.373997 2.000179
## n=500 0.4979046 1.0646659 1.248795 1.624880 2.036934
## n=600 0.4167380 0.9594940 1.222973 1.538267 1.972503
## n=700 0.4474413 0.7043636 1.318657 1.298269 2.086695
## n=800 0.3558261 0.9032952 1.340988 1.480559 1.852471
## n=900 0.4184591 0.8440885 1.312351 1.481778 1.959888
## n=1000 0.5482327 0.8486597 1.325121 1.557654 2.174067
##
## , , sd=1, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5699594 0.6347600 1.240574 1.409578 2.102247
## n=200 0.3411974 0.9859774 1.127496 1.654986 2.005527
## n=300 0.5568138 0.7710960 1.309760 1.640136 1.868297
## n=400 0.5892899 0.8630584 1.420656 1.488524 2.106760
## n=500 0.4565660 0.9875469 1.169863 1.642037 2.125870
## n=600 0.6074920 0.8311391 1.144387 1.561809 2.127146
## n=700 0.5311162 0.8793250 1.213227 1.758464 2.019087
## n=800 0.4907978 0.8137087 1.243065 1.544169 1.890569
## n=900 0.5006696 0.9126527 1.185916 1.551420 2.011440
## n=1000 0.5132049 0.8425093 1.323331 1.665086 2.067430
##
## , , sd=1.5, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.4100172 1.0050342 1.3586338 1.594119 2.030257
## n=200 0.7621120 1.1129076 1.2050693 1.766159 2.052134
## n=300 0.4495498 0.9131922 1.3998834 1.844595 2.048539
## n=400 0.5189146 0.9392205 0.9085613 1.502305 2.159537

```

```

## n=500 0.7089984 0.6966299 1.1168839 1.472302 1.942223
## n=600 0.3991638 0.7265354 1.1572410 1.748560 1.735354
## n=700 0.4141086 0.7800295 1.3853238 1.765437 1.860513
## n=800 0.3797112 0.8637205 1.1967238 1.669677 1.904555
## n=900 0.4946693 0.9968749 1.3024325 1.747231 2.093768
## n=1000 0.6315810 0.8282354 1.1035843 1.541674 1.946093
##
## , , sd=2, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.2670268 1.0417428 0.9765324 1.779177 2.139938
## n=200 0.2064741 0.8966665 1.2011619 1.562213 1.929578
## n=300 0.2953656 0.8280923 1.4380720 1.301385 2.089569
## n=400 0.7044943 0.8050854 1.4651276 1.320472 2.051137
## n=500 0.4125427 0.8432624 0.9844522 1.838044 2.144216
## n=600 0.7200719 0.9085732 1.0035755 1.809966 2.205790
## n=700 0.5352158 0.8738860 1.2227186 1.564884 2.114450
## n=800 0.3849922 0.7982686 1.2963208 1.625716 1.980291
## n=900 0.3505717 0.7606954 1.0653161 1.494651 1.986403
## n=1000 0.5160520 0.8734081 1.3842792 1.766722 2.085573
##
## , , sd=1, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5968253 1.0425781 1.207359 1.494982 1.755818
## n=200 0.6013728 0.8169956 1.089477 1.759701 2.050444
## n=300 0.4185383 0.8722548 1.202220 1.632546 2.054984
## n=400 0.5761413 0.9439121 1.215469 1.611440 2.019500
## n=500 0.6625234 0.8265606 1.301594 1.551010 1.973081
## n=600 0.6338517 0.9970815 1.257844 1.716586 2.047527
## n=700 0.4954136 0.8538731 1.202218 1.656197 1.921142
## n=800 0.4847987 0.8606272 1.280399 1.681161 1.939804
## n=900 0.4581839 0.8963640 1.363804 1.640401 2.005699
## n=1000 0.4964781 0.9341992 1.243164 1.696117 1.985360
##
## , , sd=1.5, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5659700 0.7903221 1.254210 1.886166 1.790293
## n=200 0.5821258 0.6511703 1.343681 1.782181 1.894019
## n=300 0.6550444 0.9367739 1.380756 1.705998 2.028275
## n=400 0.7048194 0.9633662 1.120887 1.688024 1.811344

```

```

## n=500 0.3923555 0.7626086 1.104116 1.537444 2.047353
## n=600 0.4717144 0.8688271 1.175261 1.826726 1.983990
## n=700 0.3871651 0.6201482 1.305263 1.669981 2.042779
## n=800 0.4834246 0.9357010 1.305622 1.523736 2.081715
## n=900 0.4433156 0.7916531 1.359010 1.580634 1.940405
## n=1000 0.4946876 0.8067392 1.480151 1.486763 1.900808
##
## , , sd=2, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.4237920 0.5144245 0.4718755 2.004104 2.186295
## n=200 0.5777614 0.7299343 1.6298909 1.499530 1.929751
## n=300 0.6276662 0.8600749 0.9626478 2.011661 2.175176
## n=400 0.5785860 0.9158230 1.1544399 1.389809 2.169336
## n=500 0.2516702 0.9907678 1.0602298 1.532465 1.682624
## n=600 0.5165529 0.8616768 1.5189127 1.652623 1.941119
## n=700 0.4674128 0.7479103 1.2133761 1.738886 1.996975
## n=800 0.5090721 0.8062749 1.3959268 1.542798 1.994769
## n=900 0.4197560 0.8280591 1.2129929 1.504396 1.947569
## n=1000 0.6527630 0.8873246 1.2662086 1.656169 2.115128
##
## , , sd=1, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.7359835 1.0087000 1.1157127 1.851686 2.286974
## n=200 0.5466285 0.8691083 0.9809655 1.677466 1.914859
## n=300 0.4092951 0.8421824 1.1834134 1.463671 2.105769
## n=400 0.4870693 0.8787365 1.3442915 1.641625 2.038680
## n=500 0.6247941 0.9737552 1.1814126 1.583607 1.981457
## n=600 0.5543342 0.9385104 1.2433161 1.676785 1.916412
## n=700 0.4178732 0.8808745 1.1952781 1.530920 1.960542
## n=800 0.5618498 0.8943310 1.2475019 1.686068 1.946561
## n=900 0.5052058 0.8785736 1.2679218 1.571624 2.009535
## n=1000 0.4550461 0.8622264 1.2189931 1.640547 2.027112
##
## , , sd=1.5, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.4449720 0.8435500 1.5231357 1.528876 2.461235
## n=200 0.3352203 0.8038153 0.9593336 1.823112 2.149320
## n=300 0.4930487 0.6791231 1.0272955 1.574880 2.304307
## n=400 0.3806721 1.1714969 1.4056411 1.543264 2.258722

```

```

## n=500 0.4589277 0.7928819 1.5044037 1.632672 1.829121
## n=600 0.4884702 0.6958911 1.2883971 1.606977 2.109358
## n=700 0.5038252 0.8396545 1.4939591 1.598702 2.198315
## n=800 0.5515391 0.8256058 1.2838778 1.644795 2.020077
## n=900 0.4102779 0.7769137 1.2511958 1.567744 1.925168
## n=1000 0.6885496 0.9734467 1.2688095 1.525439 1.946295
##
## , , sd=2, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.5717011 0.8796835 2.1004676 1.351107 2.598461
## n=200 1.1810891 0.4725781 0.9070217 1.490537 2.444452
## n=300 0.5900949 1.0569764 1.3134750 1.766744 2.089249
## n=400 0.7380901 0.4503391 1.0841655 1.704982 1.998472
## n=500 0.6158445 1.0384364 1.6255235 1.911106 1.868147
## n=600 0.6300586 1.1140343 1.1895053 1.822727 1.933571
## n=700 0.3625215 0.9113498 1.2522415 1.654996 1.981563
## n=800 0.4427674 0.8570775 1.1367413 1.796147 2.103874
## n=900 0.3816606 0.8423425 1.3168243 1.712630 2.102760
## n=1000 0.4265370 0.9495205 1.3021528 1.603875 1.875588
##
## , , sd=1, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.2288956 0.9259850 1.345124 1.767008 1.904891
## n=200 0.4782098 0.9611524 1.326272 1.689215 2.050280
## n=300 0.4520819 0.9920557 1.238380 1.753581 2.087970
## n=400 0.4573970 1.1576259 1.246390 1.497974 2.019858
## n=500 0.5434649 0.9485599 1.191573 1.631702 2.049623
## n=600 0.4632066 0.8962950 1.266618 1.657375 1.924914
## n=700 0.3907963 0.8164093 1.155234 1.615976 1.949318
## n=800 0.4930138 0.9572131 1.295102 1.684709 1.936302
## n=900 0.4979275 0.8736681 1.214887 1.584737 1.985072
## n=1000 0.5301261 0.8977935 1.313722 1.630559 1.935747
##
## , , sd=1.5, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.4688968 0.9435859 1.380401 1.586464 2.192555
## n=200 0.7831954 0.8507725 1.104056 1.524999 1.987862
## n=300 0.4957244 0.9953593 1.100958 1.596888 2.032147
## n=400 0.6377009 1.1108968 1.039524 1.621890 1.989716

```

```

## n=500  0.2757090 0.7425858 1.221020 1.687816 1.782811
## n=600  0.4674703 0.9888671 1.462157 1.805296 2.070940
## n=700  0.3602414 0.8718152 1.369915 1.868538 1.981321
## n=800  0.3886475 1.1178513 1.246394 1.612441 2.065348
## n=900  0.4814983 0.8016443 1.148795 1.443979 2.099358
## n=1000 0.4683020 0.8041379 1.013566 1.662726 1.873865
##
## , , sd=2, rep=5
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.08611764 0.9398614 0.9146867 1.669708 2.319204
## n=200  0.67138321 0.6832920 1.0754579 1.433456 2.068870
## n=300  0.44312742 1.2584762 1.3301563 1.650963 2.100398
## n=400  0.66455181 0.8364694 1.2167647 1.894041 1.989744
## n=500  0.41708387 0.8419375 1.1330271 1.713047 2.102631
## n=600  0.60004510 0.7267214 1.2282300 1.533603 2.044685
## n=700  0.42035536 0.8611750 1.3460957 1.744825 2.228779
## n=800  0.59792089 1.0213057 0.9896550 1.553520 1.600474
## n=900  0.51242473 0.7970967 1.1751429 1.646252 2.039864
## n=1000 0.56705287 0.8053208 1.2585926 1.673913 1.889833
##
## attr(,"class")
## [1] "Eco"    "array"
##
## $average
##      n  mu  sd      avg
## 1   100 0.00 1.0 0.5628181
## 2   100 0.00 1.5 0.4244795
## 3   100 0.00 2.0 0.3944758
## 4   100 0.25 1.0 0.8769350
## 5   100 0.25 1.5 0.8906907
## 6   100 0.25 2.0 0.8353085
## 7   100 0.50 1.0 1.2598397
## 8   100 0.50 1.5 1.3048593
## 9   100 0.50 2.0 1.2206527
## 10  100 0.75 1.0 1.5751601
## 11  100 0.75 1.5 1.5565558
## 12  100 0.75 2.0 1.7404132
## 13  100 1.00 1.0 2.0166709
## 14  100 1.00 1.5 2.0241493
## 15  100 1.00 2.0 2.3514768
## 16  200 0.00 1.0 0.4892177

```

## 17	200	0.00	1.5	0.5462168
## 18	200	0.00	2.0	0.6495468
## 19	200	0.25	1.0	0.8720167
## 20	200	0.25	1.5	0.8028606
## 21	200	0.25	2.0	0.7996067
## 22	200	0.50	1.0	1.2150492
## 23	200	0.50	1.5	1.1357791
## 24	200	0.50	2.0	1.3258080
## 25	200	0.75	1.0	1.6500989
## 26	200	0.75	1.5	1.6937863
## 27	200	0.75	2.0	1.5421762
## 28	200	1.00	1.0	2.0076074
## 29	200	1.00	1.5	1.9727911
## 30	200	1.00	2.0	2.0811684
## 31	300	0.00	1.0	0.4759540
## 32	300	0.00	1.5	0.4879391
## 33	300	0.00	2.0	0.4725716
## 34	300	0.25	1.0	0.8340291
## 35	300	0.25	1.5	0.8063792
## 36	300	0.25	2.0	0.9733540
## 37	300	0.50	1.0	1.2693292
## 38	300	0.50	1.5	1.2317647
## 39	300	0.50	2.0	1.2568465
## 40	300	0.75	1.0	1.6012659
## 41	300	0.75	1.5	1.6497061
## 42	300	0.75	2.0	1.6829863
## 43	300	1.00	1.0	2.0274458
## 44	300	1.00	1.5	2.1388580
## 45	300	1.00	2.0	2.1121160
## 46	400	0.00	1.0	0.5203035
## 47	400	0.00	1.5	0.5590698
## 48	400	0.00	2.0	0.6218474
## 49	400	0.25	1.0	0.9553801
## 50	400	0.25	1.5	0.9715426
## 51	400	0.25	2.0	0.7359215
## 52	400	0.50	1.0	1.2934233
## 53	400	0.50	1.5	1.1087563
## 54	400	0.50	2.0	1.1985429
## 55	400	0.75	1.0	1.5656764
## 56	400	0.75	1.5	1.5885842
## 57	400	0.75	2.0	1.5366603
## 58	400	1.00	1.0	2.0247618

## 59	400	1.00	1.5	2.0519664
## 60	400	1.00	2.0	2.0417733
## 61	500	0.00	1.0	0.5507736
## 62	500	0.00	1.5	0.4689769
## 63	500	0.00	2.0	0.4390092
## 64	500	0.25	1.0	0.9204453
## 65	500	0.25	1.5	0.7972986
## 66	500	0.25	2.0	0.9558140
## 67	500	0.50	1.0	1.2293479
## 68	500	0.50	1.5	1.2402327
## 69	500	0.50	2.0	1.2104054
## 70	500	0.75	1.0	1.6059639
## 71	500	0.75	1.5	1.6035074
## 72	500	0.75	2.0	1.7239084
## 73	500	1.00	1.0	2.0403021
## 74	500	1.00	1.5	1.9586539
## 75	500	1.00	2.0	1.9669104
## 76	600	0.00	1.0	0.5344870
## 77	600	0.00	1.5	0.4817211
## 78	600	0.00	2.0	0.5766933
## 79	600	0.25	1.0	0.9343605
## 80	600	0.25	1.5	0.8663548
## 81	600	0.25	2.0	0.9141000
## 82	600	0.50	1.0	1.2351578
## 83	600	0.50	1.5	1.2943275
## 84	600	0.50	2.0	1.2326393
## 85	600	0.75	1.0	1.6353797
## 86	600	0.75	1.5	1.7550350
## 87	600	0.75	2.0	1.6714372
## 88	600	1.00	1.0	1.9778364
## 89	600	1.00	1.5	1.9878951
## 90	600	1.00	2.0	2.0195340
## 91	700	0.00	1.0	0.4882859
## 92	700	0.00	1.5	0.4147847
## 93	700	0.00	2.0	0.4465893
## 94	700	0.25	1.0	0.8473095
## 95	700	0.25	1.5	0.8076536
## 96	700	0.25	2.0	0.8197369
## 97	700	0.50	1.0	1.2378238
## 98	700	0.50	1.5	1.3463535
## 99	700	0.50	2.0	1.2706179
## 100	700	0.75	1.0	1.6478257

##	101	700	0.75	1.5	1.7042384
##	102	700	0.75	2.0	1.6003720
##	103	700	1.00	1.0	1.9530612
##	104	700	1.00	1.5	1.9756043
##	105	700	1.00	2.0	2.0816926
##	106	800	0.00	1.0	0.5103950
##	107	800	0.00	1.5	0.4564280
##	108	800	0.00	2.0	0.4581157
##	109	800	0.25	1.0	0.8726540
##	110	800	0.25	1.5	0.8945214
##	111	800	0.25	2.0	0.8772444
##	112	800	0.50	1.0	1.2900639
##	113	800	0.50	1.5	1.2587603
##	114	800	0.50	2.0	1.2319263
##	115	800	0.75	1.0	1.6264243
##	116	800	0.75	1.5	1.6173179
##	117	800	0.75	2.0	1.5997479
##	118	800	1.00	1.0	1.9117509
##	119	800	1.00	1.5	2.0124269
##	120	800	1.00	2.0	1.9063757
##	121	900	0.00	1.0	0.4799391
##	122	900	0.00	1.5	0.4518301
##	123	900	0.00	2.0	0.4165744
##	124	900	0.25	1.0	0.8827002
##	125	900	0.25	1.5	0.8238516
##	126	900	0.25	2.0	0.8144564
##	127	900	0.50	1.0	1.2711118
##	128	900	0.50	1.5	1.2685788
##	129	900	0.50	2.0	1.2165253
##	130	900	0.75	1.0	1.5946823
##	131	900	0.75	1.5	1.5781026
##	132	900	0.75	2.0	1.5679413
##	133	900	1.00	1.0	2.0249983
##	134	900	1.00	1.5	2.0055806
##	135	900	1.00	2.0	2.0072967
##	136	1000	0.00	1.0	0.4844173
##	137	1000	0.00	1.5	0.5442077
##	138	1000	0.00	2.0	0.5421275
##	139	1000	0.25	1.0	0.8993163
##	140	1000	0.25	1.5	0.8237354
##	141	1000	0.25	2.0	0.8728468
##	142	1000	0.50	1.0	1.2540685

```
## 143 1000 0.50 1.5 1.2294453
## 144 1000 0.50 2.0 1.3072707
## 145 1000 0.75 1.0 1.6479383
## 146 1000 0.75 1.5 1.5642421
## 147 1000 0.75 2.0 1.6516665
## 148 1000 1.00 1.0 2.0045373
## 149 1000 1.00 1.5 1.9084779
## 150 1000 1.00 2.0 2.0280377
##
## attr("class")
## [1] "Eco"
```

8.2 Appendix B

Output of the GLS simulation without parallelisation

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type : Sequential
##
## Number of Cores Used in Parallelization : 1 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : -0.9971
## Maximum : 3.383637
## Mean : 1.260196
## Median : 1.284057
##
## Execution Time of Monte Carlo Simulation 31.50186 secs
##
## Name of The Class : Eco

## $results
## , , sd=1, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.7554538 0.5016876 1.7356494 1.517285 2.639110
## n=200  0.3903311 0.6408042 1.5194431 1.417126 3.255660
## n=300  0.8317798 0.8173362 1.2688679 1.366244 2.007371
## n=400  0.0664591 0.7887018 1.1664611 1.262650 1.601207
```

```

## n=500 0.7349604 0.8108023 1.5108027 1.624144 2.316178
## n=600 0.5483240 1.3362425 0.7466396 1.539507 1.775243
## n=700 1.1797722 0.7454456 1.5538655 1.596843 1.851457
## n=800 0.7388026 0.4151828 1.2355987 2.121480 1.617091
## n=900 0.3375558 0.5516673 0.9502070 1.037452 2.600048
## n=1000 0.2812896 0.8840261 1.4690083 1.667241 2.124077
##
## , , sd=1.5, rep=1
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 1.07566007 0.97703407 0.3943886 1.0410972 1.357513
## n=200 0.52383817 1.45057710 1.6489542 1.5278402 1.404216
## n=300 0.71906564 0.40330063 2.2602221 0.8222903 2.070699
## n=400 1.04256675 1.22061498 0.7880425 1.1109698 1.966145
## n=500 -0.07188688 1.41714029 1.4742135 1.9709113 3.005006
## n=600 0.87050124 1.16986970 0.8726417 1.1228528 1.821454
## n=700 -0.25848227 0.49203602 1.0950518 1.6813917 1.675552
## n=800 0.39160576 0.03376035 1.3268501 1.6087353 2.523682
## n=900 0.01281269 0.82989922 1.1917932 2.1448010 2.249365
## n=1000 1.00643264 0.23351513 1.7264966 1.7730168 1.716113
##
## , , sd=2, rep=1
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.1989645 0.9356581 1.650652 1.4279687 2.842366
## n=200 0.6770072 0.8534528 1.886844 1.1371663 2.086750
## n=300 0.9021613 1.3441820 1.592364 1.5281411 1.918323
## n=400 0.3678666 0.6749776 1.494146 1.0124086 2.125654
## n=500 -0.3538140 1.4641727 1.702553 2.0690877 1.771409
## n=600 0.2521821 0.8727736 1.281437 1.6461148 1.787155
## n=700 -0.5283604 1.1573474 1.301522 1.6284337 2.040922
## n=800 0.7162405 1.7569821 1.089098 1.8510202 2.008889
## n=900 -0.1466358 1.2663258 0.669074 0.9226198 1.983874
## n=1000 0.4922426 0.5592436 1.209339 0.5571104 2.177159
##
## , , sd=1, rep=2
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.93794659 0.3042632 0.8558104 1.395916 2.488330
## n=200 0.46865388 0.8026164 0.6673771 1.754976 2.196959
## n=300 0.21497718 0.6857743 1.4560339 1.591513 1.441236
## n=400 0.09099797 0.1139196 0.9989419 1.530588 1.901867

```

```

## n=500    0.26214018 1.0189866 0.9483061 1.929226 2.028602
## n=600    0.36515870 1.1206859 1.3254734 2.154065 2.398600
## n=700    0.84564182 0.6471012 1.4065464 1.768395 2.419217
## n=800    0.85872788 0.7206308 1.3704309 1.246906 2.218637
## n=900   -0.14990817 1.2264243 1.6872835 1.774656 2.095731
## n=1000   0.96179814 0.6660319 1.2241939 1.846178 2.384093
##
## , , sd=1.5, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.7971403 0.6846694 1.4747453 1.376802 1.289544
## n=200  0.8722172 1.5062628 0.8667520 1.455997 3.187951
## n=300  1.1649019 1.7666435 0.9995124 1.495770 1.687811
## n=400  0.8440923 0.5933976 0.8222845 1.806278 2.256190
## n=500  0.5786570 1.1103728 1.9378712 1.644442 2.421673
## n=600  0.7493392 1.1661280 1.2064266 1.645635 1.537470
## n=700  0.8992186 1.0192023 1.0847103 1.817849 1.520385
## n=800  0.5199020 0.9399592 1.7241364 1.260431 1.886227
## n=900  1.0116356 1.1141843 1.4591379 2.097425 2.420061
## n=1000 1.1299320 0.6024359 0.3630059 2.020767 1.720216
##
## , , sd=2, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 -0.01919950 0.10389729 -0.7181519 2.1975727 2.688336
## n=200  1.02896044 -0.06349928 1.7339415 1.9835128 1.816331
## n=300 -0.18610242 1.60434976 1.6070085 -0.3271213 1.931503
## n=400  1.41709802 1.20750797 1.5523077 1.6424103 1.171583
## n=500  1.23639159 0.54645719 1.6774379 1.3658300 3.363879
## n=600  0.39037287 1.42422862 0.5759413 2.1932974 1.854756
## n=700 -0.34830352 0.85511492 1.2548947 2.4213875 2.016490
## n=800 -0.08188814 1.27109005 1.7893126 0.9299721 1.717897
## n=900 -0.07990839 0.66567655 1.8002084 1.7681873 1.740396
## n=1000 0.88195270 0.50717811 1.2185675 2.1915303 1.744440
##
## , , sd=1, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.2500595 1.3250897 0.9547569 2.293587 1.819439
## n=200  0.5471549 0.9566464 1.3296827 1.775252 1.804901
## n=300  1.0456930 1.4996928 1.4615289 1.399473 2.154331
## n=400  0.6342614 0.5517242 1.4421203 1.787110 2.553724

```

```

## n=500 0.5785579 1.0079606 1.1506459 1.573662 1.915940
## n=600 0.3589415 0.7270598 1.5240060 1.239381 1.971919
## n=700 0.7345540 0.2262226 1.2611827 1.103715 2.039746
## n=800 0.5015531 1.9152089 1.6893004 1.561901 2.198732
## n=900 0.1721376 1.0750490 1.1722128 1.835734 1.946778
## n=1000 0.5581321 0.7179216 1.0656947 1.773012 2.400244
##
## , , sd=1.5, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.75515002 1.0532149 1.3675520 2.458531 2.511075
## n=200 -0.04934648 0.5496542 2.0479018 1.675846 1.142718
## n=300 1.00181000 1.6369489 0.7753943 2.638377 1.829983
## n=400 0.38708014 1.2533585 1.4714135 1.435519 2.018202
## n=500 0.37250451 0.7996507 1.2444854 1.892440 1.427025
## n=600 0.26431978 1.3546424 1.2058852 1.796262 2.147378
## n=700 1.12511433 0.6271908 0.8411113 1.638469 1.970271
## n=800 0.52208688 0.6764603 1.6023554 1.526346 1.906584
## n=900 0.22266670 0.5468360 1.5856621 2.134440 1.515266
## n=1000 0.20373892 1.9340035 1.2128314 1.337933 1.380179
##
## , , sd=2, rep=3
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 -0.5134825 -0.9971000 0.02609116 2.483622 2.1737654
## n=200 0.8564846 0.5270190 2.19874069 1.115522 2.0483072
## n=300 0.1357223 1.1346463 0.67885853 1.989233 2.3651937
## n=400 -0.1132654 1.5174377 -0.39911038 2.430706 2.7068852
## n=500 0.2692164 0.9373865 0.45341757 1.488270 2.0855372
## n=600 0.2767981 1.9185888 1.10986631 1.779951 2.2288551
## n=700 0.4175472 0.7176046 0.78665976 1.500209 2.3208915
## n=800 0.2236755 0.7400305 1.63497839 2.006660 2.2390314
## n=900 1.3383189 1.7077645 1.90542412 2.962268 1.5069244
## n=1000 0.2337000 0.4510541 0.72055597 2.240282 0.9580619
##
## , , sd=1, rep=4
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 1.3469860 -0.05560687 1.0932454 1.903516 2.156825
## n=200 0.3208913 0.96409270 1.0863966 1.386434 1.835235
## n=300 0.6078219 0.39871289 1.3616965 2.100878 1.933667
## n=400 0.6771807 1.05598360 1.5240453 2.261056 2.531636

```

```

## n=500  0.4919199  1.10827112  1.5332686  1.517603  1.843585
## n=600  0.5637876  0.85089642  1.8731358  1.617511  2.200501
## n=700  0.3582705  0.99200307  0.8214109  1.560444  2.128348
## n=800  0.5240053  0.94117803  1.2620739  1.546574  1.873964
## n=900  0.6295721  0.83325772  0.9917639  1.333128  1.971597
## n=1000 0.6248638  1.00187859  1.3589893  1.665165  1.630167
##
## , , sd=1.5, rep=4
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100  0.14190040  2.01821137  0.8517502  2.0724785  2.762258
## n=200  0.82223920  0.95236727  2.3576613  2.7525450  2.656997
## n=300  0.41468489  0.81135204  1.7321312  2.2129173  1.586342
## n=400  0.47456049  1.28481297  1.6045288  1.6357618  2.517795
## n=500  0.19596935 -0.09776629  1.3816823  1.1821549  1.806055
## n=600  0.04919382  1.33474572  1.2062848  0.0429013  2.320981
## n=700  0.34423613  0.48523985  1.9450566  1.5101308  2.451672
## n=800  1.32565822  0.84774132  1.2387826  2.3976442  1.870370
## n=900  0.47854733  0.51916320  1.1726107  1.6816128  1.705770
## n=1000 0.82817973  0.80225623  1.5175430  1.4264566  1.863020
##
## , , sd=2, rep=4
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100  0.8525966  1.1749948  3.38363655  1.248533  3.057232
## n=200  0.5182802  0.4838462  1.41934260  2.241456  2.358571
## n=300  1.0733668  2.0578513  0.96974173  2.245360  1.290691
## n=400  0.8011078  0.4410973  0.90802926  1.296971  2.153393
## n=500  0.2470769  0.7742284  2.22385975  2.482988  1.599441
## n=600  1.1095269  0.4747139  0.06967184  1.576253  1.038778
## n=700  0.3443959 -0.7980158  0.88632336  1.763624  1.642792
## n=800  0.3212259 -0.1889027  1.12345703  2.337993  1.622583
## n=900  0.8488329  1.3243651  0.42830492  1.324446  2.120547
## n=1000 1.6411901  0.1763252  1.43712633  1.095596  1.415895
##
## , , sd=1, rep=5
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100  0.1106945  1.3324074  1.283149  1.795427  1.479256
## n=200  0.5031188  1.5027343  1.460474  1.921770  2.617298
## n=300  0.3730896  0.6590489  1.300485  1.664153  2.229097
## n=400  0.4404047  1.6079649  1.600920  1.597121  1.219695

```

```

## n=500 0.9142947 0.8821124 0.993282 2.142944 2.017532
## n=600 0.7455402 1.3534103 1.252659 1.654961 2.019734
## n=700 0.1307611 0.5045976 1.915240 1.719891 1.816085
## n=800 0.0348579 0.4676373 1.687917 1.773396 1.871487
## n=900 0.5683141 1.0347100 1.329680 1.971053 2.031339
## n=1000 0.2944263 0.7265625 1.342830 1.397584 1.678541
##
## , , sd=1.5, rep=5
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.3492151 0.941764185 1.9927388 1.1271039 2.343358
## n=200 0.9322249 0.009236562 1.6216833 1.5246716 2.126926
## n=300 -0.2061382 0.527637441 1.1785690 1.2855802 1.955373
## n=400 -0.2080310 0.865475626 1.5810396 1.8863548 2.607916
## n=500 1.0591201 1.269133681 1.3106256 1.6893725 2.221825
## n=600 1.0529351 0.935277255 0.8883463 0.4731693 2.883712
## n=700 -0.1566764 1.060909722 1.3480097 1.5789494 2.593332
## n=800 0.3458820 0.824462226 1.0043539 1.3977273 1.529786
## n=900 0.4578455 0.443865375 1.0857179 1.1477187 1.641970
## n=1000 0.7074108 0.556170334 1.5201499 1.3028712 2.580936
##
## , , sd=2, rep=5
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.35838129 1.9134632 0.6236571 1.8050059 2.775072
## n=200 1.13555545 -0.3462561 0.5517002 1.2833010 2.082454
## n=300 -0.12779402 0.8027196 0.2661467 1.1407255 2.706796
## n=400 0.84449836 1.5107112 1.3472610 2.1182648 2.410341
## n=500 0.58335229 0.3130528 2.2292441 0.5206100 1.407526
## n=600 1.00687281 0.8306865 0.8167926 -0.4327573 1.897837
## n=700 0.90763719 0.7547756 1.2602192 1.9296762 2.212971
## n=800 1.16899109 0.5890770 1.1466515 0.6023830 1.086282
## n=900 -0.53156942 0.2064395 1.3326650 0.1082137 1.878707
## n=1000 -0.06566813 1.5126884 1.9075839 1.1067314 1.610853
##
## attr(,"class")
## [1] "Eco" "array"
##
## $average
##      n    mu  sd      avg
## 1    100 0.00 1.0 0.6802281
## 2    100 0.00 1.5 0.6238132

```


## 3	100	0.00	2.0	0.1754521
## 4	100	0.25	1.0	0.6815682
## 5	100	0.25	1.5	1.1349788
## 6	100	0.25	2.0	0.6261827
## 7	100	0.50	1.0	1.1845221
## 8	100	0.50	1.5	1.2162350
## 9	100	0.50	2.0	0.9931770
## 10	100	0.75	1.0	1.7811463
## 11	100	0.75	1.5	1.6152025
## 12	100	0.75	2.0	1.8325404
## 13	100	1.00	1.0	2.1165920
## 14	100	1.00	1.5	2.0527497
## 15	100	1.00	2.0	2.7073544
## 16	200	0.00	1.0	0.4460300
## 17	200	0.00	1.5	0.6202346
## 18	200	0.00	2.0	0.8432576
## 19	200	0.25	1.0	0.9733788
## 20	200	0.25	1.5	0.8936196
## 21	200	0.25	2.0	0.2909125
## 22	200	0.50	1.0	1.2126747
## 23	200	0.50	1.5	1.7085905
## 24	200	0.50	2.0	1.5581137
## 25	200	0.75	1.0	1.6511116
## 26	200	0.75	1.5	1.7873799
## 27	200	0.75	2.0	1.5521916
## 28	200	1.00	1.0	2.3420107
## 29	200	1.00	1.5	2.1037616
## 30	200	1.00	2.0	2.0784825
## 31	300	0.00	1.0	0.6146723
## 32	300	0.00	1.5	0.6188648
## 33	300	0.00	2.0	0.3594708
## 34	300	0.25	1.0	0.8121130
## 35	300	0.25	1.5	1.0291765
## 36	300	0.25	2.0	1.3887498
## 37	300	0.50	1.0	1.3697225
## 38	300	0.50	1.5	1.3891658
## 39	300	0.50	2.0	1.0228240
## 40	300	0.75	1.0	1.6244521
## 41	300	0.75	1.5	1.6909870
## 42	300	0.75	2.0	1.3152675
## 43	300	1.00	1.0	1.9531402
## 44	300	1.00	1.5	1.8260415

## 45	300	1.00	2.0	2.0425014
## 46	400	0.00	1.0	0.3818608
## 47	400	0.00	1.5	0.5080537
## 48	400	0.00	2.0	0.6634611
## 49	400	0.25	1.0	0.8236588
## 50	400	0.25	1.5	1.0435319
## 51	400	0.25	2.0	1.0703464
## 52	400	0.50	1.0	1.3464978
## 53	400	0.50	1.5	1.2534618
## 54	400	0.50	2.0	0.9805268
## 55	400	0.75	1.0	1.6877048
## 56	400	0.75	1.5	1.5749766
## 57	400	0.75	2.0	1.7001523
## 58	400	1.00	1.0	1.9616256
## 59	400	1.00	1.5	2.2732495
## 60	400	1.00	2.0	2.1135711
## 61	500	0.00	1.0	0.5963746
## 62	500	0.00	1.5	0.4268728
## 63	500	0.00	2.0	0.3964446
## 64	500	0.25	1.0	0.9656266
## 65	500	0.25	1.5	0.8997062
## 66	500	0.25	2.0	0.8070595
## 67	500	0.50	1.0	1.2272611
## 68	500	0.50	1.5	1.4697756
## 69	500	0.50	2.0	1.6573025
## 70	500	0.75	1.0	1.7575159
## 71	500	0.75	1.5	1.6758640
## 72	500	0.75	2.0	1.5853571
## 73	500	1.00	1.0	2.0243674
## 74	500	1.00	1.5	2.1763168
## 75	500	1.00	2.0	2.0455584
## 76	600	0.00	1.0	0.5163504
## 77	600	0.00	1.5	0.5972578
## 78	600	0.00	2.0	0.6071506
## 79	600	0.25	1.0	1.0776590
## 80	600	0.25	1.5	1.1921326
## 81	600	0.25	2.0	1.1041983
## 82	600	0.50	1.0	1.3443827
## 83	600	0.50	1.5	1.0759169
## 84	600	0.50	2.0	0.7707418
## 85	600	0.75	1.0	1.6410851
## 86	600	0.75	1.5	1.0161640

## 87	600	0.75	2.0	1.3525717
## 88	600	1.00	1.0	2.0731992
## 89	600	1.00	1.5	2.1421991
## 90	600	1.00	2.0	1.7614761
## 91	700	0.00	1.0	0.6497999
## 92	700	0.00	1.5	0.3906821
## 93	700	0.00	2.0	0.1585833
## 94	700	0.25	1.0	0.6230740
## 95	700	0.25	1.5	0.7369157
## 96	700	0.25	2.0	0.5373653
## 97	700	0.50	1.0	1.3916491
## 98	700	0.50	1.5	1.2627879
## 99	700	0.50	2.0	1.0979239
## 100	700	0.75	1.0	1.5498575
## 101	700	0.75	1.5	1.6453580
## 102	700	0.75	2.0	1.8486661
## 103	700	1.00	1.0	2.0509706
## 104	700	1.00	1.5	2.0422424
## 105	700	1.00	2.0	2.0468134
## 106	800	0.00	1.0	0.5315894
## 107	800	0.00	1.5	0.6210270
## 108	800	0.00	2.0	0.4696490
## 109	800	0.25	1.0	0.8919676
## 110	800	0.25	1.5	0.6644767
## 111	800	0.25	2.0	0.8336554
## 112	800	0.50	1.0	1.4490643
## 113	800	0.50	1.5	1.3792957
## 114	800	0.50	2.0	1.3566995
## 115	800	0.75	1.0	1.6500513
## 116	800	0.75	1.5	1.6381768
## 117	800	0.75	2.0	1.5456057
## 118	800	1.00	1.0	1.9559821
## 119	800	1.00	1.5	1.9433300
## 120	800	1.00	2.0	1.7349365
## 121	900	0.00	1.0	0.3115343
## 122	900	0.00	1.5	0.4367016
## 123	900	0.00	2.0	0.2858076
## 124	900	0.25	1.0	0.9442217
## 125	900	0.25	1.5	0.6907896
## 126	900	0.25	2.0	1.0341143
## 127	900	0.50	1.0	1.2262295
## 128	900	0.50	1.5	1.2989843

```

## 129 900 0.50 2.0 1.2271353
## 130 900 0.75 1.0 1.5904047
## 131 900 0.75 1.5 1.8411995
## 132 900 0.75 2.0 1.4171470
## 133 900 1.00 1.0 2.1290987
## 134 900 1.00 1.5 1.9064864
## 135 900 1.00 2.0 1.8460896
## 136 1000 0.00 1.0 0.5441020
## 137 1000 0.00 1.5 0.7751388
## 138 1000 0.00 2.0 0.6366835
## 139 1000 0.25 1.0 0.7992841
## 140 1000 0.25 1.5 0.8256762
## 141 1000 0.25 2.0 0.6412979
## 142 1000 0.50 1.0 1.2921433
## 143 1000 0.50 1.5 1.2680054
## 144 1000 0.50 2.0 1.2986345
## 145 1000 0.75 1.0 1.6698362
## 146 1000 0.75 1.5 1.5722088
## 147 1000 0.75 2.0 1.4382501
## 148 1000 1.00 1.0 2.0434242
## 149 1000 1.00 1.5 1.8520927
## 150 1000 1.00 2.0 1.5812816
##
## attr(,"class")
## [1] "Eco"

```

Output of the GLS simulation with parallelisation

```
##
## Repetition(nrep)      : 5
##
## Parallelization Type : Multisession
##
## Number of Cores Used in Parallelization : 4 out of 8
##
## Input Parameters : c("n", "100", "1000", "100") c("mu", "0", "1", "0.25") c("sd", "1",
##
## Simulation Length : 750
## Minumum : -0.9971
## Maximum : 3.383637
## Mean : 1.260196
## Median : 1.284057
##
## Execution Time of Monte Carlo Simulation 29.47388 secs
##
## Name of The Class : Eco

## $results
## , , sd=1, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  0.7554538 0.5016876 1.7356494 1.517285 2.639110
## n=200  0.3903311 0.6408042 1.5194431 1.417126 3.255660
## n=300  0.8317798 0.8173362 1.2688679 1.366244 2.007371
## n=400  0.0664591 0.7887018 1.1664611 1.262650 1.601207
## n=500  0.7349604 0.8108023 1.5108027 1.624144 2.316178
## n=600  0.5483240 1.3362425 0.7466396 1.539507 1.775243
## n=700  1.1797722 0.7454456 1.5538655 1.596843 1.851457
## n=800  0.7388026 0.4151828 1.2355987 2.121480 1.617091
## n=900  0.3375558 0.5516673 0.9502070 1.037452 2.600048
## n=1000 0.2812896 0.8840261 1.4690083 1.667241 2.124077
##
## , , sd=1.5, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100  1.07566007 0.97703407 0.3943886 1.0410972 1.357513
## n=200  0.52383817 1.45057710 1.6489542 1.5278402 1.404216
## n=300  0.71906564 0.40330063 2.2602221 0.8222903 2.070699
## n=400  1.04256675 1.22061498 0.7880425 1.1109698 1.966145
```

```

## n=500 -0.07188688 1.41714029 1.4742135 1.9709113 3.005006
## n=600 0.87050124 1.16986970 0.8726417 1.1228528 1.821454
## n=700 -0.25848227 0.49203602 1.0950518 1.6813917 1.675552
## n=800 0.39160576 0.03376035 1.3268501 1.6087353 2.523682
## n=900 0.01281269 0.82989922 1.1917932 2.1448010 2.249365
## n=1000 1.00643264 0.23351513 1.7264966 1.7730168 1.716113
##
## , , sd=2, rep=1
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.1989645 0.9356581 1.650652 1.4279687 2.842366
## n=200 0.6770072 0.8534528 1.886844 1.1371663 2.086750
## n=300 0.9021613 1.3441820 1.592364 1.5281411 1.918323
## n=400 0.3678666 0.6749776 1.494146 1.0124086 2.125654
## n=500 -0.3538140 1.4641727 1.702553 2.0690877 1.771409
## n=600 0.2521821 0.8727736 1.281437 1.6461148 1.787155
## n=700 -0.5283604 1.1573474 1.301522 1.6284337 2.040922
## n=800 0.7162405 1.7569821 1.089098 1.8510202 2.008889
## n=900 -0.1466358 1.2663258 0.669074 0.9226198 1.983874
## n=1000 0.4922426 0.5592436 1.209339 0.5571104 2.177159
##
## , , sd=1, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.93794659 0.3042632 0.8558104 1.395916 2.488330
## n=200 0.46865388 0.8026164 0.6673771 1.754976 2.196959
## n=300 0.21497718 0.6857743 1.4560339 1.591513 1.441236
## n=400 0.09099797 0.1139196 0.9989419 1.530588 1.901867
## n=500 0.26214018 1.0189866 0.9483061 1.929226 2.028602
## n=600 0.36515870 1.1206859 1.3254734 2.154065 2.398600
## n=700 0.84564182 0.6471012 1.4065464 1.768395 2.419217
## n=800 0.85872788 0.7206308 1.3704309 1.246906 2.218637
## n=900 -0.14990817 1.2264243 1.6872835 1.774656 2.095731
## n=1000 0.96179814 0.6660319 1.2241939 1.846178 2.384093
##
## , , sd=1.5, rep=2
##
##          mu=0    mu=0.25    mu=0.5    mu=0.75    mu=1
## n=100 0.7971403 0.6846694 1.4747453 1.376802 1.289544
## n=200 0.8722172 1.5062628 0.8667520 1.455997 3.187951
## n=300 1.1649019 1.7666435 0.9995124 1.495770 1.687811
## n=400 0.8440923 0.5933976 0.8222845 1.806278 2.256190

```

```

## n=500 0.5786570 1.1103728 1.9378712 1.644442 2.421673
## n=600 0.7493392 1.1661280 1.2064266 1.645635 1.537470
## n=700 0.8992186 1.0192023 1.0847103 1.817849 1.520385
## n=800 0.5199020 0.9399592 1.7241364 1.260431 1.886227
## n=900 1.0116356 1.1141843 1.4591379 2.097425 2.420061
## n=1000 1.1299320 0.6024359 0.3630059 2.020767 1.720216
##
## , , sd=2, rep=2
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 -0.01919950 0.10389729 -0.7181519 2.1975727 2.688336
## n=200 1.02896044 -0.06349928 1.7339415 1.9835128 1.816331
## n=300 -0.18610242 1.60434976 1.6070085 -0.3271213 1.931503
## n=400 1.41709802 1.20750797 1.5523077 1.6424103 1.171583
## n=500 1.23639159 0.54645719 1.6774379 1.3658300 3.363879
## n=600 0.39037287 1.42422862 0.5759413 2.1932974 1.854756
## n=700 -0.34830352 0.85511492 1.2548947 2.4213875 2.016490
## n=800 -0.08188814 1.27109005 1.7893126 0.9299721 1.717897
## n=900 -0.07990839 0.66567655 1.8002084 1.7681873 1.740396
## n=1000 0.88195270 0.50717811 1.2185675 2.1915303 1.744440
##
## , , sd=1, rep=3
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.2500595 1.3250897 0.9547569 2.293587 1.819439
## n=200 0.5471549 0.9566464 1.3296827 1.775252 1.804901
## n=300 1.0456930 1.4996928 1.4615289 1.399473 2.154331
## n=400 0.6342614 0.5517242 1.4421203 1.787110 2.553724
## n=500 0.5785579 1.0079606 1.1506459 1.573662 1.915940
## n=600 0.3589415 0.7270598 1.5240060 1.239381 1.971919
## n=700 0.7345540 0.2262226 1.2611827 1.103715 2.039746
## n=800 0.5015531 1.9152089 1.6893004 1.561901 2.198732
## n=900 0.1721376 1.0750490 1.1722128 1.835734 1.946778
## n=1000 0.5581321 0.7179216 1.0656947 1.773012 2.400244
##
## , , sd=1.5, rep=3
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100 0.75515002 1.0532149 1.3675520 2.458531 2.511075
## n=200 -0.04934648 0.5496542 2.0479018 1.675846 1.142718
## n=300 1.00181000 1.6369489 0.7753943 2.638377 1.829983
## n=400 0.38708014 1.2533585 1.4714135 1.435519 2.018202

```

```

## n=500    0.37250451 0.7996507 1.2444854 1.892440 1.427025
## n=600    0.26431978 1.3546424 1.2058852 1.796262 2.147378
## n=700    1.12511433 0.6271908 0.8411113 1.638469 1.970271
## n=800    0.52208688 0.6764603 1.6023554 1.526346 1.906584
## n=900    0.22266670 0.5468360 1.5856621 2.134440 1.515266
## n=1000   0.20373892 1.9340035 1.2128314 1.337933 1.380179
##
## , , sd=2, rep=3
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 -0.5134825 -0.9971000 0.02609116 2.483622 2.1737654
## n=200 0.8564846 0.5270190 2.19874069 1.115522 2.0483072
## n=300 0.1357223 1.1346463 0.67885853 1.989233 2.3651937
## n=400 -0.1132654 1.5174377 -0.39911038 2.430706 2.7068852
## n=500 0.2692164 0.9373865 0.45341757 1.488270 2.0855372
## n=600 0.2767981 1.9185888 1.10986631 1.779951 2.2288551
## n=700 0.4175472 0.7176046 0.78665976 1.500209 2.3208915
## n=800 0.2236755 0.7400305 1.63497839 2.006660 2.2390314
## n=900 1.3383189 1.7077645 1.90542412 2.962268 1.5069244
## n=1000 0.2337000 0.4510541 0.72055597 2.240282 0.9580619
##
## , , sd=1, rep=4
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 1.3469860 -0.05560687 1.0932454 1.903516 2.156825
## n=200 0.3208913 0.96409270 1.0863966 1.386434 1.835235
## n=300 0.6078219 0.39871289 1.3616965 2.100878 1.933667
## n=400 0.6771807 1.05598360 1.5240453 2.261056 2.531636
## n=500 0.4919199 1.10827112 1.5332686 1.517603 1.843585
## n=600 0.5637876 0.85089642 1.8731358 1.617511 2.200501
## n=700 0.3582705 0.99200307 0.8214109 1.560444 2.128348
## n=800 0.5240053 0.94117803 1.2620739 1.546574 1.873964
## n=900 0.6295721 0.83325772 0.9917639 1.333128 1.971597
## n=1000 0.6248638 1.00187859 1.3589893 1.665165 1.630167
##
## , , sd=1.5, rep=4
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 0.14190040 2.01821137 0.8517502 2.0724785 2.762258
## n=200 0.82223920 0.95236727 2.3576613 2.7525450 2.656997
## n=300 0.41468489 0.81135204 1.7321312 2.2129173 1.586342
## n=400 0.47456049 1.28481297 1.6045288 1.6357618 2.517795

```



```

## n=500 0.19596935 -0.09776629 1.3816823 1.1821549 1.806055
## n=600 0.04919382 1.33474572 1.2062848 0.0429013 2.320981
## n=700 0.34423613 0.48523985 1.9450566 1.5101308 2.451672
## n=800 1.32565822 0.84774132 1.2387826 2.3976442 1.870370
## n=900 0.47854733 0.51916320 1.1726107 1.6816128 1.705770
## n=1000 0.82817973 0.80225623 1.5175430 1.4264566 1.863020
##
## , , sd=2, rep=4
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 0.8525966 1.1749948 3.38363655 1.248533 3.057232
## n=200 0.5182802 0.4838462 1.41934260 2.241456 2.358571
## n=300 1.0733668 2.0578513 0.96974173 2.245360 1.290691
## n=400 0.8011078 0.4410973 0.90802926 1.296971 2.153393
## n=500 0.2470769 0.7742284 2.22385975 2.482988 1.599441
## n=600 1.1095269 0.4747139 0.06967184 1.576253 1.038778
## n=700 0.3443959 -0.7980158 0.88632336 1.763624 1.642792
## n=800 0.3212259 -0.1889027 1.12345703 2.337993 1.622583
## n=900 0.8488329 1.3243651 0.42830492 1.324446 2.120547
## n=1000 1.6411901 0.1763252 1.43712633 1.095596 1.415895
##
## , , sd=1, rep=5
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 0.1106945 1.3324074 1.283149 1.795427 1.479256
## n=200 0.5031188 1.5027343 1.460474 1.921770 2.617298
## n=300 0.3730896 0.6590489 1.300485 1.664153 2.229097
## n=400 0.4404047 1.6079649 1.600920 1.597121 1.219695
## n=500 0.9142947 0.8821124 0.993282 2.142944 2.017532
## n=600 0.7455402 1.3534103 1.252659 1.654961 2.019734
## n=700 0.1307611 0.5045976 1.915240 1.719891 1.816085
## n=800 0.0348579 0.4676373 1.687917 1.773396 1.871487
## n=900 0.5683141 1.0347100 1.329680 1.971053 2.031339
## n=1000 0.2944263 0.7265625 1.342830 1.397584 1.678541
##
## , , sd=1.5, rep=5
##
##          mu=0      mu=0.25      mu=0.5  mu=0.75      mu=1
## n=100 0.3492151 0.941764185 1.9927388 1.1271039 2.343358
## n=200 0.9322249 0.009236562 1.6216833 1.5246716 2.126926
## n=300 -0.2061382 0.527637441 1.1785690 1.2855802 1.955373
## n=400 -0.2080310 0.865475626 1.5810396 1.8863548 2.607916

```

```

## n=500    1.0591201 1.269133681 1.3106256 1.6893725 2.221825
## n=600    1.0529351 0.935277255 0.8883463 0.4731693 2.883712
## n=700   -0.1566764 1.060909722 1.3480097 1.5789494 2.593332
## n=800    0.3458820 0.824462226 1.0043539 1.3977273 1.529786
## n=900    0.4578455 0.443865375 1.0857179 1.1477187 1.641970
## n=1000   0.7074108 0.556170334 1.5201499 1.3028712 2.580936
##
## , , sd=2, rep=5
##
##          mu=0      mu=0.25      mu=0.5      mu=0.75      mu=1
## n=100    0.35838129 1.9134632 0.6236571 1.8050059 2.775072
## n=200    1.13555545 -0.3462561 0.5517002 1.2833010 2.082454
## n=300   -0.12779402 0.8027196 0.2661467 1.1407255 2.706796
## n=400    0.84449836 1.5107112 1.3472610 2.1182648 2.410341
## n=500    0.58335229 0.3130528 2.2292441 0.5206100 1.407526
## n=600    1.00687281 0.8306865 0.8167926 -0.4327573 1.897837
## n=700    0.90763719 0.7547756 1.2602192 1.9296762 2.212971
## n=800    1.16899109 0.5890770 1.1466515 0.6023830 1.086282
## n=900   -0.53156942 0.2064395 1.3326650 0.1082137 1.878707
## n=1000  -0.06566813 1.5126884 1.9075839 1.1067314 1.610853
##
## attr(,"class")
## [1] "Eco"    "array"
##
## $average
##      n  mu  sd      avg
## 1   100 0.00 1.0 0.6802281
## 2   100 0.00 1.5 0.6238132
## 3   100 0.00 2.0 0.1754521
## 4   100 0.25 1.0 0.6815682
## 5   100 0.25 1.5 1.1349788
## 6   100 0.25 2.0 0.6261827
## 7   100 0.50 1.0 1.1845221
## 8   100 0.50 1.5 1.2162350
## 9   100 0.50 2.0 0.9931770
## 10  100 0.75 1.0 1.7811463
## 11  100 0.75 1.5 1.6152025
## 12  100 0.75 2.0 1.8325404
## 13  100 1.00 1.0 2.1165920
## 14  100 1.00 1.5 2.0527497
## 15  100 1.00 2.0 2.7073544
## 16  200 0.00 1.0 0.4460300

```

## 17	200	0.00	1.5	0.6202346
## 18	200	0.00	2.0	0.8432576
## 19	200	0.25	1.0	0.9733788
## 20	200	0.25	1.5	0.8936196
## 21	200	0.25	2.0	0.2909125
## 22	200	0.50	1.0	1.2126747
## 23	200	0.50	1.5	1.7085905
## 24	200	0.50	2.0	1.5581137
## 25	200	0.75	1.0	1.6511116
## 26	200	0.75	1.5	1.7873799
## 27	200	0.75	2.0	1.5521916
## 28	200	1.00	1.0	2.3420107
## 29	200	1.00	1.5	2.1037616
## 30	200	1.00	2.0	2.0784825
## 31	300	0.00	1.0	0.6146723
## 32	300	0.00	1.5	0.6188648
## 33	300	0.00	2.0	0.3594708
## 34	300	0.25	1.0	0.8121130
## 35	300	0.25	1.5	1.0291765
## 36	300	0.25	2.0	1.3887498
## 37	300	0.50	1.0	1.3697225
## 38	300	0.50	1.5	1.3891658
## 39	300	0.50	2.0	1.0228240
## 40	300	0.75	1.0	1.6244521
## 41	300	0.75	1.5	1.6909870
## 42	300	0.75	2.0	1.3152675
## 43	300	1.00	1.0	1.9531402
## 44	300	1.00	1.5	1.8260415
## 45	300	1.00	2.0	2.0425014
## 46	400	0.00	1.0	0.3818608
## 47	400	0.00	1.5	0.5080537
## 48	400	0.00	2.0	0.6634611
## 49	400	0.25	1.0	0.8236588
## 50	400	0.25	1.5	1.0435319
## 51	400	0.25	2.0	1.0703464
## 52	400	0.50	1.0	1.3464978
## 53	400	0.50	1.5	1.2534618
## 54	400	0.50	2.0	0.9805268
## 55	400	0.75	1.0	1.6877048
## 56	400	0.75	1.5	1.5749766
## 57	400	0.75	2.0	1.7001523
## 58	400	1.00	1.0	1.9616256

## 59	400	1.00	1.5	2.2732495
## 60	400	1.00	2.0	2.1135711
## 61	500	0.00	1.0	0.5963746
## 62	500	0.00	1.5	0.4268728
## 63	500	0.00	2.0	0.3964446
## 64	500	0.25	1.0	0.9656266
## 65	500	0.25	1.5	0.8997062
## 66	500	0.25	2.0	0.8070595
## 67	500	0.50	1.0	1.2272611
## 68	500	0.50	1.5	1.4697756
## 69	500	0.50	2.0	1.6573025
## 70	500	0.75	1.0	1.7575159
## 71	500	0.75	1.5	1.6758640
## 72	500	0.75	2.0	1.5853571
## 73	500	1.00	1.0	2.0243674
## 74	500	1.00	1.5	2.1763168
## 75	500	1.00	2.0	2.0455584
## 76	600	0.00	1.0	0.5163504
## 77	600	0.00	1.5	0.5972578
## 78	600	0.00	2.0	0.6071506
## 79	600	0.25	1.0	1.0776590
## 80	600	0.25	1.5	1.1921326
## 81	600	0.25	2.0	1.1041983
## 82	600	0.50	1.0	1.3443827
## 83	600	0.50	1.5	1.0759169
## 84	600	0.50	2.0	0.7707418
## 85	600	0.75	1.0	1.6410851
## 86	600	0.75	1.5	1.0161640
## 87	600	0.75	2.0	1.3525717
## 88	600	1.00	1.0	2.0731992
## 89	600	1.00	1.5	2.1421991
## 90	600	1.00	2.0	1.7614761
## 91	700	0.00	1.0	0.6497999
## 92	700	0.00	1.5	0.3906821
## 93	700	0.00	2.0	0.1585833
## 94	700	0.25	1.0	0.6230740
## 95	700	0.25	1.5	0.7369157
## 96	700	0.25	2.0	0.5373653
## 97	700	0.50	1.0	1.3916491
## 98	700	0.50	1.5	1.2627879
## 99	700	0.50	2.0	1.0979239
## 100	700	0.75	1.0	1.5498575

101 700 0.75 1.5 1.6453580
102 700 0.75 2.0 1.8486661
103 700 1.00 1.0 2.0509706
104 700 1.00 1.5 2.0422424
105 700 1.00 2.0 2.0468134
106 800 0.00 1.0 0.5315894
107 800 0.00 1.5 0.6210270
108 800 0.00 2.0 0.4696490
109 800 0.25 1.0 0.8919676
110 800 0.25 1.5 0.6644767
111 800 0.25 2.0 0.8336554
112 800 0.50 1.0 1.4490643
113 800 0.50 1.5 1.3792957
114 800 0.50 2.0 1.3566995
115 800 0.75 1.0 1.6500513
116 800 0.75 1.5 1.6381768
117 800 0.75 2.0 1.5456057
118 800 1.00 1.0 1.9559821
119 800 1.00 1.5 1.9433300
120 800 1.00 2.0 1.7349365
121 900 0.00 1.0 0.3115343
122 900 0.00 1.5 0.4367016
123 900 0.00 2.0 0.2858076
124 900 0.25 1.0 0.9442217
125 900 0.25 1.5 0.6907896
126 900 0.25 2.0 1.0341143
127 900 0.50 1.0 1.2262295
128 900 0.50 1.5 1.2989843
129 900 0.50 2.0 1.2271353
130 900 0.75 1.0 1.5904047
131 900 0.75 1.5 1.8411995
132 900 0.75 2.0 1.4171470
133 900 1.00 1.0 2.1290987
134 900 1.00 1.5 1.9064864
135 900 1.00 2.0 1.8460896
136 1000 0.00 1.0 0.5441020
137 1000 0.00 1.5 0.7751388
138 1000 0.00 2.0 0.6366835
139 1000 0.25 1.0 0.7992841
140 1000 0.25 1.5 0.8256762
141 1000 0.25 2.0 0.6412979
142 1000 0.50 1.0 1.2921433

```
## 143 1000 0.50 1.5 1.2680054
## 144 1000 0.50 2.0 1.2986345
## 145 1000 0.75 1.0 1.6698362
## 146 1000 0.75 1.5 1.5722088
## 147 1000 0.75 2.0 1.4382501
## 148 1000 1.00 1.0 2.0434242
## 149 1000 1.00 1.5 1.8520927
## 150 1000 1.00 2.0 1.5812816
##
## attr("class")
## [1] "Eco"
```

Eidesstattliche Versicherung

Ich versichere an Eides statt durch meine Unterschrift, dass ich die vorstehende Arbeit selbständig und ohne fremde Hilfe angefertigt und alle Stellen, die ich wörtlich oder annähernd wörtlich aus Veröffentlichungen entnommen habe, als solche kenntlich gemacht habe, mich auch keiner anderen als der angegebenen Literatur oder sonstiger Hilfsmittel bedient habe. Die Arbeit hat in dieser oder ähnlicher Form noch keiner anderen Prüfungsbehörde vorgelegen.

Essen, den _____

The image shows three handwritten signatures in black ink. The first signature on the left is 'A. Langnau', the middle one is 'Öcal Kaptan', and the right one is 'Sunyoung Ji'. They are written in a cursive, flowing style.

Alexander Langnau, Öcal Kaptan, Sunyoung Ji