ME 760 Homework 4

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1. Verify the contour integral $\int_C [2xy^2dx + 2x^2ydy + dz]$ is independent of the path. Evaluate this integral between the points $(0, 0, 0)$ and (a, b, c).

   A line integral with the vector function of the form $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is path indenpendent if $curl\boldsymbol{F} = 0$. The curl of the vector is found and evaluated below using equation 1.

$$curl\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0 \tag{1}$$

   Expansion of the above yields the following:

$$curl\boldsymbol{F} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\boldsymbol{i} - \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\boldsymbol{j} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\boldsymbol{k} \tag{2}$$

   The two non-trivial partial derivatives are shown below:

$$\frac{\partial Q}{\partial x} = 4xy$$

$$\frac{\partial P}{\partial y} = 4xy$$

   Substituting the values into equation 2, yields $curl\boldsymbol{F} = 0$, thus the contour integral is independent of the path.

   Evaluation of this leads to $f = x^2 + y^2 + z$. Thus evaluating using the form $f(B) - (A)$ yields:

$$\boxed{a^2b^2 + c}$$

2. Given the parametric form of a cone $r(u,v) = [u\cos v, u\sin v, cu]$

   (a) find an explicit representation of the form $z = f(x,y)$

   (b) find and identify the paramer curves defind as $u = const$ and $v = const$

   (c) find the normal vector N to the conical surface

   (a) Let $x, y, z = u\cos v, u\sin v, cu$ respectively. Squaring x and y and summing them yields: $x^2 + y^2 = u^2\left(\cos^2 v + \sin^2 v\right)$. Solving for u from this: $u = \sqrt{x^2 + y^2}$. Using this to substitute into z results in the explicity form of the equation:

$$\boxed{z = c\sqrt{x^2 + y^2}} \tag{3}$$

   (b) From (a), it is seen that $\boxed{u = \sqrt{x^2 + y^2}}$. v is found from the division of x and y. Using this and trigonemetric definitions yields: $\boxed{v = tan^{-1}\frac{x}{y}}$.

   (c) The normal vector is found using equation 4

$$\nabla f = 0 \tag{4}$$

   Differentiation of the explicit form with respect to x, y, z results in the normal vector:

$$\boxed{\boldsymbol{N} = \frac{2xc}{\sqrt{x^2 + y^2}}\boldsymbol{i} + \frac{2yc}{\sqrt{x^2 + y^2}}\boldsymbol{j} - \boldsymbol{k}}$$

3. In class we discussed surface integrals without regard to orientation. By reparameterizing the surface integral could be written as

$$I = \int\int_{s} G(s)\, dS = \int\int_{R} G(r(u,v))\, |N(u,v)|dudv$$

(a) Consider the case $G = z$ adn teh surface S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$. Use polar coordinates and evaluate the right hand side of the above reult.

(b) The surface S is also given explicitly by $z = f(x,y) = \sqrt{9 - x^2 - y^2}$. For such cases the surface integral can be rewritten as

$$\int\int_{S} G(r)\, dA = \int\int_{R*} G(x,y,f(x,y))\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy \qquad (5)$$

Evaluate the right-hand side fo this result.