ME 760 Homework 1

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1. For $\vec{a} \neq 0$, what does $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{b}$ imply about the vectors $\vec{b}$ and $\vec{c}$? Give a geometric interpretation. Hint: if $\vec{a}$, $\vec{b}$, and $\vec{c}$ all begin at the same point, where do the ends of the vectors $\vec{b}$ and $\vec{c}$ lie?

   This implies that the sum of $\vec{b}$ and $\vec{c}$ is perpindicular to $\vec{a}$. This is realized through rearranging the equation as $\vec{a} \cdot \left(\vec{b} - \vec{c}\right) = 0$. This however, doesn't imply that $\vec{b} = \vec{c}$.

2. In practical work the following formulat is quite useful.

$$|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a}) \left(\vec{b} \cdot \vec{b}\right) - \left(\vec{a} \cdot \vec{b}\right)^2} \tag{1}$$

Give a proof.

By definition $\vec{a} \times \vec{b} = \|a\| \|b\| \sin\theta$ and $\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos\theta$ Substituting the above definitions and squaring both sides produces:

$$\|a\|^2 \|b\|^2 \sin^2\theta = \|a\|^2 \|b\|^2 - \|a\|^2 \|b\|^2 \cos^2\theta$$

Rearranging and factoring to utilize trigonometric identities:

$$\|a\|^2 \|b\|^2 \left(\sin^2\theta + \cos^2\theta\right) = \|a\|^2 \|b\|^2$$

Using $\sin^2\theta + \cos^2\theta = 1$, the equations are equivalent.

$$\|a\|^2 \|b\|^2 = \|a\|^2 \|b\|^2$$

3. Determine the angles in the triangle formed by the three vertices $P_1 = (2, 2, 2)$, $P_2 = (3, 1, 1)$, and $P_3 = (3, 3, 3)$.

   Begin by determining the three vectors that connect the three points:

   - $\vec{v_1} = P_1 - P_2 = \langle 1, -1, -1 \rangle$
   - $\vec{v_2} = P_3 - P_2 = \langle 0, 2, 2 \rangle$
   - $\vec{v_3} = P_1 - P_3 = \langle -1, -1, -1 \rangle$

   Using the definition of the dot product, $\vec{a} \cdot \vec{b} = \|a\|\|b\| cos\theta$, the angles can be determined by:

$$\theta = \arccos \frac{\vec{a} \cdot \vec{b}}{\|a\|\|b\|} \tag{2}$$

$$\boxed{\begin{array}{l} \theta_{1-2} = \arccos \frac{-4}{\sqrt{3}\sqrt{8}} = 2.52rad \\ \theta_{2-3} = \arccos \frac{-4}{\sqrt{8}\sqrt{3}} = 2.52rad \\ \theta_{3-1} = \arccos \frac{1}{\sqrt{3}\sqrt{3}} = 1.23rad \end{array}}$$

4. What is the equation of the plane containing the triangle of the previous problem?

Equation of the plane can be expressed from a point and a vector orthogonal to the plane, using $\vec{v_1}$ and $\vec{v_2}$ from the previous problem to find a normal vector:

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{v_{11}} & \vec{v_{13}} \\ \vec{v_{21}} & \vec{v_{23}} \end{vmatrix} - \begin{vmatrix} \vec{v_{12}} & \vec{v_{13}} \\ \vec{v_{22}} & \vec{v_{23}} \end{vmatrix} + \begin{vmatrix} \vec{v_{22}} & \vec{v_{23}} \\ \vec{v_{22}} & \vec{v_{23}} \end{vmatrix} \tag{3}$$

Substituting values into equation 3 and substituiting yields:

$$\vec{n} = 0\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

Using equation 4 where a, b, and c come from $\vec{n}$ and x_0, y_0, and z_0 are values from P_1.

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0 \tag{4}$$

Substituiting values in results in the equation of the plane being:

$$\boxed{2y - 2z = 0}$$

5. Show that the three vectors $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{w} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$ are coplanar.

Three vectors are can be showed to be coplanar if the scalar triple products evaluates to 0

$$\left[\vec{a}, \vec{b}, \vec{c}\right] = \vec{a} \cdot \left(\vec{b} \times \vec{c}\right) \tag{5}$$

Equation 5 is equivalant to the determinant below:

$$d = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \tag{6}$$

Substituting values from $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 4 & -1 \end{vmatrix}$$

The determinant of which is evaluated as:

$$d = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{7}$$

Substituting values from $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$:

$$d = 1 \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix}$$

Upon evaluation, $d = 0$, therefore the vectors are coplanar.

6. For general vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ show by expanding in component form:

$$\left(\vec{a} \times \vec{b}\right) \times \vec{c} = (\vec{a} \cdot \vec{c})\,\vec{b} - \left(\vec{b} \cdot \vec{c}\right)\vec{a}$$

Decomposing the right side into component form and expanding results in:

$$(a_x c_x + a_y c_y + a_z c_z)\, b_x - (b_x c_x + b_y c_y + b_z c_z)\, a_x$$

$$(a_x c_x + a_y c_y + a_z c_z)\, b_y - (b_x c_x + b_y c_y + b_z c_z)\, a_y$$

$$(a_x c_x + a_y c_y + a_z c_z)\, b_z - (b_x c_x + b_y c_y + b_z c_z)\, a_z$$

Simplification and combing of terms yields:

$$(a_y c_y + a_z c_z)\, b_x - (b_y c_y + b_z c_z)\, a_x$$

$$(a_x c_x + a_z c_z)\, b_y - (b_x c_x + b_z c_z)\, a_y$$

$$(a_x c_x + a_y c_y)\, b_z - (b_x c_x + b_y c_y)\, a_z$$

Moving to the left side and computing $\vec{a} \times \vec{b}$ yields:

$$\vec{d} = \vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\,\hat{\imath} - (a_x b_z - b_x a_z)\,\hat{\jmath} + (a_x b_y - b_x a_y)\,\hat{k}$$

Computing the last cross product and simplifying:

$$\begin{aligned}\vec{d} \times \vec{c} = ((a_x c_x + a_y c_y + a_z c_z)\, b_x - (b_x c_x + b_y c_y + b_z c_z)\, a_x)\,\hat{\imath} - \\ ((a_x c_x + a_y c_y + a_z c_z)\, b_y - (b_x c_x + b_y c_y + b_z c_z)\, a_y)\,\hat{\jmath} + \\ ((a_x c_x + a_y c_y + a_z c_z)\, b_z - (b_x c_x + b_y c_y + b_z c_z)\, a_z)\,\hat{k}\end{aligned}$$

Thus the equation is valid and shows that the cross product is non-associative

7. Line L passes through the two points $P_1 = (-2, -2, -2)$ and $P_2 = (-1, -1, -1)$.

   a) What is the equation of this line?

   b) What are the coordinates of the two points of intersection of this line with a sphere of radius 2 centered at $(2, 2, 2)$?

   a) The vector parallel to the line is $\vec{v} = \langle 111 \rangle$. Thus the equation of the line can be represented using the vector and point P_1 by the following set of parametric equations:

$$\boxed{\vec{r} = \langle -2, -2, -2 \rangle + t\langle 1, 1, 1 \rangle}$$

   b) The previous solution can be rewrote into component form and substituted into equation 8, standard form of sphere equation.

$$(x - x_c)^2 + (y - y_c)^2 + (z - z_c)^2 = r^2 \tag{8}$$

   Substituiting and changing into terms of t yields:

$$[-2t - (x_c - 1)]^2 + [-2t - (y_c - 1)]^2 + [-2t - (z_c - 1)]^2 = r^2$$

   After substituting values for the sphere center and radius, terms are combined and order to solve for t using the quadratic equation.

$$3t^2 - 24t + 44 = 0$$

   This results in $t = 5.15$ and $t = 2.85$. Substituting these back into the parametric set of equations results in two intersection points:

$$\boxed{(3.15, 3.15, 3.15)\ and\ (0.85, 0.85, 0.85)}$$

8. Two fixed points A and B have position vectors $\vec{a}$ and $\vec{b}$.

(a) Identify the plane P given by $\left(\vec{a}-\vec{b}\right)\cdot\vec{r}=\left(a^2-b^2\right)/2$ where a and b are the magnitudes of $\vec{a}$ and $\vec{b}$.

(b) Show that the equation

$$|\left(\vec{a}-\vec{r}\right)\cdot\left(\vec{b}-\vec{r}\right)|=0$$

describes a sphere S of radius $R=|\vec{a}-\vec{b}|/2$. Hint: Cast this equation into the standard form $\left(\vec{r}-\vec{c}\right)\left(\vec{r}-\vec{c}\right)=R^2$. where you should find $c=\left(\vec{a}+\vec{b}\right)/2$

(c) Show that the intersection of P and S is also the intersection of two spheres centered on A and B each of radius $|\vec{a}-\vec{b}|/\sqrt{2}$. Hint Add and substract the equations for P and S and cas the results into the standard form $|\vec{r}-\vec{d}|^2=R^2$.