

ME 760 Homework 2

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1. For the array

$$C = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

calculate:

(a) C^2

(b) $C^T C$

(c) CC^T

(a) Each element in the product can be expressed from equation 1 where i and j are the row and column indices respectively.

$$C_{i,j}^2 = \sum_{n=1}^3 C_{i,n} C_{n,j} \quad (1)$$

The subsequent calculations are seen below:

$$C_{1,1}^2 = 4 * 4 + 6 * 6 + 2 * 2 = 56$$

$$C_{1,2}^2 = 4 * 6 + 6 * 0 + 2 * 3 = 30$$

$$C_{1,3}^2 = 4 * 2 + 6 * 3 + 2 * -1 = 24$$

$$C_{2,1}^2 = 6 * 4 + 0 * 6 + 2 * 2 = 30$$

$$C_{2,2}^2 = 6 * 6 + 0 * 0 + 3 * 3 = 45$$

$$C_{2,3}^2 = 6 * 2 + 0 * 3 + 3 * -1 = 9$$

$$C_{3,1}^2 = 2 * 4 + 3 * 6 + -1 * 2 = 24$$

$$C_{3,2}^2 = 2 * 6 + 3 * 0 + -1 * 3 = 9$$

$$C_{3,3}^2 = 2 * 2 + 3 * 3 + -1 * -1 = 14$$

The resulting matrix therefore is:

$$C^2 = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$$

(b) Calculation of the transpose, C^T of C is realized through equation 2.

$$C_{i,j}^T = C_{j,i} \quad (2)$$

Computation of C^T , results in $C^T = C$. Thus the calculation of CC^T is identical to that of (a). Thus the resulting matrix is:

$$C^T C = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$$

(c) Similar to (b), the calculation of CC^T is identical to C^2 . Thus the resulting matrix is:

$$CC^T = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}$$

2. Solve $\mathbf{Ax} = \mathbf{b}$ for the following set of linear equations:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 8 & 6 \\ -2 & 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 40 \end{pmatrix}$$

(a) by Gauss Elimination

(b) by using Cramer's Rule

(c) by finding the inverse $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

(a) The goal of Gaussian Elimination is to represent the system in reduced row echelon form (RREF), the following steps are used to reduce \mathbf{A} to RREF:

i. Divide row 2 by 8:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ -2 & 4 & -6 & 40 \end{array} \right)$$

ii. Multiply row 1 by 2 and add to row 3:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 6 & -8 & 58 \end{array} \right)$$

iii. Multiple row 2 by 6 and subtract from row 3

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 0 & -12.5 & 62.5 \end{array} \right)$$

iv. Divide row 3 by -12.5:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 0 & 1 & -5 \end{array} \right)$$

v. Multiply row 3 by 0.75 and subtract from row 2:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array} \right)$$

vi. Subtract row 2 and add row 3 to row 1:

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array} \right)$$

Thus leaving the solution vector to be:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

- (b) Use of Cramer's Rule requires computation of the determinate of multiple matrices and division to compute the solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{\det(\mathbf{A})}{\det(\mathbf{A}_x)} \\ \frac{\det(\mathbf{A})}{\det(\mathbf{A}_y)} \\ \frac{\det(\mathbf{A})}{\det(\mathbf{A}_z)} \end{pmatrix} \quad (3)$$

Where \mathbf{A}_x , \mathbf{A}_y , and \mathbf{A}_z represent a combination of the \mathbf{A} and \mathbf{b} comprised by replacing the associated vector in \mathbf{A} with \mathbf{b} .

$$\mathbf{A}_x = \begin{pmatrix} 9 & 1 & -1 \\ -6 & 8 & 6 \\ 40 & 4 & -6 \end{pmatrix}$$

$$\mathbf{A}_y = \begin{pmatrix} 1 & 9 & -1 \\ 0 & -6 & 6 \\ -2 & 40 & -6 \end{pmatrix}$$

$$\mathbf{A}_z = \begin{pmatrix} 1 & 1 & 9 \\ 0 & 8 & -6 \\ -2 & 4 & 40 \end{pmatrix}$$

With the determinate being found using equation 4.

$$\det(\mathbf{A}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad (4)$$

Evaluation of the four determinates yields:

$$\det(\mathbf{A}) = -100$$

$$\det(\mathbf{A}_x) = -100$$

$$\det(\mathbf{A}_y) = -300$$

$$\det(\mathbf{A}_z) = 500$$

Substituting into equation 3 yields:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

- (c) Finding the inverse is done by augmenting the identity matrix onto \mathbf{A} and reducing \mathbf{A} to the identity. The augmented matrix is:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 8 & 6 & 0 & 1 & 0 \\ -2 & 4 & -6 & 0 & 0 & 1 \end{array} \right)$$

The matrix inverse is found with the following steps:

i. Divide row 2 by 8

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ -2 & 4 & -6 & 0 & 0 & 1 \end{array} \right)$$

ii. Add 2 times row 1 to row 3

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ 0 & 6 & -8 & 2 & 0 & 1 \end{array} \right)$$

iii. Subtract 6 times row 2 from row 3, then divide by -12.5

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array} \right)$$

iv. Subtract 0.75 times row 3 from row 2

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.12 & 0.08 & 0.06 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array} \right)$$

v. Add row 3 and subtract row 2 from row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.72 & -0.02 & -0.14 \\ 0 & 1 & 0 & 0.12 & 0.08 & 0.06 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array} \right)$$

Thus \mathbf{A}^{-1} is:

$$\begin{pmatrix} 0.72 & -0.02 & -0.14 \\ 0.12 & 0.08 & 0.06 \\ -0.16 & 0.06 & -0.08 \end{pmatrix}$$

From this \mathbf{x} can be found by $\mathbf{A}^{-1}\mathbf{b}$:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.72 & -0.02 & -0.14 \\ 0.12 & 0.08 & 0.06 \\ -0.16 & 0.06 & -0.08 \end{pmatrix} \begin{pmatrix} 9 \\ -6 \\ 40 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

3. Determine the angles in the triangle formed by the three vertices $P_1 = (2, 2, 2)$, $P_2 = (3, 1, 1)$, and $P_3 = (3, 3, 3)$.

Begin by determining the three vectors that connect the three points:

- $\vec{v}_1 = P_1 - P_2 = \langle 1, -1, -1 \rangle$
- $\vec{v}_2 = P_3 - P_2 = \langle 0, 2, 2 \rangle$
- $\vec{v}_3 = P_1 - P_3 = \langle -1, -1, -1 \rangle$

Using the definition of the dot product, $\vec{a} \cdot \vec{b} = \|a\|\|b\|\cos\theta$, the angles can be determined by:

$$\theta = \arccos \frac{\vec{a} \cdot \vec{b}}{\|a\|\|b\|} \quad (5)$$

$\begin{aligned} \theta_{1-2} &= \arccos \frac{-4}{\sqrt{3}\sqrt{8}} = 2.52rad \\ \theta_{2-3} &= \arccos \frac{-4}{\sqrt{8}\sqrt{3}} = 2.52rad \\ \theta_{3-1} &= \arccos \frac{1}{\sqrt{3}\sqrt{3}} = 1.23rad \end{aligned}$
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4. Find the spectra and eigenvectors for the two matrixes below. Show your work.

$$\mathbf{A} = \begin{pmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}$$