

ME 760 Homework 6

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1. Under what conditions is $(ax + by) dx + (kx + ly) dy$ an exact differential? Here a, b, k and l are constants. Solve the exact equation.

A differential is exact if it can be represented in the form $\frac{dy}{dx}N(x, y) + M(x, y) = 0$. The test for exactness is seen in equation 1.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1)$$

From this it can be seen that for the equation to be exact, $b = k$.

By first integrating M with respect to x , $u(x, y) = \frac{ax^2}{2} + byx + p(y)$. Differentiating this function with respect to y and setting equal to N yields: $p'(y) = ly$, given that b and k are equal. Integrating $p(y)$ and substituting, it is found that:

$$F(x, y) = ax^2/2 + byx + ly^2/2 = C$$

2. Show that $x^a y^b$ is an integrating factor for the following equation and then find its solution.

$$(b+1)x \frac{dy}{dx} + (a+1)y = 0$$

Begin by multiplying the entire equation by the integrating factor:

$$(b+1)x^{a+1}y^b dy + (a+1)x^a y^{b+1} dx = 0$$

This can be seen to be in the form of an exact equation, with $M = (a+1)x^a y^{b+1}$ and $N = (b+1)x^{a+1}y^b$. Thus differentiating M with respect to y and N with respect to x .

$$\begin{aligned} b(a+1)x^a y^b + (a+1)x^a y^b \\ a(b+1)x^a y^b + (b+1)x^a y^b \end{aligned}$$

As such, the integrating factor applies if and only if $a = b$.

Begin by integrating M with respect to x . $u(x, y) = \frac{ax^{a+1}}{a+1}y^{b+1} + \frac{x^{a+1}y^{b+1}}{a+1} + p(y)$. Differentiating this and setting equal to N yields that $p'(y) = 0$. Thus combining yields:

$$\boxed{F(x, y) = \frac{ax^{a+1}}{a+1}y^{b+1} + \frac{x^{a+1}y^{b+1}}{a+1} + C}$$

3. Find the solution of

(a) $y'(x) = x(4 - y)$

(b) $y'(x) + 6x^2y = e^{-2x^3}/x^2$ with $y(1) = 0$

(a) By separating variables, this quickly becomes $dy(4 - y)^{-1} = xdx$. By integrating both sides, this becomes $-\ln(4 - y) = \frac{x^2}{2} + C_1$. Solving for y explicitly this yields:

$$y = C_2 - e^{-x^2/2}$$

(b) This is a first order, linear differential equation, as such the integrating factor is $q(x) = \exp(\int 6x^2 dx) = \exp(2x^3)$. Thus multiplying and integrating yields:

$$y(x) = \frac{C}{e^{2x^3}} - \frac{1}{xe^{2x^3}}$$

After applying the boundary condition of $y(1) = 0$:

$$y(x) = \frac{1}{e^{2x^3}} - \frac{1}{xe^{2x^3}}$$

4. Show if the following two sets of functions are linearly independent on the positive x axis

(a) $[\cos x, \sin x, \sin 2x]$

(b) $[\exp(x) \cos x, \exp(x) \sin x, \exp(x)]$

(a) The functions can be shown to be linearly independent by using showing that the Wronskian is non-zero for any point in the interval. The Wronskian is found below:

$$W = \begin{vmatrix} \cos x & \sin x & \sin 2x \\ -\sin x & \cos x & 2 \cos 2x \\ -\cos x & -\sin x & -4 \sin 2x \end{vmatrix}$$

Computing this yields $W = -3 \sin 2x$. Clearly, these functions are linearly independent.

(b) Again the functions can be shown to be linearly independent by having a non-zero Wronskian. The matrix used to determine the Wronskian is seen below:

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x & e^x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x & e^x \\ -2e^x \sin x & 2e^x \cos x & e^x \end{vmatrix}$$

Evaluating this yields $W = e^x - 1$ as such, the functions are linearly independent for all positive x values.

5. Solve the ode $y'' + 4y' + 3y = 65 \cos 2x$ by

(a) The method of undetermined coefficients

(b) The variation of parameters method

(a) Begin by guessing a possible solution of the form $A \cos 2x + B \sin 2x$. This can be differentiated and plugged into the ode. Doing so and reducing down yields $(-16A - 8B) \cos 2x + (8A - 16B) \sin 2x = 65 \cos 2x$. The two sides of the equation can be used to form two equations:

$$(-16A - 8B) = 65$$

$$(8A - 16B) = 0$$

Solving these equations simultaneously and plugging into the guess:

$$y(x) = 3.25 \cos 2x + 6.5 \sin 2x$$

6. What is the solutions to the initial value problem $y'''' - y = 0$ with $y(0) = -1$, $y'(0) = 7$, $y''(0) = -1$, and $y'''(0) = 7$.

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