ME 760 Homework 2

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1. For the array

$$C = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

calculate:

(a) $\boldsymbol{C}^2$

(b) $\boldsymbol{C}^T\boldsymbol{C}$

(c) $\boldsymbol{C}\boldsymbol{C}^T$

(a) Each element in the product can be expressed from equation 1 where i and j are the row and column indices respectively.

$$\boldsymbol{C}^2_{i,j} = \sum_{n=1}^{3} C_{i,n}C_{n,j} \tag{1}$$

The subsuquent calculations are seen below:

$$\begin{aligned}
\boldsymbol{C}^2_{1,1} &= 4*4+6*6+2*2 = 56 \\
\boldsymbol{C}^2_{1,2} &= 4*6+6*0+2*3 = 30 \\
\boldsymbol{C}^2_{1,3} &= 4*2+6*3+2*-1 = 24 \\
\boldsymbol{C}^2_{2,1} &= 6*4+0*6+2*2 = 30 \\
\boldsymbol{C}^2_{2,2} &= 6*6+0*0+3*3 = 45 \\
\boldsymbol{C}^2_{2,3} &= 6*2+0*3+3*-1 = 9 \\
\boldsymbol{C}^2_{3,1} &= 2*4+3*6+-1*2 = 24 \\
\boldsymbol{C}^2_{3,2} &= 2*6+3*0+-1*3 = 9 \\
\boldsymbol{C}^2_{3,3} &= 2*2+3*3+-1*-1 = 14
\end{aligned}$$

The resulting matrix therefore is:

$$\boxed{\boldsymbol{C}^2 = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}}$$

(b) Calculation of the transpose, $\boldsymbol{C}^T$ of $\boldsymbol{C}$ is realized through equation 2.

$$C^T_{i,j} = C_{j,i} \tag{2}$$

Computation of $\boldsymbol{C}^T$, results in $\boldsymbol{C}^T = \boldsymbol{C}$. Thus the calculation of $\boldsymbol{C}\boldsymbol{C}^T$ is identical to that of (a). Thus the resulting matrix is:

$$\boxed{\boldsymbol{C}^T\boldsymbol{C} = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}}$$

(c) Similiar to (b), the calculation of $\boldsymbol{C}\boldsymbol{C}^T$ is identical to $\boldsymbol{C}^2$. Thus the resulting matrix is:

$$\boxed{\boldsymbol{C}\boldsymbol{C}^T = \begin{bmatrix} 56 & 30 & 24 \\ 30 & 45 & 9 \\ 24 & 9 & 14 \end{bmatrix}}$$

2. Solve $\boldsymbol{Ax} = \boldsymbol{b}$ for the following set of linear equations:

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 8 & 6 \\ -2 & 4 & -6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \\ 40 \end{pmatrix}$$

(a) by Gauss Elimination

(b) by using Cramer's Rule

(c) by finding the inverse $\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$

(a) The goal of Gaussian Elimination is to represent the system in reduced row echelon form (RREF), the following steps are used to reduce $\boldsymbol{A}$ to RREF:

i. Divide row 2 by 8:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ -2 & 4 & -6 & 40 \end{array}\right)$$

ii. Multiply row 1 by 2 and add to row 3:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 6 & -8 & 58 \end{array}\right)$$

iii. Multiple row 2 by 6 and subtract from row 3

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 0 & -12.5 & 62.5 \end{array}\right)$$

iv. Divide row 3 by -12.5:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0.75 & -0.75 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

v. Multiply row 3 by 0.75 and subtract from row 2:

$$A = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

vi. Subtract row 2 and add row 3 to row 1:

$$A = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -5 \end{array}\right)$$

Thus leaving the solution vector to be:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

(b) Use of Cramer's Rule requires computation of the determinate of multiple matrices and division to compute the solution:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \dfrac{det(\boldsymbol{A})}{det(\boldsymbol{A_x})} \\ \dfrac{det(\boldsymbol{A})}{det(\boldsymbol{A_y})} \\ \dfrac{det(\boldsymbol{A})}{det(\boldsymbol{A_z})} \end{pmatrix} \tag{3}$$

Where $\boldsymbol{A_x}$, $\boldsymbol{A_y}$, and $\boldsymbol{A_z}$ represent a combination of the $\boldsymbol{A}$ and $\boldsymbol{b}$ comprised by replacing the associated vector in $\boldsymbol{A}$ with $\boldsymbol{b}$.

$$\boldsymbol{A_x} = \begin{pmatrix} 9 & 1 & -1 \\ -6 & 8 & 6 \\ 40 & 4 & -6 \end{pmatrix}$$

$$\boldsymbol{A_y} = \begin{pmatrix} 1 & 9 & -1 \\ 0 & -6 & 6 \\ -2 & 40 & -6 \end{pmatrix}$$

$$\boldsymbol{A_x} = \begin{pmatrix} 1 & 1 & 9 \\ 0 & 8 & -6 \\ -2 & 4 & 40 \end{pmatrix}$$

With the determinate being found using equation 4.

$$det(\boldsymbol{A}) = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \tag{4}$$

Evaluation of the four determinates yields:

$$\begin{aligned} det(\boldsymbol{A}) &= -100 \\ det(\boldsymbol{A_x}) &= -100 \\ det(\boldsymbol{A_y}) &= -300 \\ det(\boldsymbol{A_z}) &= 500 \end{aligned}$$

Substituting into equation 3 yields:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

(c) Finding the inverse is done by augmenting the identiy matrix onto $\boldsymbol{A}$ and reducing $\boldsymbol{A}$ to the identity. The augmented matrix is:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 8 & 6 & 0 & 1 & 0 \\ -2 & 4 & -6 & 0 & 0 & 1 \end{array}\right)$$

The matrix inverse is found with the following steps:

i. Divide row 2 by 8

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ -2 & 4 & -6 & 0 & 0 & 1 \end{array}\right)$$

ii. Add 2 times row 1 to row 3

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ 0 & 6 & -8 & 2 & 0 & 1 \end{array}\right)$$

iii. Subtract 6 times row 2 from row 3, then divide by -12.5

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0.75 & 0 & 0.125 & 0 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array}\right)$$

iv. Subtract 0.75 times row 3 from row 2

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0.12 & 0.08 & 0.06 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array}\right)$$

v. Add row 3 and subtract row 2 from row 1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.72 & -0.02 & -0.14 \\ 0 & 1 & 0 & 0.12 & 0.08 & 0.06 \\ 0 & 0 & 1 & -0.16 & 0.06 & -0.08 \end{array}\right)$$

Thus $\boldsymbol{A}^{-1}$ is:

$$\begin{pmatrix} 0.72 & -0.02 & -0.14 \\ 0.12 & 0.08 & 0.06 \\ -0.16 & 0.06 & -0.08 \end{pmatrix}$$

From this $\boldsymbol{x}$ can be found by $\boldsymbol{A}^{-1}\boldsymbol{b}$:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0.72 & -0.02 & -0.14 \\ 0.12 & 0.08 & 0.06 \\ -0.16 & 0.06 & -0.08 \end{pmatrix} \begin{pmatrix} 9 \\ -6 \\ 40 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -5 \end{pmatrix}}$$

3. For each of the sets below determine if the set constitutes a vector space or not. Give a reason for your answer. If you conclude the set is a vector space, determine its dimension and provide a basis.

   (a) all vectors in R^3 satisfying $v_1 - 3v_2 + 2v_3 =$ where v_i are teh components of a vector $\boldsymbol{v}$

   (b) all functions $y\left(x\right) = a\cos x + b\sin x$ with arbitrary real constants a and b

   (c) all skew-symmetric 2x2 matrices

   (d) all 2x2 matrices with $a_{11} + a_{22} = 0$

   (e) all $m \times m$ matrices with positive elements

   A vector space is defined by the following paramenters:

   - $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$
   - $(\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z} = \boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z})$
   - $\boldsymbol{x} + 0 = 0 + \boldsymbol{x}$
   - $\boldsymbol{x} + -\boldsymbol{x} = 0$
   - $r\left(s\boldsymbol{x}\right) = \left(rs\right)\boldsymbol{x}$
   - $\left(r + s\right)\boldsymbol{x} = r\boldsymbol{x} + s\boldsymbol{x}$
   - $r\left(\boldsymbol{x} + \boldsymbol{y}\right) = r\boldsymbol{x} + r\boldsymbol{y}$
   - $1\boldsymbol{x} = \boldsymbol{x}$

   (a) This set constitutes a vector space as all eight constraints can be shown to be valid. This space has a dimension of 3 and a basis of:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

   (b) Again, this is a vector space. With dimension of 1 and a basis of:

$$\left\{ \begin{matrix} \cos x \\ \sin x \end{matrix} \right\}$$

   (c) Is a vector space with dimension of 2

   (d) Is a vector space with dimension of 2

   (e) A vector space is defined with dimension of m.

4. Find the spectra and eigenvectors for the two matrixes below. Show you work.

$$\boldsymbol{A} = \begin{pmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{A} = \begin{pmatrix} a & 1 & 0 \\ 1 & a & 1 \\ 0 & 1 & a \end{pmatrix}$$

The spectra of each matrix is defined as the set of its eignevalues. Calculation of the eignevalues is realized through teh characteristic equation and found by equation 5.

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0 \tag{5}$$

Substituting values in and calculation of the determinate yields the following characteristic equations abd spectra:

$$(3-\lambda)(4-\lambda)(1-\lambda) = 0$$

with a spectra of:

$$\boxed{\lambda(\boldsymbol{A}) = \begin{Bmatrix} 3 & 4 & 1 \end{Bmatrix}}$$

Finding the determinate and expanding the characteristic polynomial yields:

$$(a-\lambda)^3 - 2(a-\lambda) = 0$$

Factoring this yields the following solutions:

$$\boxed{\lambda(\boldsymbol{A}) = \begin{Bmatrix} a & a \pm \sqrt{2} \end{Bmatrix}}$$

Eigenvector for each of the associated eigenvalues must satisfy:

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \tag{6}$$

Rearrangment to solve for $\boldsymbol{v}$:

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) \cdot \boldsymbol{v} = 0 \tag{7}$$

Each of the aforementioned eigenvalues are substituted into equation 7 and solve yields the non-unique eigenvectors:

(a) $\lambda = 1$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{pmatrix} 2 & 5 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

As can be seen, $\boldsymbol{v}$ is underdeterminate, as such any value for $\boldsymbol{v}_3$ can be used, 2 is chosen. Using this and substituiting into the other equation yields:

$$\boxed{\boldsymbol{v}(\lambda = 1) = \begin{Bmatrix} 7 \\ -4 \\ 2 \end{Bmatrix}}$$

(b) $\lambda = 3$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{pmatrix} 0 & 5 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & -2 \end{pmatrix}$$

Again, the system is underdeterminate, thus $\boldsymbol{v}_1 =$ is chosen for simplicity, resulting in the following eigenvector

$$\boxed{\boldsymbol{v}\,(\lambda = 3) = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}}$$

(c) $\lambda = 4$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{pmatrix} -1 & 5 & 3 \\ 0 & 0 & 6 \\ 0 & 0 & -3 \end{pmatrix}$$

It can clearly be seen this system is underdeterminate, as such $\boldsymbol{v}_2 = 1$ is chosen for simplicity. With the resulting eigenvector of:

$$\boxed{\boldsymbol{v}\,(\lambda = 4) = \begin{Bmatrix} 5 \\ 1 \\ 0 \end{Bmatrix}}$$

Similiar to the previous problem, each eignevalue is substituted in the associated eigenvector

(a) $\lambda = a$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Choosing $\boldsymbol{v}_1 = 1$, the eigenvector becomes:

$$\boxed{\boldsymbol{v}\,(\lambda = a) = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}}$$

(b) $\lambda = a \pm \sqrt{2}$

$$(\boldsymbol{A} - \lambda \boldsymbol{I}) = \begin{pmatrix} \pm\sqrt{2} & 1 & 0 \\ 1 & \pm\sqrt{2} & 1 \\ 0 & 1 & \pm\sqrt{2} \end{pmatrix}$$

Simplification of both eignevalue solutions yields a trivial eigenvector of:

$$\boxed{\boldsymbol{v}\left(\lambda = a \pm \sqrt{2}\right) = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}}$$