ME 760 Homework 6

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1. Under what conditions is $(ax + by)\,dx + (kx + ly)\,dy$ an exact differential? Here $a, b, kandl$ are constants. Solve the exact equation.

   A differential is exact if it can be represented in the form $\frac{dy}{dx}N(x,y) + M(x,y) = 0$. The test for exactness is seen in equation 1.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{1}$$

   From this it can be seen that for the equation to be exact, $b = k$.

   By first integrating M with respect to x, $u(x,y) = \frac{ax^2}{2} + byx + p(y)$. Differentiating this function with respect to y and setting equal to N yields: $p\prime(y) = ly$, given that b and k are equal. Integrating $p(y)$ and substituting, it is found that:

$$\boxed{F(x,y) = ax^2/2 + byx + ly^2/2 = C}$$

2. Show that $x^a y^b$ is an integrating factor for the following equation and then find its solution.

$$(b+1)x\frac{dy}{dx} + (a+1)y = 0$$

Begin by multiplying the entire equation by the integrating factor:

$$(b+1)x^{a+1}y^b dy + (a+1)x^a y^{b+1} dx = 0$$

This can be seen to be in the form of an exact equation, with $M = (a+1)x^a y^{b+1}$ and $N = (b+1)x^{a+1}y^b$. Thus differentiating M with respect to y and N with respect to x.

$$b(a+1)x^a y^b + (a+1)x^a y^b$$
$$a(b+1)x^a y^b + (b+1)x^a y^b$$

As such, the integrating factor applies if and only if $a = b$.

Begin by integrating M with respect to x. $u(x,y) = \frac{ax^{a+1}}{a+1}y^{b+1} + \frac{x^{a+1}y^{b+1}}{a+1} + p(y)$. Differentiating this and setting equal to N yields that $p(y) = 0$. Thus combining yields:

$$\boxed{F(x,y) = \frac{ax^{a+1}}{a+1}y^{b+1} + \frac{x^{a+1}y^{b+1}}{a+1} + C}$$

3. Find the solution of

   (a) $y\prime(x) = x(4 - y)$

   (b) $y\prime(x) + 6x^2y = e^{-2x^3}/x^2$ with $y(1) = 0$

   (a) By seperating variables, this quickly becomes $dy(4-y)^{-1} = xdx$. By integrating both sides, this becomes $-ln(4 - y) = \frac{x^2}{2} + C_1$. Solving for y explicitly this yields:

$$\boxed{y = C_2 - e^{-x^2/2}}$$

   (b) This is a first order, linear differential equation, as such the integrating factor is $q(x) = exp(\int 6x^2dx) = exp(2x^3)$. Thus multipling and integrating yields:

$$y(x) = \frac{C}{e^{2x^3}} - \frac{1}{xe^{2x^3}}$$

   After applying the boundary condition of $y(1) = 0$:

$$\boxed{y(x) = \frac{1}{e^{2x^3}} - \frac{1}{xe^{2x^3}}}$$

4. Show if the following two sets of functions are linearly independent on the positive x axis

   (a) $[\cos x, \sin x, \sin 2x]$
   (b) $[exp(x)\cos x, exp(x)\sin x, exp(x)]$

   (a) The functions can be shown to be linearly independent by using showing that the Wronskian is non-zero for any point in the interval. The Wronskian is found below:

$$W = \begin{vmatrix} \cos x & \sin x & \sin 2x \\ -\sin x & \cos x & 2\cos 2x \\ -\cos x & -\sin x & -4\sin 2x \end{vmatrix}$$

   Computing this yields $W = -3\sin 2x$. Clearly, these functions are linearly independent.

   (b) Again the functions can be shown to be linearly independent by having a non-zero Wronskian. The matrix used to determine the Wronskian is seen below:

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x & e^x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x & e^x \\ -2e^x \sin x & 2e^x \cos x & e^x \end{vmatrix}$$

   Evaluating this yields $W = e^x - 1$ as such, the functions are linearly independent for all positive x values.

5. Solve the ode $y'' + 4y' + 3y = 65\cos 2x$ by

(a) The method of undertermined coefficients

(b) The variation of parameters method

(a) Begin by guessing a possible solution of the form $A\cos 2x + B\sin 2x$. This can be differentiated and plugged into the ode. Doing so and reducing down yields $(-16A - 8B)\cos 2x + (8A - 16B)\sin 2x = 65\cos 2x$. The two sides of the equation can be used to form two equations:

$$(-16A - 8B) = 65$$
$$(8A - 16B) = 0$$

Solving these equations simultaneously and plugging into the guess:

$$\boxed{y(x) = 3.25\cos 2x + 6.5\sin 2x + C}$$

(b) The particular solution of using the variation of parameters mehtod is found using equation 2.

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W} dx + y_2 \int \frac{y_1 g(x)}{W} dx \tag{2}$$

Given that $g(x) = \cos 2x$, it can be assumed that $y_1 = \cos 2x$ and $y_2 = \sin 2x$. By doing so the Wronskian is found below:

$$W = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

Plugging these values into 2 and integrating yields:

$$\boxed{y(x) = \frac{65x}{2}\sin 2x + \frac{65}{16}\sin 2x \sin 4x + \frac{65}{16}\cos 2x \cos 4x + C}$$

6. What is the solutions to the initial value problem $y'''' - y = 0$ with $y(0) = -1$, $y'(0) = 7$, $y''(0) = -1$, and $y'''(0) = 7$.

   The general solution is found to be:

$$y(x) = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

   Apply the four initial value problems results in the following equations:

$$\begin{aligned} c_1 + c_2 + c_3 &= -1 \\ c_1 - c_2 + c_4 &= 7 \\ c_1 + c_2 - c_3 &= -1 \\ c_1 - c_2 - c_4 &= 7 \end{aligned}$$

   From this it can be seen that c_3 and c_4 must be 0. This reduces to 2 equations, with two unknowns, solving these leads to:

$$\boxed{y(x) = 3e^x - 4e^{-x}}$$

7. What is the solutions to the initial value problem $y'''' - y = 0$ with $y(0) = -1$, $y'(0) = 7$, $y''(0) = -1$, and $y'''(0) = 7$.