ME 760 Homework 2

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1. For the array

$$C = \begin{pmatrix} 4 & 6 & 2 \\ 6 & 0 & 3 \\ 2 & 3 & -1 \end{pmatrix}$$

calculate:

(a) $\boldsymbol{C}^2$

(b) $\boldsymbol{C}^T\boldsymbol{C}$

(c) $\boldsymbol{C}\boldsymbol{C}^T$

2. In practical work the following formulat is quite useful.

$$|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a}) \left(\vec{b} \cdot \vec{b}\right) - \left(\vec{a} \cdot \vec{b}\right)^2} \tag{1}$$

Give a proof.

By definition $\vec{a} \times \vec{b} = \|a\| \|b\| \sin \theta$ and $\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos \theta$ Substituting the above definitions and squaring both sides produces:

$$\|a\|^2 \|b\|^2 \sin^2 \theta = \|a\|^2 \|b\|^2 - \|a\|^2 \|b\|^2 \cos^2 \theta$$

Rearranging and factoring to utilize trigonometric identities:

$$\|a\|^2 \|b\|^2 \left(\sin^2 \theta + \cos^2 \theta\right) = \|a\|^2 \|b\|^2$$

Using $\sin^2 \theta + \cos^2 \theta = 1$, the equations are equivalent.

$$\|a\|^2 \|b\|^2 = \|a\|^2 \|b\|^2$$

3. Determine the angles in the triangle formed by the three vertices $P_1 = (2, 2, 2)$, $P_2 = (3, 1, 1)$, and $P_3 = (3, 3, 3)$.

   Begin by determining the three vectors that connect the three points:

   - $\vec{v_1} = P_1 - P_2 = \langle 1, -1, -1 \rangle$
   - $\vec{v_2} = P_3 - P_2 = \langle 0, 2, 2 \rangle$
   - $\vec{v_3} = P_1 - P_3 = \langle -1, -1, -1 \rangle$

   Using the definition of the dot product, $\vec{a} \cdot \vec{b} = \|a\|\|b\|cos\theta$, the angles can be determined by:

$$\theta = \arccos \frac{\vec{a} \cdot \vec{b}}{\|a\|\|b\|} \tag{2}$$

$$\boxed{\begin{array}{l} \theta_{1-2} = \arccos \frac{-4}{\sqrt{3}\sqrt{8}} = 2.52rad \\ \theta_{2-3} = \arccos \frac{-4}{\sqrt{8}\sqrt{3}} = 2.52rad \\ \theta_{3-1} = \arccos \frac{1}{\sqrt{3}\sqrt{3}} = 1.23rad \end{array}}$$

4. What is the equation of the plane containing the triangle of the previous problem?

Equation of the plane can be expressed from a point and a vector orthogonal to the plane, using $\vec{v_1}$ and $\vec{v_2}$ from the previous problem to find a normal vector:

$$\vec{n} = \vec{v_1} \times \vec{v_2} = \begin{vmatrix} \vec{v_{11}} & \vec{v_{13}} \\ \vec{v_{21}} & \vec{v_{23}} \end{vmatrix} - \begin{vmatrix} \vec{v_{12}} & \vec{v_{13}} \\ \vec{v_{22}} & \vec{v_{23}} \end{vmatrix} + \begin{vmatrix} \vec{v_{22}} & \vec{v_{23}} \\ \vec{v_{22}} & \vec{v_{23}} \end{vmatrix} \tag{3}$$

Substituting values into equation 3 and substituiting yields:

$$\vec{n} = 0\hat{\imath} + 2\hat{\jmath} - 2\hat{k}$$

Using equation 4 where a, b, and c come from $\vec{n}$ and x_0, y_0, and z_0 are values from P_1.

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0 \tag{4}$$

Substituiting values in results in the equation of the plane being:

$$\boxed{2y - 2z = 0}$$