

# ME 760 Homework 6

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1. Under what conditions is  $(ax + by) dx + (kx + ly) dy$  an exact differential? Here  $a, b, k$  and  $l$  are constants. Solve the exact equation.

A differential is exact if it can be represented in the form  $\frac{dy}{dx}N(x, y) + M(x, y) = 0$ . The test for exactness is seen in equation 1.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad (1)$$

From this it can be seen that for the equation to be exact,  $b = k$ .

By first integrating  $M$  with respect to  $x$ ,  $u(x, y) = \frac{ax^2}{2} + byx + p(y)$ . Differentiating this function with respect to  $y$  and setting equal to  $N$  yields:  $p'(y) = ly$ , given that  $b$  and  $k$  are equal. Integrating  $p(y)$  and substituting, it is found that:

$$F(x, y) = ax^2/2 + byx + ly^2/2 = C$$

2. Show that  $x^a y^b$  is an integrating factor for the following equation and then find its solution.

$$(b+1)x \frac{dy}{dx} + (a+1)y = 0$$

Begin by multiplying the entire equation by the integrating factor:

$$(b+1)x^{a+1}y^b dy + (a+1)x^a y^{b+1} dx = 0$$

This can be seen to be in the form of an exact equation, with  $M = (a+1)x^a y^{b+1}$  and  $N = (b+1)x^{a+1}y^b$ . Thus differentiating  $M$  with respect to  $y$  and  $N$  with respect to  $x$ .

$$\begin{aligned} b(a+1)x^a y^b + (a+1)x^a y^b \\ a(b+1)x^a y^b + (b+1)x^a y^b \end{aligned}$$

As such, the integrating factor applies if and only if  $a = b$ .

3. Find the solution of

(a)  $y'(x) = x(4 - y)$

(b)  $y'(x) + 6x^2y = e^{-2x^3}/x^2$  with  $y(1) = 0$

(a) By separating variables, this quickly becomes  $dy(4 - y)^{-1} = xdx$ . By integrating both sides, this becomes  $-\ln(4 - y) = \frac{x^2}{2} + C_1$ . Solving for  $y$  explicitly this yields:

$$y = C_2 - e^{-x^2/2}$$

(b) This is a first order, linear differential equation, as such the integrating factor is  $q(x) = \exp(\int 6x^2 dx) = \exp(2x^3)$ . Thus multiplying and integrating yields:

$$y(x) = \frac{C}{e^{2x^3}} - \frac{1}{xe^{2x^3}}$$

After applying the boundary condition of  $y(1) = 0$ :

$$y(x) = \frac{1}{e^{2x^3}} - \frac{1}{xe^{2x^3}}$$

4. Show if the following two sets of functions are linearly independent on the positive x axis

(a)  $[\cos x, \sin x, \sin 2x]$

(b)  $[\exp(x) \cos x, \exp(x) \sin x, \exp(x)]$

(a) The functions can be shown to be linearly independent by using showing that the Wronskian is non-zero for any point in the interval. The Wronskian is found below:

$$W = \begin{vmatrix} \cos x & \sin x & \sin 2x \\ -\sin x & \cos x & 2 \cos 2x \\ -\cos x & -\sin x & -4 \sin 2x \end{vmatrix}$$

Computing this yields  $W = -3 \sin 2x$ . Clearly, these functions are linearly independent.

(b) Again the functions can be shown to be linearly independent by having a non-zero Wronskian. The matrix used to determine the Wronskian is seen below:

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x & e^x \\ -e^x \sin x + e^x \cos x & e^x \cos x + e^x \sin x & e^x \\ -2e^x \sin x & 2e^x \cos x & e^x \end{vmatrix}$$

Evaluating this yields  $W = e^x - 1$  as such, the functions are linearly independent for all positive x values.

5. Solve the ode  $y'' + 4y' + 3y = 65 \cos 2x$  by

(a) The method of undetermined coefficients

(b) The variation of parameters method

(a) Begin by guessing a possible solution of the form  $A \cos 2x + B \sin 2x$ . This can be differentiated and plugged into the ode. Doing so and reducing down yields  $(-16A - 8B) \cos 2x + (8A - 16B) \sin 2x = 65 \cos 2x$ . The two sides of the equation can be used to form two equations:

$$(-16A - 8B) = 65$$

$$(8A - 16B) = 0$$

Solving these equations simultaneously and plugging into the guess:

$$y(x) = 3.25 \cos 2x + 6.5 \sin 2x$$

6. What is the solutions to the initial value problem  $y'''' - y = 0$  with  $y(0) = -1$ ,  $y'(0) = 7$ ,  $y''(0) = -1$ , and  $y'''(0) = 7$ .