ME 760 Homework 4

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1. Verify the contour integral $\int_C \left[2xy^2dx + 2x^2ydy + dz\right]$ is independent of the path. Evaluate this integral between the points $(0, 0, 0)$ and (a, b, c).

   A line integral with the vector function of the form $\boldsymbol{F} = P\boldsymbol{i} + Q\boldsymbol{j} + R\boldsymbol{k}$ is path indenpendent if $curl\boldsymbol{F} = 0$. The curl of the vector is found and evaluated below using equation 1.

$$curl\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0 \tag{1}$$

   Expansion of the above yields the following:

$$curl\boldsymbol{F} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\boldsymbol{i} - \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\boldsymbol{j} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\boldsymbol{k} \tag{2}$$

   The two non-trivial partial derivatives are shown below:

$$\frac{\partial Q}{\partial x} = 4xy$$

$$\frac{\partial P}{\partial y} = 4xy$$

   Substituting the values into equation 2, yields $curl\boldsymbol{F} = 0$, thus the contour integral is independent of the path.

   Evaluation of this leads to $f = x^2 + y^2 + z$. Thus evaluating using the form $f(B) - (A)$ yields:

$$\boxed{a^2b^2 + c}$$

2. Given the parametric form of a cone $r(u, v) = [u \cos v, u \sin v, cu]$

   (a) find an explicit representation of the form $z = f(x, y)$

   (b) find and identify the paramer curves defind as $u = const$ and $v = const$

   (c) find the normal vector N to the conical surface

   (a) Let $x, y, z = u \cos v, u \sin v, cu$ respectively. Squaring x and y and summing them yields: $x^2 + y^2 = u^2 \left(\cos^2 v + \sin^2 v\right)$. Solving for u from this: $u = \sqrt{x^2 + y^2}$. Using this to substitute into z results in the explicity form of the equation:

$$\boxed{z = c\sqrt{x^2 + y^2}} \tag{3}$$

   (b) From (a), it is seen that $\boxed{u = \sqrt{x^2 + y^2}}$. v is found from the division of x and y. Using this and trigonemetric definitions yields: $\boxed{v = tan^{-1}\frac{x}{y}}$.

   (c) The normal vector is found using equation 4

$$\nabla f = 0 \tag{4}$$

   Differentiation of the explicit form with respect to x, y, z results in the normal vector:

$$\boxed{\boldsymbol{N} = \frac{2xc}{\sqrt{x^2 + y^2}}\boldsymbol{i} + \frac{2yc}{\sqrt{x^2 + y^2}}\boldsymbol{j} - \boldsymbol{k}}$$

3. In class we discussed surface integrals without regard to orientation. By reparameterizing the surface integral could be written as

$$I = \int\int_s G(s)\, dS = \int\int_R G(r(u,v))\, |N(u,v)|\, dudv$$

   (a) Consider the case $G = z$ adn teh surface S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$. Use polar coordinates and evaluate the right hand side of the above reult.

   (b) The surface S is also given explicitly by $z = f(x,y) = \sqrt{9 - x^2 - y^2}$. For such cases the surface integral can be rewritten as

$$\int\int_S G(r)\, dA = \int\int_{R*} G(x,y,f(x,y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dxdy \qquad (5)$$

   Evaluate the right-hand side fo this result.

   (a) Rewriting the function in polar coordinates yields: $r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 9$

   (b)

4. Evaluate $\int\int_S F \cdot \hat{n} dA$ using the divergnece theorem when:

   (a) $F = [x^3, y^3, z^3]$ and the surface is the sphere $x^2 + y^2 + z^2 = 9$

   (b) $F = [9x, ycosh^2 x, -zsinh^2 x]$ and teh surface is teh ellipsoid $4x^2+y^2+9z^2 = 36$

   The evaluation of the integral becomes significantly simplified when by using the equality in equation 6

$$\int\int_S F \cdot \hat{n} dA = \int\int\int_V \nabla F \cdot dV \tag{6}$$

   (a) The $div\boldsymbol{F}$ is found to be $3x^2 + 3y^2 + 3z^2$, or when converted to sphereical coordinates, $\rho^4 \sin\phi$. Thus applying the bounds of integration for a sphere in sphereical coordinates:

$$I = \int_0^3 \int_0^\pi \int_0^{2\pi} \rho^4 \sin\phi d\theta d\phi d\rho$$

   Integrating the above yields $\boxed{\frac{729\pi}{5}}$

   (b) The $div\boldsymbol{F} = 10$ is found using the trig identity of $\cosh^2 x - \sinh^2 x = 1$ The bounds of integration are left in rectangular form, with the resulting integration being:

$$I = 10 \int_0^9 \int_0^{\sqrt{36-4x^2}} \int_0^{\frac{\sqrt{36-4x^2-y^2}}{9}} dzdydx$$

   Computing the first integral results in

$$I = \frac{10}{9} \int_0^9 \int_0^{\sqrt{36-4x^2}} \sqrt{36-4x^2-y^2} dydx$$

   Using integration tables found in Larson and Edwards Ninth Edition, the second integration is found to using the following form

$$\int \sqrt{a^2-u^2} du = \frac{1}{2}\left(u\sqrt{a^2-u^2} + a^2 \sin^{-1}\frac{u}{a}\right) + C \tag{7}$$

   Implementing this and evaluating leads to the final integral of the form

$$I = \frac{5\pi}{18} \int_0^9 36 - 4x^2 dx$$

   Evaluation of this leads to $\boxed{180\pi}$

5. Consider the vector function $F = [e^z, e^z \sin y, e^z \cos y]$ and the surface $z = y^2, 0 \leq x \leq 4, 0 \leq y \leq 2$. Stoke's theorem states that

$$\int\int_S (\nabla \times F) \cdot \hat{n} dS = \oint_C F \cdot dr$$

   (a) Evaluate the left-hand side of this result

   (b) Evaluate the right-hand side of this result

   (a) Evaluation of the left-hand side begins with the calculation of $curlF$, found using 1. This is found to be: $-2e^z \sin y\boldsymbol{i} + e^z\boldsymbol{j}$. The normal vector is found from Larson and Edwards Ninth Edition using equation 8.

$$\hat{n} = -g_x(x,y)\boldsymbol{i} - g_y(x,y)\boldsymbol{j} + \boldsymbol{k} \tag{8}$$

   Where g_x and g_y are the partial derivatives of surface S wrt x, y respectively. Finding $\hat{n}$ and computing the dot product yields: $2ye^z$. Computing the integral yields $\boxed{16e^z}$.

   (b) The curve must first be parameterized into t. Letting $y = t$, the vector becomes $\langle 0, t, e^z \cos t \rangle$ Using the form $\int_0^{2\pi} Mdx+Ndy+Pdz$, the integral can be rewrote into it's parametrized form.

$$\int_0^{2\pi} t + e^{2z} \cos^2 t dt$$

   Evaluation of this yields $\boxed{\pi + \frac{e^{2\pi}}{3}}$. I believe my parameterization is incorrect, but am unsure where.