ME 760 Homework 3

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1. Find the matrix $\boldsymbol{A}$ for each of the indicated linear transformation $y = Ax$. Find its eigenvalues and eigenvectors.

   - Reflection about the x-axis in R^2. Here $x = [xy]$
   - Orthogonal projection of R^3 onto the plane $x = y$. Here $x = [xyz]$

   - The reflection about the x-axis is given as:

   $$\boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

   Its eigenvalues are ± 1 and eigenvectors of $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ respectively.

   - Transformation into $y = x$ is realized through:

   $$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   This results in eigenvalues of which are ± 1 and eigenvectors of $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^T$ respectively.

2. Prove taht trace of a square real of complex matrix A equals the sum of its eigenvalues. This fact is often useful check on the accuracy of eigenvalue calculations. Demonstrate with an example of your choosing.

   Proof: For any square matrix there exists a matrix $\boldsymbol{P}$ such that $\boldsymbol{J} = \boldsymbol{PAP^{-1}}$ where $\boldsymbol{J}$ is Jordan Conical form. Which has eigenvalues for its diagonal elements. Thus:

$$tr\left(\boldsymbol{D}\right) = \sum_{i}^{n} \lambda_i = \sum_{j}^{n} D_{jj}$$

   Using definitions of the trace: $tr\left(\boldsymbol{D}\right) = tr\left(\boldsymbol{PAP^{-1}}\right) = tr\left(\boldsymbol{APP^{-1}}\right) = tr\left(\boldsymbol{A}\right)$

   Example: (Using matrix from probelm 6) The eigenvalues can be found from the characteristic polynomial:

$$\left(\lambda - 16\right)\left(\lambda + 8\right)\left(\lambda - 4\right) = 0$$

   Giving rise to the eigenvalues to be equal to $[16, -8, 4]$. The sum of which is 12 The trace of a matrix is the sum of the diagonals, for which $tr\left(A\right) = 12$.

3. Prove that the eigenvectors of a real symmetric matrix corresponding to different eigenvalues are orthogonal.

   Let a, b be two unique eigenvalues with corresponding eigenvectors $\boldsymbol{v}, \boldsymbol{w}$. $\boldsymbol{v}, \boldsymbol{w}$ are shown to be orthogonal when the dot product, $\boldsymbol{v} \cdot \boldsymbol{w} = 0$

$$\begin{aligned} a\left(\boldsymbol{v} \cdot \boldsymbol{w}\right) &= a\boldsymbol{v} \cdot \boldsymbol{w} \\ &= \boldsymbol{Av} \cdot \boldsymbol{w} \\ &= \boldsymbol{v} \cdot \boldsymbol{A^T w} \\ &= \boldsymbol{v} \cdot \boldsymbol{Aw} \\ &= \boldsymbol{v} \cdot \boldsymbol{bw} \\ &= b\left(\boldsymbol{v} \cdot \boldsymbol{w}\right) \end{aligned}$$

   This is made possible using the identity of $\boldsymbol{A} = \boldsymbol{A^T}$. Rearranging this results in:

$$\left(a - b\right)\left(\boldsymbol{v} \cdot \boldsymbol{w}\right) = 0$$

   Therefore they are orthogonal.

4. Do there exist real symmetric 3x3 matrices that are also orthogonal? If so give an example.

   Yes, they exist. As by definition they don't contradict each other. An example is:

$$\begin{pmatrix} \frac{-7}{9} & \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{-1}{9} & \frac{8}{9} \\ \frac{4}{9} & \frac{8}{9} & \frac{-1}{9} \end{pmatrix}$$

5. Prove that hermitian, skew-hermitian, and unitary matrices are normal. A matrix is normal if $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$

   - Hermitian By definition $\boldsymbol{A} = \boldsymbol{A}^*$. As such:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^* &= \boldsymbol{A}^2 \\ &= \boldsymbol{A}^*\boldsymbol{A} \end{aligned}$$

   - Skew-Hermitian By definition $-\boldsymbol{A} = \boldsymbol{A}^*$. As such:

$$\begin{aligned} -\boldsymbol{A}\boldsymbol{A}^* &= -\boldsymbol{A}^2 \\ &= -\boldsymbol{A}^*\boldsymbol{A} \end{aligned}$$

   - Unitary As teh complex conjugate is the inverse of $\boldsymbol{A}$, $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{I}$ and the converse is true. Thus $\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A}^*\boldsymbol{A}$.

6. Find the linear transformation that diagonalizes:

$$\begin{pmatrix} 16 & 0 & 0 \\ 48 & -8 & 0 \\ 84 & -24 & 4 \end{pmatrix}$$

Finding the eigenvalues from the characteristic polynomial:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} 16-\lambda & 0 & 0 \\ 48 & -8-\lambda & 0 \\ 84 & -24 & 4-\lambda \end{pmatrix}$$

The factorized characteristic polynomial is therefore:

$$(16-\lambda)(-8-\lambda)(4-\lambda) = 0$$

Leading to eigenvalues of $\begin{bmatrix} 16 & -8 & 4 \end{bmatrix}$.
The eigenvalues are used to compute the eigenvectors, $\boldsymbol{Y} = (\boldsymbol{A} - \lambda \boldsymbol{I})\, \boldsymbol{x} = 0$:

- $\lambda = 4$

$$\boldsymbol{Y} = \begin{pmatrix} 12 & 0 & 0 \\ 48 & -12 & 0 \\ 84 & -24 & 0 \end{pmatrix}$$

Reducing to RREF results in the eigenvector $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$

- $\lambda = -8$

$$\boldsymbol{Y} = \begin{pmatrix} 8 & 0 & 0 \\ 48 & 0 & 0 \\ 84 & -24 & -4 \end{pmatrix}$$

Reducing to RREF results in the eigenvector $\begin{bmatrix} 0 & \frac{1}{2} & 1 \end{bmatrix}^T$

- $\lambda = 16$

$$\boldsymbol{Y} = \begin{pmatrix} 0 & 0 & 0 \\ 48 & -24 & 0 \\ 84 & -24 & -16 \end{pmatrix}$$

Reducing to RREF results in the eigenvector $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}^T$

Combining these result in the transformation matrix:

$$\boxed{\begin{pmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{2}{3} \\ 1 & 1 & 1 \end{pmatrix}}$$

7. Use the power method to find teh alrgest eigenvalue to 5 sig figs of:

$$\begin{pmatrix} 3 & 5 & 3 \\ 0 & 4 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

The power method is defined as

$$\lim_{n\to\infty} \frac{||x^n||}{||x^{n-1}||} \tag{1}$$

x	$\boldsymbol{A}x^n$	$\mu(A)$
0	$\begin{pmatrix} 1.0000 \\ 0.3000 \\ 0.0000 \end{pmatrix}$	4.6573
1	$\begin{pmatrix} 0.9662 \\ 0.2577 \\ 0.0000 \end{pmatrix}$	4.3120
2	$\begin{pmatrix} 0.9710 \\ 0.2390 \\ 0.0000 \end{pmatrix}$	4.2179
3	$\begin{pmatrix} 0.9740 \\ 0.2267 \\ 0.0000 \end{pmatrix}$	4.1554
4	$\begin{pmatrix} 0.9759 \\ 0.2182 \\ 0.0000 \end{pmatrix}$	4.1124
5	$\begin{pmatrix} 0.9772 \\ 0.2122 \\ 0.0000 \end{pmatrix}$	4.0821
6	$\begin{pmatrix} 0.9781 \\ 0.2080 \\ 0.0000 \end{pmatrix}$	4.0604
7	$\begin{pmatrix} 0.9788 \\ 0.2049 \\ 0.0000 \end{pmatrix}$	4.0446
8	$\begin{pmatrix} 0.9793 \\ 0.2026 \\ 0.0000 \end{pmatrix}$	4.0331
9	$\begin{pmatrix} 0.9796 \\ 0.2009 \\ 0.0000 \end{pmatrix}$	4.0246
10	$\begin{pmatrix} 0.9799 \\ 0.1997 \\ 0.0000 \end{pmatrix}$	4.0184
11	$\begin{pmatrix} 0.9800 \\ 0.1988 \\ 0.0000 \end{pmatrix}$	4.0137

12	$\begin{pmatrix} 0.9802 \\ 0.1981 \\ 0.0000 \end{pmatrix}$	4.0103
13	$\begin{pmatrix} 0.9803 \\ 0.1976 \\ 0.0000 \end{pmatrix}$	4.0077
14	$\begin{pmatrix} 0.9804 \\ 0.1972 \\ 0.0000 \end{pmatrix}$	4.0057
15	$\begin{pmatrix} 0.9804 \\ 0.1970 \\ 0.0000 \end{pmatrix}$	4.0043
16	$\begin{pmatrix} 0.9805 \\ 0.1967 \\ 0.0000 \end{pmatrix}$	4.0032
17	$\begin{pmatrix} 0.9805 \\ 0.1966 \\ 0.0000 \end{pmatrix}$	4.0024
18	$\begin{pmatrix} 0.9805 \\ 0.1965 \\ 0.0000 \end{pmatrix}$	4.0018
19	$\begin{pmatrix} 0.9805 \\ 0.1964 \\ 0.0000 \end{pmatrix}$	4.0014
20	$\begin{pmatrix} 0.9805 \\ 0.1963 \\ 0.0000 \end{pmatrix}$	4.0010
21	$\begin{pmatrix} 0.9806 \\ 0.1963 \\ 0.0000 \end{pmatrix}$	4.0008
22	$\begin{pmatrix} 0.9806 \\ 0.1962 \\ 0.0000 \end{pmatrix}$	4.0006
23	$\begin{pmatrix} 0.9806 \\ 0.1962 \\ 0.0000 \end{pmatrix}$	4.0004
24	$\begin{pmatrix} 0.9806 \\ 0.1962 \\ 0.0000 \end{pmatrix}$	4.0003
25	$\begin{pmatrix} 0.9806 \\ 0.1962 \\ 0.0000 \end{pmatrix}$	4.0002
26	$\begin{pmatrix} 0.9806 \\ 0.1962 \\ 0.0000 \end{pmatrix}$	4.0002