

ME 760 Homework 1

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1. For $\vec{a} \neq 0$, what does $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ imply about the vectors \vec{b} and \vec{c} ? Give a geometric interpretation. Hint: if \vec{a} , \vec{b} , and \vec{c} all begin at the same point, where do the ends of the vectors \vec{b} and \vec{c} lie?

This implies that the sum of \vec{b} and \vec{c} is perpendicular to \vec{a} . This is realized through rearranging the equation as $\vec{a} \cdot (\vec{b} - \vec{c}) = 0$. This however, doesn't imply that $\vec{b} = \vec{c}$.

2. In practical work the following formulat is quiter useful.

$$|\vec{a} \times \vec{b}| = \sqrt{(\vec{a} \cdot \vec{a}) (\vec{b} \cdot \vec{b}) - (\vec{a} \cdot \vec{b})^2} \quad (1)$$

Give a proof.

3. Determine the angles in the triangle formed by the three vertices $P_1 = (2, 2, 2)$, $P_2 = (3, 1, 1)$, and $P_3 = (3, 3, 3)$.

Begin by determining the three vectors that connect the three points:

- $\vec{v}_1 = P_1 - P_2 = \langle 1, -1, -1 \rangle$
- $\vec{v}_2 = P_3 - P_2 = \langle 0, 2, 2 \rangle$
- $\vec{v}_3 = P_1 - P_3 = \langle -1, -1, -1 \rangle$

Using the definition of the dot product, $\vec{a} \cdot \vec{b} = \|a\|\|b\|\cos\theta$, the angles can be determined by:

$$\theta = \arccos \frac{\vec{a} \cdot \vec{b}}{\|a\|\|b\|} \quad (2)$$

$\begin{aligned} \theta_{1-2} &= \arccos \frac{-4}{\sqrt{3}\sqrt{8}} = 2.52rad \\ \theta_{2-3} &= \arccos \frac{-4}{\sqrt{8}\sqrt{3}} = 2.52rad \\ \theta_{3-1} &= \arccos \frac{1}{\sqrt{3}\sqrt{3}} = 1.23rad \end{aligned}$
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4. What is the equation of the plane containing the triangle of the previous problem?

Equation of the plane can be expressed from a point and a vector orthogonal to the plane, using \vec{v}_1 and \vec{v}_2 from the previous problem to find a normal vector:

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{v}_{11} & \vec{v}_{13} \\ \vec{v}_{21} & \vec{v}_{23} \end{vmatrix} - \begin{vmatrix} \vec{v}_{12} & \vec{v}_{13} \\ \vec{v}_{22} & \vec{v}_{23} \end{vmatrix} + \begin{vmatrix} \vec{v}_{22} & \vec{v}_{23} \\ \vec{v}_{22} & \vec{v}_{23} \end{vmatrix} \quad (3)$$

Substituting values into equation 3 and substituting yields:

$$\vec{n} = 0\hat{i} + 2\hat{j} - 2\hat{k}$$

Using equation 4 where a , b , and c come from \vec{n} and x_0 , y_0 , and z_0 are values from P_1 .

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad (4)$$

Substituting values in results in the equation of the plane being:

$$\boxed{2y - 2z = 0}$$

5. Show that the three vectors $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{w} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - \hat{\mathbf{k}}$ are coplanar.

Three vectors are can be showed to be coplanar if the scalar triple products evaluates to 0

$$[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) \quad (5)$$

Equation 5 is equivalent to the determinant below:

$$d = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad (6)$$

Substituting values from \mathbf{u} , \mathbf{v} , and \mathbf{w} :

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \\ 2 & 4 & -1 \end{vmatrix}$$

The determinant of which is evaluated as:

$$d = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} \quad (7)$$

Substituting values from \mathbf{u} , \mathbf{v} , and \mathbf{w} :

$$d = 1 \begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ 2 & 4 \end{vmatrix}$$

Upon evaluation, $d = 0$, therefore the vectors are coplanar.

6. For general vectors \vec{a} , \vec{b} , and \vec{c} show by expanding in component form:

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

7. Line L passes through the two points $P_1 = (-2, -2, -2)$ and $P_2 = (-1, -1, -1)$.
- a) What is the equation of this line?
 - b) What are the coordinates of the two points of intersection of this line with a sphere of radius 2 centered at $(2, 2, 2)$?
 - a) The vector parallel to the line is $\vec{v} = \langle 111 \rangle$. Thus the equation of the line can be represented using the vector and point P_1 by the following set of parametric equations:

$$\boxed{\vec{r} = \langle -2, -2, -2 \rangle + t \langle 1, 1, 1 \rangle}$$

- b) The previous solution can be rewrote into component form and substituted into the