

ME 760 Homework 4

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1. Verify the contour integral $\int_C [2xy^2dx + 2x^2ydy + dz]$ is independent of the path. Evaluate this integral between the points $(0, 0, 0)$ and (a, b, c) .

A line integral with the vector function of the form $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is path independent if $\text{curl}\mathbf{F} = 0$. The curl of the vector is found and evaluated below using equation 1.

$$\text{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = 0 \quad (1)$$

Expansion of the above yields the following:

$$\text{curl}\mathbf{F} = \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\mathbf{i} - \frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\mathbf{j} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\mathbf{k} \quad (2)$$

The two non-trivial partial derivatives are shown below:

$$\frac{\partial Q}{\partial x} = 4xy$$

$$\frac{\partial P}{\partial y} = 4xy$$

Substituting the values into equation 2, yields $\text{curl}\mathbf{F} = 0$, thus the contour integral is independent of the path.

Evaluation of this leads to $f = x^2 + y^2 + z$. Thus evaluating using the form $f(B) - f(A)$ yields:

$$\boxed{a^2b^2 + c}$$

2. Given the parametric form of a cone $r(u, v) = [u \cos v, u \sin v, cu]$

- (a) find an explicit representation of the form $z = f(x, y)$
 - (b) find and identify the parametric curves defined as $u = \text{const}$ and $v = \text{const}$
 - (c) find the normal vector N to the conical surface
- (a) Let $x, y, z = u \cos v, u \sin v, cu$ respectively. Squaring x and y and summing them yields: $x^2 + y^2 = u^2 (\cos^2 v + \sin^2 v)$. Solving for u from this: $u = \sqrt{x^2 + y^2}$. Using this to substitute into z results in the explicit form of the equation:

$$\boxed{z = c\sqrt{x^2 + y^2}} \quad (3)$$

- (b) From (a), it is seen that $\boxed{u = \sqrt{x^2 + y^2}}$. v is found from the division of x and y . Using this and trigonometric definitions yields: $\boxed{v = \tan^{-1} \frac{x}{y}}$.

- (c) The normal vector is found using equation 4

$$\nabla f = 0 \quad (4)$$

Differentiation of the explicit form with respect to x, y, z results in the normal vector:

$$\boxed{\mathbf{N} = \frac{2xc}{\sqrt{x^2 + y^2}} \mathbf{i} + \frac{2yc}{\sqrt{x^2 + y^2}} \mathbf{j} - \mathbf{k}}$$

3. In class we discussed surface integrals without regard to orientation. By reparametrizing the surface integral could be written as

$$I = \int \int_s G(s) dS = \int \int_R G(r(u, v)) |N(u, v)| du dv$$

- (a) Consider the case $G = z$ and the surface S is the hemisphere $x^2 + y^2 + z^2 = 9$ with $z \geq 0$. Use polar coordinates and evaluate the right hand side of the above result.
- (b) The surface S is also given explicitly by $z = f(x, y) = \sqrt{9 - x^2 - y^2}$. For such cases the surface integral can be rewritten as

$$\int \int_S G(r) dA = \int \int_{R^*} G(x, y, f(x, y)) \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dx dy \quad (5)$$

Evaluate the right-hand side for this result.

- (a) Rewriting the function in polar coordinates yields: $r^2 \cos^2 \theta + r^2 \sin^2 \theta + z^2 = 9$. The N vector is found using the cross of r_θ and r_r .
- (b) Assuming $G = z$ from the previous problem, this explicitly becomes $\sqrt{9 - x^2 - y^2}$. Computing the partials and rearranging the square root:

$$\int \int \sqrt{9 - x^2 - y^2} \sqrt{1 + \frac{4x^2 + 4y^2}{9 - x^2 - y^2}} dy dx$$

Finding the bounds of integration results in

$$\int_0^3 \int_0^{\sqrt{9-x^2}} \sqrt{9 - x^2 - y^2} \sqrt{1 + \frac{4x^2 + 4y^2}{9 - x^2 - y^2}} dy dx$$

4. Evaluate $\int \int_S \mathbf{F} \cdot \hat{n} dA$ using the divergence theorem when:

(a) $\mathbf{F} = [x^3, y^3, z^3]$ and the surface is the sphere $x^2 + y^2 + z^2 = 9$

(b) $\mathbf{F} = [9x, y \cosh^2 x, -z \sinh^2 x]$ and the surface is the ellipsoid $4x^2 + y^2 + 9z^2 = 36$

The evaluation of the integral becomes significantly simplified when by using the equality in equation 6

$$\int \int_S \mathbf{F} \cdot \hat{n} dA = \int \int \int_V \nabla \cdot \mathbf{F} \cdot dV \quad (6)$$

(a) The $\text{div} \mathbf{F}$ is found to be $3x^2 + 3y^2 + 3z^2$, or when converted to spherical coordinates, $\rho^4 \sin \phi$. Thus applying the bounds of integration for a sphere in spherical coordinates:

$$I = \int_0^3 \int_0^\pi \int_0^{2\pi} \rho^4 \sin \phi d\theta d\phi d\rho$$

Integrating the above yields $\boxed{\frac{729\pi}{5}}$

(b) The $\text{div} \mathbf{F} = 10$ is found using the trig identity of $\cosh^2 x - \sinh^2 x = 1$. The bounds of integration are left in rectangular form, with the resulting integration being:

$$I = 10 \int_0^9 \int_0^{\sqrt{36-4x^2}} \int_0^{\frac{\sqrt{36-4x^2-y^2}}{9}} dz dy dx$$

Computing the first integral results in

$$I = \frac{10}{9} \int_0^9 \int_0^{\sqrt{36-4x^2}} \sqrt{36-4x^2-y^2} dy dx$$

Using integration tables found in Larson and Edwards Ninth Edition, the second integration is found to use the following form

$$\int \sqrt{a^2 - u^2} du = \frac{1}{2} \left(u \sqrt{a^2 - u^2} + a^2 \sin^{-1} \frac{u}{a} \right) + C \quad (7)$$

Implementing this and evaluating leads to the final integral of the form

$$I = \frac{5\pi}{18} \int_0^9 36 - 4x^2 dx$$

Evaluation of this leads to $\boxed{180\pi}$

5. Consider the vector function $F = [e^z, e^z \sin y, e^z \cos y]$ and the surface $z = y^2, 0 \leq x \leq 4, 0 \leq y \leq 2$. Stoke's theorem states that

$$\int \int_S (\nabla \times F) \cdot \hat{n} dS = \oint_C F \cdot dr$$

- (a) Evaluate the left-hand side of this result
 - (b) Evaluate the right-hand side of this result
- (a) Evaluation of the left-hand side begins with the calculation of $\text{curl} F$, found using 1. This is found to be: $-2e^z \sin y \mathbf{i} + e^z \mathbf{j}$. The normal vector is found from Larson and Edwards Ninth Edition using equation 8.

$$\hat{n} = -g_x(x, y) \mathbf{i} - g_y(x, y) \mathbf{j} + \mathbf{k} \quad (8)$$

Where g_x and g_y are the partial derivatives of surface S wrt x, y respectively. Finding \hat{n} and computing the dot product yields: $2ye^z$. Computing the integral yields $\boxed{16e^z}$.

- (b) The curve must first be parameterized into t . Letting $y = t$, the vector becomes $\langle 0, t, e^z \cos t \rangle$ Using the form $\int_0^{2\pi} Mdx + Ndy + Pdz$, the integral can be rewrote into it's parametrized form.

$$\int_0^{2\pi} t + e^{2z} \cos^2 t dt$$

Evaluation of this yields $\boxed{\pi + \frac{e^{2\pi}}{3}}$. I believe my parameterization is incorrect, but am unsure where.