ME 760 Homework 4

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1. Estimate an upper bound for the spectral radius of the following matrix. Compare this bound to the actual spectral radius. Show your work.

$$\boldsymbol{A} = \begin{pmatrix} 7 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

The spectral radius is the largest eigenvalue, explicitly found by $\|\boldsymbol{A} - \lambda \boldsymbol{I}\| = 0$ Expading this and evaluating results in a largest eigenvalue of 8.

An approximation for the spectral radius is found by using the power method, seen in Eqation 1. The approximation will be considered to be convereged when no change is seen in 3 decimal place.

$$\mu\left(A\right) = \lim_{n \to \infty} \frac{\|x^n\|}{\|x^{n-1}} \tag{1}$$

Where x^n is found by $x^n = \boldsymbol{A}x^{n-1}$.

x	$\boldsymbol{A}x^n$	$\mu\left(A\right)$
0	$\begin{pmatrix} 1.0000 \\ 0.3000 \\ 0.0000 \end{pmatrix}$	7.6348
1	$\begin{pmatrix} 0.9169 \\ 0.3013 \\ 0.2620 \end{pmatrix}$	7.9498
2	$\begin{pmatrix} 0.9062 \\ 0.3015 \\ 0.2966 \end{pmatrix}$	7.9944
3	$\begin{pmatrix} 0.9047 \\ 0.3015 \\ 0.3009 \end{pmatrix}$	7.9993
4	$\begin{pmatrix} 0.9046 \\ 0.3015 \\ 0.3014 \end{pmatrix}$	7.9999

The approximation and the analytically calculated spectral radius are the same.

2. Given the curve C: $\boldsymbol{r}\left(u\right) = \boldsymbol{i}cosu + \boldsymbol{j}2sinu$, find:

   - a tengent vector, and the corresponding unit vector
   - tangent and unit tanget at the point P: $(1/2, sqrt3, 0)$
   - the equation of the line throguh P that is tangent th the curve. Sketch the curve and the tangent.

   - Differentiation of each component results in teh tangent vector of:

$$\boxed{\boldsymbol{r}' = -\boldsymbol{i}sinu + \boldsymbol{j}2cosu}$$

   - Using the result of a and Eq. 2, the unit tangent vector is:

$$\hat{\boldsymbol{r}'} = \frac{\boldsymbol{r}'}{\|\boldsymbol{r}'} \tag{2}$$

$$\boxed{\hat{\boldsymbol{r}'} = \frac{-\boldsymbol{i}sinu + \boldsymbol{j}2cosu}{\sqrt{\sin^2 u + 4\cos^2 u}}}$$

   - Evaluation of both the tangent and unit tangent vectors at point P results in:

$$\boxed{\boldsymbol{r}' = -0.47\boldsymbol{i} - 0.32\boldsymbol{j}}$$

$$\boxed{\hat{\boldsymbol{r}'} = -0.83\boldsymbol{i} - 0.55\boldsymbol{j}}$$

   - The tangent line is d

3. Find the length of teh circulate helix $\boldsymbol{r}(u) = \boldsymbol{i}a\cos u + \boldsymbol{j}a\sin u + \boldsymbol{k}u$ from $(a, 0, 0)$ to $(a, 0, 2\pi)$.

   The path length is found using equation 3:

$$L = \int_0^{2\pi} \|\boldsymbol{r}'\| du \tag{3}$$

   Thus differentiating each component:

$$\boldsymbol{r}' = -\boldsymbol{i}a\sin u + \boldsymbol{j}a\cos u + \boldsymbol{k}$$

   Finding the magnitude of the resultant vector yields:

$$\|\boldsymbol{r}'\| = a\sqrt{2}$$

   Plugging this into equation 3 and evaluating results in the length being:

$$\boxed{2a\pi\sqrt{2}}$$

4. Sketch $\boldsymbol{r}(t) = \boldsymbol{i}(R\sin\omega t + \omega R t) + \boldsymbol{j}(R\cos\omega t + R)$ taking $R = 1$ and $\omega = 1$. This curve is called a cycloid and is the path of a point on the rim of a wheel of radius R that rolls without slipping aloing the x-axis. Find the velocity and acceleration at the minimum and maximum y-values of the curve.

   The function was plotted in python using a range from 0 to 4π.

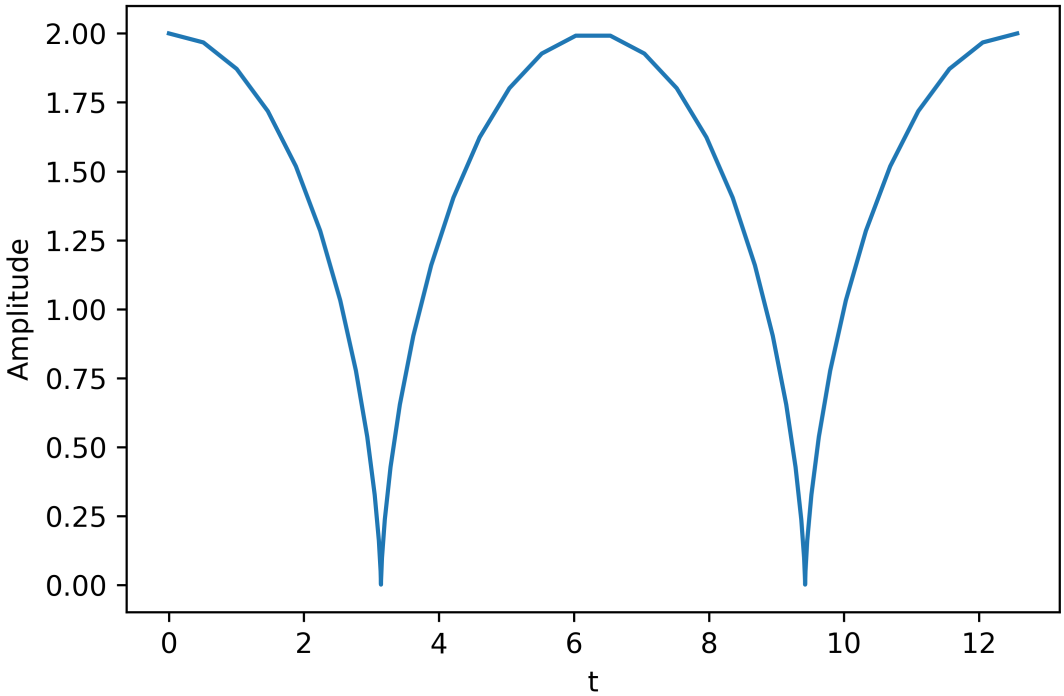

Figure 1: Sketch of cycloid with $R = 1$ and $\omega = 1$.

The minimum and maximum y-values are found at $\pi/2$ and 0 respectively.

Velocity is the first derivative and acceleration is the second derivative, equations 4 and 5 indicate these respectively.

$$\boldsymbol{v} = \boldsymbol{r}' = (\cos t + 1)\,\boldsymbol{i} - \boldsymbol{j}\sin t \tag{4}$$

$$\boldsymbol{a} = \boldsymbol{r}'' = -\boldsymbol{i}\cos t - \boldsymbol{j}\cos t \tag{5}$$

Evaluation of the velocity at the minimum and maximum result in:

$$\boxed{\boldsymbol{v}_{max} = 2\boldsymbol{i}}$$

$$\boxed{\boldsymbol{v}_{min} = \boldsymbol{i} - \boldsymbol{j}}$$

$$\boxed{\boldsymbol{a}_{max} = -\boldsymbol{j}}$$

$$\boxed{\boldsymbol{a}_{min} = -\boldsymbol{i}}$$

5. The flow of heat in a temperature field takes place in the direction of the maximum decrease of temperature. For the temperature field $T\left(x,y,z\right) = z/\left(x^2+y^2\right)$ find the direction and magnitude of the heat flow in general and explicitly at the point $(0,1,2)$.

   The direction of maximum decrease is given as: $-\nabla T\left(x,y,z\right)$. With the magnitudeof heat flow being $\sqrt{a^2+b^2+c^2}$ where a,b,c are direction coefficients.

   - General solution:
     - Maximum decrease:

$$\boxed{-\nabla T = -\left(\frac{2xz}{\left(x^2+y^2\right)^2}\boldsymbol{i} + \frac{2yz}{\left(x^2+y^2\right)^2}\boldsymbol{j} + \frac{1}{x^2+y^2}\boldsymbol{k}\right)}$$

     - Magnitude:

$$\boxed{\|\nabla T\| = \sqrt{\frac{4x^2z^2}{\left(x^2+y^2\right)^4} + \frac{4y^2z^2}{\left(x^2+y^2\right)^4} + \frac{1}{\left(x^2+y^2\right)^2}}}$$

   - Explicit:
     - Maxumum decrease:

$$\boxed{4\boldsymbol{j}+\boldsymbol{k}}$$

     - Magnitude:

$$\boxed{\|\nabla T\| = \sqrt{17}}$$

6. Find the unit normal

   - to the surface $ax + by + cz = d =$ at any point P
   - to the surface $x^2 + y^2 + z^2 = 26$ at the point $(1, 4, 3)$

   The unit normal vector is described by Equation 6.

$$\hat{n} = \frac{\nabla f}{\|\nabla f\|} \tag{6}$$

   - Using Equation 6 the resulting vector.

$$\boxed{\hat{\boldsymbol{n}} = \frac{a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}}{\sqrt{a^2 + b^2 + c^2}}}$$

   - The gradient of the function is first calculated: $\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$. Thus leading to the magnitude as $\|\nabla f\| = 2\sqrt{x^2 + y^2 + z^2} = \sqrt{26}$. Thus the unit normal vector is:

$$\boxed{\hat{\boldsymbol{n}} = \frac{26}{\sqrt{26}} (\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})}$$

7. Find the divergence of $(-\boldsymbol{i}y + \boldsymbol{j}x)\,/\,(x^2 + y^2)$.

   By definition the divergence of $\boldsymbol{v}$, $div\boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v}$. Expanding this into component form yields:

$$\nabla \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y} \tag{7}$$

   Computing the partial derivatives yields:

$$\begin{aligned}\frac{\partial \boldsymbol{v}}{\partial x} &= \frac{2xy}{(x^2+y^2)^2}\\ \frac{\partial \boldsymbol{v}}{\partial y} &= \frac{-2xy}{(x^2+y^2)^2}\end{aligned}$$

   Thus the divergence is 0.

8. Prove that $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$.

   Expanding into component form an rewriting into an easier way to visualize the cross product:

$$\nabla \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \tag{8}$$

   Computing the cross product:

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \boldsymbol{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \boldsymbol{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) \tag{9}$$

   Expanding the dot product and multiplying out yields:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = \left(\frac{\partial z}{\partial y \partial x} - \frac{\partial y}{\partial z \partial x}\right) - \left(\frac{\partial z}{\partial x \partial y} - \frac{\partial x}{\partial z \partial y}\right) + \left(\frac{\partial y}{\partial x \partial z} - \frac{\partial x}{\partial y \partial z}\right) \tag{10}$$

   Simplification of Equation 10 it can be seen that$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$