ME 760 Homework 4

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1. Estimate an upper bound for th spectral radius of the following matrix. OCmpare this bound to the actual spectral radius. Show your work.

$$\boldsymbol{A} = \begin{pmatrix} 7 & 0 & 3 \\ 2 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

2. Given the curve C: $\boldsymbol{r}\,(u) = \boldsymbol{i}cosu + \boldsymbol{j}2sinu$, find:

- a tengent vector, and the corresponding unit vector
- tangent and unit tanget at the point P: $(1/2, sqrt3, 0)$
- the equation of the line throguh P that is tangent th the curve. Sketch the curve and the tangent.

- Differentiation of each component results in teh tangent vector of:

$$\boxed{\boldsymbol{r}' = -\boldsymbol{i}sinu + \boldsymbol{j}2cosu}$$

- Using the result of a and Eq. 1, the unit tangent vector is:

$$\hat{\boldsymbol{r}'} = \frac{\boldsymbol{r}'}{\|\boldsymbol{r}'} \tag{1}$$

$$\boxed{\hat{\boldsymbol{r}'} = \frac{-\boldsymbol{i}sinu + \boldsymbol{j}2cosu}{\sqrt{\sin^2 u + 4\cos^2 u}}}$$

- Evaluation of both the tangent and unit tangent vectors at point P results in:

$$\boxed{\boldsymbol{r}' = -0.47\boldsymbol{i} - 0.32\boldsymbol{j}}$$

$$\boxed{\hat{\boldsymbol{r}'} = -0.83\boldsymbol{i} - 0.55\boldsymbol{j}}$$

- The tangent line is d

3. Find the length of teh circulate helix $\boldsymbol{r}(u) = \boldsymbol{i}a\cos u + \boldsymbol{j}a\sin u + \boldsymbol{k}u$ from $(a, 0, 0)$ to $(a, 0, 2\pi)$.

   The path length is found using equation 2:

$$L = \int_0^{2\pi} \|\boldsymbol{r}'\| du \tag{2}$$

   Thus differentiating each component:

$$\boldsymbol{r}' = -\boldsymbol{i}a\sin u + \boldsymbol{j}a\cos u + \boldsymbol{k}$$

   Finding the magnitude of the resultant vector yields:

$$\|\boldsymbol{r}'\| = a\sqrt{2}$$

   Plugging this into equation 2 and evaluating results in the length being:

$$\boxed{2a\pi\sqrt{2}}$$

4. Sketch $\boldsymbol{r}(t) = \boldsymbol{i}(R\sin\omega t + \omega Rt) + \boldsymbol{j}(R\cos\omega t + R)$ taking $R = 1$ and $\omega = 1$. This curve is called a cycloid and is the path of a point on the rim of a wheel of radius R that rolls without slipping aloing the x-axis. Find the velocity and acceleration at the minimum and maximum y-values of the curve.

   The function was plotted in python using a range from 0 to 4π.

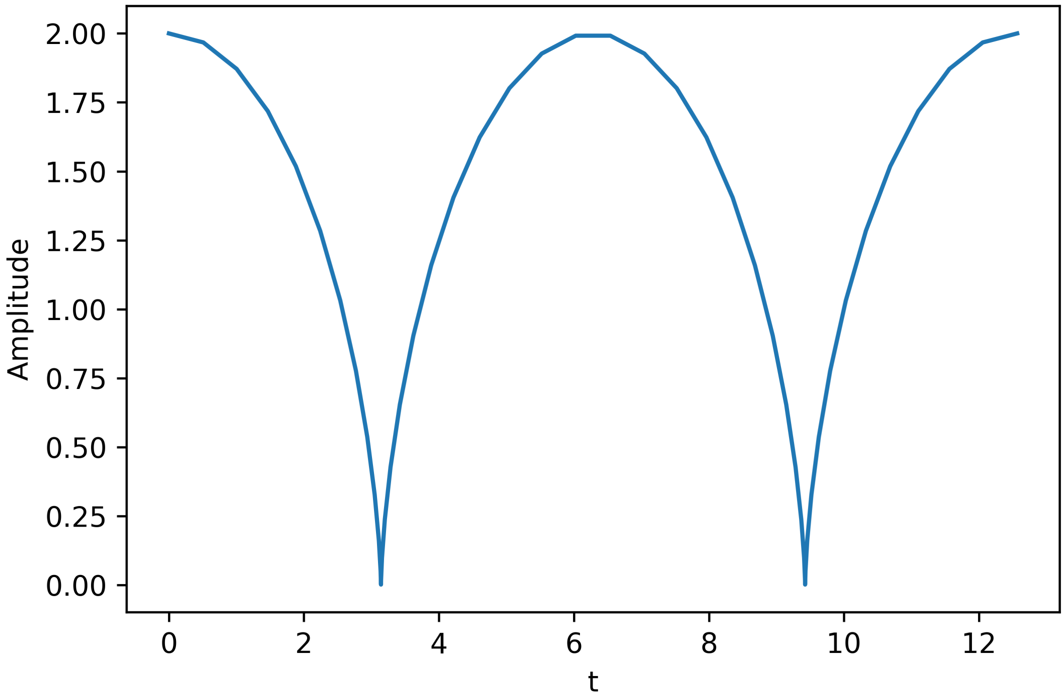

Figure 1: Sketch of cycloid with $R = 1$ and $\omega = 1$.

The minimum and maximum y-values are found at $\pi/2$ and 0 respectively.

Velocity is the first derivative and acceleration is the second derivative, equations 3 and 4 indicate these respectively.

$$\boldsymbol{v} = \boldsymbol{r}' = (\cos t + 1)\,\boldsymbol{i} - \boldsymbol{j}\sin t \tag{3}$$

$$\boldsymbol{a} = \boldsymbol{r}'' = -\boldsymbol{i}\cos t - \boldsymbol{j}\cos t \tag{4}$$

Evaluation of the velocity at the minimum and maximum result in:

$$\boxed{\boldsymbol{v}_{max} = 2\boldsymbol{i}}$$

$$\boxed{\boldsymbol{v}_{min} = \boldsymbol{i} - \boldsymbol{j}}$$

$$\boxed{\boldsymbol{a}_{max} = -\boldsymbol{j}}$$

$$\boxed{\boldsymbol{a}_{min} = -\boldsymbol{i}}$$

5. The flow of heat in a temperature field takes place in the direction of the maximum decrease of temperature. For the temperature field $T(x, y, z) = z/(x^2 + y^2)$ find teh direction and magnitude of the heat flow in general and explicitly at the point $(0, 1, 2)$.

   The direction

   - General solution:

6. Find the unit normal

   - to the surface $ax + by + cz = d =$ at any point P
   - to the surface $x^2 + y^2 + z^2 = 26$ at the point $(1, 4, 3)$

   The unit normal vector is described by Equation 5.

$$\hat{n} = \frac{\nabla f}{\|\nabla f\|} \tag{5}$$

   - Using Equation 5 the resulting vector.

$$\boxed{\hat{\boldsymbol{n}} = \frac{a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}}{\sqrt{a^2 + b^2 + c^2}}}$$

   - The gradient of the function is first calculated: $\nabla f = 2x\boldsymbol{i} + 2y\boldsymbol{j} + 2z\boldsymbol{k}$. Thus leading to the magnitude as $\|\nabla f\| = 2\sqrt{x^2 + y^2 + z^2} = \sqrt{26}$. Thus the unit normal vector is:

$$\boxed{\hat{\boldsymbol{n}} = \frac{26}{\sqrt{26}}\left(\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k}\right)}$$

7. Find the divergence of $(-\boldsymbol{i}y + \boldsymbol{j}x) / (x^2 + y^2)$.

   By definition the divergence of $\boldsymbol{v}$, $div\boldsymbol{v} \equiv \nabla \cdot \boldsymbol{v}$. Expanding this into component form yields:

$$\nabla \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial x} + \frac{\partial \boldsymbol{v}}{\partial y} \tag{6}$$

   Computing the partial derivatives yields:

$$\frac{\partial \boldsymbol{v}}{\partial x} = \frac{2xy}{\left(x^2 + y^2\right)^2}$$
$$\frac{\partial \boldsymbol{v}}{\partial y} = \frac{-2xy}{\left(x^2 + y^2\right)^2}$$

   Thus the divergence is 0.

8. Prove that $\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$.

   Expanding into component form an rewriting into an easier way to visualize the cross product:

$$\nabla \cdot \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \tag{7}$$

   Computing the cross product:

$$\nabla \times \boldsymbol{v} = \boldsymbol{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \boldsymbol{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \boldsymbol{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) \tag{8}$$

   Expanding the dot product and multiplying out yields:

$$\nabla \cdot (\nabla \times \boldsymbol{v}) = \left(\frac{\partial z}{\partial y \partial x} - \frac{\partial y}{\partial z \partial x}\right) - \left(\frac{\partial z}{\partial x \partial y} - \frac{\partial x}{\partial z \partial y}\right) + \left(\frac{\partial y}{\partial x \partial z} - \frac{\partial x}{\partial y \partial z}\right) \tag{9}$$

   Simplification of Equation 9 it can be seen that$\nabla \cdot (\nabla \times \boldsymbol{v}) = 0$