Zero Knowledge Proofs

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Idea (ZKP)

Want to convince to someone that you know something without giving away information about what you know

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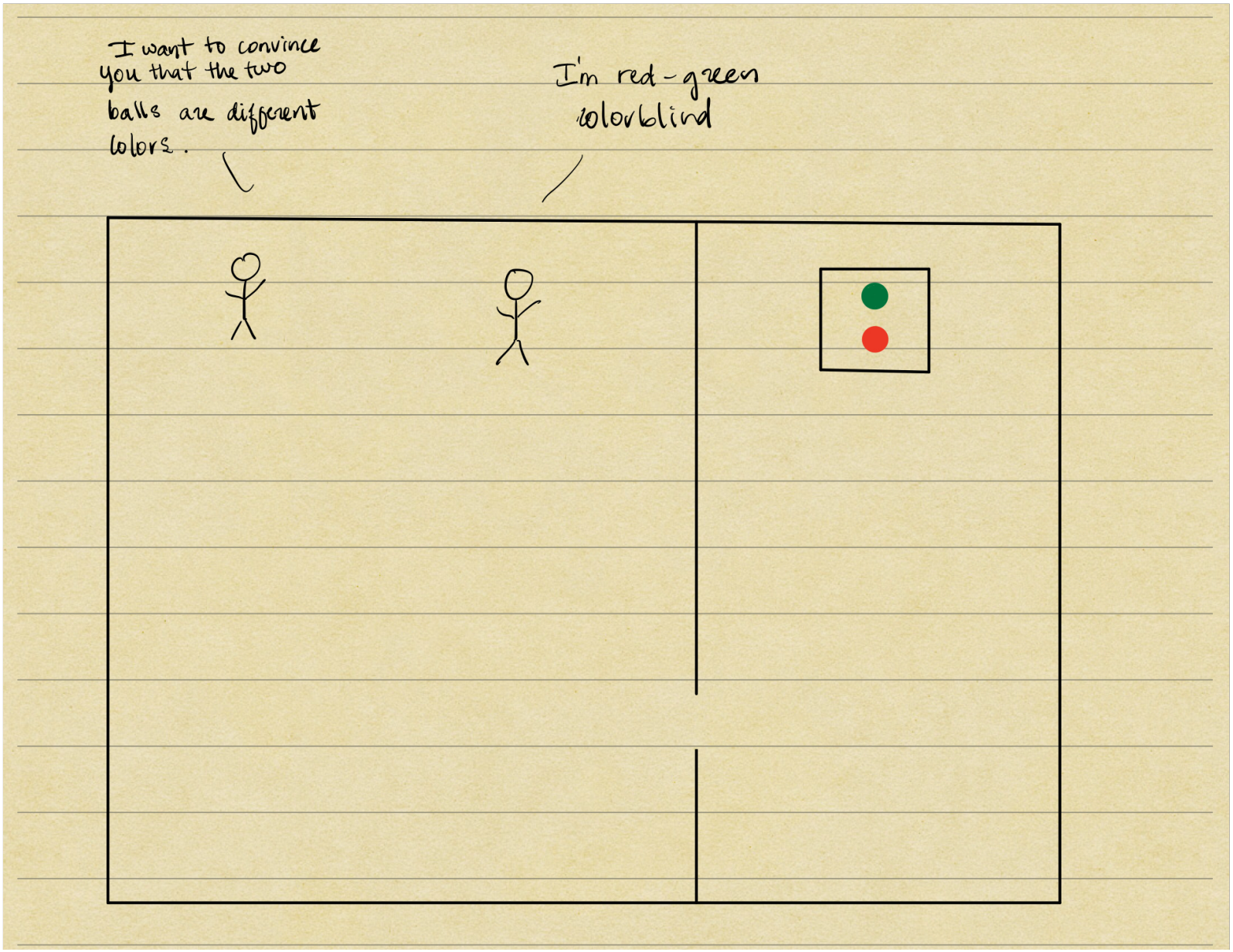
- Identification Schemes and Entity Authentication


I want to convince you that the two balls are different colors.
I'm red-green colorblind

Definitions

- **Prover**: *One proving that they have knowledge of something*
- **Verifier**: *One verifying that prover really does have knowledge*
- **Completeness**: *If statement is true, honest verifier will in fact be convinced it is true by honest prover*
- **Soundness**: *If statement is false, no cheating prover can convince honest verifier it is true, except with small probability (soundness error)*
- **Zero Knowledge**: *If statement is true, no verifier learns anything other than the fact that the statement is true*

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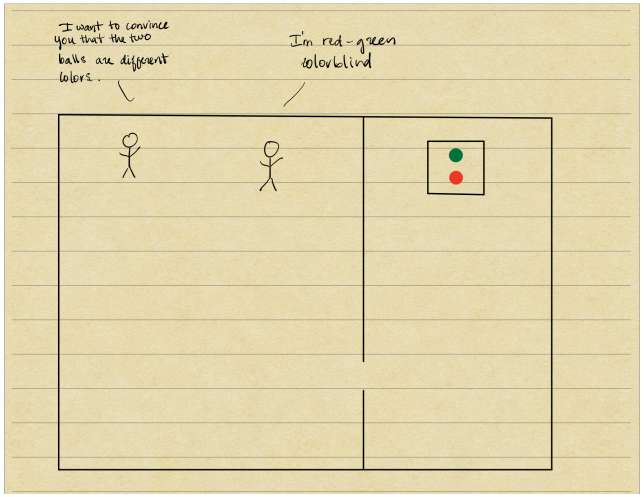
- In context of general cryptographic protocols, **Soundness** might be rephrased as *"Someone who is able to impersonate the honest prover with relatively high probability must know the secret"*
- Can formalize above definitions/scheme through Turing machines



Example revisited:

- **Prover**: Person (you) who is trying to convince friend that the colors differ
- **Verifier**: Friend who is colorblind
- **Completeness**: If there really is a difference in the colors of the balls, then for a non-colorblind prover, answering "yes" or "no" to whether or not the ball was swapped will be trivial
- **Soundness**: If there was no difference in the color of the balls, then it would be hard to be correct every time (P(correct) $\sim \frac{1}{2^n}$ with n trials)
- **Zero Knowledge**: Answering "yes" or "no" to whether the balls were switched gives away no information to your colorblind friend about what color each ball is.

Example: Feige - Fiat - Shamir Identification Scheme Preliminaries

Problem (Computational Composite Quadratic Residues)

Instance: A positive integer n that is the product of two unknown distinct primes p and q where $p, q \equiv 3 \pmod 4$, and an integer $x \in \mathbb{Z}_n^*$ such that the Jacobi symbol $\left(\frac{x}{n}\right) = 1$.

Question: Find $y \in \mathbb{Z}_n^*$ such that $y^2 \equiv \pm x \pmod n$.

Setup:

- $n = p \cdot q$ is public while p, q are private
- Choose random $S_1, ..., S_k \in \mathbb{Z}_n$
- For $1 \leq j \leq k$, compute $I_j = \pm 1/S_j^2 \pmod n$, where the sign is chosen randomly
- $\mathbf{I} = (I_1, ..., I_k)$ is the public key and $\mathbf{S} = (S_1, ..., S_k)$ is the private key

Example: Feige - Fiat - Shamir Identification Scheme

Recall $\mathbf{I} = (I_1, ..., I_k)$ is the public key and $\mathbf{S} = (S_1, ..., S_k)$ is the private key

Protocol (Feige - Fiat - Shamir Identification Scheme)

Repeat the following steps until verifier is convinced:

com *Prover chooses random* $R \in \mathbb{Z}_n$ *and computes* $X = \pm R^2 \pmod n$ *with random sign, and sends this* X *to verifier*

chal *Verifier sends prover a random boolean vector* $\mathbf{E} = (E_1, ..., E_k) \in \{0, 1\}^k$

resp *Prover computes* $Y = R \prod_{\{j:E_j=1\}} S_j \pmod n$ *and sends this to verifier*

verif *Verifier verifies that* $X = \pm Y^2 \prod_{\{j:E_j=1\}} I_j \pmod n$. *If so, then verifier "accepts", otherwise verifier "rejects".*

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The point is that the prover wants to convince the verifier that the prover does in fact know the inverses of the S_j^2s, meaning that the prover does know S_j for each j, and that the prover does not want to simply tell the verifier the S_js.

Example: Feige - Fiat - Shamir Identification Scheme Completeness

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resp Prover computes $Y = R \prod_{\{j:E_j=1\}} S_j \pmod{n}$ and sends this to verifier

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Completeness

$Y^2 \prod_{\{j:E_j=1\}} I_j \equiv (R \prod_{\{j:E_j=1\}} S_j)^2 (\prod_{\{j:E_j=1\}} I_j) \pmod{n}$

$\equiv R^2 (\prod_{\{j:E_j=1\}} S_j^2 I_j)) \equiv \pm R^2 \equiv \pm X \pmod{n}$

Example: Feige - Fiat - Shamir Identification Scheme Soundness

Recall:

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verif Verifier verifies that $X = \pm Y^2 \prod_{\{j:E_j=1\}} I_j \pmod{n}$. If so, then verifier "accepts", otherwise verifier "rejects".

A dishonest prover may try to guess the challenge $\mathbf{E} = (E_1, ..., E_k) \in \{0,1\}^k$ ahead of time. Then given a particular $\mathbf{E}$, choosing X to fool the verifier is straight forward:

Soundness

Suppose the adversary can guess ahead of time $\mathbf{E}$, in step 1. Then the adversary chooses $X \equiv Y^2(\prod_{\{j:E_j=1\}} I_j)$ for a random $Y \in \mathbb{Z}_n$, and sends X to the verifier as the adversary's commitment. The verifier sends back the correctly guessed $\mathbf{E}$ as the challenge, and the adversary sends back Y as the response. Then the verifier verifies that $Y^2(\prod_{\{j:E_j=1\}} I_j) \equiv X$, as this was how X was initially chosen by the adversary.

Example: Feige - Fiat - Shamir Identification Scheme Soundness continued

Recall:

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Soundness continued

The ability of the adversary to fool the verifier depends upon correctly guessing $\mathbf{E} = (E_1, ..., E_k) \in \{0,1\}^k$ ahead of time. The probability of correctly guessing $\mathbf{E}$ is $\frac{1}{2^k}$, and so the probability of correctly guessing $\mathbf{E}$ for t rounds of the protocol is $\frac{1}{2^{tk}}$, which is exceedingly low.

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Final notes: