Universal Property of Module of Kähler Differentials

$$\begin{tikzcd}
 & \Omega_{S/R} \arrow[d, "\exists!\, e',\ e'(df) := ef."] \\
S \arrow[ur, "d"] \arrow[r, "e"'] & M
\end{tikzcd}$$

Proposition: If $S = R[x_1, \dots, x_r]$, then

$$\Omega_{S/R} = \bigoplus_{i=1}^{r} S\, dx_i .$$

$$\underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{\text{ith coordinate}} \longmapsto dx_i$$

$$\begin{tikzcd} S^r \arrow[r, shift left] & \Omega_{S/R} \arrow[l, shift left] \end{tikzcd}$$

$$\begin{pmatrix} \partial_1 \\ \vdots \\ \partial_r \end{pmatrix}$$

$$\begin{tikzcd} & \Omega_{S/R} \arrow[d, "\exists!\, \partial_i,\ \partial_i(dx_j) := \frac{\partial}{\partial x_i} x_j = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}"] \\ S \arrow[ur, "d"] \arrow[r, "\frac{\partial}{\partial x_i}"'] & S \end{tikzcd}$$

Note:

$$\begin{tikzcd} S & & \Omega_{S/R} \\ R \arrow[u] & \rightsquigarrow & S \arrow[u] \end{tikzcd}$$

<u>Algebras</u> <u>Modules</u>

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\phi"] & S \arrow[r, mapsto, "d"] \arrow[d, mapsto, "\phi"] & \Omega_{S/R} \arrow[d] \\ R' \arrow[r] & S' \arrow[r, mapsto, "d'"'] & \Omega_{S'/R'} \end{tikzcd}$$

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d] & S \arrow[d] \\ R' \arrow[r] & S' \end{tikzcd}$$

$$\begin{tikzcd} & \Omega_{S/R} \arrow[dd, "\exists!\,\phi'"] \\ S \arrow[ur, "d"] \arrow[r] & S' \arrow[r, "d'"'] & \Omega_{S'/R'} \end{tikzcd}$$

Proposition (Base Change):

$$\begin{tikzcd}
 & \Omega(R'\otimes_R S)/R' \arrow[dd, leftrightarrow, "\wr"] \arrow[dd, "d(a'\otimes b) = a'\otimes db"] \\
R'\otimes_R S \arrow[ur, "d"] \arrow[dr, "1\otimes d"'] & \\
 & R'\otimes_R \Omega_{S/R}
\end{tikzcd}$$

$$\begin{tikzcd}
 & & R'\otimes_R \Omega(R\otimes_R S)/R \arrow[ddl, "a'\otimes db = a'db = d(a'\otimes b)"] \\
 & \Omega(R\otimes_R S)/R \arrow[ur] \arrow[d, "(1\otimes d)\wr = d\wr"] & \\
R\otimes_R S \arrow[uur] \arrow[r] & R'\otimes_R S \arrow[r, "d"'] & \Omega(R'\otimes_R S)/R'
\end{tikzcd}$$

Proposition (Tensor Products):

$$T \otimes_{S_i} \Omega_{S_i/R} = \Big(\big(\bigotimes_{j \neq i}{}_R\, S_j\big) \otimes_R S_i\Big) \otimes_{S_i} \Omega_{S_i/R}$$

$$= \Big(\bigotimes_{j \neq i}{}_R\, S_j\Big) \otimes_R \big(S_i \otimes_{S_i} \Omega_{S_i/R}\big)$$

$$= \Big(\bigotimes_{j \neq i}{}_R\, S_j\Big) \otimes_R \Omega_{S_i/R}.$$

$$\Omega := \bigoplus_i \big(T \otimes_{S_i} \Omega_{S_i/R}\big) = \bigoplus_i \Big(\big(\bigotimes_{j \neq i}{}_R\, S_j\big) \otimes_R \Omega_{S_i/R}\Big)$$

write $d_i : S_i \to \Omega_{S_i/R}$

$c \in T$ may be written $\sum_i^{\text{finite}} \otimes b_i$.

$$1 \otimes d_i : T = \Big(\bigotimes_{j \neq i}{}_R\, S_j\Big) \otimes_R S_i \longrightarrow \Big(\bigotimes_{j \neq i}{}_R\, S_j\Big) \otimes_R \Omega_{S_i/R}$$

$$\begin{tikzcd}
 & \Omega_{T/R} \arrow[d, "\exists!\,\alpha"] \\
T \arrow[ur, "d_T"] \arrow[r, "\sum_i 1 \otimes d_i"'] & \Omega
\end{tikzcd}$$

$$\alpha(d_T t) = \alpha\Big(d_T\Big(\sum_i^{\text{finite}} \otimes_R b_i\Big)\Big)$$

$$:= \sum_i^{\text{finite}} 1 \otimes_R db_i.$$

$$
\begin{tikzcd}
 & T \arrow[r, "d_T"] & \Omega_{T/R} \\
S_i \arrow[ur, "\tau"] \arrow[rr, "d_i"'] & & \Omega_{S_i/R} \arrow[u, "\exists!\lambda_i"', "\lambda_i(dg) := d_T(\tau(g)) = d_T(1 \otimes b_i)"] \arrow[r] & T \otimes_{S_i} \Omega_{S_i/R}
\end{tikzcd}
$$

β_i, $\beta_i(1 \otimes_{S_i} db_i) := d_T(1 \otimes_R db_i)$

then $\bigoplus_i \beta_i : \bigoplus_i T \otimes_{S_i} \Omega_{S_i/R} =: \Omega \longrightarrow \Omega_{T/R}$

is the required inverse of α.

Corollary:

Let $T' := R[x_1, \dots, x_r]$.

Then $T = S \otimes_R T'$ $(= S \otimes_R R[x_1, \dots, x_r])$.

Then $\Omega_{T/R} \simeq (T \otimes_S \Omega_{S/R}) \oplus (T \otimes_{T'} \Omega_{T'/R})$.

$\Omega_{T'/R} \simeq \bigoplus_{i=1}^{r} T' dx_i$.

$$T \otimes_{T'} \Omega_{T'/R} = T \otimes_{T'} \left(\bigoplus_{i=1}^{r} T' dx_i \right)$$

$$= \bigoplus_{i=1}^{r} (T \otimes_{T'} T') dx_i = \bigoplus_{i=1}^{r} T dx_i.$$

Thus

$$\Omega_{T/R} \simeq (T \otimes_S \Omega_{S/R}) \oplus (T \otimes_{T'} \Omega_{T'/R})$$

$$= (T \otimes_S \Omega_{S/R}) \oplus \left(\bigoplus_i T dx_i \right).$$

Corollary (Coequalizers):

Note $\pi : S_2 \to S_2/I =: T$

$$I/I^2 \xrightarrow{d} T \otimes_{S_2} \Omega_{S_2/R} \to \Omega_{T/R} \to 0$$

$$\Omega_{T/R} \simeq T \otimes_{S_2} \Omega_{S_2/R} \Big/ \mathrm{im}(d)$$

$$= T \otimes_{S_2} \Omega_{S_2/R} \Big/ \langle d(\psi(b) - \psi'(b)) \rangle.$$

Proposition (Localization):

Recall: If $L := S[x_1, \dots, x_r]$ a polynomial ring over an R-alg. S, then

$$\Omega_{L/R} = (L \otimes_S \Omega_{S/R}) \oplus (\bigoplus_i L\,dx_i).$$

So that

$$\Omega_{S[x]/R} = (S[x] \otimes_S \Omega_{S/R}) \oplus (S[x]\,dx).$$

$$\pi : S[x] \to S[x]/I := T, \quad I = \langle sx - 1 \rangle,$$

$$I/I^2 \xrightarrow{d} T \otimes_{S[x]} \Omega_{S[x]/R} \xrightarrow{D_\pi} \Omega_{T/R} \to 0$$

$$\Omega_{T/R} = \Omega_{S[u^{-1}]/R} =$$

$$= T \otimes_{S[x]} \Omega_{S[x]/R} / \mathrm{im}(d)$$

$$= T \otimes_{S[x]} (S[x] \otimes_S \Omega_{S/R} \oplus S[x]\,dx) / \mathrm{im}(d)$$

$$= (T \otimes_{S[x]} S[x] \otimes_S \Omega_{S/R} \oplus T \otimes_{S[x]} S[x]\,dx) / \mathrm{im}(d)$$

$$= (T \otimes_S \Omega_{S/R} \oplus T\,dx) / \mathrm{im}(d)$$

$$= \left(S[U^{-1}] \otimes_S \Omega_{S/R} \oplus S[U^{-1}]dx\right) \Big/ S[U^{-1}]\, d(sx-1).$$

Fact: For $M = N \oplus R\cdot g$ with N an R-module, $g \in M$ is not torsion, then

$$N \oplus R\cdot g \,/ \langle n+g \rangle \simeq N$$

for any $n \in N$.

$$d(sx-1) = x\,ds + s\,dx$$

To finish, if $\mathcal{D}$ is the diagram of R-algebras whose objects are the localizations $S[s^{-1}]$ for $s \in U$, with maps

$S[s^{-1}] \to S[(st)^{-1}]$ the natural localization map for $s, t \in U$, then

$$S[U^{-1}] = \varinjlim \mathcal{D}$$

so that by Theorem (colimits), we have

$$\Omega_{\varinjlim B/R} = \Omega_{S[U^{-1}]/R} = \varinjlim_{s \in U} S[U^{-1}] \otimes_{S[s^{-1}]} \Omega_{S[s^{-1}]/R}.$$

But for any S-module M,

$$M[U^{-1}] = \varinjlim_{s \in U} S[U^{-1}] \otimes_{S[s^{-1}]} M[s^{-1}].$$

Proposition (Direct Products):

$$\begin{tikzcd} & \Omega_{\prod S_i/R} \arrow[d] \\ \prod_i S_i \arrow[ur] \arrow[r, "D"'] & M \end{tikzcd}$$

$$\begin{tikzcd} & \Omega_{S_i/R} \arrow[d] \\ S_i = e_i\left(\prod_i S_i\right) \arrow[ur] \arrow[r, "D_{e_i}"'] & e_i(M) := M_i \end{tikzcd}$$

Thus $\prod \Omega_{S_i/R}$ has the universal property that characterizes $\Omega_{\prod S_i/R}$.