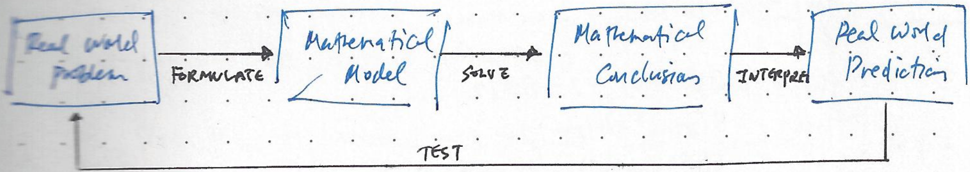
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$
Black-Scholes
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$
Wave Equation
But, by similar triangles
$$i\hbar\frac{\partial}{\partial t}|\Psi(\mathbf{r},t)\rangle = \hat{H}|\Psi(\mathbf{r},t)\rangle$$
Schrödinger Equation
$$V - E + F = 2.$$
Euler's polyhedron formula
Ellipse
Parabola
$$a^2 + b^2 = c^2$$
Pythagorean Theorem
$$\int_L f(z)dz$$
new origin
x=2 y=3
The Hyperbola
$$y = \frac{3x+10}{x-2}$$
XY=16
and therefore the value of
Special Relativity
$$E = mc^2$$
Fig. 195. The so-called Normal Curve as an Approximation to the Binomial Distribution
Fig. 35. The First Six Pythagorean Numbers

## Lesson 2 - Functions

- A mathematical model is a mathematical description of a real-world phenomenon (such as the size of a population, the speed of a falling object, the cost of a product). The purpose of the model is to understand the phenomenon and perhaps to make predictions about future behavior




Given a real-world problem, the first step is formulate a mathematical model by identifying and naming the independent & dependent variables and making assumptions that simplify the phenomenon enough to make it mathematically tractable. We can formulate the model using physical laws and/or data-experimentation. The second step is to apply the mathematics that we know (such as calculus!) to the model in order to reach some conclusions. In the third step, we interpret those conclusions and make predictions. Finally we test our predictions against new data. If we need to, we can refine our model and start the cycle again

A mathematical model is never a completely accurate representation of a physical situation. It is an idealization. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions.

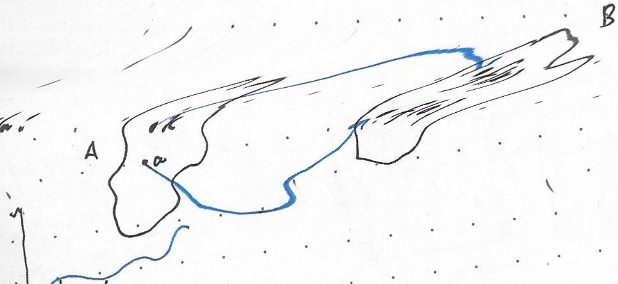
Functions are very useful for creating models because they help us establish relationships between different variables.

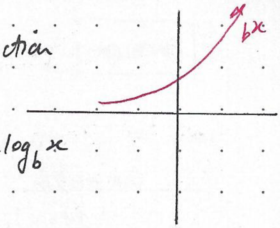
**Defn**: A function $f$ is a rule/map that assigns each element $x$ in a set $A$ exactly one element, called $f(x)$, in a set $B$. The set $A$ is called the domain of the function and the range of $f$ is the set of all possible values of $f(x)$ as $x$ varies through the domain.

Here are some helpful ways to think of a function:

① Machine: $x \mapsto \boxed{f} \to f(x)$

② Arrow diagram:






For $b>0$ and $b \neq 1$, a function of the form $f(x) = b^x$ is an exponential function

The inverse of an exponential function is a logarithmic function: $f(x) = \log_b x$

We will often use $e$ as our base in calculus

ROBLEM 4

(1) $5^x = 7 \Rightarrow x \ln 5 = \ln 7 \Rightarrow x = \dfrac{\ln(7)}{\ln(5)} \approx 1.21$

(2) $\ln(x+1) = 5 \Rightarrow e^5 = x+1 \Rightarrow x = e^5 - 1$

ROBLEM 5

(1) $\dfrac{e^x - e^{-x}}{2} = 1 \Rightarrow e^x - e^{-x} - 2 = 0$ $\quad \cancel{(e^x - \sqrt{3})(e^x + \sqrt{3})} = 0$

$e^{2x} - 1 - 2e^x = 0$

$\Rightarrow e^{2x} - 2e^x - 1 = 0$

$\cancel{(e^x - 1)(e^x - 1) = 0} \rightarrow \cancel{e^x = 1} \Rightarrow \boxed{\cancel{x = 0}}$

(2) $2\ln(\sqrt{x}) - \ln(1-x) = 2$

$\ln x - \ln(1-x) = 2 \Rightarrow \ln\left(\dfrac{x}{1-x}\right) = 2$

$\dfrac{x}{1-x} = e^2 \Rightarrow x = \dfrac{e^2}{1+e^2} \approx 0.88$ (plug back in)

$(e^x)^2 - 2e^x - 1 = 0 \Rightarrow u = e^x \Rightarrow u^2 - 2u - 1 = 0$

$u = \dfrac{2 \pm \sqrt{4+4}}{2} = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$

$\Rightarrow e^x = 1 + \sqrt{2}$ ($e^x$ cannot be negative)

$x = \ln(1+\sqrt{2}) \approx 0.881$

---

n 5 - Arithmetic, Composition & Transformations

aving learned about some important functions, we now investigate the question of how to get more mplicated functions by combining these.

ere are 3 basic ways we can combine functions to get new ones:

(1) Arithmetic

$(f+g)(x) = f(x) + g(x)$

$(f-g)(x) = f(x) - g(x)$

$(fg)(x) = f(x)g(x)$

$(f/g)(x) = f(x)/g(x)$ (excluding where $g(x) = 0$)

The domain of the functions above is the intersection of the domains of $f$ and $g$.

# Lesson 6 - Inverse Functions

In mathematics the term inverse is used to describe functions that reverse one another in the sense that each undoes the effect of the other.

PROBLEM 1

- $g(f(x)) = \sqrt[3]{f(x)-1} = \sqrt[3]{(x^3+1)-1} = x$ $\qquad x \mapsto g(f(x)) \mapsto x$
- $f(g(y)) = [g(y)]^3+1 = (\sqrt[3]{y-1})^3+1 = y$ $\qquad y \mapsto f(g(y)) \mapsto y$

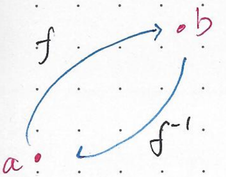
If the functions $f$ and $g$ satisfy the two conditions

$g(f(x)) = x$ for every $x$ in the domain of $f$

$f(g(y)) = y$ for every $y$ in the domain of $g$

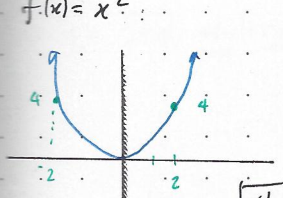
then $f$ and $g$ are inverse functions.

If $g$ is the inverse of $f$, we typically will express it as $f^{-1}(y)$.
For example: the inverse of $f(x) = 2x$ is $f^{-1}(x) = \frac{1}{2}x \Rightarrow f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$




$$\boxed{\begin{array}{l} \text{domain of } f = \text{range of } f^{-1} \\ \text{range of } f = \text{domain of } f^{-1} \end{array}}$$

Since $f^{-1}$ must also be a function, it must assign inputs to a single output. Consider $f(x) = x^2$:



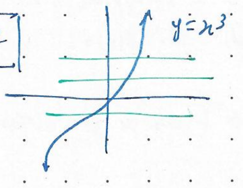

The inverse of $x^2$ maps 4 to 2 places ($-2$ and $2$) which is a problem.

To avoid this, we force functions which need an inverse to be one-to-one.

**Thm** A function has an inverse $\Leftrightarrow$ it is one-to-one.

* **Horiz-Line Test** A function has an inverse function $\Leftrightarrow$ the graph is cut at most once by a horizontal line.

PROBLEM 2




How do we calculate $f^{-1}$?

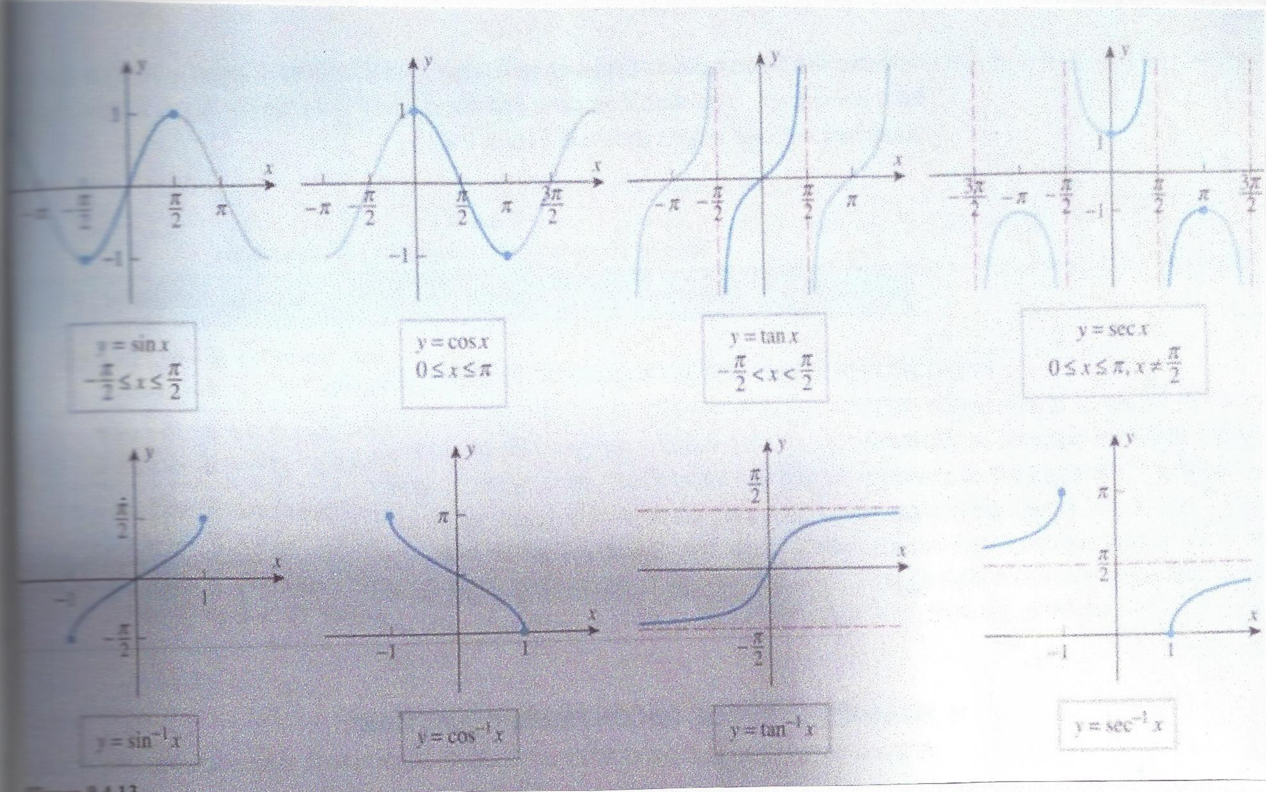
Step 1: Write down the equation $y = f(x)$

Step 2: Switch the $x \leftrightarrow y$

Step 3: Solve for $y$

or

reflect about the line $y = x$




Here are some helpful rules for calculating limits:

If $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist, then:

(1) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$

(2) $\lim_{x \to c} [f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$

(3) $\lim_{x \to c} K f(x) = K \lim_{x \to c} f(x)$

(4) $\lim_{x \to c} [f(x) g(x)] = [\lim_{x \to c} f(x)] [\lim_{x \to c} g(x)]$

(5) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if $\lim_{x \to c} g(x) \neq 0$

(6) $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n$

(7) $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to c} f(x)}$

we will define this soon

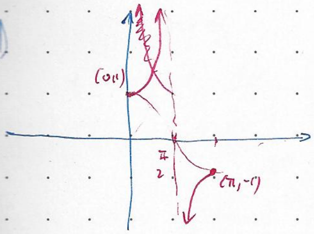
$\boxed{\text{Direct Substitution Property}}$ If $f(x)$ is "nice enough", then $\lim_{x \to a} f(x) = f(a)$, $\lim_{\substack{x \to a^- \\ x \to a^+}} f(x) = f(a)$

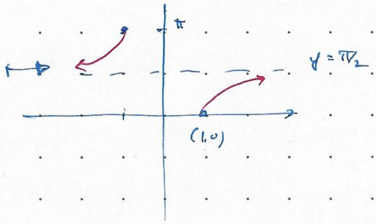
Inverse Trig Function (II)

$x(t) = \tan^{-1}(t^2) \Rightarrow v(t) = \frac{1}{1+t^4} \cdot 2t \; ; \; v(5/2) = \frac{2(5/2)}{1+(\frac{5}{2})^4} = \frac{5}{1+\frac{625}{16}} = \frac{80}{641}$

$y = \cos^{-1}x \rightarrow \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}} \Big|_{x=\frac{\sqrt{3}}{2}} \rightarrow \frac{-1}{\sqrt{1-\frac{3}{4}}} = \frac{-1}{\sqrt{\frac{1}{4}}} = -2 \Rightarrow m = \frac{1}{2}$

point: $(-\sqrt{3}/2, \frac{4\pi}{3}) \Rightarrow (y - \frac{4\pi}{3}) = \frac{1}{2}(x + \frac{\sqrt{3}}{2})$




$y = \sec^{-1}x \Rightarrow x = \sec(y)$

$1 = \sec(y)\tan(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec(y)\cdot\tan(y)}$

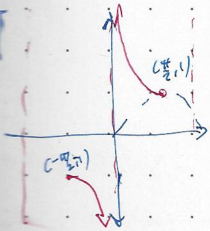
$\sec(y) = x \; ; \; \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2-1}$

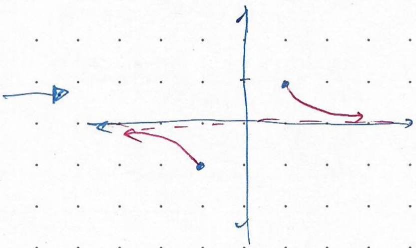
$\rightarrow \frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}}$

The graph of $\sec^{-1}x$ shows that the slope is always positive: Therefore:

$$\frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{-1}{x\sqrt{x^2-1}}, & x < -1 \end{cases} \quad |x| > 1$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, \; |x| > 1.}$$

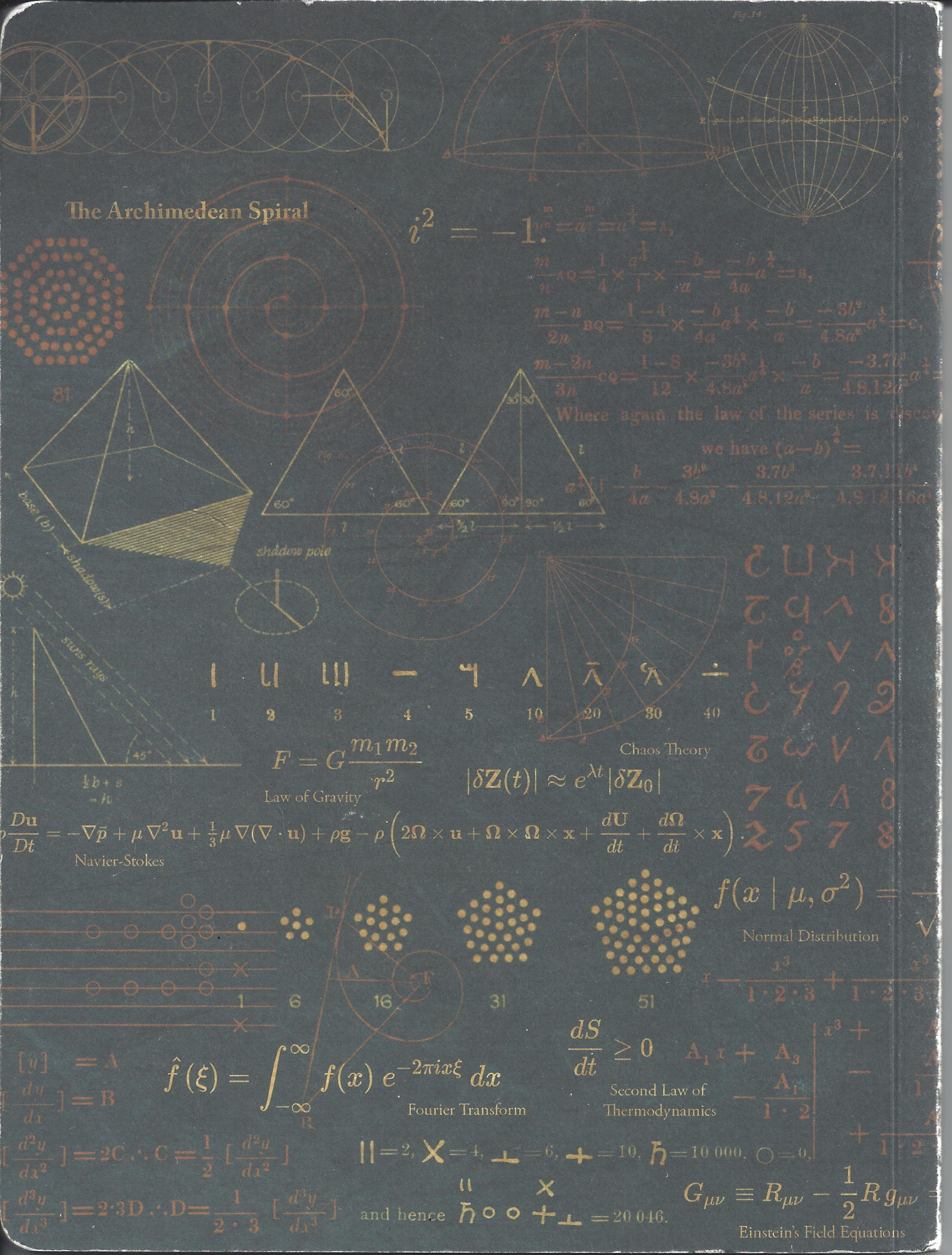


$y = \csc^{-1}x \Rightarrow x = \csc y \rightarrow 1 = -\csc y \cot(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{\csc y \cot(y)} = \frac{-1}{x \pm\sqrt{x^2-1}}$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \frac{-1}{x\sqrt{x^2-1}}, & x > 1 \\ \frac{1}{x\sqrt{x^2-1}}, & x < -1 \end{cases} \Rightarrow \boxed{\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2-1}}, \; |x| > 1}$$

$x = \cot y \Rightarrow 1 = -\csc^2(y)\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-1}{\csc^2(\cot^{-1}(x))} = \frac{-1}{1+x^2}$



The Archimedean Spiral
$i^2 = -1.$
Where again the law of the series is discov
we have $(a-b)^{\frac{1}{2}} =$
shadow pole
shadow(s)
suns rays
1 2 3 4 5 10 20 30 40
$F = G\frac{m_1 m_2}{r^2}$
Law of Gravity
Chaos Theory
$|\delta \mathbf{Z}(t)| \approx e^{\lambda t}|\delta \mathbf{Z}_0|$
$\frac{D\mathbf{u}}{Dt} = -\nabla\bar{p} + \mu\nabla^2\mathbf{u} + \frac{1}{3}\mu\nabla(\nabla\cdot\mathbf{u}) + \rho\mathbf{g} - \rho\left(2\mathbf{\Omega}\times\mathbf{u} + \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{x} + \frac{d\mathbf{U}}{dt} + \frac{d\mathbf{\Omega}}{dt}\times\mathbf{x}\right)$
Navier-Stokes
$f(x \mid \mu, \sigma^2) =$
Normal Distribution
1 6 16 31 51
$\frac{dS}{dt} \geq 0$
Second Law of Thermodynamics
$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$
Fourier Transform
$[y] = A$
$\left[\frac{dy}{dx}\right] = B$
$\left[\frac{d^2y}{dx^2}\right] = 2C \therefore C = \frac{1}{2}\left[\frac{d^2y}{dx^2}\right]$
$\left[\frac{d^3y}{dx^3}\right] = 2\cdot 3D \therefore D = \frac{1}{2\cdot 3}\left[\frac{d^3y}{dx^3}\right]$
and hence
$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}R\, g_{\mu\nu}$
Einstein's Field Equations