

Methods For Solving L0-Norm Minimization Problems In Compressed Sensing

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October 24, 2015

Abstract

Compressed Sensing is a signal processing technique for under-sampling and optimal recovery of a signal, in particular images. It exploits the fact that natural images are sparse in some domain like the Discrete Fourier Transform or Wavelets. The sparsity of the signal can be exploited via optimization to recover it from far fewer samples than required by the Nyquist-Shannon sampling theorem. The sparsity constraint, however, is of a combinatorial nature and is NP-hard. We explore various methods for solving this problem via convex optimization. We plan to use two classes of methods: Orthogonal Matching Pursuit and Rank Minimization to solve this problem.

1 Introduction

A common goal of signal processing is to reconstruct a signal from a series of sampling measurements. In general, this task is impossible because there is no way to reconstruct a signal during the times that the signal is not measured. Nevertheless, with prior knowledge or assumptions about the signal, it turns out to be possible to perfectly reconstruct a signal from a series of measurements. Over time, engineers have improved their understanding of which assumptions are practical and how they can be generalized.

An early breakthrough in signal processing was the Nyquist-Shannon sampling theorem. It states that if the signal's highest frequency is less than half of the sampling rate, then the signal can be reconstructed perfectly. The main idea is that with prior knowledge about constraints on the signal's frequencies, fewer samples are needed to reconstruct the signal.

Around 2004, Emmanuel Cands, Terence Tao, and David Donoho proved that given knowledge about a signal's sparsity, the signal may be reconstructed with even fewer samples than the sampling theorem requires. This idea is the basis of compressed sensing.

2 The Compressed Sensing Problem

Given a signal \mathbf{x} , we have its representation in a basis Ψ given by

$$\mathbf{x} = \Psi\alpha$$

where α are the coefficients of \mathbf{x} in the basis represented by the columns of Ψ . \mathbf{x} being sparse in the basis Ψ means that the cardinality of α is much less than the number of elements in it. This is the case with natural images and signals in the Fourier, Wavelet or Discrete Cosine Transform domain.

We consider sampling linear combinations of the components of the signal. This means the sampled version of the signal available to us is

$$\mathbf{y} = \Phi\mathbf{x}$$

The matrix Φ is typically a short and fat matrix, so that the number of columns in \mathbf{y} is much less than the number of elements in \mathbf{x} . This gives rise to the 'compressive' nature of the measurements. We can reconstruct the signal from less measurements than the number of elements in the signal itself, reducing

measurement times and speeding up the acquisition process. In practice the matrix is usually binary, with the result that only some of the elements of \mathbf{x} appear in the elements of \mathbf{y} . One has to make sure that the basis Ψ is ‘incoherent’ with the measurement Φ , meaning that the measurement vectors (columns of Φ) are themselves not sparse in the basis Ψ . This is usually done by choosing the matrix Φ randomly.

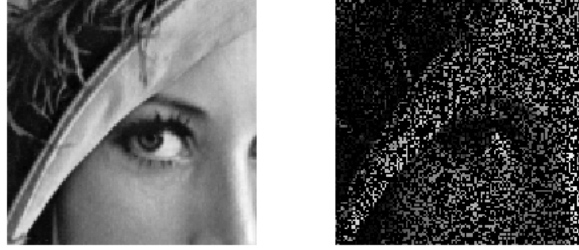


Figure 1: An example of compressed sensing in images. Right: The output of the sampling represented as an image. Left: The output from a reconstruction algorithm. Figure taken from <https://www.ceremade.dauphine.fr/~peyre/matlab/sparsity/>

3 Solving Compressed Sensing

The compressed sensing problem is then to reconstruct \mathbf{x} from \mathbf{y} , Φ and Ψ . We need to solve for \mathbf{x} , given that

$$\mathbf{y} = \Phi\Psi\alpha$$

This is an under-determined linear system. The optimization problem we construct to solve this is

$$\hat{\alpha} = \min_{\alpha} \|\mathbf{y} - \Phi\Psi\alpha\| < \epsilon$$

such that $\|\alpha\|_0 \leq K$. The parameter K controls the sparsity of the estimated α .

However, this is not a convex program since the L_0 norm in the constraint is not convex. We aim to use approximation methods to solve this problem.

3.1 Matching Pursuit

Matching pursuit [2] is a type of sparse approximation which involves finding the “best matching” projections of multidimensional data onto a dictionary \mathcal{D} . The basic idea is to represent the signal $\mathcal{Y}(x)$ as a weighted sum of functions $f_n(x)$ (called atoms) taken from \mathcal{D} :

$$\mathcal{Y}(x) = \sum_{n=0}^N a_n f_n(x)$$

In our setting the dictionary is the basis in which our signal is sparse in (like DCT or Fourier coefficients, the matrix $\Phi\Psi$). We restrict the number of coefficients to N to enforce sparsity. Matching pursuit provides a way to find the ‘best’ representative atoms so that the signal is represented faithfully while honouring the sparsity constraint.

As seen in the above, the algorithm picks the dictionary element that is the most coherent with the residual function at a given iteration. This tries to ensure that we pick the least number of elements from the dictionary, thus accounting for the sparsity of the signal. A major drawback of this algorithm is that if the dictionary atoms are not orthogonal, it can pick the same atom twice. The modification that corrects this is called Orthogonal Matching Pursuit.

Data: Signal: $\mathcal{Y}(x)$, dictionary \mathcal{D}
Result: List of coefficients: $(a_n, f_n(x))$.
Initialization
 $R_1(x) \leftarrow \mathcal{Y}(x)$
 $n \leftarrow 1$;
while $\|R_n(x)\| < \text{threshold}$ **do**
 $f_n(x) \leftarrow \arg \max_{f_i(x) \in \mathcal{D}} \|R_n(x) - f_i(x)\|$ $a_n \leftarrow \|R_n(x) - f_i(x)\|$
 $R_{n+1}(x) \leftarrow R_n(x) - a_n f_n(x)$
 $n \leftarrow n + 1$
end

Algorithm 1: Matching Pursuit

3.2 Orthogonal Matching Pursuit

Orthogonal Matching Pursuit solves the above problem by building a ‘support set’ of atoms, that is, a set of atoms that have already been used in the reconstruction of the signal. The algorithm goes as shown in Algorithm 2.

Data: Signal: $\mathcal{Y}(x)$, dictionary \mathcal{D}
Result: List of coefficients: $(a_n, f_n(x))$.
Initialization
 $R_1(x) \leftarrow \mathcal{Y}(x)$
 $n \leftarrow 1$;
 $\mathcal{S} \leftarrow \Phi$
while $\|R_n(x)\| < \text{threshold}$ **do**
 $f_n(x) \leftarrow \arg \max_{f_i(x) \in \mathcal{D}} \|R_n(x) - f_i(x)\|$ $\mathcal{S} = \mathcal{S} \cup f_n(x)$
 $\mathbf{a} \leftarrow \arg \min_{w \in \mathbb{R}^k} \|\mathcal{Y}(x) - \sum_{f_i(x) \in \mathcal{S}} w_i f_i(x)\|$
 $R_{n+1}(x) \leftarrow \mathcal{Y}(x) - \sum a_n f_n(x)$
 $n \leftarrow n + 1$
end

Algorithm 2: Orthogonal Matching Pursuit

Since the residual calculation in the algorithm makes sure that the residual at each step is orthogonal to all the atoms in the support set, no atom can be picked twice by OMP. This makes OMP a good choice for low-computation sparse signal recovery.

3.3 Difference-of-Convex Functions Approximation

This approach involves relieving the non-convexity of the constraint by constructing a convex series of approximations to it. This convex series of constraints approaches the non-convex L0 norm constraint as a parameter approaches zero.

Given the L0 norm constraint $\|x\|_0 = \sum \text{sgn}(|x_i|)$ we construct the following underestimate for $\text{sgn}(|z|)$:

$$\begin{aligned} \psi(x, t) &= \min\left(1, \frac{|x|}{t}\right) \\ &= \frac{|x|}{t} - \frac{(x-t)^+ + (-x-t)^+}{t} \\ &= \frac{h(x, 0)}{t} - \frac{h(x, t)}{t} \end{aligned}$$

where $z^+ = \max(z, 0)$ and $h(x, t) = (x-t)^+ + (-x-t)^+$. $h(x, t)$ is a convex function of x , hence this is a

difference-of-convex functions. Summing over all components of the representation of the signal, we have

$$\begin{aligned}\phi(x, t) &= \sum \psi(x_i, t) \\ &= \frac{\|x\|_1}{t} - \frac{g(x, t)}{t}\end{aligned}$$

where $g(x, t) = \sum h(x_i, t)$. Next, we construct a convex underestimate for this by linearizing the negative convex function by constructing its affine approximation around the current estimate. Consider a $\xi \in \partial g(x, t)$. We have, at a given y

$$\phi(x, t) \leq u(x, y, \xi, t) := \frac{\|x\|_1}{t} - \frac{1}{t} [g(y, t) - \xi^T(x - y)]$$

Using this the compressed sensing problem can be cast into the following series of optimization problems indexed by t :

$$(P_t(y, \xi)) \quad \min \|\mathbf{y} - \Phi \Psi \mathbf{x}\|$$

such that

$$\frac{\|x\|_1}{t} - \frac{1}{t} [g(y, t) - \xi^T(x - y)] \leq K$$

We can get rid of the non-smooth L1 norm by casting the problem as:

$$(CP_t(y, \xi)) \quad \min \|\mathbf{y} - \Phi \Psi \mathbf{x}\|$$

such that

$$\begin{aligned}\frac{z}{t} - \frac{1}{t} [g(y, t) - \xi^T(x - y)] &\leq K \\ -x_i &\leq z_i \leq x_i\end{aligned}$$

It can be proved that as $t \rightarrow 0$, the solution to this series of convex programs converges to the solution to the L0 norm optimization problem (see the analysis section).

Data: Signal: $\mathcal{Y}(x)$, dictionary $\mathcal{D} \equiv \Phi \Psi, \epsilon > 0$

Result: α

Initialization

$t \leftarrow t_0 > 0$

$x \leftarrow x^0$ as starting guess

$\xi^0 \in \partial g(x^0, t)$

$k \leftarrow 0$

while $\|x^k - x^{k-1}\| > \epsilon$ **do**

 | Solve $(CP_t(x^k, \xi^k))$, set solution to x^{k+1}, z^{k+1} $k \leftarrow k + 1$

end

Algorithm 3: SCA Method

The entire algorithm can then be written as shown in Algorithm 3. The problem $(CP_t(x^k, \xi^k))$ can be solved using KKT conditions.

3.4 Penalty Decomposition Methods

General rank minimization problems with rank appearing in either objective function or constraint can be solved as lower dimensional vector optimization problems [6]. As a consequence, we establish that a class of rank minimization problems have closed form solutions. Using this result, we then propose penalty decomposition methods for general rank minimization problems. Penalty Decomposition methods work by formulating a penalty function, minimizing which amounts to minimizing our objective function.

The Compressed Sensing problem can be reformulated as:

$$\min_{(X, Y)} \{f(X) : X - Y = 0, X \in \mathcal{X}, Y \in \mathcal{Y}\}$$

where $f(X)$ is the C.S objective function and Y is defined as:

$$Y := \{Y \in \mathcal{Y} \mid \text{rank}(Y) \leq r\}$$

Given a penalty parameter $\rho \geq 0$, the associated quadratic penalty function for the above C.S. problem is defined as:

$$Q_\rho := f(X) + \frac{\rho}{2} \|X - Y\|_F^2$$

We use *Block Coordinate Descent* to minimize the above penalty function. Using results from [6], the minimizer X is related to the minimizer x of the following vector optimization problem as $X = UD(x)V^T$ where $D(x)$ is a diagonal matrix with elements of x as the diagonal entries and $U\Sigma V^T$ is the *Singular Value Decomposition* of Y .

$$Q_\rho := f(x) + \frac{\rho}{2} \|x - y\|_2^2$$

where

$$y := \{y \in \mathcal{Y} \mid \|y\|_0 \leq r\}$$

The L0 norm minimization problem thus becomes tractable under certain conditions (which are discussed in the *Analysis* section later) using this methodology

4 Results

We test out the compressed sensing problem as solved by all three methods on the standard Barbara image. The image at two resolutions is shown below:



We extract 8×8 patches from the image and apply compressed sensing on them. We extract compressed measurements from patches by multiplying the vectorized patch with a fixed random $64 \times 64f$ matrix where f is the level of compression we achieve. We thus aim to reconstruct the entire patch from measurements that are less in number by a factor of f . We then add the reconstructed patches back to form the output image. Results are shown below:

4.1 Orthogonal Matching Pursuit

OMP results for $f = 0.1$, $f = 0.2$ and $f = 0.6$ respectively, on the reduced image:



Clearly, the level of reconstructed detail increases as we increase the number of measurements, as expected. OMP results for $f = 0.1$, $f = 0.2$ and $f = 0.6$ respectively, on the full resolution image:



4.2 Difference-of-Convex Functions Approximation

SCA results for $f = 0.1$, $f = 0.2$ and $f = 0.6$ respectively, on the reduced image:



Clearly, the level of reconstructed detail increases as we increase the number of measurements, as expected.

It is interesting to note that **all** of the pixels in the $f = 0.2$ output are within ± 15 units of the ground truth. The mean absolute error is around 1.8, which on a scale of 255 is 0.72%. On the contrary, the corresponding figure for the OMP output is 10, which is more than 6 times our error. SCA results for $f = 0.1$, $f = 0.2$ and $f = 0.6$ respectively, on the full resolution image:



As we see here, SCA can recover as much detail with a fraction $f = 0.2$ of measurements as OMP can at $f = 0.6$. In fact, in the full resolution case, we see that SCA recovers the original image almost **exactly** in the $f = 0.2$ case. This is considerable savings on exposure time, about a factor of 5 over traditional imaging, and a factor of 3 over OMP.

4.3 PD Methods

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5 Analysis

Now that we showed empirically that the methods work, we present proofs that under certain conditions, the problem under consideration is actually solved exactly by our algorithms.

5.1 Difference-of-Convex Functions Approximation

Consider the generic problem given by

$$\hat{\alpha} = \min_{\alpha} \|\mathbf{y} - \mathbf{A}\alpha\| < \epsilon$$

such that $\|\alpha\|_0 \leq K$. We constructed an underestimate for the L0 norm of α parameterized by t , given by $\phi(\alpha, t)$. We have, $\|\alpha\|_0, \phi(\alpha, t) \geq 0$.

Now consider the feasible sets of the two problems. Let F_0 be the feasible set of the original problem and F_t be the feasible set of problem P_t . Thus,

$$F_0 = \{\alpha \in \mathbb{R}^n : \|\alpha\|_0 \leq K\} \text{ and } F_t = \{\alpha \in \mathbb{R}^n : \phi(\alpha, t) \leq K\}$$

It is easy to see that both of these are closed sets in \mathbb{R} . Therefore, since $\|\alpha\|_0 \geq \phi(\alpha, t)$, we have,

$$F_0 \subseteq F_t \text{ for all } t > 0$$

Now, we know, $\phi(\alpha, t_2) \leq \phi(\alpha, t_1)$ for any $0 < t_1 \leq t_2$. Therefore, $F_0 \subset F_{t_1} \subseteq F_{t_2}$ for any $0 < t_1 \leq t_2$. Given this we know that $\lim_{t \rightarrow 0^+} F_t$ exists and $F_0 \subset \lim_{t \rightarrow 0^+} F_t$. Next, by the definition of $\psi(z, t)$ as the underestimate for $|sgn(z)|$, we know $\lim_{t \rightarrow 0} \psi(z, t) = |sgn(z)|$. This gives, by summation over components of α , $\lim_{t \rightarrow 0} \phi(\alpha, t) = \|\alpha\|_0$. This, therefore, means $\lim_{t \rightarrow 0} F_t = F_0$ because of the inclusion property proved above. Now, the objective function, being an L2 norm of a linear function of the optimizing variable, is convex. Thus, the problem P_1 with a feasible set that is the superset of another problem P_2 with the same objective function can potentially achieve a lower optimal value than P_2 . Thus, if $v(P)$ denotes the optimal value of the problem P , we have $v(P_t) \leq v(P_0)$. Now, since we proved that the constraint of P_t converges to the constraint of P_0 as $t \rightarrow 0$, we have $\lim_{t \rightarrow 0} v(P_t) = v(P)$.

Next, the $u(\alpha, \alpha_0, \xi, t)$ was constructed for the DC function $\phi(\alpha, t)$ around the point α_0 , with $\phi(\alpha, t) \leq u(\alpha, \alpha_0, \xi, t)$. Following the same logic as above, the feasible sets of CP_t , given by

$$F_t(\alpha_0, \xi) = \{\alpha \in \mathbb{R}^n : u(\alpha, \alpha_0, \xi, t) \leq K\}$$

follow $F_t(\alpha_0, \xi) \subseteq F_t$ for all $t > 0$, $\xi \in \partial\phi(\alpha_0, t)$. This means, $v(P_t(\alpha_0, \xi)) \geq v(P_t)$ for all $t > 0$, $\xi \in \partial\phi(\alpha_0, t)$.

We further relax the non-smoothness of the constraint in $P_t(\alpha_0, \xi)$ by introducing additional variables z_i , to form the problem $CP_t(\alpha_0, \xi)$. This step does not change the problem and forces $z_i = |\alpha_i|$. The constraint introduced, $-z_i \leq \alpha_i \leq z_i$ is equivalent to saying $|\alpha_i| \leq z_i$. The feasible sets over x for $CP_t(\alpha_0, \xi)$, are, then, smaller subsets of the feasible sets over x for $P_t(\alpha_0, \xi)$ for any $|x_i| < z_i$ (since the objective function is a non-decreasing function as α_i moves away from the optimum). Hence, the minimum always lies in the difference of the feasible sets for $P_t(\alpha_0, \xi)$ and $CP_t(\alpha_0, \xi)$, except when $|x_i| = z_i$, when both minima are equal. Thus, if there exists a solution to $P_t(\alpha_0, \xi)$, given by x^* , there is a solution to $CP_t(\alpha_0, \xi)$, given by x^*, z^* where $z_i = |x_i|$. Thus $P_t(\alpha_0, \xi)$ and $CP_t(\alpha_0, \xi)$ are equivalent optimization problems. We use the fact that $CP_t(\alpha_0, \xi)$ is smooth to derive a condition on the optimum for $CP_t(\alpha_0, \xi)$, and therefore the optimum of $P_t(\alpha_0, \xi)$.

Now that we have shown the relation between P and P_t , we will analyse the optimum of P_t . Let α^* be a local optimum for P_t . The KKT conditions (formulated with respect to the sub-gradient) imply that there exists a non-negative μ^* such that

$$0 \in 2A^T A\alpha^* + A^T y + \mu^* \partial\phi(\alpha^*, t),$$

$$\mu^*(\phi(\alpha^*, t) - K) = 0$$

A point α^* satisfying these is called a KKT point for the problem P_t .

The corresponding conditions for the problem $CP_t(\alpha_0, \xi)$ can be written, with $(\tilde{\alpha}, \tilde{z})$ solving $CP_t(\alpha_0, \xi)$, there exist non-negative multipliers $\tilde{\zeta}, \tilde{\beta}, \tilde{\mu}$ such that

$$2A^T A \tilde{\alpha} + A^T y + \tilde{\zeta} - \tilde{\beta} - \frac{1}{t} \tilde{\mu} \xi = 0,$$

$$\frac{1}{t} \tilde{\mu} e - \tilde{\zeta} - \tilde{\beta} = 0,$$

$$\tilde{\mu} \left(\frac{1}{t} e^T \tilde{z} - \frac{1}{t} g(\alpha_0, t) - K \right) = 0,$$

$$\tilde{\zeta}_i (z_i - \tilde{\alpha}_i) = 0,$$

$$\tilde{\beta}_i (z_i + \tilde{\alpha}_i) = 0$$

We already showed that $\tilde{\alpha}_i = \tilde{z}_i$. This implies, $\tilde{\alpha}_i > 0 \implies \tilde{\beta}_i = 0$ and $\tilde{\alpha}_i < 0 \implies \tilde{\zeta}_i = 0$. Thus, $\tilde{\zeta}_i = (1/t) \tilde{\mu}$ when $\tilde{\alpha}_i > 0$ and $\tilde{\beta}_i = (1/t) \tilde{\mu}$ when $\tilde{\alpha}_i < 0$. When $\tilde{\alpha}_i = 0$, $\tilde{\zeta} - \tilde{\beta} \in [-(1/t) \tilde{\mu}, (1/t) \tilde{\mu}]$. This implies that $\tilde{\zeta} - \tilde{\beta} - (1/t) \tilde{\mu} \xi \in \tilde{\mu} \partial \phi(\tilde{\alpha}_i, t)$. Then, the first equation in the $CP_t(\alpha_0, \xi)$ KKT conditions implies the first equation in the P_t KKT conditions, and the constraint conditions in $CP_t(\alpha_0, \xi)$ give the constraint conditions in P_t , if $\tilde{\alpha} = \alpha_0$. Thus, we prove that if there exists a solution $(\alpha_0, |\alpha_0|)$ to $CP_t(\alpha_0, \xi)$ for some $\xi \in g(\alpha_0, t)$, that is the solution for P_t .

With this, we can generate a series of subproblems by solving $CP_t(\alpha^k, \xi^k)$, and setting the solution to α^{k+1} . If the difference $\alpha^{k+1} - \alpha^k$ is small (in terms of some norm), we can say that α^k approximately solves $CP_t(\alpha^k, \xi^k)$. By the above proof, we have that α^k is a KKT point for P_t . As we proved at the beginning of the section, the problem P_t converges to the original compressed sensing problem P as $t \rightarrow 0$. This means, choosing a small t we almost exactly solve the compressed sensing problem with our successive convex approximations.

5.2 PD Methods

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6 Discussion

6.1 What is it that SCA considers but OMP ignores?

OMP is a greedy algorithm, that is, it picks the ‘best’ matching atom for the residual at each stage as it goes through the dictionary. While the OMP output is optimal among the subset of atoms that OMP chooses, there is no guarantee that the subset of atoms OMP picks is optimal. In that sense, OMP doesn’t look at all the vectors and the entire data ‘jointly’. SCA, on the other hand, operates on coefficients of all atoms (the α s) together, and hence is able to guarantee both optimality of the subset picked and the optimality of the output image within the picked subset.

6.2 What is it that PD considers but OMP ignores?

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7 Conclusions and Learning

We presented new methods for replacing the current heuristic-like OMP method for solving the compressed sensing problem. We showed our method can produce better results at higher compression levels in measurements. Using our method, it is possible to cut down on exposure time for a frame (number of measurements being less), and hence, image events that are fast-changing sharply. This can be used to create high spatial- and time-resolution videos at the same time. Because of the constraint function being a better approximation of the L0 norm than the approximation using just the L1 norm, our methods are expected to perform better even than methods using the L1 norm. The SCA method could be further improved by ‘annealing’ t , that is, first choosing a large value to come close to the L1 norm, (which is convex and hence tractable), and slowly decreasing t to reach the L0 norm problem.

8 Acknowledgements

The authors would like to thank the course instructor for having given them the opportunity to explore applications of convex optimization in their fields of interest. They would also like to acknowledge the importance of the Carnegie Mellon University Linux Time-share Pool in creating the results presented in this paper. This would not have been possible without that much computing power.

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