CS663 Assignment-4 Question-3

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Part A

Consider a matrix $\mathbf{A}$ of size $m \times n$. Define $\mathbf{P} = \mathbf{A}^T\mathbf{A}$ and $\mathbf{Q} = \mathbf{A}\mathbf{A}^T$.

Now, if $\mathbf{A}$ has dimensions $m \times n$, $\mathbf{P}$ has dimensions $n \times n$. Hence if we want to evaluate $\mathbf{y}^T\mathbf{P}\mathbf{y}$ for a (column) vector $\mathbf{y}$, $\mathbf{y}$ must have dimensions $n \times 1$. This means the product $\mathbf{y}^T\mathbf{P}\mathbf{y}$ will be a scalar.

Now we have

$$\mathbf{y}^T\mathbf{P}\mathbf{y} = \mathbf{y}^T\mathbf{A}^T\mathbf{A}\mathbf{y}$$

Putting $\mathbf{Z} = \mathbf{A}\mathbf{y}$, and noticing that $\mathbf{Z}^T = \mathbf{y}^T\mathbf{A}^T$, we have

$$\mathbf{y}^T\mathbf{P}\mathbf{y} = \mathbf{Z}^T\mathbf{Z}$$

Thus the given expression is the dot product of a vector with itself, which is always non-negative. Thus,

$$\mathbf{y}^T\mathbf{P}\mathbf{y} \geq 0$$

Similarly, we have for a $n \times 1$ vector $\mathbf{z}$, with $\mathbf{Y} = \mathbf{A}\mathbf{z}$

$$\mathbf{z}^T\mathbf{Q}\mathbf{z} = \mathbf{Y}^T\mathbf{Y}$$

which is again the dot product of a vector with itself, which is always non-negative. Thus,

$$\mathbf{z}^T\mathbf{Q}\mathbf{z} \geq 0$$

Now consider an eigenvector $\mathbf{z}$ of $\mathbf{A}^T\mathbf{A}$, where $\mathbf{z}$ is a $m \times 1$ vector. We must have for some λ

$$\mathbf{A}^T\mathbf{A}\mathbf{z} = \lambda\mathbf{z}$$

A pre-multiplication by $\mathbf{z}^T$ yields

$$\mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z} = \mathbf{z}^T\lambda\mathbf{z}$$

which means

$$\mathbf{z}^T\mathbf{A}^T\mathbf{A}\mathbf{z} = \lambda\mathbf{z}^T\mathbf{z}$$

Now the left hand side is positive, as proved above. $\mathbf{z}^T\mathbf{z}$ is positive, because it is the dot product of a vector with itself. Hence, λ must be non-negative.

Similarly for $\mathbf{A}\mathbf{A}^T$, consider an eigenvector $\mathbf{z}$ of dimension $m\times 1$. Multiplying the eigenvalue equation on the left by $\mathbf{z}^T$ we get for some λ

$$\mathbf{z}^T\mathbf{A}\mathbf{A}^T\mathbf{z} = \lambda\mathbf{z}^T\mathbf{z}$$

which again means λ is non-negative.

Hence the eigenvalues of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}\mathbf{A}^T$ are non-negative.

Part B

If $\mathbf{u}$ is an eigenvector of $\mathbf{P}$ with eigenvalue λ, we have

$$\mathbf{A}^T\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$$

Pre-multiply by $\mathbf{A}^T$ to give

$$\mathbf{A}\mathbf{A}^T\mathbf{A}\mathbf{u} = \mathbf{A}\lambda\mathbf{u}$$

implying

$$\mathbf{A}\mathbf{A}^T\mathbf{A}\mathbf{u} = \lambda\mathbf{A}\mathbf{u}$$

Thus $\mathbf{Au}$ is an eigenvector of $\mathbf{Q}$ with eigenvalue λ.

Similarly, if $\mathbf{v}$ is an eigenvector of $\mathbf{Q}$ with eigenvalue μ, we have by pre-multiplying with $\mathbf{A}^T$

$$\mathbf{A}^T\mathbf{A}\mathbf{A}^T\mathbf{v} = \mu\mathbf{A}^T\mathbf{v}$$

Thus $\mathbf{A}^T\mathbf{v}$ is an eigenvector of $\mathbf{P}$ with eigenvalue μ. Clearly $\mathbf{u}$ has n elements and $\mathbf{v}$ has m elements.

Part C

If $\mathbf{v}_i$ is an eigenvector of $\mathbf{Q}$, we have

$$\mathbf{A}\mathbf{A}^T\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

Defining $\mathbf{u}_i$ as

$$\mathbf{u}_i = \frac{\mathbf{A}^T\mathbf{v}_i}{||\mathbf{A}^T\mathbf{v}_i||}$$

and substituting it in the previous equation yields

$$\mathbf{A}\mathbf{u}_i = \frac{\lambda_i}{||\mathbf{A}^T\mathbf{v}_i||}\mathbf{v}_i$$

Define γ_i as

$$\gamma_i = \frac{\lambda_i}{||\mathbf{A}^T\mathbf{v}_i||}$$

Clearly γ_i are non-negative because λ_i are non-negative and the denominator being a norm is non-negative as well. Thus there exist some real, non-negative γ_i which satisfy

$$\mathbf{A}\mathbf{u}_i = \gamma_i\mathbf{u}_i$$

Part D

Consider the product $\mathbf{U}\Gamma\mathbf{V}^T$ with the matrices as defined in the question. We have

$$\mathbf{U}\Gamma = [\mathbf{v}_1|\mathbf{v}_2|\cdots|\mathbf{v}_m]\begin{pmatrix}\gamma_1 & 0 & \cdots & 0\\ 0 & \gamma_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \gamma_m\end{pmatrix}$$

Thus

$$\mathbf{U}\Gamma = [\gamma_1\mathbf{v}_1|\gamma_2\mathbf{v}_2|\cdots|\gamma_m\mathbf{v}_m]$$

From the previous part each $\gamma_i\mathbf{v}_i$ can be written as

$$\gamma_i\mathbf{v}_i = \mathbf{A}\mathbf{u}_i$$

Thus

$$\mathbf{U}\Gamma = [\mathbf{A}\mathbf{u}_1|\mathbf{A}\mathbf{u}_2|\cdots|\mathbf{A}\mathbf{u}_n]$$

which means

$$\mathbf{U}\Gamma\mathbf{V}^T = [\mathbf{A}\mathbf{u}_1|\mathbf{A}\mathbf{u}_2|\cdots|\mathbf{A}\mathbf{u}_n][\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]^T$$

implying

$$\mathbf{U}\Gamma\mathbf{V}^T = \mathbf{A}[\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n][\mathbf{u}_1|\mathbf{u}_2|\cdots|\mathbf{u}_n]^T$$

Now given that

$$\mathbf{u}_i\mathbf{u}_j = \delta_{ij}$$

we have

$$A = \mathbf{U}\Gamma\mathbf{V}^T$$

where Γ is a matrix that has the non-negative γ_i on its diagonal elements and zeros on the off-diagonal elements. Thus we are done.