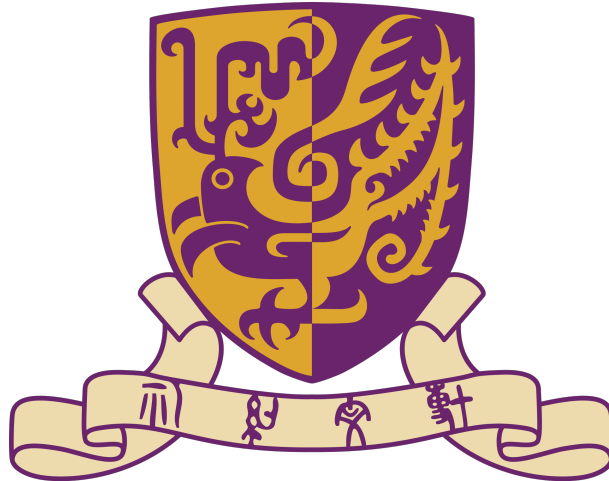


New Approach of Numerical Relativity

Implementation, tests and applications



Alan Tsz-Lok Lam

Department of Physics
The Chinese University of Hong Kong

This thesis is submitted for the degree of
Master of Philosophy in Physics

July 2021

I would like to dedicate this thesis to my loving parents . . .

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Alan Tsz-Lok Lam
July 2021

Acknowledgements

And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

Table of contents

List of figures	xv
List of tables	xvii
I Methodology: Formulations	1
1 Formulations of Einstein Field Equations	3
1.1 Introduction to general relativity	3
1.1.1 Einstein Field Equation	3
1.1.2 Gravitational Wave	3
1.2 The 3+1 decomposition of spacetime	3
1.2.1 Foliation of spacetime	3
1.2.2 Derivative operator	5
1.2.3 Extrinsic curvature	6
1.2.4 The Gauss, Codazzi and Ricci equations	7
1.2.5 Constraint and evolution equations	7
1.2.6 The Arnowitt, Deser and Misner equations	9
1.3 Conformal Decomposition	10
1.3.1 Conformal transformation of the spatial metric	10
1.3.2 Conformal transformation of the extrinsic curvature	11
1.3.3 Conformal transverse-traceless decomposition	13
1.4 Gauge Condition	14
1.4.1 Maximal Slicing	14
1.4.2 Generalized Dirac Gauge	16
1.5 Constrained scheme for the Einstein equations	17
1.5.1 Isenberg–Wilson–Mathews approximation	18
1.5.2 The Fully Constrained Formulation	20

2	Formulations of the Relativistic Hydrodynamics	23
2.1	Hydrodynamics	23
2.1.1	Perfect fluid	23
2.1.2	Rest-mass conservation	23
2.1.3	Energy and momentum conservation	23
2.1.4	Conservative form of hydrodynamics equations	23
2.1.5	Conserved to Primitive variables conversion	23
2.2	The reference-metric formulism	23
II	Methodology: Numerical methods	25
3	Numerical Methods	27
3.1	Multigrid Method for elliptic equations	28
3.1.1	Overview	28
3.1.2	Smoother	28
3.1.3	Restriction and Prolongation	28
3.1.4	Multigrid cycle	28
3.1.5	Grid Structure	28
3.1.6	Ghost cells and refinement boundary	28
3.2	Elliptic system in fully constrained formulation	28
3.2.1	Elliptic divergence cleaning for h^{ij}	28
3.2.2	Maximal slicing condition	28
3.2.3	Elliptic sector	28
3.3	High-resolution shock-capturing (HRSC) methods	28
3.3.1	Approximate Riemann solver	28
3.3.2	Reconstruction scheme	28
3.3.3	Finite volume discretization in curvilinear coordinate	28
3.4	Time discretization	28
3.4.1	Explicit Runge-Kutta Method	28
3.4.2	Courant-Friedrichs <i>Lewy</i> Conditions	28
3.4.3	Hyperbolic system in fully constrained formulation	28
3.4.4	Time Discretization	28
4	Numerical Tests and Result	29
4.1	Multigrid solver tests	29

4.1.1	Newtonian gravitational field	29
4.2	General relativistic hydrodynamics in dynamic spacetime	29
4.2.1	Non-rotating neutron star	29
4.2.2	Rapidly rotating neutron star	29
References		31
Appendix A Useful relations for implementation of constrained scheme		35
A.1	The elliptic equations in constrained scheme	35
A.2	Generalized Dirac gauge conditions	35
Appendix B Reference flat metric in 3D		37

List of figures

List of tables

Part I

Methodology: Formulations

Chapter 1

Formulations of Einstein Field Equations

1.1 Introduction to general relativity

1.1.1 Einstein Field Equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

1.1.2 Gravitational Wave

1.2 The 3+1 decomposition of spacetime

Due to the complexity and nonlinearity of Einstein field equations, it is extremely difficult to obtain analytical solution even for the simplest dynamical evolution systems. Therefore, the accurate discription of the such systems can only be derived through numerical simulation. For this, we need to reformulate the Einstein equations as an initial-value problem or Cauchy problem. In this chapter, we will introduction the Arnowitt-Deser-Misner (ADM) formulation, which is the foundation of the 3+1 numerical relativity. In particular, we will focus on the constrained scheme for the Einstein equations.

1.2.1 Foliation of spacetime

In the 3+1 decomposition, the spacetime manifold \mathcal{M} is foliated into a set of non-intersecting spacelike hypersurfaces Σ_t parameterized by the coordinate time t [20]. We

denote a future-directed timelike unit four-vector n^μ normal to the hypersurface Σ_t (i.e. $n_\mu \propto \nabla_\mu t$). The induced spacetime metric $\gamma_{\mu\nu}$ on each hypersurface can then be defined as

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu. \quad (1.2)$$

Thus, we can construct spatial projection tensor $\gamma^\mu{}_\nu$ and time projection tensor $N^\mu{}_\nu$ as

$$\gamma^\mu{}_\nu := \delta^\mu{}_\nu + n^\mu n_\nu, \quad N^\mu{}_\nu := -n^\mu n_\nu, \quad (1.3)$$

which decompose any generic four-vector U^μ into spatial part $\gamma^\mu{}_\nu U^\nu$ and timelike part $N^\mu{}_\nu U^\nu$. Therefore, we can decompose the timelike vector field

$$t^\mu = \alpha n^\mu + \beta^\mu \quad (1.4)$$

into two components as

$$\alpha := -t^\mu n_\mu, \quad \beta^\mu := t^\nu \gamma^\mu{}_\nu, \quad (1.5)$$

where the lapse function α measures the physical proper time ($\alpha \Delta t$) between two neighboring spatial hypersurface Σ_t and $\Sigma_{t+\Delta t}$, and the shift vector β^i measures the changes of spatial coordinates on $\Sigma_{t+\Delta t}$.

Here, we summarise several useful relations. The timelike normal vector n^μ and its corresponding one-form n_μ can be expressed as

$$n^\mu = \frac{1}{\alpha} (1, \beta^i), \quad n_\mu = (\alpha, \vec{0}). \quad (1.6)$$

The generic line element in 3+1 decomposition is given by

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + \beta_i dx^i dt + \gamma_{ij} dx^i dx^j \quad (1.7)$$

The covariant and contravariant components of the metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^j \\ \beta^i & \gamma^{ij} \end{pmatrix}. \quad (1.8)$$

From equation(1.8), we can conclude that

$$\sqrt{-g} = \alpha \sqrt{\gamma}, \quad (1.9)$$

where $g := \det(g_{\mu\nu})$ and $\gamma := \det(\gamma_{ij})$.

1.2.2 Derivative operator

With the 3+1 decomposition, we can now construct the 3-dimensional covariant derivative D_α associated with $\gamma_{\mu\nu}$ by projecting the 4-dimensional covariant derivative ∇_α onto Σ_t , which is given by

$$D_\alpha T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} = \gamma_\alpha^\beta \gamma_{\rho_1}^{\mu_1} \gamma_{\rho_2}^{\mu_2} \cdots \gamma_{\nu_1}^{\sigma_1} \gamma_{\nu_2}^{\sigma_2} \cdots \nabla_\beta T^{\rho_1\rho_2\cdots}_{\sigma_1\sigma_2\cdots}, \quad (1.10)$$

for arbitrary tensor $T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ on spatial hypersurface Σ_t . Using equation(1.10), it can be shown that the covariant derivative of $\gamma_{\mu\nu}$ vanishes

$$\begin{aligned} D_\alpha \gamma_{\mu\nu} &= \gamma_\alpha^\beta \gamma_\rho^\mu \gamma_\nu^\sigma \nabla_\beta (g_{\rho\sigma} + n_\rho n_\sigma) \\ &= \gamma_\alpha^\beta \gamma_\rho^\mu \gamma_\nu^\sigma (n_\rho \nabla_\beta n_\sigma + n_\sigma \nabla_\beta n_\rho) = 0 \end{aligned} \quad (1.11)$$

The components of 3-dimensional connection coefficients $\Gamma^\alpha_{\mu\nu}$ in coordinate basis can be expressed as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \gamma^{\alpha\beta} (\partial_\nu \gamma_{\beta\mu} + \partial_\mu \gamma_{\beta\nu} - \partial_\beta \gamma_{\mu\nu}). \quad (1.12)$$

Here, the upper left index ⁽⁴⁾ marks the 4-dimensional tensors while the unmarked one represents purely spatial 3-dimensional tensors. Similarly, the 3-dimensional Riemann tensor $R^\alpha_{\beta\mu\nu}$ associated with $\gamma_{\mu\nu}$ is defined by requiring that

$$2D_{[\nu} D_{\mu]} W_\beta = W_\alpha R^\alpha_{\beta\mu\nu}, \quad R^\alpha_{\beta\mu\nu} n_\alpha = 0, \quad (1.13)$$

which can be explicitly expressed in coordinate basis as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\nu\rho} \Gamma^\rho_{\beta\mu}. \quad (1.14)$$

The 3-dimensional Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined in a similar manner as their 4-dimensional counterparts

$$R_{\mu\nu} := R^\alpha_{\mu\alpha\nu}, \quad R := R^\mu_{\mu}. \quad (1.15)$$

Since $R^\alpha_{\beta\mu\nu}$ is purely spatial and can be computed by the spatial derivatives of the spatial metric alone, it only contains about information about the curvature intrinsic to the

hypersurface Σ_t , but cannot contain all the information of ${}^{(4)}R^\alpha{}_{\beta\mu\nu}$ which includes time derivative of the 4-dimensional metric. The missing information can be found in a purely spatial symmetric tensor called the extrinsic curvature $K_{\mu\nu}$.

1.2.3 Extrinsic curvature

The extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the spatial metric $\gamma_{\mu\nu}$. Therefore, the spatial metric and extrinsic curvature $(\gamma_{\mu\nu}, K_{\mu\nu})$ are equivalent to the positions and velocities in classical mechanics, which describe the instantaneous state of the gravitational field. It can be obtained by projecting of the gradient of the normal vector $\gamma_\mu{}^\lambda \gamma_\nu{}^\rho \nabla_\lambda n_\rho$ into the hypersurface Σ_t , and then taking the negative expression of the symmetric part

$$\begin{aligned} K_{\mu\nu} &:= -\gamma_\mu{}^\lambda \gamma_\nu{}^\rho \nabla_\lambda n_\rho \\ &= -\gamma_\mu{}^\lambda (\delta_\nu{}^\rho + n_\nu n^\rho) \nabla_\lambda n_\rho \\ &= -\gamma_\mu{}^\lambda \nabla_\lambda n_\nu, \end{aligned} \tag{1.16}$$

where the identity $n^\rho \nabla_\lambda n_\rho = 0$ is used.

We can also define an spatial acceleration a_ν

$$a_\nu := n^\mu \nabla_\mu n_\nu, \tag{1.17}$$

satisfying the identities

$$a_\nu = D_\nu \ln \alpha, \tag{1.18}$$

to rewrite equation(1.16) as

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu \tag{1.19}$$

Finally, we can write the extrinsic curvature $K_{\mu\nu}$ as the Lie derivative of the spatial metric along the local normal n^μ

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}. \tag{1.20}$$

Using equation(1.4), we can express the Lie derivative \mathcal{L}_n as

$$\mathcal{L}_n = \frac{1}{\alpha} (\mathcal{L}_t - \mathcal{L}_\beta), \quad (1.21)$$

and thus obtain the evolution equation for the spatial metric

$$\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}. \quad (1.22)$$

1.2.4 The Gauss, Codazzi and Ricci equations

To express the Einstein field equations in term of the spatial variables $(\gamma_{\mu\nu}, K_{\mu\nu})$ we defined previous, we first have to relate 3-dimensional Riemann tensor $R^\alpha_{\beta\mu\nu}$ on Σ_t to the 4-dimensional Riemann tensor ${}^{(4)}R^\alpha_{\beta\mu\nu}$ on \mathcal{M} , The relation between $R^\alpha_{\beta\mu\nu}$ and the full spatial projection of ${}^{(4)}R^\alpha_{\beta\mu\nu}$ is given by the *Gauss' equation*

$$R_{\alpha\beta\mu\nu} + K_{\alpha\mu}K_{\beta\nu} - K_{\alpha\nu}K_{\beta\mu} = \gamma_\alpha{}^\rho \gamma_\beta{}^\sigma \gamma_\mu{}^\lambda \gamma_\nu{}^\delta {}^{(4)}R_{\rho\sigma\lambda\delta}, \quad (1.23)$$

while the projection of ${}^{(4)}R^\alpha_{\beta\mu\nu}$ with one index projected in the normal direction is given by the *Codazzi equation*

$$D_\nu K_{\mu\alpha} - D_\mu K_{\nu\alpha} = \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \gamma_\alpha{}^\lambda n^\delta {}^{(4)}R_{\rho\sigma\lambda\delta}. \quad (1.24)$$

Finally, by projecting two indices of ${}^{(4)}R_{\rho\sigma\lambda\delta}$ in the normal direction, we can relate it to the time derivative of $K_{\mu\nu}$

$$\mathcal{L}_n K_{\mu\nu} = n^\alpha n^\beta \gamma_\mu{}^\lambda \gamma_\nu{}^\delta {}^{(4)}R_{\alpha\beta\lambda\delta} - \frac{1}{\alpha} D_\mu D_\nu \alpha - K_\nu{}^\lambda K_{\mu\lambda}, \quad (1.25)$$

which is called the *Ricci equation*.

1.2.5 Constraint and evolution equations

Using the Gauss, Codazzi and Ricci equations, the Einstein fields equations can be decomposed into a set of evolution equations and a set of constraint equations of $(\gamma_{\mu\nu}, K_{\mu\nu})$.

To begin with, we define the following matter quantities

$$S_{\mu\nu} := \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}, \quad (1.26)$$

$$S_\mu := -\gamma^\alpha_\mu n^\beta T_{\alpha\beta}, \quad (1.27)$$

$$S := S^\mu_\mu, \quad (1.28)$$

$$E := n^\alpha n^\beta T_{\alpha\beta}, \quad (1.29)$$

which decompose the stress-energy tensor as

$$T_{\mu\nu} = En_\mu n_\nu + S_\mu n_\nu + S_\nu n_\mu + S_{\mu\nu}. \quad (1.30)$$

By contracting the α, μ indices in equation(1.23), we can obtain

$$R_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu \left({}^{(4)}R_{\alpha\beta} + n^\rho n^\sigma {}^{(4)}R_{\alpha\rho\beta\sigma} \right) + K_{\mu\lambda} K_\nu{}^\lambda - K_{\mu\nu} K, \quad (1.31)$$

where $K := K^\mu_\mu$ is the trace of the extrinsic curvature, called the *mean curvature*. Further contracting the μ, ν indices in equation(1.31), the contracted Gauss' equation becomes

$$2n^\mu n^\nu G_{\mu\nu} = R + K^2 - K_{\mu\nu} K^{\mu\nu}, \quad (1.32)$$

Using the Einstein equation(1.1), we can obtain the *Hamiltonian constraint*

$$R + K^2 - K_{\mu\nu} K^{\mu\nu} = 16\pi E. \quad (1.33)$$

Similarly, by contracting α, ν indices in equation(1.24), the Codazzi equation yields

$$D_\nu K_\mu{}^\nu - D_\mu K = -8\pi \gamma^\alpha_\mu n^\beta T_{\alpha\beta}, \quad (1.34)$$

and thus obtain the *momentum constrain equation*

$$D_\nu K_\mu{}^\nu - D_\mu K = 8\pi S_\mu. \quad (1.35)$$

The Ricci equation (1.25) can be rewritten using equation(1.31) to

$$\mathcal{L}_n K_{\mu\nu} = R_{\mu\nu} - \gamma^\rho_\mu \gamma^\sigma_\nu {}^{(4)}R_{\rho\sigma} - 2K_{\mu\lambda} K_\nu{}^\lambda + K K_{\mu\nu} - \frac{1}{\alpha} D_\mu D_\nu \alpha. \quad (1.36)$$

Using the Einstein equations (1.1)

$$\begin{aligned}\gamma_\mu{}^\rho \gamma_\nu{}^\sigma {}^{(4)}R_{\rho\sigma} &= 8\pi \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \left(T_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} T^\mu{}_\mu \right) \\ &= 8\pi \left[S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} (S - E) \right]\end{aligned}\tag{1.37}$$

and equation(1.21), we can finally obtain the evolution equation for $K_{\mu\nu}$ as

$$\mathcal{L}_t K_{\mu\nu} = -D_\mu D_\nu \alpha + \alpha (R_{\mu\nu} - 2K_{\mu\lambda} K_\nu{}^\lambda + K K_{\mu\nu}) - 8\pi \alpha \left[S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} (S - E) \right] + \mathcal{L}_\beta K_{\mu\nu}.\tag{1.38}$$

1.2.6 The Arnowitt, Deser and Misner equations

The Lie derivative in the evolution equations (1.22) and (1.38) can be expressed in terms of coordinate basis as

$$\mathcal{L}_t K_{\mu\nu} = \partial_t K_{\mu\nu}\tag{1.39}$$

$$\mathcal{L}_\beta K_{\mu\nu} = \beta^\lambda D_\lambda K_{\mu\nu} + K_{\mu\lambda} D_\nu \beta^\lambda + K_{\lambda\nu} D_\mu \beta^\lambda\tag{1.40}$$

As the result, the Einstein field equations (1.1) in the standard 3+1 decomposition can be decomposed into a set of constraint equations and evolution equation of (γ_{ij}, K_{ij}) in terms of coordinate basis, which are referred to as the Arnowitt, Deser and Misner (ADM) equations [1, 35]

$$R + K^2 - K_{ij} K^{ij} = 16\pi E, \tag{Hamiltonian constraint} \tag{1.41}$$

$$D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i, \tag{momentum constraint} \tag{1.42}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \tag{spatial metric evolution} \tag{1.43}$$

$$\begin{aligned}\partial_t K_{ij} &= -D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K_j{}^k + K K_{ij}) \\ &\quad - 8\pi \alpha \left[S_{ij} - \frac{1}{2} \gamma_{ij} (S - E) \right] \\ &\quad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k.\end{aligned}\tag{extrinsic curvature evolution} \tag{1.44}$$

1.3 Conformal Decomposition

The conformal decomposition factors out a scalar component from a spatial metric. It was first developed for initial data problems in general relativity [19, 33, 34, 36], and then used in reformulating evolution equations in the 3+1 formulation. In this section, we will discuss the conformal decomposition of spatial metric and extrinsic curvature in numerical relativity.

1.3.1 Conformal transformation of the spatial metric

We consider the conformal transformation of the spatial metric γ_{ij} as

$$\tilde{\gamma}_{ij} = \psi^{-4} \gamma_{ij}, \quad (1.45)$$

where $\tilde{\gamma}_{ij}$ is the *conformal metric* and ψ is a positive scaling factor satisfying

$$\psi := \det \left(\frac{\gamma}{f} \right), \quad \gamma := \det (\gamma_{ij}), \quad f := \det (f_{ij}) \quad (1.46)$$

for a time independent flat metric f_{ij} (i.e. $\det (\tilde{\gamma}_{ij}) = f$ by construction).

Thus, the *inverse conformal metric* is given by

$$\tilde{\gamma}^{ij} := \psi^4 \gamma^{ij}. \quad (1.47)$$

Substituting the conformal transformation (1.45) into equation (1.12), we can obtain the transformation law for 3-dimensional connection coefficient

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + 2 \left(\delta^i_j \tilde{D}_k \ln \psi + \delta^i_k \tilde{D}_j \ln \psi - \tilde{\gamma}_{jk} \tilde{\gamma}^{il} \tilde{D}_l \ln \psi \right). \quad (1.48)$$

From now on, we denote all objects associated with the conformal metric $\tilde{\gamma}^{ij}$ with a tilde symbol. Similarly, the transformation for Ricci tensor and scalar curvature are given by

$$\begin{aligned} R_{ij} = \tilde{R}_{ij} - 2 \left(\tilde{D}_i \tilde{D}_j \ln \psi + \tilde{\gamma}_{ij} \tilde{\gamma}^{lm} \tilde{D}_l \tilde{D}_m \ln \psi \right) \\ + 4 \left[\left(\tilde{D}_i \ln \psi \right) \left(\tilde{D}_j \ln \psi \right) - \tilde{\gamma}_{ij} \tilde{\gamma}^{lm} \tilde{D}_l \left(\tilde{D}_m \ln \psi \right) \right] \end{aligned} \quad (1.49)$$

$$R = \psi^{-4} \tilde{R} - 8 \psi^{-5} \tilde{D}^2 \psi, \quad (1.50)$$

where $\tilde{D}^2 = \tilde{\gamma}^{ij} \tilde{D}_i \tilde{D}_j$ denotes the Laplace operator associated with $\tilde{\gamma}_{ij}$. Therefore, using equation(1.50), the Hamiltonian constraint (1.41) becomes

$$8\tilde{D}^2\psi - \psi\tilde{R} - \psi^5 K^2 + \psi^2 K_{ij} K^{ij} = -16\pi\psi^5 E. \quad (1.51)$$

1.3.2 Conformal transformation of the extrinsic curvature

Traceless decomposition

Before we perform the conformal transformation to the extrinsic curvature K_{ij} , it is convenient to split K_{ij} into the trace part

$$K := \gamma^{ij} K_{ij}, \quad (1.52)$$

and its traceless part

$$A_{ij} := K_{ij} - \frac{1}{3}\gamma_{ij}K, \quad \text{tr}_\gamma A_{ij} = \gamma^{ij}A_{ij} = 0. \quad (1.53)$$

Therefore, we can obtain the traceless decomposition of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \quad K^{ij} = A^{ij} + \frac{1}{3}\gamma^{ij}K. \quad (1.54)$$

The evolution equations of the spatial metric (1.43) in conformal decomposition formulation can hence be written as

$$\partial_t \psi = \beta^i \tilde{D}_i \psi - \frac{1}{6}\psi \left(\alpha K - \tilde{D}_i \beta^i \right), \quad (\text{conformal factor evolution}) \quad (1.55)$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha\psi^{-4} A_{ij} + \tilde{\gamma}_{jk} \tilde{D}_i \beta^k + \tilde{\gamma}_{ik} \tilde{D}_j \beta^k - \frac{2}{3}\tilde{\gamma}_{ij} \tilde{D}_k \beta^k, \quad (\text{conformal metric evolution}) \quad (1.56)$$

and the constraint equations become

$$\tilde{D}^2\psi - \frac{1}{8}\psi\tilde{R} + \left(\frac{1}{8}A_{ij}A^{ij} - \frac{1}{12}K^2 + 2\pi E \right) \psi^5 = 0, \quad (\text{Hamiltonian constraint}) \quad (1.57)$$

$$\tilde{D}_i (\psi^{10} A^{ij}) - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi\psi^{10} S^i. \quad (\text{momentum constraint}) \quad (1.58)$$

Conformal transformation of the traceless part

We consider the transformation

$$A^{ij} := \psi^a \tilde{A}^{ij}, \quad (1.59)$$

for some undetermined exponent α . Here, we discuss two natural choices of a : $a = -4$ and $a = -10$.

“Time-evolution” scaling: $a = -4$. This choice of scaling was considered by Nakamura in 1994 [22], It comes naturally from the evolution equation of conformal metric (1.56), where the $\psi^{-4}A_{ij}$ term suggests the conformal transformation of A_{ij} to have the same scaling factor as the conformal spatial metric (1.47)

$$\tilde{A}^{ij} := \psi^4 A^{ij}, \quad (1.60)$$

where the indices of \tilde{A}^{ij} and \tilde{A}_{ij} are lowered and raised by the conformal metric $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ respectively (i.e. $\tilde{A}_{ij} = \tilde{\gamma}_{il}\tilde{\gamma}_{jm}\tilde{A}^{lm} = \psi^{-4}A_{ij}$). The evolution equations of conformal spatial metric therefore become

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{jk} \tilde{D}_i \beta^k + \tilde{\gamma}_{ik} \tilde{D}_j \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_k \beta^k. \quad (1.61)$$

The Hamiltonian constraint and momentum constraint in this scaling is rewritten as

$$\tilde{D}^2 \psi = \frac{1}{8} \psi \tilde{R} - \left(2\pi E - \frac{1}{12} K^2 + \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} \right) \psi^5, \quad (1.62)$$

$$\tilde{D}_i \left(\psi^6 \tilde{A}^{ij} \right) - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i. \quad (1.63)$$

“Momentum-constraint” scaling: $a = -10$. Another possible choice of scaling factor $a = -10$ originates from the momentum constraint equation (1.58), which was first suggested by Lichnerowicz in 1944 [19]. we define

$$\hat{A}^{ij} := \psi^{10} A^{ij}, \quad (1.64)$$

and thus

$$\hat{A}_{ij} = \psi^2 A_{ij}. \quad (1.65)$$

Here, we use hat symbol to separate the "momentum-constraint" scaling from the tilde symbol in "time-evolution" scaling. These two scaling are related by

$$\hat{A}^{ij} = \psi^6 \tilde{A}^{ij}, \quad \hat{A}_{ij} = \psi^6 \tilde{A}_{ij}, \quad \hat{A}_{ij} \hat{A}^{ij} = \psi^{12} \tilde{A}_{ij} \tilde{A}^{ij} \quad (1.66)$$

Therefore, the constraint equations can be written as

$$\tilde{D}^2 \psi = \frac{1}{8} \psi \tilde{R} - \left(2\pi E - \frac{1}{12} K^2 \right) \psi^5 - \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7}, \quad (1.67)$$

$$\tilde{D}_i \hat{A}^{ij} - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i. \quad (1.68)$$

Equation (1.68) is known as *Lichnerowicz equation*. Although equation (1.63) and equation (1.68) are equivalent, they have different mathematical properties if they are treated as a partial differential equation for ψ . According to the maximal principle, the local uniqueness of the solutions depends on the sign of the exponent of ψ in the quadratic extrinsic curvature A^2 term [12, 17, 25, 29, 31]. Equation (1.63) suffers from the mathematical nonuniqueness problems due to the positive exponent (+5), while the negative exponent (−7) in equation (1.68) guarantees the local uniqueness of the solutions.

1.3.3 Conformal transverse-traceless decomposition

Using the "momentum-constraint" scaling mentioned previously, we can further decompose the symmetric, traceless tensor \hat{A}^{ij} as

$$\hat{A}^{ij} = \hat{A}_{TT}^{ij} + \hat{A}_L^{ij}, \quad (1.69)$$

where \hat{A}_{TT}^{ij} is the transverse-traceless part which is divergenceless

$$\tilde{D}_j \hat{A}_L^{ij} = 0, \quad (1.70)$$

and \hat{A}_L^{ij} is the longitudinal part satisfying

$$\hat{A}_L^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{D}_k X^k \equiv \left(\tilde{L} X \right)^{ij}. \quad (1.71)$$

The vector X^i here is the vector potential and \tilde{L} is the *longitudinal operator* or *conformal Killing operator* associated with $\tilde{\gamma}$ which gives a symmetric, traceless tensor. The divergence

of \hat{A}^{ij} becomes

$$\tilde{D}_j \hat{A}^{ij} = \tilde{D}^2 X^i + \frac{1}{3} \tilde{D}^i (\tilde{D}_j X^j) + \tilde{R}^i_j X^j \equiv \tilde{\Delta}_L, \quad (1.72)$$

where $\tilde{\Delta}_L$ is the *vector Laplacian*. Thus, the momentum constraint in the conformal transverse-traceless (CTT) decomposition yields

$$\tilde{\Delta}_L X^i - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i, \quad (1.73)$$

and the corresponding Hamiltonian constraint is the same as equation (1.67).

1.4 Gauge Condition

In the 3+1 formulation of spacetime, the gauge freedom is preserved. One can freely choose the lapse function α and shift vector β^i . In this section, we introduce the *maximal slicing* condition and *generalized Dirac gauge*, which is used in the formulation of the constrained scheme in the next section.

1.4.1 Maximal Slicing

By taking the trace of the evolution equation for extrinsic curvature (1.44), we can obtain an elliptic equation for the lapse α

$$D^2 \alpha = -\partial_t K + \alpha [K_{ij} K^{ij} + 4\pi (E + S)] + \beta^i D_i K. \quad (1.74)$$

One well-known choice to further simplify this equation is the *maximal slicing* condition

$$K = 0 = \partial_t K, \quad (1.75)$$

which corresponds to the vanishing mean curvature in all hypersurface Σ_t . This type of slicing was first introduced by Lichnerowicz [19] and then made popularized by York [29, 30]. Under this condition, the enclosed volume inside some hypersurface Σ_t becomes maximal, hence the name *maximal slicing*. With this choice of condition, equation (1.74) reduces to

$$D^2 \alpha = \alpha [A_{ij} A^{ij} + 4\pi (E + S)], \quad (1.76)$$

which is independent of the shift β^i . Alternatively, we can combine the conformally decomposed Hamiltonian equation (1.67) to obtain

$$\tilde{D}^2(\alpha\psi) = \alpha\psi \left[\frac{7}{8}\psi^{-8}\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{8}\tilde{R} + 2\pi\psi^4(E + 2S) \right]. \quad (1.77)$$

The *maximal slicing* condition not only helps decouple the constraint equations, but also has some nice physical properties: it is a natural extension of Newtonian gravity and has singularity avoidance property.

Natural generalization to Newtonian limit

Consider the Newtonian limit of a weak and static gravitational field

$$\alpha \approx 1 + \Phi, \quad \gamma_{ij} \approx (1 + 2\Phi) f_{ij}, \quad K_{ij} = 0, \quad \text{for } \Phi \ll 1, \quad (1.78)$$

where Φ is the Newtonian gravitational potential, and non-relativistic matter

$$S \ll E, \quad E \approx \rho_0, \quad (1.79)$$

where ρ_0 is the proper rest mass energy density. Equation (1.76) reduces to Poisson equation for the newtonian gravitational potential

$$\nabla^2\Phi = 4\pi\rho_0. \quad (1.80)$$

Singularity avoidance

Another interesting property of maximal slicing is singularity avoidance. Equation (1.16) shows that under maximal slicing condition, the normal observers move like irrotational and incompressible fluid elements

$$\nabla_\mu n^\mu = 0, \quad (1.81)$$

which implies that focusing of the timelike unit normal vector field is prohibited. It can also be seen from equation (1.43), which yields the following continuity equation for volume element $\sqrt{\gamma}$ in maximal slicing condition

$$\partial_t\sqrt{\gamma} = \partial_i(\sqrt{\gamma}\beta^i). \quad (1.82)$$

This suggests that as long as a regular shift vector is chosen and the initial condition for $\gamma_{\mu\nu}$ is regular, γ is regular forever.

1.4.2 Generalized Dirac Gauge

The *Dirac gauge* was first introduced by Dirac in 1959 [16], and then generalized by the Meudon group [8]. We define the *generalized Dirac gauge* as

$$\mathcal{D}_j \tilde{\gamma}^{ij} = 0, \quad (1.83)$$

which fully specify the coordinates in the hypersurface Σ_t including the initial one. Here, \mathcal{D} denotes the covariant derivative associated with the flat background metric f_{ij} defined in equation (1.46), which relates \tilde{D} by

$$\tilde{D}_k T^{i_1 \dots i_p}_{j_1 \dots j_q} = \mathcal{D}_k T^{i_1 \dots i_p}_{j_1 \dots j_q} + \sum_{r=1}^p \Delta^{i_r}_{lk} T^{i_1 \dots l \dots i_p}_{j_1 \dots j_q} + \sum_{r=1}^q \Delta^l_{j_r k} T^{i_1 \dots i_p}_{j_1 \dots l \dots j_q}, \quad (1.84)$$

where Δ^k_{ij} is given by

$$\Delta^k_{ij} := \frac{1}{2} \tilde{\gamma}^{kl} (\mathcal{D}_i \tilde{\gamma}_{lj} + \mathcal{D}_j \tilde{\gamma}_{il} - \mathcal{D}_l \tilde{\gamma}_{ij}). \quad (1.85)$$

We can further define the potentials h^{ij} as the deviation of the conformal metric from the flat fiducial metric

$$h^{ij} := \tilde{\gamma}^{ij} - f^{ij}. \quad (1.86)$$

Thus, the generalized Dirac is equivalent to

$$\mathcal{D}_j h^{ij} = 0, \quad (1.87)$$

Under such gauge condition, the conformal Ricci tensor \tilde{R}^{ij} is simplified drastically as

$$\tilde{R}^{ij} = \frac{1}{2} \mathcal{D}^2 h^{ij} + \tilde{R}^{ij}_*, \quad (1.88)$$

where \tilde{R}_*^{ij} is the quadratic part of \tilde{R}^{ij}

$$\tilde{R}_*^{ij} := \frac{1}{2} \left[h^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} - \mathcal{D}_l h^{ik} \mathcal{D}_k h^{jl} - \tilde{\gamma}_{kl} \tilde{\gamma}^{mn} \mathcal{D}_m h^{ik} \mathcal{D}_n h^{jl} \right. \quad (1.89)$$

$$\left. + \tilde{\gamma}_{nl} \mathcal{D}_k h^{mn} (\tilde{\gamma}^{ik} \mathcal{D}_m h^{jl} + \tilde{\gamma}^{jk} \mathcal{D}_m h^{il}) + \frac{1}{2} \tilde{\gamma}^{ik} \tilde{\gamma}^{kl} \mathcal{D}_l h^{mn} \mathcal{D}_k \tilde{\gamma}_{mn} \right], \quad (1.90)$$

and $\mathcal{D}^2 = \mathcal{D}^i \mathcal{D}_i$ is the Laplacian operator associated with the flat metric with $\mathcal{D}^i := f^{ij} \mathcal{D}_j$. The curvature scalar \tilde{R} of the conformal metric does not contain any second order derivative of $\tilde{\gamma}_{ij}$ thanks to the gauge condition

$$\tilde{R} = \frac{1}{4} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_l \tilde{\gamma}_{ij} - \frac{1}{2} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_j \tilde{\gamma}_{il}. \quad (1.91)$$

Note that the generalized gauge condition result in transverse-traceless (TT) gauge asymptotically, which are well adapted to the treatment of gravitational radiation.

1.5 Constrained scheme for the Einstein equations

Although the ADM equations (eqs. (1.41) to (1.44)) formulate the Einstein equations into a well-defined initial-value problem based on spacetime foliation, it is known to be unsuitable for numerical simulation due to the unbounded growth of numerical error (same for conformal decomposition formulation). Several reformulations were developed to maintain stable evolutions. Most of schemes such as BSSN [2, 27], CCZ4 [7], Z4c [6] schemes are based on free-evolution approach where the hyperbolic-type dynamical equations are evolved without enforcing the constraints. The constraint equations are solved only to obtain the initial data and serve as an indicator for numerical errors during the simulation. On the other hand, the fully constrained approach minimizes the hyperbolic-type evolution equations by solving the constraint equations at each time step. Though the elliptic nature of the constraint equations makes it computationally costly to solve, elliptic equations are more stable than hyperbolic equations, and the constraint-violating modes appeared in the free-evolution scheme vanish by construction in fully constrained evolution.

With the efficient elliptic solvers developed, our relativistic hydrodynamics code **Gmunu** employs the Isenberg–Wilson–Mathews (also known as conformal flatness condition) approximation to the general relativity and the fully constrained formulation. In this section, we introduce the formulations and implementations under the constrained scheme.

1.5.1 Isenberg–Wilson–Mathews approximation

The Isenberg–Wilson–Mathews approximation (IWM), also known as conformal flatness condition (CF), is a (gravitational-)waveless approximation to general relativity which was independently developed by Isenberg in 1978 [18] as well as Wilson and Mathews in 1989 [32]. In IWM, the spatial metric is approxiated to be conformally flat (hence the name conformal flatness condition)

$$\gamma_{ij} = \psi^4 f_{ij}, \quad \text{or equivalently} \quad \tilde{\gamma}_{ij} = f_{ij}, \quad (1.92)$$

under maximal slicing condition $K = 0 = \partial_t K$ (eq. (1.75)) (i.e. $\partial_t \tilde{\gamma}_{ij} = 0$ and $\tilde{D}_i = \mathcal{D}_i$ by construction). The equations for the conformal factor and extrinsic curvature can be obtained from equations (1.55) and (1.56) respectively

$$\partial_t \psi = \beta^i \mathcal{D}_i \psi + \frac{\psi}{6} \mathcal{D}_i \beta^i \quad (1.93)$$

$$K_{ij} = \frac{\psi^4}{2\alpha} \left(f_{jk} \mathcal{D}_i \beta^k + f_{ik} \mathcal{D}_j \beta^k - \frac{2}{3} f_{ij} \mathcal{D}_k \beta^k \right). \quad (1.94)$$

The CFC approximation reduces the ADM equations in a set of coupled non-linear elliptic equations for ψ (from eq. (1.62)), α (from eq. (1.77)), and β^i (from eq. (1.63))

$$\mathcal{D}^2 \psi = - \left(2\pi E + \frac{1}{8} K_{ij} K^{ij} \right) \psi^5, \quad (1.95)$$

$$\mathcal{D}^2 (\alpha \psi) = \alpha \psi^5 \left[\frac{7}{8} K_{ij} K^{ij} + 2\pi (E + 2S) \right], \quad (1.96)$$

$$\Delta_L \beta^i = 16\pi \alpha \psi^4 S^i - 2\psi^{10} K^{ij} \mathcal{D}_j (\alpha \psi^{-6}), \quad (1.97)$$

where $\Delta_L \beta^i := \mathcal{D}^2 \beta^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j \beta^j$ is the *vector Laplacian* (eq. (1.72)) associated with the flat metric f_{ij} . Although the solutions (ψ, α, β^i) to the CFC system do not satisfy all the conformal 3+1 equations in general, it is exact for spherically symmetric spacetime [11] and correct at the 1-PN order in the post-Newtonian expansion of general relativity. The CFC scheme has been successful in numerical simulation for various astrophysical simulation [3–5, 14, 15, 21, 23, 26] and obtained good approximation to full general relativity in axisymmetric rotating neutron stars [9, 14, 15] and in rotating iron core collapses [24].

Extended CFC scheme

Despite the success of CFC scheme in astrophysical simulation, it suffers from the non-uniqueness issue as we discussed in section 1.3.2. A solution to this has been found [12, 26, 28], which involves an extra elliptic equation of the vector potential from CTT decomposition (see section 1.3.3). The resultant scheme is called *extended conformal flatness approximation* (XCFC), and the metric can be solved by following

$$\Delta_L X^i = 8\pi f^{ij} (S^*)_j, \quad (1.98)$$

$$\mathcal{D}^2 \psi = -2\pi E^* \psi^{-1} - \frac{1}{8} \psi^{-7} f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij}, \quad (1.99)$$

$$\mathcal{D}^2 (\alpha \psi) = (\alpha \psi) \left[2\pi (E^* + 2S^*) \psi^{-2} + \frac{7}{8} \psi^{-8} f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij} \right], \quad (1.100)$$

$$\Delta_L \beta^i = 16\pi \alpha \psi^{-6} f^{ij} (S^*)_j - 2\hat{A}^{ij} \mathcal{D}_j (\alpha \psi^{-6}), \quad (1.101)$$

where the hydrodynamical conserved quantities are defined as

$$E^* := \psi^6 E, \quad S^* := \psi^6 S, \quad (S^*)_i := \psi^6 S_i, \quad (1.102)$$

and the extra vector elliptic equation (1.98) comes from the momentum constraint (eq. (1.73)) by taking CFC approximation and $\hat{A}_{TT}^{ij} \approx 0$ in CTT decomposition (eq. (1.69)), which gives

$$\hat{A}^{ij} \approx \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3} f^{ij} \mathcal{D}_k X^k \equiv (LX)^{ij}. \quad (1.103)$$

Unlike the coupled system in IWM approximation, the XCFC system is decoupled in a hierarchical way, and hence guarantees local uniqueness. Once the hydrodynamical conserved quantities are known, the metric equations can be solved accordingly as follows

Step 1: Solve equation (1.98) for the vector potential X^i from the conserved variables $(S^*)_i$.

Step 2: Obtain \hat{A}^{ij} from equation (1.103).

Step 3: Solve equation (1.99) for the conformal factor ψ .

Step 4: Work out S^* using conformal factor ψ .

Step 5: Solve equation (1.100) for the lapse function α .

Step 6: Solve equation (1.101) for the shift vector β^i .

1.5.2 The Fully Constrained Formulation

The fully constrained formulation (FCF) was first proposed by the Meudon group [8], and then later modified by the Valencia's group [10, 12, 13]. By choosing the maximal slicing condition (section 1.4.1) and the generalized Dirac gauge (section 1.4.2), the conformal 3+1 decomposition can be recasted in a coupled elliptic-hyperbolic system, where all the constraint equations are utilized to minimize the number of hyperbolic equations in the system.

Hyperbolic sector

The Valencia's group performed first-order reduction [13] to the reduced FCF system [8] by introducing auxiliary variable

$$w^{ij}_k := \mathcal{D}_k h^{ij}, \quad (1.104)$$

Thus, the hyperbolic sector in the FCF is recasted as a first-order system in $(h^{ij}, \hat{A}^{ij}, w^{ij}_k)$ as follows

$$\partial_t h^{ij} = 2\alpha\psi^{-6}\hat{A}^{ij} + \beta^k w^{ij}_k - \tilde{\gamma}^{ik}\mathcal{D}_k\beta^j - \tilde{\gamma}^{kj}\mathcal{D}_k\beta^i + \frac{2}{3}\tilde{\gamma}^{ij}\mathcal{D}_k\beta^k, \quad (1.105)$$

$$\begin{aligned} \partial_t \hat{A}^{ij} = & \mathcal{D}_k \left(\frac{1}{2}\alpha\psi^2\tilde{\gamma}^{kl}w^{ij}_l + \beta^k\hat{A}^{ij} \right) - \hat{A}^{kj}\mathcal{D}_k\beta^i - \hat{A}^{ik}\mathcal{D}_k\beta^j + \frac{2}{3}\hat{A}^{ij}\mathcal{D}_k\beta^k + 2\alpha\psi^{-6}\tilde{\gamma}^{kl}\hat{A}^{ik}\hat{A}^{jl} \\ & - 8\pi\alpha\psi^6 \left(\psi^4 S^{ij} - \frac{1}{3}\tilde{\gamma}^{ij}S \right) + \alpha\psi^2 \left(\tilde{R}^{ij}_* - \frac{1}{3}\tilde{R} \right) + 8\alpha \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} - \frac{1}{3}\tilde{\gamma}^{ij}\tilde{\gamma}^{kl} \right) \mathcal{D}_k\psi\mathcal{D}_l\psi \\ & + 4\psi \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} + \tilde{\gamma}^{il}\tilde{\gamma}^{jk} - \frac{2}{3}\tilde{\gamma}^{kl}\tilde{\gamma}^{ij} \right) \mathcal{D}_k\psi\mathcal{D}_l\alpha - \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} - \frac{1}{3}\tilde{\gamma}^{ij}\tilde{\gamma}^{kl} \right) \mathcal{D}_k\mathcal{D}_l(\alpha\psi^2) \\ & - \frac{1}{2}(\tilde{\gamma}^{ik}w^{lj}_k + \tilde{\gamma}^{lk}w^{il}_k) \mathcal{D}_l(\alpha\psi^2), \end{aligned} \quad (1.106)$$

$$\partial_t w^{ij}_k = \mathcal{D}_k \left(2\alpha\psi^{-6}\hat{A}^{ij} + \beta^k w^{ij}_k - \tilde{\gamma}^{ik}\mathcal{D}_k\beta^j - \tilde{\gamma}^{kj}\mathcal{D}_k\beta^i + \frac{2}{3}\tilde{\gamma}^{ij}\mathcal{D}_k\beta^k \right), \quad (1.107)$$

where \tilde{R}^{ij}_* and \tilde{R} defined in section 1.4.2. This evolution system is proved to be strongly hyperbolic [13] under following conditions

$$\alpha \neq 0, \quad \alpha^2 - \beta^i\beta_i > 0, \quad (1.108)$$

and the characteristic speed is given by

$$\lambda_{\pm}^i = -\beta^i \pm \alpha\psi^{-2}\tilde{\gamma}^{ii}. \quad (1.109)$$

Elliptic sector

The elliptic sector in the FCF scheme is given by

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_lX^i + \frac{1}{3}\tilde{\gamma}^{ik}\mathcal{D}_k\mathcal{D}_lX^l + \Delta^i{}_{kl}(LX)^{kl} = 8\pi\tilde{\gamma}^{ij}(S^*)_j - \Delta^i{}_{kl}\hat{A}_{TT}^{kl}, \quad (1.110)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\psi = -2\pi E^*\psi^{-1} - \frac{1}{8}\psi^{-7}\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij} + \frac{1}{8}\psi\tilde{R}, \quad (1.111)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l(\alpha\psi) = (\alpha\psi) \left[2\pi\psi^{-2}(E^* + 2S^*) + \frac{7}{8}\psi^{-8}\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij} + \frac{\tilde{R}}{8} \right], \quad (1.112)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i + \frac{1}{3}\tilde{\gamma}^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l = 16\pi\alpha\psi^{-6}\tilde{\gamma}^{ij}(S^*)_j + 2\hat{A}^{ij}\mathcal{D}_j(\alpha\psi^{-6}) - 2\alpha\psi^{-6}\Delta^i{}_{kl}\hat{A}^{kl}. \quad (1.113)$$

Note that the FCF scheme is a natural generalization of XCFC scheme since it reduces to XCFC if the approximation $h^{ij} = \hat{A}_{TT}^{ij} = 0$ is made.

Chapter 2

Formulations of the Relativistic Hydrodynamics

2.1 Hydrodynamics

2.1.1 Perfect fluid

2.1.2 Rest-mass conservation

2.1.3 Energy and momentum conservation

2.1.4 Conservative form of hydrodynamics equations

2.1.5 Conserved to Primitive variables conversion

2.2 The reference-metric formulism

Part II

Methodology: Numerical methods

Chapter 3

Numerical Methods

3.1 Multigrid Method for elliptic equations

3.1.1 Overview

3.1.2 Smoother

3.1.3 Restriction and Prolongation

3.1.4 Multigrid cycle

3.1.5 Grid Structure

3.1.6 Ghost cells and refinement boundary

3.2 Elliptic system in fully constrained formulation

3.2.1 Elliptic divergence cleaning for h^{ij}

3.2.2 Maximal slicing condition

3.2.3 Elliptic sector

3.3 High-resolution shock-capturing (HRSC) methods

3.3.1 Approximate Riemann solver

3.3.2 Reconstruction scheme

3.3.3 Finite volume discretization in curvilinear coordinate

3.4 Time discretization

3.4.1 Explicit Runge-Kutta Method

Chapter 4

Numerical Tests and Result

4.1 Multigrid solver tests

4.1.1 Newtonian gravitational field

4.2 General relativistic hydrodynamics in dynamic space-time

4.2.1 Non-rotating neutron star

4.2.2 Rapidly rotating neutron star

References

- [1] R Amowitt, S Deser, and CW Misner. “The dynamics of general relativity”. In: *Gravitation: An Introduction to Current Research* (1962), pp. 227–265.
- [2] Thomas W Baumgarte and Stuart L Shapiro. “Numerical integration of Einstein’s field equations”. In: *Physical Review D* 59.2 (1998), p. 024007.
- [3] A Bauswein, N Stergioulas, and H-T Janka. “Revealing the high-density equation of state through binary neutron star mergers”. In: *Physical Review D* 90.2 (2014), p. 023002.
- [4] Andreas Bauswein, Stéphane Goriely, and H-T Janka. “Systematics of dynamical mass ejection, nucleosynthesis, and radioactively powered electromagnetic signals from neutron-star mergers”. In: *The Astrophysical Journal* 773.1 (2013), p. 78.
- [5] Andreas Bauswein et al. “Equation-of-state dependence of the gravitational-wave signal from the ring-down phase of neutron-star mergers”. In: *Physical Review D* 86.6 (2012), p. 063001.
- [6] Sebastiano Bernuzzi and David Hilditch. “Constraint violation in free evolution schemes: Comparing the BSSNOK formulation with a conformal decomposition of the Z4 formulation”. In: *Physical Review D* 81.8 (2010), p. 084003.
- [7] Carles Bona et al. “General-covariant evolution formalism for numerical relativity”. In: *Physical Review D* 67.10 (2003), p. 104005.
- [8] Silvano Bonazzola et al. “Constrained scheme for the Einstein equations based on the Dirac gauge and spherical coordinates”. In: *Physical Review D* 70.10 (2004), p. 104007.
- [9] Gregory B Cook, Stuart L Shapiro, and Saul A Teukolsky. “Testing a simplified version of Einstein’s equations for numerical relativity”. In: *Physical Review D* 53.10 (1996), p. 5533.
- [10] Isabel Cordero-Carrión, Pablo Cerdá-Durán, and José Mari ´a Ibáñez. “Gravitational waves in dynamical spacetimes with matter content in the fully constrained formulation”. In: *Physical Review D* 85.4 (2012), p. 044023.
- [11] Isabel Cordero-Carrion, José Mari ´a Ibáñez, and Juan Antonio Morales-Lladosa. “Maximal slicings in spherical symmetry: local existence and construction”. In: *Journal of mathematical physics* 52.11 (2011), p. 112501.
- [12] Isabel Cordero-Carrión et al. “Improved constrained scheme for the Einstein equations: An approach to the uniqueness issue”. In: *Physical Review D* 79.2 (2009), p. 024017.

- [13] Isabel Cordero-Carrión et al. “Mathematical issues in a fully constrained formulation of the Einstein equations”. In: *Physical Review D* 77.8 (2008), p. 084007.
- [14] Harald Dimmelmeier, Jose A Font, and Ewald Müller. “Relativistic simulations of rotational core collapse II. Collapse dynamics and gravitational radiation”. In: *Astronomy & Astrophysics* 393.2 (2002), pp. 523–542.
- [15] Harald Dimmelmeier, Nikolaos Stergioulas, and José A Font. “Non-linear axisymmetric pulsations of rotating relativistic stars in the conformal flatness approximation”. In: *Monthly Notices of the Royal Astronomical Society* 368.4 (2006), pp. 1609–1630.
- [16] PAM Dirac. “Fixation of coordinates in the Hamiltonian theory of gravitation”. In: *Physical Review* 114.3 (1959), p. 924.
- [17] Lawrence C Evans. “Partial differential equations and Monge-Kantorovich mass transfer”. In: *Current developments in mathematics* 1997.1 (1997), pp. 65–126.
- [18] James A Isenberg. “Waveless approximation theories of gravity”. In: *International Journal of Modern Physics D* 17.02 (2008), pp. 265–273.
- [19] André Lichnerowicz. *L’intégration des équations de la gravitation relativiste et le problème des n-corps*. Gauthier-Villars, 1944.
- [20] Charles W Misner, Kip S Thorne, and John Archibald Wheeler. *Gravitation*. Macmillan, 1973.
- [21] Bernhard Müller. “The dynamics of neutrino-driven supernova explosions after shock revival in 2D and 3D”. In: *Monthly Notices of the Royal Astronomical Society* 453.1 (2015), pp. 287–310.
- [22] Takashi Nakamura. “3D numerical relativity.” In: *Relativistic Cosmology* (1994), pp. 155–182.
- [23] Roland Oechslin, Stephan Rosswog, and Friedrich-Karl Thielemann. “Conformally flat smoothed particle hydrodynamics application to neutron star mergers”. In: *Physical Review D* 65.10 (2002), p. 103005.
- [24] Christian D Ott et al. “Rotating collapse of stellar iron cores in general relativity”. In: *Classical and Quantum Gravity* 24.12 (2007), S139.
- [25] Murray H Protter and Hans F Weinberger. *Maximum principles in differential equations*. Springer Science & Business Media, 2012.
- [26] Motoyuki Saijo. “The collapse of differentially rotating supermassive stars: conformally flat simulations”. In: *The Astrophysical Journal* 615.2 (2004), p. 866.
- [27] Masaru Shibata and Takashi Nakamura. “Evolution of three-dimensional gravitational waves: Harmonic slicing case”. In: *Physical Review D* 52.10 (1995), p. 5428.
- [28] Masaru Shibata and Koji Uryū. “Merger of black hole-neutron star binaries: Non-spinning black hole case”. In: *Physical Review D* 74.12 (2006), p. 121503.
- [29] Larry L Smarr. *Sources of Gravitational Radiation: Proceedings of the Battelle Seattle Workshop*. CUP Archive, 1979.
- [30] Larry Smarr and James W York Jr. “Radiation gauge in general relativity”. In: *Physical Review D* 17.8 (1978), p. 1945.

- [31] Michael Eugene Taylor. *Partial differential equations. 3, Nonlinear equations*. Springer, 1991.
- [32] James R Wilson and Grant J Mathews. “Relativistic hydrodynamics.” In: *Frontiers in numerical relativity* (1989), pp. 306–314.
- [33] James W York Jr. “Conformally invariant orthogonal decomposition of symmetric tensors on Riemannian manifolds and the initial-value problem of general relativity”. In: *Journal of Mathematical Physics* 14.4 (1973), pp. 456–464.
- [34] James W York Jr. “Gravitational degrees of freedom and the initial-value problem”. In: *Physical Review Letters* 26.26 (1971), p. 1656.
- [35] James W York Jr. “Kinematics and dynamics of general relativity”. In: *Sources of gravitational radiation* (1979), pp. 83–126.
- [36] James W York Jr. “Role of conformal three-geometry in the dynamics of gravitation”. In: *Physical review letters* 28.16 (1972), p. 1082.

Appendix A

Useful relations for implementation of constrained scheme

A.1 The elliptic equations in constrained scheme

A.2 Generalized Dirac gauge conditions

Appendix B

Reference flat metric in 3D

