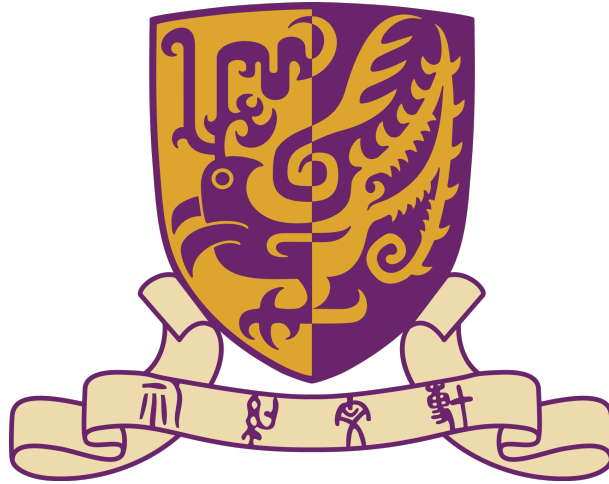


New Approach of Numerical Relativity

Implementation, tests and applications



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I would like to dedicate this thesis to my loving parents . . .

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Alan Tsz-Lok Lam
July 2021

Acknowledgements

And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

Table of contents

List of figures	xv
List of tables	xvii
I Methodology: Formulations	1
1 Formulations of Einstein Field Equations	3
1.1 Introduction to general relativity	3
1.1.1 Einstein Field Equation	3
1.1.2 Gravitational Wave	3
1.2 The 3+1 decomposition of spacetime	3
1.2.1 Foliation of spacetime	3
1.2.2 Derivative operator	5
1.2.3 Extrinsic curvature	6
1.2.4 The Gauss, Codazzi and Ricci equations	7
1.2.5 Constraint and evolution equations	7
1.2.6 The Arnowitt, Deser and Misner equations	9
1.3 Conformal Decomposition	10
1.3.1 Conformal transformation of the spatial metric	10
1.3.2 Conformal transformation of the extrinsic curvature	11
1.3.3 Conformal transverse-traceless decomposition	13
1.4 Gauge Condition	14
1.4.1 Maximal Slicing	14
1.4.2 Generalized Dirac Gauge	16
1.5 Constrained scheme for the Einstein equations	17
1.5.1 Isenberg–Wilson–Mathews approximation	18
1.5.2 The Meudon-Valencia Fully Constrained Formulation	20

1.5.3	The fully-constrained scheme in Gmunu	22
2	Formulations of the Relativistic Hydrodynamics	25
2.1	Relativistic Hydrodynamics	25
2.1.1	Perfect fluid	25
2.1.2	Rest-mass conservation	26
2.1.3	Energy and momentum conservation	27
2.1.4	Conservative form of hydrodynamics equations	28
2.1.5	Conserved to Primitive variables conversion	28
2.2	The reference-metric formalism	30
2.2.1	General relativistic hydrodynamics (GRHD) equations	31
II	Methodology: Numerical methods and tests	33
3	Numerical Methods	35
3.1	High-resolution shock-capturing (HRSC) methods	36
3.1.1	Approximate Riemann solver	36
3.1.2	Reconstruction scheme	36
3.1.3	Finite volume discretization in curvilinear coordinate	36
3.2	Atmosphere treatment	36
3.2.1	Positivity preserving limit	36
3.3	Time discretization	36
3.3.1	Explicit Runge-Kutta Method	36
3.3.2	Courant-Friedrichs <i>Lewy</i> Conditions	36
3.4	Multigrid Method for elliptic equations	36
3.4.1	Overview	36
3.4.2	Smoother	36
3.4.3	Restriction and Prolongation	36
3.4.4	Multigrid cycle	36
3.4.5	Grid Structure	36
3.4.6	Ghost cells and refinement boundary	36
4	Numerical Tests and Results	37
4.1	Multigrid solver tests	37
4.1.1	Newtonian gravitational field	37
4.2	General relativistic hydrodynamics in dynamic spacetime	37

4.2.1	Non-rotating neutron star	37
4.2.2	Rapidly rotating neutron star	37
References		39
Appendix A Useful relations for implementation of constrained scheme		43
A.1	The elliptic equations in constrained scheme	43
A.2	Generalized Dirac gauge conditions	43
Appendix B Reference flat metric in 3D		45

List of figures

List of tables

Part I

Methodology: Formulations

Chapter 1

Formulations of Einstein Field Equations

1.1 Introduction to general relativity

1.1.1 Einstein Field Equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1.1}$$

1.1.2 Gravitational Wave

1.2 The 3+1 decomposition of spacetime

Due to the complexity and nonlinearity of Einstein field equations, it is extremely difficult to obtain analytical solution even for the simplest dynamical evolution systems. Therefore, the accurate discription of the such systems can only be derived through numerical simulation. For this, we need to reformulate the Einstein equations as an initial-value problem or Cauchy problem. In this chapter, we will introduction the Arnowitt-Deser-Misner (ADM) formulation, which is the foundation of the 3+1 numerical relativity. In particular, we will focus on the constrained scheme for the Einstein equations.

1.2.1 Foliation of spacetime

In the 3+1 decomposition, the spacetime manifold \mathcal{M} is foliated into a set of non-intersecting spacelike hypersurfaces Σ_t parameterized by the coordinate time t [\[25\]](#). We

denote a future-directed timelike unit four-vector n^μ normal to the hypersurface Σ_t (i.e. $n_\mu \propto \nabla_\mu t$). The induced spacetime metric $\gamma_{\mu\nu}$ on each hypersurface can then be defined as

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu. \quad (1.2)$$

Thus, we can construct spatial projection tensor $\gamma^\mu{}_\nu$ and time projection tensor $N^\mu{}_\nu$ as

$$\gamma^\mu{}_\nu := \delta^\mu{}_\nu + n^\mu n_\nu, \quad N^\mu{}_\nu := -n^\mu n_\nu, \quad (1.3)$$

which decompose any generic four-vector U^μ into spatial part $\gamma^\mu{}_\nu U^\nu$ and timelike part $N^\mu{}_\nu U^\nu$. Therefore, we can decompose the timelike vector field

$$t^\mu = \alpha n^\mu + \beta^\mu \quad (1.4)$$

into two components as

$$\alpha := -t^\mu n_\mu, \quad \beta^\mu := t^\nu \gamma^\mu{}_\nu, \quad (1.5)$$

where the lapse function α measures the physical proper time ($\alpha \Delta t$) between two neighboring spatial hypersurface Σ_t and $\Sigma_{t+\Delta t}$, and the shift vector β^i measures the changes of spatial coordinates on $\Sigma_{t+\Delta t}$.

Here, we summarise several useful relations. The timelike normal vector n^μ and its corresponding one-form n_μ can be expressed as

$$n^\mu = \frac{1}{\alpha} (1, \beta^i), \quad n_\mu = (\alpha, \vec{0}). \quad (1.6)$$

The generic line element in 3+1 decomposition is given by

$$ds^2 = -(\alpha^2 - \beta^i \beta_i) dt^2 + \beta_i dx^i dt + \gamma_{ij} dx^i dx^j \quad (1.7)$$

The covariant and contravariant components of the metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^j \\ \beta^i & \gamma^{ij} \end{pmatrix}. \quad (1.8)$$

From equation(1.8), we can conclude that

$$\sqrt{-g} = \alpha \sqrt{\gamma}, \quad (1.9)$$

where $g := \det(g_{\mu\nu})$ and $\gamma := \det(\gamma_{ij})$.

1.2.2 Derivative operator

With the 3+1 decomposition, we can now construct the 3-dimensional covariant derivative D_α associated with $\gamma_{\mu\nu}$ by projecting the 4-dimensional covariant derivative ∇_α onto Σ_t , which is given by

$$D_\alpha T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots} = \gamma_\alpha^\beta \gamma_{\rho_1}^{\mu_1} \gamma_{\rho_2}^{\mu_2} \cdots \gamma_{\nu_1}^{\sigma_1} \gamma_{\nu_2}^{\sigma_2} \cdots \nabla_\beta T^{\rho_1\rho_2\cdots}_{\sigma_1\sigma_2\cdots}, \quad (1.10)$$

for arbitrary tensor $T^{\mu_1\mu_2\cdots}_{\nu_1\nu_2\cdots}$ on spatial hypersurface Σ_t . Using equation(1.10), it can be shown that the covariant derivative of $\gamma_{\mu\nu}$ vanishes

$$\begin{aligned} D_\alpha \gamma_{\mu\nu} &= \gamma_\alpha^\beta \gamma_\rho^\mu \gamma_\nu^\sigma \nabla_\beta (g_{\rho\sigma} + n_\rho n_\sigma) \\ &= \gamma_\alpha^\beta \gamma_\rho^\mu \gamma_\nu^\sigma (n_\rho \nabla_\beta n_\sigma + n_\sigma \nabla_\beta n_\rho) = 0 \end{aligned} \quad (1.11)$$

The components of 3-dimensional connection coefficients $\Gamma^\alpha_{\mu\nu}$ in coordinate basis can be expressed as

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \gamma^{\alpha\beta} (\partial_\nu \gamma_{\beta\mu} + \partial_\mu \gamma_{\beta\nu} - \partial_\beta \gamma_{\mu\nu}). \quad (1.12)$$

Here, the upper left index ⁽⁴⁾ marks the 4-dimensional tensors while the unmarked one represents purely spatial 3-dimensional tensors. Similarly, the 3-dimensional Riemann tensor $R^\alpha_{\beta\mu\nu}$ associated with $\gamma_{\mu\nu}$ is defined by requiring that

$$2D_{[\nu} D_{\mu]} W_\beta = W_\alpha R^\alpha_{\beta\mu\nu}, \quad R^\alpha_{\beta\mu\nu} n_\alpha = 0, \quad (1.13)$$

which can be explicitly expressed in coordinate basis as

$$R^\alpha_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\beta\nu} - \Gamma^\alpha_{\nu\rho} \Gamma^\rho_{\beta\mu}. \quad (1.14)$$

The 3-dimensional Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined in a similar manner as their 4-dimensional counterparts

$$R_{\mu\nu} := R^\alpha_{\mu\alpha\nu}, \quad R := R^\mu_{\mu}. \quad (1.15)$$

Since $R^\alpha_{\beta\mu\nu}$ is purely spatial and can be computed by the spatial derivatives of the spatial metric alone, it only contains about information about the curvature intrinsic to the

hypersurface Σ_t , but cannot contain all the information of ${}^{(4)}R^\alpha{}_{\beta\mu\nu}$ which includes time derivative of the 4-dimensional metric. The missing information can be found in a purely spatial symmetric tensor called the extrinsic curvature $K_{\mu\nu}$.

1.2.3 Extrinsic curvature

The extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the spatial metric $\gamma_{\mu\nu}$. Therefore, the spatial metric and extrinsic curvature $(\gamma_{\mu\nu}, K_{\mu\nu})$ are equivalent to the positions and velocities in classical mechanics, which describe the instantaneous state of the gravitational field. It can be obtained by projecting of the gradient of the normal vector $\gamma_\mu{}^\lambda \gamma_\nu{}^\rho \nabla_\lambda n_\rho$ into the hypersurface Σ_t , and then taking the negative expression of the symmetric part

$$\begin{aligned} K_{\mu\nu} &:= -\gamma_\mu{}^\lambda \gamma_\nu{}^\rho \nabla_\lambda n_\rho \\ &= -\gamma_\mu{}^\lambda (\delta_\nu{}^\rho + n_\nu n^\rho) \nabla_\lambda n_\rho \\ &= -\gamma_\mu{}^\lambda \nabla_\lambda n_\nu, \end{aligned} \tag{1.16}$$

where the identity $n^\rho \nabla_\lambda n_\rho = 0$ is used.

We can also define an spatial acceleration a_ν

$$a_\nu := n^\mu \nabla_\mu n_\nu, \tag{1.17}$$

satisfying the identities

$$a_\nu = D_\nu \ln \alpha, \tag{1.18}$$

to rewrite equation(1.16) as

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu \tag{1.19}$$

Finally, we can write the extrinsic curvature $K_{\mu\nu}$ as the Lie derivative of the spatial metric along the local normal n^μ

$$K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu}. \tag{1.20}$$

Using equation(1.4), we can express the Lie derivative \mathcal{L}_n as

$$\mathcal{L}_n = \frac{1}{\alpha} (\mathcal{L}_t - \mathcal{L}_\beta), \quad (1.21)$$

and thus obtain the evolution equation for the spatial metric

$$\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}. \quad (1.22)$$

1.2.4 The Gauss, Codazzi and Ricci equations

To express the Einstein field equations in term of the spatial variables $(\gamma_{\mu\nu}, K_{\mu\nu})$ we defined previous, we first need to relate 3-dimensional Riemann tensor $R^\alpha_{\beta\mu\nu}$ on Σ_t to the 4-dimensional Riemann tensor ${}^{(4)}R^\alpha_{\beta\mu\nu}$ on \mathcal{M} , The relation between $R^\alpha_{\beta\mu\nu}$ and the full spatial projection of ${}^{(4)}R^\alpha_{\beta\mu\nu}$ is given by the *Gauss' equation*

$$R_{\alpha\beta\mu\nu} + K_{\alpha\mu}K_{\beta\nu} - K_{\alpha\nu}K_{\beta\mu} = \gamma_\alpha{}^\rho \gamma_\beta{}^\sigma \gamma_\mu{}^\lambda \gamma_\nu{}^\delta {}^{(4)}R_{\rho\sigma\lambda\delta}, \quad (1.23)$$

while the projection of ${}^{(4)}R^\alpha_{\beta\mu\nu}$ with one index projected in the normal direction is given by the *Codazzi equation*

$$D_\nu K_{\mu\alpha} - D_\mu K_{\nu\alpha} = \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \gamma_\alpha{}^\lambda n^\delta {}^{(4)}R_{\rho\sigma\lambda\delta}. \quad (1.24)$$

Finally, by projecting two indices of ${}^{(4)}R_{\rho\sigma\lambda\delta}$ in the normal direction, we can relate it to the time derivative of $K_{\mu\nu}$

$$\mathcal{L}_n K_{\mu\nu} = n^\alpha n^\beta \gamma_\mu{}^\lambda \gamma_\nu{}^\delta {}^{(4)}R_{\alpha\beta\lambda\delta} - \frac{1}{\alpha} D_\mu D_\nu \alpha - K_\nu{}^\lambda K_{\mu\lambda}, \quad (1.25)$$

which is called the *Ricci equation*.

1.2.5 Constraint and evolution equations

Using the Gauss, Codazzi and Ricci equations, the Einstein fields equations can be decomposed into a set of evolution equations and a set of constraint equations of $(\gamma_{\mu\nu}, K_{\mu\nu})$.

To begin with, we define the following matter quantities

$$S_{\mu\nu} := \gamma^\alpha_\mu \gamma^\beta_\nu T_{\alpha\beta}, \quad (1.26)$$

$$S_\mu := -\gamma^\alpha_\mu n^\beta T_{\alpha\beta}, \quad (1.27)$$

$$S := S^\mu_\mu, \quad (1.28)$$

$$E := n^\alpha n^\beta T_{\alpha\beta}, \quad (1.29)$$

which decompose the stress-energy tensor as

$$T_{\mu\nu} = En_\mu n_\nu + S_\mu n_\nu + S_\nu n_\mu + S_{\mu\nu}. \quad (1.30)$$

By contracting the α, μ indices in equation(1.23), we can obtain

$$R_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu \left({}^{(4)}R_{\alpha\beta} + n^\rho n^\sigma {}^{(4)}R_{\alpha\rho\beta\sigma} \right) + K_{\mu\lambda} K_\nu{}^\lambda - K_{\mu\nu} K, \quad (1.31)$$

where $K := K^\mu_\mu$ is the trace of the extrinsic curvature, called the *mean curvature*. Further contracting the μ, ν indices in equation(1.31), the contracted Gauss' equation becomes

$$2n^\mu n^\nu G_{\mu\nu} = R + K^2 - K_{\mu\nu} K^{\mu\nu}, \quad (1.32)$$

Using the Einstein equation(1.1), we can obtain the *Hamiltonian constraint*

$$R + K^2 - K_{\mu\nu} K^{\mu\nu} = 16\pi E. \quad (1.33)$$

Similarly, by contracting α, ν indices in equation(1.24), the Codazzi equation yields

$$D_\nu K_\mu{}^\nu - D_\mu K = -8\pi \gamma^\alpha_\mu n^\beta T_{\alpha\beta}, \quad (1.34)$$

and thus obtain the *momentum constrain equation*

$$D_\nu K_\mu{}^\nu - D_\mu K = 8\pi S_\mu. \quad (1.35)$$

The Ricci equation (1.25) can be rewritten using equation(1.31) to

$$\mathcal{L}_n K_{\mu\nu} = R_{\mu\nu} - \gamma^\rho_\mu \gamma^\sigma_\nu {}^{(4)}R_{\rho\sigma} - 2K_{\mu\lambda} K_\nu{}^\lambda + K K_{\mu\nu} - \frac{1}{\alpha} D_\mu D_\nu \alpha. \quad (1.36)$$

Using the Einstein equations (1.1)

$$\begin{aligned}\gamma_\mu{}^\rho \gamma_\nu{}^\sigma {}^{(4)}R_{\rho\sigma} &= 8\pi \gamma_\mu{}^\rho \gamma_\nu{}^\sigma \left(T_{\rho\sigma} - \frac{1}{2} g_{\rho\sigma} T^\mu{}_\mu \right) \\ &= 8\pi \left[S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} (S - E) \right]\end{aligned}\tag{1.37}$$

and equation(1.21), we can finally obtain the evolution equation for $K_{\mu\nu}$ as

$$\mathcal{L}_t K_{\mu\nu} = -D_\mu D_\nu \alpha + \alpha (R_{\mu\nu} - 2K_{\mu\lambda} K_\nu{}^\lambda + K K_{\mu\nu}) - 8\pi \alpha \left[S_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} (S - E) \right] + \mathcal{L}_\beta K_{\mu\nu}.\tag{1.38}$$

1.2.6 The Arnowitt, Deser and Misner equations

The Lie derivative in the evolution equations (1.22) and (1.38) can be expressed in terms of coordinate basis as

$$\mathcal{L}_t K_{\mu\nu} = \partial_t K_{\mu\nu}\tag{1.39}$$

$$\mathcal{L}_\beta K_{\mu\nu} = \beta^\lambda D_\lambda K_{\mu\nu} + K_{\mu\lambda} D_\nu \beta^\lambda + K_{\lambda\nu} D_\mu \beta^\lambda\tag{1.40}$$

As the result, the Einstein field equations (1.1) in the standard 3+1 decomposition can be decomposed into a set of constraint equations and evolution equation of (γ_{ij}, K_{ij}) in terms of coordinate basis, which are referred to as the Arnowitt, Deser and Misner (ADM) equations [1, 44]

$$R + K^2 - K_{ij} K^{ij} = 16\pi E, \tag{Hamiltonian constraint} \tag{1.41}$$

$$D_j (K^{ij} - \gamma^{ij} K) = 8\pi S^i, \tag{momentum constraint} \tag{1.42}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \tag{spatial metric evolution} \tag{1.43}$$

$$\begin{aligned}\partial_t K_{ij} &= -D_i D_j \alpha + \alpha (R_{ij} - 2K_{ik} K_j{}^k + K K_{ij}) \\ &\quad - 8\pi \alpha \left[S_{ij} - \frac{1}{2} \gamma_{ij} (S - E) \right] \\ &\quad + \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k.\end{aligned}\tag{extrinsic curvature evolution} \tag{1.44}$$

1.3 Conformal Decomposition

The conformal decomposition factors out a scalar component from a spatial metric. It was first developed for initial data problems in general relativity [23, 42, 43, 45], and then used in reformulating evolution equations in the 3+1 formulation. In this section, we will discuss the conformal decomposition of spatial metric and extrinsic curvature in numerical relativity.

1.3.1 Conformal transformation of the spatial metric

We consider the conformal transformation of the spatial metric γ_{ij} as

$$\tilde{\gamma}_{ij} = \psi^{-4} \gamma_{ij}, \quad (1.45)$$

where $\tilde{\gamma}_{ij}$ is the *conformal metric* and ψ is a positive scaling factor satisfying

$$\psi := \det \left(\frac{\gamma}{f} \right), \quad \gamma := \det (\gamma_{ij}), \quad f := \det (f_{ij}) \quad (1.46)$$

for a time independent flat metric f_{ij} (i.e. $\det (\tilde{\gamma}_{ij}) = f$ by construction).

Thus, the *inverse conformal metric* is given by

$$\tilde{\gamma}^{ij} := \psi^4 \gamma^{ij}. \quad (1.47)$$

Substituting the conformal transformation (1.45) into equation (1.12), we can obtain the transformation law for 3-dimensional connection coefficient

$$\Gamma^i_{jk} = \tilde{\Gamma}^i_{jk} + 2 \left(\delta^i_j \tilde{D}_k \ln \psi + \delta^i_k \tilde{D}_j \ln \psi - \tilde{\gamma}_{jk} \tilde{\gamma}^{il} \tilde{D}_l \ln \psi \right). \quad (1.48)$$

From now on, we denote all objects associated with the conformal metric $\tilde{\gamma}^{ij}$ with a tilde symbol. Similarly, the transformation for Ricci tensor and scalar curvature are given by

$$\begin{aligned} R_{ij} = & \tilde{R}_{ij} - 2 \left(\tilde{D}_i \tilde{D}_j \ln \psi + \tilde{\gamma}_{ij} \tilde{\gamma}^{lm} \tilde{D}_l \tilde{D}_m \ln \psi \right) \\ & + 4 \left[\left(\tilde{D}_i \ln \psi \right) \left(\tilde{D}_j \ln \psi \right) - \tilde{\gamma}_{ij} \tilde{\gamma}^{lm} \tilde{D}_l \left(\tilde{D}_m \ln \psi \right) \right] \end{aligned} \quad (1.49)$$

$$R = \psi^{-4} \tilde{R} - 8 \psi^{-5} \tilde{D}^2 \psi, \quad (1.50)$$

where $\tilde{D}^2 = \tilde{\gamma}^{ij} \tilde{D}_i \tilde{D}_j$ denotes the Laplace operator associated with $\tilde{\gamma}_{ij}$. Therefore, using equation(1.50), the Hamiltonian constraint (1.41) becomes

$$8\tilde{D}^2\psi - \psi\tilde{R} - \psi^5 K^2 + \psi^2 K_{ij} K^{ij} = -16\pi\psi^5 E. \quad (1.51)$$

1.3.2 Conformal transformation of the extrinsic curvature

Traceless decomposition

Before we perform the conformal transformation to the extrinsic curvature K_{ij} , it is convenient to split K_{ij} into the trace part

$$K := \gamma^{ij} K_{ij}, \quad (1.52)$$

and its traceless part

$$A_{ij} := K_{ij} - \frac{1}{3}\gamma_{ij}K, \quad \text{tr}_\gamma A_{ij} = \gamma^{ij}A_{ij} = 0. \quad (1.53)$$

Therefore, we can obtain the traceless decomposition of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \quad K^{ij} = A^{ij} + \frac{1}{3}\gamma^{ij}K. \quad (1.54)$$

The evolution equations of the spatial metric (1.43) in conformal decomposition formulation can hence be written as

$$\partial_t \psi = \beta^i \tilde{D}_i \psi - \frac{1}{6}\psi \left(\alpha K - \tilde{D}_i \beta^i \right), \quad (\text{conformal factor evolution}) \quad (1.55)$$

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha\psi^{-4}A_{ij} + \tilde{\gamma}_{jk}\tilde{D}_i\beta^k + \tilde{\gamma}_{ik}\tilde{D}_j\beta^k - \frac{2}{3}\tilde{\gamma}_{ij}\tilde{D}_k\beta^k, \quad (\text{conformal metric evolution}) \quad (1.56)$$

and the constraint equations become

$$\tilde{D}^2\psi - \frac{1}{8}\psi\tilde{R} + \left(\frac{1}{8}A_{ij}A^{ij} - \frac{1}{12}K^2 + 2\pi E \right) \psi^5 = 0, \quad (\text{Hamiltonian constraint}) \quad (1.57)$$

$$\tilde{D}_i (\psi^{10} A^{ij}) - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi\psi^{10} S^i. \quad (\text{momentum constraint}) \quad (1.58)$$

Conformal transformation of the traceless part

We consider the transformation

$$A^{ij} := \psi^a \tilde{A}^{ij}, \quad (1.59)$$

for some undetermined exponent α . Here, we discuss two natural choices of a : $a = -4$ and $a = -10$.

“Time-evolution” scaling: $a = -4$. This choice of scaling was considered by Nakamura in 1994 [28], It comes naturally from the evolution equation of conformal metric (1.56), where the $\psi^{-4}A_{ij}$ term suggests the conformal transformation of A_{ij} to have the same scaling factor as the conformal spatial metric (1.47)

$$\tilde{A}^{ij} := \psi^4 A^{ij}, \quad (1.60)$$

where the indices of \tilde{A}^{ij} and \tilde{A}_{ij} are lowered and raised by the conformal metric $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ respectively (i.e. $\tilde{A}_{ij} = \tilde{\gamma}_{il}\tilde{\gamma}_{jm}\tilde{A}^{lm} = \psi^{-4}A_{ij}$). The evolution equations of conformal spatial metric therefore become

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha \tilde{A}_{ij} + \tilde{\gamma}_{jk} \tilde{D}_i \beta^k + \tilde{\gamma}_{ik} \tilde{D}_j \beta^k - \frac{2}{3} \tilde{\gamma}_{ij} \tilde{D}_k \beta^k. \quad (1.61)$$

The Hamiltonian constraint and momentum constraint in this scaling is rewritten as

$$\tilde{D}^2 \psi = \frac{1}{8} \psi \tilde{R} - \left(2\pi E - \frac{1}{12} K^2 + \frac{1}{8} \tilde{A}_{ij} \tilde{A}^{ij} \right) \psi^5, \quad (1.62)$$

$$\tilde{D}_i \left(\psi^6 \tilde{A}^{ij} \right) - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i. \quad (1.63)$$

“Momentum-constraint” scaling: $a = -10$. Another possible choice of scaling factor $a = -10$ originates from the momentum constraint equation (1.58), which was first suggested by Lichnerowicz in 1944 [23]. we define

$$\hat{A}^{ij} := \psi^{10} A^{ij}, \quad (1.64)$$

and thus

$$\hat{A}_{ij} = \psi^2 A_{ij}. \quad (1.65)$$

Here, we use hat symbol to separate the "momentum-constraint" scaling from the tilde symbol in "time-evolution" scaling. These two scaling are related by

$$\hat{A}^{ij} = \psi^6 \tilde{A}^{ij}, \quad \hat{A}_{ij} = \psi^6 \tilde{A}_{ij}, \quad \hat{A}_{ij} \hat{A}^{ij} = \psi^{12} \tilde{A}_{ij} \tilde{A}^{ij} \quad (1.66)$$

Therefore, the constraint equations can be written as

$$\tilde{D}^2 \psi = \frac{1}{8} \psi \tilde{R} - \left(2\pi E - \frac{1}{12} K^2 \right) \psi^5 - \frac{1}{8} \hat{A}_{ij} \hat{A}^{ij} \psi^{-7}, \quad (1.67)$$

$$\tilde{D}_i \hat{A}^{ij} - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i. \quad (1.68)$$

Equation (1.68) is known as *Lichnerowicz equation*. Although equation (1.63) and equation (1.68) are equivalent, they have different mathematical properties if they are treated as a partial differential equation for ψ . According to the maximal principle, the local uniqueness of the solutions depends on the sign of the exponent of ψ in the quadratic extrinsic curvature A^2 term [15, 20, 31, 37, 39]. Equation (1.63) suffers from the mathematical nonuniqueness problems due to the positive exponent (+5), while the negative exponent (−7) in equation (1.68) guarantees the local uniqueness of the solutions.

1.3.3 Conformal transverse-traceless decomposition

Using the "momentum-constraint" scaling mentioned previously, we can further decompose the symmetric, traceless tensor \hat{A}^{ij} as

$$\hat{A}^{ij} = \hat{A}_{TT}^{ij} + \hat{A}_L^{ij}, \quad (1.69)$$

where \hat{A}_{TT}^{ij} is the transverse-traceless part which is divergenceless

$$\tilde{D}_j \hat{A}_L^{ij} = 0, \quad (1.70)$$

and \hat{A}_L^{ij} is the longitudinal part satisfying

$$\hat{A}_L^{ij} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3} \tilde{\gamma}^{ij} \tilde{D}_k X^k \equiv \left(\tilde{L} X \right)^{ij}. \quad (1.71)$$

The vector X^i here is the vector potential and \tilde{L} is the *longitudinal operator* or *conformal Killing operator* associated with $\tilde{\gamma}$ which gives a symmetric, traceless tensor. The divergence

of \hat{A}^{ij} becomes

$$\tilde{D}_j \hat{A}^{ij} = \tilde{D}^2 X^i + \frac{1}{3} \tilde{D}^i (\tilde{D}_j X^j) + \tilde{R}^i_j X^j \equiv \tilde{\Delta}_L, \quad (1.72)$$

where $\tilde{\Delta}_L$ is the *vector Laplacian*. Thus, the momentum constraint in the conformal transverse-traceless (CTT) decomposition yields

$$\tilde{\Delta}_L X^i - \frac{2}{3} \psi^6 \tilde{D}^i K = 8\pi \psi^{10} S^i, \quad (1.73)$$

and the corresponding Hamiltonian constraint is the same as equation (1.67).

1.4 Gauge Condition

In the 3+1 formulation of spacetime, the gauge freedom is preserved. One can freely choose the lapse function α and shift vector β^i . In this section, we introduce the *maximal slicing* condition and *generalized Dirac gauge*, which is used in the formulation of the constrained scheme in the next section.

1.4.1 Maximal Slicing

By taking the trace of the evolution equation for extrinsic curvature (1.44), we can obtain an elliptic equation for the lapse α

$$D^2 \alpha = -\partial_t K + \alpha [K_{ij} K^{ij} + 4\pi (E + S)] + \beta^i D_i K. \quad (1.74)$$

One well-known choice to further simplify this equation is the *maximal slicing* condition

$$K = 0 = \partial_t K, \quad (1.75)$$

which corresponds to the vanishing mean curvature in all hypersurface Σ_t . This type of slicing was first introduced by Lichnerowicz [23] and then made popularized by York [37, 38]. Under this condition, the enclosed volume inside some hypersurface Σ_t becomes maximal, hence the name *maximal slicing*. With this choice of condition, equation (1.74) reduces to

$$D^2 \alpha = \alpha [A_{ij} A^{ij} + 4\pi (E + S)], \quad (1.76)$$

which is independent of the shift β^i . Alternatively, we can combine the conformally decomposed Hamiltonian equation (1.67) to obtain

$$\tilde{D}^2(\alpha\psi) = \alpha\psi \left[\frac{7}{8}\psi^{-8}\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{8}\tilde{R} + 2\pi\psi^4(E + 2S) \right]. \quad (1.77)$$

The *maximal slicing* condition not only helps decouple the constraint equations, but also has some nice physical properties: it is a natural extension of Newtonian gravity and has singularity avoidance property.

Natural generalization to Newtonian limit

Consider the Newtonian limit of a weak and static gravitational field

$$\alpha \approx 1 + \Phi, \quad \gamma_{ij} \approx (1 + 2\Phi) f_{ij}, \quad K_{ij} = 0, \quad \text{for } \Phi \ll 1, \quad (1.78)$$

where Φ is the Newtonian gravitational potential, and non-relativistic matter

$$S \ll E, \quad E \approx \rho_0, \quad (1.79)$$

where ρ_0 is the proper rest mass energy density. Equation (1.76) reduces to Poisson equation for the Newtonian gravitational potential

$$\nabla^2\Phi = 4\pi\rho_0. \quad (1.80)$$

Singularity avoidance

Another interesting property of maximal slicing is singularity avoidance. Equation (1.16) shows that under maximal slicing condition, the normal observers move like irrotational and incompressible fluid elements

$$\nabla_\mu n^\mu = 0, \quad (1.81)$$

which implies that focusing of the timelike unit normal vector field is prohibited. It can also be seen from equation (1.43), which yields the following continuity equation for volume element $\sqrt{\gamma}$ in maximal slicing condition

$$\partial_t\sqrt{\gamma} = \partial_i(\sqrt{\gamma}\beta^i). \quad (1.82)$$

This suggests that as long as a regular shift vector is chosen and the initial condition for $\gamma_{\mu\nu}$ is regular, γ is regular forever.

1.4.2 Generalized Dirac Gauge

The *Dirac gauge* was first introduced by Dirac in 1959 [19], and then generalized by the Meudon group [10]. We define the *generalized Dirac gauge* as

$$\mathcal{D}_j \tilde{\gamma}^{ij} = 0, \quad (1.83)$$

which fully specify the coordinates in the hypersurface Σ_t including the initial one. Here, \mathcal{D} denotes the covariant derivative associated with the flat background metric f_{ij} defined in equation (1.46), which relates \tilde{D} by

$$\tilde{D}_k T^{i_1 \dots i_p}_{j_1 \dots j_q} = \mathcal{D}_k T^{i_1 \dots i_p}_{j_1 \dots j_q} + \sum_{r=1}^p \Delta^{i_r}_{lk} T^{i_1 \dots l \dots i_p}_{j_1 \dots j_q} + \sum_{r=1}^q \Delta^l_{j_r k} T^{i_1 \dots i_p}_{j_1 \dots l \dots j_q}, \quad (1.84)$$

where Δ^k_{ij} is given by

$$\Delta^k_{ij} := \frac{1}{2} \tilde{\gamma}^{kl} (\mathcal{D}_i \tilde{\gamma}_{lj} + \mathcal{D}_j \tilde{\gamma}_{il} - \mathcal{D}_l \tilde{\gamma}_{ij}). \quad (1.85)$$

We can further define the potentials h^{ij} as the deviation of the conformal metric from the flat fiducial metric

$$h^{ij} := \tilde{\gamma}^{ij} - f^{ij}. \quad (1.86)$$

Thus, the generalized Dirac is equivalent to

$$\mathcal{D}_j h^{ij} = 0, \quad (1.87)$$

Under such gauge condition, the conformal Ricci tensor \tilde{R}^{ij} is simplified drastically as

$$\tilde{R}^{ij} = \frac{1}{2} \mathcal{D}^2 h^{ij} + \tilde{R}^{ij}_*, \quad (1.88)$$

where \tilde{R}_*^{ij} is the quadratic part of \tilde{R}^{ij}

$$\tilde{R}_*^{ij} := \frac{1}{2} \left[h^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} - \mathcal{D}_l h^{ik} \mathcal{D}_k h^{jl} - \tilde{\gamma}_{kl} \tilde{\gamma}^{mn} \mathcal{D}_m h^{ik} \mathcal{D}_n h^{jl} \right. \quad (1.89)$$

$$\left. + \tilde{\gamma}_{nl} \mathcal{D}_k h^{mn} (\tilde{\gamma}^{ik} \mathcal{D}_m h^{jl} + \tilde{\gamma}^{jk} \mathcal{D}_m h^{il}) + \frac{1}{2} \tilde{\gamma}^{ik} \tilde{\gamma}^{kl} \mathcal{D}_l h^{mn} \mathcal{D}_k \tilde{\gamma}_{mn} \right], \quad (1.90)$$

and $\mathcal{D}^2 = \mathcal{D}^i \mathcal{D}_i$ is the Laplacian operator associated with the flat metric with $\mathcal{D}^i := f^{ij} \mathcal{D}_j$. The curvature scalar \tilde{R} of the conformal metric does not contain any second order derivative of $\tilde{\gamma}_{ij}$ thanks to the gauge condition

$$\tilde{R} = \frac{1}{4} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_l \tilde{\gamma}_{ij} - \frac{1}{2} \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \mathcal{D}_j \tilde{\gamma}_{il}. \quad (1.91)$$

Note that the generalized gauge condition result in transverse-traceless (TT) gauge asymptotically, which are well adapted to the treatment of gravitational radiation.

1.5 Constrained scheme for the Einstein equations

Although the ADM equations (eqs. (1.41) to (1.44)) formulate the Einstein equations into a well-defined initial-value problem based on spacetime foliation, it is known to be unsuitable for numerical simulation due to the unbounded growth of numerical error (same for conformal decomposition formulation). Several reformulations were developed to maintain stable evolutions. Most of schemes such as BSSN [4, 35], CCZ4 [9], Z4c [8] schemes are based on free-evolution approach where the hyperbolic-type dynamical equations are evolved without enforcing the constraints. The constraint equations are solved only to obtain the initial data and serve as an indicator for numerical errors during the simulation. On the other hand, the fully constrained approach minimizes the hyperbolic-type evolution equations by solving the constraint equations at each time step. Though the elliptic nature of the constraint equations makes it computationally costly to solve, elliptic equations are more stable than hyperbolic equations, and the constraint-violating modes appeared in the free-evolution scheme vanish by construction in fully constrained evolution.

With the efficient elliptic solvers developed, our relativistic hydrodynamics code **Gmunu** employs the Isenberg–Wilson–Mathews (also known as conformal flatness condition) approximation to the general relativity and the fully constrained formulation. In this section, we introduce the formulations and implementations under the constrained scheme.

1.5.1 Isenberg–Wilson–Mathews approximation

The Isenberg–Wilson–Mathews approximation (IWM), also known as conformal flatness condition (CF), is a (gravitational-)waveless approximation to general relativity which was independently developed by Isenberg in 1978 [22] as well as Wilson and Mathews in 1989 [41]. In IWM, the spatial metric is approxiated to be conformally flat (hence the name conformal flatness condition)

$$\gamma_{ij} = \psi^4 f_{ij}, \quad \text{or equivalently} \quad \tilde{\gamma}_{ij} = f_{ij}, \quad (1.92)$$

under maximal slicing condition $K = 0 = \partial_t K$ (eq. (1.75)) (i.e. $\partial_t \tilde{\gamma}_{ij} = 0$ and $\tilde{D}_i = \mathcal{D}_i$ by construction). The equations for the conformal factor and extrinsic curvature can be obtained from equations (1.55) and (1.56) respectively

$$\partial_t \psi = \beta^i \mathcal{D}_i \psi + \frac{\psi}{6} \mathcal{D}_i \beta^i \quad (1.93)$$

$$K_{ij} = \frac{\psi^4}{2\alpha} \left(f_{jk} \mathcal{D}_i \beta^k + f_{ik} \mathcal{D}_j \beta^k - \frac{2}{3} f_{ij} \mathcal{D}_k \beta^k \right). \quad (1.94)$$

The CFC approximation reduces the ADM equations in a set of coupled non-linear elliptic equations for ψ (from eq. (1.62)), α (from eq. (1.77)), and β^i (from eq. (1.63))

$$\mathcal{D}^2 \psi = - \left(2\pi E + \frac{1}{8} K_{ij} K^{ij} \right) \psi^5, \quad (1.95)$$

$$\mathcal{D}^2 (\alpha \psi) = \alpha \psi^5 \left[\frac{7}{8} K_{ij} K^{ij} + 2\pi (E + 2S) \right], \quad (1.96)$$

$$\Delta_L \beta^i = 16\pi \alpha \psi^4 S^i - 2\psi^{10} K^{ij} \mathcal{D}_j (\alpha \psi^{-6}), \quad (1.97)$$

where $\Delta_L \beta^i := \mathcal{D}^2 \beta^i + \frac{1}{3} \mathcal{D}^i \mathcal{D}_j \beta^j$ is the *vector Laplacian* (eq. (1.72)) associated with the flat metric f_{ij} . Although the solutions (ψ, α, β^i) to the CFC system do not satisfy all the conformal 3+1 equations in general, it is exact for spherically symmetric spacetime [14] and correct at the 1-PN order in the post-Newtonian expansion of general relativity. The CFC scheme has been successful in numerical simulation for various astrophysical simulation [5–7, 17, 18, 27, 29, 33] and obtained good approximation to full general relativity in axisymmetric rotating neutron stars [11, 17, 18] and in rotating iron core collapses [30].

Extended CFC scheme

Despite the success of CFC scheme in astrophysical simulation, it suffers from the non-uniqueness issue as we discussed in section 1.3.2. A solution to this has been found [15, 33, 36], which involves an extra elliptic equation of the vector potential from CTT decomposition (see section 1.3.3). The resultant scheme is called *extended conformal flatness approximation* (XCFC), and the metric can be solved by following

$$\Delta_L X^i = 8\pi f^{ij} (S^*)_j, \quad (1.98)$$

$$\mathcal{D}^2 \psi = -2\pi E^* \psi^{-1} - \frac{1}{8} \psi^{-7} f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij}, \quad (1.99)$$

$$\mathcal{D}^2 (\alpha \psi) = (\alpha \psi) \left[2\pi (E^* + 2S^*) \psi^{-2} + \frac{7}{8} \psi^{-8} f_{il} f_{jm} \hat{A}^{lm} \hat{A}^{ij} \right], \quad (1.100)$$

$$\Delta_L \beta^i = 16\pi \alpha \psi^{-6} f^{ij} (S^*)_j - 2\hat{A}^{ij} \mathcal{D}_j (\alpha \psi^{-6}), \quad (1.101)$$

where the hydrodynamical conserved quantities are defined as

$$E^* := \psi^6 E, \quad S^* := \psi^6 S, \quad (S^*)_i := \psi^6 S_i, \quad (1.102)$$

and the extra vector elliptic equation (1.98) comes from the momentum constraint (eq. (1.73)) by taking CFC approximation and $\hat{A}_{TT}^{ij} \approx 0$ in CTT decomposition (eq. (1.69)), which gives

$$\hat{A}^{ij} \approx \mathcal{D}^i X^j + \mathcal{D}^j X^i - \frac{2}{3} f^{ij} \mathcal{D}_k X^k \equiv (LX)^{ij}. \quad (1.103)$$

Unlike the coupled system in IWM approximation, the XCFC system is decoupled in a hierarchical way, and hence guarantees local uniqueness. Once the hydrodynamical conserved quantities are known, the metric equations can be solved accordingly as follows

Step 1: Solve equation (1.98) for the vector potential X^i from the conserved variables $(S^*)_i$.

Step 2: Obtain \hat{A}^{ij} from equation (1.103).

Step 3: Solve equation (1.99) for the conformal factor ψ .

Step 4: Work out S^* using conformal factor ψ .

Step 5: Solve equation (1.100) for the lapse function α .

Step 6: Solve equation (1.101) for the shift vector β^i .

1.5.2 The Meudon-Valencia Fully Constrained Formulation

The fully constrained formulation (FCF) was first proposed by the Meudon group [10], and then later modified by the Valencia's group [13, 15, 16]. By choosing the maximal slicing condition (section 1.4.1) and the generalized Dirac gauge (section 1.4.2), the conformal 3+1 decomposition can be recasted in a coupled elliptic-hyperbolic system, where all the constraint equations are utilized to minimize the number of hyperbolic equations in the system. Here, we summarize the gauge conditions and algebraic constraint for the scheme

$$K := \gamma_{ij} K^{ij} = 0, \quad (\text{maximal slicing condition}) \quad (1.104)$$

$$\mathcal{D}_j h^{ij} = 0, \quad (\text{generalized Dirac gauge}) \quad (1.105)$$

$$\det \tilde{\gamma}_{ij} = f, \quad (\text{conformal decomposition}) \quad (1.106)$$

$$K^{ij} = \psi^{-10} \hat{A}^{ij}, \quad \hat{A}^{ij} = \hat{A}_{TT}^{ij} + (LX)^{ij}. \quad (\text{conformal decomposition}) \quad (1.107)$$

Note that although it would be more general to use the *Killing operator* \tilde{L} associated with $\tilde{\gamma}_{ij}$ in equation (1.107), the *Killing operator* L associated with flat metric f_{ij} is used since it is easier to handle technically. Under such gauge conditions, the ADM equations are decomposed into a coupled hyperbolic-elliptic system.

Hyperbolic sector

The Valencia's group performed first-order reduction [16] to the reduced FCF system [10] by introducing auxiliary variable

$$w^{ij}_k := \mathcal{D}_k h^{ij}, \quad (1.108)$$

Thus, the hyperbolic sector in the FCF is recasted as a first-order system in $(h^{ij}, \hat{A}^{ij}, w^{ij}_k)$ as follows

$$\partial_t h^{ij} = 2\alpha\psi^{-6}\hat{A}^{ij} + \beta^k w^{ij}_k - \tilde{\gamma}^{ik}\mathcal{D}_k\beta^j - \tilde{\gamma}^{kj}\mathcal{D}_k\beta^i + \frac{2}{3}\tilde{\gamma}^{ij}\mathcal{D}_k\beta^k, \quad (1.109)$$

$$\begin{aligned} \partial_t \hat{A}^{ij} = & \mathcal{D}_k \left(\frac{1}{2}\alpha\psi^2\tilde{\gamma}^{kl}w^{ij}_l + \beta^k\hat{A}^{ij} \right) - \hat{A}^{kj}\mathcal{D}_k\beta^i - \hat{A}^{ik}\mathcal{D}_k\beta^j + \frac{2}{3}\hat{A}^{ij}\mathcal{D}_k\beta^k + 2\alpha\psi^{-6}\tilde{\gamma}^{kl}\hat{A}^{ik}\hat{A}^{jl} \\ & - 8\pi\alpha\psi^6 \left(\psi^4 S^{ij} - \frac{1}{3}\tilde{\gamma}^{ij}S \right) + \alpha\psi^2 \left(\tilde{R}^{ij}_* - \frac{1}{3}\tilde{R} \right) + 8\alpha \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} - \frac{1}{3}\tilde{\gamma}^{ij}\tilde{\gamma}^{kl} \right) \mathcal{D}_k\psi\mathcal{D}_l\psi \\ & + 4\psi \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} + \tilde{\gamma}^{il}\tilde{\gamma}^{jk} - \frac{2}{3}\tilde{\gamma}^{kl}\tilde{\gamma}^{ij} \right) \mathcal{D}_k\psi\mathcal{D}_l\alpha - \left(\tilde{\gamma}^{ik}\tilde{\gamma}^{jl} - \frac{1}{3}\tilde{\gamma}^{ij}\tilde{\gamma}^{kl} \right) \mathcal{D}_k\mathcal{D}_l(\alpha\psi^2) \\ & - \frac{1}{2}(\tilde{\gamma}^{ik}w^{lj}_k + \tilde{\gamma}^{lk}w^{il}_k)\mathcal{D}_l(\alpha\psi^2), \end{aligned} \quad (1.110)$$

$$\partial_t w^{ij}_k = \mathcal{D}_k \left(2\alpha\psi^{-6}\hat{A}^{ij} + \beta^k w^{ij}_k - \tilde{\gamma}^{ik}\mathcal{D}_k\beta^j - \tilde{\gamma}^{kj}\mathcal{D}_k\beta^i + \frac{2}{3}\tilde{\gamma}^{ij}\mathcal{D}_k\beta^k \right), \quad (1.111)$$

where \tilde{R}^{ij}_* and \tilde{R} defined in section 1.4.2. This evolution system is proved to be strongly hyperbolic [16] under following conditions

$$\alpha \neq 0, \quad \alpha^2 - \beta^i\beta_i > 0, \quad (1.112)$$

and the characteristic speed is given by

$$\lambda_{\pm}^i = -\beta^i \pm \alpha\psi^{-2}\tilde{\gamma}^{ii}. \quad (1.113)$$

In Meudon-Valencia's approach, only the transverse-traceless (TT) part of the hyperbolic equations are evolved. Each tenorial hyperbolic equation is reduced to two scalar equations under TT decomposition [10] and solved using spectral method, and partially implicit Runge-Kutta methods (PIRK) is used to update w^{ij}_k to maintain numerical stability [12, 13]. To fulfill the constraint in equation (??), the trace part of h^{ij} was calculated by the following iteration procedure:

Step 1: Calculate $h := \det \tilde{\gamma}^{ij} - f$.

Step 2: Solve the Poisson $\mathcal{D}^2\Phi = h$.

Step 3: Update h^{ij} by $h^{ij}_{(new)} = h^{ij}_{(old)} + \frac{1}{2}(hf^{ij} - \mathcal{D}^i\mathcal{D}^j\Phi)$.

Step 4: Repeat step 1 to 3 until the $\det \tilde{\gamma}^{ij} = f$ is satisfied.

Elliptic sector

The elliptic sector in the FCF scheme is given by

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_lX^i + \frac{1}{3}\tilde{\gamma}^{ik}\mathcal{D}_k\mathcal{D}_lX^l + \Delta^i{}_{kl}(LX)^{kl} = 8\pi\tilde{\gamma}^{ij}(S^*)_j - \Delta^i{}_{kl}\hat{A}_{TT}^{kl}, \quad (1.114)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\psi = -2\pi E^*\psi^{-1} - \frac{1}{8}\psi^{-7}\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij} + \frac{1}{8}\psi\tilde{R}, \quad (1.115)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l(\alpha\psi) = (\alpha\psi) \left[2\pi\psi^{-2}(E^* + 2S^*) + \frac{7}{8}\psi^{-8}\tilde{\gamma}_{il}\tilde{\gamma}_{jm}\hat{A}^{lm}\hat{A}^{ij} + \frac{\tilde{R}}{8} \right], \quad (1.116)$$

$$\tilde{\gamma}^{kl}\mathcal{D}_k\mathcal{D}_l\beta^i + \frac{1}{3}\tilde{\gamma}^{ik}\mathcal{D}_k\mathcal{D}_l\beta^l = 16\pi\alpha\psi^{-6}\tilde{\gamma}^{ij}(S^*)_j + 2\hat{A}^{ij}\mathcal{D}_j(\alpha\psi^{-6}) - 2\alpha\psi^{-6}\Delta^i{}_{kl}\hat{A}^{kl}, \quad (1.117)$$

which can be solved following the same routine as XCFC scheme (section 1.5.1). Note that the FCF scheme is a natural generalization of XCFC scheme since it reduces to XCFC if the approximation $h^{ij} = \hat{A}_{TT}^{ij} = 0$ is made.

passive FCF

The elliptic sector in FCF scheme is technically more difficult to solve. Therefore, the Valencia's group an approximation scheme based on FCF, called passive FCF [13], which neglected the backreaction of the GWs onto the dynamics of the system. Unlike the XCFC which sets the gravitational radiation to zero, the gravitational wave can be evolved in passive FCF scheme by keeping the hyperbolic sector in FCF scheme, while solving the elliptic sector in XCFC scheme.

1.5.3 The fully-constrained scheme in Gmunu

In Gmunu, the fully-constrained scheme is solved in a slightly different approach. Instead of solving eqs. (1.109) to (1.111), we omit the auxiliary variable w^{ij}_k and evolve the following

hyperbolic equations for (h^{ij}, A^{ij})

$$\partial_t h^{ij} = 2\alpha\psi^{-6}\hat{A}^{ij} + \beta^k \mathcal{D}_k h^{ij} - \tilde{\gamma}^{ik} \mathcal{D}_k \beta^j - \tilde{\gamma}^{kj} \mathcal{D}_k \beta^i + \frac{2}{3} \tilde{\gamma}^{ij} \mathcal{D}_k \beta^k, \quad (1.118)$$

$$\begin{aligned} \partial_t \hat{A}^{ij} = & \frac{1}{2} \alpha \psi^2 \tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l h^{ij} + \beta^k \mathcal{D}_k \hat{A}^{ij} - \hat{A}^{kj} \mathcal{D}_k \beta^i - \hat{A}^{ik} \mathcal{D}_k \beta^j + \frac{5}{3} \hat{A}^{ij} \mathcal{D}_k \beta^k + 2\alpha\psi^{-6} \tilde{\gamma}^{kl} \hat{A}^{ik} \hat{A}^{jl} \\ & - 8\pi\alpha\psi^6 \left(\psi^4 S^{ij} - \frac{1}{3} \tilde{\gamma}^{ij} S \right) + \alpha\psi^2 \left(\tilde{R}_*^{ij} - \frac{1}{3} \tilde{R} \right) + 8\alpha \left(\tilde{\gamma}^{ik} \tilde{\gamma}^{jl} - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \right) \mathcal{D}_k \psi \mathcal{D}_l \psi \\ & + 4\psi \left(\tilde{\gamma}^{ik} \tilde{\gamma}^{jl} + \tilde{\gamma}^{il} \tilde{\gamma}^{jk} - \frac{2}{3} \tilde{\gamma}^{kl} \tilde{\gamma}^{ij} \right) \mathcal{D}_k \psi \mathcal{D}_l \alpha - \left(\tilde{\gamma}^{ik} \tilde{\gamma}^{jl} - \frac{1}{3} \tilde{\gamma}^{ij} \tilde{\gamma}^{kl} \right) \mathcal{D}_k \mathcal{D}_l (\alpha\psi^2) \\ & - \frac{1}{2} \left(\tilde{\gamma}^{ik} \mathcal{D}_k h^{lj} + \tilde{\gamma}^{lk} \mathcal{D}_k h^{il} - \tilde{\gamma}^{kl} \mathcal{D}_k h^{ij} \right) \mathcal{D}_l (\alpha\psi^2), \end{aligned} \quad (1.119)$$

$$(1.120)$$

directly without performing TT composition. Note that although the hyperbolic system is *unconditionally unstable* if second-order explicit *Runge-Kutta* (RK) method used, it has shown to be numerically stable using fourth-order explicit RK method [12, 34]. In *Gmunu*, the hyperbolic sector in FCF scheme is evolving simulataneously with the hydrodynamical system using fourth-order *strong stability-preserving Runge-Kutta* (ssprk) methods. After the time integration is done, we need to modify $(h_{old}^{ij}, \hat{A}_{old}^{ij})$ so that the gauge conditions can be satisfied. The generalized Dirac gauge condition is then enforced by *elliptic divergence cleaning* method as follows

Step 1: Solve the *vector Laplacian* equation

$$\Delta_L W^i = \mathcal{D}_j h_{(old)}^{ij}. \quad (1.121)$$

Step 2: Update h^{ij} by

$$h_{(new)}^{ij} = h_{(old)}^{ij} - (LW)^{ij}. \quad (1.122)$$

This treatment is inspired by the divergence cleaning of magnetic monopole in magnetohydrodynamics [2, 3, 40]. In *Gmunu*, a fourth-order central finite difference scheme is used for $\mathcal{D}_j h_{old}^{ij}$, and a second-order central finite difference scheme is used for $\Delta_L W^i$ and $(LW)^{ij}$. To enforce the algebraic constraint (1.106), we modify the trace part of h^{ij} using the same routine described in section 1.5.2.

For \hat{A}^{ij} , it can be modified by the following procedures:

Step 1: Obtain the traceless part of $\hat{A}_{(old)}^{ij}$ by

$$\hat{A}_{(1)}^{ij} = \hat{A}_{(old)}^{ij} - \frac{1}{3} f^{ij} f_{kl} \hat{A}_{(old)}^{kl}. \quad (1.123)$$

Step 2: Calculate the vector potential X^i using momentum constraint

$$\tilde{\gamma}^{kl} \mathcal{D}_k \mathcal{D}_l X^i + \frac{1}{3} \tilde{\gamma}^{ik} \mathcal{D}_k \mathcal{D}_l X^l + \Delta^i{}_{kl} (LX)^{kl} = 8\pi \tilde{\gamma}^{ij} (S^*)_j - \Delta^i{}_{kl} \hat{A}_{(1)}^{kl} - \mathcal{D}_j \hat{A}_{(1)}^{ij}. \quad (1.124)$$

Note that differing from equation (1.114), $\hat{A}_{(1)}^{ij}$ is used on the right-hand-side instead of \hat{A}_{TT}^{ij} . Thus, the vector potential X^i here is an additional longitudinal part for \hat{A}^{ij} to satisfy the momentum constraint.

Step 3: Update \hat{A}^{ij} by

$$\hat{A}_{(new)}^{ij} = \hat{A}_{(1)}^{ij} + (LX)^{ij}. \quad (1.125)$$

The remaining unknowns (ψ, α, β^i) can then be solved accordingly by adopting the same procedure in section 1.5.2.

Chapter 2

Formulations of the Relativistic Hydrodynamics

2.1 Relativistic Hydrodynamics

In the previous chapter, we explored the formulation of the left hand side of the Einstein equations. In this chapter, we focus on the right hand side of the equations, specifically the 3+1 treatment of perfect fluid in conservative form [24, 32].

2.1.1 Perfect fluid

In this section, we consider relativistic *perfect fluid* (i.e. no viscous effects and heat fluxes). The energy-momentum tensor of perfect fluid is given by

$$T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}, \quad (2.1)$$

where ρ is the rest-mass density, u^μ is the fluid 4-velocity, $h = 1 + \epsilon + \frac{p}{\rho}$ is the enthalpy, p is the pressure, and ϵ is the specific internal energy.

The *Lorentz factor* W of the fluid with respects to the Eulerian observers is given by the projection of fluid 4-velocity u^μ to n^μ

$$W := -n_\mu u^\mu = \alpha u^t. \quad (2.2)$$

Thus, the spatial 4-velocity of a fluid measured by Eulerian observers is

$$v^\mu := \frac{\gamma_\nu^\mu u^\nu}{W}, \quad (2.3)$$

or in component form

$$v^t = 0, \quad v^i = \frac{u^i}{W} + \frac{\beta^i}{\alpha}, \quad (2.4)$$

$$v_t = \beta_i v^i, \quad v_i = \frac{u_i}{W}. \quad (2.5)$$

As a result, the *Lorentz factor* can also be written as

$$W = \frac{1}{\sqrt{1 - v_i v^i}}. \quad (2.6)$$

The dynamics of fluid are governed by the equations of rest-mass conservation and energy momentum conservation. The system is closed by the equation of state (EOS) of the fluid, which reflects the thermodynamic properties of the fluid at a macroscopic level. Here, we consider two common analytic equation of state as follows:

1. Ideal-gas

$$p = (\Gamma - 1)\rho\epsilon, \quad (2.7)$$

where Γ is the polytropic index of the gas.

2. Polytropic EOS

$$p = K\rho^\Gamma, \quad (2.8)$$

where Γ again is the polytropic index the gas, and K is a constant of proportionality.

2.1.2 Rest-mass conservation

The general-relativistic conservation of rest-mass is given by

$$\nabla_\mu J^\mu = 0, \quad J^\mu := \rho u^\mu, \quad (2.9)$$

where J^μ is the rest-mass current density. Under 3+1 decomposition, the rest-mass conservation can be written as

$$\partial_t (\sqrt{\gamma} D) + \partial_i [\sqrt{\gamma} D (\alpha v^i - \beta^i)] = 0, \quad (2.10)$$

where

$$D := \rho \alpha u^t = \rho W, \quad (2.11)$$

is the conserved quantity.

2.1.3 Energy and momentum conservation

Other than the conservation of rest-mass, the energy and momentum conservation of the fluid must also be satisfied

$$\nabla_\nu T^{\mu\nu} = 0. \quad (2.12)$$

Following the 3+1 decomposition of stress-energy tensor introduced in equation (1.30), the fluid quantities in the case of perfect fluid become

$$S^{\mu\nu} = \rho h W^2 v^\mu v^\nu + p \gamma^{\mu\nu}, \quad (2.13)$$

$$S^\mu = \rho h W^2 v^\mu, \quad (2.14)$$

$$E = \rho h W^2 - p. \quad (2.15)$$

The conservation of energy and momentum hence leads to

$$\partial_t (\sqrt{\gamma} S_j) + \partial_i [\sqrt{\gamma} (\alpha S^i_j - \beta^i S_j)] = \sqrt{\gamma} \left(\frac{1}{2} \alpha S^{ik} \partial_j \gamma_{ik} + S_i \partial_j \beta^i - E \partial_j \alpha \right), \quad (2.16)$$

$$\partial_t (\sqrt{\gamma} E) + \partial_i [\sqrt{\gamma} (\alpha S^i - \beta^i E)] = \sqrt{\gamma} (\alpha K_{ij} S^{ij} - S^i \partial_i \alpha). \quad (2.17)$$

Equation (2.17) can also be written as

$$\partial_t (\sqrt{\gamma} \tau) + \partial_i [\sqrt{\gamma} (\alpha (S^i - D v^i) - \beta^i \tau)] = \sqrt{\gamma} (\alpha K_{ij} S^{ij} - S^i \partial_i \alpha), \quad (2.18)$$

to obtain better numerical accuracy, where the conserved quantity τ is defined as

$$\tau := E - D. \quad (2.19)$$

2.1.4 Conservative form of hydrodynamics equations

Here, we summarize eqs. (2.10), (2.16) and (2.18) and express in conservative form

$$\partial_t (\sqrt{\gamma} \mathbf{Q}) \partial_i (\sqrt{\gamma} \mathbf{F}^i) = \sqrt{\gamma} \mathbf{S}, \quad (2.20)$$

where the conserved variables \mathbf{Q} , flux terms \mathbf{F}^i and source terms \mathbf{S} are

$$\mathbf{Q} = \begin{pmatrix} D \\ S_j \\ \tau \end{pmatrix} = \begin{pmatrix} \rho W \\ \rho h W^2 v_j \\ \rho h W^2 - p - D \end{pmatrix}, \quad (2.21)$$

$$\mathbf{F}^i = \begin{pmatrix} D(\alpha v^i - \beta^i) \\ S_j(\alpha v^i - \beta^i) + \alpha \delta^i_j p \\ \tau \text{left}(\alpha v^i - \beta^i + \alpha p v^i) \end{pmatrix}, \quad (2.22)$$

$$\mathbf{S} = \begin{pmatrix} 0 \\ \frac{1}{2} \alpha S^{ik} \partial_j \gamma_{ik} + S_i \partial_j \beta^i - E \partial_j \alpha \\ \alpha K_{ij} S^{ij} - S^i \partial_i \alpha \end{pmatrix}, \quad (2.23)$$

which is commonly known as the "*Valencia formulation*" of the relativistic-hydrodynamics equations.

The characteristic speeds of such system is given by

$$\lambda_0 = \alpha v^i - \beta^i \quad (\text{triple eigenvalue}), \quad (2.24)$$

$$\lambda_{\pm} = \frac{\alpha}{1 - v^2 c_s^2} \left\{ v^i (1 - c_s^2) \pm c_s^2 \sqrt{(1 - v^2) [\gamma^i_i (1 - v^2 c_s^2) - v^{i^2} (1 - c_s^2)]} \right\}, \quad (2.25)$$

where λ_{\pm} measures the propagation speeds of the *acoustic waves* of the system, and λ_0 corresponds to the propagation speed of *matter waves*.

2.1.5 Conserved to Primitive variables conversion

It is not trivial to convert the conserved variables (D, S_i, τ) introduced in section 2.1.4 to primitive variables (ρ, W, v^i, p) in relativistic hydrodynamics. One must solve non-linear equations numerically. In **Gmunu**, we adopted the routine used in [21]. The implementation details are included here:

Step 1: Calculate the rescaled variables and some useful relations which are fixed during the iterations

$$S := \sqrt{S_i S^i}, \quad (2.26)$$

$$r := \frac{S}{D}, \quad q := \frac{\tau}{D}, \quad k := \frac{S}{\tau + D}. \quad (2.27)$$

Step 2: Determine the bounds of the root

$$z_- := \frac{k/2}{\sqrt{1 - k^2/4}}, \quad z_+ := \frac{k}{\sqrt{1 - k^2}}. \quad (2.28)$$

Step 3: In the interval $[z_-, z_+]$, we solve

$$f(z) = z - \frac{r}{\hat{h}(z)}, \quad (2.29)$$

where

$$\hat{h}(z) = (1 + \hat{\epsilon}(z))(1 + \hat{a}(z)), \quad (2.30)$$

$$\hat{\epsilon}(z) = \hat{W}(z)q - zr + \frac{z^2}{1 + \hat{W}(z)}, \quad \hat{a}(z) = \frac{\hat{p}(z)}{\hat{\rho}(z)(1 + \hat{\epsilon}(z))}, \quad (2.31)$$

$$\hat{p}(z) = p(\hat{\rho}(z), \hat{\epsilon}(z)), \quad \hat{\rho}(z) = \frac{D}{\hat{W}(z)}, \quad \hat{W}(z) = \sqrt{1 + z^2}. \quad (2.32)$$

Equation (2.29) is solved using Illinois algorithm in **Gmunu**. Note that during the iterations, we enforce the density ρ and the specific energy ϵ fall into the validity region of the EOS, i.e., we evaluate the updated ρ and ϵ with $\hat{\rho}(z) = \max(\min(\rho_{\max, \hat{\rho}}, \rho_{\min}))$ and $\hat{\epsilon}(z) = \max(\min(\epsilon_{\max, \hat{\epsilon}}, \epsilon_{\min}))$.

Step 4: With the root z_0 of eq. (2.29), we can then work out the primitive variables (ρ, ϵ, p) respectively with the equations used in step 3. For the velocity v^i , it can be obtained with z by

$$\hat{v}^i(z) = \frac{S^i/D}{\hat{h}(z)\hat{W}(z)}. \quad (2.33)$$

2.2 The reference-metric formalism

The reference-metric formalism, a generalization of the "*Valencia*" formulation, was original proposed by Montero [26] in 2014 for general relativistic hydrodynamics (GRHD) and later extended to ideal magnetohydrodynamics. With the conformal decomposition introduced in section 1.3.1, the determinant of the spatial metric can be rewritten as

$$\begin{aligned}\sqrt{\gamma} &= \psi^6 \sqrt{\tilde{\gamma}} \\ &= \sqrt{\hat{\gamma}} \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}},\end{aligned}\tag{2.34}$$

where $\hat{\gamma}_{ij}$ is the *time-independent reference metric* which can be chosen to fit the mesh grid in curvilinear coordinate. Therefore, the "*Valencia*" formulation in equation (2.24) can be generalized as

$$\partial_t \mathbf{q} + \hat{\nabla}_i \mathbf{f}^i = \mathbf{s},\tag{2.35}$$

where $\hat{\nabla}_i$ is the covariant derivatives associated with the time-independent reference metric $\hat{\gamma}_{ij}$, $(\mathbf{q}, \mathbf{f}^i, \mathbf{s})$ are the *conformally rescaled* conserved variables, fluxes and source terms respectively.

Comparing to the "*Valencia*" formulation, the hydrodynamical equations in reference-metric approach are numerically more accurate in curvilinear coordinates in multi-dimensional simulation, especially when symmetry is imposed. Equation (2.34) can be further written as

$$\partial_t \mathbf{q} + \frac{1}{\sqrt{\hat{\gamma}}} \partial_j \left(\sqrt{\hat{\gamma}} \mathbf{f}^j \right) = \mathbf{s} + \mathbf{s}_{geom},\tag{2.36}$$

where \mathbf{s}_{geom} are the *geometrical* source terms which contain the 3-Christoffel symbols $\hat{\Gamma}^i_{jk}$ associated with the reference metric $\hat{\gamma}_{ij}$.

Note that the momentum conservation in this expression satisfy to *machine precision* rather than to the level of truncation error due to the fact that the geometrical source terms \mathbf{s}_{geom} are identically vanishing for the components associated with ignorable coordinates in the metric.

2.2.1 General relativistic hydrodynamics (GRHD) equations

The equations of relativistic hydrodynamics in reference-metric approach is given by

$$\partial_t (q_D) + \frac{1}{\sqrt{\hat{\gamma}}} \partial_j \left[\sqrt{\hat{\gamma}} (f_D)^j \right] = 0, \quad (2.37)$$

$$\partial_t (q_{S_i}) + \frac{1}{\sqrt{\hat{\gamma}}} \partial_j \left[\sqrt{\hat{\gamma}} (f_{S_i})^j \right] = s_{S_i} + \hat{\Gamma}^l_{ik} (f_{S_l})^k, \quad (2.38)$$

$$\partial_t (q_\tau) + \frac{1}{\sqrt{\hat{\gamma}}} \partial_j \left[\sqrt{\hat{\gamma}} (f_\tau)^j \right] = s_\tau, \quad (2.39)$$

$$(2.40)$$

The conformally rescaled conserved quantities \mathbf{q} are

$$q_D := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} D = \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (\rho W), \quad (2.41)$$

$$q_{S_i} := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} S_i = \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (\rho h W^2 v_i), \quad (2.42)$$

$$q_\tau := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} \tau = \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (\rho h W^2 - p - D). \quad (2.43)$$

The flux terms \mathbf{f}^i are

$$(f_D)^i := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (D \hat{v}^i), \quad (2.44)$$

$$(f_{S_j})^i := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (S_j \hat{v}^i + \delta^i_j \alpha p), \quad (2.45)$$

$$(f_\tau)^i := \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} (\tau \hat{v}^i + \alpha p v^i), \quad (2.46)$$

where $\hat{v}^i := \alpha v^i - \beta^i$.

Finally, the source terms \mathbf{s} are

$$s_D = 0, \quad (2.47)$$

$$s_{S_i} = \alpha \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} \left[-T^{00} \alpha \partial_i \alpha + T^0_k \hat{\nabla}_i \beta^k + \frac{1}{2} (T^{00} \beta^j \beta^k + 2T^{0j} \beta^k + T^{jk}) \hat{\nabla}_i \gamma_{jk} \right], \quad (2.48)$$

$$s_\tau = \alpha \psi^6 \sqrt{\tilde{\gamma}/\hat{\gamma}} \left[T^{00} (K_{ij} \beta^i \beta^j - \beta^k \partial_k \alpha) + 2T^{0j} (2K_{jk} \beta^k - \partial_j \alpha) + T^{ij} K_{ij} \right]. \quad (2.49)$$

Part II

Methodology: Numerical methods and tests

Chapter 3

Numerical Methods

In this chapter, we are going to discuss the details of the numerical method used in Gmunu. In the fully-constrained scheme, the evolution of relativistic hydrodynamics in dynamical spacetime consists of a set of coupled hyperbolic-elliptic system. Firstly, we discuss the numerical method adopted for the hyperbolic system in Gmunu, including the high-resolution shock-capturing (HRSC) method and the finite volume method. Then, we discuss the multigrid solver in Gmunu, which is used to solve the elliptic system.

3.1 High-resolution shock-capturing (HRSC) methods

3.1.1 Approximate Riemann solver

3.1.2 Reconstruction scheme

3.1.3 Finite volume discretization in curvilinear coordinate

3.2 Atmosphere treatment

3.2.1 Positivity preserving limit

3.3 Time discretization

3.3.1 Explicit Runge-Kutta Method

3.3.2 Courant-Friedrichs *Lewy Conditions*

3.4 Multigrid Method for elliptic equations

3.4.1 Overview

3.4.2 Smoother

3.4.3 Restriction and Prolongation

3.4.4 Multigrid cycle

3.4.5 Grid Structure

3.4.6 Ghost cells and refinement boundary

Chapter 4

Numerical Tests and Results

4.1 Multigrid solver tests

4.1.1 Newtonian gravitational field

4.2 General relativistic hydrodynamics in dynamic space-time

4.2.1 Non-rotating neutron star

4.2.2 Rapidly rotating neutron star

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Appendix A

Useful relations for implementation of constrained scheme

A.1 The elliptic equations in constrained scheme

A.2 Generalized Dirac gauge conditions

Appendix B

Reference flat metric in 3D

