New Approach of Numerical Relativity

Implementation, tests and applications

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This thesis is submitted for the degree of
Master of Philosophy in Physics

July 2021

I would like to dedicate this thesis to my loving parents . . .

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

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July 2021

Acknowledgements

And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

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Chapter 2

Formulations of Einstein Field Equations

2.1 Introduction

Due to the complexity and nonlinearity of Einstein field equations, it is extremely difficult to obtain analytical solution even for the simplest dynamical evolution systems. Therefore, the accurate discription of the such systems can only be derived through numerical simulation. For this, we need to reformulate the Einstein equations as an initial-value problem or Cauchy problem. In this chapter, we will introduction the Arnowitt-Deser-Misner (ADM) formulation, which is the foundation of the 3+1 numerical relativity. In particular, we will focus on the constrained scheme for the Einstein equations.

2.2 The 3+1 decomposition of spacetime

2.2.1 Foliation of spacetime

In the 3+1 decomposition, the spacetime manifold $\mathcal{M}$ is foliated into a set of non-intersecting spacelike hypersurfaces Σ_t parameterized by the coordinate time t. We denote a future-directed timelike unit four-vector n^μ normal to the hypersurface Σ_t (i.e. $n_\mu \propto \nabla_\mu t$). The induced spacetime metric $\gamma_{\mu\nu}$ on each hypersurfuce can then be defined as

$$\gamma_{\mu\nu} \coloneqq g_{\mu\nu} + n_\mu n_\nu. \tag{2.1}$$

Thus, we can construct spatial projection tensor $\gamma^{\mu}{}_{\nu}$ and time projection tensor $N^{\mu}{}_{\nu}$ as

$$\gamma^{\mu}{}_{\nu} := \delta^{\mu}{}_{\nu} + n^{\mu} n_{\nu}, \quad N^{\mu}{}_{\nu} := -n^{\mu} n_{\nu}, \tag{2.2}$$

which decompose any generic four-vector U^{μ} into spatial part $\gamma^{\mu}{}_{\nu} U^{\nu}$ and timelike part $N^{\mu}{}_{\nu} U^{\nu}$. Therefore, we can decompose the timelike vector field $t^{\mu} = (\partial/\partial t)^{\mu} = \alpha n^{\mu} + \beta^{\mu}$ into two components as

$$\alpha := -t^{\mu} n_{\mu}, \quad \beta^{\mu} := t^{\nu} \gamma^{\mu}{}_{\nu}, \tag{2.3}$$

where the lapse function α measures the physical proper time ($\alpha \Delta t$) between two neighboring spatial hypersurface Σ_t and $\Sigma_{t+\Delta t}$, and the shift vector β^i measures the changes of spatial coordinates on $\Sigma_{t+\Delta t}$.
Here, we summarise several useful relations. The timelike normal vector n^{μ} and its corresponding one-form n_{μ} can be expressed as

$$n^{\mu} = \frac{1}{\alpha} \left(1, \beta^i\right), \quad n_{\mu} = \left(\alpha, \vec{0}, \right). \tag{2.4}$$

The generic line element in 3+1 decomposition is given by

$$ds^2 = -\left(\alpha^2 - \beta^i \beta_i\right) dt^2 + \beta_i dx^i dt + \gamma_{ij} dx^i dx^j \tag{2.5}$$

The covariant and contravariant components of the metric can be written as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i \beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^j \\ \beta^i & \gamma^{ij} \end{pmatrix}. \tag{2.6}$$

From equation(2.6), we can conclude that

$$\sqrt{-g} = \alpha \sqrt{\gamma}, \tag{2.7}$$

where $g := \det\left(g_{\mu\nu}\right)$ and $\gamma := \det\left(\gamma_{ij}\right)$.

2.2.2 Derivative operator

With the 3+1 decomposition, we can now construct the 3-dimensional covariant derivative D_{α} associated with $\gamma_{\mu\nu}$ by projecting the 4-dimensional covariant derivative

∇_α onto Σ_t, which is given by

$$D_\alpha T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \gamma_\alpha{}^\beta \gamma_{\rho_1}{}^{\mu_1} \gamma_{\rho_2}{}^{\mu_2} \ldots \gamma_{\nu_1}{}^{\sigma_1} \gamma_{\nu_2}{}^{\sigma_2} \ldots \nabla_\beta T^{\rho_1\rho_2\cdots}{}_{\sigma_1\sigma_2\ldots}, \tag{2.8}$$

for arbitrary tensor $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}$ on spatial hypersurface Σ_t. Using equation(2.8), it can be shown that the corvariant derivative of $\gamma_{\mu\nu}$ vanishes

$$\begin{aligned} D_\alpha \gamma_{\mu\nu} &= \gamma_\alpha{}^\beta \gamma_\rho{}^\mu \gamma_\nu{}^\sigma \nabla_\beta \left(g_{\rho\sigma} + n_\rho n_\sigma \right) \\ &= \gamma_\alpha{}^\beta \gamma_\rho{}^\mu \gamma_\nu{}^\sigma \left(n_\rho \nabla_\beta n_\sigma + n_\sigma \nabla_\beta n_\rho \right) = 0 \end{aligned} \tag{2.9}$$

The components of 3-dimensional connection coefficients $\Gamma^\alpha{}_{\mu\nu}$ in coordinate basis can be expressed as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} \gamma^{\alpha\beta} \left(\partial_\nu \gamma_{\beta\mu} + \partial_\mu \gamma_{\beta\nu} - \partial_\beta \gamma_{\mu\nu} \right). \tag{2.10}$$

Here, the upper left index $^{(4)}$ marks the 4-dimensional tensors while the unmarked one represents purely spatial 3-dimensional tensors. Similarly, the 3-dimensional Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ associated with $\gamma_{\mu\nu}$ is defined by requiring that

$$2 D_{[\nu} D_{\mu]} W_\beta = W_\alpha R^\alpha{}_{\beta\mu\nu}, \quad R^\alpha{}_{\beta\mu\nu} n_\alpha = 0, \tag{2.11}$$

which can be explicitly expressed in coordinate basis as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\mu\rho} \Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\nu\rho} \Gamma^\rho{}_{\beta\mu}. \tag{2.12}$$

The 3-dimensional Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined in a similar manner as their 4-dimensional counterparts

$$R_{\mu\nu} := R^\alpha{}_{\mu\alpha\nu}, \quad R := R^\mu{}_\mu. \tag{2.13}$$

Since $R^\alpha{}_{\beta\mu\nu}$ is purely spatial and can be computed by the spatial derivatives of the spatial metric alone, it only contains about information about the curvature intrinsic to the hypersurface Σ_t, but cannot contain all the information of ${}^{(4)}R^\alpha{}_{\beta\mu\nu}$ which includes time derivative of the 4-dimensional metric. The missing information can be found in a purely spatial symmetric tensor called the extrinsic curvature $K_{\mu\nu}$.

2.2.3 Extrinsic curvature

The extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the spatial metric $\gamma_{\mu\nu}$. Therefore, the spatial metric and extrinsic curvature $\left(\gamma_{\mu\nu}, K_{\mu\nu}\right)$ are equivalent to the positions and velocities in classical mechanics, which describe the instantaneous state of the gravitational field. It can be obtained by projecting of the gradient of the normal vector $\gamma_\mu{}^\lambda\gamma_\nu{}^\rho\nabla_\lambda n_\rho$ into the hypersurface Σ_t, and then taking the negative expression of the symmetric part

$$\begin{aligned} K_{\mu\nu} &:= -\,\gamma_\mu{}^\lambda\gamma_\nu{}^\rho\nabla_\lambda n_\rho \\ &= -\,\gamma_\mu{}^\lambda\left(\delta_\nu{}^\rho + n_\nu n^\rho\right)\nabla_\lambda n_\rho \\ &= -\,\gamma_\mu{}^\lambda\nabla_\lambda n_\nu, \end{aligned} \tag{2.14}$$

where the identity $n^\rho\nabla_\lambda n_\rho = 0$ is used.
We can also define an spatial acceleration a_ν

$$a_\nu := n^\mu\nabla_\mu n_\nu, \tag{2.15}$$

satisfying the identities

$$a_\nu = D_\nu \ln\alpha, \tag{2.16}$$

to rewrite equation(2.14) as

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu \tag{2.17}$$

Finally, we can write the extrinsci curvature $K_{\mu\nu}$ as the Lie derivative of the spatial metric along the local normal n^μ

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n\gamma_{\mu\nu} \tag{2.18}$$

2.2.4 The Arnowitt, Deser and Misner equations

In the standard 3+1 decomposition, the Einstein fields equations can be decomposed into a set of evolution equations and a set of constraint equations, which are referred to as the Arnowitt, Deser and Misner (ADM) equations. The evolution equations govern the evolution of $\left(\gamma_{ij}, K_{ij}\right)$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \qquad \text{(spatial metric evolution)}$$

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A.2 Generalized Dirac gauge conditions

Appendix B

Reference flat metric in 3D