New Approach of Numerical Relativity

Implementation, tests and applications

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I would like to dedicate this thesis to my loving parents ...

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This dissertation contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Alan Tsz-Lok Lam
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Acknowledgements

And I would like to acknowledge ...

Abstract

This is where you write your abstract ...

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Chapter 2

Formulations of Einstein Field Equations

2.1 Introduction

Due to the complexity and nonlinearity of Einstein field equations, it is extremely difficult to obtain analytical solution even for the simplest dynamical evolution systems. Therefore, the accurate discription of the such systems can only be derived through numerical simulation. For this, we need to reformulate the Einstein equations as an initial-value problem or Cauchy problem. In this chapter, we will introduction the Arnowitt-Deser-Misner (ADM) formulation, which is the foundation of the 3+1 numerical relativity. In particular, we will focus on the constrained scheme for the Einstein equations.

2.2 The 3+1 decomposition of spacetime

2.2.1 Foliation of spacetime

In the 3+1 decomposition, the spacetime manifold $\mathcal{M}$ is foliated into a set of non-intersecting spacelike hypersurfaces Σ_t parameterized by the coordinate time t [7]. We denote a future-directed timelike unit four-vector n^μ normal to the hypersurface Σ_t (i.e. $n_\mu \propto \nabla_\mu t$). The induced spacetime metric $\gamma_{\mu\nu}$ on each hypersurfuce can then be defined as

$$\gamma_{\mu\nu} \coloneqq g_{\mu\nu} + n_\mu n_\nu. \tag{2.1}$$

Thus, we can construct spatial projection tensor ${\gamma^\mu}_\nu$ and time projection tensor ${N^\mu}_\nu$ as

$$ {\gamma^\mu}_\nu \coloneqq {\delta^\mu}_\nu + n^\mu n_\nu, \qquad\qquad {N^\mu}_\nu \coloneqq -n^\mu n_\nu, \tag{2.2}$$

which decompose any generic four-vector U^μ into spatial part ${\gamma^\mu}_\nu U^\nu$ and timelike part ${N^\mu}_\nu U^\nu$. Therefore, we can decompose the timelike vector field

$$ t^\mu = \alpha n^\mu + \beta^\mu \tag{2.3}$$

into two components as

$$ \alpha \coloneqq -t^\mu n_\mu, \qquad\qquad \beta^\mu \coloneqq t^\nu {\gamma^\mu}_\nu, \tag{2.4}$$

where the lapse function α measures the physical proper time $(\alpha\Delta t)$ between two neighboring spatial hypersurface Σ_t and $\Sigma_{t+\Delta t}$, and the shift vector β^i measures the changes of spatial coordinates on $\Sigma_{t+\Delta t}$.

Here, we summarise several useful relations. The timelike normal vector n^μ and its corresponding one-form n_μ can be expressed as

$$ n^\mu = \frac{1}{\alpha}\left(1, \beta^i\right), \qquad\qquad n_\mu = \left(\alpha, \vec{0},\right). \tag{2.5}$$

The generic line element in 3+1 decomposition is given by

$$ ds^2 = -\left(\alpha^2 - \beta^i\beta_i\right)dt^2 + \beta_i dx^i dt + \gamma_{ij}dx^i dx^j \tag{2.6}$$

The covariant and contravariant components of the metric can be written as

$$ g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^i\beta_i & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \qquad\qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{\alpha^2} & \beta^j \\ \beta^i & \gamma^{ij} \end{pmatrix}. \tag{2.7}$$

From equation(2.7), we can conclude that

$$ \sqrt{-g} = \alpha\sqrt{\gamma}, \tag{2.8}$$

where $g \coloneqq \det\left(g_{\mu\nu}\right)$ and $\gamma \coloneqq \det\left(\gamma_{ij}\right)$.

2.2.2 Derivative operator

With the 3+1 decomposition, we can now construct the 3-dimensional covariant derivative D_α associated with $\gamma_{\mu\nu}$ by projecting the 4-dimensional covariant derivative ∇_α onto Σ_t, which is given by

$$D_\alpha T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots} = \gamma_\alpha{}^\beta\gamma_{\rho_1}{}^{\mu_1}\gamma_{\rho_2}{}^{\mu_2}\ldots\gamma_{\nu_1}{}^{\sigma_1}\gamma_{\nu_2}{}^{\sigma_2}\ldots\nabla_\beta T^{\rho_1\rho_2\cdots}{}_{\sigma_1\sigma_2\ldots}, \tag{2.9}$$

for arbitrary tensor $T^{\mu_1\mu_2\cdots}{}_{\nu_1\nu_2\ldots}$ on spatial hypersurface Σ_t. Using equation(2.9), it can be shown that the corvariant derivative of $\gamma_{\mu\nu}$ vanishes

$$\begin{aligned} D_\alpha\gamma_{\mu\nu} &= \gamma_\alpha{}^\beta\gamma_\rho{}^\mu\gamma_\nu{}^\sigma\nabla_\beta\left(g_{\rho\sigma}+n_\rho n_\sigma\right) \\ &= \gamma_\alpha{}^\beta\gamma_\rho{}^\mu\gamma_\nu{}^\sigma\left(n_\rho\nabla_\beta n_\sigma+n_\sigma\nabla_\beta n_\rho\right)=0 \end{aligned} \tag{2.10}$$

The components of 3-dimensional connection coefficients $\Gamma^\alpha{}_{\mu\nu}$ in coordinate basis can be expressed as

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\gamma^{\alpha\beta}\left(\partial_\nu\gamma_{\beta\mu}+\partial_\mu\gamma_{\beta\nu}-\partial_\beta\gamma_{\mu\nu}\right). \tag{2.11}$$

Here, the upper left index $^{(4)}$ marks the 4-dimensional tensors while the unmarked one represents purely spatial 3-dimensional tensors. Similarly, the 3-dimensional Riemann tensor $R^\alpha{}_{\beta\mu\nu}$ associated with $\gamma_{\mu\nu}$ is defined by requiring that

$$2D_{[\nu}D_{\mu]}W_\beta = W_\alpha R^\alpha{}_{\beta\mu\nu}, \qquad R^\alpha{}_{\beta\mu\nu}n_\alpha = 0, \tag{2.12}$$

which can be explicitly expressed in coordinate basis as

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu\Gamma^\alpha{}_{\beta\nu}-\partial_\nu\Gamma^\alpha{}_{\beta\mu}+\Gamma^\alpha{}_{\mu\rho}\Gamma^\rho{}_{\beta\nu}-\Gamma^\alpha{}_{\nu\rho}\Gamma^\rho{}_{\beta\mu}. \tag{2.13}$$

The 3-dimensional Ricci tensor $R_{\mu\nu}$ and Ricci scalar R are defined in a similar manner as their 4-dimensional counterparts

$$R_{\mu\nu} \coloneqq R^\alpha{}_{\mu\alpha\nu}, \qquad R \coloneqq R^\mu{}_\mu. \tag{2.14}$$

Since $R^\alpha{}_{\beta\mu\nu}$ is purely spatial and can be computed by the spatial derivatives of the spatial metric alone, it only contains about information about the curvature intrinsic to the hypersurface Σ_t, but cannot contain all the information of ${}^{(4)}R^\alpha{}_{\beta\mu\nu}$ which includes time derivative of the 4-dimensional metric. The missing information can be found in a purely spatial symmetric tensor called the extrinsic curvature $K_{\mu\nu}$.

2.2.3 Extrinsic curvature

The extrinsic curvature $K_{\mu\nu}$ is related to the time derivative of the spatial metric $\gamma_{\mu\nu}$. Therefore, the spatial metric and extrinsic curvature $(\gamma_{\mu\nu}, K_{\mu\nu})$ are equivalent to the positions and velocities in classical mechanics, which describe the instantaneous state of the gravitational field. It can be obtained by projecting of the gradient of the normal vector ${\gamma_\mu}^\lambda {\gamma_\nu}^\rho \nabla_\lambda n_\rho$ into the hypersurface Σ_t, and then taking the negative expression of the symmetric part

$$\begin{aligned} K_{\mu\nu} &:= -\, {\gamma_\mu}^\lambda {\gamma_\nu}^\rho \nabla_\lambda n_\rho \\ &= -\, {\gamma_\mu}^\lambda \left({\delta_\nu}^\rho + n_\nu n^\rho\right) \nabla_\lambda n_\rho \\ &= -\, {\gamma_\mu}^\lambda \nabla_\lambda n_\nu, \end{aligned} \tag{2.15}$$

where the identity $n^\rho \nabla_\lambda n_\rho = 0$ is used.
We can also define an spatial acceleration a_ν

$$a_\nu := n^\mu \nabla_\mu n_\nu, \tag{2.16}$$

satisfying the identities

$$a_\nu = D_\nu \ln \alpha, \tag{2.17}$$

to rewrite equation(2.15) as

$$K_{\mu\nu} = -\nabla_\mu n_\nu - n_\mu a_\nu \tag{2.18}$$

Finally, we can write the extrinsic curvature $K_{\mu\nu}$ as the Lie derivative of the spatial metric along the local normal n^μ

$$K_{\mu\nu} = -\frac{1}{2}\mathcal{L}_n \gamma_{\mu\nu}. \tag{2.19}$$

Using equation(2.3), we can express the Lie derivative $\mathcal{L}_n$ as

$$\mathcal{L}_n = \frac{1}{\alpha}\left(\mathcal{L}_t - \mathcal{L}_\beta\right), \tag{2.20}$$

and thus obtain the evolution equation for the spatial metric

$$\mathcal{L}_t \gamma_{\mu\nu} = -2\alpha K_{\mu\nu} + \mathcal{L}_\beta \gamma_{\mu\nu}. \tag{2.21}$$

2.2.4 The Gauss, Codazzi and Ricci equations

To express the Einstein field equations in term of the spatial variables $(\gamma_{\mu\nu}, K_{\mu\nu})$ we defined previous, we first have to relate 3-dimensional Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$ on Σ_t to the 4-dimensional Riemann tensor ${}^{(4)}R^{\alpha}{}_{\beta\mu\nu}$ on $\mathcal{M}$, The relation between $R^{\alpha}{}_{\beta\mu\nu}$ and the full spatial projection of ${}^{(4)}R^{\alpha}{}_{\beta\mu\nu}$ is given by the *Gauss' equation*

$$R_{\alpha\beta\mu\nu} + K_{\alpha\mu}K_{\beta\nu} - K_{\alpha\nu}K_{\beta\mu} = \gamma_{\alpha}{}^{\rho}\gamma_{\beta}{}^{\sigma}\gamma_{\mu}{}^{\lambda}\gamma_{\nu}{}^{\delta\,(4)}R_{\rho\sigma\lambda\delta}, \tag{2.22}$$

while the projection of ${}^{(4)}R^{\alpha}{}_{\beta\mu\nu}$ with one index projected in the normal direction is given by the *Codazzi equation*

$$D_{\nu}K_{\mu\alpha} - D_{\mu}K_{\nu\alpha} = \gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma}\gamma_{\alpha}{}^{\lambda}n^{\delta\,(4)}R_{\rho\sigma\lambda\delta}. \tag{2.23}$$

Finally, by projecting two indices of ${}^{(4)}R_{\rho\sigma\lambda\delta}$ in the normal direction, we can relate it to the time derivative of $K_{\mu\nu}$

$$\mathcal{L}_n K_{\mu\nu} = n^{\alpha}n^{\beta}\gamma_{\mu}{}^{\lambda}\gamma_{\nu}{}^{\delta\,(4)}R_{\alpha\beta\lambda} - \frac{1}{\alpha}D_{\mu}D_{\nu}\alpha - K_{\nu}{}^{\lambda}K_{\mu\lambda}, \tag{2.24}$$

which is called the *Ricci equation*.

2.2.5 Constraint and evolution equations

Using the Gauss, Codazzi and Ricci equations, the Einstein fields equations can be decomposed into a set of evolution equations and a set of constraint equations of $(\gamma_{\mu\nu}, K_{\mu\nu})$. To begin with, we define the following matter quantities

$$S_{\mu\nu} :=\gamma^{\alpha}{}_{\mu}\gamma^{\beta}{}_{\nu}T_{\alpha\beta}, \tag{2.25}$$
$$S_{\mu} := -\gamma^{\alpha}{}_{\mu}n^{\beta}T_{\alpha\beta}, \tag{2.26}$$
$$S :=S^{\mu}{}_{\mu}, \tag{2.27}$$
$$E :=n^{\alpha}n^{\beta}T_{\alpha\beta}, \tag{2.28}$$

which decompose the stress-energy tensor as

$$T_{\mu\nu} = En_{\mu}n_{\nu} + S_{\mu}n_{\nu} + S_{\nu}n_{\mu} + S_{\mu\nu}. \tag{2.29}$$

By contracting the α, μ indices in equation(2.22), we can obtain

$$R_{\mu\nu} = \gamma_{\mu}{}^{\alpha}\gamma_{\nu}{}^{\beta}\left({}^{(4)}R_{\alpha\beta} + n^{\rho}n^{\sigma(4)}R_{\alpha\rho\beta\sigma}\right) + K_{\mu\lambda}K_{\nu}{}^{\lambda} - K_{\mu\nu}K, \tag{2.30}$$

where $K \coloneqq K^{\mu}{}_{\mu}$ is the trace of the extrinsic curvature, called the *mean curvature*. Further contracting the μ, ν indices in equation(2.30), the contracted Gauss' equation becomes

$$2n^{\mu}n^{\nu}G_{\mu\nu} = R + K^2 - K_{\mu\nu}K^{\mu\nu}, \tag{2.31}$$

Using the Einstein equation(1.1), we can obtain the *Hamiltonian constraint*

$$R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi E. \tag{2.32}$$

Similarly, by contracting α, ν indices in equation(2.23), the Codazzi equation yields

$$D_{\nu}K_{\mu}{}^{\nu} - D_{\mu}K = -8\pi\gamma^{\alpha}{}_{\mu}n^{\beta}T_{\alpha\beta}, \tag{2.33}$$

and thus obtain the *momentum constrain equation*

$$D_{\nu}K_{\mu}{}^{\nu} - D_{\mu}K = 8\pi S_{\mu}. \tag{2.34}$$

The Ricci equation (2.24) can be rewritten using equation(2.30) to

$$\mathcal{L}_n K_{\mu\nu} = R_{\mu\nu} - \gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma(4)}R_{\rho\sigma} - 2K_{\mu\lambda}K_{\nu}{}^{\lambda} + KK_{\mu\nu} - \frac{1}{\alpha}D_{\mu}D_{\nu}\alpha. \tag{2.35}$$

Using the Einstein equations (1.1)

$$\begin{aligned}\gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma(4)}R_{\rho\sigma} &= 8\pi\gamma_{\mu}{}^{\rho}\gamma_{\nu}{}^{\sigma}\left(T_{\rho\sigma} - \frac{1}{2}g_{\rho\sigma}T^{\mu}{}_{\mu}\right)\\ &= 8\pi\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S - E\right)\right]\end{aligned} \tag{2.36}$$

and equation(2.20), we can finally obtain the evolution equation for $K_{\mu\nu}$ as

$$\mathcal{L}_t K_{\mu\nu} = -D_{\mu}D_{\nu}\alpha + \alpha\left(R_{\mu\nu} - 2K_{\mu\lambda}K_{\nu}{}^{\lambda} + KK_{\mu\nu}\right) - 8\pi\alpha\left[S_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\left(S - E\right)\right] + \mathcal{L}_{\beta}K_{\mu\nu}. \tag{2.37}$$

2.2.6 The Arnowitt, Deser and Misner equations

The Lie derivative in the evolution equations (2.21) and (2.37) can be expressed in terms of coordinate basis as

$$\mathcal{L}_t K_{\mu\nu} = \partial_t K_{\mu\nu} \tag{2.38}$$

$$\mathcal{L}_\beta K_{\mu\nu} = \beta^\lambda D_\lambda K_{\mu\nu} + K_{\mu\lambda} D_\nu \beta^\lambda + K_{\lambda\nu} D_\mu \beta^\lambda \tag{2.39}$$

As the result, the Einstein field equaitons (1.1) in the standard 3+1 decomposition can be decomposed into a set of constraint equations and evolution equation of (γ_{ij}, K_{ij}) in terms of coordinate basis, which are referred to as the Arnowitt, Deser and Misner (ADM) equations [1, 15]

$$R + K^2 - K_{ij}K^{ij} = 16\pi E, \qquad \text{(Hamiltonian constraint)} \tag{2.40}$$

$$D_j \left(K^{ij} - \gamma^{ij} K\right) = 8\pi S^i, \qquad \text{(momentum constraint)} \tag{2.41}$$

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i, \qquad \text{(spatial metric evolution)} \tag{2.42}$$

$$\begin{aligned} \partial_t K_{ij} = &- D_i D_j \alpha + \alpha \left(R_{ij} - 2K_{ik} K_j{}^k + K K_{ij}\right) \\ &- 8\pi\alpha \left[S_{ij} - \frac{1}{2}\gamma_{ij} \left(S - E\right)\right] \qquad \text{(extrinsic curvature evolution)} \\ &+ \beta^k D_k K_{ij} + K_{ik} D_j \beta^k + K_{kj} D_i \beta^k. \end{aligned} \tag{2.43}$$

2.3 Conformal Decomposition

The conformal decompostion factors out a scalar component from a spatial metric. It was first developped for initial data problems in general relativity [6, 13, 14, 16], and then used in reformulating evolution equations in the 3+1 formulation. In this section, we will discuss the conformal decomposition of spatial metric and extrinsic curvature in numerical relativity.

2.3.1 Conformal transformation of the spatial metric

We consider the conformal transformation of the spatial metric γ_{ij} as

$$\tilde{\gamma}_{ij} = \psi^{-4} \gamma_{ij}, \tag{2.44}$$

where $\tilde{\gamma}_{ij}$ is the *conformal metric* and ψ is a positive scaling factor satisfying

$$\psi := \det\left(\frac{\gamma}{f}\right), \qquad \gamma := \det(\gamma_{ij}), \qquad f := \det(f_{ij}) \tag{2.45}$$

for a time independent flat metric f_{ij} (i.e. $\det(\tilde{\gamma}_{ij}) = f$ by construction).
Thus, the *inverse conformal metric* is given by

$$\tilde{\gamma}^{ij} := \psi^4 \gamma^{ij}. \tag{2.46}$$

Substituting the conformal transformation (2.44) into equation(2.11), we can obtain the transformation law for 3-dimensional connection coefficient

$$\Gamma^i{}_{jk} = \tilde{\Gamma}^i{}_{jk} + 2\left(\delta^i{}_j \tilde{D}_k \ln\psi + \delta^i{}_k \tilde{D}_j \ln\psi - \tilde{\gamma}_{jk}\tilde{\gamma}^{il}\tilde{D}_l \ln\psi\right). \tag{2.47}$$

From now on, we denote all objects associated with the conformal metric $\tilde{\gamma}^{ij}$ with a tilde symbol. Similarly, the transformation for Ricci tensor and scalar curvature are given by

$$\begin{aligned} R_{ij} =& \tilde{R}_{ij} - 2\left(\tilde{D}_i\tilde{D}_j \ln\psi + \tilde{\gamma}_{ij}\tilde{\gamma}^{lm}\tilde{D}_l\tilde{D}_m \ln\psi\right) \\ &+ 4\left[\left(\tilde{D}_i \ln\psi\right)\left(\tilde{D}_j \ln\psi\right) - \tilde{\gamma}_{ij}\tilde{\gamma}^{lm}\tilde{D}_l\left(\tilde{D}_l \ln\psi\right)\left(\tilde{D}_m \ln\psi\right)\right] \end{aligned} \tag{2.48}$$

$$R = \psi^{-4}\tilde{R} - 8\psi^{-5}\tilde{D}^2\psi, \tag{2.49}$$

where $\tilde{D}^2 = \tilde{\gamma}^{ij}\tilde{D}_i\tilde{D}_j$ denotes the Laplace operator associated with $\tilde{\gamma}_{ij}$. Therefore, using equation(2.49), the Hamiltonian constraint (2.40) becomes

$$8\tilde{D}^2\psi - \psi\tilde{R} - \psi^5 K^2 + \psi^2 K_{ij}K^{ij} = -16\pi\psi^5 E. \tag{2.50}$$

2.3.2 Conformal transformation of the extrinsic curvature

Traceless decomposition

Before we perform the conformal transformation to the extrinsic curvature K_{ij}, it is convenient to split K_{ij} into the trace part

$$K := \gamma^{ij}K_{ij}, \tag{2.51}$$

and its traceless part

$$A_{ij} \coloneqq K_{ij} - \frac{1}{3}\gamma_{ij}K, \qquad \mathrm{tr}_\gamma\, A_{ij} = \gamma^{ij}A_{ij} = 0. \tag{2.52}$$

Therefore, we can obtain the tracelss decomposition of the extrinsic curvature

$$K_{ij} = A_{ij} + \frac{1}{3}\gamma_{ij}K, \qquad K^{ij} = A^{ij} + \frac{1}{3}\gamma^{ij}K. \tag{2.53}$$

The evolution equations of the spatial metric (2.42) in conformal deposition formulation can hence be written as

$$\partial_t\psi = \beta^i\tilde{D}_i\psi - \frac{1}{6}\psi\left(\alpha K - \tilde{D}_i\beta^i\right), \qquad \text{(conformal factor evolution)} \tag{2.54}$$

$$\partial_t\tilde{\gamma}_{ij} = -2\alpha\psi^{-4}A_{ij} + \tilde{\gamma}_{jk}\tilde{D}_i\beta^k + \tilde{\gamma}_{ik}\tilde{D}_j\beta^k - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}_k\beta^k, \qquad \text{(conformal metric evolution)} \tag{2.55}$$

and the constraint equations become

$$\tilde{D}^2\psi - \frac{1}{8}\psi\tilde{R} + \left(\frac{1}{8}A_{ij}A^{ij} - \frac{1}{12}K^2 + 2\pi E\right)\psi^5 = 0, \qquad \text{(Hamiltonian constraint)} \tag{2.56}$$

$$\tilde{D}_i\left(\psi^{10}A^{ij}\right) - \frac{2}{3}\psi^6\tilde{D}^iK = 8\pi\psi^{10}S^i. \qquad \text{(momentum constraint)} \tag{2.57}$$

Conformal transformation of the traceless part

We consider the transformation

$$A^{ij} \coloneqq \psi^a\tilde{A}^{ij}, \tag{2.58}$$

for some undetermined exponent α. Here, we discuss two natural choices of a: $a = -4$ and $a = -10$.

"Time-evolution" scaling: $a = -4$. This choice of scaling was considered by Nakamura in 1994 [8], It comes naturally from the evolution equation of conformal metric (2.55), where the $\psi^{-4}A_{ij}$ term suggests the conformal transformation of A_{ij} to have the same scaling factor as the conformal spatial metric (2.46)

$$\tilde{A}^{ij} \coloneqq \psi^4 A^{ij}, \tag{2.59}$$

where the indices of $\tilde{A}^{ij}$ and $\tilde{A}_{ij}$ are lowered and raised by the conformal metric $\tilde{\gamma}_{ij}$ and $\tilde{\gamma}^{ij}$ respectively (i.e. $\tilde{A}_{ij} = \tilde{\gamma}_{il}\tilde{\gamma}_{jm}\tilde{A}^{lm} = \psi^{-4}A_{ij}$). The evolution equations of conformal spatial metric therefore become

$$\partial_t \tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{jk}\tilde{D}_i\beta^k + \tilde{\gamma}_{ik}\tilde{D}_j\beta^k - \frac{1}{3}\tilde{\gamma}_{ij}\tilde{D}_k\beta^k. \tag{2.60}$$

The Hamiltonian constraint and momentum constraint in this scaling is rewritten as

$$\tilde{D}^2\psi = \frac{1}{8}\psi\tilde{R} - \left(2\pi E - \frac{1}{12}K^2 + \frac{1}{8}\tilde{A}_{ij}\tilde{A}^{ij}\right)\psi^5, \tag{2.61}$$

$$\tilde{D}_i\left(\psi^6\tilde{A}^{ij}\right) - \frac{2}{3}\psi^6\tilde{D}^iK = 8\pi\psi^{10}S^i. \tag{2.62}$$

"Momentum-constraint" scaling: $a = -10$. Another possible choice of scaling factor $a = -10$ originates from the momentum constraint equation (2.57), which was first suggested by Lichnerowicz in 1944 [6]. we define

$$\hat{A}^{ij} := \psi^{10}A^{ij}, \tag{2.63}$$

and thus

$$\hat{A}_{ij} = \psi^2 A_{ij}. \tag{2.64}$$

Here, we use hat symbol to separate the "momentum-constraint" scaling from the tilde symbol in "time-evolution" scaling. These two scaling are related by

$$\hat{A}^{ij} = \psi^6\tilde{A}^{ij}, \qquad \hat{A}_{ij} = \psi^6\tilde{A}_{ij}, \qquad \hat{A}_{ij}\hat{A}^{ij} = \psi^{12}\tilde{A}_{ij}\tilde{A}^{ij} \tag{2.65}$$

Therefore, the constraint equations can be written as

$$\tilde{D}^2\psi = \frac{1}{8}\psi\tilde{R} - \left(2\pi E - \frac{1}{12}K^2\right)\psi^5 - \frac{1}{8}\hat{A}_{ij}\hat{A}^{ij}\psi^{-7}, \tag{2.66}$$

$$\tilde{D}_i\hat{A}^{ij} - \frac{2}{3}\psi^6\tilde{D}^iK = 8\pi\psi^{10}S^i. \tag{2.67}$$

Equation(2.67) is known as *Lichnerowicz equation.* Although equation (2.62) and equation (2.67) are equivalent, they have different mathematical properties if they are treated as a partial differential equation for ψ. According to the maximal principle, the local uniqueness of the solutions depends on the sign of the exponent of ψ in the quadratic extrinsic curvature A^2 term [3, 5, 9, 10, 12]. Equation (2.62) suffers from the mathematical nonuniqueness problems due to the positive exponent (+5), while the negative exponent (−7) in equation (2.67) guarantees the local uniquess of the solutions.

2.3.3 Conformal transverse-traceless decomposition

Using the "momentum-constraint” scaling mentioned previously, we can further decompose the symmetric, traceless tensor $\hat{A}^{ij}$ as

$$\hat{A}^{ij} = \hat{A}^{ij}_{TT} + \hat{A}^{ij}_{L}, \tag{2.68}$$

where $\hat{A}^{ij}_{TT}$ is the transverse-traceless part which is divergenceless

$$\tilde{D}_j \hat{A}^{ij}_{L} = 0, \tag{2.69}$$

and $\hat{A}^{ij}_{L}$ is the longitudinal part satisfying

$$\hat{A}^{ij}_{L} = \tilde{D}^i X^j + \tilde{D}^j X^i - \frac{2}{3}\tilde{\gamma}^{ij}\tilde{D}_k X^k \equiv \left(\tilde{L}W\right)^{ij}. \tag{2.70}$$

The vector X^i here is the vector potential and $\tilde{L}$ is the *longitudinal operator* or *conformal Killing operator* associated with $\tilde{\gamma}$ which gives a symmetric, traceless tensor. The divergence of $\hat{A}^{ij}$ becomes

$$\tilde{D}_j \hat{A}^{ij} = \tilde{D}^2 X^i + \frac{1}{3}\tilde{D}^i\left(\tilde{D}_j X^j\right) + \tilde{R}^i{}_j X^j \equiv \tilde{\Delta}_L, \tag{2.71}$$

where $\tilde{\Delta}_L$ is the *vector Laplacian*. Thus, the momentum constraint in the conformal transverse-traceless (CTT) decomposition yields

$$\tilde{D}^2 X^i + \frac{1}{3}\tilde{D}^i\left(\tilde{D}_j X^j\right) + \tilde{R}^i{}_j X^j - \frac{2}{3}\psi^6 \tilde{D}^i K = 8\pi\psi^{10} S^i, \tag{2.72}$$

with the Hamiltonian constraint same as equation (2.66).

2.4 Gauge Condition

In the 3+1 formulation of spacetime, the gauge freedom is preserved. One can freely choose the lapse function α and shift vector β^i. In this section, we introduce the *maximal slicing* condition and *generalized Dirac gauge*, which is used in the formulation of the constrained scheme in the next section.

2.4.1 Maximal Slicing

By taking the trace of the evolution equation for extrinsic curvature (2.43), we can obtain an elliptic equation for the lapse α

$$D^2\alpha = -\partial_t K + \alpha\left[K_{ij}K^{ij} + 4\pi\left(E+S\right)\right] + \beta^i D_i K. \tag{2.73}$$

One well-known choice to further simplify this equation is the *maximal slicing* condition

$$K = 0 = \partial_t K, \tag{2.74}$$

which corresponds to the vanishing mean curvature in all hypersurface Σ_t. This type of slicing was first introduced by Lichnerowicz [6] and then made popularized by York [10, 11]. Under this condition, the enclosed volume inside some hypersurface Σ_t becomes maximal, hence the name *maximal slicing*. With this choice of condition, equation (2.73) reduces to

$$D^2\alpha = \alpha\left[A_{ij}A^{ij} + 4\pi\left(E+S\right)\right], \tag{2.75}$$

which is independent of the shift β^i. Alternatively, we can combine the conformally decomposed Hamiltonian equation (2.66) to obtain

$$\tilde{D}^2\left(\alpha\psi\right) = \alpha\psi\left[\frac{7}{8}\psi^{-8}\hat{A}_{ij}\hat{A}^{ij} + \frac{1}{8}\tilde{R} + 2\pi\psi^4\left(E+2S\right)\right]. \tag{2.76}$$

The *maximal slicing* condition not only helps decouple the constraint equations, but also has some nice physical properties: it is a natural extension of Newtonian gravity and has singularity avoidance property.

Natural generalization to Newtonian limit

Consider the Newtonain limit of a weak and static gravitational field

$$\alpha \approx 1+\Phi, \qquad \gamma_{ij} \approx \left(1+2\Phi\right)f_{ij}, \qquad K_{ij} = 0, \qquad \text{for } \Phi \ll 1, \tag{2.77}$$

where Φ is the Newtonian gravitational potential, and non-relativistic matter

$$S \ll E, \qquad\qquad E \approx \rho_0, \tag{2.78}$$

where ρ_0 is the proper rest mass energy density. Equation (2.75) reduces to Poisson equation for the newtonian gravitational potential

$$\nabla^2\Phi = 4\pi\rho_0. \tag{2.79}$$

Singularity avoidance

Another interesting property of maximal slicing is singularity avoidance. Equation (2.15) shows that under maximal slicing condition, the normal observers move like irrotational and incompressible fluid elements

$$\nabla_\mu n^\mu = 0, \tag{2.80}$$

which implies that focusing of the timelike unit normal vector field is prohibited. It can also be seen from equation (2.42), which yields the following continuity equation for volume element $\sqrt{\gamma}$ in maximal slicing condition

$$\partial_t\sqrt{\gamma} = \partial_i\left(\sqrt{\gamma}\beta^i\right). \tag{2.81}$$

This suggests that as long as a regular shift vector is chosen and the initial condition for $\gamma_{\mu\nu}$ is regular, γ is regular forever.

2.4.2 Generalized Dirac Gauge

The *Dirac gauge* was first introduced by Dirac in 1959 [4], and then generalized by the Meudon group [2]. We define the *generalized Dirac gauge* as

$$\mathcal{D}_j\tilde{\gamma}^{ij} = 0, \tag{2.82}$$

which fully specify the coordinates in the hypersurface Σ_t including the initial one. Here, $\mathcal{D}$ denotes the covariant derivative associated with the flat background metric

f_{ij} defined in equation (2.45), which relates $\tilde{D}$ by

$$\tilde{D}_k T^{i_1\dots i_p}{}_{j_1\dots j_q} = \mathcal{D}_k T^{i_1\dots i_p}{}_{j_1\dots j_q} + \sum_{r=1}^{p} \Delta^{i_r}{}_{lk} T^{i_1\dots l\dots i_p}{}_{j_1\dots j_q} + \sum_{r=1}^{q} \Delta^{l}{}_{j_r k} T^{i_1\dots i_p}{}_{j_1\dots l\dots j_q}, \tag{2.83}$$

where $\Delta^k{}_{ij}$ is given by

$$\Delta^k{}_{ij} := \frac{1}{2}\tilde{\gamma}^{kl}\left(\mathcal{D}_i\tilde{\gamma}_{lj} + \mathcal{D}_j\tilde{\gamma}_{il} - \mathcal{D}_l\tilde{\gamma}_{ij}\right). \tag{2.84}$$

We can further define the potentials h^{ij} as the deviation of the conformal metric from the flat fiducial metric

$$h^{ij} := \tilde{\gamma}^{ij} - f^{ij}. \tag{2.85}$$

Thus, the generalized Dirac is equivalent to

$$\mathcal{D}_j h^{ij} = 0, \tag{2.86}$$

Under such gauge condition, the conformal Ricci tensor $\tilde{R}^{ij}$ is simplified drastically as

$$\tilde{R}^{ij} = \frac{1}{2}\mathcal{D}^2 h^{ij} + \tilde{R}^{ij}_{*}, \tag{2.87}$$

where $\tilde{R}^{ij}_{*}$ is the quadratic part of $\tilde{R}^{ij}$

$$\tilde{R}^{ij}_{*} :- \frac{1}{2}\Big[h^{kl}\mathcal{D}_k\mathcal{D}_l h^{ij} - \mathcal{D}_l h^{ik}\mathcal{D}_k h^{jl} - \tilde{\gamma}_{kl}\tilde{\gamma}^{mn}\mathcal{D}_m h^{ik}\mathcal{D}_n h^{jl} \tag{2.88}$$

$$+ \tilde{\gamma}_{nl}\mathcal{D}_k h^{mn}\left(\tilde{\gamma}^{ik}\mathcal{D}_m h^{jl} + \tilde{\gamma}^{jk}\mathcal{D}_m h^{il}\right) + \frac{1}{2}\tilde{\gamma}^{ik}\tilde{\gamma}^{kl}\mathcal{D}_l h^{mn}\mathcal{D}_k\tilde{\gamma}_{mn}\Big], \tag{2.89}$$

and $\mathcal{D}^2 = f^{ij}\mathcal{D}_i\mathcal{D}_j$ is the Laplacian operator associated with the flat metric. The curvature scalar $\tilde{R}$ of the conformal metric does not contain any second order derivative of $\tilde{\gamma}_{ij}$ thanks to the gauge condition

$$\tilde{R} = \frac{1}{4}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_l\tilde{\gamma}_{ij} - \frac{1}{2}\tilde{\gamma}^{kl}\mathcal{D}_k h^{ij}\mathcal{D}_j\tilde{\gamma}_{il}. \tag{2.90}$$

Note that the generalized gauge condition result in transverse-traceless (TT) gauge asymtotically, which are well adapted to the treatment of gravitational radiation.

2.5 Constrained scheme for the Einstein equations

2.5.1 Isenberg–Wilson–Mathews Approximation

The

2.5.2 The Fully Constrained Formulation

Chapter 3

Formulations of the Relativistic Hydrodynamics

3.1 The 3+1 "Valencia" formulation

3.2 The reference-metric formulism

3.3 Conserved to Primitive variables conversion

Chapter 4

NUmerical Methods and Tests

4.1 Multigrid Method for elliptic equations

4.1.1 Overview

4.1.2 Smoother

4.1.3 Restriction and Prolongation

4.1.4 Multigrid cycle

4.1.5 Grid Structure

4.1.6 Ghost cells and refinement boundary

4.1.7 Numerical Tests

4.2 Numerical method for hydrodynamics

4.2.1 Finite volume methods

4.2.2 Time Discretization

4.2.3 Atmosphere Treatment

4.2.4 Numerical Tests

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Appendix A

Useful relations for implementation of constrained scheme

A.1 The elliptic equations in constrained scheme

A.2 Generalized Dirac gauge conditions

Appendix B

Reference flat metric in 3D