# HW4b_answer

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**EE 242, Win 22**
**Homework 4b**

**HW4b Topics**: LTI systems, complex numbers and exponentials, Fourier series

**HW4b References**: Lectures 16-17

**NOTE**: You will notice that some selected problems are given answers through "Show that" statements. Make sure you show all your work clearly for every problem.
Throughout the assignment, $u(t), u[n]$ are the unit step functions in continuous and discrete time, respectively.
**HW4b Problems**:

1. For each system shown, state whether or not the given observed input/output pair $x(t), y(t)$ indicates the system is *not* time-invariant. Explain your reasoning either way.

   (a) $\cos(\pi t) \xrightarrow{T_a} 3\sin(\pi(t+1))$

   (b) $e^{-2t}\cos(\pi t) \xrightarrow{T_b} e^{-2t}\sin(\pi t)u(t)$

   (c) $a^{-n} \xrightarrow{T_c} ba^{-n-2}, |a| > 1, b \in \mathbb{R}$

   (d) $\cos(\frac{\pi}{4}n) + e^{-\frac{\pi}{2}|n|} \xrightarrow{T_d} \sin(\frac{\pi}{4}n)$

a) $\cos(\pi t) \rightarrow 3\sin(\pi(t+1))$ $\omega = \pi$, freq preserved $\rightarrow$ LTI $\rightarrow$ TI

b) $e^{-2t}\cos(\pi t) \rightarrow e^{-2t}\sin(\pi t)\, u(t)$ × high light part is LTI because ω presence, then check u(t)

$y(t) = u(t) \cdot x(t)$

$\left.\begin{array}{l} y(t-t_0) = u(t-t_0)\, x(t-t_0) \\ T\{x(t-t_0)\} = u(t)\, x(t-t_0) \end{array}\right\}$ × same not TI

c) $a^{-n} \rightarrow ba^{-n-2}$ $|a|>1$, $b \in R$

$ba^{-n-2} = ba^{-2}a^{-n}$

$y = T\{x(n)\} = ba^{-2}x(n)$

$\left.\begin{array}{l} y(n-n_0) = ba^{-2}x(n-n_0) \\ T\{x(n-n_0)\} = ba^{-2}x(n-n_0) \end{array}\right\}$ same $\rightarrow$ TI

d) $\cos(\frac{\pi}{4}n) + e^{-\frac{\pi}{2}|n|} \longrightarrow \sin(\frac{\pi}{4}n)$
$\cos(\frac{\pi}{4}n - \frac{\pi}{2}) = \cos(\frac{\pi}{4}(n-2))$

$T\{x(n)\} = y(n) = x(n-2) - e^{-\frac{\pi}{2}|n-2|}$

$\left.\begin{array}{l} y(n-n_0) = x(n-2-n_0) - e^{-\frac{\pi}{2}|n-n_0-2|} \\ T\{x(n-n_0)\} = x(n-2-n_0) - e^{-\frac{\pi}{2}|n-2|} \end{array}\right\}$ not same

× TI

2. Consider an LTI system with an impulse response $h[n] = a^n u[n]$ for a complex number $a$, i.e. $a \in \mathbb{C}$. For each of the following $a$:

- Compute $|a|$
- Compute $\angle a$ in radians ($-\pi$ to $\pi$)
- Show whether or not the system is BIBO stable

(a) $a = j$

(b) $a = \frac{1+2j}{4}$

(c) $a = \frac{2}{1+j}$

a) $|a| = \sqrt{1^2} = 1$

$\angle a = \arctan\left(\frac{1}{0}\right) = \frac{\pi}{2}$

$a = e^{j\frac{\pi}{2}}$

$h[n] = \underbrace{e^{j\frac{\pi}{2}n}}_{\text{periodic}} u[n]$ $\qquad \int_0^\infty \left|e^{j\frac{\pi}{2}n}\right| dn = \infty$

not BIBO stable

b) $|a| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{4}$

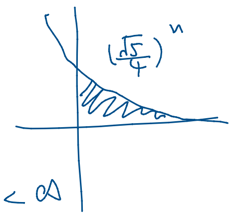
$\angle a = \arctan\left(\frac{2}{1}\right) = 1.1075$

$a = \frac{\sqrt{5}}{4} e^{j1.1075}$

$h[n] = \left(\frac{\sqrt{5}}{4}\right)^n e^{j1.1075n} u[n]$

$\int_0^\infty \left(\frac{\sqrt{5}}{4}\right)^n e^{j1.1075n} dn = B < \infty$

BIBO stable




c) $a = \frac{2}{1+j} = 1 - j$

$|a| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\angle a = \tan^{-1}\left(\frac{-1}{1}\right) = \frac{-\pi}{4}$

$a = \sqrt{2}\, e^{j\frac{-\pi}{4}}$

$h[n] = \sqrt{2}^n e^{j\frac{-\pi}{4}n} u[n]$

$\int_0^\infty \left|\sqrt{2}^n e^{j\frac{-\pi}{4}n}\right| dn = \infty$

not stable

3. Compute the output signal $y$ for the following paired inputs and LTI system impulse responses $x$ and $h$, respectively:

(a) $x(t) = e^{-\frac{t}{3}}u(t)$, $h(t) = u(t) + \delta(t-1)$

(b) $x(t) = e^{-t}\cos(\frac{\pi}{2}t)u(t+1)$, $h(t) = e^{-2t}u(t)$

(c) $x[n] = e^{-(j\frac{\pi}{4}+1)n}u[n]$, $h[n] = u[n]$

a) $y = e^{-\frac{t}{3}} u(t) * (u(t) + \delta(t-1))$

$= e^{-t/3} u(t) * u(t) + e^{-t/3} u(t) * \delta(t-1)$

$= 3(1 - e^{-\frac{1}{3}t}) u(t) + e^{-(t-1)/3} u(t-1)$

$e^{-2t+2\tau}$

b) $y = e^{-t}\cos(\frac{\pi}{2}t) u(t+1) * e^{-2t} u(t)$

$= \int_{-\infty}^{\infty} e^{-\tau}\cos(\frac{\pi}{2}\tau) u(\tau+1) \; e^{-2(t-\tau)} u(t-\tau)\, d\tau$

$e^{-2t}\int^{t} e^{-\tau}\cos(\frac{\pi}{2}\tau) e^{2\tau} d\tau \cdot u(t+1)$

$$= \int_{-\infty} e \quad \cos(\tfrac{\pi}{2}\tau)\, u(\tau+1) \quad e \qquad u(t-\tau)\, d\tau$$

$$= e^{-2t} \int_{-1}^{t} e^{-\tau} \cos(\tfrac{\pi}{2}\tau)\, e^{2\tau}\, d\tau \quad \cdot \; u(t+1)$$

$$= e^{-2t} u(t+1) \int^{t} e^{\tau} \cos(\tfrac{\pi}{2}\tau)\, d\tau = e^{-2t} u(t+1) \frac{2(e^{t+1}(\pi \sin(\frac{\pi}{2}t) + 2\cos(\frac{\pi}{2}t)) + \pi)}{e(4+\pi^2)}$$

(c) $x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} e^{-(j\frac{\pi}{4}+1)k}\, u[k]\, u[n-k]$$

$$= \sum_{k=0}^{k=n} \left(e^{-(j\frac{\pi}{4}+1)}\right)^{k} u[n]$$

$$= \frac{1 - e^{-(j\frac{\pi}{4}+1)(n+1)}}{1 - e^{-(j\frac{\pi}{4}+1)}}\, u[n]$$

4. For a given fundamental frequency $\omega_0$ and fourier coefficient sequence $a_j$, compute the original time-domain signal $x(t) = \sum_n a_n e^{j\omega_n n t}$ in reduced form (in this case, no complex exponentials). If a fourier coefficient is not stated for a given index, assume that it is zero.

(a) $\omega_0 = \frac{\pi}{4}, a_{-1} = \frac{1}{2j}e^{-j\frac{\pi}{3}}, a_1 = -\frac{1}{2j}e^{-j\frac{\pi}{3}}$

(b) $\omega_0 = \frac{\pi}{2}, a_{-3} = 2, a_0 = 1 a_3 = 2(1+j)$

a)

$$\frac{1}{2j} e^{+j\frac{\pi}{3}} e^{j\frac{\pi}{4}(-1)t} - \frac{1}{2j} e^{-j\frac{\pi}{3}} e^{j\frac{\pi}{4}\cdot 1\, t}$$

$$= \frac{1}{2j}\left(e^{-j(\frac{\pi}{4}t - \frac{\pi}{3})} - e^{j(\frac{\pi}{4}t - \frac{\pi}{3})}\right)$$

$$= \frac{-1}{2j}\left(e^{j(\frac{\pi}{4}t - \frac{\pi}{3})} - e^{-j(\frac{\pi}{4}t - \frac{\pi}{3})}\right)$$

$$= -\sin(\tfrac{\pi}{4}t - \tfrac{\pi}{3})$$

b)

$$2e^{j\frac{\pi}{2}\cdot -3t} + 1 + 2(1+j)\, e^{j\frac{\pi}{2}\cdot 3t} \qquad 2 = 4 \cdot \tfrac{1}{2}$$

$$= \cancel{2e^{j\frac{\pi}{2}\cdot -3t}} + 1 + \cancel{2e^{j\frac{\pi}{2}3t}} + 2j\, e^{j\frac{\pi}{2}3t}$$

$$\ldots (\cos(\tfrac{3\pi}{2}t) + j\sin(\tfrac{3\pi}{2}t))$$

$$= \cancel{2e} + 1 + \cancel{2e} + 2je$$

$$= 4 \cdot \cos\left(\frac{3\pi}{2} t\right) + 1 + 2j\left(\cos\left(\frac{3\pi}{2} t\right) + j\sin\left(\frac{3\pi}{2} t\right)\right)$$

$$= 4\cos\left(\frac{3\pi}{2} t\right) + 1 + 2j\cos\left(\frac{3\pi}{2} t\right) - 2\sin\left(\frac{3\pi}{2} t\right)$$