HW5

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1. State whether the following signals $x(t)$ have fourier series representations. In either case, show mathematically why this is the case:

(a) $x(t)=\sum_{k=-\infty}^{\infty} j^k \delta(t-k)$

(b) $x(t)=\sin(\pi t)e^{-|t|}$

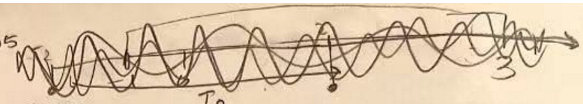
$T=1$ $\quad \omega=\frac{2\pi}{T}=2\pi$

a) for $y(t)=\sum_{k=-\infty}^{\infty}\delta(t-k)$ $\quad C_k=\frac{1}{T}=1$ $\quad y(t)=\sum_{-\infty}^{\infty} e^{jk2\pi t}$

$x(t)=(\sum_{-\infty}^{\infty} j^k)y(t)$ $\quad C_k=j^k\frac{1}{T}$, $T=1$ $\quad C_k=j^k$

$x(t)=\sum_{-\infty}^{\infty} j^k e^{jk\frac{\pi}{2}t}$ $\quad T=4$ for j^k, $\omega_0=\frac{2\pi}{4}=\frac{\pi}{2}$

HW5



Part 1 FT CT

1. a. $x(t)=\sum_{k=-\infty}^{\infty} j^k \delta(t-k)$

① Graph

$k=-2$ $\quad -1\,\delta(t+2)$

$k=-1$ $\quad -j\,\delta(t+1)$

$k=0$ $\quad 1\,\delta(t)$

$k=1$ $\quad j\,\delta(t-1)$

$k=2$ $\quad -1\,\delta(t-2)$

$k=3$ $\quad -j\,\delta(t-3)$

$k=4$ $\quad 1\,\delta(t-4)$

$-j \quad 1 \quad j \quad -1$

$T_0=4$

$\omega_0=\pi/2$

② $T_0=4 \rightarrow \omega_0=2\pi/4=\pi/2$

③ -1 to 3 Bounded (Don't matter what's the num)

④ $a_k=\frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt$

$=\frac{1}{4}\int_{-1}^{3}\left(-j\delta(t+1)+\delta(t)+j\delta(t-1)-\delta(t-2)\right)e^{-jk(\frac{\pi}{2})t}\,dt$

$=\frac{1}{4}\int e^{-jk(\frac{\pi}{2})t}\cdot(-j\delta(t+1))\,dt \Rightarrow -\frac{1}{4}je^{(-j)k\frac{\pi}{2}(-1)} \Rightarrow -\frac{1}{4}je^{jk(\pi/2)} \Rightarrow -\frac{j e^{jk(\pi/2)}}{4} \Rightarrow \frac{1}{4j}e^{jk(\pi/2)}$

$\frac{1}{4}\int e^{-jk(\frac{\pi}{2})t}\cdot\delta(t)\,dt \Rightarrow \frac{1}{4}e^{-jk(\pi/2)(0)} \Rightarrow \frac{1}{4}$

$\frac{1}{4}\int e^{-jk(\frac{\pi}{2})t}\cdot j\delta(t-1)\,dt \Rightarrow j\frac{1}{4}\int e^{-jk(\frac{\pi}{2}t)}\delta(t-1) \Rightarrow \frac{j}{4}e^{-jk(\pi/2)} \Rightarrow -\frac{1}{4j}e^{-jk(\pi/2)}$

$\frac{1}{4}\int e^{-jk(\frac{\pi}{2})t}\cdot(-\delta(t-2))\,dt \Rightarrow \frac{1}{4}e^{-j(k)(\pi/2)2} \Rightarrow \frac{1}{4}e^{-jk(\pi)}$

$\frac{1}{4j}\left(e^{jk(\pi/2)}-e^{-jk(\pi/2)}\right) \Rightarrow \frac{1}{2}\sin(\pi/2\,t)$

$a_0=1/4$

$a_2=\frac{1}{2}$

$a_1=\frac{1}{4j}=a_{-1}$

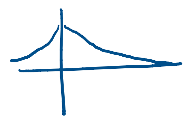
b) $x(t)$ need to be periodic to repesent into fourier series

$\sin(\pi t)$ is period.

$x(t)$ needs to be per...

$\sin(\pi t)$ is period.

$e^{-|t|}$ is not periodic $\rightarrow$



$x(t)$ decay, not FS representation

2. Find the fundamental frequency ω_0 and fourier coefficients c_k for this signal:
$x(t) = 4 + 5\cos(3t)\cos(t) + e^{j4t} + 3e^{-j7t}$

$\omega_2 = 3 \quad \omega_3 = 1 \quad \omega_4 = 4 \quad \omega_5 = 7$

$\omega_0 = GCD(3, 1, 4, 7) = 1$

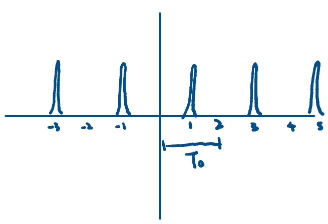
$x(t) = 4e^{0t} + 5 \cdot \frac{1}{2} \cdot \frac{1}{2}\left(e^{j3t} + e^{-j3t}\right)\left(e^{jt} + e^{-jt}\right) + e^{j4t} + 3e^{-j7t}$

$= 4e^{0t} + \frac{5}{4}\left(e^{j4t} + e^{j2t} + e^{-j2t} + e^{-j4t}\right) + e^{j4t} + 3e^{-j7t}$

$= 4e^{0t} + \frac{9}{4}e^{j4t} + \frac{5}{4}e^{j2t} + \frac{5}{4}e^{-j2t} + \frac{5}{4}e^{-j4t} + 3e^{-j7t}$

$C_0 = 4. \quad C_4 = \frac{9}{4} \quad C_2 = \frac{5}{4} \quad C_{-2} = \frac{5}{4} \quad C_{-4} = \frac{5}{4} \quad C_{-7} = 3$

3. Solve for the fundamental frequency ω_0 and fourier series coefficients c_k of the signal $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - 1 + 2n)$.
Use the fourier coefficient equation $c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}dt$.



$T_0 = 2 \quad \boxed{\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi}$

$C_k = \frac{1}{2}\int_0^2 \delta(t-1)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{2}\int_0^2 e^{-jk\omega}\, dt$

$\boxed{C_k = \frac{1}{2}e^{-j\pi}}$

$C_k = \frac{1}{2}e^{-jk\pi} = (-1)^k \frac{1}{2}$

4. An LTI system S is described by a transfer function $H(j\omega)$ that alters the fourier coefficients of its inputs by the following rule:

$$H(j\omega) = \begin{cases} 2 & |\omega| \leq 4 \\ 0 & \text{else} \end{cases}$$

For a given input signal $x(t)$ with fourier coefficients c_k, the system's output $y(t)$ will have fourier coefficients $c'_k = H(jk\omega_0)c_k$.

(a) Compute ω_0, T_0 and all c_k for the input signal $x(t) = -1 + 3\cos(t)\sin(t) - 2\cos(4t)\cos(t) + 3e^{j3t} + 4e^{-j5t}$

(b) Compute the output signal $y(t) = S\{x(t)\}$ for the input given in part (a).

(c) Repeat parts (a) and (b) for an new input $x(t) = -1 + 3\cos(3t)sin(3t) - 2\cos(12t)\cos(3t) + 3e^{j9t} + 4e^{-j15t}$

a) $x(t) = -1 + 3\cos(t)\sin(t) - 2\cos(4t)\cos(t) + 3e^{j3t} + 4e^{-j5t}$

$\rightarrow x(t) = -1e^{0t} + 3 \cdot \frac{1}{2} \cdot \frac{1}{2j}\left(e^{jt} + e^{-jt}\right)\left(e^{jt} - e^{-jt}\right) - 2 \cdot \frac{1}{2} \cdot \frac{1}{2}\left(e^{j4t} + e^{-j4t}\right)\left(e^{jt} + e^{-jt}\right) + 3e^{j3t} + 4e^{-j5t}$

$= -1 + \frac{3}{4j}\left(e^{j2t} \cancel{-1+1} - e^{-j2t}\right) - \frac{1}{2}\left(e^{j5t} + e^{j3t} + e^{-j3t} + e^{-j5t}\right) + 3e^{j3t} + 4e^{-j5t}$

$= -1 + \frac{3}{4j}e^{j2t} - \frac{3}{4j}e^{-j2t} - \frac{1}{2}e^{j5t} \ldots e^{j3t} \ldots e^{-j3t} \ldots e^{-j5t}$

$\boxed{GCD(2, 5, 3) = 1 = \omega_0}$

$= -1 + \frac{3}{4j}(e \quad \cancel{-1} + 1 - e \quad) - \frac{1}{2}(e \quad + e \quad + e \quad + e \quad) + 3e \quad + 4e$

$= -1 + \frac{3}{4j}e^{j2t} - \frac{3}{4j}e^{-j2t} - \frac{1}{2}e^{j5t} + 2.5e^{j3t} - \frac{1}{2}e^{-j3t} + 3.5e^{-j5t}$

$\boxed{\begin{array}{l} GCD(2,5,3) = 1 = \omega_0 \\ T_0 = \frac{2\pi}{\omega} = 2\pi \\ \omega_0 = 1 \end{array}}$

$C_0 = -1 \quad C_2 = \frac{3}{4j} \quad C_{-2} = -\frac{3}{4j} \quad C_5 = -\frac{1}{2} \quad C_3 = 2.5 \quad C_{-3} = -\frac{1}{2} \quad C_{-5} = 3.5$

b) $\omega = 1 \leq 4$, $H(j\omega) = 2$ $\quad y(t) = H(j\omega)x(t) = 2x(t)$

$y(t) = -2 + \frac{3}{2j}e^{j2t} - \frac{3}{2j}e^{-j2t} + 5e^{j3t} - e^{-j3t}$

c) $x(t) = -1 + 3\cos(5t)\sin(3t) - 2\cos(12t)\cos(3t) + 3e^{j9t} + 4e^{-j15t}$

$= -1 + \frac{3}{4j}(e^{j3t} + e^{-j3t})(e^{j3t} - e^{-j3t}) - \frac{1}{2}(e^{j12t} + e^{-j12t})(e^{j3t} + e^{-j3t}) + 3e^{j9t} + 4e^{-j15t}$

$= -1 + \frac{3}{4j}(e^{j6t} - e^{-j6t}) - \frac{1}{2}(e^{j15t} + e^{j9t} + e^{-j9t} + e^{-j15t}) + 3e^{j9t} + 4e^{-j15t}$

$= -1 + \frac{3}{4j}e^{j6t} - \frac{3}{4j}e^{-j6t} - \frac{1}{2}e^{j15t} + 2.5e^{j9t} - \frac{1}{2}e^{-j9t} + 3.5e^{-j15t}$

$GCD(6,15,9) = 3 = \omega_0$ $\quad 3 < 4 \rightarrow H(j\omega) = 2$

$T_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{3}$

$= -1 + \frac{3}{4j}e^{j3(2)t} - \frac{3}{4j}e^{j3(-2)t} - \frac{1}{2}e^{j3(5)t} + 2.5e^{j3(3)t} - \frac{1}{2}e^{j3(-3)t} + 3.5e^{j3(-5)t}$

$C_0 = -1 \quad C_2 = \frac{3}{4j} \quad C_{-2} = -\frac{3}{4j} \quad C_5 = -\frac{1}{2} \quad C_3 = 2.5 \quad C_{-3} = -\frac{1}{2} \quad C_{-5} = 3.5$

c) 2)

$H(j\omega) = 2 \quad 3 < 4$

$y(t) = 2 \cdot x(t) = -2$

5. A discrete time periodic signal $x[n]$ has a fundamental period $N = 5$.
The nonzero fourier coefficients of $x[n]$ are $c_0 = 4, c_2 = c^*_{-2} = 3e^{j\frac{\pi}{3}}, c_5 = c^*_{-5} = \frac{2}{3}e^{j\frac{\pi}{6}}$

(a) Express $x[n]$ in the form $x[n] = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 n}$

(b) Express $x[n]$ in the form $x[n] = \sum_{k=0}^{\infty} A_k \sin(k\omega_0 n + \phi_k)$

$C_{-2} = 3e^{-j\frac{\pi}{3}} \quad C_{-5} = \frac{2}{3}e^{-j\frac{\pi}{6}} \quad N = 5 \quad \omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5}$

a)

$x[n] = 4e^{j0n} + 3e^{j\frac{\pi}{3}}e^{j2\frac{2\pi}{5}n} + 3e^{-j\frac{\pi}{3}}e^{j(-2)\frac{2\pi}{5}n} + \frac{2}{3}e^{j\frac{\pi}{6}}e^{j5\frac{2\pi}{5}n} + \frac{2}{3}e^{-j\frac{\pi}{6}}e^{j(-5)\frac{2\pi}{5}n}$

b)

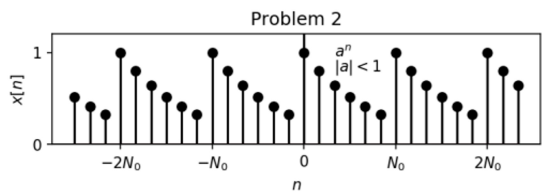
$x[n] = 4e^{j0\omega_0 n} + 3(e^{j(\frac{4\pi}{5}n + \frac{\pi}{3})} + e^{-j(\frac{4\pi}{5}n + \frac{\pi}{3})}) + \frac{2}{3}(e^{j(2\pi n + \frac{\pi}{6})} + e^{-j(2\pi n + \frac{\pi}{6})})$

$= 4e^{j0\omega_0 n} + 6\cos(\frac{4\pi}{5}n + \frac{\pi}{3}) + \frac{4}{3}\cos(2\pi n + \frac{\pi}{6})$

$= 4\sin(\quad + \frac{\pi}{2}) + 6\sin(2 \cdot \frac{2\pi}{5} \cdot n + \frac{5\pi}{6}) + \frac{4}{3}\sin(5 \cdot \frac{2\pi}{5} \cdot n + \frac{2\pi}{3})$

$$= 4\sin\left(+ \frac{\pi}{2} \right) + 6\sin\left(2\cdot\frac{2\pi}{5}\cdot n + \frac{5\pi}{6}\right) + \frac{4}{3}\sin\left(5\cdot\frac{2\pi}{5}\cdot n + \frac{2\pi}{3}\right)$$

6. Compute the fourier series for the periodic signal shown in figure 1. Solve for the general case of $N_0 > 0$ and $|a| < 1$, then solve for $N_0 = 6$ and $a = \frac{2}{3}$



$\omega_0 = \frac{2\pi}{N}$

Figure 1: Problem 2 periodic signal.

$$C_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} a^n e^{-jk\frac{2\pi}{N_0}n}$$

$$= \frac{1}{N_0} \sum_{n=0}^{N_0-1} \left(a e^{-jk\frac{2\pi}{N_0}}\right)^n$$

$$= \frac{1}{N_0} \frac{1-\left(ae^{-jk\frac{2\pi}{N_0}}\right)^{N_0}}{1 - ae^{-jk\frac{2\pi}{N_0}}}$$

$$= \frac{1}{N_0} \frac{1-\left(ae^{-jk\frac{2\pi}{N_0}}\right)^{N_0}}{1 - ae^{-jk\frac{2\pi}{N_0}}}$$

$$x[n] = \sum_{k=0}^{N_0-1} \frac{1}{N_0} \frac{1-\left(ae^{-jk\frac{2\pi}{N_0}}\right)^{N_0}}{1 - ae^{-jk\frac{2\pi}{N_0}}} e^{jk\frac{2\pi}{N_0}n}$$

$N_0 = 6 \quad a = \frac{2}{3}$

$$x[n] = \frac{1}{6} \sum_{k=0}^{5} \frac{1-\left(\frac{2}{3}e^{-jk\frac{\pi}{3}}\right)^6}{1-\frac{2}{3}e^{-jk\pi/3}} e^{jk\frac{\pi}{3}n}$$

$$= 0.456 + (0.13-0.11j)e^{j\frac{\pi}{3}n} + (0.096-0.042j)e^{j\frac{2\pi}{3}n}$$

$$+ 0.0912\, e^{j\pi n} + (0.096+0.042j)e^{jk\frac{4\pi}{3}n} + (0.13+0.11j)e^{j\frac{5\pi}{3}n}$$

7. An N_0-periodic signal $x[n]$ has a fourier series representation given by $x[n] = \sum_{k\in[N_0]} c_k e^{jk\omega_0 n}$. Show that Parseval's theorem is true for this signal, namely that

$$\frac{1}{N_0}\sum_{n\in[N_0]} |x[n]|^2 = \sum_{k\in[N_0]} |c_k|^2$$

What does Parseval's theorem say about power and energy in the time-domain and frequency-domain representations of a signal?

7. $\because\ x[n] \cdot x^*[n] = |x[n]|^2$

$\therefore\ \frac{1}{N_0}\sum_{n\in N_0} |x[n]|^2 = \frac{1}{N_0}\sum_{n\in N_0} x[n]\cdot x^*[n]$

$\because\ x[n] = \sum_{k\in[N_0]} c_k e^{jk\omega_0 n}$

$x^*[n] = \sum_{k\in[N_0]} c_k^* e^{-jk\omega_0 n}$

$$\therefore\ \frac{1}{N_0}\sum_{n\in[N_0]} |x[n]|^2 = \frac{1}{N_0}\sum_{n\in N_0} x[n]\cdot \sum_{k\in[N_0]} c_k^* e^{-jk\omega_0 n}$$

$$= \sum_{k\in[N_0]} c_k^* \cdot \underbrace{\frac{1}{N_0}\sum_{n\in N_0} x[n]\cdot e^{-jk\omega_0 n}}_{c_k}$$

$$= \sum_{k\in[N_0]} |c_k|^2$$

Parseval's theorem states that power and energy of a a signal is equal to those of the signal in frequency domain