hW6

2022年3月4日 22:12

HW6 Topics: CTFT and DTFT

NOTE: You will notice that some selected problems are given answers through "Show that" statements. Make sure you show all your work clearly for every problem.
Throughout the assignment, $u(t)$ is the unit step function, $r(t) = tu(t)$ and $p(t) = u(t) - u(t-1)$

Problem 1

Consider the signal $x(t)$, which consists of a single rectangular pulse of unit height, is symmetric about the origin, and has a total width T_1.

rect$\left(\frac{t}{T}\right)$, symitry, length is T

(a) Sketch $x(t)$.

(b) Sketch $\tilde{x}(t)$, which is a periodic repetition of $x(t)$ with period $T_0 = 3T_1/2$.

(c) Compute $X(\omega)$, the Fourier transform of $x(t)$. Sketch $|X(\omega)|$ for $|\omega| \leq 6\pi/T_1$.

(d) Compute a_k, the Fourier series coefficients of $\tilde{x}(t)$. Sketch a_k for $k = 0, \pm1, \pm2, \pm3$.

(e) Using your answers to (c) and (d), verify that, for this example,

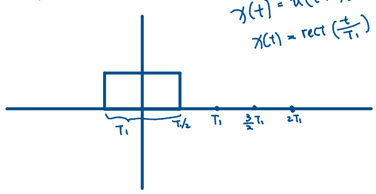
$$a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega=(2\pi k)/T_0}$$

(f) Write a statement that indicates how the Fourier series for a periodic function can be obtained if the Fourier transform of one period of this periodic function is given.

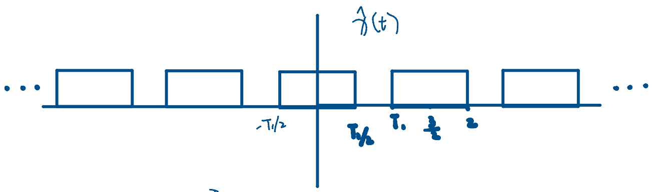
a)

$x(t) = u(t+T_1/2) - u(t-T_1/2)$

$x(t) = \text{rect}\left(\frac{t}{T_1}\right)$



b>



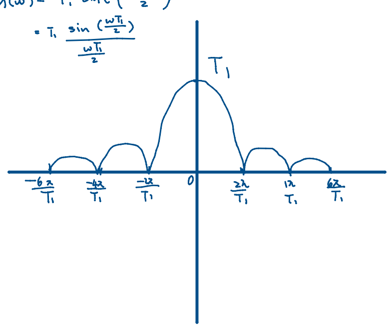
c) $X(\omega) = T_1 \text{sinc}\left(\frac{\omega T_1}{2}\right)$

$= T_1 \frac{\sin\left(\frac{\omega T_1}{2}\right)}{\frac{\omega T_1}{2}}$

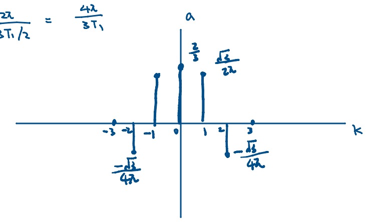
$\frac{\omega T_1}{2} = k\pi$

$\omega = \frac{2k\pi}{T_1} \leq \frac{6\pi}{T_1}$

$K = -3, -2, -1, 0, 1, 2, 3$



d) $c_k = \frac{1}{3T_1/2} \int_{-T_1/2}^{T_1/2} 1 \, e^{-jk\omega_0 t} \, dt$, $\omega_0 = \frac{2\pi}{3T_1/2} = \frac{4\pi}{3T_1}$



$C_0 = \frac{2}{3T_1}\left(\frac{T_1}{2} + \frac{T_1}{2}\right) = \frac{2}{3} = a_0$

$C_k = \frac{2}{3T_1}\left(\frac{1}{jk\omega_0} e^{jk\omega_0 \frac{T_1}{2}} - \frac{1}{jk\omega_0} e^{-jk\omega_0 \frac{T_1}{2}}\right)$

$= \frac{4}{3T_1 \cdot k\omega_0} \cdot \sin\left(k\omega_0 \frac{T_1}{2}\right)$, $\omega_0 = \frac{4\pi}{3T_1}$ $\omega_0^{-1} = \frac{3T_1}{4\pi}$

$\frac{\cancel{4}^{2}\pi}{3\cancel{T_1}} \cdot \frac{\cancel{T_1}}{\cancel{2}_1} = \frac{2\pi}{3}$

$= \frac{4}{3\pi k} \cdot \frac{3\pi}{4\pi} \sin\left(\frac{2\pi k}{3}\right)$

$= \frac{1}{k\pi} \sin\left(\frac{2\pi k}{3}\right) = a_F$

$a_1 = \frac{\sqrt{3}}{2\pi} = a_{-1}$ $a_2 = \frac{-\sqrt{3}}{4\pi} = a_{-2}$ $a_3 = a_{-3} = 0$

e) $a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega = (2\pi k)/T_0}$

$= \frac{1}{3\cancel{T_1}/2} \cdot \cancel{T_1} \cdot \frac{\sin\left(\frac{2\pi k}{3T_1/2} \frac{T_1}{2}\right)}{\frac{2\pi k \cdot \cancel{2}}{3\cancel{T_1}} \frac{\cancel{T_1}}{\cancel{2}}}$

$= \frac{2}{3} \frac{\sin\left(\frac{2\pi k}{3}\right)}{\frac{2}{3}\pi k}$

$= \frac{1}{\pi k} \sin\left(\frac{2\pi k}{3}\right) = \frac{1}{k\pi} \sin\left(\frac{2\pi k}{3}\right)$ same to the eqation I dotained in d)

f) (f) To find a_k, substuting $w = (2\pi k)/T_0$ in to the known fourier transform, and dividing the result by T_0 (fundamental period).

we first find FS coifficients use the fuction $a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega = (2\pi k)/T_0}$

than plug a_k into $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ to find the equation of FS

Problem 2

Consider the periodic signal $\tilde{x}(t)$ in Figure P2–1, which is composed solely of impulses.

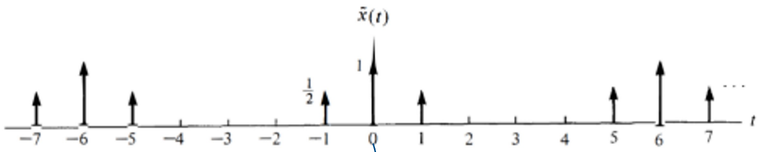


Figure P2-1

$T_0 = 6$

(a) What is the fundamental period T_0?

(b) Find the Fourier series of $\tilde{x}(t)$.

(c) Find the Fourier transform of the signals in Figures P2-2 and P2-3.

(d) $\tilde{x}(t)$ can be expressed as either $x_1(t)$ periodically repeated or $x_2(t)$ periodically repeated, i.e.,

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_1(t - kT_1), \quad \text{or} \qquad \text{(P2-1)}$$

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} x_2(t - kT_2) \qquad \text{(P2-2)}$$

Determine T_1 and T_2

(e) Verify that the Fourier series of $\tilde{x}(t)$ is composed of scaled samples of either $X_1(\omega)$ or $X_2(\omega)$.

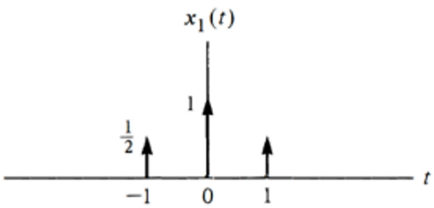


Figure P2-2

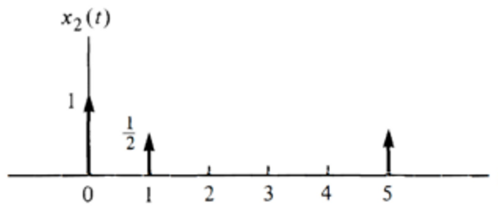


Figure P2-3

a) $T_0 = 6$

b) $$a_k = \frac{1}{T_0} X(\omega) \Big|_{\omega=(2\pi k)/T_0}$$

$$x_1(t) = \underbrace{\tfrac{1}{2}\delta(t+1)}_{\frac{1}{2}e^{j\omega 1}} + \underbrace{\delta(t)}_{1} + \underbrace{\tfrac{1}{2}\delta(t-1)}_{\frac{1}{2}e^{-j\omega}}$$

$$X_1(\omega) = \frac{1}{2}e^{j\omega 1} + 1 + \frac{1}{2}e^{-j\omega}$$

$$= \frac{1}{2}\left(e^{j\omega} + e^{-j\omega}\right) + 1$$

$$= \cos(\omega) + 1$$

$$a_k = \frac{1}{6}\left(\cos(\omega) + 1\right)\Big|_{\omega = \frac{2\pi k}{6}} = \frac{\pi k}{3}$$

$$= \frac{1}{6}\left(\cos\left(\frac{\pi k}{3}\right) + 1\right)$$

$$x(t) = \frac{1}{6}\sum_{k=-\infty}^{\infty} \left(\cos\left(\frac{\pi k}{3}\right)+1\right) e^{jk\frac{\pi}{3}t}$$

c) $x_1(t) = \frac{1}{2}\delta(t+1) + \delta(t) + \frac{1}{2}\delta(t-1)$

$X_1(\omega) = \frac{1}{2}e^{j\omega 1} + 1 + \frac{1}{2}e^{-j\omega}$

$= \frac{1}{2}(e^{j\omega} + e^{-j\omega}) + 1$

$= \cos(\omega) + 1$

$x_2(t) = \delta(t) + \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t-5)$

$X_2(\omega) = 1 + \frac{1}{2}e^{-j\omega} + \frac{1}{2}e^{-5j\omega}$

$= 1 + \frac{1}{2}(e^{-j\omega} + e^{-5j\omega})$

d) $T_1 = 6$ $T_2 = 6$ they are the same because period is 6.

e) $a_k^2 = \frac{1}{6}\left(1 + \frac{1}{2}(e^{-j\omega} + e^{-5j\omega})\right)\Big|_{\omega = \frac{2\pi k}{6} = \frac{\pi k}{3}}$

$a_k^2 = \frac{1}{6}(1 + \frac{1}{2}(e^{-j\frac{\pi k}{3}} + e^{-5j\frac{\pi k}{3}})$

$a_1^2 = 0.25$ $a_2^2 = 0.083$ $a_3^2 = 0$ ....

$a_k^1 = \frac{1}{6}(\cos(\omega) + 1)\Big|_{\omega = \frac{2\pi k}{6} = \frac{\pi k}{3}}$

$= \frac{1}{6}(\cos(\frac{\pi k}{3}) + 1)$

$a_1^1 = 0.25$ $a_2^1 = 0.083$ $a_3^1 = 0$ .....

$a_k = a_k^1 = a_k^2$, the cofficent of FS generate from $X_1(\omega)$ and $X_2(\omega)$ is same as FS coefficent of $\hat{x}(t)$

Therefore, FS of $\hat{x}(t)$ is composed of scaled sample of $x_1(\omega)$ $x_2(\omega)$

(e) $\because$ $x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{6}(\frac{1}{2}e^{jk w_0} + 1 + \frac{1}{2}e^{-jk w_0})e^{jk w_0 t}$

$X_1(w) = \frac{1}{2}e^{jw} + 1 + \frac{1}{2}e^{-jw}$

$X_2(w) = 1 + \frac{1}{2}e^{-jw} + \frac{1}{2}e^{-5jw}$

$\therefore$ as k and t varies, we can always find a w of scaled $X_1(w)$ or $X_2(w)$ to make either of them equal to $X(t)$.

$\therefore$ $x(t)$ is composed of scaled samples of either $X_1(w)$ or $X_2(w)$

Problem 3

Determine the Fourier series coefficients for the three periodic sequences shown in Figures P3-1 to P3-3. Since these three sequences all have the same nonzero values over one period, we suggest that you first determine an expression for the envelope of the Fourier series coefficients and then sample this envelope at the appropriate spacings in each case.

(a)

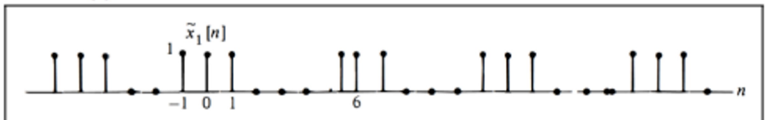


Figure P3-1

(b)

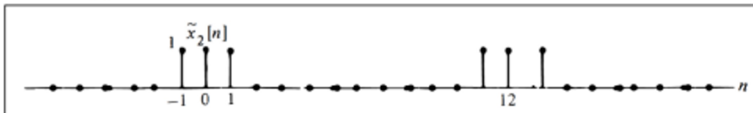


Figure P3-2

$\text{rect}[\frac{n}{2}]$

$C_k = \frac{1}{N_0}\sum_{n=-1}^{1}(u[n+1] - u[n-1])e^{-jk\omega_0 n}$

$= \frac{1}{N_0}(e^{jk\omega_0} + e^{-jk\omega_0} + 0.5)$

$= \frac{2}{N_0}(\cos(k\omega_0) + 0.5)$, $\omega_0 = \frac{2\pi}{N_0}$

$= \frac{2}{N_0}(\cos(k\frac{2\pi}{N_0}) + 0.5)$

a) $N_0 = 6$

$C_k = \frac{2}{6}(\cos(\frac{2\pi}{6}k) + 0.5)$

$= \frac{1}{3}(\cos(\frac{\pi}{3} \cdot k) + 0.5)$

b) $N_0 = 12$

$C_k = \frac{2}{12}(\cos(\frac{2\pi}{12}k) + 0.5)$

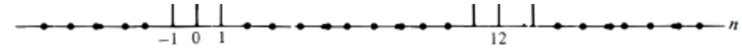


Figure P3-2

(c)

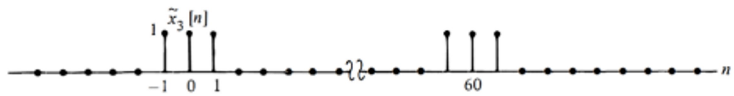


Figure P3-3

b) $N_0 = 12$

$C_k = \frac{2}{12}\left(\cos\left(\frac{2\pi}{12}k\right)+0.5\right)$

$= \frac{1}{6}\left(\cos\left(\frac{\pi}{6}k\right)+0.5\right)$

c) $N_0 = 60$

$C_k = \frac{2}{60}\left(\cos\left(\frac{2\pi}{60}k\right)+0.5\right)$

$= \frac{1}{30}\left(\cos\left(\frac{\pi}{30}k\right)+0.5\right)$

$$X(\omega) = \frac{\sin\left(\frac{3}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)}$$

$$a_k = \frac{1}{N_0} X(\omega)\Big|_{\omega=\frac{2\pi k}{N_0}} = \frac{1}{N_0}\frac{\sin\left(\frac{3}{2}\frac{2\pi k}{N_0}\right)}{\sin\left(\frac{\pi k}{N_0}\right)}$$

a) $N_0 = 6$

$a_k = \frac{1}{6}\frac{\sin\left(\frac{\pi k}{2}\right)}{\sin\left(\frac{\pi k}{6}\right)}$

b) $N_0 = 12$

$a_k = \frac{1}{12}\frac{\sin\left(\frac{\pi k}{4}\right)}{\sin\left(\frac{\pi k}{12}\right)}$

c) $N_0 = 60$

$a_k = \frac{1}{60}\frac{\sin\left(\frac{\pi k}{20}\right)}{\sin\left(\frac{\pi k}{60}\right)}$

same result.

Problem 4

Consider a discrete-time system with impulse response

$$h[n] = (\tfrac{1}{2})^n u[n]$$

Determine the response to each of the following inputs:

(a) $x[n] = (-1)^n = e^{j\pi n}$ for all n

(b) $x[n] = e^{j(\pi n/4)}$ for all n

(c) $x[n] = \cos\left(\frac{\pi n}{4} + \frac{\pi}{8}\right)$ for all n

a) $x[n] * h[n] = \sum_{k=-\infty}^{\infty} e^{j\pi(n-k)} \left(\frac{1}{2}\right)^k u(k)$

$= \sum_{k=0}^{\infty} e^{j\pi(n-k)} \left(\frac{1}{2}\right)^k$

$= e^{j\pi n} \sum_{k=0}^{\infty} \left(\left(\frac{1}{e}\right)^{j\pi} \cdot \frac{1}{2}\right)^k$

$= e^{j\pi n} \cdot \frac{2}{3} = \frac{2}{3} \cdot e^{j\pi n}$

b) $x[n] * h[n] = \sum_{k=0}^{\infty} e^{j\frac{\pi}{4}(n-k)} \left(\frac{1}{2}\right)^k$

$e^{j\frac{\pi}{4}n} \sum^{\infty} e^{-j\frac{\pi}{4}k} \left(\frac{1}{2}\right)^k$

b) $x[n] * h[n] = \sum_{k=0} e^{j\frac{\pi}{4}(n-k)} \left(\frac{1}{2}\right)^k$

$$= e^{j\frac{\pi}{4}n} \sum_{k=0}^{\infty} e^{-j\frac{\pi}{4}k} \left(\tfrac{1}{2}\right)^k$$

$$= e^{j\frac{\pi}{4}n} \sum_{k=0}^{\infty} \left(\frac{1}{e^{j\frac{\pi}{4}}} \cdot \frac{1}{2}\right)^k$$

$$= e^{j\frac{\pi}{4}n} \left(\frac{1 - \left(\frac{1}{2e^{j\frac{\pi}{4}}}\right)^{\infty+1}}{1 - \frac{1}{2e^{j\frac{\pi}{4}}}} \right)$$

$$= e^{j\frac{\pi}{4}n} \cdot 1.3572 \cdot e^{-0.05005j}$$

$$= 1.3572 \, e^{j\left(\frac{\pi}{4}n - 0.05005\right)}$$

c) $x[n] * h[n]$

$$x[n] = \frac{1}{2} e^{j\left(\frac{\pi n}{4} + \frac{\pi}{8}\right)} + \frac{1}{2} e^{-j\left(\frac{\pi n}{4} + \frac{\pi}{8}\right)}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(e^{j\left(\frac{\pi(n-k)}{4} + \frac{\pi}{8}\right)} + e^{-j\left(\frac{\pi(n-k)}{4} + \frac{\pi}{8}\right)} \right) \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \left(e^{\frac{j\pi n}{4} - \frac{j\pi k}{4} + \frac{j\pi}{8}} \right) \left(\frac{1}{2}\right)^k + \frac{1}{2} \sum_{k=0}^{\infty} \left(e^{-\frac{j\pi n}{4} + \frac{j\pi k}{4} - \frac{j\pi}{8}} \right) \left(\frac{1}{2}\right)^k$$

$$= \frac{1}{2} \cdot e^{\left(\frac{j\pi n}{4} + \frac{j\pi}{8}\right)} \sum_{k=0}^{\infty} \left(\frac{1}{2e^{j\pi/4}}\right)^k + \frac{1}{2} e^{-\left(\frac{j\pi n}{4} + \frac{j\pi}{8}\right)} \sum_{k=0}^{\infty} \left(\frac{e^{j\pi/4}}{2}\right)^k$$

$$= \frac{1}{2} \frac{1}{1 - \frac{e^{-j\pi/4}}{2}} \cdot e^{\left(\frac{j\pi n}{4} + \frac{j\pi}{8}\right)} + \frac{1}{2} e^{-\left(\frac{j\pi n}{4} + \frac{j\pi}{8}\right)} \cdot \frac{1}{1 - \frac{e^{j\pi/4}}{2}}$$