Quiz2-3 review

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impulse : (same to $x[n_0]$)

$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$

$\int_{\frac{\pi}{2}}^{\pi} \sin(t)\,\delta(t-\frac{3\pi}{2})\,dt$

$= \sin(\frac{\pi}{2})\int_{\frac{\pi}{2}}^{\pi} \delta(t-\frac{3\pi}{2})\,dt = 1$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$

$\delta(t) = \frac{d}{dt}u(t)$ $\qquad \delta[n] = u[n] - u[n-1]$

$\int_{t=-\infty}^{t=\infty} \delta(at)\,dt = \frac{1}{|a|}$

$y(t) = \int_{-\infty}^{\infty} x(t)\,\delta(3t-3)\,dt$

$= \int_{-\infty}^{\infty} x(t)\,\delta(3(t-1))\,dt$

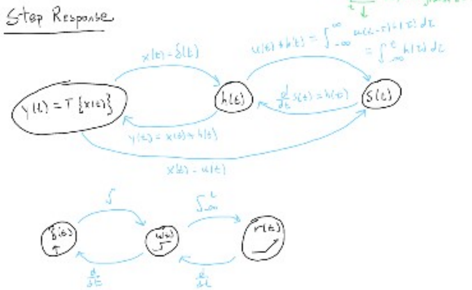
$= \frac{1}{3}\int_{-\infty}^{\infty} x(\tau)\,\delta(\tau-1)\,d\tau$

$= \frac{x(1)}{3}$

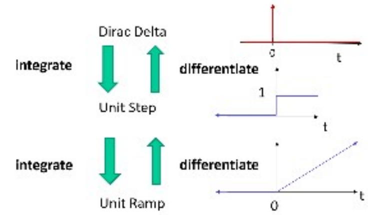
(a) $s(t) = r(t+2) - 2r(t) + 2r(t-1) - r(t-2)$

Answer: $h(t) = \frac{d}{dt}s(t) = u(t+2) - 2u(t) + 2u(t-1) - u(t-2)$

Step Response



Signal Relationships



For DT, we used sum instead of integral, and first difference instead of differentiating

convolution :

$\delta \rightarrow LTI \rightarrow h$

$y[n] = x[n-3]$:

$x[n] \longrightarrow LTI \rightarrow x[n-3]$

$h_1[n] = \delta[n-3]$

Step response : $s(t) = u(t) * h(t) = \int h(\tau)u(t-\tau)d\tau$

$u(t) \rightarrow LTI \rightarrow u(t) * h(t)$

$\delta[n-2] \rightarrow LTI \rightarrow u[n-8] + u[n-2]$

$\delta[n] \rightarrow LTI \rightarrow (u[n-6] + u[n] = h)$

$r(t) = u(t) * u(t) \qquad u(t) = u(t) * \delta(t)$

$h_3(t) = \int_{t-1}^{t} \delta(v)\,dv \qquad v = t-\tau$

$\tau = t - v$

$= \int_{1}^{0} \delta(t-\tau)\,-d\tau \qquad d\tau = -dv$

$= \int_{0}^{1} \delta(t-\tau)\,d\tau$

$= u(t) - u(t-1)$ when deal with $y = \int \ldots$

$a(2x_1[n]-1) + b(2x_2[n]-1)$

$2(ax_1 + bx_2) - 1$

* To find h, system need to be LTI

$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \rightarrow \boxed{LTI} \rightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$

$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \boxed{LTI} \rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$

Notation

shifting.

$\int x(t)\,\delta(t)\,dt = x(0)$

$\int x(t)\,\delta(t-1)\,dt = x(1)$

$x[n] * \delta[n-n_0] = x[n-n_0]$ (delay input)

$x_b[n] = 0.8^n u[n] \Rightarrow y_b(n) = 2\left(0.8^{n+1}u(n+1)\right)$

Echo

- *Commutative* – can flip either signal
  $x(t) * h(t) = h(t) * x(t)$
- *Associative* – order doesn't matter
  $x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$
- *Distributive* – divide and conquer
  $x(t) * (h_1(t) + h_2(t)) = (x(t) * h_1(t)) + (x(t) * h_2(t))$

Discrete-time is Tricky (III)

- Keeping track of frequencies/periods for discrete-time
  $x[n] = A\cos(\omega_S n + \phi)$
  - ω_s, $N_s = \frac{2\pi}{\omega_s}$: frequency and period of sinusoid if it were in CT
    - N_s: the number of samples before $x[n]$ repeats itself
  - If N_S is rational $\Leftrightarrow x[n]$ is periodic
    - What is the fundamental period N_0 of the DT signal?
    - We need to find some $m \in \mathbb{Z}^+$ such that $N = \frac{2\pi}{\omega_s}m = N_s m$ is an integer
    - The smallest such N is the fundamental frequency, N_0

Results so far…

$u(t) * u(t) = r(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$

(t) $u(t) * e^{-at}u(t) = \frac{1}{a}(1 - e^{-at})u(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{a}(1 - e^{-at}), & t \geq 0 \end{cases}$

$u(t) * e^{at} = \frac{1}{a}e^{at}$

$p(t) * p(t) = \wedge(t) = \begin{cases} 0 & t < 0, t > 2 \\ t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \end{cases}$

Math solve convolution :

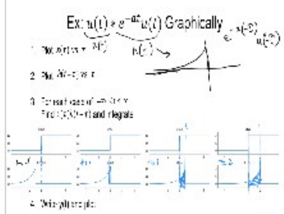
$h(\tau)$: simply $\rightarrow h(t-\tau)$

$x(t)$: complex

Graph :

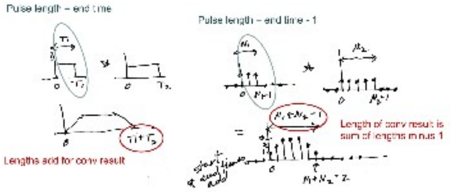
$x(t)$ stay

$h(t) \rightarrow$ inverse and shift. to



Start and end time :

DT vs. CT convolution (cont.)



Start times & end times add for both

propertys :

System Property Testing using Impulse Response $h(t)$

- Since the impulse response $h(t)$ fully specifies an LTI system, we can use it in property testing
- Gives **additional tools** to test system properties (easy tests for LTI systems)
- Key results:
  - Memoryless ↔ $h(t) = T\{\delta(t)\} = A\delta(t)$
  - Causal system ↔ $h(t) = 0$ for $t < 0$
  - Stable system ↔ $\int_{-\infty}^{\infty}|h(t)|dt < \infty$ (absolutely integrable)
  - Invertible ↔ there exists $h_I(t)$ such that $h(t) * h_I(t) = \delta(t)$

Tests are the same for DT, with a sum instead of an integral

Complex number:

$z = a + jb$ $\quad |z| = r = \sqrt{a^2+b^2}$ $\quad \angle z = \phi = \tan^{-1}\frac{b}{a}$

$j = \sqrt{-1}$

$a = Re\{z\}$

$b = Im\{z\}$

polar: $z = re^{j\phi} = r(\cos(\phi) + j\sin(\phi))$

draw the circle:

Signal in Freq domain:

CT:

$x(t) = Ce^{st} = Ce^{\alpha t}e^{j\omega t}$

$e^{j\omega t} \to \cos(\omega t) + j\sin(\omega t)$ (even, odd)

$\alpha = 0$ period; $\neq 0$ aperiod

$S = \alpha + j\omega$

$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$

$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$

$e^{j\omega t} \to \boxed{LTI} \to Ae^{j\omega t}$ $\qquad e^{st} \to LTI \to H(s)e^{st}$

Checking Signal Properties: $x(t) = Ce^{st}$

- When is a complex exponential bounded? $|x(t)| \le B < \infty \ \forall t$ — when $\alpha = 0$; $|Ce^{\alpha t}||e^{j\omega t}|$ bounded
- When is a complex exponential a power signal? $\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2dt < \infty$ — $|e^{\alpha t}|^2 = |e^{2\alpha t}|$ when $\alpha = 0$
- When does a complex exponential have finite energy? $\int_{-\infty}^{\infty}|x(t)|^2dt < \infty$ — $\int_{-\infty}^{\infty}|e^{2\alpha t}||e^{j\omega t}| < \infty$; depends on C^2: $C^2\int_{-\infty}^{\infty}e^{2\alpha t}\,dt < \infty$, $\alpha = 0$

$x(t) \to \boxed{LTI} \to y(t) = x(t) * h(t)$

$\delta(t) \to \boxed{LTI} \to h(t)$ Impulse response

$u(t) \to \boxed{LTI} \to s(t)$ Step response

$e^{st} \to \boxed{LTI} \to H(s)e^{st}$ Exponential response, H(s) = transfer function

s is a complex number $s = \sigma + j\omega$

Continuous-time Fourier Series (for Periodic Signals only)

- Signal Synthesis: building signals with CEs
  $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ → Sum of multiple CEs
- Signal Analysis
  $c_k = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t}dt$ → take apart $x(t)$ c_k
  - The $< T_0 >$ means integrate over one period
  - For *simple sinusoids*, just use Euler's formula

Analysis square wave: $c_0 = \frac{1}{2}$ $\quad c_k = \frac{1}{k\pi}\sin\left(\frac{k\pi}{2}\right)$

$\omega_0 = \frac{2\pi}{T_0}$ — if impulse: $c_k = \frac{1}{T_0}$

Fourier Series for Sinusoids: Analysis by Synthesis

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ $\qquad \omega_0 = \frac{2\pi}{T_0}$

Steps:
1. Find fundamental frequency, ω_0
   $T_0 = LCM(T_a, T_b, \ldots)$ $\qquad \omega_0 = \frac{2\pi}{T_0} = GCD(\omega_a, \omega_b, \ldots)$
2. Expand sinusoids into CEs via Euler's formula:
   $\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$
3. Write CEs in terms of $k\omega_0$
4. Read off Fourier coefficients, c_k (complex numbers)

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Results So Far: Easy Cases for $x(t), T_0 = \frac{2\pi}{\omega_0}$

- Constant term → c_0 $\quad 3e^{j\omega(0)t} = 3$
- Lonely CE $(Ae^{jk\omega_0 t})$ → $c_k = A$
- Sinusoids (Euler)
  $\cos(k\omega_0 t) = \frac{1}{2}e^{jk\omega_0 t} + \frac{1}{2}e^{-jk\omega_0 t}$ $\quad c_k = \frac{1}{2} = c_{-k}$
  $\sin(k\omega_0 t) = \frac{1}{2j}e^{jk\omega_0 t} - \frac{1}{2j}e^{-jk\omega_0 t}$ $\quad c_k = \frac{1}{2j}$ $\quad c_{-k} = -\frac{1}{2j}$
- More generally
  $x(t) = A + \sum_{k=1}^{\infty} r_k\cos(k\omega_0 t + \theta_k)$
  $c_0 = A \quad c_k = \frac{r_k}{2}e^{j\theta_k} \quad c_{-k} = \frac{r_k}{2}e^{-j\theta_k} \quad k = 1, 2, \ldots$

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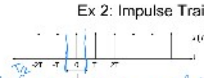
Fourier Series Properties

	$x(t)$	$\leftrightarrow$	c_k	
Linear ops	$y(t) = ax(t) + b$	$\leftrightarrow$	$c_k^y = ac_k,\ k \neq 0,\ c_0^y = ac_0 + b$	Added constant only affects DC term
Time scale, a > 0	$y(t) = x(at)$	$\leftrightarrow$	$c_k^y = c_k,\ \omega_0^y = a\omega_0$	Same c_k, scale ω_0
reverse	$y(t) = x(-t)$	$\leftrightarrow$	$c_k^y = c_{-k}$	reverse
Shift in time t_0	$y(t) = x(t - t_0)$	$\leftrightarrow$	$c_k^y = c_k e^{-jk\omega_0 t_0}$	Add linear phase term $jk\omega_0 t_0$
	$ax_1(t) + bx_2(t)$	$\leftrightarrow$	$c_k^y = ac_k^{x_1} + bc_k^{x_2}$	Adjusting k if ω_0 changes

$\frac{\omega_n}{\omega_0} \times k_n$

Ex 2: Impulse Train

$x(t) = \sum_{n=-\infty}^{\infty}\delta(t - nT)$



$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \to \int \delta(t)e^{\ldots}dt$ sifting property

$c_k = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)\,dt = \frac{1}{T}$ — c_k does not depend on k

i.e. All c_k are the same weight

$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$

1 + r + r^2 + ... + r^n = SUM(k=0...n) r^k = (1 - r^{n+1})/(1 - r).
r + r^2 + r^3 + ... + r^n = SUM(k=1...n) r^k = (r - r^{n+1})/(1 - r).

$e^{-jk\pi} = (-1)^k$

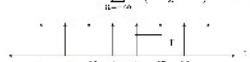
3. $h[n] = (h_2 * h_3 - h_1) * h_4$
$h[n] = h_2 * h_3 * h_4 - h_1 * h_4$

$h_2 * h_3$
$= e^{-n}u[n] * e^{-n}u[n-1]$
$= \sum_{k=0}^{\infty} e^{-k}u[k]\, e^{-(n-k)}u[n-k-1]$
$= \sum_{k=0}^{n-1} e^{-k}e^{-(n-k)} \cdot u[n-1]$

Ex 3: Shifted Impulse Train

$y(t) = x(t - t_0) \to c_k^y = c_k e^{-jk\omega_0 t_0}$ $\quad \sum_{n=-\infty}^{\infty}\delta(t - nT)$

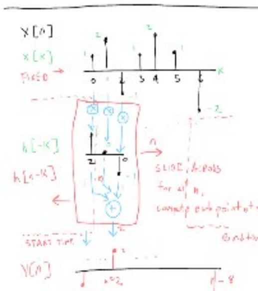
$x(t) = \sum_{n=-\infty}^{\infty}\delta\left(t - \frac{T}{2} - nT\right)$ $\quad c_k = \frac{1}{T}$



$\omega_0 = \frac{2\pi}{T}$

$c_k^y = c_k e^{-jk\omega_0 t_0}$
$= \frac{1}{T}e^{-jk\frac{2\pi}{T}\frac{T}{2}} \to \frac{1}{T}e^{-jk\pi} \to (-1)^k$
$\to \boxed{(-1)^k\frac{1}{T}}$

$e^{-jk\pi} = (-1)^k$



→ Start time $y[n]$ = sum of start times of $x[n], h[n]$
→ End time $y[n]$ = sum of end times
→ length of $y[n]$ = sum of lengths
* one index is -1

$\sum_{n=0}^{\infty} c = \infty,\ c \in \mathbb{R}$

$\sum_{n=-\infty}^{\infty} c = \infty$, c is constant

$\sum_{n=-N}^{N}|a|^2 = a^2(2N+1)$

$\sum_{n=0}^{\infty}(a^n)^2 = \sum_{n=0}^{\infty}(a^2)^n = \frac{1}{1-a^2}$

$\sum_{n=0}^{N}|n|^2 = \frac{N(N+1)(2N+1)}{6}$

.. For each system shown, state whether or not the given observed input/output pair $x(t), y(t)$ indicates the system is *not* time-invariant. Explain your reasoning either way.

(a) $\cos(\pi t) \xrightarrow{T_a} 3\sin(\pi(t+1))$
Answer: No.
A linear gain and a constant delay are not inconsistent with an LTI system. This in/out pair does not break LTI behavior rules.

(b) $e^{-2t}\cos(\pi t) \xrightarrow{T_b} e^{-2t}\sin(\pi t)u(t)$
Answer: Yes.
The appearance of $u(t)$ in the output indicates that this is not LTI.

(c) $a^{-n} \xrightarrow{T_c} ba^{-n-2}$, $|a| > 1, b \in \mathbb{R}$
Answer: No.
$y[n] = ba^{-n-2} = ba^{-2}a^{-n}$, so this is just a linear gain. The in/out pair does not indicate the system isn't LTI.

(d) $\cos(\frac{\pi}{4}n) + e^{-\frac{\pi}{2}|n|} \xrightarrow{T_d} \sin(\frac{\pi}{4}n)$
Answer: Yes.
No LTI system can selectively zero-out the $e^{\frac{\pi}{2}|n|}$ term. This would be possible if there was no $|\cdot|$ operation in the exponent and if there was a phase lag introduced in the output signal.

∅ All formulas

CT vs. DT LCCDE

Assume we have an LTI system described a LCCDE:

CT
$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{m=0}^{M} b_m \frac{d^m x(t)}{dt^m}$$

DT
$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$$

Use the differentiation [or time shift] property of the FT:

$$\frac{d^k x(t)}{dt^k} \leftrightarrow (j\omega)^k X(\omega) \qquad x[n-k] \leftrightarrow e^{-j\omega k} X(\omega)$$

Take FT of both sides of the LCCDE:

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{m=0}^{M} b_m (j\omega)^m X(\omega) \qquad \sum_{k=0}^{N} a_k e^{-j\omega k} Y(\omega) = \sum_{m=0}^{M} b_m e^{-j\omega m} X(\omega)$$

Pull out common terms and get:

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M} b_m (j\omega)^m}{\sum_{k=0}^{N} a_k (j\omega)^k} \qquad H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{\sum_{k=0}^{N} a_k e^{-j\omega k}}$$

Example: Freq derivative

- Find the FT of $v(t) = te^{-2t}u(t)$
- From the formulas, $x(t) = e^{-2t}u(t)$ has FT $X(\omega) = \frac{1}{2+j\omega}$
- Property: $tx(t) \leftrightarrow j\frac{d}{d\omega}X(\omega)$

$$V(\omega) = j\frac{d}{d\omega}\left(\frac{1}{2+j\omega}\right) = \frac{-j}{(2+j\omega)^2}$$

Quotient rule

Useful Result: Parseval's Relation

- Power of a CT periodic signal

$$P = \frac{1}{T_0}\int_{<T_0>} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |c_k|^2$$

- Power of a DT periodic signal

$$P = \frac{1}{N_0}\sum_{<N_0>} |x[n]|^2 = \sum_{<N_0>} |c_k|^2$$

Fourier Series - CT vs. DT

- Signal Synthesis: building signals with CEs

Infinite # of c_k: $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$ Finite # of c_k: $$x[n] = \sum_{<N_0>} c_k e^{jk\omega_0 n}$$

- Signal Analysis

Integral over time variable: $$c_k = \frac{1}{T}\int_{<T_0>} x(t)e^{-jk\omega_0 t}dt$$ Sum over integer indices: $$c_k = \frac{1}{N_0}\sum_{<N_0>} x[n]e^{-jk\omega_0 n}$$

$<T_0>$ and $<N_0>$ indicate integrate/sum over length of one period

$$T_0 = \frac{2\pi}{\omega_0},\ N_0 = \frac{2\pi}{\omega_0}$$

Recall: Discrete-time Periodicity is Tricky (II)

- There is a finite unique range of frequencies
  - For periodic $x[n] = Ce^{j\omega_0 n} = Ce^{j\omega_0 n}e^{j2\pi n} = Ce^{j(\omega_0+2\pi)n}$
  - In general, frequency $\omega_0 \equiv \omega_0 + m2\pi$
  - Often the range is given as $(-\pi, \pi]$, but it could be $[0, 2\pi)$
- Period of 1 gives a constant, so the smallest period is 2 ($\omega_0 = \pi$) → the highest frequency is π!

$$x[n] = e^{j\pi n} = (-1)^n$$

- For periodic $x[n] = Ce^{jk\omega_0 n}$ with $N_0 = \frac{2\pi}{\omega_0}$

$$(k+N_0)\omega_0 = \left(k + \frac{2\pi}{\omega_0}\right)\omega_0 = k\omega_0 + 2\pi \equiv k\omega_0$$

There are N_0 unique harmonics!

Signals that have a FT

Condition 1: If the signal has finite energy, then it has a FT. $E\{x(t)\} < \infty$ FT ✓

Alternate Conditions for Existence of FT:

Dirichlet Conditions: If the following conditions hold:
1. The signal $f(t)$ is absolutely integrable $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$
2. Over any finite interval, $f(t)$ has a finite number of maxima and minima
3. In any finite interval, $f(t)$ has a finite number of discontinuities, each of which is finite

→ Then the FT representation converges to $f(t)$ except at the points of discontinuity, where it converges to the average value on either side

Eigenfunctions & LTI Systems

$$H(\omega) = H(s)|_{s=j\omega}$$

$e^{j\omega t} \rightarrow \boxed{H(\omega)} \rightarrow H(\omega)e^{j\omega t}$

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \rightarrow \boxed{H(\omega)} \rightarrow y(t) = \sum_{k=-\infty}^{\infty} c_k H(k\omega_0) e^{jk\omega_0 t}$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \rightarrow \boxed{H(\omega)} \rightarrow y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)H(\omega)e^{j\omega t}d\omega$

$H(\omega)$ specifies frequency scaling

CT Fourier Transform | DT Fourier Transform

Aperiodic signals (CT) | Aperiodic signals (DT)

CT — Synthesis (Inverse FT): Infinite frequency range
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Analysis (FT): Integrate over continuous time
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

DT — Synthesis (Inverse FT): Finite (2π) frequency range
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

Analysis (FT): Sum over integer time
$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

DT frequency domain is less intuitive, so we'll start with CT FTs, except we need to know a little for the lab

A Few FT Properties

	$f(t)$ – time domain	$F(\omega)$ – Fourier Transform	Reason/Intuition
Scaling	$af(t)$	$aF(\omega)$	Linearity
Additivity	$f(t) + g(t)$	$F(\omega) + G(\omega)$	
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$	Shift in time → linear phase term in frequency
Time-scaling	$f(at)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$	Inverse relationship between time/frequency
Conjugation	$f^*(t)$	$F^*(-\omega)$	Frequency reversal, and $j \to -j$
Convolution	$f(t) * g(t)$	$F(\omega)G(\omega)$	Convolution in time → multiplication in frequency

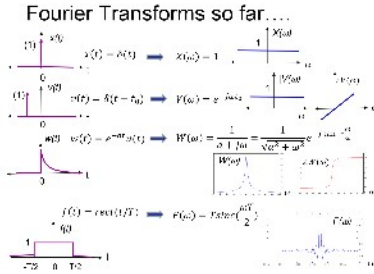
Fourier Transforms so far....

$x(t) = \delta(t) \Rightarrow X(\omega) = 1$

$v(t) = \delta(t - t_0) \Rightarrow V(\omega) = e^{-j\omega t_0}$

$w(t) = e^{-at}u(t) \Rightarrow W(\omega) = \frac{1}{a+j\omega} = \frac{1}{\sqrt{a^2+\omega^2}}e^{-j\tan^{-1}(\omega/a)}$

$f(t) = \text{rect}(t/T) \Rightarrow F(\omega) = T\text{sinc}\left(\frac{\omega T}{2}\right)$



How to Plot $\text{sinc}(x) = \frac{\sin(x)}{x}$

Value at $x = 0$: $\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1$

From L'Hôpital's rule

Zero crossings: $x \neq 0$, $\sin(x) = 0 \rightarrow x = k\pi$

$\sin(x) = 0$ $\text{sinc}(x)$

$$FT\left[\text{rect}\left(\frac{t}{T}\right)\right] = T\text{sinc}\left(\frac{\omega T}{2}\right)$$

$f(t) = \text{rect}\left(\frac{t}{T}\right)$

$$\frac{\omega T}{2} = k\pi \rightarrow \omega = \frac{2k\pi}{T},\ k = \ldots, -2, -1, 1, 2, \ldots$$

Notice: 2π Periodicity of DT FT

$X(\omega + 2\pi) = X(\omega)$

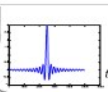
$x[n] = \delta[n] \Rightarrow X(\omega) = 1$		$X(\omega + 2\pi) = 1 = X(\omega)$
$v[n] = \delta[n - n_0] \Rightarrow V(\omega) = e^{-j\omega n_0}$		$V(\omega + 2\pi) = e^{-j(\omega+2\pi)n_0} = e^{-j\omega n_0}e^{-j2\pi n_0} = e^{-j\omega n_0} = V(\omega)$
$y[n] = \delta[n + n_0] + \delta[n - n_0] \Rightarrow Y(\omega) = 2\cos(\omega n_0)$		$Y(\omega + 2\pi) = 2\cos((\omega + 2\pi)n_0) = 2\cos(\omega n_0) = Y(\omega)$
$z[n] = a^n u[n] \Rightarrow Z(\omega) = \frac{1}{1 - ae^{-j\omega}}$		$Z(\omega + 2\pi) = \frac{1}{1 - ae^{-j(\omega+2\pi)}} = \frac{1}{1 - ae^{-j\omega}} = Z(\omega)$

Handy check: If your answer is not periodic with 2π then it is not a valid DT FT & you made a math error

Common examples

$$x(t) = \frac{W}{2\pi}\text{sinc}\left(\frac{W}{2}t\right) \Leftrightarrow F(\omega) = \text{rect}(\omega/W)$$

$$B\text{sinc}\left(\frac{B}{2}t\right) \Rightarrow 2\pi \cdot \text{rect}(\omega/B)$$



Fourier Transforms so far....

$x(t) = \delta(t) \Leftrightarrow X(\omega) = 1$

$v(t) = \delta(t - t_0) \Leftrightarrow V(\omega) = e^{-j\omega t_0}$

$y(t) = \frac{1}{2\pi} \Leftrightarrow Y(\omega) = \delta(\omega)$

$? \Leftrightarrow Z(\omega) = \delta(\omega - \omega_0)$

$= \frac{1}{2\pi}e^{j\omega_0 t}$

FT of General Periodic Signals

We know that
$$FT\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

Let $x(t)$ be a periodic signal with fundamental frequency ω_0 and FS representation
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

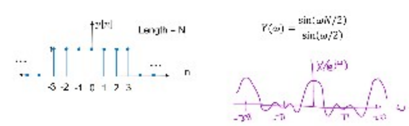
Because the FT is a linear operator, it follows that
$$X(\omega) = FT\{x(t)\} = \sum_{k=-\infty}^{\infty} a_k FT\{e^{jk\omega_0 t}\} = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

The FT of a periodic signal is a sum of lonely impulses

How to Plot sin(ax)/sin(x)

$x[n]$ Length = N

$$Y(\omega) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$



Zero crossings when $\sin(\omega N/2) = 0$

$$\frac{\omega N}{2} = \pm k\pi \rightarrow \omega = \pm\frac{k2\pi}{N}$$

For $N=7$, $\omega = \pm\frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$

Periodic signal FTs: Summary so far…

$1 \Rightarrow 2\pi\delta(\omega)$

$e^{j\omega_0 t} \Rightarrow 2\pi\delta(\omega - \omega_0)$

$\cos(\omega_0 t) \Rightarrow \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$

$\sin(\omega_0 t) \Rightarrow j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0)$

Periodic signals have lonely impulses in the frequency domain

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \Rightarrow \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

Digression: Sinusoid in Freq <--> Lonely Impulses in Time

$$x(t) = \frac{1}{2}\delta(t+5) + \frac{1}{2}\delta(t-5)$$
$$X(\omega) = \frac{1}{2}e^{j5\omega} + \frac{1}{2}e^{-j5\omega} = \cos(5\omega)$$

$x(t)$ & $X(\omega)$: real & even

$$y(t) = -\frac{1}{2}\delta(t+5) + \frac{1}{2}\delta(t-5)$$
$$Y(\omega) = -\frac{1}{2}e^{j5\omega} + \frac{1}{2}e^{-j5\omega}$$
$$= -\frac{j}{2j}\left(e^{j5\omega} - e^{-j5\omega}\right) = -j\sin(5\omega)$$

$y(t)$: real & odd; $Y(\omega)$: imag & odd

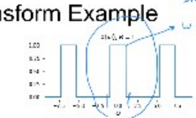
Inverse Transform Example

$x[n] \rightarrow X(\omega)$

$$X(\omega) = \begin{cases} 1 & |\omega| < B \\ 0 & B < |\omega| < \pi \end{cases}$$

$-B < \omega < B$

Sinc $= \frac{\sin(x)}{x}$



$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$
$$\int_{-B}^{B} (1)\, e^{j\omega n}d\omega \rightarrow \frac{1}{2\pi}\cdot\frac{1}{jn}\cdot e^{j\omega n}\Big|_{-B}^{B}$$
$$= \frac{1}{2\pi jn}\left(e^{jBn} - e^{-jBn}\right) = \frac{1}{\pi}\frac{\sin(Bn)}{n}\cdot\frac{B}{B} = \frac{B}{\pi}\cdot\frac{\sin(Bn)}{Bn}$$
$$= \boxed{\frac{B}{\pi}\text{Sinc}(Bn)}$$

Find the Fourier Transform of $f(t) = e^{-a|t|}$, $a > 0$

Break the signal into two pieces: $f(t) = e^{-a(-t)}u(-t) + e^{-at}u(t)$

Use additivity & time reverse (scaling) property

Fourier Series vs. Fourier Transform

$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$ → LTI → $y(t) = \sum_{k=-\infty}^{\infty} c_k H(jk\omega_0) e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$

$T_0 = 2\pi/\omega_0$

$\sum_k A_k \cos(k\omega_0 t + \theta_k)$ → LTI → $\sum_k A_k |H(jk\omega_0)| \cos(k\omega_0 t + \theta_k + \angle H(jk\omega_0))$

sin sin

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$ → LTI → $y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega$

$H(j\omega)$ specifies frequency scaling

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Example 3

x(t) → $H(j\omega)$ → $y(t)$

$H(j\omega) = \begin{cases} 1 & |\omega| < 3\pi \\ 0 & |\omega| > 3\pi \end{cases}$

$h(t) = 3\text{sinc}(3\pi t)$

$1 \cdot e^{j0t} \cdot e$

$x_2(t) = 1 + \sin(20t)$ $w=20$

$y_2(t) = H(0) \cdot 1 + |H(20)| \sin(20t + \angle H(20))$

1   0 $\boxed{y_2(t) = 1}$

Let's Focus on Digital Filters to Start

- General equation: $\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]$

Note: This equation is for causal filters. We can start m before zero if the system does not need to be real-time.

- Two main categories of digital filters
  - Finite impulse response (FIR): N=0

    $y[n] = \sum_{m=0}^{M} b_m x[n-m]$ $h[n] = \sum_{m=0}^{M} b_m \delta[n-m]$
  - Infinite impulse response (IIR): everything else
    - The $y[n-k]$ terms introduce feedback, which makes the impulse response infinite length

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Digression

$x(t-t_0) = e^{-j\omega t_0} X(\omega)$

$x[n-n_0] = e^{-j\omega n_0} X(\omega)$

- FT property: frequency shifting
  - CT: $e^{j\omega_0 t} x(t) \leftrightarrow X(\omega - \omega_0)$
  - DT: $e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$
- For DT, note that:

  $h_2[n] = (-1)^n h_1[n] = e^{-j\pi n} h_1[n]$

  $H_2(\omega) = H_1(\omega - \pi)$

  so multiplying by $(-1)^n$ shifts the center from 0 to π, changing an LPF to an HPF or vice versa
- Rewriting $(-a)^n u[n] = (-1)^n a^n u[n]$ explains the simple IIR results

$a > 0 \rightarrow$ HPF

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AM with Sinusoidal Carrier

also called AM-Double-Side-Band (DSB)

x(t) should be bandlimited, Low pass filter to ensure this

Convolution

$c(t) = \cos(\omega_c t)$

$C(\omega) = \pi[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)]$

$y(t) = x(t)\cos(\omega_c t)$

$Y(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega)$

$= \frac{1}{2}\left(X(\omega + \omega_c) + X(\omega - \omega_c)\right)$

$\frac{1}{2}$ $w_c >> w_m$ $-w_c$ w_c

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Nyquist Sampling Theorem

For a band-limited signal x(t) with X(ω) = 0 for |ω| > ω_B. In order to uniquely reconstruct x(t) after sampling, the sampling frequency ω_s must be greater than $2\omega_B$.

$-\omega_B$ ω_B $\frac{2\pi}{\omega_B} = T_s$

$2\omega_B < \omega_s = \frac{2\pi}{T_s}$ or $T_s < \frac{\pi}{\omega_B}$ Nyquist rate $= 2\omega_B$

Also called Nyquist–Shannon Sampling Theorem

ω_s

Terminology:
- $2\omega_B$ is the **Nyquist rate**, it is a property of a signal
- $\frac{\omega_s}{2}$ is the **Nyquist frequency**, it is a property of the sampler
- When $\omega_B > \frac{\omega_s}{2}$ then we say we have **aliasing**, or we are **undersampling**

$\omega_B > \frac{\omega_s}{2}$

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Simple Linear Phase System

The phase of the frequency response tells you about time delays

scales   shift

$y(t) = Mx(t - \alpha)$

System that scales the input by M and delays it by α has the transfer function:

$H(\omega) = Me^{-j\alpha\omega}$

magnitude $= M$ phase $= -\alpha\omega$

and the impulse response: $h(t) = M\delta(t - \alpha)$

Describe these systems: Freq Domain ω rad

1) $H(\omega) = e^{-j8\omega}$ scales by 1, delays by 8 $M=1$, $\alpha = 8$
2) $H(\omega) = e^{j4\omega}$ scales by 1, advances by 4   advance
3) $H(\omega) = 1/3$ scales by 1/3, no delay $M = 1/3$
4) $H(\omega) = 5e^{-j\omega\pi}$ scales by 5, delays by π M delay

Constant magnitude for all frequencies → all pass filter

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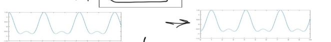
Why Linear Phase?

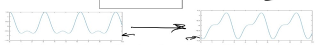
$x(t) = u(2t)$
$x(t-1) = u(2(t-1))$

- Linear phase means that all frequency components shift by the same amount, so they combine in the same way

$\cos(2t) + 0.5\cos(4t)$ → $H(\omega) = e^{-j\alpha\omega}$ → $\cos(2(t-\alpha)) + 0.5\cos(4(t-\alpha))$ shift



$\cos(2t) + 0.5\cos(4t)$ → $H(\omega) = e^{-j\alpha\omega^2}$ → $\cos(2(t-2\alpha)) + 0.5\cos(4(t-4\alpha))$



- Human hearing is not sensitive to minor phase differences, but large distortions sound bad

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Ex1: Ideal Low Pass Filter

Consider the periodic signal:

$x(t) = 10 + \cos(t) + \cos(2t) + \cos(10t) + \cos(20t)$

Find the outputs when x(t) is input to these two filters

$|H(\omega)|$ -4π 0 4π ω

$H_2(\omega) = \begin{cases} 2e^{j\omega} & |\omega| < 4\pi \\ 0 & |\omega| > 4\pi \end{cases}$ $G_1 = 2$ shift $= +1$

All frequencies |ω|>4π are removed

$y_2(t) = 2y_1(t+1)$

$= 20 + 2\cos(t+1) + 2\cos(2(t+1)) + 2\cos(10(t+1))$

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Back to FT Properties

	$f(t)$ ↔ Time Domain	$F(\omega)$ ↔ Fourier Transform		
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$	Shift in time → add a linear phase term in frequency	Duals
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$	Multiply by a CE → shift in frequency	
Convolution	$f(t) * g(t)$	$F(\omega)G(\omega)$	Convolve in time → multiply in frequency	Duals
Multiplication	$f(t)g(t)$	$\frac{1}{2\pi}F(\omega) * G(\omega)$	Multiply in time → Convolve in frequency	

Frequency shift is a special case of multiplication with $g(t) = e^{j\omega_0 t}$

Synchronous Demodulation

$y(t) = x(t)\cos(\omega_c t)$

$Y(\omega) = \frac{1}{2}[X(\omega + \omega_c) + X(\omega - \omega_c)]$

$c(t) = \cos(\omega_c t)$

$C(\omega) = \pi[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)]$

$w(t) = y(t)\cos(\omega_c t)$

$W(\omega) = \frac{1}{2}[Y(\omega + \omega_c) + Y(\omega - \omega_c)]$

$= \frac{1}{4}X(\omega + 2\omega_c) + \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega - 2\omega_c)$

ω_{LPF}

$\omega_M \le \omega_{LPF} \le 2\omega_c - \omega_M$

Recover x(t) from y(t) via LPF H(ω) with gain 2 and cutoff frequency ω_{LPF}

Reconstructing Original Signal

$1/T_s$ ω_c $X_s(\omega)$ w_c w_h

→ x(t) is a band-limited signal with X(ω) = 0 for |ω| > ω_B

Band

$w > \omega_B$ or $w < -\omega_B$

$X(\omega) = 0$

Ideal low pass filter with gain T_s and cutoff frequency ω_c

$\omega_B \le \omega_c \le \omega_s - \omega_B$

$H(\omega)$ T_s $X_r(\omega)$

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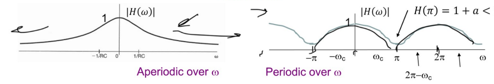
Compare CT & DT: Practical 1st-order LPF

Continuous Time		Discrete Time
$y(t) + a\frac{dy(t)}{dt} = x(t)$		$y[n] + ay[n-1] = (1+a)x[n]$
$h(t) = \frac{1}{a}e^{-t/a}u(t)$		$h[n] = (1+a)(-a)^n u[n]$
$H(\omega) = \frac{1/a}{\frac{1}{a} + j\omega}$	Not quite equivalent	$H(\omega) = \frac{1+a}{1+ae^{-j\omega}}$ $a < 0$



$H(\pi) = 1 + a < 1$

Aperiodic over ω   Periodic over ω

Ex 2: Different Ideal Filters

ICE $x(t)$ T_1

Consider another periodic signal:

x(t) = square wave (50% on/off), frequency $\omega_0 = 2\pi$; $T_0 = 1$; $T_1 = 1/4$

$a_k = \frac{1}{2}\text{sinc}(k\pi/2)$ $k \neq 0, a_0 = \frac{1}{2}$

Find the outputs when x(t) is input to these filters

LPF   BPF   HPF

$H_1(\omega) = \begin{cases} 1 & |\omega| < 3\pi \\ 0 & |\omega| > 3\pi \end{cases}$ $H_2(\omega) = \begin{cases} 2e^{j\omega} & 3\pi < |\omega| < 7\pi \\ 0 & otherwise \end{cases}$ $H_3(\omega) = \begin{cases} 1 & |\omega| > \pi \\ 0 & |\omega| < \pi \end{cases}$

$y_1(t)$ $|k\omega_0| < 3\pi$ $y_2(t)$ $y_3(t)$

a_0, a_1, a_{-1}

$a_0 = \frac{1}{2}$, $a_1 = \frac{1}{2}\text{sinc}(\pi/2) = \frac{1}{2}\frac{\sin(\pi/2)}{\pi/2} = \frac{1}{\pi}$

$a_{-1} = \frac{1}{2}\text{sinc}(-\pi/2) = \frac{1}{2}\frac{\sin(-\pi/2)}{-\pi/2} = \frac{1}{\pi}$

$= \boxed{y_1(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi t)}$

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Aside: Duality & FT Properties

formulas

Reminder: Duality doesn't exactly hold for the DTFT.

Duality	$f(t)$	$F(\omega)$	
	$F(t)$	$2\pi f(-\omega)$	
Convolution	$f(t) * g(t)$	$F(\omega)G(\omega)$	Multiplication, convolution duals
Multiplication	$f(t)g(t)$	$\frac{1}{2\pi}F(\omega) * G(\omega)$	
Time shift	$f(t - t_0)$	$F(\omega)e^{-j\omega t_0}$	Time & freq shift duals
Frequency shift	$f(t)e^{j\omega_0 t}$	$F(\omega - \omega_0)$	
Time multiply	$tf(t)$	$j\frac{d}{d\omega}F(\omega)$	Multiply-derivative duals
Differentiation	$\frac{d}{dt}f(t)$	$j\omega F(\omega)$	

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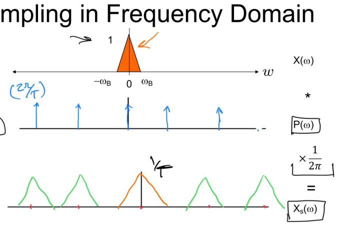
- Sampling in time: $x_s(t) = x(t)p(t)$ sampling
  - Multiply by an impulse train in time: $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$
  - Convolve with an impulse train in frequency: $P(\omega) = \frac{2\pi}{T_s}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$

$X_s(\omega) = \frac{1}{2\pi}X(\omega) * P(\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$ sampling frequency

Sampling in Frequency Domain

ICE



ω_B=X(jω) bandwidth   X(ω) $-\omega_B$ 0 ω_B ω

$\omega_s = \frac{2\pi}{T}$ sampling frequency $(2\pi/T)$

$\ast$ $P(\omega)$ $\times \frac{1}{2\pi}$ $=$ $X_s(\omega)$

$\frac{2\pi}{T} \cdot \frac{1}{2\pi} = \frac{1}{T}$ $1/T$

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