Problem 1

(a) $X(w) = \int_{-\infty}^{\infty} x(t) e^{-jwt} \, dt$

$= \int_{-\infty}^{\infty} \delta(t-t_0) e^{-jwt} dt$

impulse function only exist at $t = t_0$

$x(w) = e^{-jwt} \big|_{t=t_0} = e^{-jwt_0}$

$F[\delta(t)] = 1$

(b) $\cos(\pi t) = \frac{1}{2}[e^{j\pi t} + e^{j\pi t}]$

$X(w) = \frac{1}{2} \cdot [2\pi \delta(w-\pi) + 2\pi \delta(w+\pi)]$

$\sin(t) = \frac{1}{2j}[e^{jt} + e^{-jt}]$

$X(w) = j\pi[\delta(w+1) - \delta(w-1)]$

$F[x(t)] = \pi\delta(w-\pi) + \pi\delta(w+\pi) + j\pi\delta(w+1) - j\pi\delta(w-1)$

(c) $X(t) = e^{-2(t-1)} u(t-1)$

$X(w) = \int_1^{\infty} e^{-2(t-1)} \cdot e^{-jwt}$

$= e^2 \int_1^{\infty} e^{-2t} \cdot e^{-jwt}$ $\qquad e^{-(2+jw)t}$

$= e^2 \left| -\frac{e^{-(2+jw)t}}{2+jw} \right|_1^{\infty}$

$= \frac{e^2 \cdot e^{-(2+jw)}}{2+jw} = \frac{e^{-jw}}{2+jw}$

(d) $e^{-2|t|} = \frac{4}{4+w^2}$

$x(t-t_0) \xleftrightarrow{FT} e^{-jwt_0} \cdot X(w)$

$$F[x(t)] = e^{-jw} \cdot \left(\frac{4}{4+w^2}\right)$$

Problem 2:

(a) $F_1(jw) = \frac{1}{3+jw} + \frac{1}{4-jw}$

$$\frac{1}{a+jw} \Leftarrow e^{-at}u(t)$$
$$\frac{1}{a-jw} \Leftarrow e^{at}u(-t)$$

$$F_1(t) = e^{-3t}u(t) + e^{4t}u(-t)$$

(b) $F(jw) = \cos(4w + \frac{\pi}{3})$

$$= \cos(\tfrac{\pi}{3})\cos(4w) - \sin(\tfrac{\pi}{3})\sin(4w)$$
$$= \frac{1}{2}\cos(4w) - \frac{\sqrt{3}}{2}\sin(4w)$$
$$= \frac{1}{2}\left[\sqrt{\tfrac{\pi}{2}}\,\delta(t-4) + \sqrt{\tfrac{\pi}{2}}\,\delta(t+4)\right] + \frac{\sqrt{3}}{2}\left[j\sqrt{\tfrac{\pi}{2}}\,\delta(t+4) - j\,\delta(t-4)\right]$$

Problem 3.

$$g(t) = \frac{\sin 2t}{\pi t}$$
$$\frac{\sin 2t}{\pi t} = \frac{x_1(t)}{\pi t} \cdot \cos(t)$$
$$x_1(t)\cos(t) = \sin(2t)$$
$$\cancel{*}\ \frac{1}{2}\sin(2t) = \sin(t)\cos(t)$$
$$x(t) = \frac{2\sin(t)}{\pi t}$$

Problem 4

(a) $x[n] = u[n-2] - u[n-6]$

$$u[n-2] \xrightarrow{DTFT} e^{-jw2} \cdot \left[\frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} (w-2\pi k) \right]$$
$$u[n-6] \xrightarrow{DTFT} e^{-jw6} \cdot \left[\frac{1}{1-e^{jw}} + \sum_{k=-\infty}^{\infty} (w-2\pi k) \right]$$

$$X(w) = e^{-jw2} \cdot \left[\frac{1}{1-e^{-jw}} + \sum_{k=-\infty}^{\infty} (w-2\pi k) \right] + e^{-jw6} \left[\frac{1}{1-e^{jw}} + \sum_{k=\infty}^{\infty} (w-2\pi k) \right]$$

(b) $x[n] = (\frac{1}{2})^{-n} u[n-1]$

$$u[n] \xrightarrow{DTFT} \frac{1}{1-e^{-jw}} + \sum_{-\infty}^{\infty} \pi \delta(w-2\pi k)$$

$$a^n u[n] \xrightarrow{DTFT} \frac{1}{1-a \cdot e^{jw}}$$

$$a^{n-1} u[n-1] \xrightarrow{DTFT} \frac{e^{-jw}}{1-a \cdot e^{jw}}$$

$$a^{-n-1} u[-n+1] \xrightarrow{DTFT} \frac{e^{jw}}{1-a e^{jw}}$$
$$a^{-n} u[-n-1] \xrightarrow{DTFT} \frac{a \cdot e^{jw}}{1-a \cdot e^{jw}}$$

$$x[n] \xrightarrow{DTFT} \frac{0.5 e^{jw}}{1-0.5 e^{jw}}$$

(d) $x[n] = 2^n \sin(\frac{\pi}{4} n) u[-n]$

$$2^{-n} \cdot u[n] \xrightarrow{DTFT} \frac{1}{1-0.5 e^{-jw}}$$

$$\sin(n\theta) = \frac{e^{jn\theta} - e^{jn\theta}}{2j}$$
$$\sin(n\theta) [(\tfrac{1}{2})^n u(n)] = \frac{1}{2j} [e^{jn\theta} [(\tfrac{1}{2})^n u(n)] - e^{jn\theta}$$
$$[(\tfrac{1}{2})^n u(n)]$$
$$\xrightarrow{DTFT}$$

$$\frac{1}{2j}\left[\frac{1}{1-0.5e^{-j(w-\theta)}}-\frac{1}{1-0.5e^{-j(w+\theta)}}\right]$$

$$\left(\frac{1}{2}\right)^n \sin(n\theta)\, u[n] = \frac{1}{2j}\left[\frac{1}{1-0.5e^{-j(w-\theta)}}-\frac{1}{1-0.5e^{-j(w+\theta)}}\right]$$

replace n with $-n$

$$\frac{1}{2j}\left[\frac{1}{1-0.5e^{-j(w-\theta)}}-\frac{1}{1-0.5e^{j(w+\theta)}}\right]$$

Problem 5

(a) $$x[n] = \frac{1}{2\pi}\int_{2\pi} x(e^{jw})\cdot e^{jwn}\,dw$$

$$x[n] = \frac{1}{2\pi}\int_{-\frac{3\pi}{4}}^{-\frac{\pi}{4}} e^{jwn}\,dw + \frac{1}{2\pi}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} e^{jwn}\,dw$$

$$= \frac{1}{\pi\cdot n}\cdot\left[\sin\left(\frac{3\pi n}{4}\right) - \sin\left(\frac{\pi n}{4}\right)\right]$$

(b) $$x[n] = \frac{1}{2\pi}\int_{2\pi}\left[1+3e^{jw}+2e^{-j2w}-4e^{-3jw}\right] e^{jw}\,dw$$

$$= \delta[n] + 3\delta[n-1] + 28\delta[n-2] - 488[n-3] + \delta[n-10]$$

(e) $$x[n] = \sum_{k=n} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \xrightarrow{F.T} \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(w - \frac{2\pi k}{N}\right)$$

$$x[n] = \sum_{n=0}^{4} \frac{(-1)^k}{2\pi} e^{jk\left(\frac{\pi}{2}\right)n}$$

$$= \frac{1}{2\pi}\left[1 - e^{\frac{j\pi n}{2}} + e^{j\pi n} - e^{\frac{j3\pi n}{2}}\right]$$