Derivation of control equations in Region 2, 2.5 and 3

Course: Wind Turbine Control Online Course (June 3, 2021)

1 Derivation of Optimal Torque Constant in Region 2

Objective: Derive an expression for the optimal generator torque that optimises the power output and has the following form:

$$Q_g = K\Omega_r^2, \tag{1}$$

where the expression for K consists entirely of known quantities.

Table 1: Variables used in the derivation

Variable	Description	Units
A	Rotor area	m^2
C_p	Rotor power coefficient	-
n_g	Gearbox ratio	-
K	Optimal generator torque constant	Nm/(rad/s)^2
λ_{opt}	Optimal tip-speed ratio (TSR)	-
Ω_r	Rotor speed	rad/s
Ω_g	Generator speed	rad/s
P_{aero}	Rotor aerodynamic power	W
P_{gen}	Generated power	W
P_{HSS}	Power in high-speed shaft	W
P_{wind}	Power in the wind	W
Q_g	Generator torque on high speed shaft side	Nm
Q_{gLSS}	Generator torque on low speed shaft side	Nm
R	Rotor radius	m
ρ	Air density	kg/m^3
θ_{opt}	Pitch angle at optimal TSR	deg
V	Wind speed	m/s

Derivation: All variables are defined in Table 1. We begin with the expression for the aerodynamic power in the rotor:

$$P_{aero} = P_{wind}C_p(\theta_{opt}, \lambda_{opt}), \tag{2a}$$
$$= \frac{1}{2}\rho AV^3 C_p(\theta_{opt}, \lambda_{opt}). \tag{2b}$$

Notice that the power co-efficient C_p is at maximum for given $\theta_{opt}, \lambda_{opt}$. By solving the definition of the tip-speed ratio ($\lambda = R\Omega_r/V$) for V and substituting that into Eq. (2b), we arrive at the following equation:

$$P_{aero} = \frac{\rho AR^3 C_p(\theta_{opt}, \lambda_{opt})}{2\lambda_{opt}^3}\Omega_r^3. \tag{3}$$

Let us assume that the drive train efficiency is 100% but the generator has a certain efficiency η. The expression for the generated power is therefore

$$P_{gen} = \eta P_{HSS}. \tag{4}$$

Because the gearbox is 100% efficient, the power available in the rotor is perfectly transformed to power in the high-speed shaft:

$$P_{aero} = P_{HSS}. \tag{5}$$

Combining Eqs. (3) and (5) yields

$$P_{gen} = \eta \frac{\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\lambda_{opt}^3} \Omega_r^3. \tag{6}$$

As mentioned above, we seek an expression for the generator torque that maximises the generator power in (6). Thus, let's replace the generator power with an expression for the torque using the fact that power is the product of torque and angular velocity:

$$P_{gen} = Q_g \Omega_g = \eta \frac{\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\lambda_{opt}^3} \Omega_r^3. \tag{7}$$

where the Q_g is the generator torque on the high speed shaft side. The generator speed (Ω_g) and the rotor speed (Ω_r) are related through the gearbox ratio (n_g), $\Omega_g = n_g \cdot \Omega_r$, so we can substitutes that into Eq. (7)

$$Q_g \Omega_g = Q_g \cdot n_g \cdot \Omega_r = \eta \frac{\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\lambda_{opt}^3} \Omega_r^3. \tag{8}$$

If we define another variable Q_{gLSS} for the generator torque on the low speed shaft side, we then have $Q_{gLSS} = Q_g \cdot n_g$. So Eq. (8) can be written as

$$Q_{gLSS} \cdot \Omega_r = \eta \frac{\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\lambda_{opt}^3} \Omega_r^3. \tag{9}$$

From Eq. (8), we divide both sides by $\Omega_r \cdot n_g$ to get the following equation for the generator torque on the high speed shaft side:

$$\boxed{Q_g = \frac{\eta \rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\, n_g\, \lambda_{opt}^3} \Omega_r^2} \tag{10}$$

Similarly, from Eq. (9), we divide both sides by Ω_r to get the following equation for the generator torque on the low speed shaft side:

$$\boxed{Q_{gLSS} = \frac{\eta \rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\, \lambda_{opt}^3} \Omega_r^2} \tag{11}$$

Equation (10) and (11) are the expressions for generator torque on the high speed shaft side and on the low speed shaft side, respectively. They are in the desired form

$Q_g = K_{HSS}\Omega_r^2$ and $Q_{gLSS} = K_{LSS}\Omega_r^2$. Thus, from the Eq. (10) the expression for K —the optimal torque constant in Region 1—is given by

$$\boxed{K = K_{HSS} = \frac{\eta\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\, n_g\, \lambda_{opt}^3}} \tag{12}$$

Or, from the Eq. (11) the expression for K —the optimal torque constant in Region 1—is given by

$$\boxed{K = K_{LSS} = \frac{\eta\rho A R^3 C_p(\theta_{opt}, \lambda_{opt})}{2\, \lambda_{opt}^3}} \tag{13}$$

2 Derivation of PI Torque Controller in Region 2

Objective: Derive the PI torque controller equation and calculate the Proportional Gain (k_{Pg}) and Integral Gain (k_{Ig}) for the PI generator torque controller, such that the drive-train dynamics is characterised by the natural frequency ω_{Ω_g} and damping ratio ζ_{Ω_g}

First, we start from the open-loop system.

Figure 1 shows the schematic diagram of a open-loop wind turbine drive train dynamic. The equation of motion of the simplified wind turbine drive train dynamic can be ex-

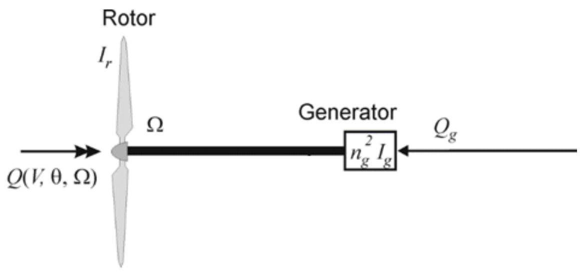

Figure 1: The schematic diagram of an open-loop wind turbine drive train system.

pressed in the following equation:

$$(I_r + n_g^2 I_g)\dot{\Omega} = Q(V, \Omega, \theta) - \frac{1}{\eta} Q_g \tag{14}$$

where, Q is the aerodynamic torque, which is the function of wind speed (V), rotor speed (Ω) and blade pitch angle (θ). Q_g is the generator torque, which is our control input. I_r and I_g are rotor inertia and equivalent generator inertia respectively. n_g is the gearbox ratio. η is the generator efficiency.

Second, we move on to the closed-loop system.

Figure 2 shows the schematic diagram of a closed-loop wind turbine drive train dynamic system with a PI-controller.

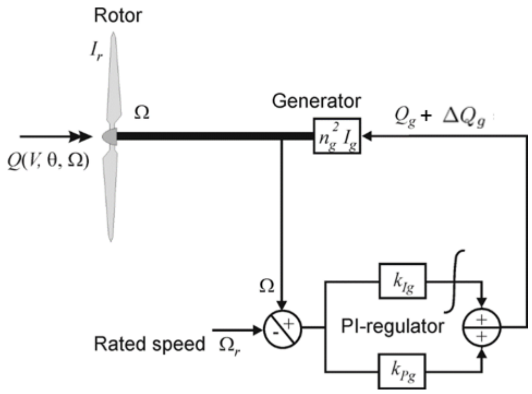

Figure 2: The schematic diagram of a closed-loop wind turbine drive train system.

The equation of motion of this closed-loop system can be expressed in the following equation:

$$(I_r + n_g^2 I_g)\dot{\Omega} = Q(V, \Omega, \theta) - \frac{1}{\eta} Q_g \tag{15}$$

It is worth to mention that Equation (15) is a non-linear equation and it should be linearized for a controller design around rated rotor speed at an operating point defined as $op = (V_{op}, \Omega_r, \theta_{op}, Q_g\Big|_{op})$.

Let's define some derivation variables from the operating point:

$$\Omega = \Omega_r + \dot{\phi},\ V = V_{op} + \Delta v,\ \theta = \theta_{op} + \Delta\theta, Q_g = Q_g\Big|_{op} + \Delta Q_g, \tag{16}$$

Substituting (16) into (15) yields:

$$(I_r + n_g^2 I_g)(\dot{\Omega}_r + \ddot{\phi}) = Q\left(V_{op} + \Delta v, \Omega_r + \dot{\phi}, \theta_{op} + \Delta\theta\right) - \frac{1}{\eta}\left(Q_g\Big|_{op} + \Delta Q_g\right) \tag{17}$$

where, $\dot{\phi}$ is a small variation of the rotor speed, ΔQ_g is the output from the PI torque controller.

Apply Talyor expansion on the non-linear term:

$$\begin{aligned} Q\left(V_{op} + \Delta v, \Omega_r + \dot{\phi}, \theta_{op} + \Delta\theta\right) &= Q(V_{op}, \Omega_r, \theta_{op}) + \cdots \\ &+ \frac{\partial Q}{\partial V}\Big|_{op} \Delta v + \frac{\partial Q}{\partial \Omega}\Big|_{op} \dot{\phi} + \frac{\partial Q}{\partial \theta}\Big|_{op} \Delta\theta + O(\Delta v, \dot{\phi}, \Delta\theta) \end{aligned} \tag{18}$$

Notice that $O(\Delta v, \dot{\phi}, \Delta\theta)$ denotes the higher order terms that is omitted in this case. The pitch angle deviation $\Delta\theta$ is kept at zero in Region 2. The term $\frac{\partial Q}{\partial V}\Big|_{op} \Delta v$ is a external force to the drive-train system and does not affect the derivation of the system characteristics such as natural frequency and damping ratio, thus, omitted for brevity.

Substituting Eq. (18) into Eq. (17) and re-arranging the terms, we arrive at the following equations:

$$\underbrace{(I_r + n_g^2 I_g)\dot{\Omega}_r} + (I_r + n_g^2 I_g)\ddot{\phi} = \underbrace{Q(V_{op}, \Omega_r, \theta_{op})} + \frac{\partial Q}{\partial \Omega}\Big|_{op} \dot{\phi} - \frac{1}{\eta}\underbrace{Q_g\big|_{op}} - \frac{1}{\eta}\Delta Q_g \quad (19)$$

The terms marked with the curly braces form the steady-state equation:

$$(I_r + n_g^2 I_g)\dot{\Omega}_r = Q(V_{op}, \Omega_r, \theta_{op}) - \frac{1}{\eta}Q_g\big|_{op} \quad (20)$$

and can be cancelled out. In the end, the linearized closed-loop equation for a small variation of the rotor speed can be expressed as the following equation:

$$\boxed{(I_r + n_g^2 I_g)\ddot{\phi} - \frac{\partial Q}{\partial \Omega}\Big|_{op} \dot{\phi} + \frac{1}{\eta}\Delta Q_g = 0} \quad (21)$$

Introduce PI gains

According to the PI control theory, the PI feedback value on the generator torque (ΔQ_g) can be expressed as the following equation:

$$\Delta Q_g = k_{Pg}(\Omega - \Omega_r) + k_{Ig}\int_0^t (\Omega - \Omega_r)\ dt \quad (22)$$

$$= k_{Pg}\dot{\phi} + k_{Ig}\phi \quad (23)$$

where, k_{Pg} and k_{Ig} are Proportional and Integral gain respectively. They are the values of our interest and need to be calculated or, commonly saying, tuned.

Substituting Eq. 23 into Eq. 21 and re-arranging the terms, the Eq. 21 can be re-written into the following equation:

$$(I_r + n_g^2 I_g)\ddot{\phi} + (\frac{1}{\eta}k_{Pg} - \frac{\partial Q}{\partial \Omega}\Big|_{op})\dot{\phi} + \frac{1}{\eta}k_{Ig}\phi = 0 \quad (24)$$

The Eq. 24 is in the form of typical Equations of Motion for a second order dynamical system. The term, $I_r + n_g^2 I_g$, represents the mass/inertia of the system, the term, $\frac{1}{\eta}k_{Pg} - \frac{\partial Q}{\partial \Omega}\Big|_{op}$, represents the damping value of the system and the term, $\frac{1}{\eta}k_{Ig}$, represents the stiffness value of the system. Usually, we ignore the term, $\frac{\partial Q}{\partial \Omega}\Big|_{op}$, and the Eq. 24 is further simplified as:

$$(I_r + n_g^2 I_g)\ddot{\phi} + \frac{1}{\eta}k_{Pg}\dot{\phi} + \frac{1}{\eta}k_{Ig}\phi = 0 \quad (25)$$

The characteristics of the system, which are nature frequency ($\omega_{\Omega g}$) and damping ratio ($\zeta_{\Omega g}$) are defined by the mass/inertia, stiffness and damping terms. They are expressed as the function of k_{Pg} and k_{Ig} follows:

$$\omega_{\Omega g} = \sqrt{\frac{\frac{1}{\eta} k_{Ig}}{I_r + n_g^2 I_g}} \tag{26}$$

$$\zeta_{\Omega g} = \frac{\frac{1}{\eta} k_{Pg}}{2(I_r + n_g^2 I_g)\omega_{\Omega g}} \tag{27}$$

Once the desired characteristic of the dynamic system (nature frequency and damping ratio) are selected, the PI gain values are calculated as:

$$k_{Pg} = 2\eta\zeta_{\Omega g}\omega_{\Omega g}(I_r + n_g^2 I_g) \tag{28}$$

$$k_{Ig} = \eta(I_r + n_g^2 I_g)\omega_{\Omega g}^2 \tag{29}$$

How do we select the "correct" nature frequency and damping ratio? - Rule of the thumb:

The step response of a 2nd order dynamical system is characterized by the the nature frequency and damping ratio. Figure 3 shows the system response with respect to the nature frequency and damping ratio. In general, the larger the nature frequency will

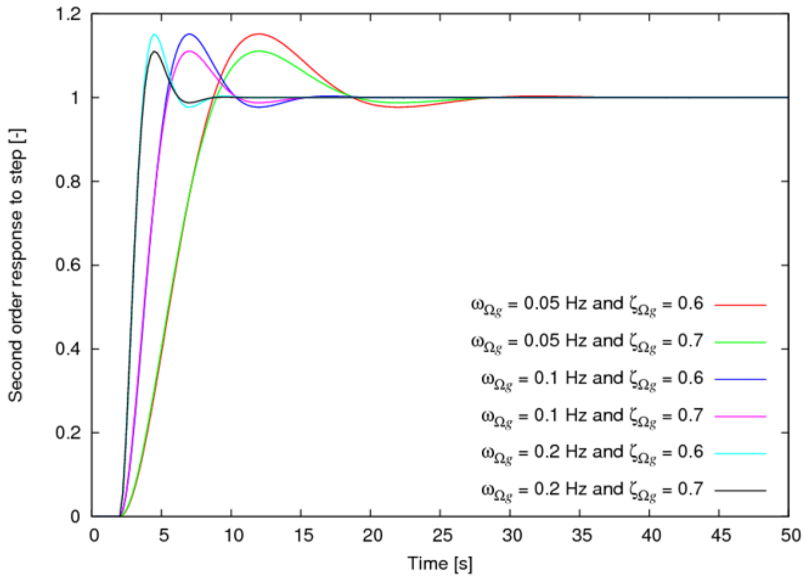

Figure 3: Step response of a second order system.

result in the faster response, and the larger damping ratio will make the system stabilized quickly. When designing the wind turbine controller, it is desired to choose the nature frequency of the controller smaller than the lowest frequency of the wind turbine system, which is usually the tower frequency, with enough stability margin. As a rule of the thumb, we usually choose $\omega_{\Omega g} = 0.1$ Hz and damping ratio $\zeta_{\Omega g} = [0.6, 0.7]$.

3 Derivation of PI Blade Pitch Controller in Region 3

In region 3, the blade pitch PI controller is commonly tuned in a way such that the rotor speed regulation mode is decoupled from other turbine structural modes. Namely, the natural frequency and damping ratio of the speed regulation mode are often selected low enough to avoid resonance with the first fore-aft tower mode but high enough to achieve good control performance.

Proposition 1. *The natural frequency and damping ratio of the rotor speed regulation mode are ω_Ω and ζ_Ω if the blade pitch PI controller is tuned as follows:*

$$\begin{aligned}\Delta\theta &= k_P(\Omega - \Omega_r) + k_I \int_0^t (\Omega - \Omega_r)dt, && \text{(30a)}\\ &= k_P\dot{\phi} + k_I\phi, && \text{(30b)}\end{aligned}$$

where the parameters are listed in Table 2 and k_P and k_I are defined as follows:

$$\begin{aligned}k_P &= \frac{2\zeta_\Omega\omega_\Omega(I_r + n_g^2 I_g) - \frac{1}{\eta}\frac{\partial Q_g}{\partial\Omega}\Big|_0}{-\frac{\partial Q}{\partial\theta}\Big|_0}, && \text{(30c)}\\ k_I &= \frac{\omega_\Omega^2(I_r + n_g^2 I_g)}{-\frac{\partial Q}{\partial\theta}\Big|_0}. && \text{(30d)}\end{aligned}$$

Proof. The tuning method relies upon a few assumptions: (i) rigid turbine, (ii) quasi-steady aerodynamics, (iii) no filters on the rotor speed measurements and (iv) pitch actuators dynamics is ignored. Based upon these assumptions, the drive-train dynamics is modelled as follows:

$$(I_r + n_g^2 I_g)\dot{\Omega} = Q(V, \Omega, \theta) - \frac{1}{\eta}Q_g(\Omega). \tag{31}$$

Notice that (31) is a nonlinear equation. A nonlinear system model is often less straight-forward to be analysed and used for control design purposes. Thus, linear approximation of a nonlinear system at a given operating point is commonly employed, that is similar to the first order Taylor expansion at the point of interest. Firstly, applying the first order Taylor expansion at the operating point $(\Omega_r, V_{\text{op}}, \theta_{\text{op}})$ to the terms in (31):

$$\begin{aligned}&\Omega \approx \Omega_r + \dot{\phi}, && \text{(32a)}\\ &V \approx V_{\text{op}} + \Delta v, && \text{(32b)}\\ &\theta \approx \theta_{\text{op}} + \Delta\theta, && \text{(32c)}\\ &Q(V, \Omega, \theta) \approx Q(V_{\text{op}}, \Omega_r, \theta_{\text{op}}) + \frac{\partial Q}{\partial V}\bigg|_0 \Delta v + \frac{\partial Q}{\partial \Omega}\bigg|_0 \dot{\phi} + \frac{\partial Q}{\partial \theta}\bigg|_0 \Delta\theta, && \text{(32d)}\\ &Q_g(\Omega) \approx Q_g(\Omega_r) + \frac{\partial Q_g}{\partial \Omega}\bigg|_0 \dot{\phi}. && \text{(32e)}\end{aligned}$$

Notice that the linearisation is often evaluated about $\theta_{\text{op}} = 0$ for simplicity reason, thus, the notation $\frac{\partial Q}{\partial\theta}\big|_{(\Omega_r, V_{\text{op}}, \theta_{\text{op}}=0)}$ is represented by $\frac{\partial Q}{\partial\theta}\big|_0$ for brevity. Secondly, based

Variable	Description	Unit
k_P	Proportional gain of the PI controller	$\mathrm{rad}\ (\mathrm{rad} \cdot \mathrm{s}^{-1})^{-1}$
k_I	Integral gain of the PI controller	$\mathrm{rad} \cdot \mathrm{rad}^{-1}$
ζ_Ω	Damping ratio of speed regulation mode	-
ω_Ω	Natural frequency of speed regulation mode	$\mathrm{rad} \cdot \mathrm{s}^{-1}$
I_r	Rotation inertia of rotor	kg-m^2
I_g	Rotation inertia of generator	kg-m^2
n_g	Gearbox ratio	-
η	Generator efficiency	-
ϕ	Azimuth angle	rad
Ω	Rotor speed	rad/s
Ω_r	Rotor speed at steady-state/rated	rad/s
$\dot{\phi}$	Rotor speed deviation from ϕ	rad/s
θ	Pitch angle	deg
θ_{op}	Pitch angle at steady-state	deg
$\Delta\theta$	Pitch angle variation from θ_{op}	deg
V	Wind speed	m/s
V_{op}	Wind speed at steady-state	m/s
Δv	Wind speed variation from V_{op}	m/s
Q	Aerodynamic torque	N-m
Q_g	Generator torque	N-m
$\frac{\partial Q}{\partial V}$	Variation of aerodynamic torque to wind speed	N-s
$\frac{\partial Q}{\partial \Omega}$	Variation of aerodynamic torque to rotor speed	N-m-s/rad
$\frac{\partial Q}{\partial \theta}$	Variation of aerodynamic torque to pitch angle	N-m/deg
$\frac{\partial Q_g}{\partial \Omega}$	Variation of generator torque to rotor speed	N-m-s/rad

Table 2: Nomenclature.

upon (31), at steady-state $(\Omega_r, V_{\text{op}}, \theta_{\text{op}})$, the drive-train motion is as follows:

$$(I_r + n_g^2 I_g)\dot{\Omega}_r = Q(V_{\text{op}}, \Omega_r, \theta_{\text{op}}) - \frac{1}{\eta} Q_g(\Omega_r), \tag{33a}$$

$$\dot{\Omega}_r = 0. \tag{33b}$$

Thus, substituting (32) and (33) into (31) yields the drive-train dynamics around the operating point $(\Omega_r, V_{\text{op}}, \theta_{\text{op}})$:

$$\left(I_r + n_g^2 I_g\right)\ddot{\phi} = \left.\frac{\partial Q}{\partial V}\right|_0 \Delta v + \left.\frac{\partial Q}{\partial \Omega}\right|_0 \dot{\phi} + \left.\frac{\partial Q}{\partial \theta}\right|_0 \Delta\theta - \frac{1}{\eta}\left.\frac{\partial Q_g}{\partial \Omega}\right|_0 \dot{\phi}, \tag{34}$$

The Proposition 1 can be proved by the sufficient condition. Let the blade pitch PI controller be defined as (30). Substituting (30b) into (34) yields:

$$(I_r + n_g^2 I_g)\ddot{\phi} + \left(\frac{1}{\eta}\left.\frac{\partial Q_g}{\partial \Omega}\right|_0 - \left.\frac{\partial Q}{\partial \Omega}\right|_0 - \left.\frac{\partial Q}{\partial \theta}\right|_0 k_P\right)\dot{\phi} - \left.\frac{\partial Q}{\partial \theta}\right|_0 k_I \phi = 0. \tag{35}$$

Notice that the variation term $\left.\frac{\partial Q}{\partial V}\right|_0 \Delta v$ is omitted from the equation (35) because the wind speed term is a driving force/disturbance to the closed-loop speed regulation loop, thus, its absence does not affect the modal analysis of the speed regulation loop. In addition, the variation term $\left.\frac{\partial Q}{\partial \Omega}\right|_0$ sometimes is assumed to be negligible for simplicity reason.

Subsequently, for natural frequency and damping ratio of the speed regulation mode to be ω_Ω and ζ_Ω, the closed-loop speed regulation system needs to possesse the characteristics of a damped and oscillatory dynamical system in a canonical form:

$$\ddot{\phi} + 2\zeta_\Omega \omega_\Omega \dot{\phi} + \omega_\Omega^2 \phi = 0. \tag{36}$$

Based upon (35) and (36), one can prove that the proportional and integral gains of the blade pitch controller k_P and k_I are defined as (30). $\square$