Monte Carlo Method

- Sampling-based computational method
- Goal: compute $\alpha = EX$

  $X_1, X_2, \cdots$ i.i.d.

  $\mathrm{var}\, X_1 < \infty$

  $n^{\frac{1}{2}}(\bar{X}_n - EX) \Rightarrow \sigma N(0,1)$

  $\bar{X}_n \overset{D}{\approx} EX + \frac{\sigma}{\sqrt{n}} N(0,1)$

  × 10 accuracy
  needs × 100 data

- Square-root convergence rate

  dimensional-insensitive

  $\sigma^2 = \int_{-\infty}^{\infty} h^2(x)\,dx - \left(\int_{-\infty}^{\infty} h(x)\,dx\right)^2$

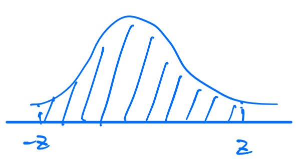
  $X = h(U)$

  $EX = \int_0^1 h(x)\,dx$

  $\Rightarrow$ smoothness doesn't matter

Confidence Interval

$$P\left(-z \le n^{\frac{1}{2}} \frac{\bar{X}_n - EX}{\sigma} \le z\right) \to P(-z \le N(0,1) \le z)$$



$$n^{\frac{1}{2}} \frac{\bar{X}_n - EX}{\sigma} \le z$$

$$\bar{X}_n - EX \le \frac{z\sigma}{\sqrt{n}}$$

$$\Rightarrow \bar{X}_n - \frac{z\sigma}{\sqrt{n}} \le EX \le \bar{X}_n + \frac{z\sigma}{\sqrt{n}}$$

unknown

$$P\left(EX \in \left[\bar{X}_n - \frac{z\sigma}{\sqrt{n}}, \bar{X}_n - \frac{z\sigma}{\sqrt{n}}\right]\right) \to 0.9$$

$z = 1.645$

$$S_n = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2}$$

$$S_n \xrightarrow{a.s.} \sigma$$

Use instead:

$$\left[\bar{X}_n - \frac{2S_n}{\sqrt{n}},\ \bar{X}_n - \frac{2S_n}{\sqrt{n}}\right]$$

This holds because

$$n^{\frac{1}{2}}\frac{\bar{X}_n - EX}{S_n} \overset{?}{\Rightarrow} N(0,1)$$

$$\underbrace{n^{\frac{1}{2}}\left(\frac{\bar{X}_n - EX}{\sigma}\right)}_{\Downarrow\ N(0,1)} \cdot \underbrace{\left(\frac{\sigma}{S_n}\right)}_{\downarrow P\ 1}$$

Slutsky's Lemma

- $X_1, X_2, \cdots, X_n$ i.i.d.
- Compute $\bar{X}_n$, S_n
- Our C.I. will be

$$\left[\bar{X}_n - \frac{2S_n}{\sqrt{n}},\ \bar{X}_n - \frac{2S_n}{\sqrt{n}}\right]$$

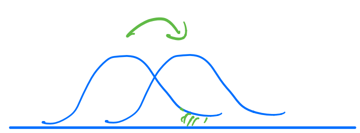
$P(A)$

- crude Monte Carlo
- importance sampling

$$P(A) = E_Q I(A) L$$

likelihood ratio

$$E_P W = E_Q W L$$



$$E_f h(x) = \int h(x) f(x)\,dx$$
$$= \int h(x)\frac{f(x)}{g(x)} g(x)\,dx$$
$$= E_g h(x)\frac{f_X(x)}{g(x)} \leftarrow L$$

- What is the best choice of Q?

$$g^* = \frac{h(x) f_X(x)}{\alpha}$$

$$E_P W = \int_\Omega W(\omega) P(d\omega) \qquad (W \geq 0)$$

$$= \int_\Omega \underbrace{Q^*(d\omega)}_{} \; \underbrace{\int_\Omega W(\omega) P(d\omega)}_{= L^*} \quad \text{// } E_P W \quad (\text{deterministic})$$

$$\parallel$$

integrates to 1

$$\frac{W(\omega) P(d\omega)}{\underbrace{\int_\Omega W(\omega') P(d\omega')}_{}} = Q^*(d\omega)$$

Under this choice Q^*,

variance $= 0$

If $W = I(A)$,

$$Q^*(d\omega) = \frac{I_A(\omega) P(d\omega)}{P(A)}$$

$$= P(d\omega \mid A)$$

Zero-variance change-of-measure for computing $P(A)$ is to use the conditional distribution.

Control Variates

Settings:
- Compute $\alpha = EX$
- EY is known
- $C = Y - EY$; $EC = 0$

$$X(\lambda) = X - \lambda C$$

$$EX(\lambda) = \alpha$$

$$n^{\frac{1}{2}} (\overline{X_n}(\lambda) - EX) \Rightarrow \sigma(\lambda) N(0,1)$$

$$\sigma^2(\lambda) = \text{Var } X(\lambda)$$

minimize

$$= \text{Var } X - 2\lambda \, \text{Cov}(X, \lambda) + \lambda^2 \, \text{Var } C$$

$$\lambda^* = \frac{Cov(X, \lambda)}{Var\, C}$$

$$\sigma^2(\lambda^*) = Var\, X \left(1 - \underbrace{\frac{Cov(X,C)^2}{Var\, X\; Var\, C}}_{\rho^2_{X,C} \leq 1} \right)$$

$$L^2 = \{ X : EX^2 < \infty \}$$

$$\langle X, Y \rangle = EXY$$

$$\langle X, Y \rangle^2 \leq \|X\|^2 \cdot \|Y\|^2$$

$$(EXY)^2 \leq EX^2 \cdot EY^2$$

$$\tilde{X} = X - EX \quad ; \quad \tilde{Y} = Y - EY$$

$$\underbrace{(E\tilde{X}\tilde{Y})^2}_{Cov(X,Y)^2} \leq \underbrace{E\tilde{X}^2}_{Var X} \cdot \underbrace{E\tilde{Y}^2}_{Var Y}$$

$$\Rightarrow \quad \rho \uparrow \qquad \sigma(\lambda) \downarrow$$

$$X(\lambda) = X - \lambda^* C$$

$$\lambda_n = \frac{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)(C_i - \bar{C}_n)}{\frac{1}{n-1} \sum_{i=1}^{n} (C_i - \bar{C})^2} \xrightarrow{a.s.} \lambda^*$$

Is the comparison we are doing an accurate characterization of efficiency?

- Var X
- Also need to consider the expected time to "draw" X

Goal: Compute $\alpha = EX$

$X_1, X_2, \cdots, X_n$

$\tau_1, \tau_2, \cdots, \tau_n$

τ_i = computer time required to generate X_i

(X_i, τ_i) i.i.d. pairs

$$N(c) = \max \{ n \geq 0 : \tau_1 + \tau_2 + \cdots + \tau_n \leq c \}$$

computational budget

$$\alpha(c) = \bar{X}_{N(c)} = \frac{S_{N(c)}}{N(c)}$$

$$S_n = X_1 + X_2 + \cdots + X_n$$

$$E\tau_1 < \infty \ ; \quad \bar{\tau}_n \xrightarrow{a.s.} E\tau_1$$

$$\frac{\tau_1 + \cdots + \tau_{N(c)}}{N(c)} \leq \frac{c}{N(c)} \leq \frac{\tau_1 + \cdots + \tau_{N(c)+1}}{N(c)+1} \left(\frac{N(c)+1}{N(c)} \right)$$

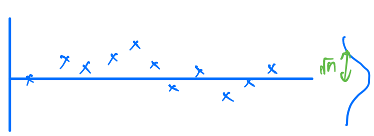
If $Z_n \to Z_\infty$

do I have $Z_{N(c)} \to Z_\infty$?

$$\frac{S_n - nEX_1}{\sqrt{n}} \Rightarrow \sigma N(0,1)$$

$$\frac{S_{T_n} - T_n EX_1}{\sqrt{n}} \overset{?}{\Rightarrow} \sigma N(0,1) \qquad \text{NO.}$$

$$X_i = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$$



by CLT

$S_n = 0$ infinitely often

T_n = n'th time random walk returns to 0

LHS $= 0$ not $\to \sigma N(0,1)$ CLT does NOT hold !

A: $N(c) \to \infty$ a.s. as $c \to \infty$

B: $\bar{\tau}_n \to E\tau_1$ a.s. as $n \to \infty$

$$P(A) = 1 = P(B)$$

For $\omega \in A \cap B$

$$N(c, \omega) \to \infty$$

$$\bar{\tau}_n(\omega) \to E\omega_1$$

We must also have

$$\bar{\tau}_{N(c,\omega)}(\omega) \to E\tau_1$$

C: $\bar{\tau}_{N(c)} \to E\tau_1$ $\qquad C \supseteq A \cap B$

$P(C) = 1$

$$\frac{\tau_1 + \cdots + \tau_{N(c)}}{N(c)} \le \frac{c}{N(c)} \le \frac{\tau_1 + \cdots + \tau_{N(c)+1}}{N(c)+1} \left(\frac{N(c)+1}{N(c)}\right)$$

$$\underset{E\tau_1}{\downarrow} \qquad\qquad \underset{E\tau_1}{\downarrow} \qquad\qquad \underset{1}{\downarrow}$$

Result: $\frac{N(c)}{c} \to \frac{1}{E\tau_1}$ a.s.

CLT ?

show on Friday

$$N(c) \approx \lambda c \qquad \left(\lambda \overset{\circ}{=} \frac{1}{E\tau_1}\right)$$

$$x(c) \approx \bar{X}_{\lfloor \lambda c \rfloor}$$

$$\lfloor \lambda c \rfloor^{\frac{1}{2}} \left(\bar{X}_{\lfloor \lambda c \rfloor} - EX\right) \Rightarrow \sigma N(0,1)$$

$$c^{\frac{1}{2}} \left(\bar{X}_{\lfloor \lambda c \rfloor} - EX\right) \Rightarrow \lambda^{-\frac{1}{2}} \sigma N(0,1)$$

$$\bar{X}_{N(c)} \overset{?}{\approx} EX + \frac{\lambda^{-\frac{1}{2}}\sigma}{\sqrt{c}} N(0,1)$$

$$\lambda^{-1} \sigma^2 = E\tau_1 \cdot \operatorname{var} X$$