Convergence in Distribution

$P(X_n \le x) \to P(X_\infty \le x)$

$X_n \in C[0,\infty)$

The following are equivalent :

$(X_n \in \mathbb{R})$

(1) $X_n \Rightarrow X_\infty$ as $n \to \infty$

(2) $Ef(X_n) \to Ef(X_\infty)$ for each bounded continuous f

i.e. $f \in bC$ $\leftarrow$ bounded continuous

(3) $\exists$ a prob space supporting a sequence $(X_n' : 1 \le n \le \infty)$ s.t.

i) $X_n' \overset{D}{=} X_n$, $1 \le n \le \infty$

ii) $X_n' \to X_\infty'$ a.s. as $n \to \infty$

- $X_n \in S$ metric space

Def: We say that $(X_n : n \ge 1)$ converges weakly to X_∞ (i.e. $X_n \Rightarrow X_\infty$)

iff

$$Ef(X_n) \to Ef(X_\infty)$$

for all $f \in bC$

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bC: $f: S \to \mathbb{R}$

whenever

$$d(X_n, X_\infty) \to 0$$

then

$$f(X_n) \to f(X_\infty)$$

f continuous

e.g. $S = C[0,1]$

$$d(x,y) = \max_{0 \le t \le 1} |x(t) - y(t)|$$

$$d(x_n, y_n) \to 0$$

iff

$$\max_{0 \le t \le 1} |X_n(t) - X_\infty(t)| \to 0$$

e.g. $f(x) = \int_0^1 x(t)\,dt$

$$|f(X_n) - f(X_\infty)| \le \int_0^1 |X_n(t) - X_\infty(t)|\,dt$$
$$\le \int_0^1 d(X_n, X_\infty)\,dt$$
$$= d(X_n, X_\infty) \to 0 \quad \text{continuous} \checkmark$$

e.g. $f(x) = \max_{0 \le t \le 1} x(t)$

$$|f(X_n) - f(X_\infty)| \le d(X_n, X_t) \to 0 \quad \text{continuous} \checkmark$$

Theorem: S complete separable metric space

$$X_n \Rightarrow X_\infty$$

iff

$\exists$ a prob space supporting

$(X_n' : 1 \le n \le \infty)$ s.t.

a) $X_n \overset{D}{=} X_n'$, $1 \le n \le \infty$

b) $X_n' \to X_\infty'$ a.s.

$[\, d(X_n', X_\infty') \to 0 \text{ a.s.} \,]$

known as "Skorohod representation theorem"

$$\left[\begin{array}{c} X_n' \xrightarrow{a.s.} X_\infty' \\ f(X_n') \xrightarrow{a.s.} f(X_\infty') \end{array} \right]$$

"path-by-path" argument

<u>Result</u>: $X_n \Rightarrow X_\infty$, $X_n \in S$ "continuous mapping principle"

$h: S \to S'$ continuous

Then,

$h(X_n) \Rightarrow h(X_\infty)$, $h(X_n) \in S'$

<u>Proof</u>: WTS $Ef(h(X_n)) \xrightarrow{?} Ef(h(X_\infty))$ $\forall f \in bC$

know: $Eg(X_n) \to Eg(X_\infty)$ $\forall g \in bC$ (definition $X_n \Rightarrow X_\infty$)

$g = f \circ h$

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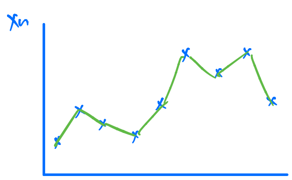
e.g.



$h(X_n) = \max_{0 \le t \le 1} X_n(t)$

$X_n \Rightarrow X_\infty$

$h(X_n) \Rightarrow h(X_\infty)$

$\max_{0 \le t \le 1} X_n(t) \Rightarrow \underbrace{\max_{0 \le t \le 1} X_\infty(t)}_{\text{easier to compute}}$

<u>Extended Result</u>:

$X_n \Rightarrow X_\infty$

$h: S \to S'$

$P(X_\infty \in D_h) = 0$ ← set of discontinuity of h

Then,

$h(X_n) \Rightarrow h(X_\infty)$

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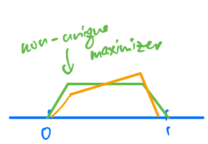
e.g. $h(x) = \operatorname{argmax}_{0 \le t \le 1} x(t)$



Generalization of CLT

- $X_1, X_2, \cdots, X_n$ i.i.d $\mathbb{R}^d$ - valued
- $E\|X_1\|^2 < \infty$
- $S_n = X_1 + \cdots + X_n$ column vector

$$\frac{S_n - nEX_1}{\sqrt{n}} \Rightarrow N(0, C)$$

$$C = EX_1X_1^T - EX_1 EX_1^T$$

- $X_1, X_2, \cdots, X_n$ $\mathbb{R}^d$ - valued (not i.i.d.)
- $E\|X_1\|^2 < \infty$
- $S_n = X_1 + \cdots + X_n$

cov of LHS

$$\Rightarrow \quad S_n - ES_n \overset{D}{\approx} N(0, ES_nS_n^T - ES_nES_n^T)$$

Caveats:

- No single X_i should "dominate" the sum
- Finite variance
- Too much dependence destroys the normal approx.

$d = 1$, $X_1, X_2, \cdots$ scalar

$$\frac{S_n - ES_n}{\sqrt{\text{Var} S_n}} \Rightarrow N(0,1)$$

- $X_{i+1} = \rho X_i + Z_{i+1}$, $|\rho| < 1$
  $\text{Var} Z_i < \infty$
  $Z_1, Z_2, \cdots$ i.i.d.
- $X_0, X_1, \cdots$ correlated
- $S_n = X_0 + \cdots + X_n$
- $ES_n = ?$, $\text{Var} S_n = ?$

$$X_i = \rho X_{i-1} + z_i$$
$$= \rho^2 X_{i-2} + \rho z_{i-1} + z_i$$
$$= \cdots$$
$$= \rho^i X_0 + \sum_{j=0}^{i-1} \rho^j z_{i-j}$$

$$S_n = \sum_{j=0}^{n} X_j$$
$$= \sum_{j=0}^{n} \left[\rho^j X_0 + \sum_{k=0}^{j-1} \rho^k z_{j-k} \right]$$

S_n is a linear combo of $z_1, z_2, \cdots, z_n$ (i.i.d.)

$$\Rightarrow \quad S_n \overset{D}{\approx} N(ES_n, \operatorname{var} S_n)$$

How good an approximation is the Normal approximation?

- $X_1, X_2, \cdots$ i.i.d., $\operatorname{var} X_1 < \infty$
- $\dfrac{S_n - nEX_1}{\sqrt{n}\,\sigma} \Rightarrow N(0,1)$

$$P\left(\frac{S_n - nEX_1}{\sqrt{n}\,\sigma} \le z \right) - P(N(0,1) \le z)$$

$$\to \frac{K_3}{6\sqrt{n}} (x^2 - 1)\phi(x) + O\left(\frac{1}{n}\right) \qquad \text{"edgeworth expansion"}$$

$$K_3 = \frac{E[(X_1 - EX_1)^3]}{\sigma^3} \qquad \text{"skewness"}$$

$\phi(x)$: pdf of $N(0,1)$