Slutsky's Lemma

- $W_n \Rightarrow W_\infty$
- $Y_n \xrightarrow{P} c$ (c fixed)

Then,

$$W_n + Y_n \Rightarrow W_\infty + c$$

$$W_n Y_n \Rightarrow c W_n$$

Delta Method

$X_1, X_2, \cdots$ i.i.d. real value r.v.

$$n^{\frac{1}{2}}\left(g(\bar{X}_n) - g(EX_1)\right) \Rightarrow g'(EX_1)\,\sigma N(0,1)$$

Provided g is continuously differentiable at EX_1 and $\sigma^2 = \mathrm{Var}\,X_1 < \infty$

Proof:

$$n^{\frac{1}{2}}\left(g(\bar{X}_n) - g(EX_1)\right)$$

b/w $\bar{X}_n$ & EX_1

$$= n^{\frac{1}{2}}\, g'(\xi_n)(\bar{X}_n - EX_1)$$

$$= \underbrace{g'(EX_1)\, n^{\frac{1}{2}} (\bar{X}_n - EX_1)}_{} \xRightarrow{CLT} g'(EX_1)\,\sigma N(0,1)$$

$$+ \underbrace{(g'(\xi_n) - g'(EX_1))}_{\Downarrow P \atop 0} \underbrace{n^{\frac{1}{2}}(\bar{X}_n - EX_1)}_{\Downarrow \atop W_\infty}$$

Slutsky's Lemma

$$\Rightarrow g'(EX_1)\,\sigma N(0,1)$$

Now let's look at extremes of r.v.'s

- $X_1, X_2, \cdots$ i.i.d. real-valued r.v.
- $M_n = \max_{1 \le i \le n} X_i$

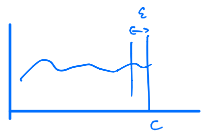
$$P(M_n > x) = 1 - P(M_n \le x)$$

$$= 1 - P(X_1 \le x, \dots, X_n \le x)$$

$$= 1 - P(X_1 \le x)^n$$

$$= 1 - (1 - \underbrace{P(X_1 > x)}_{\text{want to get } z_n/n})^n$$

$$(1 - z/n)^n \to e^{-z} \quad \text{as} \quad n \to \infty$$

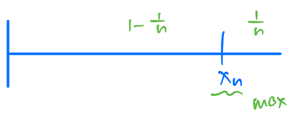


"bounded support"

$$M_n \xrightarrow{P} c$$

assume $P(X_1 > x) = (1+x)^{-\alpha}$, $\quad x > 0, \ \alpha > 0$

"heavy-tailed r.v."



$$P(X_1 > x_n) = (1 + x_n)^{-\alpha} = \frac{z}{n}$$ ← tail quantity

$$1 + x_n = \left(\frac{z}{n}\right)^{-\frac{1}{\alpha}} = z^{-\frac{1}{\alpha}} n^{\frac{1}{\alpha}}$$

$$\tilde{x}_n = z^{-\frac{1}{\alpha}} n^{\frac{1}{\alpha}}$$?

$$P(X_1 > \tilde{x}_n) = \frac{z}{n} + o\left(\frac{1}{n}\right)$$

$$P(M_n > \tilde{x}_n) = 1 - \left(1 - \frac{z}{n} + o\left(\frac{1}{n}\right)\right)^n \to 1 - e^{-z}$$

$$P(M_n > n^{\frac{1}{\alpha}} \underbrace{z^{-\frac{1}{\alpha}}}) \to 1 - e^{-z}$$

$$y = z^{-\frac{1}{\alpha}}$$

$$P\left(\frac{M_n}{n^{1/\alpha}} > y\right) \to 1 - \exp(-y^{-\frac{1}{\alpha}})$$

$$\frac{M_n}{n^{1/\alpha}} \Rightarrow W_\infty$$

$$P(W_n > y) = 1 - \exp(-y^{-\frac{1}{\alpha}})$$

"Theory of extreme values"

$X_1, X_2, \ldots, X_n$

$M_n = \max_{1 \le i \le n} X_i$

$\frac{M_n - a_n}{b_n} \Rightarrow W_\infty \leftarrow$ only 3 possible types

"maximal domain of attraction"

- $\mathbb{R}^d$ - valued
- multivariate CLT ?
- $\Omega = C[0,\infty)$

What does it mean by $X_n \Rightarrow X_\infty$?

Metric Spaces

S

$d: S \times S \to \mathbb{R}_+$ "metric"

- $d(x,y) = d(y,x)$
- $d(x,x) = 0$
- $d(x,y) \le d(x,z) + d(z,y)$

$\mathbb{R}$ $\quad d(x,y) = |x-y|$

$\mathbb{R}^d$ $\quad d(x,y) = \|x-y\|$

$C[0,1]$ $\quad d(x,y) = \sup_{0 \le t \le 1} |x(t) - y(t)|$

$C[0,\infty)$ $\quad$ "

$P(\mathbb{R}^d)$

"Almost sure" convergence

"Convergence with prob 1"

"Almost everywhere" convergence

same

- $X_n \in \mathbb{R}$

$X_n \to X_\infty$ a.s. iff $P(A) = 1$,

$A = \{ w : X_n(w) \to X_\infty(w) \text{ as } n \to \infty \}$

- X_n S-valued

$d(X_n, X_\infty) \to 0$ a.s.

iff

$X_n \to X_\infty$ a.s.

Converge in Probability

$X_n \xrightarrow{P} X_\infty$ (X_n S-valued)

iff

$d(X_n, X_\infty) \xrightarrow{P} 0$

Convergence in Distribution

$P(X_n \le x) \longrightarrow P(X_\infty \le x)$

$X_n \in C[0, \infty)$

The following are equivalent :

($X_n \in \mathbb{R}$)

(1) $X_n \Rightarrow X_\infty$ as $n \to \infty$

(2) $E f(X_n) \to E f(X_\infty)$ for each bounded continuous f

i.e. $f \in bC$ ← bounded continuous

(3) $\exists$ a prob space supporting a sequence (X_n' : $1 \le n \le \infty$) s.t.

i) $X_n' \overset{D}{=} X_n$, $1 \le n \le \infty$

ii) $X_n' \to X_\infty'$ a.s. as $n \to \infty$

Why does ① imply ③ ?

- X c.d.f F

  $X \overset{D}{=} F^{-1}(U)$

- why :

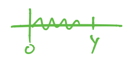
  $P(F^{-1}(U) \le x)$ since F is mono $\nearrow$

  $= P(F(F^{-1}(U)) \le F(x))$

  $= P(U \le F(x))$

  $\quad\quad\quad\;\; \overset{\shortparallel}{y}$

  $= F(x)$



$$\left(\begin{array}{l} F^{-1}(x) = \sup\{y : F(y) \le x\} \\ \quad F^{-1}(U) \overset{D}{=} X \end{array} \right.$$

no $\nearrow$ exception

$X_i \,,\; F_i$

$F_i^{-1}(U) \overset{D}{=} X_i$

$x_i' \overset{\Delta}{=} F_i^{-1}(U)$

$F_n(x) \to F_\infty(x)$

implies

$F_n^{-1}(x) \to F_\infty^{-1}(x)$

$F_n^{-1}(U) \to F_\infty^{-1}(U)$ a.s. (for all choices of u)

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③ implies ② ?

$X_n' \xrightarrow{a.s.} X_\infty'$

$f(X_n') \xrightarrow{a.s.} f(X_\infty')$

$A = \{w : f(X_n'(w)) \to f(X_\infty'(w)) \text{ as } n \to \infty\}$

$B = \{w : X_n'(w) \to X_\infty'(w) \text{ as } n \to \infty\}$,

$P(B) = 1$

$a_n \to a_\infty$

$f(a_n) \to f(a_\infty)$

$A \supseteq B$

$\Rightarrow \quad P(A) = 1$

Bounded Convergence Thm:

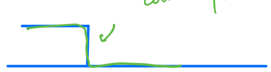
$$\begin{array}{ccc} E f(X_n') & \to & E f(X_\infty') \\ \| D & & \| D \\ X_n & & X_\infty \end{array}$$

$$E f(X_n) \to E f(X_\infty)$$

---

3 implies 1 ?

cont. fcn



x

$$P(X_n \le x) \to P(X_\infty \le x)$$