- $P(S_n > na) \quad (a > EX_1)$
- $S_n = X_1 + \cdots + X_n$ i.i.d.
- X_i's are light-tailed

$$E\exp(\theta X_1) < \infty$$

$$P(S_n > na) = E_{\theta(a)} I(S_n > na)\exp(-\theta(a) S_n + n\varphi(\theta))$$

Under $P_{\theta(a)}$ $X_1, \cdots, X_n$ i.i.d with

$$E_{\theta(a)} X_1 = a$$

Recall upper bound

$$P(S_n > na) \leq \exp(-nI(a))$$

$$I(a) = a\theta(a) - \varphi(\theta(a))$$

$$\overline{\lim_{n\to\infty}} \frac{1}{n}\log P(S_n > na) \leq -I(a)$$

want:

$$\frac{1}{n}\log P(S_n > na) \to -I(a)$$

need:

$$\underline{\lim_{n\to\infty}} \frac{1}{n}\log P(S_n > na) \geq -I(a)$$

Recall that

$$P(S_n > na) = E_{\theta(a)} I(S_n > na)\exp(-\theta(a) S_n + n\varphi(\theta))$$

$$P(S_n > na) = \exp(-nI(a)) \cdot E_{\theta(a)} I(S_n > na)\exp(-\theta(a)\underbrace{(S_n - na)}_{>0})$$

need upper bound

We know:

$$E_\theta X_1 = \varphi'(\theta)$$

Claim:

$$\mathrm{Var}_\theta X_1 = \varphi''(\theta)$$

Why:

$$\varphi(\theta) = \log E e^{\theta X_1}$$

$$\varphi'(\theta) = \frac{\frac{d}{d\theta} E e^{\theta X_1}}{E e^{\theta X_1}} = \frac{E X_1 e^{\theta X_1}}{E e^{\theta X_1}}$$

$$\varphi''(\theta) = \frac{E X_1^2 e^{\theta X_1}}{E e^{\theta X_1}} - \left(\frac{E X_1 e^{\theta X_1}}{E e^{\theta X_1}} \right)^2$$

$$= E_\theta X_1^2 - (E_\theta X_1)^2 = \mathrm{Var}_\theta(X_1)$$

Under $P_{\theta(a)}$,

$$\frac{S_n - na}{\sqrt{n}} \Rightarrow \sigma(\theta(a)) N(0,1)$$

where $\sigma^2(\theta) = \varphi^{(2)}(\theta)$

say z is a function of n

$$P_{\theta(a)}(na \le S_n \le na + z\sigma(\theta(a))\sqrt{n})$$

$$\to P(0 \le N(0,1) \le z)$$

$$P_{\theta(a)}(na \le S_n \le na + n^{\frac{2}{3}}) \to \frac{1}{2}$$

Then,

$$E_{\theta(a)} I(S_n > na) \exp(-\theta(a)(S_n - na))$$

$$\ge E_{\theta(a)} I(na < S_n \le na + n^{\frac{2}{3}}) \exp(-\theta(a) n^{\frac{2}{3}})$$

$$= \exp(-\theta(a) n^{\frac{2}{3}}) P_{\theta(a)}(na < S_n \le na + n^{\frac{2}{3}})$$

$\log \cdot \frac{1}{n} \to 0$ $\qquad$ $\frac{1}{2}$

So,

$$\lim_{n\to\infty} \frac{1}{n} \log E_{\theta(a)} I(S_n > na) \exp(-\theta(a)(S_n - na)) \geq 0$$

$$P(S_n > na) = \exp(-n I(a)) \cdot E_{\theta(a)} I(S_n > na) \exp(-\theta(a)(S_n - na))$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \log P(S_n > na) \geq -I(a)$$

We proved

$$\frac{1}{n} \log P(S_n > na) \to -I(a)$$

$$\Rightarrow \quad P(S_n > na) \approx \exp(-n I(a))$$

Imagine

$$P(S_n > a) = \exp(-n I(a) + n^{\frac{3}{4}})$$

consistent w/ $\frac{1}{n} \log(-) \to -I(a)$

but the $P(S_n > na)$ approx is poor

what's correct is :

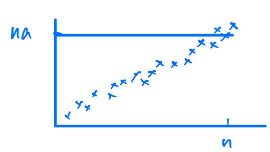
$$P(S_n > na) \sim \frac{1}{\sqrt{2\pi n \sigma^2(\theta(a))}} \exp(-n I(a)) \quad \text{as } n \to \infty$$

$$P(X_1 \in A_1, \cdots, X_k \in A_k \mid S_n > na \}$$

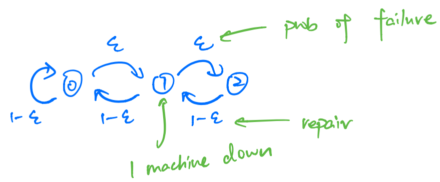
$$\to \prod_1^k P_{\theta(x)}(X_i \in A_i)$$

conditional on $\{ S_n > na \}$, $X_1, \cdots, X_k$

behave as i.i.d. observations from $P_{\theta(a)}(X_i \in A_i)$



" conspiratorial behavior " $\overset{\text{v.s.}}{\Longleftrightarrow}$ " principle of single big jump " e.g.



$0 \to 1 \to 2 \quad : \quad \varepsilon^2$

$0 \to 1 \to 0 \to 1 \to 2 \quad : \quad \varepsilon^3(1-\varepsilon) = O(\varepsilon^3) < \varepsilon^2$

When rare event occurs, it often follows a well-defined way.

Calculus-based Probability

v.s.

Measure-theoretic Probability

WLLN $\longrightarrow$ SLLN

$A = \{ w : \frac{1}{n}\sum_{i=1}^{n} X_i(w) \to EX_1 \text{ as } n\to\infty \}$

$P(A) = 1$ (SLLN)

$X_1, X_2, \cdots$ i.i.d.

sample space $\Omega = \mathbb{R}^\infty$

$w = (x_1, x_2, \cdots)$

$X_i(w) = x_i$

Need to define $P(A)$, $A \in \Omega$

$P(A) = \int_A \prod_{i=1}^{\infty} f(x_i)\, dx_1 dx_2 \cdots$

pdf f has support on $[0,1]$

change of measure

in $\mathbb{R}^d$: $\int_A \prod_{i=1}^{d} f(x_i)\, dx_1 \cdots dx_d = E\, I(A) \prod_{i=1}^{d} f(U_i)$

$$\Rightarrow \quad \int_A \prod_{i=1}^{\infty} f(x_i)\, dx_1 \cdots dx_d \cdots \quad = \quad E\, I(A) \prod_{i=1}^{\infty} f(U_i)$$

$$\prod_{i=1}^{\infty} f(U_i) = \lim_{n\to\infty} \prod_{i=1}^{n} f(U_i)$$

$$= \lim_{n\to\infty} \exp\left(\sum_{i=1}^{n} \log f(U_i) \right)$$

$$\frac{1}{n} \sum_{i=1}^{n} \log f(U_i) \longrightarrow E \log f(U_1)$$

$$< \log E f(U_1) = 0$$

$$E f(U_1) = \int_0^1 f(x)\, dx = 1$$

$$\Rightarrow \quad \sum_{i=1}^{n} \log f(U_i) \rightarrow -\infty \qquad \text{as} \quad n \to \infty$$

$$\prod_{i=1}^{\infty} f(U_i) \rightarrow 0$$

This is clearly wrong. Calculus-based probability doesn't work!