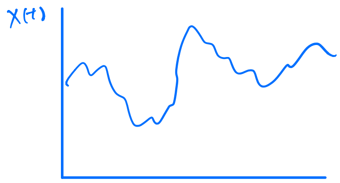
e.g. track position of a particle



$\Omega = C[0, \infty)$

infinite-dimensional sample space

$P(A)$ $\quad A \subseteq C[0, \infty)$

$\Rightarrow$ NOT calculus-based probability

Random field

$X(x, y)$

$X(x, y, z)$

$X(x, y, z, t)$

Problem of Points

2 gamblers, $1 pot

1st person wins 5 games get $1

$(4, 3) \Rightarrow$ interrupted $\Rightarrow$ how to split ?

$\begin{cases} \text{expectation} \\ \text{law of large numbers} \end{cases}$

Repeat the game n times

$\Rightarrow \sum_{i=1}^{n} \mathbb{I}(X_i = 1) \geq n\alpha$ A wins A pays α proportion of the pot

$\sum_{i=1}^{n} \mathbb{I}(X_i = 2) \geq n(1-\alpha)$

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(X_i = 1) \xrightarrow{LLN} E\,\mathbb{I}(X_i = 1) = P(X_i = 1) = p$$

$$\Rightarrow p = P(A \text{ wins} \mid (4,3))$$

$$= \frac{1}{2} P(5,3) + \frac{1}{2} P(4,4)$$

$$= \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{3}{4}$$

Law of Large Numbers (LLN)

$X_1, X_2, \cdots$ i.i.d. $E|X_1| < \infty$

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to EX_1$$

Proof WLLN when $\text{var}\, X_1 < \infty$:

$X_i : \Omega \to \mathbb{R}$

$X_i(\omega)$ r.v.'s are functions

different ways of convergence for deterministic functions

$$\left\{ \begin{array}{ll} f_n(x) \to f_\infty(x) & \text{"pointwise"} \\ \sup_{x\in\mathbb{R}} |f_n(x) - f_\infty(x)| \to 0 & \text{"uniform"} \\ \int_{-\infty}^{\infty} |f_n(x) - f_\infty(x)|^p \, dx \to 0 & \text{"}L^p\text{"} \end{array} \right.$$

Convergence in Probability :

$$X_n \xrightarrow{P} X_\infty$$

iff

$\forall\ \varepsilon > 0$,

$$P(|X_n - X_\infty| > \varepsilon) \to 0$$

as $n \to \infty$

WLLN:

want to prove $P(|\overline{X}_n - EX_1| > \varepsilon) \to 0$

To do this, let's first derive some inequalities.

Markov Inequality

- $W \geq 0$; $EW < \infty$
- $P(W > w) \leq \frac{EW}{w}$

Proof :

$P(W > w) = E\, I(W > w)$

On $\{W > w\}$, $\frac{W}{w} \geq 1$

$$\Rightarrow E\, I(W > w) \leq E\, \frac{W}{w} I(W \geq w)$$

$$\leq \frac{EW}{w}$$

Application 1. Chebyshev's inequality

- X ; $\mathrm{var}\, X < \infty$
- $P(|X - EX| > x) \leq \frac{\mathrm{Var}\, X}{x^2}$

Proof:

$$P(|X - EX| > x) = P(\underbrace{|X - EX|^2}_{W \text{ , Markov's inequality}} > x^2)$$

Application 2. Exponential inequality

"light-tail r.v."

- X ; $E\exp(\theta x) < \infty$; $\theta > 0$
- $P(X > x) \leq \frac{E\exp(\theta X)}{e^{\theta x}} = e^{-\theta x} E\exp(\theta X)$

exponential decay in right tail

Proof:

$$P(X > x) = P(\underbrace{e^{\theta X}}_{W \text{ , Markov's inequality}} > e^{\theta x})$$

$X_1, X_2, \cdots$ i.i.d., $\operatorname{var} X_1 < \infty$

$$P(|\bar{X}_n - EX| > \varepsilon) \le \frac{1}{\varepsilon^2} \frac{\operatorname{Var} X_1}{n} \to 0$$

Chebyshev's

$\Rightarrow$ WLLN: $\bar{X}_n \xrightarrow{P} EX$

$X_1, X_2, \cdots$ stationary, $\operatorname{var} X_1 < \infty$

$$(X_1, X_2, \cdots) \overset{D}{=} (X_i, X_{i+2}, \cdots)$$

$$X_1 \overset{D}{=} X_i \qquad \text{time-invariant}$$

$$(X_1, X_{1+j}) \overset{D}{=} (X_i, X_{i+j})$$

e.g. ① $X_1, X_2, \cdots$ i.i.d

② $X_1 = X_i$ $\qquad \frac{1}{n}\sum_{i=1}^n X_i = X_1 \not\to EX_1$ (perfect corr)

$\Rightarrow \frac{1}{n}\sum_{i=1}^n X_i \xrightarrow{P} EX_1$ holds

if $\operatorname{cov}(X_1, X_n) \to 0$ as $n \to \infty$

Proof: $\operatorname{Var} \bar{X}_n \to 0$?

$$\frac{1}{n^2}\operatorname{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2}\sum_1^n\sum_1^n \operatorname{Cov}(X_i, X_j)$$

$c(i-j)$?

$$= \frac{1}{n^2}\Big[\underbrace{n \operatorname{Var} X_i}_{\to 0} + \underbrace{2\sum_1^n (n-i)\, c(i)}_{?}\Big]$$

covariance lag i

$\operatorname{Cov}(X_1, X_{1+i})$

$$\Rightarrow \frac{1}{n^2}\sum_{i=1}^n |n-i|\,|c(i)|$$

$$= \frac{1}{n}\sum_{i=1}^n \underbrace{\frac{|n-i|}{n}}_{\le 1} |c(i)|$$

$$\le \frac{1}{n}\sum_1^n |c(i)| \to 0 \quad \text{by assumption}$$