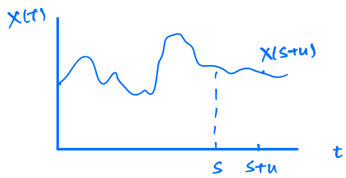
Conditional Expectation



$\Omega = C[0,\infty)$ — infinite collection of rv's

$P(X(s+u) \in A \mid X(r) : 0 \le r \le s)$

Calculus - based probability :

$$E[Y \mid \vec{Z} = \vec{z}] = \int_{-\infty}^{\infty} y \, f_{Y|\vec{Z}}(y \mid \vec{z}) \, dy$$

$$= \int_{-\infty}^{\infty} \frac{f_{Y,\vec{Z}}(y, \vec{z})}{f_{\vec{Z}}(\vec{z})} \, dy$$

joint involved in infinite r.v. doesn't work

e.g. $W_1, W_2, \cdots$ i.i.d. $f_W(\cdot)$, $w \in [0,1]$

claim: joint $\prod^{\infty} f_W(w_i) = 0$

$$\frac{1}{n} \log\left(\prod_{1}^{n} f_W(u_i)\right) = \frac{1}{n} \sum_{1}^{n} \log f_W(u_i)$$

Jensen's inequality — w supported on [0,1] so it integrates to 1 on [0,1]

$$\xrightarrow{a.s.} E \log f_W(u_i) \le \log E f_W(u_i) = \log \int_0^1 f_W(u) \, du = 0$$

$<$

$$\log\left(\prod_{i=1}^{n} f(u_i)\right) \approx n E \log f(u) \to -\infty \quad \text{a.s. as } n \to \infty$$

$$\prod_{i=1}^{n} f(u_i) \to 0 \quad \text{a.s.}$$

Prediction Theory

Prob space supporting $Y, \vec{Z}$

Given : observation of $\vec{Z}$

predict Y

$Y = h(\vec{Z})$

deterministic

$\hat{Y}$ "close to" Y : minimize $\|\hat{Y} - Y\|_p$

W , $E|W|^p < \infty$, $p \geq 1$

$\|W\|_p = (E|W|^p)^{\frac{1}{p}}$

"norm" on $L^p = \{ Z : E|Z|^p < \infty \}$

- $\|cW\|_p$
- $\|W_1 + W_2\|_p \leq \|W_1\|_p + \|W_2\|_p$

$\rightarrow$ distance btw W_1 and W_2

Find a r.v. $\hat{Y} = g^*(\vec{Z})$ which minimizes

$$\| Y - g(\vec{Z}) \|_p$$

over all deterministic $g(\cdot)$ s.t. $E\|\vec{Z}\|^p < \infty$ (i.e. $g(\vec{Z}) \in L^p$)

e.g. $p = 2$

L^2 , $\|W\|_2 = (EW^2)^{\frac{1}{2}}$

Find $\hat{Y} = g^*(Z)$ minimizing

$$(E(Y - g(Z))^2)^{\frac{1}{2}} = \|Y - g(Z)\|_2$$

over all deterministic $g(\cdot)$ s.t. $E\|Z\|^2 < \infty$ (i.e. $g(\vec{Z}) \in L^2$)

<u>Intro Geometry</u>

$W_1, W_2 \in L^2$

$\langle W_1, W_2 \rangle \triangleq E W_1 W_2$

$\langle W, W \rangle = \|W\|_2^2$

$\mathcal{H} : \{ g(Z) \in L^2 : g \text{ is deterministic} \}$

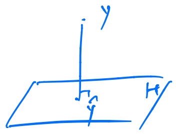
- linear subspace of L^2

$W_i : g_i(Z) \rightarrow W_\infty \in L^2$

$(\ \|W_i - W_\infty\|_2 \rightarrow 0$ as $i \rightarrow \infty\ )$

$W_i \rightarrow W_\infty$ in $\|\cdot\|_2$

$\Rightarrow$ "closed linear subspace of L^2



"Hilbert Space"
Projection Theorem

$\langle y, \hat{y}, w \rangle = 0$

Define:

$E[y \mid \vec{z}] = \hat{y}$

$E\, y^2 < \infty$

↳ extend to $E|y| < \infty$

- $E[y_1 + y_2 \mid z] = E[y_1 \mid z] + E[y_2 \mid z]$
- $y \geq 0$

  $E[y \mid z] \geq 0$
- $E[y\, g(z) \mid z] = g(z)\, E[y \mid z]$
- $E[y \mid v] = E[E[y \mid z] \mid v]$ "Tower Property"