Calculus - based Probability

v.s.

Measure - theoretic Probability

↳ allows us to assign probs to infinite - dimensional sample spaces.

- SLLN

  $X_1, X_2, \cdots$ i.i.d.

  $A = \{ w : \frac{1}{n} \sum_{i=1}^{n} X_i(w) \to EX_1 \text{ as } n \to \infty \}$ ← infinite - dim events

  $\Omega = \mathbb{R}^\infty$

  $w = (x_1, x_2, \cdots)$

- Ω ⊆ 2^Ω ← all subsets
- $P : F \to [0,1]$ measurable events
  - $P(\Omega) = 1$
  - $0 \le P(A) \le 1$
  - If A_i's are disjoint events, then

    $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ "countable additivity"

- F : σ - algebra

  1) $\phi, \Omega \in F$

  2) closed under complements : $A^c := \Omega \setminus A \in F$

  3) closed under countable unions : $A_i \in F, i \in N \Rightarrow \bigcup_{i=1}^{\infty} A_i \in F$

SLLN

$X_1, X_2, \cdots$ i.i.d.

- $\Omega = \mathbb{R}^\infty$
- Finite dimensional :

  $P((x_1, x_2, \cdots, x_n) \in A) = \int_A f(x_1) f(x_2) \cdots f(x_n) \, dx_1 \cdots dx_n$

- Kolmogorov Extension Theorem

  $\mathbb{R}^d$ extends uniquely to $\Omega = \mathbb{R}^\infty$

- $A \subseteq F$

SLLN: $P(A) = 1$ if $E|X_1| < \infty$

<u>Measurable Theoretic Probability</u>

BCT

DCT

MCT ("Monotone Convergence Theorem")

- $X_1, X_2, \cdots \geq 0$
- $X_n(w) \nearrow$ in n $\forall w$
- $EX_n \nearrow EX_\infty$ (finite or infinite)

Fatou's Lemma

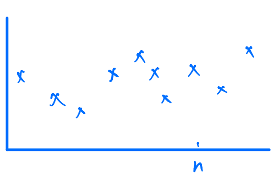
- $X_1, X_2, \cdots \geq 0$
- $E \underline{\lim}_{n\to\infty} X_n \leq \underline{\lim}_{n\to\infty} EX_n$

Fubini's Theorem

- $X_1, X_2, \cdots \geq 0$
- $E \sum_{i=1}^{\infty} X_i = \sum_{i=1}^{\infty} EX_i$

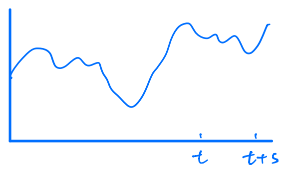
  equiv. to

- $(X(t) : t > 0)$ non-neg
- $E \int_0^1 X(t)\, dt = \int_0^1 EX(t)\, dt$ (can be $\infty = \infty$)



$P(X_{n+1} = y \mid X_0 = x_0, \cdots, X_n = x_n)$

$= P(X_{n+1} = y \mid X_n = x_n)$ "Markov Property"



$$P(X(t+s) = y \mid X(u) : 0 \le u \le t)$$

infinite number of variables
(conditioning on this doesn't make sense w/ calculus-based prob)

$$= P(X(t+s) = y \mid X(t))$$

"Markov Property" in cont. time

- Kolmogorov 1930

$$E[Y \mid \vec{X}]$$

infinite dim

- Von Neumann

L^2 prediction theory

SLLN

$$A = \{\omega : \frac{1}{n}\sum_1^n X_i(\omega) \to EX_1 \text{ as } n\to\infty\}$$

only finite many n's for which

$$|\bar{X}_n - EX_1| > \varepsilon$$

$$A_\varepsilon = \{\omega : |\bar{X}_n(\omega) - EX_1| > \varepsilon \text{ for only finite many } n\}$$

$$P(A_\varepsilon) \overset{?}{=} 1$$

$$P(\bigcap_{\varepsilon>0} A_\varepsilon) \overset{?}{=} 1$$

- $\operatorname{Var} X_1 < \infty$

$$\omega \in A_\varepsilon \text{ iff } \sum_{n=1}^\infty I(|\bar{X}_n(\omega) - EX_1| > \varepsilon) < \infty$$

$$\left.\begin{matrix} Z \ge 0 \\ EZ < \infty \end{matrix}\right] \text{ implies } Z < \infty \text{ a.s.}$$

$$\sum_{i=1}^\infty P(|\bar{X}_n - EX_1| > \varepsilon) \le \sum_1^\infty \frac{\operatorname{Var} X_i}{n\varepsilon^2} \quad \text{(Chebyshev)}$$

$$\sum \frac{1}{n} \to \infty \quad \text{(doesn't work)}$$

- $EX_1^4 < \infty$

$$\sum_{i=1}^{\infty} P(|\bar{X}_n - EX_1|^4 > \varepsilon^4) \le \frac{E|\bar{X}_n - EX_1|^4}{\varepsilon^4} \quad \text{(Markov)}$$

CLT: $n^{\frac{1}{2}}(\bar{X}_n - EX_1) \Rightarrow \sigma N(0,1)$

$$n^2(\bar{X}_n - EX_1)^4 \Rightarrow \sigma^4 N(0,1)^4$$

$$n^2 E(\bar{X}_n - EX_1)^4 \Rightarrow \sigma^4 \underbrace{EN(0,1)^4}_{=3}$$

$$E(\bar{X}_n - EX_1)^4 \Rightarrow O(\tfrac{1}{n^2})$$

$$\Rightarrow \sum_{i=1}^{\infty} P(|\bar{X}_n - EX_1| > \varepsilon) = \sum_{i=1}^{\infty} O(\tfrac{1}{n^2}) < \infty$$

Now, we know

$$P(A_\varepsilon) = 1$$

for each $\varepsilon > 0$

What about $\bigcap_{\varepsilon>0} A_\varepsilon$?

$\forall n \ge 1$

$|a_n - a| > \frac{1}{n}$ (ε) only finites

$$\Rightarrow \bigcap_{\varepsilon>0} A_\varepsilon = \bigcap_{n=1}^{\infty} A_{\frac{1}{n}} \in \mathcal{F}$$

$$P(A) = EI(A) = EI(\bigcap_1^\infty A_{\frac{1}{n}})$$

$$I(A \cap B) = I(A)\, I(B)$$

$$E\prod_{i=1}^{\infty} I(A_{\frac{1}{n}})$$

$$= E \lim_{l\to\infty} \prod_1^l I(A_{\frac{1}{n}})$$

$$\overset{BCT}{=} \lim_{l\to\infty} E \prod_1^l I(A_{\frac{1}{n}})$$

$$= \lim_{l\to\infty} EI(\bigcap_1^l A_{\frac{1}{n}})$$

$$= \lim_{l\to\infty} P(\bigcap_1^l A_{\frac{1}{n}})$$

$$= 1$$

<u>Result</u> : $X_1, X_2, \cdots$ i.i.d.

$EX_1^4 < \infty$

Then,

$$P(A) = 1$$

where

$$A = \{ w : \bar{X}_n(w) \to EX_1 \text{ as } n \to \infty \}$$

usually written as

$$\bar{X}_n \to EX_1 \quad a.s.$$

actually holds if $E|X_1| < \infty$

<u>Ergodic Theorem</u>

- $X_1, X_2, \cdots$ stationary
- $E|X_1| < \infty$

Then,

$$\frac{1}{n} \sum_{i=1}^{\infty} X_i \to w \quad a.s.$$

where w can be random. Furthermore,

$$Ew = EX_1$$

If w is deterministic, then $w = EX_1$

When can we show w is deterministic ?

We know that

$$\bar{X}_n \xrightarrow{P} EX_1$$

[when the X_i's are stationary, $EX_1^2 < \infty$, and

$cov(X_i, X_n) \to 0$ as $n \to \infty$]

Since $w = EX_1$.