WLLN

- $X_1, X_2, \cdots$ stationary
- $E|X_1|^2 < \infty$
- $c(n) : \operatorname{cov}(X_1, X_{1+n}) \to 0$ as $n \to \infty$

$$\Rightarrow \quad \bar{X}_n \xrightarrow{P} EX_1$$

$c(n) \to 0$

"asymptotic independence"

"asymptotic loss of memory"

"mixing"

Newsvendor Model

eg. a newsvedor at airport

x = order quantity

$$W(x, \underbrace{D}_{\text{demand}}) = r \min(x, D) + s\underbrace{[x-D]^+}_{\max(x-D,\,0)} - cx$$

Choose $x \overset{\text{maximize}}{\Rightarrow} \begin{cases} EW(x, D) \\ \text{median of } W(x, D) \end{cases}$

LLN $\Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n} W(x, D_i) \xrightarrow{P} EW(x, D)$

should maximize $EW(x, D)$

assume continuous

$$E\,W(x, D) = r\int_0^x y f(y)\,dy + rx\,P(D > x)$$

$$+ \; s\int_0^x (x-y)\, f(y)\,dy - cx$$

$$\frac{d}{dx} = \cancel{rx f(x)} + r\,P(D>x) - \cancel{rx f(x)}$$

$$+ \frac{d}{dx}\left(sx\int_0^x f(y)\,dy - s\int_0^x y f(y)\,dy - cx \right)$$

$$\parallel$$

$$s\int_0^x f(y)\,dy + \cancel{sx f(x)} - \cancel{sx f(x)} - c$$

$$= r\,P(D > x) + s\int_0^x f(y)\,dy - c$$

Set $\frac{d}{dx} = 0$

$$\Rightarrow \quad P(D \le x^*) = \frac{r-c}{r-s}$$

Investment

reckless investment $1+\delta$

risky investment W_i

$$V_n = V_0 R_1 R_2 \cdots R_n$$

e.g. $R_i = f(1+\delta) + (1-f) W_i$ ← i.i.d

① Maximize Expected value

$$E V_n = V_0 \, E R_1 \, E R_2 \cdots E R_n$$

$$= V_0 (\underbrace{E R_1}_{\text{maximize}})^n$$

$$E W_1 > 1+\delta \Rightarrow f^* = 0$$

$$E V_n = V_0 (E W_1)^n$$

⊛ $$\frac{1}{n}\log V_n = \frac{1}{n}\log V_0 + \underbrace{\frac{1}{n}\sum_{i=1}^{n}\log R_i}_{\xrightarrow{P}\ E\log R_i}$$

$$\frac{1}{n}\log V_n \xrightarrow{P} \frac{1}{n}\log V_0 + E\log\big(f(1+\delta) + (1-f)W_1\big)$$

choose f to maximize

$$E\log R_1 < \log ER_1$$

Jensen's inequality

$\log(x)$ concave

$$\begin{cases} E\varphi(R_1) < \varphi(ER_1) & \varphi \text{ concave} \\ E\varphi(R_1) > \varphi(ER_1) & \varphi \text{ convex} \end{cases}$$

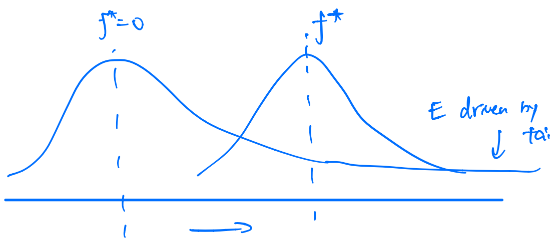
Proof: $\varphi(\bar{R}_n) = \varphi\left(\frac{1}{n}R_1 + \cdots + \frac{1}{n}R_n\right)$

$\leq \sum_{i=1}^{n} \frac{1}{n}\varphi(R_i)$ convex

↳ $E\varphi R_i$

$\Rightarrow \varphi(ER_1) \leq E\varphi(R_i)$

$$V_n \approx V_0 \exp\big(nE\log(f^*(1+\delta) + (1-f^*)W_1)\big)$$



"diversified"

$\Rightarrow$ maximize expected log-return rather than expected return

"kelly investing"

Central Limit Theorem (CLT)

- $S_n = X_1 + \cdots + X_n$
- $X_1, X_2, \cdots$ i.i.d.
- $\operatorname{var} X_1 < \infty$

$$\bar{X}_n \xrightarrow{P} EX_1$$

$$\Rightarrow n^{\frac{1}{2}}\left(\bar{X}_n - EX_1\right) \Rightarrow \sigma N(0,1)$$

↑ rate of convergence

↑ $sd(X_1)$

$$\bar{X}_n - EX_1 \Rightarrow \frac{\sigma}{\sqrt{n}} N(0,1)$$

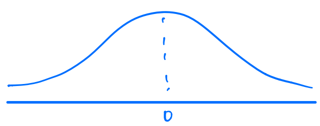
weak convergence

convergence in distribution

$\phi(x)$: pdf of $N(0,1)$

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\Phi(x) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt$$



Convergence in Distribution

$$Z_n \Rightarrow Z_\infty$$

iff

$$P(Z_n \le z) \to P(Z_\infty \le z)$$

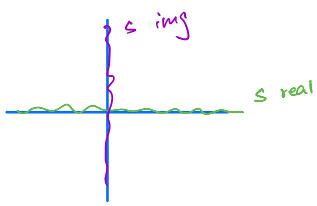
as $n \to \infty$ at every z which is a continuous point of $P(Z_n \le z)$

$$P\left(n^{\frac{1}{2}} \frac{(\bar{X}_n - EX)}{\sigma} \le z\right) \to \Phi(z) \quad \text{as } n \to \infty$$

Leplace transform

Fourier transform

$$E e^{sX} = C_X(s) \quad \leftarrow \text{complex-valued function}$$



λ real

$$E e^{\lambda X} = M(\lambda) \qquad \text{moment-generating function}$$

imaginary

$$E e^{i\theta X} = C(\theta) \qquad \text{characteristic function}$$

$$\int_{-\infty}^{\infty} e^{i\theta x} f_X(x) \, dx$$

↳ always well-defined for $\forall \theta$

$$\int_{-\infty}^{\infty} |e^{i\theta x} f_X(x)| dx \le \int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

For m.g.f. though $\int_{-\infty}^{\infty} e^{\lambda x} (1+x)^{-p} dx = \infty \quad \forall \lambda \neq 0$

Extension of CLT:

$$\bar{X}_n \xrightarrow{P} EX_1$$

$$n^{\frac{1}{2}} (\bar{X}_n - EX_1) \Rightarrow \sigma N(0,1)$$

confidence interval for EX_1

$$S_n \xrightarrow{P} \sigma = \sqrt{\mathrm{Var} X_1}$$

$$S_n = \underbrace{\sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \underbrace{\bar{X}_n}_{\downarrow\, EX_1})^2}}_{\downarrow\, \sqrt{\mathrm{Var} X_1}}$$

" sample standard deviation "

CLT for S_n ?

$\Rightarrow$ " Delta Method "

$$S_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2}$$

$$\approx g\Big(\underbrace{\frac{1}{n} \sum_{i=1}^{n} X_i^2}_{\downarrow P\ EX_1^2} , \underbrace{\frac{1}{n} \sum_{i=1}^{n} X_i}_{\downarrow P\ EX_1} \Big)$$

$$g(x,y) = \sqrt{x - y^2}$$

$$Z_i = (X_i^2, X_i)^T$$

$$Z_1, \dots, Z_n$$

$$S_n \approx g(\bar{z}_n)$$

$$\sigma = g(Ez)$$

$$S_n - \sigma = g(\bar{z}_n) - g(Ez)$$

$$\approx \nabla g(Ez) \cdot (\bar{z}_n - Ez)$$

$$= \bar{y}_n$$

$$y_i = \nabla g(Ez) \begin{pmatrix} x_i^2 - Ex_1^2 \\ x_i - Ex_1 \end{pmatrix}$$

$\uparrow$

i.i.d. scalar mean 0 r.v.'s

$$n^{1/2}\, \bar{y}_n \Rightarrow \beta\, N(0,1)$$

$$\beta = sd(y_1) = \sqrt{E y_1^2}$$