Central Limit Theorem

LLN

- $X_1, X_2, \cdots$ i.i.d. , $\text{var } X_1 < \infty$
- $S_n = X_1 + X_2 + \cdots$
- $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \xrightarrow{P} EX_1$ LLN
- $n^{\frac{1}{2}} (\bar{X}_n - EX_1) \Rightarrow \sigma N(0,1)$ CLT

Newvendor Settings:

$$x^* \longrightarrow \text{maximize } EW(x, D)$$

$$P\left(\sum_{i=1}^{n} W(x^*, D_i) > 100 \right), \quad n = 365$$

$$\frac{S_n - nEX_1}{\sqrt{n}\,\sigma} \Rightarrow N(0,1) \qquad \text{CLT}$$

$$P\left(\frac{\sum_{i=1}^{n} W(x^*, D_i) - nEW(x^*, D)}{\sqrt{n}\,\sigma} > \frac{100 - nEW(x^*, D)}{\sqrt{n}\,\sigma} \right)$$

$$\approx P\left(N(0,1) > \frac{100 - 365 EW(x^*, D)}{\sqrt{365}\,\sigma} \right)$$

$\begin{cases} \text{Moment Generating Function} \\ \text{Characteristic Function} \end{cases}$

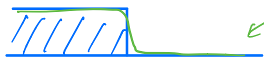
$$E e^{i\theta X} = C_X(\theta)$$

$$= E \cos(\theta X) + i E \sin(\theta X)$$

$$E \cos(\theta X) = \begin{cases} \int_{-\infty}^{\infty} \cos(\theta x) f_X(x)\, dx & \text{continuos} \\ \sum_x \cos(\theta x) P(X = x) & \text{discrete} \end{cases}$$

$$= \int_{-\infty}^{\infty} \cos(\theta x) F_X(dx)$$

$$\int_{-\infty}^{\infty} g(x)\, F_X(dx) \qquad \forall g \text{ continuous}$$



approximate all continuous functions

$$P(X \le 10) = \int \mathbb{I}(X \le 10)\, F_X(dx)$$

knows the characteristic function of X

equiv. to.

knows the distribution of X

$Z_1, Z_2, \cdots$

If $C_{Z_n}(\theta) \to f_\infty(\theta)$ where

$f_\infty(\theta)$ is continous in the neighbor of origin,

then

- $f_\infty(\theta) = E\exp(i\theta Z_\infty)$ for some $\sim Z_\infty$
- $Z_n \Rightarrow Z_\infty$

M.G.F.

$$E e^{\theta X} = \int_{-\infty}^{\infty} e^{\theta x} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx \qquad X \overset{D}{=} N(0,1)$$

$$= e^{\theta^2/2}$$

$$E\exp(\theta N(\mu, \sigma^2)) = \exp\left(\theta\mu + \frac{\theta^2\sigma^2}{2}\right)$$

C.F.

$$E\exp(i\theta N(0,1)) = \exp\left(-\frac{\theta^2}{2}\right)$$

$$E\exp(i\theta N(\mu, \sigma^2)) = \exp\left(i\theta\mu - \frac{\theta^2\sigma^2}{2}\right)$$

$$E \exp\left(i\theta \left(\frac{S_n - nEX_1}{\sqrt{n}\,\sigma}\right)\right) \overset{?}{\longrightarrow} \exp\left(-\frac{\theta^2}{2}\right) \quad \swarrow \text{ Want to prove}$$

Let $\tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i$, $\tilde{X}_i = (X_i - EX_1)/\sigma$ (centering)

$$\Rightarrow \quad E \exp\left(i \frac{\theta}{\sqrt{n}} \sum_{i=1}^{n} \tilde{X}_i\right)$$

$$= E \prod_{i=1}^{n} \exp\left(i \frac{\theta}{\sqrt{n}} \tilde{X}_i\right)$$

$$\overset{i.i.d.}{=} \prod_{i=1}^{n} E \exp\left(i \frac{\theta}{\sqrt{n}} \tilde{X}_i\right)$$

$$= \left(E \exp\left(i \frac{\theta}{\sqrt{n}} \tilde{X}_i\right)\right)^n$$

$$E \exp\left(i \frac{\theta}{\sqrt{n}} \tilde{X}_i\right) = 1 + \frac{i\theta}{\sqrt{n}} \underbrace{E\tilde{X}_i}_{0} - \frac{\sigma^2}{2n} \underbrace{E\tilde{X}_i^2}_{1} + \frac{-i}{3!} \frac{\sigma^3}{n^{3/2}} E\tilde{X}_i^3 + \cdots$$

$$= \left(1 - \frac{\sigma^2}{2n} + O(n^{-3/2})\right)^n \longrightarrow e^{-\sigma^2/2} \qquad \left(1 + \frac{z}{n}\right)^n \to e^z$$

Making it rigorous:

$$\underbrace{E \exp\left(i \frac{\theta}{\sqrt{n}} \tilde{X}_1\right)}_{\phi\left(\frac{\theta}{\sqrt{n}}\right)} = 1 - \frac{\theta^2}{2n} + o\left(\frac{1}{n}\right)$$

$$\begin{cases} E \cos\left(\frac{\theta}{\sqrt{n}} \tilde{X}_1\right) = 1 - \frac{\theta^2}{2n} + o\left(\frac{1}{n}\right) \\ E \sin\left(\frac{\theta}{\sqrt{n}} \tilde{X}_1\right) = o\left(\frac{1}{n}\right) \end{cases}$$

WTS $\Rightarrow$ $\phi\left(\frac{\theta}{\sqrt{n}}\right) = \phi(0) - \frac{\theta^2}{2n} + o\left(\frac{1}{n}\right)$

$$\phi\left(\frac{\theta}{\sqrt{n}}\right) = \phi(0) + \phi'(0)\frac{\theta}{\sqrt{n}} + \phi''(0)\frac{\theta^2}{2n} + o\left(\frac{1}{n}\right)$$

WTS $\Rightarrow$ ϕ is twice differentiable at 0

$\phi'(0) = 0$

$\phi''(0) = 1$

...

$$\phi\left(\frac{\theta}{\sqrt{n}}\right) = E \cos\left(\frac{\theta}{\sqrt{n}} \bar{X}_1\right)$$

$$\frac{\phi(\gamma) - \phi(0)}{\gamma} \overset{?}{\longrightarrow} \phi'(0)$$

$$\phi(\gamma) = E\cos(\gamma \tilde{X}_1)$$

$$\frac{\phi(\gamma) - \phi(0)}{\gamma} = E\left[\frac{\cos(\gamma \tilde{X}_1) - \cos(0 \cdot \tilde{X}_1)}{\gamma} \right]$$

$\downarrow$

$\tilde{X}_1 \cdot 1$

$$E \lim_{\gamma \to 0} \left(\frac{\cos(\gamma \tilde{X}_1) - \cos(0 \cdot \tilde{X}_1)}{\gamma} \right) = E\tilde{X}_1 \quad (= 0)$$

Bounded Convergence Theorem (BCT)

- $W_n \xrightarrow{P} W_\infty$
- $\exists \ c < \infty$ s.t.

$$P(|W_n| \le c) = 1$$

Bound

Then,

$$EW_n \to EW_\infty$$

Proof:

$\to 2c\,P(|W_n - W_\infty| > \varepsilon) \to 0$

$$E(W_n - W_\infty) = E(W_n - W_\infty)\,\mathbb{I}(|W_n - W_\infty| > \varepsilon)$$

$$+ \left| E(W_n - W_\infty)\,\mathbb{I}(|W_n - W_\infty| \le \varepsilon) \right|$$

$$\hookrightarrow \le \varepsilon P(|W_n - W_\infty| \le \varepsilon)$$

$$\le \varepsilon$$

Dominated Convergence Theorem

- $W_n \xrightarrow{P} W_\infty$
- $\exists$ a r.v. Y s.t.

$$|W_n(\omega)| \le Y(\omega), \quad \forall\, \omega \in \Omega$$

and $EY < \infty$

Then,

$$EW_n \to EW_\infty$$

Going back to previous equation,

$$\left| \frac{\cos(r\tilde{X}_1) - \cos(0\tilde{X}_1)}{r} \right|$$

$$= \left| \tilde{X}_1 \frac{d}{dy} \cos(\xi) \right| \qquad \xi \text{ lies in btw } 0 \text{ and } r\tilde{X}_1$$

$$\le |\tilde{X}_1|^2 \qquad \text{integrable r.v.}$$

- $E X_1^2 < \infty$
- $(E X_1)^2 \le E X_1^2$
- $E X_1 < \infty$

$\Rightarrow$ $\phi'(0)$ exists and $\phi'(0) = 0$

$\phi''(0)$ exists and $\phi''(0) = 1$ (apply again)

$$n^{\frac{1}{2}}(\bar{X}_n - E X_1) \Rightarrow \sigma N(0,1)$$

What can we say about $g(\bar{X}_n)$?

A $\begin{cases} \bar{X}_n \xrightarrow{P} E X_1 \\ g(\bar{X}_n) \xrightarrow{P} g(E X_1) \quad \text{if } g \text{ is continuous at } EX \end{cases}$

Delta method :

A $\quad n^{\frac{1}{2}}(g(\bar{X}_n) - g(E X_1)) \Rightarrow \eta N(0,1)$

$$\eta^2 = \text{var}(\nabla g(E X_1) \cdot X_1)$$

row vector column vector

- $W_n \Rightarrow W_\infty$
- $Y_n \xrightarrow{P} 0$

Then

$$W_n + Y_n \Rightarrow W_\infty \qquad \text{Slutsky's Lemma}$$

$$W_n \cdot Y_n \xrightarrow{P} 0$$

If

$$Y_n' \xrightarrow{P} 1,$$

Then

$$W_n Y_n' \Rightarrow W_\infty$$

Proof: $P(W_n + Y_n \le x) \xrightarrow{?} P(W_\infty \le x)$

Slutsky's Lemma

at x's that are continuity pts at $P(W_\infty \le \cdot)$

$Y_n \to 0$

$\Downarrow$

$P(|Y_n| > \varepsilon) \to 0$

$$P(W_n + Y_n \le x)$$

$$= P(W_n + Y_n \le x, |Y_n| \le \varepsilon) + P(W_n + Y_n \le x, |Y_n| > \varepsilon)$$

$$P(W_n + Y_n \le x, |Y_n| \le \varepsilon) \le P(W_n \le x + \varepsilon, |Y_n| \le \varepsilon) \le P(W_n \le x + \varepsilon)$$

$$\downarrow$$

$$P(W_\infty \le x + \varepsilon)$$

$$P(W_n + Y_n \le x, |Y_n| > \varepsilon) \le P(|Y_n| > \varepsilon) \to 0$$

$$\overline{\lim_{n\to\infty}} P(W_n + Y_n \le x) \le P(W_\infty \le x + \varepsilon)$$

$$\le P(W_\infty \le x)$$

$$\underline{\lim_{n\to\infty}} P(W_n + Y_n \le x) \ge P(W_n \le x)$$

$a_n \to a_\infty$

iff

$\overline{\lim}\, a_n \le a_n$

$\underline{\lim}\, a_n \ge a_n$