Interest Rate Model Calibration and Pricing Caps and Floors with QuantLib

Financial Algorithm- Final Project

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01 Hull White Model and Black Karasinski Model Calibration

首先會透過swaption 的market data 做calibration

WHY swaption?

會用swaption而不是cap/floor的原因有以下:

1. Swaption 流動性好
2. 交易量大
3. Information set 豐富

經過前面的環境設定，之後再define一個function建立swaptions

```
In [4]: def create_swaption_helpers(data, index, term_structure, engine):

            swaptions = []
            fixed_leg_tenor = ql.Period(1, ql.Years)
            fixed_leg_daycounter = ql.Actual360()
            floating_leg_daycounter = ql.Actual360()

            for d in data:
                vol_handle = ql.QuoteHandle(ql.SimpleQuote(d.volatility))
                helper = ql.SwaptionHelper(ql.Period(d.start, ql.Years),
                                           ql.Period(d.length, ql.Years),
                                           vol_handle,
                                           index,
                                           fixed_leg_tenor,
                                           fixed_leg_daycounter,
                                           floating_leg_daycounter,
                                           term_structure
                                           )
                helper.setPricingEngine(engine)
                swaptions.append(helper)
            return swaptions
```

01 Hull White Model and Black Karasinski Model Calibration

(二)Hull & White(1993)

- Hull & White 模型的短期利率連續極限如下，

$$dr = [\theta(t) - \alpha r]\cdot dt + \sigma \cdot dz \quad \text{......(8.4)}$$

- 在 HW 模型中只有二個參數，可由市場上交易的利率選擇權價格來推估。
  - 假設市場上有 M 個零息債券選擇權，其價格分別為 Market_i，I=1…M。令 $\text{Model}(\alpha，\sigma)_i$ 為由 HW 模型所求的的價格，則如下校準參數，

$$\min_{\alpha,\sigma}\sqrt{\sum_{i=1}^{M}\left(\frac{\text{model}_i(\alpha,\sigma)-\text{market}_i}{\text{market}_i}\right)^2}$$

define 一個function做calibrate

```
In [5]: def calibration_report(swaptions, data):

    print("-"*82)
    print("%15s %15s %15s %15s %15s" % \
    ("Model Price", "Market Price", "Implied Vol", "Market Vol", "Rel Error"))
    print("-"*82)


    cum_err = 0.0
    for i, s in enumerate(swaptions):
        model_price = s.modelValue()
        market_vol = data[i].volatility
        black_price = s.blackPrice(market_vol)
        rel_error = model_price/black_price - 1.0
        implied_vol = s.impliedVolatility(model_price,
                                          1e-5, 50, 0.0, 0.50)
        rel_error2 = implied_vol/market_vol-1.0
        cum_err += rel_error2*rel_error2

        print("%15.5f %15.5f %15.5f %15.5f %15.5f" % \
        (model_price, black_price, implied_vol, market_vol, rel_error))
    print("-"*82)
    print("Cumulative Error : %15.5f" % math.sqrt(cum_err))
```

01 Hull White Model and Black Karasinski Model Calibration

In [6]:
```
model = ql.HullWhite(term_structure);
engine = ql.JamshidianSwaptionEngine(model)
swaptions = create_swaption_helpers(data, index, term_structure, engine)

optimization_method = ql.LevenbergMarquardt(1.0e-8,1.0e-8,1.0e-8)
end_criteria = ql.EndCriteria(10000, 100, 1e-6, 1e-8, 1e-8)
model.calibrate(swaptions, optimization_method, end_criteria)

a, sigma = model.params()
print("alpha = %6.5f, sigma = %6.5f" % (a, sigma))
calibration_report(swaptions, data)
```

```
alpha = 0.04915, sigma = 0.00584
----------------------------------------------------------------------
    Model Price    Market Price     Implied Vol      Market Vol      Rel Error
----------------------------------------------------------------------
        0.00880         0.00951         0.10620         0.11480       -0.07488
        0.00967         0.01007         0.10632         0.11080       -0.04039
        0.00866         0.00871         0.10635         0.10700       -0.00606
        0.00650         0.00623         0.10644         0.10210        0.04234
        0.00354         0.00332         0.10659         0.10000        0.06561
----------------------------------------------------------------------
Cumulative Error :          0.11594
```

由圖可知經過calibration，得到所需參數

$$a = 0.04915$$

$$\sigma = 0.00584$$

得到Model Price、Market Price、Implied vol、Market vol

由此計算模型價格與市場價格誤差

01 Hull White Model and Black Karasinski Model Calibration

(四)Black & Karasinski(BK, 1991)

◆ BK 模型的短期利率連續極限如下，

$$d\ln r = [\theta(t) - \alpha(t)\ln r]\cdot dt + \sigma(t)\cdot dz \quad (8.8)$$

➢ 令波動性為常數可簡化模型成為，

$$d\ln r(t) = [\theta(t) - \alpha\ln r]\cdot dt + \sigma\cdot dz$$

In [7]:
```
model = ql.BlackKarasinski(term_structure);
engine = ql.TreeSwaptionEngine(model, 100)
swaptions = create_swaption_helpers(data, index, term_structure, engine)

optimization_method = ql.LevenbergMarquardt(1.0e-8,1.0e-8,1.0e-8)
end_criteria = ql.EndCriteria(10000, 100, 1e-6, 1e-8, 1e-8)
model.calibrate(swaptions, optimization_method, end_criteria)

a, sigma =  model.params()
print("alpha = %6.5f, sigma = %6.5f" % (a, sigma))
```
```
alpha = 0.04162, sigma = 0.11782
```

In [8]:
```
calibration_report(swaptions, data)
```

Model Price	Market Price	Implied Vol	Market Vol	Rel Error
0.00874	0.00951	0.10550	0.11480	-0.08097
0.00967	0.01007	0.10633	0.11080	-0.04026
0.00867	0.00871	0.10655	0.10700	-0.00423
0.00651	0.00623	0.10665	0.10210	0.04443
0.00355	0.00332	0.10675	0.10000	0.06714

Cumulative Error : 0.12147

由圖可知經過calibration，得到所需參數

$$a = 0.04162$$

$$\sigma = 0.11782$$

得到Model Price、Market Price、Implied vol、Market vol

由此計算模型價格與市場價格誤差

02 Pricing Caps with Constant volatility

這之後會以定價Caps為主題，分別以兩種cases作探討

1. Pricing Caps with Constant volatility
2. Pricing Caps with volatility surface

WHY?

因為Caps是由多個caplets組成，每一個caplet都應該要有不同的volatility，因此會先以constant volatility再延生到volatility surface

02 Pricing Caps with Constant volatility

透過QuantLib評價Caps的步驟:

1. construct interest rate term structure for discounting
2. construct interest rate term structure for the floating leg
3. construct the pricing engine to value caps

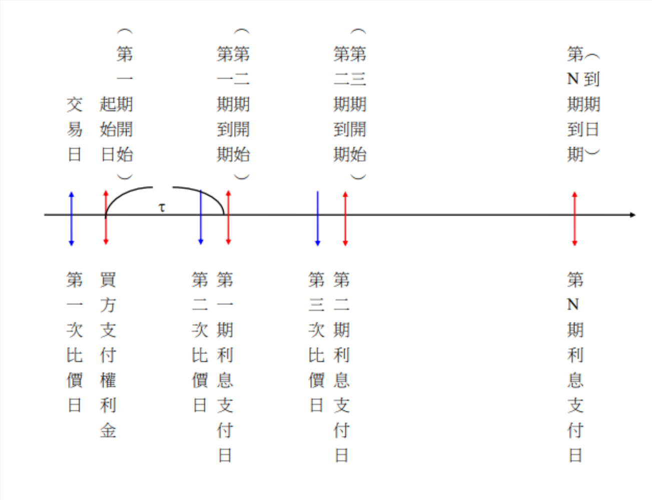
```
In [9]: calc_date = ql.Date(14, 6, 2021)
        ql.Settings.instance().evaluationDate = calc_date
```

```
In [10]: dates = [ql.Date(14,6,2021), ql.Date(14,9,2021),
                  ql.Date(14,12,2021), ql.Date(14,6,2022),
                  ql.Date(14,6,2024), ql.Date(14,6,2026),
                  ql.Date(15,6,2031), ql.Date(16,6,2036),
                  ql.Date(16,6,2041), ql.Date(14,6,2051)
                  ]
         yields = [0.000000, 0.006616, 0.007049, 0.007795,
                   0.009599, 0.011203, 0.015068, 0.017583,
                   0.018998, 0.020080]
         day_count = ql.ActualActual()
         calendar = ql.Taiwan()
         interpolation = ql.Linear()
         compounding = ql.Compounded
         compounding_frequency = ql.Annual

         term_structure = ql.ZeroCurve(dates, yields, day_count, calendar,
                                       interpolation, compounding, compounding_frequency)

         ts_handle = ql.YieldTermStructureHandle(term_structure)
```

02 Pricing Caps with Constant volatility

construct 一個十年期，每三個月比價一次的caps

```python
In [11]: start_date = ql.Date(14, 6, 2021)
         end_date = ql.Date(14, 6 , 2031)
         period = ql.Period(3, ql.Months)
         calendar = ql.Taiwan()
         buss_convention = ql.ModifiedFollowing
         rule = ql.DateGeneration.Forward
         end_of_month = False

         schedule = ql.Schedule(start_date, end_date, period,
                                calendar, buss_convention, buss_convention,
                                rule, end_of_month)
```



02 Pricing Caps with Constant volatility

```
In [12]: ibor_index = ql.USDLibor(ql.Period(3, ql.Months), ts_handle)
         ibor_index.addFixing(ql.Date(10,6,2021), 0.0065560)

         ibor_leg = ql.IborLeg([1000000], schedule, ibor_index)

In [13]: strike = 0.02
         cap = ql.Cap(ibor_leg, [strike])

         vols = ql.QuoteHandle(ql.SimpleQuote(0.547295))
         engine = ql.BlackCapFloorEngine(ts_handle, vols)

         cap.setPricingEngine(engine)

In [14]: cap.NPV()

Out[14]: 54408.95638684406
```

有了以上materials，即可計算出cap=54408.95

03 Pricing Caps with volatility surface

接著計算每個caplet都有不同的volatility

Construct a volatility matrix with different expires and strikes

In [15]:
```
strikes = [0.01,0.015, 0.02]
expiries = [ql.Period(i, ql.Years) for i in range(1,11)] + [ql.Period(12, ql.Years)]
vols = ql.Matrix(len(expiries), len(strikes))
data = [[47.27, 55.47, 64.07, 70.14, 72.13, 69.41, 72.15, 67.28, 66.08, 68.64, 65.83],
   [46.65,54.15,61.47,65.53,66.28,62.83,64.42,60.05,58.71,60.35,55.91],
   [46.6,52.65,59.32,62.05,62.0,58.09,59.03,55.0,53.59,54.74,49.54]
   ]

for i in range(vols.rows()):
    for j in range(vols.columns()):
        vols[i][j] = data[j][i]/100.0
```

03 Pricing Caps with volatility surface

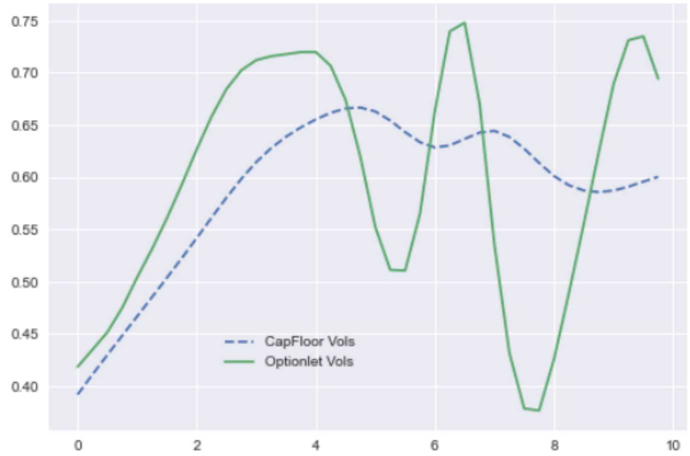
透過ql.OptionletStripper1 將每個caplet/floorlet volatility從capfloor vol中分開，再透過ql.StrippedOptionletAdapter組成新的optionlet vol才能建構term structure

透過matplotlib畫出capfloor vol and optionlet vol

In [17]:
```
optionlet_surf = ql.OptionletStripper1(capfloor_vol, ibor_index)
ovs_handle = ql.OptionletVolatilityStructureHandle(
    ql.StrippedOptionletAdapter(optionlet_surf)
)
```

In [18]:
```
tenors = np.arange(0,10,0.25)
strike = 0.015
capfloor_vols = [capfloor_vol.volatility(t, strike) for t in tenors]
opionlet_vols = [ovs_handle.volatility(t, strike) for t in tenors]

plt.plot(tenors, capfloor_vols, "--", label="CapFloor Vols")
plt.plot(tenors, opionlet_vols,"-", label="Optionlet Vols")
plt.legend(bbox_to_anchor=(0.5, 0.25))
```

Out[18]: <matplotlib.legend.Legend at 0x2076ebf8910>



03 Pricing Caps with volatility surface

```
In [19]: engine2 = ql.BlackCapFloorEngine(ts_handle, ovs_handle)
         cap.setPricingEngine(engine2)

In [20]: cap.NPV()

Out[20]: 54427.26567992578
```

即可將optionlet volatility surface評價caps or floors

由此可知cap=54427.26


Thanks