1 Existence of Concave c Function

To show that (5) defines a sequence of continuously differentiable strictly increasing concave functions $\{c_T, c_{T-1}, \ldots, c_{T-k}\}$, we start with a definition. We will say that a function $n(z)$ is 'nice' if it satisfies

1. $n(z)$ is well-defined iff $z > 0$
2. $n(z)$ is strictly increasing
3. $n(z)$ is strictly concave
4. $n(z)$ is $\mathbf{C}^3$
5. $n(z) < 0$
6. $\lim_{z \downarrow 0} n(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} n'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $v_T(m) = u(m)$ and $u(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta \, \mathbb{E}_t \left[\mathbf{\Phi}_{t+1}^{1-\rho} v_{t+1}(\mathcal{R}_{t+1} a + \boldsymbol{\xi}_{t+1}) \right]. \tag{1}$$

Since there is a positive probability that $\boldsymbol{\xi}_{t+1}$ will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{v}_t(m, c)$ as

$$\underline{v}_t(m, c) = u(c) + \mathfrak{v}_t(m - c) \tag{2}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (5) can be written as

$$v_t(m) = \max_c \, \underline{v}_t(m, c). \tag{3}$$

$\underline{v}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c \downarrow 0} \underline{v}_t(m, c) = \lim_{c \uparrow m} \underline{v}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{v}_t(m,c)}{\partial c^2} < 0$, $\lim_{c \downarrow 0} \frac{\partial \underline{v}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c \uparrow m} \frac{\partial \underline{v}_t(m,c)}{\partial c} = -\infty$. It follows that the $c_t(m)$ defined by

$$c_t(m) = \underset{0 < c < m}{\arg \max} \, \underline{v}_t(m, c) \tag{4}$$

exists and is unique, and (5) has an internal solution that satisfies

$$u'(c_t(m)) = \mathfrak{v}_t'(m - c_t(m)). \tag{5}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $c_t(m)$ and $a_t(m) = m - c_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (5) we can conclude that $c_t(m)$ is continuously differentiable and

$$c_t'(m) = \frac{\mathfrak{v}_t''(a_t(m))}{u''(c_t(m)) + \mathfrak{v}_t''(a_t(m))}. \tag{6}$$

Similarly we can easily show that $c_t(m)$ is twice continuously differentiable (as is $a_t(m)$) (See Appendix 2.) This implies that $v_t(m)$ is nice, since $v_t(m) = u(c_t(m)) + \mathfrak{v}_t(a_t(m))$.

2 $\, c_t(m)$ is Twice Continuously Differentiable

First we show that $c_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $u'(c_t(y)) - u'(c_t(m)) = \mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))$ and $\frac{a_t(y) - a_t(m)}{dm} = 1 - \frac{c_t(y) - c_t(m)}{dm}$,

$$\frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)} = \left(\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)} \right) \frac{c_t(y) - c_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim\limits_{dm \to +0} \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} < 0$ and $\lim\limits_{dm \to +0} \frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ are satisfied. Then $\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)}}{\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathfrak{v}'_t(a_t(y)) - \mathfrak{v}'_t(a_t(m))}{a_t(y) - a_t(m)}}.$$

This implies that the right-derivative, $c_t'^+(m)$ is well-defined and

$$c_t'^+(m) = \frac{\mathfrak{v}''_t(a_t(m))}{u''(c_t(m)) + \mathfrak{v}''_t(a_t(m))}.$$

Similarly we can show that $c_t'^+(m) = c_t'^-(m)$, which means $c_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $c_t'(m)$ exists and is continuous. $c_t'(m)$ is differentiable because $\mathfrak{v}''_t$ is $\mathbf{C}^1$, $c_t(m)$ is $\mathbf{C}^1$ and $u''(c_t(m)) + \mathfrak{v}''_t(a_t(m)) < 0$. $c_t''(m)$ is given by

$$c_t''(m) = \frac{a_t'(m)\mathfrak{v}'''_t(a_t)\left[u''(c_t) + \mathfrak{v}''_t(a_t)\right] - \mathfrak{v}''_t(a_t)\left[c_t'u'''(c_t) + a_t'\mathfrak{v}'''_t(a_t)\right]}{\left[u''(c_t) + \mathfrak{v}''_t(a_t)\right]^2}. \tag{7}$$

Since $\mathfrak{v}''_t(a_t(m))$ is continuous, $c_t''(m)$ is also continuous.

3 $\,\mathcal{T}$ Is a Contraction Mapping

We must show that our operator $\mathcal{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}z\}$ be continuous for any F−bounded z, $\{\mathcal{T}z\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\{\mathcal{T}x\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta\, \mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho} x\,(m_{t+1}) \right] \right\}$$

$$\{\mathcal{T}y\}(m_t) \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta\, \mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho} y\,(m_{t+1}) \right] \right\},$$

so $x(\bullet) \leq y(\bullet)$ implies $\{\mathcal{T}x\}(m_t) \leq \{\mathcal{T}y\}(m_t)$ by inspection.[1]

Condition (2) requires that $\{\mathcal{T}0\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$. By definition,

$$\{\mathcal{T}0\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently $u(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is F-bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \tag{8}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of F, $\{\mathcal{T}0\}(m_t) = u(\bar{\kappa} m_t)$ is clearly F-bounded.

Finally, we turn to condition (3), $\{\mathcal{T}(z + \zeta F)\}(m_t) \leq \{\mathcal{T}z\}(m_t) + \zeta \alpha F(m_t)$. The proof will be more compact if we define $\check{c}$ and $\check{a}$ as the consumption and assets functions[2] associated with $\mathcal{T}z$ and $\hat{c}$ and $\hat{a}$ as the functions associated with $\mathcal{T}(z + \zeta F)$; using this notation, condition (3) can be rewritten

$$u(\hat{c}) + \beta\{E(z + \zeta F)\}(\hat{a}) \leq u(\check{c}) + \beta\{Ez\}(\check{a}) + \zeta \alpha F.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$u(\hat{c}) + \beta\{Ez\}(\hat{a}) \leq u(\check{c}) + \beta\{Ez\}(\check{a}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$u(\hat{c}) + \beta\{E(z + \zeta F)\}(\hat{a}) \leq u(\hat{c}) + \beta\{Ez\}(\hat{a}) + \zeta \alpha F$$
$$\beta\{E(z + \zeta F)\}(\hat{a}) \leq \beta\{Ez\}(\hat{a}) + \zeta \alpha F$$
$$\beta \zeta\{EF\}(\hat{a}) \leq \zeta \alpha F$$
$$\beta\{EF\}(\hat{a}) \leq \alpha F$$
$$\beta\{EF\}(\hat{a}) < F.$$

where the last line follows because $0 < \alpha < 1$ by assumption.[3]

Using $F(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{a}(m_t)$, this condition is

$$\beta \mathbb{E}_t[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_t R_{t+1} + \boldsymbol{\xi}_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta \mathbb{E}_t \boldsymbol{\Phi}_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (20), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta \mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_t R_{t+1} + \boldsymbol{\xi}_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1 - \beth}. \tag{9}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

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[1]For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

[2]Section 2.8 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[3]The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (9) is bounded from above:

$$(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_tR_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_tR_{t+1})^{1-\rho}\right]-m_t^{1-\rho}$$

$$\leq(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_tR_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}$$

$$=(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_tR_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)$$

$$=(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_tR_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)$$

$$<(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.$$

We can thus conclude that equation (9) will certainly hold for any:

$$\eta>\underline{\eta}=\frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth}\tag{10}$$

which is a positive finite number under our assumptions.

The proof that $\mathcal{T}$ defines a contraction mapping under the conditions (32) and (29) is now complete.

3.1 $\mathcal{T}$ and v

In defining our operator $\mathcal{T}$ we made the restriction $\underline{\kappa}m_t\leq c_t\leq\bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (33)) that $\underline{\kappa}_tm_t\leq c_t(m_t)\leq\bar{\kappa}_tm_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (5) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (10). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho}<1$$

$$(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1})>1$$

$$(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\mathbf{P}_{\mathsf{R}})^{-1})>1$$

where we have used (31) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho>1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\mathbf{P}_{\mathsf{R}})^{-1}\approx 1-\wp^{1/\rho}\mathbf{P}_{\mathsf{R}}$ so that it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\mathbf{P}_{\mathsf{R}}>1$$

$$(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_\mathsf{R}^{1-\rho} < 1$$
$$\beta\wp^{1/\rho}\text{Þ}_\mathsf{R}^{1-\rho} < 1.$$

Calling the weak return patience factor $\text{Þ}_\mathsf{R}^\wp = \wp^{1/\rho}\text{Þ}_\mathsf{R}$ and recalling that the WRIC was $\text{Þ}_\mathsf{R}^\wp < 1$, the expression on the LHS above is $\beta\text{Þ}_\mathsf{R}^{-\rho}$ times the WRPFacDefn. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_\mathsf{R} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\text{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\text{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\text{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathcal{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (5) converge, they converge to the same unique v defined by $\mathcal{T}$.[4]

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[4]It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

References

HIRAGUCHI, RYOJI (2003): "On the Convergence of Consumption Rules," *Manuscript, Johns Hopkins University.*