Theoretical Foundations of Buffer Stock Saving

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Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
    - Calibration
    - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

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The Gap This Paper Fills

Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income

- No Liquidity Constraints

- CRRA Utility

- (Problem with Liquidity Constraints Is A Limiting Case)

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Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
    - $\Rightarrow$ $\exists$ A Unique Consumption Function $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
    - Requires A Key 'Impatience' Condition To Hold
    - Good News
        - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

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The Perfect Foresight Problem
The Real Problem

Limit as horizon T goes to infinity of

$$a_t = m_t - c_t$$
$$k_{t+1} = a_t$$
$$b_{t+1} = k_{t+1}R$$
$$p_{t+1} = p_t \underbrace{\Gamma \Psi_{t+1}}_{\equiv \Gamma_{t+1}}$$
$$m_{t+1} = b_{t+1} + p_{t+1}\xi_{t+1},$$

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \qquad (1)$$

- $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$; $\mathbb{E}_t[\Psi_{t+n}] = \mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n > 0$; $\beta < 1, \rho > 1$

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Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

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Benchmark: Perfect Foresight Model

Definitions:

Absolute Patience Factor	Þ	$=$	$(R\beta)^{1/\rho}$
Return Patience Factor	Þ_R	$=$	$\text{Þ}/R$
Perfect Foresight Growth Patience Factor	Þ_Γ	$=$	$\text{Þ}/\Gamma$

Name	Condition	Implication
(AIC) Absolute Impatience Condition	$\text{Þ} \quad < \quad 1$	c $\downarrow$ over time
(RIC) Return Impatience Condition	$\text{Þ}_R \quad < \quad 1$	c/a $\downarrow$ over time
(GIC) Growth Impatience Condition	$\text{Þ}_\Gamma \quad < \quad 1$	c/p $\downarrow$ over time

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When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

$$\Gamma \quad < \quad R \tag{2}$$

Return Impatience Condition:

$$Þ_R \quad < \quad R \tag{3}$$

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What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  'Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)':

$$\beta\Gamma^{1-\rho} \ < \ 1 \qquad (4)$$

- Without RIC, Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

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Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$
\text{FVAC:} \quad 0 < \overbrace{\beta \underline{\Gamma}^{1-\rho}}^{\equiv \underline{\beth} \; = \; \text{VAF}} < 1 \tag{5}
$$
$$
0 < \beta < \underline{\Gamma}^{\rho-1}.
$$

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Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$\text{FVAC:} \quad 0 < \overbrace{\beta \underline{\Gamma}^{1-\rho}}^{\equiv \underline{\beth} \; = \; \text{VAF}} < 1 \tag{6}$$
$$0 < \beta < \underline{\Gamma}^{\rho-1}.$$

'Weak Return Impatience Condition' (WRIC)

$$0 \leq \; \wp^{1/\rho} \mathbf{P}_{\mathsf{R}} \; < 1 \tag{7}$$

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Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\Gamma} \equiv \Gamma \underline{\psi} < \Gamma$$

Adjusted Growth Patience Factor:

$$\text{GPF-Nrm: } \mathbf{P}_{\underline{\Gamma}} = \mathbf{P}/\underline{\Gamma} = \mathbb{E}[\mathbf{P}/(\Gamma \Psi)] \tag{8}$$

Growth Impatience Condition:

$$\text{GIC-Nrm: } \mathbf{P}_{\underline{\Gamma}} < 1, \tag{9}$$

Why? Because it can be shown that

$$\lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right] = \mathbf{P}_{\underline{\Gamma}} \tag{10}$$

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Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

Five Propositions

1. $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}/c_t] = \text{Þ}$
2. $\lim_{m_t \to 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$
3. $\exists$ a unique target value of m, called $\check{m}$
4. $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \Gamma - \epsilon$
5. $\left(\frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$

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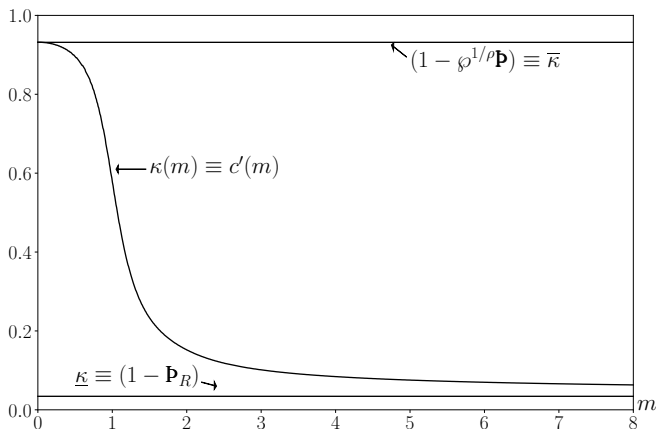Features Of the Solution
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The Target Saving Figure

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Bounds On the Consumption Function

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The Marginal Propensity to Consume

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The Invariant Distribution
Balanced Growth Equilbrium

Convergence To The Invariant Distribution

Szeidl (2013) Proves Existence of an Invariant Distribution of m, c, a, etc.

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The Invariant Distribution
Balanced Growth Equilibrium

Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t \ = \ \Gamma \qquad (11)$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $\grave{c}(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} c(m; \wp) = \grave{c}(m) \qquad (12)$$

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The MPC Out Of Permanent Shocks

https://www.econ2.jhu.edu/people/ccarroll/papers/MPCPerm.pdf

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- MPCP < 1
- But not a lot less:
    - 0.75 to 0.95 (annual rate) for wide range of parameter values

- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents $m \to \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

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