

1 Unique And Stable Target and Steady State Points

This appendix proves Theorems 2-4 and the following Lemmas:

Lemma 1. *If \check{m} and \hat{m} both exist, then $\check{m} \leq \hat{m}$.*

Lemma 2. *If \check{m} and \tilde{m} both exist, then $\check{m} \leq \tilde{m}$.*

1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t , $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and \mathcal{R}_t are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

1.3.1 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (44), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(R/\Gamma_{t+1})\mathbf{P}_R] \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\
&< 1
\end{aligned} \tag{1}$$

where the inequality reflects imposition of the GIC-Nrm (32).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} c'(m) = 0$ (see equation (36)) means that the limit of the RHS of (1) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{2}$$

so that $\zeta(\hat{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\zeta'(m_t) &\equiv \left(\frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\
&= \bar{\mathcal{R}} (1 - c'(m_t)) - 1.
\end{aligned} \tag{3}$$

Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (20) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}} (1 - c'(m_t)) - 1 &< \bar{\mathcal{R}} (1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}} \mathbf{P}_R - 1 \\
&= \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \frac{\mathbf{P}}{\mathbf{R}} \right] - 1 \\
&= \underbrace{\mathbb{E}_t \left[\frac{\mathbf{P}}{\Gamma \psi} \right]}_{=\mathbf{P}_\Gamma} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\mathbf{P}_\Gamma < 1$.

If RIC fails. Under RIC, recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}} (1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that $\zeta'(m_t)$ from (3) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \right] < 1. \quad (4)$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\mathbf{P}}{\Gamma \psi} \right]}^{\mathbf{P}_\Gamma} < 1 < \overbrace{\frac{\mathbf{P}}{\mathbf{R}}}^{\mathbf{P}_\mathbf{R}},$$

and multiplying all three elements by \mathbf{R}/\mathbf{P} gives

$$\mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \right] < \mathbf{R}/\mathbf{P} < 1$$

which satisfies our requirement in (4).

1.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[\bar{m}_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\Gamma_{t+1} ((\mathbf{R}/\Gamma_{t+1})a(m_t) + \xi_{t+1}) / \Gamma}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathbf{R}/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \end{aligned}$$

$$\begin{aligned}
&= \lim_{m_t \uparrow \infty} \left[\frac{(R/\Gamma)a(m_t) + 1}{m_t} \right] \\
&= (R/\Gamma)\mathbf{P}_R \\
&= \mathbf{P}_\Gamma \\
&< 1
\end{aligned} \tag{5}$$

where the last two lines are merely a restatement of the GIC (28).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define $\zeta(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{6}$$

so that $\zeta(\hat{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\zeta'(m_t) &\equiv \left(\frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\
&= (R/\Gamma)(1 - c'(m_t)) - 1.
\end{aligned} \tag{7}$$

Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (\mathbf{RIC}).

If RIC holds. Equation (20) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned}
\mathcal{R}(1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\
&= (R/\Gamma)\mathbf{P}_R - 1
\end{aligned}$$

which is negative because the GIC says $\mathbf{P}_\Gamma < 1$.

If RIC fails. Under \mathbf{RIC} , recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}(1 - c'(m_t)) < \mathcal{R}$$

which means that $\zeta'(m_t)$ from (7) is guaranteed to be negative if

$$\mathcal{R} \equiv (R/\Gamma) < 1. \tag{8}$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHCW also fails (that is, (8) holds).

1.5 Proof of Theorem 4

Proof of the existence of the final kind of stability is straightforward.

$$\begin{aligned}
\log \mathbf{m}_{t+1} - \log \mathbf{m}_t &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \log \Gamma \psi_{t+1} m_{t+1} - \log m_t \\
&= \log \Gamma \psi_{t+1} ((m_t - c_t) \mathbf{R} / \Gamma \psi_{t+1} + \xi_{t+1}) - \log m_t \\
&= \log ((m_t - c_t) \mathbf{R} + \Gamma \psi_{t+1} \xi_{t+1}) - \log m_t \\
&= \log \Gamma \psi_{t+1} + \log ((m_t - c_t) \mathcal{R}_{t+1} + \xi_{t+1}) - \log m_t
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= \log \Gamma + \mathbb{E}_t [\log \psi_{t+1}] + \mathbb{E}_t \log \left(\frac{(m_t - c_t) \mathcal{R}_{t+1} + \xi_{t+1}}{m_t} \right) \\
\mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= \log \Gamma + \mathbb{E}_t [\log \psi_{t+1}] + \mathbb{E}_t \log ((1 - c_t/m_t) \mathcal{R}_{t+1} + \xi_{t+1}/m_t).
\end{aligned}$$

$$\begin{aligned}
(d/dm_t) \Delta \log \mathbf{m}_{t+1} &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \left(\frac{\mathcal{R}_{t+1} (1 - \kappa(m_t))}{(m_t - c_t) \mathcal{R}_{t+1} + \xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \mathcal{R}_{t+1}^{-1} \xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \Gamma \psi_{t+1} \xi_{t+1} / \mathbf{R}} \right) - 1/m_t
\end{aligned} \tag{10}$$

As argued elsewhere, expected growth approaches a maximum of ∞ as $m_t \downarrow 0$. As $m_t \uparrow \infty$, the contribution of the term involving ξ becomes negligible and the expenditure ratio $e(m) = c(m)/m$ approaches $\underline{\kappa}$, so the limiting growth factor approaches

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t [\Delta \log \mathbf{m}_{t+1}] &= \log \Gamma + \mathbb{E}_t [\log \psi_{t+1}] + \mathbb{E}_t \log ((1 - \underline{\kappa}) \mathcal{R}_{t+1}) \\
&= \log \Gamma + \mathbb{E}_t [\log \psi_{t+1}] + \mathbb{E}_t (\log \mathbf{P} - (\log \Gamma + \log \psi_{t+1})) \\
&= \log \mathbf{P}.
\end{aligned} \tag{11}$$

The same arguments as before will guarantee continuity over the range of feasible values of m . So all that remains to prove is that $\mathbb{E}_t [\Delta \log \mathbf{m}_{t+1}]$ is strictly decreasing as m rises. Differentiating with respect to m_t yields

$$\begin{aligned}
(d/dm_t) \mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= (d/dm_t) \mathbb{E}_t \log ((1 - c_t/m_t) \mathcal{R}_{t+1} + \xi_{t+1}/m_t) \\
&= \mathbb{E}_t \left(\frac{-\xi_{t+1}/m_t^2 - \mathcal{R}_{t+1} e'(m_t)}{\xi_{t+1}/m_t + \mathcal{R}_{t+1} (1 - e(m_t))} \right) \\
&= \mathbb{E}_t \left(\frac{-\xi_{t+1}/m_t - m_t \mathcal{R}_{t+1} e'(m_t)}{\xi_{t+1} + m_t \mathcal{R}_{t+1} (1 - e(m_t))} \right) \\
&< \mathbb{E}_t \left(\frac{-\underline{\xi}/m_t - m_t \mathcal{R}_{t+1} e'(m_t)}{\bar{\xi} + m_t \mathcal{R}_{t+1} (1 - e(m_t))} \right)
\end{aligned} \tag{12}$$

and for every realization of ξ and \mathcal{R} , the numerator of this expression is negative and the denominator is positive. So the expression's expectation must be negative.

1.6 Proof of Lemmas 1-2

1.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$\begin{aligned}\check{m}_{t+1}(a) &= 1 + a\mathcal{R} \\ \hat{m}_{t+1}(a) &= 1 + a \underbrace{\mathcal{R}/\underline{\psi}}_{\bar{\mathcal{R}} > \mathcal{R}}\end{aligned}\tag{13}$$

so that we can implicitly define the target and pseudo-steady-state points as

$$\begin{aligned}\hat{m} &= \hat{m}_{t+1}(\hat{m} - c(\hat{m})) \\ \check{m} &= \check{m}_{t+1}(\check{m} - c(\check{m}))\end{aligned}\tag{14}$$

Then subtract:

$$\begin{aligned}\hat{m} - \check{m} &= (\hat{a}\underline{\psi}^{-1} - \check{a})\mathcal{R} \\ &= (a(\hat{m})\underline{\psi}^{-1} - a(\check{m}))\mathcal{R} \\ &= (a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m} + \check{m} - \hat{m})))\mathcal{R} \\ &\approx (a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})a'(\hat{m})))\mathcal{R} \\ (\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{< \mathbf{P}_\Gamma < 1}) &= (\underline{\psi}^{-1} - 1)\hat{a}\mathcal{R}\end{aligned}\tag{15}$$

The RHS of this equation is strictly positive because $\underline{\psi}^{-1} > 1$ and both \hat{a} and \mathcal{R} are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.¹

1.6.2 The m achieving Expected-Log-Balanced-Growth Is Smaller than Pseudo-Steady-State m

Expected log balanced growth occurs when

$$\begin{aligned}\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \Gamma \mathbf{m}_t \\ \mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \Gamma \mathbf{p}_t m_t \\ \mathbb{E}_t[\log \psi_{t+1} m_{t+1}] &= \log \Gamma m_t \\ \mathbb{E}_t[\log (a(m_t)\mathcal{R} + \psi_{t+1}\xi_{t+1}\Gamma)] &= \log \Gamma m_t \\ \mathbb{E}_t[\log (a(m_t)\mathcal{R} + \psi_{t+1}\xi_{t+1})] &= \log m_t\end{aligned}\tag{16}$$

and we call the m that satisfies this equation \tilde{m} .

Now we use the fact that the expectation of the log is less than the log of the

¹The use of the first order Taylor approximation could be substituted, clumsily, with the average of a' over the interval to remove the approximation in the derivations above.

expectation,

$$\begin{aligned}
\log \mathbb{E}_t[a(\tilde{m}_t)\mathcal{R} + \psi_{t+1}\xi_{t+1}] &< \log \tilde{m}_t \\
\log(a(\tilde{m}_t)\mathcal{R} + 1) &< \log \tilde{m}_t \\
a(\tilde{m}_t)\mathcal{R} + 1 &< \tilde{m}_t
\end{aligned} \tag{17}$$

Finally, subtract \check{m} from both sides,

$$\begin{aligned}
a(\tilde{m}_t)\mathcal{R} + 1 - (a(\check{m}_t)\mathcal{R} + 1) &< \tilde{m}_t - \check{m}_t \\
a(\tilde{m}_t) - a(\tilde{m}_t + \check{m} - \tilde{m}_t))\mathcal{R} &< \tilde{m}_t - \check{m}_t \\
a(\tilde{m}_t) - (a(\tilde{m}_t) + (\check{m} - \tilde{m}_t)a'(\tilde{m}))\mathcal{R} &< \tilde{m}_t - \check{m} \\
(\tilde{m}_t - \check{m})a'(\tilde{m})\mathcal{R} &< \tilde{m}_t - \check{m} \\
\underbrace{a'(\tilde{m})\mathcal{R}}_{< \mathbf{P}_\Gamma} &< 1
\end{aligned} \tag{18}$$

which again holds because $\mathbf{P}_\Gamma < 1$ (and, as above, a proof that does not require the Taylor approximation is available but more cumbersome).