

# 1 Existence of Concave $c$ Function

To show that (5) defines a sequence of continuously differentiable strictly increasing concave functions  $\{c_T, c_{T-1}, \dots, c_{T-k}\}$ , we start with a definition. We will say that a function  $n(z)$  is ‘nice’ if it satisfies

1.  $n(z)$  is well-defined iff  $z > 0$
2.  $n(z)$  is strictly increasing
3.  $n(z)$  is strictly concave
4.  $n(z)$  is  $\mathbf{C}^3$
5.  $n(z) < 0$
6.  $\lim_{z \downarrow 0} n(z) = -\infty$ .

(Notice that an implication of niceness is that  $\lim_{z \downarrow 0} n'(z) = \infty$ .)

Assume that some  $v_{t+1}$  is nice. Our objective is to show that this implies  $v_t$  is also nice; this is sufficient to establish that  $v_{t-n}$  is nice by induction for all  $n > 0$  because  $v_T(m) = u(m)$  and  $u(m) = m^{1-\rho}/(1-\rho)$  is nice by inspection.

Now define an end-of-period value function  $v_t(a)$  as

$$v_t(a) = \beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho} v_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})]. \quad (1)$$

Since there is a positive probability that  $\xi_{t+1}$  will attain its minimum of zero and since  $\mathcal{R}_{t+1} > 0$ , it is clear that  $\lim_{a \downarrow 0} v_t(a) = -\infty$  and  $\lim_{a \downarrow 0} v'_t(a) = \infty$ . So  $v_t(a)$  is well-defined iff  $a > 0$ ; it is similarly straightforward to show the other properties required for  $v_t(a)$  to be nice. (See Hiraguchi (2003).)

Next define  $\underline{v}_t(m, c)$  as

$$\underline{v}_t(m, c) = u(c) + v_t(m - c) \quad (2)$$

which is  $\mathbf{C}^3$  since  $v_t$  and  $u$  are both  $\mathbf{C}^3$ , and note that our problem’s value function defined in (5) can be written as

$$v_t(m) = \max_c \underline{v}_t(m, c). \quad (3)$$

$\underline{v}_t$  is well-defined if and only if  $0 < c < m$ . Furthermore,  $\lim_{c \downarrow 0} \underline{v}_t(m, c) = \lim_{c \uparrow m} \underline{v}_t(m, c) = -\infty$ ,  $\frac{\partial^2 \underline{v}_t(m, c)}{\partial c^2} < 0$ ,  $\lim_{c \downarrow 0} \frac{\partial \underline{v}_t(m, c)}{\partial c} = +\infty$ , and  $\lim_{c \uparrow m} \frac{\partial \underline{v}_t(m, c)}{\partial c} = -\infty$ . It follows that the  $c_t(m)$  defined by

$$c_t(m) = \arg \max_{0 < c < m} \underline{v}_t(m, c) \quad (4)$$

exists and is unique, and (5) has an internal solution that satisfies

$$u'(c_t(m)) = v'_t(m - c_t(m)). \quad (5)$$

Since both  $u$  and  $v_t$  are strictly concave, both  $c_t(m)$  and  $a_t(m) = m - c_t(m)$  are strictly increasing. Since both  $u$  and  $v_t$  are three times continuously differentiable, using (5) we can conclude that  $c_t(m)$  is continuously differentiable and

$$c'_t(m) = \frac{v''_t(a_t(m))}{u''(c_t(m)) + v''_t(a_t(m))}. \quad (6)$$

Similarly we can easily show that  $c_t(m)$  is twice continuously differentiable (as is  $a_t(m)$ ) (See Appendix 2.) This implies that  $v_t(m)$  is nice, since  $v_t(m) = u(c_t(m)) + v_t(a_t(m))$ .

## 2 $c_t(m)$ is Twice Continuously Differentiable

First we show that  $c_t(m)$  is  $\mathbf{C}^1$ . Define  $y$  as  $y \equiv m + dm$ . Since  $u'(c_t(y)) - u'(c_t(m)) = v'_t(a_t(y)) - v'_t(a_t(m))$  and  $\frac{a_t(y) - a_t(m)}{dm} = 1 - \frac{c_t(y) - c_t(m)}{dm}$ ,

$$\frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} = \left( \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} \right) \frac{c_t(y) - c_t(m)}{dm}$$

Since  $c_t$  and  $a_t$  are continuous and increasing,  $\lim_{dm \rightarrow +0} \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} < 0$  and  $\lim_{dm \rightarrow +0} \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$  are satisfied. Then  $\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$  for sufficiently small  $dm$ . Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)}}{\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)}}.$$

This implies that the right-derivative,  $c_t^{'+}(m)$  is well-defined and

$$c_t^{'+}(m) = \frac{v''_t(a_t(m))}{u''(c_t(m)) + v''_t(a_t(m))}.$$

Similarly we can show that  $c_t^{'+}(m) = c_t^{-'}(m)$ , which means  $c_t'(m)$  exists. Since  $v_t$  is  $\mathbf{C}^3$ ,  $c_t'(m)$  exists and is continuous.  $c_t'(m)$  is differentiable because  $v''_t$  is  $\mathbf{C}^1$ ,  $c_t(m)$  is  $\mathbf{C}^1$  and  $u''(c_t(m)) + v''_t(a_t(m)) < 0$ .  $c_t''(m)$  is given by

$$c_t''(m) = \frac{a'_t(m)v''_t(a_t) [u''(c_t) + v''_t(a_t)] - v''_t(a_t) [c'_t u'''(c_t) + a'_t v'''_t(a_t)]}{[u''(c_t) + v''_t(a_t)]^2}. \quad (7)$$

Since  $v''_t(a_t(m))$  is continuous,  $c_t''(m)$  is also continuous.

## 3 $\mathcal{T}$ Is a Contraction Mapping

We must show that our operator  $\mathcal{T}$  satisfies all of Boyd's conditions.

Boyd's operator  $\mathcal{T}$  maps from  $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$  to  $\mathcal{C}(\mathcal{A}, \mathcal{B})$ . A preliminary requirement is therefore that  $\{\mathcal{T}z\}$  be continuous for any  $F$ -bounded  $z$ ,  $\{\mathcal{T}z\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$ . This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\begin{aligned} \{\mathcal{T}x\}(m_t) & \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho} x(m_{t+1})] \right\} \\ \{\mathcal{T}y\}(m_t) & \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho} y(m_{t+1})] \right\}, \end{aligned}$$

so  $x(\bullet) \leq y(\bullet)$  implies  $\{\mathcal{T}x\}(m_t) \leq \{\mathcal{T}y\}(m_t)$  by inspection.<sup>1</sup>

Condition (2) requires that  $\{\mathcal{J}0\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ . By definition,

$$\{\mathcal{J}0\}(m_t) = \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ \left( \frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently  $u(\bar{\kappa}m_t)$ . Thus, condition (2) will hold if  $(\bar{\kappa}m_t)^{1-\rho}$  is  $F$ -bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \quad (8)$$

for some real scalar  $\eta > 0$  whose value will be determined in the course of the proof. Under this definition of  $F$ ,  $\{\mathcal{J}0\}(m_t) = u(\bar{\kappa}m_t)$  is clearly  $F$ -bounded.

Finally, we turn to condition (3),  $\{\mathcal{T}(z + \zeta F)\}(m_t) \leq \{\mathcal{T}z\}(m_t) + \zeta \alpha F(m_t)$ . The proof will be more compact if we define  $\check{c}$  and  $\check{a}$  as the consumption and assets functions<sup>2</sup> associated with  $\mathcal{T}z$  and  $\hat{c}$  and  $\hat{a}$  as the functions associated with  $\mathcal{T}(z + \zeta F)$ ; using this notation, condition (3) can be rewritten

$$u(\hat{c}) + \beta \{E(z + \zeta F)\}(\hat{a}) \leq u(\check{c}) + \beta \{Ez\}(\check{a}) + \zeta \alpha F.$$

Now note that if we force the  $\cup$  consumer to consume the amount that is optimal for the  $\wedge$  consumer, value for the  $\cup$  consumer must decline (at least weakly). That is,

$$u(\hat{c}) + \beta \{Ez\}(\hat{a}) \leq u(\check{c}) + \beta \{Ez\}(\check{a}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned} u(\hat{c}) + \beta \{E(z + \zeta F)\}(\hat{a}) &\leq u(\hat{c}) + \beta \{Ez\}(\hat{a}) + \zeta \alpha F \\ \beta \{E(z + \zeta F)\}(\hat{a}) &\leq \beta \{Ez\}(\hat{a}) + \zeta \alpha F \\ \beta \zeta \{EF\}(\hat{a}) &\leq \zeta \alpha F \\ \beta \{EF\}(\hat{a}) &\leq \alpha F \\ \beta \{EF\}(\hat{a}) &< F. \end{aligned}$$

where the last line follows because  $0 < \alpha < 1$  by assumption.<sup>3</sup>

Using  $F(m) = \eta + m^{1-\rho}$  and defining  $\hat{a}_t = \hat{a}(m_t)$ , this condition is

$$\beta \mathbb{E}_t[\Phi_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta \mathbb{E}_t \Phi_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing **PF-FVAC** (equation (20), which says  $\beth < 1$ ) can be rewritten as:

$$\eta > \frac{\beta \mathbb{E}_t[\Phi_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho}}{1 - \beth}. \quad (9)$$

But since  $\eta$  is an arbitrary constant that we can pick, the proof thus reduces to showing

<sup>1</sup>For a fixed  $m_t$ , recall that  $m_{t+1}$  is just a function of  $c_t$  and the stochastic shocks.

<sup>2</sup>Section 2.8 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

<sup>3</sup>The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (9) is bounded from above:

$$\begin{aligned}
& (1 - \wp)\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + \wp\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1})^{1-\rho}] - m_t^{1-\rho} \\
& \leq (1 - \wp)\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] \\
& \quad + \wp\beta R^{1-\rho}((1 - \bar{\kappa})m_t)^{1-\rho} - m_t^{1-\rho} \\
& = (1 - \wp)\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] \\
& \quad + m_t^{1-\rho} \left( \wp\beta R^{1-\rho} \left( \wp^{1/\rho} \frac{(R\beta)^{1/\rho}}{R} \right)^{1-\rho} - 1 \right) \\
& = (1 - \wp)\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + m_t^{1-\rho} \left( \underbrace{\wp^{1/\rho} \frac{(R\beta)^{1/\rho}}{R}}_{<1 \text{ by WRIC}} - 1 \right) \\
& < (1 - \wp)\beta \mathbb{E}_t [\Phi_{t+1}^{1-\rho}(\theta/(1 - \wp))^{1-\rho}] = \beth(1 - \wp)^\rho \theta^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (9) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1 - \wp)^\rho \theta^{1-\rho}}{1 - \beth} \quad (10)$$

which is a positive finite number under our assumptions.

The proof that  $\mathcal{T}$  defines a contraction mapping under the conditions (32) and (29) is now complete.

### 3.1 $\mathcal{T}$ and $v$

In defining our operator  $\mathcal{T}$  we made the restriction  $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$ . However, in the discussion of the consumption function bounds, we showed only (in (33)) that  $\underline{\kappa}_t m_t \leq c_t(m_t) \leq \bar{\kappa}_t m_t$ . (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (5) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace  $\bar{\kappa}$  with  $\bar{\kappa}_{T-1}$  and the **WRIC** must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (10). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
& \wp\beta(R(1 - \bar{\kappa}_{T-1}))^{1-\rho} < 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - \bar{\kappa}_{T-1}) > 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - (1 + \wp^{1/\rho}\mathbf{P}_R)^{-1}) > 1
\end{aligned}$$

where we have used (31) for  $\bar{\kappa}_{T-1}$  (and in the second step the reversal of the inequality occurs because we have assumed  $\rho > 1$  so that we are exponentiating both sides by the negative number  $1 - \rho$ ). To see that this is a weak condition, note that for small values

of  $\wp$  this expression can be further simplified using  $(1 + \wp^{1/\rho} \mathbf{P}_R)^{-1} \approx 1 - \wp^{1/\rho} \mathbf{P}_R$  so that it becomes

$$\begin{aligned} (\wp\beta)^{1/(1-\rho)} \wp^{1/\rho} \mathbf{P}_R &> 1 \\ (\wp\beta) \wp^{(1-\rho)/\rho} \mathbf{P}_R^{1-\rho} &< 1 \\ \beta \wp^{1/\rho} \mathbf{P}_R^{1-\rho} &< 1. \end{aligned}$$

Calling the weak return patience factor  $\mathbf{P}_R^\wp = \wp^{1/\rho} \mathbf{P}_R$  and recalling that the **WRIC** was  $\mathbf{P}_R^\wp < 1$ , the expression on the LHS above is  $\beta \mathbf{P}_R^{-\rho}$  times the WRPFacDefn. Since we usually assume  $\beta$  not far below 1 and parameter values such that  $\mathbf{P}_R \approx 1$ , this condition is clearly not very different from the **WRIC**.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique  $v(m)$ . But since  $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n} = \underline{\kappa}$  and  $\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa}$ , it must be the case that the  $v(m)$  toward which these  $v_{T-n}$ 's are converging is the *same*  $v(m)$  that was the endpoint of the contraction defined by our operator  $\mathcal{T}$ . Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (5) converge, they converge to the same unique  $v$  defined by  $\mathcal{T}$ .<sup>4</sup>

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<sup>4</sup>It seems likely that convergence of the value functions for the original problem could be proven even if only the **WRIC** were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

## References

HIRAGUCHI, RYOJI (2003): “On the Convergence of Consumption Rules,” *Manuscript*,  
*Johns Hopkins University*.