Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds foundations for rigorous and intuitive understanding of 'buffer stock' saving models (Bewley (1977)-like models with a wealth target), pairing each theoretical result with quantitative illustrations. After describing conditions under which a consumption function exists, the paper articulates stricter 'Growth Impatience' conditions that guarantee alternative forms of stability — either at the population level, or for individual consumers. Together, the numerical tools and analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem

JEL codes D81, D91, E21

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The paper's results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper's own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to Will Du, James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at the Johns Hopkins University, a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights, and at a presentation at the Australian National University.

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1 Introduction

In the presence of realistic transitory and permanent shocks to income *a la* Friedman (1957) and Muth (1960), only one further ingredient is required to construct a microeconomically testable model of consumption: A description of preferences. Zeldes (1989) was the first to calibrate a quantitatively plausible example; his paper spawned a literature showing that such models' predictions can match household life cycle data reasonably well, whether or not explicit liquidity constraints are imposed.[1]

A connected literature in macroeconomic theory, starting with Bewley (1977), has derived limiting properties of related infinite-horizon problems – but only in models more complex than the case with just shocks and preferences, because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et al. (1989)) cannot be applied when utility and/or marginal utility are unbounded. Many proof methods also rule out permanent shocks *a la* Friedman (1957) and Muth (1960) (and Zeldes).[2]

This paper's first contribution is to articulate conditions under which the infinite-horizon Friedman-Muth(-Zeldes) problem (without complications like a consumption floor or liquidity constraints) defines a contraction mapping whose limit is useful (neither zero nor ∞) as the horizon recedes. A 'Finite Value of Autarky Condition' is mostly sufficient (the other requirement, the 'Weak Return Impatience Condition',[3] is unlikely to bind). Because the infinite-horizon solution is the limit of finite-horizon recursions, many intermediate results are also useful for solving finite-horizon problems.

But the main theoretical contribution is to identify, for the infinite-horizon case, conditions under which 'stable' points exist (wealth can be predicted to move toward a 'target'; alternatively, there is a 'balanced growth' equilibrium) either for individual consumers or for the aggregate. The requirement for stability is always that the model's parameters satisfy a 'Growth Impatience Condition' whose details depend on the quantity whose stability is of interest. A model with stable points qualifies as a 'buffer stock' model.[4]

Even without a formal proof of its existence, buffer stock saving has been intuitively understood to underlie central quantitative results in heterogeneous agent macroeconomics; for example, the logic of target saving is central to the claim by Krueger, Mitman, and Perri (2016) in the *Handbook of Macroeconomics* that such models explain why, during the Great Recession, middle-class consumers cut their spending more than the poor or the rich. The theory below provides the rigorous basis for this claim: Learning that the future has become more uncertain does not change the urgent imperatives of

[1]See Carroll (1997) or Gourinchas and Parker (2002) for arguments that models with only 'natural' constraints (see below) match a wide variety of facts; for a model with explicit constraints that produces very similar results, see, e.g. Cagetti (2003).

[2]See the fuller discussion at the end of Section 2.1.

[3]This is a generalization of a condition in Ma, Stachurski, and Toda (2020).

[4]Buffer stock models are neither a subset nor a superset of Bewley (1977) models. But closed economies in which capital accumulation is limited by declining marginal productivity are always 'buffer stock' economies, because the declining marginal product of capital guarantees that a Growth Impatience Condition will hold in equilibrium (see below). The less obvious applications are to populations (or economies) whose marginal saving behavior does not determine the relevant interest rate, or in which the marginal product of capital does not fall as capital is accumulated (again, see below).

the poor (their high $u'(c)$ means they — optimally — have little room to maneuver). And, increased labor income uncertainty does not much change the behavior of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to respond to uncertainty by substantially reducing their spending.

Analytical derivations required for the proofs also explain many other results familiar from the numerical literature.

The paper begins by describing sufficient conditions for the problem to define a sensible (nondegenerate) limiting consumption function (and explains how the model relates to those previously considered). The conditions are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. This analysis establishes limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; using these limits, the contraction mapping theorem is proven.

The next theoretical contribution demonstrates that a model with an 'artificial' liquidity constraint (it prohibits borrowing by consumers who could certainly repay) is a limiting case of the unconstrained model. The analytical appeal of the unconstrained model is that it is both mathematically convenient (e.g., the consumption function is twice continuously differentiable), and arbitrarily close (cf. Section 2.11) to less tractable models. This congenial environment makes proofs easier (if we define a proposition as holding (in the limit) if it continues to hold as the horizon extends to infinity).

In proving the remaining theorems, the next section examines key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Next, the central theorems articulate conditions under which different measures of 'growth impatience' imply useful conclusions about points of stability ('target' or 'balanced growth' points).

The final section elaborates the conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, a small open economy populated by buffer stock consumers has a balanced growth equilibrium in which growth rates of consumption, income, and wealth match the exogenous growth rate of permanent income (equivalent, here, to productivity growth). In the terms of Schmitt-Grohé and Uribe (2003), buffer stock saving is an appealing method of 'closing' a small open economy model, because it requires no ad-hoc assumptions. Not even liquidity constraints.[5]

[5]The paper's insights are instantiated in the Econ-ARK toolkit, whose buffer stock saving module flags parametric choices under which a problem is degenerate or under which stable ratios of wealth to income may not exist.

2 The Problem

2.1 Setup

The infinite horizon solution is the (limiting) first-period solution to a sequence of finite-horizon problems as the horizon (the last period of life) becomes arbitrarily distant.

That is, for the value function, fixing a terminal date T, we are interested in $\mathbf{v}_{T-n}$ in the sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, \ldots, \mathbf{v}_{T-n}\}$. We will say that the problem has a 'nondegenerate' infinite horizon solution if, corresponding to that $\mathbf{v}$, as $n \uparrow \infty$ there is a limiting consumption function $\mathrm{c}(m) = \lim_{n\uparrow\infty} \mathrm{c}_{T-n}$ which is neither $\mathrm{c}(m) = 0$ everywhere (for all m) nor $\mathrm{c}(m) = \infty$ everywhere (a 'useful' solution).

Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max\ \mathbb{E}_t \left[\sum_{i=0}^{n} \beta^i \mathrm{u}(\mathbf{c}_{t+i}) \right]$$

where the Constant Relative Risk Aversion (CRRA) utility function

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$$

exhibits relative risk aversion $\rho > 1$.[6] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ and permanent noncapital income $\mathbf{p}_t$, which both are positive,

$$\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty), \tag{1}$$

and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{2}$$

In the usual exposition, a dynamic budget constraint (DBC) combines several distinct events that jointly determine next period's $\mathbf{m}$ (given this period's choices); for the detailed analysis here, it will be useful to disarticulate and describe every separate step:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{k}_{t+1} &= \mathbf{a}_t \\
\mathbf{b}_{t+1} &= \mathbf{k}_{t+1}\mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma\psi_{t+1}}_{\equiv\Gamma_{t+1}} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1}\xi_{t+1},
\end{aligned}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which translate one-for-one into capital $\mathbf{k}_{t+1}$ at the beginning of the next period, which (before the consumption choice) grows by a fixed interest factor $\mathsf{R} = (1+\mathsf{r})$, so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[7] $\mathbf{m}_{t+1}$ ('market resources') is

[6] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is omitted because it would be cumbersome.

[7] Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \;\forall\; n \geq 1$). Permanent noncapital income in $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1 \;\forall\; n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).

Following Zeldes (1989), in future periods $t+n \;\forall\; n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{3}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \;\forall\; n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.[8] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (3) and $\mathcal{F}_\theta$). For quick identification in tables and graphs, we will call this the 'Friedman/Muth' model because it is a specific implementation of Friedman (1957)'s ideas as interpreted by Muth (1960).

The model looks more special than it is. In particular, a positive probability of zero-income events may seem objectionable (despite empirical support).[9] But a nonzero minimum value of ξ (motivated, say, by the existence of unemployment insurance) could be handled by capitalizing the PDV of minimum income into current market assets,[10] transforming that model back into this one. And no key results would change if the transitory shocks were persistent but mean-reverting, instead of IID. Also, the assumption of a positive point mass for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.

2.2 Comparison to Existing Literature

This model differs from Bewley's (1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that $\mathrm{u}(0)$ is well-defined, or that $\mathrm{u}'(0)$ is well-defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds to be of dominant importance in microdata.[11] It differs from Deaton (1991) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income to reach zero.

[8] Rabault (2002) and Li and Stachurski (2014) analyze cases where the shock processes have unbounded support.

[9] We calibrate this probability to 0.005 to match data from the Panel Study of Income Dynamics (Carroll (1992)).

[10] So long as unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in Section 2.12.

[11] MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000). Much of the literature instead incorporates highly 'persistent' but not completely permanent shocks, but Daly, Hryshko, and Manovskii (2016) show that when measurement problems are handled correctly data yields serial correlation coefficients $0.98 - 1.00$; and Hryshko and Manovskii (2020) suggests that survey data support the same conclusion.

It differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[12,13] Li and Stachurski (2014) show how to allow unbounded returns by using policy function iteration, but also impose constraints.

The paper with perhaps the most commonalities is by Ma, Stachurski, and Toda (2020), henceforth MST, who establish the existence and uniqueness of a solution to a general income fluctuation problem in a Markovian setting. The most important differences are that MST impose liquidity constraints, assume that $\mathrm{u}'(0) = 0$, and that expected marginal utility of income is finite ($\mathbb{E}[\mathrm{u}'(Y)] < \infty$). These assumptions are not consistent with the combination of CRRA utility and income dynamics used here, whose joint properties are key to the results.[14]

2.3 The Problem Can Be Normalized By Permanent Income

We establish a bit more notation by reviewing the familiar result that in such problems (CRRA utility, permanent shocks) the number of states can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Value in the last period is $\mathrm{u}(\mathbf{m}_T)$; using (in the last line in (4)) the fact that for our CRRA utility function, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, and generically defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with $m = \mathbf{m}/\mathbf{p}$), consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{c_{T-1}} \; \mathrm{u}(\mathbf{p}_{T-1}c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho}\left\{\max_{c_{T-1}} \; \mathrm{u}(c_{T-1}) + \beta\, \mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)]\right\}.
\end{aligned} \tag{4}$$

Now, in a one-time deviation from the notational convention established in the last sentence, define nonbold ‘normalized value’ not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \; \mathrm{u}(\mathrm{c}_t) + \beta\, \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
k_{t+1} &= a_t/\Gamma_{t+1} \\
b_{t+1} &= k_{t+1}\mathsf{R} = (\mathsf{R}/\Gamma_{t+1})a_t \;\; = \;\; \mathcal{R}_{t+1}a_t \\
m_{t+1} &= b_{t+1} + \xi_{t+1},
\end{aligned} \tag{5}$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a ‘permanent-income-growth-normalized’ return factor, and

[12] Similar restrictions are made in the well known papers by Scheinkman and Weiss (1986), Clarida (1987), and Chamberlain and Wilson (2000). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

[13] Alvarez and Stokey (1998) relaxed the bounds on the return function, but they address only the deterministic case.

[14] The incorporation of permanent shocks rules out application of the tools of Matkowski and Nowak (2011), who followed and corrected an error in the fundamental work on the local contraction mapping method developed in Rincón-Zapatero and Rodríguez-Palmero (2003). Martins-da Rocha and Vailakis (2010) provide a correction to Rincón-Zapatero and Rodríguez-Palmero (2003), that works under easier conditions to verify, but only addresses the deterministic case.

the reformulated problem's first order condition is[15]

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho}c_{t+1}^{-\rho}]. \tag{6}$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (5), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem (5), we will have solutions to the original problem for any $t < T$ from:

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t),$$
$$\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t\mathrm{c}_t(m_t).$$

2.4 Definition of a Nondegenerate Solution

The problem has a nondegenerate solution if as the horizon n gets arbitrarily large the solution in the first period of life $\mathrm{c}_{T-n}(m)$ gets arbitrarily close to a limiting $\mathrm{c}(m)$:

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m) \tag{7}$$

that satisfies

$$0 < \mathrm{c}(m) < \infty \tag{8}$$

for every $0 < m < \infty$.

2.5 Perfect Foresight Benchmarks

The familiar analytical solution to the perfect foresight model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

2.5.1 The Finite Human Wealth Condition (FHWC)

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (2) imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{9}$$

where $\mathbf{b}$ is nonhuman wealth, and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$ 'human wealth' is

$$\mathbf{h}_t = \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + \cdots + \mathcal{R}^{t-T}\mathbf{p}_t$$

[15] Leaving aside their assumptions about the marginal utility function and liquidity constraints, it is tempting to view this as a special case of the model of MST, with our $\mathcal{R}_{t+1} = \mathsf{R}/\Gamma_{t+1}$ (defined below equation (5)) corresponding to their stochastic rate of return on capital and the VAF $\beta\Gamma_{t+1}^{1-\rho}$ defined below (31) corresponding to their stochastic discount factor. A caveat is that, here, $\mathcal{R}_{t+1}$ and the modified discount factor are intimately connected (through Γ_{t+1}), which has profound effects.

$$= \underbrace{\left(\frac{1-\mathcal{R}^{-(T-t+1)}}{1-\mathcal{R}^{-1}}\right)}_{\equiv h_t} \mathbf{p}_t. \tag{10}$$

For $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, need the Finite Human Wealth Condition ('FHWC'):

$$\text{FHWC:} \quad \underbrace{\Gamma/\mathsf{R}}_{\equiv\mathcal{R}^{-1}} < 1 \qquad . \tag{11}$$

Intuitively, finite human wealth requires a growth rate of (noncapital) income smaller than the interest rate at which that income is being discounted.

2.5.2 The Return Impatience Condition (RIC)

Without constraints, the consumption Euler equation always holds; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$\mathbf{c}_{t+1}/\mathbf{c}_t = \underbrace{(\mathsf{R}\beta)^{1/\rho}}_{\equiv\text{Þ}} \tag{12}$$

where the archaic letter 'thorn' represents what we will call the 'Absolute Patience Factor' (APF) because, if the 'absolute impatience condition' (AIC) holds,[16]

$$\text{AIC:} \quad \text{Þ} < 1, \tag{13}$$

the consumer's level of spending will be too large to sustain indefinitely. We call such a consumer 'absolutely impatient.'

A 'Return Patience Factor' (RPF) relates absolute patience to the return factor:

$$\text{RPF:} \quad \text{Þ}_{\mathsf{R}} \equiv \text{Þ}/\mathsf{R} \tag{14}$$

and since consumption is growing by Þ but discounted by R:

$$\mathrm{PDV}_t(\mathbf{c}) = \left(\frac{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}{1-\text{Þ}_{\mathsf{R}}}\right)\mathbf{c}_t \tag{15}$$

from which the IBC (9) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv\underline{\kappa}_t}(\mathbf{b}_t + \mathbf{h}_t) \tag{16}$$

defining a normalized finite-horizon perfect foresight consumption function:

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n} - 1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{T-n} \tag{17}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) — it answers the question 'if the consumer had an extra unit of resources, how much more spending would occur?' ($\bar{\mathrm{c}}$'s

[16] Impatience conditions have figured in intertemporal optimization problems since the beginning, e.g. in Ramsey (1928). These issues are so central that it would be hopeless to attempt to cite conditions in every other paper that correspond to conditions named and briefly exposited here. I make no claim to novelty for any condition aside from those implicated in my theorems, whose forerunners *will* be articulated.

overbar signfies that $\bar{c}$ will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is a lower bound for the MPC).

The denominator of (16) is the reason that, for $\underline{\kappa}$ to be strictly positive as $n = T - t$ goes to infinity, we must impose the Return Impatience Condition (RIC):

$$\text{RIC:}\quad \text{Þ}_{\mathsf{R}} < 1, \tag{18}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{Þ}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{19}$$

The RIC thus implies that the consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{c}(m) = 0$). We call a consumer who satisfies the RIC 'return impatient.'

2.5.3 Nondegenerate PF Unconstrained Solution Requires FHWC and RIC

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting upper bound consumption function will be

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \tag{20}$$

and so in order to rule out the degenerate limiting solution $\bar{c}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (11).

2.5.4 The Growth Impatience Condition (GIC)

By analogy to the RPF, we define a 'growth patience factor' (GPF) as

$$\text{GPF:}\qquad \text{Þ}_{\Gamma} = \text{Þ}/\Gamma, \tag{21}$$

and define a 'growth impatience condition' (GIC)

$$\text{GIC:}\qquad \text{Þ}_{\Gamma} < 1 \tag{22}$$

which is equivalent to (25) (exponentiate both sides by $1/\rho$).

2.5.5 Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)

Under 'autarky,' capital markets do not exist; the consumer has no choice but to spend permanent noncapital income $\mathbf{p}$ in every period. Because $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, the value the consumer would achieve is

$$\begin{aligned}\mathbf{v}_t^{\text{autarky}} &= \mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma) + \beta^2\mathrm{u}(\mathbf{p}_t\Gamma^2) + \ldots \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1 - (\beta\Gamma^{1-\rho})^{T-t+1}}{1 - \beta\Gamma^{1-\rho}}\right)\end{aligned}$$

which (for $\Gamma > 0$) asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if any of these equivalent conditions holds:

$$
\begin{aligned}
\text{PF-FVAC:}\quad \overbrace{\beta\Gamma^{1-\rho}}^{\equiv\beth} &< 1 \\
\beta\mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma \equiv \mathcal{R} \\
\text{Þ}_{\mathsf{R}} &< \mathcal{R}^{1-1/\rho},
\end{aligned}
$$

where we call $\beth$[17] the 'Perfect Foresight Value Of Autarky Factor' (PF-VAF), and the variants of (23) constitute alternative versions of the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC; they guarantee that a perfect-foresight consumer who always spends all permanent income 'has finite autarky value.'[18]

If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied.[19] Likewise, if the FHWC and the GIC are both satisfied, PF-FVAC follows:

$$
\begin{aligned}
\text{Þ} &< \Gamma < \mathsf{R} \\
\text{Þ}_{\mathsf{R}} &< \Gamma/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} < 1
\end{aligned}
\tag{24}
$$

(the last line holds because FHWC $\Rightarrow 0 \leq (\Gamma/\mathsf{R}) < 1$ and $\rho > 1 \Rightarrow 0 < 1 - 1/\rho < 1$).

The first panel of Table **??** summarizes: The PF-Unconstrained model has a nondegenerate limiting solution if we impose the RIC and FHWC (these conditions are necessary as well as sufficient). Together the PF-FVAC and the FHWC imply the RIC. If we impose the GIC and the FHWC, both the PF-FVAC and the RIC follow, so GIC+FHWC are also sufficient. But there are circumstances under which the RIC and FHWC can hold while the PF-FVAC fails ('~~PF-FVAC~~'). For example, if $\Gamma = 0$, the problem is a standard 'cake-eating' problem with a nondegenerate solution under the RIC (when the consumer has access to capital markets).

More useful than this prose or a table is the diagrammatic relation of the conditions shown in Figure **??**. Each node represents a quantity considered in the foregoing analysis. The arrow associated with each inequality reflects imposition of that condition. For example, one way we wrote the PF-FVAC in equation (23) is $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, so imposition of the PF-FVAC is captured by the diagonal arrow connecting Þ and $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$. Traversing the boundary of the diagram clockwise starting at Þ involves imposing first the GIC then the FHWC, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply the PF-FVAC. Reversal of a condition reverses the arrow's direction; so, for example, the bottom-most arrow going to $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ imposes ~~FHWC~~; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram in a clockwise direction from Þ through Γ to $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ to R, revealing that imposition of GIC and FHWC (and, redundantly, FHWC again) let us conclude that the RIC holds because the starting point is Þ and

[17] This is another kind of discount factor, so we use the Hebrew 'bet' which is a cognate of the Greek 'beta'.

[18] This is related to the key impatience condition in Alvarez and Stokey (1998).

[19] Divide both sides of the second inequality in (23) by R:

$$\text{Þ}/\mathsf{R} < (\Gamma/\mathsf{R})^{1-1/\rho} \tag{23}$$

and FHWC $\Rightarrow$ the RHS is < 1 because $(\Gamma/\mathsf{R}) < 1$ (and the RHS is raised to a positive power (because $\rho > 1$)).

the endpoint is R. (Consult Appendix K for a detailed exposition of diagrams of this type).

$$\begin{tikzcd}
\text{Þ} \arrow[r, "\text{GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] & \Gamma \arrow[d, "\text{FHWC}"] \arrow[dl, phantom, "\neq" near start] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] \arrow[ur, phantom, "\neq" near start] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 1 Perfect Foresight Relation of GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is one way of writing the PF-FVAC, equation (23)

2.5.6 PF Constrained Solution Exists Under RIC or {~~RIC~~, GIC}

We next sketch the perfect foresight constrained solution because it is a useful benchmark (and limit) for the unconstrained problem with uncertainty (our ultimate interest).

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound, $b_t = 0$ (if it were relevant at any higher point, it would certainly be relevant here). The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ it is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (6):

$$1^{-\rho} > \mathsf{R}\beta\Gamma^{-\rho}1^{-\rho}, \tag{25}$$

which is just a restatement of the GIC.

We now examine implications of possible configurations of the conditions.

~~GIC~~ and RIC. If the GIC fails but the RIC (18) holds, Appendix A shows that, for some $0 < m_\# < 1$, an unconstrained consumer behaving according to (20) would choose $c < m$ for all $m > m_\#$. In this case the solution to the constrained consumer's problem is simple: For any $m \geq m_\#$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one.

If the consumer were somehow[20] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using $\grave{\bullet}$ for the version of a function $\bullet$ in the presence of constraints (and recalling that $\bar{\mathrm{c}}(m)$ is the unconstrained perfect foresight solution):

$$\grave{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases}$$

GIC and RIC. More useful is the case where the return impatience and GIC conditions both hold. In this case Appendix A shows that the limiting constrained consumption function is piecewise linear, with $\grave{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m^1_{\#} > 1$, and with discrete declines in the MPC at a set of kink points $\{m^1_{\#}, m^2_{\#}, \ldots\}$. As $m \uparrow \infty$ the constrained consumption function $\grave{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\grave{\boldsymbol{\kappa}}(m) \equiv \grave{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$.[21] Similarly, the value function $\grave{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer.

This logic holds even when the finite human wealth condition fails (~~FHWC~~), because the constraint prevents the (limiting) consumer[22] from borrowing against unbounded human wealth to finance unbounded current consumption. Under these circumstances, the consumer who starts with any $b_t > 0$ will, over time, run those resources down so that by some finite number of periods τ in the future the consumer will reach $b_{t+\tau} = 0$, and thereafter will set $\mathbf{c} = \mathbf{p}$ for eternity (which the PF-FVAC says yields finite value). Using the same steps as for equation (23), value of the interim program is also finite:

$$\mathbf{v}_{t+\tau} = \Gamma^{\tau(1-\rho)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-(t+\tau)+1}}{1-\beta\Gamma^{1-\rho}}\right).$$

So, even under ~~FHWC~~, the limiting consumer's value for any finite m will be the sum of two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+\tau} = 0$, and the finite component due to the value of consuming all $\mathbf{p}_{t+\tau}$ thereafter.

GIC and ~~RIC~~. The most peculiar possibility occurs when the RIC fails. Under these circumstances the FHWC must also fail (Appendix A), and the constrained consumption function is nondegenerate. (See appendix Figure 8 for a numerical example). While $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\grave{\mathrm{c}}(m)$ is finite, strictly positive, and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where ~~RIC~~ would suggest a degenerate limit of $\grave{\mathrm{c}}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\grave{\mathrm{c}}(m) = \infty$.

Tables 3 and **??** (and appendix table 5) codify.

We now examine the case with uncertainty but without constraints, which will turn out to be a close parallel to the model with constraints but without uncertainty.

---

[20] "Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

[21] See Carroll, Holm, and Kimball (2019) for details.

[22] That is, one obeying $\mathrm{c}(m) = \lim_{n\uparrow\infty} \mathrm{c}_{T-n}(m)$.

2.6 Uncertainty-Modified Conditions

2.6.1 A Normalized Growth Impatience Condition (GIC-Nrm)

When uncertainty is introduced, the expectation of beginning-of-period bank balances b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})] = a_t(\mathsf{R}/\Gamma)\,\mathbb{E}_t[\psi_{t+1}^{-1}]$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is greater than one. Now define

$$\underline{\psi} \equiv \left(\mathbb{E}[\psi^{-1}]\right)^{-1} \tag{26}$$

which satisfies $\underline{\psi} < 1$ (thanks again to Mr. Jensen), so it is convenient to define

$$\underline{\Gamma} \equiv \Gamma\underline{\psi} < \Gamma$$

because this allows us to write uncertainty-adjusted versions of equations and conditions in a manner exactly parallel to those for the perfect foresight case; for example, we define a normalized Growth Patience Factor (GPF-Nrm):

$$\text{GPF-Nrm:}\;\; \text{Þ}_{\underline{\Gamma}} = \text{Þ}/\underline{\Gamma} = \mathbb{E}[\text{Þ}/(\Gamma\psi)] \tag{27}$$

and a normalized version of the Growth Impatience Condition:

$$\text{GIC-Nrm:}\;\; \text{Þ}_{\underline{\Gamma}} < 1, \tag{28}$$

that is stronger than the perfect foresight version (22) because $\underline{\Gamma} < \Gamma$.

2.6.2 The Finite Value of Autarky Condition (FVAC)

Analogously to (23), value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned}\mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + \cdots + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}\ldots\Gamma_T)\right] \\ &= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-\left(\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]\right)^{T-t+1}}{1-\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right),\end{aligned}$$

suggesting the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = \left(\mathbb{E}[\psi^{1-\rho}]\right)^{1/(1-\rho)}, \tag{29}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ. Defining

$$\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}}, \tag{30}$$

$\mathbf{v}_t$ will be positive and finite as T approaches ∞ if

$$\begin{aligned}\text{FVAC:}\;\; 0 < \overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\beth}}\;=\;\text{VAF}} &< 1 \\ 0 < \beta &< \underline{\underline{\Gamma}}^{\rho-1}.\end{aligned} \tag{31}$$

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

We call (31) the 'finite value of autarky condition' because it guarantees that value is finite for a consumer who always consumes their (now stochastic) permanent income (and $\underline{\underline{\beth}}$ is the 'Value of Autarky Factor' (or 'VAF')).[23] For nondegenerate ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (23) because $\underline{\underline{\Gamma}} < \Gamma$.[24]

2.7 The Baseline Numerical Solution

Figure 2, familiar from the literature, depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and earlier periods under baseline parameter values listed in Table 2. (The 45 degree line is $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge to a nondegenerate $\mathrm{c}(m)$. Our next purpose is to show that this appearance is not deceptive.

2.8 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem defines a sequence of continuously differentiable strictly increasing strictly concave[25] functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, \ldots\}$.

[23] In a stationary environment — that is, with $\underline{\underline{\Gamma}} = 1$ — this corresponds to an impatience condition imposed by Ma, Stachurski, and Toda (2020); but their remaining conditions do not correspond to those here, because their problem differs and their method of proof differs.

[24] Rewrite (31) as

$$\begin{aligned}
\beta\mathsf{R} &< \mathsf{R}\underline{\underline{\Gamma}}^{\rho-1} \\
(\beta\mathsf{R})^{1/\rho} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
\text{Þ}_\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho}
\end{aligned}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

[25] With one obvious exception: $\mathrm{c}_T(m)$ is linear (and so only weakly concave).

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Factor	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Factor	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	Þ	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	Þ_{R}	$\equiv$	$\text{Þ}/\mathsf{R}$	0.961
Growth Patience Factor	Þ_{Γ}	$\equiv$	$\text{Þ}/\Gamma$	0.970
Normalized Growth Patience Factor	$\text{Þ}_{\underline{\Gamma}}$	$\equiv$	$\text{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Factor	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941
Weak Impatience Factor	$\wp^{1/\rho}\text{Þ}$	$\equiv$	$(\wp\beta\mathsf{R})^{1/\rho}$	0.071

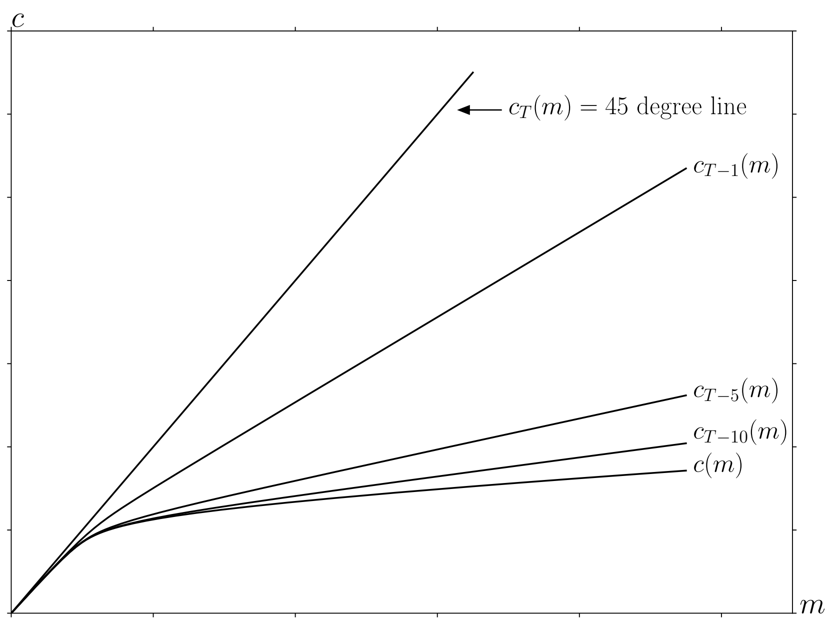

Figure 2 Convergence of the Consumption Rules

The straightforward but tedious proof is relegated to Appendix B. For present purposes, the most important point is that the income process induces what Aiyagari (1994) dubbed a 'natural borrowing constraint': $\mathrm{c}_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero income over the remaining horizon, risking the possibility of a requirement to spend zero, yielding negative infinite utility. To avoid this calamity, the consumer never spends everything. Zeldes (1989) seems to have been the first to argue, based on his numerical results, that the natural borrowing constraint was a quantitatively plausible alternative to 'artificial' or 'ad hoc' borrowing constraints.[26]

Strict concavity and continuous differentiability of the consumption function are key elements in many of the arguments below, but are not characteristics of models with 'artificial' borrowing constraints (we will see below that the analytical convenience of these features is considerable).

2.9 Bounds for the Consumption Functions

The consumption functions depicted in Figure 2 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which will be needed in the contraction mapping proof.[27]

Assume (as justified above) that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathrm{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion.)

Under our imposed assumption that human wealth is finite, the MPC bound as wealth approaches infinity is easy to understand: As the *proportion* of consumption that will be financed out of human wealth approaches zero, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero. In the course of proving this, Appendix G provides a useful recursive expression (used below) for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+1}^{-1}. \tag{32}$$

2.9.1 The Weak RIC Condition (WRIC)

Appendix equation (85) presents a parallel expression for the limiting maximal MPC as $m_t \downarrow 0$:

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho} \text{Þ}_{\mathsf{R}} \bar{\kappa}_{t+1}^{-1} \tag{33}$$

[26]The same (numerical) point applies for infinite horizon models (calibrated to actual empirical data on household income dynamics); cf. Carroll (1992).

[27]Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; Ma and Toda (2020) show that these results generalize to the limits derived here if capital income is added to the model.

where $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ satisfies:

$$\text{WRIC:}\ \ 1 \geq \wp^{1/\rho}\text{Þ}_{\mathsf{R}}, \tag{34}$$

a condition we dub the 'Weak Return Impatience Condition' because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$). The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathrm{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{35}$$

because consumption starts at zero and is continuously differentiable, is strictly concave,[28] and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is in Appendix D).

2.10 The Problem Is a Contraction Mapping Under WRIC and FVAC

As mentioned above, standard theorems in the contraction mapping literature following Stokey et. al. (1989) require utility or marginal utility to be bounded over the space of possible values of m, which does not hold here because the possibility (however unlikely) of an unbroken string of zero-income events through the end of the horizon means that utility (and marginal utility) are unbounded as $m \downarrow 0$. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., Matkowski and Nowak (2011)), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that incorporates permanent shocks.[29]

Fortunately, Boyd (1990) provided a weighted contraction mapping theorem that Alvarez and Stokey (1998) showed could be used to address the homogeneous case (of which CRRA is an example) in a deterministic framework; later, Durán (2003) showed how to extend the Boyd (1990) approach to the stochastic case.

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A},\mathcal{B})$ where $\mathcal{C}(\mathcal{A},\mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $F \in \mathcal{C}(\mathcal{A},\mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then $\bullet$ is F-bounded if the F-norm of $\bullet$,*

$$\| \bullet \|_F = \sup_m \left[\frac{|\bullet(m)|}{F(m)}\right], \tag{36}$$

is finite.

For $\mathcal{C}_F(\mathcal{A},\mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A},\mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

[28] Carroll and Kimball (1996)

[29] See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

Boyd's Weighted Contraction Mapping Theorem. *Let* $\mathsf{T} : \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$ *such that*[30,31]

$$
\begin{aligned}
&(1)\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \le \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \le \{\mathsf{T}\mathrm{y}\} \\
&(2)\{\mathsf{T}\mathbf{0}\} \in \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \\
&(3)\textit{There exists some real } 0 < \alpha < 1 \textit{ such that} \\
&\quad \{\mathsf{T}(\mathrm{w} + \zeta F)\} \le \{\mathsf{T}\mathrm{w}\} + \zeta\alpha F \quad \textit{holds for all real } \zeta > 0 \ .
\end{aligned}
$$

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{>0}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t) = \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1} + \xi_{t+1})\right].$$

Using this, we introduce the mapping $\mathcal{T} : \mathcal{C}_{F}(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$.[32]

We can show that our operator $\mathcal{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first is the WRIC necessary for convergence of the maximal MPC, equation (34) above. More serious is the Finite Value of Autarky Condition, equation (31). (We discuss the interpretation of these restrictions in detail in Section 2.12 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. *$\mathcal{T}$ is a contraction mapping if the weak return impatience condition* (34) *and the finite value of autarky condition* (31) *hold.*

Intuitively, Boyd's theorem shows that if you can find a F that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with F, the normalized problem defines a contraction mapping. The intuition for the FVAC condition is that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly the sustainable amount (say, by consuming $(\mathsf{r}/\mathsf{R})b_0 - \epsilon$ for some small $\epsilon > 0$).

The cumbersome details of the proof are relegated to Appendix D. Given that the value function converges, Appendix E.2 shows that the consumption functions converge.[33]

2.11 The Liquidity Constrained Solution as a Limit

This section explains why a related problem commonly considered in the literature (e.g., by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The 'related' problem makes two changes to the problem defined above:

---

[30] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[31] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \le \mathrm{y}$ is short for $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)\ \forall \bullet \in \mathcal{A}$. In this notation, $\zeta\alpha F$ in (3) is a *function* which can be applied to any argument $\bullet$ (because F is a function).

[32] Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous F-bounded functions z) and the compactness of $[\underline{\kappa} m_t, \bar{\kappa} m_t]$.

[33] MST's proof is for convergence of the consumption policy function directly, rather than of the value function, which is why their conditions are on u', which governs behavior.

1. An ‘artificial’ liquidity constraint is imposed: $a_t \geq 0$
2. The probability of zero-income events is zero: $\wp = 0$

The essence of the argument is simple. Imposing the artificial constraint without changing $\wp$ would not change behavior at all: The possibility of earning zero income over the remaining horizon already prevents the consumer from ending the period with zero assets. So, for precautionary reasons, the consumer will save something.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the *probability* that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the *amount* of precautionary saving induced by the zero-income events approaches zero as $\wp \downarrow 0$. But “zero” is the amount of precautionary saving that would be induced by a zero-probability event for the impatient liquidity constrained consumer.

Another way to understand this is just to think of the liquidity constraint reflecting a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ends the period with (weakly) negative assets.

See Appendix H for the formal proof justifying the foregoing intuitive discussion.[34]

The conditions required for convergence and nondegeneracy are thus strikingly similar between the liquidity constrained perfect foresight model and the model with uncertainty but no explicit constraints: The liquidity constrained perfect foresight model is just the limiting case of the model with uncertainty as the degree of all three kinds of uncertainty (zero-income events, other transitory shocks, and permanent shocks) approaches zero.

2.12 Relations Between Parametric Restrictions

The full relationship among conditions is represented in Figure 3. Though the diagram looks complex, it is merely a modified version of the earlier diagram (Figure **??**) with further (mostly intermediate) inequalities inserted. (Arrows with a “because” now label relations that always hold under the model’s assumptions.)[35]

The ‘weakness’ of the additional condition sufficient for contraction beyond the FVAC, the WRIC, can be seen by asking ‘under what circumstances would the FVAC hold but the WRIC fail?’ Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \ 1 \ < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{37}$$

If we require $\mathsf{R} \geq 1$, the WRIC is redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that (with $\rho > 1$ and $\beta < 1$) the RIC (and WRIC) must hold. But neither theory nor evidence demand that $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC’s relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration

[34] It seems likely that a similar argument would apply even in the context of a model like that of MST, perhaps with some weak restrictions on returns.

[35] Again, readers unfamiliar with such diagrams should see Appendix K for a more detailed exposition.

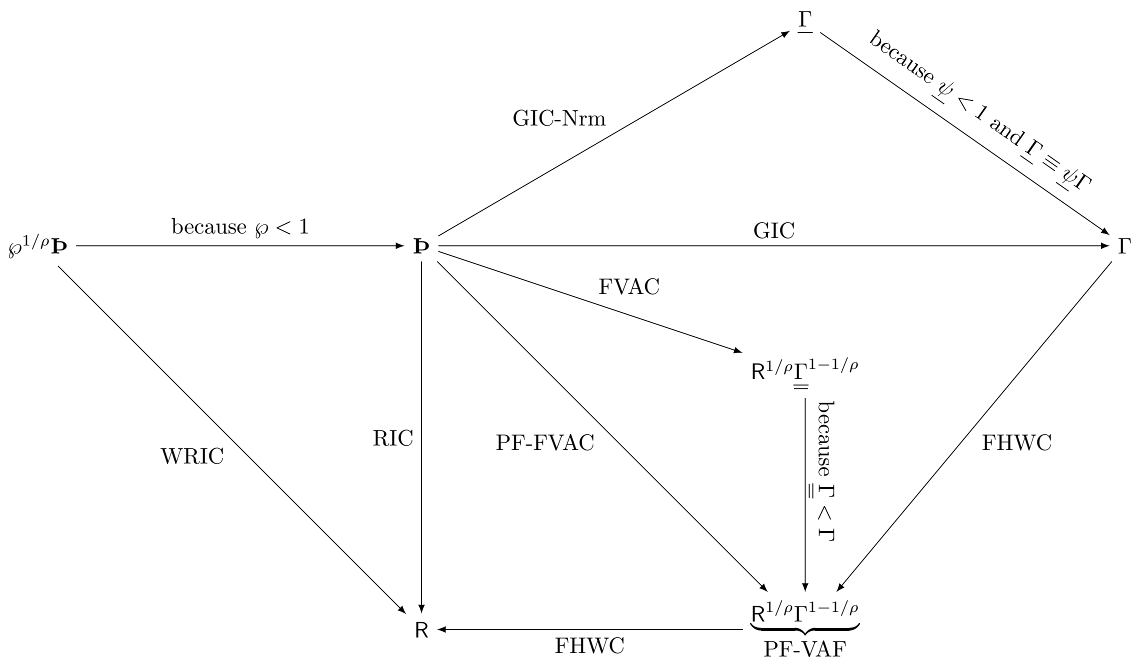

Figure 3 Relation of All Inequality Conditions
See Table 2 for Numerical Values of Nodes Under Baseline Parameters

that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (37) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind.

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which Section 2.11 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.12.1 When the RIC Fails

In the perfect foresight problem (Section 2.5.3), the RIC was necessary for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the much weaker WRIC is sufficient for nondegeneracy (assuming that the FVAC holds). We can directly derive the features the problem must exhibit (given the FVAC) under ~~RIC~~ (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} \;<\; \left(\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1}\right)^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< \underline{\underline{\psi}}
\end{aligned} \tag{38}$$

but since $\underline{\underline{\psi}} < 1$ (cf. the argument below (29)), this requires $\mathsf{R}/\Gamma < 1$; so, given the FVAC, the RIC can fail only if human wealth is unbounded. As an illustration of the usefulness of our diagrams, note that this algebraically complicated conclusion could be easily reached diagrammatically in figure 3 by starting at the R node and imposing ~~RIC~~, reversing the RIC arrow and then traversing the diagram along any clockwise path to the PF-VAF node at which point we realize that we *cannot* impose the FHWC because that would let us conclude $\mathsf{R} > \mathsf{R}$.

As in the perfect foresight constrained problem, unbounded limiting human wealth (~~FHWC~~) here does not lead to a degenerate limiting consumption function (finite human wealth is not a condition required for the convergence theorem). But, from equation (32) and the discussion surrounding it, an implication of ~~RIC~~ is that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, in the special {~~RIC~~,~~FHWC~~} case (unavailable in the perfect foresight model) the presence of uncertainty both permits unlimited human wealth (in the $n \uparrow \infty$ limit) and at the same time prevents unlimited human wealth from resulting in (limiting) infinite consumption (at any finite m). Intutively, in the presence of uncertainty, pathological patience (which in the perfect foresight model results in a limiting consumption function of $\mathrm{c}(m) = 0$ for finite m) plus unbounded human wealth (which the perfect foresight model prohibits because it leads to a limiting consumption function $\mathrm{c}(m) = \infty$ for any finite m) combine to yield a unique finite limiting (as $n \uparrow \infty$) level of consumption and MPC for any finite value of m. Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {GIC,~~RIC~~} case. There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.[36]

2.12.2 When the RIC Holds

FHWC. If the RIC and FHWC both hold, a perfect foresight solution exists (see 2.5.3 above). As $m \uparrow \infty$ the limiting consumption function and value function become arbitrarily close to those in the perfect foresight model, because human wealth pays for a vanishingly small portion of spending. This will be the main case analyzed in detail below.

[36] Ma and Toda (2020) derive conditions under which the limiting MPC is zero in an even more general case where there is also capital income risk.

~~FHWC~~. The more exotic case is where FHWC does not hold; in the perfect foresight model, {RIC,~~FHWC~~} is the degenerate case with limiting $\bar{\mathrm{c}}(m) = \infty$. Here, since the FVAC implies that the PF-FVAC holds (traverse Figure 3 clockwise from Þ by imposing FVAC and continue to the PF-VAF node): Reversing the arrow connecting the R and PF-VAF nodes implies that under ~~FHWC~~:

$$\begin{aligned} &\overbrace{\text{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma}^{\text{PF-FVAC}} \\ &\text{Þ} < \Gamma \end{aligned}$$

where the transition from the first to the second lines is justified because ~~FHWC~~ $\Rightarrow (\mathsf{R}/\Gamma)^{1/\rho} < 1$. So, {RIC, ~~FHWC~~} implies the GIC holds. However, we are not entitled to conclude that the GIC-Nrm holds: $\text{Þ} < \Gamma$ does not imply $\text{Þ} < \underline{\psi}\Gamma$ where $\underline{\psi} < 1$.

We have now established the principal points of comparison between the perfect foresight solutions and the solutions under uncertainty; these are codified in the remaining parts of Tables 3 and **??**.

3 Analysis of the Converged Consumption Function

Figures 4-6 capture the main properties of the converged consumption rule when the RIC, GIC-Nrm, and FHWC all hold.[37]

Figure 4 shows the expected growth factors for consumption, the level of market resources, and the market resources ratio, $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ and $\mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]$, and $\mathbb{E}_t[m_{t+1}/m_t]$, for a consumer behaving according to the converged consumption rule, while Figures 5—6 illustrate theoretical bounds for the consumption function and the MPC.

Three points are worth highlighting.

First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to Þ, indicated by the lower bound in Figure 4, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{Þ}_{\mathsf{R}})$ (see Figure 5) — the same as the perfect foresight MPC. Second, as m_t approaches zero the consumption growth factor approaches ∞ (Figure 4) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}})$ (Figure 5). Third, there is a value of the market resources ratio $m_t = \check{m}$ at which the expected growth rate of the level of market resources $\mathbf{m}$ matches the expected growth rate of permanent income Γ, and a different (larger) target ratio $\hat{m}$ where $\mathbb{E}[m_{t+1}/m_t] = 1$ and the expected growth rate of consumption is lower than Γ. Thus, at the individual level, this model does not have a single m at which $\mathbf{p}$, $\mathbf{m}$, and $\mathbf{c}$ all grow at the same rate.

3.1 Limits as m Approaches Infinity

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

[37] These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

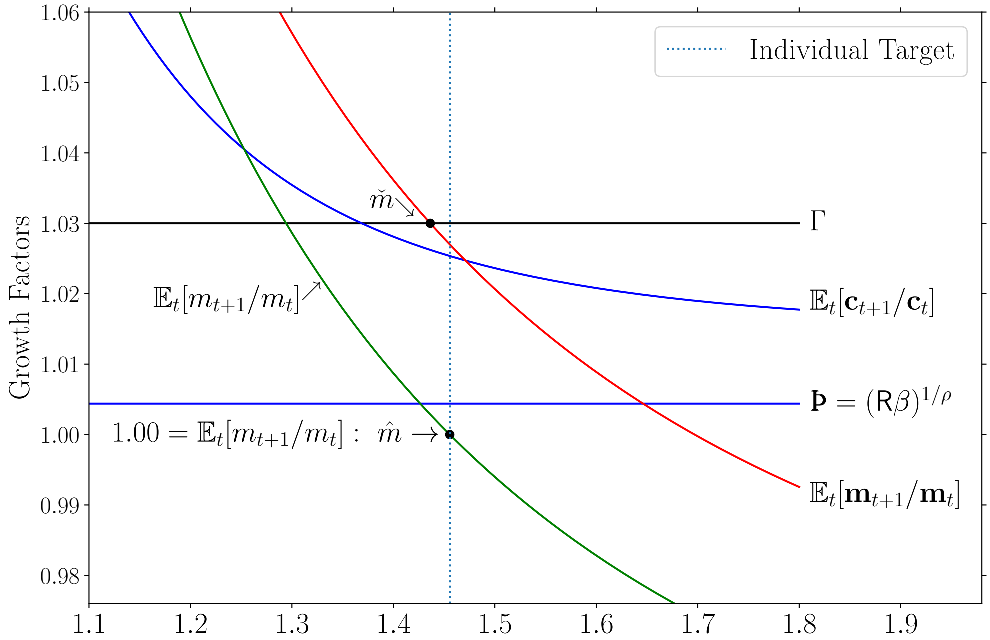

Figure 4 'Stable' (Target; Balanced Growth) m Values Defined By Expected Growth

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathrm{c}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption. Our imposition of the RIC guarantees that $\underline{\kappa} > 0$, so this solution satisfies our definition of nondegeneracy, and because this solution is always available it defines a lower bound on both the consumption and value functions.

Assuming the FHWC holds, the infinite horizon perfect foresight solution (20) constitutes an upper bound on consumption in the presence of uncertainty, since the introduction of uncertainty strictly decreases the level of consumption at any m (Carroll and Kimball (1996)). Thus, we can write

$$\begin{aligned}\underline{\mathrm{c}}(m) <& \mathrm{c}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathrm{c}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m).\end{aligned} \tag{39}$$

But

$$\begin{aligned}\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m-1+h)/m \\ &= 1,\end{aligned}$$

so as $m \uparrow \infty, \mathrm{c}(m)/\underline{\mathrm{c}}(m) \rightarrow 1$, and the continuous differentiability and strict concavity of $\mathrm{c}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

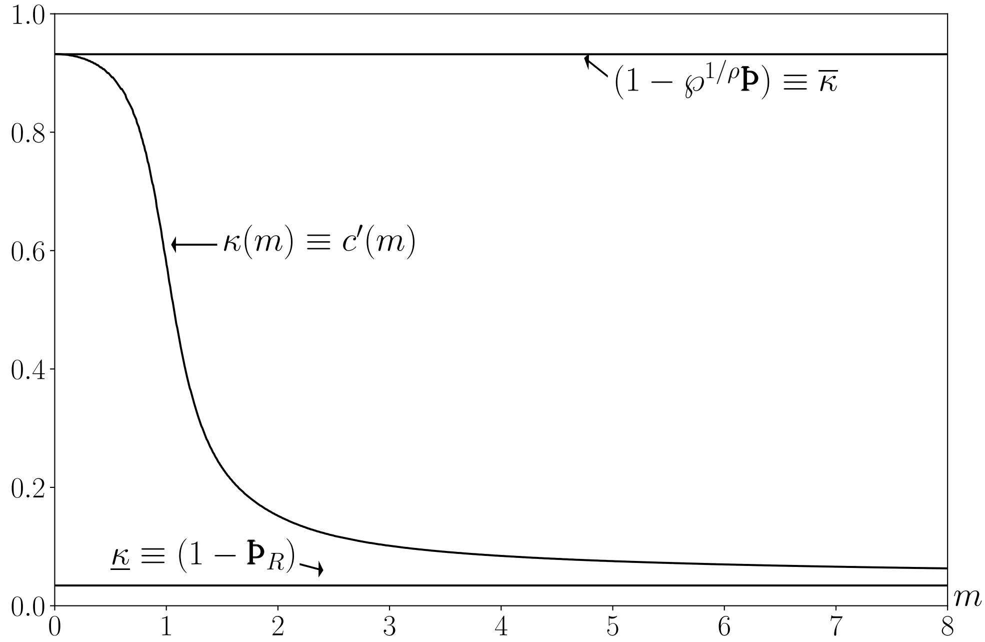

Figure 5 Limiting MPC's

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

Figure 5 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t \uparrow \infty} \mathbb{E}_t[\Gamma_{t+1} c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\Gamma_{t+1} \underline{c}_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1} c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1} \bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \uparrow \infty} \Gamma_{t+1} \underline{\mathrm{c}}(m_{t+1})/\bar{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1} \bar{\mathrm{c}}(m_{t+1})/\underline{\mathrm{c}}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1} m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m_t) = m_t - \mathrm{c}(m_t)$),

$$\begin{aligned} \lim_{m_t \uparrow \infty} \Gamma_{t+1} m_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t} \right) \\ &= (\mathsf{R}\beta)^{1/\rho} = \text{Þ} \end{aligned} \tag{40}$$

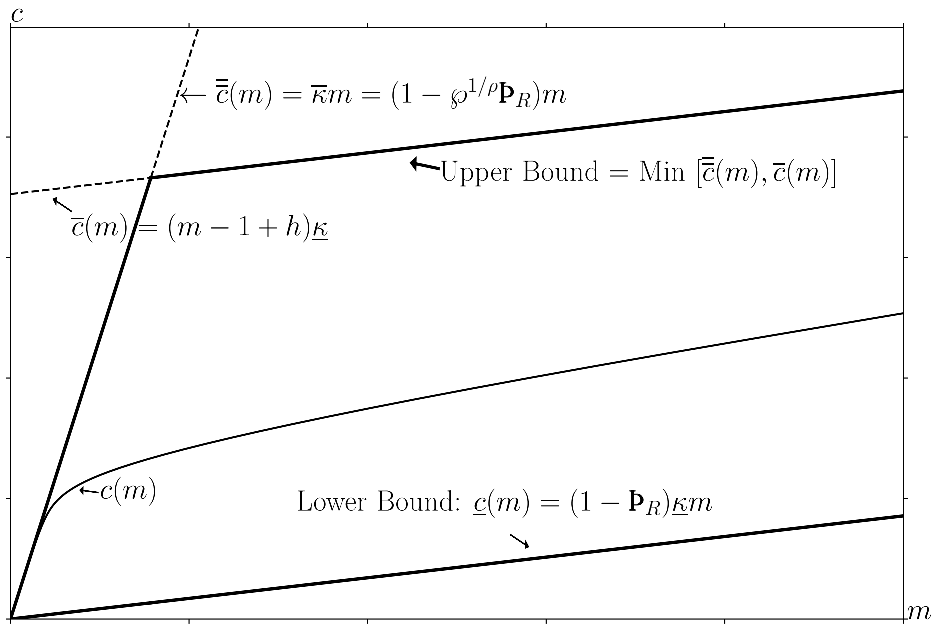

Figure 6 Upper and Lower Bounds on the Consumption Function

because $\lim_{m_t \uparrow \infty} \mathrm{a}'(m) = \mathbf{Þ}_{\mathsf{R}}$[38] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$\mathbf{Þ} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \mathbf{Þ}$$

so as cash goes to infinity, consumption growth approaches its value $\mathbf{Þ}$ in the perfect foresight model.

3.2 Limits as m Approaches Zero

Equation (33) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m \downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho}\mathbf{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with

[38] This is because $\lim_{m_t \uparrow \infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} \mathrm{c}'(m_t) = \mathbf{Þ}_{\mathsf{R}}$.

L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} \mathrm{c}'(m) = \lim_{m \downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 5 visually confirms that the numerical solution obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned}
\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&\quad + (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&> (1-\wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\
&= \infty
\end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.[39]

3.3 Unique 'Stable' Points

Theorems whose substance is described here (and whose details are in an appendix) articulate alternative (but closely related) stability criteria.

3.3.1 'Individual Target Wealth' $\hat{m}$

One kind of 'stable' point is a 'target' value $\hat{m}$ such that if $m_t = \hat{m}$, then $\mathbb{E}_t[m_{t+1}] = m_t$. Existence of such a target turns out to require the GIC-Nrm condition.

Theorem 2. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC-Nrm all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{41}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that

$$\begin{aligned}
\forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\
\forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t.
\end{aligned} \tag{42}$$

[39] None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

Since $m_{t+1} = (m_t - \mathrm{c}(m_t))\mathcal{R}_{t+1} + \xi_{t+1}$, the implicit equation for $\hat{m}$ is

$$\begin{aligned} \mathbb{E}_t[(\hat{m} - \mathrm{c}(\hat{m}))\mathcal{R}_{t+1} + \xi_{t+1}] &= \hat{m} \\ (\hat{m} - \mathrm{c}(\hat{m}))\underbrace{\mathcal{R}\,\mathbb{E}_t[\psi^{-1}]}_{\equiv\bar{\mathcal{R}}} + 1 &= \hat{m}. \end{aligned} \tag{43}$$

3.3.2 Individual Balanced-Growth 'pseudo steady state' $\check{m}$

Our second definition of 'stability' derives from a traditional question in macroeconomic models: whether there is a 'balanced growth' equilibrium in which aggregate variables (income, consumption, market resources) all grow forever by the same factor Γ. For our model, Figure 4 showed that there is no single m for which $\mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t = \mathbb{E}_t[\mathbf{c}_{t+1}]/\mathbf{c}_t = \Gamma$ for an individual consumer. Nevertheless, the next section will show that economies populated by heterogeneous collections of such consumers can exhibit balanced growth in the aggregate, and in the cross-section of households.

As an input to that analysis, we show here that if the GIC holds, the problem will exhibit what we call a balanced growth 'pseudo-steady-state' point, by which we mean that there is some $\check{m}$ such that, for all $m_t > \check{m}$, $\mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] < \Gamma$, and conversely if $m_t < \check{m}$ then $\mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] > \Gamma$.

The critical m will be the value $\check{m}$ at which $\mathbf{m}$ growth matches Γ:

$$\begin{aligned} \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &= \mathbb{E}_t[\mathbf{p}_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t[m_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/(m_t\mathbf{p}_t) &= \mathbb{E}_t[\mathbf{p}_t\Gamma\psi_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t\left[\psi_{t+1}\underbrace{((m_t - \mathrm{c}(m_t)\mathsf{R}/(\Gamma\psi_{t+1})) + \xi_{t+1})}_{m_{t+1}}\right]/m_t &= 1 \\ \mathbb{E}_t\left[(\check{m} - \mathrm{c}(\check{m}))\overbrace{\mathsf{R}/\Gamma}^{\mathcal{R}} + \psi_{t+1}\xi_{t+1}\right] &= \check{m} \\ (\check{m} - \mathrm{c}(\check{m}))\mathcal{R} + 1 &= \check{m}. \end{aligned} \tag{44}$$

The only difference between (44) and (43) is the substitution of $\mathcal{R}$ for $\bar{\mathcal{R}}$.[40,41]

Theorem 3 formally states the relevant proposition.

Theorem 3. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-*

[40] A third candidate 'stable point' is the $\check{m}$ where $\mathbb{E}_t[\log m_{t+1}] = \log \Gamma m_t$; this can be conveniently rewritten as $\mathbb{E}_t\left[\log\left((\check{m} - \mathrm{c}(\check{m}))\mathcal{R} + \psi_{t+1}\xi_{t+1}\right)\right] = \log \check{m}_t$. Because the expectation of the log of a stochastic variable is less than the log of the expectation, this will be satisfied by an $\check{m} < \check{m}$. For our purposes, little would be gained by an analysis of this point parallel to those of the other points of stability; but to accommodate potential uses, the toolkit computes the value of this point.

[41] Our choice to call to this the individual problem's 'individual balanced-growth pseudo-steady-state' $\check{m}$ is motivated by what happens in the case where all draws of all future shocks just happen to take on their expected value of 1.0. (They unexpectedly always take on their expected values). In that infinitely improbable case, the economy *would* exhibit balanced growth.

$$\mathbb{E}_t[m_{t+1}/m_t|\psi_{t+1} = \xi_{t+1} = 1] = \Gamma\left(\check{m} - \mathrm{c}(\check{m})\mathcal{R} + 1\right)/m = \Gamma$$

to-income ratio $\check{m} > 0$ such that

$$\mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{45}$$

Moreover, $\check{m}$ is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t > \Gamma \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t < \Gamma. \end{aligned} \tag{46}$$

The proofs of the theorems are almost completely parallel; to save space, they are relegated to Appendix M. They involve three steps:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1}/m_t]$
   - This follows from existence and continuity of the constituents
2. Existence of the equilibrium point
   - This follows from existence of upper and lower bound limiting MPCs, existence and continuity of the decision rule, and the Intermediate Value Theorem
3. Monotonicity of $\mathbb{E}_t[m_{t+1} - m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1} - m_t]$
   - This follows from concavity of the consumption function

3.4 Example With Balanced-Growth $\check{m}$ But No Target $\hat{m}$

Because the equations defining target and pseudo-steady-state m, (43) and (44), differ only by substitution of $\mathcal{R}$ for $\bar{\mathcal{R}} = \mathcal{R}\,\mathbb{E}[\psi^{-1}]$, if there are no permanent shocks ($\psi \equiv 1$), the conditions are identical. For many parameterizations (e.g., under the baseline parameter values used for constructing figure 4), $\hat{m}$ and $\check{m}$ will not differ much.

An illuminating exception is exhibited in Figure 7, which modifies the baseline parameter values by quadrupling the variance of the permanent shocks, enough to cause failure of the GIC-Nrm; now there is no target wealth level $\hat{m}$ (precautionary motives keep $\mathrm{c}(m)$ everywhere below the level that would keep expected m constant).

The pseudo-steady-state still exists because it turns off realizations of the permanent shock. But the next section will show that an aggregate balanced growth equilibrium will exist even when realizations of the permanent shock are not turned off: The required condition for aggregate balanced growth is the regular, not the normalized, GIC.

Before we get to the formal arguments below, the key insight can be understood here by considering the evolution of an economy that starts, at date t, with the entire population at $m_t = \check{m}$, but then evolves according to the model's assumed dynamics between t and $t+1$. Equation (44) will still hold, so for this first period, at least, the economy will exhibit balanced growth: the growth factor for aggregate $\mathbf{m}$ will match the growth factor for permanent income Γ. It is true that there will be people for whom $b_{t+1} = a_t\mathsf{R}/(\Gamma\psi_{t+1})$ is boosted by a small draw of ψ_{t+1}. But their contribution to the *level* of the aggregate variable is given by $\mathbf{b}_{t+1} = b_{t+1}\psi_{t+1}$, so their $\mathbf{b}_{t+1}$ is reweighted by an amount that exactly unwinds that boosting. This is how it is possible for each individual consumer to have a target wealth ratio of infinity, and yet for the economy as

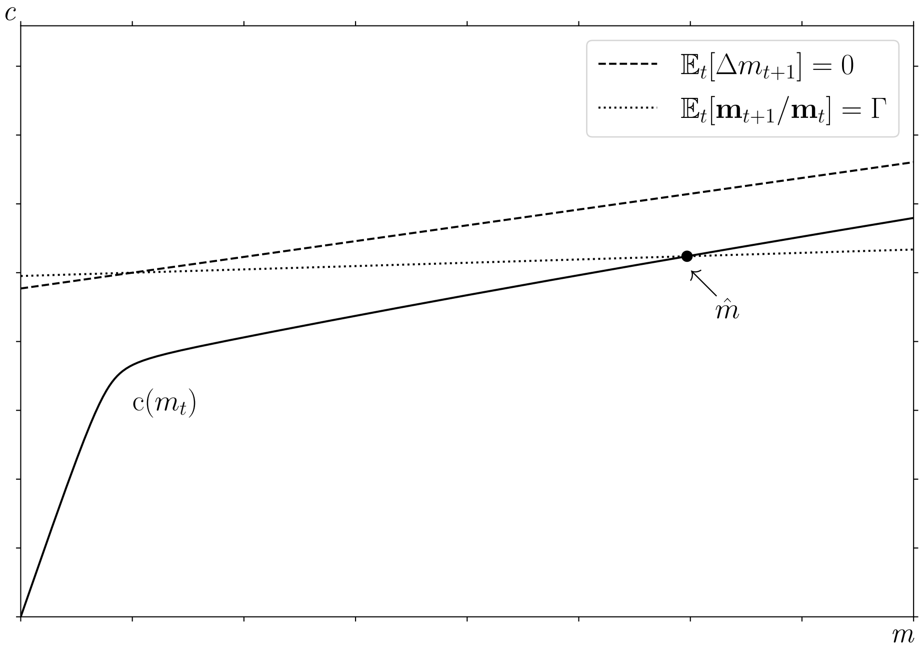

Figure 7 {FVAC,GIC,~~GIC-Nrm~~}: No $\hat{m}$ Exists But $\check{m}$ Does

a whole to have a balanced growth equilibrium with a finite aggregate balanced growth steady state (which, for symmetry with the pseudo-steady-state, we will call $\check{M}$). This is not numerically the same as the individual pseudo-steady-state ratio $\check{m}$ because the nonlinearities involved in simulation and aggregation have consequences.[42]

4 Invariant Relationships

Assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above.

Szeidl (2013) proved that such a population will be characterized by invariant distributions of m, c, and a under the condition[43]

$$\text{Þ}_{\Gamma} < e^{\mathbb{E}[\log \psi]} \tag{47}$$

[42]Still, the pseudo-steady-state can be calculated immediately from the policy function without any simulation, and therefore would likely serve as an excellent and low-cost starting point for the numerical simulation process.

[43]Szeidl (2013)'s equation (9), in our notation, is:

$$\begin{aligned}
\mathbb{E} \log \mathsf{R}(1-\kappa) &< \mathbb{E} \log \Gamma\psi \\
\mathbb{E} \log \mathsf{R}\text{Þ}_{\mathsf{R}} &< \mathbb{E} \log \Gamma\psi \\
\log \text{Þ}_{\Gamma} &< \mathbb{E} \log \psi
\end{aligned}$$

which, exponentiated, yields (47).

which is stronger than our GIC condition $\text{Þ}_\Gamma < 1$ (under the imposed assumption $\mathbb{E}[\log \psi] < 0$).[44]

Harmenberg (2021) substitutes a clever change of probability-measure into Szeidl's proof to show that under the GIC, invariant *permanent-income-weighted* distributions exist.[45] This allows him to prove a conjecture from an earlier draft of this paper (Carroll (2020, Submitted)) that under the GIC, aggregate consumption grows at the same rate Γ as aggregate noncapital income in the long run (with the corollary that aggregate assets and market resources grow at that same rate). Harmenberg (2021) shows[46] that his reformulation of the problem can reduce costs of calculation enormously.

The remainder of this section briefly draws out some implications of these points.

4.1 Individual Balanced Growth of Income, Consumption, and Wealth

Say that $\mathbb{M}[\bullet]$ yields the mean of its argument in the population (as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future). Using boldface capitals for aggregates, the growth factor for aggregate noncapital income is:

$$\begin{aligned}\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}\left[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t\right]/\mathbb{M}\left[\mathbf{p}_t\xi_t\right] \\ &= \Gamma\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

Consider an economy that satisfies the Szeidl impatience condition (47) and has existed for long enough by date t that we can consider it as Szeidl-converged. In such an economy a microeconomist with a population-representative panel dataset could calculate the growth rate of consumption for each individual household, and take the average:

$$\begin{aligned}\mathbb{M}\left[\Delta \log \mathbf{c}_{t+1}\right] &= \mathbb{M}\left[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t\mathbf{p}_t\right] \\ &= \mathbb{M}\left[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t\right] + \mathbb{M}\left[\log c_{t+1} - \log c_t\right].\end{aligned} \tag{48}$$

Because this economy is Szeidl-converged, distributions of c_t and c_{t+1} will be identical, so that the second term in (48) disappears; hence, mean cross-sectional growth rates of consumption and permanent income are the same:

$$\mathbb{M}\left[\Delta \log \mathbf{c}_{t+1}\right] = \mathbb{M}\left[\Delta \log \mathbf{p}_{t+1}\right] = \log \Gamma. \tag{49}$$

In a Harmenberg-invariant economy (and therefore also any Szeidl-invariant economy), a similar proposition holds in the cross-section as a direct implication of the fact that a

[44] Under our default (though not required) assumption that $\log \psi \sim \mathcal{N}(-\sigma_\psi^2/2, \sigma_\psi^2)$; the GIC-Nrm in this case, is $\text{Þ}_\Gamma < \exp(-\sigma^2)$, so if the GIC-Nrm holds then Szeidl's condition will hold.

[45] Harmenberg in a nutshell: If ψ is described by density function $f_\psi(\psi)$, define the *permanent-income-neutral measure* by $\tilde{f}_\psi(\psi) \equiv \psi f_\psi(\psi)$. Following derivations parallel to those in the previous foonote, the condition for the existence of invariant permanent-income-weighted distributions is

$$\log \text{Þ}_\Gamma < \tilde{\mathbb{E}} \log \psi$$

where the $\sim$ over the expectations operator indicates that the expectation is being taken with respect to $f_\psi(\psi)$. But by assumption $\tilde{\mathbb{E}} \log \psi = 0$ so exponentiating both sides of this equation turns it into $\text{Þ}_\Gamma < 1$ which is the GIC.

[46] The test of the Harmenberg method currently implemented in the Econ-ARK toolkit shows that Harmenberg's method reduces the standard error of the simulated aggregate consumption series by a factor of more than 8 (using identical computational resources); see the last part of `test_Harmenbergs_method.sh`.

constant proportion of total permanent income is accounted for by the successive sets of consumers with any particular m. This fact is one way of interpreting Harmenberg's definition of the density of the permanent-income-invariant distribution of m; call this density $p(m)$.[47] Call $\mathbf{c}_t(m)$ the total amount of consumption at date t by persons with market resources m, and note that in the invariant economy this is given by the converged consumption function $\mathrm{c}(m)$ multiplied by the amount of permanent income accruing to such people $p(m)\mathbf{P}_t$. Since $p(m)$ is invariant and aggregate permanent income grows according to $\mathbf{P}_{t+1} = \Gamma\mathbf{P}_t$,

$$\begin{aligned}\log \mathbf{c}_{t+1}(m) - \log \mathbf{c}_t(m) &= \log \mathrm{c}(m)p(m)\mathbf{P}_{t+1} - \log \mathrm{c}(m)p(m)\mathbf{P}_t \\ &= \log \Gamma\end{aligned}$$

will hold for any m.

4.2 Aggregate Balanced Growth and Idiosyncratic Covariances

Harmenberg has a brief discussion demonstrating that the covariance between the individual consumption ratio c and the idiosyncratic component of permanent income $\mathbf{p}$ does not shrink to zero over time; so covariances constitute another potential measurement that can be used for construction of microfoundations.

Consider a date-t Harmenberg-converged economy, and define the mean value of the consumption ratio as $\mathbf{c}_{t+n} \equiv \mathbb{M}[c_{t+n}]$. Normalizing period-$t$ aggregate permanent income to $\mathbf{P}_t = 1$, total consumption at $t+1$ and $t+2$ are

$$\begin{aligned}\mathbf{C}_{t+1} &= \mathbb{M}[c_{t+1}\mathbf{p}_{t+1}] = \mathbf{c}_{t+1}\Gamma^1 + \mathrm{cov}_{t+1}(c_{t+1}, \mathbf{p}_{t+1}) \\ \mathbf{C}_{t+2} &= \mathbb{M}[c_{t+2}\mathbf{p}_{t+2}] = \mathbf{c}_{t+2}\Gamma^2 + \mathrm{cov}_{t+2}(c_{t+2}, \mathbf{p}_{t+2})\end{aligned} \tag{50}$$

and Harmenberg's proof that $\mathbf{C}_{t+2} - \Gamma\mathbf{C}_{t+1} = 0$ allows us to subtract $\Gamma\mathbf{C}_{t+1}$ from $\mathbf{C}_{t+2}$ to obtain:

$$(\mathbf{c}_{t+2} - \mathbf{c}_{t+1})\,\Gamma^2 = \Gamma\,\mathrm{cov}_{t+1} - \mathrm{cov}_{t+2}\,. \tag{51}$$

In the special case of a Szeidl-invariant economy, $\mathbf{c}_{t+2} = \mathbf{c}_{t+1}$, implying that the economy exhibits balanced growth in the covariance:

$$\mathrm{cov}_{t+2} = \Gamma\,\mathrm{cov}_{t+1}\,. \tag{52}$$

The more interesting case is when the economy is Harmenberg- but not Szeidl-invariant. In that case, if the cov and the $\mathbf{c}$ terms have constant growth factors Ω_{cov} and $\Omega_{\mathbf{c}}$,[48] an equation corresponding to (51) will hold in $t+n$:

$$\begin{aligned}(\overbrace{\Omega_{\mathbf{c}}^n\mathbf{c}_t}^{\mathbf{c}_{t+n}} - \Omega_c^{n-1}\mathbf{c}_t)\Gamma^n &= \left(\Gamma\Omega_{\mathrm{cov}}^{n-1} - \Omega_{\mathrm{cov}}^n\right)\mathrm{cov}_t \\ (\Omega_{\mathbf{c}}\Gamma)^{n-1}(\Omega_{\mathbf{c}} - 1)\mathbf{c}_t\Gamma &= \Omega_{\mathrm{cov}}^{n-1}(\Gamma - \Omega_{\mathrm{cov}})\mathrm{cov}_t\end{aligned} \tag{53}$$

---

[47] In his notation, this is $\tilde{\psi}^m$.

[48] This 'if' is a conjecture, not something proven by Harmenberg (or anyone else). But see appendix N for an example of a Harmenberg-invariant economy in which simulations suggest this proposition holds.

so for the LHS and RHS to grow at the same rates we need

$$\Omega_{\text{cov}} = \Omega_{\mathbf{c}}\Gamma. \tag{54}$$

This is intuitive. In the Szeidl-invariant economy, this just reproduces our result above that the covariance exhibits balanced growth because $\Omega_{\mathbf{c}} = 1$. The revised result just says that in the Harmenberg case where the mean value $\mathbf{c}$ of the consumption ratio c can grow, the covariance must rise in proportion to any ongoing expansion of $\mathbf{c}$ (as well as in proportion to the growth in $\mathbf{p}$).

4.3 Implications for Microfoundations

Thus we have microeconomic propositions, for both growth rates and for covariances of observable variables,[49] that can be tested in either cross-section or panel microdata to judge (and calibrate) the microfoundations that should hold for a macroeconomic analysis that requires balanced growth for its conclusions.

At first blush, these points are reassuring; one of the most persuasive arguments for the agenda of building microfoundations of macroeconomics is that newly available 'big data' allow us to measure cross-sectional covariances with great precision, so that we can use microeconomic natural experiments to disentangle questions that are hopelessly entangled in aggregate time-series data. Knowing that such covariances ought to be a stable feature of a stably growing economy is therefore encouraging.

But this discussion also highlights an uncomfortable point: In the model as specified, permanent income does not have a limiting distribution; it becomes ever more dispersed as the economy with infinite-horizon consumers continues to exist indefinitely.

A few microeconomic data sources attempt direct measurement of 'permanent income'; Carroll, Slacalek, Tokuoka, and White (2017), for example, show that their assumptions about the magnitude of permanent shocks (and mortality; see below) yield a distribution of permanent income that roughly matches answers in the U.S. *Survey of Consumer Finances* ('SCF') to a survey question designed to elicit a direct measure of permanent income. That paper uses those results to calibrate a model that matches empirical facts about the distribution of permanent income and wealth, and to show that the model also does a reasonable job of matching empirical facts about the marginal propensity to consume. The quantitative credibility of the argument depends on the model's match to the distribution of permanent income inequality, which would not be possible in a model without a nondegenerate steady-state distribution of permanent income.

For macroeconomists who want to build microfoundations by comparing the microeconomic implications of their models to micro data (directly – not in ratios to permanent income, which microeconomic datasets other than the SCF rarely attempt to measure), it would be something of a challenge to determine how to construct microeconomically comparable simulated microeconomic data from a model with no limiting distribution of permanent income.

Death can solve this problem.

[49] Parallel results to those for consumption can be obtained for other measures like market assets.

4.4 Mortality

Most heterogeneous-agent models incorporate a constant positive probability of death, following Blanchard (1985) and Yaari (1965). In the Blanchardian model, if the probability of death exceeds a threshold that depends on the size of the permanent shocks, Carroll, Slacalek, Tokuoka, and White (2017) show that the limiting distribution of permanent income has a finite variance. Blanchard (1985) assumes the existence of a universal annuitization scheme in which estates of dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, most notably that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor. That is, if the 'pure' time preference factor is β and probability of remaining alive (not dead) is $\cancel{\mathcal{D}}$, then the assumption that no utility accrues after death makes the effective discount factor $\underline{\beta} = \beta\cancel{\mathcal{D}}$ while the enhancement to the rate of return from the annuity scheme yields an effective interest factor $\bar{\mathsf{R}} = \mathsf{R}/\cancel{\mathcal{D}}$ (recall that because of white-noise mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\underline{\beta}\bar{\mathsf{R}}$ is unchanged from its value in the infinite horizon model:

$$\underline{\beta}\bar{\mathsf{R}} = \left(\beta\cancel{\mathcal{D}}\mathsf{R}/\cancel{\mathcal{D}}\right)^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} = \mathbf{Þ}. \tag{55}$$

The only adjustments this requires to the analysis above are therefore to the few elements that involve a role for the interest factor distinct from its contribution to $\mathbf{Þ}$ (principally, the RIC, which becomes $\mathbf{Þ}/\bar{\mathsf{R}}$).

Blanchard (1985)'s innovation was useful not only for the insight it provided but also because the principal alternative, the Life Cycle model of Modigliani (1966), was computationally challenging given then-available technologies. Despite its (considerable) conceptual value, Blanchard's analytical solution is now rarely used because essentially all modern modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard is to follow Modigliani in constructing a realistic description of income over the life cycle and assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2001, 2016)).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it represents something of a polar alternative to Blanchard: Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean; the estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium (for consumers without a life cycle income profile). If $\cancel{\mathcal{D}}$ is the probability of remaining alive, the condition changes from the plain GIC to a looser mortality-adjusted GIC:

$$\cancel{\mathfrak{D}}\text{Þ}_{\Gamma} < 1. \tag{56}$$

With no income growth, what is required to prohibit unbounded growth in aggregate wealth is the condition that prevents the per-capita wealth-to-permanent-income ratio of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite horizon ('life cycle') versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to Friedman (1957) and standard specifications of preferences — and without plausible (but computationally and mathematically inconvenient) complications like liquidity constraints — have attractive properties (like continuous differentiability of the consumption function, and analytical limiting MPC's as resources approach their minimum and maximum possible values).

The main focus of the paper, though, is on the limiting solution of the finite horizon model as the horizon extends to infinity. The paper shows that the simple model has other appealing features: A 'Finite Value of Autarky' condition guarantees convergence of the consumption function, under the mild additional requirement of a 'Weak Return Impatience Condition' that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model's relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite horizon versions of such models is in heterogeneous agent macroeconomics. The paper articulates intuitive 'Growth Impatience Conditions' under which populations of such agents, with Blanchardian (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for many results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{c}(m)$ that arise under various parametric conditions. Results are summarized in table 5.

A.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails (~~GIC~~, $1 < \mathbf{Þ}/\Gamma$). Under ~~GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[50]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \\
1 &< \mathsf{R}\beta\Gamma^{-\rho} \\
\mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\Gamma).
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \mathbf{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\grave{c}_{T-n}(m) = 0(b_t + h). \tag{57}$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. ~~FHWC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{c}_{T-n}(m) = 0$ or $\grave{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[51]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer

[50] The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{c}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[51] The knife-edge case is where $\mathbf{Þ} = \Gamma$, in which case the two quantities counterbalance and the limiting function is $\grave{c}(m) = \min[m, 1]$.

would choose $c = m$ from Equation (20):

$$\begin{aligned} m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\ m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\ m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right) \end{aligned} \tag{58}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[52] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[53] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

A.2 If GIC Holds

Imposition of the GIC reverses the inequality in (57), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b_{\#}^{n}$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^{n}$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b_{\#}^{0}$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^{1}$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^{0} = 1 + b_{\#}^{0} = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^{0} = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\mathbf{Þ}_{\Gamma}$, so consumption n periods in the past must have been

$$c_{\#}^{n} = \mathbf{Þ}_{\Gamma}^{-n} c_t = \mathbf{Þ}_{\Gamma}^{-n}. \tag{59}$$

[52] Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[53] As an illustration, consider a consumer for whom $\mathbf{Þ} = 1$, $\mathsf{R} = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}_{t-n}^{t} &= c_{t-n}(1+\text{Þ}/\mathsf{R}+\cdots+(\text{Þ}/\mathsf{R})^{n}) \\
&= c_{\#}^{n}(1+\text{Þ}_{\mathsf{R}}+\cdots+\text{Þ}_{\mathsf{R}}^{n}) \\
&= \text{Þ}_{\Gamma}^{-n}\left(\frac{1-\text{Þ}_{\mathsf{R}}^{n+1}}{1-\text{Þ}_{\mathsf{R}}}\right) \\
&= \left(\frac{\text{Þ}_{\Gamma}^{-n}-\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h_{\#}^{n} = 1+\cdots+\mathcal{R}^{-n} \tag{60}$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^{t} \quad = b_{\#}^{n}+h_{\#}^{n}$$

from which we can solve for the $b_{\#}^{n}$ such that the consumer with $b_{t-n}=b_{\#}^{n}$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t=0$:

$$b_{\#}^{n} = \mathbb{C}_{t-n}^{t} - \overbrace{\left(\frac{1-\mathcal{R}^{-(n+1)}}{1-\mathcal{R}^{-1}}\right)}^{h_{\#}^{n}}. \tag{61}$$

Defining $m_{\#}^{n}=b_{\#}^{n}+1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m_{\#}^{n}, c_{\#}^{n}\}$ for integer values of $n\geq 0$ (and setting $\grave{c}(m)=m$ for $m<1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m=m_{\#}^{n}-\epsilon$ is discretely higher than for a consumer with assets $m=m_{\#}^{n}+\epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (61) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{62}$$

A.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1}<1$, in which case the second term in (61) limits to a constant as $n\uparrow\infty$, and (62) reduces to a requirement that

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}-(\text{Þ}_{\mathsf{R}}/\text{Þ}_{\Gamma})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) \quad = \infty$$

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}-\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}\right) = \infty$$
$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right) = \infty.$$

Given the GIC $\text{Þ}_{\Gamma}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \Gamma$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{63}$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (62) requires

$$\lim_{n\to\infty}\left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}-\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right)+\left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1}\right) = \infty$$
$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1}-\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) = \infty$$

If RIC Holds. When the RIC holds, rearranging (64) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\Gamma}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) = \infty$$

and for this to be true we need

$$\begin{aligned} \text{Þ}_{\Gamma}^{-1} &> \mathcal{R}^{-1} \\ \Gamma/\text{Þ} &> \Gamma/\mathsf{R} \\ 1 &> \text{Þ}/\mathsf{R} \end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n} = \lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{64}$$

which with a bit of algebra[54] can be shown to asymptote to the MPC in the perfect

[54]Calculate the limit of

$$\left(\frac{\text{Þ}_{\Gamma}^{-n}}{\text{Þ}_{\Gamma}^{-n}/(1-\text{Þ}_{\mathsf{R}})-(1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right) = \left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}})+\mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{65}$$

foresight model:[55]

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 1 - \text{\textbf{Þ}}_{\mathsf{R}}. \tag{66}$$

If RIC Fails. Consider now the ~~RIC~~ case, $\text{\textbf{Þ}}_{\mathsf{R}} > 1$. We can rearrange (64)as

$$\lim_{n\to\infty} \left(\frac{\text{\textbf{Þ}}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{\textbf{Þ}}_{\mathsf{R}}-1)} - \frac{\mathcal{R}^{-1}(\text{\textbf{Þ}}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{\textbf{Þ}}_{\mathsf{R}}-1)} \right) \mathcal{R}^{-n} - \left(\frac{\text{\textbf{Þ}}_{\Gamma}^{-n}}{\text{\textbf{Þ}}_{\mathsf{R}}-1} \right) = \infty. \tag{67}$$

which makes clear that with ~~FHWC~~ $\Rightarrow \mathcal{R}^{-1} > 1$ and ~~RIC~~ $\Rightarrow \text{\textbf{Þ}}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (64) will be positive if

$$\begin{aligned} \text{\textbf{Þ}}_{\mathsf{R}}\mathcal{R}^{-1} - \text{\textbf{Þ}}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{\textbf{Þ}}_{\mathsf{R}} - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \text{\textbf{Þ}}_{\mathsf{R}} \\ \Gamma &> \text{\textbf{Þ}} \end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{\textbf{Þ}}_{\Gamma}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \text{\textbf{Þ}}_{\Gamma}^{-1} \\ \Gamma/\mathsf{R} &> \Gamma/\text{\textbf{Þ}} \\ \text{\textbf{Þ}}/\mathsf{R} &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\Gamma/\text{\textbf{Þ}}$ while the $b_{\#}$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/\mathsf{R} \;>\; \Gamma/\text{\textbf{Þ}} \tag{68}$$

because ~~RIC~~ $\Rightarrow \text{\textbf{Þ}} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (64) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{69}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 8 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m-1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into

[55]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

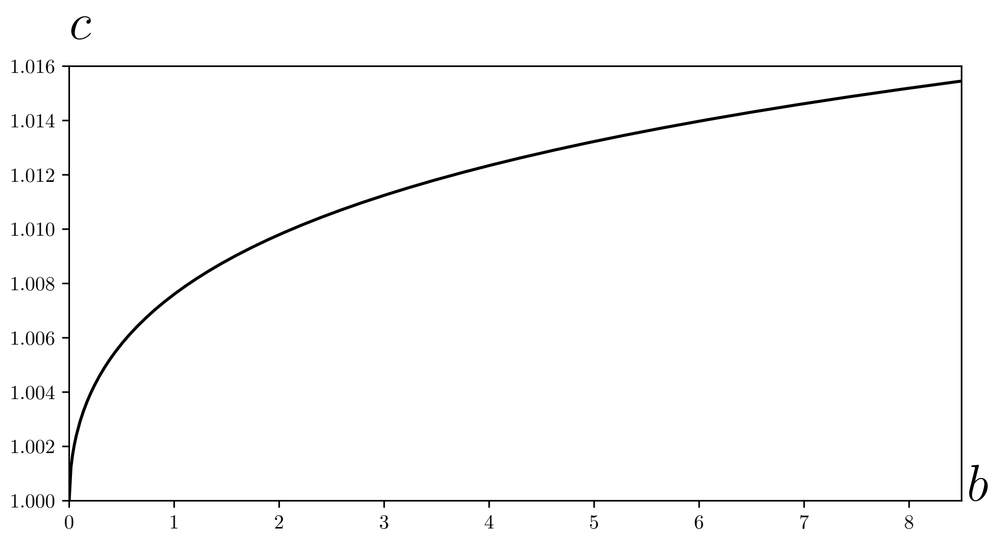

Figure 8 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

B Existence of Concave c Function

To show that (5) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, \ldots, \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$

5. $\mathrm{n}(z) < 0$
6. $\lim_{z \downarrow 0} \ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta \, \mathbb{E}_t \left[\Gamma_{t+1}^{1-\rho} \mathrm{v}_{t+1}(\mathcal{R}_{t+1} a + \xi_{t+1}) \right]. \tag{70}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0} \mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0} \mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c) \tag{71}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (5) can be written as

$$\mathrm{v}_t(m) = \max_c \ \underline{\mathrm{v}}_t(m, c). \tag{72}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0} \underline{\mathrm{v}}_t(m, c) = \lim_{c\uparrow m} \underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m} \frac{\partial \underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \underset{0<c<m}{\arg\max} \ \underline{\mathrm{v}}_t(m, c) \tag{73}$$

exists and is unique, and (5) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{74}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (74) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{75}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y)) - \mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm\to+0} \frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm\to+0} \frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}'_t(\mathrm{a}_t(y))-\mathfrak{v}'_t(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m) = \frac{\mathfrak{v}''_t(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}'_t(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}'_t(m)$ exists and is continuous. $\mathrm{c}'_t(m)$ is differentiable because $\mathfrak{v}''_t$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}''_t(\mathrm{a}_t(m)) < 0$. $\mathrm{c}''_t(m)$ is given by

$$\mathrm{c}''_t(m) = \frac{a'_t(m)\mathfrak{v}'''_t(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}''_t(a_t)\right] - \mathfrak{v}''_t(a_t)\left[c'_t\mathrm{u}'''(c_t) + a'_t\mathfrak{v}'''_t(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}''_t(a_t)\right]^2}. \tag{76}$$

Since $\mathfrak{v}''_t(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}''_t(m)$ is also continuous.

D $\mathfrak{T}$ Is a Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_{F}(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\begin{aligned}\{\mathfrak{T}\mathrm{x}\}(m_t) \text{ is } &\max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}(m_{t+1})\right]\right\} \\ \{\mathfrak{T}\mathrm{y}\}(m_t) \text{ is } &\max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}(m_{t+1})\right]\right\},\end{aligned}$$

so $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)$ implies $\{\mathfrak{T}\mathrm{x}\}(m_t) \le \{\mathfrak{T}\mathrm{y}\}(m_t)$ by inspection.[56]

[56] For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

Condition (2) requires that $\{\mathfrak{T}\mathbf{0}\} \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. By definition,

$$\{\mathfrak{T}\mathbf{0}\}(m_t) = \max_{c_t \in [\underline{\kappa} m_t, \bar{\kappa} m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa} m_t)$. Thus, condition (2) will hold if $(\bar{\kappa} m_t)^{1-\rho}$ is $\digamma$-bounded. We use the bounding function

$$\digamma(m) = \eta + m^{1-\rho}, \tag{77}$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of $\digamma$, $\{\mathfrak{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly $\digamma$-bounded.

Finally, we turn to condition (3), $\{\mathfrak{T}(\mathrm{z} + \zeta\digamma)\}(m_t) \leq \{\mathfrak{T}\mathrm{z}\}(m_t) + \zeta\alpha\digamma(m_t)$. The proof will be more compact if we define $\check{\mathrm{c}}$ and $\check{\mathrm{a}}$ as the consumption and assets functions[57] associated with $\mathfrak{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathfrak{T}(\mathrm{z} + \zeta\digamma)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta\digamma)\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}) + \zeta\alpha\digamma.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\check{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z} + \zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\{\mathsf{E}(\mathrm{z} + \zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\zeta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \zeta\alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &< \digamma.
\end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.[58]

Using $\digamma(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (23), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1 - \beth}. \tag{78}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

[57] Section 2.8 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[58] The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (78) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (78) will certainly hold for any:

$$\eta>\underline{\eta}=\frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{79}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (34) and (31) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (35)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (5) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (79). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho}&<1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1})&>1\\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{\Thorn}_{\mathsf{R}})^{-1})&>1
\end{aligned}$$

where we have used (33) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho>1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{\Thorn}_{\mathsf{R}})^{-1}\approx 1-\wp^{1/\rho}\text{\Thorn}_{\mathsf{R}}$ so that it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{\Thorn}_{\mathsf{R}}>1$$

$$
\begin{aligned}
(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.
\end{aligned}
$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (5) converge, they converge to the same unique v defined by $\mathfrak{T}$.[59]

E Convergence in Euclidian Space

E.1 Convergence of v_t

Boyd's theorem shows that $\mathfrak{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathfrak{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathfrak{T}$, since $\mathrm{v}^*(m) = \mathfrak{T}\mathrm{v}^*(m)$,

$$
\left\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\right\|_{\digamma} \leq \alpha^{n-1}\left\|\mathrm{v}_T - \mathrm{v}^*\right\|_{\digamma}. \tag{80}
$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$ and $\kappa = \left\|\mathrm{v}_T - \mathrm{v}^*\right\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathcal{A}, \mathcal{B})$. It follows that

$$
\left|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right| \leq \kappa\alpha^{n-1}\left|\digamma(m)\right|. \tag{81}
$$

Then we obtain

$$
\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{82}
$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

E.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

[59] It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

Consider any convergent subsequence $\{\mathrm{c}_{T-n(i)}(m)\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m))+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)})+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \tag{83}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{84}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

F Equality of c and p Growth with Transitory Shocks

Section 4.1 asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\hat{m}$ is the invariant mean level of m we can define an 'average marginal propensity to save away from $\hat{m}$' function:

$$\bar{\mathrm{a}}'(\Delta) = \Delta^{-1}\int_{\hat{m}}^{\hat{m}+\Delta} \mathrm{a}'(z)dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$, so if we define $\hat{a}$ as the value of a corresponding to $m = \hat{m}$, we can write

$$a_{t+1,i} = \hat{a} + (m_{t+1,i} - \hat{m})\bar{\mathrm{a}}'(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \hat{m})$$

so

$$\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \mathrm{cov}_t\left(\bar{\mathrm{a}}'(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \hat{m}), \Gamma\mathbf{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho} < \bar{\mathrm{a}}'(m) < \text{Þ}_{\mathsf{R}}$,

$$|\mathrm{cov}_t((\wp\mathsf{R}\beta)^{1/\rho}a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(\text{Þ}a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC-Nrm says that $\text{Þ} < \Gamma$, while the FHWC says that $\Gamma < \mathsf{R}$; combining these facts we get:

$$|\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma|\mathrm{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathbb{M}[\mathcal{R}_{t+1,i}]$ and consider a first order Taylor expansion of $\bar{\mathrm{a}}'(m_{t+1,i})$ around $\hat{m}_{t+1,i} = \breve{\mathcal{R}} a_{t,i} + 1$,

$$\bar{\mathrm{a}}'_{t+1,i} \approx \bar{\mathrm{a}}'(\hat{m}_{t+1,i}) + \bar{\mathrm{a}}''(\hat{m}_{t+1,i})\,(m_{t+1,i} - \hat{m}_{t+1,i})\,.$$

The problem comes from the $\bar{\mathrm{a}}''$ term (which we implicitly define as the derivative of $\bar{\mathrm{a}}'$). The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\mathrm{a}}'$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\mathrm{a}}''$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

G The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (6) can be rewritten

$$\begin{aligned}
\mathrm{e}_t(m_t)^{-\rho} &= \beta\mathsf{R}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t)+\Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t}\right)\right)^{-\rho}\right] \\
&= \quad (1-\wp)\beta\mathsf{R}m_t^{\rho}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0\right] \\
&\quad + \wp\beta\mathsf{R}^{1-\rho}\,\mathbb{E}_t\left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t))\frac{m_t - \mathrm{c}_t(m_t)}{m_t}\right)^{-\rho} \mid \xi_{t+1} = 0\right].
\end{aligned}$$

Consider the first conditional expectation in (6), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m\downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned}(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1}\\ \bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}\\ \bar{\kappa}_t^{-1} &= 1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}.\end{aligned}$$

As noted in the main text, we need the WRIC (34) for this to be a convergent sequence:

$$0 \le \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{85}$$

Since $\bar{\kappa}_T = 1$, iterating (85) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{86}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (85) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} \quad = \beta\mathsf{R}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\,(\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho}\right]$$

and using L'Hôpital's rule $\lim_{m_t\to\infty}\mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty}\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned}(m_t\underline{\kappa}_t)^{-\rho} &= \beta\mathsf{R}\big(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\big)^{-\rho}\\ \underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\text{Þ}_{\mathsf{R}}=(1-\underline{\kappa})}\ \underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1}\end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \le \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1+\underline{\kappa}_{t+1}^{-1}\text{Þ}_{\mathsf{R}} \tag{87}$$

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n\uparrow\infty}\kappa_{T-n}^{-1} \tag{88}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty}\underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t\underbrace{\left(1+\text{Þ}_{\mathsf{R}}+\text{Þ}_{\mathsf{R}}^2+\cdots\right)}_{=1+\text{Þ}_{\mathsf{R}}(1+\text{Þ}_{\mathsf{R}}\underline{\kappa}_{t+2}^{-1})\ldots} = c_t\underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{89}$$

H The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{90}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{91}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (91) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{92}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (92), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\theta}}^{\bar{\theta}} (\mathrm{c}_T(a+\theta/(1-\wp)))^{-\rho} d\mathcal{F}_\theta\right], \tag{93}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{94}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0}\mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (92) and (94) are identical, this means that $\lim_{\wp\downarrow 0}\mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (93) is $\lim_{a\downarrow 0}\wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0}\mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0}\mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0}\mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (90). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (86) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

I Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (Carroll (2006)): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[60]

$$\begin{aligned}
\mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))] \\
\vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}.
\end{aligned}$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (95) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\begin{aligned}
\mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}_{t+1}^m\mathcal{R}_{t+1}] \\
&= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m] \\
\mathfrak{c}_t^a &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}_{t+1}^m]/\mathrm{u}''(\mathfrak{c}_t)
\end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$\begin{aligned}
c &= \mathfrak{m} - a \\
\mathfrak{c}^a + 1 &= \mathfrak{m}^a
\end{aligned}$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$\begin{aligned}
\mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1+\mathfrak{c}^a)
\end{aligned}$$

[60] The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

J The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, \cdots\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[61] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_\#, m^2_\#, \cdots\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathrm{c}(m)$ for integer values of n (e.g., $c^n_\# = \mathbf{Þ}_\Gamma^{-n}$) are continuous functions of n; the conclusion that $\mathrm{c}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\check{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[62]

This strategy generates a smooth limiting consumption function — except at the remaining kink point defined by $\{m^0_\#, c^0_\#\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_\#$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_\#} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_\#} \mathrm{c}'(m) = \kappa^0_\# < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\check{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\check{\mathrm{c}}(m)$ between $m^0_\#$ and $m^{\underline{n}}_\#$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

[61] Unless $\beta \approx +0$.

[62] In practice, we calculate the first and second derivatives of c and use piecewise polynomial approximation methods that match the function at these points.

1. $\hat{c}(m^0_\#) = c^0_\#$
2. $\hat{c}'(m^0_\#) = 1$
3. $\hat{c}'(m^{\underline{n}}_\#) = (d\mathrm{c}^n_\#/dn)(d\mathrm{m}^n_\#/dn)^{-1}|_{n=\underline{n}}$
4. $\hat{c}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\#/dn^2)(d^2\mathrm{m}^n_\#/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\check{c}(m)$ and fidelity of that function to the $\mathrm{c}(m)$ (see the actual code for details).

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \le m^0_\# & m \\ m^0_\# < m < m^{\underline{n}}_\# & \check{c}(m) \\ m^{\underline{n}}_\# < m & \mathrm{c}(m) \end{cases} \tag{95}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\check{c}(m) \ge \mathrm{c}(m)$, implicitly defining $m = e^\mu$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^\mu)/\mathrm{c}_T(e^\mu)) \tag{96}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \check{c}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi'_t(\mu) = \left(\frac{-\left(\frac{\mathrm{c}'_t\mathrm{c}_T - \mathrm{c}_t\mathrm{c}'_T}{\mathrm{c}_T^2} e^\mu\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{97}$$

which can be solved for

$$\mathrm{c}'_t = (\mathrm{c}_t\mathrm{c}'_T/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi'_t. \tag{98}$$

Similarly, we can solve (96) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right)\mathrm{c}_T(m). \tag{99}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

K Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures **??**, 3).

$$\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \textbf{Þ} \arrow[rr, "\text{GIC}(\equiv \textbf{Þ} < \Gamma)"] \arrow[dr, "(\textbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] & \arrow[d, phantom, "\neq"] & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}$$

Figure 9 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

K.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 9, whose three nodes represent values of the absolute patience factor **Þ**, the permanent-income growth factor Γ, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from **Þ** to Γ, reflects imposition of the GIC condition (clicking GIC should take you to its definition; definitions of other conditions are also linked below)).[63] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the **Þ** node and impose the GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that **Þ** $< \mathsf{R}$.

One could also impose **Þ** $< \mathsf{R}$ directly (without imposing GIC and FHWC) by following the downward-sloping diagonal arrow exiting **Þ**. Although alternate routes from one node to another all justify the same core conclusion (**Þ** $< \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[64] to indicate that the diagram is not commutative.[65]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, ~~RIC~~ $\equiv$ **Þ** $> \mathsf{R}$, would be represented by moving the

[63]For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[64]For a popular introduction to category theory, see Riehl (2017).

[65]But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

arrowhead from the bottom right to the top left of the line segment connecting **Þ** and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and Γ, but the Γ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed GIC (that is, if we imposed ~~GIC~~), that would take us to the **Þ** node, and we could deduce $\mathsf{R} > \text{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the **Þ** node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~.[66]

Under these conventions, Figure **??** in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

$$
\begin{tikzcd}
\text{Þ} \arrow[r, "\text{GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] & \Gamma \arrow[d, "\text{FHWC}"] \arrow[dl, phantom, "\neq", pos=0.3] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] \arrow[ur, phantom, "\neq", pos=0.3] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 10 Relation of GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (23)

This diagram can be interpreted, for example, as saying that, starting at the **Þ** node, it is possible to derive the PF-FVAC[67] by imposing both the GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition

[66]The corresponding algebra is

$$
\begin{aligned}
\cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\
\cancel{\text{GIC}}: &\quad \Gamma < \text{Þ} \\
\Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \text{Þ},
\end{aligned}
$$

.

[67]in the form $\text{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\text{Þ} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FHWC}: &\quad \Gamma < \mathsf{R} \\ \cancel{\text{RIC}}: &\quad \mathsf{R} < \text{Þ} \\ &\quad \Gamma < \text{Þ} \end{aligned} \tag{100}$$

which leads to the negation of both of the conditions leading into Þ. ~~GIC~~ is obtained directly as the last line in (100) and ~~PF-FVAC~~ follows if we start by multiplying the Return Patience Factor (RPF$=\text{Þ}/\mathsf{R}$) by the FHWF ($=\Gamma/\mathsf{R}$) raised to the power $1/\rho-1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$\begin{aligned} 1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \text{ from FHWC}} \\ 1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right) \\ \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma &< \text{Þ} \end{aligned}$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

L When Is Consumption Growth Declining in m?

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

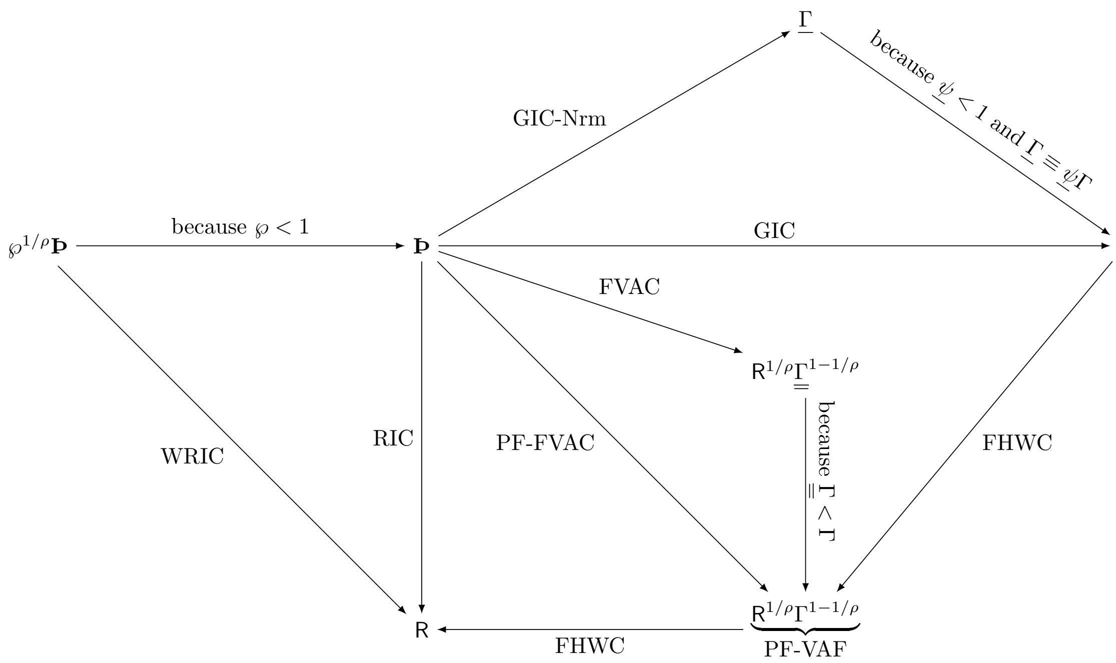

Figure 11 Relation of All Inequality Conditions

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t)-\mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{101}$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (101) in brackets can be manipulated to yield

$$\begin{aligned}\mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\ &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}.\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned}1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho}_{t+1}] \\ 0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}\boldsymbol{\Upsilon}'_{t+1}] \\ &= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) \\ \mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]\end{aligned} \tag{102}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (102) that (101) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (102), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

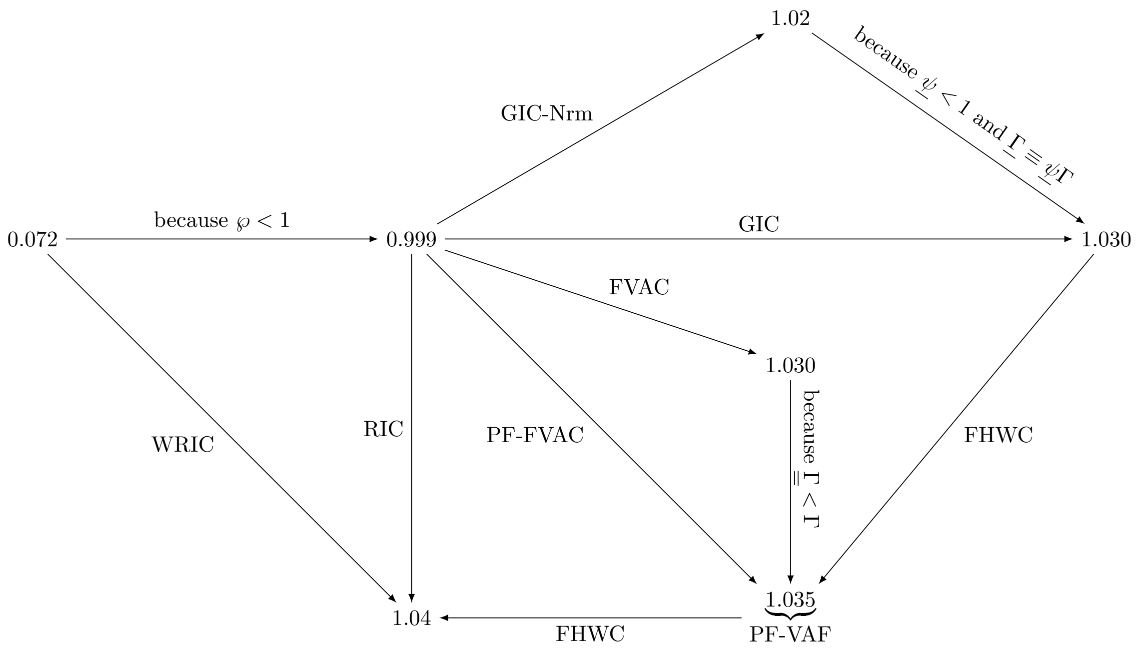

Figure 12 Numerical Relation of All Inequality Conditions

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\mathrm{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

M Unique, Stable Target and Steady State Points

This appendix proves Theorems 2-3 and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

M.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

M.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

M.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

M.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (40), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\Thorn}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\Thorn}/\Gamma_{t+1}] \\
&< 1
\end{aligned} \tag{103}$$

where the inequality reflects imposition of the GIC-Nrm (28).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (32)) means that the limit of the RHS of (103) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

M.3.2 Existence of m > 1 where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{104}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{105}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (19) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (105) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1. \tag{106}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (106).

M.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

M.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in M.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

M.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection M.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t}\right] \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} \qquad (107)\\
&= \text{Þ}_{\Gamma} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (22).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,\end{aligned} \tag{108}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t)\right] \\ &= (\mathsf{R}/\Gamma)\left(1 - \mathrm{c}'(m_t)\right) - 1.\end{aligned} \tag{109}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails ($\cancel{\text{RIC}}$).

If RIC holds. Equation (19) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}\mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} - 1\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\Gamma} < 1$.

If RIC fails. Under $\cancel{\text{RIC}}$, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (109) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1. \tag{110}$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (110) holds).

M.5 A Third Measure

A footnote in Section 3 mentions the possibility of calculating growth in the expectation of the log of m rather than the expectation of the ratio. Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the ratio.

$$\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right) = \log(\Gamma\psi_{t+1}m_{t+1}) - \log m_t$$

$$
\begin{aligned}
&= \log \Gamma(a_t\mathcal{R} + \psi_{t+1}\xi_{t+1}) - \log m_t \\
&= \log \Gamma(a_t\mathcal{R} + 1 + (\psi_{t+1}\xi_{t+1} - 1)) - \log m_t
\end{aligned}
$$

Now define $\tilde{m}_{t+1} = a_t\mathcal{R} + 1$, and compute the expectation:

$$
\begin{aligned}
\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\mathbf{m}_t)] &= \mathbb{E}_t\left[\log \Gamma(\tilde{m}_{t+1} + (\psi_{t+1}\xi_{t+1} - 1))\right] - \log m_t \\
&= \log \Gamma + \mathbb{E}_t\left[\log(\tilde{m}_{t+1}(1 + \tilde{m}_{t+1}^{-1}(\psi_{t+1}\xi_{t+1} - 1))\right] - \log m_t \\
&= \underbrace{\log \Gamma + \log \tilde{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t\left[\log(1 + \tilde{m}_{t+1}^{-1}(\psi_{t+1}\xi_{t+1} - 1))\right]
\end{aligned}
$$

and exponentiating tells us that

$$
\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \exp(\mathbb{E}_t\left[\log(1 + \tilde{m}_{t+1}^{-1}(\psi_{t+1}\xi_{t+1} - 1))\right]) \tag{111}
$$

and this latter factor is a number that approaches 1 from below as m_t rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected ratio.

M.6 Proof of Lemma

M.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$
\begin{aligned}
\check{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\
\hat{\mathrm{m}}_{t+1}(a) &= 1 + a\underbrace{\mathcal{R}/\underline{\psi}}_{\bar{\mathcal{R}} > \mathcal{R}}
\end{aligned} \tag{112}
$$

so that we can implicitly define the target and pseudo-steady-state points as

$$
\begin{aligned}
\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\
\check{m} &= \check{\mathrm{m}}_{t+1}(\check{m} - \mathrm{c}(\check{m}))
\end{aligned} \tag{113}
$$

Then subtract:

$$
\begin{aligned}
\hat{m} - \check{m} &= \left(\hat{a}\underline{\psi}^{-1} - \check{a}\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1} - a(\check{m})\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\
&\approx \left(a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\
(\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\text{Þ}_\Gamma<1}) &= (\underline{\psi}^{-1} - 1)\hat{a}\mathcal{R}
\end{aligned} \tag{114}
$$

The RHS of this equation is strictly positive because $\underline{\psi}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[68]

[68] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a' over the interval to remove the approximation in the derivations above.

M.6.2 The m Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State m

Expected log balanced growth occurs when

$$\begin{aligned} \mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \Gamma \mathbf{m}_t \\ \mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \Gamma \mathbf{p}_t m_t \\ \mathbb{E}_t[\log \psi_{t+1} m_{t+1}] &= \log \Gamma m_t \\ \mathbb{E}_t[\log (a(m_t)\mathsf{R} + \psi_{t+1}\xi_{t+1}\Gamma)] &= \log \Gamma m_t \\ \mathbb{E}_t[\log (a(m_t)\mathcal{R} + \psi_{t+1}\xi_{t+1})] &= \log m_t \end{aligned} \tag{115}$$

and we call the m that satisfies this equation $\tilde{m}$.

Subtract the definition of $\check{m}$ from that of $\tilde{m}$:

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \psi_{t+1}\xi_{t+1})]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m} \tag{116}$$

Now we use the fact that the expectation of the log is less than the log of the expectation,

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \psi_{t+1}\xi_{t+1})]) < (a(\tilde{m})\mathcal{R} + 1) \tag{117}$$

so

$$\begin{aligned} \exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + 1)]) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\ (a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\ (a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\ (a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{\mathrm{a}}')\mathcal{R} &< \tilde{m} - \check{m} \\ (\tilde{m} - \check{m})\bar{\mathrm{a}}'\mathcal{R} &< \tilde{m} - \check{m} \\ \underbrace{\bar{\mathrm{a}}'\mathcal{R}}_{<\text{Þ}_\Gamma} &< 1 \end{aligned} \tag{118}$$

where we are interpreting $\bar{\mathrm{a}}'$ as the mean of the value of a' over the interval between $\tilde{m}$ and $\check{m}$.

N Balanced Growth in $\mathfrak{c}$ and $\mathrm{cov}(c, \mathbf{p})$

Section 4.2 demonstrates some propositions under the assumption that, when an economy satisfies the GIC, there will be constant growth factors $\Omega_{\mathfrak{c}}$ and Ω_{cov} respectively for $\mathfrak{c}$ (the average value of the consumption ratio) and $\mathrm{cov}(c, \mathbf{p})$. In the case of a Szeidl-invariant economy, the main text shows that these are $\Omega_{\mathfrak{c}} = 1$ and $\Omega_{\mathrm{cov}} = \Gamma$. If the economy is Harmenberg- but not Szeidl-invariant, no proof is offered that these growth factors will be constant.

N.1 $\log c$ and $\log(-\operatorname{cov}(c, \mathbf{p}))$ Grow Linearly

Figures 13 and 14 plot the results of simulations of an economy that satisfies Harmenberg- but not Szeidl-invariance with a population of 4 million agents over the last 1000 periods (of a 2000 period simulation). The first figure shows that $\log \mathbf{c}$ increases apparently linearly. The second figure shows that $\log(-\operatorname{cov}(c, \boldsymbol{p}))$ also increases apparently linearly. (These results are produced by the notebook `ApndxBalancedGrowthcNrmAndCov.ipynb`).

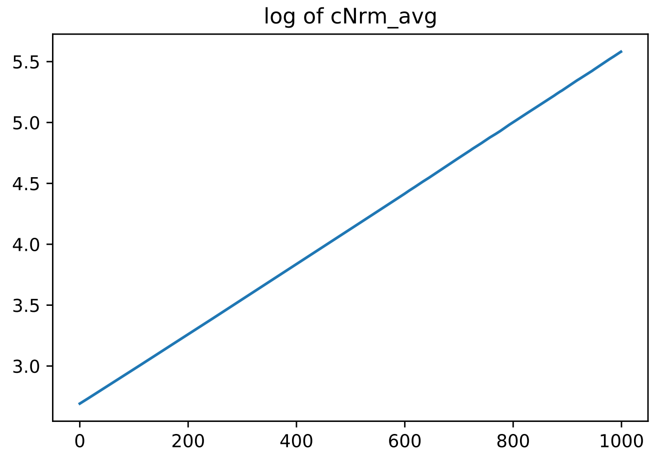

Figure 13 log $\mathfrak{c}$ Appears to Grow Linearly

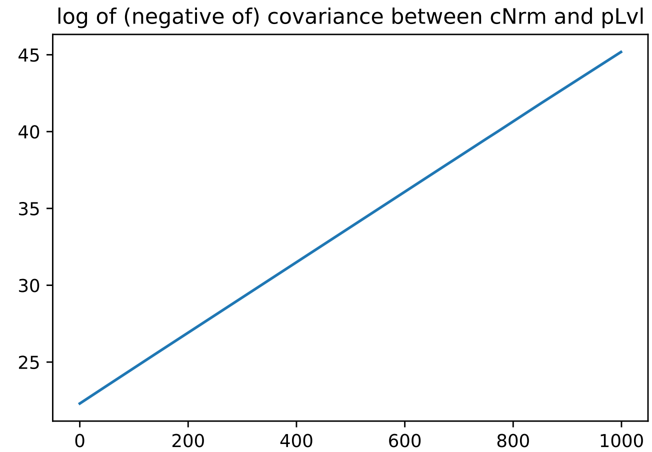

Figure 14 log $-\operatorname{cov}(c, \mathbf{p})$ Appears to Grow Linearly

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Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$\Gamma/\mathsf{R} < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.	$\Gamma/\mathsf{R} < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\mathbf{Þ} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $\mathbf{c}_{t+1} < \mathbf{c}_t$	$\mathbf{Þ} < 1$ *If wealth is large enough,* the *expectation* of consumption next period will be smaller than this period's consumption: $\lim_{m_t \to \infty} \mathbb{E}_t[\mathbf{c}_{t+1}] < \mathbf{c}_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\mathbf{Þ}/\mathsf{R} < 1$ The growth factor for consumption $\mathbf{Þ}$ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite: $\mathrm{c}'(m) = 1 - \mathbf{Þ}/\mathsf{R} < 1$	$\wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $\mathrm{c}'(m) < 1 - \wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$
Growth Impatience Conditions	
GIC	GIC-Nrm
$\mathbf{Þ}/\Gamma < 1$ For an unconstrained PF consumer, the ratio of $\mathbf{c}$ to $\mathbf{p}$ will fall over time. For constrained, guarantees the constraint eventually binds. Guarantees $\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = \mathbf{Þ}_\Gamma$	$\mathbf{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality stronger than GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \mathbf{Þ}_{\underline{\Gamma}}$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta\Gamma^{1-\rho} < 1$ equivalently $\mathbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Consumption Model(s)	Conditions	Comments
$\bar{\mathrm{c}}(m)$: PF Unconstrained	RIC, FHWC[°]	RIC$\Rightarrow$ $\lvert \mathrm{v}(m)\rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m)\rvert$
$\underline{\mathrm{c}}(m) = \underline{\kappa} m$		PF model with no human wealth ($h = 0$)
Section 2.5.3:		RIC prevents $\bar{\mathrm{c}}(m) = \underline{\mathrm{c}}(m) = 0$
Section 2.5.3:		FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
Eq (23):		PF-FVAC+FHWC $\Rightarrow$ RIC
Eq (24):		GIC+FHWC $\Rightarrow$ PF-FVAC
$\grave{\mathrm{c}}(m)$: PF Constrained	~~GIC~~, RIC	FHWC holds ($\Gamma < \text{Þ} < \mathsf{R} \Rightarrow \Gamma < \mathsf{R}$)
Section 2.5.6:		$\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m > m_{\#} < 1$
		(~~RIC~~ would yield $m_{\#} = 0$ so $\grave{\mathrm{c}}(m) = 0$)
Appendix A:	GIC,RIC	$\lim_{m\to\infty} \grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m), \lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
		kinks where horizon to $b = 0$ changes*
Appendix A:	GIC,~~RIC~~	$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0$
		kinks where horizon to $b = 0$ changes*
$\mathrm{c}(m)$: Friedman/Muth	Section 3.1,	$\underline{\mathrm{c}}(m) < \mathrm{c}(m) < \bar{\mathrm{c}}(m)$
	Section 3.2	$\underline{\mathrm{v}}(m) < \mathrm{v}(m) < \bar{\mathrm{v}}(m)$
Section 2.10:	FVAC, WRIC	Sufficient for Contraction
Section 2.12:		WRIC is weaker than RIC
Figure 3:		FVAC is stronger than PF-FVAC
Section 2.12.2:		~~FHWC~~+RIC $\Rightarrow$GIC, $\lim_{m\to\infty} \boldsymbol{\kappa}(m) = \underline{\kappa}$
Section 2.12.1:		~~RIC~~ $\Rightarrow$~~FHWC~~, $\lim_{m\to\infty} \boldsymbol{\kappa}(m) = 0$
Section 3.3:		"Buffer Stock Saving" Conditions
Section 3.3.2:		GIC $\Rightarrow \exists 0 < \check{m} < \infty$
Section 3.3.1:		GIC-Nrm $\Rightarrow \exists 0 < \hat{m} < \infty$

[‡]For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines a unique optimal value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < \mathrm{v}(m) < 0$. °RIC, FHWC are necessary as well as sufficient for the perfect foresight case. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc. **In the Friedman/Muth model, the RIC+FHWC are sufficient, but *not* necessary for nondegeneracy

Table 5 Taxonomy of Perfect Foresight Liquidity Constrained Outcomes

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~GIC~~	$1 < \text{Þ}/\Gamma$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \Gamma$); $\grave{\mathrm{c}}(m) = \bar{\mathrm{c}}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	$\grave{\mathrm{c}}(m)$ is degenerate: $\grave{\mathrm{c}}(m) = 0$
GIC	$\text{Þ}/\Gamma < 1$	Constraint binds in finite time for any m
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC may or may not hold
		$\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m) - \grave{\mathrm{c}}(m) = 0$
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	~~FHWC~~
		$\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GICholds, the constraint will bind in finite time.