

1 Existence of Concave c Function

To show that (5) defines a sequence of continuously differentiable strictly increasing concave functions $\{c_T, c_{T-1}, \dots, c_{T-k}\}$, we start with a definition. We will say that a function $n(z)$ is ‘nice’ if it satisfies

1. $n(z)$ is well-defined iff $z > 0$
2. $n(z)$ is strictly increasing
3. $n(z)$ is strictly concave
4. $n(z)$ is \mathbf{C}^3
5. $n(z) < 0$
6. $\lim_{z \downarrow 0} n(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z \downarrow 0} n'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $v_T(m) = u(m)$ and $u(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $v_t(a)$ as

$$v_t(a) = \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})]. \quad (1)$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} v_t(a) = -\infty$ and $\lim_{a \downarrow 0} v'_t(a) = \infty$. So $v_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $v_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{v}_t(m, c)$ as

$$\underline{v}_t(m, c) = u(c) + v_t(m - c) \quad (2)$$

which is \mathbf{C}^3 since v_t and u are both \mathbf{C}^3 , and note that our problem’s value function defined in (5) can be written as

$$v_t(m) = \max_c \underline{v}_t(m, c). \quad (3)$$

\underline{v}_t is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c \downarrow 0} \underline{v}_t(m, c) = \lim_{c \uparrow m} \underline{v}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{v}_t(m, c)}{\partial c^2} < 0$, $\lim_{c \downarrow 0} \frac{\partial \underline{v}_t(m, c)}{\partial c} = +\infty$, and $\lim_{c \uparrow m} \frac{\partial \underline{v}_t(m, c)}{\partial c} = -\infty$. It follows that the $c_t(m)$ defined by

$$c_t(m) = \arg \max_{0 < c < m} \underline{v}_t(m, c) \quad (4)$$

exists and is unique, and (5) has an internal solution that satisfies

$$u'(c_t(m)) = v'_t(m - c_t(m)). \quad (5)$$

Since both u and v_t are strictly concave, both $c_t(m)$ and $a_t(m) = m - c_t(m)$ are strictly increasing. Since both u and v_t are three times continuously differentiable, using (5) we can conclude that $c_t(m)$ is continuously differentiable and

$$c'_t(m) = \frac{v''_t(a_t(m))}{u''(c_t(m)) + v''_t(a_t(m))}. \quad (6)$$

Similarly we can easily show that $c_t(m)$ is twice continuously differentiable (as is $a_t(m)$) (See Appendix 2.) This implies that $v_t(m)$ is nice, since $v_t(m) = u(c_t(m)) + v_t(a_t(m))$.

2 $c_t(m)$ is Twice Continuously Differentiable

First we show that $c_t(m)$ is \mathbf{C}^1 . Define y as $y \equiv m + dm$. Since $u'(c_t(y)) - u'(c_t(m)) = v'_t(a_t(y)) - v'_t(a_t(m))$ and $\frac{a_t(y) - a_t(m)}{dm} = 1 - \frac{c_t(y) - c_t(m)}{dm}$,

$$\frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} = \left(\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} \right) \frac{c_t(y) - c_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm \rightarrow +0} \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} < 0$ and $\lim_{dm \rightarrow +0} \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ are satisfied. Then $\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ for sufficiently small dm . Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)}}{\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{v'_t(a_t(y)) - v'_t(a_t(m))}{a_t(y) - a_t(m)}}.$$

This implies that the right-derivative, $c_t'^+(m)$ is well-defined and

$$c_t'^+(m) = \frac{v_t''(a_t(m))}{u''(c_t(m)) + v_t''(a_t(m))}.$$

Similarly we can show that $c_t'^+(m) = c_t'^-(m)$, which means $c_t'(m)$ exists. Since v_t is \mathbf{C}^3 , $c_t'(m)$ exists and is continuous. $c_t'(m)$ is differentiable because v_t'' is \mathbf{C}^1 , $c_t(m)$ is \mathbf{C}^1 and $u''(c_t(m)) + v_t''(a_t(m)) < 0$. $c_t''(m)$ is given by

$$c_t''(m) = \frac{a_t'(m)v_t'''(a_t) [u''(c_t) + v_t''(a_t)] - v_t''(a_t) [c_t'u'''(c_t) + a_t'v_t'''(a_t)]}{[u''(c_t) + v_t''(a_t)]^2}. \quad (7)$$

Since $v_t''(a_t(m))$ is continuous, $c_t''(m)$ is also continuous.

3 \mathcal{T} Is a Contraction Mapping

We must show that our operator \mathcal{T} satisfies all of Boyd's conditions.

Boyd's operator \mathcal{T} maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}z\}$ be continuous for any F -bounded z , $\{\mathcal{T}z\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\begin{aligned} \{\mathcal{T}x\}(m_t) & \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} x(m_{t+1})] \right\} \\ \{\mathcal{T}y\}(m_t) & \text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} y(m_{t+1})] \right\}, \end{aligned}$$

so $x(\bullet) \leq y(\bullet)$ implies $\{\mathcal{T}x\}(m_t) \leq \{\mathcal{T}y\}(m_t)$ by inspection.¹

Condition (2) requires that $\{\mathcal{J}0\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$. By definition,

$$\{\mathcal{J}0\}(m_t) = \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

the solution to which is patently $u(\bar{\kappa}m_t)$. Thus, condition (2) will hold if $(\bar{\kappa}m_t)^{1-\rho}$ is F -bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \quad (8)$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of F , $\{\mathcal{J}0\}(m_t) = u(\bar{\kappa}m_t)$ is clearly F -bounded.

Finally, we turn to condition (3), $\{\mathcal{T}(z + \zeta F)\}(m_t) \leq \{\mathcal{T}z\}(m_t) + \zeta \alpha F(m_t)$. The proof will be more compact if we define \check{c} and \check{a} as the consumption and assets functions² associated with $\mathcal{T}z$ and \hat{c} and \hat{a} as the functions associated with $\mathcal{T}(z + \zeta F)$; using this notation, condition (3) can be rewritten

$$u(\hat{c}) + \beta \{E(z + \zeta F)\}(\hat{a}) \leq u(\check{c}) + \beta \{Ez\}(\check{a}) + \zeta \alpha F.$$

Now note that if we force the \cup consumer to consume the amount that is optimal for the \wedge consumer, value for the \cup consumer must decline (at least weakly). That is,

$$u(\hat{c}) + \beta \{Ez\}(\hat{a}) \leq u(\check{c}) + \beta \{Ez\}(\check{a}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned} u(\hat{c}) + \beta \{E(z + \zeta F)\}(\hat{a}) &\leq u(\hat{c}) + \beta \{Ez\}(\hat{a}) + \zeta \alpha F \\ \beta \{E(z + \zeta F)\}(\hat{a}) &\leq \beta \{Ez\}(\hat{a}) + \zeta \alpha F \\ \beta \zeta \{EF\}(\hat{a}) &\leq \zeta \alpha F \\ \beta \{EF\}(\hat{a}) &\leq \alpha F \\ \beta \{EF\}(\hat{a}) &< F. \end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.³

Using $F(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{a}(m_t)$, this condition is

$$\beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta \mathbb{E}_t \Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing **PF-FVAC** (equation (23), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho}}{1 - \beth}. \quad (9)$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

¹For a fixed m_t , recall that m_{t+1} is just a function of c_t and the stochastic shocks.

²Section 2.8 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

³The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (9) is bounded from above:

$$\begin{aligned}
& (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + \wp\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1})^{1-\rho}] - m_t^{1-\rho} \\
& \leq (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + \wp\beta \mathbf{R}^{1-\rho}((1 - \bar{\kappa})m_t)^{1-\rho} - m_t^{1-\rho} \\
& = (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + m_t^{1-\rho} \left(\wp\beta \mathbf{R}^{1-\rho} \left(\wp^{1/\rho} \frac{(\mathbf{R}\beta)^{1/\rho}}{\mathbf{R}} \right)^{1-\rho} - 1 \right) \\
& = (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + m_t^{1-\rho} \left(\underbrace{\wp^{1/\rho} \frac{(\mathbf{R}\beta)^{1/\rho}}{\mathbf{R}}}_{<1 \text{ by WRIC}} - 1 \right) \\
& < (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1 - \wp))^{1-\rho}] = \underline{\mathfrak{Z}}(1 - \wp)^\rho \underline{\theta}^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (9) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\underline{\mathfrak{Z}}(1 - \wp)^\rho \underline{\theta}^{1-\rho}}{1 - \underline{\mathfrak{Z}}} \quad (10)$$

which is a positive finite number under our assumptions.

The proof that \mathcal{T} defines a contraction mapping under the conditions (35) and (32) is now complete.

3.1 \mathcal{T} and v

In defining our operator \mathcal{T} we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (36)) that $\underline{\kappa}_t m_t \leq c_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (5) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the **WRIC** must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (10). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{aligned}
& \wp\beta(\mathbf{R}(1 - \bar{\kappa}_{T-1}))^{1-\rho} < 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - \bar{\kappa}_{T-1}) > 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - (1 + \wp^{1/\rho}\mathbf{P}_\mathbf{R})^{-1}) > 1
\end{aligned}$$

where we have used (34) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1 - \rho$). To see that this is a weak condition, note that for small values of \wp this expression can be further simplified using $(1 + \wp^{1/\rho}\mathbf{P}_\mathbf{R})^{-1} \approx 1 - \wp^{1/\rho}\mathbf{P}_\mathbf{R}$ so that it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\mathbf{P}_\mathbf{R} > 1$$

$$\begin{aligned}
(\wp\beta)\wp^{(1-\rho)/\rho}\mathbf{P}_R^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\mathbf{P}_R^{1-\rho} &< 1.
\end{aligned}$$

Calling the weak return patience factor $\mathbf{P}_R^\wp = \wp^{1/\rho}\mathbf{P}_R$ and recalling that the **WRIC** was $\mathbf{P}_R^\wp < 1$, the expression on the LHS above is $\beta\mathbf{P}_R^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\mathbf{P}_R \approx 1$, this condition is clearly not very different from the **WRIC**.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $v(m)$. But since $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $v(m)$ toward which these v_{T-n} 's are converging is the *same* $v(m)$ that was the endpoint of the contraction defined by our operator \mathcal{T} . Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (5) converge, they converge to the same unique v defined by \mathcal{T} .⁴

⁴It seems likely that convergence of the value functions for the original problem could be proven even if only the **WRIC** were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

References

HIRAGUCHI, RYOJI (2003): “On the Convergence of Consumption Rules,” *Manuscript*,
Johns Hopkins University.