1 Convergence in Euclidian Space

1.1 Convergence of v_t

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathcal{T}$, since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\left\| \mathrm{v}_{T-n+1} - \mathrm{v}^* \right\|_{\digamma} \leq \alpha^{n-1} \left\| \mathrm{v}_T - \mathrm{v}^* \right\|_{\digamma} . \tag{1}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}\left(\mathcal{A}, \mathcal{B}\right)$ and $\kappa = \left\| \mathrm{v}_T - \mathrm{v}^* \right\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}\left(\mathcal{A}, \mathcal{B}\right)$. It follows that

$$\left| \mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m) \right| \leq \kappa \alpha^{n-1} \left| \digamma(m) \right| . \tag{2}$$

Then we obtain

$$\lim_{n \to \infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{3}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa} m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathcal{T}\mathrm{v}_{T-1} \leq \mathcal{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\left\{ \mathrm{v}_{T-n+1}(m) \right\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

1.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\left\{ \mathrm{c}_{T-n+1}(m) \right\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{ \mathrm{c}_{T-n(i)}(m) \right\}$ of $\left\{ \mathrm{c}_{T-n+1}(m) \right\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m)) + \beta\, \mathbb{E}_{T-n(i)}[\boldsymbol{\phi}_{T-n(i)+1}^{1-\rho} \mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\, \mathbb{E}_{T-n(i)}[\boldsymbol{\phi}_{T-n(i)+1}^{1-\rho} \mathrm{v}_{T-n(i)+1}(m)], \tag{4}$$

for any $c_{T-n(i)} \in [\underline{\kappa} m, \bar{\kappa} m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\, \mathbb{E}_t[\boldsymbol{\phi}_t^{1-\rho} \mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\, \mathbb{E}_t[\boldsymbol{\phi}_t^{1-\rho} \mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\, \mathbb{E}_t[\boldsymbol{\phi}_{t+1}^{1-\rho} \mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\, \mathbb{E}_t[\boldsymbol{\phi}_{t+1}^{1-\rho} \mathrm{v}(m)]. \tag{5}$$

Hence, $c^* \in \underset{c_{T-n(i)} \in [\underline{\kappa} m, \bar{\kappa} m]}{\arg\max} \left\{ \mathrm{u}(c_{T-n(i)}) + \beta\, \mathbb{E}_t[\boldsymbol{\phi}_{t+1}^{1-\rho} \mathrm{v}(m)] \right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.