Theoretical Foundations of Buffer Stock Saving

Chris Carroll

Johns Hopkins University

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Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
    - Calibration
    - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

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The Gap This Paper Fills

Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income

- No Liquidity Constraints

- CRRA Utility

- (Problem with Liquidity Constraints Is A Limiting Case)

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Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
    - $\Rightarrow$ $\exists$ A Unique Consumption Function $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
    - Requires A Key 'Impatience' Condition To Hold
    - Good News
        - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

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The Perfect Foresight Problem
The Real Problem

Limit as horizon T goes to infinity of

$$v_{T-n} = \max \ \mathbb{E}_t \left[\sum_{i=0}^{n} \beta^i u(c_{t+i}) \right]$$

$$a_t = m_t - c_t$$
$$k_{t+1} = a_t$$
$$b_{t+1} = k_{t+1}R$$
$$p_{t+1} = p_t \underbrace{\mathbf{\Phi}\Psi_{t+1}}_{\equiv \mathbf{\Phi}_{t+1}}$$
$$m_{t+1} = b_{t+1} + p_{t+1}\boldsymbol{\xi}_{t+1},$$

$$\boldsymbol{\xi}_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \boldsymbol{\theta}_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{1}$$

- $u(\bullet) = \bullet^{1-\rho}/(1-\rho); \ \mathbb{E}_t[\Psi_{t+n}] = \mathbb{E}_t[\boldsymbol{\xi}_{t+n}] = 1 \ \forall \ n > 0; \ \beta < 1, \rho > 1$

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Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

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Benchmark: Perfect Foresight Model

Definitions:

Absolute Patience Factor	$\mathbf{Þ}$	$=$	$(R\beta)^{1/\rho}$
Return Patience Factor	$\mathbf{Þ}_R$	$=$	$\mathbf{Þ}/R$
Perfect Foresight Growth Patience Factor	$\mathbf{Þ}_\Phi$	$=$	$\mathbf{Þ}/\Phi$

Name	Condition		Implication
(AIC) Absolute Impatience Condition	$\mathbf{Þ}$	$< \quad 1$	c $\downarrow$ over time
(RIC) Return Impatience Condition	$\mathbf{Þ}_R$	$< \quad 1$	c/a $\downarrow$ over time
(GIC) Growth Impatience Condition	$\mathbf{Þ}_\Phi$	$< \quad 1$	c/p $\downarrow$ over time

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When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

$$\Phi \quad < \quad R \tag{2}$$

Return Impatience Condition:

$$\mathbb{P}_R \quad < \quad R \tag{3}$$

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When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHWC) condition:

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What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  'Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)':

$$\beta \Phi^{1-\rho} \; < \; 1 \qquad\qquad (4)$$

- Without RIC, Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

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Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$\text{FVAC:} \ \ 0 < \overbrace{\beta\underline{\underline{\Phi}}^{1-\rho}}^{\equiv\underline{\underline{\beth}} \ = \ \text{VAF}} < 1 \tag{5}$$

$$0 < \beta < \underline{\underline{\Phi}}^{\rho-1},$$

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Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$\text{FVAC: } 0 < \overbrace{\beta\underline{\underline{\Phi}}^{1-\rho}}^{\equiv\underline{\underline{\beth}} \,=\, \text{VAF}} < 1 \tag{6}$$
$$0 < \beta < \underline{\underline{\Phi}}^{\rho-1},$$

'Weak Return Impatience Condition' (WRIC)

$$0 \le \wp^{1/\rho}\Phi_{\mathsf{R}} < 1 \tag{7}$$

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Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\boldsymbol{\Phi}} \equiv \boldsymbol{\Phi}\underline{\Psi} < \boldsymbol{\Phi}$$

Adjusted Growth Patience Factor:

$$\text{GPF-Mod:} \quad \Þ_{\underline{\boldsymbol{\Phi}}} = \Þ/\underline{\boldsymbol{\Phi}} = \mathbb{E}[\Þ/(\boldsymbol{\Phi}\Psi)] \tag{8}$$

Growth Impatience Condition:

$$\text{GIC-Mod:} \quad \Þ_{\underline{\boldsymbol{\Phi}}} < 1, \tag{9}$$
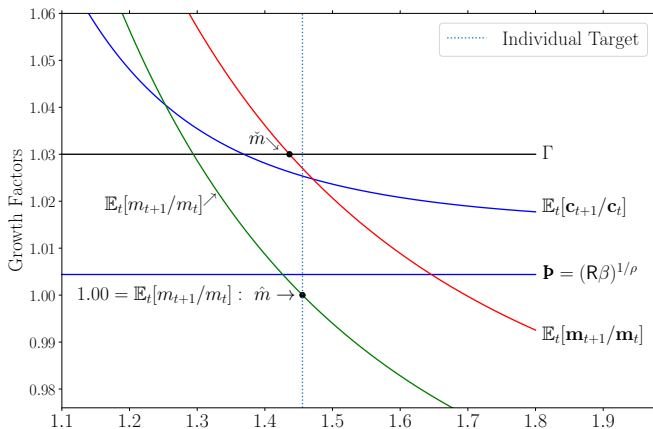
Why? Because it can be shown that

$$\lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right] = \Þ_{\underline{\boldsymbol{\Phi}}} \tag{10}$$

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Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

Five Propositions

1. $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}/c_t] = \textbf{Þ}$
2. $\lim_{m_t \to 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$
3. $\exists$ a unique target value of m, called $\check{m}$
4. $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \boldsymbol{\Phi} - \epsilon$
5. $\left(\frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$
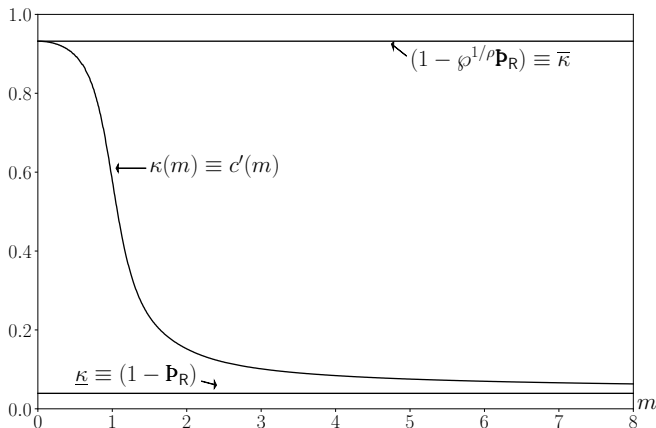
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Bounds On the Consumption Function

c

$\leftarrow \overline{\overline{c}}(m) = \overline{\kappa}m = (1 - \wp^{1/\rho}\mathbf{P}_{\mathsf{R}})m$

$\leftarrow$Upper Bound = Min $[\overline{\overline{c}}(m), \overline{c}(m)]$

$\overline{c}(m) = (m - 1 + h)\underline{\kappa}$

$\leftarrow c(m)$

Lower Bound: $\underline{c}(m) = (1 - \mathbf{P}_{\mathsf{R}})\underline{\kappa}m$

m

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The Marginal Propensity to Consume

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The Invariant Distribution
Balanced Growth Equilbrium

Convergence To The Invariant Distribution

Szeidl (2013) Proves Existence of an Invariant Distribution of m, c, a, etc.

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The Invariant Distribution
Balanced Growth Equilibrium

Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t \;\; = \;\; \Phi \tag{11}$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $\grave{c}(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} c(m; \wp) = \grave{c}(m) \tag{12}$$

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The MPC Out Of Permanent Shocks

https://www.econ2.jhu.edu/people/ccarroll/papers/MPCPerm.pdf

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- MPCP < 1
- But not a lot less:
    - 0.75 to 0.95 (annual rate) for wide range of parameter values

- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents $m \to \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

MATKOWSKI, JANUSZ, AND ANDRZEJ S. NOWAK (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.

SZEIDL, ADAM (2013): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University*, Available at http://www.personal.ceu.hu/staff/Adam_Szeidl/papers/invariant_revision.pdf.