Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds foundations for rigorous and intuitive understanding of 'buffer stock' saving models (Bewley (1977)-like models with a wealth target), pairing each theoretical result with quantitative illustrations. After describing conditions under which a consumption function exists, the paper articulates stricter 'Growth Impatience' conditions that guarantee alternative forms of stability — either at the population level, or for individual consumers. Together, the numerical tools and analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords	Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem
JEL codes	D81, D91, E21

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The paper's results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper's own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to Will Du, James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at the Johns Hopkins University, a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights, and at a presentation at the Australian National University.

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1 Introduction

In the presence of realistic transitory and permanent shocks to income *a la* Friedman (1957) and Muth (1960), only one further ingredient is required to construct a microeconomically testable model of consumption: A description of preferences. Zeldes (1989) was the first to calibrate a quantitatively plausible example; his paper spawned a literature showing that such models' predictions can match household life cycle data reasonably well, whether or not explicit liquidity constraints are imposed.[1]

A connected literature in macroeconomic theory, starting with Bewley (1977), has derived limiting properties of related infinite-horizon problems, but only in models more complex than the case with just shocks and preferences. The extra complexity has been imposed because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et al. (1989)) cannot be applied when utility and/or marginal utility are unbounded. Many proof methods also rule out permanent shocks *a la* Friedman (1957), Muth (1960), and Zeldes (1989).[2]

This paper's first contribution is to articulate conditions under which the infinite-horizon Friedman-Muth(-Zeldes) problem (without complications like a consumption floor or liquidity constraints) defines a contraction mapping problem whose limit is sensible as the horizon approaches infinity. A 'Finite Value of Autarky Condition' is mostly sufficient (the other imposed condition, the 'Weak Return Impatience Condition',[3] is unlikely to bind). Because the infinite horizon solution is the limit of finite-horizon recursions, many intermediate results are also useful for solving finite-horizon problems.

But the paper's main theoretical contribution is to identify, for the infinite-horizon case, conditions under which 'stable' values of the wealth-to-permanent-income ratio exist, either for individual consumers (a consumer's wealth can be predicted to move toward a 'target' ratio) or for the aggregate (the economy as a whole moves toward a 'balanced growth' equilibrium). The requirement for stability is always that the model's parameters satisfy a 'Growth Impatience Condition' whose details depend on the quantity whose stability is of interest. A model that exhibits stability of either kind qualifies as a 'buffer stock' model.[4]

Even without a formal proof of its existence, buffer stock saving has been intuitively understood to underlie central quantitative results in heterogeneous agent macroeconomics; for example, the logic of target saving is central to the claim by Krueger, Mitman, and Perri (2016) in the *Handbook of Macroeconomics* that such models explain why, during the Great Recession, middle-class consumers cut their spending more than the

[1] See Carroll (1997) or Gourinchas and Parker (2002) for arguments that models with only 'natural' constraints (see below) match a wide variety of facts; for a model with explicit constraints that produces very similar results, see, e.g., Cagetti (2003).

[2] See the fuller discussion at the end of Section 2.1.

[3] This is a generalization of a condition in Ma, Stachurski, and Toda (2020).

[4] Such models are neither a subset nor a superset of Bewley (1977) models. But closed economies in which capital results from saving and has declining marginal productivity are always 'buffer stock' economies under some definition of that term, because capital accumulation causes interest rates to fall, which guarantees that a Growth Impatience Condition will hold in equilibrium (see below). The more interesting applications are to populations (or economies) whose marginal saving behavior does not determine the relevant interest rate, or in which the marginal product of capital does not fall as capital is accumulated (again, see below).

poor or the rich. The theory below provides the rigorous basis for this claim: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$ means they — optimally — have little room to maneuver). And, increased labor income uncertainty does not much change the behavior of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to respond by reducing their spending.

Analytical derivations for the proofs also explain many other results familiar from the numerical literature.

The paper begins by defining sufficient conditions for the problem to define a useful (nondegenerate) limiting consumption function (and explains how the model relates to those previously considered). The conditions are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. This analysis establishes limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; using these limits, the contraction mapping theorem is proven.

The next theoretical contribution demonstrates that a corresponding model with an 'artificial' liquidity constraint (a model that prohibits borrowing by consumers who could certainly repay) is a limiting case of the model without constraints. The analytical appeal of the unconstrained model is that it is both mathematically convenient (e.g., the consumption function is twice continuously differentiable), and arbitraily close (cf. Section **??**) to less tractable models. The congenial environment makes the proof easier, and we define the analogous proposition as holding (in the limit) if it continues to hold as the horizon extends to infinity.

In proving the remaining theorems, the next section examines the key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Next, the central theorems articulate conditions under which different measures of 'growth impatience' imply useful conclusions about points of stability ('target' or 'balanced growth' points).

The final section elaborates the conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, a small open economy populated by buffer stock consumers has a balanced growth equilibrium in which growth rates of consumption, income, and wealth match the exogenous growth rate of permanent income (equivalent, here, to productivity growth). In the terms of Schmitt-Grohé and Uribe (2003), buffer stock saving is an appealing method of 'closing' a small open economy model, because it requires no ad-hoc assumptions. Not even liquidity constraints.[5]

[5]The paper's insights are instantiated in the Econ-ARK toolkit, whose buffer stock saving module flags parametric choices under which a problem is degenerate or under which stable ratios of wealth to income may not exist.

2 The Problem

2.1 Setup

The infinite horizon solution is the (limiting) first-period solution to a sequence of finite-horizon problems as the horizon (the last period of life) becomes arbitrarily distant.

That is, for the value function, fixing a terminal date T, we are interested in the term $\mathbf{v}_{T-n}$ in the sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, \ldots, \mathbf{v}_{T-n}\}$. We will say that the problem has a 'nondegenerate' infinite horizon solution if, corresponding to that $\mathbf{v}$, as $n \uparrow \infty$ there is a limiting consumption function $\mathrm{c}(m) = \lim_{n\uparrow\infty} \mathrm{c}_{T-n}$ which is neither $\mathrm{c}(m) = 0$ everywhere (for all m) nor $\mathrm{c}(m) = \infty$ everywhere.

Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max \ \mathbb{E}_t \left[\sum_{i=0}^{n} \beta^i \mathrm{u}(\mathbf{c}_{t+i}) \right]$$

where the Constant Relative Risk Aversion (CRRA) utility function

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{1}$$

exhibits relative risk aversion $\rho > 1$.[6] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ and permanent noncapital income $\mathbf{p}_t$, which both are positive,

$$\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty), \tag{2}$$

and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{3}$$

In the usual treatment, a dynamic budget constraint (DBC) incorporates several elements that jointly determine next period's $\mathbf{m}$ (given this period's choices); for the detailed analysis here, it will be useful to disarticulate and describe every step:

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{k}_{t+1} &= \mathbf{a}_t \\ \mathbf{b}_{t+1} &= \mathbf{k}_t \mathsf{R} \\ \mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1}, \end{aligned} \tag{4}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which translate one-for-one into capital $\mathbf{k}_{t+1}$ at the beginning of the next period, which (before the consumption choice) grows by a fixed interest factor $\mathsf{R} = (1+\mathsf{r})$, so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[7] $\mathbf{m}_{t+1}$ ('market resources') is

[6] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is omitted because it would be cumbersome.

[7] Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n \geq 1$). Permanent noncapital income in $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1 \ \forall \ n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).

Following Zeldes (1989), in future periods $t+n \ \forall \ n \geq 1$ there is a small probability $\wp$ that income will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{5}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \ \forall \ n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.[8] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (5) and $\mathcal{F}_\theta$). For quick identification in tables and graphs, we will call this the 'Friedman/Muth' model because it is a specific implementation of the Friedman (1957) model as interpreted by Muth (1960).

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[8] Rabault (2002) and Li and Stachurski (2014) analyze cases where the shock processes have unbounded support.

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