Theoretical Foundations of Buffer Stock Saving

Chris Carroll

Johns Hopkins University

September 12, 2019

Powered By

Econ ARK

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
    - Calibration
    - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Drawbacks of Numerical Solutions

A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- *Very* Hard To Teach!

I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

The Gap This Paper Fills

Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income

- No Liquidity Constraints

- CRRA Utility

- (Problem with Liquidity Constraints Is A Limiting Case)

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

The Gap This Paper Fills

Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income
- No Liquidity Constraints
- CRRA Utility
- (Problem with Liquidity Constraints Is A Limiting Case)

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
    - $\Rightarrow$ $\exists$ A Unique Consumption Function $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
    - Requires A Key 'Impatience' Condition To Hold
    - Good News
        - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
    - $\Rightarrow$ $\exists$ A Unique Consumption Function $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
    - Requires A Key 'Impatience' Condition To Hold
    - Good News
        - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Motivation

Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
  - $\Rightarrow$ $\exists$ A Unique Consumption Function $c(m)$
- There Is A 'Target' Ratio Of Assets to Permanent Income
  - Requires A Key 'Impatience' Condition To Hold
  - Good News
    - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Limit as horizon T goes to infinity of

$$a_t = m_t - c_t$$
$$k_{t+1} = a_t$$
$$b_{t+1} = k_{t+1}R$$
$$p_{t+1} = p_t \underbrace{\Gamma \psi_{t+1}}_{\equiv \Gamma_{t+1}}$$
$$m_{t+1} = b_{t+1} + p_{t+1}\xi_{t+1},$$

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1 - \wp) & \text{with probability } (1 - \wp) \end{cases} \quad (1)$$

- $u(\bullet) = \bullet^{1-\rho}/(1 - \rho);\ \mathbb{E}_t[\psi_{t+n}] = \mathbb{E}_t[\xi_{t+n}] = 1\ \forall\ n > 0;\ \beta < 1, \rho > 1$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

Fortunately, the Conclusions Are Simple!

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Benchmark: Perfect Foresight Model

Definitions:

Absolute Patience Factor	Þ	$= (R\beta)^{1/\rho}$
Return Patience Factor	Þ_R	$= \text{Þ}/R$
Perfect Foresight Growth Patience Factor	Þ_Γ	$= \text{Þ}/\Gamma$

Name	Condition		Implication
( ) Absolute Impatience Condition	Þ	$< \quad 1$	$c \downarrow$ over time
( ) Return Impatience Condition	Þ_R	$< \quad 1$	$c/a \downarrow$ over time
( ) Growth Impatience Condition	Þ_Γ	$< \quad 1$	$c/\textbf{\textit{p}} \downarrow$ over time

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

When Does A Useful Limiting Solution Exist?

Finite Human Wealth () condition:

$$\Gamma \quad < \quad R \tag{2}$$

Return Impatience Condition:

$$\mathbb{P}_R \quad < \quad R \tag{3}$$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

When Does A Useful Limiting Solution Exist?

Finite Human Wealth () condition:

$$\Gamma \quad < \quad R \tag{2}$$

Return Impatience Condition:

$$\text{Þ}_R \quad < \quad R \tag{3}$$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

What If There Are Liquidity Constraints?

- is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  'Perfect Foresight Finite Value of Autarky Condition ()':

$$\beta \Gamma^{1-\rho} < 1 \qquad (4)$$

- Without , Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

What If There Are Liquidity Constraints?

- is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is
  'Perfect Foresight Finite Value of Autarky Condition ()':

$$\beta\Gamma^{1-\rho} \; < \; 1 \qquad (4)$$

- Without , Constraints Are Irrelevant
  - Because Wealth Always Wants To Rise, So Constraint Never Binds

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$
: \; 0 < \overbrace{\beta \underline{\underline{\Gamma}}^{1-\rho}}^{\equiv \underline{\underline{\beth}} \, =} < 1 \tag{5}
$$
$$
0 < \beta < \underline{\underline{\Gamma}}^{\rho-1}.
$$

Introduction
The Problem
Features Of The Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$: \ 0 < \overbrace{\beta \underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\exists}}=} < 1 \tag{6}$$
$$0 < \beta < \underline{\underline{\Gamma}}^{\rho-1}.$$

'Weak Return Impatience Condition' ()

$$0 \leq \ \wp^{1/\rho}\mathbf{P}_{\mathrm{R}} \ < 1 \tag{7}$$

Introduction
The Problem
Features Of The Solution
A Small Open Buffer Stock Economy
Conclusions

The Perfect Foresight Problem
The Real Problem

Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\Gamma} \equiv \Gamma \underline{\psi} < \Gamma$$

Adjusted Growth Patience Factor:

$$: \ \Phi_{\underline{\Gamma}} = \Phi / \underline{\Gamma} = \mathbb{E}[\Phi/(\Gamma\psi)] \tag{8}$$

Growth Impatience Condition:

$$: \ \Phi_{\underline{\Gamma}} < 1, \tag{9}$$

Why? Because it can be shown that

$$\lim_{m_t \to \infty} \mathbb{E}_t \left[\frac{m_{t+1}}{m_t} \right] = \Phi_{\underline{\Gamma}} \tag{10}$$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

Five Propositions

1. $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}/c_t] = \text{Þ}$

2. $\lim_{m_t \to 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$

3. $\exists$ a unique target value of m, called $\check{m}$

4. $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \Gamma - \epsilon$

5. $\left(\frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$

Introduction
The Problem
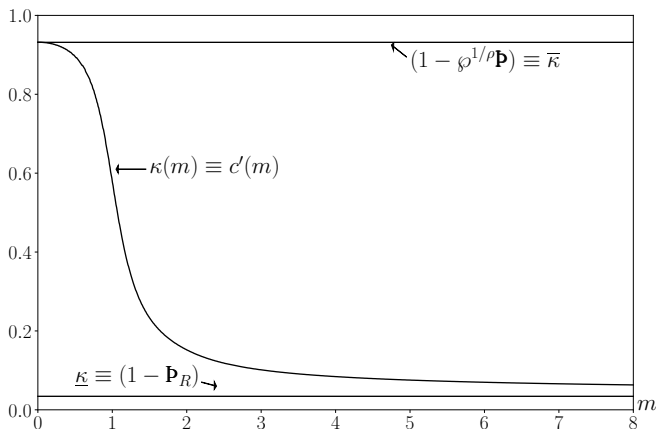Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

The Target Saving Figure

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

Bounds On the Consumption Function

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

The Marginal Propensity to Consume

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

Five Propositions
The Target Saving Figure
Bounds On The Consumption Function
The Consumption Function and Target Wealth

The Consumption Function and Target Wealth

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Invariant Distribution
Balanced Growth Equilbrium

Convergence To The Invariant Distribution

Szeidl (2013) Proves Existence of an Invariant Distribution of m, c, a, etc.

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Invariant Distribution
Balanced Growth Equilibrium

Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t = \Gamma \qquad (11)$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $c(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} c(m; \wp) = c(m) \qquad (12)$$

Introduction
The Problem
Features Of The Solution
A Small Open Buffer Stock Economy
Conclusions

The Invariant Distribution
Balanced Growth Equilibrium

Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t = \Gamma \tag{11}$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $c(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} c(m; \wp) = c(m) \tag{12}$$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Invariant Distribution
Balanced Growth Equilibrium

Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t = \Gamma \tag{11}$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving $\approx$ Liquidity Constraints
- If $\mathrm{c}(m)$ is solution for constrained consumer,

$$\lim_{\wp \downarrow 0} \mathrm{c}(m; \wp) = \mathrm{c}(m) \tag{12}$$

Introduction
The Problem
Features Of the Solution
A Small Open Buffer Stock Economy
Conclusions

The Invariant Distribution
Balanced Growth Equilbrium

The MPC Out Of Permanent Shocks

https://www.econ2.jhu.edu/people/ccarroll/papers/MPCPerm.pdf

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- MPCP < 1
- But not a lot less:
    - 0.75 to 0.95 (annual rate) for wide range of parameter values

- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with)
  - Growth Impatience Condition Prevents $m \to \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

MATKOWSKI, JANUSZ, AND ANDRZEJ S. NOWAK (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.

SZEIDL, ADAM (2013): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University*, Available at http://www.personal.ceu.hu/staff/Adam_Szeidl/papers/invariant_revision.pdf.