1 Unique, Stable Target and Steady State Points

This appendix proves Theorems 2-3 and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

1.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (39), but dropping the $\mathbf{\Phi}_{t+1}$ from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \mathbb{E}_t[(\mathsf{R}/\mathbf{\Phi}_{t+1})\mathbf{Þ}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\mathbf{Þ}/\mathbf{\Phi}_{t+1}] \qquad (1)\\
&< 1
\end{aligned}$$

where the inequality reflects imposition of the GIC-Mod (26).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$ (see equation (30)) means that the limit of the RHS of (1) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \qquad (2)\\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right)\right] \qquad (3)\\
&= \bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1.
\end{aligned}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \mathbf{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1$$

$$
\begin{aligned}
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathbf{\Phi}\mathbf{\Psi}}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathbf{\Phi}\mathbf{\Psi}}\right]}_{=\text{Þ}_{\underline{\mathbf{\Phi}}}} - 1
\end{aligned}
$$

which is negative because the GIC-Mod says $\text{Þ}_{\underline{\mathbf{\Phi}}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (3) is guaranteed to be negative if

$$
\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\mathbf{\Phi}\mathbf{\Psi}}\right] < 1. \tag{4}
$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$
\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\mathbf{\Phi}\mathbf{\Psi}}\right]}^{\text{Þ}_{\underline{\mathbf{\Phi}}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},
$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$
\mathbb{E}_t\left[\frac{\mathsf{R}}{\mathbf{\Phi}\mathbf{\Psi}}\right] < \mathsf{R}/\text{Þ} < 1
$$

which satisfies our requirement in (4).

1.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathbf{\Phi}_{t+1} \left((\mathsf{R}/\mathbf{\Phi}_{t+1}) \mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1} \right) / \mathbf{\Phi}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathsf{R}/\mathbf{\Phi}) \mathrm{a}(m_t) + \mathbf{\Psi}_{t+1} \boldsymbol{\xi}_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\mathbf{\Phi}) \mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\mathbf{\Phi}) \mathbf{Þ}_{\mathsf{R}} \qquad (5) \\
&= \mathbf{Þ}_{\mathbf{\Phi}} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (19).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}] - m_t$ is monotonically decreasing.

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] = 1 \qquad (6) \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\mathbf{\Psi}_{t+1} m_{t+1}/m_t] > 1,
\end{aligned}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t} \right) \boldsymbol{\zeta}(m_t) &= \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) \left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \mathbf{\Psi}_{t+1} \boldsymbol{\xi}_{t+1} - m_t \right) \right] \qquad (7) \\
&= (\mathsf{R}/\mathbf{\Phi}) \left(1 - \mathrm{c}'(m_t) \right) - 1.
\end{aligned}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\mathcal{R} \left(1 - \mathrm{c}'(m_t) \right) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\mathbf{\Phi}) \mathbf{Þ}_{\mathsf{R}} - 1
\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\mathbf{\Phi}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\,(1 - \mathrm{c}'(m_t)) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (7) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\mathbf{\Phi}) < 1. \tag{8}$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (8) holds).

1.5 A Third Measure

A footnote in Section 3 mentions reasons why it may be useful to calculate $\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\log \mathbf{m}_t)]$. Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the growth factor.

$$\begin{aligned}
\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right) &= \log(\mathbf{\Phi}\mathbf{\Psi}_{t+1}m_{t+1}) - \log m_t \\
&= \log \mathbf{\Phi}(a_t\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}) - \log m_t \\
&= \log \mathbf{\Phi}(a_t\mathcal{R} + 1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)) - \log m_t
\end{aligned}$$

Now define $\breve{m}_{t+1} = a_t\mathcal{R} + 1$, and compute the expectation:

$$\begin{aligned}
\mathbb{E}_t[\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right)] &= \mathbb{E}_t\left[\log \mathbf{\Phi}(\breve{m}_{t+1} + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1))\right] - \log m_t \\
&= \log \mathbf{\Phi} + \mathbb{E}_t\left[\log \breve{m}_{t+1}\left(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1}\right)\right] - \log m_t \\
&= \underbrace{\log \mathbf{\Phi} + \log \breve{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]
\end{aligned}$$

and exponentiating tells us that

$$\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]\exp(\mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \tag{9}$$

and this latter factor is a number that approaches 1 from below as m_t rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected growth factor. This implies that the m at which 'balanced growth' can be expected in the log, $\tilde{m}$, exceeds the corresponding point for the ratio, $\breve{m}$.

Furthermore, in the limit as $\mathbf{m}_t$ gets arbitrarily large, if the RIC holds and thus $\underline{\kappa} > 0$, a_{t+1} rises without bound, as does $\breve{m}_{t+1} = a_{t+1}\mathcal{R} + 1$, so the approximation $\log(1+\epsilon) \approx \epsilon$ becomes arbitrarily good. Consequently, the last term on the RHS of (9) can be approximated as

$$\begin{aligned}
\mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) &\approx \mathbb{E}_t\left[(\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \\
&= 0
\end{aligned}$$

This demonstrates that

$$\lim_{\mathbf{m}_t\uparrow\infty} \exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \tag{10}$$

1.6 Proof of Lemma

1.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$\begin{aligned} \check{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\ \hat{\mathrm{m}}_{t+1}(a) &= 1 + a \underbrace{\mathcal{R}/\underline{\mathbf{\Psi}}}_{\bar{\mathcal{R}}>\mathcal{R}} \end{aligned} \tag{11}$$

so that we can implicitly define the target and pseudo-steady-state points as

$$\begin{aligned} \hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\ \check{m} &= \check{\mathrm{m}}_{t+1}(\check{m} - \mathrm{c}(\check{m})) \end{aligned} \tag{12}$$

Then subtract:

$$\begin{aligned} \hat{m} - \check{m} &= \left(\hat{a}\underline{\mathbf{\Psi}}^{-1} - \check{a}\right)\mathcal{R} \\ &= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - a(\check{m})\right)\mathcal{R} \\ &= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\ &\approx \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\ (\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\text{\textbf{Þ}}_{\mathbf{\Phi}}<1}) &= (\underline{\mathbf{\Psi}}^{-1} - 1)\hat{a}\mathcal{R} \end{aligned} \tag{13}$$

The RHS of this equation is strictly positive because $\underline{\mathbf{\Psi}}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[1]

1.6.2 The m Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State m

Expected log balanced growth occurs when

$$\begin{aligned} \mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \mathbf{\Phi}\mathbf{m}_t \\ \mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \mathbf{\Phi}\mathbf{p}_t m_t \\ \mathbb{E}_t[\log \mathbf{\Psi}_{t+1} m_{t+1}] &= \log \mathbf{\Phi} m_t \\ \mathbb{E}_t[\log(a(m_t)\mathsf{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\mathbf{\Phi})] &= \log \mathbf{\Phi} m_t \\ \mathbb{E}_t[\log(a(m_t)\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})] &= \log m_t \end{aligned} \tag{14}$$

and we call the m that satisfies this equation $\tilde{m}$.

Subtract the definition of $\check{m}$ from that of $\tilde{m}$:

$$\exp(\mathbb{E}_t[\log(a(\tilde{m})\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m} \tag{15}$$

Now we use the fact that the expectation of the log is less than the log of the

[1] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a′ over the interval to remove the approximation in the derivations above.

expectation,

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) < (a(\tilde{m})\mathcal{R} + 1) \tag{16}$$

so

$$\begin{aligned}
\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + 1)]) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{\mathrm{a}}')\mathcal{R} &< \tilde{m} - \check{m} \\
(\tilde{m} - \check{m})\bar{\mathrm{a}}'\mathcal{R} &< \tilde{m} - \check{m} \\
\underbrace{\bar{\mathrm{a}}'\mathcal{R}}_{<\text{Þ}_{\boldsymbol{\Phi}}} &< 1
\end{aligned} \tag{17}$$

where we are interpreting $\bar{\mathrm{a}}'$ as the mean of the value of a' over the interval between $\tilde{m}$ and $\check{m}$.