1 Unique And Stable Target and Steady State Points

This appendix proves Theorems 2-4 and the following Lemmas:

Lemma 1. *If* $\check{m}$ *and* $\hat{m}$ *both exist, then* $\check{m} \leq \hat{m}$.

Lemma 2. *If* $\check{m}$ *and* $\tilde{m}$ *both exist, then* $\check{m} \leq \tilde{m}$.

1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

1.3.1 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (45), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned}\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{Þ}_{\mathsf{R}}]\end{aligned}$$

$$\begin{aligned} &= \mathbb{E}_t[\text{Þ}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \tag{1}$$

where the inequality reflects imposition of the GIC-Nrm (33).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (37)) means that the limit of the RHS of (1) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1, \end{aligned} \tag{2}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\ &= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1. \end{aligned} \tag{3}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (21) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\ &= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\ &= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1 \end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (3) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1. \tag{4}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (4).

1.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t)+1}{m_t} \right] \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} \qquad (5)\\
&= \text{Þ}_{\Gamma} \\
&< 1
\end{aligned}
$$

where the last two lines are merely a restatement of the GIC (29).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \qquad (6)\\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned}
$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$
\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t)\right] \qquad (7)\\
&= (\mathsf{R}/\Gamma)\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned}
$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (21) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$
\begin{aligned}
\mathcal{R}\,(1 - \mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} - 1
\end{aligned}
$$

which is negative because the GIC says $\text{Þ}_{\Gamma} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\mathcal{R}\,(1 - \mathrm{c}'(m_t)) < \mathcal{R}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (7) is guaranteed to be negative if

$$
\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1. \qquad (8)
$$

But we showed in Section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (8) holds).

1.5 Proof of Theorem 4

Proof of the existence of the final kind of stability is straightforward.

$$
\begin{aligned}
\log \mathbf{m}_{t+1} - \log \mathbf{m}_t &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \log \Gamma \psi_{t+1} m_{t+1} - \log m_t \\
&= \log \Gamma \psi_{t+1} \left((m_t - c_t)\mathsf{R}/\Gamma \psi_{t+1} + \xi_{t+1} \right) - \log m_t \\
&= \log \left((m_t - c_t)\mathsf{R} + \Gamma \psi_{t+1} \xi_{t+1} \right) - \log m_t \\
&= \log \Gamma \psi_{t+1} + \log \left((m_t - c_t)\mathcal{R}_{t+1} + \xi_{t+1} \right) - \log m_t \\
\mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \log \left(\frac{(m_t - c_t)\mathcal{R}_{t+1} + \xi_{t+1}}{m_t} \right) \\
\mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \log \left((1 - c_t/m_t)\mathcal{R}_{t+1} + \xi_{t+1}/m_t \right).
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
(d/dm_t)\Delta \log \mathbf{m}_{t+1} &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \left(\frac{\mathcal{R}_{t+1}(1 - \kappa(m_t))}{(m_t - c_t)\mathcal{R}_{t+1} + \xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \mathcal{R}_{t+1}^{-1}\xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \Gamma \psi_{t+1}\xi_{t+1}/\mathsf{R}} \right) - 1/m_t
\end{aligned} \tag{10}
$$

As argued elsewhere, expected growth approaches a maximum of ∞ as $m_t \downarrow 0$. As $m_t \uparrow \infty$, the contribution of the term involving ξ becomes negligible and the expenditure ratio $\mathrm{e}(m) = \mathrm{c}(m)/m$ approaches $\underline{\kappa}$, so the limiting growth factor approaches

$$
\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\Delta \log \mathbf{m}_{t+1}] &= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \log \left((1 - \underline{\kappa})\mathcal{R}_{t+1} \right) \\
&= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \left(\log \text{\textbf{Þ}} - (\log \Gamma + \log \psi_{t+1}) \right) \\
&= \log \text{\textbf{Þ}}.
\end{aligned} \tag{11}
$$

The same arguments as before will guarantee continuity over the range of feasible values of m. So all that remains to prove is that $\mathbb{E}_t[\Delta \log \mathbf{m}_{t+1}]$ is strictly decreasing as m rises. Differentiating with respect to m_t yields

$$
\begin{aligned}
(d/dm_t)\, \mathbb{E}_t \Delta \log \mathbf{m}_{t+1} &= (d/dm_t)\, \mathbb{E}_t \log \left((1 - c_t/m_t)\mathcal{R}_{t+1} + \xi_{t+1}/m_t \right) \\
&= \mathbb{E}_t \left(\frac{-\xi_{t+1}/m_t^2 - \mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\xi_{t+1}/m_t + \mathcal{R}_{t+1}(1 - \mathrm{e}(m_t))} \right) \\
&= \mathbb{E}_t \left(\frac{-\xi_{t+1}/m_t - m_t\mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\xi_{t+1} + m_t\mathcal{R}_{t+1}(1 - \mathrm{e}(m_t))} \right) \\
&< \mathbb{E}_t \left(\frac{-\underline{\xi}/m_t - m_t\mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\bar{\xi} + m_t\mathcal{R}_{t+1}(1 - \mathrm{e}(m_t))} \right)
\end{aligned} \tag{12}
$$

and for every realization of ξ and $\mathcal{R}$, the numerator of this expression is negative and the denominator is positive. So the expression's expectation must be negative.

1.6 Proof of Lemmas 1-2

1.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$
\begin{aligned}
\check{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\
\hat{\mathrm{m}}_{t+1}(a) &= 1 + a\underbrace{\mathcal{R}/\underline{\psi}}_{\bar{\mathcal{R}}>\mathcal{R}}
\end{aligned}
\tag{13}
$$

so that we can implicitly define the target and pseudo-steady-state points as

$$
\begin{aligned}
\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\
\check{m} &= \check{\mathrm{m}}_{t+1}(\check{m} - \mathrm{c}(\check{m}))
\end{aligned}
\tag{14}
$$

Then subtract:

$$
\begin{aligned}
\hat{m} - \check{m} &= \left(\hat{a}\underline{\psi}^{-1} - \check{a}\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1} - a(\check{m})\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\
&\approx \left(a(\hat{m})\underline{\psi}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\
(\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\text{Þ}_{\Gamma}<1}) &= (\underline{\psi}^{-1} - 1)\hat{a}\mathcal{R}
\end{aligned}
\tag{15}
$$

The RHS of this equation is strictly positive because $\underline{\psi}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[1]

1.6.2 The m achieving Expected-Log-Balanced-Growth Is Smaller than Pseudo-Steady-State m

Expected log balanced growth occurs when

$$
\begin{aligned}
\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \Gamma \mathbf{m}_t \\
\mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \Gamma \mathbf{p}_t m_t \\
\mathbb{E}_t[\log \psi_{t+1} m_{t+1}] &= \log \Gamma m_t \\
\mathbb{E}_t[\log \left(a(m_t)\mathsf{R} + \psi_{t+1}\xi_{t+1}\Gamma\right)] &= \log \Gamma m_t \\
\mathbb{E}_t[\log \left(a(m_t)\mathcal{R} + \psi_{t+1}\xi_{t+1}\right)] &= \log m_t
\end{aligned}
\tag{16}
$$

and we call the m that satisfies this equation $\tilde{m}$.

Now we use the fact that the expectation of the log is less than the log of the

[1] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a$'$ over the interval to remove the approximation in the derivations above.

expectation,

$$\begin{aligned} \log \mathbb{E}_t[a(\tilde{m}_t)\mathcal{R} + \psi_{t+1}\xi_{t+1}] &< \log \tilde{m}_t \\ \log\left(a(\tilde{m}_t)\mathcal{R} + 1\right) &< \log \tilde{m}_t \\ a(\tilde{m}_t)\mathcal{R} + 1 &< \tilde{m}_t \end{aligned} \tag{17}$$

Finally, subtract $\check{m}$ from both sides,

$$\begin{aligned} a(\tilde{m}_t)\mathcal{R} + 1 - (a(\check{m}_t)\mathcal{R} + 1) &< \tilde{m}_t - \check{m}_t \\ a(\tilde{m}_t) - a(\tilde{m}_t + \check{m} - \tilde{m}_t))\mathcal{R} &< \tilde{m}_t - \check{m}_t \\ a(\tilde{m}_t) - (a(\tilde{m}_t) + (\check{m} - \tilde{m}_t)a'(\tilde{m}))\mathcal{R} &< \tilde{m}_t - \check{m} \\ (\tilde{m}_t - \check{m})a'(\tilde{m})\mathcal{R} &< \tilde{m}_t - \check{m} \\ \underbrace{a'(\tilde{m})\mathcal{R}}_{<\text{Þ}_\Gamma} &< 1 \end{aligned} \tag{18}$$

which again holds because $\text{Þ}_\Gamma < 1$ (and, as above, a proof that does not require the Taylor approximation is available but more cumbersome).