

# 1 Convergence in Euclidian Space

## 1.1 Convergence of $v_t$

Boyd's theorem shows that  $\mathcal{T}$  defines a contraction mapping in a  $F$ -bounded space. We now show that  $\mathcal{T}$  also defines a contraction mapping in Euclidian space.

Calling  $v^*$  the unique fixed point of the operator  $\mathcal{T}$ , since  $v^*(m) = \mathcal{T}v^*(m)$ ,

$$\|v_{T-n+1} - v^*\|_F \leq \alpha^{n-1} \|v_T - v^*\|_F. \quad (1)$$

On the other hand,  $v_T - v^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$  and  $\kappa = \|v_T - v^*\|_F < \infty$  because  $v_T$  and  $v^*$  are in  $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ . It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa \alpha^{n-1} |F(m)|. \quad (2)$$

Then we obtain

$$\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v^*(m). \quad (3)$$

Since  $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$ ,  $v_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < v_T(m)$ . On the other hand,  $v_{T-1} \leq v_T$  means  $\mathcal{T}v_{T-1} \leq \mathcal{T}v_T$ , in other words,  $v_{T-2}(m) \leq v_{T-1}(m)$ . Inductively one gets  $v_{T-n}(m) \geq v_{T-n-1}(m)$ . This means that  $\{v_{T-n+1}(m)\}_{n=1}^\infty$  is a decreasing sequence, bounded below by  $v^*$ .

## 1.2 Convergence of $c_t$

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions  $\{c_{T-n+1}(m)\}_{n=1}^\infty$ .

Consider any convergent subsequence  $\{c_{T-n(i)}(m)\}$  of  $\{c_{T-n+1}(m)\}_{n=1}^\infty$  converging to  $c^*$ . By the definition of  $c_{T-n}(m)$ , we have

$$u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\Phi_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)] \geq u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\Phi_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)], \quad (4)$$

for any  $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$ . Now letting  $n(i)$  go to infinity, it follows that the left hand side converges to  $u(c^*) + \beta \mathbb{E}_t[\Phi_t^{1-\rho} v(m)]$ , and the right hand side converges to  $u(c_{T-n(i)}(m)) + \beta \mathbb{E}_t[\Phi_t^{1-\rho} v(m)]$ . So the limit of the preceding inequality as  $n(i)$  approaches infinity implies

$$u(c^*) + \beta \mathbb{E}_t[\Phi_{t+1}^{1-\rho} v(m)] \geq u(c_{T-n(i)}(m)) + \beta \mathbb{E}_t[\Phi_{t+1}^{1-\rho} v(m)]. \quad (5)$$

Hence,  $c^* \in \arg \max_{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]} \{u(c_{T-n(i)}(m)) + \beta \mathbb{E}_t[\Phi_{t+1}^{1-\rho} v(m)]\}$ . By the uniqueness of  $c(m)$ ,  $c^* = c(m)$ .