# 1 Unique And Stable Target and Steady State Points

This appendix proves Theorems 2 and 3 and

**Lemma 1.** *If both $\breve{m}$ and $\hat{m}$ exist, then $\hat{m} < \breve{m}$.*

## 1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$

- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

### 1.1.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; Theorem 1). (Indeed, Appendix C shows that $c(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all $t$, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if $\xi_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both $a_{t-1}$ and $\mathcal{R}_t$ are strictly positive. With $m_t$ and $m_{t+1}$ both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

### 1.1.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)

2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

4. The Intermediate Value Theorem

*Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$.*

**If RIC holds.** Logic exactly parallel to that of Section 3.1 leading to equation (45), but dropping the $\Gamma_{t+1}$ from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{Þ}_\mathsf{R}] \\
&= \mathbb{E}_t[\text{Þ}/\Gamma_{t+1}] \\
&< 1
\end{aligned} \tag{1}$$

where the inequality reflects imposition of the GIC-Nrm (33).

**If RIC fails.** When the RIC fails, the fact that $\lim_{m\uparrow\infty} c'(m) = 0$ (see equation (37)) means that the limit of the RHS of (1) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

*Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$.*

Paralleling the logic for $c$ in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to $m_t$ is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

*Intermediate Value Theorem.* If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

*1.1.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{2}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right) \boldsymbol{\zeta}(m_t) = \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t)\right] \tag{3}$$

$$= \bar{\mathcal{R}}\left(1 - c'(m_t)\right) - 1.$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

**If RIC holds**. Equation (21) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\bar{\mathcal{R}}\left(1 - c'(m_t)\right) - 1 < \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1$$

$$= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1$$

$$= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1$$

$$= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

**If RIC fails.** Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} c'(m) = 0$. Concavity of the consumption function means that $c'$ is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\left(1 - c'(m_t)\right) < \bar{\mathcal{R}}$$

which means that $\zeta'(m_t)$ from (3) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1. \tag{4}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_\Gamma} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_\mathsf{R}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (4).

## 1.2 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$

- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$

- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

### 1.2.1 Existence and Continuity of The Ratio

Since by assumption $0 < \underline{\psi} \le \psi_{t+1} \le \bar{\psi} < \infty$, our proof in 1.1.1 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

### 1.2.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \le \psi_{t+1} \le \bar{\psi} < \infty$, our proof in Subsection 1.1.1 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to $m_t$ is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to $m_t$ is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as $m_t$ goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$
\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty}\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t}\right] \\
&= (\mathsf{R}/\Gamma)\text{Þ}_\mathsf{R} \tag{5} \\
&= \text{Þ}_\Gamma
\end{aligned}
$$

$$< 1$$

where the last two lines are merely a restatement of the GIC (29).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

*1.2.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned}
\tag{6}
$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) = \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t)\right] \tag{7}
$$
$$
= (\mathsf{R}/\Gamma)\,(1 - c'(m_t)) - 1.
$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

**If RIC holds**. Equation (21) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$
\mathcal{R}\,(1 - c'(m_t)) - 1 < \mathcal{R}(1 - \underbrace{(1 - Þ_{\mathsf{R}})}_{\underline{\kappa}}) - 1
$$
$$
= (\mathsf{R}/\Gamma)Þ_{\mathsf{R}} - 1
$$

which is negative because the GIC says $Þ_\Gamma < 1$.

**If RIC fails.** Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} c'(m) = 0$. Concavity of the consumption function means that $c'$ is a decreasing function, so everywhere

$$
\mathcal{R}\,(1 - c'(m_t)) < \mathcal{R}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (7) is guaranteed to be negative if

$$
\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1. \tag{8}
$$

But we showed in Section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (8) holds).