Main Appendices

A c Functions Exist, are Concave, and Differentible

To show that (5) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, \ldots, \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$
6. $\lim_{z\downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \boldsymbol{\xi}_{t+1})\right].$$

Since there is a positive probability that $\boldsymbol{\xi}_{t+1}$ will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0}\mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0}\mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

Next define $\underline{\mathrm{v}}_t(m, c)$ as

$$\underline{\mathrm{v}}_t(m, c) = \mathrm{u}(c) + \mathfrak{v}_t(m - c) \tag{1}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (5) can be written as

$$\mathrm{v}_t(m) = \max_c\ \underline{\mathrm{v}}_t(m, c). \tag{2}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m, c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m, c) = -\infty$, $\frac{\partial^2\underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \arg\max_{0<c<m}\ \underline{\mathrm{v}}_t(m, c) \tag{3}$$

exists and is unique, and (5) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{4}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (4) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{5}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix B.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

B $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y) - \mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y) - \mathrm{a}_t(m)}\right) \frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm \to +0} \frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm \to +0} \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y) - \mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{6}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

C $\mathfrak{T}$ Is a Contraction Mapping

We must show that our operator $\mathfrak{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is therefore that $\{\mathfrak{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathfrak{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\{\mathcal{T}\mathrm{x}\}(m_t) \text{ is } \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\}$$
$$\{\mathcal{T}\mathrm{y}\}(m_t) \text{ is } \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \left\{\mathrm{u}(c_t)+\beta\,\mathbb{E}_t\left[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\},$$

so $\mathrm{x}(\bullet)\leq\mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t)\leq\{\mathcal{T}\mathrm{y}\}(m_t)$ by inspection.[1]

Condition (2) requires that $\{\mathcal{T}\mathbf{0}\}\in\mathcal{C}_{F}\left(\mathcal{A},\mathcal{B}\right)$. By definition,

$$\{\mathcal{T}\mathbf{0}\}(m_t)=\max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]}\left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right)+\beta 0\right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa}m_t)$. Thus, condition (2) will hold if $(\bar{\kappa}m_t)^{1-\rho}$ is F-bounded, which it is if we use the bounding function

$$F(m)=\eta+m^{1-\rho}, \tag{7}$$

defined in the main text.

Finally, we turn to condition (3), $\{\mathcal{T}(\mathrm{z}+\zeta F)\}(m_t)\leq\{\mathcal{T}\mathrm{z}\}(m_t)+\zeta\alpha F(m_t)$. The proof will be more compact if we define $\check{\mathrm{c}}$ and $\breve{\mathrm{a}}$ as the consumption and assets functions[2] associated with $\mathcal{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathcal{T}(\mathrm{z}+\zeta F)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}})\leq\mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}})+\zeta\alpha F.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})\leq\mathrm{u}(\check{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}})+\beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha F\\
\beta\{\mathsf{E}(\mathrm{z}+\zeta F)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}})+\zeta\alpha F\\
\beta\zeta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \zeta\alpha F\\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &\leq \alpha F\\
\beta\{\mathsf{E}F\}(\hat{\mathrm{a}}) &< F.
\end{aligned}$$

where the last line follows because $0<\alpha<1$ by assumption.[3]

Using $F(m)=\eta+m^{1-\rho}$ and defining $\hat{a}_t=\hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\mathbf{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\boldsymbol{\xi}_{t+1})^{1-\rho}]-m_t^{1-\rho}<\eta(1-\underbrace{\beta\,\mathbb{E}_t\,\mathbf{\Phi}_{t+1}^{1-\rho}}_{=\beth})$$

[1]For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

[2]Section 2.8 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[3]The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

which by imposing PF-FVAC (equation (20), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\boldsymbol{\xi}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}}{1-\beth}. \tag{8}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing that the numerator of (8) is bounded from above:

$$\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]\\
&+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]\\
&+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\boldsymbol{\theta}_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}^{1-\rho}(\underline{\boldsymbol{\theta}}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\boldsymbol{\theta}}^{1-\rho}.
\end{aligned}$$

We can thus conclude that equation (8) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1-\wp)^{\rho}\underline{\boldsymbol{\theta}}^{1-\rho}}{1-\beth} \tag{9}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (32) and (**??**) is now complete.

C.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (33)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (5) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (9). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$\begin{gathered}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho} < 1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1}) > 1
\end{gathered}$$

$$(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}) > 1$$

where we have used (31) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1} \approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$\begin{aligned}
(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}} &> 1 \\
(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1 \\
\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} &< 1.
\end{aligned}$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping in $F-$bounded space with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty}\underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathcal{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (5) converge, they converge to the same unique v defined by $\mathcal{T}$.[4]

D Convergence in Euclidian Space

D.1 Convergence of v_t

Boyd's theorem shows that $\mathcal{T}$ defines a contraction mapping in an F-bounded space. We now show that $\mathcal{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathcal{T}$, since $\mathrm{v}^*(m) = \mathcal{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_F \leq \alpha^{n-1}\|\mathrm{v}_T - \mathrm{v}^*\|_F. \tag{10}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_F(\mathcal{A},\mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A},\mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1}|F(m)|. \tag{11}$$

Then we obtain

$$\lim_{n\to\infty}\mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{12}$$

[4]It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

D.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\begin{aligned}\mathrm{u}(c_{T-n(i)}(m))+&\beta\,\mathbb{E}_{T-n(i)}[\mathbf{\Phi}_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)]\\ &\geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_{T-n(i)}[\mathbf{\Phi}_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)],\end{aligned} \tag{13}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\mathbf{\Phi}_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\mathbf{\Phi}_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{14}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\mathbf{\Phi}_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

E Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions.

Results are summarized in table 1.

E.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails (~~GIC~~, $1 < \text{Þ}/\mathbf{\Phi}$). Under ~~GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\mathbf{\Phi}$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[5]

$$
\begin{aligned}
1 &< (\mathsf{R}\beta)^{1/\rho}\mathbf{\Phi}^{-1} \\
1 &< \mathsf{R}\beta\mathbf{\Phi}^{-\rho} \\
\mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\mathbf{\Phi}).
\end{aligned}
$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer's consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHWC holds. If the RIC fails ($1 < \text{Þ}_{\mathsf{R}}$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$
\grave{\mathrm{c}}_{T-n}(m) = 0(b_t + h). \tag{15}
$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHWC fails. ~~FHWC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\grave{\mathrm{c}}_{T-n}(m) = 0$ or $\grave{\mathrm{c}}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞.[6]

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the RIC (and the FHWC can be shown to be a consequence of ~~GIC~~ and RIC). In this case, the consumer's optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer would choose $c = m$ from Equation (17):

$$
\begin{aligned}
m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\
m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\
m_{\#} &= (h - 1)\left(\frac{\underline{\kappa}}{1 - \underline{\kappa}}\right)
\end{aligned} \tag{16}
$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.[7] For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m; for such m, the constrained consumer is obliged to choose $\grave{\mathrm{c}}(m) = m$.[8] For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{\mathrm{c}}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

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[5]The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\mathbf{\Phi})$ which is $c_{\#} = \mathbf{\Phi}/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

[6]The knife-edge case is where $\text{Þ} = \mathbf{\Phi}$, in which case the two quantities counterbalance and the limiting function is $\grave{\mathrm{c}}(m) = \min[m, 1]$.

[7]Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by ~~GIC~~.

[8]As an illustration, consider a consumer for whom $\text{Þ} = 1$, $\mathsf{R} = 1.01$ and $\mathbf{\Phi} = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\mathbf{\Phi} < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

E.2 If GIC Holds

Imposition of the GIC reverses the inequality in (15), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t, but perpetually thereafter.

Now define $b^n_\#$ as the b_t such that an unconstrained consumer holding $b_t = b^n_\#$ would behave so as to arrive in period $t+n$ with $b_{t+n} = 0$ (with $b^0_\#$ trivially equal to 0); for example, a consumer with $b_{t-1} = b^1_\#$ was on the 'cusp' of being constrained in period $t-1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m^0_\# = 1 + b^0_\# = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c^0_\# = 1$.

We can construct the entire 'prehistory' of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to $\text{\TH}_{\boldsymbol{\Phi}}$, so consumption n periods in the past must have been

$$c^n_\# = \text{\TH}_{\boldsymbol{\Phi}}^{-n} c_t = \text{\TH}_{\boldsymbol{\Phi}}^{-n}. \tag{17}$$

The PDV of consumption from $t-n$ until t can thus be computed as

$$\begin{aligned}
\mathbb{C}^t_{t-n} &= c_{t-n}(1 + \text{\TH}/\mathsf{R} + \cdots + (\text{\TH}/\mathsf{R})^n) \\
&= c^n_\#(1 + \text{\TH}_{\mathsf{R}} + \cdots + \text{\TH}_{\mathsf{R}}^n) \\
&= \text{\TH}_{\boldsymbol{\Phi}}^{-n} \left(\frac{1 - \text{\TH}_{\mathsf{R}}^{n+1}}{1 - \text{\TH}_{\mathsf{R}}}\right) \\
&= \left(\frac{\text{\TH}_{\boldsymbol{\Phi}}^{-n} - \text{\TH}_{\mathsf{R}}}{1 - \text{\TH}_{\mathsf{R}}}\right)
\end{aligned}$$

and note that the consumer's human wealth between $t-n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post-t income) is

$$h^n_\# = 1 + \cdots + \mathcal{R}^{-n} \tag{18}$$

while the intertemporal budget constraint says

$$\mathbb{C}^t_{t-n} = b^n_\# + h^n_\#$$

from which we can solve for the $b^n_\#$ such that the consumer with $b_{t-n} = b^n_\#$ would unconstrainedly plan (in period $t-n$) to arrive in period t with $b_t = 0$:

$$b^n_\# = \mathbb{C}^t_{t-n} - \overbrace{\left(\frac{1 - \mathcal{R}^{-(n+1)}}{1 - \mathcal{R}^{-1}}\right)}^{h^n_\#}. \tag{19}$$

Defining $m^n_\# = b^n_\# + 1$, consider the function $\grave{c}(m)$ defined by linearly connecting the points $\{m^n_\#, c^n_\#\}$ for integer values of $n \geq 0$ (and setting $\grave{c}(m) = m$ for $m < 1$). This function will return, for any value of m, the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^{n} - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^{n} + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (19) for the entire domain of positive real values of b, we need $b_{\#}^{n}$ to become arbitrarily large with n. That is, we need

$$\lim_{n\to\infty} b_{\#}^{n} = \infty. \tag{20}$$

E.2.1 If FHWC Holds

The FHWC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (19) limits to a constant as $n \uparrow \infty$, and (20) reduces to a requirement that

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n} - (\text{Þ}_{\mathsf{R}}/\text{Þ}_{\boldsymbol{\Phi}})^{n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n} - \mathcal{R}^{-n}\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{1-\text{Þ}_{\mathsf{R}}} \right) &= \infty.
\end{aligned}$$

Given the GIC $\text{Þ}_{\boldsymbol{\Phi}}^{-1} > 1$, this will hold iff the RIC holds, $\text{Þ}_{\mathsf{R}} < 1$. But given that the FHWC $\mathsf{R} > \boldsymbol{\Phi}$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m\to\infty} \grave{\mathrm{c}}(m) - \bar{\mathrm{c}}(m) = 0. \tag{21}$$

E.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (20) requires

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{\mathcal{R}^{-n}\text{Þ}_{\mathsf{R}} - \text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}}-1} \right) + \left(\frac{1-\mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1}-1} \right) &= \infty \\
\lim_{n\to\infty} \left(\frac{\text{Þ}_{\mathsf{R}}}{\text{Þ}_{\mathsf{R}}-1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1} \right) \mathcal{R}^{-n} - \left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}}-1} \right) &= \infty
\end{aligned}$$

If RIC Holds. When the RIC holds, rearranging (22) gives

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{1-\text{Þ}_{\mathsf{R}}}\right)-\mathcal{R}^{-n}\left(\frac{\text{Þ}_{\mathsf{R}}}{1-\text{Þ}_{\mathsf{R}}}+\frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1}-1}\right) \quad = \infty$$

and for this to be true we need

$$\begin{aligned} \text{Þ}_{\boldsymbol{\Phi}}^{-1} &> \mathcal{R}^{-1} \\ \boldsymbol{\Phi}/\text{Þ} &> \boldsymbol{\Phi}/\mathsf{R} \\ 1 &> \text{Þ}/\mathsf{R} \end{aligned}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n\to\infty}\kappa_{\#}^{n}=\lim_{n\to\infty}\left(\frac{c_{\#}^{n}}{b_{\#}^{n}}\right) \tag{22}$$

which with a bit of algebra[9] can be shown to asymptote to the MPC in the perfect foresight model:[10]

$$\lim_{m\to\infty}\grave{\boldsymbol{\kappa}}(m)=1-\text{Þ}_{\mathsf{R}}. \tag{24}$$

If RIC Fails. Consider now the ~~RIC~~ case, $\text{Þ}_{\mathsf{R}} > 1$. We can rearrange (22)as

$$\lim_{n\to\infty}\left(\frac{\text{Þ}_{\mathsf{R}}(\mathcal{R}^{-1}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}-\frac{\mathcal{R}^{-1}(\text{Þ}_{\mathsf{R}}-1)}{(\mathcal{R}^{-1}-1)(\text{Þ}_{\mathsf{R}}-1)}\right)\mathcal{R}^{-n}-\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\mathsf{R}}-1}\right) \quad = \infty. \tag{25}$$

which makes clear that with ~~FHWC~~ $\Rightarrow \mathcal{R}^{-1} > 1$ and ~~RIC~~ $\Rightarrow \text{Þ}_{\mathsf{R}} > 1$ the numerators and denominators of both terms multiplying $\mathcal{R}^{-n}$ can be seen transparently to be positive. So, the terms multiplying $\mathcal{R}^{-n}$ in (22) will be positive if

$$\begin{aligned} \text{Þ}_{\mathsf{R}}\mathcal{R}^{-1}-\text{Þ}_{\mathsf{R}} &> \mathcal{R}^{-1}\text{Þ}_{\mathsf{R}}-\mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \text{Þ}_{\mathsf{R}} \\ \boldsymbol{\Phi} &> \text{Þ} \end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving $\mathcal{R}^{-n}$ goes to $+\infty$ faster than the term involving $-\text{Þ}_{\boldsymbol{\Phi}}^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \text{Þ}_{\boldsymbol{\Phi}}^{-1} \\ \boldsymbol{\Phi}/\mathsf{R} &> \boldsymbol{\Phi}/\text{Þ} \\ \text{Þ}/\mathsf{R} &> 1 \end{aligned}$$

---

[9]Calculate the limit of

$$\left(\frac{\text{Þ}_{\boldsymbol{\Phi}}^{-n}}{\text{Þ}_{\boldsymbol{\Phi}}^{-n}/(1-\text{Þ}_{\mathsf{R}})-(1-\mathcal{R}^{-1}\mathcal{R}^{-n})/(1-\mathcal{R}^{-1})}\right)=\left(\frac{1}{1/(1-\text{Þ}_{\mathsf{R}})+\mathcal{R}^{-n}\mathcal{R}^{-1}/(1-\mathcal{R}^{-1})}\right) \tag{23}$$

[10]For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.

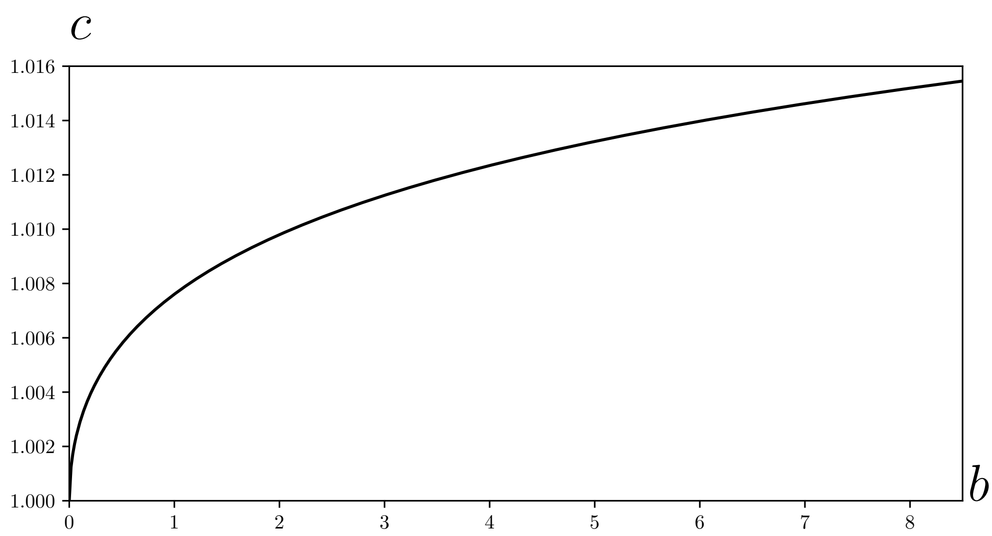

Figure 1 Appendix: Nondegenerate c Function with ~~FHWC~~ and ~~RIC~~

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_{\#}^{n}$ term is increasing backward in time at rate dominated in the limit by $\boldsymbol{\Phi}/\text{\textbf{Þ}}$ while the $b_{\#}$ term is increasing at a rate dominated by $\boldsymbol{\Phi}/\mathsf{R}$ term and

$$\boldsymbol{\Phi}/\mathsf{R} \quad > \quad \boldsymbol{\Phi}/\text{\textbf{Þ}} \tag{26}$$

because $\cancel{\text{RIC}} \Rightarrow \text{\textbf{Þ}} > \mathsf{R}$.

Consequently, while $\lim_{n\uparrow\infty} b_{\#}^{n} = \infty$, the limit of the *ratio* $c_{\#}^{n}/b_{\#}^{n}$ in (22) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that ~~RIC~~ implies a limiting MPC of zero,

$$\lim_{m\to\infty} \grave{\boldsymbol{\kappa}}(m) = 0, \tag{27}$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\grave{\mathrm{c}}(m) = 0$. (Figure 1 presents an example for $\rho = 2$, $\mathsf{R} = 0.98$, $\beta = 1.00$, $\boldsymbol{\Phi} = 0.99$; note that the horizontal axis is bank balances $b = m - 1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHWC holds. If so, the RIC automatically holds, and the solution limits into the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

F The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{28}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \mathbf{\Phi} = 1$, and there are no permanent shocks, $\mathbf{\Psi} = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} \mathrm{v}_T(a+\boldsymbol{\theta}) d\mathcal{F}_{\boldsymbol{\theta}} \right\}. \tag{29}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} \mathrm{u}'(\mathrm{c}_T(a+\boldsymbol{\theta})) d\mathcal{F}_{\boldsymbol{\theta}},$$

and the solution to (29) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{30}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (30), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\boldsymbol{\theta}}}^{\bar{\boldsymbol{\theta}}} (\mathrm{c}_T(a+\boldsymbol{\theta}/(1-\wp)))^{-\rho} d\mathcal{F}_{\boldsymbol{\theta}}\right], \tag{31}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{32}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (30) and (32) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (31) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (28). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (55) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

G Unique and Stable Target and Steady State Points

This appendix proves Theorems 1-2 and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

G.1 Proof of Theorem 1

Theorem 1. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC-Mod all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \tag{33}$$

Moreover, $\hat{m}$ is a point of 'stability' in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\hat{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{34}$$

The elements of the proof of Theorem 1 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

G.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \boldsymbol{\xi}_t$, even if $\boldsymbol{\xi}_t$ takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

G.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

G.3.1 Existence of m where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (39), but dropping the $\boldsymbol{\Phi}_{t+1}$ from the RHS, establishes that

$$\begin{aligned}\lim_{m_t\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1}}{m_t}\right] \\ &= \mathbb{E}_t[(\mathsf{R}/\boldsymbol{\Phi}_{t+1})\text{Þ}_{\mathsf{R}}] \\ &= \mathbb{E}_t[\text{Þ}/\boldsymbol{\Phi}_{t+1}] \\ &< 1\end{aligned} \tag{35}$$

where the inequality reflects imposition of the GIC-Mod (26).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (30)) means that the limit of the RHS of (35) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Mod and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

G.3.2 Existence of m > 1 where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

G.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,\end{aligned} \tag{36}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\xi}_{t+1} - m_t\right)\right] \\ &= \bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1.\end{aligned} \tag{37}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\ &= \mathbb{E}_t\left[\frac{\mathsf{R}}{\boldsymbol{\Phi}\boldsymbol{\Psi}}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\ &= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\boldsymbol{\Phi}\boldsymbol{\Psi}}\right]}_{=\text{Þ}_{\underline{\boldsymbol{\Phi}}}} - 1\end{aligned}$$

which is negative because the GIC-Mod says $\text{Þ}_{\underline{\boldsymbol{\Phi}}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\left(1 - \mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (37) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\boldsymbol{\Phi}\boldsymbol{\Psi}}\right] < 1. \tag{38}$$

But the combination of the GIC-Mod holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\boldsymbol{\Phi}\boldsymbol{\Psi}}\right]}^{\text{Þ}_{\underline{\boldsymbol{\Phi}}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\boldsymbol{\Phi}\boldsymbol{\Psi}}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (38).

G.4 Proof of Theorem 2

Theorem 2. *For the nondegenerate solution to the problem defined in Section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-*

to-income ratio $\check{m} > 0$ such that

$$\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{39}$$

Moreover, $\check{m}$ is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t > \mathbf{\Phi} \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t < \mathbf{\Phi}. \end{aligned} \tag{40}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

G.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in G.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t]$.

G.4.2 Existence of a stable point

Since by assumption $0 < \underline{\mathbf{\Psi}} \leq \mathbf{\Psi}_{t+1} \leq \bar{\mathbf{\Psi}} < \infty$, our proof in Subsection G.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathbf{\Phi}_{t+1}\left((\mathsf{R}/\mathbf{\Phi}_{t+1})\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1}\right)/\mathbf{\Phi}}{m_t}\right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\mathbf{\Phi})\mathrm{a}(m_t) + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}}{m_t}\right] \\ &= \lim_{m_t \uparrow \infty}\left[\frac{(\mathsf{R}/\mathbf{\Phi})\mathrm{a}(m_t) + 1}{m_t}\right] \\ &= (\mathsf{R}/\mathbf{\Phi})\mathbf{\text{Þ}}_{\mathsf{R}} \\ &= \mathbf{\text{Þ}}_{\mathbf{\Phi}} \\ &< 1 \end{aligned} \tag{41}$$

where the last two lines are merely a restatement of the GIC (19).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\mathbf{\Psi}_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

G.4.3 $\mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] < 1\\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] = 1\\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\boldsymbol{\Psi}_{t+1}m_{t+1}/m_t] > 1,\end{aligned} \tag{42}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)\left(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - m_t\right)\right]\\ &= (\mathsf{R}/\boldsymbol{\Phi})\left(1 - \mathrm{c}'(m_t)\right) - 1.\end{aligned} \tag{43}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (16) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}\mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1\\ &= (\mathsf{R}/\boldsymbol{\Phi})\text{Þ}_{\mathsf{R}} - 1\end{aligned}$$

which is negative because the GIC says $\text{Þ}_{\boldsymbol{\Phi}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\left(1 - \mathrm{c}'(m_t)\right) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (43) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\boldsymbol{\Phi}) < 1. \tag{44}$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (44) holds).

G.5 A Third Measure

A footnote in Section 3 mentions reasons why it may be useful to calculate $\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\log\mathbf{m}_t)]$. Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the growth factor.

$$\begin{aligned}\log\left(\mathbf{m}_{t+1}/\mathbf{m}_t\right) &= \log(\boldsymbol{\Phi}\boldsymbol{\Psi}_{t+1}m_{t+1}) - \log m_t\\ &= \log\boldsymbol{\Phi}(a_t\mathcal{R} + \boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}) - \log m_t\\ &= \log\boldsymbol{\Phi}(a_t\mathcal{R} + 1 + (\boldsymbol{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)) - \log m_t\end{aligned}$$

Now define $\breve{m}_{t+1} = a_t\mathcal{R} + 1$, and compute the expectation:

$$\begin{aligned}
\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\mathbf{m}_t)] &= \mathbb{E}_t\left[\log \mathbf{\Phi}(\breve{m}_{t+1} + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1))\right] - \log m_t \\
&= \log \mathbf{\Phi} + \mathbb{E}_t\left[\log \breve{m}_{t+1}\left(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1}\right)\right] - \log m_t \\
&= \underbrace{\log \mathbf{\Phi} + \log \breve{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]
\end{aligned}$$

and exponentiating tells us that

$$\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \exp(\mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \tag{45}$$

and this latter factor is a number that approaches 1 from below as m_t rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected growth factor. This implies that the m at which 'balanced growth' can be expected in the log, $\tilde{m}$, exceeds the corresponding point for the ratio, $\breve{m}$.

Furthermore, in the limit as $\mathbf{m}_t$ gets arbitrarily large, if the RIC holds and thus $\underline{\kappa} > 0$, a_{t+1} rises without bound, as does $\breve{m}_{t+1} = a_{t+1}\mathcal{R} + 1$, so the approximation $\log(1+\epsilon) \approx \epsilon$ becomes arbitrarily good. Consequently, the last term on the RHS of (45) can be approximated as

$$\begin{aligned}
\mathbb{E}_t\left[\log(1 + (\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) &\approx \mathbb{E}_t\left[(\mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1} - 1)\breve{m}_{t+1}^{-1})\right]) \\
&= 0
\end{aligned}$$

This demonstrates that

$$\lim_{\mathbf{m}_t \uparrow \infty} \exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \tag{46}$$

G.6 Proof of Lemma

G.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$\begin{aligned}
\breve{\mathrm{m}}_{t+1}(a) &= 1 + a\mathcal{R} \\
\hat{\mathrm{m}}_{t+1}(a) &= 1 + a\,\underbrace{\mathcal{R}/\underline{\mathbf{\Psi}}}_{\bar{\mathcal{R}} > \mathcal{R}}
\end{aligned} \tag{47}$$

so that we can implicitly define the target and pseudo-steady-state points as

$$\begin{aligned}
\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m} - \mathrm{c}(\hat{m})) \\
\breve{m} &= \breve{\mathrm{m}}_{t+1}(\breve{m} - \mathrm{c}(\breve{m}))
\end{aligned} \tag{48}$$

Then subtract:

$$
\begin{aligned}
\hat{m} - \check{m} &= \left(\hat{a}\underline{\mathbf{\Psi}}^{-1} - \check{a}\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - a(\check{m})\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m} + \check{m} - \hat{m}))\right)\mathcal{R} \\
&\approx \left(a(\hat{m})\underline{\mathbf{\Psi}}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\
(\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{<\mathbf{Þ}_{\mathbf{\Phi}}<1}) &= (\underline{\mathbf{\Psi}}^{-1} - 1)\hat{a}\mathcal{R}
\end{aligned} \tag{49}
$$

The RHS of this equation is strictly positive because $\underline{\mathbf{\Psi}}^{-1} > 1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1 - \mathcal{R}a') > 0$. So the equation can only hold if $\hat{m} - \check{m} > 0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[11]

G.6.2 The m Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State m

Expected log balanced growth occurs when

$$
\begin{aligned}
\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \mathbf{\Phi}\mathbf{m}_t \\
\mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \mathbf{\Phi}\mathbf{p}_t m_t \\
\mathbb{E}_t[\log \mathbf{\Psi}_{t+1} m_{t+1}] &= \log \mathbf{\Phi} m_t \\
\mathbb{E}_t[\log (a(m_t)\mathsf{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1}\mathbf{\Phi})] &= \log \mathbf{\Phi} m_t \\
\mathbb{E}_t[\log (a(m_t)\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})] &= \log m_t
\end{aligned} \tag{50}
$$

and we call the m that satisfies this equation $\tilde{m}$.

Subtract the definition of $\check{m}$ from that of $\tilde{m}$:

$$
\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m} \tag{51}
$$

Now we use the fact that the expectation of the log is less than the log of the expectation,

$$
\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \mathbf{\Psi}_{t+1}\boldsymbol{\xi}_{t+1})]) < (a(\tilde{m})\mathcal{R} + 1) \tag{52}
$$

so

$$
\begin{aligned}
\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + 1)]) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\
(a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{\mathrm{a}}')\mathcal{R} &< \tilde{m} - \check{m} \\
(\tilde{m} - \check{m})\bar{\mathrm{a}}'\mathcal{R} &< \tilde{m} - \check{m} \\
\underbrace{\bar{\mathrm{a}}'\mathcal{R}}_{<\mathbf{Þ}_{\mathbf{\Phi}}} &< 1
\end{aligned} \tag{53}
$$

[11] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a$'$ over the interval to remove the approximation in the derivations above.

where we are interpreting $\bar{\mathrm{a}}'$ as the mean of the value of a' over the interval between $\tilde{m}$ and $\check{m}$.

H The Limiting MPC's

For $m_t > 0$ we can define $\mathrm{e}_t(m_t) = \mathrm{c}_t(m_t)/m_t$ and $\mathrm{a}_t(m_t) = m_t - \mathrm{c}_t(m_t)$ and the Euler equation (6) can be rewritten

$$\begin{aligned}
\mathrm{e}_t(m_t)^{-\rho} = \beta \mathsf{R}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(m_{t+1}) \left(\frac{\overbrace{\mathsf{R}\mathrm{a}_t(m_t) + \mathbf{\Phi}_{t+1}\boldsymbol{\xi}_{t+1}}^{=m_{t+1}\mathbf{\Phi}_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \\
= \quad (1-\wp)\beta \mathsf{R} m_t^{\rho}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(m_{t+1}) m_{t+1}\mathbf{\Phi}_{t+1} \right)^{-\rho} \middle| \, \boldsymbol{\xi}_{t+1} > 0 \right] \\
+ \wp \beta \mathsf{R}^{1-\rho}\, \mathbb{E}_t \left[\left(\mathrm{e}_{t+1}(\mathcal{R}_{t+1}\mathrm{a}_t(m_t)) \frac{m_t - \mathrm{c}_t(m_t)}{m_t} \right)^{-\rho} \middle| \, \boldsymbol{\xi}_{t+1} = 0 \right].
\end{aligned}$$

Consider the first conditional expectation in (6), recalling that if $\boldsymbol{\xi}_{t+1} > 0$ then $\boldsymbol{\xi}_{t+1} \equiv \boldsymbol{\theta}_{t+1}/(1-\wp)$. Since $\lim_{m \downarrow 0} \mathrm{a}_t(m) = 0$, $\mathbb{E}_t[(\mathrm{e}_{t+1}(m_{t+1})m_{t+1}\mathbf{\Phi}_{t+1})^{-\rho} \mid \boldsymbol{\xi}_{t+1} > 0]$ is contained within bounds defined by $(\mathrm{e}_{t+1}(\underline{\boldsymbol{\theta}}/(1-\wp))\mathbf{\Phi}\underline{\mathbf{\Psi}}\underline{\boldsymbol{\theta}}/(1-\wp))^{-\rho}$ and $(\mathrm{e}_{t+1}(\bar{\boldsymbol{\theta}}/(1-\wp))\mathbf{\Phi}\bar{\mathbf{\Psi}}\bar{\boldsymbol{\theta}}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned}
(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}.
\end{aligned}$$

As noted in the main text, we need the WRIC (32) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{54}$$

Since $\bar{\kappa}_T = 1$, iterating (54) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty}\bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{55}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving $\boldsymbol{\xi}_{t+1}$ in (54) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} \quad = \beta \mathsf{R}\, \mathbb{E}_t \left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\, (\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho} \right]$$

and using L'Hôpital's rule $\lim_{m_t \to \infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t \to \infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\rho} &= \beta \mathsf{R} \big(\underline{\kappa}_{t+1} \mathsf{R} (1 - \underline{\kappa}_t) m_t \big)^{-\rho} \\ \underbrace{\mathsf{R}^{-1} \text{Þ}}_{\equiv \text{Þ}_{\mathsf{R}} = (1 - \underline{\kappa})} \ \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \text{Þ}_{\mathsf{R}} \tag{56}$$

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \kappa_{T-n}^{-1} \tag{57}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n \to \infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1 + \text{Þ}_{\mathsf{R}} + \text{Þ}_{\mathsf{R}}^2 + \cdots \right)}_{= 1 + \text{Þ}_{\mathsf{R}}(1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+2}^{-1}) \ldots} \quad = c_t \underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t) \underline{\kappa}_t \tag{58}$$

I Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures **??**, 3).

I.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 2, whose three nodes represent values of the absolute patience factor **Þ**, the permanent-income growth factor $\boldsymbol{\Phi}$, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition

$$\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \textbf{Þ} \arrow[rr, "\text{GIC}(\equiv \textbf{Þ} < \Gamma)"] \arrow[dr, "(\textbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] & \arrow[d, phantom, "\neq"] & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}$$

Figure 2 Appendix: Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

(like, the uppermost arrow, pointing from $\textbf{Þ}$ to $\boldsymbol{\Phi}$, reflects imposition of the GIC condition (clicking GIC should take you to its definition; definitions of other conditions are also linked below)).[12] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the $\textbf{Þ}$ node and impose the GIC and then the FHWC, and see that imposition of these conditions allows us to conclude that $\textbf{Þ} < \mathsf{R}$.

One could also impose $\textbf{Þ} < \mathsf{R}$ directly (without imposing GIC and FHWC) by following the downward-sloping diagonal arrow exiting $\textbf{Þ}$. Although alternate routes from one node to another all justify the same core conclusion ($\textbf{Þ} < \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[13] to indicate that the diagram is not commutative.[14]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, ~~RIC~~ $\equiv \textbf{Þ} > \mathsf{R}$, would be represented by moving the arrowhead from the bottom right to the top left of the line segment connecting $\textbf{Þ}$ and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and $\boldsymbol{\Phi}$, but the $\boldsymbol{\Phi}$ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed GIC (that is, if we imposed ~~GIC~~), that would take us to the $\textbf{Þ}$ node, and we could deduce $\mathsf{R} > \textbf{Þ}$. However, we would

[12]For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[13]For a popular introduction to category theory, see Riehl (2017).

[14]But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

have to stop traversing the diagram at this point, because the arrow exiting from the $\mathbf{Þ}$ node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~GIC~~) requires us also to reverse the final condition, giving us ~~RIC~~.[15]

Under these conventions, Figure **??** in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

$$\begin{tikzcd}
\mathbf{Þ} \arrow[r, "\text{GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}" description] \arrow[dr, phantom, shift left=6ex, "\neq"] \arrow[dr, phantom, shift right=6ex, "\neq"] & \Gamma \arrow[d, "\text{FHWC}"] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 3 Appendix: Relation of GIC, FHWC, RIC, and PFVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{Þ} < \mathsf{R}^{1/\rho}\boldsymbol{\Phi}^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (20)

This diagram can be interpreted, for example, as saying that, starting at the $\mathbf{Þ}$ node, it is possible to derive the PF-FVAC[16] by imposing both the GIC and the FHWC; or by imposing RIC and ~~FHWC~~. Or, starting at the $\boldsymbol{\Phi}$ node, we can follow the imposition of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\mathbf{Þ} < \boldsymbol{\Phi}$. Algebraically,

$$\begin{aligned} \text{FHWC}: \quad & \boldsymbol{\Phi} < \mathsf{R} \\ \cancel{\text{RIC}}: \quad & \mathsf{R} < \mathbf{Þ} \\ & \boldsymbol{\Phi} < \mathbf{Þ} \end{aligned} \tag{59}$$

---

[15]The corresponding algebra is

$$\begin{aligned} \cancel{\text{FHWC}}: \quad & \mathsf{R} < \boldsymbol{\Phi} \\ \cancel{\text{GIC}}: \quad & \boldsymbol{\Phi} < \mathbf{Þ} \\ \Rightarrow \cancel{\text{RIC}}: \quad & \mathsf{R} < \mathbf{Þ}, \end{aligned}$$

.

[16]in the form $\mathbf{Þ} < (\mathsf{R}/\boldsymbol{\Phi})^{1/\rho}\boldsymbol{\Phi}$

which leads to the negation of both of the conditions leading into $\mathbf{Þ}$. ~~GIC~~ is obtained directly as the last line in (59) and ~~PF-FVAC~~ follows if we start by multiplying the Return Patience Factor (RPF=$\mathbf{Þ}/\mathsf{R}$) by the FHWF (=$\mathbf{\Phi}/\mathsf{R}$) raised to the power $1/\rho-1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\mathbf{\Phi}/\mathsf{R})^{1/\rho-1}}^{>1 \quad \text{from FHWC}} \\
1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\mathbf{\Phi})^{1/\rho}\mathsf{R}\mathbf{\Phi}/\mathsf{R}}\right) \\
\mathsf{R}^{1/\rho}\mathbf{\Phi}^{1-1/\rho} = (\mathsf{R}/\mathbf{\Phi})^{1/\rho}\mathbf{\Phi} &< \mathbf{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

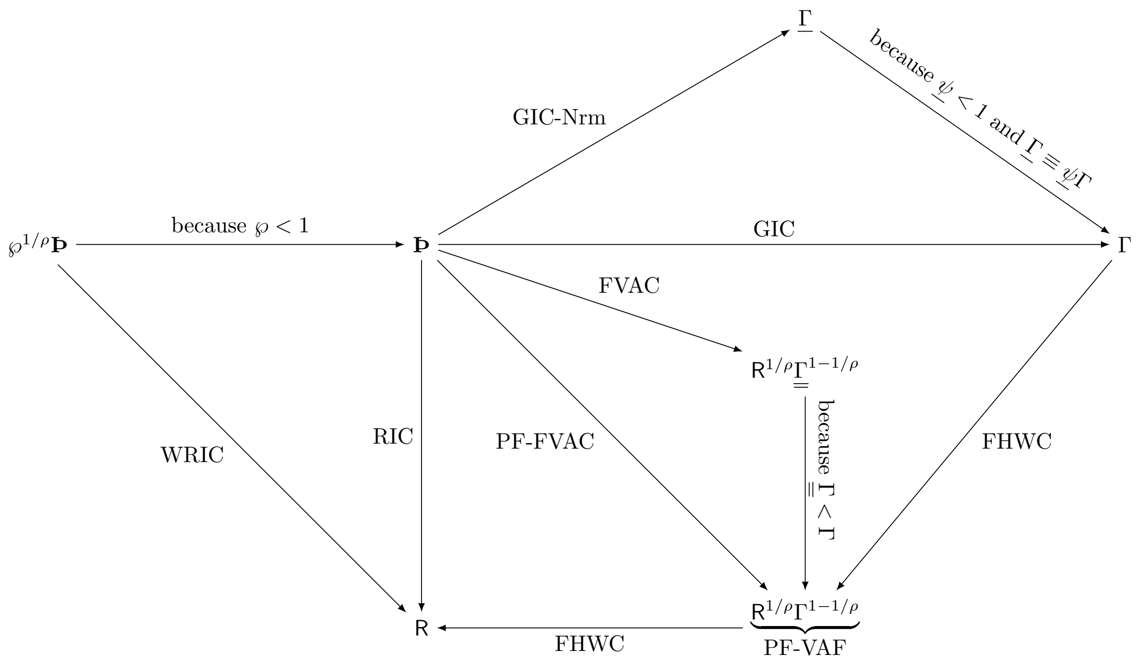

Figure 4 Appendix: Relation of All Inequality Conditions

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

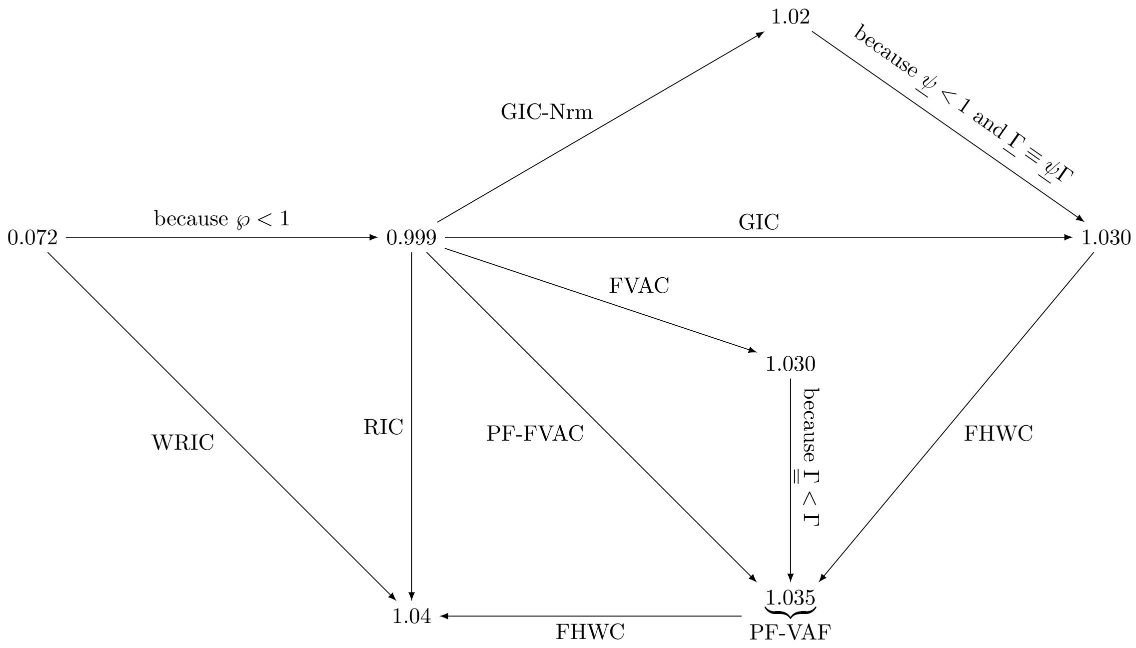

Figure 5 Appendix: Numerical Relation of All Inequality Conditions

J Apparent Balanced Growth in $\mathfrak{c}$ and $\mathrm{cov}(c, \mathbf{p})$

Section 4.2 demonstrates some propositions under the assumption that, when an economy satisfies the GIC, there will be constant growth factors $\Omega_{\mathfrak{c}}$ and Ω_{cov} respectively for $\mathfrak{c}$ (the average value of the consumption ratio) and $\mathrm{cov}(c, \mathbf{p})$. In the case of a Szeidl-invariant economy, the main text shows that these are $\Omega_{\mathfrak{c}} = 1$ and $\Omega_{\mathrm{cov}} = \mathbf{\Phi}$. If the economy is Harmenberg- but not Szeidl-invariant, no proof is offered that these growth factors will be constant.

J.1 $\log c$ and $\log(\mathrm{cov}(c, \mathbf{p}))$ Grow Linearly

Figures 6 and 7 plot the results of simulations of an economy that satisfies Harmenberg- but not Szeidl-invariance with a population of 4 million agents over the last 1000 periods (of a 2000 period simulation).[17] The first figure shows that $\log \mathfrak{c}$ increases apparently linearly. The second figure shows that $\log(-\mathrm{cov}(c, \mathbf{p}))$ also increases apparently linearly. (These results are produced by the notebook `ApndxBalancedGrowthcNrmAndCov.ipynb`).

[17] For an exposition of our implementation of Harmenberg's method, see this supplemental appendix.

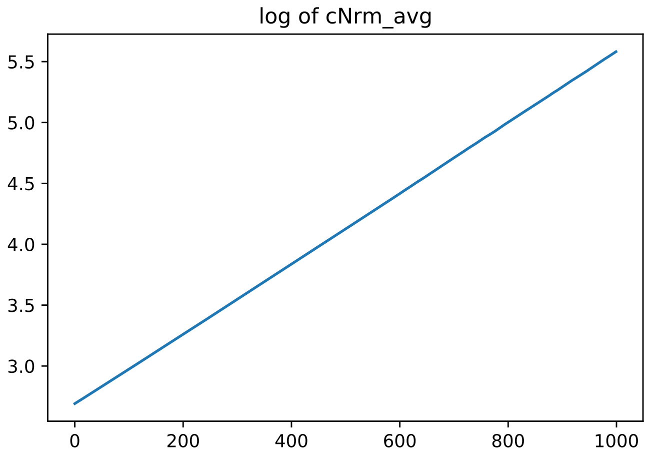

Figure 6 Appendix: log $\mathfrak{c}$ Appears to Grow Linearly

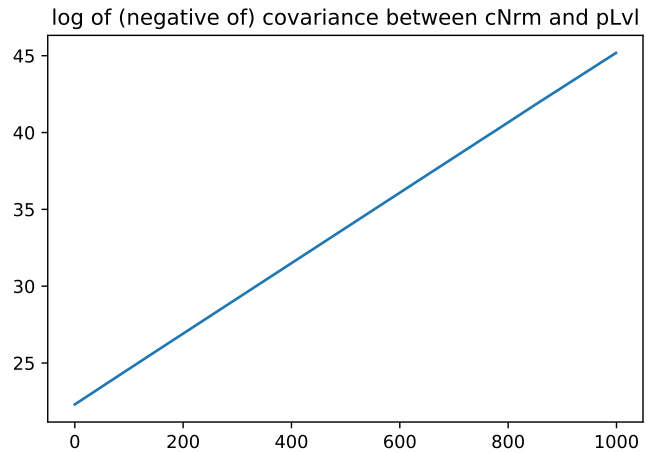

Figure 7 Appendix: log $(-\text{cov}(c, \mathbf{p}))$ Appears to Grow Linearly

Supplemental Appendices

K Equality of c and p Growth with Transitory Shocks

Section 4.1 asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\hat{m}$ is the invariant mean level of m we can define an 'average marginal propensity to save away from $\hat{m}$' function:

$$\bar{\text{a}}'(\Delta) = \Delta^{-1} \int_{\hat{m}}^{\hat{m}+\Delta} \text{a}'(z)dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\mathbf{\Psi}_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$, so if we define $\hat{a}$ as the value of a corresponding to $m = \hat{m}$, we can write

$$a_{t+1,i} = \hat{a} + (m_{t+1,i} - \hat{m})\bar{\text{a}}'(\overbrace{\mathcal{R}a_{t,i} + \boldsymbol{\xi}_{t+1,i}}^{m_{t+1,i}} - \hat{m})$$

so

$$\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \text{cov}_t\left(\bar{\text{a}}'(\mathcal{R}a_{t,i} + \boldsymbol{\xi}_{t+1,i} - \hat{m}), \mathbf{\Phi}\mathbf{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho} < \bar{\text{a}}'(m) < \text{\textbf{Þ}}_{\mathsf{R}}$,

$$|\text{cov}_t((\wp\mathsf{R}\beta)^{1/\rho}a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\text{cov}_t(\text{\textbf{Þ}}a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC-Mod says that $\text{\textbf{Þ}} < \mathbf{\Phi}$, while the FHWC says that $\mathbf{\Phi} < \mathsf{R}$; combining these facts we get:

$$|\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \mathbf{\Phi}|\text{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $\check{\mathbf{A}}\mathbf{\Phi}^n$ term which is growing steadily by the factor $\mathbf{\Phi}$). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \mathbf{\Phi}$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathbb{M}\left[\mathcal{R}_{t+1,i}\right]$ and consider a first order Taylor expansion of $\bar{\text{a}}'(m_{t+1,i})$ around $\hat{m}_{t+1,i} = \breve{\mathcal{R}}a_{t,i} + 1$,

$$\bar{\text{a}}'_{t+1,i} \approx \bar{\text{a}}'(\hat{m}_{t+1,i}) + \bar{\text{a}}''(\hat{m}_{t+1,i})\left(m_{t+1,i} - \hat{m}_{t+1,i}\right).$$

The problem comes from the $\bar{\text{a}}''$ term (which we implicitly define as the derivative of $\bar{\text{a}}'$). The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\text{a}}'$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\text{a}}''$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

L Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (**?**): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[18]

$$
\begin{aligned}
\mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\boldsymbol{\xi}_{t+1}))] \\
\vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\boldsymbol{\xi}_{t+1}))^{-\rho}\right]\right)^{-1/\rho}.
\end{aligned}
$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (60) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$
\begin{aligned}
\mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}^a_t &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1})\boldsymbol{\Phi}_{t+1}\mathrm{c}^m_{t+1}\mathcal{R}_{t+1}] \\
&= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}] \\
\mathfrak{c}^a_t &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}]/\mathrm{u}''(\mathfrak{c}_t)
\end{aligned}
$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$
\begin{aligned}
c &= \mathfrak{m} - a \\
\mathfrak{c}^a + 1 &= \mathfrak{m}^a
\end{aligned}
$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$
\begin{aligned}
\mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1 + \mathfrak{c}^a)
\end{aligned}
$$

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

M The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, \cdots\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[19] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

[18] The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

[19] Unless $\beta \approx +0$.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_\#, m^2_\#, \cdots\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathrm{c}(m)$ for integer values of n (e.g., $c^n_\# = \text{\TH}_{\boldsymbol{\Phi}}^{-n}$) are continuous functions of n; the conclusion that $\mathrm{c}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\check{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[20]

This strategy generates a smooth limiting consumption function — except at the remaining kink point defined by $\{m^0_\#, c^0_\#\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_\#$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_\#} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_\#} \mathrm{c}'(m) = \kappa^0_\# < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\check{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\check{\mathrm{c}}(m)$ between $m^0_\#$ and $m^{\underline{n}}_\#$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

1. $\hat{\mathrm{c}}(m^0_\#) = c^0_\#$
2. $\hat{\mathrm{c}}'(m^0_\#) = 1$
3. $\hat{\mathrm{c}}'(m^{\underline{n}}_\#) = (d\mathrm{c}^n_\# / dn)(d\mathrm{m}^n_\# / dn)^{-1}|_{n=\underline{n}}$
4. $\hat{\mathrm{c}}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\# / dn^2)(d^2\mathrm{m}^n_\# / dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\check{\mathrm{c}}(m)$ and fidelity of that function to the $\mathrm{c}(m)$ (see the actual code for details).

[20]In practice, we calculate the first and second derivatives of c and use piecewise polynomial approximation methods that match the function at these points.

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m_{\#}^{0} & m \\ m_{\#}^{0} < m < m_{\#}^{n} & \check{\mathrm{c}}(m) \\ m_{\#}^{n} < m & \mathrm{c}(m) \end{cases} \tag{60}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\check{\mathrm{c}}(m) \geq \mathrm{c}(m)$, implicitly defining $m = e^{\mu}$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^{\mu})/\mathrm{c}_T(e^{\mu})) \tag{61}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \check{\mathrm{c}}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi_t'(\mu) = \left(\frac{-\left(\frac{\mathrm{c}_t'\mathrm{c}_T - \mathrm{c}_t\mathrm{c}_T'}{\mathrm{c}_T^2} e^{\mu}\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{62}$$

which can be solved for

$$\mathrm{c}_t' = (\mathrm{c}_t\mathrm{c}_T'/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi_t'. \tag{63}$$

Similarly, we can solve (61) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right) \mathrm{c}_T(m). \tag{64}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

N When Is Consumption Growth Declining in m?

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \boldsymbol{\Phi}_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \boldsymbol{\xi}_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\boldsymbol{\Phi}_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{65}$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}_{t+1}' \equiv \mathrm{c}'(m_{t+1})$), since $\boldsymbol{\Phi}_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (65) in brackets can be manipulated to yield

$$\begin{aligned}
\mathrm{c}_t \boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Phi}_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\
&= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}.
\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned}
1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho}_{t+1}] \\
0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}\boldsymbol{\Upsilon}'_{t+1}] \\
&= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) \\
\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]
\end{aligned} \tag{66}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (66) that (65) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (66), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\mathrm{cov}_t\left((\boldsymbol{\Phi}\boldsymbol{\Psi}_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to $\boldsymbol{\xi}_{t+1}$ satisfy the proposition, since a higher draw of $\boldsymbol{\xi}$ both reduces $c^{-\rho-1}_{t+1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of $\boldsymbol{\Psi}_{t+1}$ will reduce m_{t+1}, thus increasing both $c^{-\rho-1}_{t+1}$ and c'_{t+1}. On the other hand, the $c^{-\rho-1}_{t+1}$ term is multiplied by $\boldsymbol{\Phi}\boldsymbol{\Psi}_{t+1}$, so the effect of a higher $\boldsymbol{\Psi}_{t+1}$ could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock $\boldsymbol{\Psi}_{t+1}$ can also lead a negative correlation.)

O Harmenberg's Method

Harmenberg defines a 'density kernel' describing the law of motion for the normalized state $\pi(m_{t+1}, m_t, \boldsymbol{\Psi}_{t+1})$ and defines $f_{\boldsymbol{\Psi}}$ as the density of the permanent shock distribution; we correspondingly define $F_{\boldsymbol{\Psi}}$ as the CDF of permanent shocks.

The joint cumulative distribution boldface $\boldsymbol{\chi}$ for m_{t+1} and p_{t+1} as a function of the stochastic variables and the joint marginal distributions (nonbold χ) in period t is:

$$\boldsymbol{\chi}_{t+1}(m_{t+1},\, p_{t+1}) = \\ \int_{\underline{m}_{t+1}}^{m_{t+1}} \int_{\underline{p}_{t+1}}^{p_{t+1}} \int_{p_t} \int_{m_t} \pi_t\left(m_{t+1}, m_t, \boldsymbol{\Psi}_{t+1}\right) \chi_t(m_t, p_t) dm_t dp_t dF_{\boldsymbol{\Psi}(p_{t+1})} dm_{t+1}$$

where

$$\begin{aligned} \boldsymbol{\Psi}_{t+1} &= p_{t+1}/(\boldsymbol{\Phi} p_t) \\ F_{\boldsymbol{\Psi}(p_{t+1})} &= \int_{\underline{\boldsymbol{\Psi}}}^{\boldsymbol{\Psi}(p_{t+1})} f_{\boldsymbol{\Psi}} d\boldsymbol{\Psi} \\ dF_{\boldsymbol{\Psi}(p_{t+1})} &= \left(\frac{d(p_{t+1}/(\boldsymbol{\Phi} p_t))}{dp_{t+1}}\right) f_{\boldsymbol{\Psi}}(\boldsymbol{\Psi}_{t+1}) dp_{t+1} \\ &= \left(\frac{1}{\boldsymbol{\Phi} p_t}\right) f_{\boldsymbol{\Psi}}(\boldsymbol{\Psi}_{t+1}) dp_{t+1} \end{aligned}$$

Harmenberg (2021)'s first equation corresponds to the marginal density obtained by differentiating the above CDF with respect to m_{t+1} and p_{t+1} and says that the 'Markov operator that maps a distribution $\chi \in D(m \times p)$ to the next-period χ' (that is, defines the dynamics of the joint marginal distribution of m and p) is given by

$$\chi_{t+1}(m_{t+1}, p_{t+1}) = \int_{p_t} \left[\int_{m_t} \pi_t \left(m_{t+1}, m_t, \overbrace{\frac{p_{t+1}}{\boldsymbol{\Phi} p_t}}^{\boldsymbol{\Psi}_{t+1}} \right) f_{\boldsymbol{\Psi}}(\frac{p_{t+1}}{\boldsymbol{\Phi} p_t}) \chi_t(m_t, p_t) \left(\frac{1}{\boldsymbol{\Phi} p_t}\right) dm_t \right] dp_t \tag{67}$$

A somewhat awkward notational scheme allows us to define an almost completely parallel representation of the corresponding discrete transition process:

1. $\{\mathtt{i}, \mathtt{j}, \mathtt{k}\}$ index elements of vectors identifying possible values of $\boldsymbol{\Psi}_{t+1}$, m_t, and m_{t+1}
   - We use the capital of the Roman letter to count the number of possible entries
   - e.g., there are K possible different values for $\mathrm{m} : \{\mathrm{m}_t^0, \mathrm{m}_t^1, ..., \mathrm{m}_t^K\}$
2. $\mathtt{n}$ indexes the level of permanent income now: $\mathrm{p}_t[\mathtt{n}]$
3. $\Pi_t[\mathtt{k}, \mathtt{j}, \mathtt{i}]$ indicates the probability of making a transition from value $\mathtt{j}$ of m_t to value $\mathtt{k}$ of m_{t+1} given that the realization of $\boldsymbol{\Psi}_{t+1}$ is $\boldsymbol{\Psi}_{t+1}[\mathtt{i}]$
4. $\digamma_{\boldsymbol{\Psi}}[\mathtt{i}]$ is the probability of drawing the $\mathtt{i}$'th value of $\boldsymbol{\Psi}_{t+1}$

then for a person whose location is described in period t as being at permanent income level $\mathrm{p}_t^{\mathtt{n}}$ and market resources ratio $m_t[\mathtt{j}]$, the elements of the matrix in the next period are given by:

$$\mathrm{X}_{t+1}[\mathtt{k}, \mathtt{q}] = \sum_{\mathtt{n}} \left(\sum_{\mathtt{j}} \Pi_t[\mathtt{k}, \mathtt{j}, \iota(\mathtt{n}, \mathtt{q})] \digamma_{\boldsymbol{\Psi}}[\iota(\mathtt{n}, \mathtt{q})] \mathrm{X}_t[\mathtt{j}, \mathtt{n}] \right) \tag{68}$$

where $\mathrm{m}_{t+1}[\mathtt{k}]$ is the $\mathtt{k}$'th element of the vector m_{t+1}, $\mathrm{p}_{t+1}[\mathtt{q}]$ is the $\mathtt{q}$'th element of p_{t+1}, and $\iota(\mathtt{n},\mathtt{q})$ is a function that calculates the index value of $\mathtt{i}$ that would achieve the transition from $\mathrm{p}_t^{\mathtt{n}}$ to $\mathrm{p}_{t+1}[\mathtt{q}]$.[21,22] Harmenberg defines

$$\chi_t^m(m_t) := \int_{p_t} \chi_t(m_t, p_t) dp_t \tag{69}$$

which measures the population density of persons whose market resources are m_t. In matrix terms, the corresponding representation is:

$$\mathrm{X}_t^m[\mathtt{j}] = \sum_{\mathtt{n}} \mathrm{X}_t[\mathtt{j},\mathtt{n}] \tag{70}$$

which makes it easy to see that $\mathrm{X}_t^m[\mathtt{j}]$ just measures the total probability mass associated with all possible levels of permanent income for agents at $\mathrm{m}_t^{\mathtt{j}}$. That is, it tells us *how many agents* have $m_t = \mathrm{m}_t^{\mathtt{j}}$.

In order to compute the absolute aggregate *amount* of market resources $\mathbf{m}_t$ (boldface indicates levels) owned by people with a market resources *ratio* of $\mathrm{m}_t^{\mathtt{j}}$, we need to know the total *amount* of permanent income accruing to those people:

$$\mathbf{m}_t[\mathtt{j}] = \mathrm{m}_t^{\mathtt{j}} \underbrace{\sum_{\mathtt{n}} \mathrm{p}_t^{\mathtt{n}} \mathrm{X}_t[\mathtt{j},\mathtt{n}]}_{\equiv \tilde{\mathrm{X}}^m[\mathtt{j}]\mathbf{\Phi}^t} \tag{71}$$

where $\tilde{\mathrm{X}}^m[\mathtt{j}]$ is what Harmenberg calls the Permanent Income Weighted measure. Under the assumption that aggregate permanent income was 1.0 in period 0 and has grown by the factor $\mathbf{\Phi}$ thereafter, the following direct formula for $\tilde{\mathrm{X}}^m$ can be seen to capture the *proportion of aggregate permanent income earned* by people at the given m_t:

$$\tilde{\mathrm{X}}^m[\mathtt{j}] = \mathbf{\Phi}^{-t} \sum_{\mathtt{n}} \mathrm{p}_t^{\mathtt{n}} \mathrm{X}_t[\mathtt{j},\mathtt{n}] \tag{72}$$

With this in hand, it is a simple matter to compute the total aggregate mass of $\mathbf{M}$:

$$\mathbf{M}_t = \mathbf{\Phi}^t \sum_{\mathtt{j}} \mathrm{m}_t^{\mathtt{j}} \tilde{\mathrm{X}}^m[\mathtt{j}] \tag{73}$$

or for that matter consumption:

$$\mathbf{C}_t = \mathbf{\Phi}^t \sum_{\mathtt{j}} \mathrm{c}(\mathrm{m}_t^{\mathtt{j}}) \tilde{\mathrm{X}}^m[\mathtt{j}] \tag{74}$$

Harmenberg formulates his corresponding propositions in the continuous description

---

[21] Of course, unless the is highly unlikely that $\iota(\mathtt{n},\mathtt{q})$ would yield an integer unless the elements of $\mathtt{q}$ were chosen as the unique elements defined by all possible combinations of p_t and $\digamma_{\mathbf{\Psi}}$; one option is to allocate probabilities in proportion to distance to the nearest integer values of $\iota(\mathtt{n},\mathtt{q})$ above and below the current point. (There are other options, which may not be meaningfully better).

[22] The $1/\mathbf{\Phi} p_t$ is present in the continuous version but not the discrete version because the latter captures masses and the former captures densities.

of the problem using probability measures, e.g.:

$$\tilde{\chi}_t^m(m_t) = \int_{p_t} p_t \chi(m_t, p_t) dp_t \tag{75}$$

$$\mathbf{C}_t = \mathbf{\Phi}^t \int_{m_t} \mathrm{c}(m_t) \tilde{\chi}^m(m_t) dm_t \tag{76}$$

The point here is 'that the permanent-income-weighted distribution is a sufficient statistic for computing aggregate consumption, aggregate savings, and similar aggregate variables'. Thus, rather than requiring us to keep track of the multidimensional joint distribution over m and $\mathbf{p}$, we need only know the distribution of permanent-income-weighted m.

The crucial last step is to define the law of motion for the weighted system. In the continuous formulation, Harmenberg shows (his Theorem 1) that, if we define a 'permanent-income-shock-weighted' version of the original permanent shock distribution[23] as

$$\tilde{f}_{\mathbf{\Psi}}(\mathbf{\Psi}_{t+1}) = \mathbf{\Psi}_{t+1} f_{\mathbf{\Psi}}(\mathbf{\Psi}_{t+1}) \tag{77}$$

then the laws of motion are respectively given by

$$\begin{aligned} \chi_{t+1}^m(m_{t+1}) &= \int_{m_t} \int_{\mathbf{\Psi}_{t+1}} \pi(m_{t+1}, m_t, \mathbf{\Psi}_{t+1}) \chi_t^m(m_t) f_{\mathbf{\Psi}}(\mathbf{\Psi}_{t+1}) d\mathbf{\Psi}_{t+1} dm_t \\ \tilde{\chi}_{t+1}^m(m_{t+1}) &= \int_{m_t} \int_{\mathbf{\Psi}_{t+1}} \pi(m_{t+1}, m_t, \mathbf{\Psi}_{t+1}) \tilde{\chi}_t^m(m_t) \tilde{f}_{\mathbf{\Psi}}(\mathbf{\Psi}_{t+1}) d\mathbf{\Psi}_{t+1} dm_t \end{aligned} \tag{78}$$

(where the difference between the two is the presence or absence of the $\sim$ accent in three places).

The key steps in the proof are the change in variables in which $\mathbf{\Psi}_{t+1} = p_{t+1}/(\mathbf{\Phi} p_t)$ and a change in the order of integration which is permitted by Fubini's theorem.

Omitting the $\mathbf{\Phi}$ term (equivalently, setting it to $\mathbf{\Phi} = 1$), the discrete version of the proof is

$$\tilde{\mathrm{X}}^m[\mathtt{k}] = \sum_{\mathtt{q}} \mathrm{p}_{t+1}^{\mathtt{q}} \mathrm{X}_{t+1}[\mathtt{k}, \mathtt{q}] \tag{79}$$

$$\approx \sum_{\mathtt{q}} \sum_{\mathtt{j}} \sum_{\mathtt{n}} \mathrm{p}_{t+1}^{\mathtt{q}} \Pi(\mathtt{k}, \mathtt{j}, \mathtt{q}) \tilde{F}_{\mathbf{\Psi}}(\iota(\mathtt{n}, \mathtt{q})) \mathrm{X}_t(\mathtt{j}, \mathtt{n}) \tag{80}$$

and the $\approx$ captures the fact that in the discrete context the necessity to allocate masses to points on the grid will lead to approximation error.

The change of variables is accomplished by realizing that just as there was an ι that gave us the appropriate $\mathbf{\Psi}$ required to get from p_t to p_{t+1}, we can define a $\varrho(\mathtt{n}, \mathtt{i})$ that lets us approximate the $\mathtt{q}$ needed as an argument to Π and the index for $\mathrm{p}_{t+1}^{\mathtt{q}}$. Now we

[23](Harmenberg calls this the 'permanent-income-neutral measure,' which is slightly confusing as it does not involve the level of permanent income but only the shocks thereto).

can sum over the permanent shocks indexed by $\mathtt{i}$:

$$
\begin{aligned}
\mathrm{X}^m[\mathtt{k}] &= \sum_{\mathtt{j}}\sum_{\mathtt{i}}\sum_{\mathtt{n}} \mathrm{p}_{t+1}^{\varrho(\mathtt{n},\mathtt{i})}\Pi(\mathtt{k},\mathtt{j},\varrho(\mathtt{n},\mathtt{i}))\digamma_{\boldsymbol{\Psi}}(\mathtt{i})\mathrm{X}_t(\mathtt{j},\mathtt{n}) && (81)\\
&= \sum_{\mathtt{j}}\sum_{\mathtt{i}} \mathrm{p}_{t+1}^{\varrho(\mathtt{n},\mathtt{i})}\Pi(\mathtt{k},\mathtt{j},\varrho(\mathtt{n},\mathtt{i}))\digamma_{\boldsymbol{\Psi}}(\mathtt{i})\mathrm{X}_t^m(\mathtt{j}) && (82)
\end{aligned}
$$

where he second line follows because the summation of $\mathrm{X}_t(\mathtt{j},\mathtt{n})$ is over $\mathtt{n}$ is exactly the step that yields $\mathrm{X}^m(\mathrm{m}_t^{\mathtt{j}})$.

The steps for the permanent-income-weighted version of the proposition are identical, with the substitution of weighted for unweighted versions of the various probability objects.

Table 1 Appendix: Perfect Foresight Liquidity Constrained Taxonomy

For constrained $\grave{c}$ and unconstrained $\bar{c}$ consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
~~GIC~~	$1 < \text{Þ}/\boldsymbol{\Phi}$	Constraint never binds for $m \geq 1$
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC holds ($\mathsf{R} > \boldsymbol{\Phi}$); $\grave{c}(m) = \bar{c}(m)$ for $m \geq 1$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	$\grave{c}(m)$ is degenerate: $\grave{c}(m) = 0$
GIC	$\text{Þ}/\boldsymbol{\Phi} < 1$	Constraint binds in finite time $\forall\ m$
and RIC	$\text{Þ}/\mathsf{R} < 1$	FHWC may or may not hold $\lim_{m\uparrow\infty} \bar{c}(m) - \grave{c}(m) = 0$ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = \underline{\kappa}$
and ~~RIC~~	$1 < \text{Þ}/\mathsf{R}$	~~FHWC~~ $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GICholds, the constraint will bind in finite time.