

1 Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\dot{c}(m)$ that arise under various parametric conditions. Results are summarized in table 1.

1.1 If ~~GIC~~ Fails

A consumer is ‘growth patient’ if the perfect foresight growth impatience condition fails (~~GIC~~, $1 < \mathbf{P}/\Gamma$). Under ~~GIC~~ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathbf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return \mathbf{R} :¹

$$\begin{aligned} 1 &< (\mathbf{R}\beta)^{1/\rho}\Gamma^{-1} \\ 1 &< \mathbf{R}\beta\Gamma^{-\rho} \\ u'(1) &< \mathbf{R}\beta u'(\Gamma). \end{aligned}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer’s consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHC holds. If the ~~RIC~~ fails ($1 < \mathbf{P}_R$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\dot{c}_{T-n}(m) = 0(b_t + h). \quad (1)$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHC fails. ~~FHC~~ implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\dot{c}_{T-n}(m) = 0$ or $\dot{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞ .²

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying ~~GIC~~ we must impose the ~~RIC~~ (and the ~~FHC~~ can be shown to be a consequence of ~~GIC~~ and ~~RIC~~). In this case, the consumer’s optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer

¹The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $u'(c_\#) = \mathbf{R}\beta u'(\Gamma)$ which is $c_\# = \Gamma/(\mathbf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\dot{c}(m) = \min[m, c_\# + (m - c_\#)\underline{\kappa}]$.

²The knife-edge case is where $\mathbf{P} = \Gamma$, in which case the two quantities counterbalance and the limiting function is $\dot{c}(m) = \min[m, 1]$.

would choose $c = m$ from Equation (20):

$$\begin{aligned} m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\ m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\ m_{\#} &= (h - 1) \left(\frac{\underline{\kappa}}{1 - \underline{\kappa}} \right) \end{aligned} \tag{2}$$

which (under these assumptions) satisfies $0 < m_{\#} < 1$.³ For $m < m_{\#}$ the unconstrained consumer would choose to consume more than m ; for such m , the constrained consumer is obliged to choose $\hat{c}(m) = m$.⁴ For any $m > m_{\#}$ the constraint will never bind and the consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

(Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it to study the tail behavior of the wealth distribution.)

1.2 If GIC Holds

Imposition of the GIC reverses the inequality in (1), and thus reverses the conclusion: A consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer will be constrained, not only in period t , but perpetually thereafter.

Now define $b_{\#}^n$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^n$ would behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b_{\#}^0$ trivially equal to 0); for example, a consumer with $b_{t-1} = b_{\#}^1$ was on the ‘cusp’ of being constrained in period $t - 1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding (because the consumer would have desired, but been unable, to enter period t with negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and thereafter) with resources of $m_t = m_{\#}^0 = 1 + b_{\#}^0 = 1$: The consumer cannot spend more (because constrained), and will not choose to spend less (because impatient), than $c_t = c_{\#}^0 = 1$.

We can construct the entire ‘prehistory’ of this consumer leading up to t as follows. Maintaining the assumption that the constraint has never bound in the past, c must have been growing according to \mathbf{P}_{Γ} , so consumption n periods in the past must have been

$$c_{\#}^n = \mathbf{P}_{\Gamma}^{-n} c_t = \mathbf{P}_{\Gamma}^{-n}. \tag{3}$$

³Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by GIC.

⁴As an illustration, consider a consumer for whom $\mathbf{P} = 1$, $R = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

The PDV of consumption from $t - n$ until t can thus be computed as

$$\begin{aligned}\mathbb{C}_{t-n}^t &= c_{t-n}(1 + \mathbf{P}/R + \cdots + (\mathbf{P}/R)^n) \\ &= c_{\#}^n(1 + \mathbf{P}_R + \cdots + \mathbf{P}_R^n) \\ &= \mathbf{P}_\Gamma^{-n} \left(\frac{1 - \mathbf{P}_R^{n+1}}{1 - \mathbf{P}_R} \right) \\ &= \left(\frac{\mathbf{P}_\Gamma^{-n} - \mathbf{P}_R}{1 - \mathbf{P}_R} \right)\end{aligned}$$

and note that the consumer's human wealth between $t - n$ and t (the relevant time horizon, because from t onward the consumer will be constrained and unable to access post- t income) is

$$h_{\#}^n = 1 + \cdots + \mathcal{R}^{-n} \quad (4)$$

while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^t = b_{\#}^n + h_{\#}^n$$

from which we can solve for the $b_{\#}^n$ such that the consumer with $b_{t-n} = b_{\#}^n$ would unconstrainedly plan (in period $t - n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^n = \mathbb{C}_{t-n}^t - \overbrace{\left(\frac{1 - \mathcal{R}^{-(n+1)}}{1 - \mathcal{R}^{-1}} \right)}^{h_{\#}^n}. \quad (5)$$

Defining $m_{\#}^n = b_{\#}^n + 1$, consider the function $\mathring{c}(m)$ defined by linearly connecting the points $\{m_{\#}^n, c_{\#}^n\}$ for integer values of $n \geq 0$ (and setting $\mathring{c}(m) = m$ for $m < 1$). This function will return, for any value of m , the optimal value of c for a liquidity constrained consumer with an infinite horizon. The function is piecewise linear with 'kink points' where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets $m = m_{\#}^n - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^n + \epsilon$ because the latter consumer will spread a marginal dollar over more periods before exhausting it.

In order for a unique consumption function to be defined by this sequence (5) for the entire domain of positive real values of b , we need $b_{\#}^n$ to become arbitrarily large with n . That is, we need

$$\lim_{n \rightarrow \infty} b_{\#}^n = \infty. \quad (6)$$

1.2.1 If *FHWC* Holds

The *FHWC* requires $\mathcal{R}^{-1} < 1$, in which case the second term in (5) limits to a constant as $n \uparrow \infty$, and (6) reduces to a requirement that

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n} - (\mathbf{P}_R/\mathbf{P}_\Gamma)^n \mathbf{P}_R}{1 - \mathbf{P}_R} \right) = \infty$$

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n} - \mathcal{R}^{-n} \mathbf{P}_\mathbf{R}}{1 - \mathbf{P}_\mathbf{R}} \right) &= \infty \\ \lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n}}{1 - \mathbf{P}_\mathbf{R}} \right) &= \infty.\end{aligned}$$

Given the **GIC** $\mathbf{P}_\Gamma^{-1} > 1$, this will hold iff the **RIC** holds, $\mathbf{P}_\mathbf{R} < 1$. But given that the **FHWC** $\mathbf{R} > \Gamma$ holds, the **GIC** is stronger (harder to satisfy) than the **RIC**; thus, the **FHWC** and the **GIC** together imply the **RIC**, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m \rightarrow \infty} \bar{c}(m) - \bar{c}(m) = 0. \quad (7)$$

1.2.2 If FHWC Fails

If the **FHWC** fails, matters are a bit more complex.

Given failure of **FHWC**, (6) requires

$$\begin{aligned}\lim_{n \rightarrow \infty} \left(\frac{\mathcal{R}^{-n} \mathbf{P}_\mathbf{R} - \mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\mathbf{R} - 1} \right) + \left(\frac{1 - \mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1} - 1} \right) &= \infty \\ \lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\mathbf{R}}{\mathbf{P}_\mathbf{R} - 1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1} - 1} \right) \mathcal{R}^{-n} - \left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\mathbf{R} - 1} \right) &= \infty\end{aligned}$$

If RIC Holds. When the **RIC** holds, rearranging (8) gives

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n}}{1 - \mathbf{P}_\mathbf{R}} \right) - \mathcal{R}^{-n} \left(\frac{\mathbf{P}_\mathbf{R}}{1 - \mathbf{P}_\mathbf{R}} + \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1} - 1} \right) = \infty$$

and for this to be true we need

$$\begin{aligned}\mathbf{P}_\Gamma^{-1} &> \mathcal{R}^{-1} \\ \Gamma/\mathbf{P} &> \Gamma/\mathbf{R} \\ 1 &> \mathbf{P}/\mathbf{R}\end{aligned}$$

which is merely the **RIC** again. So the problem has a solution if the **RIC** holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n \rightarrow \infty} \kappa_{\#}^n = \lim_{n \rightarrow \infty} \left(\frac{c_{\#}^n}{b_{\#}^n} \right) \quad (8)$$

which with a bit of algebra⁵ can be shown to asymptote to the MPC in the perfect

⁵Calculate the limit of

$$\left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\Gamma^{-n}/(1 - \mathbf{P}_\mathbf{R}) - (1 - \mathcal{R}^{-1}\mathcal{R}^{-n})/(1 - \mathcal{R}^{-1})} \right) = \left(\frac{1}{1/(1 - \mathbf{P}_\mathbf{R}) + \mathcal{R}^{-n}\mathcal{R}^{-1}/(1 - \mathcal{R}^{-1})} \right) \quad (9)$$

foresight model:⁶

$$\lim_{m \rightarrow \infty} \dot{\kappa}(m) = 1 - \mathbf{P}_R. \quad (10)$$

If RIC Fails. Consider now the **RIC** case, $\mathbf{P}_R > 1$. We can rearrange (8) as

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_R(\mathcal{R}^{-1} - 1)}{(\mathcal{R}^{-1} - 1)(\mathbf{P}_R - 1)} - \frac{\mathcal{R}^{-1}(\mathbf{P}_R - 1)}{(\mathcal{R}^{-1} - 1)(\mathbf{P}_R - 1)} \right) \mathcal{R}^{-n} - \left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_R - 1} \right) = \infty. \quad (11)$$

which makes clear that with **EHWC** $\Rightarrow \mathcal{R}^{-1} > 1$ and **RIC** $\Rightarrow \mathbf{P}_R > 1$ the numerators and denominators of both terms multiplying \mathcal{R}^{-n} can be seen transparently to be positive. So, the terms multiplying \mathcal{R}^{-n} in (8) will be positive if

$$\begin{aligned} \mathbf{P}_R \mathcal{R}^{-1} - \mathbf{P}_R &> \mathcal{R}^{-1} \mathbf{P}_R - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \mathbf{P}_R \\ \Gamma &> \mathbf{P} \end{aligned}$$

which is merely the **GIC** which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving \mathcal{R}^{-n} goes to $+\infty$ faster than the term involving $-\mathbf{P}_\Gamma^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \mathbf{P}_\Gamma^{-1} \\ \Gamma/R &> \Gamma/\mathbf{P} \\ \mathbf{P}/R &> 1 \end{aligned}$$

which merely confirms the starting assumption that the **RIC** fails.

What is happening here is that the $c_\#^n$ term is increasing backward in time at rate dominated in the limit by Γ/\mathbf{P} while the $b_\#$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/R > \Gamma/\mathbf{P} \quad (12)$$

because **RIC** $\Rightarrow \mathbf{P} > R$.

Consequently, while $\lim_{n \uparrow \infty} b_\#^n = \infty$, the limit of the *ratio* $c_\#^n/b_\#^n$ in (8) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the **RIC** fails. It remains true that **RIC** implies a limiting MPC of zero,

$$\lim_{m \rightarrow \infty} \dot{\kappa}(m) = 0, \quad (13)$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\dot{c}(m) = 0$. (Figure 1 presents an example for $\rho = 2$, $R = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m - 1$; the part of the consumption function below the depicted points is uninteresting — $c = m$ — so not worth plotting).

We can summarize as follows. Given that the **GIC** holds, the interesting question is whether the **FHWC** holds. If so, the **RIC** automatically holds, and the solution limits into

⁶For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.



Figure 1 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

the solution to the unconstrained problem as $m \uparrow \infty$. But even if the ~~FHWC~~ fails, the problem has a well-defined and nondegenerate solution, whether or not the ~~RIC~~ holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

References

- MA, QINGYIN, AND ALEXIS AKIRA TODA (2020): “A Theory of the Saving Rate of the Rich,” .
- STACHURSKI, JOHN, AND ALEXIS AKIRA TODA (2019): “An Impossibility Theorem for Wealth in Heterogeneous-Agent Models with Limited Heterogeneity,” *Journal of Economic Theory*, 182, 1–24.

Table 1 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained \bar{c} and unconstrained \bar{c} consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
GIC and RIC and RIC	$1 < \mathbf{P}/\Gamma$ $\mathbf{P}/R < 1$ $1 < \mathbf{P}/R$	Constraint never binds for $m \geq 1$ FHWC holds ($R > \Gamma$); $\dot{c}(m) = \bar{c}(m)$ for $m \geq 1$ $\dot{c}(m)$ is degenerate: $\dot{c}(m) = 0$
GIC and RIC	$\mathbf{P}/\Gamma < 1$ $\mathbf{P}/R < 1$	Constraint binds in finite time for any m FHWC may or may not hold $\lim_{m \uparrow \infty} \bar{c}(m) - \dot{c}(m) = 0$ $\lim_{m \uparrow \infty} \dot{\kappa}(m) = \underline{\kappa}$
and RIC	$1 < \mathbf{P}/R$	FHWC $\lim_{m \uparrow \infty} \dot{\kappa}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and ~~RIC~~ both hold, while the third row indicates that when the ~~GIC~~ and the ~~RIC~~ both fail, the consumption function is degenerate; the next row indicates that whenever the ~~GIC~~ holds, the constraint will bind in finite time.