1 Unique And Stable Target and Steady State Points

This appendix proves Theorems 2-3 and:

Lemma 1. *If $\check{m}$ and $\hat{m}$ both exist, then $\check{m} \leq \hat{m}$.*

1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the WRIC and FVAC; Theorem 1).

Section 2.8 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

This follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

1.3.1 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

If RIC holds. Logic exactly parallel to that of Section 3.1 leading to equation (44), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{Þ}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{Þ}/\Gamma_{t+1}] \qquad (1)
\end{aligned}$$

$$< 1$$

where the inequality reflects imposition of the GIC-Nrm (32).

If RIC fails. When the RIC fails, the fact that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$ (see equation (36)) means that the limit of the RHS of (1) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m\uparrow\infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

1.3.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for c in Section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t\downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{2}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{3}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (20) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\left(1-\mathrm{c}'(m_t)\right) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (3) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1. \tag{4}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\boldsymbol{\text{Þ}}}{\Gamma\psi}\right]}^{\boldsymbol{\text{Þ}}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\boldsymbol{\text{Þ}}}{\mathsf{R}}}^{\boldsymbol{\text{Þ}}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\boldsymbol{\text{Þ}}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\boldsymbol{\text{Þ}} < 1$$

which satisfies our requirement in (4).

1.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

1.4.1 Existence and Continuity of the Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in 1.2 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

1.4.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection 1.2 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t)+\xi_{t+1}\right)/\Gamma}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t)+\psi_{t+1}\xi_{t+1}}{m_t}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{m_t \uparrow \infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t} \right] \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} \\
&= \text{Þ}_{\Gamma} \\
&< 1
\end{aligned} \tag{5}
$$

where the last two lines are merely a restatement of the GIC (28).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.4.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{6}
$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$
\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t)\right] \\
&= (\mathsf{R}/\Gamma)\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{7}
$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (20) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.9.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$
\begin{aligned}
\mathcal{R}\,(1 - \mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} - 1
\end{aligned}
$$

which is negative because the GIC says $\text{Þ}_{\Gamma} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\mathcal{R}\,(1 - \mathrm{c}'(m_t)) < \mathcal{R}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (7) is guaranteed to be negative if

$$
\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1. \tag{8}
$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (8) holds).

1.5 Proof of Theorem ??

Proof of the existence of the final kind of stability is not straightforward.

$$\begin{aligned}
\log \mathbf{m}_{t+1} - \log \mathbf{m}_t &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \log \Gamma \psi_{t+1} m_{t+1} - \log m_t \\
&= \log \Gamma \psi_{t+1} \left((m_t - c_t)\mathsf{R}/\Gamma \psi_{t+1} + \xi_{t+1} \right) - \log m_t \\
&= \log \left((m_t - c_t)\mathsf{R} + \Gamma \psi_{t+1} \xi_{t+1} \right) - \log m_t \\
&= \log \left((m_t - c_t)\mathsf{R}(1 + \left(\frac{\Gamma \psi_{t+1} \xi_{t+1}}{(m_t - c_t)\mathsf{R}} \right) \right) - \log m_t \\
&= \log \mathsf{R} + \log(m_t - c_t) + \log \left(1 + \left(\frac{\Gamma \psi_{t+1} \xi_{t+1}}{(m_t - c_t)\mathsf{R}} \right) \right) - \log m_t
\end{aligned} \tag{9}$$

The derivative of log growth with respect to m is therefore

$$\begin{aligned}
(d/dm_t)\Delta \log \mathbf{m}_{t+1} &= \left(\frac{\mathrm{a}'(m_t)}{(m_t - c_t)} \right) - \frac{1}{m_t} + \left(\frac{(m_t - c_t)\mathsf{R}(\mathrm{a}'(m_t)\mathsf{R})}{(m_t - c_t)\mathsf{R} + \Gamma \psi_{t+1} \xi_{t+1}} \right) \\
&= \left(\frac{m_t \mathrm{a}'(m_t)}{m_t(m_t - c_t)} \right) - \frac{m_t - c_t}{m_t(m_t - c_t)} + \\
&= \left(\frac{m_t \mathrm{a}'(m_t) - (m_t - c_t)}{m_t(m_t - c_t)} \right) +
\end{aligned} \tag{10}$$

$$\begin{aligned}
(d/dm_t)\Delta \log \mathbf{m}_{t+1} &= \log \mathbf{p}_{t+1} m_{t+1} - \log \mathbf{p}_t m_t \\
&= \left(\frac{\mathcal{R}_{t+1}(1 - \kappa(m_t))}{(m_t - c_t)\mathcal{R}_{t+1} + \xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \mathcal{R}_{t+1}^{-1}\xi_{t+1}} \right) - 1/m_t \\
&= \left(\frac{(1 - \kappa(m_t))}{(m_t - c_t) + \Gamma \psi_{t+1} \xi_{t+1}/\mathsf{R}} \right) - 1/m_t.
\end{aligned} \tag{11}$$

As argued elsewhere, expected growth approaches a maximum of ∞ as $m_t \downarrow 0$. As $m_t \uparrow \infty$, the contribution of the term involving ξ becomes negligible and the expenditure ratio $\mathrm{e}(m) = \mathrm{c}(m)/m$ approaches $\underline{\kappa}$, so the limiting growth factor approaches

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\Delta \log \mathbf{m}_{t+1}] &= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \log \left((1 - \underline{\kappa})\mathcal{R}_{t+1} \right) \\
&= \log \Gamma + \mathbb{E}_t[\log \psi_{t+1}] + \mathbb{E}_t \left(\log \text{Þ} - (\log \Gamma + \log \psi_{t+1}) \right) \\
&= \log \text{Þ}.
\end{aligned} \tag{12}$$

The same arguments as before will guarantee continuity over the range of feasible values of m. So all that remains to prove is that $\mathbb{E}_t[\Delta \log \mathbf{m}_{t+1}]$ is strictly decreasing as

m rises. Differentiating with respect to m_t yields

$$
\begin{aligned}
(d/dm_t)\,\mathbb{E}_t\,\Delta\log\mathbf{m}_{t+1} &= (d/dm_t)\,\mathbb{E}_t\log\left((1-c_t/m_t)\mathcal{R}_{t+1}+\xi_{t+1}/m_t\right) \\
&= \mathbb{E}_t\left(\frac{-\xi_{t+1}/m_t^2-\mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\xi_{t+1}/m_t+\mathcal{R}_{t+1}(1-\mathrm{e}(m_t))}\right) \\
&= \mathbb{E}_t\left(\frac{-\xi_{t+1}/m_t-m_t\mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\xi_{t+1}+m_t\mathcal{R}_{t+1}(1-\mathrm{e}(m_t))}\right) \\
&< \mathbb{E}_t\left(\frac{-\underline{\xi}/m_t-m_t\mathcal{R}_{t+1}\mathrm{e}'(m_t)}{\bar{\xi}+m_t\mathcal{R}_{t+1}(1-\mathrm{e}(m_t))}\right)
\end{aligned}
\tag{13}
$$

and for every realization of ξ and $\mathcal{R}$, the numerator of this expression is negative and the denominator is positive. So the expression's expectation must be negative.

1.6 Proof of Lemmas 1-??

1.6.1 Pseudo-Steady-State m Is Smaller than Target m

Designate

$$
\begin{aligned}
\check{\mathrm{m}}_{t+1}(a) &= 1+a\mathcal{R} \\
\hat{\mathrm{m}}_{t+1}(a) &= 1+a\underbrace{\mathcal{R}/\underline{\psi}}_{\bar{\mathcal{R}}>\mathcal{R}}
\end{aligned}
\tag{14}
$$

so that we can implicitly define the target and pseudo-steady-state points as

$$
\begin{aligned}
\hat{m} &= \hat{\mathrm{m}}_{t+1}(\hat{m}-\mathrm{c}(\hat{m})) \\
\check{m} &= \check{\mathrm{m}}_{t+1}(\check{m}-\mathrm{c}(\check{m}))
\end{aligned}
\tag{15}
$$

Then subtract:

$$
\begin{aligned}
\hat{m}-\check{m} &= \left(\hat{a}\underline{\psi}^{-1}-\check{a}\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1}-a(\check{m})\right)\mathcal{R} \\
&= \left(a(\hat{m})\underline{\psi}^{-1}-(a(\hat{m}+\check{m}-\hat{m}))\right)\mathcal{R} \\
&\approx \left(a(\hat{m})\underline{\psi}^{-1}-(a(\hat{m})+(\check{m}-\hat{m})\mathrm{a}'(\hat{m}))\right)\mathcal{R} \\
(\hat{m}-\check{m})(1-\underbrace{a'(\hat{m})\mathcal{R}}_{<\boldsymbol{\text{Þ}}_\Gamma<1}) &= (\underline{\psi}^{-1}-1)\hat{a}\mathcal{R}
\end{aligned}
\tag{16}
$$

The RHS of this equation is strictly positive because $\underline{\psi}^{-1}>1$ and both $\hat{a}$ and $\mathcal{R}$ are positive; while on the LHS, $(1-\mathcal{R}a')>0$. So the equation can only hold if $\hat{m}-\check{m}>0$. That is, the target ratio exceeds the pseudo-steady-state ratio.[1]

[1] The use of the first order Taylor approximation could be substituted, cumbersomely, with the average of a' over the interval to remove the approximation in the derivations above.

1.6.2 The m achieving Expected-Log-Balanced-Growth Is Smaller than Pseudo-Steady-State m

Expected log balanced growth occurs when

$$\begin{aligned}
\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \Gamma \mathbf{m}_t \\
\mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \Gamma \mathbf{p}_t m_t \\
\mathbb{E}_t[\log \psi_{t+1} m_{t+1}] &= \log \Gamma m_t \\
\mathbb{E}_t[\log (a(m_t)\mathsf{R} + \psi_{t+1}\xi_{t+1}\Gamma)] &= \log \Gamma m_t \\
\mathbb{E}_t[\log (a(m_t)\mathcal{R} + \psi_{t+1}\xi_{t+1})] &= \log m_t
\end{aligned} \tag{17}$$

and we call the m that satisfies this equation $\acute{m}$.

Now we use the fact that the expectation of the log is less than the log of the expectation,

$$\begin{aligned}
\log \mathbb{E}_t[a(\acute{m}_t)\mathcal{R} + \psi_{t+1}\xi_{t+1}] &< \log \acute{m}_t \\
\log (a(\acute{m}_t)\mathcal{R} + 1) &< \log \acute{m}_t \\
a(\acute{m}_t)\mathcal{R} + 1 &< \acute{m}_t
\end{aligned} \tag{18}$$

Finally, subtract $\check{m}$ from both sides,

$$\begin{aligned}
a(\acute{m}_t)\mathcal{R} + 1 - (a(\check{m}_t)\mathcal{R} + 1) &< \acute{m}_t - \check{m}_t \\
a(\acute{m}_t) - a(\acute{m}_t + \check{m} - \acute{m}_t))\mathcal{R} &< \acute{m}_t - \check{m}_t \\
a(\acute{m}_t) - (a(\acute{m}_t) + (\check{m} - \acute{m}_t)a'(\acute{m}))\mathcal{R} &< \acute{m}_t - \check{m} \\
(\acute{m}_t - \check{m})a'(\acute{m})\mathcal{R} &< \acute{m}_t - \check{m} \\
\underbrace{a'(\acute{m})\mathcal{R}}_{<\text{\textbf{Þ}}_\Gamma} &< 1
\end{aligned} \tag{19}$$

which again holds because $\text{\textbf{Þ}}_\Gamma < 1$ (and, as above, a proof that does not require the Taylor approximation is available but more cumbersome).