

# 1 Unique, Stable Target and Steady State Points

This appendix proves Theorems 2-3 and:

**Lemma 1.** *If  $\check{m}$  and  $\hat{m}$  both exist, then  $\check{m} \leq \hat{m}$ .*

## 1.1 Proof of Theorem 2

The elements of the proof of Theorem 2 are:

- Existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$  is monotonically decreasing

## 1.2 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed sufficient conditions (the **WRIC** and **FVAC**; Theorem 1).

Section 2.8 shows that for all  $t$ ,  $a_{t-1} = m_{t-1} - c_{t-1} > 0$ . Since  $m_t = a_{t-1}\mathcal{R}_t + \xi_t$ , even if  $\xi_t$  takes on its minimum value of 0,  $a_{t-1}\mathcal{R}_t > 0$ , since both  $a_{t-1}$  and  $\mathcal{R}_t$  are strictly positive. With  $m_t$  and  $m_{t+1}$  both strictly positive, the ratio  $\mathbb{E}_t[m_{t+1}/m_t]$  inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

## 1.3 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ .

This follows from:

1. Existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$  (just proven)
2. Existence a point where  $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where  $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

### 1.3.1 Existence of $m$ where $\mathbb{E}_t[m_{t+1}/m_t] < 1$

**If RIC holds.** Logic exactly parallel to that of Section 3.1 leading to equation (39), but dropping the  $\Phi_{t+1}$  from the RHS, establishes that

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(R/\Phi_{t+1})\mathbf{P}_R] \\ &= \mathbb{E}_t[\mathbf{P}/\Phi_{t+1}] \\ &< 1 \end{aligned} \tag{1}$$

where the inequality reflects imposition of the **GIC-Nrm** (26).

**If RIC fails.** When the **RIC** fails, the fact that  $\lim_{m \uparrow \infty} c'(m) = 0$  (see equation (30)) means that the limit of the RHS of (1) as  $m \uparrow \infty$  is  $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$ . In the next step of this proof, we will prove that the combination **GIC-Nrm** and **RIC** implies  $\bar{\mathcal{R}} < 1$ .

So we have  $\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$  whether the **RIC** holds or fails.

### 1.3.2 Existence of $m > 1$ where $\mathbb{E}_t[m_{t+1}/m_t] > 1$

Paralleling the logic for  $c$  in Section 3.2: the ratio of  $\mathbb{E}_t[m_{t+1}]$  to  $m_t$  is unbounded above as  $m_t \downarrow 0$  because  $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$ .

*Intermediate Value Theorem.* If  $\mathbb{E}_t[m_{t+1}/m_t]$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

### 1.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.

Now define  $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$  and note that

$$\begin{aligned} \zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1, \end{aligned} \tag{2}$$

so that  $\zeta(\hat{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned} \zeta'(m_t) &\equiv \left( \frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[ \left( \frac{d}{dm_t} \right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}} (1 - c'(m_t)) - 1. \end{aligned} \tag{3}$$

Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the **RIC** holds or fails.

**If RIC holds.** Equation (16) indicates that if the **RIC** holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.9.1 that if the **RIC** holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\bar{\mathcal{R}} (1 - c'(m_t)) - 1 < \bar{\mathcal{R}} (1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1$$

$$\begin{aligned}
&= \bar{\mathcal{R}} \mathbf{p}_R - 1 \\
&= \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Phi \Psi} \frac{\mathbf{p}}{\mathbf{R}} \right] - 1 \\
&= \underbrace{\mathbb{E}_t \left[ \frac{\mathbf{p}}{\Phi \Psi} \right]}_{=\mathbf{p}_\Phi} - 1
\end{aligned}$$

which is negative because the **GIC-Nrm** says  $\mathbf{p}_\Phi < 1$ .

**If RIC fails.** Under **RIC**, recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\bar{\mathcal{R}} (1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that  $\zeta'(m_t)$  from (3) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Phi \Psi} \right] < 1. \quad (4)$$

But the combination of the **GIC-Nrm** holding and the **RIC** failing can be written:

$$\underbrace{\mathbb{E}_t \left[ \frac{\mathbf{p}}{\Phi \Psi} \right]}_{\mathbf{p}_\Phi} < 1 < \underbrace{\frac{\mathbf{p}_R}{\mathbf{R}}}_{\mathbf{p}_R},$$

and multiplying all three elements by  $\mathbf{R}/\mathbf{p}$  gives

$$\mathbb{E}_t \left[ \frac{\mathbf{R}}{\Phi \Psi} \right] < \mathbf{R}/\mathbf{p} < 1$$

which satisfies our requirement in (4).

## 1.4 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of  $\mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where  $\mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\Psi_{t+1}m_{t+1} - m_t]$  is monotonically decreasing

### 1.4.1 Existence and Continuity of the Ratio

Since by assumption  $0 < \underline{\Psi} \leq \Psi_{t+1} \leq \bar{\Psi} < \infty$ , our proof in 1.2 that demonstrated existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$  implies existence and continuity of  $\mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t]$ .

#### 1.4.2 Existence of a stable point

Since by assumption  $0 < \underline{\Psi} \leq \Psi_{t+1} \leq \bar{\Psi} < \infty$ , our proof in Subsection 1.2 that the ratio of  $\mathbb{E}_t[m_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$  implies that the ratio  $\mathbb{E}_t[\Psi_{t+1}m_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$ .

The limit of the expected ratio as  $m_t$  goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\Phi_{t+1} ((R/\Phi_{t+1})a(m_t) + \xi_{t+1}) / \Phi}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{(R/\Phi)a(m_t) + \Psi_{t+1}\xi_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[ \frac{(R/\Phi)a(m_t) + 1}{m_t} \right] \\
&= (R/\Phi)\mathbf{P}_R \\
&= \mathbf{P}_\Phi \\
&< 1
\end{aligned} \tag{5}$$

where the last two lines are merely a restatement of the **GIC** (19).

The Intermediate Value Theorem says that if  $\mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t]$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

#### 1.4.3 $\mathbb{E}_t[\Psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define  $\zeta(m_t) \equiv \mathbb{E}_t[\Psi_{t+1}m_{t+1}] - m_t$  and note that

$$\begin{aligned}
\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t] < 1 \\
\zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t] = 1 \\
\zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\Psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{6}$$

so that  $\zeta(\hat{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned}
\zeta'(m_t) &\equiv \left( \frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[ \left( \frac{d}{dm_t} \right) (\mathcal{R}(m_t - c(m_t)) + \Psi_{t+1}\xi_{t+1} - m_t) \right] \\
&= (R/\Phi)(1 - c'(m_t)) - 1.
\end{aligned} \tag{7}$$

Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the **RIC** holds or fails (**RIC**).

**If **RIC** holds.** Equation (16) indicates that if the **RIC** holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.9.1 that if the **RIC** holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\begin{aligned}
\mathcal{R}(1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\
&= (R/\Phi)\mathbf{P}_R - 1
\end{aligned}$$

which is negative because the **GIC** says  $\mathbf{P}_\Phi < 1$ .

**If RIC fails.** Under **RIC**, recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\mathcal{R} (1 - c'(m_t)) < \mathcal{R}$$

which means that  $\zeta'(m_t)$  from (7) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathbf{R}/\Phi) < 1. \quad (8)$$

But we showed in Section 2.6 that the only circumstances under which the problem has a nondegenerate solution while the **RIC** fails were ones where the **FHWC** also fails (that is, (8) holds).

## 1.5 A Third Measure

A footnote in Section 3 mentions reasons why it may be useful to calculate  $\mathbb{E}_t[\log(\mathbf{m}_{t+1}/\log \mathbf{m}_t)]$ . Here we show that one way of doing that is to calculate a nonlinear adjustment factor for the expectation of the growth factor.

$$\begin{aligned} \log(\mathbf{m}_{t+1}/\mathbf{m}_t) &= \log(\Phi \Psi_{t+1} m_{t+1}) - \log m_t \\ &= \log \Phi (a_t \mathcal{R} + \Psi_{t+1} \xi_{t+1}) - \log m_t \\ &= \log \Phi (a_t \mathcal{R} + 1 + (\Psi_{t+1} \xi_{t+1} - 1)) - \log m_t \end{aligned}$$

Now define  $\check{m}_{t+1} = a_t \mathcal{R} + 1$ , and compute the expectation:

$$\begin{aligned} \mathbb{E}_t[\log(\mathbf{m}_{t+1}/\mathbf{m}_t)] &= \mathbb{E}_t[\log \Phi (\check{m}_{t+1} + (\Psi_{t+1} \xi_{t+1} - 1))] - \log m_t \\ &= \log \Phi + \mathbb{E}_t[\log \check{m}_{t+1} (1 + (\Psi_{t+1} \xi_{t+1} - 1) \check{m}_{t+1}^{-1})] - \log m_t \\ &= \underbrace{\log \Phi + \log \check{m}_{t+1} - \log m_t}_{\equiv \log \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]} + \mathbb{E}_t[\log(1 + (\Psi_{t+1} \xi_{t+1} - 1) \check{m}_{t+1}^{-1})] \end{aligned}$$

and exponentiating tells us that

$$\exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \exp(\mathbb{E}_t[\log(1 + (\Psi_{t+1} \xi_{t+1} - 1) \check{m}_{t+1}^{-1})]) \quad (9)$$

and this latter factor is a number that approaches 1 from below as  $m_t$  rises. Thus the expected growth rate of the log is smaller than the log of the growth rate of the expected growth factor. This implies that the  $m$  at which ‘balanced growth’ can be expected in the log,  $\check{m}$ , exceeds the corresponding point for the ratio,  $\check{m}$ .

Furthermore, in the limit as  $\mathbf{m}_t$  gets arbitrarily large, if the **RIC** holds and thus  $\kappa > 0$ ,  $a_{t+1}$  rises without bound, as does  $\check{m}_{t+1} = a_{t+1} \mathcal{R} + 1$ , so the approximation  $\log(1 + \epsilon) \approx \epsilon$  becomes arbitrarily good. Consequently, the last term on the RHS of (9) can be approximated as

$$\begin{aligned} \mathbb{E}_t[\log(1 + (\Psi_{t+1} \xi_{t+1} - 1) \check{m}_{t+1}^{-1})] &\approx \mathbb{E}_t[(\Psi_{t+1} \xi_{t+1} - 1) \check{m}_{t+1}^{-1}] \\ &= 0 \end{aligned}$$

This demonstrates that

$$\lim_{\mathbf{m}_t \uparrow \infty} \exp(\mathbb{E}_t[\log \mathbf{m}_{t+1}/\mathbf{m}_t]) = \mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t] \quad (10)$$

## 1.6 Proof of Lemma

### 1.6.1 Pseudo-Steady-State $m$ Is Smaller than Target $m$

Designate

$$\begin{aligned}\check{m}_{t+1}(a) &= 1 + a\mathcal{R} \\ \hat{m}_{t+1}(a) &= 1 + a \underbrace{\mathcal{R}/\underline{\Psi}}_{\mathcal{R} > \mathcal{R}}\end{aligned}\tag{11}$$

so that we can implicitly define the target and pseudo-steady-state points as

$$\begin{aligned}\hat{m} &= \hat{m}_{t+1}(\hat{m} - c(\hat{m})) \\ \check{m} &= \check{m}_{t+1}(\check{m} - c(\check{m}))\end{aligned}\tag{12}$$

Then subtract:

$$\begin{aligned}\hat{m} - \check{m} &= (\hat{a}\underline{\Psi}^{-1} - \check{a})\mathcal{R} \\ &= (a(\hat{m})\underline{\Psi}^{-1} - a(\check{m}))\mathcal{R} \\ &= (a(\hat{m})\underline{\Psi}^{-1} - (a(\hat{m} + \check{m} - \hat{m})))\mathcal{R} \\ &\approx (a(\hat{m})\underline{\Psi}^{-1} - (a(\hat{m}) + (\check{m} - \hat{m})a'(\hat{m})))\mathcal{R} \\ (\hat{m} - \check{m})(1 - \underbrace{a'(\hat{m})\mathcal{R}}_{< \mathbf{P}_{\Phi} < 1}) &= (\underline{\Psi}^{-1} - 1)\hat{a}\mathcal{R}\end{aligned}\tag{13}$$

The RHS of this equation is strictly positive because  $\underline{\Psi}^{-1} > 1$  and both  $\hat{a}$  and  $\mathcal{R}$  are positive; while on the LHS,  $(1 - \mathcal{R}a') > 0$ . So the equation can only hold if  $\hat{m} - \check{m} > 0$ . That is, the target ratio exceeds the pseudo-steady-state ratio.<sup>1</sup>

### 1.6.2 The $m$ Achieving Individual Expected-Log-Balanced-Growth Is Smaller than the Individual Pseudo-Steady-State $m$

Expected log balanced growth occurs when

$$\begin{aligned}\mathbb{E}_t[\log \mathbf{m}_{t+1}] &= \log \Phi \mathbf{m}_t \\ \mathbb{E}_t[\log \mathbf{p}_{t+1} m_{t+1}] &= \log \Phi \mathbf{p}_t m_t \\ \mathbb{E}_t[\log \Psi_{t+1} m_{t+1}] &= \log \Phi m_t \\ \mathbb{E}_t[\log (a(m_t)\mathcal{R} + \Psi_{t+1}\xi_{t+1}\Phi)] &= \log \Phi m_t \\ \mathbb{E}_t[\log (a(m_t)\mathcal{R} + \Psi_{t+1}\xi_{t+1})] &= \log m_t\end{aligned}\tag{14}$$

and we call the  $m$  that satisfies this equation  $\tilde{m}$ .

Subtract the definition of  $\tilde{m}$  from that of  $\check{m}$ :

$$\exp(\mathbb{E}_t[\log (a(\tilde{m})\mathcal{R} + \Psi_{t+1}\xi_{t+1})]) - (a(\check{m})\mathcal{R} + 1) = \tilde{m} - \check{m}\tag{15}$$

Now we use the fact that the expectation of the log is less than the log of the

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<sup>1</sup>The use of the first order Taylor approximation could be substituted, clumsily, with the average of  $a'$  over the interval to remove the approximation in the derivations above.

expectation,

$$\exp(\mathbb{E}_t[\log(a(\tilde{m})\mathcal{R} + \Psi_{t+1}\boldsymbol{\xi}_{t+1})]) < (a(\tilde{m})\mathcal{R} + 1) \quad (16)$$

so

$$\begin{aligned} \exp(\mathbb{E}_t[\log(a(\tilde{m})\mathcal{R} + 1)]) - (a(\tilde{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\ (a(\tilde{m})\mathcal{R} + 1) - (a(\check{m})\mathcal{R} + 1) &< \tilde{m} - \check{m} \\ (a(\tilde{m}) - a(\tilde{m} + \check{m} - \tilde{m}))\mathcal{R} &< \tilde{m} - \check{m} \\ (a(\tilde{m}) - (a(\tilde{m}) + (\check{m} - \tilde{m})\bar{a}'))\mathcal{R} &< \tilde{m} - \check{m} \\ (\tilde{m} - \check{m})\bar{a}'\mathcal{R} &< \tilde{m} - \check{m} \\ \underbrace{\bar{a}'\mathcal{R}}_{< \mathbf{P}_\Phi} &< 1 \end{aligned} \quad (17)$$

where we are interpreting  $\bar{a}'$  as the mean of the value of  $a'$  over the interval between  $\tilde{m}$  and  $\check{m}$ .