1 Convergence in Euclidian Space

1.1 Convergence of v_t

Boyd's theorem shows that $\mathscr{T}$ defines a contraction mapping in a $\digamma$-bounded space. We now show that $\mathscr{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathscr{T}$, since $\mathrm{v}^*(m) = \mathscr{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_{\digamma} \leq \alpha^{n-1} \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma}. \tag{1}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_{\digamma}(\mathscr{A},\mathscr{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_{\digamma} < \infty$ because v_T and v^* are in $\mathcal{C}_{\digamma}(\mathscr{A},\mathscr{B})$. It follows that

$$\left|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)\right| \leq \kappa\alpha^{n-1} \left|\digamma(m)\right|. \tag{2}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{3}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathscr{T}\mathrm{v}_{T-1} \leq \mathscr{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

1.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

Consider any convergent subsequence $\left\{\mathrm{c}_{T-n(i)}(m)\right\}$ of $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $\mathrm{c}_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m)) + \beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \tag{4}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{5}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.