## 1 Convergence in Euclidian Space

## 1.1 Convergence of $v_t$

Boyd's theorem shows that  $\mathcal{T}$  defines a contraction mapping in a  $\mathcal{F}$ -bounded space. We now show that  $\mathcal{T}$  also defines a contraction mapping in Euclidian space.

Calling v\* the unique fixed point of the operator  $\mathcal{T}$ , since v\*(m) =  $\mathcal{T}$ v\*(m),

$$\|\mathbf{v}_{T-n+1} - \mathbf{v}^*\|_F \le \alpha^{n-1} \|\mathbf{v}_T - \mathbf{v}^*\|_F. \tag{1}$$

On the other hand,  $\mathbf{v}_T - \mathbf{v}^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$  and  $\kappa = \|\mathbf{v}_T - \mathbf{v}^*\|_F < \infty$  because  $\mathbf{v}_T$  and  $\mathbf{v}^*$  are in  $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ . It follows that

$$|\mathbf{v}_{T-n+1}(m) - \mathbf{v}^*(m)| \le \kappa \alpha^{n-1} |F(m)|.$$
 (2)

Then we obtain

$$\lim_{n \to \infty} \mathbf{v}_{T-n+1}(m) = \mathbf{v}^*(m). \tag{3}$$

Since  $\mathbf{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$ ,  $\mathbf{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathbf{v}_T(m)$ . On the other hand,  $\mathbf{v}_{T-1} \leq \mathbf{v}_T$  means  $\Im \mathbf{v}_{T-1} \leq \Im \mathbf{v}_T$ , in other words,  $\mathbf{v}_{T-2}(m) \leq \mathbf{v}_{T-1}(m)$ . Inductively one gets  $\mathbf{v}_{T-n}(m) \geq \mathbf{v}_{T-n-1}(m)$ . This means that  $\{\mathbf{v}_{T-n+1}(m)\}_{n=1}^{\infty}$  is a decreasing sequence, bounded below by  $\mathbf{v}^*$ .

## 1.2 Convergence of $c_t$

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions  $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$ .

Consider any convergent subsequence  $\{c_{T-n(i)}(m)\}$  of  $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$  converging to  $c^*$ . By the definition of  $c_{T-n}(m)$ , we have

$$u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)] \ge u(c_{T-n(i)}) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)],$$
(4)

for any  $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$ . Now letting n(i) go to infinity, it follows that the left hand side converges to  $\mathbf{u}(c^*) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho}\mathbf{v}(m)]$ , and the right hand side converges to  $\mathbf{u}(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho}\mathbf{v}(m)]$ . So the limit of the preceding inequality as n(i) approaches infinity implies

$$u(c^*) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}v(m)] \ge u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}v(m)].$$
 (5)

Hence,  $c^* \in \underset{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]}{\arg \max} \left\{ \mathbf{u}(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathbf{v}(m)] \right\}$ . By the uniqueness of  $\mathbf{c}(m)$ ,  $c^* = \mathbf{c}(m)$ .