? Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max\ \mathbb{E}_t \left[\sum_{i=0}^{n} \beta^i \mathrm{u}(\mathbf{c}_{t+i}) \right]$$

Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\grave{\mathrm{c}}(m)$ that arise under various parametric conditions. Results are summarized in table **??**.

A.1 If GIC Fails

A consumer is 'growth patient' if the perfect foresight growth impatience condition fails ($\cancel{\text{GIC}}$, $1 < \text{Þ}/\Gamma$). Under $\cancel{\text{GIC}}$ the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (\mathsf{R}\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R:[1]

$$\begin{aligned} 1 &< (\mathsf{R}\beta)^{1/\rho}\Gamma^{-1} \\ 1 &< \mathsf{R}\beta\Gamma^{-\rho} \\ \mathrm{u}'(1) &< \mathsf{R}\beta\mathrm{u}'(\Gamma). \end{aligned}$$

Reference to (**??**)

B Existence of a Concave Consumption Function

To show that (**??**) defines a sequence of continuously differentiable strictly increasing concave functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, \ldots, \mathrm{c}_{T-k}\}$, we start with a definition. We will say that a function $\mathrm{n}(z)$ is 'nice' if it satisfies

1. $\mathrm{n}(z)$ is well-defined iff $z > 0$
2. $\mathrm{n}(z)$ is strictly increasing
3. $\mathrm{n}(z)$ is strictly concave
4. $\mathrm{n}(z)$ is $\mathbf{C}^3$
5. $\mathrm{n}(z) < 0$

[1] The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $\mathrm{u}'(c_{\#}) = \mathsf{R}\beta\mathrm{u}'(\Gamma)$ which is $c_{\#} = \Gamma/(\mathsf{R}\beta)^{1/\rho}$ and the consumption function will be defined by $\grave{\mathrm{c}}(m) = \min[m, c_{\#} + (m - c_{\#})\underline{\kappa}]$.

6. $\lim_{z\downarrow 0}\ \mathrm{n}(z) = -\infty$.

(Notice that an implication of niceness is that $\lim_{z\downarrow 0} \mathrm{n}'(z) = \infty$.)

Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because $\mathrm{v}_T(m) = \mathrm{u}(m)$ and $\mathrm{u}(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

Now define an end-of-period value function $\mathfrak{v}_t(a)$ as

$$\mathfrak{v}_t(a) = \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})\right]. \tag{1}$$

Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a\downarrow 0}\mathfrak{v}_t(a) = -\infty$ and $\lim_{a\downarrow 0}\mathfrak{v}_t'(a) = \infty$. So $\mathfrak{v}_t(a)$ is well-defined iff $a > 0$; it is similarly straightforward to show the other properties required for $\mathfrak{v}_t(a)$ to be nice. (See Hiraguchi (**?**).)

Next define $\underline{\mathrm{v}}_t(m,c)$ as

$$\underline{\mathrm{v}}_t(m,c) = \mathrm{u}(c) + \mathfrak{v}_t(m-c) \tag{2}$$

which is $\mathbf{C}^3$ since $\mathfrak{v}_t$ and u are both $\mathbf{C}^3$, and note that our problem's value function defined in (**??**) can be written as

$$\mathrm{v}_t(m) = \max_c\ \underline{\mathrm{v}}_t(m,c). \tag{3}$$

$\underline{\mathrm{v}}_t$ is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c\downarrow 0}\underline{\mathrm{v}}_t(m,c) = \lim_{c\uparrow m}\underline{\mathrm{v}}_t(m,c) = -\infty$, $\frac{\partial^2\underline{\mathrm{v}}_t(m,c)}{\partial c^2} < 0$, $\lim_{c\downarrow 0}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = +\infty$, and $\lim_{c\uparrow m}\frac{\partial\underline{\mathrm{v}}_t(m,c)}{\partial c} = -\infty$. It follows that the $\mathrm{c}_t(m)$ defined by

$$\mathrm{c}_t(m) = \underset{0<c<m}{\arg\max}\ \underline{\mathrm{v}}_t(m,c) \tag{4}$$

exists and is unique, and (**??**) has an internal solution that satisfies

$$\mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(m - \mathrm{c}_t(m)). \tag{5}$$

Since both u and $\mathfrak{v}_t$ are strictly concave, both $\mathrm{c}_t(m)$ and $\mathrm{a}_t(m) = m - \mathrm{c}_t(m)$ are strictly increasing. Since both u and $\mathfrak{v}_t$ are three times continuously differentiable, using (5) we can conclude that $\mathrm{c}_t(m)$ is continuously differentiable and

$$\mathrm{c}_t'(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}. \tag{6}$$

Similarly we can easily show that $\mathrm{c}_t(m)$ is twice continuously differentiable (as is $\mathrm{a}_t(m)$) (See Appendix C.) This implies that $\mathrm{v}_t(m)$ is nice, since $\mathrm{v}_t(m) = \mathrm{u}(\mathrm{c}_t(m)) + \mathfrak{v}_t(\mathrm{a}_t(m))$.

C $\mathrm{c}_t(m)$ is Twice Continuously Differentiable

First we show that $\mathrm{c}_t(m)$ is $\mathbf{C}^1$. Define y as $y \equiv m + dm$. Since $\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m)) = \mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))$ and $\frac{\mathrm{a}_t(y)-\mathrm{a}_t(m)}{dm} = 1 - \frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$,

$$\frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} = \left(\frac{\mathrm{u}'(\mathrm{c}_t(y)) - \mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y)) - \mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}\right)\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm}$$

Since c_t and a_t are continuous and increasing, $\lim_{dm\to+0}\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} < 0$ and $\lim_{dm\to+0}\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ are satisfied. Then $\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)} < 0$ for sufficiently small dm. Hence we obtain a well-defined equation:

$$\frac{\mathrm{c}_t(y)-\mathrm{c}_t(m)}{dm} = \frac{\frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}{\frac{\mathrm{u}'(\mathrm{c}_t(y))-\mathrm{u}'(\mathrm{c}_t(m))}{\mathrm{c}_t(y)-\mathrm{c}_t(m)} + \frac{\mathfrak{v}_t'(\mathrm{a}_t(y))-\mathfrak{v}_t'(\mathrm{a}_t(m))}{\mathrm{a}_t(y)-\mathrm{a}_t(m)}}.$$

This implies that the right-derivative, $\mathrm{c}_t'^{+}(m)$ is well-defined and

$$\mathrm{c}_t'^{+}(m) = \frac{\mathfrak{v}_t''(\mathrm{a}_t(m))}{\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m))}.$$

Similarly we can show that $\mathrm{c}_t'^{+}(m) = \mathrm{c}_t'^{-}(m)$, which means $\mathrm{c}_t'(m)$ exists. Since $\mathfrak{v}_t$ is $\mathbf{C}^3$, $\mathrm{c}_t'(m)$ exists and is continuous. $\mathrm{c}_t'(m)$ is differentiable because $\mathfrak{v}_t''$ is $\mathbf{C}^1$, $\mathrm{c}_t(m)$ is $\mathbf{C}^1$ and $\mathrm{u}''(\mathrm{c}_t(m)) + \mathfrak{v}_t''(\mathrm{a}_t(m)) < 0$. $\mathrm{c}_t''(m)$ is given by

$$\mathrm{c}_t''(m) = \frac{a_t'(m)\mathfrak{v}_t'''(a_t)\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right] - \mathfrak{v}_t''(a_t)\left[c_t'\mathrm{u}'''(c_t) + a_t'\mathfrak{v}_t'''(a_t)\right]}{\left[\mathrm{u}''(c_t) + \mathfrak{v}_t''(a_t)\right]^2}. \tag{7}$$

Since $\mathfrak{v}_t''(\mathrm{a}_t(m))$ is continuous, $\mathrm{c}_t''(m)$ is also continuous.

D Proof that $\mathcal{T}$ Is a Contraction Mapping

We must show that our operator $\mathcal{T}$ satisfies all of Boyd's conditions.

Boyd's operator T maps from $\mathcal{C}_F(\mathcal{A},\mathcal{B})$ to $\mathcal{C}(\mathcal{A},\mathcal{B})$. A preliminary requirement is therefore that $\{\mathcal{T}\mathrm{z}\}$ be continuous for any $F-$bounded z, $\{\mathcal{T}\mathrm{z}\} \in \mathcal{C}(\mathbb{R}_{++},\mathbb{R})$. This is not difficult to show; see Hiraguchi (?).

Consider condition (1). For this problem,

$$\begin{aligned} \{\mathcal{T}\mathrm{x}\}(m_t) \text{ is } & \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{x}\left(m_{t+1}\right)\right]\right\} \\ \{\mathcal{T}\mathrm{y}\}(m_t) \text{ is } & \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \left\{\mathrm{u}(c_t) + \beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{y}\left(m_{t+1}\right)\right]\right\}, \end{aligned}$$

so $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)$ implies $\{\mathcal{T}\mathrm{x}\}(m_t) \le \{\mathcal{T}\mathrm{y}\}(m_t)$ by inspection.[2]

Condition (2) requires that $\{\mathcal{T}\mathbf{0}\} \in \mathcal{C}_F\left(\mathcal{A},\mathcal{B}\right)$. By definition,

$$\{\mathcal{T}\mathbf{0}\}(m_t) = \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \left\{\left(\frac{c_t^{1-\rho}}{1-\rho}\right) + \beta 0\right\}$$

the solution to which is patently $\mathrm{u}(\bar{\kappa}m_t)$. Thus, condition (2) will hold if $\left(\bar{\kappa}m_t\right)^{1-\rho}$ is F-bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \tag{8}$$

[2]For a fixed m_t, recall that m_{t+1} is just a function of c_t and the stochastic shocks.

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of $\digamma$, $\{\mathfrak{T}\mathbf{0}\}(m_t) = \mathrm{u}(\bar{\kappa} m_t)$ is clearly $\digamma$-bounded.

Finally, we turn to condition (3), $\{\mathfrak{T}(\mathrm{z}+\zeta\digamma)\}(m_t) \leq \{\mathfrak{T}\mathrm{z}\}(m_t)+\zeta\alpha\digamma(m_t)$. The proof will be more compact if we define $\check{\mathrm{c}}$ and $\breve{\mathrm{a}}$ as the consumption and assets functions[3] associated with $\mathfrak{T}\mathrm{z}$ and $\hat{\mathrm{c}}$ and $\hat{\mathrm{a}}$ as the functions associated with $\mathfrak{T}(\mathrm{z}+\zeta\digamma)$; using this notation, condition (3) can be rewritten

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}) + \zeta\alpha\digamma.$$

Now note that if we force the $\smile$ consumer to consume the amount that is optimal for the $\wedge$ consumer, value for the $\smile$ consumer must decline (at least weakly). That is,

$$\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) \leq \mathrm{u}(\check{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\breve{\mathrm{a}}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned}
\mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \mathrm{u}(\hat{\mathrm{c}}) + \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\{\mathsf{E}(\mathrm{z}+\zeta\digamma)\}(\hat{\mathrm{a}}) &\leq \beta\{\mathsf{E}\mathrm{z}\}(\hat{\mathrm{a}}) + \zeta\alpha\digamma \\
\beta\zeta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \zeta\alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &\leq \alpha\digamma \\
\beta\{\mathsf{E}\digamma\}(\hat{\mathrm{a}}) &< \digamma.
\end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.[4]

Using $\digamma(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{\mathrm{a}}(m_t)$, this condition is

$$\beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta\,\mathbb{E}_t\,\Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (??), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}\right] - m_t^{1-\rho}}{1-\beth}. \tag{9}$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

[3]Section **??** proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

[4]The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

that the numerator of (9) is bounded from above:

$$
\begin{aligned}
&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\hat{a}_t\mathcal{R}_{t+1})^{1-\rho}\right]-m_t^{1-\rho}\\
\leq&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+\wp\beta\mathsf{R}^{1-\rho}((1-\bar{\kappa})m_t)^{1-\rho}-m_t^{1-\rho}\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\wp\beta\mathsf{R}^{1-\rho}\left(\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)^{1-\rho}-1\right)\\
=&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}((1-\bar{\kappa})m_t\mathcal{R}_{t+1}+\theta_{t+1}/(1-\wp))^{1-\rho}\right]+m_t^{1-\rho}\left(\underbrace{\wp^{1/\rho}\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}}_{<1\text{ by WRIC}}-1\right)\\
<&(1-\wp)\beta\,\mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1-\wp))^{1-\rho}\right]=\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}.
\end{aligned}
$$

We can thus conclude that equation (9) will certainly hold for any:

$$\eta>\underline{\eta}=\frac{\beth(1-\wp)^{\rho}\underline{\theta}^{1-\rho}}{1-\beth} \tag{10}$$

which is a positive finite number under our assumptions.

The proof that $\mathfrak{T}$ defines a contraction mapping under the conditions (**??**) and (**??**) is now complete.

D.1 $\mathfrak{T}$ and v

In defining our operator $\mathfrak{T}$ we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the discussion of the consumption function bounds, we showed only (in (**??**)) that $\underline{\kappa}_t m_t \leq \mathrm{c}_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the MPC's.) We have therefore not proven (yet) that the sequence of value functions (**??**) defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger (but still quite weak) condition. The place where these conditions have force is in the step at (10). Consideration of the prior two equations reveals that a sufficient stronger condition is

$$
\begin{aligned}
\wp\beta(\mathsf{R}(1-\bar{\kappa}_{T-1}))^{1-\rho}&<1\\
(\wp\beta)^{1/(1-\rho)}(1-\bar{\kappa}_{T-1})&>1\\
(\wp\beta)^{1/(1-\rho)}(1-(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1})&>1
\end{aligned}
$$

where we have used (**??**) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality occurs because we have assumed $\rho>1$ so that we are exponentiating both sides by the negative number $1-\rho$). To see that this is a weak condition, note that for small values of $\wp$ this expression can be further simplified using $(1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}})^{-1}\approx 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ so that it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\text{Þ}_{\mathsf{R}}>1$$

$$(\wp\beta)\wp^{(1-\rho)/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} < 1$$
$$\beta\wp^{1/\rho}\text{Þ}_{\mathsf{R}}^{1-\rho} < 1.$$

Calling the weak return patience factor $\text{Þ}_{\mathsf{R}}^{\wp} = \wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ and recalling that the WRIC was $\text{Þ}_{\mathsf{R}}^{\wp} < 1$, the expression on the LHS above is $\beta\text{Þ}_{\mathsf{R}}^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\text{Þ}_{\mathsf{R}} \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $\mathrm{v}(m)$. But since $\lim_{n\to\infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $\mathrm{v}(m)$ toward which these v_{T-n}'s are converging is the *same* $\mathrm{v}(m)$ that was the endpoint of the contraction defined by our operator $\mathfrak{T}$. Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (**??**) converge, they converge to the same unique v defined by $\mathfrak{T}$.[5]

E Convergence in Euclidian Space

E.1 Convergence of v_t

Boyd's theorem shows that $\mathfrak{T}$ defines a contraction mapping in a F-bounded space. We now show that $\mathfrak{T}$ also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator $\mathfrak{T}$, since $\mathrm{v}^*(m) = \mathfrak{T}\mathrm{v}^*(m)$,

$$\|\mathrm{v}_{T-n+1} - \mathrm{v}^*\|_F \leq \alpha^{n-1} \|\mathrm{v}_T - \mathrm{v}^*\|_F . \tag{11}$$

On the other hand, $\mathrm{v}_T - \mathrm{v}^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ and $\kappa = \|\mathrm{v}_T - \mathrm{v}^*\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$. It follows that

$$|\mathrm{v}_{T-n+1}(m) - \mathrm{v}^*(m)| \leq \kappa\alpha^{n-1} |F(m)| . \tag{12}$$

Then we obtain

$$\lim_{n\to\infty} \mathrm{v}_{T-n+1}(m) = \mathrm{v}^*(m). \tag{13}$$

Since $\mathrm{v}_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $\mathrm{v}_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < \mathrm{v}_T(m)$. On the other hand, $\mathrm{v}_{T-1} \leq \mathrm{v}_T$ means $\mathfrak{T}\mathrm{v}_{T-1} \leq \mathfrak{T}\mathrm{v}_T$, in other words, $\mathrm{v}_{T-2}(m) \leq \mathrm{v}_{T-1}(m)$. Inductively one gets $\mathrm{v}_{T-n}(m) \geq \mathrm{v}_{T-n-1}(m)$. This means that $\{\mathrm{v}_{T-n+1}(m)\}_{n=1}^{\infty}$ is a decreasing sequence, bounded below by v^*.

E.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{\mathrm{c}_{T-n+1}(m)\}_{n=1}^{\infty}$.

[5]It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

Consider any convergent subsequence $\{c_{T-n(i)}(m)\}$ of $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to c^*. By the definition of $c_{T-n}(m)$, we have

$$\mathrm{u}(c_{T-n(i)}(m))+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)] \geq \mathrm{u}(c_{T-n(i)})+\beta\,\mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho}\mathrm{v}_{T-n(i)+1}(m)], \tag{14}$$

for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left hand side converges to $\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$, and the right hand side converges to $\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_t^{1-\rho}\mathrm{v}(m)]$. So the limit of the preceding inequality as $n(i)$ approaches infinity implies

$$\mathrm{u}(c^*) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)] \geq \mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]. \tag{15}$$

Hence, $c^* \in \underset{c_{T-n(i)}\in[\underline{\kappa}m,\bar{\kappa}m]}{\arg\max} \left\{\mathrm{u}(c_{T-n(i)}) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}(m)]\right\}$. By the uniqueness of $\mathrm{c}(m)$, $c^* = \mathrm{c}(m)$.

F Equality of Aggregate Consumption Growth and Income Growth with Transitory Shocks

Section **??** asserted that in the absence of permanent shocks it is possible to prove that the growth factor for aggregate consumption approaches that for aggregate permanent income. This section establishes that result.

First define $\mathrm{a}(m)$ as the function that yields optimal end-of-period assets as a function of m.

Suppose the population starts in period t with an arbitrary value for $\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$. Then if $\breve{m}$ is the invariant mean level of m we can define a 'mean MPS away from $\breve{m}$' function :

$$\bar{\acute{\mathrm{a}}}(\Delta) = \Delta^{-1}\int_{\breve{m}}^{\breve{m}+\Delta} \mathrm{a}'(z)dz$$

where the combination of the bar and the ´ are meant to signify that this is the average value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at $\mathcal{R}$, if we define a as the value of a corresponding to $m = \breve{m}$, we can write

$$a_{t+1,i} = a + (m_{t+1,i} - \breve{m})\bar{\acute{\mathrm{a}}}(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i}}^{m_{t+1,i}} - \breve{m})$$

so

$$\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \mathrm{cov}_t\left(\bar{\acute{\mathrm{a}}}(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \breve{m}), \Gamma\mathbf{p}_{t,i}\right).$$

But since $\mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho} < \bar{\acute{\mathrm{a}}}(m) < \text{\textbf{Þ}}_{\mathsf{R}}$,

$$|\mathrm{cov}_t((\wp\mathsf{R}\beta)^{1/\rho}a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\mathrm{cov}_t(\text{\textbf{Þ}}a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

and for the version of the model with no permanent shocks the GIC-Nrm says that $\text{\textbf{Þ}} < \Gamma$, while the FHWC says that $\Gamma < \mathsf{R}$

$$|\mathrm{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma|\mathrm{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n\to\infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\breve{\mathcal{R}} = \mathbb{M}\left[\mathcal{R}_{t+1,i}\right]$ and consider a first order Taylor expansion of $\bar{\text{a}}(m_{t+1,i})$ around $\hat{m}_{t+1,i} = \breve{\mathcal{R}}a_{t,i} + 1$,

$$\bar{\text{a}}_{t+1,i} \approx \bar{\text{a}}(\hat{m}_{t+1,i}) + \bar{\text{a}}'(\hat{m}_{t+1,i})\left(m_{t+1,i} - \hat{m}_{t+1,i}\right).$$

The problem comes from the $\bar{\text{a}}'$ term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level $\bar{\text{a}}$. We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on $\bar{\text{a}}'$ (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

G The Limiting MPC's

For $m_t > 0$ we can define $\text{e}_t(m_t) = \text{c}_t(m_t)/m_t$ and $\text{a}_t(m_t) = m_t - \text{c}_t(m_t)$ and the Euler equation (**??**) can be rewritten

$$\begin{aligned}
\text{e}_t(m_t)^{-\rho} &= \beta \mathsf{R}\,\mathbb{E}_t\left[\left(\text{e}_{t+1}(m_{t+1})\left(\frac{\overbrace{\mathsf{R}\text{a}_t(m_t)+\Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t}\right)\right)^{-\rho}\right] \\
&= \quad (1-\wp)\beta\mathsf{R}m_t^{\rho}\,\mathbb{E}_t\left[\left(\text{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1}\right)^{-\rho} \mid \xi_{t+1} > 0\right] \\
&\quad + \wp\beta\mathsf{R}^{1-\rho}\,\mathbb{E}_t\left[\left(\text{e}_{t+1}(\mathcal{R}_{t+1}\text{a}_t(m_t))\frac{m_t - \text{c}_t(m_t)}{m_t}\right)^{-\rho} \mid \xi_{t+1} = 0\right].
\end{aligned}$$

Consider the first conditional expectation in (**??**), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1-\wp)$. Since $\lim_{m\downarrow 0}\text{a}_t(m) = 0$, $\mathbb{E}_t[(\text{e}_{t+1}(m_{t+1})m_{t+1}\Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(\text{e}_{t+1}(\underline{\theta}/(1-\wp))\Gamma\underline{\psi}\underline{\theta}/(1-\wp))^{-\rho}$ and $(\text{e}_{t+1}(\bar{\theta}/(1-\wp))\Gamma\bar{\psi}\bar{\theta}/(1-\wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1-\wp)$ goes to zero as m_t^{ρ} goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta\wp\mathsf{R}^{1-\rho}\bar{\kappa}_{t+1}^{-\rho}(1-\bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ, we can conclude that

$$\bar{\kappa}_t = \wp^{-1/\rho}(\beta\mathsf{R})^{-1/\rho}\mathsf{R}(1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

$$\underbrace{\wp^{1/\rho}\overbrace{\mathsf{R}^{-1}(\beta\mathsf{R})^{1/\rho}}^{\text{Þ}_{\mathsf{R}}}}_{\equiv\wp^{1/\rho}\text{Þ}_{\mathsf{R}}}\bar{\kappa}_t = (1-\bar{\kappa}_t)\bar{\kappa}_{t+1}$$

which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned}
(\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_t)^{-1} &= (1-\bar{\kappa}_t)^{-1}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1}(1-\bar{\kappa}_t) &= \wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1} \\
\bar{\kappa}_t^{-1} &= 1+\wp^{1/\rho}\text{Þ}_{\mathsf{R}}\bar{\kappa}_{t+1}^{-1}.
\end{aligned}$$

As noted in the main text, we need the WRIC (**??**) for this to be a convergent sequence:

$$0 \le \wp^{1/\rho}\text{Þ}_{\mathsf{R}} < 1, \tag{16}$$

Since $\bar{\kappa}_T = 1$, iterating (16) backward to infinity (because we are interested in the limiting consumption function) we obtain:

$$\lim_{n\to\infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1-\wp^{1/\rho}\text{Þ}_{\mathsf{R}} \tag{17}$$

and we will therefore call $\bar{\kappa}$ the 'limiting maximal MPC.'

The minimal MPC's are obtained by considering the case where $m_t \uparrow \infty$. If the FHWC holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (16) can be neglected, leading to a revised limiting Euler equation

$$(m_t \mathrm{e}_t(m_t))^{-\rho} \;\; = \beta\mathsf{R}\,\mathbb{E}_t\left[(\mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1})\,(\mathsf{R}\mathrm{a}_t(m_t)))^{-\rho}\right]$$

and we know from L'Hôpital's rule that $\lim_{m_t\to\infty} \mathrm{e}_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t\to\infty} \mathrm{e}_{t+1}(\mathrm{a}_t(m_t)\mathcal{R}_{t+1}) = \underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned}
(m_t\underline{\kappa}_t)^{-\rho} &= \beta\mathsf{R}\big(\underline{\kappa}_{t+1}\mathsf{R}(1-\underline{\kappa}_t)m_t\big)^{-\rho} \\
\underbrace{\mathsf{R}^{-1}\text{Þ}}_{\equiv\text{Þ}_{\mathsf{R}}=(1-\underline{\kappa})}\;\underline{\kappa}_t &= (1-\underline{\kappa}_t)\underline{\kappa}_{t+1}
\end{aligned}$$

and the same sequence of derivations used above yields the conclusion that if the RIC $0 \le \text{Þ}_{\mathsf{R}} < 1$ holds, then a recursive formula for the minimal marginal propensity to consume is given by

$$\underline{\kappa}_t^{-1} = 1+\underline{\kappa}_{t+1}^{-1}\text{Þ}_{\mathsf{R}} \tag{18}$$

so that $(\{\underline{\kappa}_{T-n}^{-1}\})_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n\uparrow\infty} \kappa_{T-n}^{-1} \tag{19}$$

as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n\to\infty} \underline{\kappa}_{T-n}^{-1} = \infty$ and so the limiting MPC is $\underline{\kappa} = 0$.

For the purpose of constructing the limiting perfect foresight consumption function, it is useful further to note that the PDV of consumption is given by

$$c_t \underbrace{\left(1+\text{Þ}_{\mathsf{R}}+\text{Þ}_{\mathsf{R}}^2+\cdots\right)}_{=1+\text{Þ}_{\mathsf{R}}(1+\text{Þ}_{\mathsf{R}}\underline{\kappa}_{t+2}^{-1})\ldots} \;\; = c_t\underline{\kappa}_{T-n}^{-1}.$$

which, combined with the intertemporal budget constraint, yields the usual formula for the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \tag{20}$$

H The Perfect Foresight Liquidity Constrained Solution as a Limit

Formally, suppose we change the description of the problem by making the following two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

and we designate the solution to this consumer's problem $\grave{\mathrm{c}}_t(m)$. We will henceforth refer to this as the problem of the 'restrained' consumer (and, to avoid a common confusion, we will refer to the consumer as 'constrained' only in circumstances when the constraint is actually binding).

Redesignate the consumption function that emerges from our original problem for a given fixed $\wp$ as $\mathrm{c}_t(m;\wp)$ where we separate the arguments by a semicolon to distinguish between m, which is a state variable, and $\wp$, which is not. The proposition we wish to demonstrate is

$$\lim_{\wp \downarrow 0} \mathrm{c}_t(m;\wp) = \grave{\mathrm{c}}_t(m). \tag{21}$$

We will first examine the problem in period $T-1$, then argue that the desired result propagates to earlier periods. For simplicity, suppose that the interest, growth, and time-preference factors are $\beta = \mathsf{R} = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$; the results below are easily generalized to the full-fledged version of the problem.

The solution to the restrained consumer's optimization problem can be obtained as follows. Assuming that the consumer's behavior in period T is given by $\mathrm{c}_T(m)$ (in practice, this will be $\mathrm{c}_T(m) = m$), consider the unrestrained optimization problem

$$\grave{\mathrm{a}}^*_{T-1}(m) = \arg\max_a \left\{ \mathrm{u}(m-a) + \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{v}_T(a+\theta) d\mathcal{F}_\theta \right\}. \tag{22}$$

As usual, the envelope theorem tells us that $\mathrm{v}'_T(m) = \mathrm{u}'(\mathrm{c}_T(m))$ so the expected marginal value of ending period $T-1$ with assets a can be defined as

$$\grave{\mathfrak{v}}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} \mathrm{u}'(\mathrm{c}_T(a+\theta)) d\mathcal{F}_\theta,$$

and the solution to (22) will satisfy

$$\mathrm{u}'(m-a) = \grave{\mathfrak{v}}'_{T-1}(a). \tag{23}$$

$\grave{\mathrm{a}}^*_{T-1}(m)$ therefore answers the question "With what level of assets would the restrained consumer like to end period $T-1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note that the restrained consumer's income process remains different from the process for the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\grave{\mathrm{a}}_{T-1}(m) = \max[0, \grave{\mathrm{a}}^*_{T-1}(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (**?**)) that

$$m^1_{\#} = \left(\grave{\mathfrak{v}}'_{T-1}(0)\right)^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T-1$.

Analogously to (23), defining

$$\mathfrak{v}'_{T-1}(a;\wp) \equiv \left[\wp a^{-\rho} + (1-\wp)\int_{\underline{\theta}}^{\bar{\theta}} (\mathrm{c}_T(a+\theta/(1-\wp)))^{-\rho} d\mathcal{F}_\theta\right], \tag{24}$$

the Euler equation for the original consumer's problem implies

$$(m-a)^{-\rho} = \mathfrak{v}'_{T-1}(a;\wp) \tag{25}$$

with solution $\mathrm{a}^*_{T-1}(m;\wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. Since the LHS of (23) and (25) are identical, this means that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. That is, for any fixed value of $m > m^1_{\#}$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m, the consumers must have the same c, so the consumption functions are identical in the limit.

Now consider values $m \leq m^1_{\#}$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (24) is $\lim_{a\downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a\downarrow 0}(m-a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m^1_{\#}$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp\downarrow 0} \mathfrak{v}'_{T-1}(a;\wp) = \grave{\mathfrak{v}}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\grave{\mathrm{a}}^*_{T-1}(m) < 0$, and we showed earlier that $\lim_{\wp\downarrow 0} \mathrm{a}^*_{T-1}(m;\wp) = \grave{\mathrm{a}}^*_{T-1}(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m^1_{\#}$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp\downarrow 0} \mathrm{c}_{T-1}(m;\wp) = \grave{\mathrm{c}}_{T-1}(m)$ which is the period $T-1$ version of (21). But given equality of the period $T-1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (17) for the maximal marginal propensity to consume satisfies

$$\lim_{\wp \downarrow 0} \bar{\kappa} \quad = 1,$$

which makes sense because the marginal propensity to consume for a constrained restrained consumer is 1 by our definitions of 'constrained' and 'restrained.'

I Endogenous Gridpoints Solution Method

The model is solved using an extension of the method of endogenous gridpoints (?): A grid of possible values of end-of-period assets $\vec{a}$ is defined, and at these points, marginal end-of-period-t value is computed as the discounted next-period expected marginal utility of consumption (which the Envelope theorem says matches expected marginal value). The results are then used to identify the corresponding levels of consumption at the beginning of the period:[6]

$$\begin{aligned}
\mathrm{u}'(\mathfrak{c}_t(\vec{a})) &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}'(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))] \\
\vec{c}_t \equiv \mathfrak{c}_t(\vec{a}) &= \left(\mathsf{R}\beta\,\mathbb{E}_t\left[(\Gamma_{t+1}\mathrm{c}_{t+1}(\mathcal{R}_{t+1}\vec{a}+\xi_{t+1}))^{-\rho}\right]\right)^{-1/\rho}.
\end{aligned}$$

The dynamic budget constraint can then be used to generate the corresponding m's:

$$\vec{m}_t \quad = \vec{a} + \vec{c}_t.$$

An approximation to the consumption function could be constructed by linear interpolation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be made (for a given number of gridpoints) if the interpolation is constructed so that it also matches the marginal propensity to consume at the gridpoints. Differentiating (26) with respect to a (and dropping policy function arguments for simplicity) yields a marginal propensity to *have consumed* $\mathfrak{c}^a$ at each gridpoint:

$$\begin{aligned}
\mathrm{u}''(\mathfrak{c}_t)\mathfrak{c}^a_t &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\Gamma_{t+1}\mathrm{c}^m_{t+1}\mathcal{R}_{t+1}] \\
&= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}] \\
\mathfrak{c}^a_t &= \mathsf{R}\beta\,\mathbb{E}_t[\mathrm{u}''(\Gamma_{t+1}\mathrm{c}_{t+1})\mathsf{R}\mathrm{c}^m_{t+1}]/\mathrm{u}''(\mathfrak{c}_t)
\end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from the marginal propensity to have consumed by noting that, if we define $\mathfrak{m}(a) = \mathfrak{c}(a) - a$,

$$\begin{aligned}
c &= \mathfrak{m} - a \\
\mathfrak{c}^a + 1 &= \mathfrak{m}^a
\end{aligned}$$

which, together with the chain rule $\mathfrak{c}^a = \mathrm{c}^m\mathfrak{m}^a$, yields the MPC from

$$\begin{aligned}
\mathrm{c}^m(\mathfrak{c}^a + 1) &= \mathfrak{c}^a \\
\mathrm{c}^m &= \mathfrak{c}^a/(1+\mathfrak{c}^a)
\end{aligned}$$

[6]The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

and we call the vector of MPC's at the $\vec{m}_t$ gridpoints $\vec{\kappa}_t$.

J The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $\mathrm{c}_T(m) = m$ and constructing $\{\mathrm{c}_{T-1}, \mathrm{c}_{T-2}, \cdots\}$ by time iteration (a method that will converge to $\mathrm{c}(m)$ by standard theorems). But $\mathrm{c}_T(m) = m$ is very far from the final converged consumption rule $\mathrm{c}(m)$,[7] and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked:' The slope of the consumption function changes discretely at the points $\{m^1_\#, m^2_\#, \cdots\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $\mathrm{c}(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $\mathrm{c}(m)$ for integer values of n (e.g., $c^n_\# = \text{\TH}_\Gamma^{-n}$) are continuous functions of n; the conclusion that $\mathrm{c}(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\breve{\mathrm{c}}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.[8]

This strategy generates a smooth limiting consumption function — except at the remaining kink point defined by $\{m^0_\#, c^0_\#\}$. Below this point, the solution must match $\mathrm{c}(m) = m$ because the constraint is binding. At $m = m^0_\#$ the MPC discretely drops (that is, $\lim_{m \uparrow m^0_\#} \mathrm{c}'(m) = 1$ while $\lim_{m \downarrow m^0_\#} \mathrm{c}'(m) = \kappa^0_\# < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\breve{\mathrm{c}}(m)$ above some $n \geq \underline{n}$, but to replace $\breve{\mathrm{c}}(m)$ between $m^0_\#$ and $m^{\underline{n}}_\#$ with the unique polynomial function $\hat{\mathrm{c}}(m)$ that satisfies the following criteria:

[7] Unless $\beta \approx +0$.

[8] In practice, we calculate the first and second derivatives of c and use piecewise polynomial approximation methods that match the function at these points.

1. $\hat{\mathrm{c}}(m^0_\#) = c^0_\#$

2. $\hat{\mathrm{c}}'(m^0_\#) = 1$

3. $\hat{\mathrm{c}}'(m^{\underline{n}}_\#) = (d\mathrm{c}^n_\#/dn)(d\mathrm{m}^n_\#/dn)^{-1}|_{n=\underline{n}}$

4. $\hat{\mathrm{c}}''(m^{\underline{n}}_\#) = (d^2\mathrm{c}^n_\#/dn^2)(d^2\mathrm{m}^n_\#/dn^2)^{-1}|_{n=\underline{n}}$

where $\underline{n}$ is chosen judgmentally in a way calculated to generate a good compromise between smoothness of the limiting consumption function $\check{\mathrm{c}}(m)$ and fidelity of that function to the $\mathrm{c}(m)$ (see the actual code for details).

We thus define the terminal function as

$$\mathrm{c}_T(m) = \begin{cases} 0 < m \leq m^0_\# & m \\ m^0_\# < m < m^{\underline{n}}_\# & \check{\mathrm{c}}(m) \\ m^{\underline{n}}_\# < m & \mathrm{c}(m) \end{cases} \tag{26}$$

Since the precautionary motive implies that in the presence of uncertainty the optimal level of consumption is below the level that is optimal without uncertainty, and since $\check{\mathrm{c}}(m) \geq \mathrm{c}(m)$, implicitly defining $m = e^\mu$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - \mathrm{c}_t(e^\mu)/\mathrm{c}_T(e^\mu)) \tag{27}$$

which must be a number between $-\infty$ and $+\infty$ (since $0 < \mathrm{c}_t(m) < \check{\mathrm{c}}(m)$ for $m > 0$). This function turns out to be much better behaved (as a numerical observation; no formal proof is offered) than the level of the optimal consumption rule $\mathrm{c}_t(m)$. In particular, $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi'_t(\mu) = \left(\frac{-\left(\frac{\mathrm{c}'_t\mathrm{c}_T - \mathrm{c}_t\mathrm{c}'_T}{\mathrm{c}^2_T} e^\mu\right)}{1 - \mathrm{c}_t/\mathrm{c}_T} \right) \tag{28}$$

which can be solved for

$$\mathrm{c}'_t = (\mathrm{c}_t\mathrm{c}'_T/\mathrm{c}_T) - ((\mathrm{c}_T - \mathrm{c}_t)/m)\chi'_t. \tag{29}$$

Similarly, we can solve (27) for

$$\mathrm{c}_t(m) = \left(1 - e^{\chi_t(\log m)}\right) \mathrm{c}_T(m). \tag{30}$$

Thus, having approximated χ_t, we can recover from it the level and derivative(s) of c_t.

K Relational Diagrams for the Inequality Conditions

This appendix explains in detail the paper's 'inequalities' diagrams (Figures ??, ??).

$$
\begin{tikzcd}
((\mathsf{R}\beta)^{1/\rho} \equiv)\ \textbf{Þ} \arrow[rr, "\text{GIC}(\equiv \textbf{Þ} < \Gamma)"] \arrow[dr, "(\textbf{Þ} < \mathsf{R} \equiv)\ \text{RIC}"'] & {} \arrow[d, phantom, "\neq"] & \Gamma \arrow[dl, "\text{FHWC}\ (\equiv \Gamma < \mathsf{R})"] \\
& \mathsf{R} &
\end{tikzcd}
$$

Figure 1 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

K.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 1, whose three nodes represent values of the absolute patience factor **Þ**, the permanent-income growth factor Γ, and the riskfree interest factor R. The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from **Þ** to Γ, reflects imposition of the PF-GICNrm condition (clicking PF-GICNrm should take you to its definition; definitions of other conditions are also linked below)).[9] Annotations inside parenthetical expressions containing $\equiv$ are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β, and Γ are all in $\mathbb{R}_{++}$, and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the **Þ** node and impose the PF-GICNrm and then the FHWC, and see that imposition of these conditions allows us to conclude that $\textbf{Þ} < \mathsf{R}$.

One could also impose $\textbf{Þ} < \mathsf{R}$ directly (without imposing PF-GICNrm and FHWC) by following the downward-sloping diagonal arrow exiting **Þ**. Although alternate routes from one node to another all justify the same core conclusion ($\textbf{Þ} < \mathsf{R}$, in this case), $\neq$ symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in category theory diagrams,[10] to indicate that the diagram is not commutative.[11]

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the RIC, $\cancel{\text{RIC}} \equiv \textbf{Þ} > \mathsf{R}$, would be represented by moving the

[9]For convenience, the equivalent ($\equiv$) mathematical statement of each condition is expressed nearby in parentheses.

[10]For a popular introduction to category theory, see **?**.

[11]But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

arrowhead from the bottom right to the top left of the line segment connecting $\mathbf{Þ}$ and R.

If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow connecting R and Γ, but the Γ node would then have no exiting arrows so no further deductions could be made. However, if we *also* reversed PF-GICNrm (that is, if we imposed ~~PF-GICNrm~~), that would take us to the $\mathbf{Þ}$ node, and we could deduce $\mathsf{R} > \mathbf{Þ}$. However, we would have to stop traversing the diagram at this point, because the arrow exiting from the $\mathbf{Þ}$ node points back to our starting point, which (if valid) would lead us to the conclusion that $\mathsf{R} > \mathsf{R}$. Thus, the reversal of the two earlier conditions (imposition of ~~FHWC~~ and ~~PF-GICNrm~~) requires us also to reverse the final condition, giving us ~~RIC~~.[12]

Under these conventions, Figure **??** in the main text presents a modified version of the diagram extended to incorporate the PF-FVAC (reproduced here for convenient reference).

$$
\begin{tikzcd}
\mathbf{Þ} \arrow[rr, "\text{GIC}"] \arrow[dd, "\text{RIC}"'] \arrow[ddrr, "\text{PF-FVAC}"] & & \Gamma \arrow[dd, "\text{FHWC}"] \arrow[dl, phantom, "\neq"] \\
& \arrow[dl, phantom, "\neq"] & \\
\mathsf{R} \arrow[rr, "\cancel{\text{FHWC}}"'] & & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}
$$

Figure 2 Relation of PF-GICNrm, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the PF-FVAC, (**??**)

This diagram can be interpreted, for example, as saying that, starting at the $\mathbf{Þ}$ node, it is possible to derive the PF-FVAC[13] by imposing both the PF-GICNrm and the FHWC;

[12]The corresponding algebra is

$$
\begin{aligned}
\cancel{\text{FHWC}}: &\quad \mathsf{R} < \Gamma \\
\cancel{\text{PF-GICNrm}}: &\quad \Gamma < \mathbf{Þ} \\
\Rightarrow \cancel{\text{RIC}}: &\quad \mathsf{R} < \mathbf{Þ},
\end{aligned}
$$

.

[13]in the form $\mathbf{Þ} < (\mathsf{R}/\Gamma)^{1/\rho}\Gamma$

or by imposing RIC and ~~FHWC~~. Or, starting at the Γ node, we can follow the imposition of the FHWC (twice — reversing the arrow labeled ~~FHWC~~) and then ~~RIC~~ to reach the conclusion that $\text{Þ} < \Gamma$. Algebraically,

$$
\begin{aligned}
\text{FHWC}: &\quad \Gamma < \mathsf{R} \\
\cancel{\text{RIC}}: &\quad \mathsf{R} < \text{Þ} \\
&\quad \Gamma < \text{Þ}
\end{aligned} \tag{31}
$$

which leads to the negation of both of the conditions leading into Þ. ~~PF-GICNrm~~ is obtained directly as the last line in (31) and ~~PF-FVAC~~ follows if we start by multipling the Return Patience Factor (RPF=$\text{Þ}/\mathsf{R}$) by the FHWF ($=\Gamma/\mathsf{R}$) raised to the power $1/\rho - 1$, which is negative since we imposed $\rho > 1$. FHWC implies FHWF < 1 so when FHWF is raised to a negative power the result is greater than one. Multiplying the RPF (which exceeds 1 because ~~RIC~~) by another number greater than one yields a product that must be greater than one:

$$
\begin{aligned}
1 &< \overbrace{\left(\frac{(\mathsf{R}\beta)^{1/\rho}}{\mathsf{R}}\right)}^{>1 \text{ from } \cancel{\text{RIC}}} \overbrace{(\Gamma/\mathsf{R})^{1/\rho-1}}^{>1 \quad \text{from FHWC}} \\
1 &< \left(\frac{(\mathsf{R}\beta)^{1/\rho}}{(\mathsf{R}/\Gamma)^{1/\rho}\mathsf{R}\Gamma/\mathsf{R}}\right) \\
\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} = (\mathsf{R}/\Gamma)^{1/\rho}\Gamma &< \text{Þ}
\end{aligned}
$$

which is one way of writing ~~PF-FVAC~~.

The complexity of this algebraic calculation illustrates the usefulness of the diagram, in which one merely needs to follow arrows to reach the same result.

After the warmup of constructing these conditions for the perfect foresight case, we can represent the relationships between all the conditions in both the perfect foresight case and the case with uncertainty as shown in Figure **??** in the paper (reproduced here).

Finally, the next diagram substitutes the values of the various objects in the diagram under the baseline parameter values and verifies that all of the asserted inequality conditions hold true.

L When Is Consumption Growth Declining in m?

Figure **??** depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

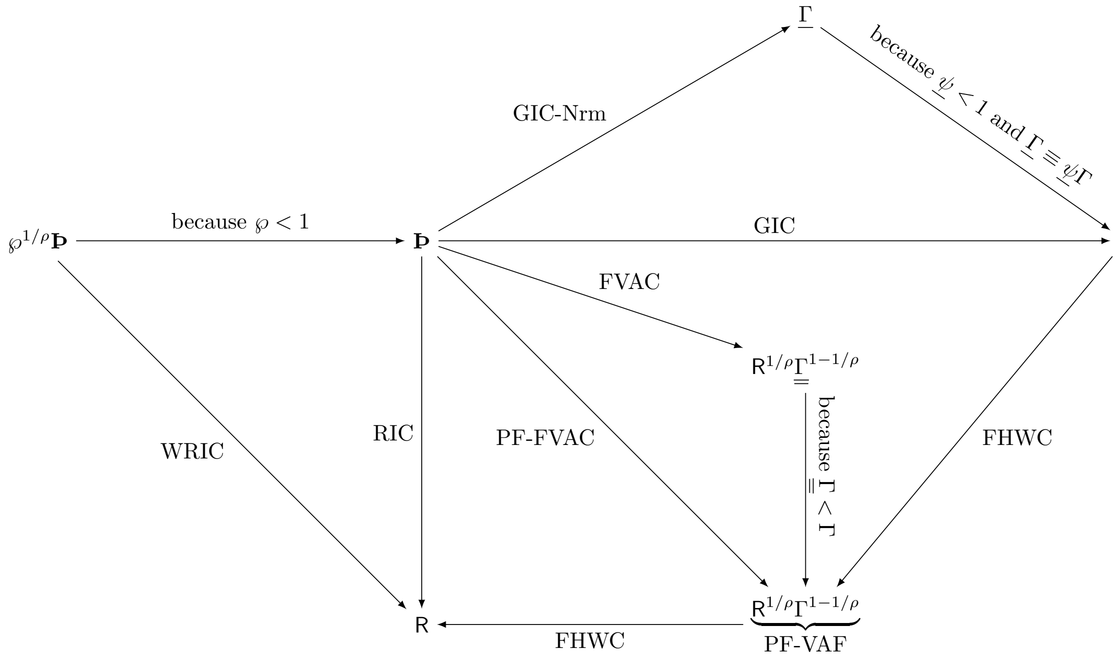

Figure 3 Relation of All Inequality Conditions

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{32}$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $\mathrm{c}'_{t+1} \equiv \mathrm{c}'(m_{t+1})$), since $\Gamma_{t+1}\mathcal{R}_{t+1} = \mathsf{R}$, the portion of the LHS of equation (32) in brackets can be manipulated to yield

$$\begin{aligned}\mathrm{c}_t\boldsymbol{\Upsilon}'_{t+1} &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\Gamma_{t+1}\mathrm{c}_{t+1}/\mathrm{c}_t \\ &= \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}.\end{aligned}$$

Now differentiate the Euler equation with respect to m_t:

$$\begin{aligned}1 &= \mathsf{R}\beta\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho}_{t+1}] \\ 0 &= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}\boldsymbol{\Upsilon}'_{t+1}] \\ &= \mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]\,\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] + \mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) \\ \mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] &= -\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1})/\,\mathbb{E}_t[\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}]\end{aligned}$$

but since $\boldsymbol{\Upsilon}_{t+1} > 0$ we can see from (33) that (32) is equivalent to

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \boldsymbol{\Upsilon}'_{t+1}) > 0$$

which, using (33), will be true if

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}^{-\rho-1}_{t+1}, \mathrm{c}'_{t+1}\mathrm{a}'_t\mathsf{R} - \mathrm{c}'_t\boldsymbol{\Upsilon}_{t+1}) > 0$$

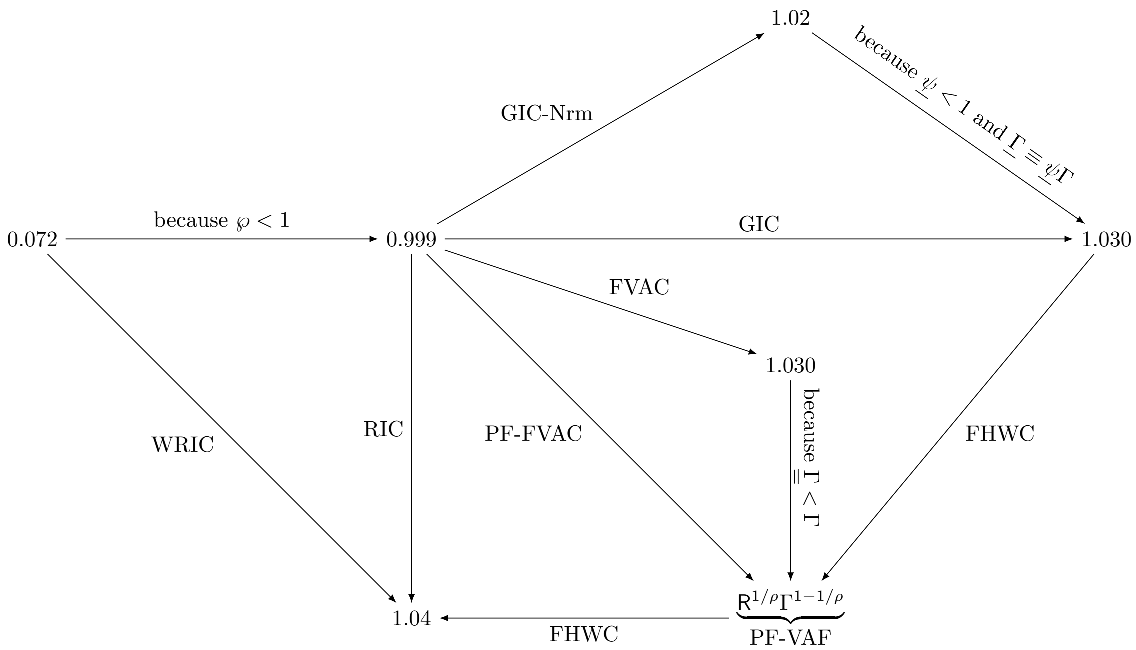

Figure 4 Numerical Relation of All Inequality Conditions

which in turn will be true if both

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \mathrm{c}'_{t+1}) > 0$$

and

$$\mathrm{cov}_t(\boldsymbol{\Upsilon}_{t+1}^{-\rho-1}, \boldsymbol{\Upsilon}_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\mathrm{cov}_t\left((\Gamma\psi_{t+1}\mathrm{c}(m_{t+1}))^{-\rho-1}, \mathrm{c}'(m_{t+1})\right) > 0.$$

The two shocks cause two kinds of variation in m_{t+1}. Variations due to ξ_{t+1} satisfy the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal propensity to consume. However, permanent shocks have conflicting effects. On the one hand, a higher draw of ψ_{t+1} will reduce m_{t+1}, thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1}. On the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1} could be to decrease the first term in the covariance, leading to a negative covariance with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a negative correlation.)

M Unique And Stable Target and Steady State Points

This appendix proves Theorems **??** and **??** and

Lemma 1. *If both $\check{m}$ and $\hat{m}$ exist, then $\hat{m} < \check{m}$.*

M.1 Proof of Theorem ??

The elements of the proof of Theorem ?? are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

M.1.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; Theorem ??). (Indeed, Appendix C shows that $\mathrm{c}(m)$ is not just continuous, but twice continuously differentiable.)

Section ?? shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

M.1.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$.

If RIC holds. Logic exactly parallel to that of Section ?? leading to equation (??), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t\left[\frac{\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1}}{m_t}\right] \\
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{\textbf{Þ}}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{\textbf{Þ}}/\Gamma_{t+1}] \qquad (33) \\
&< 1
\end{aligned}$$

where the inequality reflects imposition of the GIC-Nrm (??).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$ (see equation (??)) means that the limit of the RHS of (33) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$.

Paralleling the logic for c in Section **??**: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

Intermediate Value Theorem. If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.1.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned} \tag{34}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned} \tag{35}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails.

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}$$

which is negative because the GIC-Nrm says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under $\cancel{\text{RIC}}$, recall that $\lim_{m \uparrow \infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (35) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < 1. \tag{36}$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (36).

M.2 Proof of Theorem ??

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

M.2.1 Existence and Continuity of The Ratio

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in M.1.1 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

M.2.2 Existence of a stable point

Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in Subsection M.1.1 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t\uparrow\infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{\Gamma_{t+1}\left((\mathsf{R}/\Gamma_{t+1})\mathrm{a}(m_t) + \xi_{t+1}\right)/\Gamma}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \mathbb{E}_t\left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + \psi_{t+1}\xi_{t+1}}{m_t}\right] \\
&= \lim_{m_t\uparrow\infty} \left[\frac{(\mathsf{R}/\Gamma)\mathrm{a}(m_t) + 1}{m_t}\right] \\
&= (\mathsf{R}/\Gamma)\text{Þ}_{\mathsf{R}} \qquad (37)\\
&= \text{Þ}_{\Gamma} \\
&< 1
\end{aligned}$$

where the last two lines are merely a restatement of the GIC (??).

The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

M.2.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ *is monotonically decreasing.*

Define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned} \boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\ \boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\ \boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1, \end{aligned} \tag{38}$$

so that $\boldsymbol{\zeta}(\hat{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}(m_t - \mathrm{c}(m_t)) + \psi_{t+1}\xi_{t+1} - m_t)\right] \\ &= (\mathsf{R}/\Gamma)\,(1 - \mathrm{c}'(m_t)) - 1. \end{aligned} \tag{39}$$

Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (**??**) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section **??** that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$\begin{aligned} \mathcal{R}\,(1 - \mathrm{c}'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \text{\textbf{Þ}}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\ &= (\mathsf{R}/\Gamma)\text{\textbf{Þ}}_{\mathsf{R}} - 1 \end{aligned}$$

which is negative because the GIC says $\text{\textbf{Þ}}_{\Gamma} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R}\,(1 - \mathrm{c}'(m_t)) < \mathcal{R}$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (39) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathsf{R}/\Gamma) < 1. \tag{40}$$

But we showed in Section **??** that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHWC also fails (that is, (40) holds).

M.3 Proof of Lemma