

# Theoretical Foundations of Buffer Stock Saving

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# Drawbacks of Numerical Solutions

## A Black Box

- Can Construct Solution to Model Without Really Understanding It
- Hard Even To Be Sure Your Numerical Solution Is *Right*
- Little Intuition for How Results Might Change With
  - Calibration
  - Structure
- *Very Hard To Teach!*

## I Am A *Big* Fan Of Numerical Methods

- Have Done A Good Deal Of Work With Them Myself
- But As A Result, Have Felt All These Drawbacks Keenly

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Foundations For Microeconomic Household's Problem With

- Uncertain Labor Income
- No Liquidity Constraints
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# Key Result

Restrictions On Parameter Values Such That

- Problem Defines A Contraction Mapping
  - $\Rightarrow \exists$  A Unique Consumption Function  $c(m)$
- There Is A 'Target' Ratio Of Assets To Permanent Income
  - Requires A Key 'Impatience' Condition To Hold
  - Good News
    - Condition Is Weaker (Easier To Satisfy) Than Previous Papers Imposed



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Limit as horizon  $T$  goes to infinity of

$$\mathbf{a}_t = \mathbf{m}_t - \mathbf{c}_t \quad (1)$$

$$\mathbf{b}_{t+1} = \mathbf{a}_t R$$

$$\mathbf{p}_{t+1} = \mathbf{p}_t \underbrace{\Gamma^{\psi_{t+1}}}_{\equiv \Gamma_{t+1}}$$

$$\mathbf{m}_{t+1} = \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1},$$

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1 - \wp) & \text{with probability } (1 - \wp) \end{cases} \quad (2)$$

- $u(\bullet) = \bullet^{1-\rho}/(1-\rho)$ ;  $\mathbb{E}_t[\psi_{t+n}] = \mathbb{E}_t[\xi_{t+n}] = 1 \ \forall \ n > 0$ ;  $\beta < 1, \rho > 1$

# Surely This Problem Has Been Solved?

No.

- Can't Use Stokey et. al. theorems because CRRA utility
- Lit thru Matkowski and Nowak (2011) Can't Handle Permanent Shocks
- Must Use Boyd's 'Weighted' Contraction Mapping Theorem
- Surprisingly Subtle

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## Benchmark: Perfect Foresight Model

### Definitions:

|  |                     |   |                     |
|--|---------------------|---|---------------------|
| Absolute Patience Factor                 | $\mathbf{P}$        | = | $(R\beta)^{1/\rho}$ |
| Return Patience Factor                   | $\mathbf{P}_R$      | = | $\mathbf{P}/R$      |
| Perfect Foresight Growth Patience Factor | $\mathbf{P}_\Gamma$ | = | $\mathbf{P}/\Gamma$ |

| Name                                | Condition               | Implication                         |
|-------------------------------------|-------------------------|-------------------------------------|
| (AIC) Absolute Impatience Condition | $\mathbf{P} < 1$        | $c \downarrow$ over time            |
| (RIC) Return Impatience Condition   | $\mathbf{P}_R < 1$      | $c/a \downarrow$ over time          |
| (GIC) Growth Impatience Condition   | $\mathbf{P}_\Gamma < 1$ | $c/\mathbf{p} \downarrow$ over time |

# When Does A Useful Limiting Solution Exist?

Finite Human Wealth (FHW) condition:

$$\Gamma < R \quad (3)$$

Return Impatience Condition:

$$\mathbf{P}_R < R \quad (4)$$



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## What If There Are Liquidity Constraints?

- FHWC is *not* necessary for solution to exist
- Other Key Condition For Useful Solution is  
'Perfect Foresight Finite Value of Autarky Condition (PF-FVAC)':

$$\beta \Gamma^{1-\rho} < 1 \quad (5)$$

- Without RIC, Constraints Are Irrelevant
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# Liquidity Constraints and Uncertainty

- Introduce permanent shocks to income
- Finite Value of Autarky Condition Becomes

$$\begin{aligned}
 0 &< \overbrace{\beta \Gamma^{1-\rho}}^{\equiv \underline{\Gamma}} < 1 \\
 0 &< \beta < \underline{\Gamma}^{\rho-1}
 \end{aligned}
 \tag{6}$$

## Contraction Mapping Requirements

Finite Value of Autarky Condition: Same As In Liq Constr Problem!

$$\begin{aligned}
 0 &< \overbrace{\beta \underline{\Gamma}^{1-\rho}}^{\underline{\Gamma}} < 1 \\
 0 &< \beta < \underline{\Gamma}^{\rho-1}
 \end{aligned} \tag{7}$$

'Weak Return Impatience Condition' (WRIC)

$$0 \leq \delta^{1/\rho} \mathbf{P}_R < 1 \tag{8}$$

## Requirement For Existence Of A Target

Definitions: 'Uncertainty-Adjusted' Growth:

$$\underline{\Gamma} \equiv \Gamma \psi < \Gamma \quad (9)$$

Adjusted Growth Patience Factor:

$$\mathbf{P}_{\underline{\Gamma}} = \mathbf{P} / \underline{\Gamma} = \mathbb{E}[\mathbf{P} / (\Gamma \psi)] \quad (10)$$

Growth Impatience Condition:

$$\mathbf{P}_{\underline{\Gamma}} < 1, \quad (11)$$

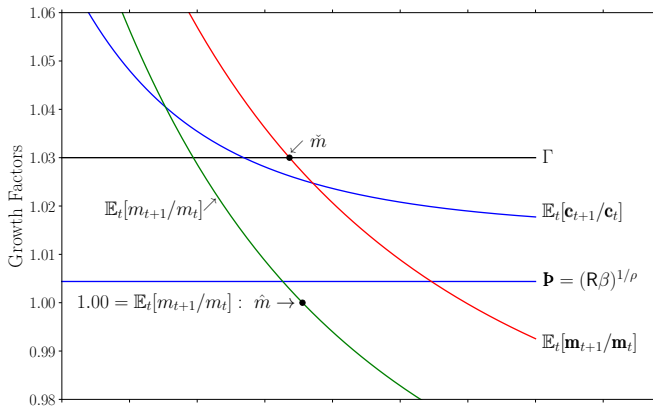
Why? Because it can be shown that

$$\lim_{m_t \rightarrow \infty} \mathbb{E}_t \left[ \frac{m_{t+1}}{m_t} \right] = \mathbf{P}_{\underline{\Gamma}} \quad (12)$$

# Five Propositions

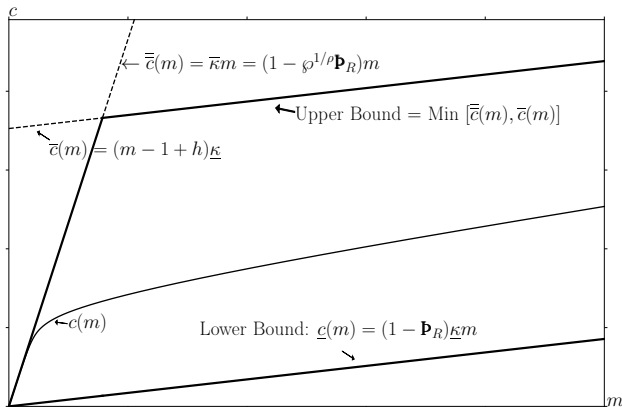
- 1  $\lim_{m_t \rightarrow \infty} \mathbb{E}_t[c_{t+1}/c_t] = \mathbf{P}$
- 2  $\lim_{m_t \rightarrow 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$
- 3  $\exists$  a unique target value of  $m$ , called  $\check{m}$
- 4  $\mathbb{E}_t[c_{t+1}/c_t | m_t = \check{m}] = \Gamma - \epsilon$
- 5  $\left( \frac{d\mathbb{E}_t[c_{t+1}/c_t]}{dm_t} \right) < 0$

# The Target Saving Figure

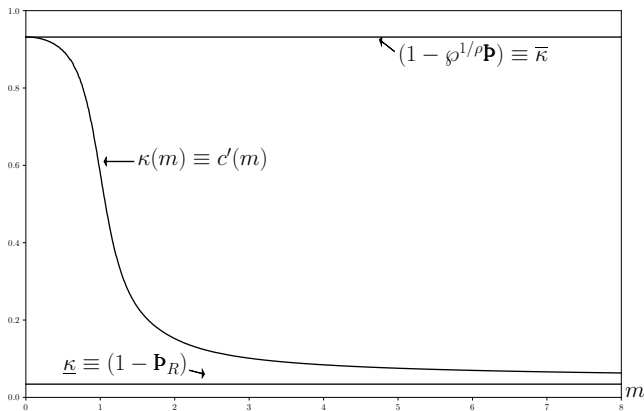




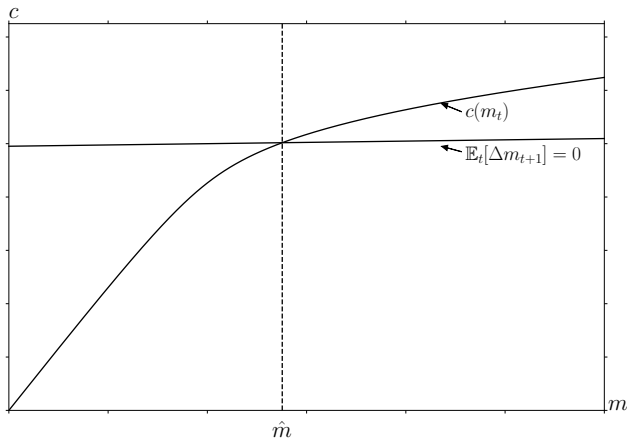
# Bounds On the Consumption Function



# The Marginal Propensity to Consume

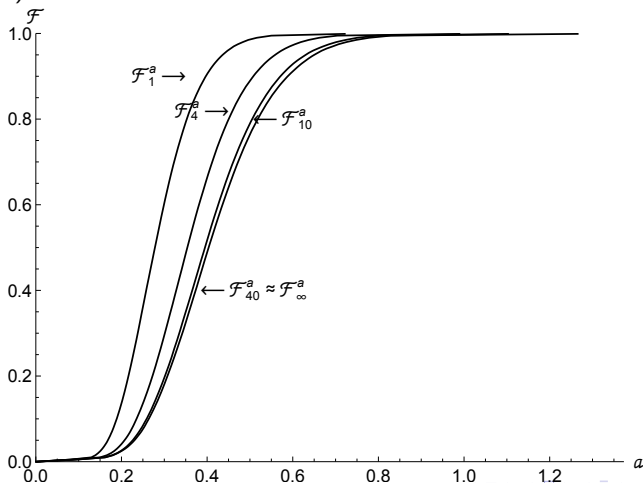


# The Consumption Function and Target Wealth



# Convergence To The Invariant Distribution

Szeidl (2013) Proves Existence of an Invariant Distribution of  $m, c, a$ , etc.



# Balanced Growth Equilibrium

Achieved When Cross Section Distribution Reaches Invariance

$$Y_{t+1}/Y_t = C_{t+1}/C_t = \Gamma \quad (13)$$

Fisherian Separation Fails, Even Without Liquidity Constraints!

Insight:

- Precautionary Saving  $\approx$  Liquidity Constraints
- If  $\hat{c}(m)$  is solution for constrained consumer,

$$\lim_{\varphi \downarrow 0} c(m; \varphi) = \hat{c}(m) \quad (14)$$

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# The MPC Out Of Permanent Shocks

<https://www.econ2.jhu.edu/people/ccarroll/papers/MPCPerm.pdf>

Lots of Recent Papers Trying to Measure the MPCP

Paper Proves:

- $MPCP < 1$
- But not a lot less:
  - 0.75 to 0.95 (annual rate) for wide range of parameter values



- Defined Conditions Under Which Widely Used Problem Has Solution
  - Finite Value of Autarky Condition Guarantees Contraction (with WRIC)
  - Growth Impatience Condition Prevents  $m \rightarrow \infty$
- Economy Of Buffer Stock Consumers Exhibits Balanced Growth
  - Even In Absence of General Equilibrium Adj of Interest Rate

MATKOWSKI, JANUSZ, AND ANDRZEJ S. NOWAK (2011): "On Discounted Dynamic Programming With Unbounded Returns," *Economic Theory*, 46, 455–474.

SZEIDL, ADAM (2013): "Stable Invariant Distribution in Buffer-Stock Saving and Stochastic Growth Models," *Manuscript, Central European University*, Available at [http://www.personal.ceu.hu/staff/Adam\\_Szeidl/papers/invariant\\_revision.pdf](http://www.personal.ceu.hu/staff/Adam_Szeidl/papers/invariant_revision.pdf).