Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds theoretical foundations for rigorous and intuitive understanding of 'buffer stock' saving models, pairing each theoretical result with a quantitative illustration. After describing conditions under which a consumption function exists, the paper shows that a 'target' buffer stock exists only under strictly stronger conditions. Under these 'buffer stock' conditions, the average growth rate of microeconomic consumers' spending matches the average growth rate of their permanent income; conditions required for aggregate 'balanced growth' can be looser. Together, the (provided) numerical tools and (proven) analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem

JEL codes D81, D91, E21

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All figures and numerical results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (**?**); for reference to the toolkit itself see Acknowleding Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for

1 Introduction

In the presence of empirically realistic transitory and permanent shocks to income *a la* **?**, only one more ingredient is required to construct a testable model of optimal consumption: A description of preferences. Modelers usually assume geometric discounting of a constant relative risk aversion utility function, because, starting with Zeldes (**?**), a large literature has constructed numerical solutions whose quantitative predictions match microeconomic evidence reasonably well; results are similar whether or not liquidity constraints are imposed.[1]

A companion theoretical literature has derived analytical properties of limiting "true" (nonnumerical) mathematical solutions – but only for models more complex than the case with just shocks and utility. The extra complexity has been required because standard contraction mapping theorems (beginning with **?** and including those building on Stokey et. al. (**?**)) cannot be applied when the utility function and resources are both unbounded (see the fuller discussion at the end of section 2.1).

This paper's first technical contribution is to articulate the (loose) conditions under which the simple problem (without shortcuts like a consumption floor or liquidity constraints) defines a contraction mapping with a nondegenerate consumption function. The interesting requirement is a 'Finite Value of Autarky' condition. The paper's second theoretical contribution is to specify the conditions under which the resulting consumption function implies existence of a 'target' wealth-to-permanent-income ratio (the model exhibits 'buffer stock' saving behavior). Buffer stock saving arises when the model's parameters satisfy a "Growth Impatience Condition" that relates preferences and uncertainty to predictable income growth.

Even without a formal proof, target saving has been intuitively understood to underlie central quantitative results from the heterogeneous agent macroeconomics literature; for example, the logic of target saving is central to the explanation by **?** of the fact that, during the Great Recession, middle-class consumers cut their consumption more than the poor or the rich. The theory below explains why: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $\mathrm{u}'(c)$ means they have little room to maneuver). And, increased labor income uncertainty does not change the behavior of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to reduce their discretionary spending.

Conveniently, elements required for the convergence proof also provide analytical foundations for many other results familiar from the numerical literature. Analytical

[1] For life cycle models without constraints, see **?**; **?**; for an infinite horizon model without constraints that resembles **?**'s constrained model, see **?**.

funding Econ-ARK's extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper's own references as **?**. Thanks to Alexis Akira Toda, James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at Johns Hopkins University and a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights.

derivations of these results are provided along the way to the proofs, paired with numerically computed illustrations (made using the open-source Econ-ARK toolkit). The main insights of the paper are instantiated in the toolkit, whose buffer stock saving module flags parametric choices under which a problem fails to define a contraction mapping; a target level of wealth does not exist; or the solution is otherwise surprising.

The paper proceeds in three parts.

The first part articulates the conditions required for the problem to define a nondegenerate limiting consumption function, and explains how the model relates to those previously considered in the literature. The conditions required for convergence are interestingly parallel to those required for the liquidity constrained perfect foresight model; that parallel is explored and explained. Next, the paper derives limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; then the theorem is proven explaining when the problem defines a contraction mapping. Finally, a related class of commonly-used models (exemplified by Deaton (**?**)) is shown to constitute a particular limit of this paper's model.

The next section examines five key properties of the model. First, as cash approaches infinity the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as cash approaches zero the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Third, a theorem explains why, if the consumer is 'growth impatient,' a unique target cash-to-permanent-income ratio will exist. Fourth, at the target cash ratio, the expected growth rate of consumption is slightly less than the expected growth rate of permanent (noncapital) income. Finally, the expected growth rate of consumption is declining in the level of cash. The first four propositions are proven under general assumptions about parameter values; the last is shown to hold if there are no transitory shocks, but may fail in extreme cases if there are both transitory and permanent shocks.

The final section discusses conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, an economy populated by buffer stock consumers converges to a balanced growth equilibrium in which consumption, income, and wealth all converge to the exogenous growth rate of permanent income (equivalent, here, to productivity). In the terms of **?**, buffer stock saving is a method of 'closing' a small open economy model, and one that is attractive because it requires no controversial assumptions: Not even liquidity constraints.

2 The Problem

2.1 Setup

We are interested in the infinite-horizon solution, which we define as the limit of the solution to a sequence of finite-horizon problems as the horizon becomes arbitrarily large. For expositional purposes, it will be useful to fix a terminal date T and consider the problem of a consumer who begins life in period $T-1$, then the problem of a consumer who begins life in $T-2$, and so on. Informally, we will say that the problem has a 'nondegenerate' infinite horizon solution if there is a limiting consumption function $\mathrm{c} = \lim_{n\uparrow\infty} \mathrm{c}_{T-n}$ which is neither zero everywhere nor infinity everywhere.

Specifically, a consumer born n periods before date T solves the problem

$$\max\ \mathbb{E}_t\left[\sum_{i=0}^{n}\beta^i \mathrm{u}(\mathbf{c}_{t+i})\right]$$

where

$$\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho) \tag{1}$$

is a constant relative risk aversion utility function with $\rho > 1$.[2] The consumer's initial condition is defined by market resources $\mathbf{m}_t$ (**?**'s 'cash-on-hand') and permanent noncapital income $\mathbf{p}_t$, which both start out strictly positive, $\{\mathbf{p}_t, \mathbf{m}_t\} \in (0,\infty)$, and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \tag{2}$$

In the usual treatment, a dynamic budget constraint (DBC) incorporates several elements that determine next period's $\mathbf{m}$ (given this period's choices); for the detailed analysis here, it will be useful to disarticulate the steps so that individual ingredients can be separately examined:

$$\begin{aligned}
\mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\
\mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R} \\
\mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma\psi_{t+1}}_{\equiv\Gamma_{t+1}} \\
\mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1}\xi_{t+1},
\end{aligned} \tag{3}$$

where $\mathbf{a}_t$ indicates the consumer's assets at the end of period t, which grow by a fixed interest factor $\mathsf{R} = (1+\mathsf{r})$ between periods, so that $\mathbf{b}_{t+1}$ is the consumer's financial ('bank') balances before next period's consumption choice;[3] $\mathbf{m}_{t+1}$ ('market resources') is the sum of financial wealth $\mathbf{b}_{t+1}$ and noncapital income $\mathbf{p}_{t+1}\xi_{t+1}$ (permanent noncapital income $\mathbf{p}_{t+1}$ multiplied by a mean-one iid transitory income shock factor ξ_{t+1}; transitory shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1\ \forall\ n \geq 1$). Permanent noncapital income in $t+1$ is equal to its previous value, multiplied by a growth factor Γ, modified by a mean-one iid shock ψ_{t+1}, $\mathbb{E}_t[\psi_{t+n}] = 1\ \forall\ n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).[4]

Following **?**, in future periods $t+n\ \forall\ n \geq 1$ there is a small probability $\wp$ that income

[2] The main results also hold for logarithmic utility which is the limit as $\rho \to 1$ but incorporating the logarithmic special case in the proofs is cumbersome and therefore omitted.

[3] Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see **?**, **?**, and **?** for the implications of capital income risk for the distribution of wealth and other questions not considered here.

[4] Hereafter for brevity we occasionally drop time subscripts, e.g. $\mathbb{E}[\psi^{-\rho}]$ signifies $\mathbb{E}_t[\psi_{t+1}^{-\rho}]$.

will be zero (a 'zero-income event'),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1-\wp) & \text{with probability } (1-\wp) \end{cases} \tag{4}$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \ \forall\ n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.[5] Call the cumulative distribution functions $\mathcal{F}_\psi$ and $\mathcal{F}_\theta$ (where $\mathcal{F}_\xi$ is derived trivially from (4) and $\mathcal{F}_\theta$).

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable (though it has empirical support).[6] However, it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,[7] transforming that model back into this one. And no key results would change if the transitory shocks were persistent but mean-reverting, instead of IID.

This model differs from Bewley's (**?**) classic formulation in several ways. The CRRA utility function does not satisfy Bewley's assumption that u(0) is well defined, or that u′(0) is well defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (**?**) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds are quantitatively important in micro data[8] and which since **?** have been understood to be far more consequential for household welfare than are transitory fluctuations. It differs from Deaton (**?**) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* **?**); and the transitory shocks here can occasionally cause income to reach zero.[9] It differs from models found in Stokey et. al. (**?**) because neither liquidity constraints nor bounds on utility or marginal utility are imposed.[10,11] **?** show how to allow unbounded returns by using policy function iteration, but also impose constraints.

The paper with the most in common with this one is **?**, henceforth MST, who establish the existence and uniqueness of a solution to a general income fluctuation problem in a Markovian setting. The most important differences are that MST impose liquidity constraints, assume that $\mathrm{u}'(0) = 0$, and assume that for every possible realization of income Y, expected marginal utility is finite ($\mathbb{E}[\mathrm{u}'(Y)] < \infty \ \forall\ Y$). These assumptions are not consistent with the CRRA utility function and income processes used here, whose properties are key to the derivation of the results.[12]

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[5] See **?** and **?** for analyses of cases where the shock processes have unbounded support.

[6] We will calibrate this probability to 0.005 percent to match data from the Panel Study of Income Dynamics (**?**).

[7] So long as unemployment benefits are proportional to $\mathbf{p}_t$; see the discussion in section 2.11.

[8] MaCurdy (**?**); Abowd and Card (**?**); Carroll and Samwick (**?**); Jappelli and Pistaferri (**?**); Storesletten, Telmer, and Yaron (**?**); **?**

[9] Below it will become clear that the Deaton model is a particular limit of this paper's model.

[10] Similar restrictions to those in the cited literature are made in the well known papers by Scheinkman and Weiss (**?**), Clarida (**?**), and **?**. See **?** for an elegant analysis of a related but simpler continuous-time model.

[11] **?** relaxed the bounds on the return function, but they address only the deterministic case.

[12] The incorporation of permanent shocks rules out application of the tools of **?**, who followed and corrected an error

2.2 The Problem Can Be Rewritten in Ratio Form

We establish a bit more notation by reviewing the standard result that in problems of this class (CRRA utility, permanent shocks) the number of relevant state variables can be reduced from two ($\mathbf{m}$ and $\mathbf{p}$) to one ($m = \mathbf{m}/\mathbf{p}$). Generically defining nonbold variables as the boldface counterpart normalized by $\mathbf{p}_t$ (as with m), assume that value in the last period of life is $\mathrm{u}(\mathbf{m}_T)$, and consider the problem in the second-to-last period,

$$\begin{aligned}
\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} \ \mathrm{u}(\mathbf{c}_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{m}_T)] \\
&= \max_{c_{T-1}} \ \mathrm{u}(\mathbf{p}_{T-1}c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\mathbf{p}_T m_T)] \\
&= \mathbf{p}_{T-1}^{1-\rho}\left\{\max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\,\mathbb{E}_{T-1}[\mathrm{u}(\Gamma_T m_T)]\right\}, \qquad (5)
\end{aligned}$$

where the last line follows because for the CRRA utility function (1), $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$.

Now, in a one-time deviation from the notational convention established in the last paragraph, define nonbold 'normalized value' not as $\mathbf{v}_t/\mathbf{p}_t$ but as $\mathrm{v}_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{\{\mathrm{c}\}_t^T} \ \mathrm{u}(\mathrm{c}_t) + \beta\,\mathbb{E}_t[\Gamma_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \qquad (6) \\
a_t &= m_t - c_t \\
b_{t+1} &= (\mathsf{R}/\Gamma_{t+1})a_t \ = \ \mathcal{R}_{t+1}a_t \\
m_{t+1} &= b_{t+1} + \xi_{t+1},
\end{aligned}$$

where $\mathcal{R}_{t+1} \equiv (\mathsf{R}/\Gamma_{t+1})$ is a 'growth-normalized' return factor, and the new problem's first order condition is[13]

$$c_t^{-\rho} = \mathsf{R}\beta\,\mathbb{E}_t[\Gamma_{t+1}^{-\rho}c_{t+1}^{-\rho}]. \qquad (7)$$

Since $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$, defining $\mathrm{v}_{T-1}(m_{T-1})$ from (6), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to all earlier periods, so that if we solve the normalized one-state-variable problem (6), we will have solutions to the original problem for any $t < T$ from:

$$\begin{aligned}
\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t), \\
\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t\mathrm{c}_t(m_t).
\end{aligned}$$

in the fundamental work on the local contraction mapping method developed in ?. ? provides a correction to ?, and provides conditions that are easier to verify, but again only addresses the deterministic case.

[13] Leaving aside their assumptions about the utility function and liquidity constraints, the normalized problem can be viewed as a special case of the model of ?, with $\mathcal{R}_{t+1}$ corresponding to their stochastic rate of return on capital and $\beta\Gamma_{t+1}^{1-\rho}$ to their stochastic discount factor, and with the special property that $\mathcal{R}_{t+1}$ and the modified discount factor are correlated because Γ_{t+1} plays a role in each.

2.3 Definition of a Nondegenerate Solution

Formally, we say the problem has a nondegenerate solution if as the horizon n gets arbitrarily large the solution in the first period of life $\mathrm{c}_{T-n}(m)$ defines a unique $\mathrm{c}(m)$:

$$\mathrm{c}(m) \equiv \lim_{n\to\infty} \mathrm{c}_{T-n}(m) \tag{8}$$

that satisfies

$$0 < \mathrm{c}(m) < \infty \tag{9}$$

for every $0 < m < \infty$. ('Degenerate' limits will be cases where the limiting consumption function is $\mathrm{c}(m) = 0$ or $\mathrm{c}(m) = \infty$.)

2.4 Perfect Foresight Benchmarks

The familiar analytical solution to the perfect foresight model, obtained by setting $\wp = 0$ and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (2) imply an exactly-holding intertemporal budget constraint (IBC):

$$\mathrm{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\mathrm{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \tag{10}$$

where $\mathbf{b}$ is nonhuman wealth and $\mathbf{h}_t$ is 'human wealth,' and with a constant $\mathcal{R} \equiv \mathsf{R}/\Gamma$,

$$\begin{aligned} \mathbf{h}_t &= \mathbf{p}_t + \mathcal{R}^{-1}\mathbf{p}_t + \mathcal{R}^{-2}\mathbf{p}_t + ... + \mathcal{R}^{t-T}\mathbf{p}_t \\ &= \underbrace{\left(\frac{1-\mathcal{R}^{-(T-t+1)}}{1-\mathcal{R}^{-1}}\right)}_{\equiv h_t} \mathbf{p}_t \end{aligned} \tag{11}$$

This equation shows that in order for $h \equiv \lim_{n\to\infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition ('FHWC'):

$$\underbrace{\Gamma/\mathsf{R}}_{\equiv \mathcal{R}^{-1}} < 1. \tag{12}$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 PF Unconstrained Solution Exists Under RIC and FHWC

The consumption Euler equation holds in every period; with $\mathrm{u}'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$\mathbf{c}_{t+1}/\mathbf{c}_t = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ} \tag{13}$$

where the Old English letter 'thorn' represents what we will call the 'Absolute Patience Factor,' or APF:[14]

$$\text{\textbf{Þ}} = (\mathsf{R}\beta)^{1/\rho} \tag{14}$$

The sense in which **Þ** captures patience is that if the 'absolute impatience condition' (AIC) holds,

$$\text{\textbf{Þ}} < 1, \tag{15}$$

the consumer will choose to spend an amount too large to sustain indefinitely (the level of consumption must fall over time). We call such a consumer 'absolutely impatient.'

We next define a 'Return Patience Factor' (RPF) that relates absolute patience to the return factor:

$$\text{\textbf{Þ}}_{\mathsf{R}} \equiv \text{\textbf{Þ}}/\mathsf{R} \tag{16}$$

and since consumption is growing by **Þ** but discounted by R:

$$\mathrm{PDV}_t(\mathbf{c}) = \left(\frac{1-\text{\textbf{Þ}}_{\mathsf{R}}^{T-t+1}}{1-\text{\textbf{Þ}}_{\mathsf{R}}}\right)\mathbf{c}_t \tag{17}$$

from which the IBC (10) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1-\text{\textbf{Þ}}_{\mathsf{R}}}{1-\text{\textbf{Þ}}_{\mathsf{R}}^{T-t+1}}\right)}^{\equiv \underline{\kappa}_t}(\mathbf{b}_t + \mathbf{h}_t) \tag{18}$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathrm{c}}_{T-n}(m_{T-n}) = (\overbrace{m_{T-n}-1}^{\equiv b_{T-n}} + h_{T-n})\underline{\kappa}_{T-n} \tag{19}$$

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) because it answers the question 'if the consumer had an extra unit of wealth, how much more would be spent.' ($\bar{\mathrm{c}}$'s overbar signfies that this function will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, the underbar for κ indicates that $\underline{\kappa}$ is a lower bound).

Equation (18) makes plain that for the limiting MPC $\underline{\kappa}$ to be strictly positive as $n = T - t$ goes to infinity we must impose the Return Impatience Condition (RIC):

$$\text{\textbf{Þ}}_{\mathsf{R}} < 1, \tag{20}$$

so that

$$0 < \underline{\kappa} \equiv 1 - \text{\textbf{Þ}}_{\mathsf{R}} = \lim_{n\to\infty} \underline{\kappa}_{T-n}. \tag{21}$$

The RIC thus imposes a second kind of 'impatience:' The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to

[14] Impatience conditions of one kind or another have figured in intertemporal optimization problems since such problems were first formalized in economics, most notably by **?**. Discussion of these issues has remained central to the literature, and no summary could do justice to the topic; I therefore refrain from even attempting it.

spend nothing today out of an increase in current wealth. That is, the RIC rules out the degenerate limiting solution $\bar{c}(m) = 0$. We will say that a consumer who satisfies the RIC is 'return impatient.'

Given that the RIC holds, and as before defining limiting objects by the absence of a time subscript (e.g., $\bar{c}(m) = \lim_{n\uparrow\infty} \bar{c}_{T-n}(m)$), the limiting consumption function will be

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \tag{22}$$

and we now see that in order to rule out the degenerate limiting solution $\bar{c}(m) = \infty$ we need h to be finite; that is, we must impose the finite human wealth condition (12).

Finally, since the perfect foresight growth factor for consumption is Þ, the fact that $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$ allows us to write an analytical expression for value:

$$\begin{aligned} v_t &= \mathrm{u}(c_t) + \beta\mathrm{u}(c_t\text{Þ}) + \beta^2\mathrm{u}(c_t\text{Þ}^2) + ... \\ &= \mathrm{u}(c_t)\left(1 + \beta\text{Þ}^{1-\rho} + (\beta\text{Þ}^{1-\rho})^2 + ...\right) \\ &= \mathrm{u}(c_t)\left(\frac{1 - (\beta\text{Þ}^{1-\rho})^{T-t+1}}{1 - \beta\text{Þ}^{1-\rho}}\right) \end{aligned} \tag{23}$$

which asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if $\beta\text{Þ}^{1-\rho} < 1$;[15] with a bit of algebra, this requirement can be shown to be equivalent to the RIC. Thus, the same conditions that guarantee a nondegenerate limiting consumption function also guarantee a nondegenerate limiting value function (which, interestingly, will *not* be true when we incorporate uncertainty).

Conclusions so far are summarized in the first panel of Table 4: The PF-Unconstrained model will have a nondegenerate solution if we impose the RIC and FHWC.

2.4.3 PF Constrained Solution Exists Under RIC or Under {~~RIC~~,PF-GIC}

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, which obtains for a consumer who enters period t with $b_t = 0$. The constraint is 'relevant' if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today's resources $c_t = m_t = 1$, exceeds the marginal utility from doing the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (7):

$$1^{-\rho} > \mathsf{R}\beta(\Gamma)^{-\rho}1^{-\rho}. \tag{24}$$

By analogy to the return patience factor, we therefore define a 'perfect foresight growth patience factor' (PF-GPF) as

$$\text{Þ}_\Gamma = \text{Þ}/\Gamma, \tag{25}$$

and define a 'perfect foresight growth impatience condition' (PF-GIC)

$$\text{Þ}_\Gamma < 1 \tag{26}$$

which is equivalent to (24) (exponentiate both sides by $1/\rho$).

[15] This is related to a condition in ?.

~~PF-GIC~~ and RIC. If the PF-GIC fails but the RIC (20) holds, appendix **??** shows that, for some $0 < m_{\#} < 1$, an unconstrained consumer behaving according to (22) would choose $c < m$ for all $m > m_{\#}$. In this case the solution to the constrained consumer's problem is simple: For any $m \geq m_{\#}$ the constraint does not bind (and will never bind in the future) so here the constrained consumption function is identical to the unconstrained one. If the consumer were somehow[16] to arrive at an $m < m_{\#} < 1$ the constraint would bind and the consumer would consume $c = m$. Using the $\grave{\bullet}$ accent to designate version of a function in the presence of constraints:

$$\grave{\mathrm{c}}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{\mathrm{c}}(m) & \text{if } m \geq m_{\#}. \end{cases} \tag{27}$$

PF-GIC and RIC. More useful is the case where the return impatience and PF growth-impatience conditions both hold. In this case appendix **??** shows that the limiting constrained consumption function is piecewise linear, with $\grave{\mathrm{c}}(m) = m$ up to a first 'kink point' at $m^1_{\#} > 1$, and with discrete declines in the MPC at a set of kink points $\{m^1_{\#}, m^2_{\#}, ...\}$. As $m \uparrow \infty$ the constrained consumption function $\grave{\mathrm{c}}(m)$ becomes arbitrarily close to the unconstrained $\bar{\mathrm{c}}(m)$, and the marginal propensity to consume function $\grave{\boldsymbol{\kappa}}(m) \equiv \grave{\mathrm{c}}'(m)$ limits to $\underline{\kappa}$. Similarly, the value function $\grave{\mathrm{v}}(m)$ is nondegenerate and limits into the value function of the unconstrained consumer. This logic holds even when the finite human wealth condition fails (~~FHWC~~): A solution exists because the constraint prevents the consumer from borrowing against infinite human wealth to finance infinite current consumption. Under these circumstances, the consumer who starts with any amount of resources $b_t > 1$ will, over time, run those resources down so that by some finite number of periods n in the future the consumer will reach $b_{t+n} = 0$, and thereafter will set $c = m = 1$ for eternity, a policy that will (using the same steps as for equation (23)) yield value of

$$\mathbf{v}_{t+n} = \Gamma^{n(1-\rho)}\mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho})^{T-(t+n)+1}}{1-\beta\Gamma^{1-\rho}}\right),$$

which will be finite whenever any of these equivalent conditions holds:

$$\begin{aligned} \overbrace{\beta\Gamma^{1-\rho}}^{\equiv\beth} &< 1 \\ \beta\mathsf{R}\Gamma^{-\rho} &< \mathsf{R}/\Gamma \\ \mathbf{Þ} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho} \\ \mathbf{Þ}_{\Gamma} &< (\mathsf{R}/\Gamma)^{1/\rho}, \end{aligned} \tag{28}$$

where we call $\beth$[17] the 'Perfect Foresight Finite Value Of Autarky Factor' (PF-FVAF), and (28) is the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC, because it guarantees that a consumer who always spends all permanent income 'has finite autarky value.' Note that the last version of the PF-FVAC in (28) implies the PF-GIC $\mathbf{Þ}_{\Gamma} < 1$ whenever ~~FHWC~~ ($\mathsf{R} < \Gamma$) holds. So, if ~~FHWC~~, value for any finite m will be the sum of

[16]"Somehow" because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

[17]This is another kind of discount factor, so we use the Hebrew 'bet' which is a cognate of the Greek 'beta'.

two finite numbers: The component due to the unconstrained consumption choice made over the finite horizon leading up to $b_{t+n} = 0$, and the finite component due to the value of consuming all $\mathbf{p}_{t+n}$ thereafter.

PF-GIC and ~~RIC~~. The most peculiar possibility occurs when the RIC fails. Under these circumstances the FHWC must also fail (Appendix **??**), and the constrained consumption function is nondegenerate. (See appendix Figure **??** for a numerical example). While it is true that $\lim_{m\uparrow\infty} \grave{\boldsymbol{\kappa}}(m) = 0$, nevertheless the limiting constrained consumption function $\grave{c}(m)$ is strictly positive and strictly increasing in m. This result interestingly reconciles the conflicting intuitions from the unconstrained case, where ~~RIC~~ would suggest a degenerate limit of $\grave{c}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\grave{c}(m) = \infty$.

Tables 3 and 4 (and appendix table **??**) codify these points. Perhaps more useful is the intuitive representation of the relations of the conditions for the unconstrained perfect foresight case presented in Figure 1. Each node represents a quantity considered in the foregoing analysis. The arrow associated with each inequality condition reflects the imposition of that condition. For example, one way we wrote the PF-FVAC in equation (28) is $\text{Þ} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, so imposition of the PF-FVAC is captured by the diagonal arrow connecting Þ and $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$. Traversing the diagram clockwise starting at Þ involves imposing first the PF-GIC then the FHWC, and the consequent arrival at the bottom right node tells us that these two conditions jointly imply that the PF-FVAC holds. Reversal of a condition will reverse the arrow's direction; so, for example, the bottommost arrow going from R to $\mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$ imposes ~~FHWC~~; but we can cancel the cancellation and reverse the arrow. This would allow us to traverse the diagram in a clockwise direction from Þ to R, revealing that imposition of PF-GIC and FHWC (and, redundantly, FHWC again) lets us conclude that the RIC holds because the starting point is Þ and the endpoint is R, and this is a chain of inequalities. (Consult Appendix **??** for a detailed exposition of diagrams of this type).

We now turn to the case with uncertainty. The model without constraints but with uncertainty will turn out to be a close parallel to the model with constraints but without uncertainty.

2.5 Uncertainty-Modified Conditions

2.5.1 Impatience

When uncertainty is introduced, the expectation of b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t\,\mathbb{E}_t[\mathcal{R}_{t+1}] = a_t\mathcal{R}\,\mathbb{E}_t[\psi_{t+1}^{-1}] \tag{29}$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define the object

$$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{\TH} \arrow[r, "\text{PF-GIC}"] \arrow[d, "\text{RIC}"'] \arrow[dr, "\text{PF-FVAC}"] \arrow[dr, phantom, shift left=6ex, "\neq"] \arrow[dr, phantom, shift right=6ex, "\neq"] & \Gamma \arrow[d, "\text{FHWC}"] \\
\mathsf{R} \arrow[r, "\cancel{\text{FHWC}}"'] & \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}
\end{tikzcd}$$

Figure 1 Relation of PF-GIC, FHWC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\text{\TH} < \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}$, which is one way of writing the PF-FVAC, equation (28)

because this permits us to write expressions like the RHS of (29) compactly as, e.g., $a_t\mathcal{R}/\underline{\psi}^{-1}$.[18] We refer to this as the 'compensated return,' because it compensates (in a risk-neutral way) for the effect of uncertainty on the expected growth-normalized return (in the sense implicitly defined in (29)).

We can now transparently generalize the PF-GIC (26) by defining a 'compensated growth factor'

$$\underline{\Gamma} = \Gamma\underline{\psi} \tag{30}$$

and a compensated Growth Patience Pactor (GPF):

$$\text{\TH}_{\underline{\Gamma}} = \text{\TH}/\underline{\Gamma} = \mathbb{E}[\text{\TH}/\Gamma\psi] \tag{31}$$

and a straightforward derivation ((45) below) yields the conclusion that

$$\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \text{\TH}_{\underline{\Gamma}},$$

which implies that if we wish to prevent m from heading to infinity (that is, if we want m to be expected to fall for some large enough value of m) we must impose a generalized version of the Perfect Foresight Growth Impatience Condition (26); we call the 'Growth Impatience Condition' (GIC) the requirement that the Growth Patience Factor (31)

[18] One way to think of $\underline{\psi}$ is as a particular kind of a 'certainty equivalent' of the shock; this captures the intuition that mean-one shock renders a given mean level of income less valuable than if the shock did not exist, so that $\underline{\psi} < 1$.

must be less than 1:[19]

$$\text{Þ}_{\underline{\Gamma}} < 1 \tag{32}$$

which is stronger than the perfect foresight version (26) because

$$\underline{\Gamma} < \Gamma \tag{33}$$

(Jensen's inequality implies that $\underline{\psi} < 1$ for nondegenerate ψ).

2.5.2 Autarky Value

Analogously to (23), a consumer who spent exactly their permanent income every period would have value determined by the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned}
\mathbf{v}_t &= \mathbb{E}_t\left[\mathrm{u}(\mathbf{p}_t) + \beta\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}) + ... + \beta^{T-t}\mathrm{u}(\mathbf{p}_t\Gamma_{t+1}...\Gamma_T)\right] \\
&= \mathrm{u}(\mathbf{p}_t)\left(\frac{1-(\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1-\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}]}\right)
\end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$$

which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ (and $\underline{\underline{\psi}} < \underline{\psi}$ for the reasonable (though not required) case of $\rho > 2$); defining

$$\underline{\underline{\Gamma}} = \Gamma\underline{\underline{\psi}} \tag{34}$$

we can see that $\mathbf{v}_t$ will be finite as T approaches ∞ if

$$\begin{aligned}
\overbrace{\beta\underline{\underline{\Gamma}}^{1-\rho}}^{\equiv\underline{\underline{\sqsupset}}} &< 1 \\
\beta &< \underline{\underline{\Gamma}}^{\rho-1}
\end{aligned} \tag{35}$$

which we call the 'finite value of autarky' condition (FVAC) because it guarantees that value is finite for a consumer who always consumes their (now stochastic) permanent income (and we will call $\underline{\underline{\sqsupset}}$ the 'Finite Value of Autarky Factor,' FVAF).[20] For nondegenerate ψ, this condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect foresight version (28) because $\underline{\underline{\Gamma}} < \Gamma$.[21]

---

[19] Equation (32) is a bit easier to satisfy than the similar condition imposed by Deaton (**?**): $\left(\mathbb{E}[\psi^{-\rho}]\right)^{1/\rho}\,\text{Þ}_{\Gamma} < 1$ to guarantee that his problem defined a contraction mapping.

[20] In a stationary environment – that is, with $\underline{\underline{\Gamma}} = 1$ – this corresponds to an impatience condition imposed by **?**; but their remaining conditions do not correspond to those here, because their problem differs and their method of proof differs.

[21] To see this, rewrite (35) as

$$\begin{aligned}
\beta\mathsf{R} &< \mathsf{R}\underline{\underline{\Gamma}}^{\rho-1} \\
(\beta\mathsf{R})^{1/\rho} &< \mathsf{R}^{1/\rho}\Gamma^{1-1/\rho}\underline{\underline{\psi}}^{1-1/\rho}
\end{aligned}$$

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	$\wp$	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

2.6 The Baseline Numerical Solution

Figure 2, familiar from the literature, depicts the successive consumption rules that apply in the last period of life ($\mathrm{c}_T(m)$), the second-to-last period, and earlier periods under baseline parameter values listed in Table 2. (The 45 degree line is $\mathrm{c}_T(m) = m$ because in the last period of life it is optimal to spend all remaining resources.)

In the figure, the consumption rules appear to converge to a nondegenerate $\mathrm{c}(m)$. Our next purpose is to show that this appearance is not deceptive.

2.7 Concave Consumption Function Characteristics

A precondition for the main proof is that the maximization problem (6) defines a sequence of continuously differentiable strictly increasing strictly concave[22] functions $\{\mathrm{c}_T, \mathrm{c}_{T-1}, ...\}$. The straightforward but tedious proof is relegated to appendix **??**. For present purposes, the most important point is that the income process induces what **?** dubbed a 'natural borrowing constraint': $\mathrm{c}_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t+1$ with balances b_{t+1} of zero, and then might earn zero income over the remaining horizon, requiring the consumer to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. **?** seems to have been the first to argue, based on his numerical results, that the natural borrowing constraint was a quantitatively plausible alternative to 'artificial' or 'ad hoc' borrowing constraints in a life cycle model.[23]

Strict concavity and continuous differentiability of the consumption function are key elements in many of the arguments below, but are not characteristics of models with

$$\text{Þ}_\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

[22] With one obvious exception: $\mathrm{c}_T(m)$ is linear (and so only weakly concave).

[23] **?** made the same (numerical) point for infinite horizon models (calibrated to actual empirical data on household income dynamics).

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula			Approximate Calculated Value
Finite Human Wealth Factor	$\mathcal{R}^{-1}$	$\equiv$	Γ/R	0.990
PF Finite Value of Autarky Factor	$\beth$	$\equiv$	$\beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi}$	$\equiv$	$(\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma}$	$\equiv$	$\Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}}$	$\equiv$	$(\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}}$	$\equiv$	$\Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	$\mathbf{Þ}$	$\equiv$	$(\mathsf{R}\beta)^{1/\rho}$	0.999
Return Patience Factor	$\mathbf{Þ}_{\mathsf{R}}$	$\equiv$	$\mathbf{Þ}/\mathsf{R}$	0.961
PF Growth Patience Factor	$\mathbf{Þ}_{\Gamma}$	$\equiv$	$\mathbf{Þ}/\Gamma$	0.970
Growth Patience Factor	$\mathbf{Þ}_{\underline{\Gamma}}$	$\equiv$	$\mathbf{Þ}/\underline{\Gamma}$	0.980
Finite Value of Autarky Factor	$\underline{\underline{\beth}}$	$\equiv$	$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941
Weak Impatience Factor	$\wp^{1/\rho}\mathbf{Þ}$	$\equiv$	$(\wp\beta\mathsf{R})^{1/\rho}$	0.071

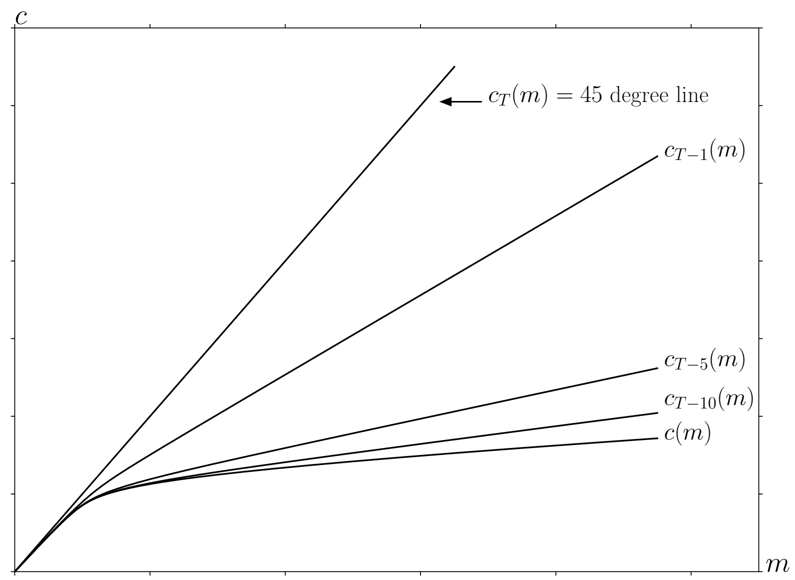


Figure 2 Convergence of the Consumption Rules

'artificial' borrowing constraints. The analytical convenience of these features is considerable, even if models with natural borrowing constraints in practice usually give similar results to those with artificial constraints.

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 2 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which also turn out to be useful in the contraction mapping proof.[24]

Assume that a continuously differentiable concave consumption function exists in period $t+1$, with an origin at $\mathrm{c}_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t+1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption that human wealth is finite, the proportion of consumption that will be financed out of human wealth approaches zero. In consequence, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero. In the course of proving this, appendix **??** provides a useful recursive expression (used below) for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \text{Þ}_{\mathsf{R}} \underline{\kappa}_{t+1}^{-1}. \tag{36}$$

2.8.1 Weak RIC Conditions

There is a parallel expression for the limiting maximal MPC as $m \downarrow 0$: appendix equation (**??**) shows that, as $m_t \uparrow \infty$,

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho} \text{Þ}_{\mathsf{R}} \bar{\kappa}_{t+1}^{-1}. \tag{37}$$

where $\left\{\bar{\kappa}_{T-n}^{-1}\right\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the 'weak return patience factor' $\wp^{1/\rho}\text{Þ}_{\mathsf{R}}$ satisfies:

$$0 \leq \wp^{1/\rho} \text{Þ}_{\mathsf{R}} < 1, \tag{38}$$

a condition that we dub the 'Weak Return Impatience Condition' (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\text{Þ}_{\mathsf{R}} < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That is why the probability of the zero income event $\wp$ appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \ \leq \mathrm{c}_t(m_t) \leq \ \bar{\kappa}_t m_t \tag{39}$$

[24] **?** show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; **?** show that these results generalize to the limits derived here if capital income is added to the model.

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave (?), and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is provided in appendix ??).

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

To prove that the consumption rules converge, we need to show that the problem defines a contraction mapping. As mentioned above, this cannot be proven using the standard theorems in the literature following Stokey et. al. (?), which require marginal utility to be bounded over the space of possible values of m, because the possibility (however unlikely) of an unbroken string of zero-income events through the end of the horizon means that marginal utility is unbounded. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of 'unbounded returns' (see, e.g., ?), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that incorporates permanent shocks.[25]

Fortunately, Boyd (?) provided a weighted contraction mapping theorem that ? showed could be used to address the homogeneous case (of which CRRA is an example) in a deterministic framework; later, ? showed how to extend the ? approach to the stochastic case.

Definition 1. *Consider any function $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ where $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the space of continuous functions from $\mathcal{A}$ to $\mathcal{B}$. Suppose $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then $\bullet$ is F-bounded if the F-norm of $\bullet$,*

$$\| \bullet \|_F = \sup_m \left[\frac{| \bullet (m)|}{F(m)} \right], \tag{40}$$

is finite.

For $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A}, \mathcal{B})$ that are F-bounded; w, x, y, and z as examples of F-bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (?) proves the following.

Boyd's Weighted Contraction Mapping Theorem. *Let $\mathsf{T} : \mathcal{C}_F(\mathcal{A}, \mathcal{B}) \to \mathcal{C}(\mathcal{A}, \mathcal{B})$ such that*[26,27]

$$\begin{aligned}
&(1)\,\mathsf{T} \textit{ is non-decreasing, i.e. } \mathrm{x} \le \mathrm{y} \Rightarrow \{\mathsf{T}\mathrm{x}\} \le \{\mathsf{T}\mathrm{y}\} \\
&(2)\,\{\mathsf{T}\mathbf{0}\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B}) \\
&(3)\,\textit{There exists some real } 0 < \alpha < 1\textit{, such that} \\
&\quad \{\mathsf{T}(\mathrm{w} + \zeta F)\} \le \{\mathsf{T}\mathrm{w}\} + \zeta \alpha F \quad \textit{holds for all real } \zeta > 0\ .
\end{aligned}$$

[25] See ? for a detailed discussion of the reasons the existing literature up through ? cannot handle the problem described here.

[26] We will usually denote the function that results from the mapping as, e.g., $\{\mathsf{T}\mathrm{w}\}$.

[27] To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element $\bullet$ in the domain of the functions in question' so that, e.g., $\mathrm{x} \le \mathrm{y}$ is short for $\mathrm{x}(\bullet) \le \mathrm{y}(\bullet)\ \forall \bullet \in \mathcal{A}$. In this notation, $\zeta\alpha F$ in (3) is a *function* which can be applied to any argument $\bullet$ (because F is a function).

Then T *defines a contraction with a unique fixed point.*

For our problem, take $\mathcal{A}$ as $\mathbb{R}_{>0}$ and $\mathcal{B}$ as $\mathbb{R}$, and define

$$\{\mathsf{E}\mathrm{z}\}(a_t) = \mathbb{E}_t\left[\Gamma_{t+1}^{1-\rho}\mathrm{z}(a_t\mathcal{R}_{t+1}+\xi_{t+1})\right].$$

Using this, we introduce the mapping $\mathfrak{T} : \mathcal{C}_{\digamma}(\mathcal{A},\mathcal{B}) \to \mathcal{C}(\mathcal{A},\mathcal{B})$,[28]

$$\{\mathfrak{T}\mathrm{z}\}(m_t) = \max_{c_t\in[\underline{\kappa}m_t,\bar{\kappa}m_t]} \mathrm{u}(c_t) + \beta\left(\{\mathsf{E}\mathrm{z}\}(m_t-c_t)\right). \tag{41}$$

We can show that our operator $\mathfrak{T}$ satisfies the conditions that Boyd requires of his operator T, if we impose two restrictions on parameter values. The first is the WRIC necessary for convergence of the maximal MPC, equation (38) above. A more serious restriction is the utility-compensated Finite Value of Autarky condition, equation (35). (We discuss the interpretation of these restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in position to state the central theorem of the paper.

Theorem 1. $\mathfrak{T}$ *is a contraction mapping if the restrictions on parameter values* (38) *and* (35) *are true (that is, if the weak return impatience condition and the finite value of autarky condition hold).*

Intuitively, Boyd's theorem shows that if you can find a $\digamma$ that is everywhere finite but goes to infinity 'as fast or faster' than the function you are normalizing with $\digamma$, the normalized problem defines a contraction mapping. The intuition for the FVAC condition is just that, with an infinite horizon, with any initial amount of bank balances b_0, in the limit your value can always be made greater than you would get by consuming exactly the sustainable amount (say, by consuming $(\mathsf{r}/\mathsf{R})b_0 - \epsilon$ for some small $\epsilon > 0$).

The details of the proof are cumbersome, and are therefore relegated to appendix **??**. Given that the value function converges, appendix **??** shows that the consumption functions converge.[29]

2.10 The Liquidity Constrained Solution as a Limit

This section explains why a related problem commonly considered in the literature (e.g., with a simpler income process, by Deaton (**?**)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability $\wp$ of the zero-income event approaches zero.

The essence of the argument is easy to state. As noted above, the possibility of earning zero income over the remainder of the horizon prevents the consumer from ending the current period with zero assets because with some finite probability the consumer would be forced to consume zero, which would be infinitely painful.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the probability that it will turn out to matter. As $\wp \downarrow 0$, that

[28]Note that the existence of the maximum is assured by the continuity of $\{\mathsf{E}\mathrm{z}\}(a_t)$ (it is continuous because it is the sum of continuous $\digamma$-bounded functions z) and the compactness of $[\underline{\kappa}m_t, \bar{\kappa}m_t]$.

[29]MST's proof is for convergence of the consumption policy function directly, rather than of the value function, which is why their conditions are on u', which governs behavior.

probability becomes arbitrarily small, so the amount of precautionary saving approaches zero. But zero precautionary saving is the amount of saving that an impatient liquidity constrained consumer with perfect foresight would choose.

Another way to understand this is just to think of the liquidity constraint as being imposed by specifying a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ends the period with (weakly) negative assets.

See appendix **??** for the formal proof justifying the foregoing intuitive discussion.

2.11 Discussion of Parametric Restrictions

The full relationship among all the conditions articulated above is represented in Figure 3. Though the diagram looks complex, it is merely a modified version of the earlier diagram with further (mostly intermediate) inequalities inserted. Again readers unfamiliar with such diagrams should see Appendix **??**) for a more detailed explanation.

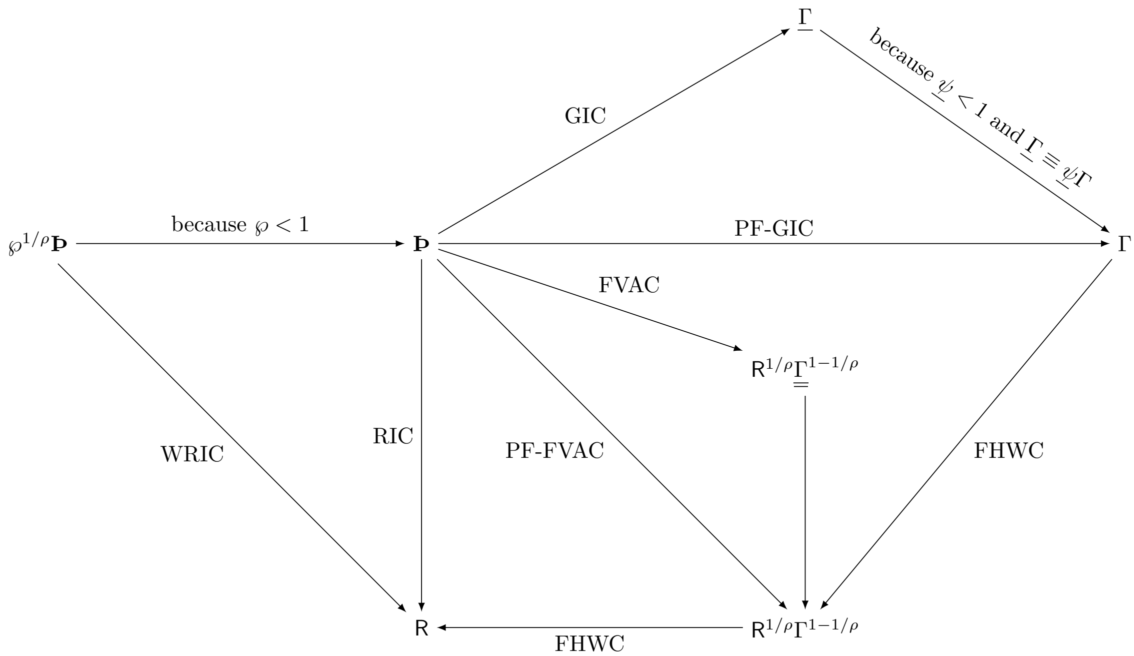


Figure 3 Relation of All Inequality Conditions
See Table 2 for Numerical Values of Nodes Under Baseline Parameters

2.11.1 Discussion of the RIC

In the perfect foresight unconstrained problem (section 2.4.2), the RIC was required for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the RIC is neither necessary nor sufficient for a nondegenerate solution. We thus begin our discussion by asking what features the problem must exhibit (given the FVAC) if the RIC fails (that is, $\mathsf{R} < (\mathsf{R}\beta)^{1/\rho}$):

$$\begin{aligned}
\mathsf{R} &< \overbrace{(\mathsf{R}\beta)^{1/\rho} \;<\; (\mathsf{R}(\Gamma\underline{\underline{\psi}})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
\mathsf{R} &< (\mathsf{R}/\Gamma)^{1/\rho}\Gamma\underline{\underline{\psi}}^{1-1/\rho} \\
\mathsf{R}/\Gamma &< (\mathsf{R}/\Gamma)^{1/\rho}\underline{\underline{\psi}}^{1-1/\rho} \\
(\mathsf{R}/\Gamma)^{1-1/\rho} &< \underline{\underline{\psi}}^{1-1/\rho}
\end{aligned} \tag{42}$$

but since $\underline{\underline{\psi}} < 1$ and $0 < 1 - 1/\rho < 1$ (because we have assumed $\rho > 1$), this inequality can hold only if $\mathsf{R}/\Gamma < 1$; that is, given the FVAC, the RIC can fail only if human wealth is unbounded. Unbounded human wealth is permitted here, as in the perfect foresight liquidity constrained problem. But, from equation (36), an implication of $\cancel{\text{RIC}}$ is that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Thus, interestingly, the presence of uncertainty both permits unlimited human wealth and at the same time prevents that unlimited wealth from resulting in infinite consumption. That is, in the presence of uncertainty, pathological patience (which in the perfect foresight model with finite wealth results in consumption of zero) plus infinite human wealth (which the perfect foresight model prohibits by assumption because it leads to infinite consumption) combine to yield a unique finite MPC for any finite value of m.[30] Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the {PF-GIC,$\cancel{\text{RIC}}$} case (for detailed analysis of this case see appendix **??**). There, too, the tension between infinite human wealth and pathological patience was resolved with a nondegenerate consumption function whose limiting MPC was zero.

2.11.2 The WRIC

The 'weakness' of the additional requirement for contraction, the weak RIC, can be seen by asking 'under what circumstances would the FVAC hold but the WRIC fail?' Algebraically, the requirement is

$$\beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho} < \; 1 \; < (\wp\beta)^{1/\rho}/\mathsf{R}^{1-1/\rho}. \tag{43}$$

If there were no conceivable parameter values that could satisfy both of these inequalities, the WRIC would have no force. And if we require $\mathsf{R} \geq 1$, the WRIC is indeed redundant because now $\beta < 1 < \mathsf{R}^{\rho-1}$, so that the RIC (and WRIC) must hold.

But neither theory nor evidence demands that we assume $\mathsf{R} \geq 1$. We can therefore approach the question of the WRIC's relevance by asking just how low R must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\wp = 0.10$. In that case (43) reduces to

$$\beta < 1 < (0.1\beta/\mathsf{R})^{1/2}$$

[30] **?** prove that the limiting MPC is zero in an even more general case where there is also capital income risk.

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathsf{R} < \beta/10$$

so that for example if $\beta = 0.96$ we would need $\mathsf{R} < 0.096$ (that is, a perpetual riskfree rate of return of worse than -90 percent a year) in order for the WRIC to bind. The relevance of the WRIC is indeed "Weak."

Perhaps the best way of thinking about this is to note that the space of parameter values for which the WRIC is relevant shrinks out of existence as $\wp \to 0$, which section 2.10 showed was the precise limiting condition under which behavior becomes arbitrarily close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\wp = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\wp = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of 'bridge' between the liquidity constrained and the unconstrained problems as $\wp$ moves from 0 to 1.

2.11.3 When the GIC Fails

If both the GIC and the RIC hold, the arguments above establish that as $m \uparrow \infty$ the limiting consumption function asymptotes to the consumption function for the perfect foresight unconstrained function. The more interesting case is where the GIC fails. A solution that satisfies the combination FVAC and ~~GIC~~ is depicted in Figure 4. The consumption function is shown along with the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus that identifies the 'sustainable' level of spending at which m is expected to remain unchanged. The diagram suggests a fact that is confirmed by deeper analysis: Under the depicted configuration of parameter values (see the code for details), the consumption function never reaches the $\mathbb{E}_t[\Delta m_{t+1}] = 0$ locus; indeed, when the RIC holds but the GIC does not, the consumption function's limiting slope $(1 - \text{\TH}/\mathsf{R})$ is shallower than that of the sustainable consumption locus $(1 - \underline{\Gamma}/\mathsf{R})$,[31] so the gap between the two actually increases with m in the limit. Although a nondegenerate consumption function exists, a target level of m does not (or, rather, the target is $m = \infty$), because no matter how wealthy a consumer becomes, the consumer will always spend less than the amount that would keep m stable (in expectation).

We have now established the points of comparison promised above between the perfect foresight solutions and the solutions under uncertainty; these are codified in the remaining parts of Tables 3 and 4.

3 Analysis of the Converged Consumption Function

Figures 5 and 6a,b capture the main properties of the converged consumption rule when the RIC, GIC, and FHWC all hold.[32] Figure 5 shows the expected consumption growth

[31] This is because $\mathbb{E}_t[m_{t+1}] = \mathbb{E}_t[\mathcal{R}_{t+1}(m_t - c_t)] + 1$; solve $m = (m - c)\mathcal{R}\underline{\psi}^{-1} + 1$ for c and differentiate.

[32] These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

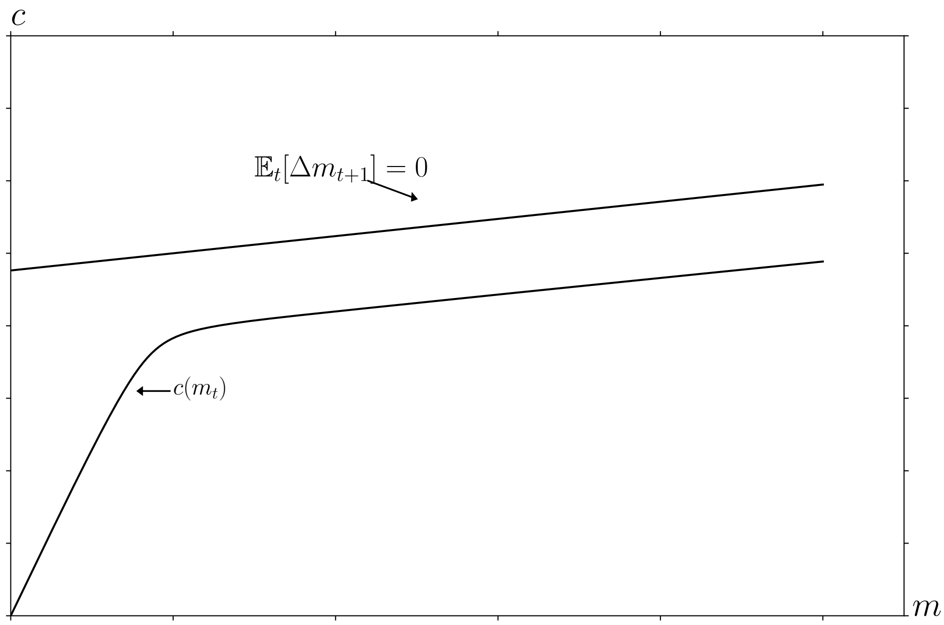


Figure 4 Example Solution when FVAC Holds but GIC Does Not

factor $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ for a consumer behaving according to the converged consumption rule, while Figures 6a,b illustrate theoretical bounds for the consumption function and the marginal propensity to consume.

Five features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$ the expected consumption growth factor goes to Þ, indicated by the lower bound in Figure 5, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \text{Þ}_{\mathsf{R}})$ (Figure 6), the same as the perfect foresight MPC. Second, as $m_t \downarrow 0$ the consumption growth factor approaches ∞ (Figure 5) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}})$ (Figure 6). Third (Figure 5), there is a target cash-on-hand-to-income ratio $\check{m}$ such that if $m_t = \check{m}$ then $\mathbb{E}_t[m_{t+1}] = m_t$, and (as indicated by the arrows of motion on the $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ curve), the model's dynamics are 'stable' around the target in the sense that if $m_t < \check{m}$ then cash-on-hand will rise (in expectation), while if $m_t > \check{m}$, it will fall (in expectation). Fourth (Figure 5), at the target m, the expected rate of growth of consumption is slightly less than the expected growth rate of permanent noncapital income. The final proposition suggested by Figure 5 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t. This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present.[33]

[33]Throughout the remaining analysis I make a final assumption that is not strictly justified by the foregoing. We have seen that the finite-horizon consumption functions $\mathrm{c}_{T-n}(m)$ are twice continuously differentiable and strictly concave, and that they converge to a continuous function $\mathrm{c}(m)$. It does not strictly follow that the limiting function $\mathrm{c}(m)$ is twice continuously differentiable, but I will assume that it is.

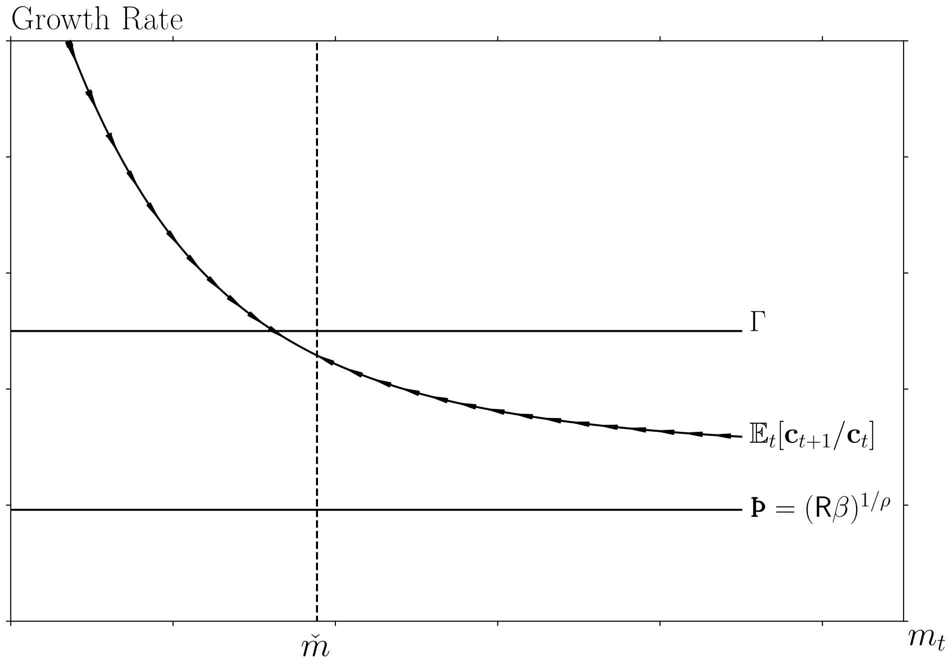


Figure 5 Target m, Expected Consumption Growth, and Permanent Income Growth

3.1 Limits as m approaches Infinity

Define

$$\underline{\mathrm{c}}(m) = \underline{\kappa} m$$

which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} = 0 \ \forall \ n \geq 1$); clearly $\underline{\mathrm{c}}(m) < \mathrm{c}(m)$, since allowing the possibility of future noncapital income cannot reduce current consumption.[34]

Assuming the FHWC holds, the infinite horizon perfect foresight solution (22) constitutes an upper bound on consumption in the presence of uncertainty, since Carroll and Kimball (**?**) show that the introduction of uncertainty strictly decreases the level of consumption at any m.

Thus, we can write

$$\begin{aligned} \underline{\mathrm{c}}(m) <& \mathrm{c}(m) < \bar{\mathrm{c}}(m) \\ 1 <& \mathrm{c}(m)/\underline{\mathrm{c}}(m) < \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m). \end{aligned} \tag{44}$$

[34] We will assume the RIC holds here and subsequently so that $\underline{\kappa} > 0$; the situation is a bit more complex when the RIC does not hold. In that case the bound on consumption is given by the spending that would be undertaken by a consumer who faced binding liquidity constraints. Detailed analysis of this special case is not sufficiently interesting to warrant inclusion in the paper.

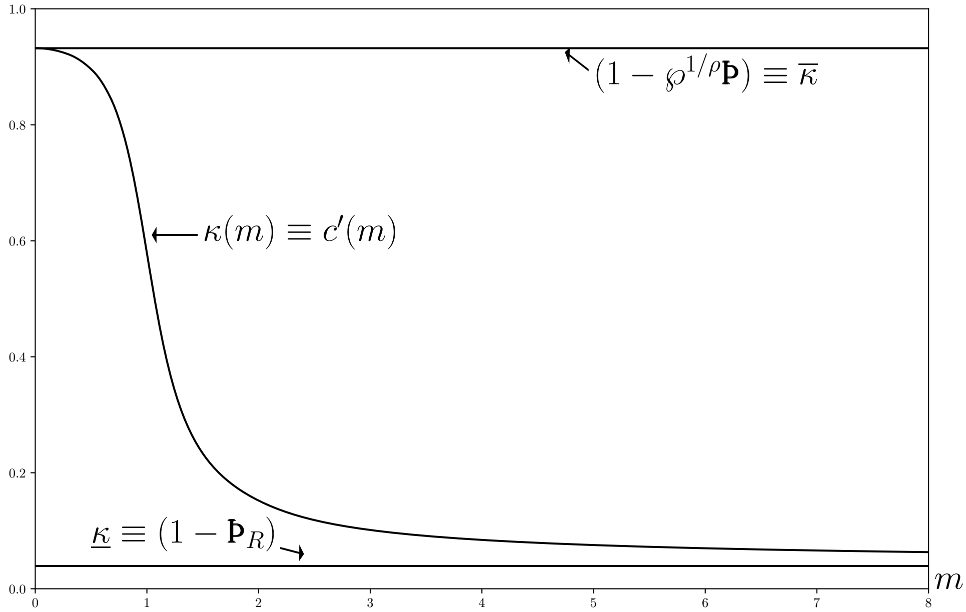


Figure 6 Limiting MPC's

But

$$\begin{aligned}\lim_{m\uparrow\infty} \bar{\mathrm{c}}(m)/\underline{\mathrm{c}}(m) &= \lim_{m\uparrow\infty} (m-1+h)/m \\ &= 1,\end{aligned}$$

so as $m \uparrow \infty, \mathrm{c}(m)/\underline{\mathrm{c}}(m) \to 1$, and the continuous differentiability and strict concavity of $\mathrm{c}(m)$ therefore implies

$$\lim_{m\uparrow\infty} \mathrm{c}'(m) = \underline{\mathrm{c}}'(m) = \bar{\mathrm{c}}'(m) = \underline{\kappa}$$

because any other fixed limit would eventually lead to a level of consumption either exceeding $\bar{\mathrm{c}}(m)$ or lower than $\underline{\mathrm{c}}(m)$.

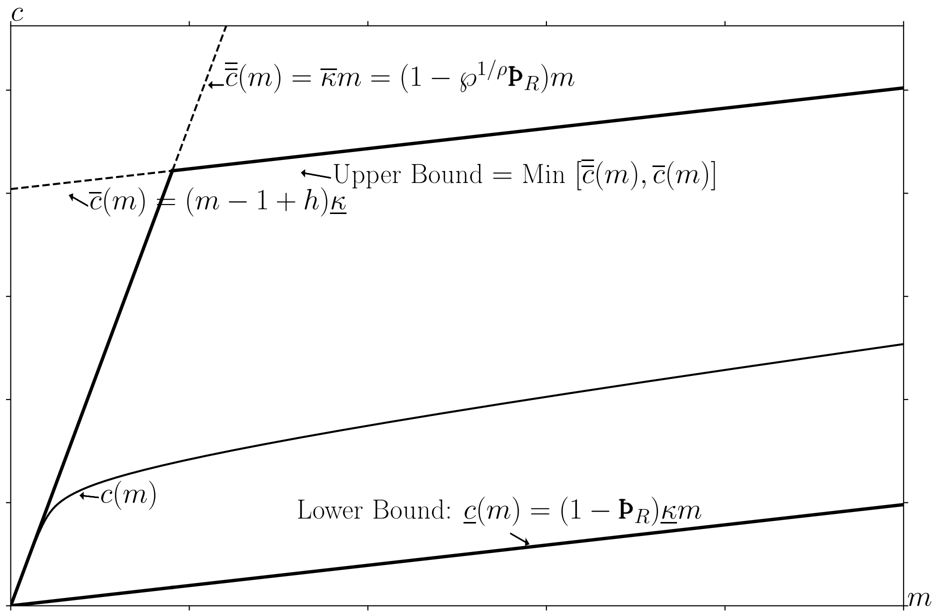
Figure 6 confirms these limits visually. The top plot shows the converged consumption function along with its upper and lower bounds, while the lower plot shows the marginal propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

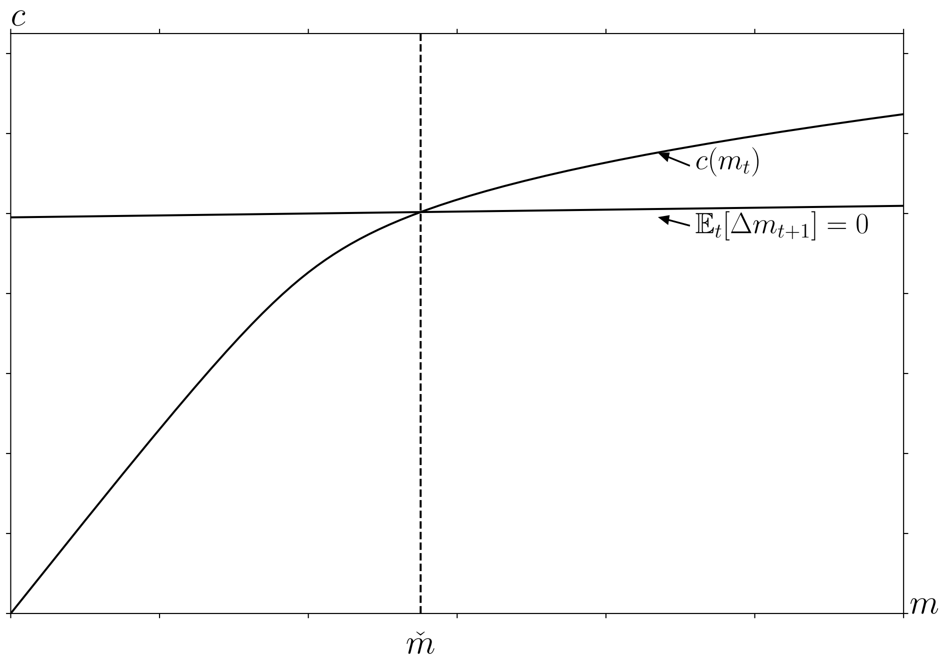
$$\lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t\uparrow\infty} \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t].$$

But

$$\mathbb{E}_t[\Gamma_{t+1}\underline{c}_{t+1}/\bar{c}_t] \le \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \le \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$



(a) Bounds



(b) Target m

Figure 7 The Consumption Function

and

$$\lim_{m_t\uparrow\infty} \Gamma_{t+1}\underline{c}(m_{t+1})/\bar{c}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}\bar{c}(m_{t+1})/\underline{c}(m_t) = \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while (for convenience defining $\mathrm{a}(m) = m_t - \mathrm{c}(m_t)$),

$$\begin{aligned} \lim_{m_t\uparrow\infty} \Gamma_{t+1}m_{t+1}/m_t &= \lim_{m_t\uparrow\infty} \left(\frac{\mathsf{R}\mathrm{a}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t}\right) \\ &= (\mathsf{R}\beta)^{1/\rho} = \text{Þ} \end{aligned} \tag{45}$$

because $\lim_{m_t\uparrow\infty} \mathrm{a}'(m) = \text{Þ}_{\mathsf{R}}$[35] and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1-\wp))/m_t$ which goes to zero as m_t goes to infinity.

Hence we have

$$\text{Þ} \leq \lim_{m_t\uparrow\infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \text{Þ}$$

so as cash goes to infinity, consumption growth approaches its value Þ in the perfect foresight model.

3.2 Limits as m Approaches Zero

Now consider the limits of behavior as m_t gets arbitrarily small.

Equation (37) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathsf{R}^{-1}(\wp\mathsf{R}\beta)^{1/\rho}.$$

Defining $\mathrm{e}(m) = \mathrm{c}(m)/m$ as before we have

$$\lim_{m\downarrow 0} \mathrm{e}(m) = (1 - \wp^{1/\rho}\text{Þ}_{\mathsf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m\downarrow 0} \mathrm{c}'(m) = \lim_{m\downarrow 0} \mathrm{e}(m) = \bar{\kappa}.$$

Figure 6 confirms that the numerical solution method obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned} \lim_{m_t\downarrow 0} \mathbb{E}_t\left[\left(\frac{\mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)}\right)\Gamma_{t+1}\right] &> \lim_{m_t\downarrow 0} \mathbb{E}_t\left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})}{\bar{\kappa}m_t}\right)\Gamma_{t+1}\right] \\ &= \wp \lim_{m_t\downarrow 0} \mathbb{E}_t\left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa}m_t}\right)\Gamma_{t+1}\right] \\ &\quad + (1-\wp) \lim_{m_t\downarrow 0} \mathbb{E}_t\left[\left(\frac{\underline{\mathrm{c}}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \theta_{t+1}/(1-\wp))}{\bar{\kappa}m_t}\right)\Gamma_{t+1}\right] \\ &> (1-\wp) \lim_{m_t\downarrow 0} \mathbb{E}_t\left[\left(\frac{\underline{\mathrm{c}}(\theta_{t+1}/(1-\wp))}{\bar{\kappa}m_t}\right)\Gamma_{t+1}\right] \end{aligned}$$

[35] This is because $\lim_{m_t\uparrow\infty} \mathrm{a}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathrm{c}(m_t)/m_t = 1 - \lim_{m_t\uparrow\infty} \mathrm{c}'(m_t) = \text{Þ}_{\mathsf{R}}$.

$$= \infty$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.

3.3 There Exists Exactly One Target Cash-on-Hand Ratio, which is Stable

We now prove the existence of a target cash-on-hand-to-income ratio $\check{m}$ towards which an agent's m is expected to move. (The $\vee$ accent invokes the fact that this is the value that other m's 'point to.')

Theorem 2. *For the problem defined in section 2.1, if the GIC (32), and WRIC (38) hold, there exists a unique cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \tag{46}$$

Moreover, $\check{m}$ is stable in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}),\ \mathbb{E}_t[m_{t+1}] > m_t \\ \forall m_t \in (\check{m}, \infty),\ \mathbb{E}_t[m_{t+1}] < m_t. \end{aligned} \tag{47}$$

The elements of the proof are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

3.3.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the conditions (the WRIC and FVAC) that theorem 1 establishes are sufficient for its existence. (Indeed, Appendix **??** shows that $\mathrm{c}(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all t, $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and $\mathcal{R}_t$ are strictly positive under our foregoing assumptions. With $m_t > 0$, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

3.3.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

The logic in section 3.2 showing that $\lim_{m_t \downarrow 0} \mathbb{E}_t[c_{t+1}/c_t] = \infty$ transparently implies the same proposition for m_t: $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$ so the ratio is unbounded.

The limit as m_t goes to infinity is

$$\lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] = \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1}}{m_t} \right]$$

$$
\begin{aligned}
&= \mathbb{E}_t[(\mathsf{R}/\Gamma_{t+1})\text{Þ}_{\mathsf{R}}] \\
&= \mathbb{E}_t[\text{Þ}/\Gamma_{t+1}] \\
&< 1
\end{aligned}
\tag{48}
$$

where the last two lines are merely a restatement of the GIC (32).

The Intermediate Value Theorem tells us that if $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

3.3.3 $\mathbb{E}_t[m_{t+1}] - m_t$ *is monotonically decreasing.*

Now define $\boldsymbol{\zeta}(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$
\begin{aligned}
\boldsymbol{\zeta}(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\
\boldsymbol{\zeta}(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\
\boldsymbol{\zeta}(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,
\end{aligned}
\tag{49}
$$

so that $\boldsymbol{\zeta}(\check{m}) = 0$. Our goal is to prove that $\boldsymbol{\zeta}(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$
\begin{aligned}
\boldsymbol{\zeta}'(m_t) \equiv \left(\frac{d}{dm_t}\right)\boldsymbol{\zeta}(m_t) &= \mathbb{E}_t\left[\left(\frac{d}{dm_t}\right)(\mathcal{R}_{t+1}(m_t - \mathrm{c}(m_t)) + \xi_{t+1} - m_t)\right] \\
&= \bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1.
\end{aligned}
\tag{50}
$$

Note that the statement of theorem 2 did not require the RIC to hold. Now, we show that (given our other assumptions) $\boldsymbol{\zeta}'(m)$ is decreasing (but for different reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (21) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < \mathrm{c}'(m_t) < 1$ so that

$$
\begin{aligned}
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \text{Þ}_{\mathsf{R}})}_{\underline{\kappa}}) - 1 \\
&= \bar{\mathcal{R}}\text{Þ}_{\mathsf{R}} - 1 \\
&= \mathbb{E}_t\left[\frac{\mathsf{R}}{\Gamma\psi}\frac{\text{Þ}}{\mathsf{R}}\right] - 1 \\
&= \underbrace{\mathbb{E}_t\left[\frac{\text{Þ}}{\Gamma\psi}\right]}_{=\text{Þ}_{\underline{\Gamma}}} - 1
\end{aligned}
$$

which is negative because the GIC says $\text{Þ}_{\underline{\Gamma}} < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m\uparrow\infty} \mathrm{c}'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$
\bar{\mathcal{R}}\,(1 - \mathrm{c}'(m_t)) < \bar{\mathcal{R}}
$$

which means that $\boldsymbol{\zeta}'(m_t)$ from (50) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathsf{R}}{\Gamma \psi} \right] < 1 \tag{51}$$

But the combination of the GIC holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\text{Þ}}{\Gamma \psi} \right]}^{\text{Þ}_{\underline{\Gamma}}} < 1 < \overbrace{\frac{\text{Þ}}{\mathsf{R}}}^{\text{Þ}_{\mathsf{R}}},$$

and multiplying all three elements by $\mathsf{R}/\text{Þ}$ gives

$$\mathbb{E}_t \left[\frac{\mathsf{R}}{\Gamma \psi} \right] < \mathsf{R}/\text{Þ} < 1$$

which satisfies our requirement in (51).

The foregoing arguments rely on the continuous differentiability of $\mathrm{c}(m)$, so they do not directly go through for a problem in which the existence of liquidity constraints can lead to discrete changes in the slope $\mathrm{c}'(m)$ at particular values of m. But we can use the fact that the constrained model is the limit of the baseline model as $\wp \downarrow 0$ to conclude that there is likely a unique target cash level even in the constrained model.

If consumers are sufficiently impatient, the limiting target level in the constrained model will be $\check{m} = \mathbb{E}_t[\xi_{t+1}] = 1$. That is, if a consumer starting with $m = 1$ will save nothing, $\mathrm{a}(1) = 0$, then the target level of m in the constrained model will be 1; if a consumer with $m = 1$ would choose to save something, then the target level of m will be greater than the expected level of income.

3.4 Expected Consumption Growth at Target m Is Less than Expected Permanent Income Growth

In Figure 5 the intersection of the target cash-on-hand ratio locus at $\check{m}$ with the expected consumption growth curve lies below the intersection with the horizontal line representing the growth rate of expected permanent income. This can be proven as follows.

Strict concavity of the consumption function implies that if $\mathbb{E}_t[m_{t+1}] = \check{m} = m_t$ then

$$\begin{aligned}
\mathbb{E}_t \left[\frac{\Gamma_{t+1} \mathrm{c}(m_{t+1})}{\mathrm{c}(m_t)} \right] &< \mathbb{E}_t \left[\left(\frac{\Gamma_{t+1}(\mathrm{c}(\check{m}) + \mathrm{c}'(\check{m})(m_{t+1} - \check{m}))}{\mathrm{c}(\check{m})} \right) \right] \\
&= \mathbb{E}_t \left[\Gamma_{t+1} \left(1 + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})} \right) (m_{t+1} - \check{m}) \right) \right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})} \right) \mathbb{E}_t \left[\Gamma_{t+1} \left(m_{t+1} - \check{m} \right) \right] \\
&= \Gamma + \left(\frac{\mathrm{c}'(\check{m})}{\mathrm{c}(\check{m})} \right) \left[\mathbb{E}_t[\Gamma_{t+1}] \underbrace{\mathbb{E}_t[m_{t+1} - \check{m}]}_{=0} + \mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) \right]
\end{aligned} \tag{52}$$

and since $m_{t+1} = (\mathsf{R}/\Gamma_{t+1})\mathrm{a}(\check{m}) + \xi_{t+1}$ and $\mathrm{a}(\check{m}) > 0$ it is clear that $\mathrm{cov}_t(\Gamma_{t+1}, m_{t+1}) < 0$ which implies that the entire term added to Γ in (52) is negative, as required.

3.5 Is Expected Consumption Growth Is a Declining Function of m_t?

Figure 5 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\boldsymbol{\Upsilon}(m_t) \equiv \Gamma_{t+1}\mathrm{c}(\mathcal{R}_{t+1}\mathrm{a}(m_t) + \xi_{t+1})/\mathrm{c}(m_t) = \mathbf{c}_{t+1}/\mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t)\,\mathbb{E}_t[\underbrace{\boldsymbol{\Upsilon}(m_t)}_{\equiv \boldsymbol{\Upsilon}_{t+1}}] < 0$$

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t\left[\Gamma_{t+1}\left(\frac{\mathrm{c}'(m_{t+1})\mathcal{R}_{t+1}\mathrm{a}'(m_t)\mathrm{c}(m_t) - \mathrm{c}(m_{t+1})\mathrm{c}'(m_t)}{\mathrm{c}(m_t)^2}\right)\right] < 0. \tag{53}$$

Appendix **??** shows that the proposition holds true if there are only transitory (and no permanent) shocks. But in the presence of permanent shocks, the software archive associated with this paper presents an example in which this perverse effect dominates. However, extreme assumptions were required (in particular, a very small probability of the zero-income shock) and the region in which $\boldsymbol{\Upsilon}'_{t+1} > 0$ is tiny. In practice, for plausible parametric choices, $\mathbb{E}_t[\boldsymbol{\Upsilon}'_{t+1}] < 0$ should generally hold.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

This section examines the behavior of large collections of buffer-stock consumers with identical parameter values. Such a collection can be thought of as either a subset of the population within a single country (say, members of a given education or occupation group), or as the whole population in a small open economy.[36]

Until now, for simplicity, we have assumed infinite horizons; and we will continue to omit mortality in section 4.1 because its incorporation does not meaningfully modify any derivations. But a reason for introducing mortality will appear at the end of section 4.2, so the implications of alternative assumptions about mortality are briefly examined in Section 4.3.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above. Szeidl (**?**) proves that such a population will

[36] We will continue to take the aggregate interest rate as exogenous and constant. It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

be characterized by an invariant distribution of m that induces invariant distributions for c and a; designate these $\mathcal{F}^m$, $\mathcal{F}^a$, and $\mathcal{F}^c$.[37]

4.1 Consumption and Income Growth at the Household Level

The operator $\mathbb{M}[\bullet]$ yields the mean value of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption, and would find

$$\begin{aligned}
\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1}\mathbf{p}_{t+1} - \log c_t \mathbf{p}_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\
&= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\
&= (\gamma - \sigma_\psi^2/2)
\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of $\mathcal{F}^c$ (?) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$.

Thus, in a population that has reached its invariant distribution, the growth rate of idiosyncratic log consumption matches the growth rate of idiosyncratic log permanent income.

4.2 Balanced Growth of Aggregate Income, Consumption, and Wealth

Attanasio and Weber (?) point out that concavity of the consumption function (or other nonlinearities) can imply that it is quantitatively important to distinguish between the growth rate of average consumption and the average growth rate of consumption.[38] We have just examined the average growth rate; we now examine the growth rate of the average.

Using boldface capital letters for aggregate variables, the growth factor for aggregate income is given by:

$$\begin{aligned}
\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t] / \mathbb{M}[\mathbf{p}_t \xi_t] \\
&= \Gamma
\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ.

From the perspective of period t, current aggregate assets are nonstochastic, $\mathbf{A}_t = \mathbb{M}[a_t \mathbf{p}_t]$, but next period's assets are stochastic,

$$\begin{aligned}
\mathbf{A}_{t+1} &= \mathbb{M}[a_{t+1}\mathbf{p}_{t+1}] \\
&= \Gamma\mathbb{M}[(a_t + (a_{t+1} - a_t))\mathbf{p}_t \psi_{t+1}]
\end{aligned}$$

[37] Szeidl's proof supplants simulation evidence of ergodicity that appeared in an earlier version of this paper.

[38] Since we assume number of the households are normalized to 1, aggregate and average variables are identical.

$$= \Gamma\left(\underbrace{\mathbb{M}[a_t\mathbf{p}_t\psi_{t+1}]}_{=\mathbf{A}_t} + \underbrace{\mathbb{M}[(a_{t+1}-a_t)]}_{=0\ \textbf{(?)}}\mathbb{M}[\mathbf{p}_t\psi_{t+1}] + \text{cov}_t(a_{t+1}-a_t, \mathbf{p}_t\psi_{t+1})\right)$$

$$\mathbf{A}_{t+1}/\mathbf{A}_t = \Gamma\left(1 + \frac{\text{cov}(a_{t+1}, \mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right)$$

Unfortunately, it is clear that the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1}.

Matters are simpler if there are no permanent shocks; see Appendix **??** for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

One way of thinking about this problem is that it reflects the fact that, under our assumptions, the $\boldsymbol{p}$ variable does not have an ergodic distribution; the distribution of permanent income becomes forever wider and wider over time, because our consumers never die and each immortal person is subject to symmetric shocks to their $\log \mathbf{p}$.

4.3 Mortality and Redistribution

Most heterogeneous agent models incorporate a constant positive probability of death, following **?**. **?** show that for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, the limiting distribution of permanent income has a finite variance. In such cases, numerical results confirm the intuition that, under appropriate impatience conditions, the growth rate of aggregate assets ends up matching the growth rate of permanent income (though a formal proof has been elusive).

But the assumption of finite lifetimes requires us to specify what happens to the assets of the dying consumers.

4.3.1 Blanchard Lives

?'s solution is an annuitization scheme in which estates of the dying are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, the most notable of which is that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor: If the probability of remaining alive is $\aleph$, then assuming that no utility accrues after death makes the effective discount factor $\hat{\beta} = \beta\aleph$; but the enhancement to the rate of return from the annuity scheme yields an effective interest rate of $\hat{\mathsf{R}}/\aleph$. Combining these, the effective patience factor in the new economy $\hat{\text{Þ}}$ is unchanged from its value in the infinite horizon model:

$$\hat{\text{Þ}} \equiv (\beta\aleph\mathsf{R}/\aleph)^{1/\rho} = (\mathsf{R}\beta)^{1/\rho} \equiv \text{Þ}. \tag{54}$$

The only adjustments this requires to the analysis from prior parts of this paper are therefore to the few elements that involve a role for R distinct from its contribution to Þ (principally, the RIC).

4.3.2 Modigliani Lives

?'s innovation was useful not only for the insight it provided but also because the prevailing alternative, the Life Cycle model of **?**, was difficult to use with the then-available computational technologies. Leaving aside its (considerable) conceptual value, there is no need for Blanchard's analytical solution today, when all quantitatively serious modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway. Computational models can can easily handle assumptions more realistic than Blanchard's about the disposition of assets at death.

The simplest such models are those that follow Modigliani in assuming there is no bequest motive; any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, **??**).

Some bequest wealth may be absorbed by an estate tax; modelers have made a variety of assumptions about how any post-estate-tax residue is distributed. We again consider the simplest choice, because it also represents something of a polar alternative to Blanchard: Without a bequest motive, there are no behavioral effects of a 100 percent estate tax, and we assume that the revenues are used to fund marginal government expenditures that yield utility in a form that is separable from utility from personal consumption – say, for public goods (or, equivalently, the resources are thrown in the ocean). In that case, the estate-related wealth effectively vanishes from the economy.

This approach alters the conditions under which the economy has a target wealth-to-income ratio. Effectively, the return on aggregate wealth is lower than the contingent-on-survival return on wealth at the individual level. The condition under which an aggregate target wealth-to-income ratio exists is then obtained simply by multiplying the Growth Impatience Condition by the probability $\aleph$ that the surivor's assets will still exist:

$$\aleph\text{Þ}_{\Gamma} < 1. \tag{55}$$

Intuitively, the condition required to prohibit unbounded growth in the aggregate wealth-to-income ratio is the condition that prevents the wealth-to-income ratio of individual consumers from growing faster than the rate at which mortality diminishes their collective wealth-to-income ratio.

Section 2.11.3 showed that the individual's problem can have a nondegenerate consumption rule for consumers who fail to satisfy the individual version of the GIC. The GIC-Agg therefore provides a bound on preferences which can accommodate a population in which individual consumers have no upper bound on target wealth, but the aggregate economy will nevertheless settle down to an equilibrium aggregate wealth-to-income ratio. Further analysis of these matters is beyond the scope of this paper, but the above-mentioned work of **?** presents an example of the application of this point (and the associated toolkit reports the results not only of tests of the individual but also the aggregate versions of the GIC).

5 Conclusions

This paper provides theoretical foundations for many characteristics of buffer stock saving models that have heretofore been observed in numerical solutions but not proven. Perhaps the most important such proposition is the existence of a target cash-to-permanent-income ratio toward which actual resources will move. The intuition provided by the existence of such a target can be a powerful aid to understanding a host of numerical results, including the existence of a 'balanced growth' equilibrium in small open economies populated by buffer stock consumers.

Another contribution is integration of the paper's results with the open-source Econ-ARK toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHWC)	
$\Gamma/\mathsf{R} < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.	$\Gamma/\mathsf{R} < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\mathbf{Þ} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $c_{t+1} < c_t$	$\mathbf{Þ} < 1$ *If wealth is large enough,* the *expectation* of consumption next period will be smaller than this period's consumption: $\lim_{m_t \to \infty} \mathbb{E}_t[c_{t+1}] < c_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\mathbf{Þ}/\mathsf{R} < 1$ The growth factor for consumption $\mathbf{Þ}$ must be smaller than the discounting factor R, so that the PDV of current and future consumption will be finite: $\mathrm{c}'(m) = 1 - \mathbf{Þ}/\mathsf{R} < 1$	$\wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $\mathrm{c}'(m) < 1 - \wp^{1/\rho}\mathbf{Þ}/\mathsf{R} < 1$
Growth Impatience Conditions	
PF-GIC	GIC
$\mathbf{Þ}/\Gamma < 1$ Guarantees that for an unconstrained consumer, the ratio of consumption to permanent income will fall over time. For a constrained consumer, guarantees the constraint will eventually be binding.	$\mathbf{Þ}\,\mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality, stronger than the PF-GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \to \infty} \mathbb{E}_t[m_{t+1}/m_t] = \mathbf{Þ}_{\underline{\Gamma}}$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta\Gamma^{1-\rho} < 1$ equivalently $\mathbf{Þ}/\Gamma < (\mathsf{R}/\Gamma)^{1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta\Gamma^{1-\rho}\,\mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ, $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Model	Conditions	Comments/Logic
PF Unconstrained Section 2.4.2	RIC, FHWC[°]	RIC $\Rightarrow \lvert \mathrm{v}(m) \rvert < \infty$; FHWC $\Rightarrow 0 < \lvert \mathrm{v}(m) \rvert$ RIC prevents $\bar{\mathrm{c}}(m) = 0$ FHWC prevents $\bar{\mathrm{c}}(m) = \infty$
PF Constrained Section 2.4.3 Appendix ??	~~PF-GIC~~, RIC	FHWC must hold ($\Gamma < \text{Þ} < \mathsf{R} \Rightarrow \Gamma < \mathsf{R}$) Identical to solution to PF Unconstrained for $m > m_{\#}$ for some $0 < m_{\#} < 1$; $\mathrm{c}(m) = m$ for $m \leq m_{\#}$ (~~RIC~~ would yield $m_{\#} = 0$ so degenerate $\mathrm{c}(m) = 0$)
	PF-GIC,RIC	$\lim_{m \to \infty} \mathring{c}(m) = \bar{c}(m), \lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$ kinks at points where horizon to $b = 0$ changes*
	PF-GIC,~~RIC~~	$\lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = 0$ kinks at points where horizon to $b = 0$ changes*
Buffer Stock Model Section 2.5	FVAC, WRIC	FHWC $\Rightarrow \lim_{m \to \infty} \mathring{c}(m) = \bar{c}(m), \lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$ ~~FHWC~~+RIC $\Rightarrow \lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = \underline{\kappa}$ ~~FHWC~~+~~RIC~~ $\Rightarrow \lim_{m \to \infty} \mathring{\boldsymbol{\kappa}}(m) = 0$ GIC guarantees finite target wealth ratio FVAC is stronger than PF-FVAC WRIC is weaker than RIC

‡For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines the unique value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < \mathrm{v}(m) < 0$. °RIC, FHWC are necessary as well as sufficient. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc.