

Theoretical Foundations of Buffer Stock Saving

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Abstract

This paper builds foundations for rigorous and intuitive understanding of ‘buffer stock’ saving models (Bewley (1977)-like models that have a wealth target), pairing each theoretical result with quantitative illustrations. After describing conditions under which a consumption function exists, the paper articulates stricter ‘Growth Impatience’ conditions that guarantee alternative forms of stability – either at the population level, or for individual consumers. Together, the numerical tools and analytical results constitute a comprehensive toolkit for understanding buffer stock models.

Keywords Precautionary saving, buffer stock saving, marginal propensity to consume, permanent income hypothesis, income fluctuation problem

JEL codes D81, D91, E21

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A dashboard allows users to see the consequences of alternative parametric choices in a live interactive framework; a corresponding Jupyter Notebook uses the Econ-ARK/HARK toolkit to produce all of the paper’s figures (warning: the notebook may take several minutes to launch).

All figures and numerical results can be automatically reproduced using the Econ-ARK/HARK toolkit, which can be cited per our references (Carroll, Kaufman, Kazil, Palmer, and White (2018)); for reference to the toolkit itself see Acknowledging Econ-ARK. Thanks to the Consumer Financial Protection Bureau for funding the original creation of the Econ-ARK toolkit; and to the Sloan Foundation for funding Econ-ARK’s extensive further development that brought it to the point where it could be used for this project. The toolkit can be cited with its digital object identifier, 10.5281/zenodo.1001067, as is done in the paper’s own references as Carroll, Kaufman, Kazil, Palmer, and White (2018). Thanks to Will Du, James Feigenbaum, Joseph Kaboski, Miles Kimball, Qingyin Ma, Misuzu Otsuka, Damiano Sandri, John Stachurski, Adam Szeidl, Alexis Akira Toda, Metin Uyanik, Mateo Velásquez-Giraldo, Weifeng Wu, Jiaxiong Yao, and Xudong Zheng for comments on earlier versions of this paper, John Boyd for help in applying his weighted contraction mapping theorem, Ryoji Hiraguchi for extraordinary mathematical insight that improved the paper greatly, David Zervos for early guidance to the literature, and participants in a seminar at the Johns Hopkins University, a presentation at the 2009 meetings of the Society of Economic Dynamics for their insights, and at a presentation at the Australian National University.

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1 Introduction

In the presence of realistic transitory and permanent shocks to income *a la* Friedman (1957) and Muth (1960), only one further ingredient is required to construct a microeconomically testable model of consumption: A description of preferences. Zeldes (1989) was the first to calibrate a quantitatively plausible example; his paper spawned a literature showing that such models' predictions can match household life cycle data reasonably well, whether or not explicit liquidity constraints are imposed.¹

A connected literature in macroeconomic theory, starting with Bewley (1977), has derived limiting properties of related infinite-horizon problems, but only in models more complex than the case with just shocks and preferences. The extra complexity has been imposed because standard contraction mapping theorems (beginning with Bellman (1957) and including those building on Stokey et. al. (1989)) cannot be applied when utility and/or marginal utility are unbounded. Many proof methods also rule out permanent shocks *a la* Friedman (1957), Muth (1960), and Zeldes (1989).²

This paper's first technical contribution is to articulate conditions under which the infinite-horizon version of the original Friedman-Muth-Zeldes problem (without complications like a consumption floor or liquidity constraints) defines a contraction mapping whose limiting solution is nondegenerate as the horizon approaches infinity. This requires generalization of a condition in Ma, Stachurski, and Toda (2020), which we call the '**Finite Value of Autarky Condition**' (the other necessary condition, the '**Weak Return Impatience Condition**' is unlikely to bind). Because the proof constructs the infinite horizon solution as the limit of finite-horizon recursions, many intermediate results are also useful for the solution of the finite horizon problem.

But the paper's main theoretical contribution is to identify conditions under which 'stable' values of the wealth-to-permanent-income ratio exist in the infinite horizon case, either for individual consumers (a consumer's wealth can be predicted to move toward a 'target' ratio) or for the aggregate (the economy as a whole moves toward a 'balanced growth' equilibrium with a constant ratio of aggregate wealth to income). The requirement for stability is always that the model's parameters satisfy a 'Growth Impatience Condition,' whose details depend on the quantity whose stability is of interest. A model that exhibits stability of either kind qualifies as a 'buffer stock' model.³

Even without a formal proof of its existence, buffer stock saving has been intuitively understood to underlie central quantitative results in heterogeneous agent macroeconomics; for example, the logic of target saving is central to the claim by Krueger, Mitman, and Perri (2016) in the *Handbook of Macroeconomics* that such models explain why,

¹See Carroll (1997) or Gourinchas and Parker (2002) for arguments that models with only 'natural' constraints (see below) match a wide variety of facts; for a model with explicit constraints that produces very similar results, see, e.g., Cagetti (2003).

²See the fuller discussion at the end of section 2.1.

³Such models are neither a subset nor a superset of Bewley (1977) models. But closed economies in which capital results from saving and has declining marginal productivity are always 'buffer stock' economies under some definition of that term, because capital accumulation causes interest rates to fall, which guarantees that a Growth Impatience Condition will hold in equilibrium (see below). The more interesting applications are to populations (or economies) whose marginal saving behavior does not determine the relevant interest rate, or in which the marginal product of capital does not fall as capital is accumulated (again, see below).

during the Great Recession, middle-class consumers cut their spending more than the poor or the rich. The theory below provides the rigorous basis for this claim: Learning that the future has become more uncertain does not change the urgent imperatives of the poor (their high $u'(c)$ means they – optimally – have little room to maneuver). And, increased labor income uncertainty does not much change the behavior of the rich because it poses little risk to their consumption. Only people in the middle have both the motivation and the wiggle-room to respond by reducing their spending.

Analytical derivations for the proofs also provide foundations for many other results familiar from the numerical literature.

The paper’s first part begins by defining sufficient conditions for the problem to define a useful (nondegenerate) limiting consumption function (and explains how the model relates to those previously considered in the literature). The conditions are interestingly parallel to those required for the **liquidity constrained perfect foresight model**; that parallel is explored and explained. This analysis establishes limiting properties of the consumption function as resources approach infinity, and as they approach their lower bound; using these limits, the contraction mapping theorem is proven.

The next theoretical contribution is to show that a corresponding model with an ‘artificial’ liquidity constraint (that is, a model that prohibits borrowing even by consumers who could certainly repay) is a particular limiting case of the model without constraints. The analytical appeal of the unconstrained model is that it is both mathematically convenient (e.g., the consumption function is twice continuously differentiable), and arbitrarily close (cf. section 2.10) to less tractable models that have elsewhere been tackled with less convenient methods. For future authors, the approach here models a strategy of proving interesting propositions in a congenial environment, and then appealing to a limiting argument to establish the analogous proposition in an explicitly constrained but formally more unwieldy environment.

In proving the remaining theorems, the **next section** examines the key properties of the model. First, as **cash approaches infinity** the expected growth rate of consumption and the marginal propensity to consume (MPC) converge to their values in the perfect foresight case. Second, as **cash approaches zero** the expected growth rate of consumption approaches infinity, and the MPC approaches a simple analytical limit. Next, the central theorems articulate conditions under which different measures of ‘growth impatience’ imply useful conclusions about points of stability (‘target’ or ‘balanced growth’ points).

The final section elaborates the conditions under which, even with a fixed aggregate interest rate that differs from the time preference rate, a small open economy populated by buffer stock consumers has a balanced growth equilibrium in which growth rates of consumption, income, and wealth match the exogenous growth rate of permanent income (equivalent, here, to productivity growth). In the terms of Schmitt-Grohé and Uribe (2003), buffer stock saving is an appealing method of ‘closing’ a small open economy model, because it requires no ad-hoc assumptions. Not even liquidity constraints.⁴

⁴The paper’s insights are instantiated in the **Econ-ARK** toolkit, whose **buffer stock saving module** flags parametric choices under which a problem is degenerate or under which stable ratios of wealth to income may not exist.

2 The Problem

2.1 Setup

The infinite horizon solution is the (limiting) first-period solution to a sequence of finite-horizon problems as the horizon (the last period of life) becomes arbitrarily distant.

That is, for the value function, fixing a terminal date T , we are interested in the term \mathbf{v}_{T-n} in the sequence of value functions $\{\mathbf{v}_T, \mathbf{v}_{T-1}, \dots, \mathbf{v}_{T-n}\}$. We will say that the problem has a ‘nondegenerate’ infinite horizon solution if, corresponding to that value function, as $n \uparrow \infty$ there is a limiting consumption function $c(m) = \lim_{n \uparrow \infty} c_{T-n}$ which is neither $c(m) = 0$ everywhere (for all m) nor $c(m) = \infty$ everywhere.

Concretely, a consumer born n periods before date T solves the problem

$$\mathbf{v}_{T-n} = \max \mathbb{E}_t \left[\sum_{i=0}^n \beta^i u(\mathbf{c}_{t+i}) \right]$$

where the Constant Relative Risk Aversion (CRRA) utility function

$$u(\bullet) = \bullet^{1-\rho} / (1-\rho) \quad (1)$$

exhibits relative risk aversion $\rho > 1$.⁵ The consumer’s initial condition is defined by market resources \mathbf{m}_t and permanent noncapital income \mathbf{p}_t , which both are positive,

$$\{\mathbf{p}_t, \mathbf{m}_t\} \in (0, \infty), \quad (2)$$

and the consumer cannot die in debt,

$$\mathbf{c}_T \leq \mathbf{m}_T. \quad (3)$$

In the usual treatment, a dynamic budget constraint (DBC) incorporates several elements that jointly determine next period’s \mathbf{m} (given this period’s choices); for the detailed analysis here, it will be useful to disarticulate the steps:

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{b}_{t+1} &= \mathbf{a}_t R \\ \mathbf{p}_{t+1} &= \mathbf{p}_t \underbrace{\Gamma^{\psi_{t+1}}}_{\equiv \Gamma_{t+1}} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{p}_{t+1} \xi_{t+1}, \end{aligned} \quad (4)$$

where \mathbf{a}_t indicates the consumer’s assets at the end of period t , which grow by a fixed interest factor $R = (1 + r)$ between periods, so that \mathbf{b}_{t+1} is the consumer’s financial (‘bank’) balances before next period’s consumption choice;⁶ \mathbf{m}_{t+1} (‘market resources’) is the sum of financial wealth \mathbf{b}_{t+1} and noncapital income $\mathbf{p}_{t+1} \xi_{t+1}$ (permanent noncapital income \mathbf{p}_{t+1} multiplied by a mean-one iid transitory income shock factor ξ_{t+1} ; transitory

⁵The main results also hold for logarithmic utility which is the limit as $\rho \rightarrow 1$ but incorporating the logarithmic special case in the proofs is omitted because it would be cumbersome.

⁶Allowing a stochastic interest factor is straightforward but adds little insight for our purposes; however, see Benhabib, Bisin, and Zhu (2015), Ma and Toda (2020), and Ma, Stachurski, and Toda (2020) for the implications of capital income risk for the distribution of wealth and other interesting questions not considered here.

shocks are assumed to satisfy $\mathbb{E}_t[\xi_{t+n}] = 1 \forall n \geq 1$). Permanent noncapital income in $t + 1$ is equal to its previous value, multiplied by a growth factor Γ , modified by a mean-one iid shock ψ_{t+1} , $\mathbb{E}_t[\psi_{t+n}] = 1 \forall n \geq 1$ satisfying $\psi \in [\underline{\psi}, \bar{\psi}]$ for $0 < \underline{\psi} \leq 1 \leq \bar{\psi} < \infty$ (and $\underline{\psi} = \bar{\psi} = 1$ is the degenerate case with no permanent shocks).

Following Zeldes (1989), in future periods $t + n \forall n \geq 1$ there is a small probability \wp that income will be zero (a ‘zero-income event’),

$$\xi_{t+n} = \begin{cases} 0 & \text{with probability } \wp > 0 \\ \theta_{t+n}/(1 - \wp) & \text{with probability } (1 - \wp) \end{cases} \quad (5)$$

where θ_{t+n} is an iid mean-one random variable ($\mathbb{E}_t[\theta_{t+n}] = 1 \forall n > 0$) whose distribution satisfies $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} \leq 1 \leq \bar{\theta} < \infty$.⁷ Call the cumulative distribution functions \mathcal{F}_ψ and \mathcal{F}_θ (where \mathcal{F}_ξ is derived trivially from (5) and \mathcal{F}_θ). For quick identification in tables and graphs, we will call this the ‘Friedman/Muth’ model because it is a specific implementation of the Friedman (1957) model as interpreted by Muth (1960).

The model looks more special than it is. In particular, the assumption of a positive probability of zero-income events may seem objectionable (though it has empirical support).⁸ But it is easy to show that a model with a nonzero minimum value of ξ (motivated, for example, by the existence of unemployment insurance) can be redefined by capitalizing the present discounted value of minimum income into current market assets,⁹ transforming that model back into this one. And no key results would change if the transitory shocks were persistent but mean-reverting, instead of IID. Also, the assumption of a positive point mass for the worst realization of the transitory shock is inessential, but simplifies the proofs and is a powerful aid to intuition.¹⁰

This model differs from Bewley’s (1977) classic formulation in several ways. The CRRA utility function does not satisfy Bewley’s assumption that $u(0)$ is well defined, or that $u'(0)$ is well defined and finite; indeed, neither the value function nor the marginal value function will be bounded. It differs from Schectman and Escudero (1977) in that they impose liquidity constraints and positive minimum income. It differs from both of these in that it permits permanent growth in income, and also permanent shocks to income, which a large empirical literature finds to be of dominant importance in micro data.¹¹ It differs from Deaton (1991) because liquidity constraints are absent; there are separate transitory and permanent shocks (*a la* Muth (1960)); and the transitory shocks here can occasionally cause income to reach zero.¹² It differs from models found in Stokey et. al. (1989) because neither liquidity constraints nor bounds on utility or

⁷Rabault (2002) and Li and Stachurski (2014) analyze cases where the shock processes have unbounded support.

⁸We calibrate this probability to 0.005 to match data from the Panel Study of Income Dynamics (Carroll (1992)).

⁹So long as unemployment benefits are proportional to \mathbf{p}_t ; see the discussion in section 2.11.

¹⁰A positive density over a positive interval above the lower bound would work instead, but would be cumbersome.

¹¹MaCurdy (1982); Abowd and Card (1989); Carroll and Samwick (1997); Jappelli and Pistaferri (2000); Storesletten, Telmer, and Yaron (2004); Blundell, Low, and Preston (2008); much of the literature instead incorporates highly ‘persistent’ but not permanent shocks, but Daly, Hryshko, and Manovskii (2016) show that when measurement problems are handled correctly data yields serial correlation coefficients 0.98-1.00; and Hryshko and Manovskii (2020) suggests that survey data support the same conclusion when other measurement problems are addressed.

¹²Below it will become clear that the Deaton model is a particular limit of this paper’s model.

201 marginal utility are imposed.^{13,14} Li and Stachurski (2014) show how to allow unbounded
 201 returns by using policy function iteration, but also impose constraints.

203 The paper with perhaps the most in common with this one is Ma, Stachurski, and
 203 Toda (2020), henceforth MST, who establish the existence and uniqueness of a solution
 203 to a general income fluctuation problem in a Markovian setting. The most important
 203 differences are that MST impose liquidity constraints, assume that $u'(0) = 0$, and that
 203 expected marginal utility of income is finite ($\mathbb{E}[u'(Y)] < \infty$). These assumptions are
 203 not consistent with the combination of CRRA utility and income dynamics used here,
 203 whose joint properties are key to the results.¹⁵

206 2.2 The Problem Can Be Normalized By Permanent Income

We establish a bit more notation by reviewing the familiar result that in such problems (CRRA utility, permanent shocks) the number of states can be reduced from two (\mathbf{m} and \mathbf{p}) to one ($m = \mathbf{m}/\mathbf{p}$). Value in the last period of life is $u(\mathbf{m}_T)$; using (in the last line in (6) below) the fact that for our CRRA utility function, $u(xy) = x^{1-\rho}u(y)$, and generically defining nonbold variables as the boldface counterpart normalized by \mathbf{p}_t (as with $m = \mathbf{m}/\mathbf{p}$), consider the problem in the second-to-last period,

$$\begin{aligned} \mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) &= \max_{\mathbf{c}_{T-1}} u(\mathbf{c}_{T-1}) + \beta \mathbb{E}_{T-1}[u(\mathbf{m}_T)] \\ &= \max_{c_{T-1}} u(\mathbf{p}_{T-1}c_{T-1}) + \beta \mathbb{E}_{T-1}[u(\mathbf{p}_T m_T)] \\ &= \mathbf{p}_{T-1}^{1-\rho} \left\{ \max_{c_{T-1}} u(c_{T-1}) + \beta \mathbb{E}_{T-1}[u(\Gamma_T m_T)] \right\}. \end{aligned} \quad (6)$$

Now, in a one-time deviation from the notational convention established in the last sentence, define nonbold ‘normalized value’ not as $\mathbf{v}_t/\mathbf{p}_t$ but as $v_t = \mathbf{v}_t/\mathbf{p}_t^{1-\rho}$, because this allows us to exploit features of the related problem,

$$\begin{aligned} v_t(m_t) &= \max_{\{c\}_t^T} u(c_t) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} v_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ b_{t+1} &= (R/\Gamma_{t+1})a_t = \mathcal{R}_{t+1}a_t \\ m_{t+1} &= b_{t+1} + \xi_{t+1}, \end{aligned} \quad (7)$$

233 where $\mathcal{R}_{t+1} \equiv (R/\Gamma_{t+1})$ is a ‘growth-normalized’ return factor, and the new problem’s
 233 first order condition is¹⁶

¹³Similar restrictions are made in the well known papers by Scheinkman and Weiss (1986), Clarida (1987), and Chamberlain and Wilson (2000). See Toche (2005) for an elegant analysis of a related but simpler continuous-time model.

¹⁴Alvarez and Stokey (1998) relaxed the bounds on the return function, but they address only the deterministic case.

¹⁵The incorporation of permanent shocks rules out application of the tools of Matkowski and Nowak (2011), who followed and corrected an error in the fundamental work on the local contraction mapping method developed in Rincón-Zapatero and Rodríguez-Palmero (2003). Martins-da Rocha and Vailakis (2010) provide a correction to Rincón-Zapatero and Rodríguez-Palmero (2003), that works under easier conditions to verify, but only addresses the deterministic case.

¹⁶Leaving aside their assumptions about the marginal utility function and liquidity constraints, it is tempting to view this as a special case of the model of MST, with our $\mathcal{R}_{t+1} = R/\Gamma_{t+1}$ (defined below equation (7)) corresponding to their

$$c_t^{-\rho} = R\beta \mathbb{E}_t[\Gamma_{t+1}^{-\rho} c_{t+1}^{-\rho}]. \quad (8)$$

Since $v_T(m_T) = u(m_T)$, defining $v_{T-1}(m_{T-1})$ from (7), we obtain

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho} v_{T-1}(\underbrace{\mathbf{m}_{T-1}/\mathbf{p}_{T-1}}_{=m_{T-1}}).$$

This logic induces to earlier periods; if we solve the normalized one-state-variable problem (7), we will have solutions to the original problem for any $t < T$ from:

$$\begin{aligned} \mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t^{1-\rho} v_t(m_t), \\ \mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) &= \mathbf{p}_t c_t(m_t). \end{aligned}$$

256 2.3 Definition of a Nondegenerate Solution

The problem has a nondegenerate solution if as the horizon n gets arbitrarily large the solution in the first period of life $c_{T-n}(m)$ gets arbitrarily close to a limiting $c(m)$:

$$c(m) \equiv \lim_{n \rightarrow \infty} c_{T-n}(m) \quad (9)$$

that satisfies

$$0 < c(m) < \infty \quad (10)$$

267 for every $0 < m < \infty$.

269 2.4 Perfect Foresight Benchmarks

272 The familiar analytical solution to the perfect foresight model, obtained by setting $\wp = 0$
272 and $\underline{\theta} = \bar{\theta} = \underline{\psi} = \bar{\psi} = 1$, allows us to define some remaining notation and terminology.

274 2.4.1 Human Wealth

The dynamic budget constraint, strictly positive marginal utility, and the can't-die-in-debt condition (3) imply an exactly-holding intertemporal budget constraint (IBC):

$$\text{PDV}_t(\mathbf{c}) = \overbrace{\mathbf{m}_t - \mathbf{p}_t}^{\mathbf{b}_t} + \overbrace{\text{PDV}_t(\mathbf{p})}^{\mathbf{h}_t}, \quad (11)$$

where \mathbf{b} is nonhuman wealth, and with a constant $\mathcal{R} \equiv R/\Gamma$ ‘human wealth’ is

$$\begin{aligned} \mathbf{h}_t &= \mathbf{p}_t + \mathcal{R}^{-1} \mathbf{p}_t + \mathcal{R}^{-2} \mathbf{p}_t + \dots + \mathcal{R}^{t-T} \mathbf{p}_t \\ &= \underbrace{\left(\frac{1 - \mathcal{R}^{-(T-t+1)}}{1 - \mathcal{R}^{-1}} \right)}_{\equiv h_t} \mathbf{p}_t. \end{aligned} \quad (12)$$

stochastic rate of return on capital and the FVAF $\beta\Gamma_{t+1}^{1-\rho}$ defined below (39) corresponding to their stochastic discount factor. But a caveat is that, here, \mathcal{R}_{t+1} and the modified discount factor are intimately (through Γ_{t+1}).

In order for $h \equiv \lim_{n \rightarrow \infty} h_{T-n}$ to be finite, we must impose the Finite Human Wealth Condition (‘FHC’):

$$\underbrace{\Gamma/R}_{\equiv R^{-1}} < 1. \quad (13)$$

Intuitively, for human wealth to be finite, the growth rate of (noncapital) income must be smaller than the interest rate at which that income is being discounted.

2.4.2 When Does the Perfect Foresight Unconstrained Solution Exist?

Without constraints, the consumption Euler equation always holds; with $u'(\mathbf{c}) = \mathbf{c}^{-\rho}$,

$$\mathbf{c}_{t+1}/\mathbf{c}_t = (R\beta)^{1/\rho} \equiv \mathbf{P} \quad (14)$$

where the archaic letter ‘**thorn**’ represents what we will call the ‘Absolute Patience Factor,’ or APF:

$$\mathbf{P} = (R\beta)^{1/\rho}. \quad (15)$$

The sense in which \mathbf{P} captures patience is that if the ‘absolute impatience condition’ (AIC) holds,¹⁷

$$\mathbf{P} < 1, \quad (16)$$

the consumer will choose to spend an amount too large to sustain indefinitely. We call such a consumer ‘absolutely impatient.’

We next define a ‘Return Patience Factor’ (RPF) that relates absolute patience to the return factor:

$$\mathbf{P}_R \equiv \mathbf{P}/R \quad (17)$$

and since consumption is growing by \mathbf{P} but discounted by R :

$$\text{PDV}_t(\mathbf{c}) = \left(\frac{1 - \mathbf{P}_R^{T-t+1}}{1 - \mathbf{P}_R} \right) \mathbf{c}_t \quad (18)$$

from which the IBC (11) implies

$$\mathbf{c}_t = \overbrace{\left(\frac{1 - \mathbf{P}_R}{1 - \mathbf{P}_R^{T-t+1}} \right)}^{\equiv \kappa_t} (\mathbf{b}_t + \mathbf{h}_t) \quad (19)$$

which defines a normalized finite-horizon perfect foresight consumption function

$$\bar{\mathbf{c}}_{T-n}(m_{T-n}) = \overbrace{(m_{T-n} - 1 + h_{T-n})}^{\equiv b_{T-n}} \underline{\kappa}_{T-n} \quad (20)$$

¹⁷Impatience conditions have figured in intertemporal optimization problems since the beginning, e.g. in Ramsey (1928). These issues are so central that it would be hopeless to attempt to cite conditions in every other paper that correspond to conditions named and briefly explicated here. I make no claim to novelty for any condition aside from those implicated in my theorems, whose parallels *will* be articulated.

where $\underline{\kappa}_t$ is the marginal propensity to consume (MPC) – it answers the question ‘if the consumer had an extra unit of resources, how much more spending would occur.’ (\bar{c} ’s overbar signifies that \bar{c} will be an upper bound as we modify the problem to incorporate constraints and uncertainty; analogously, $\underline{\kappa}$ is a lower bound for the MPC).

The denominator of equation (19) is the reason that for $\underline{\kappa}$ to be strictly positive as $n = T - t$ goes to infinity we must impose the Return Impatience Condition (RIC):

$$\mathbf{P}_R < 1, \quad (21)$$

so that

$$0 < \underline{\kappa} \equiv 1 - \mathbf{P}_R = \lim_{n \rightarrow \infty} \underline{\kappa}_{T-n}. \quad (22)$$

The RIC thus imposes a second kind of ‘impatience:’ The consumer cannot be so pathologically patient as to wish, in the limit as the horizon approaches infinity, to spend nothing today out of an increase in current wealth (the RIC rules out the degenerate limiting solution $\bar{c}(m) = 0$). We call a consumer who satisfies the RIC ‘return impatient.’

Given that the RIC holds, and (as before) defining limiting objects by the absence of a time subscript, the limiting upper bound consumption function will be

$$\bar{c}(m) = (m + h - 1)\underline{\kappa}, \quad (23)$$

and so in order to rule out the degenerate limiting solution $\bar{c}(m) = \infty$ we need h to be finite; that is, we must impose the Finite Human Wealth Condition (13).

Because $u(xy) = x^{1-\rho}u(y)$ we can write a useful analytical expression for the value the consumer would achieve by spending permanent income \mathbf{p} in every period:

$$\begin{aligned} \mathbf{v}_t^{\text{autarky}} &= u(\mathbf{p}_t) + \beta u(\mathbf{p}_t \Gamma) + \beta^2 u(\mathbf{p}_t \Gamma^2) + \dots \\ &= u(\mathbf{p}_t) (1 + \beta \Gamma^{1-\rho} + (\beta \Gamma^{1-\rho})^2 + \dots) \\ &= u(\mathbf{p}_t) \left(\frac{1 - (\beta \Gamma^{1-\rho})^{T-t+1}}{1 - \beta \Gamma^{1-\rho}} \right) \end{aligned} \quad (24)$$

which (for $\Gamma > 0$) asymptotes to a finite number as $n = T - t$ approaches $+\infty$ if any of these equivalent conditions holds:

$$\begin{aligned} &\overbrace{\beta \Gamma^{1-\rho}}^{\equiv \beth} < 1 \\ &\beta R \Gamma^{-\rho} < R/\Gamma \\ &\mathbf{P}_R < (\Gamma/R)^{1-1/\rho}, \end{aligned} \quad (25)$$

where we call \beth ¹⁸ the ‘Perfect Foresight Value Of Autarky Factor’ (PF-VAF), and the variants of (25) constitute alternative versions of the Perfect Foresight Finite Value of Autarky Condition, PF-FVAC; they guarantee that a consumer who always spends all permanent income ‘has finite autarky value’ (in the perfect foresight case).¹⁹

If the FHWC is satisfied, the PF-FVAC implies that the RIC is satisfied: Divide both

¹⁸This is another kind of discount factor, so we use the Hebrew ‘bet’ which is a cognate of the Greek ‘beta’.

¹⁹This is related to the key impatience condition in Alvarez and Stokey (1998).

sides of the second inequality in (25) by R :

$$\mathbf{D}/R < (\Gamma/R)^{1-1/\rho} \quad (26)$$

380 and FHCW \Rightarrow the RHS is < 1 because $(\Gamma/R) < 1$ (and the RHS is raised to a positive
380 power (because $\rho > 1$)).

Likewise, if the FHCW and the GIC are both satisfied, PF-FVAC follows:

$$\begin{aligned} \mathbf{D} &< \Gamma < R \\ \mathbf{D}_R &< \Gamma/R < (\Gamma/R)^{1-1/\rho} < 1 \end{aligned} \quad (27)$$

387 where the last line holds because FHCW $\Rightarrow 0 \leq (\Gamma/R) < 1$ and $\rho > 1 \Rightarrow 0 < 1-1/\rho < 1$.

402 The first panel of Table 4 summarizes: The PF-Unconstrained model has a non-
402 degenerate limiting solution if we impose the RIC and FHCW (these conditions are
402 necessary as well as sufficient). Together the PF-FVAC and the FHCW imply the RIC,
402 so PF-FVAC and FHCW are jointly sufficient. If we impose the GIC and the FHCW,
402 both the PF-FVAC and the RIC follow, so GIC+FHCW are also sufficient. But there
402 are circumstances under which the RIC and FHCW can hold while the PF-FVAC fails
402 (which we write ~~PF-FVAC~~). For example, if $\Gamma = 0$, the problem is a standard ‘cake-
402 eating’ problem with a nondegenerate solution under the RIC.

404 Perhaps more useful than this prose or the table, the relations of the conditions for the
404 unconstrained perfect foresight case are presented diagrammatically in Figure 1. Each
404 node represents a quantity considered in the foregoing analysis. The arrow associated
404 with each inequality reflects the imposition of that condition. For example, one way we
404 wrote the PF-FVAC in equation (25) is $\mathbf{D} < R^{1/\rho}\Gamma^{1-1/\rho}$, so imposition of the PF-FVAC is
404 captured by the diagonal arrow connecting \mathbf{D} and $R^{1/\rho}\Gamma^{1-1/\rho}$. Traversing the boundary
404 of the diagram clockwise starting at \mathbf{D} involves imposing first the GIC then the FHCW,
404 and the consequent arrival at the bottom right node tells us that these two conditions
404 jointly imply that the PF-FVAC holds. Reversal of a condition will reverse the arrow’s
404 direction; so, for example, the bottommost arrow going from R to $R^{1/\rho}\Gamma^{1-1/\rho}$ imposes
404 ~~FHCW~~; but we can cancel the cancellation and reverse the arrow. This would allow us
404 to traverse the diagram in a clockwise direction from \mathbf{D} to R , revealing that imposition
404 of GIC and FHCW (and, redundantly, FHCW again) let us conclude that the RIC holds
404 because the starting point is \mathbf{D} and the endpoint is R . (Consult Appendix K for a
404 detailed exposition of diagrams of this type).

434 2.4.3 PF Constrained Solution Exists Under RIC or Under{~~RIC~~, GIC}

438 We next examine the perfect foresight constrained solution because it is a useful bench-
438 mark (and limit) for the unconstrained problem with uncertainty (examined next).

If a liquidity constraint requiring $b \geq 0$ is ever to be relevant, it must be relevant at the lowest possible level of market resources, $m_t = 1$, defined by the lower bound for entering the period, $b_t = 0$ (if it were relevant at any higher point, it would certainly be relevant at this point). The constraint is ‘relevant’ if it prevents the choice that would otherwise be optimal; at $m_t = 1$ the constraint is relevant if the marginal utility from spending all of today’s resources $c_t = m_t = 1$, exceeds the marginal utility from doing



Figure 1 PF Unconstrained Model: Relation of GIC, FHC, RIC, and PF-FVAC

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{P} < R^{1/\rho}\Gamma^{1-1/\rho}$, which is one way of writing the PF-FVAC, equation (25)

the same thing next period, $c_{t+1} = 1$; that is, if such choices would violate the Euler equation (8):

$$1^{-\rho} > R\beta(\Gamma)^{-\rho}1^{-\rho}. \quad (28)$$

By analogy to the RPF, we therefore define a ‘growth patience factor’ (GPF) as

$$\mathbf{P}_\Gamma = \mathbf{P}/\Gamma, \quad (29)$$

and define a ‘growth impatience condition’ (GIC)

$$\mathbf{P}_\Gamma < 1 \quad (30)$$

which is equivalent to (28) (exponentiate both sides by $1/\rho$).

We now examine implications of possible configurations of the conditions.

GIC and RIC. If the GIC fails but the RIC (21) holds, appendix A shows that, for some $0 < m_\# < 1$, an unconstrained consumer behaving according to (23) would choose $c < m$ for all $m > m_\#$. In this case the solution to the constrained consumer’s problem is simple: For any $m \geq m_\#$ the constraint does not bind (and will never bind in the future); for such m the constrained consumption function is identical to the unconstrained one. If the consumer were somehow²⁰ to arrive at an $m < m_\# < 1$ the constraint would bind and the consumer would consume $c = m$. Using the \bullet accent to designate the version of a function \bullet in the presence of constraints (and recalling that $\bar{c}(m)$ is the unconstrained

²⁰“Somehow” because $m < 1$ could only be obtained by entering the period with $b < 0$ which the constraint forbids.

484 perfect foresight solution):

$$\dot{c}(m) = \begin{cases} m & \text{if } m < m_{\#} \\ \bar{c}(m) & \text{if } m \geq m_{\#}. \end{cases} \quad (31)$$

487 *GIC and RIC.* More useful is the case where the return impatience and GIC conditions
 487 both hold. In this case appendix A shows that the limiting constrained consumption
 487 function is piecewise linear, with $\dot{c}(m) = m$ up to a first ‘kink point’ at $m_{\#}^1 > 1$, and
 487 with discrete declines in the MPC at a set of kink points $\{m_{\#}^1, m_{\#}^2, \dots\}$. As $m \uparrow \infty$ the
 487 constrained consumption function $\dot{c}(m)$ becomes arbitrarily close to the unconstrained
 487 $\bar{c}(m)$, and the marginal propensity to consume function $\dot{\kappa}(m) \equiv \dot{c}'(m)$ limits to $\underline{\kappa}$.²¹
 487 Similarly, the value function $\dot{v}(m)$ is nondegenerate and limits into the value function of
 487 the unconstrained consumer.

This logic holds even when the finite human wealth condition fails (~~FHWC~~), because
 the constraint prevents the (limiting) consumer²² from borrowing against unbounded
 human wealth to finance unbounded current consumption. Under these circumstances,
 the consumer who starts with any amount of resources $b_t > 1$ will, over time, run those
 resources down so that by some finite number of periods τ in the future the consumer
 will reach $b_{t+\tau} = 0$, and thereafter will set $\mathbf{c} = \mathbf{p}$ for eternity (which the PF-FVAC
 says yields finite value). Using the same steps as for equation (24), value of the interim
 program is also finite:

$$\mathbf{v}_{t+\tau} = \Gamma^{\tau(1-\rho)} \mathbf{u}(\mathbf{p}_t) \left(\frac{1 - (\beta \Gamma^{1-\rho})^{T-(t+\tau)+1}}{1 - \beta \Gamma^{1-\rho}} \right).$$

496 So, if ~~FHWC~~, the limiting consumer’s value for any finite m will be the sum of two finite
 496 numbers: The component due to the unconstrained consumption choice made over the
 496 finite horizon leading up to $b_{t+\tau} = 0$, and the finite component due to the value of
 496 consuming all $\mathbf{p}_{t+\tau}$ thereafter.

507 *GIC and ~~RHC~~.* The most peculiar possibility occurs when the RIC fails. Under these
 507 circumstances the FHWC must also fail (Appendix A), and the constrained consumption
 507 function is nondegenerate. (See appendix Figure 8 for a numerical example). While it is
 507 true that $\lim_{m \uparrow \infty} \dot{\kappa}(m) = 0$, nevertheless the limiting constrained consumption function
 507 $\dot{c}(m)$ is strictly positive and strictly increasing in m . This result interestingly reconciles
 507 the conflicting intuitions from the unconstrained case, where ~~RHC~~ would suggest a
 507 degenerate limit of $\dot{c}(m) = 0$ while ~~FHWC~~ would suggest a degenerate limit of $\dot{c}(m) = \infty$.

510 Tables 3 and 4 (and appendix table 5) codify.

512 We now examine the case with uncertainty but without constraints, which will turn
 512 out to be a close parallel to the model with constraints but without uncertainty.

²¹See Carroll, Holm, and Kimball (2019) for details.

²²That is, one obeying $\mathbf{c} = \lim_{n \uparrow \infty} \mathbf{c}_{T-n}$.

514 2.5 Uncertainty-Modified Conditions

515 2.5.1 Impatience

When uncertainty is introduced, the expectation of beginning-of-period bank balances b_{t+1} can be rewritten as:

$$\mathbb{E}_t[b_{t+1}] = a_t \mathbb{E}_t[(R/\Gamma_{t+1})] = a_t (R/\Gamma) \mathbb{E}_t[\psi_{t+1}^{-1}] \quad (32)$$

where Jensen's inequality guarantees that the expectation of the inverse of the permanent shock is strictly greater than one. It will be convenient to define

$$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1} \quad (33)$$

which satisfies $\underline{\psi} < 1$ (thanks to Mr. Jensen), so we can define

$$\underline{\Gamma} \equiv \Gamma \underline{\psi} < \Gamma \quad (34)$$

which is useful because it allows us to write uncertainty-adjusted versions of equations and conditions in a manner exactly parallel to those for the perfect foresight case; for example, we define a normalized Growth Patience Pactor (GPF-Nrm):

$$\mathbf{p}_{\underline{\Gamma}} = \mathbf{p}/\underline{\Gamma} = \mathbb{E}[\mathbf{p}/(\Gamma\psi)] \quad (35)$$

and a normalized version of the Growth Impatience Condition, GIC-Nrm:

$$\mathbf{p}_{\underline{\Gamma}} < 1 \quad (36)$$

556 which is stronger than the perfect foresight version (30) because $\underline{\Gamma} < \Gamma$ (cf (34)).

558 2.5.2 Autarky Value

Analogously to (24), value for a consumer who spent exactly their permanent income every period would reflect the product of the expectation of the (independent) future shocks to permanent income:

$$\begin{aligned} \mathbf{v}_t &= \mathbb{E}_t [\mathbf{u}(\mathbf{p}_t) + \beta \mathbf{u}(\mathbf{p}_t \Gamma_{t+1}) + \dots + \beta^{T-t} \mathbf{u}(\mathbf{p}_t \Gamma_{t+1} \dots \Gamma_T)] \\ &= \mathbf{u}(\mathbf{p}_t) \left(\frac{1 - (\beta \Gamma^{1-\rho} \mathbb{E}[\psi^{1-\rho}])^{T-t+1}}{1 - \beta \Gamma^{1-\rho} \mathbb{E}[\psi^{1-\rho}]} \right) \end{aligned}$$

which invites the definition of a utility-compensated equivalent of the permanent shock,

$$\underline{\underline{\psi}} = (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)} \quad (37)$$

571 which will satisfy $\underline{\underline{\psi}} < 1$ for $\rho > 1$ and nondegenerate ψ . Defining

$$\underline{\underline{\Gamma}} = \Gamma \underline{\underline{\psi}} \quad (38)$$

we can see that \mathbf{v}_t will be finite as T approaches ∞ if

$$\begin{aligned} \overbrace{\beta \underline{\underline{\Gamma}}^{1-\rho}}^{\equiv \underline{\underline{\psi}}} &< 1 \\ \beta &< \underline{\underline{\Gamma}}^{\rho-1} \end{aligned} \quad (39)$$

Table 1 Microeconomic Model Calibration

Calibrated Parameters			
Description	Parameter	Value	Source
Permanent Income Growth Factor	Γ	1.03	PSID: Carroll (1992)
Interest Factor	R	1.04	Conventional
Time Preference Factor	β	0.96	Conventional
Coefficient of Relative Risk Aversion	ρ	2	Conventional
Probability of Zero Income	\wp	0.005	PSID: Carroll (1992)
Std Dev of Log Permanent Shock	σ_ψ	0.1	PSID: Carroll (1992)
Std Dev of Log Transitory Shock	σ_θ	0.1	PSID: Carroll (1992)

599 which we call the ‘finite value of autarky condition’ because it guarantees that value
599 is finite for a consumer who always consumes their (now stochastic) permanent income
599 (and we will call $\underline{\underline{\Gamma}}$ the ‘Value of Autarky Factor,’ VAF).²³ For nondegenerate ψ , this
599 condition is stronger (harder to satisfy in the sense of requiring lower β) than the perfect
599 foresight version (25) because $\underline{\underline{\Gamma}} < \Gamma$.²⁴

602 2.6 The Baseline Numerical Solution

609 Figure 2, familiar from the literature, depicts the successive consumption rules that
609 apply in the last period of life ($c_T(m)$), the second-to-last period, and earlier periods
609 under baseline parameter values listed in Table 2. (The 45 degree line is $c_T(m) = m$
609 because in the last period of life it is optimal to spend all remaining resources.)

628 In the figure, the consumption rules appear to converge to a nondegenerate $c(m)$. Our
628 next purpose is to show that this appearance is not deceptive.

630 2.7 Concave Consumption Function Characteristics

633 A precondition for the main proof is that the maximization problem (7) defines a
633 sequence of continuously differentiable strictly increasing strictly concave²⁵ functions
633 $\{c_T, c_{T-1}, \dots\}$. The straightforward but tedious proof is relegated to appendix B. For

²³In a stationary environment – that is, with $\underline{\underline{\Gamma}} = 1$ – this corresponds to an impatience condition imposed by Ma, Stachurski, and Toda (2020); but their remaining conditions do not correspond to those here, because their problem differs and their method of proof differs.

²⁴To see this, rewrite (39) as

$$\begin{aligned}\beta R &< R \underline{\underline{\Gamma}}^{\rho-1} \\ (\beta R)^{1/\rho} &< R^{1/\rho} \Gamma^{1-1/\rho} \underline{\underline{\psi}}^{1-1/\rho} \\ \mathbf{P}_\Gamma &< (R/\Gamma)^{1/\rho} \underline{\underline{\psi}}^{1-1/\rho}\end{aligned}$$

where the last equation is the same as the PF-FVAC condition except that the RHS is multiplied by $\underline{\underline{\psi}}^{1-1/\rho}$ which is strictly less than 1.

²⁵With one obvious exception: $c_T(m)$ is linear (and so only weakly concave).

Table 2 Model Characteristics Calculated from Parameters

Description	Symbol and Formula	Approximate Calculated Value
Finite Human Wealth Factor	$\mathcal{R}^{-1} \equiv \Gamma/R$	0.990
PF Finite Value of Autarky Factor	$\sqsupset \equiv \beta\Gamma^{1-\rho}$	0.932
Growth Compensated Permanent Shock	$\underline{\psi} \equiv (\mathbb{E}[\psi^{-1}])^{-1}$	0.990
Uncertainty-Adjusted Growth	$\underline{\Gamma} \equiv \Gamma\underline{\psi}$	1.020
Utility Compensated Permanent Shock	$\underline{\underline{\psi}} \equiv (\mathbb{E}[\psi^{1-\rho}])^{1/(1-\rho)}$	0.990
Utility Compensated Growth	$\underline{\underline{\Gamma}} \equiv \Gamma\underline{\underline{\psi}}$	1.020
Absolute Patience Factor	$\mathfrak{P} \equiv (R\beta)^{1/\rho}$	0.999
Return Patience Factor	$\mathfrak{P}_R \equiv \mathfrak{P}/R$	0.961
Growth Patience Factor	$\mathfrak{P}_\Gamma \equiv \mathfrak{P}/\Gamma$	0.970
Normalized Growth Patience Factor	$\mathfrak{P}_{\underline{\Gamma}} \equiv \mathfrak{P}/\underline{\Gamma}$	0.980
Finite Value of Autarky Factor	$\sqsubseteq \equiv \beta\Gamma^{1-\rho}\underline{\underline{\psi}}^{1-\rho}$	0.941
Weak Impatience Factor	$\wp^{1/\rho}\mathfrak{P} \equiv (\wp\beta R)^{1/\rho}$	0.071



Figure 2 Convergence of the Consumption Rules

present purposes, the most important point is that the income process induces what Aiyagari (1994) dubbed a ‘natural borrowing constraint’: $c_t(m) < m$ for all periods $t < T$ because a consumer who spent all available resources would arrive in period $t + 1$ with balances b_{t+1} of zero, and then might earn zero income over the remaining horizon, requiring the consumer to spend zero, incurring negative infinite utility. To avoid this disaster, the consumer never spends everything. Zeldes (1989) seems to have been the first to argue, based on his numerical results, that the natural borrowing constraint was a quantitatively plausible alternative to ‘artificial’ or ‘ad hoc’ borrowing constraints.²⁶ Strict concavity and continuous differentiability of the consumption function are key elements in many of the arguments below, but are not characteristics of models with ‘artificial’ borrowing constraints. The analytical convenience of these features is considerable – a point that may appeal to theorists when they realize (cf. section H below) that the solution to this congenial problem is arbitrarily close to the solution to the constrained but less wieldy problem with explicit constraints.

2.8 Bounds for the Consumption Functions

The consumption functions depicted in Figure 2 appear to have limiting slopes as $m \downarrow 0$ and as $m \uparrow \infty$. This section confirms that impression and derives those slopes, which will be needed in the contraction mapping proof.²⁷

Assume that a continuously differentiable concave consumption function exists in period $t + 1$, with an origin at $c_{t+1}(0) = 0$, a minimal MPC $\underline{\kappa}_{t+1} > 0$, and maximal MPC $\bar{\kappa}_{t+1} \leq 1$. (If $t + 1 = T$ these will be $\underline{\kappa}_T = \bar{\kappa}_T = 1$; for earlier periods they will exist by recursion from the following arguments.)

The MPC bound as wealth approaches infinity is easy to understand: In this case, under our imposed assumption that human wealth is finite, the proportion of consumption that will be financed out of human wealth approaches zero. In consequence, the proportional difference between the solution to the model with uncertainty and the perfect foresight model shrinks to zero. In the course of proving this, appendix G provides a useful recursive expression (used below) for the (inverse of the) limiting MPC:

$$\underline{\kappa}_t^{-1} = 1 + \mathbf{P}_R \underline{\kappa}_{t+1}^{-1}. \quad (40)$$

2.8.1 Weak RIC Conditions

Appendix equation (88) presents a parallel expression for the limiting maximal MPC as $m_t \downarrow 0$:

$$\bar{\kappa}_t^{-1} = 1 + \wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_{t+1}^{-1} \quad (41)$$

²⁶The same (numerical) point applies for infinite horizon models (calibrated to actual empirical data on household income dynamics); cf. Carroll (1992).

²⁷Benhabib, Bisin, and Zhu (2015) show that the consumption function becomes linear as wealth approaches infinity in a model with capital income risk and liquidity constraints; Ma and Toda (2020) show that these results generalize to the limits derived here if capital income is added to the model.

where $\{\bar{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is a decreasing convergent sequence if the ‘weak return patience factor’ $\wp^{1/\rho}\mathbf{P}_R$ satisfies:

$$0 \leq \wp^{1/\rho}\mathbf{P}_R < 1, \quad (42)$$

a condition we dub the ‘Weak Return Impatience Condition’ (WRIC) because with $\wp < 1$ it will hold more easily (for a larger set of parameter values) than the RIC ($\mathbf{P}_R < 1$).

The essence of the argument is that as wealth approaches zero, the overriding consideration that limits consumption is the (recursive) fear of the zero-income events. (That is why the probability of the zero income event \wp appears in the expression.)

We are now in position to observe that the optimal consumption function must satisfy

$$\underline{\kappa}_t m_t \leq c_t(m_t) \leq \bar{\kappa}_t m_t \quad (43)$$

because consumption starts at zero and is continuously differentiable (as argued above), is strictly concave,²⁸ and always exhibits a slope between $\underline{\kappa}_t$ and $\bar{\kappa}_t$ (the formal proof is in appendix D).

2.9 Conditions Under Which the Problem Defines a Contraction Mapping

As mentioned above, standard theorems in the contraction mapping literature following Stokey et. al. (1989) require utility or marginal utility to be bounded over the space of possible values of m , which does not hold here because the possibility (however unlikely) of an unbroken string of zero-income events through the end of the horizon means that utility (and marginal utility) are unbounded as $m \downarrow 0$. Although a recent literature examines the existence and uniqueness of solutions to Bellman equations in the presence of ‘unbounded returns’ (see, e.g., Matkowski and Nowak (2011)), the techniques in that literature cannot be used to solve the problem here because the required conditions are violated by a problem that incorporates permanent shocks.²⁹

Fortunately, Boyd (1990) provided a weighted contraction mapping theorem that Alvarez and Stokey (1998) showed could be used to address the homogeneous case (of which CRRA is an example) in a deterministic framework; later, Durán (2003) showed how to extend the Boyd (1990) approach to the stochastic case.

Definition 1. Consider any function $\bullet \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ where $\mathcal{C}(\mathcal{A}, \mathcal{B})$ is the space of continuous functions from \mathcal{A} to \mathcal{B} . Suppose $F \in \mathcal{C}(\mathcal{A}, \mathcal{B})$ with $\mathcal{B} \subseteq \mathbb{R}$ and $F > 0$. Then \bullet is F -bounded if the F -norm of \bullet ,

$$\|\bullet\|_F = \sup_m \left[\frac{|\bullet(m)|}{F(m)} \right], \quad (44)$$

is finite.

For $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ defined as the set of functions in $\mathcal{C}(\mathcal{A}, \mathcal{B})$ that are F -bounded; w, x, y, and z as examples of F -bounded functions; and using $\mathbf{0}(m) = 0$ to indicate the function that returns zero for any argument, Boyd (1990) proves the following.

²⁸Carroll and Kimball (1996)

²⁹See Yao (2012) for a detailed discussion of the reasons the existing literature up through Matkowski and Nowak (2011) cannot handle the problem described here.

Boyd's Weighted Contraction Mapping Theorem. *Let $T : \mathcal{C}_F(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{B})$ such that^{30,31}*

- (1) T is non-decreasing, i.e. $x \leq y \Rightarrow \{Tx\} \leq \{Ty\}$
- (2) $\{T0\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$
- (3) *There exists some real $0 < \alpha < 1$ such that*

$$\{T(w + \zeta F)\} \leq \{Tw\} + \zeta \alpha F \quad \text{holds for all real } \zeta > 0 \quad .$$

736 *Then T defines a contraction with a unique fixed point.*

For our problem, take \mathcal{A} as $\mathbb{R}_{>0}$ and \mathcal{B} as \mathbb{R} , and define

$$\{Ez\}(a_t) = \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} z(a_t \mathcal{R}_{t+1} + \xi_{t+1})] .$$

Using this, we introduce the mapping $\mathcal{T} : \mathcal{C}_F(\mathcal{A}, \mathcal{B}) \rightarrow \mathcal{C}(\mathcal{A}, \mathcal{B})$,³²

$$\{\mathcal{T}z\}(m_t) = \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} u(c_t) + \beta (\{Ez\}(m_t - c_t)) . \quad (45)$$

778 We can show that our operator \mathcal{T} satisfies the conditions that Boyd requires of his
 778 operator T , if we impose two restrictions on parameter values. The first is the WRIC nec-
 778 essary for convergence of the maximal MPC, equation (42) above. More serious is the
 778 Finite Value of Autarky condition, equation (39). (We discuss the interpretation of these
 778 restrictions in detail in section 2.11 below.) Imposing these restrictions, we are now in
 778 position to state the central theorem of the paper.

785 **Theorem 1.** *\mathcal{T} is a contraction mapping if the restrictions on parameter values (42)*
 785 *and (39) are true (that is, if the weak return impatience condition and the finite value*
 785 *of autarky condition hold).*

788 Intuitively, Boyd's theorem shows that if you can find a F that is everywhere finite
 788 but goes to infinity 'as fast or faster' than the function you are normalizing with F ,
 788 the normalized problem defines a contraction mapping. The intuition for the FVAC
 788 condition is just that, with an infinite horizon, with any initial amount of bank balances
 788 b_0 , in the limit your value can always be made greater than you would get by consuming
 788 exactly the sustainable amount (say, by consuming $(r/R)b_0 - \epsilon$ for some small $\epsilon > 0$).

793 The details of the proof are cumbersome, and are therefore relegated to appendix D.
 793 Given that the value function converges, appendix E.2 shows that the consumption
 793 functions converge.³³

³⁰We will usually denote the function that results from the mapping as, e.g., $\{Tw\}$.

³¹To non-theorists, this notation may be slightly confusing; the inequality relations in (1) and (3) are taken to mean 'for any specific element \bullet in the domain of the functions in question' so that, e.g., $x \leq y$ is short for $x(\bullet) \leq y(\bullet) \forall \bullet \in \mathcal{A}$. In this notation, $\zeta \alpha F$ in (3) is a *function* which can be applied to any argument \bullet (because F is a function).

³²Note that the existence of the maximum is assured by the continuity of $\{Ez\}(a_t)$ (it is continuous because it is the sum of continuous F -bounded functions z) and the compactness of $[\underline{\kappa}m_t, \bar{\kappa}m_t]$.

³³MST's proof is for convergence of the consumption policy function directly, rather than of the value function, which is why their conditions are on u' , which governs behavior.

2.10 The Liquidity Constrained Solution as a Limit

This section explains why a related problem commonly considered in the literature (e.g., by Deaton (1991)), with a liquidity constraint and a positive minimum value of income, is the limit of the problem considered here as the probability \wp of the zero-income event approaches zero.

The ‘related’ problem makes two changes to the problem defined above:

1. An ‘artificial’ liquidity constraint is imposed: $a_t \geq 0$
2. The probability of zero-income events is zero: $\wp = 0$

The essence of the argument is simple. Imposing the artificial constraint without changing \wp would not change behavior at all: The possibility of earning zero income over the remaining horizon already prevents the consumer from ending the period with zero assets. So, for precautionary reasons, the consumer will save something.

But the *extent* to which the consumer feels the need to make this precautionary provision depends on the *probability* that it will turn out to matter. As $\wp \downarrow 0$, that probability becomes arbitrarily small, so the *amount* of precautionary saving induced by the zero-income events approaches zero as $\wp \downarrow 0$. But “zero” is the amount of precautionary saving that would be induced by a zero-probability event for the impatient liquidity constrained consumer.

Another way to understand this is just to think of the liquidity constraint reflecting a component of the utility function that is zero whenever the consumer ends the period with (strictly) positive assets, but negative infinity if the consumer ends the period with (weakly) negative assets.

See appendix H for the formal proof justifying the foregoing intuitive discussion.³⁴

The conditions required for convergence and nondegeneracy are thus strikingly similar between the liquidity constrained perfect foresight model and the model with uncertainty but no explicit constraints: The liquidity constrained perfect foresight model is just the limiting case of the model with uncertainty as the degree of all three kinds of uncertainty (zero-income events, other transitory shocks, and permanent shocks) approaches zero.

2.11 Discussion of Parametric Restrictions

The full relationship among all the conditions is represented in Figure 3. Though the diagram looks complex, it is merely a modified version of the earlier diagram (Figure 1) with further (mostly intermediate) inequalities inserted. (Arrows with a “because” are a new element to label relations that always hold under the model’s assumptions.) Again readers unfamiliar with such diagrams should see Appendix K) for a more detailed explanation.

³⁴It seems likely that a similar argument would apply even in the context of a model like that of MST, perhaps with some weak restrictions on returns.

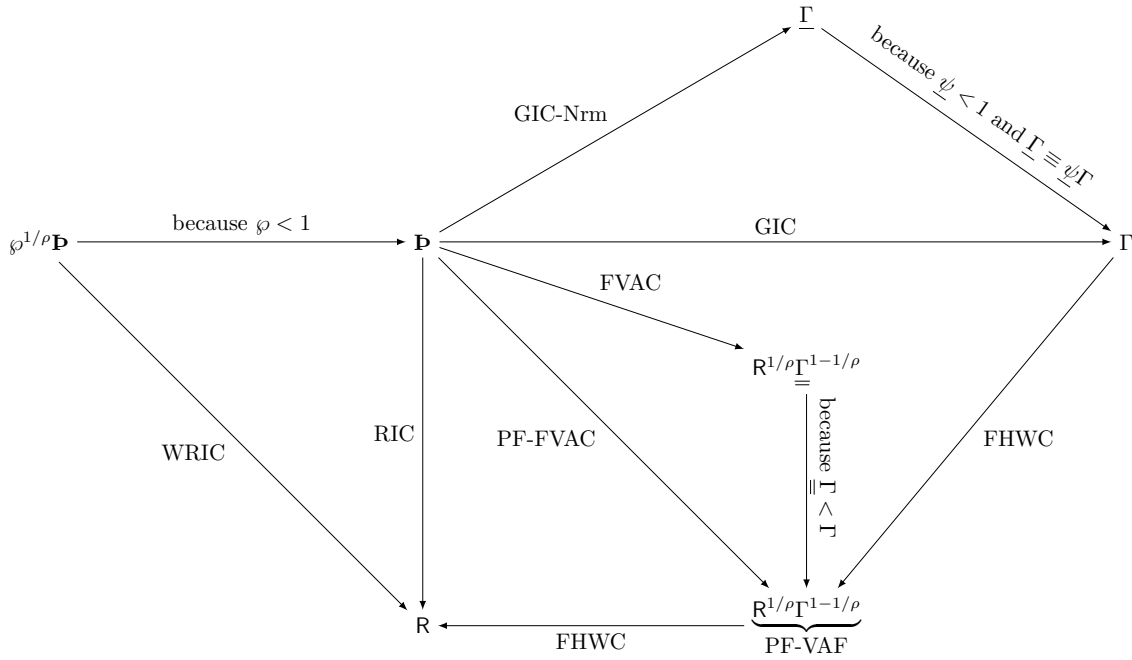


Figure 3 Relation of All Inequality Conditions

See Table 2 for Numerical Values of Nodes Under Baseline Parameters

848 2.11.1 The WRIC

The ‘weakness’ of the additional condition sufficient for contraction beyond the FVAC, the WRIC, can be seen by asking ‘under what circumstances would the FVAC hold but the WRIC fail?’ Algebraically, the requirement is

$$\beta \Gamma^{1-\rho} \underline{\underline{\psi}}^{1-\rho} < 1 < (\varphi \beta)^{1/\rho} / \mathbf{R}^{1-1/\rho}. \quad (46)$$

If we require $\mathbf{R} \geq 1$, the WRIC is redundant because now $\beta < 1 < \mathbf{R}^{\rho-1}$, so that (with $\rho > 1$ and $\beta < 1$) the RIC (and WRIC) must hold. But neither theory nor evidence demands that we assume $\mathbf{R} \geq 1$. We can therefore approach the question of the WRIC’s relevance by asking just how low \mathbf{R} must be for the condition to be relevant. Suppose for illustration that $\rho = 2$, $\underline{\underline{\psi}}^{1-\rho} = 1.01$, $\Gamma^{1-\rho} = 1.01^{-1}$ and $\varphi = 0.10$. In that case (46) reduces to

$$\beta < 1 < (0.1\beta/\mathbf{R})^{1/2}$$

but since $\beta < 1$ by assumption, the binding requirement is that

$$\mathbf{R} < \beta/10$$

874 so that for example if $\beta = 0.96$ we would need $\mathbf{R} < 0.096$ (that is, a perpetual riskfree
874 rate of return of worse than -90 percent a year) in order for the WRIC to bind.

888 Perhaps the best way of thinking about this is to note that the space of parameter
888 values for which the WRIC is relevant shrinks out of existence as $\varphi \rightarrow 0$, which section
888 2.10 showed was the precise limiting condition under which behavior becomes arbitrarily

close to the liquidity constrained solution (in the absence of other risks). On the other hand, when $\varphi = 1$, the consumer has no noncapital income (so that the FHWC holds) and with $\varphi = 1$ the WRIC is identical to the RIC; but the RIC is the only condition required for a solution to exist for a perfect foresight consumer with no noncapital income. Thus the WRIC forms a sort of ‘bridge’ between the liquidity constrained and the unconstrained problems as φ moves from 0 to 1.

2.11.2 When the RIC Fails

In the perfect foresight problem (section 2.4.2), the RIC was necessary for existence of a nondegenerate solution. It is surprising, therefore, that in the presence of uncertainty, the much weaker WRIC is sufficient for nondegeneracy (assuming that the FVAC holds). We can directly derive the features the problem must exhibit (given the FVAC) under $\underline{\text{RIC}}$ (that is, $R < (R\beta)^{1/\rho}$):

$$\begin{aligned}
 R &< \overbrace{(R\beta)^{1/\rho} < (R(\Gamma\underline{\psi})^{\rho-1})^{1/\rho}}^{\text{implied by FVAC}} \\
 R &< (R/\Gamma)^{1/\rho} \Gamma \underline{\psi}^{1-1/\rho} \\
 R/\Gamma &< (R/\Gamma)^{1/\rho} \underline{\psi}^{1-1/\rho} \\
 R/\Gamma &< \underline{\underline{\psi}}
 \end{aligned} \tag{47}$$

but since $\underline{\psi} < 1$ (cf. the argument below (37)), this requires $R/\Gamma < 1$; so, given the FVAC, the $\underline{\text{RIC}}$ can fail only if human wealth is unbounded. As an illustration of the usefulness of our diagrams, note that this algebraically complicated conclusion could be easily reached diagrammatically in figure 3 by starting at the R node and imposing $\underline{\text{RIC}}$ (reversing the RIC arrow) and then traversing the diagram along any clockwise path to the PF-VAF node at which point we realize that we *cannot* impose the FHWC because that would let us conclude $R > R$.

As in the perfect foresight constrained problem, unbounded limiting human wealth ($\underline{\text{FHWC}}$) here does not lead to a degenerate limiting consumption function (finite human wealth is not a condition required for the convergence theorem). But, from equation (40) and the discussion surrounding it, an implication of $\underline{\text{RIC}}$ is that $\lim_{m \uparrow \infty} c'(m) = 0$. Thus, interestingly, in the special $\{\underline{\text{RIC}}, \underline{\text{FHWC}}\}$ case (unavailable in the perfect foresight model) the presence of uncertainty both permits unlimited human wealth and at the same time prevents unlimited human wealth from resulting in infinite consumption at any finite m . Intuitively, in the presence of uncertainty, pathological patience (which in the perfect foresight model results in a limiting consumption function of $c(m) = 0$ for finite m) plus unbounded human wealth (which the perfect foresight model prohibits because it leads to a limiting consumption function $c(m) = \infty$ for any finite m) combine to yield a unique finite level of consumption and the MPC for any finite value of m . Note the close parallel to the conclusion in the perfect foresight liquidity constrained model in the $\{\text{GIC}, \underline{\text{RIC}}\}$ case. There, too, the tension between infinite human wealth and

912 pathological patience was resolved with a nondegenerate consumption function whose
 912 limiting MPC was zero.³⁵

914 2.11.3 When the RIC Holds

917 **FHWC**. If the RIC and FHWC both hold, a perfect foresight solution exists (see 2.4.2
 917 above). As $m \uparrow \infty$ the limiting consumption function and value function become
 917 arbitrarily close to those in the perfect foresight model, because human wealth pays
 917 for a vanishingly small portion of spending. This will be the main case analyzed in
 917 detail below.

FHWC. The more exotic case is where FHWC does not hold; in the perfect foresight
 model, $\{\text{RIC}, \text{FHWC}\}$ is the degenerate case with limiting $\bar{c}(m) = \infty$. Here, since the
 FVAC implies that the PF-FVAC holds (traverse Figure 3 clockwise from **P** by imposing
 FVAC and continue to the PF-VAF node), reversing the arrow connecting the R and
 PF-VAF nodes implies that under **FHWC**:

$$\begin{array}{c} \text{PF-FVAC} \\ \hline \mathbf{P} < (R/\Gamma)^{1/\rho} \Gamma \\ \mathbf{P} < \Gamma \end{array}$$

937 where the transition from the first to the second lines is justified because **FHWC** \Rightarrow
 937 $(R/\Gamma)^{1/\rho} < 1$. So, $\{\text{RIC}, \text{FHWC}\}$ implies the GIC holds. However, we are not entitled
 937 to conclude that the GIC-Nrm holds: $\mathbf{P} < \Gamma$ does not imply $\mathbf{P} < \underline{\psi}\Gamma$ where $\underline{\psi} < 1$.

950 We have now established the principal points of comparison between the perfect fore-
 950 sight solutions and the solutions under uncertainty; these are codified in the remaining
 950 parts of Tables 3 and 4.

1008 3 Analysis of the Converged Consumption Function

1018 Figures 4-6 capture the main properties of the converged consumption rule when the
 1018 RIC, GIC-Nrm, and FHWC all hold.³⁶ Figure 4 shows the expected growth factors
 1018 for consumption and market resources, $\mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t]$ and $\mathbb{E}_t[\mathbf{m}_{t+1}/\mathbf{m}_t]$, for a consumer
 1018 behaving according to the converged consumption rule along with the expected growth
 1018 of the m ratio, while Figures 5 and 6 illustrate theoretical bounds for the consumption
 1018 function and the marginal propensity to consume.

1052 Three features of behavior are captured, or suggested, by the figures. First, as $m_t \uparrow \infty$
 1052 the expected consumption growth factor goes to **P**, indicated by the lower bound in
 1052 Figure 4, and the marginal propensity to consume approaches $\underline{\kappa} = (1 - \mathbf{P}_R)$ (Figure 5),
 1052 the same as the perfect foresight MPC. Second, as $m_t \downarrow 0$ the consumption growth factor
 1052 approaches ∞ (Figure 4) and the MPC approaches $\bar{\kappa} = (1 - \wp^{1/\rho} \mathbf{P}_R)$ (Figure 5). Third,
 1052 there is a value $m = \hat{m}$ at which the expected growth rate of **m** matches the expected

³⁵Ma and Toda (2020) derive conditions under which the limiting MPC is zero in an even more general case where there is also capital income risk.

³⁶These figures reflect the converged rule corresponding to the parameter values indicated in Table 2.

Table 3 Definitions and Comparisons of Conditions

Perfect Foresight Versions	Uncertainty Versions
Finite Human Wealth Condition (FHC)	
$\Gamma/R < 1$ The growth factor for permanent income Γ must be smaller than the discounting factor R for human wealth to be finite.	$\Gamma/R < 1$ The model's risks are mean-preserving spreads, so the PDV of future income is unchanged by their introduction.
Absolute Impatience Condition (AIC)	
$\mathbf{P} < 1$ The unconstrained consumer is sufficiently impatient that the level of consumption will be declining over time: $\mathbf{c}_{t+1} < \mathbf{c}_t$	$\mathbf{P} < 1$ <i>If wealth is large enough, the expectation of consumption next period will be smaller than this period's consumption:</i> $\lim_{m_t \rightarrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}] < \mathbf{c}_t$
Return Impatience Conditions	
Return Impatience Condition (RIC)	Weak RIC (WRIC)
$\mathbf{P}/R < 1$ The growth factor for consumption \mathbf{P} must be smaller than the discounting factor R , so that the PDV of current and future consumption will be finite: $c'(m) = 1 - \mathbf{P}/R < 1$	$\wp^{1/\rho} \mathbf{P}/R < 1$ If the probability of the zero-income event is $\wp = 1$ then income is always zero and the condition becomes identical to the RIC. Otherwise, weaker. $c'(m) < 1 - \wp^{1/\rho} \mathbf{P}/R < 1$
Growth Impatience Conditions	
GIC	GIC-Nrm
$\mathbf{P}/\Gamma < 1$ For an unconstrained PF consumer, the ratio of \mathbf{c} to \mathbf{p} will fall over time. For constrained, guarantees the constraint eventually binds. Guarantees $\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1} m_{t+1}/m_t] = \mathbf{P}_\Gamma$	$\mathbf{P} \mathbb{E}[\psi^{-1}]/\Gamma < 1$ By Jensen's inequality stronger than GIC. Ensures consumers will not expect to accumulate m unboundedly. $\lim_{m_t \rightarrow \infty} \mathbb{E}_t[m_{t+1}/m_t] = \mathbf{P}_\Gamma$
Finite Value of Autarky Conditions	
PF-FVAC	FVAC
$\beta \Gamma^{1-\rho} < 1$ equivalently $\mathbf{P} < R^{1/\rho} \Gamma^{1-1/\rho}$ The discounted utility of constrained consumers who spend their permanent income each period should be finite.	$\beta \Gamma^{1-\rho} \mathbb{E}[\psi^{1-\rho}] < 1$ By Jensen's inequality, stronger than the PF-FVAC because for $\rho > 1$ and nondegenerate ψ , $\mathbb{E}[\psi^{1-\rho}] > 1$.

Table 4 Sufficient Conditions for Nondegenerate[‡] Solution

Consumption Model(s)	Conditions	Comments
$\bar{c}(m)$: PF Unconstrained $\underline{c}(m) = \underline{\kappa}m$ Section 2.4.2: Section 2.4.2: Eq (26): Eq (27):	RIC, FHW ^o	RIC $\Rightarrow v(m) < \infty$; FHW ^o $\Rightarrow 0 < v(m) $ PF model with no human wealth ($h = 0$) RIC prevents $\bar{c}(m) = \underline{c}(m) = 0$ FHW ^o prevents $\bar{c}(m) = \infty$ PF-FVAC+FHW ^o \Rightarrow RIC GIC+FHW ^o \Rightarrow PF-FVAC
$\dot{c}(m)$: PF Constrained Section 2.4.3: Appendix A: Appendix A:	GIC , RIC GIC, RIC GIC, RIC	FHW ^o holds ($\Gamma < \mathbf{P} < R \Rightarrow \Gamma < R$) $\dot{c}(m) = \bar{c}(m)$ for $m > m_{\#} < 1$ (RIC would yield $m_{\#} = 0$ so $\dot{c}(m) = 0$) $\lim_{m \rightarrow \infty} \dot{c}(m) = \bar{c}(m)$, $\lim_{m \rightarrow \infty} \dot{\kappa}(m) = \underline{\kappa}$ kinks where horizon to $b = 0$ changes* $\lim_{m \rightarrow \infty} \dot{\kappa}(m) = 0$ kinks where horizon to $b = 0$ changes*
$c(m)$: Friedman/Muth Section 2.9: Section 2.11.1: Figure 3: Section 2.11.3: Section 2.11.2: Section 3.3: Section 3.3.2: Section 3.3.1:	Section 3.1, Section 3.2 FVAC, WRIC	$\underline{c}(m) < c(m) < \bar{c}(m)$ $\underline{v}(m) < v(m) < \bar{v}(m)$ Sufficient for Contraction WRIC is weaker than RIC FVAC is stronger than PF-FVAC FHW^o +RIC \Rightarrow GIC, $\lim_{m \rightarrow \infty} \kappa(m) = \underline{\kappa}$ RIC \Rightarrow FHW^o , $\lim_{m \rightarrow \infty} \kappa(m) = 0$ “Buffer Stock Saving” Conditions GIC $\Rightarrow \exists \quad 0 < \check{m} < \infty$ GIC-Nrm $\Rightarrow \exists \quad 0 < \hat{m} < \infty$

[‡]For feasible m satisfying $0 < m < \infty$, a nondegenerate limiting consumption function defines a unique optimal value of c satisfying $0 < c(m) < \infty$; a nondegenerate limiting value function defines a corresponding unique value of $-\infty < v(m) < 0$. ^oRIC, FHW^o are necessary as well as sufficient for the perfect foresight case. *That is, the first kink point in $c(m)$ is $m_{\#}$ s.t. for $m < m_{\#}$ the constraint will bind now, while for $m > m_{\#}$ the constraint will bind one period in the future. The second kink point corresponds to the m where the constraint will bind two periods in the future, etc. **In the Friedman/Muth model, the RIC+FHW^o are sufficient, but *not* necessary for nondegeneracy

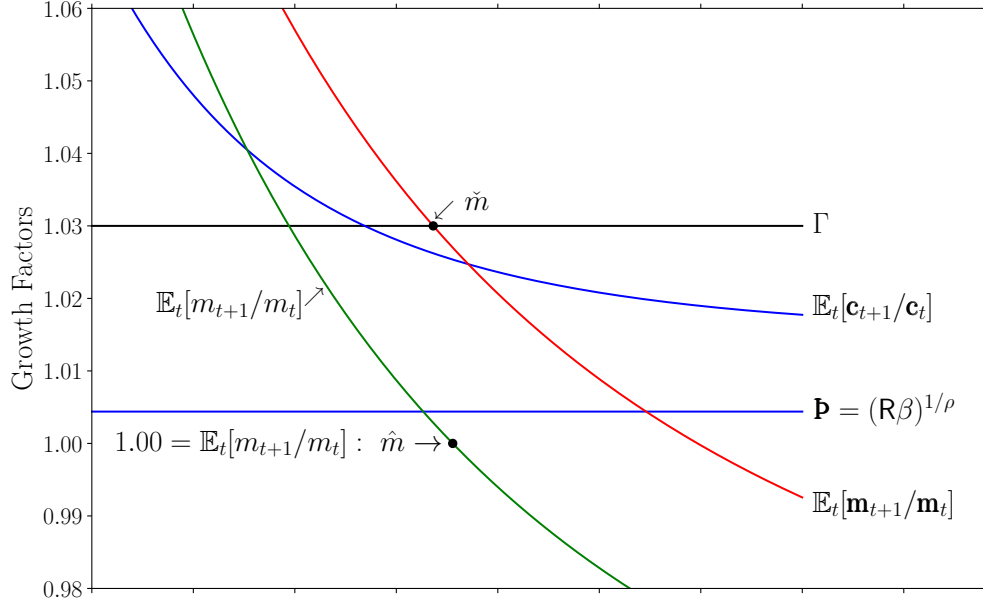


Figure 4 ‘Stable’ m Values and Expected Growth Factors

1052 growth rate of permanent income Γ , and a different (lower) value where the expected
1052 growth rate of consumption at $m = \hat{m}$ is lower than Γ . Thus, at the individual level,
1052 this model does not have a ‘balanced growth’ equilibrium in which all model variables
1052 are expected to grow at the same rate.³⁷

1058 3.1 Limits as m approaches Infinity

Define

$$\underline{c}(m) = \underline{\kappa}m$$

1067 which is the solution to an infinite-horizon problem with no noncapital income ($\xi_{t+n} =$
1067 $0 \forall n \geq 1$); clearly $\underline{c}(m) < c(m)$, since allowing the possibility of future noncapital
1067 income cannot reduce current consumption. Our imposition of the RIC guarantees that
1067 $\underline{\kappa} > 0$, so this solution satisfies our definition of nondegeneracy, and because this solution
1067 is always available it defines a lower bound on both the consumption and value functions.

Assuming the FHC holds, the infinite horizon perfect foresight solution (23) constitutes an upper bound on consumption in the presence of uncertainty, since the introduction of uncertainty strictly decreases the level of consumption at any m (Carroll and

³⁷ A final proposition suggested by Figure 4 is that the expected consumption growth factor is declining in the level of the cash-on-hand ratio m_t . This turns out to be true in the absence of permanent shocks, but in extreme cases it can be false if permanent shocks are present; see appendix L.

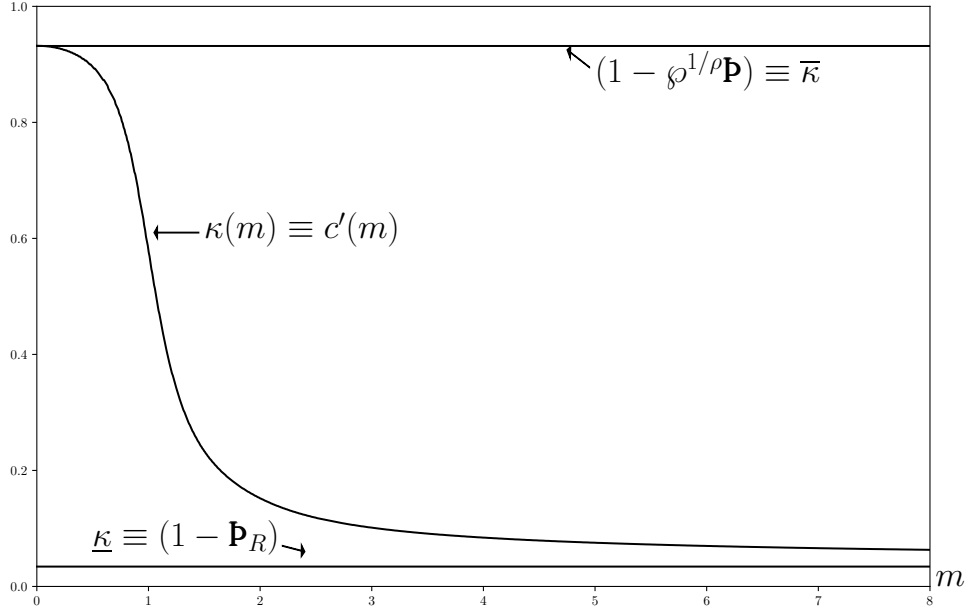


Figure 5 Limiting MPC's

Kimball (1996)). Thus, we can write

$$\begin{aligned} \underline{c}(m) < c(m) < \bar{c}(m) \\ 1 < c(m)/\underline{c}(m) < \bar{c}(m)/\underline{c}(m). \end{aligned} \tag{48}$$

But

$$\begin{aligned} \lim_{m \uparrow \infty} \bar{c}(m)/\underline{c}(m) &= \lim_{m \uparrow \infty} (m - 1 + h)/m \\ &= 1, \end{aligned}$$

so as $m \uparrow \infty$, $c(m)/\underline{c}(m) \rightarrow 1$, and the continuous differentiability and strict concavity of $c(m)$ therefore implies

$$\lim_{m \uparrow \infty} c'(m) = \underline{c}'(m) = \bar{c}'(m) = \underline{\kappa}$$

1089 because any other fixed limit would eventually lead to a level of consumption either
1089 exceeding $\bar{c}(m)$ or lower than $\underline{c}(m)$.

1093 Figure 5 confirms these limits visually. The top plot shows the converged consumption
1093 function along with its upper and lower bounds, while the lower plot shows the marginal
1093 propensity to consume.

Next we establish the limit of the expected consumption growth factor as $m_t \uparrow \infty$:

$$\lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] = \lim_{m_t \uparrow \infty} \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t].$$



Figure 6 Upper and Lower Bounds on The Consumption Function

But

$$\mathbb{E}_t[\Gamma_{t+1}c_{t+1}/\bar{c}_t] \leq \mathbb{E}_t[\Gamma_{t+1}c_{t+1}/c_t] \leq \mathbb{E}_t[\Gamma_{t+1}\bar{c}_{t+1}/\underline{c}_t]$$

and

$$\lim_{m_t \uparrow \infty} \Gamma_{t+1}\underline{c}(m_{t+1})/\bar{c}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1}\bar{c}(m_{t+1})/\underline{c}(m_t) = \lim_{m_t \uparrow \infty} \Gamma_{t+1}m_{t+1}/m_t,$$

while (for convenience defining $a(m_t) = m_t - c(m_t)$),

$$\begin{aligned} \lim_{m_t \uparrow \infty} \Gamma_{t+1}m_{t+1}/m_t &= \lim_{m_t \uparrow \infty} \left(\frac{\text{Ra}(m_t) + \Gamma_{t+1}\xi_{t+1}}{m_t} \right) \\ &= (\text{R}\beta)^{1/\rho} = \mathbf{P} \end{aligned} \quad (49)$$

1129 because $\lim_{m_t \uparrow \infty} a'(m) = \mathbf{P}_R$ ³⁸ and $\Gamma_{t+1}\xi_{t+1}/m_t \leq (\Gamma\bar{\psi}\bar{\theta}/(1 - \wp))/m_t$ which goes to zero
1129 as m_t goes to infinity.

Hence we have

$$\mathbf{P} \leq \lim_{m_t \uparrow \infty} \mathbb{E}_t[\mathbf{c}_{t+1}/\mathbf{c}_t] \leq \mathbf{P}$$

1137 so as cash goes to infinity, consumption growth approaches its value \mathbf{P} in the perfect
1137 foresight model.

³⁸This is because $\lim_{m_t \uparrow \infty} a(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} c(m_t)/m_t = 1 - \lim_{m_t \uparrow \infty} c'(m_t) = \mathbf{P}_R$.

3.2 Limits as m Approaches Zero

Equation (41) shows that the limiting value of $\bar{\kappa}$ is

$$\bar{\kappa} = 1 - \mathbf{R}^{-1}(\wp \mathbf{R} \beta)^{1/\rho}.$$

Defining $e(m) = c(m)/m$ as before we have

$$\lim_{m \downarrow 0} e(m) = (1 - \wp^{1/\rho} \mathbf{P}_{\mathbf{R}}) = \bar{\kappa}.$$

Now using the continuous differentiability of the consumption function along with L'Hôpital's rule, we have

$$\lim_{m \downarrow 0} c'(m) = \lim_{m \downarrow 0} e(m) = \bar{\kappa}.$$

Figure 5 confirms that the numerical solution obtains this limit for the MPC as m approaches zero.

For consumption growth, as $m \downarrow 0$ we have

$$\begin{aligned} \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{c(m_{t+1})}{c(m_t)} \right) \Gamma_{t+1} \right] &> \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{c}(\mathcal{R}_{t+1} a(m_t) + \xi_{t+1})}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\ &= \wp \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{c}(\mathcal{R}_{t+1} a(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\ &\quad + (1 - \wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{c}(\mathcal{R}_{t+1} a(m_t) + \theta_{t+1}/(1 - \wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\ &> (1 - \wp) \lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{c}(\theta_{t+1}/(1 - \wp))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right] \\ &= \infty \end{aligned}$$

where the second-to-last line follows because $\lim_{m_t \downarrow 0} \mathbb{E}_t \left[\left(\frac{\underline{c}(\mathcal{R}_{t+1} a(m_t))}{\bar{\kappa} m_t} \right) \Gamma_{t+1} \right]$ is positive, and the last line follows because the minimum possible realization of θ_{t+1} is $\underline{\theta} > 0$ so the minimum possible value of expected next-period consumption is positive.³⁹

3.3 Unique ‘Stable’ Points

Two theorems, whose proofs are sketched here and detailed in an appendix, articulate alternative (but closely related) stability criteria for the model.

3.3.1 ‘Individual Target Wealth’

One definition of a ‘stable’ point is an \hat{m} such that if $m_t = \hat{m}$, then $\mathbb{E}_t[m_{t+1}] = m_t$. Existence of such a target turns out to require the GIC-Nrm condition.

³⁹None of the arguments in either of the two prior sections depended on the assumption that the consumption functions had converged. With more cumbersome notation, each derivation could have been replaced by the corresponding finite-horizon versions. This strongly suggests that it should be possible to extend the circumstances under which the problem can be shown to define a contraction mapping to the union of the parameter values under which {RIC,FHWC} hold and {FVAC,WRIC} hold. That extension is not necessary for our purposes here, so we leave it for future work.

Theorem 2. For the nondegenerate solution to the problem defined in section 2.1 when FVAC, WRIC, and GIC-Nrm all hold, there exists a unique cash-on-hand-to-permanent-income ratio $\hat{m} > 0$ such that

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \quad (50)$$

Moreover, \hat{m} is a point of ‘wealth stability’ in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}), \quad \mathbb{E}_t[m_{t+1}] &> m_t \\ \forall m_t \in (\hat{m}, \infty), \quad \mathbb{E}_t[m_{t+1}] &< m_t. \end{aligned} \quad (51)$$

Since $m_{t+1} = (m_t - c(m_t))\mathcal{R}_{t+1} + \xi_{t+1}$, the implicit equation for \hat{m} is

$$\mathbb{E}_t[(\hat{m} - c(\hat{m}))\mathcal{R}_{t+1} + \xi_{t+1}] = \hat{m} \quad (52)$$

$$(\hat{m} - c(\hat{m})) \underbrace{\mathcal{R} \mathbb{E}_t[\psi^{-1}]}_{\equiv \bar{\mathcal{R}}} + 1 = \hat{m} \quad (53)$$

3.3.2 Collective Stability and the Expected-Balanced-Growth State

A traditional question in macroeconomic models is whether there is a ‘balanced growth’ equilibrium in which aggregate variables (income, consumption, market resources) all grow forever at the same rate. For the model here, we have already seen in Figure 4 that there is no single m for which $\mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t = \mathbb{E}_t[\mathbf{c}_{t+1}]/\mathbf{c}_t = \Gamma$ for an individual consumer. Nevertheless, analysis below will show that economies populated by collections of such consumers can exhibit balanced growth in the aggregate.

As an input to that analysis, we show here that if the GIC holds, the problem will have a point of what we call ‘collective stability,’ by which we mean that there is some \check{m} such that, for all $\check{m}_t > \check{m}$, $\mathbb{E}_t[\check{m}_{t+1}/\check{m}_t] < \Gamma$, and conversely for $\check{m}_t < \check{m}$. (‘Collective’ is meant to capture the fact that calculating the expectation of the levels of future \mathbf{m} and \mathbf{p} before dividing \mathbf{m} by \mathbf{p} is akin to examining aggregate values in a population).

The critical \check{m} will be the value at which \mathbf{m} growth matches Γ :

$$\begin{aligned} \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &= \mathbb{E}_t[\mathbf{p}_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t[m_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/(m_t\mathbf{p}_t) &= \mathbb{E}_t[\mathbf{p}_t\Gamma\psi_{t+1}]/\mathbf{p}_t \\ \mathbb{E}_t \left[\psi_{t+1} \underbrace{((m_t - c(m_t))\mathcal{R}/(\Gamma\psi_{t+1})) + \xi_{t+1}}_{m_{t+1}} \right] / m_t &= 1. \end{aligned} \quad (54)$$

$$\begin{aligned} \mathbb{E}_t \left[(\check{m} - c(\check{m})) \underbrace{\mathcal{R}}_{\bar{\mathcal{R}}} / \Gamma + \psi_{t+1}\xi_{t+1} \right] &= \check{m} \\ (\check{m} - c(\check{m}))\bar{\mathcal{R}} + 1 &= \check{m}. \end{aligned} \quad (55)$$

The only difference between (55) and (52) is the substitution of \mathcal{R} for $\bar{\mathcal{R}}$.

We will refer to \check{m} as the problem’s Expected-Balanced-Growth State, a term motivated by the fact that an economy composed of consumers all of whom had $m_t = \check{m}$,

would exhibit balanced growth if all consumers happened to continually draw permanent and transitory shocks equal to their expected values of 1.0 forever.⁴⁰

Theorem 3 formally states the relevant proposition.

Theorem 3. *For the nondegenerate solution to the problem defined in section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-to-income ratio $\check{m} > 0$ such that*

$$\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \quad (56)$$

Moreover, \check{m} is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &> \Gamma \\ \forall m_t \in (\check{m}, \infty), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &< \Gamma. \end{aligned} \quad (57)$$

The proofs of the two theorems are almost completely parallel; to save space, they are relegated to Appendix M. In sum, they involve three steps:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1}/m_t]$
 - This follows from existence and continuity of the constituents
2. Existence of the equilibrium point
 - This follows from the upper and lower bound limiting MPC's, existence and continuity, and the Intermediate Value Theorem
3. Monotonicity of $\mathbb{E}_t[m_{t+1} - m_t]$ or $\mathbb{E}_t[m_{t+1}\psi_{t+1} - m_t]$
 - This follows from concavity of the consumption function

3.3.3 Example Where There Is An Expected-Balanced-Growth State But No Target

Because the equations defining target and pseudo-steady-state m , (52) and (54), differ only by substitution of \mathcal{R} for $\bar{\mathcal{R}} = \mathcal{R}\mathbb{E}[\psi^{-1}]$, if there are no permanent shocks ($\psi \equiv 1$), the conditions are identical. For many parameterizations (e.g., under the baseline parameter values used for constructing figure 4), \hat{m} and \check{m} will not differ much.

An illuminating exception is exhibited in figure 7, which modifies the baseline parameter values by quadrupling the variance of the permanent shocks, enough to cause failure of the GIC-Nrm; now there is no target wealth level \hat{m} (consumption remains everywhere below the level that would keep expected m constant).

The pseudo-steady-state still exists because it turns off realizations of the permanent shock. But an aggregate balanced growth equilibrium can exist even when realizations of the permanent shock are implemented exactly as specified in the model. The key insight can be understood by considering the evolution of an economy that starts, at date t ,

40

$$\mathbb{E}_t[m_{t+1}/m_t | \psi_{t+1} = \xi_{t+1} = 1] = \Gamma(\check{m} - c(\check{m})\mathcal{R} + 1)/m = \Gamma$$

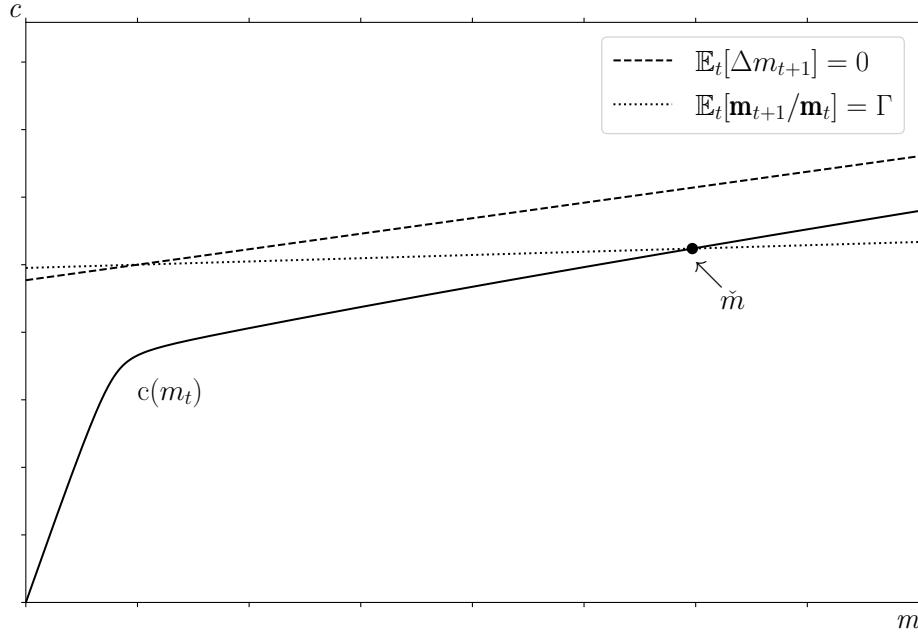


Figure 7 $\{\text{FVAC}, \text{GIC}, \text{GIC-Nrm}\}$: No \hat{m} Exists But \check{m} Does

with the entire population at $m_t = \check{m}$, but then evolves according to the model's correct assumed dynamics between t and $t + 1$. Equation (54) will still hold for this economy, so for this first period, at least, the economy will exhibit balanced growth: the growth factor for aggregate \mathbf{m} will match the growth factor for permanent income Γ . It is true that there will be people for whom $b_{t+1} = a_t R / (\Gamma \psi_{t+1})$ is boosted by a small draw of ψ_{t+1} . But their contribution to the aggregate variable is given by $\mathbf{b}_{t+1} = b_{t+1} \psi_{t+1}$, so their \mathbf{b}_{t+1} reweighted by an amount that exactly unwinds that boosting.

The surprising consequence is that, if the GIC holds but the GIC-Nrm fails, it is possible to construct an aggregate economy composed of consumers all of whom have target wealth of $\hat{m} = \infty$, but in which the aggregate economy still exhibits balanced growth with a finite ratio of aggregate wealth to income. (For an example, see the software archive for the paper).

This is a good introduction to a more explicit discussion of aggregation.

4 The Aggregate and Idiosyncratic Relationship Between Consumption Growth and Income Growth

A large (infinite) collection of small (infinitesimal) buffer-stock consumers with identical parameter values can be thought of as a subset of the population within a single country

(say, members of a given education or occupation group), or as the whole population in a small open economy with an exogenous (constant) interest rate.⁴¹

Until now for convenience we have assumed infinite horizons, with the implicit understanding that Poisson mortality could be handled by adjusting the effective discount factor for mortality. On that basis, section 4.1 continues to omit mortality. But a reason for explicitly introducing mortality will appear at the end of section 4.2, so implications of alternative assumptions about mortality are briefly examined in Section 4.3.

Formally, we assume a continuum of *ex ante* identical households on the unit interval, with constant total mass normalized to one and indexed by $i \in [0, 1]$, all behaving according to the model specified above. Szeidl (2013) proves that whenever the GIC holds such a population will be characterized by invariant distributions of m , c , and a ;⁴² designate these \mathcal{F}^m , \mathcal{F}^a , and \mathcal{F}^c .

4.1 Consumption and Income Growth at the Household Level

The operator $\mathbb{M}[\bullet]$ yields the mean of its argument in the population, as distinct from the expectations operator $\mathbb{E}[\bullet]$ used above, which represents beliefs about the future.

An economist with a microeconomic dataset could calculate the average growth rate of idiosyncratic consumption in a cross section of an economy that had converged at date t , and would find

$$\begin{aligned}\mathbb{M}[\Delta \log \mathbf{c}_{t+1}] &= \mathbb{M}[\log c_{t+1} \mathbf{p}_{t+1} - \log c_t \mathbf{p}_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t + \log c_{t+1} - \log c_t] \\ &= \mathbb{M}[\log \mathbf{p}_{t+1} - \log \mathbf{p}_t] + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma_\psi^2/2) + \mathbb{M}[\log c_{t+1} - \log c_t] \\ &= (\gamma - \sigma_\psi^2/2)\end{aligned}$$

where $\gamma = \log \Gamma$ and the last equality follows because the invariance of \mathcal{F}^c (Szeidl (2013)) means that $\mathbb{M}[\log c_{t+n}] = \mathbb{M}[\log c_t]$. Thus, the same GIC that guaranteed the existence of an ‘individual pseudo-steady-state’ value of m at the microeconomic level guarantees both that there will be an invariant distribution of the population across values of the model variables and that in that invariant distribution the mean growth rates of all idiosyncratic variables are the same (see Szeidl (2013) for details).

⁴¹It is also possible, and only slightly more difficult, to solve for the steady-state of a closed-economy version of the model where the interest rate is endogenous.

⁴²Szeidl (2013)’s equation (9), in our notation, is:

$$\begin{aligned}\mathbb{E} \log \mathbf{R}(1 - \kappa) &< \mathbb{E} \log \Gamma \psi \\ \mathbb{E} \log \mathbf{R} \mathbf{p}_\mathbf{R} &< \mathbb{E} \log \Gamma \psi \\ \log \mathbf{p}_\mathbf{R} &< \mathbb{E} \log \psi\end{aligned}$$

and under our assumption that $\log \psi \sim \mathcal{N}(-\sigma_\psi^2/2, \sigma_\psi^2)$ we can exponentiate both sides to obtain the GIC, $\mathbf{p}_\mathbf{R} < 1$. If the permanent income shocks are not lognormally distributed the expression must be tested in Szeidl’s original form.

4.2 Balanced Growth of Aggregate Income, Consumption, and Wealth

Using boldface capital letters for aggregates, the growth factor for aggregate income is:

$$\begin{aligned}\mathbf{Y}_{t+1}/\mathbf{Y}_t &= \mathbb{M}[\xi_{t+1}\Gamma\psi_{t+1}\mathbf{p}_t]/\mathbb{M}[\mathbf{p}_t\xi_t] \\ &= \Gamma\end{aligned}$$

because of the independence assumptions we have made about ξ and ψ .

From the perspective of period t ,

$$\begin{aligned}\mathbf{A}_{t+1} &= \mathbb{M}[a_{t+1}\mathbf{p}_{t+1}] \\ &= \Gamma\mathbb{M}[(a_t + (a_{t+1} - a_t))\mathbf{p}_t\psi_{t+1}] \\ &= \Gamma\left(\underbrace{\mathbb{M}[a_t\mathbf{p}_t\psi_{t+1}]}_{=\mathbf{A}_t} + \underbrace{\mathbb{M}[(a_{t+1} - a_t)]\mathbb{M}[\mathbf{p}_t\psi_{t+1}]}_{=0 \text{ (Szeidl (2013))}} + \text{cov}_t(a_{t+1} - a_t, \mathbf{p}_t\psi_{t+1})\right) \\ \mathbf{A}_{t+1}/\mathbf{A}_t &= \Gamma\left(1 + \frac{\text{cov}(a_{t+1}, \mathbf{p}_t\psi_{t+1})}{\mathbb{M}[a_t\mathbf{p}_t]}\right)\end{aligned}$$

Unfortunately, the covariance term in the numerator, while generally small, will not in general be zero. This is because the realization of the permanent shock ψ_{t+1} has a nonlinear effect on a_{t+1} . Matters are simpler if there are no permanent shocks; see Appendix F for a proof that in that case the growth rate of assets (and other variables) does eventually converge to the growth rate of aggregate permanent income.

One way of thinking about the problem here is that it may reflect the fact that, under our assumptions, permanent income \mathbf{p} does not have an ergodic distribution; its distribution of becomes forever wider over time, because our consumers never die and each immortal person is perpetually subject to symmetric shocks to their log \mathbf{p} .

This is why we need to introduce mortality.

4.3 Mortality and Redistribution

Most heterogeneous-agent models incorporate a constant positive probability of death, following Blanchard (1985). In such a Blanchardian model, for probabilities of death that exceed a threshold that depends on the size of the permanent shocks, Carroll, Slacalek, Tokuoka, and White (2017) show that the limiting distribution of permanent income has a finite variance, which is a useful step in the direction of taming the problems caused by an unbounded distribution of \mathbf{p} . Numerical results in that paper confirm the intuition that, under appropriate impatience conditions, balanced growth arises (though a formal proof remains elusive).

Even with those (numerical) results in hand, the centrality of mortality assumptions to the existence and nature of steady states requires them to be discussed briefly here.

4.3.1 Blanchard Lives

Blanchard (1985) assumes the existence of an annuitization scheme in which estates of

dying consumers are redistributed to survivors in proportion to survivors' wealth, giving the recipients a higher effective rate of return. This treatment has several analytical advantages, most notably that the effect of mortality on the time preference factor is the exact inverse of its effect on the (effective) interest factor: If the probability of remaining alive is \aleph , then assuming that no utility accrues after death makes the effective discount factor $\hat{\beta} = \beta\aleph$, while the enhancement to the rate of return from the annuity scheme yields an effective interest rate of \hat{R}/\aleph (recall that because of Poisson mortality, the average wealth of the two groups is identical). Combining these, the effective patience factor in the new economy $\hat{\mathbf{P}}$ is unchanged from its value in the infinite horizon model:

$$\hat{\mathbf{P}} \equiv (\beta\aleph R/\aleph)^{1/\rho} = (R\beta)^{1/\rho} \equiv \mathbf{P}. \quad (58)$$

The only adjustments this requires to the analysis from prior parts of this paper are therefore to the few elements that involve a role for R distinct from its contribution to \mathbf{P} (principally, the RIC).

The numerical finding that the covariance term above is approximately zero allows us to conclude again that the key requirement for aggregate balanced growth is presumably the GIC.

4.3.2 Modigliani Lives

Blanchard (1985)'s innovation was useful not only for the insight it provided but also because the principal alternative, the Life Cycle model of Modigliani (1966), was computationally challenging given the then-available technologies. Aside from its (considerable) conceptual value, there is no need for Blanchard's analytical solution today, when serious modeling incorporates uncertainty, constraints, and other features that rule out analytical solutions anyway.

The simplest alternative to Blanchard's mortality is to follow Modigliani in assuming that any wealth remaining at death occurs accidentally (not implausible, given the robust finding that for the great majority of households, bequests amount to less than 2 percent of lifetime earnings, Hendricks (2001, 2016)).

Even if bequests are accidental, a macroeconomic model must make some assumption about how they are disposed of: As windfalls to heirs, estate tax proceeds, etc. We again consider the simplest choice, because it again represents something of a polar alternative to Blanchard: Without a bequest motive, there are no behavioral effects of a 100 percent estate tax; we assume such a tax is imposed and that the revenues are effectively thrown in the ocean; the estate-related wealth effectively vanishes from the economy.

The chief appeal of this approach is the simplicity of the change it makes in the condition required for the economy to exhibit a balanced growth equilibrium. If \aleph is the probability of remaining alive, the condition changes from the plain GIC to a looser mortality-adjusted GIC:

$$\aleph\mathbf{P}_\Gamma < 1. \quad (59)$$

With no income growth, the condition required to prohibit unbounded growth in aggregate wealth would be the condition that prevents the per-capita wealth to income

ratio of surviving consumers from growing faster than the rate at which mortality diminishes their collective population. With income growth, the aggregate wealth-to-income ratio will head to infinity only if a cohort of consumers is patient enough to make the desired rate of growth of wealth fast enough to counteract combined erosive forces of mortality and productivity.

5 Conclusions

Numerical solutions to optimal consumption problems, in both life cycle and infinite horizon contexts, have become standard tools since the first reasonably realistic models were constructed in the late 1980s. One contribution of this paper is to show that finite horizon (‘life cycle’) versions of the simplest such models, with assumptions about income shocks (transitory and permanent) dating back to Friedman (1957) and standard specifications of preferences – and without (plausible, but computationally and mathematically inconvenient) complications like liquidity constraints – have attractive analytical properties (like continuous differentiability of the consumption function, and analytical limiting MPC’s as resources approach their minimum and maximum possible values), and that (more widely used) models with liquidity constraints can be viewed as a particular limiting case of this simpler model.

The main focus of the paper, though, is on the limiting solution of the finite horizon model as the horizon extends to infinity. The paper shows that the simple model has additional attractive properties: A ‘Finite Value of Autarky’ condition guarantees convergence of the consumption function, under the mild additional requirement of a ‘Weak Return Impatience Condition’ that will never bind for plausible parameterizations, but provides intuition for the bridge between this model and models with explicit liquidity constraints. The paper also provides a roadmap for the model’s relationships to the perfect foresight model without and with constraints. The constrained perfect foresight model provides an upper bound to the consumption function (and value function) for the model with uncertainty, which explains why the conditions for the model to have a nondegenerate solution closely parallel those required for the perfect foresight constrained model to have a nondegenerate solution.

The main use of infinite horizon versions of such models is in heterogeneous agent macroeconomics. The paper articulates intuitive ‘Growth Impatience Conditions’ under which populations of such agents, with Blanchardian (tighter) or Modiglianian (looser) mortality will exhibit balanced growth. Finally, the paper provides the analytical basis for many results about buffer-stock saving models that are so well understood that even without analytical foundations researchers uncontroversially use them as explanations of real-world phenomena like the cross-sectional pattern of consumption dynamics in the Great Recession.

The paper’s results are all easily reproducible [interactively on the web](#) or [on any standard computer system](#). Such reproducibility reflects the paper’s use of the open-source [Econ-ARK](#) toolkit, which is used to generate all of the quantitative results of the paper, and which integrally incorporates all of the analytical insights of the paper.

Appendices

A Perfect Foresight Liquidity Constrained Solution

Under perfect foresight in the presence of a liquidity constraint requiring $b \geq 0$, this appendix taxonomizes the varieties of the limiting consumption function $\dot{c}(m)$ that arise under various parametric conditions. Results are summarized in table 5.

A.1 If GIC Fails

A consumer is ‘growth patient’ if the perfect foresight growth impatience condition fails (\mathcal{GIC}^* , $1 < \mathbf{P}/\Gamma$). Under \mathcal{GIC}^* the constraint does not bind at the lowest feasible value of $m_t = 1$ because $1 < (R\beta)^{1/\rho}/\Gamma$ implies that spending everything today (setting $c_t = m_t = 1$) produces lower marginal utility than is obtainable by reallocating a marginal unit of resources to the next period at return R .⁴³

$$\begin{aligned} 1 &< (R\beta)^{1/\rho}\Gamma^{-1} \\ 1 &< R\beta\Gamma^{-\rho} \\ u'(1) &< R\beta u'(\Gamma). \end{aligned}$$

Similar logic shows that under these circumstances the constraint will never bind at $m = 1$ for a constrained consumer with a finite horizon of n periods, so for $m \geq 1$ such a consumer’s consumption function will be the same as for the unconstrained case examined in the main text.

RIC fails, FHCW holds. If the RIC fails ($1 < \mathbf{P}_R$) while the finite human wealth condition holds, the limiting value of this consumption function as $n \uparrow \infty$ is the degenerate function

$$\dot{c}_{T-n}(m) = 0(b_t + h). \quad (60)$$

(that is, consumption is zero for any level of human or nonhuman wealth).

RIC fails, FHCW fails. \mathcal{FHCW} implies that human wealth limits to $h = \infty$ so the consumption function limits to either $\dot{c}_{T-n}(m) = 0$ or $\dot{c}_{T-n}(m) = \infty$ depending on the relative speeds with which the MPC approaches zero and human wealth approaches ∞ .⁴⁴

Thus, the requirement that the consumption function be nondegenerate implies that for a consumer satisfying \mathcal{GIC}^* we must impose the RIC (and the FHCW can be shown to be a consequence of \mathcal{GIC}^* and RIC). In this case, the consumer’s optimal behavior is easy to describe. We can calculate the point at which the unconstrained consumer

⁴³The point at which the constraint would bind (if that point could be attained) is the $m = c$ for which $u'(c_\#) = R\beta u'(\Gamma)$ which is $c_\# = \Gamma/(R\beta)^{1/\rho}$ and the consumption function will be defined by $\dot{c}(m) = \min[m, c_\# + (m - c_\#)\kappa]$.

⁴⁴The knife-edge case is where $\mathbf{P} = \Gamma$, in which case the two quantites counterbalance and the limiting function is $\dot{c}(m) = \min[m, 1]$.

78 would choose $c = m$ from equation (23):

$$\begin{aligned}
m_{\#} &= (m_{\#} - 1 + h)\underline{\kappa} \\
m_{\#}(1 - \underline{\kappa}) &= (h - 1)\underline{\kappa} \\
m_{\#} &= (h - 1) \left(\frac{\underline{\kappa}}{1 - \underline{\kappa}} \right)
\end{aligned} \tag{61}$$

92 which (under these assumptions) satisfies $0 < m_{\#} < 1$.⁴⁵ For $m < m_{\#}$ the unconstrained
92 consumer would choose to consume more than m ; for such m , the constrained consumer
92 is obliged to choose $\bar{c}(m) = m$.⁴⁶ For any $m > m_{\#}$ the constraint will never bind and the
92 consumer will choose to spend the same amount as the unconstrained consumer, $\bar{c}(m)$.

94 (Stachurski and Toda (2019) obtain a similar lower bound on consumption and use it
94 to study the tail behavior of the wealth distribution.)

96 A.2 If GIC Holds

103 Imposition of the GIC reverses the inequality in (60), and thus reverses the conclusion: A
103 consumer who starts with $m_t = 1$ will desire to consume more than 1. Such a consumer
103 will be constrained, not only in period t , but perpetually thereafter.

114 Now define $b_{\#}^n$ as the b_t such that an unconstrained consumer holding $b_t = b_{\#}^n$ would
114 behave so as to arrive in period $t + n$ with $b_{t+n} = 0$ (with $b_{\#}^0$ trivially equal to 0); for
114 example, a consumer with $b_{t-1} = b_{\#}^1$ was on the ‘cusp’ of being constrained in period
114 $t - 1$: Had b_{t-1} been infinitesimally smaller, the constraint would have been binding
114 (because the consumer would have desired, but been unable, to enter period t with
114 negative, not zero, b). Given the GIC, the constraint certainly binds in period t (and
114 thereafter) with resources of $m_t = m_{\#}^0 = 1 + b_{\#}^0 = 1$: The consumer cannot spend
114 more (because constrained), and will not choose to spend less (because impatient), than
114 $c_t = c_{\#}^0 = 1$.

118 We can construct the entire ‘prehistory’ of this consumer leading up to t as follows.
118 Maintaining the assumption that the constraint has never bound in the past, c must
118 have been growing according to \mathbf{P}_{Γ} , so consumption n periods in the past must have
120 been

$$c_{\#}^n = \mathbf{P}_{\Gamma}^{-n} c_t = \mathbf{P}_{\Gamma}^{-n}. \tag{62}$$

⁴⁵Note that $0 < m_{\#}$ is implied by RIC and $m_{\#} < 1$ is implied by GIC.

⁴⁶As an illustration, consider a consumer for whom $\mathbf{P} = 1$, $R = 1.01$ and $\Gamma = 0.99$. This consumer will save the amount necessary to ensure that growth in market wealth exactly offsets the decline in human wealth represented by $\Gamma < 1$; total wealth (and therefore total consumption) will remain constant, even as market wealth and human wealth trend in opposite directions.

129 The PDV of consumption from $t - n$ until t can thus be computed as

$$\begin{aligned}\mathbb{C}_{t-n}^t &= c_{t-n}(1 + \mathbf{P}/R + \dots + (\mathbf{P}/R)^n) \\ &= c_{\#}^n(1 + \mathbf{P}_R + \dots + \mathbf{P}_R^n) \\ &= \mathbf{P}_{\Gamma}^{-n} \left(\frac{1 - \mathbf{P}_R^{n+1}}{1 - \mathbf{P}_R} \right) \\ &= \left(\frac{\mathbf{P}_{\Gamma}^{-n} - \mathbf{P}_R}{1 - \mathbf{P}_R} \right)\end{aligned}$$

133 and note that the consumer's human wealth between $t - n$ and t (the relevant time
133 horizon, because from t onward the consumer will be constrained and unable to access
135 post- t income) is

$$h_{\#}^n = 1 + \dots + \mathcal{R}^{-n} \quad (63)$$

139 while the intertemporal budget constraint says

$$\mathbb{C}_{t-n}^t = b_{\#}^n + h_{\#}^n$$

144 from which we can solve for the $b_{\#}^n$ such that the consumer with $b_{t-n} = b_{\#}^n$ would
146 unconstrainedly plan (in period $t - n$) to arrive in period t with $b_t = 0$:

$$b_{\#}^n = \mathbb{C}_{t-n}^t - \overbrace{\left(\frac{1 - \mathcal{R}^{-(n+1)}}{1 - \mathcal{R}^{-1}} \right)}^{h_{\#}^n}. \quad (64)$$

160 Defining $m_{\#}^n = b_{\#}^n + 1$, consider the function $\mathring{c}(m)$ defined by linearly connecting the
160 points $\{m_{\#}^n, c_{\#}^n\}$ for integer values of $n \geq 0$ (and setting $\mathring{c}(m) = m$ for $m < 1$). This
160 function will return, for any value of m , the optimal value of c for a liquidity constrained
160 consumer with an infinite horizon. The function is piecewise linear with 'kink points'
160 where the slope discretely changes; for infinitesimal ϵ the MPC of a consumer with assets
160 $m = m_{\#}^n - \epsilon$ is discretely higher than for a consumer with assets $m = m_{\#}^n + \epsilon$ because
160 the latter consumer will spread a marginal dollar over more periods before exhausting
160 it.

165 In order for a unique consumption function to be defined by this sequence (64) for the
165 entire domain of positive real values of b , we need $b_{\#}^n$ to become arbitrarily large with
167 n . That is, we need

$$\lim_{n \rightarrow \infty} b_{\#}^n = \infty. \quad (65)$$

169 A.2.1 If FHC Holds

171 The FHC requires $\mathcal{R}^{-1} < 1$, in which case the second term in (64) limits to a constant
175 as $n \uparrow \infty$, and (65) reduces to a requirement that

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_{\Gamma}^{-n} - (\mathbf{P}_R/\mathbf{P}_{\Gamma})^n \mathbf{P}_R}{1 - \mathbf{P}_R} \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n} - \mathcal{R}^{-n} \mathbf{P}_\mathbf{R}}{1 - \mathbf{P}_\mathbf{R}} \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n}}{1 - \mathbf{P}_\mathbf{R}} \right) = \infty.$$

Given the GIC $\mathbf{P}_\Gamma^{-1} > 1$, this will hold iff the RIC holds, $\mathbf{P}_\mathbf{R} < 1$. But given that the FHWC $\mathbf{R} > \Gamma$ holds, the GIC is stronger (harder to satisfy) than the RIC; thus, the FHWC and the GIC together imply the RIC, and so a well-defined solution exists. Furthermore, in the limit as n approaches infinity, the difference between the limiting constrained consumption function and the unconstrained consumption function becomes vanishingly small, because the date at which the constraint binds becomes arbitrarily distant, so the effect of that constraint on current behavior shrinks to nothing. That is,

$$\lim_{m \rightarrow \infty} \bar{c}(m) - \bar{c}(m) = 0. \quad (66)$$

A.2.2 If FHWC Fails

If the FHWC fails, matters are a bit more complex.

Given failure of FHWC, (65) requires

$$\lim_{n \rightarrow \infty} \left(\frac{\mathcal{R}^{-n} \mathbf{P}_\mathbf{R} - \mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\mathbf{R} - 1} \right) + \left(\frac{1 - \mathcal{R}^{-(n+1)}}{\mathcal{R}^{-1} - 1} \right) = \infty$$

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\mathbf{R}}{\mathbf{P}_\mathbf{R} - 1} - \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1} - 1} \right) \mathcal{R}^{-n} - \left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\mathbf{R} - 1} \right) = \infty$$

213

If RIC Holds. When the RIC holds, rearranging (67) gives

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_\Gamma^{-n}}{1 - \mathbf{P}_\mathbf{R}} \right) - \mathcal{R}^{-n} \left(\frac{\mathbf{P}_\mathbf{R}}{1 - \mathbf{P}_\mathbf{R}} + \frac{\mathcal{R}^{-1}}{\mathcal{R}^{-1} - 1} \right) = \infty$$

and for this to be true we need

$$\mathbf{P}_\Gamma^{-1} > \mathcal{R}^{-1}$$

$$\Gamma / \mathbf{P} > \Gamma / \mathbf{R}$$

$$1 > \mathbf{P} / \mathbf{R}$$

which is merely the RIC again. So the problem has a solution if the RIC holds. Indeed, we can even calculate the limiting MPC from

$$\lim_{n \rightarrow \infty} \kappa_\#^n = \lim_{n \rightarrow \infty} \left(\frac{c_\#^n}{b_\#^n} \right) \quad (67)$$

which with a bit of algebra⁴⁷ can be shown to asymptote to the MPC in the perfect

⁴⁷Calculate the limit of

$$\left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_\Gamma^{-n} / (1 - \mathbf{P}_\mathbf{R}) - (1 - \mathcal{R}^{-1} \mathcal{R}^{-n}) / (1 - \mathcal{R}^{-1})} \right) = \left(\frac{1}{1 / (1 - \mathbf{P}_\mathbf{R}) + \mathcal{R}^{-n} \mathcal{R}^{-1} / (1 - \mathcal{R}^{-1})} \right) \quad (68)$$

foresight model:⁴⁸

$$\lim_{m \rightarrow \infty} \dot{\kappa}(m) = 1 - \mathbf{P}_R. \quad (69)$$

If RIC Fails. Consider now the \mathbf{RIC}^* case, $\mathbf{P}_R > 1$. We can rearrange (67) as

$$\lim_{n \rightarrow \infty} \left(\frac{\mathbf{P}_R(\mathcal{R}^{-1} - 1)}{(\mathcal{R}^{-1} - 1)(\mathbf{P}_R - 1)} - \frac{\mathcal{R}^{-1}(\mathbf{P}_R - 1)}{(\mathcal{R}^{-1} - 1)(\mathbf{P}_R - 1)} \right) \mathcal{R}^{-n} - \left(\frac{\mathbf{P}_\Gamma^{-n}}{\mathbf{P}_R - 1} \right) = \infty. \quad (70)$$

which makes clear that with $\mathbf{EHW}^* \Rightarrow \mathcal{R}^{-1} > 1$ and $\mathbf{RIC}^* \Rightarrow \mathbf{P}_R > 1$ the numerators and denominators of both terms multiplying \mathcal{R}^{-n} can be seen transparently to be positive. So, the terms multiplying \mathcal{R}^{-n} in (67) will be positive if

$$\begin{aligned} \mathbf{P}_R \mathcal{R}^{-1} - \mathbf{P}_R &> \mathcal{R}^{-1} \mathbf{P}_R - \mathcal{R}^{-1} \\ \mathcal{R}^{-1} &> \mathbf{P}_R \\ \Gamma &> \mathbf{P} \end{aligned}$$

which is merely the GIC which we are maintaining. So the first term's limit is $+\infty$. The combined limit will be $+\infty$ if the term involving \mathcal{R}^{-n} goes to $+\infty$ faster than the term involving $-\mathbf{P}_\Gamma^{-n}$ goes to $-\infty$; that is, if

$$\begin{aligned} \mathcal{R}^{-1} &> \mathbf{P}_\Gamma^{-1} \\ \Gamma/R &> \Gamma/\mathbf{P} \\ \mathbf{P}/R &> 1 \end{aligned}$$

which merely confirms the starting assumption that the RIC fails.

What is happening here is that the $c_\#^n$ term is increasing backward in time at rate dominated in the limit by Γ/\mathbf{P} while the $b_\#$ term is increasing at a rate dominated by Γ/R term and

$$\Gamma/R > \Gamma/\mathbf{P} \quad (71)$$

because $\mathbf{RIC}^* \Rightarrow \mathbf{P} > R$.

Consequently, while $\lim_{n \uparrow \infty} b_\#^n = \infty$, the limit of the *ratio* $c_\#^n/b_\#^n$ in (67) is zero. Thus, surprisingly, the problem has a well defined solution with infinite human wealth if the RIC fails. It remains true that \mathbf{RIC}^* implies a limiting MPC of zero,

$$\lim_{m \rightarrow \infty} \dot{\kappa}(m) = 0, \quad (72)$$

but that limit is approached gradually, starting from a positive value, and consequently the consumption function is *not* the degenerate $\dot{c}(m) = 0$. (Figure 8 presents an example for $\rho = 2$, $R = 0.98$, $\beta = 1.00$, $\Gamma = 0.99$; note that the horizontal axis is bank balances $b = m - 1$; the part of the consumption function below the depicted points is uninteresting $- c = m$ - so not worth plotting).

We can summarize as follows. Given that the GIC holds, the interesting question is whether the FHW holds. If so, the RIC automatically holds, and the solution limits into

⁴⁸For an example of this configuration of parameters, see the notebook `doApndxLiqConstr.nb` in the Mathematica software archive.



Figure 8 Nondegenerate Consumption Function with ~~FHWC~~ and ~~RIC~~

the solution to the unconstrained problem as $m \uparrow \infty$. But even if the FHWC fails, the problem has a well-defined and nondegenerate solution, whether or not the RIC holds.

Although these results were derived for the perfect foresight case, we know from work elsewhere in this paper and in other places that the perfect foresight case is an upper bound for the case with uncertainty. If the upper bound of the MPC in the perfect foresight case is zero, it is not possible for the upper bound in the model with uncertainty to be greater than zero, because for any $\kappa > 0$ the level of consumption in the model with uncertainty would eventually exceed the level of consumption in the absence of uncertainty.

Ma and Toda (2020) characterize the limits of the MPC in a more general framework that allows for capital and labor income risks in a Markovian setting with liquidity constraints, and find that in that much more general framework the limiting MPC is also zero.

B Existence of a Concave Consumption Function

To show that (7) defines a sequence of continuously differentiable strictly increasing concave functions $\{c_T, c_{T-1}, \dots, c_{T-k}\}$, we start with a definition. We will say that a function $n(z)$ is ‘nice’ if it satisfies

1. $n(z)$ is well-defined iff $z > 0$
2. $n(z)$ is strictly increasing
3. $n(z)$ is strictly concave
4. $n(z)$ is \mathbf{C}^3

- 21 5. $n(z) < 0$
 23 6. $\lim_{z \downarrow 0} n(z) = -\infty$.

29 (Notice that an implication of niceness is that $\lim_{z \downarrow 0} n'(z) = \infty$.)

34 Assume that some v_{t+1} is nice. Our objective is to show that this implies v_t is also
 34 nice; this is sufficient to establish that v_{t-n} is nice by induction for all $n > 0$ because
 34 $v_T(m) = u(m)$ and $u(m) = m^{1-\rho}/(1-\rho)$ is nice by inspection.

38 Now define an end-of-period value function $\mathbf{v}_t(a)$ as

$$\mathbf{v}_t(a) = \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} v_{t+1}(\mathcal{R}_{t+1}a + \xi_{t+1})]. \quad (73)$$

47 Since there is a positive probability that ξ_{t+1} will attain its minimum of zero and since
 47 $\mathcal{R}_{t+1} > 0$, it is clear that $\lim_{a \downarrow 0} \mathbf{v}_t(a) = -\infty$ and $\lim_{a \downarrow 0} \mathbf{v}'_t(a) = \infty$. So $\mathbf{v}_t(a)$ is well-
 47 defined iff $a > 0$; it is similarly straightforward to show the other properties required for
 47 $\mathbf{v}_t(a)$ to be nice. (See Hiraguchi (2003).)

51 Next define $\underline{v}_t(m, c)$ as

$$\underline{v}_t(m, c) = u(c) + \mathbf{v}_t(m - c) \quad (74)$$

55 which is \mathbf{C}^3 since \mathbf{v}_t and u are both \mathbf{C}^3 , and note that our problem's value function
 57 defined in (7) can be written as

$$v_t(m) = \max_c \underline{v}_t(m, c). \quad (75)$$

65 \underline{v}_t is well-defined if and only if $0 < c < m$. Furthermore, $\lim_{c \downarrow 0} \underline{v}_t(m, c) =$
 65 $\lim_{c \uparrow m} \underline{v}_t(m, c) = -\infty$, $\frac{\partial^2 \underline{v}_t(m, c)}{\partial c^2} < 0$, $\lim_{c \downarrow 0} \frac{\partial \underline{v}_t(m, c)}{\partial c} = +\infty$, and $\lim_{c \uparrow m} \frac{\partial \underline{v}_t(m, c)}{\partial c} = -\infty$. It
 67 follows that the $c_t(m)$ defined by

$$c_t(m) = \arg \max_{0 < c < m} \underline{v}_t(m, c) \quad (76)$$

72 exists and is unique, and (7) has an internal solution that satisfies

$$u'(c_t(m)) = \mathbf{v}'_t(m - c_t(m)). \quad (77)$$

80 Since both u and \mathbf{v}_t are strictly concave, both $c_t(m)$ and $a_t(m) = m - c_t(m)$ are strictly
 80 increasing. Since both u and \mathbf{v}_t are three times continuously differentiable, using (77)
 82 we can conclude that $c_t(m)$ is continuously differentiable and

$$c'_t(m) = \frac{\mathbf{v}''_t(a_t(m))}{u''(c_t(m)) + \mathbf{v}''_t(a_t(m))}. \quad (78)$$

89 Similarly we can easily show that $c_t(m)$ is twice continuously differentiable (as is $a_t(m)$)
 89 (See Appendix C.) This implies that $v_t(m)$ is nice, since $v_t(m) = u(c_t(m)) + \mathbf{v}_t(a_t(m))$.

92 C $c_t(m)$ is Twice Continuously Differentiable

100 First we show that $c_t(m)$ is \mathbf{C}^1 . Define y as $y \equiv m + dm$. Since $u'(c_t(y)) - u'(c_t(m)) =$
 100 $\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))$ and $\frac{a_t(y) - a_t(m)}{dm} = 1 - \frac{c_t(y) - c_t(m)}{dm}$,

$$\frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)} = \left(\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)} \right) \frac{c_t(y) - c_t(m)}{dm}$$

117 Since c_t and a_t are continuous and increasing, $\lim_{dm \rightarrow +0} \frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} < 0$ and
 117 $\lim_{dm \rightarrow +0} \frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ are satisfied. Then $\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)} < 0$ for
 117 sufficiently small dm . Hence we obtain a well-defined equation:

$$\frac{c_t(y) - c_t(m)}{dm} = \frac{\frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)}}{\frac{u'(c_t(y)) - u'(c_t(m))}{c_t(y) - c_t(m)} + \frac{\mathbf{v}'_t(a_t(y)) - \mathbf{v}'_t(a_t(m))}{a_t(y) - a_t(m)}}.$$

127 This implies that the right-derivative, $c_t^+(m)$ is well-defined and

$$c_t^+(m) = \frac{\mathbf{v}''_t(a_t(m))}{u''(c_t(m)) + \mathbf{v}''_t(a_t(m))}.$$

141 Similarly we can show that $c_t^+(m) = c_t^-(m)$, which means $c'_t(m)$ exists. Since \mathbf{v}_t is
 141 \mathbf{C}^3 , $c'_t(m)$ exists and is continuous. $c'_t(m)$ is differentiable because \mathbf{v}''_t is \mathbf{C}^1 , $c_t(m)$ is \mathbf{C}^1
 147 and $u''(c_t(m)) + \mathbf{v}''_t(a_t(m)) < 0$. $c''_t(m)$ is given by

$$c''_t(m) = \frac{a'_t(m) \mathbf{v}'''_t(a_t) [u''(c_t) + \mathbf{v}''_t(a_t)] - \mathbf{v}''_t(a_t) [c'_t u'''(c_t) + a'_t \mathbf{v}'''_t(a_t)]}{[u''(c_t) + \mathbf{v}''_t(a_t)]^2}. \quad (79)$$

150 Since $\mathbf{v}''_t(a_t(m))$ is continuous, $c''_t(m)$ is also continuous.

153 D Proof that \mathcal{T} Is a Contraction Mapping

157 We must show that our operator \mathcal{T} satisfies all of Boyd's conditions.

164 Boyd's operator \mathcal{T} maps from $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$ to $\mathcal{C}(\mathcal{A}, \mathcal{B})$. A preliminary requirement is
 164 therefore that $\{\mathcal{T}z\}$ be continuous for any F -bounded z , $\{\mathcal{T}z\} \in \mathcal{C}(\mathbb{R}_{++}, \mathbb{R})$. This is
 164 not difficult to show; see Hiraguchi (2003).

Consider condition (1). For this problem,

$$\begin{aligned} \{\mathcal{T}x\}(m_t) &\text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \{u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} x(m_{t+1})]\} \\ \{\mathcal{T}y\}(m_t) &\text{ is } \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \{u(c_t) + \beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho} y(m_{t+1})]\}, \end{aligned}$$

181 so $x(\bullet) \leq y(\bullet)$ implies $\{\mathcal{T}x\}(m_t) \leq \{\mathcal{T}y\}(m_t)$ by inspection.⁴⁹

Condition (2) requires that $\{\mathcal{T}0\} \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$. By definition,

$$\{\mathcal{T}0\}(m_t) = \max_{c_t \in [\underline{\kappa}m_t, \bar{\kappa}m_t]} \left\{ \left(\frac{c_t^{1-\rho}}{1-\rho} \right) + \beta 0 \right\}$$

⁴⁹For a fixed m_t , recall that m_{t+1} is just a function of c_t and the stochastic shocks.

the solution to which is patently $u(\bar{\kappa}m_t)$. Thus, condition (2) will hold if $(\bar{\kappa}m_t)^{1-\rho}$ is F -bounded. We use the bounding function

$$F(m) = \eta + m^{1-\rho}, \quad (80)$$

for some real scalar $\eta > 0$ whose value will be determined in the course of the proof. Under this definition of F , $\{\mathcal{T}\mathbf{0}\}(m_t) = u(\bar{\kappa}m_t)$ is clearly F -bounded.

Finally, we turn to condition (3), $\{\mathcal{T}(z + \zeta F)\}(m_t) \leq \{\mathcal{T}z\}(m_t) + \zeta\alpha F(m_t)$. The proof will be more compact if we define \check{c} and \check{a} as the consumption and assets functions⁵⁰ associated with $\mathcal{T}z$ and \hat{c} and \hat{a} as the functions associated with $\mathcal{T}(z + \zeta F)$; using this notation, condition (3) can be rewritten

$$u(\hat{c}) + \beta\{\mathbf{E}(z + \zeta F)\}(\hat{a}) \leq u(\check{c}) + \beta\{\mathbf{E}z\}(\check{a}) + \zeta\alpha F.$$

Now note that if we force the \smile consumer to consume the amount that is optimal for the \wedge consumer, value for the \smile consumer must decline (at least weakly). That is,

$$u(\hat{c}) + \beta\{\mathbf{E}z\}(\hat{a}) \leq u(\check{c}) + \beta\{\mathbf{E}z\}(\check{a}).$$

Thus, condition (3) will certainly hold under the stronger condition

$$\begin{aligned} u(\hat{c}) + \beta\{\mathbf{E}(z + \zeta F)\}(\hat{a}) &\leq u(\hat{c}) + \beta\{\mathbf{E}z\}(\hat{a}) + \zeta\alpha F \\ \beta\{\mathbf{E}(z + \zeta F)\}(\hat{a}) &\leq \beta\{\mathbf{E}z\}(\hat{a}) + \zeta\alpha F \\ \beta\zeta\{\mathbf{E}F\}(\hat{a}) &\leq \zeta\alpha F \\ \beta\{\mathbf{E}F\}(\hat{a}) &\leq \alpha F \\ \beta\{\mathbf{E}F\}(\hat{a}) &< F. \end{aligned}$$

where the last line follows because $0 < \alpha < 1$ by assumption.⁵¹

Using $F(m) = \eta + m^{1-\rho}$ and defining $\hat{a}_t = \hat{a}(m_t)$, this condition is

$$\beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho} < \eta(1 - \underbrace{\beta \mathbb{E}_t \Gamma_{t+1}^{1-\rho}}_{=\beth})$$

which by imposing PF-FVAC (equation (25), which says $\beth < 1$) can be rewritten as:

$$\eta > \frac{\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \xi_{t+1})^{1-\rho}] - m_t^{1-\rho}}{1 - \beth}. \quad (81)$$

But since η is an arbitrary constant that we can pick, the proof thus reduces to showing

⁵⁰Section 2.7 proves existence of a continuously differentiable consumption function, which implies the existence of a corresponding continuously differentiable assets function.

⁵¹The remainder of the proof could be reformulated using the second-to-last line at a small cost to intuition.

256 that the numerator of (81) is bounded from above:

$$\begin{aligned}
& (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + \wp\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\hat{a}_t \mathcal{R}_{t+1})^{1-\rho}] - m_t^{1-\rho} \\
& \leq (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + \wp\beta \mathbf{R}^{1-\rho}((1 - \bar{\kappa})m_t)^{1-\rho} - m_t^{1-\rho} \\
& = (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + m_t^{1-\rho} \left(\wp\beta \mathbf{R}^{1-\rho} \left(\wp^{1/\rho} \frac{(\mathbf{R}\beta)^{1/\rho}}{\mathbf{R}} \right)^{1-\rho} - 1 \right) \\
& = (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}((1 - \bar{\kappa})m_t \mathcal{R}_{t+1} + \theta_{t+1}/(1 - \wp))^{1-\rho}] + m_t^{1-\rho} \left(\underbrace{\wp^{1/\rho} \frac{(\mathbf{R}\beta)^{1/\rho}}{\mathbf{R}}}_{<1 \text{ by WRIC}} - 1 \right) \\
& < (1 - \wp)\beta \mathbb{E}_t [\Gamma_{t+1}^{1-\rho}(\underline{\theta}/(1 - \wp))^{1-\rho}] = \beth(1 - \wp)^\rho \underline{\theta}^{1-\rho}.
\end{aligned}$$

261 We can thus conclude that equation (81) will certainly hold for any:

$$\eta > \underline{\eta} = \frac{\beth(1 - \wp)^\rho \underline{\theta}^{1-\rho}}{1 - \beth} \quad (82)$$

263 which is a positive finite number under our assumptions.

267 The proof that \mathcal{T} defines a contraction mapping under the conditions (42) and (39) is
267 now complete.

269 D.1 \mathcal{T} and \mathbf{v}

279 In defining our operator \mathcal{T} we made the restriction $\underline{\kappa}m_t \leq c_t \leq \bar{\kappa}m_t$. However, in the
279 discussion of the consumption function bounds, we showed only (in (43)) that $\underline{\kappa}_t m_t \leq$
279 $c_t(m_t) \leq \bar{\kappa}_t m_t$. (The difference is in the presence or absence of time subscripts on the
279 MPC's.) We have therefore not proven (yet) that the sequence of value functions (7)
279 defines a contraction mapping.

Fortunately, the proof of that proposition is identical to the proof above, except that
we must replace $\bar{\kappa}$ with $\bar{\kappa}_{T-1}$ and the WRIC must be replaced by a slightly stronger
(but still quite weak) condition. The place where these conditions have force is in the
step at (82). Consideration of the prior two equations reveals that a sufficient stronger
condition is

$$\begin{aligned}
& \wp\beta(\mathbf{R}(1 - \bar{\kappa}_{T-1}))^{1-\rho} < 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - \bar{\kappa}_{T-1}) > 1 \\
& (\wp\beta)^{1/(1-\rho)}(1 - (1 + \wp^{1/\rho}\mathbf{P}_\mathbf{R})^{-1}) > 1
\end{aligned}$$

where we have used (41) for $\bar{\kappa}_{T-1}$ (and in the second step the reversal of the inequality
occurs because we have assumed $\rho > 1$ so that we are exponentiating both sides by the
negative number $1 - \rho$). To see that this is a weak condition, note that for small values
of \wp this expression can be further simplified using $(1 + \wp^{1/\rho}\mathbf{P}_\mathbf{R})^{-1} \approx 1 - \wp^{1/\rho}\mathbf{P}_\mathbf{R}$ so that
it becomes

$$(\wp\beta)^{1/(1-\rho)}\wp^{1/\rho}\mathbf{P}_\mathbf{R} > 1$$

$$(\wp\beta)\wp^{(1-\rho)/\rho}\mathbf{P}_R^{1-\rho} < 1$$

$$\beta\wp^{1/\rho}\mathbf{P}_R^{1-\rho} < 1.$$

Calling the weak return patience factor $\mathbf{P}_R^\wp = \wp^{1/\rho}\mathbf{P}_R$ and recalling that the WRIC was $\mathbf{P}_R^\wp < 1$, the expression on the LHS above is $\beta\mathbf{P}_R^{-\rho}$ times the WRPF. Since we usually assume β not far below 1 and parameter values such that $\mathbf{P}_R \approx 1$, this condition is clearly not very different from the WRIC.

The upshot is that under these slightly stronger conditions the value functions for the original problem define a contraction mapping with a unique $v(m)$. But since $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n} = \underline{\kappa}$ and $\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa}$, it must be the case that the $v(m)$ toward which these v_{T-n} 's are converging is the *same* $v(m)$ that was the endpoint of the contraction defined by our operator \mathcal{T} . Thus, under our slightly stronger (but still quite weak) conditions, not only do the value functions defined by (7) converge, they converge to the same unique v defined by \mathcal{T} .⁵²

E Convergence in Euclidian Space

E.1 Convergence of v_t

Boyd's theorem shows that \mathcal{T} defines a contraction mapping in a F -bounded space. We now show that \mathcal{T} also defines a contraction mapping in Euclidian space.

Calling v^* the unique fixed point of the operator \mathcal{T} , since $v^*(m) = \mathcal{T}v^*(m)$,

$$\|v_{T-n+1} - v^*\|_F \leq \alpha^{n-1} \|v_T - v^*\|_F. \quad (83)$$

On the other hand, $v_T - v^* \in \mathcal{C}_F(\mathcal{A}, \mathcal{B})$ and $\kappa = \|v_T - v^*\|_F < \infty$ because v_T and v^* are in $\mathcal{C}_F(\mathcal{A}, \mathcal{B})$. It follows that

$$|v_{T-n+1}(m) - v^*(m)| \leq \kappa \alpha^{n-1} |F(m)|. \quad (84)$$

Then we obtain

$$\lim_{n \rightarrow \infty} v_{T-n+1}(m) = v^*(m). \quad (85)$$

Since $v_T(m) = \frac{m^{1-\rho}}{1-\rho}$, $v_{T-1}(m) \leq \frac{(\bar{\kappa}m)^{1-\rho}}{1-\rho} < v_T(m)$. On the other hand, $v_{T-1} \leq v_T$ means $\mathcal{T}v_{T-1} \leq \mathcal{T}v_T$, in other words, $v_{T-2}(m) \leq v_{T-1}(m)$. Inductively one gets $v_{T-n}(m) \geq v_{T-n-1}(m)$. This means that $\{v_{T-n+1}(m)\}_{n=1}^\infty$ is a decreasing sequence, bounded below by v^* .

E.2 Convergence of c_t

Given the proof that the value functions converge, we now show the pointwise convergence of consumption functions $\{c_{T-n+1}(m)\}_{n=1}^\infty$.

⁵²It seems likely that convergence of the value functions for the original problem could be proven even if only the WRIC were imposed; but that proof is not an essential part of the enterprise of this paper and is therefore left for future work.

47 Consider any convergent subsequence $\{c_{T-n(i)}(m)\}$ of $\{c_{T-n+1}(m)\}_{n=1}^{\infty}$ converging to
 49 c^* . By the definition of $c_{T-n}(m)$, we have

$$u(c_{T-n(i)}(m)) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)] \geq u(c_{T-n(i)}) + \beta \mathbb{E}_{T-n(i)}[\Gamma_{T-n(i)+1}^{1-\rho} v_{T-n(i)+1}(m)], \quad (86)$$

51 for any $c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]$. Now letting $n(i)$ go to infinity, it follows that the left
 51 hand side converges to $u(c^*) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho} v(m)]$, and the right hand side converges to
 51 $u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_t^{1-\rho} v(m)]$. So the limit of the preceding inequality as $n(i)$ approaches
 53 infinity implies

$$u(c^*) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} v(m)] \geq u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} v(m)]. \quad (87)$$

55 Hence, $c^* \in \arg \max_{c_{T-n(i)} \in [\underline{\kappa}m, \bar{\kappa}m]} \{u(c_{T-n(i)}) + \beta \mathbb{E}_t[\Gamma_{t+1}^{1-\rho} v(m)]\}$. By the uniqueness of $c(m)$,
 55 $c^* = c(m)$.

8 F Equality of Aggregate Consumption Growth and 8 Income Growth with Transitory Shocks

13 Section 4.2 asserted that in the absence of permanent shocks it is possible to prove that
 13 the growth factor for aggregate consumption approaches that for aggregate permanent
 13 income. This section establishes that result.

15 First define $a(m)$ as the function that yields optimal end-of-period assets as a function
 15 of m .

20 Suppose the population starts in period t with an arbitrary value for $\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})$.
 20 Then if \check{m} is the invariant mean level of m we can define a ‘mean MPS away from \check{m} ’
 23 function :

$$\bar{a}(\Delta) = \Delta^{-1} \int_{\check{m}}^{\check{m}+\Delta} a'(z) dz$$

26 where the combination of the bar and the ‘ are meant to signify that this is the average
 26 value of the derivative over the interval. Since $\psi_{t+1,i} = 1$, $\mathcal{R}_{t+1,i}$ is a constant at \mathcal{R} , if
 30 we define a as the value of a corresponding to $m = \check{m}$, we can write

$$a_{t+1,i} = a + (m_{t+1,i} - \check{m}) \bar{a}(\overbrace{\mathcal{R}a_{t,i} + \xi_{t+1,i} - \check{m}}^{m_{t+1,i}})$$

36 SO

$$\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i}) = \text{cov}_t(\bar{a}(\mathcal{R}a_{t,i} + \xi_{t+1,i} - \check{m}), \Gamma \mathbf{p}_{t,i}).$$

43 But since $R^{-1}(\phi R \beta)^{1/\rho} < \bar{a}(m) < \mathbf{P}_R$,

$$|\text{cov}_t((\phi R \beta)^{1/\rho} a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < |\text{cov}_t(\mathbf{P} a_{t+1,i}, \mathbf{p}_{t+1,i})|$$

46 and for the version of the model with no permanent shocks the GIC-Nrm says that
 49 $\mathbf{P} < \Gamma$, while the FHWC says that $\Gamma < R$

$$|\text{cov}_t(a_{t+1,i}, \mathbf{p}_{t+1,i})| < \Gamma |\text{cov}_t(a_{t,i}, \mathbf{p}_{t,i})|.$$

This means that from any arbitrary starting value, the relative size of the covariance term shrinks to zero over time (compared to the $A\Gamma^n$ term which is growing steadily by the factor Γ). Thus, $\lim_{n \rightarrow \infty} \mathbf{A}_{t+n+1}/\mathbf{A}_{t+n} = \Gamma$.

This logic unfortunately does not go through when there are permanent shocks, because the $\mathcal{R}_{t+1,i}$ terms are not independent of the permanent income shocks.

To see the problem clearly, define $\bar{\mathcal{R}} = \mathbb{M}[\mathcal{R}_{t+1,i}]$ and consider a first order Taylor expansion of $\bar{a}(m_{t+1,i})$ around $\hat{m}_{t+1,i} = \bar{\mathcal{R}}a_{t,i} + 1$,

$$\bar{a}_{t+1,i} \approx \bar{a}(\hat{m}_{t+1,i}) + \bar{a}'(\hat{m}_{t+1,i})(m_{t+1,i} - \hat{m}_{t+1,i}).$$

The problem comes from the \bar{a}' term. The concavity of the consumption function implies convexity of the a function, so this term is strictly positive but we have no theory to place bounds on its size as we do for its level \bar{a} . We cannot rule out by theory that a positive shock to permanent income (which has a negative effect on $m_{t+1,i}$) could have a (locally) unboundedly positive effect on \bar{a}' (as for instance if it pushes the consumer arbitrarily close to the self-imposed liquidity constraint).

G The Limiting MPC's

For $m_t > 0$ we can define $e_t(m_t) = c_t(m_t)/m_t$ and $a_t(m_t) = m_t - c_t(m_t)$ and the Euler equation (8) can be rewritten

$$\begin{aligned} e_t(m_t)^{-\rho} &= \beta R \mathbb{E}_t \left[\left(e_{t+1}(m_{t+1}) \left(\frac{\overbrace{Ra_t(m_t) + \Gamma_{t+1}\xi_{t+1}}^{=m_{t+1}\Gamma_{t+1}}}{m_t} \right) \right)^{-\rho} \right] \\ &= (1 - \wp) \beta R m_t^\rho \mathbb{E}_t [(e_{t+1}(m_{t+1}) m_{t+1} \Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0] \\ &\quad + \wp \beta R^{1-\rho} \mathbb{E}_t \left[\left(e_{t+1}(\mathcal{R}_{t+1} a_t(m_t)) \frac{m_t - c_t(m_t)}{m_t} \right)^{-\rho} \mid \xi_{t+1} = 0 \right]. \end{aligned}$$

Consider the first conditional expectation in (8), recalling that if $\xi_{t+1} > 0$ then $\xi_{t+1} \equiv \theta_{t+1}/(1 - \wp)$. Since $\lim_{m \downarrow 0} a_t(m) = 0$, $\mathbb{E}_t[(e_{t+1}(m_{t+1}) m_{t+1} \Gamma_{t+1})^{-\rho} \mid \xi_{t+1} > 0]$ is contained within bounds defined by $(e_{t+1}(\underline{\theta}/(1 - \wp)) \Gamma \underline{\psi} \underline{\theta}/(1 - \wp))^{-\rho}$ and $(e_{t+1}(\bar{\theta}/(1 - \wp)) \Gamma \bar{\psi} \bar{\theta}/(1 - \wp))^{-\rho}$ both of which are finite numbers, implying that the whole term multiplied by $(1 - \wp)$ goes to zero as m_t^ρ goes to zero. As $m_t \downarrow 0$ the expectation in the other term goes to $\bar{\kappa}_{t+1}^{-\rho} (1 - \bar{\kappa}_t)^{-\rho}$. (This follows from the strict concavity and differentiability of the consumption function.) It follows that the limiting $\bar{\kappa}_t$ satisfies $\bar{\kappa}_t^{-\rho} = \beta \wp R^{1-\rho} \bar{\kappa}_{t+1}^{-\rho} (1 - \bar{\kappa}_t)^{-\rho}$. Exponentiating by ρ , we can conclude that

$$\begin{aligned} \bar{\kappa}_t &= \wp^{-1/\rho} (\beta R)^{-1/\rho} R (1 - \bar{\kappa}_t) \bar{\kappa}_{t+1} \\ &\underbrace{\wp^{1/\rho} R^{-1} (\beta R)^{1/\rho}}_{\equiv \wp^{1/\rho} \mathbf{P}_R} \bar{\kappa}_t = (1 - \bar{\kappa}_t) \bar{\kappa}_{t+1} \end{aligned}$$

51 which yields a useful recursive formula for the maximal marginal propensity to consume:

$$\begin{aligned} (\wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_t)^{-1} &= (1 - \bar{\kappa}_t)^{-1} \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} (1 - \bar{\kappa}_t) &= \wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_{t+1}^{-1} \\ \bar{\kappa}_t^{-1} &= 1 + \wp^{1/\rho} \mathbf{P}_R \bar{\kappa}_{t+1}^{-1}. \end{aligned}$$

56 As noted in the main text, we need the WRIC (42) for this to be a convergent sequence:

$$0 \leq \wp^{1/\rho} \mathbf{P}_R < 1, \quad (88)$$

60 Since $\bar{\kappa}_T = 1$, iterating (88) backward to infinity (because we are interested in the
63 limiting consumption function) we obtain:

$$\lim_{n \rightarrow \infty} \bar{\kappa}_{T-n} = \bar{\kappa} \equiv 1 - \wp^{1/\rho} \mathbf{P}_R \quad (89)$$

65 and we will therefore call $\bar{\kappa}$ the ‘limiting maximal MPC.’

72 The minimal MPC’s are obtained by considering the case where $m_t \uparrow \infty$. If the
72 FHCW holds, then as $m_t \uparrow \infty$ the proportion of current and future consumption that
72 will be financed out of capital approaches 1. Thus, the terms involving ξ_{t+1} in (88) can
74 be neglected, leading to a revised limiting Euler equation

$$(m_t e_t(m_t))^{-\rho} = \beta R \mathbb{E}_t [(e_{t+1}(a_t(m_t) \mathcal{R}_{t+1}) (Ra_t(m_t)))^{-\rho}]$$

76 and we know from L’Hôpital’s rule that $\lim_{m_t \rightarrow \infty} e_t(m_t) = \underline{\kappa}_t$, and $\lim_{m_t \rightarrow \infty} e_{t+1}(a_t(m_t) \mathcal{R}_{t+1}) =$
79 $\underline{\kappa}_{t+1}$ so a further limit of the Euler equation is

$$\begin{aligned} (m_t \underline{\kappa}_t)^{-\rho} &= \beta R (\underline{\kappa}_{t+1} R (1 - \underline{\kappa}_t) m_t)^{-\rho} \\ \underbrace{R^{-1} \mathbf{P}}_{\equiv \mathbf{P}_R = (1 - \underline{\kappa})} \underline{\kappa}_t &= (1 - \underline{\kappa}_t) \underline{\kappa}_{t+1} \end{aligned}$$

83 and the same sequence of derivations used above yields the conclusion that if the RIC $0 \leq$
83 $\mathbf{P}_R < 1$ holds, then a recursive formula for the minimal marginal propensity to consume
85 is given by

$$\underline{\kappa}_t^{-1} = 1 + \underline{\kappa}_{t+1}^{-1} \mathbf{P}_R \quad (90)$$

90 so that $\{\underline{\kappa}_{T-n}^{-1}\}_{n=0}^{\infty}$ is also an increasing convergent sequence, and we define

$$\underline{\kappa}^{-1} \equiv \lim_{n \uparrow \infty} \underline{\kappa}_{T-n}^{-1} \quad (91)$$

93 as the limiting (inverse) marginal MPC. If the RIC does *not* hold, then $\lim_{n \rightarrow \infty} \underline{\kappa}_{T-n}^{-1} =$
93 ∞ and so the limiting MPC is $\underline{\kappa} = 0$.

96 For the purpose of constructing the limiting perfect foresight consumption function,
96 it is useful further to note that the PDV of consumption is given by

$$\underbrace{c_t (1 + \mathbf{P}_R + \mathbf{P}_R^2 + \dots)}_{= 1 + \mathbf{P}_R (1 + \mathbf{P}_R \underline{\kappa}_{t+2}^{-1}) \dots} = c_t \underline{\kappa}_{T-n}^{-1}.$$

101 which, combined with the intertemporal budget constraint, yields the usual formula for

103 the perfect foresight consumption function:

$$c_t = (b_t + h_t)\underline{\kappa}_t \quad (92)$$

7 H The Perfect Foresight Liquidity Constrained 7 Solution as a Limit

12 Formally, suppose we change the description of the problem by making the following
15 two assumptions:

$$\begin{aligned} \wp &= 0 \\ c_t &\leq m_t, \end{aligned}$$

21 and we designate the solution to this consumer's problem $\hat{c}_t(m)$. We will henceforth refer
21 to this as the problem of the 'restrained' consumer (and, to avoid a common confusion,
21 we will refer to the consumer as 'constrained' only in circumstances when the constraint
21 is actually binding).

27 Redesignate the consumption function that emerges from our original problem for a
27 given fixed \wp as $c_t(m; \wp)$ where we separate the arguments by a semicolon to distinguish
27 between m , which is a state variable, and \wp , which is not. The proposition we wish to
29 demonstrate is

$$\lim_{\wp \downarrow 0} c_t(m; \wp) = \hat{c}_t(m). \quad (93)$$

37 We will first examine the problem in period $T - 1$, then argue that the desired result
37 propagates to earlier periods. For simplicity, suppose that the interest, growth, and
37 time-preference factors are $\beta = R = \Gamma = 1$, and there are no permanent shocks, $\psi = 1$;
37 the results below are easily generalized to the full-fledged version of the problem.

42 The solution to the restrained consumer's optimization problem can be obtained as
42 follows. Assuming that the consumer's behavior in period T is given by $c_T(m)$ (in
44 practice, this will be $c_T(m) = m$), consider the unrestrained optimization problem

$$\hat{a}_{T-1}^*(m) = \arg \max_a \left\{ u(m - a) + \int_{\underline{\theta}}^{\bar{\theta}} v_T(a + \theta) d\mathcal{F}_{\theta} \right\}. \quad (94)$$

49 As usual, the envelope theorem tells us that $v'_T(m) = u'(c_T(m))$ so the expected
51 marginal value of ending period $T - 1$ with assets a can be defined as

$$\hat{v}'_{T-1}(a) \equiv \int_{\underline{\theta}}^{\bar{\theta}} u'(c_T(a + \theta)) d\mathcal{F}_{\theta},$$

56 and the solution to (94) will satisfy

$$u'(m - a) = \hat{v}'_{T-1}(a). \quad (95)$$

64 $\hat{a}_{T-1}^*(m)$ therefore answers the question "With what level of assets would the restrained
64 consumer like to end period $T - 1$ if the constraint $c_{T-1} \leq m_{T-1}$ did not exist?" (Note
64 that the restrained consumer's income process remains different from the process for

the unrestrained consumer so long as $\wp > 0$.) The restrained consumer's actual asset position will be

$$\hat{a}_{T-1}(m) = \max[0, \hat{a}_{T-1}^*(m)],$$

reflecting the inability of the restrained consumer to spend more than current resources, and note (as pointed out by Deaton (1991)) that

$$m_{\#}^1 = (\hat{v}'_{T-1}(0))^{-1/\rho}$$

is the cusp value of m at which the constraint makes the transition between binding and non-binding in period $T - 1$.

Analogously to (95), defining

$$\mathbf{v}'_{T-1}(a; \wp) \equiv \left[\wp a^{-\rho} + (1 - \wp) \int_{\underline{\theta}}^{\bar{\theta}} (c_T(a + \theta/(1 - \wp)))^{-\rho} d\mathcal{F}_{\theta} \right], \quad (96)$$

the Euler equation for the original consumer's problem implies

$$(m - a)^{-\rho} = \mathbf{v}'_{T-1}(a; \wp) \quad (97)$$

with solution $\hat{a}_{T-1}^*(m; \wp)$. Now note that for any fixed $a > 0$, $\lim_{\wp \downarrow 0} \mathbf{v}'_{T-1}(a; \wp) = \hat{v}'_{T-1}(a)$. Since the LHS of (95) and (97) are identical, this means that $\lim_{\wp \downarrow 0} \hat{a}_{T-1}^*(m; \wp) = \hat{a}_{T-1}^*(m)$. That is, for any fixed value of $m > m_{\#}^1$ such that the consumer subject to the restraint would voluntarily choose to end the period with positive assets, the level of end-of-period assets for the unrestrained consumer approaches the level for the restrained consumer as $\wp \downarrow 0$. With the same a and the same m , the consumers must have the same c , so the consumption functions are identical in the limit.

Now consider values $m \leq m_{\#}^1$ for which the restrained consumer is constrained. It is obvious that the baseline consumer will never choose $a \leq 0$ because the first term in (96) is $\lim_{a \downarrow 0} \wp a^{-\rho} = \infty$, while $\lim_{a \downarrow 0} (m - a)^{-\rho}$ is finite (the marginal value of end-of-period assets approaches infinity as assets approach zero, but the marginal utility of consumption has a finite limit for $m > 0$). The subtler question is whether it is possible to rule out strictly positive a for the unrestrained consumer.

The answer is yes. Suppose, for some $m < m_{\#}^1$, that the unrestrained consumer is considering ending the period with any positive amount of assets $a = \delta > 0$. For any such δ we have that $\lim_{\wp \downarrow 0} \mathbf{v}'_{T-1}(a; \wp) = \hat{v}'_{T-1}(a)$. But by assumption we are considering a set of circumstances in which $\hat{a}_{T-1}^*(m) < 0$, and we showed earlier that $\lim_{\wp \downarrow 0} \hat{a}_{T-1}^*(m; \wp) = \hat{a}_{T-1}^*(m)$. So, having assumed $a = \delta > 0$, we have proven that the consumer would optimally choose $a < 0$, which is a contradiction. A similar argument holds for $m = m_{\#}^1$.

These arguments demonstrate that for any $m > 0$, $\lim_{\wp \downarrow 0} c_{T-1}(m; \wp) = \hat{c}_{T-1}(m)$ which is the period $T - 1$ version of (93). But given equality of the period $T - 1$ consumption functions, backwards recursion of the same arguments demonstrates that the limiting consumption functions in previous periods are also identical to the constrained function.

Note finally that another intuitive confirmation of the equivalence between the two problems is that our formula (89) for the maximal marginal propensity to consume

133 satisfies

$$\lim_{\varphi \downarrow 0} \bar{\kappa} = 1,$$

137 which makes sense because the marginal propensity to consume for a constrained re-
137 strained consumer is 1 by our definitions of ‘constrained’ and ‘restrained.’

10 I Endogenous Gridpoints Solution Method

13 The model is solved using an extension of the method of endogenous gridpoints (Carroll
13 (2006)): A grid of possible values of end-of-period assets \vec{a} is defined, and at these points,
13 marginal end-of-period- t value is computed as the discounted next-period expected
13 marginal utility of consumption (which the Envelope theorem says matches expected
13 marginal value). The results are then used to identify the corresponding levels of
13 consumption at the beginning of the period:⁵³

$$\begin{aligned} u'(\mathbf{c}_t(\vec{a})) &= R\beta \mathbb{E}_t[u'(\Gamma_{t+1}c_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))] \\ \vec{c}_t \equiv \mathbf{c}_t(\vec{a}) &= (R\beta \mathbb{E}_t[(\Gamma_{t+1}c_{t+1}(\mathcal{R}_{t+1}\vec{a} + \xi_{t+1}))^{-\rho}])^{-1/\rho}. \end{aligned}$$

25 The dynamic budget constraint can then be used to generate the corresponding m ’s:

$$\vec{m}_t = \vec{a} + \vec{c}_t.$$

35 An approximation to the consumption function could be constructed by linear inter-
35 polation between the $\{\vec{m}, \vec{c}\}$ points. But a vastly more accurate approximation can be
35 made (for a given number of gridpoints) if the interpolation is constructed so that it also
35 matches the marginal propensity to consume at the gridpoints. Differentiating (98) with
35 respect to a (and dropping policy function arguments for simplicity) yields a marginal
39 propensity to *have consumed* \mathbf{c}^a at each gridpoint:

$$\begin{aligned} u''(\mathbf{c}_t)\mathbf{c}_t^a &= R\beta \mathbb{E}_t[u''(\Gamma_{t+1}c_{t+1})\Gamma_{t+1}c_{t+1}^m \mathcal{R}_{t+1}] \\ &= R\beta \mathbb{E}_t[u''(\Gamma_{t+1}c_{t+1})Rc_{t+1}^m] \\ \mathbf{c}_t^a &= R\beta \mathbb{E}_t[u''(\Gamma_{t+1}c_{t+1})Rc_{t+1}^m]/u''(\mathbf{c}_t) \end{aligned}$$

and the marginal propensity to consume at the beginning of the period is obtained from
the marginal propensity to have consumed by noting that, if we define $\mathbf{m}(a) = \mathbf{c}(a) - a$,

$$\begin{aligned} c &= \mathbf{m} - a \\ \mathbf{c}^a + 1 &= \mathbf{m}^a \end{aligned}$$

48 which, together with the chain rule $\mathbf{c}^a = c^m \mathbf{m}^a$, yields the MPC from

$$\begin{aligned} c^m(\mathbf{c}^a + 1) &= \mathbf{c}^a \\ c^m &= \mathbf{c}^a / (1 + \mathbf{c}^a) \end{aligned}$$

⁵³The software can also solve a version of the model with explicit liquidity constraints, where the Envelope condition does not hold.

and we call the vector of MPC's at the \vec{m}_t gridpoints $\vec{\kappa}_t$.

J The Terminal/Limiting Consumption Function

For any set of parameter values that satisfy the conditions required for convergence, the problem can be solved by setting the terminal consumption function to $c_T(m) = m$ and constructing $\{c_{T-1}, c_{T-2}, \dots\}$ by time iteration (a method that will converge to $c(m)$ by standard theorems). But $c_T(m) = m$ is very far from the final converged consumption rule $c(m)$,⁵⁴ and thus many periods of iteration will likely be required to obtain a candidate rule that even remotely resembles the converged function.

A natural alternative choice for the terminal consumption rule is the solution to the perfect foresight liquidity constrained problem, to which the model's solution converges (under specified parametric restrictions) as all forms of uncertainty approach zero (as discussed in the main text). But a difficulty with this idea is that the perfect foresight liquidity constrained solution is 'kinked.' The slope of the consumption function changes discretely at the points $\{m_{\#}^1, m_{\#}^2, \dots\}$. This is a practical problem because it rules out the use of derivatives of the consumption function in the approximate representation of $c(m)$, thereby preventing the enormous increase in efficiency obtainable from a higher-order approximation.

Our solution is simple: The formulae in another appendix that identify kink points on $c(m)$ for integer values of n (e.g., $c_{\#}^n = \mathbf{D}_{\Gamma}^{-n}$) are continuous functions of n ; the conclusion that $c(m)$ is piecewise linear between the kink points does not require that the *terminal consumption rule* (from which time iteration proceeds) also be piecewise linear. Thus, for values $n \geq 0$ we can construct a smooth function $\check{c}(m)$ that matches the true perfect foresight liquidity constrained consumption function at the set of points corresponding to integer periods in the future, but satisfies the (continuous, and greater at non-kink points) consumption rule defined from the appendix's formulas by noninteger values of n at other points.⁵⁵

This strategy generates a smooth limiting consumption function – except at the remaining kink point defined by $\{m_{\#}^0, c_{\#}^0\}$. Below this point, the solution must match $c(m) = m$ because the constraint is binding. At $m = m_{\#}^0$ the MPC discretely drops (that is, $\lim_{m \uparrow m_{\#}^0} c'(m) = 1$ while $\lim_{m \downarrow m_{\#}^0} c'(m) = \kappa_{\#}^0 < 1$).

Such a kink point causes substantial problems for numerical solution methods (like the one we use, described below) that rely upon the smoothness of the limiting consumption function.

Our solution is to use, as the terminal consumption rule, a function that is identical to the (smooth) continuous consumption rule $\check{c}(m)$ above some $n \geq \underline{n}$, but to replace $\check{c}(m)$ between $m_{\#}^0$ and $m_{\#}^{\underline{n}}$ with the unique polynomial function $\hat{c}(m)$ that satisfies the following criteria:

⁵⁴Unless $\beta \approx +0$.

⁵⁵In practice, we calculate the first and second derivatives of c and use piecewise polynomial approximation methods that match the function at these points.

- 57 1. $\hat{c}(m_{\#}^0) = c_{\#}^0$
- 58 2. $\hat{c}'(m_{\#}^0) = 1$
- 59 3. $\hat{c}'(m_{\#}^{\underline{n}}) = (dc_{\#}^n/dn)(dm_{\#}^n/dn)^{-1}|_{n=\underline{n}}$
- 60 4. $\hat{c}''(m_{\#}^{\underline{n}}) = (d^2c_{\#}^n/dn^2)(d^2m_{\#}^n/dn^2)^{-1}|_{n=\underline{n}}$

66 where \underline{n} is chosen judgmentally in a way calculated to generate a good compromise
 66 between smoothness of the limiting consumption function $\check{c}(m)$ and fidelity of that
 66 function to the $c(m)$ (see the actual code for details).

75 We thus define the terminal function as

$$c_T(m) = \begin{cases} 0 < m \leq m_{\#}^0 & m \\ m_{\#}^0 < m < m_{\#}^{\underline{n}} & \check{c}(m) \\ m_{\#}^{\underline{n}} < m & c(m) \end{cases} \quad (98)$$

81 Since the precautionary motive implies that in the presence of uncertainty the optimal
 81 level of consumption is below the level that is optimal without uncertainty, and since
 83 $\check{c}(m) \geq c(m)$, implicitly defining $m = e^{\mu}$ (so that $\mu = \log m$), we can construct

$$\chi_t(\mu) = \log(1 - c_t(e^{\mu})/c_T(e^{\mu})) \quad (99)$$

86 which must be a number between $-\infty$ and $+\infty$ (since $0 < c_t(m) < \check{c}(m)$ for $m > 0$).
 86 This function turns out to be much better behaved (as a numerical observation; no formal
 86 proof is offered) than the level of the optimal consumption rule $c_t(m)$. In particular,
 86 $\chi_t(\mu)$ is well approximated by linear functions both as $m \downarrow 0$ and as $m \uparrow \infty$.

91 Differentiating with respect to μ and dropping consumption function arguments yields

$$\chi_t'(\mu) = \left(\frac{-\left(\frac{c_t'c_T - c_t c_T'}{c_T^2} e^{\mu}\right)}{1 - c_t/c_T} \right) \quad (100)$$

97 which can be solved for

$$c_t' = (c_t c_T' / c_T) - ((c_T - c_t)/m) \chi_t'. \quad (101)$$

111 Similarly, we can solve (99) for

$$c_t(m) = (1 - e^{\chi_t(\log m)}) c_T(m). \quad (102)$$

115 Thus, having approximated χ_t , we can recover from it the level and derivative(s) of
 115 c_t .

21 K Relational Diagrams for the Inequality Conditions

25 This appendix explains in detail the paper's ‘inequalities’ diagrams (Figures 1, 3).



Figure 9 Inequality Conditions for Perfect Foresight Model
(Start at a node and follow arrows)

K.1 The Unconstrained Perfect Foresight Model

A simple illustration is presented in Figure 9, whose three nodes represent values of the absolute patience factor \mathbf{P} , the permanent-income growth factor Γ , and the riskfree interest factor R . The arrows represent imposition of the labeled inequality condition (like, the uppermost arrow, pointing from \mathbf{P} to Γ , reflects imposition of the **PF-GICNrm** condition (clicking **PF-GICNrm** should take you to its definition; definitions of other conditions are also linked below)).⁵⁶ Annotations inside parenthetical expressions containing \equiv are there to make the diagram readable for someone who may not immediately remember terms and definitions from the main text. (Such a reader might also want to be reminded that R, β , and Γ are all in \mathbb{R}_{++} , and that $\rho > 1$).

Navigation of the diagram is simple: Start at any node, and deduce a chain of inequalities by following any arrow that exits that node, and any arrows that exit from successive nodes. Traversal must stop upon arrival at a node with no exiting arrows. So, for example, we can start at the \mathbf{P} node and impose the **PF-GICNrm** and then the **FWC**, and see that imposition of these conditions allows us to conclude that $\mathbf{P} < R$.

One could also impose $\mathbf{P} < R$ directly (without imposing **PF-GICNrm** and **FWC**) by following the downward-sloping diagonal arrow exiting \mathbf{P} . Although alternate routes from one node to another all justify the same core conclusion ($\mathbf{P} < R$, in this case), \neq symbol in the center is meant to convey that these routes are not identical in other respects. This notational convention is used in **category theory diagrams**,⁵⁷ to indicate that the diagram is not **commutative**.⁵⁸

Negation of a condition is indicated by the reversal of the corresponding arrow. For example, negation of the **RIC**, **RIC** $\equiv \mathbf{P} > R$, would be represented by moving the

⁵⁶For convenience, the equivalent (\equiv) mathematical statement of each condition is expressed nearby in parentheses.

⁵⁷For a popular introduction to category theory, see Riehl (2017).

⁵⁸But the rest of our notation does not necessarily abide by the other conventions of category theory diagrams.

44 arrowhead from the bottom right to the top left of the line segment connecting \mathbf{D} and
 44 R .

52 If we were to start at R and then impose ~~FHWC~~, that would reverse the arrow
 52 connecting R and Γ , but the Γ node would then have no exiting arrows so no further
 52 deductions could be made. However, if we *also* reversed ~~PF-GICNrm~~ (that is, if we
 52 imposed ~~PF-GICNrm~~), that would take us to the \mathbf{D} node, and we could deduce $R > \mathbf{D}$.
 52 However, we would have to stop traversing the diagram at this point, because the arrow
 52 exiting from the \mathbf{D} node points back to our starting point, which (if valid) would lead us to
 52 the conclusion that $R > R$. Thus, the reversal of the two earlier conditions (imposition
 52 of ~~FHWC~~ and ~~PF-GICNrm~~) requires us also to reverse the final condition, giving us
 52 ~~RIC~~.⁵⁹

54 Under these conventions, Figure 1 in the main text presents a modified version of
 54 the diagram extended to incorporate the PF-FVAC (reproduced here for convenient
 54 reference).

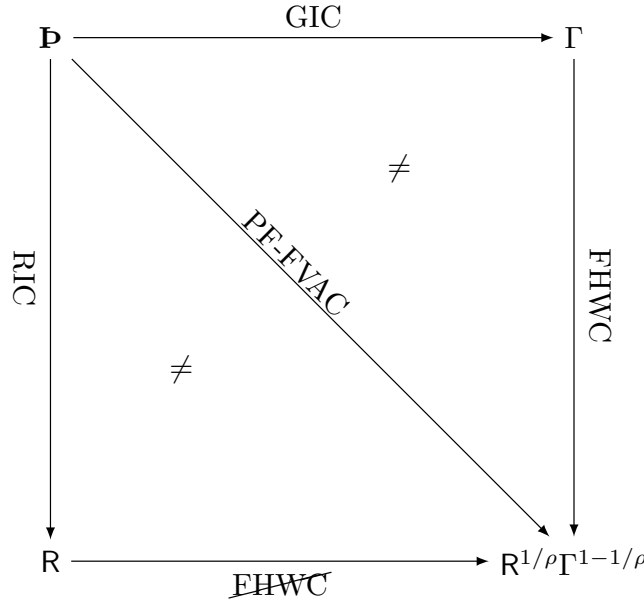


Figure 10 Relation of ~~PF-GICNrm~~, ~~FHWC~~, ~~RIC~~, and ~~PF-FVAC~~

An arrowhead points to the larger of the two quantities being compared. For example, the diagonal arrow indicates that $\mathbf{D} < R^{1/\rho}\Gamma^{1-1/\rho}$, which is an alternative way of writing the ~~PF-FVAC~~, (25)

64 This diagram can be interpreted, for example, as saying that, starting at the \mathbf{D} node, it
 64 is possible to derive the ~~PF-FVAC~~⁶⁰ by imposing both the ~~PF-GICNrm~~ and the ~~FHWC~~;

⁵⁹The corresponding algebra is

$$\begin{aligned} \text{FHWC} : & \quad R < \Gamma \\ \text{PF-GICNrm} : & \quad \Gamma < \mathbf{D} \\ \Rightarrow \text{RIC} : & \quad R < \mathbf{D}, \end{aligned}$$

⁶⁰in the form $\mathbf{D} < (R/\Gamma)^{1/\rho}\Gamma$

64 or by imposing **RIC** and **FHWC**. Or, starting at the Γ node, we can follow the imposition
64 of the **FHWC** (twice - reversing the arrow labeled **FHWC**) and then **RIC** to reach the
68 conclusion that $\mathbf{P} < \Gamma$. Algebraically,

$$\begin{aligned} \text{FHWC} : \quad & \Gamma < R \\ \text{RIC} : \quad & R < \mathbf{P} \\ & \Gamma < \mathbf{P} \end{aligned} \tag{103}$$

71 which leads to the negation of both of the conditions leading into \mathbf{P} . ~~PE-GICNrm~~ is
71 obtained directly as the last line in (103) and ~~PE-FVAC~~ follows if we start by multiplying
71 the Return Patience Factor (RPF= \mathbf{P}/R) by the FHWF(= Γ/R) raised to the power
71 $1/\rho - 1$, which is negative since we imposed $\rho > 1$. **FHWC** implies FHWF < 1 so when
71 FHWF is raised to a negative power the result is greater than one. Multiplying the RPF
71 (which exceeds 1 because **RIC**) by another number greater than one yields a product
75 that must be greater than one:

$$\begin{aligned} 1 &< \overbrace{\left(\frac{(R\beta)^{1/\rho}}{R} \right)}^{>1 \text{ from RIC}} \overbrace{(\Gamma/R)^{1/\rho-1}}^{>1 \text{ from FHWC}} \\ 1 &< \left(\frac{(R\beta)^{1/\rho}}{(R/\Gamma)^{1/\rho} R \Gamma / R} \right) \\ R^{1/\rho} \Gamma^{1-1/\rho} &= (R/\Gamma)^{1/\rho} \Gamma < \mathbf{P} \end{aligned}$$

77 which is one way of writing ~~PE-FVAC~~.

79 The complexity of this algebraic calculation illustrates the usefulness of the diagram,
79 in which one merely needs to follow arrows to reach the same result.

81 After the warmup of constructing these conditions for the perfect foresight case, we
81 can represent the relationships between all the conditions in both the perfect foresight
81 case and the case with uncertainty as shown in Figure 3 in the paper (reproduced here).

96 Finally, the next diagram substitutes the values of the various objects in the diagram
96 under the baseline parameter values and verifies that all of the asserted inequality
96 conditions hold true.

7 L When Is Consumption Growth Declining in m ?

Figure 4 depicts the expected consumption growth factor as a strictly declining function of the cash-on-hand ratio. To investigate this, define

$$\Upsilon(m_t) \equiv \Gamma_{t+1} c(\mathcal{R}_{t+1} a(m_t) + \xi_{t+1}) / c(m_t) = \mathbf{c}_{t+1} / \mathbf{c}_t$$

and the proposition in which we are interested is

$$(d/dm_t) \mathbb{E}_t[\underbrace{\Upsilon(m_t)}_{\equiv \Upsilon_{t+1}}] < 0$$



Figure 11 Relation of All Inequality Conditions

or differentiating through the expectations operator, what we want is

$$\mathbb{E}_t \left[\Gamma_{t+1} \left(\frac{c'(m_{t+1}) \mathcal{R}_{t+1} a'(m_t) c(m_t) - c(m_{t+1}) c'(m_t)}{c(m_t)^2} \right) \right] < 0. \quad (104)$$

Henceforth indicating appropriate arguments by the corresponding subscript (e.g. $c'_{t+1} \equiv c'(m_{t+1})$), since $\Gamma_{t+1} \mathcal{R}_{t+1} = \mathbf{R}$, the portion of the LHS of equation (104) in brackets can be manipulated to yield

$$\begin{aligned} c_t \mathbf{\Upsilon}'_{t+1} &= c'_{t+1} a'_t \mathbf{R} - c'_t \Gamma_{t+1} c_{t+1} / c_t \\ &= c'_{t+1} a'_t \mathbf{R} - c'_t \mathbf{\Upsilon}_{t+1}. \end{aligned}$$

Now differentiate the Euler equation with respect to m_t :

$$\begin{aligned} 1 &= \mathbf{R} \beta \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho}] \\ 0 &= \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1} \mathbf{\Upsilon}'_{t+1}] \\ &= \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1}] \mathbb{E}_t[\mathbf{\Upsilon}'_{t+1}] + \text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) \\ \mathbb{E}_t[\mathbf{\Upsilon}'_{t+1}] &= -\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) / \mathbb{E}_t[\mathbf{\Upsilon}_{t+1}^{-\rho-1}] \end{aligned}$$

but since $\mathbf{\Upsilon}_{t+1} > 0$ we can see from (105) that (104) is equivalent to

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, \mathbf{\Upsilon}'_{t+1}) > 0$$

which, using (105), will be true if

$$\text{cov}_t(\mathbf{\Upsilon}_{t+1}^{-\rho-1}, c'_{t+1} a'_t \mathbf{R} - c'_t \mathbf{\Upsilon}_{t+1}) > 0$$

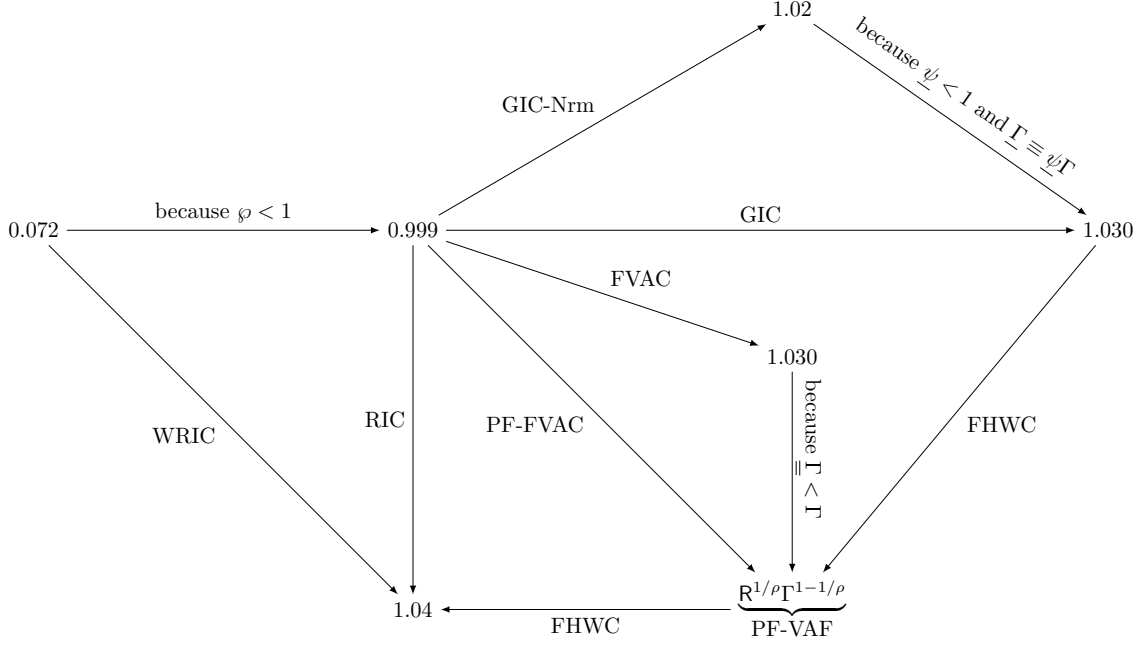


Figure 12 Numerical Relation of All Inequality Conditions

52 which in turn will be true if both

$$\text{cov}_t(\Upsilon_{t+1}^{-\rho-1}, c'_{t+1}) > 0$$

and

$$\text{cov}_t(\Upsilon_{t+1}^{-\rho-1}, \Upsilon_{t+1}) < 0.$$

The latter proposition is obviously true under our assumption $\rho > 1$. The former will be true if

$$\text{cov}_t((\Gamma\psi_{t+1}c(m_{t+1}))^{-\rho-1}, c'(m_{t+1})) > 0.$$

75 The two shocks cause two kinds of variation in m_{t+1} . Variations due to ξ_{t+1} satisfy
 75 the proposition, since a higher draw of ξ both reduces $c_{t+1}^{-\rho-1}$ and reduces the marginal
 75 propensity to consume. However, permanent shocks have conflicting effects. On the one
 75 hand, a higher draw of ψ_{t+1} will reduce m_{t+1} , thus increasing both $c_{t+1}^{-\rho-1}$ and c'_{t+1} . On
 75 the other hand, the $c_{t+1}^{-\rho-1}$ term is multiplied by $\Gamma\psi_{t+1}$, so the effect of a higher ψ_{t+1}
 75 could be to decrease the first term in the covariance, leading to a negative covariance
 75 with the second term. (Analogously, a lower permanent shock ψ_{t+1} can also lead a
 75 negative correlation.)

15 M Unique And Stable Target and Steady State Points

21 This appendix proves Theorems 2 and 3 and

28 **Lemma 1.** *If both \check{m} and \hat{m} exist, then $\hat{m} < \check{m}$.*

M.1 Proof of Theorem 2

The elements of the proof of theorem 2 are:

- Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing

M.1.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$.

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that $c(m)$ is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all t , $a_{t-1} = m_{t-1} - c_{t-1} > 0$. Since $m_t = a_{t-1}\mathcal{R}_t + \xi_t$, even if ξ_t takes on its minimum value of 0, $a_{t-1}\mathcal{R}_t > 0$, since both a_{t-1} and \mathcal{R}_t are strictly positive. With m_t and m_{t+1} both strictly positive, the ratio $\mathbb{E}_t[m_{t+1}/m_t]$ inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

M.1.2 Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$.

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] = 1$ follows from:

1. Existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ (just proven)
2. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] < 1$.

If RIC holds. Logic exactly parallel to that of section 3.1 leading to equation (49), but dropping the Γ_{t+1} from the RHS, establishes that

$$\begin{aligned}
 \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t} \right] \\
 &= \mathbb{E}_t[(\mathcal{R}/\Gamma_{t+1})\mathbf{P}_R] \\
 &= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\
 &< 1
 \end{aligned} \tag{105}$$

where the inequality reflects imposition of the GIC-Nrm (36).

If RIC fails. When the RIC fails, the fact that $\lim_{m \uparrow \infty} c'(m) = 0$ (see equation (40)) means that the limit of the RHS of (105) as $m \uparrow \infty$ is $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$. In the next step of this proof, we will prove that the combination GIC-Nrm and RIC implies $\bar{\mathcal{R}} < 1$.

So we have $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$ whether the RIC holds or fails.

79 *Existence of a point where $\mathbb{E}_t[m_{t+1}/m_t] > 1$.*

81 Paralleling the logic for c in section 3.2: the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded
 81 above as $m_t \downarrow 0$ because $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$.

85 *Intermediate Value Theorem.* If $\mathbb{E}_t[m_{t+1}/m_t]$ is continuous, and takes on values above
 85 and below 1, there must be at least one point at which it is equal to one.

86 *M.1.3 $\mathbb{E}_t[m_{t+1}] - m_t$ is monotonically decreasing.*

Now define $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$ and note that

$$\zeta(m_t) < 0 \leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \quad (??)$$

$$\zeta(m_t) = 0 \leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \quad (106)$$

$$\zeta(m_t) > 0 \leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1,$$

so that $\zeta(\hat{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned} \zeta'(m_t) &\equiv \left(\frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t} \right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}} (1 - c'(m_t)) - 1. \end{aligned} \quad (107)$$

109 Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different
 109 reasons) whether the RIC holds or fails.

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned} \bar{\mathcal{R}} (1 - c'(m_t)) - 1 &< \bar{\mathcal{R}} (1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}} \mathbf{P}_R - 1 \\ &= \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \frac{\mathbf{P}}{\mathbf{R}} \right] - 1 \\ &= \mathbb{E}_t \left[\underbrace{\frac{\mathbf{P}}{\Gamma \psi}}_{=\mathbf{P}_\Gamma} \right] - 1 \end{aligned}$$

118 which is negative because the GIC-Nrm says $\mathbf{P}_\Gamma < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\bar{\mathcal{R}} (1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that $\zeta'(m_t)$ from (107) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma \psi} \right] < 1. \quad (108)$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\overbrace{\mathbb{E}_t \left[\frac{\mathbf{P}}{\Gamma\psi} \right]}^{\mathbf{P}_\Gamma} < 1 < \overbrace{\frac{\mathbf{P}}{\mathbf{R}}}^{\mathbf{P}_\mathbf{R}},$$

and multiplying all three elements by \mathbf{R}/\mathbf{P} gives

$$\mathbb{E}_t \left[\frac{\mathbf{R}}{\Gamma\psi} \right] < \mathbf{R}/\mathbf{P} < 1$$

137 which satisfies our requirement in (108).

139 M.2 Proof of Theorem 3

142 The elements of the proof are:

- 144 • Existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- 145 • Existence of a point where $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- 146 • $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$ is monotonically decreasing

148 M.2.1 Existence and Continuity of The Ratio

151 Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in M.1.1 that demonstrated existence and continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ implies existence and continuity of $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$.

152 M.2.2 Existence of a stable point

155 Since by assumption $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$, our proof in subsection M.1.1 that the ratio of $\mathbb{E}_t[m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$ implies that the ratio $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$ to m_t is unbounded as $m_t \downarrow 0$.

The limit of the expected ratio as m_t goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{\Gamma_{t+1} ((\mathbf{R}/\Gamma_{t+1})a(m_t) + \xi_{t+1}) / \Gamma}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[\frac{(\mathbf{R}/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\ &= \lim_{m_t \uparrow \infty} \left[\frac{(\mathbf{R}/\Gamma)a(m_t) + 1}{m_t} \right] \\ &= (\mathbf{R}/\Gamma)\mathbf{P}_\mathbf{R} \\ &= \mathbf{P}_\Gamma \\ &< 1 \end{aligned} \tag{109}$$

172 where the last two lines are merely a restatement of the GIC (30).

174 The Intermediate Value Theorem says that if $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ is continuous, and takes
 174 on values above and below 1, there must be at least one point at which it is equal to
 174 one.

175 *M.2.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.*

Define $\zeta(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ and note that

$$\begin{aligned}\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,\end{aligned}\tag{110}$$

so that $\zeta(\hat{m}) = 0$. Our goal is to prove that $\zeta(\bullet)$ is strictly decreasing on $(0, \infty)$ using the fact that

$$\begin{aligned}\zeta'(m_t) &\equiv \left(\frac{d}{dm_t}\right) \zeta(m_t) = \mathbb{E}_t \left[\left(\frac{d}{dm_t}\right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\ &= (R/\Gamma) (1 - c'(m_t)) - 1.\end{aligned}\tag{111}$$

197 Now, we show that (given our other assumptions) $\zeta'(m)$ is decreasing (but for different
 197 reasons) whether the RIC holds or fails (~~RIC~~).

If RIC holds. Equation (22) indicates that if the RIC holds, then $\underline{\kappa} > 0$. We show at the bottom of Section 2.8.1 that if the RIC holds then $0 < \underline{\kappa} < c'(m_t) < 1$ so that

$$\begin{aligned}\mathcal{R} (1 - c'(m_t)) - 1 &< \mathcal{R} (1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= (R/\Gamma) \mathbf{P}_R - 1\end{aligned}$$

204 which is negative because the GIC says $\mathbf{P}_R < 1$.

If RIC fails. Under ~~RIC~~, recall that $\lim_{m \uparrow \infty} c'(m) = 0$. Concavity of the consumption function means that c' is a decreasing function, so everywhere

$$\mathcal{R} (1 - c'(m_t)) < \mathcal{R}$$

which means that $\zeta'(m_t)$ from (111) is guaranteed to be negative if

$$\mathcal{R} \equiv (R/\Gamma) < 1.\tag{112}$$

216 But we showed in section 2.5 that the only circumstances under which the problem
 216 has a nondegenerate solution while the RIC fails were ones where the FHWC also fails
 216 (that is, (112) holds).

217 M.3 Proof of Lemma

Table 5 Taxonomy of Perfect Foresight Liquidity Constrained Model Outcomes

For constrained \bar{c} and unconstrained \bar{c} consumption functions

Main Condition Subcondition	Math	Outcome, Comments or Results
GIC and RIC and RIC	$1 < \mathbf{P}/\Gamma$ $\mathbf{P}/R < 1$ $1 < \mathbf{P}/R$	Constraint never binds for $m \geq 1$ FHWC holds ($R > \Gamma$); $\dot{c}(m) = \bar{c}(m)$ for $m \geq 1$ $\dot{c}(m)$ is degenerate: $\dot{c}(m) = 0$
GIC and RIC	$\mathbf{P}/\Gamma < 1$ $\mathbf{P}/R < 1$	Constraint binds in finite time for any m FHWC may or may not hold $\lim_{m \uparrow \infty} \bar{c}(m) - \dot{c}(m) = 0$ $\lim_{m \uparrow \infty} \dot{\kappa}(m) = \underline{\kappa}$
and RIC	$1 < \mathbf{P}/R$	FHWC $\lim_{m \uparrow \infty} \dot{\kappa}(m) = 0$

Conditions are applied from left to right; for example, the second row indicates conclusions in the case where ~~GIC~~ and RIC both hold, while the third row indicates that when the GIC and the RIC both fail, the consumption function is degenerate; the next row indicates that whenever the GIC holds, the constraint will bind in finite time.

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