

# 1 Unique And Stable Target and Steady State Points

The two theorems and lemma to be proven in this appendix are:

**Theorem 2.** *For the nondegenerate solution to the problem defined in section 2.1 when FVAC, WRIC, and GIC-Nrm all hold, there exists a unique cash-on-hand-to-permanent-income ratio  $\hat{m} > 0$  such that*

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \hat{m}. \quad (1)$$

Moreover,  $\hat{m}$  is a point of ‘wealth stability’ in the sense that

$$\begin{aligned} \forall m_t \in (0, \hat{m}), \quad \mathbb{E}_t[m_{t+1}] &> m_t \\ \forall m_t \in (\hat{m}, \infty), \quad \mathbb{E}_t[m_{t+1}] &< m_t. \end{aligned} \quad (2)$$

**Theorem 3.** *For the nondegenerate solution to the problem defined in section 2.1 when FVAC, WRIC, and GIC all hold, there exists a unique pseudo-steady-state cash-on-hand-to-income ratio  $\check{m} > 0$  such that*

$$\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m}. \quad (3)$$

Moreover,  $\check{m}$  is a point of stability in the sense that

$$\begin{aligned} \forall m_t \in (0, \check{m}), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &> \Gamma \\ \forall m_t \in (\check{m}, \infty), \quad \mathbb{E}_t[\mathbf{m}_{t+1}]/\mathbf{m}_t &< \Gamma. \end{aligned} \quad (4)$$

**Lemma 1.** *If both  $\check{m}$  and  $\hat{m}$  exist, then  $\hat{m} < \check{m}$ .*

## 1.1 Proof of Theorem 2

The elements of the proof of theorem 2 are:

- Existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$
- Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[m_{t+1}] - m_t$  is monotonically decreasing

### 1.1.1 Existence and Continuity of $\mathbb{E}_t[m_{t+1}/m_t]$ .

The consumption function exists because we have imposed the sufficient conditions (the WRIC and FVAC; theorem 1). (Indeed, Appendix C shows that  $c(m)$  is not just continuous, but twice continuously differentiable.)

Section 2.7 shows that for all  $t$ ,  $a_{t-1} = m_{t-1} - c_{t-1} > 0$ . Since  $m_t = a_{t-1}\mathcal{R}_t + \xi_t$ , even if  $\xi_t$  takes on its minimum value of 0,  $a_{t-1}\mathcal{R}_t > 0$ , since both  $a_{t-1}$  and  $\mathcal{R}_t$  are strictly positive. With  $m_t$  and  $m_{t+1}$  both strictly positive, the ratio  $\mathbb{E}_t[m_{t+1}/m_t]$  inherits continuity (and, for that matter, continuous differentiability) from the consumption function.

1.1.2 *Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] = 1$ .*

Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] = 1$  follows from:

1. Existence and continuity of  $\mathbb{E}_t[m_{t+1}/m_t]$  (just proven)
2. Existence a point where  $\mathbb{E}_t[m_{t+1}/m_t] < 1$
3. Existence a point where  $\mathbb{E}_t[m_{t+1}/m_t] > 1$
4. The Intermediate Value Theorem

*Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] < 1$ .*

**If RIC holds.** Logic exactly parallel to that of section 3.1 leading to equation (49), but dropping the  $\Gamma_{t+1}$  from the RHS, establishes that

$$\begin{aligned} \lim_{m_t \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1}}{m_t} \right] \\ &= \mathbb{E}_t[(\mathcal{R}/\Gamma_{t+1})\mathbf{P}_R] \\ &= \mathbb{E}_t[\mathbf{P}/\Gamma_{t+1}] \\ &< 1 \end{aligned} \tag{5}$$

where the inequality reflects imposition of the GIC-Nrm (36).

**If RIC fails.** When the RIC fails, the fact that  $\lim_{m \uparrow \infty} c'(m) = 0$  (see equation (40)) means that the limit of the RHS of (5) as  $m \uparrow \infty$  is  $\bar{\mathcal{R}} = \mathbb{E}_t[\mathcal{R}_{t+1}]$ . In the next step of this proof, we will prove that the combination GIC-Nrm and ~~RIC~~ implies  $\bar{\mathcal{R}} < 1$ .

So we have  $\lim_{m \uparrow \infty} \mathbb{E}_t[m_{t+1}/m_t] < 1$  whether the RIC holds or fails.

*Existence of a point where  $\mathbb{E}_t[m_{t+1}/m_t] > 1$ .*

Paralleling the logic for  $c$  in section 3.2: the ratio of  $\mathbb{E}_t[m_{t+1}]$  to  $m_t$  is unbounded above as  $m_t \downarrow 0$  because  $\lim_{m_t \downarrow 0} \mathbb{E}_t[m_{t+1}] > 0$ .

*Intermediate Value Theorem.* If  $\mathbb{E}_t[m_{t+1}/m_t]$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

1.1.3  $\mathbb{E}_t[m_{t+1}] - m_t$  is monotonically decreasing.

Now define  $\zeta(m_t) \equiv \mathbb{E}_t[m_{t+1}] - m_t$  and note that

$$\begin{aligned} \zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] < 1 \\ \zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] = 1 \\ \zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[m_{t+1}/m_t] > 1, \end{aligned} \tag{6}$$

so that  $\zeta(\hat{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned} \zeta'(m_t) &\equiv \left( \frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[ \left( \frac{d}{dm_t} \right) (\mathcal{R}_{t+1}(m_t - c(m_t)) + \xi_{t+1} - m_t) \right] \\ &= \bar{\mathcal{R}} (1 - c'(m_t)) - 1. \end{aligned} \tag{7}$$

Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the RIC holds or fails.

**If RIC holds.** Equation (22) indicates that if the RIC holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.8.1 that if the RIC holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\begin{aligned} \bar{\mathcal{R}}(1 - c'(m_t)) - 1 &< \bar{\mathcal{R}}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= \bar{\mathcal{R}}\mathbf{P}_R - 1 \\ &= \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Gamma\psi} \frac{\mathbf{P}}{\mathbf{R}} \right] - 1 \\ &= \underbrace{\mathbb{E}_t \left[ \frac{\mathbf{P}}{\Gamma\psi} \right]}_{=\mathbf{P}_\Gamma} - 1 \end{aligned}$$

which is negative because the GIC-Nrm says  $\mathbf{P}_\Gamma < 1$ .

**If RIC fails.** Under ~~RIC~~, recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\bar{\mathcal{R}}(1 - c'(m_t)) < \bar{\mathcal{R}}$$

which means that  $\zeta'(m_t)$  from (11) is guaranteed to be negative if

$$\bar{\mathcal{R}} \equiv \mathbb{E}_t \left[ \frac{\mathbf{R}}{\Gamma\psi} \right] < 1. \quad (8)$$

But the combination of the GIC-Nrm holding and the RIC failing can be written:

$$\underbrace{\mathbb{E}_t \left[ \frac{\mathbf{P}}{\Gamma\psi} \right]}_{\mathbf{P}_\Gamma} < 1 < \underbrace{\frac{\mathbf{P}_R}{\mathbf{R}}}_{\bar{\mathcal{R}}},$$

and multiplying all three elements by  $\mathbf{R}/\mathbf{P}$  gives

$$\mathbb{E}_t \left[ \frac{\mathbf{R}}{\Gamma\psi} \right] < \mathbf{R}/\mathbf{P} < 1$$

which satisfies our requirement in (8).

## 1.2 Proof of Theorem 3

The elements of the proof are:

- Existence and continuity of  $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$
- Existence of a point where  $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1$
- $\mathbb{E}_t[\psi_{t+1}m_{t+1} - m_t]$  is monotonically decreasing

### 1.2.1 Existence and Continuity of The Ratio

Since by assumption  $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$ , our proof in 1.1.1 that demonstrated existence and continuity of  $\mathbb{E}_t[\bar{m}_{t+1}/m_t]$  implies existence and continuity of  $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$ .

### 1.2.2 Existence of a stable point

Since by assumption  $0 < \underline{\psi} \leq \psi_{t+1} \leq \bar{\psi} < \infty$ , our proof in subsection 1.1.1 that the ratio of  $\mathbb{E}_t[m_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$  implies that the ratio  $\mathbb{E}_t[\psi_{t+1}m_{t+1}]$  to  $m_t$  is unbounded as  $m_t \downarrow 0$ .

The limit of the expected ratio as  $m_t$  goes to infinity is most easily calculated by modifying the steps for the prior theorem explicitly:

$$\begin{aligned}
\lim_{m_t \uparrow \infty} \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] &= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{\Gamma_{t+1} ((R/\Gamma_{t+1})a(m_t) + \xi_{t+1}) / \Gamma}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \mathbb{E}_t \left[ \frac{(R/\Gamma)a(m_t) + \psi_{t+1}\xi_{t+1}}{m_t} \right] \\
&= \lim_{m_t \uparrow \infty} \left[ \frac{(R/\Gamma)a(m_t) + 1}{m_t} \right] \\
&= (R/\Gamma)\mathbf{P}_R \\
&= \mathbf{P}_\Gamma \\
&< 1
\end{aligned} \tag{9}$$

where the last two lines are merely a restatement of the GIC (30).

The Intermediate Value Theorem says that if  $\mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t]$  is continuous, and takes on values above and below 1, there must be at least one point at which it is equal to one.

### 1.2.3 $\mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$ is monotonically decreasing.

Define  $\zeta(m_t) \equiv \mathbb{E}_t[\psi_{t+1}m_{t+1}] - m_t$  and note that

$$\begin{aligned}
\zeta(m_t) < 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] < 1 \\
\zeta(m_t) = 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] = 1 \\
\zeta(m_t) > 0 &\leftrightarrow \mathbb{E}_t[\psi_{t+1}m_{t+1}/m_t] > 1,
\end{aligned} \tag{10}$$

so that  $\zeta(\hat{m}) = 0$ . Our goal is to prove that  $\zeta(\bullet)$  is strictly decreasing on  $(0, \infty)$  using the fact that

$$\begin{aligned}
\zeta'(m_t) &\equiv \left( \frac{d}{dm_t} \right) \zeta(m_t) = \mathbb{E}_t \left[ \left( \frac{d}{dm_t} \right) (\mathcal{R}(m_t - c(m_t)) + \psi_{t+1}\xi_{t+1} - m_t) \right] \\
&= (R/\Gamma) (1 - c'(m_t)) - 1.
\end{aligned} \tag{11}$$

Now, we show that (given our other assumptions)  $\zeta'(m)$  is decreasing (but for different reasons) whether the RIC holds or fails ( $\mathbf{RIC}$ ).

**If RIC holds.** Equation (22) indicates that if the RIC holds, then  $\underline{\kappa} > 0$ . We show at the bottom of Section 2.8.1 that if the RIC holds then  $0 < \underline{\kappa} < c'(m_t) < 1$  so that

$$\begin{aligned} \mathcal{R}(1 - c'(m_t)) - 1 &< \mathcal{R}(1 - \underbrace{(1 - \mathbf{P}_R)}_{\underline{\kappa}}) - 1 \\ &= (\mathbf{R}/\Gamma)\mathbf{P}_R - 1 \end{aligned}$$

which is negative because the GIC says  $\mathbf{P}_R < 1$ .

**If RIC fails.** Under ~~RIC~~, recall that  $\lim_{m \uparrow \infty} c'(m) = 0$ . Concavity of the consumption function means that  $c'$  is a decreasing function, so everywhere

$$\mathcal{R}(1 - c'(m_t)) < \mathcal{R}$$

which means that  $\zeta'(m_t)$  from (11) is guaranteed to be negative if

$$\mathcal{R} \equiv (\mathbf{R}/\Gamma) < 1. \tag{12}$$

But we showed in section 2.5 that the only circumstances under which the problem has a nondegenerate solution while the RIC fails were ones where the FHC also fails (that is, (12) holds).