

# Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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<http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf>

- Efficient Solution Methods for Canonical  $C$  problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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1. *Journal of Management Studies*, 1997, 34, 1, 1-14.

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} u(\mathbf{c}_t) + \gamma \mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \quad (4)$$

$m$ — 'market resources' (net worth plus current income)

$p-$  permanent labor income

# Trick: Normalize the Problem

$$\begin{aligned}
 v_t(m_t) &= \max_{c_t} u(c_t) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} v_{t+1}(m_{t+1})] \\
 &\text{s.t.} \\
 a_t &= m_t - c_t \\
 m_{t+1} &= \underbrace{(R/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \boldsymbol{\theta}_{t+1}.
 \end{aligned} \tag{5}$$

where nonbold variables are bold ones normalized by  $\boldsymbol{p}$ :

$$m_t = m_t / \boldsymbol{p}_t \tag{6}$$

Yields  $c_t(m)$  from which we can obtain

$$c_t(m_t, \boldsymbol{p}_t) = c_t(m_t / \boldsymbol{p}_t) \boldsymbol{p}_t \tag{7}$$

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## When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman (transitory/permanent) income process
  - e.g., AR(1)
  - But micro evidence is consistent with Friedman

# Trick: View Everything from End of Period

Define

$$v_{+s}(a_s) = \mathbb{E}_s[\beta \mathcal{G}_{s+1}^{1-\rho} v_{s+1}(\mathcal{R}_{s+1} a_s + \theta_{s+1})] \quad (8)$$

so

$$v_t(m_t) = \max_{c_t} u(c_t) + v_t(m_t - c_t) \quad (9)$$

with FOC

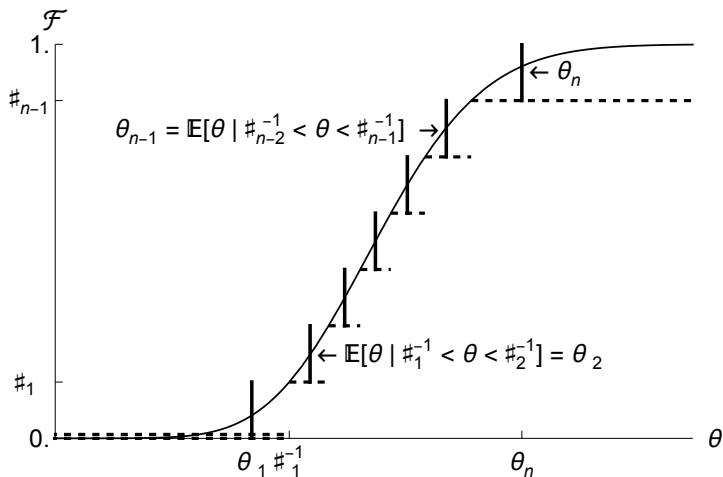
$$u^c(c_s) = v_{+s}^a(m_s - c_s). \quad (10)$$

and Envelope relation

$$u^c(c_t) = v_t^m(m_t) \quad (11)$$

# Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:





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$$\mathbf{v}'_t(a_t) = \beta \mathcal{R} \mathcal{G}_{t+1}^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n u'(c_{t+1}(\mathcal{R}_{t+1} a_t + \boldsymbol{\theta}_i)) \quad (12)$$

So for any particular  $m_{T-1}$  the corresponding  $c_{T-1}$  can be found using the FOC:

$$u^c(c_s) = v_{+s}^a(m_s - c_s). \quad (13)$$

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# Trick: Interpolate a Consumption Rule

- 1 Define a grid of points  $\vec{m}$  (indexed  $m[i]$ )
- 2 Use numerical rootfinder to solve  $u'(c) = v'_t(m[i] - c)$ 
  - The  $c$  that solves this becomes  $c[i]$
- 3 Construct interpolating function  $\hat{c}$  by linear interpolation
  - 'Connect-the-dots'

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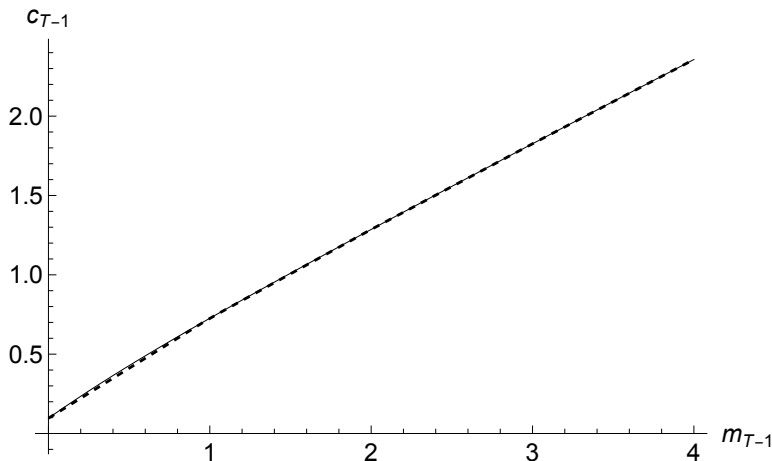
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# Trick: Interpolate a Consumption Rule

Example:  $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$  (solid is 'correct' soln)





# Problem: Numerical Rootfinding is *Slow*

Numerical search for values of  $c_{T-1}$  satisfying  $u'(c) = v'_t(m[i] - c)$  at, say, 6 gridpoints of  $\vec{m}_{T-1}$  may require hundreds or even thousands of evaluations of

$$v'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \mathcal{G}_T^{1-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n (\mathcal{R}_T a_{T-1} + \theta_i)^{-\rho}$$

# Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values  $\vec{a}$
- For each  $a[j]$  compute  $v'_t(a[j])$

Each of these  $v'_t[j]$  corresponds to a unique  $c[j]$  via FOC:

$$\begin{aligned} c[j]^{-\rho} &= v'_t(a[j]) \\ c[j] &= (v'_t(a[j]))^{-1/\rho} \end{aligned} \quad (14)$$

But the DBC says

$$\begin{aligned} a_t &= m_t - c_t \\ m[j] &= a[j] + c[j] \end{aligned} \quad (15)$$

So computing  $v'_t$  at a vector of  $\vec{a}$  values has produced for us the corresponding  $\vec{c}$  and  $\vec{m}$  values at virtually no cost!

From these we can interpolate as before to construct  $\hat{c}_t(m)$ .

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# Why Directly Approximating $v_t$ is a Bad Idea

## Principles of Approximation

- Hard to approximate things that approach  $\infty$  for relevant  $m$ 
  - Not a prob for Rep Agent models: 'relevant'  $m$ 's are  $\approx$  SS
- Hard to approximate things that are highly nonlinear

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# Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$c_t(m) = (m + h_t)\underline{\kappa}_t \quad (16)$$

for market resources  $m$  and end-of-period human wealth  $h$ .

This is why it's a good idea to approximate  $c_t$

Bonus: Easy to debug programs by setting  $\sigma^2 = 0$  and testing whether numerical solution matches analytical!

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# But What if You *Need* the Value Function?

Perfect foresight value function:

$$\begin{aligned}
 \bar{v}_t(m_t) &= u(\bar{c}_t) \mathbb{C}_t^T \\
 &= u(\bar{c}_t) \underline{\kappa}_t^{-1} \\
 &= u((\blacktriangle m_t + \blacktriangle h_t) \underline{\kappa}_t) \underline{\kappa}_t^{-1} \\
 &= u(\blacktriangle m_t + \blacktriangle h_t) \underline{\kappa}_t^{1-\rho} \underline{\kappa}_t^{-1} \\
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where the second line uses the fact demonstrated in Carroll (2023) that  $\mathbb{C}_t = \kappa_t^{-1}$ .

This can be transformed as

$$\begin{aligned}
 \bar{\lambda}_t &\equiv ((1 - \rho) \bar{v}_t)^{1/(1-\rho)} \\
 &= c_t (\mathbb{C}_t^T)^{1/(1-\rho)} \\
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# Approximate Slope Too

Carroll (2023) shows that  $c_t^m$  exists everywhere.

Define *consumed* function and its derivative as

$$\begin{aligned} c_t(a) &= (v'_t(a))^{-1/\rho} \\ c_t^a(a) &= -(1/\rho) (v'_t(a))^{-1-1/\rho} v''_t(a) \end{aligned} \tag{19}$$

and using chain rule it is easy to show that

$$c_t^m = c_t^a / (1 + c_t^a) \tag{20}$$

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# To Implement: Modify Prior Procedures in Two Ways

- 1 Construct  $\vec{c}_t^m$  along with  $\vec{c}_t$  in EGM algorithm
- 2 Approximate  $c_t(m)$  using piecewise Hermite polynomial
  - Exact match to both level and derivative at set of points

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# Problem: $\hat{c}$ Below Bottom $m$ Gridpoint and Extrapolation

Consider what happens as  $a_{T-1}$  approaches  $\underline{a}_{T-1} \equiv -\underline{\theta}\mathcal{R}_T^{-1}$ ,

$$\lim_{a \downarrow \underline{a}_{T-1}} v'_{T-1}(a) = \lim_{a \downarrow \underline{a}_{T-1}} \beta R \mathcal{G}_T^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n (a \mathcal{R}_T + \theta_i)^{-\rho} \\ = \infty$$

This means our lowest value in  $\vec{a}_{T-1}$  should be  $> \underline{a}_{T-1}$ .

Suppose we construct  $\hat{c}$  by linear interpolation:

$$\hat{c}_{T-1}(m) = \hat{c}_{T-1}(\vec{m}_{T-1}[1]) + \hat{c}'_{T-1}(\vec{m}_{T-1}[1])(m - \vec{m}_{T-1}[1])$$

True  $c$  is strictly concave  $\Rightarrow \exists m^- > \underline{m}_{T-1}$  for which

$$m^- - \hat{c}_{T-1}(m^-) < \underline{a}_{T-1}$$



# Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned}\lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}^m(m) &= \bar{\kappa}_{T-1}\end{aligned}\tag{21}$$

- ① Redefine  $\vec{a}$  *relative* to  $\underline{a}_{T-1}$
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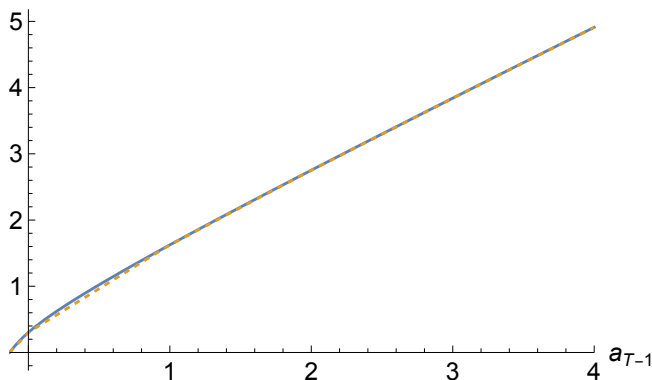
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# Trick: Improving the $a$ Grid

Grid Spacing: Uniform

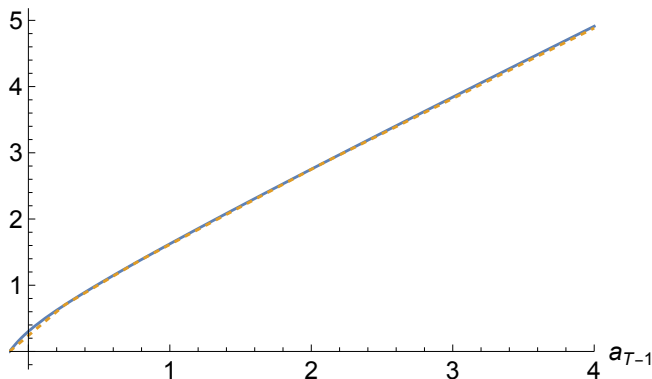
$$(u'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



# Trick: Improving the $a$ Grid

Grid Spacing: Same  $\{\underline{a}, \bar{a}\}$  But Triple Exponential  $e^{e^{\dots}}$  Growth

$$(v'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



# The Method of Moderation

- Further improves speed and accuracy of solution
- See my talk at the conference!



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# Imposing 'Artificial' Borrowing Constraints

$$v_{T-1}(m_{T-1}) = \max_{c_{T-1}} u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \mathcal{G}_T^{1-\rho} v_T(m_T)]$$

s.t.

$$a_{T-1} = m_{T-1} - c_{T-1}$$

$$m_T = \mathcal{R}_T a_{T-1} + \theta_T$$

$$a_{T-1} \geq 0.$$

Define  $\hat{c}_t^*$  as soln to unconstrained problem. Then

$$\hat{c}_{T-1}(m_{T-1}) = \min[m_{T-1}, \hat{c}_{T-1}^*(m_{T-1})]. \quad (22)$$

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# Imposing 'Artificial' Borrowing Constraints

Point where constraint makes transition from binding to not is

$$u'(m_{T-1}^{\#}) = v'_{T-1}(0.)$$

$$m_{T-1}^{\#} = (v'_{T-1}(0.))^{-1/\rho}$$

Procedure is very easy:

- Add 0. as first point in  $\vec{a}$
- $\Rightarrow \vec{m}[1] = m_{T-1}^{\#}$
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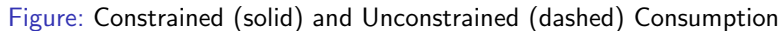
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# Recursion: Period $t$ Solution Given Period $t + 1$

## 1 Construct

$$\begin{aligned} c_{\bar{t},i} &= (v'_{+s}(a_{t,i}))^{-1/\rho}, \\ &= \left( \beta \mathbb{E}_t \left[ R \mathcal{G}_{t+1}^{-\rho} (\hat{c}_{t+1}(\mathcal{R}_{t+1} a_{t,i} + \theta_{t+1}))^{-\rho} \right] \right)^{-1/\rho}, \end{aligned} \quad (23)$$

## 2 Call the result $\vec{c}_t$ and generate the corresponding $\vec{m}_t = \vec{c}_t + \vec{a}_t$

## 3 Interpolate to create $\hat{c}_t(m)$

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# Consumption Rules $\dot{c}_{T-n}(m)$ Converge

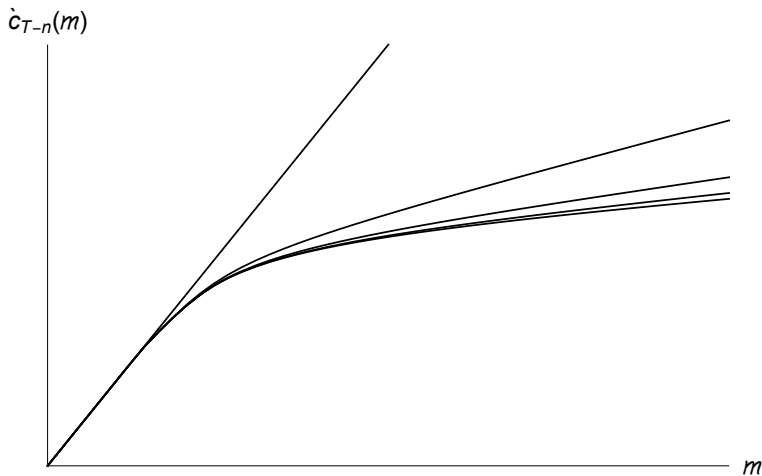


Figure: Converging  $\dot{c}_{T-n}(m)$  Functions for  $n = \{1, 5, 10, 15, 20\}$

# Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.  
The portfolio return is

$$\begin{aligned}\mathfrak{R}_{t+1} &= R(1 - \varsigma_t) + \mathbf{R}_{t+1}\varsigma_t \\ &= R + (\mathbf{R}_{t+1} - R)\varsigma_t\end{aligned}\tag{24}$$

so (setting  $\mathcal{G} = 1$ ) the maximization problem is

$$v_t(m_t) = \max_{\{c_t, \varsigma_t\}} u(c_t) + \beta \mathbb{E}_t[v_{t+1}(m_{t+1})]$$

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$$\mathbf{u}^c(c_t) = \mathbb{E}_t[\beta \mathfrak{R}_{t+1} \mathbf{u}^c(c_{t+1})]. \quad (25)$$

while the FOC with respect to the portfolio share yields

$$\begin{aligned} 0 &= \mathbb{E}_t[v_{t+1}^m(m_{t+1})(\mathbf{R}_{t+1} - R)a_t] \\ &= a_t \mathbb{E}_t[u^c(c_{t+1}(m_{t+1}))(\mathbf{R}_{t+1} - R)]. \end{aligned}$$

# Convergence

When the problem satisfies certain conditions (Carroll (2023)), it defines a ‘converged’ consumption rule with a ‘target’ ratio  $\check{m}$  that satisfies:

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m} \quad (26)$$

Define the target  $m$  implied by the consumption rule  $c_t$  as  $\check{m}_t$ .

Then a plausible metric for convergence is to define some value  $\epsilon$  and to declare the solution to have converged when

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# Trick: Coarse then Fine $\theta$

- 1 Start with coarse grid for  $\theta$  (say, 3 points)
- 2 Solve to convergence; call period of convergence  $n$
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# Life Cycle Maximization Problem

$$v_t(m_t) = \max_{c_t} u(c_t) + \beta \mathcal{L}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\Psi_{t+1} \mathcal{G}_{t+1})^{1-\rho} v_{t+1}(m_{t+1})]$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = a_t \underbrace{\left( \frac{R}{\Psi_{t+1} \mathcal{G}_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1}$$

$\mathcal{L}_t^{t+n}$  : probability to Live until age  $t + n$  given alive at age  $t$

$\hat{\beta}_t^{t+n}$  : age-varying discount factor between ages  $t$  and  $t + n$

$\Psi_t$  : mean-one shock to permanent income

$\beta$  : time-invariant 'pure' discount factor

# Details follow Cagetti (2003)

- Parameterization of Uncertainty
- Probability of Death
- Demographic Adjustments to  $\beta$

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# Empirical Wealth Profiles

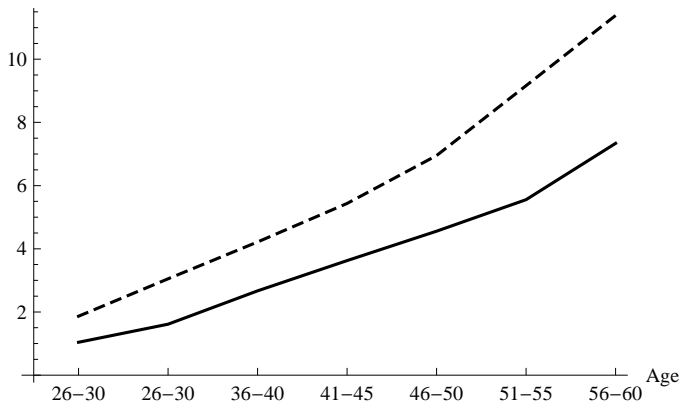


Figure:  $m$  from SCF (means (dashed) and medians (solid))

# Simulated Moments

Given a set of parameter values  $\{\rho, \Xi\}$ :

- Start at age 25 with empirical  $m$  data
- Draw shocks using calibrated  $\sigma_{\Psi}^2, \sigma_{\theta}^2$
- Consume according to solved  $c_t$

$\Rightarrow m$  distribution by age

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\_\_\_\_\_

```

GapEmpiricalSimulatedMedians[ $\rho, \beth$ ] :=
[
    ConstructcFuncLife[ $\rho, \beth$ ];
    Simulate;
    
$$\sum_i^N \omega_i |\varsigma_i^\tau - \mathbf{s}^\tau(\xi)|$$

];

```

---

$$\xi = \{\rho, \sqsupset\} \quad (28)$$

solve

$$\min_{\xi} \sum_i^N \omega_i |\varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi)| \quad (29)$$

\_\_\_\_\_

$\rho$	$\beta$
3.69	0.88
(0.047)	(0.002)

# Contour Plot

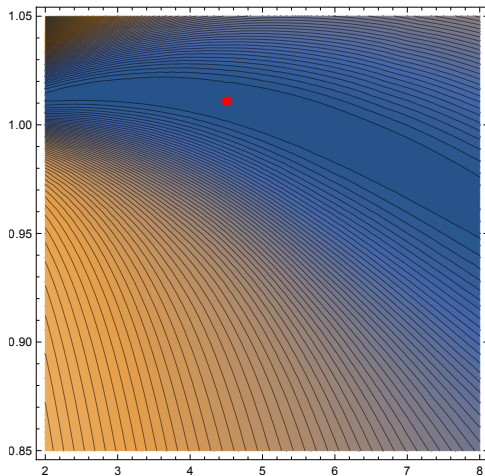


Figure: Point Estimate and Height of Minimized Function

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