Solution Methods for Microeconomic Dynamic Stochastic Optimization Problems

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Note: The GitHub repo `SolvingMicroDSOPs repo` associated with this document contains python code that produces all results, from scratch, except for the last section on indirect inference. The numerical results have been confirmed by showing that the answers that the raw python produces correspond to the answers produced by tools available in the Econ-ARK toolkit, more specifically those in the `HARK` which has full documentation. The MSM results at the end have have been superseded by tools in the `EstimatingMicroDSOPs repo`.

Abstract

These notes describe tools for solving microeconomic dynamic stochastic optimization problems, and show how to use those tools for efficiently estimating a standard life cycle consumption/saving model using microeconomic data. No attempt is made at a systematic overview of the many possible technical choices; instead, I present a specific set of methods that have proven useful in my own work (and explain why other popular methods, such as value function iteration, are a bad idea). Paired with these notes is *Mathematica*, Matlab, and Python software that solves the problems described in the text.

Keywords Dynamic Stochastic Optimization, Method of Simulated Moments, Structural Estimation, Indirect Inference

JEL codes E21, F41

PDF: https://github.com/llorracc/SolvingMicroDSOPs/blob/master/SolvingMicroDSOPs.pdf
Slides: https://github.com/econ-ark/SolvingMicroDSOPs/blob/master/SolvingMicroDSOPs-Slides.pdf
Web: https://econ-ark.github.io/SolvingMicroDSOPs
Code: https://github.com/econ-ark/SolvingMicroDSOPs/tree/master/Code
Archive: https://github.com/econ-ark/SolvingMicroDSOPs
(Contains LaTeX code for this document and software producing figures and results)

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The notes were originally written for my Advanced Topics in Macroeconomic Theory class at Johns Hopkins University; instructors elsewhere are welcome to use them for teaching purposes. Relative to earlier drafts, this version incorporates several improvements related to new results in the paper "Theoretical Foundations of Buffer Stock Saving" (especially tools for approximating the consumption and value functions). Like the last major draft, it also builds on material in "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems" published in *Economics Letters*, available at http://www.econ2.jhu.edu/people/ccarroll/EndogenousArchive.zip, and by including sample code for a method of simulated moments estimation of the life cycle model *a la* **?** and Cagetti (**?**). Background derivations, notation, and related subjects are treated in my class notes for first year macro, available at http://www.econ2.jhu.edu/people/ccarroll/public/lecturenotes/consumption. I am grateful to several generations of graduate students in helping me to refine these notes, to Marc Chan for help in updating the text and software to be consistent with **?**, to Kiichi Tokuoka for drafting the section on structural estimation, to Damiano Sandri for exceptionally insightful help in revising and updating the method of simulated moments estimation section, and to Weifeng Wu and Metin Uyanik for revising to be consistent with the 'method of moderation' and other improvements. All errors are my own. This document can be cited as **?** in the references.

Contents

1 Introduction

These lecture notes provide a gentle introduction to a particular set of solution tools for the canonical consumption-saving/portfolio allocation problem. Specifically, the notes describe and solve optimization problems for a consumer facing uninsurable idiosyncratic risk to nonfinancial income (e.g., labor or transfer income), first without and then with optimal portfolio choice,[1] with detailed intuitive discussion of the various mathematical and computational techniques that, together, speed the solution by many orders of magnitude compared to "brute force" methods. The problem is solved with and without liquidity constraints, and the infinite horizon solution is obtained as the limit of the finite horizon solution. After the basic consumption/saving problem with a deterministic interest rate is described and solved, an extension with portfolio choice between a riskless and a risky asset is also solved. Finally, a simple example shows how to use these methods (via the statistical 'method of simulated moments' ('MSM') to estimate structural parameters like the coefficient of relative risk aversion (*a la* Gourinchas and Parker (**?**) and Cagetti (**?**)).

2 The Problem

{sec:the-problem}

The usual analysis of dynamic stochastic programming problems packs a great many events (intertemporal choice, stochastic shocks, intertemporal returns, income growth, the taking of expectations, and more) into a single step in which the agent makes an optimal choice taking account of all of these elements. For the detailed analysis here, we will be careful to disarticulate everything that happens in the problem explicitly into separate steps so that each element can be scrutinized and understood in isolation.

We are interested in the behavior a consumer who begins period t with a certain amount of 'capital' $\mathbf{k}_t$, which is immediately rewarded by a return factor R_t with the proceeds deposited in a **b**ank account **b**alance:

$$\mathbf{b}_t = \mathbf{k}_t \mathsf{R}_t. \tag{1}$$ {eq:bLvl}

Simultaneously with the realization of the capital return, the consumer also receives noncapital income $\mathbf{y}_t$, which is determined by multiplying the consumer's 'permanent income' $\mathbf{p}_t$ by a transitory shock θ_t:

$$\mathbf{y}_t = \mathbf{p}_t \theta_t \tag{2}$$ {eq:yLvl}

whose whose expectation is 1 (that is, before realization of the transitory shock, the consumer's expectation is that actual income will on average be equal to permanent income $\mathbf{p}_t$).

The combination of bank balances $\mathbf{b}$ and income $\mathbf{y}$ define's the consumer's 'market

[1] See **?** and **?** for a solution to the problem of a consumer whose only risk is rate-of-return risk on a financial asset; the combined case (both financial and nonfinancial risk) is solved below, and much more closely resembles the case with only nonfinancial risk than it does the case with only financial risk.

resources' (sometimes called 'cash-on-hand,' following **?**):

$$\mathbf{m}_t = \mathbf{b}_t + \mathbf{y}_t, \tag{3}$$ {eq:mLvl}

available to be spent on consumption $\mathbf{c}_t$ for a consumer subject to a liquidity constraint that requires $\mathbf{c} \leq \mathbf{m}$.

The consumer's goal is to maximize discounted utility from consumption over the rest of a lifetime ending at date T:

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\mathbf{c}_{t+n}) \right]. \tag{4}$$ {eq:MaxProb}

Income evolves according to:

$$\begin{aligned} \mathbf{p}_{t+1} = \mathcal{G}_{t+1}\mathbf{p}_t & \quad \text{- permanent labor income dynamics} \\ \log\ \theta_{t+n} \sim \ \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) & \quad \text{- lognormal transitory shocks } \forall\ n > 0. \end{aligned} \tag{5}$$ {eq:permincgrow}

Equation (**??**) indicates that we are allowing for a predictable average profile of income growth over the lifetime $\{\mathcal{G}\}_0^T$ (to capture typical career wage paths, pension arrangements, etc).[2] Finally, the utility function is of the Constant Relative Risk Aversion (CRRA), form, $\mathrm{u}(\bullet) = \bullet^{1-\rho}/(1-\rho)$.

It is well known that this problem can be rewritten in recursive (Bellman) form

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} \ \mathrm{u}(\mathbf{c}_t) + \beta\,\mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \tag{6}$$ {eq:vrecurse}

subject to the Dynamic Budget Constraint (DBC) implicitly defined by equations (**??**)-(**??**) and to the transition equation that defines next period's initial capital as this period's end-of-period assets:

$$\mathbf{k}_{t+1} = \mathbf{a}_t. \tag{7}$$ {eq:transition.state}

3 Normalization

{sec:normalization}

The single most powerful method for speeding the solution of such models is to redefine the problem in a way that reduces the number of state variables (if at all possible). Here, the obvious idea is to see whether the problem can be rewritten in terms of the ratio of various variables to permanent noncapital ('labor') income $\mathbf{p}_t$ (henceforth for brevity, 'permanent income.')

In the last period of life, there is no future, $\mathbf{v}_{T+1} = 0$, so the optimal plan is to consume everything:

$$\mathbf{v}_T(\mathbf{m}_T, \mathbf{p}_T) = \frac{\mathbf{m}_T^{1-\rho}}{1-\rho}. \tag{8}$$ {eq:levelTm1}

[2]For expositional and pedagogical purposes, this equation assumes that there are no shocks to permanent income. A large literature finds that, in reality, permanent (or at least extremely highly persistent) shocks exist and are quite large; such shocks therefore need to be incorporated into any 'serious' model (that is, one that hopes to match and explain empirical data), but the treatment of permanent shocks clutters the exposition without adding much to the intuition, so permanent shocks are omitted from the analysis until the last section of the notes, which shows how to match the model with empirical micro data. For a full treatment of the theory including permanent shocks, see **?**.

Now define nonbold variables as the bold variable divided by the level of permanent income in the same period, so that, for example, $m_T = \mathbf{m}_T/\mathbf{p}_T$; and define $\mathrm{v}_T(m_T) = \mathrm{u}(m_T)$.[3] For our CRRA utility function, $\mathrm{u}(xy) = x^{1-\rho}\mathrm{u}(y)$, so (**??**) can be rewritten as

$$\begin{aligned} \mathbf{v}_T(\mathbf{m}_T, \mathbf{p}_T) &= \mathbf{p}_T^{1-\rho}\frac{m_T^{1-\rho}}{1-\rho} \\ &= (\mathbf{p}_{T-1}\mathcal{G}_T)^{1-\rho}\frac{m_T^{1-\rho}}{1-\rho} \\ &= \mathbf{p}_{T-1}^{1-\rho}\mathcal{G}_T^{1-\rho}\mathrm{v}_T(m_T). \end{aligned} \tag{9}$$

{eq:vT}

Now define a new optimization problem:

$$\begin{aligned} \mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \beta\,\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\ &\text{s.t.} \\ a_t &= m_t - c_t \\ k_{t+1} &= a_t \\ m_{t+1} &= \underbrace{(\mathsf{R}/\mathcal{G}_{t+1})}_{\equiv\mathcal{R}_{t+1}} k_{t+1} + \theta_{t+1}, \end{aligned} \tag{10}$$

{eq:vNormed}

where the last equation is the normalized version of the transition equation for $\mathbf{m}_{t+1}$.[4] Then it is easy to see that for $t = T-1$,

$$\mathbf{v}_{T-1}(\mathbf{m}_{T-1}, \mathbf{p}_{T-1}) = \mathbf{p}_{T-1}^{1-\rho}\mathrm{v}_{T-1}(m_{T-1})$$

and so on back to all earlier periods. Hence, if we solve the problem (**??**) which (when optimal consumption is being chosen) has only a single state variable m_t, we can obtain the levels of the value function, consumption, and all other variables from the corresponding permanent-income-normalized solution objects by multiplying each by $\mathbf{p}_t$, e.g. $\mathbf{c}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t\mathrm{c}_t(\mathbf{m}_t/\mathbf{p}_t)$ (or, for the value function, $\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \mathbf{p}_t^{1-\rho}\mathrm{v}_t(m_t)$). We have thus reduced the problem from two continuous state variables to one (and thereby enormously simplified its solution).

For some problems it will not be obvious that there is an appropriate 'normalizing' variable, but many problems can be normalized if sufficient thought is given. For example, **?** shows how a bank's optimization problem can be normalized by the level of the bank's productivity.

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[3]Nonbold value is bold value divided by $\mathbf{p}^{1-\rho}$ rather than $\mathbf{p}$.

[4]Derivation:

$$\begin{aligned} \mathbf{m}_{t+1}/\mathbf{p}_{t+1} &= (\mathbf{m}_t - \mathbf{c}_t)\mathsf{R}/\mathbf{p}_{t+1} + \mathbf{y}_{t+1}/\mathbf{p}_{t+1} \\ m_{t+1} &= \left(\frac{\mathbf{m}_t}{\mathbf{p}_t} - \frac{\mathbf{c}_t}{\mathbf{p}_t}\right)\mathsf{R}\frac{\mathbf{p}_t}{\mathbf{p}_{t+1}} + \frac{\mathbf{y}_{t+1}}{\mathbf{p}_{t+1}} \\ &= \underbrace{(m_t - c_t)}_{a_t}(\mathsf{R}/\mathcal{G}_{t+1}) + \theta_{t+1}. \end{aligned}$$

4 The Usual Theory, and A Bit More Notation

{sec:usualtheory}

4.1 Steps

The preceding analysis made some implicit assumptions about what was known when; difficulty intuiting or inferring those implicit assumptions is a common stumbling block for students trying to this material.

Generically, we want to think of the Bellman problem itself as having three steps:

1. **Arrival**: Incoming state variables (e.g., k_t) are known, but any shocks associated with the period have not been realized and decision(s) have not yet been made

2. **Decision**: All exogenous variables (like income shocks, rate of return shocks, predictable income growth $\mathcal{G}$ have been realized (so that, e.g., m_t's value is known) and the agent solves the optimization problem

3. **Continuation**: After all decisions have been made, it is possible to calculate the consequences of the decision, taking as given the 'outgoing' state variables (e.g., a) – sometimes called 'post-state' variables.

The (implicit) default assumption is often to think of the step of the problem where the agent is solving a decision problem as defining the unique moment at which the problem is defined. This is what implicitly was done above, when (for example) in (**??**) we related current value v_t to the expectation of future value v_{t+1}.

When we want to refer to a specific step within period t we will do so by preceding it by an indicator character:

Step	Indicator	Example	Explanation
Arrival	$\leftarrow$	$\mathrm{v}_{\leftarrow t}(k_t)$	value of entering t (before shocks)
Decision	(blank/none)	$\mathrm{v}_t(m_t)$	value of t-decision (after shocks)
Continuation	$\rightarrow$	$\mathrm{v}_{t\rightarrow}(a_t)$	value of outgoing state (after decision)

This notation allows us to capture the fact that the value of the consumer's circumstances can be computed at any of the three steps (and is a function of a different state variable at each step).

Note that there is no need to use these subscripts for the model's variables; while a variable like $a_t = m_t - c_t$ takes on its value in the transition from the Decision to the Continuation step, a will have only one unique value over the course of the period and therefore a notation like $a_{\leftarrow t}$ would be useless because the variable does not have a value until the continuation step is reached. Each variable in our problem has a unique value defined at some point during the period, so there is no ambiguity in referring to them with normal notation like a_t.

4.2 The Usual Theory

Using this new notation, the first order condition for (**??**) with respect to c_t is

$$\begin{aligned}
\mathrm{u}^c(c_t) &= \mathbb{E}_{t\rightarrow}[\beta \mathcal{R}_{t+1}\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}^m_{t+1}(m_{t+1})] \\
&= \mathbb{E}_{t\rightarrow}[\beta \mathsf{R} \quad\ \mathcal{G}_{t+1}^{-\rho}\mathrm{v}^m_{t+1}(m_{t+1})]
\end{aligned} \tag{11}$$ {eq:upceqEvtp1}

and because the `Envelope` theorem tells us that

$$\mathrm{v}^m_t(m_t) = \mathbb{E}_{\leftarrow t}[\beta \mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{v}^m_{t+1}(m_{t+1})] \tag{12}$$ {eq:envelope}

we can substitute the LHS of (**??**) for the RHS of (**??**) to get

$$\mathrm{u}^c(c_t) = \mathrm{v}^m_t(m_t) \tag{13}$$ {eq:Envelope}

and rolling forward one period,

$$\mathrm{u}^c(c_{t+1}) = \mathrm{v}^m_{t+1}(a_t\mathcal{R}_{t+1} + \theta_{t+1}) \tag{14}$$ {eq:upctp1EqVpxtp

and substituting the LHS in equation (**??**) finally gives us the Euler equation for consumption:

$$\mathrm{u}^c(c_t) = \mathbb{E}_{t\rightarrow}[\beta \mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{u}^c(c_{t+1})]. \tag{15}$$ {eq:cEuler}

For future reference, it may be useful here to write the distinct value functions at each step (using the transition equations from (**??**)):

$$\mathrm{v}_{\leftarrow t}(k_t) = \mathbb{E}_{\leftarrow t}[\mathrm{v}_t(\overbrace{k_t\mathcal{R}_t + \theta_t}^{=m_t})] \tag{16}$$ {eq:vBegStep}

$$\mathrm{v}_t(m_t) = \mathrm{u}(\mathrm{c}_t(m_t)) + \mathrm{v}_{t\rightarrow}(\overbrace{m_t - c_t}^{a_t}) \tag{17}$$ {eq:vMidStep}

and

$$\mathrm{v}_{t\rightarrow}(a_t) = \beta \mathrm{v}_{\leftarrow t+1}(\underbrace{k_{t+1}}_{=a_t}) \tag{18}$$ {eq:vEndStep}

where the last line illustrates the notation for addressing the beginning step of the successor period.

Putting all this together, from the perspective of the beginning of period $t+1$ we can write the 'arrival value' function and its first derivative as

$$\mathrm{v}_{\leftarrow t+1}(k_{t+1}) = \mathbb{E}_{\leftarrow t+1}[\ \ \mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(\mathcal{R}_{t+1}k_t + \theta_{t+1})] \tag{19}$$ {eq:vFuncBegtpdefn

and

$$\mathrm{v}_{t\rightarrow}(a_t) = \beta \mathrm{v}_{\leftarrow t}(a_{t+1}) \tag{20}$$ {eq:vEndtdefn}

because they return the expected $t+1$ value and marginal value associated with arriving in period $t+1$ with any given amount of **k**apital. Finally, note for future use that since $k_{t+1} = a_t$ and $\mathrm{v}_{t\rightarrow}(a_t) = \beta \mathrm{v}_{\leftarrow t+1}(k_{t+1})$, the first order condition (**??**) can now be rewritten compactly as

$$\mathrm{u}^c(c_t) = \mathrm{v}^a_{t\rightarrow}(m_t - c_t). \tag{21}$$ {eq:upEqbetaOp}

5 Solving the Next-to-Last Period

{sec:NextToLast}

For convenience assuming that $\mathcal{G} = 1$ so that the $\mathcal{G}$ terms disappear from the earlier derivations, the problem in the second-to-last period of life can now be expressed as

$$\mathrm{v}_{T-1}(m_{T-1}) = \max_{c_{T-1}} \ \mathrm{u}(c_{T-1}) + \beta\,\mathbb{E}_{T-1_{\rightarrow}}\left[\mathrm{v}_T(\underbrace{(m_{T-1} - c_{T-1})\mathcal{R}_T + \theta_T}_{m_T})\right],$$

where $\mathbb{E}_{T-1_{\rightarrow}}$ indicates that the expectation is taken as of the end of period $T-1$.

Using (1) the fact that $\mathrm{v}_T = \mathrm{u}(c_T)$; (2) the definition of $\mathrm{u}(c_T)$; (3) the definition of the expectations operator, this becomes:

$$\mathrm{v}_{T-1}(m_{T-1}) = \max_{c_{T-1}} \ \frac{c_{T-1}^{1-\rho}}{1-\rho} + \beta \int_0^\infty \frac{((m_{T-1} - c_{T-1})\mathcal{R}_T + \vartheta)^{1-\rho}}{1-\rho} d\mathcal{F}(\vartheta)$$

where $\mathcal{F}$ is the cumulative distribution function for θ_T.

The maximization implicitly defines a function $\mathrm{c}_{T-1}(m_{T-1})$ that yields optimal consumption in period $T-1$ for any specific numerical level of resources like $m_{T-1} = 1.7$. But because there is no general analytical solution to this problem, for any given m_{T-1} we must use numerical computational tools to find the c_{T-1} that maximizes the expression. This is excruciatingly slow because for every potential c_{T-1} to be considered, a definite integral over the interval $(0, \infty)$ must be calculated numerically, and numerical integration is *very* slow (especially over an unbounded domain!).

5.1 Discretizing the Distribution

Our first speedup trick is therefore to construct a discrete approximation to the lognormal distribution that can be used in place of numerical integration. That is, we want to approximate the expectation over θ of a function $g(\theta)$ by calculating its value at set of n_θ points θ_i, each of which has an associated probability weight w_i:

$$\begin{aligned}\mathbb{E}[g(\theta)] &= \int_{\underline{\theta}}^{\overline{\theta}} (\vartheta) d\mathcal{F}(\vartheta) \\ &\approx \sum_{\theta=1}^{n} w_i g(\theta_i)\end{aligned}$$

(because adding n weighted values to each other is enormously faster than general-purpose numerical integration).

Such a procedure is called a 'quadrature' method of integration; **?** survey a number of options, but for our purposes we choose the one which is easiest to understand: An 'equiprobable' approximation (that is, one where each of the values of θ_i has an equal probability, equal to $1/n_\theta$).

We calculate such an n-point approximation as follows.

Define a set of points from $\sharp_0$ to $\sharp_{n_\theta}$ on the $[0, 1]$ interval as the elements of the set

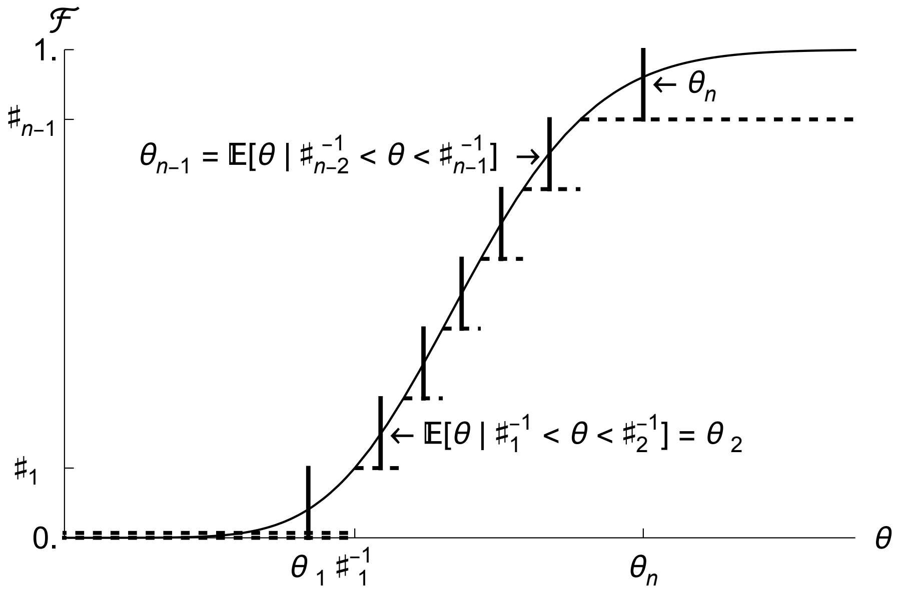

Figure 1 Equiprobable Discrete Approximation to Lognormal Distribution $\mathcal{F}$

{fig:discreteapprox}

$\sharp = \{0, 1/n, 2/n, \ldots, 1\}$.[5] Call the inverse of the θ distribution $\mathcal{F}^{-1}$, and define the points $\sharp^{-1}_i = \mathcal{F}^{-1}(\sharp_i)$. Then the conditional mean of θ in each of the intervals numbered 1 to n is:

$$\theta_i \equiv \mathbb{E}[\theta|\sharp^{-1}_{i-1} \leq \theta < \sharp^{-1}_i] = \int_{\sharp^{-1}_{i-1}}^{\sharp^{-1}_i} \vartheta \; d\mathcal{F}\;(\vartheta), \tag{22}$$

and when the integral is evaluated numerically for each i the result is a set of values of θ that correspond to the mean value in each of the n intervals.

The method is illustrated in Figure **??**. The solid continuous curve represents the "true" CDF $\mathcal{F}(\theta)$ for a lognormal distribution such that $\mathbb{E}[\theta] = 1$, $\sigma_\theta = 0.1$. The short vertical line segments represent the n_θ equiprobable values of θ_i which are used to approximate this distribution.[6]

[5]These points define intervals that constitute a partition of the domain of $\mathcal{F}$.

[6]More sophisticated approximation methods exist (e.g. Gauss-Hermite quadrature; see **?** for a discussion of other alternatives), but the method described here is easy to understand, quick to calculate, and has additional advantages briefly described in the discussion of simulation below.