The Method of Moderation

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Christopher D. Carroll[1]
JHU

Karsten Chipeniuk
RBNZ

Kiichi Tokuoka[2]
ECB

Weifeng Wu[3]
Fannie Mae

Abstract

In a risky world, a pessimist assumes the worst will happen. Someone who ignores risk altogether is an optimist. Consumption decisions are mathematically simple for both the pessimist and the optimist because both behave as if they live in a riskless world. A realist (that is, someone who wants to respond optimally to risk) faces a much more difficult problem, but (under standard conditions) will choose a level of spending somewhere between pessimist's and the optimist's. We use this fact to redefine the space in which the realist searches for optimal consumption rules. The resulting solution accurately represents the numerical consumption rule over the entire interval of feasible wealth values with remarkably few computations.

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[1]Carroll: Department of Economics, Johns Hopkins University, Baltimore, MD, http://econ.jhu.edu/people/ccarroll/, ccarroll@jhu.edu [2]Tokuoka: International Monetary Fund, Washington, DC, ktokuoka@imf.org [3]Wu: Weifeng Wu, Fanie Mae, Washington DC, .

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1 Introduction

Solving a consumption, investment, portfolio choice, or similar intertemporal optimization problem using numerical methods generally requires the modeler to choose how to represent a policy or value function. In the stochastic case, where analytical solutions are generally not available, a common approach is to use low-order polynominal splines that exactly match the function (and maybe some derivatives) at a finite set of gridpoints, and then to assume that interpolated or extrapolated versions of that spline represent the function well at the continuous infinity of unmatched points.

This paper argues that a better approach in the standard consumption problem is to rely upon the fact that without uncertainty, the optimal consumption function has a simple analytical solution. The key insight is that, under standard assumptions, the consumer who faces an uninsurable labor income risk will consume less than a consumer with the same path for expected income but who does not perceive any uncertainty as being attached to that future income. The 'realistic' consumer, who *does* perceive the risks, will engage in 'precautionary saving,' so the perfect foresight riskless solution provides an upper bound to the solution that will actually be optimal. A lower bound is provided by the behavior of a consumer who has the subjective belief that the future level of income will be the worst that it can possibly be. This consumer, too, behaves according to the convenient analytical perfect foresight solution, but his certainty is that of a pessimist perfectly confident in his pessimism.

Using results from Carroll (2023b), we show how to use these upper and lower bounds to tightly constrain the shape and characteristics of the solution to problem of the 'realist.' Imposition of these constraints can clarify and speed the solution of the realist's problem.

After showing how to use the method in the baseline case, we show how refine it to encompass an even tighter theoretical bound.

2 The Realist's Problem

We assume that truly optimal behavior in the problem facing the consumer who understands all his risks is captured by

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\mathbf{c}_{t+n}) \right]. \tag{1}$$

subject to

$$\begin{aligned} \mathbf{a}_t &= \mathbf{m}_t - \mathbf{c}_t \\ \mathbf{b}_{t+1} &= \mathbf{a}_t \mathsf{R}_{t+1} \\ \mathbf{y}_{t+1} &= \mathbf{p}_{t+1} \boldsymbol{\theta}_{t+1} \\ \mathbf{m}_{t+1} &= \mathbf{b}_{t+1} + \mathbf{y}_{t+1} \end{aligned} \tag{2}$$

where

$$
\begin{array}{rr}
\beta- & \text{pure time discount factor} \\
\mathbf{a}_t- & \text{assets after all actions have been accomplished in period } t \\
\mathbf{b}_{t+1}- & \text{`bank balances' (nonhuman wealth) at the beginning of } t+1 \\
\mathbf{c}_t- & \text{consumption in period } t \\
\mathbf{m}_t- & \text{`market resources' available for consumption (`cash-on-hand')} \\
\mathbf{p}_{t+1}- & \text{`permanent labor income' in period } t+1 \\
\mathsf{R}_{t+1}- & \text{interest factor } (1+\mathsf{r}_{t+1}) \text{ from period } t \text{ to } t+1 \\
\mathbf{y}_{t+1}- & \text{noncapital income in period } t+1.
\end{array}
$$

and the exogenous variables evolve according to

$$
\begin{aligned}
\mathbf{p}_{t+1} &= \mathcal{G}_{t+1}\mathbf{p}_t & \text{- permanent labor income dynamics} \\
\log\ \boldsymbol{\theta}_{t+n} &\sim\ \mathcal{N}(-\sigma_{\boldsymbol{\theta}}^2/2, \sigma_{\boldsymbol{\theta}}^2) & \text{- lognormal transitory shocks } \forall\ n > 0
\end{aligned} \tag{3}
$$

It turns out (see Carroll (2023a) for a proof) that this problem can be rewritten in a more convenient form in which choice and state variables are normalized by the level of permanent income, e.g., using nonbold font for normalized variables, $m_t = c_t/\boldsymbol{p}_t$. When that is done, the Bellman equation for the transformed version of the consumer's problem is

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t}\ \ \mathrm{u}(c_t) + \beta\,\mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho}\mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \underbrace{(\mathsf{R}/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \boldsymbol{\theta}_{t+1}.
\end{aligned} \tag{4}
$$

and because we have not imposed a liquidity constraint, the solution satisfies the Euler equation

$$
\mathrm{u}^c(c_t) = \mathbb{E}_t[\beta\mathsf{R}\mathcal{G}_{t+1}^{-\rho}\mathrm{u}^c(c_{t+1})]. \tag{5}
$$

For the remainder of the paper we will assume that permanent income $\boldsymbol{p}_t$ grows by a constant factor $\mathcal{G}$ and is not subject to stochastic shocks. (The generalization to the case with permanent shocks is straightforward.)

References

Carroll, Christopher D. (2023a): "Solving Microeconomic Dynamic Stochastic Optimization Problems," *Econon-ARK REMARK*.

——— (2023b): "Theoretical Foundations of Buffer Stock Saving," *Revise and Resubmit, Quantitative Economics*.