# Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf

- Efficient Solution Methods for Canonical $C$ problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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## The Basic Problem at Date $t$

$$\max\ \mathbb{E}_t\left[\sum_{n=0}^{T-t} \beth^n \mathrm{u}(\mathbf{c}_{t+n})\right]. \tag{1}$$

$$y_t = \boldsymbol{p}_t \boldsymbol{\theta}_t \tag{2}$$

$$\mathbf{p}_{t+1} = \mathcal{G}_{t+1}\mathbf{p}_t \qquad \text{- permanent labor income dynamics}$$

$$\log\ \boldsymbol{\theta}_{t+n} \sim\ \mathcal{N}(-\sigma^2_{\boldsymbol{\theta}}/2, \sigma^2_{\boldsymbol{\theta}}) \quad \text{- lognormal transitory shocks} \ \forall\ n > 0 \tag{3}$$

## Bellman Equation

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} \ \mathrm{u}(\mathbf{c}_t) + \beth \mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \tag{4}$$

$m-$ ‘market resources’ (net worth plus current income)

$\boldsymbol{p}-$ permanent labor income

# Trick: Normalize the Problem

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
m_{t+1} &= \underbrace{(\mathsf{R}/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} a_t + \boldsymbol{\theta}_{t+1}.
\end{aligned} \tag{5}
$$

where nonbold variables are bold ones normalized by $\boldsymbol{p}$:

$$m_t = \boldsymbol{m}_t / \boldsymbol{p}_t \tag{6}$$

Yields $\mathrm{c}_t(m)$ from which we can obtain

$$\boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) = \mathrm{c}_t(\boldsymbol{m}_t/\boldsymbol{p}_t)\boldsymbol{p}_t \tag{7}$$

## When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman (transitory/permanent) income process
  - e.g., AR(1)
  - But micro evidence is consistent with Friedman

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# Trick: View Everything from End of Period

Define

$$\mathrm{v}_{+s}(a_s) = \mathbb{E}_s[\beta \mathcal{G}_{s+1}^{1-\rho} \mathrm{v}_{s+1}(\mathcal{R}_{s+1} a_s + \boldsymbol{\theta}_{s+1})] \tag{8}$$

so

$$\mathrm{v}_t(m_t) = \max_{c_t} \quad u(c_t) + \mathfrak{v}_t(m_t - c_t) \tag{9}$$

with FOC

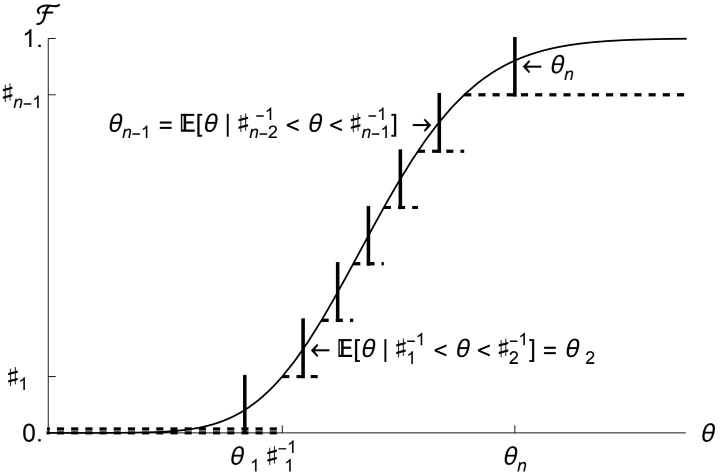
$$\mathrm{u}^c(c_s) = \mathrm{v}_{+s}^a(m_s - c_s). \tag{10}$$

and Envelope relation

$$\mathrm{u}^c(c_t) = \mathrm{v}_t^m(m_t) \tag{11}$$

# Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:



# Trick: Discretize the Risks

$$\mathfrak{v}_t'(a_t) = \beta \mathsf{R} \mathcal{G}_{t+1}^{-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} u' \left(\mathrm{c}_{t+1}(\mathcal{R}_{t+1} a_t + \boldsymbol{\theta}_i)\right) \tag{12}$$

So for any particular $m_{T-1}$ the corresponding $c_{T-1}$ can be found using the FOC:

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## Trick: Interpolate a Consumption Rule

1. Define a grid of points $\vec{m}$ (indexed $m[i]$)
2. Use numerical rootfinder to solve $u'(c) = \mathfrak{v}'_t(m[i] - c)$
   - The $c$ that solves this becomes $c[i]$
3. Construct interpolating function $\grave{c}$ by linear interpolation
   - 'Connect-the-dots'

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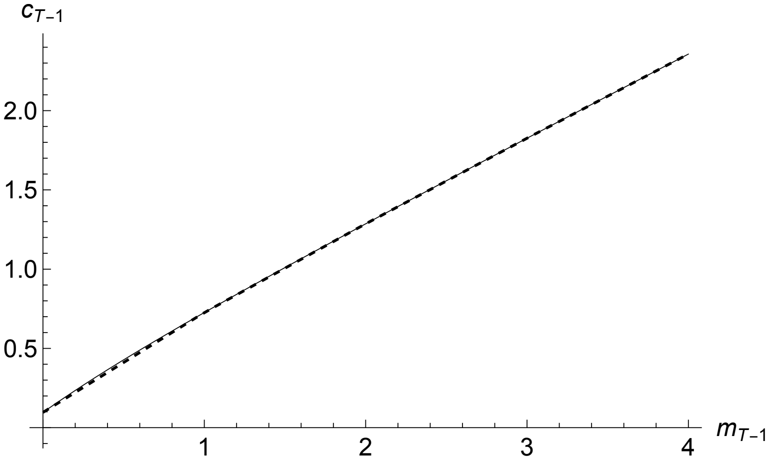
# Trick: Interpolate a Consumption Rule

Example: $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$ (solid is ‘correct’ soln)



# Problem: Numerical Rootfinding is *Slow*

Numerical search for values of $c_{T-1}$ satisfying $u'(c) = \mathfrak{v}'_t(m[i] - c)$ at, say, 6 gridpoints of $\vec{m}_{T-1}$ may require hundreds or even thousands of evaluations of

$$\mathfrak{v}'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \mathcal{G}_T^{1-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} (\mathcal{R}_T a_{T-1} + \boldsymbol{\theta}_i)^{-\rho}$$

# Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values $\vec{a}$
- For each $a[j]$ compute $\mathfrak{v}_t'(a[j])$

Each of these $\mathfrak{v}_t'[j]$ corresponds to a unique $c[j]$ via FOC:

$$\begin{aligned} c[j]^{-\rho} &= \mathfrak{v}_t'(a[j]) \\ c[j] &= \left(\mathfrak{v}_t'(a[j])\right)^{-1/\rho} \end{aligned} \tag{14}$$

But the DBC says

$$\begin{aligned} a_t &= m_t - c_t \\ m[j] &= a[j] + c[j] \end{aligned} \tag{15}$$

So computing $\mathfrak{v}_t'$ at a vector of $\vec{a}$ values has produced for us the corresponding $\vec{c}$ and $\vec{m}$ values at virtually no cost!

From these we can interpolate as before to construct $\grave{\mathrm{c}}_t(m)$.

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# Why Directly Approximating $\mathrm{v}_t$ is a Bad Idea

Principles of Approximation

- Hard to approximate things that approach $\infty$ for relevant $m$
  - Not a prob for Rep Agent models: 'relevant' $m$'s are $\approx$ SS
- Hard to approximate things that are highly nonlinear

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# Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$\mathrm{c}_t(m) = (m + \mathfrak{h}_t)\underline{\kappa}_t \tag{16}$$

for market resources $m$ and end-of-period human wealth $\mathfrak{h}$.

This is why it's a good idea to approximate $\mathrm{c}_t$

Bonus: Easy to debug programs by setting $\sigma^2 = 0$ and testing whether numerical solution matches analytical!

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# But What if You *Need* the Value Function?

Perfect foresight value function:

$$\begin{aligned}
\bar{\mathrm{v}}_t(m_t) &= \mathrm{u}(\bar{c}_t)\mathbb{C}_t^T \\
&= \mathrm{u}(\bar{c}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}((\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle \mathfrak{h}_t)\underline{\kappa}_t^{1-\rho}\underline{\kappa}_t^{-1} \\
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\end{aligned} \tag{17}$$

where the second line uses the fact demonstrated in Carroll (2023) that $\mathbb{C}_t = \kappa_t^{-1}$.

This can be transformed as

$$\begin{aligned}
\bar{\Lambda}_t &\equiv ((1-\rho)\bar{\mathrm{v}}_t)^{1/(1-\rho)} \\
&= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\
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# Approximate Slope Too

Carroll (2023) shows that $\mathrm{c}_t^m$ exists everywhere.

Define *consumed* function and its derivative as

$$\begin{aligned} \mathfrak{c}_t(a) &= (\mathfrak{v}_t'(a))^{-1/\rho} \\ \mathfrak{c}_t^a(a) &= -(1/\rho)\left(\mathfrak{v}_t'(a)\right)^{-1-1/\rho}\mathfrak{v}_t''(a) \end{aligned} \tag{19}$$

and using chain rule it is easy to show that

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# To Implement: Modify Prior Procedures in Two Ways

1. Construct $\vec{c}_t^m$ along with $\vec{c}_t$ in EGM algorithm
2. Approximate $c_t(m)$ using piecewise Hermite polynomial
    - Exact match to both level and derivative at set of points

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# Problem: $\grave{c}$ Below Bottom $m$ Gridpoint and Extrapolation

Consider what happens as $a_{T-1}$ approaches $\underline{a}_{T-1} \equiv -\underline{\boldsymbol{\theta}}\mathcal{R}_T^{-1}$,

$$\begin{aligned}\lim_{a \downarrow \underline{a}_{T-1}} \mathfrak{v}'_{T-1}(a) &= \lim_{a \downarrow \underline{a}_{T-1}} \beta \mathrm{R} \mathcal{G}_T^{-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} (a\mathcal{R}_T + \boldsymbol{\theta}_i)^{-\rho} \\ &= \infty\end{aligned}$$

This means our lowest value in $\vec{a}_{T-1}$ should be $> \underline{a}_{T-1}$.

Suppose we construct $\grave{c}$ by linear interpolation:

$$\grave{c}_{T-1}(m) = \grave{c}_{T-1}(\vec{m}_{T-1}[1]) + \grave{c}'_{T-1}(\vec{m}_{T-1}[1])(m - \vec{m}_{T-1}[1])$$

True $c$ is strictly concave $\Rightarrow \exists m^- > \underline{m}_{T-1}$ for which $m^- - \grave{c}_{T-1}(m^-) < \underline{a}_{T-1}$

# Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned} \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}^{m}_{T-1}(m) &= \bar{\kappa}_{T-1} \end{aligned} \tag{21}$$

1. Redefine $\vec{a}$ *relative* to $\underline{a}_{T-1}$
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then proceed as before.

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# Solution: Hard-Code the Bottom Point

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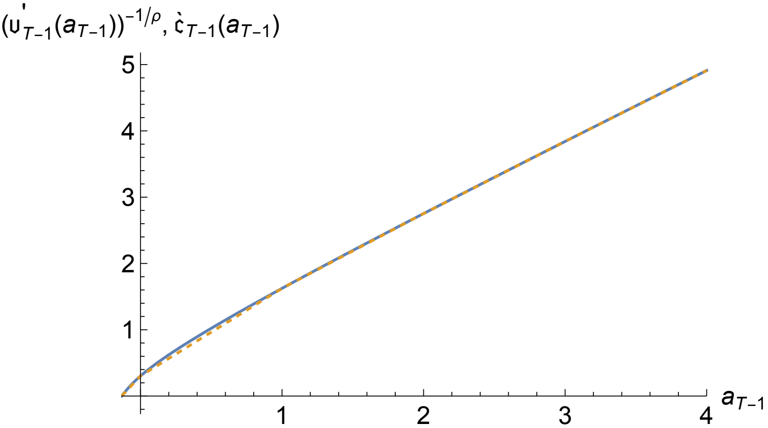
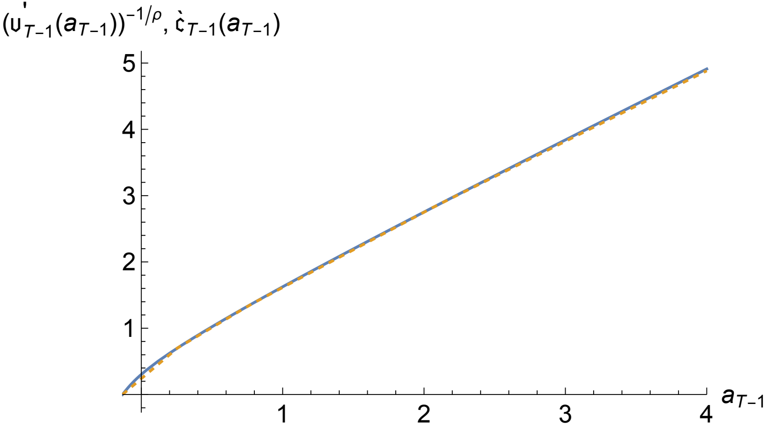
# Trick: Improving the $a$ Grid

Grid Spacing: Uniform



# Trick: Improving the $a$ Grid

Grid Spacing: Same $\{\underline{a}, \bar{a}\}$ But Triple Exponential $e^{e^{e^{\cdots}}}$ Growth



## The Method of Moderation

- Further improves speed and accuracy of solution
- See my talk at the conference!

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# Imposing 'Artificial' Borrowing Constraints

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\begin{aligned}
\mathrm{v}_{T-1}(m_{T-1}) = & == \max_{c_{T-1}} \; u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \mathcal{G}_T^{1-\rho} \mathrm{v}_T(m_T)] \\
\text{s.t.} & \\
a_{T-1} = & == m_{T-1} - c_{T-1} \\
m_T = & == \mathcal{R}_T a_{T-1} + \boldsymbol{\theta}_T \\
a_{T-1} \geq & \; 0.
\end{aligned}
$$

Define $\grave{c}_t^*$ as soln to unconstrained problem. Then

$$
\grave{c}_{T-1}(m_{T-1}) = \min[m_{T-1}, \grave{c}_{T-1}^*(m_{T-1})]. \tag{22}
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Procedure is very easy:

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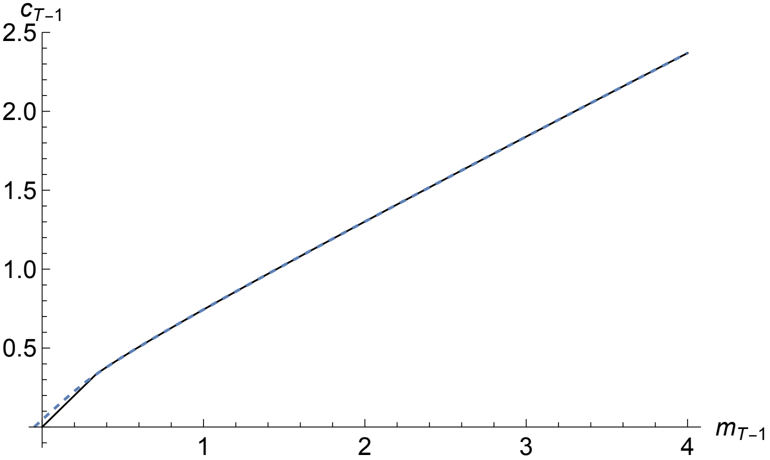


Figure: Constrained (solid) and Unconstrained (dashed) Consumption

# Recursion: Period $t$ Solution Given Period $t+1$

1. Construct

$$\begin{aligned} \mathrm{c}_{\bar{t},i} &= \left(\mathrm{v}'_{+s}(a_{t,i})\right)^{-1/\rho}, \\ &= \left(\beta \mathbb{E}_t \left[\mathsf{R}\mathcal{G}_{t+1}^{-\rho}(\grave{\mathrm{c}}_{t+1}(\mathcal{R}_{t+1}a_{t,i} + \boldsymbol{\theta}_{t+1}))^{-\rho}\right]\right)^{-1/\rho}, \end{aligned} \tag{23}$$

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# Consumption Rules $\grave{c}_{T-n}$ Converge

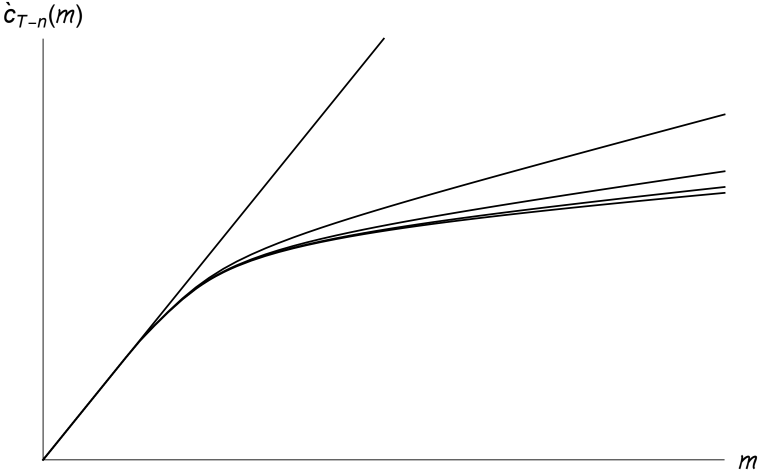


Figure: Converging $\grave{c}_{T-n}(m)$ Functions for $n = \{1, 5, 10, 15, 20\}$

# Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.
The portfolio return is

$$\begin{aligned}\mathfrak{R}_{t+1} &= \mathsf{R}(1-\varsigma_t) + \mathbf{R}_{t+1}\varsigma_t \\ &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma_t\end{aligned} \tag{24}$$

so (setting $\mathcal{G} = 1$) the maximization problem is

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The FOC with respect to $c_t$ now yields an Euler equation

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When the problem satisfies certain conditions (Carroll (2023)), it defines a 'converged' consumption rule with a 'target' ratio $\check{m}$ that satisfies:

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# Trick: Coarse then Fine $\theta$

1. Start with coarse grid for $\theta$ (say, 3 points)
2. Solve to convergence; call period of convergence $n$
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# Life Cycle Maximization Problem

$$\mathrm{v}_t(m_t) = \max_{c_t} \quad \mathrm{u}(c_t) + \beth \mathcal{L}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\boldsymbol{\Psi}_{t+1} \mathcal{G}_{t+1})^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})]$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = a_t \underbrace{\left( \frac{\mathsf{R}}{\boldsymbol{\Psi}_{t+1} \mathcal{G}_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \boldsymbol{\theta}_{t+1}$$

$\mathcal{L}_t^{t+n}$ : probability to $\mathcal{L}$ive until age $t+n$ given alive at age $t$

$\hat{\beta}_t^{t+n}$ : age-varying discount factor between ages $t$ and $t+n$

$\Psi_t$ : mean-one shock to permanent income

$\beth$ : time-invariant 'pure' discount factor

# Details follow Cagetti (2003)

- Parameterization of Uncertainty
- Probability of Death
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# Empirical Wealth Profiles

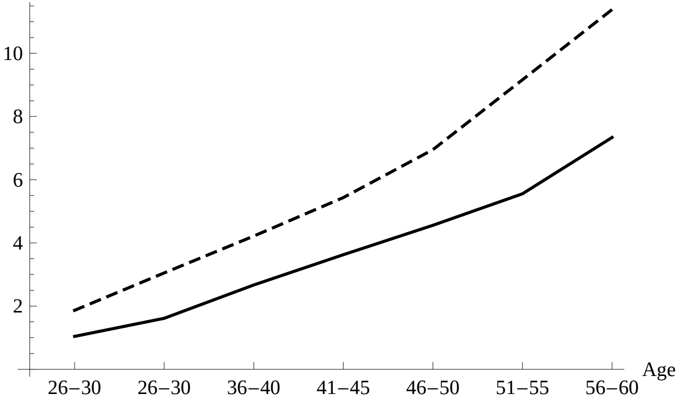


Figure: $m$ from SCF (means (dashed) and medians (solid))

# Simulated Moments

Given a set of parameter values $\{\rho, \beth\}$:

- Start at age 25 with empirical *m* data
- Draw shocks using calibrated $\sigma^2_{\mathbf{\Psi}}, \sigma^2_{\boldsymbol{\theta}}$
- Consume according to solved $c_t$

$\Rightarrow$ *m* distribution by age

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# Choose What to Simulate

GapEmpiricalSimulatedMedians[$\rho$, $\beth$]:=

[ ConstructcFuncLife[$\rho$, $\beth$];

Simulate;

$$\sum_{i}^{N} \omega_i \left| \varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi) \right|$$

];

# Calculate Match Between Theory and Data

$$\xi = \{\rho, \beth\} \tag{28}$$

solve

$$\min_{\xi} \sum_{i}^{N} \omega_i \left|\varsigma_i^{\mathcal{T}} - \mathbf{s}^{\mathcal{T}}(\xi)\right| \tag{29}$$

# Bootstrap Standard Errors (Horowitz (2001))

Yields estimates of

Table: Estimation Results

| $\rho$ | $\beth$ |
|---|---|
| 3.69 | 0.88 |
| (0.047) | (0.002) |

# Contour Plot

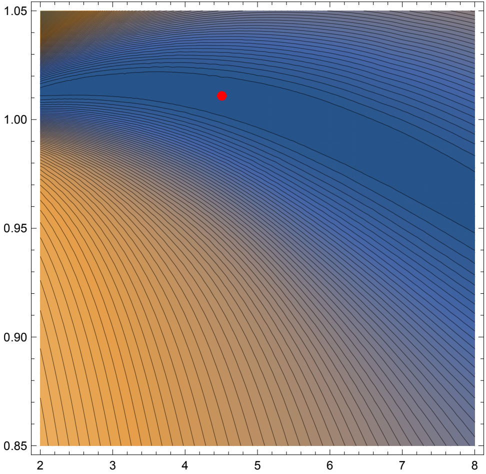


Figure: Point Estimate and Height of Minimized Function

# References I

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