

# Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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<http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf>

- Efficient Solution Methods for Canonical  $C$  problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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# The Basic Problem at Date $t$

$$\max \mathbb{E}_t \left[ \sum_{n=0}^{T-t} \beta^n u(\mathbf{c}_{t+n}) \right]. \quad (1)$$

$$y_t = \mathbf{p}_t \theta_t \quad (2)$$

$\mathbf{p}_{t+1} = \mathcal{G}_{t+1} \mathbf{p}_t$  - permanent labor income dynamics

$\log \theta_{t+n} \sim \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2)$  - lognormal transitory shocks  $\forall n > 0$ .  
(3)



# Bellman Equation

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} u(\mathbf{c}_t) + \beta \mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \quad (4)$$

$m$ — 'market resources' (net worth plus current income)

$p$ — permanent labor income

## Trick: Normalize the Problem

$$\begin{aligned}
v_t(m_t) &= \max_{c_t} u(c_t) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} v_{t+1}(m_{t+1})] \\
\text{s.t.} \quad & \\
a_t &= m_t - c_t \\
k_{t+1} &= a_t \\
m_{t+1} &= \underbrace{(R/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} k_{t+1} + \theta_{t+1},
\end{aligned} \tag{5}$$

where nonbold variables are bold ones normalized by  $\mathbf{p}$ :

$$m_t = m_t / \mathbf{p}_t \quad (6)$$

Yields  $c_t(m)$  from which we can obtain

$$c_t(m_t, \mathbf{p}_t) = c_t(m_t/\mathbf{p}_t)\mathbf{p}_t \quad (7)$$

## When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman (transitory/permanent) income process
  - e.g., AR(1)
  - But micro evidence is consistent with Friedman

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## Trick: View Everything from End of Period

## Define

$$\mathbf{v}_{+(s)}(\mathbf{a}_s) = \mathbb{E}_s[\beta \mathcal{G}_{s+1}^{1-\rho} \mathbf{v}_{\sim(s+1)}(\mathcal{R}_{s+1} \mathbf{a}_s + \theta_{s+1})] \quad (8)$$

so

$$v_t(m_t) = \max_{c_t} u(c_t) + v_t(m_t - c_t) \quad (9)$$

with FOC

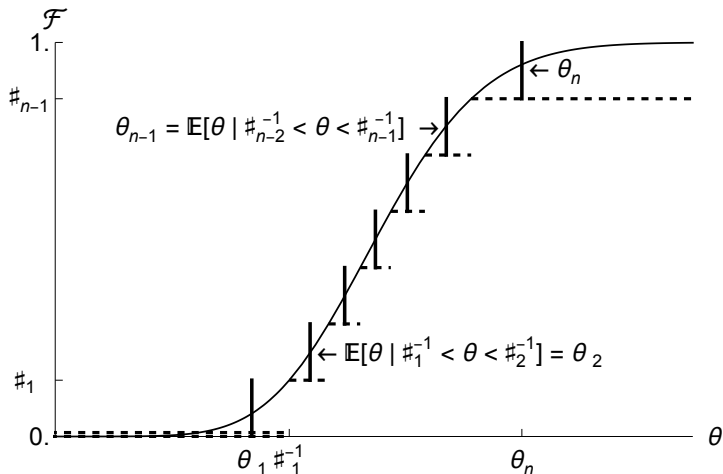
$$u^c(c_s) = v_{+(s)}^a(m_s - c_s). \quad (10)$$

and Envelope relation

$$u^c(c_t) = v_{\sim(s)}^m(m_t) \quad (11)$$

# Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:





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$$\mathbf{v}'_t(a_t) = \beta R \mathcal{G}_{t+1}^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n u'(\mathbf{c}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_i)) \quad (12)$$

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$$\mathbf{v}'_t(a_t) = \beta R \mathcal{G}_{t+1}^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n u' (c_{t+1}(\mathcal{R}_{t+1} a_t + \theta_i)) \quad (12)$$

So for any particular  $m_{T-1}$  the corresponding  $c_{T-1}$  can be found using the FOC:

$$u^c(c_s) = v_{+(s)}^a(m_s - c_s). \quad (13)$$



## Trick: Interpolate a Consumption Rule

- 1 Define a grid of points  $\vec{m}$  (indexed  $m[i]$ )
- 2 Use numerical rootfinder to solve  $u'(c) = v'_t(m[i] - c)$ 
  - The  $c$  that solves this becomes  $c[i]$
- 3 Construct interpolating function  $\hat{c}$  by linear interpolation
  - 'Connect-the-dots'

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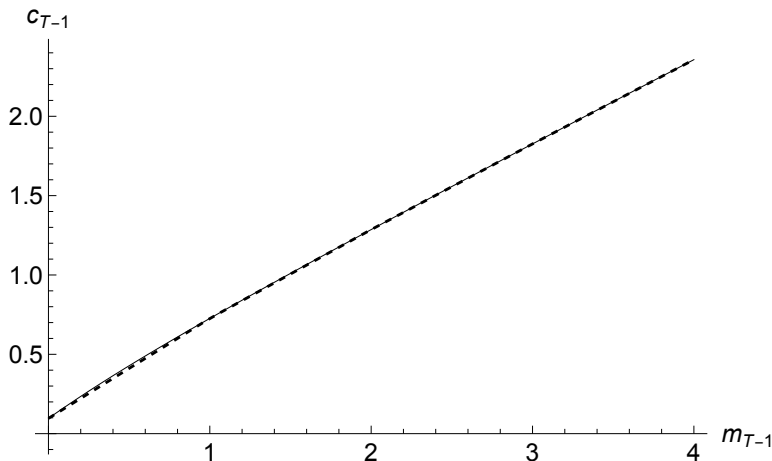
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## Trick: Interpolate a Consumption Rule

Example:  $\vec{m}_{T-1} = \{0., 1., 2., 3., 4.\}$  (solid is 'correct' soln)







# Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values  $\vec{a}$
- For each  $a[j]$  compute  $v'_t(a[j])$

Each of these  $v'_t[j]$  corresponds to a unique  $c[j]$  via FOC:

$$\begin{aligned} c[j]^{-\rho} &= v'_t(a[j]) \\ c[j] &= (v'_t(a[j]))^{-1/\rho} \end{aligned} \tag{14}$$

But the DBC says

$$\begin{aligned} a_t &= m_t - c_t \\ m[j] &= a[j] + c[j] \end{aligned} \tag{15}$$

So computing  $v'_t$  at a vector of  $\vec{a}$  values has produced for us the corresponding  $\vec{c}$  and  $\vec{m}$  values at virtually no cost!

From these we can interpolate as before to construct  $\hat{c}_t(m)$ .

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# Why Directly Approximating $v_t$ is a Bad Idea

## Principles of Approximation

- Hard to approximate things that approach  $\infty$  for relevant  $m$ 
  - Not a prob for Rep Agent models: 'relevant'  $m$ 's are  $\approx SS$
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# Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$c_t(m) = (m + h_t)\underline{\kappa}_t \quad (16)$$

for market resources  $m$  and end-of-period human wealth  $h$ .

This is why it's a good idea to approximate  $c_t$

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# But What if You *Need* the Value Function?

Perfect foresight value function:

$$\begin{aligned}\bar{v}_t(m_t) &= u(\bar{c}_t)\mathbb{C}_t^T \\ &= u(\bar{c}_t)\underline{\kappa}_t^{-1} \\ &= u((\blacktriangle m_t + \blacktriangle h_t)\underline{\kappa}_t)\underline{\kappa}_t^{-1} \\ &= u(\blacktriangle m_t + \blacktriangle h_t)\underline{\kappa}_t^{1-\rho}\underline{\kappa}_t^{-1} \\ &= u(\blacktriangle m_t + \blacktriangle h_t)\underline{\kappa}_t^{-\rho}\end{aligned}\tag{17}$$

where the second line uses the fact demonstrated in Carroll (2023) that  $\mathbb{C}_t = \kappa_t^{-1}$ .

This can be transformed as

$$\begin{aligned}\bar{\lambda}_t &\equiv ((1 - \rho)\bar{v}_t)^{1/(1-\rho)} \\ &= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\ &= (\blacktriangle m_t + \blacktriangle h_t)\underline{\kappa}_t^{-\rho/(1-\rho)}\end{aligned}$$

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# Approximate Slope Too

Carroll (2023) shows that  $c_t^m$  exists everywhere.

Define *consumed* function and its derivative as

$$\begin{aligned} c_t(a) &= (v'_t(a))^{-1/\rho} \\ c_t^a(a) &= -(1/\rho) (v'_t(a))^{-1-1/\rho} v''_t(a) \end{aligned} \tag{19}$$

and using chain rule it is easy to show that

$$c_t^m = c_t^a / (1 + c_t^a) \tag{20}$$

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# To Implement: Modify Prior Procedures in Two Ways

- 1 Construct  $\vec{c}_t^m$  along with  $\vec{c}_t$  in EGM algorithm
- 2 Approximate  $c_t(m)$  using piecewise Hermite polynomial
  - Exact match to both level and derivative at set of points

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# Problem: $\check{c}$ Below Bottom $m$ Gridpoint and Extrapolation

Consider what happens as  $a_{T-1}$  approaches  $\underline{a}_{T-1} \equiv -\underline{\theta}\mathcal{R}_T^{-1}$ ,

$$\lim_{a \downarrow \underline{a}_{T-1}} v'_{T-1}(a) = \lim_{a \downarrow \underline{a}_{T-1}} \beta R \mathcal{G}_T^{-\rho} \left( \frac{1}{n} \right) \sum_{i=1}^n (a \mathcal{R}_T + \theta_i)^{-\rho} \\ = \infty$$

This means our lowest value in  $\vec{a}_{T-1}$  should be  $> \underline{a}_{T-1}$ .

Suppose we construct  $\check{c}$  by linear interpolation:

$$\check{c}_{T-1}(m) = \check{c}_{T-1}(\vec{m}_{T-1}[1]) + \check{c}'_{T-1}(\vec{m}_{T-1}[1])(m - \vec{m}_{T-1}[1])$$

True  $c$  is strictly concave  $\Rightarrow \exists m^- > \underline{m}_{T-1}$  for which

$$m^- - \check{c}_{T-1}(m^-) < \underline{a}_{T-1}$$



# Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned}\lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} c_{T-1}^m(m) &= \bar{\kappa}_{T-1}\end{aligned}\tag{21}$$

- ① Redefine  $\vec{a}$  *relative* to  $\underline{a}_{T-1}$
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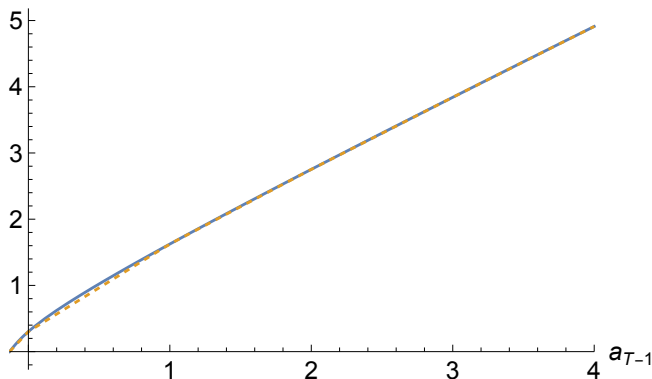
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# Trick: Improving the $a$ Grid

Grid Spacing: Uniform

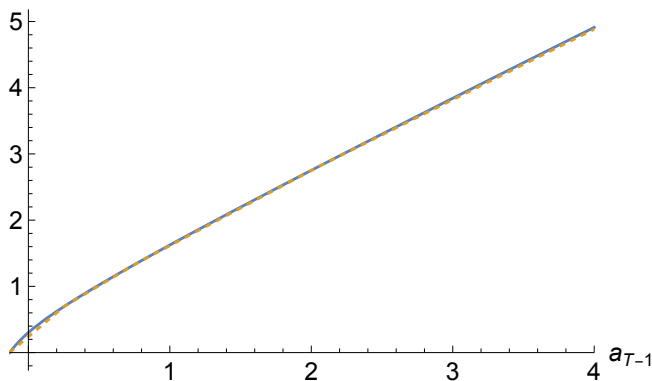
$$(u'_{T-1}(a_{T-1}))^{-1/\rho}, \dot{c}_{T-1}(a_{T-1})$$



# Trick: Improving the $a$ Grid

Grid Spacing: Same  $\{\underline{a}, \bar{a}\}$  But Triple Exponential  $e^{e^{\dots}}$  Growth

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- Further improves speed and accuracy of solution
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# Imposing 'Artificial' Borrowing Constraints

$$\begin{aligned}
 v_{T-1}(m_{T-1}) &= \max_{c_{T-1}} u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \mathcal{G}_T^{1-\rho} v_T(m_T)] \\
 &\text{s.t.} \\
 a_{T-1} &= m_{T-1} - c_{T-1} \\
 m_T &= \mathcal{R}_T a_{T-1} + \theta_T \\
 a_{T-1} &\geq 0.
 \end{aligned}$$

Define  $\hat{c}_t^*$  as soln to unconstrained problem. Then

$$\hat{c}_{T-1}(m_{T-1}) = \min[m_{T-1}, \hat{c}_{T-1}^*(m_{T-1})]. \quad (22)$$

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Point where constraint makes transition from binding to not is

$$\begin{aligned} u'(m_{T-1}^\#) &= \mathbf{v}'_{T-1}(0.) \\ m_{T-1}^\# &= (\mathbf{v}'_{T-1}(0.))^{-1/\rho} \end{aligned}$$

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# Recursion: Period $t$ Solution Given Period $t + 1$

## 1 Construct

$$\begin{aligned}
 c_{\bar{t},i} &= \left( v_{+(s)}^a(a_{t,i}) \right)^{-1/\rho}, \\
 &= \left( \beta \mathbb{E}_t \left[ \mathcal{RG}_{t+1}^{-\rho} (\hat{c}_{t+1} (\mathcal{R}_{t+1} a_{t,i} + \theta_{t+1}))^{-\rho} \right] \right)^{-1/\rho}, \quad (23)
 \end{aligned}$$

- 2 Call the result  $\vec{c}_t$  and generate the corresponding  $\vec{m}_t = \vec{c}_t + \vec{a}_t$
- 3 Interpolate to create  $\hat{c}_t(m)$

# Consumption Rules $\hat{c}_{T-n}$ Converge

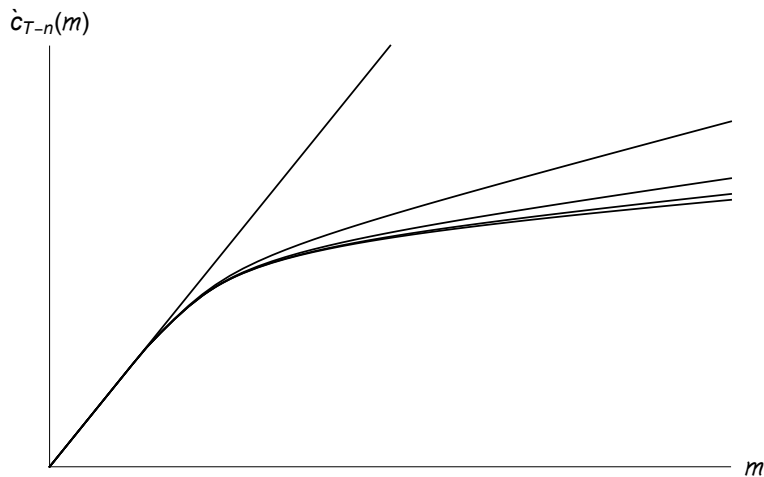


Figure: Converging  $\hat{c}_{T-n}(m)$  Functions for  $n = \{1, 5, 10, 15, 20\}$

Now the consumer has a choice between a risky and a safe asset.  
The portfolio return is

s.t.

$$m_{t+1} = (m_t - c_t)\mathfrak{R}_{t+1} + \theta_{t+1}$$

$$0 \leq \varsigma_t \leq 1,$$









# Portfolio Choice

The FOC with respect to  $c_t$  now yields an Euler equation

$$\mathbf{u}^c(c_t) = \mathbb{E}_t[\beta \mathfrak{R}_{t+1} \mathbf{u}^c(c_{t+1})]. \quad (25)$$



# Convergence

When the problem satisfies certain conditions (Carroll (2023)), it defines a ‘converged’ consumption rule with a ‘target’ ratio  $\check{m}$  that satisfies:

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Then a plausible metric for convergence is to define some value  $\epsilon$  and to declare the solution to have converged when

$$|\check{m}_{t+1} - \check{m}_t| < \epsilon \quad (27)$$

## Trick: Coarse then Fine $\theta$











Trick: Coarse then Fine  $\vec{a}_{T-1}$

- 1 Start with coarse grid for  $\vec{a}$  (say, 5 gridpoints)
- 2 Solve to convergence; call period of convergence  $n$
- 3 Construct finer grid for  $\vec{a}$  (say, 20 points)
- 4 Solve for period  $T - n - 1$  assuming  $\vec{c}_{T-n}$
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# Life Cycle Maximization Problem

$$v_t(m_t) = \max_{c_t} u(c_t) + \beta_{t+1} \mathbb{E}_t[(\psi_{t+1} \mathcal{G}_{t+1})^{1-\rho} v_{t+1}(m_{t+1})]$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = a_t \underbrace{\left( \frac{R}{\Psi_{t+1} \mathcal{G}_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1}$$

$\mathcal{L}_t^{t+n}$ : probability to  $\mathcal{L}$ ive until age  $t + n$  given alive at age  $t$

$\hat{\beta}_t^{t+n}$ : age-varying discount factor between ages  $t$  and  $t + n$

$\Psi_t$  : mean-one shock to permanent income

- ▷ : time-invariant ‘pure’ discount factor





- Parameterization of Uncertainty
- Probability of Death
- Demographic Adjustments to  $\beta$



## Simulated Moments

Given a set of parameter values  $\{\rho, \gamma\}$ :

- Start at age 25 with empirical  $m$  data
- Draw shocks using calibrated  $\sigma_{\Psi}^2, \sigma_{\theta}^2$
- Consume according to solved  $c_t$

$\Rightarrow m$  distribution by age

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## Choose What to Simulate

```

GapEmpiricalSimulatedMedians[ $\rho, \beth$ ] :=
[
    ConstructcFuncLife[ $\rho, \beth$ ];
    Simulate;
    
$$\sum_i^N \omega_i |\varsigma_i^\tau - \mathbf{s}^\tau(\xi)|$$

];

```

## Calculate Match Between Theory and Data

$$\xi = \{\rho, \sqsupset\} \quad (28)$$

solve

$$\min_{\xi} \sum_i^N \omega_i |\varsigma_i^T - \mathbf{s}^T(\xi)| \quad (29)$$





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