Structural Estimation of Dynamic Stochastic Optimizing Models of Intertemporal Choice For Dummies!

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http://www.econ2.jhu.edu/people/ccarroll/SolvingMicroDSOPs-Slides.pdf

- Efficient Solution Methods for Canonical C problem
  - CRRA utility
  - Plausible (microeconomically calibrated) uncertainty
  - Life cycle or infinite horizon
- How To Add a Second Choice Variable
- Method of Simulated Moments Estimation of Parameters

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The Basic Problem at Date t

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beth^n \mathrm{u}(\mathbf{c}_{t+n}) \right]. \tag{1}$$

$$y_t = \boldsymbol{p}_t \theta_t \tag{2}$$

$$\begin{aligned} \mathbf{p}_{t+1} &= \mathcal{G}_{t+1} \mathbf{p}_t && \text{- permanent labor income dynamics} \\ \log \ \theta_{t+n} &\sim \ \mathcal{N}(-\sigma_\theta^2/2, \sigma_\theta^2) && \text{- lognormal transitory shocks } \forall \ n > 0. \end{aligned} \tag{3}$$

Bellman Equation

$$\mathbf{v}_t(\mathbf{m}_t, \mathbf{p}_t) = \max_{\mathbf{c}_t} \; \mathrm{u}(\mathbf{c}_t) + \beth\mathbb{E}_t[\mathbf{v}_{t+1}(\mathbf{m}_{t+1}, \mathbf{p}_{t+1})] \tag{4}$$

m — 'market resources' (net worth plus current income)

$\boldsymbol{p}$ — permanent labor income

Trick: Normalize the Problem

$$
\begin{aligned}
\mathrm{v}_t(m_t) &= \max_{c_t} \quad \mathrm{u}(c_t) + \beta \mathbb{E}_t[\mathcal{G}_{t+1}^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})] \\
&\text{s.t.} \\
a_t &= m_t - c_t \\
k_{t+1} &= a_t \\
m_{t+1} &= \underbrace{(\mathsf{R}/\mathcal{G}_{t+1})}_{\equiv \mathcal{R}_{t+1}} k_{t+1} + \theta_{t+1},
\end{aligned} \tag{5}
$$

where nonbold variables are bold ones normalized by $\boldsymbol{p}$:

$$
m_t = \boldsymbol{m}_t / \boldsymbol{p}_t \tag{6}
$$

Yields $\mathrm{c}_t(m)$ from which we can obtain

$$
\boldsymbol{c}_t(\boldsymbol{m}_t, \boldsymbol{p}_t) = \mathrm{c}_t(\boldsymbol{m}_t / \boldsymbol{p}_t) \boldsymbol{p}_t \tag{7}
$$

When Doesn't Normalization Work?

- Non-CRRA utility
- Non-Friedman (transitory/permanent) income process
  - e.g., AR(1)
  - But micro evidence is consistent with Friedman

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Trick: View Everything from End of Period

Define

$$\mathrm{v}_{t_{\rightarrow}}(a_t) = \beta \mathrm{v}_{\leftarrow t}(a_{t+1}) \tag{8}$$

so

$$\mathrm{v}_t(m_t) = \max_{c_t} \quad u(c_t) + \mathfrak{v}_t(m_t - c_t) \tag{9}$$

with FOC

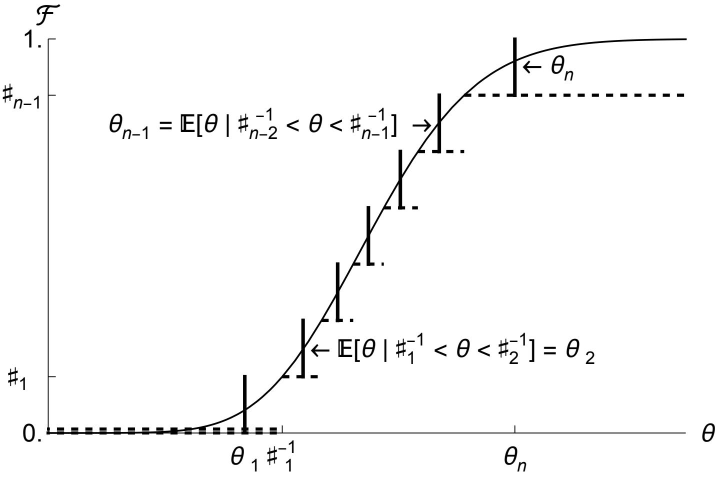
$$\mathrm{u}^c(c_t) = \mathrm{v}^a_{t_{\rightarrow}}(m_t - c_t). \tag{10}$$

and Envelope relation

$$\mathrm{u}^c(c_t) = \mathrm{v}^m_t(m_t) \tag{11}$$

Trick: Discretize the Risks

E.g. use an equiprobable 7-point distribution:



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$$\mathfrak{v}'_t(a_t) = \beta \mathsf{R} \mathcal{G}_{t+1}^{-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} u' \left(\mathrm{c}_{t+1}(\mathcal{R}_{t+1} a_t + \theta_i)\right) \tag{12}$$

So for any particular m_{T-1} the corresponding c_{T-1} can be found using the FOC:

$$\mathrm{u}^c(c_t) = \mathrm{v}^a_{t_\rightarrow}(m_t - c_t). \tag{13}$$

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Trick: Interpolate a Consumption Rule

1. Define a grid of points $\boldsymbol{m}$ (indexed $m[i]$)
2. Use numerical rootfinder to solve $u'(c) = \mathfrak{v}'_t(m[i] - c)$
   - The c that solves this becomes $c[i]$
3. Construct interpolating function $\grave{c}$ by linear interpolation
   - 'Connect-the-dots'

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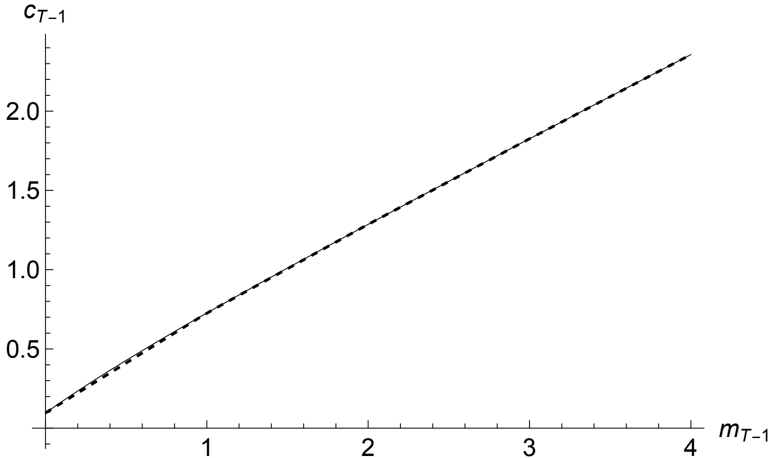
Trick: Interpolate a Consumption Rule

Example: $\boldsymbol{m}_{T-1} = \{0., 1., 2., 3., 4.\}$ (solid is 'correct' soln)



Problem: Numerical Rootfinding is *Slow*

Numerical search for values of c_{T-1} satisfying $u'(c) = \mathfrak{v}'_t(m[i] - c)$ at, say, 6 gridpoints of $\boldsymbol{m}_{T-1}$ may require hundreds or even thousands of evaluations of

$$\mathfrak{v}'_{T-1}(\overbrace{m_{T-1} - c_{T-1}}^{a_{T-1}}) = \beta_T \mathcal{G}_T^{1-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} (\mathcal{R}_T a_{T-1} + \theta_i)^{-\rho}$$

Solution: The Method of Endogenous Gridpoints

- Define vector of *end-of-period* asset values $\boldsymbol{a}$
- For each $a[j]$ compute $\mathfrak{v}'_t(a[j])$

Each of these $\mathfrak{v}'_t[j]$ corresponds to a unique $c[j]$ via FOC:

$$\begin{aligned} c[j]^{-\rho} &= \mathfrak{v}'_t(a[j]) \\ c[j] &= \left(\mathfrak{v}'_t(a[j])\right)^{-1/\rho} \end{aligned} \tag{14}$$

But the DBC says

$$\begin{aligned} a_t &= m_t - c_t \\ m[j] &= a[j] + c[j] \end{aligned} \tag{15}$$

So computing $\mathfrak{v}'_t$ at a vector of $\boldsymbol{a}$ values has produced for us the corresponding $\boldsymbol{c}$ and $\boldsymbol{m}$ values at virtually no cost!

From these we can interpolate as before to construct $\grave{\mathrm{c}}_t(m)$.

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- Define vector of *end-of-period* asset values $\boldsymbol{a}$
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Principles of Approximation

- Hard to approximate things that approach ∞ for relevant m
  - Not a prob for Rep Agent models: 'relevant' m's are $\approx$ SS
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Approximate Something That Would Be Linear in PF Case

Perfect Foresight Theory:

$$\mathrm{c}_t(m) = (m + \mathfrak{h}_t)\underline{\kappa}_t \tag{16}$$

for market resources m and end-of-period human wealth $\mathfrak{h}$.

This is why it's a good idea to approximate c_t

Bonus: Easy to debug programs by setting $\sigma^2 = 0$ and testing whether numerical solution matches analytical!

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But What if You *Need* the Value Function?

Perfect foresight value function:

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\begin{aligned}
\bar{\mathrm{v}}_t(m_t) &= \mathrm{u}(\bar{c}_t)\mathbb{C}_t^T \\
&= \mathrm{u}(\bar{c}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}((\blacktriangle m_t + \blacktriangle h_{t_\rightarrow})\underline{\kappa}_t)\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{t_\rightarrow})\underline{\kappa}_t^{1-\rho}\underline{\kappa}_t^{-1} \\
&= \mathrm{u}(\blacktriangle m_t + \blacktriangle h_{t_\rightarrow})\underline{\kappa}_t^{-\rho}
\end{aligned}
\tag{17}
$$

This can be transformed as

$$
\begin{aligned}
\bar{\Lambda}_t &\equiv ((1-\rho)\bar{\mathrm{v}}_t)^{1/(1-\rho)} \\
&= c_t(\mathbb{C}_t^T)^{1/(1-\rho)} \\
&= (\blacktriangle m_t + \blacktriangle h_{t_\rightarrow})\underline{\kappa}_t^{-\rho/(1-\rho)}
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which is linear.

If you need the value function, approximate the *inverted* value function to generate λ_t and then obtain your approximation from

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Approximate Slope Too

Carroll (2023) shows that c_t^m exists everywhere.

Define *consumed* function and its derivative as

$$\begin{aligned} \mathfrak{c}_t(a) &= (\mathfrak{v}_t'(a))^{-1/\rho} \\ \mathfrak{c}_t^a(a) &= -(1/\rho)\left(\mathfrak{v}_t'(a)\right)^{-1-1/\rho}\mathfrak{v}_t''(a) \end{aligned} \tag{19}$$

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To Implement: Modify Prior Procedures in Two Ways

1. Construct $\mathbf{c}_t^m$ along with $\mathbf{c}_t$ in EGM algorithm
2. Approximate $\mathrm{c}_t(m)$ using piecewise Hermite polynomial
    - Exact match to both level and derivative at set of points

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Problem: $\grave{c}$ Below Bottom m Gridpoint and Extrapolation

Consider what happens as a_{T-1} approaches $\underline{a}_{T-1} \equiv -\underline{\theta}\mathcal{R}_T^{-1}$,

$$\lim_{a \downarrow \underline{a}_{T-1}} \mathfrak{v}'_{T-1}(a) = \lim_{a \downarrow \underline{a}_{T-1}} \beta \mathrm{R} \mathcal{G}_T^{-\rho} \left(\frac{1}{n}\right) \sum_{i=1}^{n} (a\mathcal{R}_T + \theta_i)^{-\rho}$$
$$= \infty$$

This means our lowest value in $\boldsymbol{a}_{T-1}$ should be $> \underline{a}_{T-1}$.

Suppose we construct $\grave{c}$ by linear interpolation:

$$\grave{c}_{T-1}(m) = \grave{c}_{T-1}(\boldsymbol{m}_{T-1}[1]) + \grave{c}'_{T-1}(\boldsymbol{m}_{T-1}[1])(m - \boldsymbol{m}_{T-1}[1])$$

True c is strictly concave $\Rightarrow \exists m^{-} > \underline{m}_{T-1}$ for which
$m^{-} - \grave{c}_{T-1}(m^{-}) < \underline{a}_{T-1}$

Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned} \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}^{m}_{T-1}(m) &= \bar{\kappa}_{T-1} \end{aligned} \tag{21}$$

1. Redefine $\boldsymbol{a}$ *relative* to $\underline{a}_{T-1}$
2. Construct corresponding $\boldsymbol{m}_{T-1}$ and $\boldsymbol{c}_{T-1}$
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then proceed as before.

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5. Prepend $\bar{\kappa}_{T-1}$ to $\boldsymbol{\kappa}_{T-1}$

then proceed as before.

Solution: Hard-Code the Bottom Point

Theory says that

$$\begin{aligned} \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}_{T-1}(m) &= 0 \\ \lim_{m \downarrow \underline{m}_{T-1}} \mathrm{c}^{m}_{T-1}(m) &= \bar{\kappa}_{T-1} \end{aligned} \tag{21}$$

1. Redefine $\boldsymbol{a}$ *relative* to $\underline{a}_{T-1}$
2. Construct corresponding $\boldsymbol{m}_{T-1}$ and $\boldsymbol{c}_{T-1}$
3. Prepend $\underline{m}_{T-1}$ to $\boldsymbol{m}_{T-1}$
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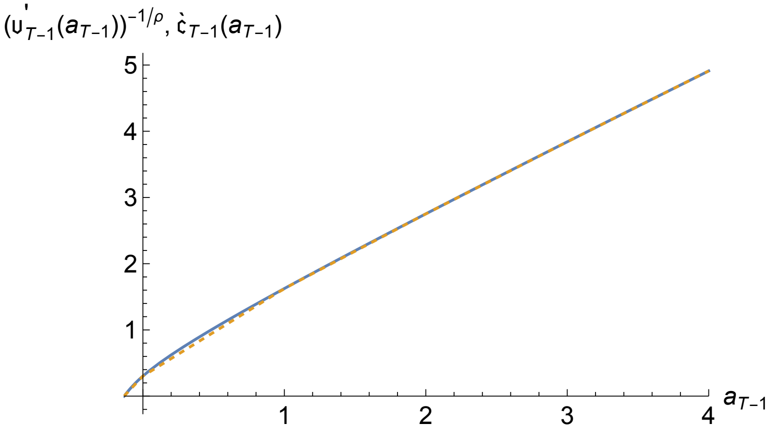
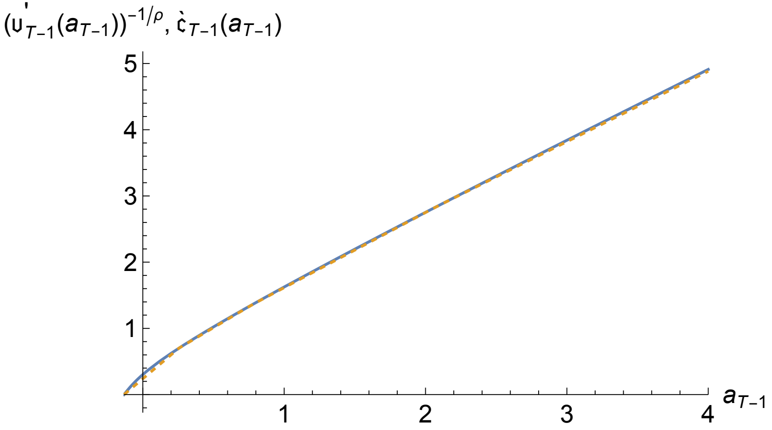
Trick: Improving the a Grid

Grid Spacing: Uniform



Trick: Improving the a Grid

Grid Spacing: Same $\{\underline{a}, \bar{a}\}$ But Triple Exponential $e^{e^{e^{\cdots}}}$ Growth



The Method of Moderation

- Further improves speed and accuracy of solution
- See my talk at the conference!

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Imposing 'Artificial' Borrowing Constraints

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\begin{aligned}
\mathrm{v}_{T-1}(m_{T-1}) &= \max_{c_{T-1}} \quad u(c_{T-1}) + \mathbb{E}_{T-1}[\beta \mathcal{G}_T^{1-\rho} \mathrm{v}_T(m_T)] \\
&\text{s.t.} \\
a_{T-1} &= m_{T-1} - c_{T-1} \\
m_T &= \mathcal{R}_T a_{T-1} + \theta_T \\
a_{T-1} &\geq 0.
\end{aligned}
$$

Define $\grave{c}_t^*$ as soln to unconstrained problem. Then

$$
\grave{c}_{T-1}(m_{T-1}) = \min[m_{T-1}, \grave{c}_{T-1}^*(m_{T-1})]. \tag{22}
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Imposing ‘Artificial’ Borrowing Constraints

Point where constraint makes transition from binding to not is

$$\begin{aligned} u'(m^{\#}_{T-1}) &= \mathfrak{v}'_{T-1}(0.) \\ m^{\#}_{T-1} &= \left(\mathfrak{v}'_{T-1}(0.)\right)^{-1/\rho} \end{aligned}$$

Procedure is very easy:

- Add 0. as first point in $\boldsymbol{a}$
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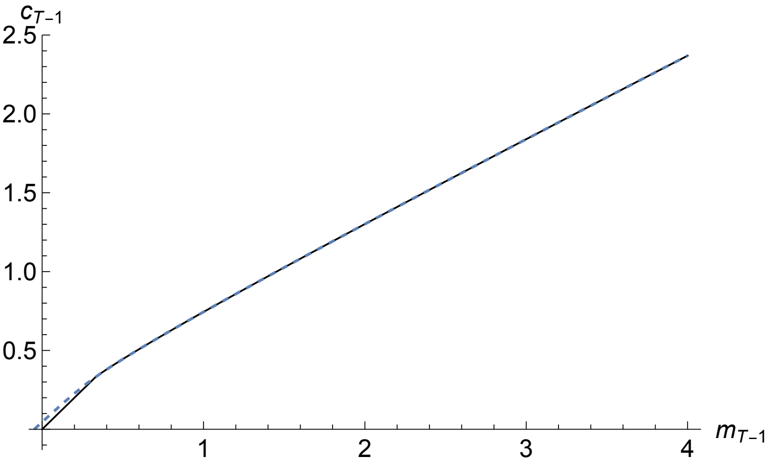

Figure: Constrained (solid) and Unconstrained (dashed) Consumption

Recursion: Period t Solution Given Period $t+1$

1. Construct

$$\begin{aligned} \mathrm{c}_{\bar{t},i} &= \left(\mathrm{v}^{a}_{t\rightarrow}(a_{t,i})\right)^{-1/\rho}, \\ &= \left(\beta \mathbb{E}_{\leftarrow t}\left[\mathsf{R}\mathcal{G}^{-\rho}_{t+1}(\grave{\mathrm{c}}_{t+1}(\mathcal{R}_{t+1}a_{t,i}+\theta_{t+1}))^{-\rho}\right]\right)^{-1/\rho}, \end{aligned} \tag{23}$$

2. Call the result $\boldsymbol{c}_t$ and generate the corresponding $\boldsymbol{m}_t = \boldsymbol{c}_t + \boldsymbol{a}_t$
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Consumption Rules $\grave{c}_{T-n}$ Converge

Figure: Converging $\grave{c}_{T-n}(m)$ Functions for $n = \{1, 5, 10, 15, 20\}$

Portfolio Choice

Now the consumer has a choice between a risky and a safe asset.

The portfolio return is

$$\begin{aligned}\mathfrak{R}_{t+1} &= \mathsf{R}(1-\varsigma_t) + \mathbf{R}_{t+1}\varsigma_t \\ &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma_t\end{aligned} \tag{24}$$

so (setting $\mathcal{G} = 1$) the maximization problem is

$$\mathrm{v}_t(m_t) = \max_{\{c_t,\varsigma_t\}} \quad \mathrm{u}(c_t) + \beta\mathbb{E}_t[\mathrm{v}_{t+1}(m_{t+1})]$$

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$$\begin{aligned}\mathfrak{R}_{t+1} &= \mathsf{R} + (\mathbf{R}_{t+1} - \mathsf{R})\varsigma_t \\ m_{t+1} &= (m_t - c_t)\mathfrak{R}_{t+1} + \theta_{t+1} \\ 0 \leq &\varsigma_t \leq 1,\end{aligned}$$

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The FOC with respect to c_t now yields an Euler equation

$$\mathrm{u}^c(c_t) = \mathbb{E}_t[\beta \mathfrak{R}_{t+1} \mathrm{u}^c(c_{t+1})]. \tag{25}$$

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Convergence

When the problem satisfies certain conditions (Carroll (2023)), it defines a 'converged' consumption rule with a 'target' ratio $\check{m}$ that satisfies:

$$\mathbb{E}_t[m_{t+1}/m_t] = 1 \text{ if } m_t = \check{m} \tag{26}$$

Define the target m implied by the consumption rule c_t as $\check{m}_t$.

Then a plausible metric for convergence is to define some value ϵ and to declare the solution to have converged when

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Trick: Coarse then Fine θ

1. Start with coarse grid for θ (say, 3 points)
2. Solve to convergence; call period of convergence n
3. Construct finer grid for θ (say, 7 points)
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Life Cycle Maximization Problem

$$\mathrm{v}_t(m_t) = \max_{c_t} \quad \mathrm{u}(c_t) + \beth \mathcal{L}_{t+1} \hat{\beta}_{t+1} \mathbb{E}_t[(\mathbf{\Psi}_{t+1} \mathcal{G}_{t+1})^{1-\rho} \mathrm{v}_{t+1}(m_{t+1})]$$

s.t.

$$a_t = m_t - c_t$$

$$m_{t+1} = a_t \underbrace{\left(\frac{\mathsf{R}}{\mathbf{\Psi}_{t+1} \mathcal{G}_{t+1}} \right)}_{\equiv \mathcal{R}_{t+1}} + \theta_{t+1}$$

$\mathcal{L}_t^{t+n}$: probability to $\mathcal{L}$ive until age $t+n$ given alive at age t

$\hat{\beta}_t^{t+n}$: age-varying discount factor between ages t and $t+n$

Ψ_t : mean-one shock to permanent income

$\beth$: time-invariant ‘pure’ discount factor

Details follow Cagetti (2003)

- Parameterization of Uncertainty
- Probability of Death
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Empirical Wealth Profiles

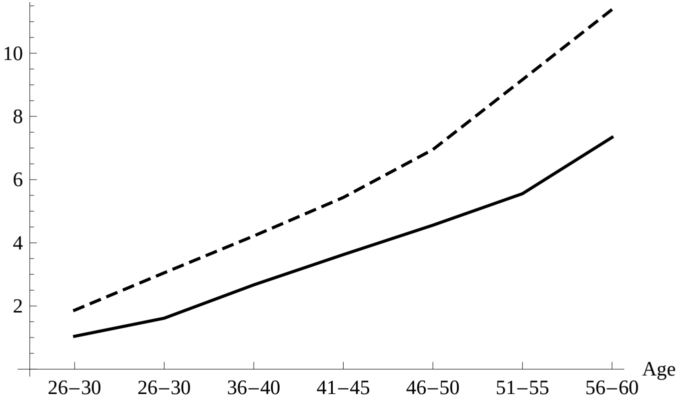

Figure: m from SCF (means (dashed) and medians (solid))

Simulated Moments

Given a set of parameter values $\{\rho, \daleth\}$:

- Start at age 25 with empirical *m* data
- Draw shocks using calibrated σ^2_{Ψ},σ^2_{θ}
- Consume according to solved c_t

$\Rightarrow$ *m* distribution by age

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Choose What to Simulate

$\texttt{GapEmpiricalSimulatedMedians}[\rho, \beth] \texttt{:=}$

$[\qquad \texttt{ConstructcFuncLife}[\rho, \beth];$

$\texttt{Simulate;}$

$$\sum_{i}^{N} \omega_i \left|\varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi)\right|$$

$];$

Calculate Match Between Theory and Data

$$\xi = \{\rho, \beth\} \tag{28}$$

solve

$$\min_{\xi} \sum_{i}^{N} \omega_i \left| \varsigma_i^{\tau} - \mathbf{s}^{\tau}(\xi) \right| \tag{29}$$

Bootstrap Standard Errors (Horowitz (2001))

Yields estimates of

Table: Estimation Results

ρ	$\beth$
3.69	0.88
(0.047)	(0.002)

Contour Plot

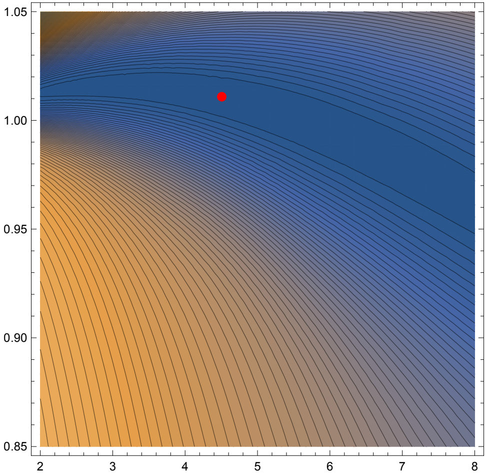

Figure: Point Estimate and Height of Minimized Function

References I

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