Solution Methods for Microeconomic Dynamic Stochastic Optimization Problems

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Note: The code associated with this document should work (though the Matlab code may be out of date), but has been superceded by the set of tools available in the Econ-ARK toolkit, more specifically the HARK Framework. The SMM estimation code at the end has specifically been superceded by the SolvingMicroDSOPs REMARK

Abstract

These notes describe tools for solving microeconomic dynamic stochastic optimization problems, and show how to use those tools for efficiently estimating a standard life cycle consumption/saving model using microeconomic data. No attempt is made at a systematic overview of the many possible technical choices; instead, I present a specific set of methods that have proven useful in my own work (and explain why other popular methods, such as value function iteration, are a bad idea). Paired with these notes is *Mathematica*, Matlab, and Python software that solves the problems described in the text.

Keywords Dynamic Stochastic Optimization, Method of Simulated Moments, Structural Estimation

JEL codes E21, F41

PDF: https://github.com/llorracc/SolvingMicroDSOPs/blob/master/SolvingMicroDSOPs.pdf
Slides: https://github.com/llorracc/SolvingMicroDSOPs/blob/master/SolvingMicroDSOPs-Slides.pdf
Web: https://llorracc.github.io/SolvingMicroDSOPs
Code: https://github.com/llorracc/SolvingMicroDSOPs/tree/master/Code
Archive: https://github.com/llorracc/SolvingMicroDSOPs
(Contains LaTeX code for this document and software producing figures and results)

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The notes were originally written for my Advanced Topics in Macroeconomic Theory class at Johns Hopkins University; instructors elsewhere are welcome to use them for teaching purposes. Relative to earlier drafts, this version incorporates several improvements related to new results in the paper "Theoretical Foundations of Buffer Stock Saving" (especially tools for approximating the consumption and value functions). Like the last major draft, it also builds on material in "The Method of Endogenous Gridpoints for Solving Dynamic Stochastic Optimization Problems" published in *Economics Letters*, available at http://www.econ2.jhu.edu/people/ccarroll/EndogenousArchive.zip, and by including sample code for a method of simulated moments estimation of the life cycle model *a la* **?** and Cagetti (**?**). Background derivations, notation, and related subjects are treated in my class notes for first year macro, available at http://www.econ2.jhu.edu/people/ccarroll/public/lecturenotes/consumption. I am grateful to several generations of graduate students in helping me to refine these notes, to Marc Chan for help in updating the text and software to be consistent with **?**, to Kiichi Tokuoka for drafting the section on structural estimation, to Damiano Sandri for exceptionally insightful help in revising and updating the method of simulated moments estimation section, and to Weifeng Wu and Metin Uyanik for revising to be consistent with the 'method of moderation' and other improvements. All errors are my own. This document can be cited as **?** in the references.

Contents

1 Introduction

Calculating the mathematically optimal amount to save is remarkably difficult. Under well-founded assumptions about the nature of risk (and attitudes toward risk), the problem cannot be solved analytically; computational solutions are the only option. To avoid having to solve this hard problem, past generations of economists showed impressive ingenuity in reformulating the question. Budding graduate students are still taught a host of tricks whose purpose is partly to avoid the resort to numerical solutions: Quadratic or Constant Absolute Risk Aversion utility, perfect markets, perfect insurance, perfect foresight, the "timeless perspective," the restriction of uncertainty to very special kinds,[1] and more.

The motivation for these reformulations is to exchange an intractable general problem for a tractable specific alternative. Unfortunately, the burgeoning literature on numerical solutions has shown that the features that yield tractability also profoundly change the essence of the solution. These tricks are excuses to solve a problem that has defined away the central difficulty: Understanding the proper role of uncertainty (and other complexities like constraints) in optimal intertemporal choice.

These points are not unique to the consumption/saving problem; the same propositions apply to almost any question that involves both intertemporal choice and uncertainty, including many aspects of the behavior of firms and governments.

These lecture notes provide a gentle introduction to a particular set of solution tools and show how they can be used to solve some canonical problems in consumption choice and portfolio allocation. Specifically, the notes describe and solve optimization problems for a consumer facing uninsurable idiosyncratic risk to nonfinancial income (e.g., labor or transfer income),[2] with detailed intuitive discussion of the various mathematical and computational techniques that, together, speed the solution by many orders of magnitude compared to "brute force" methods. The problem is solved with and without liquidity constraints, and the infinite horizon solution is obtained as the limit of the finite horizon solution. After the basic consumption/saving problem with a deterministic interest rate is described and solved, an extension with portfolio choice between a riskless and a risky asset is also solved. Finally, a simple example is presented of how to use these methods (via the statistical 'method of simulated moments' or MSM; sometimes called 'simulated method of moments' or SMM) to estimate structural parameters like the coefficient of relative risk aversion (*a la* Gourinchas and Parker (**?**) and Cagetti (**?**)).

2 The Problem

The usual analysis of dynamic stochastic programming problems packs a great many events (intertemporal choice, stochastic shocks, intertemporal returns, income growth,

[1] E.g., lognormally distributed rate-of-return risk – but no labor income risk – under CRRA utility (the **?**-**?** model).

[2] Expenditure shocks (such as for medical needs, or to repair a broken automobile) are usually treated in a manner similar to labor income shocks. See **?** and **?** for a solution to the problem of a consumer whose only risk is rate-of-return risk on a financial asset; the combined case (both financial and nonfinancial risk) is solved below, and much more closely resembles the case with only nonfinancial risk than it does the case with only financial risk.

and more) into a small number of steps and variables. For the detailed analysis here, we will be careful to disarticulate everything that happens in the problem explicitly into separate steps so that each element can be scrutinized and understood in isolation.

We are interested in the behavior a consumer who begins period t with a certain amount of 'capital' $\mathbf{k}_t$, which is immediately rewarded by a return factor R_t with the proceeds deposited in a **b**ank account **b**alance:

$$\mathbf{b}_t = \mathbf{k}_t \mathsf{R}_t. \tag{1}$$

Simultaneously with the realization of the capital return, the consumer also receives noncapital income $\mathbf{y}_t$, which is determined by multiplying the consumer's 'permanent income' $\mathbf{p}_t$ by a transitory shock $\boldsymbol{\theta}_t$:

$$\mathbf{y}_t = \mathbf{p}_t \boldsymbol{\theta}_t \tag{2}$$

whose whose expectation is 1 (that is, before realization of the transitory shock, the consumer's expectation is that actual income will on average be equal to permanent income $\mathbf{p}_t$).

The combination of bank balances $\mathbf{b}$ and income $\mathbf{y}$ define's the consumer's 'market resources' (sometimes called 'cash-on-hand', following **?**):

$$\mathbf{m}_t = \mathbf{b}_t + \mathbf{y}_t, \tag{3}$$

which are available to be spent on consumption $\mathbf{c}_t$.

The consumer's goal is to maximize discounted utility from consumption over the rest of a lifetime whose last period is date T:

$$\max \ \mathbb{E}_t \left[\sum_{n=0}^{T-t} \beta^n \mathrm{u}(\mathbf{c}_{t+n}) \right]. \tag{4}$$