1 Multiple State Variables

We now wish to consider how the problem changes if there are multiple state variables rather than just a single state variable. The example we will use will be the case where the utility from consumption depends on the size of a 'habit stock' which represents an average of past levels of consumption. Formally, the goal is to

$$\max \left\{ \sum_{s=t}^{T} \beta^{s-t} \mathrm{u}(c_s, h_{s-1}) \right\} \tag{1}$$

Now there are two state variables in the problem at time t, the level of assets m_t and the level of the habit stock h_{t-1}, where the accumulation equation for m_t is the same as before and the transition equation for habits is

$$h_t = h_{t-1} + \lambda(c_t - h_{t-1}). \tag{2}$$

That is, the habit stock at the end of this period is equal to the habit stock at the end of last period plus a proportion of the gap between the level of consumption chosen this period and the level of the habit stock from last period. In other words, habits 'catch up' to consumption at rate λ.

Assume that the utility function is given by

$$\mathrm{u}(c_t, h_{t-1}) = \frac{(c_t/h_{t-1}^{\gamma})^{1-\rho}}{1-\rho}. \tag{3}$$

Now that u_t has two arguments we need to be able to distinguish between the derivatives with respect to each argument. Our notation will be that the derivative of u_t with respect to c_t is $u_t^c(c_t, h_{t-1})$ or u_t^c for short, and analogously for u_t^h. Thus we have

$$\begin{aligned} \mathrm{u}_t^c &= (c_t h_{t-1}^{-\gamma})^{-\rho} h_{t-1}^{-\gamma} \\ &= c_t^{-\rho} h_{t-1}^{\rho\gamma-\gamma} \\ \mathrm{u}_t^h &= -\gamma (c_t h_{t-1}^{-\gamma})^{-\rho} c_t h_{t-1}^{-\gamma-1} \\ &= -\gamma c_t^{1-\rho} h_{t-1}^{\gamma\rho-\gamma-1} = -\gamma(c_t/h_{t-1})\mathrm{u}_t^c. \end{aligned} \tag{4}$$

Bellman's equation for this problem (imposing liquidity constraints again) is

$$\begin{aligned} \mathrm{v}_t(m_t, h_{t-1}) &= \max_{\{c_t\}} \{\mathrm{u}(c_t, h_{t-1}) + \mathbb{E}_t[\beta \mathrm{v}_{t+1}(m_{t+1}, h_t)]\} \\ &\text{s.t.} \\ m_{t+1} &= [m_t - c_t]\mathcal{R}_{t+1} + \theta_{t+1} \\ h_t &= h_{t-1} + \lambda(c_t - h_{t-1}) \\ c_t &\le m_t. \end{aligned}$$

As was done for utility above, define the derivatives of v_t with respect to each argument as $\mathrm{v}_t^m(m_t, h_{t-1})$ (v_t^m for short) and v_t^h. Also as above, we want to define a function which corresponds to the expectation of the value of ending period t in a given position, but now the 'position' involves both the level of assets a_t and the level of the habit stock h_t:

$$\mathrm{v}_{+(t)}(a_t, h_t) = \mathbb{E}_t[\beta v_{t+1}(\mathcal{R}_{t+1} a_t + \theta_{t+1}, h_t))] \tag{5}$$

For future reference note that the derivatives are

$$\begin{aligned} \mathrm{v}^a_{+(t)} &= \beta\,\mathbb{E}_t[\mathcal{R}_{t+1} v^m_{t+1}] \\ \mathrm{v}^h_{+(t)} &= \beta\,\mathbb{E}_t[\mathrm{v}^h_{t+1}] \end{aligned} \tag{6}$$

and the maximization problem can be rewritten

$$\begin{aligned} \mathrm{v}_t(m_t, h_{t-1}) &= \max_{\{c_t\}} \mathrm{u}(c_t, h_{t-1}) + \beta \mathrm{v}_{+(t)}(m_t - c_t, h_{t-1} + \lambda(c_t - h_{t-1})) \\ &\text{s.t.} \\ c_t &\le m_t. \end{aligned}$$

1.1 The Strategy

This problem will be solved by a generalization of the strategy used to solve the one-state problem. Previously, with a period-$t+1$ consumption function in hand, we started by calculating v^a_i at the set of n gridpoints a_i contained in the variable `aVec`. Now we will need to have a grid of possible values for habits h_j as well (in the variable `hVec`) with, say, m gridpoints. We will then calculate the values of $\mathrm{v}^a_{+(t)}$ and $\mathrm{v}^h_{+(t)}$ at all *combinations* of the a_i and h_j gridpoints, for a total of $m \times n$ datapoints. Finally, with these results in hand, we can obtain the corresponding values of consumption and construct the approximating interpolation to the consumption function.

1.2 Optimality Conditions

1.2.1 The First Order Condition for c_t

The FOC for this problem with respect to c_t is:

$$\begin{aligned} 0 &= \mathrm{u}^c_t + \beta\,\mathbb{E}_t\left[\mathrm{v}^m_{t+1}(-\mathsf{R}) + \mathrm{v}^h_{t+1}\lambda\right] \\ \mathrm{u}^c_t &= \beta\,\mathbb{E}_t[\mathcal{R}_{t+1}\mathrm{v}^m_{t+1} - \lambda \mathrm{v}^h_{t+1}] \\ &= \left[\mathrm{v}^a_{+(t)} - \lambda \mathrm{v}^h_{+(t)}\right]. \end{aligned} \tag{7}$$

Substituting the definition of u^c_t:

$$\begin{aligned} c_t^{-\rho} h_{t-1}^{\rho\gamma-\gamma} &= \left[\mathrm{v}^a_{+(t)} - \lambda \mathrm{v}^h_{+(t)}\right] \\ c_t &= \left[h_{t-1}^{\gamma-\rho\gamma}\left(\mathrm{v}^a_{+(t)} - \lambda \mathrm{v}^h_{+(t)}\right)\right]^{-1/\rho}. \end{aligned} \tag{8}$$

and the liquidity constraint implies that if the $\check{c}_t$ which satisfies this equation is larger than m_t the consumer spends m_t rather than $\check{c}_t$. The point at which the liquidity constraint becomes binding is therefore implicitly defined by the equation

$$m_t = \left[h_{t-1}^{\gamma-\rho\gamma}\left(\mathrm{v}^a_{+(t)}(0, h_{t-1} + \lambda(m_t - h_{t-1})) - \lambda \mathrm{v}^h_{+(t)}(0, h_{t-1} + \lambda(m_t - h_{t-1}))\right)\right]^{-1/\rho}, \tag{9}$$

or incorporating the transition equation for and h_{t-1} that

$$c_t = \left[h_{t-1}^{\gamma-\rho\gamma}\left(\mathrm{v}^a_{+(t)}(m_t - c_t, h_{t-1} + \lambda(c_t - h_{t-1})) - \lambda \mathrm{v}^h_{+(t)}(m_t - c_t, h_{t-1} + \lambda(c_t - h_{t-1}))\right)\right]^{-1/\rho}.$$

This equation could be solved for a given grid of m_t and h_{t-1} values to yield the two-dimensional matrix of values necessary to construct an interpolating approximation to the consumption function. However, note that the value of m_t at which the constraint becomes binding depends on the level of h_{t-1}. This causes a problem, because the built-in Mathematica interpolation algorithms require multidimensional interpolations to contain a value for every possible combination of the independent variables. Thus, we would need to add each of the m binding points to the `mVec` variable, and to pair *all* of those bindingpoints to *each* possible value in `hVec`. This would immediately increase the number of points in the m dimension by m, so the total number of gridpoints would now be $(n+m) \times m$. Thus this strategy would require increasing the size of the matrix of interpolating values by a *factor* of m with a corresponding serious slowdown in computational speed.

Fortunately, there is a way to get around this problem. To understand it, start by noticing the translation in this context of the trick used to solve the one-dimensional consumption problem. Recall that there the trick was to start with the grid of values in `aVec` and find the unique c_i, and consequently $m_i = a_i + c_i$, associated with each a_i thus constructing the set of c_i and m_i values without solving a maximization problem or numerically finding a root. Here, the most effective strategy is to use `aVec` to define a set of a_t values but use `hVec` to define a set of values for h_{t-1}, with the transition equation determining the h_t to be paired with the given a_t. That is, defining the set $\mathcal{L} = \{\{a_1, h_1\}, \{a_1, h_2\}, \ldots, \{a_m, h_{n_\theta}\}\}$, we find the set of values of c_i that satisfy

$$c_k^\ell = \left[(h_k^\ell)^{\gamma-\rho\gamma}\left(\mathrm{v}^a_{+(t)}(a_k^\ell, h_k^\ell + \lambda(c_k^\ell - h_k^\ell)) - \lambda \mathrm{v}^h_{+(t)}(a_k^\ell, h_k^\ell + \lambda(c_k^\ell - h_k^\ell)\right)\right]^{-1/\rho}.$$

and the budget constraint again gives us the m_k^ℓ associated with the given $\{a_t, h_{t-1}\}$ pair.

However, there is a problem: Mathematica's built-in two-dimensional interpolation routines require a complete set of values of c_t for *each distinct* $\{m_k, h_j\}$ pair. But the list of m_k gridpoints associated with each distinct point in `hVec` will be different. Thus to use the built-in routines it would be necessary to take the union of all the m_k produced and then construct values of c_k for each of these. If there are m values in `aVec`, this would mean solving the problem at a total of $m \times m \times n$ girdpoints, only $m \times n$ of which would have been produced during the first-round procedure.

There is a a much better solution: Manual interpolation. This works as follows. For each distinct value of $h_{t,i}$ the value of $c_{t,i}$ associated with each given $a_{t,i}$ is calculated as before, starting with $a_{1,t} = 0$ which yields the value of $m_{t,i}$ and $c_{t,i}$ at which the constraint begins to bind for this combination of $a_{t,i}$ and $h_{t,i}$. Then an `InterpolatingFunction` is constructed as before.

Thus we will have a collection of j interpolating consumption functions, one for each value in `hVec`. Denote these functions by $\grave{\hat{\mathrm{c}}}_t(m_t, j)$.

Now consider how to estimate $\mathrm{c}_t(m_t, h_t)$ for arbitrary m_t and h_t. Define $\underline{h}_t$ as the value, and $\underline{j}_t$ as the index of the nearest `hVec` point below h_t and define $\bar{h}_t$ analogously for the nearest gridpoint above h_t. Then we define our approximation to the consumption function as

$$\grave{\hat{\mathrm{c}}}_t(m_t, h_t) = \grave{\hat{\mathrm{c}}}(m_t, \underline{j}) + \left(\frac{h_t - \underline{h}_t}{\bar{h}_t - \underline{h}_t}\right) \left(\grave{\hat{\mathrm{c}}}(m_t, \bar{j}) - \grave{\hat{\mathrm{c}}}(m_t, \underline{j})\right)$$

with points outside the `hVec` points constructed by extrapolation.

Note that a liquidity constraint is again incorporated at almost zero cost: Simply make sure that the lowest value in `aVec` is 0, and append the point $\{0., 0.\}$ as the bottommost point in each of the lists of $\{m_t, c_t\}$ assoicated with the various values in `hVec`.

1.2.2 Applying the Envelope Theorem

The Envelope theorem on m_t says:

$$\begin{aligned} \mathrm{v}_t^m &= \frac{\partial \mathrm{v}_t}{\partial m_t} + \overbrace{\frac{\partial \mathrm{v}_t}{\partial c_t}}^{=0} \frac{\partial c_t}{\partial m_t} \\ \mathrm{v}_t^m &= \beta\, \mathbb{E}_t[\mathcal{R}_{t+1} \mathrm{v}_{t+1}^m] \end{aligned} \tag{10}$$

Substituting this into the FOC equation (7) gives

$$\begin{aligned} \mathrm{u}_t^c &= \mathrm{v}_t^m - \mathbb{E}_t[\beta \lambda \mathrm{v}_{t+1}^h] \\ \mathrm{v}_t^m &= \mathrm{u}_t^c + \mathbb{E}_t[\beta \lambda \mathrm{v}_{t+1}^h] \\ &= \mathrm{u}_t^c + \lambda \mathrm{v}_{+(t)}^h \end{aligned} \tag{11}$$

What if the consumer is liquidity constrained? It is useful here to rewrite Bellman's equation:

$$\mathrm{v}_t(m_t, h_{t-1}) = \mathrm{u}(c_t, h_{t-1}) + \beta\, \mathbb{E}_t \left[\mathrm{v}_{t+1}((m_t - c_t)\mathcal{R}_{t+1} + \theta_{t+1}, h_{t-1} + \lambda(c_t - h_{t-1}))\right]$$

Substituting in the fact that $c_t = m_t$ (because the consumer is constrained)

$$\mathrm{v}_t(m_t, h_{t-1}) = \mathrm{u}(c_t, h_{t-1}) + \mathbb{E}_t \left[\beta \mathrm{v}_{t+1}(\theta_{t+1}, h_{t-1} + \lambda(c_t - h_{t-1}))\right]$$

Thus $\partial \mathrm{v}_t / \partial m_t = 0$, and because the liquidity constraint implies that $\partial c_t / \partial m_t = 1$, equation (10) becomes

$$\begin{aligned} \mathrm{v}_t^m &= \frac{\partial \mathrm{v}_t}{\partial c_t} \\ &= \mathrm{u}_t^c + \beta\, \mathbb{E}_t[\lambda \mathrm{v}_{t+1}^h] \end{aligned} \tag{12}$$

which is identical to the expression (11) for v_t^m for the unconstrained consumer.

The Envelope theorem for h_{t-1} says:

$$\begin{aligned}
\mathrm{v}_t^h &= \frac{\partial \mathrm{v}_t}{\partial h_{t-1}} + \overbrace{\frac{\partial \mathrm{v}_t}{\partial c_t}}^{=0} \frac{\partial c_t}{\partial h_{t-1}} \\
&= \mathrm{u}_t^h + \beta\, \mathbb{E}_t \left[\mathrm{v}_{t+1}^h \frac{\partial h_t}{\partial h_{t-1}} \right] \\
&= \mathrm{u}_t^h + \beta\, \mathbb{E}_t[(1-\lambda)\mathrm{v}_{t+1}^h] \\
&= \mathrm{u}_t^h + (1-\lambda)\mathrm{v}_{+(t)}^h
\end{aligned}$$

What if the consumer is constrained? In that case while $\partial \mathrm{v}_t/\partial c_t \neq 0$, $\partial c_t/\partial h_{t-1} = 0$, so as with v_t^m the constraint has no effect on the expression for v_t^h.

1.3 Transformations

In the time-separable consumption problem, there was a unique transformation that allowed us to unwind much of the linearity attributable to the curvature of the utility function; this was possible because that problem has an analytical solution in the perfect foresight case, and because the problem is one with a single state variable.

In the habits model, neither of these conditions is true, and consequently the appropriate transformation strategy is not obvious. To see this, consider the last period of life and suppose there is no uncertainty. Then $\mathrm{v}_{\sim(T)}^m = \mathrm{u}_T^c$ will be of the form $c_T^{-\rho} h_{T-1}^{\rho\gamma-\gamma}$, while $\mathrm{v}_{\sim(T)}^h = \mathrm{u}^h$ will be something of the form $-\gamma c_T^{1-\rho} h_{T-1}^{\gamma\rho-\gamma-1}$. It is possible to exponentiate either of these equations to make it linear in one or the other of c or h, but not both.

However, there is a two-step procedure that solves the problem. In the first step, the object in question is transformed as in the original problem without habits. For example, the transformed utility function for the last period of life would be

$$\begin{aligned}
n_T^c &= [\mathrm{u}_T^c]^{-1/\rho} \\
&= c_T h_{T-1}^{\gamma(1/\rho-1)}.
\end{aligned} \tag{13}$$

This leaves us with an equation that is linear in c_T but nonlinear in h_{T-1}. So now we do a second transformation: dividing the n_T function by $h_{T-1}^{\gamma(1/\rho-1)}$. This yields a function $\hat{n_\theta}_T^c$ that can be transformed back into u_T^c on demand, but is itself linear in c_t. Analogously, in the last period of life the marginal value of habits is given by

$$\mathrm{v}_{\sim(T)}^h = -\gamma c_T^{1-\rho} h_{T-1}^{\gamma(\rho-1)-1}$$

so the natural normalization procedure is to define

$$\Lambda_t^h = \left[\frac{h_{t-1}^{1-\gamma(\rho-1)} \mathrm{v}_t^h}{-\gamma} \right]^{1/(1-\rho)} \tag{14}$$

and similarly for $\mathrm{v}_{+(t)}^h$.

1.4 Transforming $\mathrm{c}(\bullet, h_{t-1})$

The habits problem presents an additional difficulty: the consumption function is highly nonlinear in h_{t-1} for a given m_t. To see this, start with the first order condition for the problem in the second-to-last period of life (setting $\beta = \mathcal{R}_{t+1} = 1$ and assuming away uncertainty for transparency):

$$\begin{aligned}
\mathrm{u}^c_t &= \mathrm{u}^c_{t+1} - \lambda \mathrm{v}^h_{t+1} \\
\mathrm{u}^c_{T-1} &= \mathrm{u}^c_T - \lambda \mathrm{u}^h_T \\
&= \mathrm{u}^c_T \left(1 + \gamma\lambda(c_T/h_{T-1})\right) \\
c_{T-1}^{-\rho} &= c_T^{-\rho}\left(\frac{h_{T-1}}{h_{T-2}}\right)^{\gamma(\rho-1)} \left(1 + \gamma\lambda(c_T/h_{T-1})\right) \\
c_{T-1} &= (m_{T-1} - c_{T-1})\left(\frac{h_{T-1}}{h_{T-2}}\right)^{\gamma(1/\rho-1)} \left(1 + \gamma\lambda\left[\frac{m_{T-1} - c_{T-1}}{(1-\lambda)h_{T-2} + \lambda c_{T-1}}\right]\right)^{-1/\rho} \\
1 &= \left(\frac{m_{T-1} - c_{T-1}}{c_{T-1}}\right)\left(\frac{(1-\lambda)h_{T-2} + \lambda c_{T-1}}{h_{T-2}}\right)^{\gamma(1/\rho-1)} \left(1 + \gamma\lambda\left[\frac{m_{T-1} - c_{T-1}}{(1-\lambda)h_{T-2} + \lambda c_{T-1}}\right]\right)^{-1/\rho}
\end{aligned}$$

Now conjecture that for fixed m_{T-1}

$$\lim_{h_{T-1}\to 0} c_{T-1} = \mu h^m_{T-2} \tag{15}$$

for some $\mu < 1, \mu > 1$. Then the limit of (15) as $h_{T-2} \to \infty$ is given by

$$\begin{aligned}
1 &= (m_{T-1}/c_{T-1})\left(\frac{\lambda c_{T-1}}{h_{T-2}}\right)^{\gamma(1/\rho-1)} \left(\gamma\left[\frac{m_{T-1}}{c_{T-1}}\right]\right)^{-1/\rho} \\
&= c_{T-1}^{\gamma(1/\rho-1)-1+1/\rho} h_{T-2}^{-\gamma(1/\rho-1)} \lambda^{\gamma(1/\rho-1)} (\gamma m_{T-1})^{-1/\rho} \\
&= c_{T-1}^{(1+\gamma)(1/\rho-1)} h_{T-2}^{-\gamma(1/\rho-1)} \lambda^{\gamma(1/\rho-1)} (\gamma m_{T-1})^{-1/\rho} \\
&= c_{T-1}^{(1+\gamma)} h_{T-2}^{-\gamma} \lambda^{\gamma} (\gamma m_{T-1})^{1/(\rho-1)} \\
c_{T-1} &= h_{T-2}^{\gamma/(1+\gamma)} \lambda^{-\gamma} (\gamma m_{T-1})^{\frac{1}{(\gamma+1)(\rho-1)}}
\end{aligned}$$

which confirms the conjecture (15). On the other hand, suppose that we postulate that

$$\lim_{h_{T-2}\to\infty} c_{T-1} = \kappa. \tag{16}$$

Under this conjecture, the limit of the RHS of (15) as $h_{T-2} \to \infty$ is

$$\begin{aligned}
\kappa &= (1-\kappa)\,(1-\lambda)^{\gamma(1/\rho-1)} \\
\kappa\left(1 + (1-\lambda)^{\gamma(1/\rho-1)}\right) &= (1-\lambda)^{\gamma(1/\rho-1)} \\
\kappa &= \left(\frac{(1-\lambda)^{\gamma(1/\rho-1)}}{1 + (1-\lambda)^{\gamma(1/\rho-1)}}\right)
\end{aligned}$$

confirming conjecture (16).

This combination of results indicates that the consumption function must be globally strongly nonlinear in h_{T-1} (a nonlinearity which also arises in successive earlier periods $T-2$ and so on).

If we wish to normalize the consumption rule by something that will help to make it approximately linear, that normalizing function will need approach proportionality to $h_{t-1}^{\gamma/(1+\gamma)}$ as h_{t-1} goes to zero but approach a constant as h_{t-1} approaches infinity. The function

$$n(h_{t-1}) = h_{t-1}^{\gamma/(1+\gamma)}(h_{t-1}+\mu)^{-\gamma/(1+\gamma)} \tag{17}$$

has precisely these characteristics for $\mu > 0$. Some experimentation with possible values of μ led to the conclusion that under the baseline parameter values this function does a good job of linearizing the relationship between consumption and h_{t-1} for a value of $\mu = 0.04$.

The combined transformations for the various variables are thus

$$\begin{aligned}
\mathrm{c}^a_{\bar{t}}(a_t,h_t) &= [\mathrm{v}^a_{+(t)}(a_t,h_t)]^{-1/\rho} h_t^{\gamma(1-1/\rho)} \Lambda^m_t(m_t,h_{t-1}) \quad = [\mathrm{v}^m_t(m_t,h_{t-1})]^{-1/\rho} h_{t-1}^{\gamma(1-1/\rho)} \mathrm{c}^h_{\bar{t}}(a_t,h_t) = [-h_t^{1-\gamma} \\
\Lambda^h_t(m_t,h_{t-1}) &= [-h_{t-1}^{1-\gamma(\rho-1)}\mathrm{v}^h_t(m_t,h_{t-1})/\gamma]^{1/(1-\rho)} \\
\Lambda_t(m_t,h_{t-1}) &= [(1-\rho)\mathrm{v}_t(m_t,h_{t-1})]^{1/(1-\rho)} h_{t-1}^{\gamma} \\
\chi(m_t,h_{t-1}) &= \mathrm{c}(m_t,h_{t-1}) h_{t-1}^{-\gamma/(1+\gamma)}(h_{t-1}+\mu)^{\gamma/(1+\gamma)}.
\end{aligned} \tag{18}$$

where these are the functions that are actually approximated by interpolation and the objects of interest (like the consumption function) are obtained by reversing the transformations.

1.5 The Program

The consumption problem with habit formation is solved in `habits.m`, whose structure closely follows that of `multiperiod.m`.

Assuming the problem has been solved up to period $t+1$ (and thus we have numerical functions $\hat{\mathrm{v}}^m_{t+1}(m_{t+1},h_t)$ and $\hat{\mathrm{v}}^h_{t+1}(m_{t+1},h_t)$),

1. Form a list called `ArgArray` of all possible combinations of the values in `aVec` and `hVec`, and index the components of that list by k. Thus if there are m points in both grids we have `ArgArray`$= \{\{a_1,h_1\}, \{a_1,h_2\}, \ldots, \{a_1,h_m\}, \{a_2,h_1\}, \{a_2,h_2\}, \ldots, \{a_2,h_m\}, \{a_m,h_1\}, \{a_m,h_2\}, \ldots, \{a_m,h_m\}\}$. Designate this list as $\mathcal{L}$ with individual members $\{\ell_1,\ell_2,\ldots\ell_{m\times m}\}$. Finally, define the notation $\bullet^\ell_k$ to mean "the value of $\bullet$ associated with the kth element of the list $\mathcal{L}$; e.g. $h^\ell_1 = h_1$, $h^\ell_2 = h_2$, and $a^\ell_2 = a_1$."

   Now at each of the k locations in `ArgArray` calculate the value of $\mathrm{c}^a_{\bar{t}}$ and $\mathrm{c}^h_{\bar{t}}$ (from equations (6) and (**??**)) and (**??**) and (18),

   $$\begin{aligned}
   \mathrm{c}^a_{k,t} &= (h^\ell_k)^{\gamma(1-1/\rho)}\left(\beta\,\mathbb{E}_t\left[\mathcal{R}_{t+1}(\hat{\mathrm{v}}^m_{t+1}(\mathcal{R}_{t+1}s^\ell_k+\theta_{t+1},h^\ell_k))\right]\right)^{-1/\rho} \\
   \mathrm{c}^h_{k,t} &= \left(-(h^\ell_k)^{1-\gamma(\rho-1)}\beta\,\mathbb{E}_t\left[\hat{\mathrm{v}}^h_{t+1}(\mathcal{R}_{t+1}s^\ell_k+\theta_{t+1},h^\ell_k)\right]/\gamma\right)^{1/(1-\rho)},
   \end{aligned} \tag{19}$$

   generating lists of values $\mathrm{c}^a_{k,t}$,$\mathrm{c}^h_{k,t}$.

2. Construct interpolating functions $\hat{c}^a_t(a_t, h_t)$ and $\hat{c}^h_t(a_t, h_t)$, from which we can obtain $\hat{v}^a_{+(t)}$ and $\hat{v}^h_{+(t)}$ via the inverse of the transformations (**??**) and (18).

3. Loop over the m values of h in `hVec`, indexing them by j; for each j:
   - Loop over a_i finding the optimal c_a associated with this a_i and h_j from the formula in equation (8):

$$c_i = \left[h_j^{\gamma-\rho\gamma}(\mathrm{v}^a_{+(t)}(a_i, (1-\lambda)h_j + c_i) - \lambda \mathrm{v}^h_{+(t)}(a_i, (1-\lambda)h_j + c_i))\right]^{-1/\rho}$$

   - Construct $m_i = c_i + a_i$
   - Construct `ct[mt,j]` as a linear interpolation of the $\{m_i, c_i\}$

With $\grave{\hat{c}}_t(m_t, h_{t-1})$, $\hat{v}^a_{+(t)}$ and $\hat{v}^h_{+(t)}$ in hand we can obtain v^m_t and v^h_t from (11) and (13). Thus we have generated $\grave{\hat{c}}_t$, $\hat{v}^m_t$ and $\hat{v}^h_t$ from $\hat{v}^a_{t+1}$ and $\hat{v}^h_{t+1}$, and we can continue the iteration indefinitely.

The problem is solved in the program `habits.m`. Details of the Mathematica implementation follow those described above for `multiperiod.m` closely, and so need not be described here. The program generates a three-D figure showing the consumption rule $\mathrm{c}_t(m_t, h_{t-1})$ for the first period of 'life.' The figure behaves as one would expect: consumption is increasing in the level of resources and in the level of the habit stock.[1]

[1]For a detailed analysis of some of the properties of this habit formation model, see Carroll (**?**)