ABSTRACT HEAT EQUATION

Conference Paper · May 2002

DOI: 10.13140/2.1.4740.2244

Václav Záda

Technical University of Liberec

ABSTRACT HEAT EQUATION

Václav ZÁDA
Technical University of Liberec, Faculty of Mechatronics – KSI, 461 17 LIBEREC 1, Hálkova 6,
Czech Republic, e-mail: vaclav.zada@vslib.cz

Abstract: *The heat equation of distribution of temperature in any body without any other sources can be transformed into a more general form and then solved similarly as it is known from classical linear differential equations with constant coefficients. This is possible although the heat equation is linear partial differential equation of second order. This fact opens a question when is it possible to control systems described by partial differential equations by analogy theory to the classical linear one. It is clear that this theory must be much more abstract. In this contribution we will study only the general form of heat equation. This article is mathematically oriented.*

Key words: *Energetic Space, Friedrichs Extension, Non-expansive semigroup,*

1. Introduction

Let G be a nonempty open bounded set in R^n, where $n \geq 1$. Let $R_0 = [0, \infty)$ be a half closed Interval. We consider the initial boundary-value problem for heat equation:

$$\begin{aligned} &\Delta u = u_t && \text{on } G \times R_0, \\ &u(x,t) = 0 && \text{on } \partial G \times R_0 \quad \text{(boundary condition)}, \\ &u(x,t) = u_0(x) && \text{on } G \quad \text{(initial condition)}. \end{aligned} \tag{1}$$

This problem allows the following physical interpretation. Set $u(x,t)$ is temperature at the point x at time t. Then these equations describe the distribution of temperature in the body G without any other heat sources. The given function u_0 corresponds to the initial temperature of the body at time $t = 0$. Here

$$\Delta u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial^2 z} \tag{2}$$

$$u_t = \frac{\partial u}{\partial t}. \tag{3}$$

For precision we must remark that the heating equation is in the form

$$\frac{l}{cr} \Delta u = u_t,$$

which differs from (1) in a constant. This one is not essential, because the equation is linear. The real heating equation is described in the three dimensional space R^3, hence $n = 3$. Our description (1) is more common. We set $X = L_2(G)$ and $B = -\Delta$ with the domain $D(B) = C_0^\infty(G)$ of the definition of the operator B. Let X_E be defined as $X_E = \breve{W}_2^1(G) \ldots$ the Sobolev space.

The theorem in the next chapter inform us that for each initial temperature the problem (1) has a unique generalised solution. Heat equations represents an irreversible process in nature.

2. The Abstract Heat Equation

Generally, consider the initial-value problem:

$$u'(t) + Au(t) = 0 \quad \text{for} \quad t \in R_0, \ u(0) = u_0 . \tag{4}$$

Our aim is to prove the following theorem:

The Main Theorem : *The original equation (4) has a unique* C^1 *solution* $u : R_0 \to X$ *for every given initial value* $u_0 \in D(A)$. *This solution has a form*

$$u(t) = e^{-tA} u_0 \quad \textit{for all} \quad t \in R_0 .$$

The operator $-A$ *is the generator of the linear, nonexpansive, strongly continuous semigroup* $\{e^{-tA}\}$.

The solution from this theorem is called *a mild solution or a generalised solution*. The proof at this theorem will be done by assumptions (A1) and (A2). It can be shown these assumptions are satisfied with the energetic space X_E [1], [2]. They will be defined in the next chapters.

3. Preliminary Definitions and Assertions

The inner product in a Hilhert space will be denoted by $(.|.)$. We shall write *iff* instead of *if and only if*.
Remark: Let S be a dense subset of the Hilbert space X. If $(x | y) = 0$ for fixed $x \in X$ and every $y \in S$, then $x = 0$. In fact, since S is dense in X, there exists a sequence $\{y_n\}$ in S such that $y_n \to x$ as $n \to \infty$. Then $(x | y_n) = 0$ and letting $n \to \infty$ we obtain $(x | x) = 0$, and hence $x = 0$.

Definition. Let $T : D(T) \subseteq X \to X$ be a linear operator, where $D(T)$ is dense in the Hilbert space X. Let us define the set

$$D(T^*) = \{v \in X ; (\exists w \in X)(\forall u \in D(T)) : (Tu | v) = (u | w)\} .$$

Then we set $T^* v = w$ and we obtain in this way the *adjoint operator*

$$T^* : D(T^*) \subseteq X \to X .$$

If $(Tu | v_i) = (u | v_i)$ for $i = 1,2$ then $(u | v_1 - v_2) = 0$ for all $u \in D(T)$. But $D(T)$ is dense in X and hence $v_1 - v_2 = 0$. We see that this definition is correct.

Definition. Let $T : D(T) \subseteq X \to X$ be a linear operator on the real Hilbert space X, where the domain of the definition of T is $D(T)$. Then we define:

(1) Let $A : D(A) \subseteq X \to Y$ and $B : D(B) \subseteq X \to Y$ be operators, where X, Y are linear spaces over field K. Then we say that the operator A is an *extension* of the operator B, if and only if $Ax = Bx$ for all $x \in D(B)$ and $D(B) \subseteq D(A)$. We write it by $B \subseteq A$.

(2) The operator T is called *strongly monotone* iff

$$(Tu | u) \geq c \|u\|^2 \quad \text{for all } u \in D(T) \text{ and fixed } c > 0 .$$

(3) The operator T is called *symmetric* iff $D(T)$ is dense in X and $(Tu | v) = (u | Tv)$ for all $u, v \in D(T)$, that is $A \subseteq A^*$.

(4) The operator T is called *self-adjoint* iff $D(T)$ is dense in X and $A = A^*$.

Theorem 1 : Let $T : X \to X$ be a linear compact symmetric operator on the non-trivial separable Hilbert space X over K. Then the following hold :

a) The operator T has a complete orthonormal system of eigenvectors.
b) The eigenvalues are real, each non-zero eigenvalue has finite multiplicity.
c) Two eigenvectors that correspond to different eigenvalues are orthogonal.
d) If the operator has a countable set of eigenvalues, then these eigenvalues form a sequence $\{ l_n \}$ such that

$$l_n \to 0.$$

The proofs of this assertions are in [1].

We assume the following first assumption:

(A1) The operator $B : D(B) \subseteq X \to X$ is linear, symmetric and strongly monotone on the real separable Hilbert space X with $\dim X = \infty$.

Let $(.|.)$ denotes the *inner product* and $\|.\|$ the *norm* on X. Let the energetic inner product be defined as

$$(x \,|\, y)_E = (Bx \,|\, y) \text{ for } x, y \in D(B)$$

and corresponding energetic norm

$$\|x\|_E = (x \,|\, x)_E^{\frac{1}{2}} \text{ for } x \in X .$$

Then $(x \,|\, x)_E = 0$ implies $x = 0$ and from symmetry of B we obtain

$$(x \,|\, y)_E = (y \,|\, x)_E \text{ for } x, y \in D(B).$$

Hence $(x \,|\, y)_E$ represents an inner product on the linear space $D(B)$.

Definition. The *energetic space* X_E of the operator B has precisely of all $x \in X$ that have these properties:

1) There exists a sequence $\{x_n\}$ in $D(B)$ such that $x_n \to x$ in X.
2) The sequence $\{x_n\}$ is Cauchy with respect the energetic norm.

We set $(x \,|\, y)_E = \lim\limits_{n \to \infty} (x_n \,|\, y_n)$ for all $x, y \in X_E$, where sequences $\{x_n\}$, $\{y_n\}$ are *admissible*, that is they have properties 1) and 2) .

Theorem 2:
The energetic space X_E is a real Hilbert space with norm $(.|.)_E$, the definition set $D(B)$ of operator B is dense in X_E and the embedding $X_E \subseteq X$ is continuous, that is $\|x\| \leq \|x\|_E \big/ \sqrt{c}$ for $x \in X_E$, or in the equivalent form

$$\|x\|_E^2 \geq c\|x\|^2. \tag{5}$$

Proof. a) Let $\{x_n\}$ be an admissible sequence. We suppose firstly that $u = 0$. Since this sequence is Cauchy with respect the energetic norm and

$$\big| \|x_n\|_E - \|x_m\|_E \big| \leq \|x_n - x_m\|_E \tag{6}$$

then $\|x_n\|_E$ is Cauchy and hence exists a limit

$$a = \lim_{n\to\infty} \|x_n\|_E .$$

$$\begin{aligned}&\left|(x_n \mid x_m)_E - (x_k \mid x_k)_E\right| = \left|(x_n - x_k \mid x_m)_E + (x_k \mid x_m - x_k)_E\right| \\ &\le \|x_n - x_k\|_E \|x_m\|_E + \|x_k\|_E \|x_m - x_k\|_E < e\end{aligned} \tag{7}$$

for $n, m, k > n_0$, where n_0 is a suitable natural number. Because $x_n \to 0$ for $n \to \infty$, we get $(x_n \mid x_m)_E = (x_n \mid Bx_m) \to 0$ as $n \to \infty$. Letting $n, k \to \infty$ in (7) we obtain $\left|a^2\right| \le e$ for every positive e and so $a = 0$. Hence we have proved now that

$$\lim_{n\to\infty} \|x_n\|_E = 0 .$$

b) Let $x \in X_E$ and choose an admissible sequence $\{x_n\}$ for this element. From inequality (6) we see that the sequence $\|x_n\|_E$ is Cauchy. Let it be defined

$$\|x\|_E = \lim_{n\to\infty} \|x_n\|_E .$$

We must verify that this definition is correct. Let $\{y_n\}$ be another admissible sequence for x. Since the sequence $\{x_n - y_n\}$ is admissible for element 0, we obtain

$$\left| \|x_n\|_E - \|y_n\|_E \right| \le \|x_n - y_n\|_E \to 0 ,$$

as was proved before. Hence we have

$$\|x\|_E = \lim_{n\to\infty} \|y_n\|_E$$

and the definition of $\|x\|_E$ does not depend on the sequence – the definition is correct.

c) It is easy to verify the identity

$$(x \mid y)_E = \left(\|x + y\|_E^2 - \|x - y\|_E^2 \right) 4^{-1}. \tag{8}$$

Let $\{x_n\}, \{y_n\}$ be admissible sequences for $x, y \in X_E$. Then $\{x_n \pm y_n\}$ is admissible for $x \pm y$. By previous part b) and (8), the limit

$$(x \mid y)_E = \lim_{n\to\infty} (x_n \mid y_n)_E \tag{9}$$

exists and is independent on the chosen of the admissible sequences.

e) Let $\{x_n\}$ be an admissible sequence for arbitrary $x \in X_E$. Then by step b) we show

$$\lim_{n\to\infty} \|x_n - x\|_E = 0 .$$

Since $\{x_n\}$ is Cauchy with respect the norm $\|.\|_E$, then $\|x_m - x_n\|_E < e$ for $n, m > n_0$, where n_0 is a suitable number dependent on e. Let m be fixed. Then the sequence $\{x_m - x_n\}$ is admissible for the element $x_m - x$. Letting $n \to \infty$, we get

$$\|x_m - x\|_E \le e \text{ for } m > n_0 .$$

From our assumption $(Bu \mid u) \ge c\|u\|^2$ we obtain

$$\|x_n\|_E^2 \ge c\|x_n\|^2$$

and by letting $n \to \infty$ this implies that

$$\|x\|_E^2 \ge c\|x\|^2$$

End of proof.

Definition. The *Fridrichs extension* $A : D(A) \subseteq X \to X$ of the operator $B : D(B) \subseteq X \to X$ is defined

$$Au = B_E u \text{ for all } u \in D(A),$$

where $D(A) = \{u \in X_E ; B_E u \in X\}$.

We assume the following second assumption:
(A2) Let the operator $A : D(A) \subseteq X \to X$ be the Fridrichs extension of B with the energetic space X_E and suppose that the embedding $X_E \subseteq X$ is compact.

Lemma 3 : For the Fridrichs extension hold:
The operator $A : D(A) \subseteq X \to X$ is self-adjoint, bijective and strongly monotone. Hence exists the inverse operator $A^{-1} : X \to X$ that is linear, continuous, and self-adjoint. If the enbeding $X_E \subseteq X$ is compact, then the operator A^{-1} is compact.

Proof: Since A is a restriction of bijective operator $B_E : X_E \to X_E^*$, is A bijective too. For $u \in D(A)$

$$(Au \mid u) = \langle Au, u \rangle_E = \langle B_E u, u \rangle_E = (u \mid u)_E \ge c\|u\|^2 \quad \text{by } (5).$$

The operator $B_E^{-1} : X_E^* \to X_E$ is linear and continuous and since the embedding $X \subseteq X_E^*$ is continuous, the restriction $B_E^{-1} : X \to X_E$ is also continuous. But this operator is identical to A^{-1} and hence

$$A^{-1} : X \to X_E \tag{10}$$

is linear and continuous. Since the embedding $X_E \subseteq X$ is continuous, the operator

$$A^{-1} : X \to X \tag{11}$$

is linear and continuous.
If the embedding is compact then from (10) follows that the operator (11) is compact operator.
Let $x, y \in X$. Then

$$(A^{-1}x \mid A^{-1}y)_E = \left\langle B_E(A^{-1}x), A^{-1}y \right\rangle_E = \left\langle x, A^{-1}y \right\rangle_E = (x \mid A^{-1}y).$$

But

$$(A^{-1}x \mid A^{-1}y)_E = (A^{-1}y \mid A^{-1}x)_E$$

and so

$$(x \mid A^{-1}y)_E = (A^{-1}x \mid y) \ .$$

We see that the operator

$$A^{-1} : X \to X$$

is linear, continuous and symmetric and hence is self-adjoint. From this we show that A is self–adjoint too. Let $C = A^{-1}$. Then C is self-adjoint, hence $C = C^*$. Hence $A^* = (C^{-1})^* = (C^*)^{-1} = (C)^{-1} = A$.

End of proof.

4. Functions of operators

Now suppose that the operator $C : D(C) \subseteq X \to X$ is self-adjoint on the separable Hilbert space X and C possesses a complete orthonormal system $\{x_n\}$ of eigenvectors, where $Cx_n = l_n x_n$.
Now consider functions $F : R \times J \to R$ for any real interval J of $R.$. Then we want to define a functional of the form

$$F(C,t) = \sum_n F(l_n, t)(u_n \mid u_n)u_n .$$

Set $x(t) = F(C,t)u$ for fixed $x \in X$. We want to show that $x'(t) = F_t(C,t)u$, where

$$F_t(C,t) = \sum_n F_t(l_n, t)(u_n \mid u_n)u_n .$$

Lemma 4.

Let the operator $C : D(C) \subseteq X \to X$ be self-adjoint on the separable Hilbert space X and C possesses a complete orthonormal system $\{x_n\}$ of eigenvectors, where $Cx_n = l_n x_n$ and let function $F : R \times J \to R$ be given for any real subinterval J of $R.$.

a) Assume $\sum_n |a_n (x_n \mid x)|^2 < \infty$ for $a_n = \sup_{t \in J} |F(l_n, t)|$ and let $F(l,t)$ is continuous on J for every fixed $l \in R$. Then $x(t)$ as a function of t is continuous on J.

b) Assume $\sum_n |b_n (x_n \mid x)|^2 < \infty$ for $b_n = \sup_{t \in J} |F_t(l_n, t)|$ and the presumption of point a) for fixed $x \in X$. If $F(l,t)$ is continuously differentiable with respect t on J for every fiixed $l \in R$, then $x(t)$ as a function of t is continuously differentiable on J and relation $x'(t) = F_t(C,t)u$ is true for all $t \in J$.

Proof.

Ad a). Denote $F_n(t) = F(l_n, t)$, $c_n = (x_n \mid x)$. Then we get with using $|x + y|^2 \le 2(|x|^2 + |y|^2)$ that

$$H = \|F(C,s)x - F(C,t)x\|^2 = \sum_n |(F_n(s) - F_n(t))c_n|^2$$

$$\le \sum_{n \le k} |(F_n(s) - F_n(t))c_n|^2 + 2\sum_{n > k} |a_n c_n|^2 .$$

For arbitrary $e > 0$ there exists k such that

$$2\sum_{n > k} |a_n c_n|^2 < e.$$

But F_n is continuous on interval J and hence we obtain $H < 2e$.
Ad b). By help of the mean value theorem we have

$$\frac{F_n(s)-F_n(t)}{s-t}=F_n'(t+q(s-t))$$

for some $0<q<1$ and now we can use the same argument as in the proof of a).

End of proof.

5. Semigroups

The key relation are $S(s+t)=S(s)S(t)$ for $s,t\geq 0$, and $S(0)=I$, where I is the identical operator.

Definition. Let X be a Banach space [1] over field K. *A semigroup* $S_+=\{S(t)\}_{t\geq 0}$ on X is a family of operators $S(t):X\to X$ for $t\geq 0$, such that hold

$$S(s+t)=S(s)S(t) \text{ for } s,t\geq 0$$
$$S(0)=I.$$

The operator $G:D(G)\subseteq X\to X$ defined through

$$Tx=\lim_{t\to 0^+}t^{-1}(S(t)-I)x$$

iff this limit exists is called the *generator* of the semigroup $S_+=\{S(t)\}_{t\geq 0}$. Set for all $t\geq 0$

$$x(t)=S(t)x_0.$$

a) S_+ is called linear iff $S(t):X\to X$ is linear and continuous.

b) S_+ is called *strongly continuous* iff the function $x:[0,\infty)\to R$ is continuous for every $x_0\in X$.

c) S_+ is called *nonexpansive* iff this one is *strongly continuous* and for all $t\geq 0$ holds

$$\|S(t)x_0-S(t)y_0\|\leq\|x_0-y_0\|,$$

for all $x_0,y_0\in X$.

If we change the previous definition not only for $t\geq 0$, but for $t\in R$ we obtain a *one-parameter group*.

6. The Proof of Main Theorem

The operator A is strongly monotone*:* The operator A is a restriction of B_E. Since the operator $B_E:X_E\to X_E^*$ is bijective, so the operator A is bijective too. For all $u\in D(A)$ is

$$(Au\,|\,u)=\langle Au,u\rangle_E=\langle B_Eu,u\rangle_E=(u\,|\,u)_E\geq\|u\|^2 \tag{12}$$

by (5).

Uniqueness. Let $u_1,u_2:R_0\to X$ be two C^1 solutions of (4). Define $v(t)=u_1(t)-u_2(t)$. Then $v(0)=0$ and $0=(v(0)\,|\,v(0))$. For inner product

$$\frac{d}{dt}(v(t)\,|\,v(t))=2(v'(t)\,|\,v(t))=-2(Av(t)\,|\,v(t))\leq -c\|v\|^2\leq 0 \qquad \text{for } t\in R_0$$

since A is strongly monotone. But $0\leq(v(t)\,|\,v(t))$, hence $v(t)=0$ for all $t\in R_0$.

Lemma 5: A has complete orthonormal system of eigenvectors.

From (12)

$$(Au \mid u) \geq c\|u\|^2 \text{ for } u \in D(A).$$

If u is a solution of

$$Au = mu \tag{13}$$

and $u \neq 0$, then

$$m(u \mid u) = (Au \mid u) \geq c(u \mid u) = c\|u\|^2.$$

Hence $m \geq c > 0$.

Again from lemma 3, the operator $A^{-1} : X \to X$ is compact and symmetric.
The problem

$$(u \mid Bv) = m(u \mid v) \quad \text{for fixed } u \in X_E,\ m \in R,\ v \in D(B). \tag{14}$$

is equivalent to (13). In fact, from lemma 3 and from the fact that (14) and problem

$$Bu = mu$$

have the unique solution. (14) is identical to

$(Av \mid u) = m(u \mid v)$ for fixed $u \in X_E$ and all $v \in D(B)$. But for $u \in X_E$ and all $v \in D(B)$

$$(Av \mid u) = \langle B_E v \mid u \rangle = (v \mid u)_E = (u \mid v)_E,$$

and so (14) is equivalent to

$$\langle B_e u \mid v \rangle = m(u \mid v).$$

Since the set $D(B)$ is dense in X_E, this one is equivalent to

$$B_E u = mu.$$

Since $mu \in X$, this is equivalent to

$$Au = mu \text{ for } u \in D(A).$$

For $l = m^{-1}$ the eigenvalue problem (14) is equivalent to

$$A^{-1}u = lu,\ u \neq 0.$$

If $l = 0$ then $A^{-1}u = 0$ and hence $u = 0$. This is contradiction. So we have $l \neq 0$. The assertion follows now from Theorem 1.

The semigroup. The operator A has complete orthonormal system $\{u_n\}$ of eigenvectors with

$$Au_n = l_n u_n \text{ and } l_n \geq c > 0 \text{ for all } n = 1,2,\ldots .$$

We use the functional calculus from chapter 4. Then,

$$Au = \sum_n l_n (u_n \mid u) u_n,$$

and $u \in D(A)$ if and only if $\sum_n | l_n (u_n \mid u) u_n |^2 < \infty$. By definition,

$$e^{-tA}u = \sum_n e^{-t\,l_n}(u_n \mid u)u_n \quad \text{for each } t \in R_0,$$

where $u \in D(e^{-tA})$ if and only if $\sum_n \left| e^{-t\,l_n}(u_n \mid u) \right|^2 < \infty$.

For all $u \in X$ and $t \in R_0$

$$\left\| e^{-tA}u \right\|^2 = \sum_n \left| e^{-t\,l_n}(u_n \mid u) \right|^2 \leq \sum_n \left| (u_n \mid u) \right|^2 = \|u\|^2 .$$

Hence $X = D(e^{-tA})$ and $\left\| e^{-tA} \right\| \leq 1.$ Then from

$$e^{-(s+t)\,l_n} = e^{-s\,l_n} e^{-t\,l_n},$$

we will prove the semigroup property

$$e^{-(s+t)A} = e^{-sA}e^{-tA} \text{ for all } \quad s,t \in R_0. \tag{15}$$

To prove it, let $s,t \in R_0$ and $u \in X$. Since the operator e^{-tA} is self-adjoint, then

$$(u_n \mid e^{-tA}u) = (e^{-tA}u_n \mid u) = e^{-t\,l_n}(u_n \mid u).$$

and hence

$$e^{-sA}(e^{-tA}u) = \sum_n e^{-s\,l_n}(u_n \mid e^{-tA}u)u_n = \sum_n e^{-(s+t)\,l_n}(u_n \mid u)u_n = e^{-(s+t)A}u .$$

Denote

$$u(t) = e^{-tA}u_o .$$

For $u_0 \in X$ we have the majorant condition

$$\sum_n \left| e^{-t\,l_n}(u_n \mid u_0) \right|^2 \leq \sum_n \left| (u_n \mid u_0) \right|^2 < \infty .$$

From lemma 4 the function $t \mapsto u(t)$ is continuous on R_0 and $u(0) = u_0$.

Let $t \in R_0$ and $u_0 \in D(A)$. Then, we have

$$\sum_n \left| l_n e^{-t\,l_n}(u_n \mid u_0) \right|^2 \leq \sum_n \left| l_n (u_n \mid u_0) \right|^2 < \infty .$$

From lemma 4 we obtain that the derivative

$$u'(t) = -\sum_n l_n e^{-t\,l_n}(u_n \mid u_0)u_n$$

exists for every $t \in R_0$ and the function u' is continuous on R_0. Furthermore,

$$\sum_n \left| l_n (u_n \mid e^{-tA}u_0) \right|^2 = \sum_n \left| l_n (u_n e^{-t\,l_n} \mid u_0) \right|^2 < \infty$$

and hence $u(t) \in D(A)$. Now we show $u(t)$ is a solution of (4).

$$Au(t) = \sum_n l_n (u_n \mid e^{-tA} u_0) u_n = \sum_n l_n e^{-t\, l_n} (u_n \mid u_0) u_n = -u'(t).$$

Hence $u(t)$ is a solution of our problem.

Generator. Now we want prove the last assertion of the main theorem. Let it be defined a linear operator through

$$Mu = \lim_{t \to 0^+} t^{-1} (e^{-tM} - I) u ,$$

where $u \in D(M)$ iff this limit exists and $u \in X$. Then $M : D(M) \subseteq X \to X$. By differentiation of

$$(e^{-tA} u \mid v) = (u \mid e^{-tA} v)$$

with respect to t at point 0 we obtain

$$(Mu \mid v) = (u \mid Mv) ,$$

hence M is symmetric. Let $u = e^{-tA} u_0$ for $t \in R_0$ and $u_0 \in D(A)$. Then $u'(0) = -Au_0$ and hence $-A \subseteq M$. So we wee M is an extension of the self-adjoint operator $-A$ and it is clear that $M = -A$.

End of proof

References

[1] Zeidler, E.: Nonlinear Functional Analysis and Its Applications. Vol. 2A: Linear Monotone Operators. Springer-Verlaf, New York, 1986.

[2] Zeidler, E.: Nonlinear Functional Analysis and Its Applications. Vol. 4: Applications to Mathematical Physics. Springer-Verlaf, New York, 1995.