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Compensated compactness and applications to partial differential equations

1 INTRODUCTION

Most of the partial differential equations that arise in continuum mechanics and physics are nonlinear. The linearized equations are now quite well understood but methods for the nonlinear ones still require improvement.

Ten years ago there were mainly two methods for examining nonlinear equations (J.L. Lions [5]): one used a compactness argument while the other, the so-called monotonicity argument, relied in most cases on some kind of convexity result. Even after a systematic study of applications (G. Duvaut, J.L. Lions [2]) there were quite a lot of problems left unsolved.

A new period began with the work of J.M. Ball [1] which permitted the treatment of realistic problems in nonlinear elasticity.

The mathematical tool presented here, which mostly represents results of joint work with F. Murat, generalises this approach to general nonlinear partial differential equations and follows the same ideas: there are some nonlinear functions which are nicely behaved with respect to weak convergence. The characterization of these particular functions is not yet complete but some necessary and some sufficient conditions which have already been obtained are very helpful.

The importance of weak convergence as a mathematical model for the relation between microscopic and macroscopic quantities has been pointed out by the author in earlier work (L. Tartar [12-16]). These results were first obtained (with F. Murat) in 1974 on problems related to (what was then called) homogenization but it was only after J.M. Ball's work and the first version of F. Murat [8] for compensated compactness that I fully understood the possibilities of this method.

When I wrote the article [16] in August 1977, I was not entirely happy with the method because I did not see how to use it to solve nonlinear hyperbolic equations. A few months later it became clear, however, that entropy conditions can be handled by the method and hence it was possible to give results for a single hyperbolic equation. However, for systems of hyperbolic equations it is still a conjecture whether the desired results follow.

I sincerely believe that this is the right way to attack nonlinear partial differential equations (from mechanics or physics but maybe not from other fields) and that this tool will be of great use in open problems for such areas as free boundary problems, boundary layer theory and turbulence.

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2 COMPENSATED COMPACTNESS

Let Ω be a bounded open set in $\mathbb{R}^N$, and let

$$u_1^n, \ldots, u_N^n \longrightarrow u_1, \ldots, u_N \text{ weakly in } L^2(\Omega)$$

$$v_1^n, \ldots, v_N^n \longrightarrow v_1, \ldots, v_N \text{ weakly in } L^2(\Omega).$$

A typical result of our theory will be the following theorem.

Theorem 1 If $\operatorname{div} u^n$ is bounded in $L^2(\Omega)$ $\left(\operatorname{div} u^n = \sum_{i=1}^{N} \frac{\partial u_i^n}{\partial x_i}\right)$ and if $\operatorname{curl} v^n$ is bounded in $(L^2(\Omega))^{N^2}$ $\left((\operatorname{curl} v^n)_{ij} = \partial v_i^n/\partial x_j - \partial v_j^n/\partial x_i \right)$ then

$$\sum_{i=1}^{N} u_i^n v_i^n \rightharpoonup \sum_{i=1}^{N} u_i v_i \quad \text{in the sense of distributions.}$$

Remarks (1) Generally we do not have $u_i^n v_i^n \rightharpoonup u_i v_i$ although the theorem proves that $\Sigma\, u_i^n v_i^n \rightharpoonup \Sigma\, u_i v_i$; hence the title of these lectures (compensated compactness).

(2) As an example explaining compactness, the following is well known. If $u_i^n \rightharpoonup u_i$ in $H^1(\Omega)$ then $u_i^n \longrightarrow u_i$ strongly in $L^2(\Omega)$.

(3) If F has the property that $F(u_n, v_n) \rightharpoonup F(u,v)$ under the above hypotheses, then F has to be of the form

$$F(u,v) = C\, u.v + \text{affine function of } u,v.$$

We will begin with a simpler but related question.

Question Suppose $u^n : \Omega \longrightarrow \mathbb{R}^P$ where $\Omega \subset \mathbb{R}^N$ is a bounded open set, suppose $u^n(x) \in K$ for almost all x (where $K \subset \mathbb{R}^P$), and let $u_i^n \rightharpoonup u_i$. Do we have $u(x) \in K$ almost everywhere (a.e.)?

Recall that on bounded sets the weak topology of $L^q(\Omega)$ (where $1 < q < +\infty$) is metrizable and that closed bounded convex sets of $L^q(\Omega)$ are weakly (sequentially) compact. Moreover $v_n \rightharpoonup v$ if and only if

$$\int_\Omega v_n \phi \, dx \longrightarrow \int_\Omega v \phi \, dx \quad \text{for all } \phi \in L^{q'}(\Omega), \text{ where } \frac{1}{q} + \frac{1}{q'} = 1 .$$

Remark If $\{v_n\}$ is bounded in $L^q(\Omega)$ then $v_n \rightharpoonup v$ if and only if

$$\int_\Omega v_n \phi \, dx \to \int_\Omega v\phi \, dx \quad \text{for all} \quad \phi \in \mathcal{D}(\Omega) ,$$

where $\mathcal{D}(\Omega)$ denotes the set of all infinitely differentiable functions with compact support in Ω.

On $L^\infty(\Omega)$ we denote the weak $\star$ topology by $\sigma(L^\infty, L^1)$ and note that on bounded sets it is metrizable. Then $v_n \rightharpoonup v$ in $L^\infty(\Omega)$ weak $\star$ if and only if

$$\int_\Omega v_n \phi \, dx \to \int_\Omega v\phi \, dx \quad \text{for all} \quad \phi \in L^1(\Omega).$$

On $\mathcal{M}(\Omega)$ (the space of measures) the weak topology is defined as follows:

$$\mu_n \rightharpoonup \mu \text{ if and only if } (\mu_n, \phi) \to (\mu, \phi) \text{ for all } \phi \in \mathcal{D}(\Omega).$$

Remark On $L^1(\Omega)$ we can use this topology.

A typical result of weak convergence is contained in the following:

Example Suppose $f \in L^\infty(\mathbb{R})$ is a periodic function of period T. Let $v_n(x) = f(nx)$, then

$$v_n \rightharpoonup \alpha \quad \text{in} \quad L^\infty \text{ weak } \star$$

where

$$\alpha = \text{average of } f = \frac{1}{T}\int_0^T f(x)\,dx.$$

But as $(v_n)^2 \rightharpoonup \beta$ = average of f^2, we do not have $\beta = \alpha^2$ unless f is constant a.e. .

Actually we have $\beta \geqslant \alpha^2$ as a consequence of

Lemma 2 If F is a convex continuous real-valued function on $\mathbb{R}^p$, if $u_n \longrightarrow u$ in $(L^\infty(\Omega))^p$ weak $*$, and if $F(u_n) \longrightarrow \ell$ then $\ell \geq F(u)$ a.e..

Proof. We know that since F is convex and continuous we can write

$$F(b) = \sup_{L \in A} L(b)$$

where A is a countable set of affine functions. Therefore

$$F(u_n(x)) \geq L(u_n(x)) .$$

Passing to the limit we obtain

$$\ell(x) \geq L(u(x)) \quad \text{a.e. .}$$

Hence

$$\ell \geq \sup_{L \in A} L(u) = F(u) \quad \text{a.e. .}$$

□

The basic result (essentially due to Lyapunov [6]) which answers our first question is:

Theorem 3 (1) Let $v_n : \Omega \longrightarrow \mathbb{R}^p$ be such that

$$v_n \longrightarrow v \quad \text{in} \quad (L^\infty(\Omega))^p \text{ weak } * \quad \text{and} \quad v_n(x) \in K \quad \text{a.e. .}$$

Then $v(x) \in \overline{\text{conv}}(K)$ a.e. (where $\overline{\text{conv}}(K)$ is the closed convex hull of K).

(2) Conversely, let $v \in (L^\infty(\Omega))^p$ and $v(x) \in \overline{\text{conv}}(K)$ a.e. .

Then there exists a sequence $\{v_n\}$ such that

$$v_n \longrightarrow v \quad \text{in} \quad (L^\infty(\Omega))^p \text{ weak } * \quad \text{and} \quad v_n(x) \in K \quad \text{a.e. for all } n.$$

Proof. (1) Let L be affine and $L \geqslant 0$ on K. Then $L(v_n) \geqslant 0$ and hence $L(v) \geqslant 0$ a.e. . Hence $v(x)$ belongs to any countable intersection of half spaces containing K and thus $v(x) \in \overline{\mathrm{conv}}(K)$ a.e. .

(2) There exist characteristic functions $\chi_1^n, \dots, \chi_q^n$ of disjoint sets such that $\chi_j^n \rightharpoonup \theta_j$ where the numbers θ_j satisfy $\theta_j \geqslant 0$, $\Sigma\theta_j = 1$, and are given in advance. We can construct these functions in the following way.

Let B be the unit cube in $\mathbb{R}^p$. We make a partition $\{B_j\}$ of B such that meas $(B_j) = \theta_j$ and let

$$\chi_j = \begin{cases} 1 & \text{in } B_j \\ 0 & \text{in } \bigcup\limits_{k \neq j} B_k \end{cases},$$

extending χ_j outside B periodically.

Then

$$\chi_j^n(x) = \chi_j(nx) \rightharpoonup \theta_j .$$

Let ω be a step function taking values in conv(K): that is $\Omega = \bigcup\limits_{j=1}^{r} \Omega_j$ and $\omega\Big|_{\Omega_j} = k_j$ with $k_j \in \mathrm{conv}(K)$. It is well-known that v can be approached in the strong L^p topology by a sequence of such step functions. It is thus enough to prove that such an ω can be approached in L^∞ weak $\star$ by functions taking only values in K.

Since $k_j = \sum\limits_{\ell=1}^{q_j} \theta_{j\ell} k_{j\ell}$ with $\theta_{j\ell} \geqslant 0$ $\sum\limits_{\ell} \theta_{j\ell} = 1$, and $k_{j\ell} \in K$ we can by the above construct sequences of characteristic functions $\chi_{j\ell}^n$ such that

$$\chi_{j\ell}^n \rightharpoonup \theta_{j\ell} \quad \text{in } L^\infty \text{ weak} \star .$$

Define ϕ_n on Ω by $\phi_n\Big|_{\Omega_j} = \sum_\ell \chi_{j\ell} k_{j\ell}$; then $\phi_n \longrightarrow \omega$ weak $\star$ and ϕ_n takes only values among the $k_{j\ell}$, so that it satisfies our constraints. $\square$

<u>Example</u> Suppose we have $\phi_n \longrightarrow \phi$ and $\phi_n^2 \longrightarrow \psi$ in L^∞ weak $\star$. We take $u_n = (\phi_n,\phi_n^2)$ and K to be the parabola $\{(x,y):\ y = x^2\}$, so that $\mathrm{conv}(K) = \{(x,y):\ y \geqslant x^2\}$. The theorem says that $(\phi,\psi) \in \mathrm{conv}(K)$, that is $\psi \geqslant \phi^2$, a consequence of Lemma 1. But the theorem also says that this information is optimal, because if $\psi \geqslant \phi^2$ there exists a sequence u_n taking values in K, i.e. $u_n = (\phi_n,\phi_n^2)$, with $\phi_n \longrightarrow \phi$ and $\phi_n^2 \longrightarrow \psi$.

If we impose some other conditions on ϕ_n the set K changes and we have to compute $\overline{\mathrm{conv}}(K)$. For example if $0 \leqslant \phi_n \leqslant 1$, $K = \{(x,y):\ y = x^2$ and $0 \leqslant x \leqslant 1\}$, then $\overline{\mathrm{conv}}(K) = \{(x,y): 0 \leqslant x^2 \leqslant y \leqslant x \leqslant 1\}$ so we can only say that $0 \leqslant \phi^2 \leqslant \psi \leqslant \phi \leqslant 1$.

<u>Remark</u> If the ϕ_n are complex valued with $\phi_n \longrightarrow \phi$ and $\phi_n^2 \longrightarrow \psi$ there will be no relation between ϕ and ψ. More generally if $F_1,\ldots,F_m$ are entire holomorphic functions such that no linear combination of them is constant then the weak limits of $F_1(\phi_n),\ldots,F_m(\phi_n)$ are independent as a simple consequence of the maximum principle.

Related to Theorem 3 are some results on weak continuity and weak lower semicontinuity which will be used later in these lectures.

Let Ω be a bounded open set, let $u \in (L^\infty(\Omega))^p$ and let F be a continuous real valued function on $\mathbb{R}^p$.

<u>Question</u> When is the functional $u \longmapsto \int_\Omega F(u)dx$ (sequentially) weakly continuous or (sequentially) weakly lower semicontinuous?

Theorem 4 (1) If F is convex and continuous, then the functional is lower semicontinuous with respect to the weak * topology on $(L^\infty(\Omega))^p$.

(2) If the functional is lower semicontinuous with respect to the weak * topology, then F is convex.

Proof. (1) The first fact is well known.

(2) Let $a,b \in \mathbb{R}^p$, $\theta \in [0,1]$. We want to prove that

$$F(\theta a + (1-\theta)b) \leqslant \theta F(a) + (1-\theta)F(b) \ .$$

Construct

$$u_n = \chi_n a + (1-\chi_n)b$$

where χ_n is a characteristic function such that

$$\chi_n \longrightarrow \theta \quad \text{in} \quad L^\infty(\Omega) \quad \text{weak} \ *.$$

Then, trivially, we have

$$F(u_n) = \chi_n F(a) + (1-\chi_n)F(b).$$

Therefore

$$F(u_n) \longrightarrow \theta F(a) + (1-\theta)F(b)$$

$$(\text{and} \quad u_n \longrightarrow \theta a + (1-\theta)b).$$

But since $u \longmapsto \int_\Omega F(u)\,dx$ is lower semicontinuous, we have that

$$\varliminf_{n\to\infty} \int_\Omega F(u_n)\,dx \geqslant \int_\Omega F(u)\,dx.$$

Therefore

$$\operatorname{meas}(\Omega)(\theta F(a) + (1-\theta)F(b)) \geqslant \operatorname{meas}(\Omega)(F(\theta a + (1-\theta)b)) . \qquad \square$$

The preceding results enable us to obtain results on relaxed controls. This is illustrated by

Example (control problem)

Let u be a control function satisfying the constraint

$$|u(t)| \leqslant 1 \quad \text{a.e.} \quad \text{for} \quad t \in [0,T] .$$

Consider the system

$$\begin{cases} y'(t) = u(t) , \\ y(0) = 0 \end{cases} .$$

We seek to minimize the cost function

$$J(u) = \int_0^T (|y|^2 - |u|^2)\, dt.$$

(Note that J is not convex, so that it is not obvious that a minimizer exists. Observe, in contrast, that

$$J_1(u) = \int_0^T (|y|^2 + |u|^2)\, dt$$

is convex in u and continuous. Therefore it attains its minimum, which in this case is obviously $J_1(u) = 0$ for $y = u = 0$.)

We note that

$$J(u) > -T \quad \text{for all} \quad u.$$

(Since $\int_0^T |y|^2 dt \geqslant 0$ and $-\int_0^T |u|^2 dt \geqslant -T$, it follows that $J(u) \geqslant -T$.

But if $\int_0^T (|y|^2 - |u|^2) dt = -T$, this would imply $y = 0$ and $|u|^2 = 1$ a.e..

But $y = 0$ implies $u = 0$ a.e. which is a contradiction.)

We claim that

$$\inf_v J(v) = -T \ .$$

Choosing

$$u_n(t) = \begin{cases} +1 & \text{if } 0 \leqslant t < \frac{T}{2n} \\ -1 & \text{if } \frac{T}{2n} \leqslant t < \frac{T}{n} \end{cases}$$

and extending it periodically, we obtain

$$y_n \longrightarrow 0 \text{ strongly}$$

and

$$u_n^2 = 1 \ .$$

Therefore $\inf J(v) = -T$. But as we have seen before $y = 0$ and $u^2 = 1$ cannot be a solution of the problem, so that the minimum is not attained. Hence we are led to consider a generalized problem (a relaxed control problem).

We consider two functions u and v such that

$$(u(t))^2 \leqslant v(t) \leqslant 1 \quad \text{a.e. .}$$

If we define $K = \{(x,y) : y = x^2 \text{ and } y \leqslant 1\}$ we see that

$$(u,v) \in \overline{\mathrm{conv}}(K) \quad \text{a.e. .}$$

We define the generalized cost function

$$\tilde{J}(u,v) = \int_0^T (|y|^2 - v)dt .$$

Observe that $\tilde{J}(u,v)$ is a convex function of u and v defined on a convex set. Therefore $\tilde{J}$ attains its minimum, in fact at $u = 0$ and $v = 1$. But there is a one-to-one map from the initial problem to the generalized one defined by

$$u \longmapsto (u,u^2)$$

and

$$J(u) = \tilde{J}(u,u^2).$$

Observe that the range of the map is dense for the weak topology. Thus if we have a minimizing sequence $\{u_n\}$ so that $J(u_n) \longrightarrow \inf J(u)$, we have that $\{(u_n,u_n^2)\}$ is a minimizing sequence for $\tilde{J}$ which tends to the solution $(0,1)$ of $\min \tilde{J}$.

If we want to improve the preceding theorems we are led to introduce the following concepts.

3 GENERALIZED FUNCTIONS (parametrized measures)

Let $u_n:\Omega \longrightarrow \mathbb{R}^n$ and $u_n(x) \in K$ a.e. . We have seen before that if $u_n \rightharpoonup u$, then $u(x) \in \overline{\text{conv}}(K)$ a.e. and that this result is optimal.

Now, suppose we want to know the relationship between the limit of u_n and the limit of $F(u_n)$ where $F: \mathbb{R}^n \longrightarrow \mathbb{R}$. Construct new functions:

$$v_n : x \longmapsto v_n(x) = (u_n(x),F(u_n(x))).$$

Then

$$v_n(x) \in K' = \{(k,F(k)):k \in K\} \text{ a.e. .}$$

Suppose

$$v_n \longrightarrow v = (u,\ell); \quad \text{i.e.} \begin{cases} u_n \longrightarrow u \\ F(u_n) \longrightarrow \ell . \end{cases}$$

Then, by the preceding theory, $v(x) \in \overline{\mathrm{conv}}(K')$ a.e. . Similarly, if we have $F:\mathbb{R}^p \longrightarrow \mathbb{R}^q$ and

$$F_1(u_n),\ldots,F_q(u_n) \longrightarrow \ell_1,\ldots,\ell_q$$

then we consider

$$K'' = \{(k,F_1(k),\ldots,F_q(k)):k \in K\}$$

and we conclude that

$$(u(x),\ell_1(x),\ldots,\ell_q(x)) \in \overline{\mathrm{conv}}(K'') \quad \text{a.e. .}$$

We are therefore led to the following question. What can be said about the limits of all functions $F(u_n)$ for all functions F? The following result is in the spirit of L.C. Young [17].

Theorem 5 (1) Suppose K is bounded in $\mathbb{R}^p$ and Ω is an open set in $\mathbb{R}^N$. Let $u_n:\Omega \longrightarrow \mathbb{R}^p$ be such that $u_n(x) \in K$ a.e. . Then there exists a subsequence $\{u_m\}$ and a family of probability measures $\{\nu_x\}_{x\in\Omega}$ on $\mathbb{R}^p$ (depending measurably on x) with $\mathrm{supp}\,\nu_x \subset \overline{K}$ such that if F is a continuous function on $\mathbb{R}^p$ and

$$\overline{f}(x) = \langle \nu_x, F(\lambda) \rangle \quad \text{a.e.}$$

then

$$F(u_m) \longrightarrow \overline{f} \text{ in } L^\infty(\Omega) \text{ weak } * .$$

(2) Conversely, let $\{\nu_x\}$ be a family of probability measures with support in K. Then there exists a sequence $\{u_n\}$, where $u_n:\Omega \longrightarrow \mathbb{R}^p$ and $u_n(x) \in K$ a.e., such that for all continuous functions F on $\mathbb{R}^p$ we have

$$F(u_n) \longrightarrow \overline{f} = \langle \nu_x, F(\lambda) \rangle \quad \text{in } L^\infty(\Omega) \text{ weak } * \ .$$

Examples (1) If F = identity and if $u_n \longrightarrow u$ then

$$u(x) = \langle \nu_x, \lambda \rangle \quad \text{a.e. .}$$

(2) Let $\theta_i \geqslant 0$, $\sum_{i=1}^{q} \theta_i = 1$, and let $v(x) = a_i$ for x in the i^{th} interval (i.e. $\sum_{j=0}^{i-1} \theta_j \leqslant x < \sum_{j=0}^{i} \theta_j$). Extend v to be of period 1 on $\mathbb{R}$, and let

$$u_n(x) = v(nx) \ .$$

Since

$$F(u_n) \longrightarrow \sum_{i=1}^{q} \theta_i F(a_i) \quad L^\infty(\Omega) \text{ weak } *$$

we can take

$$\nu_x = \sum_{i=1}^{q} \theta_i \delta_{a_i} \quad \text{where } \delta_{a_i} \text{ is the Dirac measure at } a_i \ .$$

(3) Let $v:[0,1] \longrightarrow \mathbb{R}^p$ be a continuous function of period 1. Then if we define ν by

$$\langle \nu, f(\lambda) \rangle = \int_0^1 f(v(x))dx$$

and if $u_n(x) = v(nx)$, we can take $\nu_x = \nu$.

Before proving the theorem we need the following definition and proposition.

<u>Definition</u> Let $u:\Omega \longrightarrow \mathbb{R}^p$ be a measurable function satisfying $v(x) \in K$ (a bounded set) a.e. . Define a measure μ on $\Omega \times \mathbb{R}^p$ by

$$<\mu,\phi(x,\lambda)> = \int_\Omega \phi(x,v(x))\,dx$$

for all continuous functions ϕ with compact support in $\Omega \times \mathbb{R}^p$. μ is called <u>the generalized function associated to u</u> (or <u>the Radon measure associated to u</u>).

<u>Proposition 6</u> The Radon measure μ has the following properties:

(1) $\mu \geqslant 0$; i.e. $<\mu,\phi> \geqslant 0$ if $\phi \geqslant 0$.

(2) $\operatorname{supp}\mu \subset \overline{\operatorname{graph} v}$; i.e. if $\phi = 0$ on $\overline{\operatorname{graph} v}$ then $<\mu,\phi> = 0$.

(3) The projection on Ω of $\mu = \operatorname{proj}_\Omega \mu = dx$;
i.e. if $\phi(x,\lambda) = \psi(x)$ then $<\mu,\phi> = \int_\Omega \psi(x)dx$.

<u>Proof</u>. (1) and (2) are obvious.

(3) $\operatorname{proj}_\Omega \mu$ is the image of μ under the projection map $(x,\lambda) \longmapsto x$, which is defined if the map is proper on a neighbourhood of $\operatorname{supp}\mu$. This is true since $v(x) \in K$ (a bounded set) a.e. . Because $\operatorname{supp}\mu \subset \Omega \times K$ and K is bounded we can define $<\mu,\phi>$ if ϕ is continuous and has compact support in x. For if

$$\chi(\lambda) = \begin{cases} 1 & \text{in a neighbourhood of } K \\ 0 & \text{outside a large ball,} \end{cases}$$

then $\phi(x,\lambda)\chi(\lambda)$ is independent of $\lambda \in \overline{K}$. $\square$

Proof of the Theorem (1) Let μ_n be associated to u_n, i.e.

$$<\mu_n,\phi(x,\lambda)> = \int_\Omega \phi(x,u_n(x))dx \ .$$

Let us suppose for simplicity that Ω is bounded. We may then extract a subsequence μ_m such that

$$\mu_m \longrightarrow \mu \text{ weakly } * \quad \text{i.e. } <\mu_n,\phi> \longrightarrow <\mu,\phi> \text{ for all } \phi .$$

We investigate the properties of μ :

(i) $\mu \geqslant 0$ since for all $\phi \geqslant 0$, $<\mu,\phi> = \lim_{n\to\infty} <\mu_n,\phi> \geqslant 0$.

(ii) $\operatorname{supp}\mu \subset \Omega\times\overline{K}$ i.e. if $\phi = 0$ on $\Omega\times\overline{K}$ then $<\mu,\phi> = 0$. This is obvious since $<\mu_n,\phi> = 0$ (observe that $\operatorname{supp}\mu$ may no longer be contained in the closure of a graph).

(iii) $\text{proj}_\Omega \mu = dx$. For if $\phi(x,\lambda) = \psi(x)$ then

$$<\mu,\phi> = \lim_n <\mu_n,\phi> = \int_\Omega \psi(x)dx \ .$$

But it is a standard result (using the Radon-Nikodym theorem) that if μ satisfies the above three properties, then there exists a family of probability measures $\{\nu_x\}$ such that

$$\mu = \int_\Omega \nu_x\, dx \ , \text{ i.e. } <\mu,\phi(x,\lambda)> = \int_\Omega <\nu_x,\phi(x,\lambda)>dx \ .$$

Using this construction we are able to prove the theorem. If $F(u_m) \longrightarrow \ell$, then for all ψ we have

$$<\mu_m,\psi(x)F(\lambda)> \longrightarrow <\mu,\psi(x)F(\lambda)> = \int_\Omega <\nu_x,F(\lambda)>\psi(x)dx \ .$$

But also

$$<\mu_m,\psi(x)F(\lambda)> = \int_\Omega \psi(x)F(u_m(x))dx \longrightarrow \int_\Omega \psi(x)\ell(x)dx \ .$$

Therefore we conclude that

$$\ell(x) = <\nu_x,F(\lambda)> \quad \text{a.e.}$$

and the proof is complete.

(2) Now let us prove the converse. Let

$$\mu = \int_\Omega \nu_x \, dx \ ,$$

and let

$$\mathcal{M} = \{\mu : \mu \ \text{ associated with a function } \ \nu:\Omega \longrightarrow K\}$$

$$\mathfrak{M} = \{\mu : \text{(i) } \mu \geqslant 0 \ \text{ (ii) } \operatorname{supp}\mu \subset \Omega \times \overline{K} \ \text{ and (iii) } \operatorname{proj}_\Omega \mu = dx\}.$$

Our aim is to prove that $\overline{\mathcal{M}} = \mathfrak{M}$ (in the weak $\star$ topology). The proof that $\overline{\mathcal{M}} = \mathfrak{M}$ will end the proof of the converse of the theorem. Our proof will be decomposed into two steps.

<u>1st step</u>: $\overline{\mathcal{M}}$ is convex.

<u>2nd step</u>: $\overline{\operatorname{conv}(\mathcal{M})} = \mathfrak{M}$.

<u>1st step</u> Let $\mu_1,\ldots,\mu_q \in \mathcal{M}$. Then (by the definition of $\mathcal{M}$) there exist $v_1,\ldots,v_q$ such that

$$<\mu_i,\phi> = \int_\Omega \phi(x,v_i(x))dx \ .$$

We want to prove that if $\theta_i \geqslant 0$ and $\sum_{i=1}^q \theta_i = 1$ then $\mu = \sum_{i=1}^q \theta_i\mu_i \in \overline{\mathcal{M}}$.

By Theorem 3, there exist characteristic functions $\chi_{i,n}$ of disjoint sets such that

$$\chi_{i,n} \longrightarrow \theta_i \quad \text{in} \quad L^\infty(\Omega) \text{ weak } * .$$

Let

$$w_n = \sum_{i=1}^{q} \chi_{i,n}(x) v_i(x) .$$

Then

$$\phi(x,w_n(x)) = \sum_{i=1}^{q} \chi_{i,n}(x)\phi(x,v_i(x)) ,$$

and hence

$$\int_\Omega \phi(x,w_n(x))dx = \sum_{i=1}^{q} \int_\Omega \chi_{i,n}(x)\phi(x,v_i(x))dx$$

$$\longrightarrow \sum_{i=1}^{q} \int_\Omega \theta_i \phi(x,v_i(x))dx$$

$$= \sum_{i=1}^{q} \theta_i <\mu_i,\phi> .$$

Thus $\sum_{i=1}^{q} \theta_i\mu_i \in \overline{\mathcal{M}}$ and $\overline{\mathcal{M}}$ is convex.

<u>2nd step</u> We know that to find the closed convex hull of $\mathcal{M}$ we need only consider the affine continuous functions which are positive on $\mathcal{M}$ (by the Hahn-Banach theorem). But an affine continuous function is of the form

$$\mu \longmapsto <\mu,\phi_0(x,\lambda)> + \alpha$$

with ϕ_0 continuous (bounded) and $\alpha \in \mathbb{R}$, and this is positive on $\mathcal{M}$ if

$$<\mu,\phi_0(x,\lambda)> + \alpha \geqslant 0 \quad \text{for all} \quad \mu \in \mathcal{M} .$$

This is equivalent to

$$(*) \qquad \int_\Omega \phi_0(x,v(x))dx + \alpha \geqslant 0 \quad \text{for all} \quad v:\Omega \longrightarrow K.$$

Define

$$\psi_0(x) = \inf_{\lambda\in\overline{K}} \{\phi_0(x,\lambda)\} .$$

Then ψ_0 is continuous and bounded and therefore $(*)$ is equivalent to

$$(**) \qquad \int_\Omega \psi_0(x)dx + \alpha \geqslant 0 .$$

Let

$$\phi_0(x,\lambda) = \psi_0(x) + \chi_0(x,\lambda)$$

so that $\chi_0 \geqslant 0$ on $\Omega\times\overline{K}$.

Conversely if $\chi_0 \geqslant 0$ on $\Omega\times\overline{K}$ and $\int_\Omega \psi_0(x)dx + \alpha \geqslant 0$ then

$\langle\mu,\phi_0\rangle + \alpha \geqslant 0$ for all $\mu \in \mathcal{M}$. But $\operatorname{conv}(\mathcal{M}) = \{\mu: \langle\mu,\phi_0\rangle + \alpha \geqslant 0$ for all ϕ_0 and α defined as above$\}$.

Now it remains to prove that this characterises $\mathfrak{M}$.

(1) Take $\psi_0 = \alpha = 0$. Then $\langle\mu,\phi_0\rangle = \langle\mu,\chi_0\rangle \geqslant 0$ for $\chi_0 \geqslant 0$, so $\mu \geqslant 0$.

(2) If $\chi = 0$ on $\Omega\times\overline{K}$ then $\chi \geqslant 0$ and $-\chi \geqslant 0$, and thus by (1) we have both $\langle\mu,\chi\rangle \geqslant 0$ and $\langle\mu,-\chi\rangle \geqslant 0$. Therefore $\langle\mu,\chi\rangle = 0$, and hence $\operatorname{supp}\mu \subset \Omega\times\overline{K}$.

(3) Take $\alpha = -\int_\Omega \psi_0(x)dx$ and $\chi_0 = 0$.

Then

$$\langle\mu,\psi_0\rangle + \alpha \geqslant 0 \quad \text{for all} \quad \psi_0$$

implies

$$\langle \mu, \psi_0 \rangle \geqslant \int_\Omega \psi_0(x)\,dx \ .$$

Applying this to $-\psi_0$ we get

$$\langle \mu, \psi_0 \rangle \leqslant \int_\Omega \psi_0(x)\,dx \ .$$

Therefore

$$\langle \mu, \psi_0 \rangle = \int_\Omega \psi_0(x)\,dx$$

and hence $\mathrm{proj}_\Omega \mu = dx$.

If $\mu \in \mathfrak{M}$ then $\langle \mu, \phi_0 \rangle + \alpha \geqslant 0$ because $\langle \mu, \phi_0 \rangle = \langle \mu, \chi_0 \rangle + \langle \mu, \psi_0 \rangle$ and $\langle \mu, \chi_0 \rangle \geqslant 0$, $\langle \mu, \phi_0 \rangle = \int_\Omega \psi_0\,dx$. □

Remark Let $u_n \rightharpoonup u$ in $L^\infty(\Omega)$ weak $\star$. Then $u_n \longrightarrow u$ strongly (in every $L^p(\Omega)$, $p < \infty$) if and only if

$$\nu_x = \delta_{u(x)} \quad \text{a.e.},$$

where δ_y denotes the Dirac measure at y. For if $u_n \longrightarrow u$ strongly, then by Theorem 5,

$$F(u(x)) = \langle \nu_x, F(\lambda) \rangle \qquad \text{a.e.},$$

while conversely, if $\nu_x = \delta_{u(x)}$ a.e. then

$$(u_n)_i \rightharpoonup u_i \ , \ (u_n)_i^2 \rightharpoonup u_i^2 \quad \text{in } L^\infty(\Omega) \text{ weak } \star ,$$

and hence $u_n \longrightarrow u$ strongly.

Example Let

$$u(x) = \text{electrostatic potential},$$

$$E = -\operatorname{grad} u = \text{electric field},$$

and suppose that

$$u(x) = u_0(x) + \varepsilon u_1\left(\frac{x}{\varepsilon}\right)$$

where u_0, u_1 are smooth, u_1 is periodic of period T and ε is small. Then

$$E = -\operatorname{grad} u_0 - \operatorname{grad} u_1\left(\frac{x}{\varepsilon}\right) .$$

Observe that while u can be approximated by u_0, E is very different from $\operatorname{grad} u_0$ in the strong topology. However in the weak topology we have

$$\operatorname{grad} u_1\left(\frac{x}{\varepsilon}\right) \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Therefore the "natural" topology should be the weak topology (recall furthermore that on bounded sets the weak topology is metrizable). The "natural" space to work with, in this example, is the energy space associated with $\int_\Omega |E|^2 dx$. Therefore we need $E \in L^2(\Omega)$ and hence $u \in H^1(\Omega)$, so that the "natural" space should be $H^1(\Omega)$.

The constitutive equations are:

$$\begin{cases} E = -\operatorname{grad} u \\ D = aE \\ \operatorname{div} D = \rho \end{cases}$$

where D is the electric induction field, ρ is the charge density, and a

is a constant. The electrostatic energy (density) is $E.D = e$.

Suppose we can measure ρ, E, D and u; this means that

$$u_\varepsilon \longrightarrow u_0 \quad \text{in} \quad H^1(\Omega) \text{ weak},$$

$$E_\varepsilon \longrightarrow E_0 \quad \text{in} \quad L^2(\Omega) \text{ weak},$$

$$D_\varepsilon \longrightarrow D_0 \quad \text{in} \quad L^2(\Omega) \text{ weak},$$

$$\rho_\varepsilon \longrightarrow \rho_0 \quad \text{in} \quad L^2(\Omega) \text{ weak}.$$

The problem is: does

$$e_\varepsilon = E_\varepsilon . D_\varepsilon \longrightarrow e_0 = E_0 . D_0 ?$$

The problem can be solved by the compensated compactness theorem mentioned in the beginning of this course.

If an experiment depends on a function $f(E_\varepsilon, D_\varepsilon)$ different from $E_\varepsilon . D_\varepsilon$, then it will be impossible to deduce the result of a measurement of $f(E_\varepsilon, D_\varepsilon)$ from measurements of E_ε and D_ε. But the generalized function associated with (E,D) will actually give all information on the "limit" of $f(E_\varepsilon, D_\varepsilon)$.

We give now some more examples of the usefulness of the definition of a generalized function:

<u>Examples</u> (1) Suppose we have the following equation:

$$(\ast) \qquad \frac{\partial u}{\partial t} + u^2 = 0 ,$$

and suppose we "measure" $u(x,0)$, $x \in \Omega \subset \mathbb{R}^N$, and we would like to know $u(x,T)$ for a certain $T > 0$. If we can "measure" $u(x,0)$ in the strong topology then we have a good estimate on $u(x,T)$ since the explicit solution of $(\ast)$ is:

$$u(x,t) = \frac{u(x,0)}{1 + tu(x,0)} .$$

If we "measure" $u(x,0)$ in the weak topology, we have no estimate on $u(x,t)$, since the semigroup is not weakly continuous i.e.

$$u_\varepsilon(x,0) \longrightarrow u(x,0) \quad \text{in } L^\infty(\Omega) \text{ weak } \star$$

does not imply that

$$u_\varepsilon(x,T) \longrightarrow u(x,T) \quad \text{in } L^\infty(\Omega) \text{ weak } \star .$$

But if we know the generalized function μ associated with "lim" $u_\varepsilon(x,0)$, we can compute the measured value of $u(x,T)$. So if μ_ε is the generalized function associated with u_ε, and $\mu_\varepsilon \longrightarrow \mu$ in the sense of generalized functions, we have that

$$\int_\Omega \frac{u_\varepsilon(x,0)}{1 + Tu_\varepsilon(x,0)} \phi(x)dx \longrightarrow \langle\mu, \phi(x)\frac{\lambda}{1 + T\lambda}\rangle \overset{\text{def}}{=} \langle\mu_T, \lambda\phi(x)\rangle .$$

Hence $u_\varepsilon(x,T) \longrightarrow v(x,T)$ in $L^\infty(\Omega)$ weak $\star$, where

$$v(x,t) = \langle\nu_x, \frac{\lambda}{1 + \lambda T}\rangle = \int_{\mathbb{R}} \frac{\lambda}{1 + \lambda T} d\nu_x(\lambda),$$

with ν_x defined as in Theorem 5 by

$$\mu = \int_\Omega \nu_x dx .$$

(2) Consider the following equation:

$$\frac{\partial u}{\partial t} + a_\varepsilon(x)u = 0$$

where a_ε is bounded. Then

$$u(x,T) = e^{-Ta_\varepsilon(x)} u(x,0) = e^{-Ta_\varepsilon(x)} u_0(x) .$$

Therefore, in terms of the corresponding semigroup:

$$u(T) = S_\varepsilon(T)u_0 .$$

But if $a_\varepsilon \longrightarrow a$ and $S_\varepsilon(T) \longrightarrow S(T)$, in general $S(T)$ will not be a semi-group, since in general

$$e^{-Ta_\varepsilon(x)} \longrightarrow \phi(x,T) \neq e^{-Ta(x)} .$$

But if we consider the problem in the sense of generalized functions, we have:

$$a_\varepsilon \longrightarrow \mu = \int_\Omega \nu_x dx \quad \text{(in the sense of generalized functions)}$$

and

$$\phi(x,T) = \langle \nu_x, e^{-T\lambda} \rangle .$$

5 SUMMARY OF RESULTS FOR THE CASE WITHOUT DERIVATIVES

Note In the sequel we will always consider sequences in which $\varepsilon \longrightarrow 0$.

Hypotheses

(H1) $u_1^\varepsilon, \ldots, u_p^\varepsilon \longrightarrow u_1, \ldots, u_p$ in $L^\infty(\Omega)$ weak *

(H2) $u^\varepsilon(x) = (u_1^\varepsilon(x), \ldots, u_p^\varepsilon(x)) \in K$ a.e. .

The questions under consideration were:

Question

(Q1) Does $u(x) \in K$ a.e.?

(Q2) For which functions F do we have

$$F(u^\varepsilon) \longrightarrow F(u)?$$

(Q3) For which functions F do we have

if $F(u^\varepsilon) \longrightarrow \ell$ then $\ell \geqslant F(u)$?

(Q4) If $F(u^\varepsilon) \longrightarrow \ell$ (strongly), does $F(u) = \ell$?

The answers are collected together in the following theorem.

Theorem 7 Under hypotheses (H1) and (H2) we may deduce in general

(R1) that $u(x) \in K$: if and only if K is closed and convex,

(R2) that $F(u^\varepsilon) \longrightarrow F(u)$ in $L^\infty(\Omega)$ weak $\star$:if and only if F is affine on $\overline{\mathrm{conv}}(K)$,

(R3) that if $F(u^\varepsilon) \longrightarrow \ell$ in $L^\infty(\Omega)$ weak $\star$ then $\ell \geqslant F(u)$: if and only if F is convex on $\overline{\mathrm{conv}}(K)$,

(R4) that if $F(u^\varepsilon) \longrightarrow \ell$ (strongly) then $\ell = F(u)$: if and only if for all $a \in \mathbb{R}^m$ $(F:\mathbb{R}^p \longrightarrow \mathbb{R}^m)$ F is constant on $\overline{\mathrm{conv}}(F^{-1}(a) \cap \overline{K})$ i.e. if $k_1,\dots,k_r$ are such that $F(k_i) = a$ then $F = a$ on $\mathrm{conv}\{k_1,\dots,k_r\}$.

Proof. (R1) is just Theorem 2.

(R3) We have seen in Theorem 2 how to construct from characteristic functions a sequence

$$u^\varepsilon = \sum_{j=1}^{q} \chi_j^\varepsilon k^j \longrightarrow u$$

where $k^j \in K$, and such that $\chi_j^\varepsilon \longrightarrow \theta_j$ with $\theta_j \geqslant 0$, $\sum_{j=1}^{q} \theta_j = 1$. Then

$$F(u^\varepsilon) \longrightarrow \sum_{j=1}^{q} \theta_j F(k^j) = \ell \geqslant F(u) = F\left(\sum_{j=1}^{q} \theta_j k^j\right)$$

implies that F is convex (and continuous) on $\overline{\mathrm{conv}}(K)$. The converse is obvious.

(R2) is a simple consequence of (R3) since we need in this case $\ell \geqslant F(u)$ and $-\ell \geqslant -F(u)$, so that we can apply (R3) to F and $-F$ to obtain the result.

(R4) Let $\mu = \int_\Omega \nu_x dx$ be the generalized function corresponding to the sequence u^ε. We then have the following equivalences:

$$\begin{aligned} F(u^\varepsilon) \longrightarrow \ell \text{ strongly } &\Longleftrightarrow \text{ image of } \nu_x \text{ by } F \text{ is a Dirac measure } \delta_{\ell(x)} \\ &\Longleftrightarrow \operatorname{supp} \nu_x \subset F^{-1}(\ell(x)) \cap \overline{K} \\ &\Longleftrightarrow F \text{ is constant on } \overline{\mathrm{conv}}(\operatorname{supp} \nu_x) \\ &\Longleftrightarrow F(u) = \text{constant} = \langle \nu_x, F(\lambda) \rangle = \ell(x). \qquad \square \end{aligned}$$

6 WEAK CONVERGENCE AND PARTIAL DERIVATIVES

Now we turn our attention to a somewhat more complicated problem, where we have some hypotheses on the derivatives of the sequences under consideration. We state the problem in the same way as before.

Hypotheses

(H1) $u_1^\varepsilon, \ldots, u_p^\varepsilon \longrightarrow u_1, \ldots, u_p$ in $L^\infty(\Omega)$ weak $\star$

(H2) $u^\varepsilon(x) \in K$ a.e.

(H3) $\sum_{j,k} a_{ijk} \frac{\partial u_j^\varepsilon}{\partial x_k} \in$ bounded set in L^∞ for $i = 1, \ldots, q$ where the a_{ijk} are real constants and where the derivatives $\frac{\partial u_j^\varepsilon}{\partial x_k}$ are in the sense of distributions.

The questions are the same as before ((Q1)...(Q4)).

Remark A more general question whose solution will answer all the preceding ones is:

(Q1′) Find the smallest set $\tilde{K}$ such that (H1) + (H2) imply $u(x) \in \tilde{K}$ a.e. . Then (Q1) means: is $\tilde{K} = K$?

(Q2) can be interpreted as follows: add a new function $u^{\varepsilon}_{p+1} = F(u^{\varepsilon}_1,\dots,u^{\varepsilon}_p)$ and let $K_1 = \{(k,F(k)),k \in K\}$. (Q2) means: is $\tilde{K}_1 \subset \{(k,F(k)):k \in \overline{\mathrm{conv}}(K)\}$?

In the same way (Q3) means: is $\tilde{K}_1 \subset \{k,\lambda):k \in \overline{\mathrm{conv}}(K),\ \lambda \geqslant F(k)\}$? Actually the best answer will be to characterize the generalized function μ which one can obtain as a limit of sequences u^{ε} satisfying (H1) and (H2).

The partial answer we will give to these questions will rely heavily on the following definition.

Definition The sets $\mathcal{V},\mathcal{V}^{\circ}$ are defined by $\mathcal{V} = \{(\lambda,\xi):\lambda \in \mathbb{R}^p,\ \xi \in \mathbb{R}^N$ such that $\sum_{j,k} a_{ijk}\lambda_j\xi_k = 0$ for $i = 1,\dots,q\}$ and $\mathcal{V}^{\circ} = \{(\lambda,\xi) \in \mathcal{V}:\xi \neq 0\}$.

Remarks: (1) If $(\lambda,\xi) \in \mathcal{V}$ then $(\alpha\lambda,\alpha\xi) \in \mathcal{V}$ for all $\alpha \in \mathbb{R}$.

(2) If $(\lambda,\xi),(\lambda,\eta) \in \mathcal{V}$ then $(\lambda,\xi+\eta) \in \mathcal{V}$ and if $(\lambda,\xi),(\mu,\xi) \in \mathcal{V}$ then $(\lambda+\mu,\xi) \in \mathcal{V}$.

Definition $\Lambda = \{\lambda \in \mathbb{R}^p$:there exists $\xi \neq 0$ $(\lambda,\xi) \in \mathcal{V}\}$.

Remarks: (1) In the case without derivatives we have $\mathcal{V} = \mathbb{R}^p \times \mathbb{R}^N$ and $\Lambda = \mathbb{R}^p$. If $\Lambda = \mathbb{R}^p$, this means that (H3) contains very little information.

(2) In the compactness case we have $\Lambda = \{0\}$, and this happens if the list (H3) contains all the derivatives $\frac{\partial u^{\varepsilon}_j}{\partial x_k}$, which are thus all bounded.(The

answers to questions (Q1)...(Q3) are then trivial, since in this case the convergence becomes strong.)

We can now prove our first result, which gives a necessary condition in connection with (Q1). We will see later that the result can be improved.

Theorem 8 If (H1),(H2) and (H3) imply $u(x) \in K$ a.e., then K must satisfy the following conditions:

NC0: K is closed.

NC1: If $a,b \in \overline{K} = K$ and $b - a \in \Lambda$, then the segment $[a,b]$ is in K.

Proof. NC0 is obvious.

NC1: Let $a,b \in K$ and $b - a \in \Lambda$. Then by the definition of Λ, there exists $\xi \in \mathbb{R}^N$, $\xi \neq 0$, such that

$$\sum_{j,k} a_{ijk}(b-a)_j\xi_k = 0 \quad \text{for } i = 1,\dots,q.$$

For any sequence of functions $\{f^\varepsilon\}$ we construct

$$u^\varepsilon = a + (b-a)f^\varepsilon(\xi.x)$$

and we claim that

$$(*) \qquad \sum_{j,k} a_{ijk}\frac{\partial u_j^\varepsilon}{\partial x_k} = 0$$

holds.

(1) If f^ε is smooth, then

$$\frac{\partial u_j^\varepsilon}{\partial x_k} = (b-a)_j \frac{\partial}{\partial x_k}(f^\varepsilon(\xi.x))$$

$$= f^{\varepsilon\prime}(\xi.x)(b-a)_j\xi_k.$$

Therefore

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\varepsilon}{\partial x_k} = f^{\varepsilon\prime}(\xi.x) \sum_{j,k} a_{ijk}(b-a)_j \xi_k = 0$$

and $(*)$ is satisfied.

(2) By approximation, $(*)$ is true for all f^ε.

Now we make a particular choice of f^ε, namely a sequence of characteristic functions such that

$$f^\varepsilon \longrightarrow \theta \in [0,1] .$$

Then u^ε takes only the values a and b, and so $u^\varepsilon(x) \in K$ a.e.. Furthermore

$$u^\varepsilon \longrightarrow (1-\theta)a + \theta b \quad \text{in} \quad (L^\infty)^p \text{ weak } \star ,$$

so that

$$(1-\theta)a + \theta b \in K.$$

Thus

$$[a,b] \subset K. \qquad \square$$

<u>Corollary 9</u> A necessary condition on K and F so that (Q2) can be answered positively is:

If $a_1,\ldots,a_n \in K$, $\xi \neq 0$ with $(a_j - a_k,\xi) \in \mathcal{V}_0$ for all j,k then F is affine on $\mathrm{conv}\{a_1,\ldots,a_n\}$.

<u>Proof</u>. Choose functions $\{u^\varepsilon\}$ of $\xi.x$, taking only the values $a_1,\ldots,a_n$ and such that

$$F(u^\varepsilon) \longrightarrow \sum_{i=1}^{n} \theta_i F(a_i), \quad \text{where} \quad \theta_i \geqslant 0, \quad \sum_{i=1}^{n} \theta_i = 1.$$

Since we require that $F\left(\sum_{i=1}^{n} \theta_i a_i\right) = \sum_{i=1}^{n} \theta_i F(a_i)$ it follows that F is affine on $\mathrm{conv}\{a_1, \ldots, a_n\}$. □

Remark If K is convex, then in the statement of the corollary we can take $n = 2$ (i.e. we consider only a_1 and a_2).

Corollary 10 A necessary condition on K and F so that (Q3) can be answered positively is:

If $a_1, \ldots, a_n \in K$, $\xi \neq 0$ with $(a_j - a_k, \xi) \in \mathcal{V}$ for all j,k then F is convex on $\mathrm{conv}\{a_1, \ldots, a_n\}$.

Proof. This is similar to that of Corollary 9. □

Remark In the case $K = \mathbb{R}^N$, Corollary 10 implies that F is convex in any direction belonging to Λ i.e. for all $\lambda_0 \in \Lambda$, $a \in \mathbb{R}^p$, $F(a + t\lambda_0)$ is convex in t. In particular, if F is of class C^2, we have that

$$F''(a)(\lambda,\lambda) \geqslant 0 \quad \text{for all} \quad a \in \mathbb{R}^p, \quad \lambda \in \Lambda.$$

Now let us look at some examples illustrating these results.

Example 1 Let $\Omega = \mathbb{R}^N$, $p = N$, and let $v:\Omega \longrightarrow \mathbb{R}$, $u_i = \dfrac{\partial v}{\partial x_i}$. Suppose we have a sequence $\{u_i^\varepsilon\}$ such that

$$u_i^\varepsilon \rightharpoonup u_i \quad i = 1, \ldots N \quad \text{in} \quad L^\infty(\Omega) \text{ weak } *$$

and

$$F(u^\varepsilon) \rightharpoonup \ell \quad \text{in} \quad L^\infty(\Omega) \text{ weak } * .$$

We ask the question: when is $\ell \geqslant F(u)$? Observe first that

$$\frac{\partial u_i^\varepsilon}{\partial x_j} - \frac{\partial u_j^\varepsilon}{\partial x_i} = 0 \quad \text{for all} \quad i,j .$$

Thus

$$\mathcal{V} = \{(\lambda,\xi) \in \mathbb{R}^{2N} : \lambda_i \xi_j - \lambda_j \xi_i = 0 \quad \text{for all} \quad i,j\}.$$

So, adopting the convention that $a \| b$ (a parallel to b) even if $a = 0$ or $b = 0$, we have

$$\mathcal{V} = \{(\lambda,\xi) \in \mathbb{R}^{2N} : \lambda \| \xi\} .$$

Thus

$$\Lambda = \mathbb{R}^p .$$

So the necessary condition that F has to satisfy if we want $\ell \geqslant F(u)$ is that F be convex.

This result should be compared with the result of Theorem 4 for the case where the u of the theorem is replaced by $\operatorname{grad} v$. As we see the answer in the two cases is the same.

Remark If we want the result only for $|\operatorname{grad} v| \leqslant 1$, then we take K = unit ball in $\mathbb{R}^p$, and we arrive at the same answer i.e. F has to be convex on K.

Example 2 In the preceding example we were investigating when the functional $\int_\Omega F(\operatorname{grad} v)dx$ is weakly lower semicontinuous. Now we try to minimize

$$\int_\Omega F(\operatorname{grad} v, \operatorname{grad} w)dx,$$

where $\Omega = \mathbb{R}^N$, $p = 2N$, and $v,w:\Omega \longrightarrow \mathbb{R}$. To this end we construct a new function

$$u = (u_1,\ldots,u_N,u_{N+1},\ldots,u_{2N}),$$

where

$$u_i = \frac{\partial v}{\partial x_i} \quad \text{and} \quad u_{N+1} = \frac{\partial w}{\partial x_i} .$$

Thus

$$\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \frac{\partial u_{N+1}}{\partial x_j} - \frac{\partial u_{N+j}}{\partial x_i} = 0 \quad \text{for all } i,j ,$$

and so

$$\mathcal{V} = \{((\lambda,\mu),\xi):\lambda \in \mathbb{R}^N,\ \mu \in \mathbb{R}^N \text{ and } \xi \in \mathbb{R}^N \text{ such that}$$
$$\lambda_i\xi_j - \lambda_j\xi_i = \mu_i\xi_j - \mu_j\xi_i = 0 \quad \text{for all } i,j\}.$$

Using again the convention that $0 \parallel \alpha$ for all α we obtain

$$\mathcal{V} = \{((\lambda,\mu),\xi):\lambda \parallel \xi \text{ and } \mu \parallel \xi\} .$$

Hence

$$\Lambda = \{(\lambda,\mu):\lambda \parallel \mu\}$$

so the necessary condition for the functional to be weakly lower semicontinuous is that

$$F(a + t\lambda, b + t\mu) \text{ is convex in } t \quad \text{for all } \lambda \parallel \mu \quad \text{for all } a,b \in \mathbb{R}^N.$$

Similarly the necessary condition for weak continuity is

$F(a+t\lambda, b+t\mu)$ is affine in t for all $\lambda \| \mu$ and for all $a,b \in \mathbb{R}^N$.

Hence

$$\begin{cases} (1) \text{ taking } \lambda \in \mathbb{R}^N \text{ and } \mu = 0,\ F(a,b) \text{ has to be affine in } a \text{ for fixed } b \\ (2) \text{ taking } \lambda = 0 \text{ and } \mu \in \mathbb{R}^N,\ F(a,b) \text{ has to be affine in } b \text{ for fixed } a. \end{cases}$$

Therefore we conclude that

$$F = \text{affine} + \text{bilinear},$$

the bilinear part being $\sum_{i,j} \alpha_{ij} a_i b_j$ and vanishing if $a \| b$. Thus the bilinear part is a linear combination of terms $a_i b_j - a_j b_i$ for all i,j. We will see later that each F of this type is actually weakly continuous.

Example 3 Let $\Omega = \mathbb{R}^N$, $p = 2N$, and $u = (v_1,\ldots,v_N,w_1,\ldots,w_N)$ and suppose we have sequences $\begin{cases} v^\varepsilon \rightharpoonup v \\ w^\varepsilon \rightharpoonup w \end{cases}$ such that

(1) $\operatorname{div} v^\varepsilon$ is bounded in $L^\infty(\Omega)$

(2) $\operatorname{curl} w^\varepsilon$ is bounded in $(L^\infty(\Omega))^{N^2}$.

Then

$$\mathcal{V} = \{((\lambda,\mu),\xi) : \sum_{i=1}^N \lambda_i \xi_i = 0 \text{ and } \mu_i \xi_j - \mu_j \xi_i = 0 \text{ for all } i,j\}$$

$$= \{((\lambda,\mu),\xi) : \lambda \perp \xi \text{ and } \mu \| \xi\}.$$

Thus

$$\Lambda = \{(\lambda,\mu) : \lambda \perp \mu\}.$$

By Corollary 9 a necessary condition for $F(v,w)$ to be weakly continuous is that F be affine in each direction (λ,μ) with $\lambda.\mu = 0$. Considering first the case $\mu = 0$, then the case $\lambda = 0$, we conclude that

(1) F is affine in λ for fixed μ

(2) F is affine in μ for fixed λ.

Hence

$$F = \text{affine} + \sum_{i,j} \alpha_{ij}\lambda_i\mu_j.$$

But the quadratic part has to be zero if $\lambda.\mu = 0$. Therefore

$$F = \text{affine} + c\sum_i \lambda_i\mu_i .$$

We will see later that this particular function is weakly continuous.

Example 4 This is exactly the same as the previous example but we suppose $(v^\varepsilon(x),w^\varepsilon(x)) \in K$ a.e. and ask: when is K stable (i.e. when does the weak limit of $(v^\varepsilon,w^\varepsilon)$ belong to K a.e.)? We restrict ourselves to the particular case when K is defined by

$$v^\varepsilon(x) = G(w^\varepsilon(x)), \quad \text{where} \quad G:\mathbb{R}^N \longrightarrow \mathbb{R}^N.$$

We let

$$\phi^\varepsilon \longrightarrow \phi \quad \text{in} \quad W^{1,\infty}(\Omega) \text{ weak } \star .$$

Hence

$$w^\varepsilon = \operatorname{grad}\phi^\varepsilon \longrightarrow w = \operatorname{grad}\phi \quad \text{in} \quad (L^\infty(\Omega))^N \text{ weak } \star.$$

Suppose that

$$v^\varepsilon = G(\operatorname{grad}\phi^\varepsilon) \longrightarrow v \quad \text{in} \quad (L^\infty(\Omega))^N \text{ weak } *$$

and that

$$\operatorname{div} v^\varepsilon = \operatorname{div} G(\operatorname{grad}\phi^\varepsilon) \quad \text{is bounded in} \quad L^\infty(\Omega).$$

The question now is: what are the functions G such that the above hypotheses imply $v = G(\operatorname{grad}\phi)$? Theorem 8 says that a necessary condition is that if $(G(a),a)$ and $(G(b),b)$ differ by a vector in Λ then the whole line segment is in K: i.e. if

$$(G(b) - G(a)).(b - a) = 0 \quad \text{(scalar product)}$$

then

$$((1-\theta)G(a) + \theta G(b),(1-\theta)a + \theta b) \in K \quad \text{for all} \quad \theta \in [0,1].$$

Thus G has to be affine in $[a,b]$.

<u>Remark</u> We will give later (Theorem 23) the answer to this question.

<u>Example 5</u> Let $\Omega = \mathbb{R}^2$, $u_1^\varepsilon, u_2^\varepsilon \longrightarrow u_1, u_2$ in $L^\infty(\Omega)$ weak $*$, and suppose that $\frac{\partial u_1^\varepsilon}{\partial x_1}$ and $\frac{\partial u_2^\varepsilon}{\partial x_2}$ are bounded. Then

$$\mathcal{V} = \{((\lambda_1,\lambda_2),(\xi_1,\xi_2)) = ((\lambda_1,\lambda_2),\xi) : \lambda_1\xi_1 = \lambda_1\xi_2 = 0\}$$

and so

$$\Lambda = \{(\lambda_1,\lambda_2) : \lambda_1 = 0 \quad \text{or} \quad \lambda_2 = 0\}.$$

So a necessary condition for F to be weakly lower semicontinuous is that F be separately convex, and similarly a necessary condition for F to be weakly continuous is that:

$$F = \text{affine} + c\,\lambda_1\lambda_2 .$$

Finally, a necessary condition for K to be stable is that K is closed and that the intersection of K with every line parallel to an axis is an interval.

We will show later that all these conditions are also sufficient.

Example 6 $\Omega = \mathbb{R}^3$; $u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \rightharpoonup u_1, u_2, u_3$ in $L^\infty(\Omega)$ weak $\star$ and $\frac{\partial u_1^\varepsilon}{\partial x_1}$, $\frac{\partial u_2^\varepsilon}{\partial x_2}$ and $\frac{\partial u_3^\varepsilon}{\partial x_3}$ are bounded in $L^\infty(\Omega)$. Then

$$\mathcal{V} = \{((\lambda_1,\lambda_2,\lambda_3),(\xi_1,\xi_2,\xi_3)) = ((\lambda_1,\lambda_2,\lambda_3),\xi) : \lambda_1\xi_1 = \lambda_2\xi_2 = \lambda_3\xi_3 = 0\}.$$

Hence

$$\Lambda = \{(\lambda_1,\lambda_2,\lambda_3) : \text{at least one of the } \lambda_i = 0\}.$$

The necessary condition for F to be weakly continuous is that F be affine if we fix one of the λ_j, and hence

$$F = \text{affine} + \text{quadratic}.$$

But the quadratic part has to be 0 if one of the λ_j is 0. Such a quadratic form has to be 0, and thus F is affine.

In some of the above examples we saw that the necessary condition is also sufficient. In fact this is a special case of the following theorem.

Theorem 11 Suppose we have the following hypotheses

(H1′) $u_i^\varepsilon \rightharpoonup u_i$ in $L^2(\Omega)$ weak for $i = 1,\dots,p$

(H3′) $\sum_{j,k} a_{ijk} \frac{\partial u_j^\varepsilon}{\partial x_k} \in$ compact set (for the strong topology) of $H^{-1}_{loc}(\Omega)$

for $i = 1,\dots,q$.

Then if Q is quadratic and satisfies $Q(\lambda) \geqslant 0$ for all $\lambda \in \Lambda$, and if $Q(u^\varepsilon) \longrightarrow \ell$ in the sense of distributions (ℓ may be a measure), then

$$\ell \geqslant Q(u) \quad \text{(in the sense of measures)}.$$

Remarks (1) $H^{-1}(\Omega)$ is the dual of $H_0^1(\Omega)$.

(2) If Q is quadratic then to say that Q is convex in the directions of Λ is equivalent to saying that $Q(\lambda) \geqslant 0$ for all $\lambda \in \Lambda$.

(3) This theorem says that if F is quadratic then the necessary condition stated in Corollary 10 is also sufficient.

Proof. 1st step: We do a translation

$$v_i^\varepsilon = u_i^\varepsilon - u_i .$$

Then (H1′) and (H3′) imply that

(1) $v_i^\varepsilon \longrightarrow 0$ in $L^2(\Omega)$ weak

(2) $\sum_{j,k} a_{ijk} \frac{\partial v_j^\varepsilon}{\partial x_k} \in$ compact set of $H^{-1}_{loc}(\Omega)$ for $i = 1,\dots,q$.

Since Q is quadratic, there exists a bilinear form $q(a,b)$ such that

$$Q(a) = q(a,a).$$

Therefore

$$Q(v^\varepsilon) = Q(u^\varepsilon - u) = Q(u^\varepsilon) - 2q(u^\varepsilon,u) + Q(u).$$

But

$$Q(u^\varepsilon) \longrightarrow \ell \quad \text{and} \quad q(u^\varepsilon,u) \longrightarrow q(u,u),$$

the second statement holding because $q(a,b)$ is linear in a for fixed b. Therefore since $q(u,u) = Q(u)$

$$Q(v^\varepsilon) \longrightarrow \ell - Q(u) = m .$$

So the theorem is equivalent to the statement that $m \geqslant 0$.

2nd step: Next we perform a localization as follows. We let

$$w^\varepsilon = \phi v_\varepsilon \quad \text{with} \quad \phi \in \mathcal{D}(\Omega).$$

Then

(1) $\operatorname{supp} w_\varepsilon \subset$ compact set of $\mathbb{R}^N$

and

(2) $w_\varepsilon \longrightarrow 0$ in $L^2(\Omega)$ weak.

Also

$$\sum_{j,k} a_{ijk} \frac{\partial w_j^\varepsilon}{\partial x_k} = \phi \sum_{j,k} a_{ijk} \frac{\partial v_j^\varepsilon}{\partial x_k} + \sum_{j,k} a_{ijk} v_j^\varepsilon \frac{\partial \phi}{\partial x_k} .$$

The first term belongs to a compact set of $H^{-1}(\Omega)$ and the second is bounded in $L^2(\Omega)$, and hence belongs to a compact set of $H^{-1}(\Omega)$. Therefore

$$\sum_{j,k} a_{ijk} \frac{\partial w_j^\varepsilon}{\partial x_k} \in \text{compact set of } H^{-1}(\Omega).$$

Hence extracting possibly a subsequence we have

(3) $\displaystyle\sum_{j,k} a_{ijk} \frac{\partial w_j^\varepsilon}{\partial x_k} \longrightarrow 0$ in $H^{-1}(\Omega)$ strong

and

$$Q(w^\varepsilon) \longrightarrow \phi^2 m.$$

3rd step To prove $m \geqslant 0$ it is enough to show that

$$(4) \quad \lim_{\varepsilon \to 0} \int Q(w^{\varepsilon})dx \geqslant 0$$

since this will prove that

$$\lim_{\varepsilon \to 0} \int Q(\phi v^{\varepsilon})dx = \langle m, \phi^2 \rangle \geqslant 0$$

and this for all $\phi \in \mathfrak{D}(\Omega)$, implying that $m \geqslant 0$.

4th step We use the Fourier transform and Plancherel formula. Let us define

$$(5) \quad \hat{w}^{\varepsilon}_j = \mathscr{F}(w^{\varepsilon}_j) = \int_{\mathbb{R}^N} w^{\varepsilon}_j(x) e^{-2\pi i(\xi.x)} dx \ .$$

The Plancherel formula gives

$$(6) \quad \int_{\mathbb{R}^N} v(x)\overline{w}(x) = \int_{\mathbb{R}^N} \hat{v}(\xi)\overline{\hat{w}}(\xi)d\xi$$

where v and $w \in L^2(\mathbb{R}^N)$ are complex valued.

We extend Q from $\mathbb{R}^p$ to $\mathbb{C}^p$ into an Hermitian form. Recall that

$$Q(\lambda) = \sum_{j,k} q_{jk}\lambda_j\lambda_k$$

with $q_{jk} = q_{kj}$ real. We define

$$\tilde{Q}(\lambda) = \sum_{j,k} q_{jk}\lambda_j\overline{\lambda}_k \ .$$

Hence we have

(7) $\mathrm{Re}(\tilde{Q}(\lambda)) \geqslant 0$ if $\lambda \in \Lambda + i\Lambda$, since if $\lambda = \lambda_1 + i\lambda_2$ with $\lambda_1, \lambda_2 \in \Lambda$ then

$$\tilde{Q}(\lambda) = (Q(\lambda_1)+Q(\lambda_2)) + i(q(\lambda_1,\lambda_2)+q(\lambda_2,\lambda_1))$$

(where q was defined by $Q(a) = q(a,a)$) and therefore

$$\mathrm{Re}\,\tilde{Q}(\lambda) = Q(\lambda_1) + Q(\lambda_2) \geqslant 0.$$

By the Plancherel formula,

$$\int_{\mathbb{R}^N} Q(w^\varepsilon)dx = \int_{\mathbb{R}^N} \tilde{Q}(\hat{w}^\varepsilon)d\xi = \int_{\mathbb{R}^N} \mathrm{Re}\,\tilde{Q}(\hat{w}^\varepsilon)d\xi$$

so (4) is equivalent to

$$(8)\quad \varliminf_{\varepsilon\to\infty} \int_{\mathbb{R}^N} \mathrm{Re}\,\tilde{Q}(\hat{w}^\varepsilon)d\xi \geqslant 0.$$

But, since $\operatorname{supp} w_j^\varepsilon \subset$ fixed compact set C of $\mathbb{R}^N$, we have

$$\hat{w}_j^\varepsilon(\xi) = \int_C w_j^\varepsilon(x)e^{-2\pi i(\xi.x)}dx.$$

But $e^{-2\pi i(\xi.x)} \in L^2(C)$ and since $w_j^\varepsilon \longrightarrow 0$ in $L^2(\mathbb{R}^N)$ we deduce that

$$(9)\quad \hat{w}_j^\varepsilon(\xi) \longrightarrow 0 \text{ for all } \xi, \text{ and } |\hat{w}_j^\varepsilon(\xi)| \leqslant M.$$

Therefore

$$\hat{w}_j^\varepsilon \longrightarrow 0 \text{ locally (in } L^2{}_{loc}(\mathbb{R}^N) \text{ strong).}$$

Hence

$$(10)\quad \int_{|\xi|\leqslant r} \tilde{Q}(\hat{w}^\varepsilon)d\xi \longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

Taking the Fourier transform of (3) leads to:

$$(11)\quad \frac{1}{1+|\xi|} \sum_{j,k} a_{ijk}\hat{w}_j^\varepsilon(\xi)\xi_k \longrightarrow 0 \text{ in } L^2(\mathbb{R}^N) \text{ strong for } i = 1,\dots,q.$$

To finish the proof of the theorem we only need the following lemma.

Lemma 12 Suppose that $Q(\lambda) \geqslant 0$ for all $\lambda \in \Lambda$. Then for all $\alpha > 0$ there exists a constant C_α such that

$$\operatorname{Re} \tilde{Q}(\lambda) \geqslant -\alpha|\lambda|^2 - C_\alpha\left(\sum_{i=1}^{q} \left|\sum_{j,k} a_{ijk}\lambda_j\eta_k\right|^2\right)$$

for all $\lambda \in \mathbb{C}^p$, and for all $\eta \in \mathbb{R}^N$ with $|\eta| = 1$.

End of the proof of Theorem 11 We take

$$\lambda = \hat{w}^\varepsilon(\xi) \quad \text{and} \quad \eta = \frac{\xi}{|\xi|} .$$

It follows from (11) that

$$(12) \quad \frac{1}{|\xi|} \sum_{j,k} a_{ijk}\hat{w}_j^\varepsilon \xi_k \longrightarrow 0 \quad \text{in} \quad L^2(|\xi| \geqslant 1) \text{ strong.}$$

Using the lemma, we have

$$\operatorname{Re} \tilde{Q}(\hat{w}^\varepsilon(\xi)) \geqslant -\alpha|\hat{w}^\varepsilon(\xi)|^2 - C_\alpha \sum_i \left|\sum_{j,k} a_{ijk} \frac{\hat{w}_j^\varepsilon(\xi)\xi_k}{|\xi|}\right|^2 .$$

Integrating, we get

$$\int_{|\xi|\geqslant 1} \operatorname{Re} \tilde{Q}(\hat{w}^\varepsilon(\xi))d\xi \geqslant -\alpha \int_{|\xi|\geqslant 1} |\hat{w}^\varepsilon(\xi)|^2 d\xi - C_\alpha \int_{|\xi|\geqslant 1} \sum_i \left|\sum_{j,k} a_{ijk} \frac{\hat{w}_j^\varepsilon(\xi)\xi_k}{|\xi|}\right|^2 d\xi .$$

But by (9) and (12) we deduce that

$$\varliminf_{\varepsilon\to\infty} \int_{|\xi|\geqslant 1} \operatorname{Re} \tilde{Q}(\hat{w}^\varepsilon(\xi))d\xi \geqslant -M\alpha$$

and this is true for all α as small as desired. Hence

$$(13)\quad \varliminf_{\varepsilon\to\infty} \int_{|\xi|\geq 1} \operatorname{Re} \tilde{Q}(\hat{w}^{\varepsilon}(\xi))\,d\xi \geq 0.$$

Then (10) and (13) give the result. □

Proof of Lemma 12. We proceed by contradiction. Suppose there exists an $\alpha_0 < 0$ such that for all $C_\alpha = n$ there exist $\lambda^n \in \mathbb{C}^p$ with $|\lambda^n| = 1$ and $\eta^n \in \mathbb{R}^N$ with $|\eta^n| = 1$ such that

$$(*)\quad \operatorname{Re} \tilde{Q}(\lambda^n) < -\alpha_0 |\lambda^n|^2 - n \sum_i \Big| \sum_{j,k} a_{ijk} \lambda_j^n \eta_k^n \Big|^2 .$$

Extract convergent subsequences such that

$$\lambda^n \longrightarrow \lambda^\infty \quad \text{and} \quad \eta^n \longrightarrow \eta^\infty .$$

Then

$$\sum_i \Big| \sum_{j,k} a_{ijk}\, \lambda_j^n \eta_k^n \Big|^2 \leq \frac{C}{n}$$

where C is a constant. Hence, passing to the limit, we deduce that

$$\sum_{j,k} a_{ijk} \lambda_j^\infty \eta_k^\infty = 0.$$

Therefore

$$\lambda^\infty \in \Lambda + i\Lambda$$

and hence

$$\operatorname{Re} \tilde{Q}(\lambda^\infty) \geq 0.$$

But from $(*)$ we have also

$$\operatorname{Re} \tilde{Q}(\lambda^{\infty}) = \lim_{n\to\infty} \operatorname{Re} \tilde{Q}(\lambda^n) \leqslant -\alpha_0 < 0,$$

and this is a contradiction. □

Corollary 13 If Q is quadratic and satisfies $Q(\lambda) = 0$ for all $\lambda \in \Lambda$, and if $\{u^{\varepsilon}\}$ satisfies (H1) and (H3), then $Q(u^{\varepsilon}) \longrightarrow Q(u)$ in the sense of distributions.

Proof. Extract a subsequence such that

$$Q(u^{\varepsilon}) \longrightarrow \ell \quad (\ell \text{ may a priori be a measure}).$$

Applying the theorem to Q and then to $-Q$, we obtain $\ell = Q(u)$, and since this is true for all subsequences the corollary follows. □

Remark If (H1) and (H3) imply $F(u^{\varepsilon}) \rightharpoonup F(u)$ in $L^{\infty}(\Omega)$ weak $*$ we have shown that F is affine in all directions from Λ. If F is quadratic Theorem 12 shows that this condition is also sufficient.

Notation If $\lambda \in \mathbb{R}^p$, we can define a symmetric matrix $m_{ij} = \lambda_i\lambda_j$. Let $M = \{m_{ij} = \lambda_i\lambda_j : \lambda \in \Lambda\}$.

Corollary 14 If u satisfies (H1) and (H3) (u^{ε} bounded in $(L^{\infty}(\Omega))^p$), if $u_i^{\varepsilon} \rightharpoonup u_i$, $u_i^{\varepsilon}u_j^{\varepsilon} \rightharpoonup v_{ij}$ in $L^{\infty}(\Omega)$ weak $*$ and if $m_{ij} = v_{ij} - u_iu_j$ then $m \in \overline{\operatorname{conv}}(M)$. Conversely if $\bar{u} \in \mathbb{R}^p$ and $m \in \overline{\operatorname{conv}}(M)$, then there exists a sequence $\{u^{\varepsilon}\}$ such that $u^{\varepsilon} \rightharpoonup \bar{u}$, $\sum_{j,k} a_{ijk} \frac{\partial u_j^{\varepsilon}}{\partial x_k} = 0$ for $i = 1,\ldots,q$, and $u_i^{\varepsilon}u_j^{\varepsilon} \rightharpoonup \bar{u}_i\bar{u}_j + m_{ij}$.

Proof. Let Q be positive on Λ and

$$Q(w) = \sum_{i,j} q_{ij}w_iw_j \quad \text{where} \quad q_{ij} = q_{ji}.$$

Then by the theorem we have that

$$\sum_{i,j} q_{ij} v_{ij} = \lim_{\varepsilon\to\infty} Q(u^\varepsilon) \geq Q(u) = \sum_{i,j} q_{ij} u_i u_j \quad \text{a.e.,}$$

and this is equivalent to

$$\sum_{i,j} q_{ij}(v_{ij} - u_i u_j) \geq 0 \quad \text{whenever } Q \text{ is positive on } \Lambda.$$

Thus any linear form

$$s \longmapsto \sum_{i,j} q_{ij} s_{ij}$$

on the space of symmetric matrices that is positive on M is positive on $m_{ij} = v_{ij} - u_i u_j$. Thus $m \in \overline{\text{conv}}(M)$ by the Hahn-Banach theorem. □

Let us now prove the converse. We can suppose $m \in \text{conv}(M)$ so there exist $\theta_\alpha, \lambda^\alpha \in A$, with $\theta_\alpha \geq 0$, $\sum_\alpha \theta_\alpha = 1$, $\lambda^\alpha \in \Lambda$ such that

$$m_{ij} = \sum_\alpha \theta_\alpha \lambda_i^\alpha \lambda_j^\alpha .$$

Let $\xi_\alpha \in \mathbb{R}_J^N$, $\xi_\alpha \neq 0$, be such that $(\lambda^\alpha, \xi^\alpha) \in \mathcal{V}$ and try

$$u^\varepsilon = \sum_\alpha \lambda^\alpha f_\alpha\left(\frac{\xi^\alpha . x}{\varepsilon}\right)$$

where the f^α are periodic with average 0. Then the u^ε satisfy $u^\varepsilon \longrightarrow 0$ and

$$\sum_{j,k} a_{ijk} \frac{\partial u_j^\varepsilon}{\partial x_k} = 0, \quad i = 1,\dots,q.$$

Let us look at the limit of

$$u_i^\varepsilon u_j^\varepsilon = \sum_{\alpha,\beta} \lambda_i^\alpha \lambda_j^\beta f_\alpha\left(\frac{\xi^\alpha . x}{\varepsilon}\right) f_\beta\left(\frac{\xi^\beta . x}{\varepsilon}\right) .$$

Case 1 ξ_α is not parallel to ξ_β if $\alpha \neq \beta$.

Then for $\alpha \neq \beta$, $f_\alpha\left(\frac{\xi^\alpha . x}{\varepsilon}\right) f_\beta\left(\frac{\xi^\beta . x}{\varepsilon}\right) \rightharpoonup 0$. So it is enough to choose f_α so that the average of f_α^2 is θ_α to obtain $u_i^\varepsilon u_j^\varepsilon \rightharpoonup \sum_\alpha \lambda_i^\alpha \lambda_j^\alpha \theta_\alpha$.

Case 2 Some of the ξ^α are parallel.

Then A is the union of disjoint finite sets B_j, and if $\alpha,\beta \in B_j$ then ξ^α and ξ^β are parallel. In this case we have to construct the functions f_α so that $\sum_{\alpha,\beta} \lambda_i^\alpha \lambda_j^\beta f_\alpha\left(\frac{\xi . x}{\varepsilon}\right) f_\beta\left(\frac{\xi . x}{\varepsilon}\right)$ converges to $\sum_{\alpha\in\beta} \theta_\alpha \lambda_i^\alpha \lambda_j^\alpha$. This is possible by taking the f_α to be characteristic functions of disjoint intervals of $[0,1]$ of length θ_α.

Let us now return to some of the examples following Corollary 10 and see how Theorem 11 can be applied.

Example 3 (with the same notation as before)

Let

$$w^\varepsilon \rightharpoonup w \quad \text{in} \quad (L^2(\Omega))^N \text{ weak},$$

$$\operatorname{curl} w^\varepsilon \longrightarrow \operatorname{curl} w \quad \text{in} \quad H^{-1}(\Omega)^{N^2} \text{ strong}.$$

$$v^\varepsilon \rightharpoonup v \quad \text{in} \quad (L^2(\Omega))^N \text{ weak},$$

and

$$\operatorname{div} v^\varepsilon \longrightarrow \operatorname{div} v \quad \text{in} \quad H^{-1}(\Omega) \text{ strong}.$$

Recall that in this example we have

$$\Lambda = \{(\lambda,\mu) \in \mathbb{R}^N \times \mathbb{R}^N : \lambda \perp \mu\}.$$

Now by Theorem 11 we deduce that

$$u^{\varepsilon}.v^{\varepsilon} \longrightarrow u.v \quad \text{in} \quad \mathcal{D}'(\Omega)$$

and that this is the only non-affine function which is weakly continuous.

Example 5 Let $u_1^{\varepsilon}, u_2^{\varepsilon} \longrightarrow u_1, u_2$ in $L^{\infty}(\Omega)$ weak $\star$ and $\frac{\partial u_1^{\varepsilon}}{\partial x_1}$ and $\frac{\partial u_2^{\varepsilon}}{\partial x_2}$ be bounded in $L^{\infty}(\Omega)$. We found that in this case

$$\Lambda = \{(\lambda_1, \lambda_2): \text{ at least one of the } \lambda_j = 0\}.$$

Then by Theorem 11

$$u_1^{\varepsilon} u_2^{\varepsilon} \longrightarrow u_1 u_2$$

and this is the only non-affine function which is weakly continuous. Similarly the necessary condition for weak lower semicontinuity was that F is separately convex. We will now see using Theorem 11 that this condition is also sufficient. In the following three propositions we suppose that the hypotheses of Example 5 hold.

Proposition 15 If F is separately convex and continuous, then

$F(u^{\varepsilon}) \longrightarrow \ell$ implies $\ell \geqslant F(u)$ a.e. .

Proposition 16 If u^{ε} satisfies (H1), (H3) and

(H2) $u^{\varepsilon}(x) \in K$ a.e.,

if K is closed and if all intersections of K with lines parallel to the axes are intervals, then

$$u(x) \in K \quad \text{a.e.} \ .$$

Proposition 17 (1) If $u^\varepsilon \longrightarrow \mu = \int \nu_{x_1 x_2} dx_1 dx_2$ (in the sense of generalized functions), then $\nu_{x_1 x_2}$ is a probability with support in $\overline{K}$ and $\nu_{x_1 x_2}$ is a tensor product $\nu_{x_1}(\lambda_1) \otimes \nu_{x_2}(\lambda_2)$ (as a consequence we have that $\operatorname{supp} \nu$ = product of two sets).

(2) Let ν_1, ν_2 be two probability measures on $\mathbb{R}$ (independent of x). Then there exists a sequence $\{u^\varepsilon\}$ such that

$$u^\varepsilon(x) \in (\operatorname{supp} \nu_1) \times (\operatorname{supp} \nu_2) \quad \text{and} \quad u^\varepsilon \longrightarrow \mu = \int \nu_1 \otimes \nu_2 dx.$$

Proof of Proposition 17 (1) Let $\phi_1(\lambda_1)$ and $\phi_2(\lambda_2)$ be C^1 functions and let

$$v_1^\varepsilon = \phi_1(u_1^\varepsilon) \quad \text{and} \quad v_2^\varepsilon = \phi_2(u_2^\varepsilon).$$

Then we have that

$$\frac{\partial v_1^\varepsilon}{\partial x_1} \in \text{bounded set in } L^\infty(\Omega) \ ,$$

$$\frac{\partial v_2^\varepsilon}{\partial x_2} \in \text{bounded set in } L^\infty(\Omega).$$

The theorem says that:

$$\text{if} \quad v_1^\varepsilon \longrightarrow v_1 \quad \text{and} \quad v_2^\varepsilon \longrightarrow v_2, \quad \text{then} \quad v_1^\varepsilon v_2^\varepsilon \longrightarrow v_1 v_2.$$

We know by Theorem 5 that:

$$\text{if} \quad F(u_1^\varepsilon u_2^\varepsilon) \longrightarrow \ell \quad \text{then} \quad \ell(x) = \langle \nu_x, F(\lambda_1, \lambda_2) \rangle \quad \text{a.e.} \ .$$

Therefore

$$v_1(x) = <\nu_x, \phi_1(\lambda_1)> \quad \text{and} \quad v_2(x) = <\nu_x, \phi_2(\lambda_2)>$$

and by Theorem 11

$$v_1(x)v_2(x) = <\nu_x, \phi_1(\lambda_1)\phi_2(\lambda_2)> \quad \text{a.e.}.$$

Thus a.e. we have that ν_x satisfies

$$(\star) \quad <\nu_x, \phi_1(\lambda_1)\phi_2(\lambda_2)> = <\nu_x, \phi_1(\lambda_1)> <\nu_x, \phi_2(\lambda_2)>$$

and this is still true if ϕ_1 and ϕ_2 are polynomials. Therefore by a density argument we see that it is true a.e. whenever ϕ_1 and ϕ_2 are continuous, and this means exactly that

$$\nu = \nu_1 \otimes \nu_2 .$$

From $(\star)$ we deduce that

$$\operatorname{supp} \nu = (\operatorname{supp} \nu_1) \times (\operatorname{supp} \nu_2).$$

(2) Using Theorem 5 we first construct sequences of functions in one variable $\phi_1^\varepsilon(t)$ and $\phi_2^\varepsilon(t)$ converging in the sense of generalized functions to ν_1 and ν_2. Then we take $u^\varepsilon(x) = (\phi_1^\varepsilon(x_2), \phi_2^\varepsilon(x_1))$. □

<u>Proof of Proposition 16</u> Let $I_1 = \operatorname{supp} \nu_1$ and $I_2 = \operatorname{supp} \nu_2$, so that by Proposition 17 we have $I_1 \times I_2 \subset K$.

The hypotheses on K imply

$$\operatorname{conv}(I_1) \times \operatorname{conv}(I_2) \subset K ,$$

(since taking $a \in I_1$ we have by the definition of K that $\{a\} \times \operatorname{conv}(I_2) \subset K$ and then similarly, for $b \in \operatorname{conv}(I_2)$, that

$\mathrm{conv}(I_1) \times \{b\} \subset K)$.

Furthermore, since

$$\left.\begin{aligned} u_1(x) &= \langle \nu_1, \lambda_1 \rangle \in \mathrm{conv}(I_1), \\ u_2(x) &= \langle \nu_2, \lambda_2 \rangle \in \mathrm{conv}(I_2), \end{aligned}\right\}$$

it therefore follows that

$$u(x) = (u_1(x), u_2(x)) \in K \quad \text{a.e. .} \qquad \square$$

Proof of Proposition 15 We have $F(u_1^\varepsilon, u_2^\varepsilon) \longrightarrow \ell$ and we know by Theorem 5 that

$$\ell(x) = \langle \nu_x, F(\lambda_1, \lambda_2) \rangle \quad \text{a.e. .}$$

We want to show that if F is separately convex then

$$\ell(x) \geqslant F(u(x)) \quad \text{a.e. .}$$

Since

$$\nu_x = \nu_1 \otimes \nu_2 ,$$

we have

$$\begin{aligned} \ell(x) &= \langle \nu_1 \otimes \nu_2, F(\lambda_1, \lambda_2) \rangle \\ &= \langle \nu_1, \langle \nu_2, F(\lambda_1, \lambda_2) \rangle \rangle . \end{aligned}$$

Now applying Jensen's inequality (i.e. if ν is a probability measure and G is convex then $\langle \nu, G(\lambda) \rangle \geqslant G(\langle \nu, \lambda \rangle)$), letting

$$G(\lambda_2) = F(\lambda_1, \lambda_2),$$

which is convex in λ_2, and noting that

$$\langle \nu_2, \lambda_2 \rangle = u_2(x),$$

we obtain

$$\langle \nu_2, F(\lambda_1, \lambda_2) \rangle \geqslant F(\lambda_1, u_2(x)).$$

Using again Jensen's inequality and the fact that

$$\langle \nu_1, \lambda_1 \rangle = u_1(x)$$

we get

$$\ell(x) \geqslant F(u_1(x), u_2(x)) = F(u(x)).$$

$\square$

Remark (related to Proposition 17) If ν is a probability measure on $\mathbb{R}^2$ such that

$$\langle \nu, F(\lambda_1, \lambda_2) \rangle \geqslant F(\langle \nu_1, \lambda_1 \rangle, \langle \nu_2, \lambda_2 \rangle)$$

for all separately convex F, then this does not imply that ν is a tensor product.

Proof. Let $\mathfrak{N}$ be the set of these ν. We want to show that

$$\mathfrak{N} \neq \mathcal{M} = \{\text{tensor products}\}.$$

Define

$$\mathfrak{N}_0 = \{\nu \in \mathfrak{N} : \langle \nu, \lambda_1 \rangle = \langle \nu, \lambda_2 \rangle = 0\}$$

and

$$\mathcal{M}_0 = \{\nu \in \mathcal{M} : \langle \nu, \lambda_1 \rangle = \langle \nu, \lambda_2 \rangle = 0\}.$$

If we show $\mathcal{M}_0 \neq \mathfrak{N}_0$, then the proof will be complete. For this we observe

(1) $\mathfrak{M}_0$ is convex, since it is defined by $\langle\nu, F(\lambda_1,\lambda_2)\rangle \geqslant 0$ for all separately convex F, but

(2) $\mathcal{M}_0$ is not convex, as $\mu^1 = (\delta_{-1} + \delta_1) \otimes \delta_0$ and $\mu^2 = \delta_0 \otimes (\delta_{-1} + \delta_1)$ are in $\mathcal{M}_0$ although no point of the segment joining μ^1 to μ^2 is in $\mathcal{M}_0$.

□

Now, we want to give an example of the fact that under the hypotheses of Theorem 11 the necessary condition given is not sufficient.

Example Let $\Omega \subset \mathbb{R}^2$,

$$u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon \longrightarrow u_1, u_2, u_3 \quad \text{in} \quad L^\infty(\Omega) \text{ weak } \star,$$

and suppose that

$$\frac{\partial u_1^\varepsilon}{\partial x_1}, \; \frac{\partial u_2^\varepsilon}{\partial x_2}, \; \left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) u_3^\varepsilon \quad \text{are bounded in} \quad L^\infty(\Omega).$$

Then

$$\Lambda = \{(\lambda_1,\lambda_2,\lambda_3) : \text{at least 2 of the } \lambda_j \text{ are } 0\}.$$

The necessary condition for weak continuity is satisfied by $u_1^\varepsilon u_2^\varepsilon u_3^\varepsilon$ which is obviously not quadratic. So let us take the following special functions

$$\begin{cases} u_1^\varepsilon = \sin\dfrac{x_2}{\varepsilon} \longrightarrow 0 = u_1 \;, \\ u_2^\varepsilon = \cos\dfrac{x_1}{\varepsilon} \longrightarrow 0 = u_2 \;, \\ u_3^\varepsilon = \sin\dfrac{x_2 - x_1}{\varepsilon} \longrightarrow 0 = u_3 \,. \end{cases}$$

(Note that these satisfy $\dfrac{\partial u_1^\varepsilon}{\partial x_1} = \dfrac{\partial u_2^\varepsilon}{\partial x_2} = \left(\dfrac{\partial}{\partial x_1} + \dfrac{\partial}{\partial x_2}\right) u_3^\varepsilon = 0.$)

Then

$$u_1^\varepsilon u_2^\varepsilon u_3^\varepsilon = \sin\frac{x_2}{\varepsilon}\cos\frac{x_1}{\varepsilon}\left(\sin\frac{x_2}{\varepsilon}\cos\frac{x_1}{\varepsilon} - \sin\frac{x_1}{\varepsilon}\cos\frac{x_2}{\varepsilon}\right)$$

$$= \left(\sin\frac{x_2}{\varepsilon}\right)^2\left(\cos\frac{x_1}{\varepsilon}\right)^2 - \frac{1}{4}\sin\frac{2x_1}{\varepsilon}\sin\frac{2x_2}{\varepsilon} .$$

But

$$\left.\begin{aligned} \sin\frac{2x_1}{\varepsilon} &\longrightarrow 0 \\ \sin\frac{2x_2}{\varepsilon} &\longrightarrow 0 \end{aligned}\right\} \text{ imply } \frac{1}{4}\sin\frac{2x_1}{\varepsilon}\sin\frac{2x_2}{\varepsilon} \longrightarrow 0,$$

and

$$\left.\begin{aligned} \left(\sin\frac{x_2}{\varepsilon}\right)^2 &\longrightarrow \frac{1}{2} \\ \left(\cos\frac{x_1}{\varepsilon}\right)^2 &\longrightarrow \frac{1}{2} \end{aligned}\right\} \text{ imply } \left(\sin\frac{x_2}{\varepsilon}\right)^2\left(\cos\frac{x_1}{\varepsilon}\right)^2 \longrightarrow \frac{1}{4} .$$

Thus

$$u_1^\varepsilon u_2^\varepsilon u_3^\varepsilon \longrightarrow \frac{1}{4} \neq u_1u_2u_3 = 0.$$

We now turn our attention to finding some new necessary conditions, different from those of Theorem 8, and Corollaries 9 and 10. Recall that if $K = \mathbb{R}^p$ and $F:\mathbb{R}^p \longrightarrow \mathbb{R}$ is smooth, then a necessary condition for F to be weakly continuous under hypotheses (H1) and (H3) is:

<u>NC1</u>: $F''(u)(\lambda,\lambda) = 0$ for all $u \in \mathbb{R}^p$, $\lambda \in \Lambda$,

or equivalently,

<u>NC1</u>: $F''(u)(\lambda^\alpha,\lambda^\beta) = 0$ for all $u \in \mathbb{R}^p$, if $(\lambda^\alpha,\xi^\alpha) \in \mathcal{V}_0$ and $(\lambda^\beta,\xi^\beta) \in \mathcal{V}_0$ with $\operatorname{rank}(\xi^\alpha,\xi^\beta) = 1$.

In a similar way we can deduce the following theorem.

Theorem 18 Under the above hypotheses we have the following necessary conditions for sequential weak continuity of F:

NC2: $F'''(u)(\lambda^{\alpha},\lambda^{\beta},\lambda^{\gamma}) = 0$ for all $u \in \mathbb{R}^p$, if $(\lambda^{\alpha},\xi^{\alpha}),(\lambda^{\beta},\xi^{\beta})$, $(\lambda^{\gamma},\xi^{\gamma}) \in \mathcal{V}_0$ with $\operatorname{rank}(\xi^{\alpha},\xi^{\beta},\xi^{\gamma}) \leqslant 2$,

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NCr: $F^{(r+1)}(u)(\lambda^{\alpha_1},\ldots,\lambda^{\alpha_{r+1}}) = 0$ for all $u \in \mathbb{R}^p$, if $(\lambda^{\alpha_j},\xi^j) \in \mathcal{V}_0$ with $\operatorname{rank}(\xi^1,\ldots,\xi^{r+1}) \leqslant r$.

Remark Let Λ generate a subspace E of dimension $\bar{p} \leqslant p$.

(1) If $p = \bar{p}$, and $r = N$, then obviously $\operatorname{rank}(\xi^1,\ldots,\xi^{N+1}) \leqslant N$ since $\xi^j \in \mathbb{R}^N$, and therefore NCN says that

$$F^{(N+1)}(u)(\lambda^{\alpha_1},\ldots,\lambda^{\alpha_{N+1}}) = 0 \quad \text{for all } \lambda^{\alpha_j} \in \Lambda,$$

or equivalently

$$F^{(N+1)}(u)(\lambda^{\alpha_1},\ldots,\lambda^{\alpha_{N+1}}) = 0 \quad \text{for all } \lambda^{\alpha_j} \in E.$$

This means that if F is restricted to a subspace parallel to E, then all $(N+1)^{\text{th}}$ derivatives are 0, and since $E = \mathbb{R}^p$ it follows that F is a polynomial of degree at most N.

(2) If $\bar{p} < p$, then $\Lambda \subset E \neq \mathbb{R}^p$, so we can change coordinates so that $E = \{u \in \mathbb{R}^p : u_{\bar{p}+1} = \cdots = u_p = 0\}$. Hence the quadratic form $\lambda \longrightarrow \lambda_j^2$, $j \geqslant \bar{p}+1$, is 0 on E and so is 0 on Λ. By Corollary 13, this form is weakly continuous, so that $(u_j^{\varepsilon})^2 \rightharpoonup u_j^2$, and $u_j^{\varepsilon} \longrightarrow u_j$ strongly. Hence any continuous function of $(u_{\bar{p}+1},\ldots,u_p)$ is weakly continuous. In this

case a weakly sequentially continuous function is necessarily a polynomial of degree at most N in $u_1,\ldots,u_{\bar{p}}$.

Proof of Theorem 18. Let $u^\varepsilon = \bar{u} + t[\lambda^\alpha\phi^\varepsilon_\alpha(\xi^\alpha.x) + \lambda^\beta\phi^\varepsilon_\beta(\xi^\beta.x) + \lambda^\gamma\phi^\varepsilon_\gamma(\xi^\gamma.x)]$. We suppose that the functions $\phi^\varepsilon_\alpha,\phi^\varepsilon_\beta,\phi^\varepsilon_\gamma$ have zero mean. (Further conditions on the $\{\phi^\varepsilon\}$ will be imposed later in the proof.)

$$u^\varepsilon \rightharpoonup \bar{u}\,.$$

For t small, we use the Taylor expansion of F near $\bar{u}$, namely

$$(1)\quad F(u^\varepsilon) = F(\bar{u}) + tF'(\bar{u})(\lambda^\alpha\phi^\varepsilon_\alpha + \lambda^\beta\phi^\varepsilon_\beta + \lambda^\gamma\phi^\varepsilon_\gamma) + \frac{t^2}{2}F''(\bar{u})(\qquad)^2 + \frac{t^3}{3!}F'''(\bar{u})(\qquad)^3 + o(t^3).$$

But

$$(2)\quad F'(\bar{u})(\lambda^\alpha\phi^\varepsilon_\alpha + \lambda^\beta\phi^\varepsilon_\beta + \lambda^\gamma\phi^\varepsilon_\gamma) \rightharpoonup 0$$

by construction of $\{\phi^\varepsilon\}$. Also

$$F''(\bar{u})(\qquad)^2 = \sum_{a,b} F''(u)(\lambda^a,\lambda^b)\phi^\varepsilon_a(\xi^a.x)\phi^\varepsilon_b(\xi^b.x)\,,$$

where the sum is over a,b from the set $\{\alpha,\beta,\gamma\}$.

Case 1 If $\operatorname{rank}(\xi^a,\xi^b) = 2$, then we have seen that

$$\phi^\varepsilon_a(\xi^a.x)\phi^\varepsilon_b(\xi^b.x) \rightharpoonup 0\,.$$

Case 2 If $\operatorname{rank}(\xi^a,\xi^b) = 1$, then by NC1 we have

$$F''(\bar{u})(\lambda^a,\lambda^b) = 0\,.$$

Therefore (using Case 1 and Case 2)

$$(3)\quad \frac{t^2}{2}F''(\bar{u})(\qquad)^2 \longrightarrow 0 \quad \text{as} \quad \varepsilon \longrightarrow 0.$$

Concerning the term $F'''(\bar{u})(\qquad)^3$, this equals

$$\sum_{a,b,c} F'''(\bar{u})(\lambda^a,\lambda^b,\lambda^c)\phi_a^\varepsilon(\xi^a.x)\phi_b^\varepsilon(\xi^b.x)\phi_c^\varepsilon(\xi^c.x).$$

Case 1a If $\operatorname{rank}(\xi^a,\xi^b) = 1$ we know by NC1 that

$$F''(\bar{u})(\lambda^a,\lambda^b) = 0 \quad \text{for all} \quad u.$$

Differentiating this expression in the direction λ^c, we get

$$F'''(\bar{u})(\lambda^a,\lambda^b,\lambda^c) = 0.$$

Case 1b If $\operatorname{rank}(\xi^a,\xi^b,\xi^c) = 1$, then by Cases 1a, 1b

$$F'''(\bar{u})(\lambda^a,\lambda^b,\lambda^c) = 0.$$

Case 3 The only nontrivial case is therefore $\operatorname{rank}(\xi^a,\xi^b,\xi^c) = 2$. We now choose $\{\phi^\varepsilon\}$ to be such that

$$\phi_\alpha^\varepsilon(\xi^\alpha.x)\phi_\beta^\varepsilon(\xi^\beta.x)\phi_\gamma^\varepsilon(\xi^\gamma.x) \longrightarrow d \neq 0$$

where c is constant. (Such ϕ^ε exist and they can be constructed in a similar way as in the example preceding Theorem 18.)

Hence we have by (1), (2) and (3) that

$$F(u^\varepsilon) \longrightarrow F(\bar{u}) + dt^3F'''(\bar{u})(\lambda^\alpha,\lambda^\beta,\lambda^\gamma) + o(t^3)$$

where d is a non-zero constant, and since F is weakly continuous,

$$F'''(\bar{u})(\lambda^\alpha,\lambda^\beta,\lambda^\gamma) = 0 \quad \text{if} \quad \operatorname{rank}(\xi^\alpha,\xi^\beta,\xi^\gamma) \leqslant 2.$$

By exactly the same method we can prove NCr. □

As an illustration of this theorem we will give two examples.

Example (Ball [1], Reshetnyak [10,11]) Let $\Omega = \mathbb{R}^N$ and $u_{ij}^\varepsilon \longrightarrow u_{ij}$ in $L^\infty(\Omega)$ weak $\star$ where

$$u_{ij}^\varepsilon = \frac{\partial v_i^\varepsilon}{\partial x_j}, \quad i = 1,\dots,r \text{ and } j = 1,\dots,N.$$

So

$$\frac{\partial u_{ij}^\varepsilon}{\partial x_k} - \frac{\partial u_{ik}^\varepsilon}{\partial x_j} \in \text{ bounded set in } L^\infty(\Omega) \quad \text{for all } i,j,k.$$

An easy computation gives

$$\mathscr{V}_0 = \{(\lambda_{ij},\xi) : \lambda_{ij} = \mu_i \xi_j\}.$$

The necessary condition on F to be weakly continuous is then that F has to be multiaffine and alternating, and the only functions satisfying this condition are determinants extracted from the matrix u_{ij}.

It is still an open problem to know for a function F satisfying NC1,...,NCN whether F is weakly continuous, but in this particular example it can be done and we just give an idea of the proof. For this purpose it is more convenient to use the notation of differential forms.

Let

$$w_i^\varepsilon = \sum_j u_{ij}^\varepsilon dx_j .$$

The hypothesis on $\{u_{ij}^\varepsilon\}$ implies that dw_i^ε is a 2-form given by

$$dw_i^\varepsilon = \sum_{j<k} \left(\frac{\partial u_{ij}^\varepsilon}{\partial x_k} - \frac{\partial u_{ik}^\varepsilon}{\partial x_j} \right) dx_j dx_k .$$

By Corollary 13 on quadratic forms, if $w_i^\varepsilon \longrightarrow w_i$ and $w_j^\varepsilon \longrightarrow w_j$, then

$$w_i^\varepsilon \wedge w_j^\varepsilon \longrightarrow w_i \wedge w_j .$$

Furthermore the following equality holds:

$$d(w_i^\varepsilon \wedge w_j^\varepsilon) = dw_i^\varepsilon \wedge w_j^\varepsilon - w_i^\varepsilon \wedge dw_j^\varepsilon .$$

More generally we have the following result, that we just mention without proof:

<u>Lemma 19</u>

Let w^ε be an m-form with dw^ε having bounded coefficients, and let $\bar{w}^\varepsilon$ be an $\bar{m}$-form with $d\bar{w}^\varepsilon$ having bounded coefficients. Then if $w^\varepsilon \longrightarrow w$ and $\bar{w}^\varepsilon \longrightarrow \bar{w}$,

(1) $\quad w^\varepsilon \wedge \bar{w}^\varepsilon \longrightarrow w \wedge \bar{w}$

and

(2) $\quad d(w^\varepsilon \wedge \bar{w}^\varepsilon) = dw^\varepsilon \wedge \bar{w}^\varepsilon + (-1)^m w^\varepsilon \wedge d\bar{w}^\varepsilon .$

<u>Consequences</u> (1) Theorem 1 is a consequence of the above lemma, since the divergence corresponds to an N-1-form and the curl corresponds to a 1-form.

(2) The above example is also a consequence, since we know that

(a) w_i^ε is a 1-form for $i = 1,\ldots,r$ and $r \leqslant N$ and w_i^ε has bounded coefficients, and

(b) $w_i^\varepsilon \longrightarrow w$.

Therefore by the lemma we have that

$$w_1^\varepsilon \wedge w_2^\varepsilon \wedge \ldots \wedge w_r^\varepsilon \longrightarrow w_1 \wedge \ldots \wedge w_r .$$

Example (Maxwell system) The reference space is $\mathbb{R}^4$ and we denote the components of a typical element of $\mathbb{R}^4$ by (x_1,x_2,x_3,t). There are in all 12 components of E,D,B and H. The equations are

$$\begin{cases} \operatorname{div} D = \rho \\ \dfrac{\partial D}{\partial t} - \operatorname{curl} H = \vec{j} \\ \operatorname{div} B = 0 \\ \dfrac{\partial B}{\partial t} + \operatorname{curl} E = 0 \end{cases} .$$

A simple computation gives

$$\Lambda = \{(E,D,B,H): E.B = D.H = E.D - B.H = 0\}$$

so that Λ is a cone of codimension 3.

If $E^\varepsilon, D^\varepsilon, B^\varepsilon$ and H^ε are solutions of the equations and $E^\varepsilon \longrightarrow E$, $D^\varepsilon \longrightarrow D$, $B^\varepsilon \longrightarrow B$ and $H^\varepsilon \longrightarrow H$, then we have

$$\begin{cases} E^\varepsilon.B^\varepsilon \longrightarrow E.B \\ D^\varepsilon.H^\varepsilon \longrightarrow D.H \\ E^\varepsilon.D^\varepsilon - B^\varepsilon.H^\varepsilon \longrightarrow E.D - B.H \end{cases}$$

and actually it can be shown that these are the only weakly continuous functions.

In terms of differential forms, we obtain

$$w^\varepsilon = (E_1^\varepsilon dx_1 dt + E_2^\varepsilon dx_2 dt + E_3^\varepsilon dx_3 dt) + (B_1^\varepsilon dx_2 dx_3 + B_2^\varepsilon dx_3 dx_1 + B_3^\varepsilon dx_1 dx_2)$$

and

$$\bar{w}^\varepsilon = -(H_1^\varepsilon dx_1 dt + H_2^\varepsilon dx_2 dt + H_3^\varepsilon dx_3 dt) + (D_1^\varepsilon dx_2 dx_3 + D_2^\varepsilon dx_3 dx_1 + D_3^\varepsilon dx_1 dx_2) .$$

Then Maxwell's equations are equivalent to

$$\begin{cases} dw^{\varepsilon} = 0 \\ d\bar{w}^{\varepsilon} = \tilde{w}^{\varepsilon} = \rho\, dx_1 dx_2 dx_3 + (j_1 dx_2 dx_3 dt + j_2 dx_3 dx_1 dt + j_3 dx_1 dx_2 dt). \end{cases}$$

It follows that $d\tilde{w}^{\varepsilon} = 0$, which reflects the conservation of charge.

In this example the weakly continuous functions are

$$\begin{cases} w^{\varepsilon} \wedge w^{\varepsilon} = E^{\varepsilon} B^{\varepsilon} dx dt \\ \bar{w}^{\varepsilon} \wedge \bar{w}^{\varepsilon} = -D^{\varepsilon}.H^{\varepsilon} dx dt \\ w^{\varepsilon} \wedge \bar{w}^{\varepsilon} = (E^{\varepsilon} D^{\varepsilon} - B^{\varepsilon} H^{\varepsilon}) dx dt \end{cases}$$

where $dx = dx_1 dx_2 dx_3$.

In the following paragraphs we will give some sufficient conditions for weakly lower semicontinuity, one sufficient condition having been given already in Theorem 11. But we must add that up to now this question has not received a fully satisfactory answer and many problems are still open.

Proposition 20 Suppose that $F(\lambda_1, \ldots, \lambda_p)$ has the form

$$F(\lambda_1, \ldots, \lambda_p) = G(\lambda_i, \lambda_j \lambda_k)$$

where G is a function depending on $p + p^2$ variables. Suppose further that

(1) $G = G(\lambda_i, m_{jk})$ is convex (in all its arguments taken together)

(2) $\sum_{j,k} \frac{\partial G}{\partial m_{jk}} (\lambda, m) \mu_j \mu_k \geqslant 0$ for all $\mu \in \Lambda$.

Then (H1), (H3) and $F(u^{\varepsilon}) \longrightarrow \ell$ in $L^{\infty}(\Omega)$ weak $\star$ imply that $\ell \geqslant F(u)$.

Proof. Let $u_i^{\varepsilon} \longrightarrow u_i$ and $v_{ij}^{\varepsilon} = u_i^{\varepsilon} u_j^{\varepsilon} \longrightarrow v_{ij}$. Then we know by Corollary

14 that $v_{ij} - u_i u_j \in \overline{\underset{\mu \in \Lambda}{\mathrm{conv}}(\mu_i \mu_j)} = M$. Since $F(u^\varepsilon) = G(u_i^\varepsilon, v_{ij}^\varepsilon) \longrightarrow \ell$ and G is convex we deduce that

$$\ell \geqslant G(u_i, v_{ij}) = G(u_i, u_i u_j + m_{ij})$$

for some $m \in M$.

Hence by

$$\ell \geqslant G(u_i, u_i u_j + m_{ij}) \geqslant G(u_i, u_i u_j) = F(u). \qquad \square$$

Related to these sufficient conditions for lower semicontinuity there are some sufficient conditions for the stability of K; since these results are still incomplete we mention them in the form of remarks.

Remark 21 If K can be written in the following form

$$K = \{k \in \mathbb{R}^p : F_\alpha(k) \leqslant 0 \quad \text{for } \alpha \in A \text{ (a countable set) and each } F_\alpha \text{ is weakly lower semicontinuous}\}$$

then K is stable.

Proof. $u^\varepsilon(x) \in K$ a.e. implies $F_\alpha(u^\varepsilon) \leqslant 0$ for all α and since F_α is weakly lower semicontinuous we have

$$F_\alpha(u) \leqslant \ell = \underline{\lim}_{\varepsilon \to \infty} F_\alpha(u^\varepsilon) \leqslant 0 \quad \text{for all } \alpha.$$

Hence $u(x) \in K$ a.e.. $\square$

Remark 22 Suppose that

(1) For each $k \in K$, there exists a quadratic function Q_k such that $Q_k(\lambda) \geqslant 0$ for $\lambda \in \Lambda$ and $Q_k(k' - k) \leqslant 0$ for all $k' \in K$.

(2) (a maximality assumption) If $\bar{k} \in \mathbb{R}^p$ such that $Q_k(\bar{k} - k) \leqslant 0$ for

all $k \in K$, then $\overline{k} \in K$.

Then K is stable.

Proof. This is a direct consequence of Remark 21. □

Example Let $(v_1,\dots,v_N)$ and $(w_1,\dots,w_N)$ be vector fields defined on $\Omega \subset \mathbb{R}^N$ such that $\operatorname{curl} v \in$ bounded set and $\operatorname{div} w \in$ bounded set. Suppose that $(v(x),w(x)) \in K$, and consider the special case when K is defined by $w(x) = G(v(x))$.

The standard hypothesis is the monotonicity (and continuity of G) i.e. for all a,b, $(G(a)-G(b),a-b) \geqslant 0$. Under this hypothesis if $v^\varepsilon \rightharpoonup v$ and $w^\varepsilon = G(v^\varepsilon) \rightharpoonup w$ then

$$(G(v^\varepsilon)-G(b),v^\varepsilon-b) \geqslant 0,$$

and as $w^\varepsilon.v^\varepsilon \longrightarrow w.v$ we get

$$(w-G(b),v-b) \geqslant 0 \quad \text{for all } b.$$

Choose $b = v+\eta c$ with $\eta > 0$, divide by η and let η go to 0. This gives

$$(w-G(v),c) \leqslant 0 \quad \text{for all } c,$$

so that

$$w = G(v).$$

Let us now look at what is said by the necessary condition for K to be stable. As $\Lambda = \{(v,w):v.w = 0\}$ the necessary condition says that G satisfies

(H) If $(G(a)-G(b),a-b) = 0$ then G is affine on $[a,b]$.

Because of the following theorem, which characterizes such G, the condition (H) is also sufficient.

<u>Theorem 23</u> (Jensen [3]) Suppose that G is continuous on $\mathbb{R}^N$ and satisfies (H).

Then either

(1) G is monotone

or (2) $-G$ is monotone

or (3) G is affine.

<u>Remark</u> In the case where G is the derivative of a real function ϕ one can obtain a similar result where hypotheses are made on ϕ instead of ϕ'.

<u>Proposition 24</u> Let ϕ be a real C^1 function on $\mathbb{R}^N$ such that the restriction of ϕ to each straight line is either convex or concave. Then one of the following conditions holds

(1) ϕ is convex

(2) ϕ is concave

(3) ϕ is quadratic.

<u>Proof of Theorem 23</u>.

<u>1st step</u> Suppose $N = 1$. If

$$G(a) = G(b)$$

then

$$(G(a) - G(b), a - b) = 0,$$

so by hypothesis (H) G is affine on $[a,b]$ and thus constant. This shows that G is a nonincreasing or a nondecreasing function.

2nd step The first step shows that if we restrict a,b to be on a line D then $(G(a)-G(b),a-b)$ has a constant sign. This enables us to divide the set of lines into 3 disjoint subsets as follows:

(1) We say that $D \in \mathcal{D}_+$ if

$$(G(a)-G(b),a-b) > 0 \quad \text{for some} \quad a,b \in D.$$

In particular,

$$(G(a)-G(b),a-b) \geqslant 0 \quad \text{for all} \quad a,b \in D.$$

(2) We say that $D \in \mathcal{D}_-$ if

$$(G(a)-G(b),a-b) < 0 \quad \text{for all} \quad a,b \in D.$$

In particular,

$$(G(a)-G(b),a-b) \leqslant 0 \quad \text{for all} \quad a,b \in D.$$

(3) We say that $D \in \mathcal{D}_0$ if

$$(G(a)-G(b),a-b) = 0 \quad \text{for all} \quad a,b \in D.$$

We will also need the facts that, because G is continuous, $\mathcal{D}_+$ and $\mathcal{D}_-$ are open, and that G monotone is equivalent to $\mathcal{D}_-$ empty.

3rd step Suppose $N = 2$. It is enough to prove that if there exists a line $\Delta \in \mathcal{D}_+$ and a line $\Delta' \in \mathcal{D}_-$ then G is affine (if not then $\mathcal{D}_+$ or $\mathcal{D}_-$ is empty and so G or $-G$ is monotone.) As $\mathcal{D}_+$ and $\mathcal{D}_-$ are open we may assume that Δ and Δ' intersect (and take the origin as the point of intersection.)

We now use the fact that $\mathcal{D}_-$ and $\mathcal{D}_+$ are open (and that lines going through 0 form a connected set) to assert that there are 2 lines D and

D' in $\mathcal{D}_0$ intersecting at 0.

Let e_1 and e_2 be unit vectors on D and D'. Hence using (H) we have on D

$$G(\lambda e_1) = G(0) + \lambda e'_1,$$

and on D'

$$G(\lambda e_2) = G(0) + \lambda e'_2,$$

where

$$(e_1, e'_1) = (e_2, e'_2) = 0.$$

Hence

$$\begin{aligned}(G(\lambda_1 e_1) - G(\lambda_2 e_2), \lambda_1 e_1 - \lambda_2 e_2) &= (\lambda_1 e'_1 - \lambda_2 e'_2, \lambda_1 e_1 - \lambda_2 e_2)\\ &= -\lambda_1\lambda_2[(e'_1, e_2) + (e_1, e'_2)] .\end{aligned}$$

<u>1st case</u> If

$$(e'_1, e_2) + (e_1, e'_2) = 0$$

then by (H) G is affine on $[\lambda_1 e_1, \lambda_2 e_2]$ for all λ_1, λ_2 and so

$$G(\alpha e_1 + \beta e_2) = G(0) + \alpha e'_1 + \beta e'_2 .$$

<u>2nd case</u>

$$(e'_1, e_2) + (e_1, e'_2) \neq 0.$$

We may assume for instance that this quantity is strictly positive.

Let $\lambda_1 > 0$ be fixed; then the line joining $\lambda_1 e_1$ to $\lambda_2 e_2$ is in $\mathcal{D}_-$ for $\lambda_2 > 0$ and in $\mathcal{D}_+$ for $\lambda_2 < 0$. Letting λ_2 go to $\pm\infty$ we see that the line through $\lambda_1 e_1$ parallel to D' can be approached by lines in $\mathcal{D}_-$

or in $\mathcal{D}_+$ and so is in $\mathcal{D}_0$, and that the same holds for $\lambda_1 < 0$. So lines parallel to D or D' are in $\mathcal{D}_0$.

Defining

$$\bar{e} = G(e_1 + e_2) - G(0) - e'_1 - e'_2$$

we then have

$$G(\alpha e_1 + \beta e_2) = G(0) + \alpha e'_1 + \beta e'_2 + \alpha\beta\bar{e} \quad \text{for all} \quad \alpha,\beta.$$

But $(G(\lambda e) - G(0), \lambda e)$ must have a sign independent of λ for all e. For $e = \alpha e_1 + \beta e_2$ the coefficient of λ^3 is $\alpha\beta(\bar{e}, \alpha e_1 + \beta e_2)$ which must therefore be 0; so $\bar{e} = 0$ and G is affine.

4th step $N > 2$. This step is almost a direct consequence of the second step. To see this, observe that all lines parallel to a line $D \in \mathcal{D}_\nu$ ($v = +, -$ or 0) belong to the same $\mathcal{D}_\nu$ as D. [Actually it is a well known result that if G is monotone and satisfies

$$(G(a) - G(b), a - b) = 0 \quad \text{for all} \quad a,b \in D$$

then this is true of all lines parallel to D. To see this, suppose $0 \in D$ and let e_1 be a unit vector on D; then

$$G(\lambda e_1) = G(0) + \lambda e'_1$$

with

$$(e_1, e'_1) = 0.$$

As

$$(G(a + \lambda e_1) - G(\mu e_1), a + \lambda e_1 - \mu e_1) \geqslant 0 \quad \text{for all} \quad \mu$$

and the coefficient of μ^2 is 0, the coefficient of μ is 0 and this gives us

$$(G(a+\lambda e_1) - G(0), e_1) = -(e'_1, a)$$

and so

$$(G(a+\lambda e_1) - G(a+\lambda' e_1), (a+\lambda e_1) - (a+\lambda' e_1)) = 0 \quad \text{for all} \quad a, \lambda, \lambda'.]$$

Hence if $D \in \mathcal{D}_+$ and $D' \in \mathcal{D}_-$ we can take parallels D_1, D'_1 to D, D' respectively such that D_1 and D'_1 belong to the same plane, and then apply the third step to conclude that G is affine on every plane parallel to D and D'. Then by continuity, using lines near D and D', the result will be true for all nearby planes and so G is affine. □

Proof of Proposition 24. We divide the set of lines into 3 disjoint subsets as follows:

(1) We say that $D \in \mathcal{D}_+$ if ϕ is convex but not affine on D.

(2) We say that $D \in \mathcal{D}_-$ if ϕ is concave but not affine on D.

(3) We say that $D \in \mathcal{D}_0$ if ϕ is affine on D.

Then the proof follows the same lines as that for Theorem 23. Namely

(i) If in a plane a line in $\mathcal{D}_+$ and a line in $\mathcal{D}_-$ intersect then ϕ is quadratic in this plane.

(ii) 2 parallel lines belong to the same subset.

(iii) If both $\mathcal{D}_+$ and $\mathcal{D}_-$ are nonempty then ϕ is quadratic. □

7 APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS

To finish these notes, we explain in an example how the theory of compensated compactness can be applied to partial differential equations.

We consider the differential equation

$$(\star) \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

where f is a sufficiently smooth function. Let $\Omega \subset \mathbb{R}^2$. Suppose we have a sequence of solutions u^ε, i.e.

$$(\star)^\varepsilon \quad \frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon) = 0$$

and suppose that $u^\varepsilon \longrightarrow u$ in $L^\infty(\Omega)$ weak $\star$. The question is: does u satisfy the equation $(\star)$, or in other words, does $f(u^\varepsilon) \longrightarrow f(u)$ in $L^\infty(\Omega)$ weak $\star$?

<u>Proposition 25</u> The answer to the above question is in general no, unless f is an affine function.

<u>Proof</u>. Take a and b such that f is not affine on $[a,b]$. Then there exists $\theta \in (0,1)$ such that

$$f((1-\theta)a + \theta b) \neq (1-\theta)f(a) + \theta f(b).$$

Let

$$s = \frac{f(b) - f(a)}{b - a}$$

and

$$u^\varepsilon = \chi^\varepsilon(x - st),$$

where χ^ε takes values a or b, and is such that

$$\chi^\varepsilon \longrightarrow (1-\theta)a + \theta b.$$

Then u^ε is a solution of $(\star)^\varepsilon$ and

$$f(u^\varepsilon) \longrightarrow (1-\theta)f(a) + \theta f(b) \neq f((1-\theta)a + \theta b) = f(u). \qquad \square$$

But equation $(\star)$ is a nonlinear hyperbolic equation, and it is known that additional properties must be imposed for a solution: these are called entropy conditions. We will now see that if we add suitable entropy conditions, then the answer to the above question is yes.

<u>Theorem 26</u> Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and let $f \in C^1$. Let $\{u^\varepsilon\}$ be a sequence of functions satisfying $u^\varepsilon \longrightarrow u$ in $L^\infty(\Omega)$ weak $\star$, and suppose that for all convex functions ϕ, we have

$$\text{(E)} \quad \frac{\partial}{\partial t}\phi(u^\varepsilon) + \frac{\partial}{\partial x}\psi(u^\varepsilon) \in (\text{compact set of } H^{-1}(\Omega)) + (\text{bounded set of } \mathcal{M}(\Omega))$$

where

$$\psi'(\lambda) = f'(\lambda)\phi'(\lambda),$$

and where $\mathcal{M}(\Omega)$ denotes the space of measures. Then

(1) $f(u^\varepsilon) \longrightarrow f(u)$ in $L^\infty(\Omega)$ weak $\star$

and

(2) $f'(u^\varepsilon) \longrightarrow f(u)$ in $L^p(\Omega)$ strong for all $p < \infty$.

Moreover, if there is no interval on which f is affine, then

$$u^\varepsilon \longrightarrow u \quad \text{in } L^p(\Omega) \text{ strong for all } p < \infty.$$

<u>Remark 27</u> Taking $\phi(\lambda) = \lambda$ and $\psi(\lambda) = f(\lambda)$ we have in particular assumed that

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x}f(u^\varepsilon) \in (\text{compact set of } H^{-1}(\Omega)) + (\text{bounded set of } \mathcal{M}(\Omega)).$$

To be able to apply our theory to Theorem 26, we need to modify condition (E) of the theorem, but this can be done using

Lemma 28 (Murat [9]) If a sequence $\{g^\varepsilon\}$ is such that $g^\varepsilon \in$ (compact set of $H^{-1}(\Omega)$) + (bounded set of $\mathcal{M}(\Omega)$), and if $g^\varepsilon \in$ bounded set of $W^{-1,\infty}(\Omega)$, then $g^\varepsilon \in$ compact set of $H^{-1}_{loc}(\Omega)$.

Proof. Consider the following Dirichlet problem

$$\begin{cases} -\Delta v^\varepsilon = g^\varepsilon & \text{on } \Omega \\ v^\varepsilon = 0 & \text{on } \partial\Omega . \end{cases}$$

We know that for $\Omega \subset \mathbb{R}^N$,

$$\mathcal{M}(\Omega) \text{ is compactly imbedded in } W^{-1,q}(\Omega) \text{ if } q < \frac{N}{N-1} .$$

Since

$$g^\varepsilon = g^{\varepsilon_1} + g^{\varepsilon_2},$$

where $g^{\varepsilon_1} \in$ bounded set of $\mathcal{M}(\Omega)$ and $g^{\varepsilon_2} \in$ compact set of $H^{-1}(\Omega)$, it follows by the regularity theory for the Dirichlet problem that

$$v^\varepsilon = v^{\varepsilon_1} + v^{\varepsilon_2},$$

where $v^{\varepsilon_1} \in$ compact set of $W^{1,q}_{loc}(\Omega)$ and $v^{\varepsilon_2} \in$ compact set of $H^1(\Omega)$.

Since $v^\varepsilon \in$ bounded set of $W^{1,r}(\Omega)$ for large $r < \infty$ we obtain $v^\varepsilon \in$ compact set of $H^1_{loc}(\Omega)$ and hence $g^\varepsilon \in$ compact set of $H^{-1}_{loc}(\Omega)$. □

Proof of Theorem 26.

1st step Suppose $\{u^\varepsilon\}$ is a sequence of classical solutions i.e., for all convex ϕ,

$$\frac{\partial}{\partial t}\phi(u^{\varepsilon}) + \frac{\partial}{\partial x}\psi(u^{\varepsilon}) = 0,$$

and suppose that

$$u^{\varepsilon} \rightharpoonup u, \; f(u^{\varepsilon}) \rightharpoonup \xi, \; \phi(u^{\varepsilon}) \rightharpoonup v, \; \psi(u^{\varepsilon}) \rightharpoonup w.$$

Then by the compensated compactness theory (Corollary 13) we deduce that

$$(1) \qquad u^{\varepsilon}\psi(u^{\varepsilon}) - f(u^{\varepsilon})\phi(u^{\varepsilon}) \rightharpoonup uw - \xi v.$$

In the sense of generalized functions we have that

$$u^{\varepsilon} \rightharpoonup \mu = \int_{\Omega} \nu_{x,t} dxdt,$$

and

$$u(x,t) = \langle \nu_{x,t}, \lambda \rangle \quad \text{a.e.},$$

$$\xi(x,t) = \langle \nu_{x,t}, f(\lambda) \rangle \quad \text{a.e.},$$

$$v(x,t) = \langle \nu_{x,t}, \phi(\lambda) \rangle \quad \text{a.e.},$$

$$w(x,t) = \langle \nu_{x,t}, \psi(\lambda) \rangle \quad \text{a.e.} \; .$$

Hence by (1) we deduce

$$(2) \qquad \langle \nu, \lambda\psi(\lambda) - \phi(\lambda)f(\lambda) \rangle = \langle \nu, \lambda \rangle \langle \nu, \psi(\lambda) \rangle - \langle \nu, f(\lambda) \rangle \langle \nu, \phi(\lambda) \rangle$$

for all convex ϕ, where $\nu = \nu_{x,t}$.

<u>2nd step</u> We want to show that (2) implies that $\operatorname{supp}\nu$ is contained in an interval on which f is affine, that $\xi = f(u)$ and that $f'(\lambda)$ is constant on $\operatorname{supp}\nu$ (so that $f'(u^{\varepsilon}) \longrightarrow f'(u)$ strongly). Rewriting (2) using the relations $u = \langle \nu, \lambda \rangle$ and $\xi = \langle \nu, f(\lambda) \rangle$, we obtain

(3) $\quad <\nu,(\lambda - u)\psi(\lambda) - (f(\lambda) - \xi)\phi(\lambda)> = 0$ for all convex ϕ.

Take

$$\phi(\lambda) = |\lambda - u| ,$$

$$\psi(\lambda) = \begin{cases} f(u) - f(\lambda) & \text{if } \lambda \leqslant u \\ f(\lambda) - f(u) & \text{if } \lambda \geqslant u . \end{cases}$$

For this choice

$$(\lambda - u)\psi - (f(\lambda) - \xi)\phi = (\xi - f(u))|\lambda - u| ,$$

and thus we have

(4) $\quad (\xi - f(u)) <\nu,|\lambda - u|> = 0.$

Hence

$$\begin{cases} \text{If } <\nu,|\lambda - u|> \neq 0 \text{ then } \xi = f(u). \\ \text{If } <\nu,|\lambda - u|> = 0 \text{ then } \nu = \delta_u \text{ and hence } \xi = f(u). \end{cases}$$

So we have proved that $\xi = f(u)$ a.e. . Making a translation such that at x,t we have $u = f(u) = 0$, (3) becomes

(5) $\quad <\nu,\lambda\psi(\lambda) - f(\lambda)\phi(\lambda)> = 0$

and since $u = <\nu,\lambda>$ and $f(u) = \xi = <\nu,f(\lambda)>$,

(6) $\quad <\nu,\lambda> = <\nu,f(\lambda)> = 0.$

Let $\operatorname{conv}(\operatorname{supp}\nu) = [\alpha,\beta]$. Since (6) holds we have $\alpha \leqslant 0 \leqslant \beta$. Observe first that if $\alpha = 0$ then $\beta = 0$ and hence ν is a Dirac measure. So the problem is solved in this case. Therefore there is no loss of generality in supposing $\alpha < 0 < \beta$.

Let $A \in BV$ (the set of functions of bounded variation) be defined by $A' = \lambda\nu$. Because of (6) one can choose A to be 0 outside $[\alpha,\beta]$. Similarly define B by $B' = f(\lambda)\nu$ and $B = 0$ outside $[\alpha,\beta]$. Then $A(\lambda) < 0$ on $]\alpha,\beta[$ because α and β are the endpoints of the support of ν. Equation (5) gives

$$\langle A',\psi\rangle - \langle B',\phi\rangle = 0 \quad \text{for all convex } \phi.$$

Hence

$$-\langle A,\psi'\rangle + \langle B,\phi'\rangle = 0\,,$$

and so

$$\langle B - Af',\phi'\rangle = 0.$$

Here ϕ' can be the derivative of any convex function, that is ϕ' can be any increasing function. By linearity the same will be true for every difference of increasing functions and so for any smooth function. Therefore

(7) $\quad B - Af' = 0.$

But by the definitions of A and B we have

(8) $\quad f(\lambda)A' - \lambda B' = 0.$

Hence

$$(f(\lambda)A - \lambda B)' = (f(\lambda)A' - \lambda B') + (f'(\lambda)A - B) = 0.$$

Therefore

(9) $\quad f(\lambda)A - \lambda B = \text{constant}.$

But since A,B have compact support in $[\alpha,\beta]$ the constant has to be 0, i.e.

(10) $f(\lambda)A - \lambda B = 0.$

Using (10) and (7) we deduce (as $A \neq 0$ on $]\alpha,\beta[$)

(11) $f(\lambda) - \lambda f'(\lambda) = 0$ on $[\alpha,\beta]$.

Therefore

(12) $f(\lambda) = c\lambda$ on $[\alpha,\beta]$,

and therefore $f'(u^\varepsilon) \longrightarrow f'(u)$ strongly. □

Example 1 Consider the equation

$$\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\varepsilon) - \varepsilon \frac{\partial^2 u^\varepsilon}{\partial x^2} = 0$$

where the u^ε are bounded in $L^\infty(\Omega)$ and $\sqrt{\varepsilon}\, \frac{\partial u^\varepsilon}{\partial x}$ is bounded in $L^2(\Omega)$.
Then the u^ε satisfy

$$\frac{\partial}{\partial t}\phi(u^\varepsilon) + \frac{\partial}{\partial x}\psi(u^\varepsilon) - \varepsilon \frac{\partial^2}{\partial x^2}\phi(u^\varepsilon) + \varepsilon\phi''(u^\varepsilon)\left(\frac{\partial u^\varepsilon}{\partial x}\right)^2 \leqslant 0.$$

Using the above theorem we conclude that if $u^\varepsilon \longrightarrow u$, then u is a solution of

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$$

and u satisfies the entropy condition

$$\frac{\partial}{\partial t}\phi(u) + \frac{\partial}{\partial x}\psi(u) \leqslant 0.$$

Example 2 Consider the following equation where a is continuous:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) + a(u) = 0.$$

If a has the right sign, then there is a bound in L^∞ obtained using approximation by a viscosity term. But if a' does not exist, the question then is how to obtain the BV bound. The theorem shows convergence under the following hypothesis. If f is affine in an interval $[\alpha,\beta]$, then a is affine on the same interval. The same holds for the system

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial}{\partial x} f(u) + a(u,v) = 0 \\[2ex] \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x} g(v) + b(u,v) = 0. \end{cases}$$

If f is affine on $[\alpha,\beta]$ and g affine on $[\gamma,\delta]$ we assume that a and b are affine on $[\alpha,\beta] \times [\gamma,\delta]$. Then if one can obtain L^∞ bounds on $u^\varepsilon, v^\varepsilon$ by the viscosity method we obtain convergence to a solution.

Remark 30 In the case where f is convex one can obtain the theorem using only one strictly convex entropy ϕ. In this case one can prove the inequality

$$(\psi(b) - \psi(a))(b - a) \geqslant (\phi(b) - \phi(a))(f(b) - f(a)) \quad \text{for all } a,b.$$

Then using $b = u^\varepsilon$ and $a = u$ we can take the weak limit (observe that the combination $u^\varepsilon \psi(u^\varepsilon) - \phi(u^\varepsilon) f(u^\varepsilon)$ is the very one about which we know something nice) and obtain

$$(v - \phi(\mathbf{u}))(\xi - f(u)) \leqslant 0$$

(where $f(u^\varepsilon) \rightharpoonup \xi$ and $\phi(u^\varepsilon) \rightharpoonup v$). By the convexity of ϕ and f we know that $v \geqslant \phi(u)$ and $\xi \geqslant f(u)$, so we must have

$$(v - \phi(u))(\xi - f(u)) = 0 \quad \text{a.e.}\ .$$

If $\xi - f(u) \neq 0$ on a set A then $v = \phi(u)$ on A but, as ϕ is strictly convex, this implies strong convergence of u^{ε} to u on A and so $\xi = f(u)$ on A.

Remark 31 One can try the same method for systems. For instance

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial x} = 0 \\ \\ \dfrac{\partial v}{\partial t} - \dfrac{\partial}{\partial x} f(u) = 0 \end{cases} \qquad \text{with } f' > 0 \text{ for hyperbolicity.}$$

In this case there are entropies $\phi(u,v)$: they are solutions of

$$\frac{\partial^2 \phi}{\partial u^2} = f'(u)\frac{\partial^2 \phi}{\partial v^2}\ .$$

Then

$$\begin{cases} \dfrac{\partial \phi}{\partial u} = \dfrac{\partial \psi}{\partial v} \\ \\ f'(u)\dfrac{\partial \phi}{\partial v} = \dfrac{\partial \psi}{\partial u} \end{cases}$$

defines a function ψ such that

$$\frac{\partial \phi}{\partial t} - \frac{\partial}{\partial x}\psi = 0$$

for smooth solutions (and $\dfrac{\partial \phi}{\partial t} - \dfrac{\partial \psi}{\partial x} \leqslant 0$ if ϕ is convex for "good" weak solutions). Then the problem is to characterize a probability measure ν on $\mathbb{R}^2$ such that

$$\langle \nu, \phi_1\psi_2 - \phi_2\psi_1 \rangle = 0 \quad \text{for all pairs } (\phi_i, \psi_i).$$

This problem is open.

A simple conjecture is that if f is strictly convex then ν must be a Dirac measure.

COMMENTS

The interpretation of Theorem 1 using differential forms was pointed out to me by J.Robbin in 1974.

As pointed out by C.Goulaouic, there are similarities with wave front sets introduced by Hörmander to define some products of distributions and the set $\mathcal{V}_0$ used here to show that some quadratic forms are (sequentially) weakly continuous.

Theorem 23 and Proposition 24 were conjectured by J.M.Ball and myself in 1976. In October 1977 we proved the case $N = 2$ of Proposition 24 and R.Jensen found the way to extend it to $N > 2$. His proof of Theorem 23 follows our sketch.

Necessary and sufficient conditions for weak sequential lower semicontinuity were introduced by Morrey [7]. J.M.Ball showed that the characterization of weakly continuous functions could be used to prove existence theorems in nonlinear elasticity (Ball [1]).

In the case of a convex nonlinearity, by using an explicit formula for the solution, Lax proved that the semigroup generated by the equation $\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0$ is weakly continuous (Lax [4]). Theorem 26 shows that the same result is true without a convexity assumption.

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