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Article

Generating Exact Solution of Three Dimensional Coupled Unsteady Nonlinear Generalized Viscous Burgers' Equations

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Abstract: In this paper, we derive a general analytical solution of three dimensional (3D) homogenous coupled unsteady nonlinear generalized viscous burgers' equation via Hopf-Cole transformation and separation of variable method. We also find the exact solution in the case of (3+1) - dimensional, 2D coupled, (2+1) - dimensional and (1+1) - D Burgers' equations.

Keywords: Generalized Burgers' equations, Hopf-Cole transformation, separation of variable, analytic solutions

Mathematics Subject Classification (2000): 35Q35, 65N06, 34K28

1. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used to describe complex phenomena in various fields of science and engineering. As it is well known, solving a nonlinear physical system is much more difficult than solving the linear ones. The investigation of exact solutions of NLPDEs plays an important role in study of the nonlinear physical phenomena. Burgers' equation is one of the very few nonlinear PDEs that can be linearized and can be solved exactly using Hopf-Cole transformation [1]. The Burgers' equation which was formulated by Batman [2] and treated later by Burger [3], is a nice example of parabolic and hyperbolic PDEs, e.g. with viscous term it is parabolic whereas without viscous term it is hyperbolic. In later case it possesses discontinuous solutions (shock waves) due to the nonlinear term and even if smooth initial condition is considered the solution may be discontinuous after a finite time. Burgers' equation is also called the nonlinear advection-diffusion equation which can be regarded as a qualitative approximation of the Navier-Stokes equations. Burger [3] and Cole [4] found that this equation describes various kinds of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in viscous fluid. This one dimensional (1D) Burgers' equation is also used in many physical applications such as modeling of water in unsaturated oil, dynamics of soil in water, cosmology and seismology; see references [5-7].The generalized 1D Burgers' equation with complex nonlinear coefficients is considered as a model for the nonlinear kinematic wave of the debris flow. 1D coupled viscous Burgers' equation was derived by Esipov [8] to study the model of poly-dispersive sedimentation. This system of coupled equation is a simple model of sedimentation or evolution of scaled volume concentrations of two kinds of particles in fluid suspensions or colloids under the effect of gravity. Two dimensional (2D) Burgers' equation is used in many natural applications such as modeling of gas dynamics, investigating the shallow water waves [9]etc, whereas the three dimensional (3D) Burgers' equation is used as an adhesive model for the large scale structure formation in the universe [10] (i.e. distribution of galaxies on the scales roughly 1 M pc to 100 M pc, where 1 M pc=10^6 pc$\equiv 3\times 10^{24}$ cm is a unit length commonly used in cosmology).

Analytic solutions of 1D Burgers' equations are obtained by many methods such as an explicit Backlund transformation method, tanh-coth method, differential transform method, variational iteration method, homotopy analysis method. The exact solution of 2D coupled Burgers' equations was first proposed by Fletcher [1] using Hopf-Cole transformation, which he used to generate different sets of initial and boundary conditions for the numerical implementation purpose in two dimension where equations are the coupled system of equations. Many other researchers also obtained exact solution for 2D, (2+1) -dimensional Burgers' equations as well as for the (3+1)-dimensional Burgers' equations, refer [11-30]. As we know that simulation of Burgers' equation is a natural and first step towards developing methods for the computation of complex flows. In the past a few decades, it has become customary to test new approaches in computational fluid dynamics by applying them to 1D and 2 D coupled Burgers' equations as exact solutions are available in these equations; refer [31-40] but for 3D nonlinear coupled viscous Burgers equations, till now no one has tested new schemes. The reason behind this is that no one has proposed the exact solution for 3D coupled Burgers' equations.

The purpose of this paper is to derive a general analytical solution of three dimensional unsteady homogenous generalized coupled Burgers' equations. Exact solutions of (3+1) - dimensional, 2D coupled, (2+1)-dimensional and 1D Burgers' equations are also found.

2. Three dimensional Nonlinear Coupled Burgers' Equations

Consider the three dimensional unsteady non-homogenous nonlinear coupled generalized Burgers' equation of the form:

$$\frac{\partial \vec{q}}{\partial t} + \mu\left(\vec{q}.\nabla\vec{q}\right) = \kappa\left(\Delta^2\vec{q}\right) + \vec{\mathrm{h}} \qquad (1)$$

where $\vec{q} \equiv \vec{q}\left(\vec{r},t\right) = (u,v,w)$ is the velocity vector to be determined and $\vec{r} = (x,y,z) \in \Omega \subseteq \mathbb{R}^3$; $\Omega = \{(x,y,z): a \leq x \leq b,\ c \leq y \leq d, e \leq z \leq f\}$ is the computational domain and $\partial\Omega$ is its boundary; μ and κ are arbitrary positive constants;

$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$; $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three dimensional Laplace operator; $\vec{h} \equiv \vec{h}\left(\vec{r},t\right) = (h_1,h_2,h_3)$ is the external forcing term.

In the absence of the force term, i.e. when $\vec{h} = \vec{0}$, Eq. (1) becomes homogenous coupled Burgers' equation:

$$\frac{\partial \vec{q}}{\partial t} + \mu\left(\vec{q}.\nabla\vec{q}\right) = \kappa\left(\Delta^2\vec{q}\right) \qquad (2)$$

In Cartesian coordinates, Eq. (1) is written as:

$$u_t + \mu\left(uu_x + vu_y + \omega u_z\right) = \kappa\left(u_{xx} + u_{yy} + u_{zz}\right) + h_1(x,y,z,t), \qquad (3)$$
$$v_t + \mu\left(uv_x + vv_y + \omega v_z\right) = \kappa\left(v_{xx} + v_{yy} + v_{zz}\right) + h_2(x,y,z,t), \qquad (4)$$
$$\omega_t + \mu\left(u\omega_x + v\omega_y + \omega\omega_z\right) = \kappa\left(\omega_{xx} + \omega_{yy} + \omega_{zz}\right) + h_3(x,y,z,t), \qquad (5)$$

Similarly, Eq. (2) can be written as:

$$u_t + \mu\left(uu_x + vu_y + \omega u_z\right) = \kappa\left(u_{xx} + u_{yy} + u_{zz}\right), \qquad (6)$$
$$v_t + \mu\left(uv_x + vv_y + \omega v_z\right) = \kappa\left(v_{xx} + v_{yy} + v_{zz}\right), \qquad (7)$$
$$\omega_t + \mu\left(u\omega_x + v\omega_y + \omega\omega_z\right) = \kappa\left(\omega_{xx} + \omega_{yy} + \omega_{zz}\right), \qquad (8)$$

where $u(x,y,z,t)$, $v(x,y,z,t)$ and $\omega(x,y,z,t)$ are the velocity components; u_t is the unsteady term; uu_x is the nonlinear convection term; u_{xx} is the diffusion term;

$u_t = \frac{\partial u}{\partial t}$, $v_t = \frac{\partial v}{\partial t}$; $\omega_t = \frac{\partial \omega}{\partial t}$; $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_z = \frac{\partial u}{\partial z}$, $v_x = \frac{\partial v}{\partial x}$, $v_y = \frac{\partial v}{\partial y}$, $v_z = \frac{\partial v}{\partial z}$, $\omega_x = \frac{\partial \omega}{\partial x}$,

$\omega_y = \frac{\partial \omega}{\partial y}$, $\omega_z = \frac{\partial \omega}{\partial z}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{yy} = \frac{\partial^2 u}{\partial y^2}$, $u_{zz} = \frac{\partial^2 u}{\partial z^2}$, $v_{xx} = \frac{\partial^2 v}{\partial x^2}$, $v_{yy} = \frac{\partial^2 v}{\partial y^2}$, $v_{zz} = \frac{\partial^2 v}{\partial z^2}$, $\omega_{xx} = \frac{\partial^2 \omega}{\partial x^2}$,

$\omega_{yy} = \frac{\partial^2 \omega}{\partial y^2}$, $\omega_{zz} = \frac{\partial^2 \omega}{\partial z^2}$.

Given any particular solution to the non-homogenous Eq. (1) another solution can be obtained by adding any non-zero solution of the homogenous Eq. (2). Thus, the most general solution to the non-homogenous Eq. (1) can be written as a sum of a homogenous solution $\vec{q}^{\,H}$, and a particular solution $\vec{q}^{\,P}$, with solution

$$\vec{q}^{\,N} = \vec{q}^{\,H} + \vec{q}^{\,P} \tag{9}$$

the general solution.

The following steps gives $\vec{q}^{\,N}$:

(i) Find an exact solution $\vec{q}$ of homogenous Eq. (2).

(ii) Use it as an initial and boundary conditions.

(iii) Fix a forcing term $\vec{h}$.

(iv) Find any particular solution that satisfy non-homogenous Eq. (1) with the initial and boundary conditions from step (ii), call it $\vec{q}^{\,P}$.

(v) Solve the homogenous Eq. (2) using the initial and boundary conditions, call this solution $\vec{q}^{\,H}$.

(vi) The general solution to the non-homogenous Eq. (1) is $\vec{q}^{\,N} = \vec{q}^{\,H} + \vec{q}^{\,P}$.

So the main task is to find out the exact solution of the homogenous Eq. (2). In the next section we derive an analytical solution of the homogenous Burgers' equation (2) using Hopf-Cole transformation [1] and separation of variable method. The method is simple and straight forward.

3. Generating a General Analytical Solution to 3D Unsteady Homogenous Coupled Burgers' Equation

Analytical solution of three dimensional homogenous coupled Burgers' equations can be generated via Hopf-Cole transformation [1]:

$$u = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_x}{\phi} \tag{10}$$

$$v = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_y}{\phi} \tag{11}$$

$$\omega = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_z}{\phi} \tag{12}$$

Further suppose that

$$u = f_1(\phi);\ v = f_2(\phi);\ w = f_3(\phi); \tag{13}$$

From the Eqs. (7) and (13), we get

$$f_1'(\phi)\phi_t + \mu\left(f_1(\phi)\,f_1'(\phi)\phi_x + f_2(\phi)\,f_1'(\phi)\phi_y + f_3(\phi)\,f_1'(\phi)\phi_z\right) = \kappa\left[f_1''(\phi)\phi_x^{\,2} + f_1'(\phi)\phi_{xx} + f_1''(\phi)\phi_y^{\,2} + f_1'(\phi)\phi_{yy} + f_1''(\phi)\phi_z^{\,2} + f_1'(\phi)\phi_{zz}\right] \tag{14}$$

Eqs. (8) and (13) together gives:

$$f_2'(\phi)\phi_t + \mu\left(f_1(\phi)\,f_2'(\phi)\phi_x + f_2(\phi)\,f_2'(\phi)\phi_y + f_3(\phi)\,f_2'(\phi)\phi_z\right) = \kappa\left[f_2''(\phi)\phi_x^{\,2} + f_2'(\phi)\phi_{xx} + f_2''(\phi)\phi_y^{\,2} + f_2'(\phi)\phi_{yy} + f_2''(\phi)\phi_z^{\,2} + f_2'(\phi)\phi_{zz}\right] \tag{15}$$

Similarly, from Eqs. (9) and (13), we get

$$f_3'(\phi)\phi_t + \mu\left(f_1(\phi)\,f_3'(\phi)\phi_x + f_2(\phi)\,f_3'(\phi)\phi_y + f_3(\phi)\,f_3'(\phi)\phi_z\right) = \kappa\left[f_3''(\phi)\phi_x^{\,2} + f_3'(\phi)\phi_{xx} + f_3''(\phi)\phi_y^{\,2} + f_3'(\phi)\phi_{yy} + f_3''(\phi)\phi_z^{\,2} + f_3'(\phi)\phi_{zz}\right] \tag{16}$$

where $f_i'(\phi) = \dfrac{\partial f_i(\phi)}{\partial \phi}$, i=1,2,3 ; $f_i''(\phi) = \dfrac{\partial^2 f_i(\phi)}{\partial \phi^2}$, i=1,2,3.

We assume that ϕ is bounded so that $f_1'(\phi)$, $f_2'(\phi)$ and $f_3'(\phi)$ are all nonzero functions. Taking any of Eqs. (14), (15) and (16), the same solution is obtained, so we take any one equation, say Eq. (14), divide it by $f_1'(\phi)$ both sides, we get

$$\phi_t + \mu\left(f_1(\phi)\phi_x + f_2(\phi)\phi_y + f_3(\phi)\phi_z\right) = \kappa\left[\frac{f_1''(\phi)}{f_1'(\phi)}\left(\phi_x^{\,2} + \phi_y^{\,2} + \phi_z^{\,2}\right) + \left(\phi_{xx} + \phi_{yy} + \phi_{zz}\right)\right] \tag{17}$$

Now since,

$$u = f_1(\phi) = \left(\frac{-2\kappa}{\mu}\right)\frac{\phi_x}{\phi}$$

$$\Rightarrow f_1'(\phi) = \left(\frac{2\kappa}{\mu}\right)\frac{\phi_x}{\phi^2}$$

$$\Rightarrow f_1''(\phi) = \left(\frac{-4\kappa}{\mu}\right)\frac{\phi_x}{\phi^3}$$

$$\Rightarrow \frac{f_1''(\phi)}{f_1'(\phi)} = \frac{-2}{\phi} \tag{18}$$

Plugging Eqs. (17) and (18) together, it is found that ϕ is the solution of the equation:

$$\phi_t = \kappa(\phi_{xx} + \phi_{yy} + \phi_{zz}) \tag{19}$$

Eq. (19) is a linear equation (three dimensional heat equation) which we solve by separation of variable method after which is transformed back to get the desired analytical solutions u, v and ω from Eqs. (10) - (12).

Consider a general solution of Eq. (19) of the form:

$$\phi(\mathrm{x,y,z,t}) = \mathrm{a_1 + a_2x + a_3y + a_4z + a_5xy + a_6yz + a_7zx + a_8xyz + X(x)Y(y)Z(z)T(t)} \tag{20}$$

which is sum of the solution $\phi_1(\mathrm{x,y,z,t}) = \mathrm{a_1 + a_2x + a_3y + a_4z + a_5xy + a_6yz + a_7zx + a_8xyz}$ and the separable solution $\phi_2(\mathrm{x,y,z,t}) = \mathrm{X(x)Y(y)Z(z)T(t)}$, where $a_1, a_2, a_3, a_4, a_5, a_6$, a_7 and a_8 are arbitrary constants.

The separable solution $\phi_2(\mathrm{x,y,z,t})$ can be written as

$$\phi_2(\mathrm{x,y,z,t}) = \mathrm{X(x)Y(y)Z(z)T(t)} = \mathrm{W(x,y,z)T(t)} \tag{21}$$

Since ϕ_2 is a solution of Eq. (19), so replacing ϕ_2 value from Eq. (21) into Eq. (19), we get

$$\mathrm{WT'} = \kappa(W_{xx}T + W_{yy}T + W_{zz}T) \tag{22}$$

or

$$\frac{\mathrm{WT'}}{\kappa} = (\Delta \mathrm{W})\mathrm{T} \tag{23}$$

where $\mathrm{T'} = \frac{\partial \mathrm{T}}{\partial \mathrm{t}}$ and $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the three dimensional Laplace operator.

Eq. (23) can be rearranged as

$$\frac{1}{\kappa}\left(\frac{\mathrm{T}'}{\mathrm{T}}\right)=\frac{\Delta W}{W}=-\alpha^2 \tag{24}$$

where α^2 is a separation constant and negative sign is used because a decaying function of time is anticipated. Eq. (24) gives two separated equations

$$\mathrm{T}'+\alpha^2\kappa\mathrm{T}=0 \tag{25}$$

$$\Delta\mathrm{W}+\alpha^2 W=0 \tag{26}$$

Eq. (25) yields

$$\mathrm{T(t)=Ae}^{-\alpha^2\kappa\,\mathrm{t}} \tag{27}$$

Eq. (26) can be expressed as

$$\mathrm{X''YZ+XY''Z+XYZ''}+\alpha^2\mathrm{XYZ=0} \tag{28}$$

or

$$\frac{\mathrm{X}''}{X}+\frac{\mathrm{Y}''}{Y}=-\frac{Z''}{Z}-\alpha^2=-\beta^2 \tag{29}$$

where β^2 is another separation constant. Eq. (29) gives two equations

$$\frac{\mathrm{X}''}{X}+\frac{\mathrm{Y}''}{Y}=-\beta^2 \tag{30}$$

$$\mathrm{Z}''+\mu^2\mathrm{Z}=0 \tag{31}$$

where $\mu^2=\alpha^2$-β^2 . Eq. (30) can be written as

$$\frac{\mathrm{X}''}{X}=-\frac{\mathrm{Y}''}{Y}-\beta^2=-\gamma^2 \tag{32}$$

where γ^2 is a separation constant. Eq. (32) gives two equations

$$\mathrm{Y}''+\delta^2\mathrm{Y}=0 \tag{33}$$

$$\mathrm{X}''+\gamma^2\mathrm{X}''=0 \tag{34}$$

where $\delta^2=\beta^2-\gamma^2$.

Solutions of Eqs. (31), (33) and (34) are given as:

$$Z(z)=F\sin\mu z+G\cos\mu z \tag{35}$$
$$Y(y)=D\sin\delta y+E\cos\delta y \tag{36}$$
$$X(x)=B\sin\gamma x+C\cos\gamma x \tag{37}$$

where B, C, D, E, F, G are arbitrary constants.

Hence, the general solution $\phi(x,y,z,t)$ becomes

$$\phi(x,y,z,t)=a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t} \tag{38}$$

From the Eqs. (10), (11) and (12) we obtain the desired analytical solutions u, v and ω of the 3D homogenous coupled Burgers' equations, which are given by:

$$u(x,y,z,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_2+a_5y+a_7z+a_8yz+\gamma A(B\cos\gamma x-C\sin\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t}}\right) \tag{39}$$

$$v(x,y,z,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_3+a_5x+a_6z+a_8xz+\delta A(B\sin\gamma x+C\cos\gamma x)(D\cos\delta y-E\sin\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t}}\right) \tag{40}$$

$$\omega(x,y,z,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_4+a_6y+a_7x+a_8xy+\mu A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\cos\mu z-G\sin\mu z)e^{-\alpha^2\kappa t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{-\alpha^2\kappa t}}\right) \tag{41}$$

These exact solutions are all new solutions and are not found anywhere in the literature.

In particular if we take convection coefficient $\mu=1$ and diffusivity coefficient $\kappa=\frac{1}{\mathrm{Re}}$, where Re is the Reynolds number, then Eqs. (6), (7) and (8) become:

$$u_t+uu_x+vu_y+\omega u_z=\frac{1}{\mathrm{Re}}\left(u_{xx}+u_{yy}+u_{zz}\right), \tag{42}$$
$$v_t+uv_x+vv_y+\omega v_z=\frac{1}{\mathrm{Re}}\left(v_{xx}+v_{yy}+v_{zz}\right), \tag{43}$$
$$\omega_t+u\omega_x+v\omega_y+\omega\omega_z=\frac{1}{\mathrm{Re}}\left(\omega_{xx}+\omega_{yy}+\omega_{zz}\right), \tag{44}$$

whose exact solutions are given by:

$$u(x,y,z,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_2+a_5y+a_7z+a_8yz+\gamma A(B\cos\gamma x-C\sin\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (45)$$

$$v(x,y,z,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_3+a_5x+a_6z+a_8xz+\delta A(B\sin\gamma x+C\cos\gamma x)(D\cos\delta y-E\sin\delta y)(F\sin\mu z+G\cos\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (46)$$

$$\omega(x,y,z,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_4+a_6y+a_7x+a_8xy+\mu A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\cos\mu z-G\sin\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)(F\sin\mu z+G\cos\mu z)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (47)$$

We notice that the solution $u(x,y,z,t)$ given by Eq. (39) represents the exact solution of the (3+1)-dimensional generalized Burgers' equation of the form:

$$u_t+\mu\left(uu_x+uu_y+uu_z\right)=\kappa\left(u_{xx}+u_{yy}+u_{zz}\right), \quad (48)$$

The exact solutions of 2D generalized nonlinear coupled Burgers' equations:

$$u_t+\mu\left(uu_x+vu_y\right)=\kappa(u_{xx}+u_{yy}), \quad (49)$$
$$v_t+\mu\left(uv_x+vv_y\right)=\kappa(v_{xx}+v_{yy}), \quad (50)$$

can also be found as a particular case of 3D Burgers' equations, which are:

$$u(x,y,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_2+a_4y+\gamma A(B\cos\gamma x-C\sin\gamma x)(D\sin\delta y+E\cos\delta y)e^{-\alpha^2\kappa t}}{a_1+a_2x+a_3y+a_4xy+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)e^{-\alpha^2\kappa t}}\right) \quad (51)$$

$$v(x,y,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_3+a_4x+\delta A(B\sin\gamma x+C\cos\gamma x)(D\cos\delta y-E\sin\delta y)e^{-\alpha^2\kappa t}}{a_1+a_2x+a_3y+a_4xy+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)e^{-\alpha^2\kappa t}}\right) \quad (52)$$

If $\mu=1$ and $\kappa=\frac{1}{\mathrm{Re}}$, then Eqs. (49) and (50) become the most commonly used 2D homogenous coupled unsteady nonlinear Burgers' equations:

$$u_t+uu_x+vu_y=\frac{1}{\mathrm{Re}}(u_{xx}+u_{yy}), \quad (53)$$

$$v_t+uv_x+vv_y=\frac{1}{\mathrm{Re}}(v_{xx}+v_{yy}), \quad (54)$$

whose exact solutions are given by:

$$u(x,y,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_2+a_4y+\gamma A(B\cos\gamma x-C\sin\gamma x)(D\sin\delta y+E\cos\delta y)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+a_3y+a_4xy+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (55)$$

$$v(x,y,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_3+a_4x+\delta A(B\sin\gamma x+C\cos\gamma x)(D\cos\delta y-E\sin\delta y)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+a_3y+a_4xy+A(B\sin\gamma x+C\cos\gamma x)(D\sin\delta y+E\cos\delta y)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (56)$$

The solution u(x,y,t) given by Eq. (51) also represents the exact solution of the (2+1)-dimensional generalized Burgers' equation of the form:

$$u_t+\mu\left(uu_x+vu_y\right)=\kappa(u_{xx}+u_{yy}), \quad (57)$$

By the same method, the exact solution of (1+1)-D homogenous generalized Burgers' equation of the form:

$$u_t+\mu\left(uu_x\right)=\kappa u_{xx}, \quad (58)$$

is given by:

$$u(x,t)=\left(\frac{-2\kappa}{\mu}\right)\left(\frac{a_2+\gamma A(B\cos\gamma x-C\sin\gamma x)e^{-\alpha^2\kappa t}}{a_1+a_2x+A(B\sin\gamma x+C\cos\gamma x)e^{-\alpha^2\kappa t}}\right) \quad (59)$$

If we take $\mu=1$ and $\kappa=\frac{1}{\mathrm{Re}}$, then Eq. (58) becomes the most famous 1D homogenous nonlinear Burgers' equation:

$$u_t+uu_x=\frac{1}{\mathrm{Re}}u_{xx}, \quad (60)$$

whose exact solution is given as:

$$u(x,t)=\left(\frac{-2}{\mathrm{Re}}\right)\left(\frac{a_2+\gamma A(B\cos\gamma x-C\sin\gamma x)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}{a_1+a_2x+A(B\sin\gamma x+C\cos\gamma x)e^{\left(\frac{-\alpha^2}{\mathrm{Re}}\right)t}}\right) \quad (61)$$

4. Conclusions

In this study, we have shown how to generate exact solutions of 3D, 2D homogenous generalized coupled Burgers' equations and also for 1D generalized Burgers' equation with the help of Hopf-Cole transformation and separation of variables technique. The method is simple and straight forward. These generated exact solutions can be used to find various sets of initial and boundary conditions, which thus can be utilized to find the general solution of the non-homogenous part and also to test the newly developed or already existing algorithms in case of one, two and three dimensions respectively.

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