# Spectral methods for time dependent partial differential equations

2 authors, including:

Eli Turkel
Tel Aviv University
**245** PUBLICATIONS   **10,864** CITATIONS

Some of the authors of this publication are also working on these related projects:

Project   High Order Finite Difference for Wave Equation in general domains with interfaces   View project

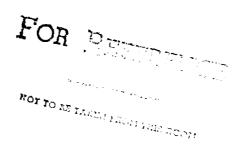Project   Time Reversed Absorbing Condition: the TRAC method   View project

# ICASE

SPECTRAL METHODS FOR TIME DEPENDENT PARTIAL
DIFFERENTIAL EQUATIONS

David Gottlieb

and

Eli Turkel

*FOR* ~~REFERENCE~~

~~NOT TO BE TAKEN FROM THIS ROOM~~

INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665

Operated by the Universities Space Research Association

# NASA

National Aeronautics and
Space Administration

**Langley Research Center**
Hampton, Virginia 23665

# SPECTRAL METHODS FOR TIME DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS

David Gottlieb[*]

Institute for Computer Applications in Science and Engineering

and

Tel-Aviv University


Eli Turkel[**]

Institute for Computer Applications in Science and Engineering

and

Tel-Aviv University

## ABSTRACT

The theory of spectral methods for time dependent partial differential equations is reviewed. When the domain is periodic Fourier methods are presented while for nonperiodic problems both Chebyshev and Legendre methods are discussed. The theory is presented for both hyperbolic and parabolic systems using both Galerkin and collocation procedures. While most of the review considers problems with constant coefficients the extension to nonlinear problems is also discussed. Some results for problems with shocks are presented.

*N84-10837*

## INTRODUCTION

We begin by describing how to construct spectral approximations to time-dependent mixed initial-boundary value problems. We shall study differential equations of the form

$$\frac{\partial u}{\partial t} = Lu + f \qquad (1.1)$$

$$u_N(0) = u_0$$

where for each $t$, $u(t)$ belongs to a Hilbert space $H$ such that $u$ satisfies homogeneous boundary conditions. For simplicity we assume that $L$ is an unbounded time independent linear operator.

Numerical methods can be characterized by specifying a finite dimensional subspace $B_N \subset H$ and a projection operator $P_N : H \to B_N$. We require that the sequence $\{P_N\}$ satisfies

$$\lim_{N \to \infty} \| P_N u - u \| = 0.$$

We shall concentrate on semi-discrete approximations to (1.1), i.e., time is still a continuous variable. Such a semi-discrete approximation can be written as

$$\frac{\partial u_N}{\partial t} = P_N L P_N u_N + P_N f$$

$$\qquad (1.2)$$

$$u_N(0) = P_N u_0$$

where $u_N \in B_N$. The numerical approximation (1.2) converges to the solution of (1.1) if

$$\lim_{N \to \infty} \| u_N - P_N u \| = 0. \qquad (1.3)$$

Combining (1.1) and (1.2) and assuming $P_N$ is independent of $t$, the error satisfies the equation

$$\frac{\partial}{\partial t}\left(u_N - P_N u\right) = P_N\, LP_N\left(u_N - P_N u\right) + P_N(LP_N - L)u. \qquad (1.4)$$

Now, $P_N LP_N$ is an operator from $B_N$ to $B_N$ and so can be viewed as a matrix. In particular, $\exp(P_N LP_N)$ is well defined. Hence the solution to (1.4) can be written as

$$u_N - P_N u = \int_0^t \exp\left(P_N LP_N(t - \tau)\right)P_N\left(LP_N - L\right)u(\tau)d\tau. \qquad (1.5)$$

We call a scheme consistent if

$$\lim_{N\to\infty} \|P_N\left(L - LP_N\right)u\| = 0 \qquad\qquad 0 < t < T, \qquad (1.6)$$

while the scheme is stable if

$$\exp\left(P_N LP_N\, t\right) < K(t), \qquad (1.7)$$

where $K$ is independent of $N$, the dimension of $B_N$, i.e., $\exp(P_N LP_N\, t)$ is uniformly bounded for all $0 < t < T$. It then follows from (1.5) that if a scheme is consistent and stable then the scheme converges.

For spectral methods we choose $B_N$ as the finite space of polynomials (or trigonometric polynomials) of degree at most $N$. The rationale behind this choice is that one can approximate arbitrary functions $f$ by such polynomials and the rate of convergence is only governed by the smoothness of the function $f$. Hence we hope to obtain highly accurate approximations to

the solutions of (1.1). Different choices of the projection operator $P_N$ lead to different subclasses of spectral methods.

In these lectures we shall only consider one-dimensional differential equations.

## 2. FOURIER METHODS

We first consider periodic problems with period $2\pi$. For this case it is natural to let $B_N$ be $\{e^{ijx}\}$, $-N \leqslant j \leqslant N$.

### (a) Galerkin Method

Let $v(x) \in H$ then

$$v(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}. \qquad (2.1)$$

The Galerkin method is characterized by the projection operator $P_N$ where

$$P_N v = \sum_{n=-N}^{N} a_n e^{inx}. \qquad (2.2)$$

We now rewrite the approximation (1.2) in the form

$$P_N\left(\frac{\partial u_N}{\partial t} - Lu_N - f\right) = 0$$

or using the definition of $P_N$

$$\left(\frac{\partial u_N}{\partial t} - Lu_N - f, e^{inx}\right) = 0 \qquad -N \leqslant n \leqslant N \qquad (2.3)$$

where

$$u_N = \sum_{n=-N}^{N} a_n(t)e^{inx}$$

$$u_N(0) = P_N u_0 .$$

This can be interpreted as a system of 2N+1 ordinary differential equations for the coefficients $a_n(t)$. Equivalently one can expand the solution to (1.1) in a finite Fourier series and then truncate $Lu_N$. Hence, the Galerkin method is equivalent to solving the system (1.1) in Fourier space rather than physical space.

An alternative basis is to expand $u(x)$ in terms of cos nx, $0 \leqslant n \leqslant N$, and sin nx, $1 \leqslant n \leqslant N-1$. This is equivalent to demanding that $a_{-n} = \overline{a}_n$ in (2.1).

### (b) Pseudospectral Method

The pseudospectral or collocation method is defined by letting $P_N$ be an interpolation operator. If $f(x)$ is a periodic function then $P_N f$ is the trigonometric interpolation of $f$ at the collocation points $x_j$, i.e.,

$$P_N f(x_j) = f(x_j) \quad \text{and} \quad P_N f \varepsilon B_N.$$

The following sets of points are the most commonly used collocation points

$$x_j = \frac{\pi j}{N} \qquad\qquad j = 0, \cdots, 2N-1 \qquad (2.4a)$$

$$y_j = \frac{2\pi j}{2N+1} \qquad\qquad j = 0, \cdots, 2N. \qquad (2.4b)$$

The $x_j$ are useful when operating with a FFT based on an even number of points while the $y_j$ are useful for an odd number of points. We shall only describe the collocation method based on the $x_j$. In this case the operator $P_N$ is given by

$$P_N f(x) = \sum_{j=0}^{2N-1} f(x_j)g_j(x).$$  (2.5)

The $g_j(x)$ are trigonometric polynomials of degree at most $N$ and $g_j(x_k) = \delta_{jk}$. These polynomials are given explicitly by

$$g_j(x) = \frac{1}{2N} \sin[N(x-x_j)]\cot \frac{x - x_j}{2} .$$  (2.6)

The fact that $g_j(x)$ is a trigonometric polynomial of degree $N$ follows from the equivalent representation

$$g_j(x) = \frac{1}{2N} \sum_{\ell=-N}^{N} \frac{1}{c_\ell} e^{i\ell(x-x_j)}$$  (2.7)

where $c_\ell = 1$ $(|\ell| \neq N)$, $c_N = c_{-N} = 2$. Thus we can represent $P_N f(x)$ either as

$$P_N f(x) = \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j)\sin[N(x-x_j)]\cot \frac{x - x_j}{2}$$  (2.8)

using (2.5) or as

$$P_N f(x) = \sum_{j=0}^{2N-1} f(x_j) \frac{1}{2N} \sum_{\ell=-N}^{N} \frac{1}{c_\ell} e^{i\ell(x-x_j)}$$

$$= \sum_{\ell=-N}^{N} \frac{1}{c_\ell} e^{i\ell x} \frac{1}{2N} \sum_{j=0}^{2N-1} f(x_j)e^{-i\ell x_j}$$  (2.9)

6

using (2.7). Defining

$$a_\ell = \frac{1}{2Nc_\ell} \sum_{j=0}^{2N-1} f(x_j)e^{-i\ell x_j} \qquad (2.10)$$

(2.9) becomes

$$P_n f = \sum_{\ell=-N}^{N} a_\ell e^{i\ell x}. \qquad (2.11)$$

When applying the pseudospectral Fourier method, either the explicit interpolatory formula (2.8) or the complex-Fourier representation (2.10) – (2.11) may be used. The operator L is a differential operator and so it is useful to obtain $\dfrac{d^k f(x_j)}{dx^k}$ in terms of $f(x_j)$. One way is simply to differentiate (2.8) and to evaluate the resulting expression at the points $x_j$.

$$\frac{d^n u_N(x_j)}{dx^n} = \sum_{k=0}^{2N-1} u_N(x_k) \frac{d^n g_k(x_j)}{dx^n} = (D_n \vec{u})_j, \qquad (2.12)$$

where $D_n$ is an $2N \times 2N$ matrix with elements

$$(D_n)_{jk} = \frac{d^n g_k(x_j)}{dx^n}$$

and $\vec{u}$ is the column vector

$$\vec{u} = \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_{2N-1}) \end{pmatrix}.$$

Explicitly,

$$(D_1)_{jk} = \begin{cases} \frac{1}{2}(-1)^{j+k} \cot \dfrac{x_j - x_k}{2} & j \neq k \\ 0 & j = k \end{cases} \qquad (2.13)$$

$$(D_2)_{jk} = \begin{cases} \frac{1}{2} (-1)^{j+k+1} \dfrac{1}{\sin^2 \frac{x_j - x_k}{2}} & j \neq k \\[2em] -\dfrac{2N^2 + 1}{6} & j = k \end{cases} \qquad . \qquad (2.14)$$

More generally

$$D_n = (D_1)^n \qquad (2.15)$$

which easily follows from the properties of $g_j(x)$. $D_1$ is a real, antisymmetric matrix. In general $D_{2k}$ is a real, symmetric matrix while $D_{2k+1}$ is a real, antisymmetric matrix.

Computationally, the evaluation of derivatives using (2.13) - (2.15) involves the multiplication of an 2N-component vector $\vec{u}$ by an $2N \times 2N$ matrix, $D_n$, which typically requires $O(N^2)$ arithmetic operations. However, since the matrix product is actually a convolutional sum, it is possible to use the FFT to evaluate (2.13) - (2.15) in only order $N \log N$ operations when $N$ is a highly composite integer (like $2^p$ or $3^q$). Nevertheless direct matrix multiplication can be quite efficient if $N$ is not too large or if a highly parallel computer is used.

It is also possible to evaluate derivatives using (2.10) - (2.11). Indeed, (2.11) gives

$$\frac{d^n P_N f}{dx^n} (x_j) = \sum_{|k| \leqslant N} (ik)^n a_k e^{ikx_j}, \qquad (2.16)$$

where $a_k$ is given by (2.10). In this approach, $a_k$ is first evaluated by (2.10) and then derivatives at $x_j$ are evaluated by (2.16). If $N$ is a

8

highly composite integer, the two discrete Fourier transforms (2.10) and (2.16) can be efficiently evaluated by the FFT algorithm in $O(N \log N)$ operations. Thus, evaluation of derivatives requires just two FFTs together with the complex multiplication by $(ik)^n$ in (2.16).

## 3. POLYNOMIAL METHODS

We now consider equation (1.1) in a finite interval $-1 < x < 1$. Since the problem is not periodic, Fourier expansions do not yield high order approximations. Instead, it is preferable to use orthogonal polynomials. We thus take $\phi_0, \cdots, \phi_N$ as the basis of $B_N$ where $\phi_j$ is a polynomial of degree $j$ and $\phi_j$ is zero at the appropriate boundaries. We only consider homogeneous boundary conditions. As before we have different spectral methods by choosing different $\{\phi_j\}$ and different projection operators.

(a) Galerkin Method

Let $f(x)$ be a sufficiently smooth function defined in $-1 < x < 1$ where $f(x)$ vanishes at the appropriate boundaries which yields a well-posed problem for (1.1). Define

$$P_N \, f(x) = \sum_{k=0}^{N} a_k \, \phi_k(x) \tag{3.1}$$

where the $a_k$'s are chosen so that

$$\int_{-1}^{1} \omega(x) \left( P_N \, f - f \right) \phi_j(x) dx = 0 \qquad j = 0, \cdots, N \tag{3.2}$$

for some nonnegative weight $\omega(x)$. We write the numerical approximation (1.2) as

$$\int_{-1}^{1} \omega(x) \left( \frac{\partial u_N}{\partial t} - L \, u_N - f \right) \phi_j(x) dx = 0 \qquad j = 0, \cdots, N \tag{3.3}$$

where

$$u_N = \sum_{k=0}^{N} a_k(t)\phi_k(x)$$

$$u_N(0) = P_N u_0.$$

As before, this gives rise to a system of N+1 ordinary differential equations for $a_k(t)$. An equivalent way is to express the p.d.e. (1.1) in $\phi$-space and then truncate after N terms.

The Legendre-Galerkin method is obtained by choosing the weight $\omega(x) = 1$. The Chebyshev-Galerkin method uses the weight $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$.

(b) Pseudospectral Chebyshev Method

In the most common pseudospectral Chebyshev method, the interpolation points in the interval (-1,1) are chosen to be the extrema

$$x_j = \cos \frac{\pi j}{N} \qquad (j = 0, \cdots, N) \qquad (3.4)$$

of the Nth-order Chebyshev polynomials $T_N(x)$. Here the Chebyshev polynomial of degree N is defined by

$$T_N(x) = \cos(N \cos^{-1} x). \qquad (3.5)$$

It follows that

$$T_N(x_j) = \cos \frac{\pi j}{N} , \qquad (3.6)$$

which indicates a close relation between the pseudospectral Chebyshev and the pseudospectral Fourier method. In order to construct the interpolant of

f(x) at the point x we define the polynomials

$$g_j(x) = \frac{(1-x^2)T_N'(x)(-1)^{j+1}}{\bar{c}_j N^2(x-x_j)} \qquad (j = 0,\cdots,N) \qquad (3.7)$$

with $\bar{c}_0 = \bar{c}_N = 2$, $\bar{c}_j = 1$ ($1 \leqslant j \leqslant N-1$). It is readily verified that

$$g_j(x_k) = \delta_{jk}.$$

The Nth-degree interpolation polynomial $P_N f(x)$ to $f(x)$ is given by

$$P_N f(x) = \sum_{j=0}^{N} f(x_j) g_j(x). \qquad (3.8)$$

A different way of representing $P_N f(x)$ is to use the identity

$$\sum_{k=0}^{N} \frac{T_k(x_j)T_k(x)}{\bar{c}_k} = \frac{(1-x^2)T_N'(x)}{2N(x-x_j)} (-1)^{j+1},$$

giving

$$\sum_{j=0}^{N} f(x_j)g_j(x) = \frac{2}{N} \sum_{j=0}^{N} \frac{f(x_j)}{\bar{c}_j} \sum_{k=0}^{N} \frac{T_k(x_j)T_k(x)}{\bar{c}_k}$$

$$= \frac{2}{N} \sum_{k=0}^{N} T_k(x) \frac{1}{\bar{c}_k} \sum_{j=0}^{N} \frac{f(x_j)T_k(x_j)}{\bar{c}_j} .$$

Thus,

$$P_N f(x) = \sum_{k=0}^{N} a_k T_k(x), \qquad (3.9)$$

where

$$a_k = \frac{2}{N} \frac{1}{\bar{c}_k} \sum_{j=0}^{N} \frac{f(x_j)T_k(x_j)}{\bar{c}_j} . \qquad (3.10)$$

It should be noted that the coefficients $a_k$ in (3.10) can be evaluated using

the FFT. In fact, using (3.6) in (3.10) gives

$$a_k = \frac{2}{N} \frac{1}{\bar{c}_k} \sum_{j=0}^{N} \frac{f(x_j)}{\bar{c}_j} \cos \frac{\pi jk}{N} .$$

(3.11)

The second step in getting a pseudospectral approximation is to express the derivatives of $P_N f$ in terms of $f(x)$ at the collocation points $x_j$. This can be done by differentiating either (3.8) or (3.9). With (3.8) we obtain

$$\frac{d^n P_N f(x)}{dx^n} = \sum_{j=0}^{N} f(x_j) \frac{d^n}{dx^n} g_j(x)$$

(3.12)

so that

$$\frac{d^n P_N f(x_k)}{dx^k} = \sum_{j=0}^{N} f(x_j)(D_n)_{kj}$$

(3.13)

where

$$(D_n)_{jk} = \frac{d^n}{dx^n} g_k(x) \Big|_{x=x_j} .$$

(3.14)

For example

$$(D_1)_{jk} = \frac{\bar{c}_j}{\bar{c}_k} \frac{(-1)^{j+k}}{x_j - x_k} \qquad (k \neq j)$$

(3.15)

$$(D_1)_{jj} = - \frac{x_j}{2(1 - x_j^2)} , \quad (D_1)_{00} = \frac{2N^2 + 1}{6} = - (D_1)_{NN}$$

and

$$D_n = (D_1)^n .$$

(3.16)

12

It should be noted from the explicit formula (3.15) that the matrix $D_1$ is not antisymmetric; also $D_2$ is not symmetric. These facts introduce both theoretical and practical difficulties in the pseudospectral Chebyshev method.

A different way to obtain an expression for the derivative of $P_N f$ is to differentiate (3.9) to get

$$\frac{d^n}{dx^n} P_N f = \sum_{k=0}^{N} a_k T_k^{(n)}(x),$$  (3.17)

where the coefficients $a_k$ are given by (3.10). For example,

$$\frac{dP_N f}{dx} = \sum_{k=0}^{N} a_k T_k'(x) = \sum_{k=0}^{N} b_k T_k(x),$$  (3.18)

where

$$b_N = 0, \quad b_{N-1} = 2N a_N$$

and

$$\bar{c}_k b_k = b_{k+2} + 2(k+1)a_{k+1} \qquad (0 < k < N-2).$$  (3.19)

In evaluating the first derivative at the collocation points $x_j$ the FFT is used to evaluate

$$\frac{d^n(P_N f)}{dx^n} = \sum_{k=0}^{N} b_k^{(n)} T_k(x)$$

where

$$b_k^{(0)} = a_k, \quad 0 < k < N, \quad b_k^{(1)} \equiv b_k,$$

and

$$\bar{c}_k \, b_k^{(n)} = b_{k+2}^{(n)} + 2(k+1)b_{k+1}^{(n-1)}$$

$$(3.20)$$

$$b_N^{(n)} = 0; \qquad\qquad b_{N-1}^{(n)} = 2Nb_N^{(n-1)}.$$

The set of points $x_j$ defined in (3.3) is not the only set used with pseudospectral Chebyshev approximation. For hyperbolic problems, a convenient alternative set of collocation points is

$$y_j = \cos\frac{\pi j}{N+1} \qquad\qquad (j = 0,\cdots,N). \qquad\qquad (3.21)$$

Two other sets of points that are sometimes used are

$$z_j^{(1)} = \cos\frac{2\pi j}{2N+1} \qquad\qquad (j = 0,\cdots,N) \qquad\qquad (3.22)$$

and

$$z_j^{(2)} = \cos\frac{\pi(2j+1)}{2N+1} \qquad\qquad (j = 0,\cdots,N). \qquad\qquad (3.23)$$

## (c) Pseudospectral Legendre Method

An attractive alternative to Chebyshev polynomial expansions is Legendre polynomial expansions. It suffices to explain how to construct a pseudospectral Legendre polynomial approximation to a derivative.

Let $x_0 = -1$, $x_N = 1$, and let $x_i (i = 1,\cdots,N-1)$ be the roots of $q_N'(x)$, where $q_N(x)$ is the Legendre polynomial of degree $N$. Given the values of any function $f(x)$ at the points $x_j$, $(j = 0,\cdots,N)$, we construct the interpolating polynomial

$$P_N \, f = \sum_{j=0}^{N} f(x_j)g_j(x) \qquad\qquad (3.24)$$

where

$$g_j(x) = -\frac{1}{\alpha_N q_N(x_j)} \frac{(1-x^2)q_N^-(x)}{x - x_j}$$

with

$$\alpha_N = N(N+1).$$

Therefore,

$$\frac{d^\ell}{dx^\ell} P_N f \bigg|_{x=x_j} = \sum_{j=0}^{N} f(x_k) \frac{\partial^\ell g_k(x_j)}{\partial x^\ell} = \sum_{k=0}^{N} (D_\ell)_{jk} f(x_k). \qquad (3.25)$$

For example

$$(D_1)_{jk} = \frac{q_N(x_j)}{q_N(x_k)} \frac{1}{x_j - x_k} \qquad\qquad (j \neq k)$$

$$(D_1)_{00} = \frac{1}{4} \alpha_N = -(D_1)_{NN} \qquad\qquad (3.26)$$

$$(D_1)_{jj} = 0 \qquad\qquad (j \neq 0, \ j \neq N).$$

The difference between the Chebyshev and Legendre methods is evident here. The matrix $D_1$ for Legendre polynomials is nearly antisymmetric, in contrast to the Chebyshev matrix given in (3.15).

By the same method, we obtain

$$(D_2)_{jk} = -2 \frac{q_N(x_j)}{q_N(x_k)} \frac{1}{(x_j - x_k)^2} \qquad \begin{array}{l} 1 \leqslant j,k \leqslant N-1, \\[4pt] j \neq k \end{array} \qquad ,$$

$$(D_2)_{jj} = -\frac{1}{3} \frac{N}{1 - x_j^2} \qquad\qquad 1 \leqslant j \leqslant N-1. \qquad (3.27)$$

This shows that $D_2 = \Lambda S \Lambda^{-1}$, where $\Lambda$ is a diagonal matrix and $S$ is symmetric.

## 4. ERROR EQUATION

The differential equation we wish to solve is

$$\frac{\partial u}{\partial t} = Lu. \tag{4.1}$$

The numerical approximation given by (1.2) is

$$\frac{\partial u_N}{\partial t} = P_N L P_N u_N. \tag{4.2}$$

We define the error equation as

$$\frac{\partial u_N}{\partial t} - L u_N = (P_N L P_N - L)u_N = (P_N L - L)u_N. \tag{4.3}$$

In the finite difference literature this is frequently called the modified equation. We shall now give explicit formulas for the right-hand-side of (4.3) for several cases of interest.

We first consider the model hyperbolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \qquad -1 \leqslant x \leqslant 1, \quad u(1,t) = 0, \quad u(x_0) = u_0(x). \tag{4.4}$$

Even though this problem has constant coefficients, nevertheless $P_N Lu_N \neq L u_N$ because of the presence of boundaries.

We first consider Chebyshev collocation at the points $x_j = \cos \frac{\pi j}{N}$, (see (3.4)). The method described previously satisfies (4.4) at the points $x_1, \cdots, x_N$ while at $x = x_0$ we impose the boundary condition. Since $u_N$ is a polynomial of degree $N$ and $\frac{\partial u_N}{\partial x}$ is a polynomial of degree $N-1$ we have that

$$\frac{\partial u_N}{\partial t} - \frac{\partial u_N}{\partial x} = Q_N(x)\tau(t), \tag{4.5}$$

where $Q_N$ is a polynomial of degree at most $N$ and $Q_N(x_j) = 0$, $j = 1, \cdots, N$, hence $Q_N(x) = (1+x)T_N'$. Comparing the coefficient of $T_N$ for both sides of (4.5) we see that

$$\tau(t) = \frac{1}{N} \frac{da_N(t)}{dt} \ .$$

Alternatively, comparing both sides of (4.5) at $x = x_0$,

$$\tau(t) = -\frac{u_x(1,t)}{N^2} \ .$$

In conclusion the error equation for the points $x_j$ is

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial t} + \frac{\dot{a}_N}{N}(1+x)T_N' = \frac{\partial u_N}{\partial x} - \frac{u_x(1,t)}{2N^2}(1+x)T_N' \ . \tag{4.6}$$

Similarly the error equation for the points $y_j$ (3.21) is

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} + \frac{\dot{a}_N \, T_{N+1}'}{N+1} = \frac{\partial u_N}{\partial t} - \frac{u_x(1,t)}{(N+1)^2} T_{N+1}', \tag{4.7}$$

and the error equation for $\{z_j^{(1)}\}$, (3.22) is

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} + \frac{1}{2}\left[T_N + \frac{(1+x)T_N'}{N}\right]\tau(t) \qquad \tau = \dot{a}_N(t) = -\frac{2u_x(1)}{1+2N} \ . \tag{4.8}$$

We now claim that (4.8) is also the error equation for the Galerkin method for (4.4). The Galerkin method satisfies (4.5) where $Q$ and $\tau$ are chosen so that

$$A = \tau(t) \int_{-1}^{1} \frac{Q(x)\left(T_j(x) - 1\right)}{\sqrt{1 - x^2}} \, d_x = 0 \qquad j = 0, \cdots, N \qquad (4.9)$$

We show that if $Q(x) = \dfrac{T_N}{2} + \dfrac{(1+x)T_N'}{2N} = \dfrac{T_0}{2} + \sum_{k=1}^{N} T_k(x)$ then (4.9) is satisfied.

If $j = 0$, $T_j - 1 = 0$ and the result is trivial.

For $j \geqslant 1$

$$A = \tau(t) \frac{1}{2} \int_{-1}^{1} \frac{T_0(T_j - 1)}{\sqrt{1 - x^2}} \, dx + \sum_{k=1}^{N} \int_{-1}^{1} \frac{T_k(T_j - 1)}{\sqrt{1 - x^2}} \, dx$$

$$= \tau(t) - \frac{1}{2} \int_{-1}^{1} \frac{T_0 \, dx}{\sqrt{1 - x^2}} + \frac{\pi}{2} \sum_{k=1}^{N} \delta_{kj} = \frac{\pi}{2} \tau(t) - 1 + \sum_{k=1}^{N} \delta_{kj} = 0.$$

It thus follows that for the constant coefficient problem (4.3) that the Galerkin Chebyshev and the Chebyshev collocation at the point $z_j^{(1)}$ are identical.

We next consider the heat equation

$$u_t = u_{xx}, \quad -1 < x < 1,$$

$$u(-1,t) = u(1,t) = 0, \quad u(x,0) = u_0(x). \qquad (4.10)$$

We now collocate at the points $x_j = \cos \dfrac{\pi j}{N}$, $j = 1, \cdots, N-1$ with boundary conditions imposed at $x_0$ and $x_N$. Similar to the above procedure we find that the error equation for the Chebyshev collocation method is

18

$$\frac{\partial u_N}{\partial t} = \frac{\partial^2 u_N}{\partial x^2} + \frac{u_{xx}(1,t)(1+x)T_N' + (-1)^{N+1} u_{xx}(-1,t)(1-x)T_N'}{2N^2} . \quad (4.11)$$

For the Galerkin method

$$\frac{\partial u_N}{\partial t} = \frac{\partial^2 u_N}{\partial x^2} + Q(x)\tau(t),$$

where $Q(x)$ is chosen to be orthogonal to $\phi_1(x)$, $j = 0, \cdots, N$ in the Chebyshev inner product. For (4.10) we choose

$\phi_j = T_j(x) - \frac{(1+x)}{2} - \frac{(-1)^j(1-x)}{2}$ . It then follows that the error equation is

$$\frac{\partial u_N}{\partial t} = \frac{\partial^2 u_N}{\partial x^2} + \left(-\frac{cx}{N} + b\right)T_N' + cT_N, \quad (4.12)$$

with

$$b = -\frac{u_{xx}(1,t) - (-1)^N u_{xx}(-1,t)}{2N^2} .$$

$$c = \frac{u_{xx}(1,t) + (-1)^N u_{xx}(-1,t)}{2(N^2 - 1)} .$$

## 5. HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

Before discussing stability properties of various schemes it is helpful to review the properties of hyperbolic partial differential equations. We begin with the model problem

$$u_t = u_x \qquad\qquad -1 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant T$$

$$(5.1)$$

$$u(x,0) = f(x), \qquad u(1,t) = g(t).$$

Integration by parts leads to the inequality

$$\frac{d}{dt} \|u\|^2 = g^2(t) - u^2(-1,t),$$

or

$$\|u^2(\cdot,T)\| + \int_0^T u^2(-1,t)dt = \|f^2\| + \int_0^T g^2(t)dt. \qquad (5.2)$$

The norms in this case are $L_2$ norms in space and the generalization to Sobolev norms is immediate. We next consider the system of equations

$$u_t = A u_x \qquad\qquad -1 < x < 1, \qquad (5.3)$$

where $A$ is a $p \times p$ constant matrix. Assuming the system is strongly hyperbolic we can diagonalize the matrix $A$. Hence we replace (5.3) by

$$u_t = D u_x \qquad\qquad -1 < x < 1, \qquad (5.4)$$

where $D = \text{diag}(d_1,d_2,\cdots,d_p) = (d^I,d^{II})$. We order the eigenvalues $d_j$ so that $d^I = (d_1,\cdots,d_k)$ is positive and $d^{II} = (d_{k+1},\cdots,d_p)$ is negative. The appropriate initial boundary value problem is

$$u_t = D u_x \qquad\qquad -1 < x < 1$$

$$u(x,0) = f(x)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.5)$$

$$u^I(1,t) = S^I u^{II}(1,t) + g^I(t)$$

$$u^{II}(-1,t) = S^{II} u^I(-1,t) + g^{II}(t),$$

where $u^I$ is the corresponding vector of length $k$ and $u^{II}$ is of length $p-k$ and $\det(S^I) \neq 0$, $\det(S^{II}) \neq 0$. We then obtain the a priori estimate

$$\|u^2(\cdot,T)\|^2 + K_1 \int_0^T [u^I(-1,t))^2 + (u^{II}(1,t))^2]dt$$

$$(5.6)$$

$$\leq \|f(x)\|^2 + K_2 \int_0^T [(g^I(t))^2 + (g^{II}(t))^2]dt,$$

for appropriate constants $K_1$, $K_2$ which depend on the matrices $D$, $S^I$, $S^{II}$. The above estimate holds only if the boundary conditions are dissipative, i.e.,

$$(S^I)^* D^I S^I + D^{II} < 0$$

$$(5.7)$$

$$(S^{II})^* D^{II} S^{II} + D^I > 0,$$

in other words, if $S^I$ and $S^{II}$ are sufficiently small. When (5.7) does not hold we sometimes can consider a new variable $v(x,t) = e^{-\alpha t} u(x,t)$ and then obtain an estimate of type (5.6) for $v$. Hence $u(x,t)$ satisfies the inequality

$$e^{\alpha T} \|u(\cdot,T)\|^2 + K_1 \int_0^T e^{2\alpha t} ((u^I(-1,t))^2 + u^{II}(1,t))^2 dt$$

$$(5.8)$$

$$\leq \|f(x)\|^2 + K_2 \int_0^T e^{2\alpha t}((g^I(t))^2 + (g^{II}(t))^2 dt$$

for constants $K_1$, $K_2$ and $\alpha \geq \alpha_0$. Sometimes we also need to consider a norm with a kernel that depends on $x$.

The above estimates were all obtained for $L^2$ (or Sobolev) norms with a weight of $K(x) = 1$. On the spectral level this is appropriate for expansions

in Legendre polynomials. Indeed, if one uses a Legendre pseudospectral method one can obtain estimates similar to (5.2), (5.6) and (5.8). However, when using a Chebyshev pseudospectral method it is more appropriate to consider Sobolev norms with a weight $K(x) = (1 - x^2)^{-1/2}$. We next show that one no longer gets a priori estimates of the type considered until now. Instead, one must rely on a weaker a priori estimate.

We again begin with the scalar equation but with homogeneous boundary conditions

$$u_t = u_x \qquad\qquad -1 < x < 1$$

$$\tag{5.9}$$

$$u(x,0) = f(x) \qquad u(1,t) = 0.$$

Following Gottlieb and Orszag [17] we consider the initial condition

$$\phi_\varepsilon(x,x_0) = \begin{cases} 0 & |x - x_0| > \varepsilon \\ 1 - \dfrac{|x - x_0|}{\varepsilon} & |x - x_0| < \varepsilon \end{cases}, \qquad (5.10)$$

i.e., $\phi_\varepsilon$ is zero everywhere except for a $\varepsilon$ neighborhood of $x_0$ where $\phi_\varepsilon$ is a triangular function with height 1. A straightforward calculation shows that

$$\|\phi_\varepsilon(x,0)\|_\omega^2 = O(\varepsilon) \qquad\qquad \omega = (1 - x^2)^{-1/2} ,$$

but that $\|\phi_\varepsilon(x,-1 + \varepsilon)\|_\omega^2 = O(\varepsilon^{1/2})$. This shows that initially the $\omega$-norm of $u$ is of order $\varepsilon^{1/2}$. However, at time $1 - \varepsilon$, $u(x,t) = f(x+t)$ and so $u(x,t)$ has $\omega$-norm of order $\varepsilon^{1/4}$. Hence, one cannot bound $u(x,t)$ uniformly in terms of the initial conditions for the Chebyshev norm with

weight $(1-x^2)^{-\frac{1}{2}}$. For this simple case one can overcome this difficulty by considering an alternative norm $\omega_1(x) = (1+x)(1-x^2)^{-\frac{1}{2}} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}$, see Gottlieb and Orszag [17]. Heuristically, this norm helps since it is no longer singular at $x = -1$. Since the differential equation (9) only allows left moving waves, no difficulties arise. However, this heuristic reasoning also demonstrates that this cure will no longer work if there is a nonhomogeneous boundary condition. Waves are now created at $x = 1$ where the kernel $\omega_1(x)$ is still singular. In particular we consider the initial boundary problem

$$u_t = u_x \qquad\qquad -1 \leqslant x \leqslant 1$$

$$(5.11)$$

$$u(x,0) = 0, \qquad u(1,t) = \phi_\varepsilon(t,t_0) \qquad \text{for some } t_0 < T.$$

By the same argument as before $\int_0^T g^2(t)dt = 0(\varepsilon)$. However, at a time $t = t_0 + \varepsilon$, $\|u(\cdot,t)\|_\omega^2$ and $\|u(\cdot,t)\|_{\omega_1}^2$ are both of order $\varepsilon^{\frac{1}{2}}$. Hence, an estimate of type (5.6) cannot exist in either the $\omega$ or the $\omega_1$ norm. Heuristically any norm which has a singular weight at either $+1$ or $-1$ will lead to difficulties, should the weight be zero at both ends its derivative is nonpositive which shows that integration by parts will not be useful.

The same example shows that for a system of equations even with homogeneous boundary conditions we cannot get an a priori estimate of type (5.8). Consider the system

$$u_t = u_x \qquad\qquad -1 \leqslant x \leqslant 1 \qquad (5.12a)$$

$$v_t = -v_x \qquad\qquad\qquad\qquad (5.12b)$$

$$u(x,0) = \phi_\epsilon(x,0) \qquad v(x,0) = 0$$

$$u(1,t) = \alpha v(1,t), \qquad v(-1,t) = \beta u(-1,t), \qquad \alpha\beta \neq 0.$$

The solution for $u$ is a left moving wave until $t = 1$ while $v$ remains zero. At time $1 + \epsilon$, $u(x,t)$ is identically zero while $v(x,t) = \phi_\epsilon(x, -1 + \epsilon)$. At this point $v = O(\epsilon^{1/2})$ in the norm $\omega = (1 - x^2)^{-1/2}$ or $\omega_2 = (1-x)(1-x^2)^{-1/2}$ which are appropriate for (5.12b). Hence, again one cannot bound $v(x,t)$ uniformly in terms of initial condition.

We finally show that the nonhomgeneous problem (5.1) is well-posed in any norm with an integrable kernel when integrations are done over $x$-$t$ space.

Define

$$Q(x) = \int_0^T u^2(x,t)dt, \tag{5.13}$$

then

$$\frac{dQ}{dx} = \int_0^T (u^2)_x \, dt = \int_0^T (u^2)_t \, dt = u^2(x,T) - f^2(x).$$

Hence

$$Q(x) = -\int_x^1 u^2(y,T)dy + \int_x^1 f^2(y)dy + \int_0^T g^2(t)dt. \tag{5.14}$$

Now integrating in space with a weight $K(x)$

$$\int_0^T \int_{-1}^1 K(x)u^2(x,t)dx \, dt = -\int_{-1}^1 K(x)dx \int_x^1 u^2(y,T)dy$$

$$+ \int_{-1}^1 K(x)dx \int_x^1 f^2(y)dy + \int_{-1}^1 K(x)dx \int_0^T g^2(t)dt,$$

or

$$\int_0^T \int_{-1}^1 K(x)u^2(x,t)dx \, dt < K_1 \int_{-1}^1 f^2(x)dx + K_1 \int_0^T g^2(t)dt. \tag{5.15}$$

Furthermore setting $x = -1$ in (5.14) we have

$$\int_0^T u^2(-1,t)dt \leq \int_{-1}^1 f^2(x)dx + \int_0^T g^2(t)dt. \tag{5.16}$$

Adding (5.15) and (5.16) we have

$$\int_0^T \int_{-1}^1 K(x)u^2(x,t)dx\ dt + \int_0^T u^2(-1,t)dt \leq K_0 \int_{-1}^1 f^2(x)dx + K_0 \int_0^T g^2(t)dt, \tag{5.17}$$

if $\int_{-1}^1 K(x)dx$ is finite. Furthermore, one usually has the inequality

$$\int_{-1}^1 f^2(x)\ dx \leq K_2 \int_{-1}^2 K(x)f^2(x)dx.$$ In particular, this is true for the

Chebyshev norm. Thus we have the a priori estimate

$$\|u\|^2_{\omega,x,t} + \|u(-1,t)\|^2_t \leq C\left[\|f\|^2_{\omega,x} + \|g\|^2_t\right]. \tag{5.18}$$

## 6. STABILITY ANALYSIS OF PSEUDOSPECTRAL SCHEMES – HYPERBOLIC EQUATIONS

In the first section we described different spectral methods. In this section we shall concentrate on the collocation method and refer the reader to [17] for the Galerkin approach.

For periodic domains the pseudospectral Fourier method is the most appropriate. We consider the model problem

$$u_t = a(x)\ u_x \qquad\qquad 0 \leq x \leq 2\pi$$

$$\tag{6.1}$$

$$u(x,0) = f(x) \qquad\qquad u \text{ is } 2\pi \text{ periodic.}$$

25

The semi-discrete Fourier is given by

$$\frac{du_N}{dt}(x_j,t) = a(x_j) \sum_{k=0}^{2N-1} (D_1)_{jk}\, u_N(x_k,t).  \tag{6.2}$$

$D_N$ is the $2N \times 2N$ matrix given by (2.13). When $a(x)$ is positive for all $x$ we have

$$\frac{d}{dt}\left[\frac{\|u_N\|^2}{a(x)}\right] = 2(D_1\, u_N, u_N).$$

Since $D_1$ is antisymmetric it follows that

$$\frac{d}{dt} \sum_{j=0}^{2N-1} \frac{u_N^2(x,t)}{a(x_1)} = 0.$$

It is then straightforward to obtain an error estimate of the type

$$\|u(t) - u_N(t)\|_{a,0} \leqslant C(1+N^2)^{1/2 - q}\, \|f\|_{a,q}\ ,$$

where $\|\cdot\|_{a,q}$ denotes the Sobolev $q$ norm with weight $a(x)$.

When $a(x)$ changes sign in the interval $(0,2\pi)$ the situation is less clear. A number of numerical studies have shown that when one uses a fixed number of modes then the numerical solution becomes unbounded as one increases $t$. We wish to stress that this does not mean that the numerical method is unstable. Stability concerns itself with a fixed time interval $0 < t < T$ and lets the number of modes increase. We first consider a concrete example (Gottlieb, Orszag, and Turkel [15])

$$u_t = a(x)u_x$$
$$a(x) = \alpha \sin x + \beta \cos x + \gamma.  \tag{6.3}$$

One can now obtain the estimate

$$\| u_N(t) \|_1 \;\leqslant\; e^{\frac{1}{2}(|\alpha| + |\beta|)t} \, \| u_N(0) \|_1 . \qquad (6.4)$$

We see explicitly that for $0 \leqslant t \leqslant T$, we have

$$\| u_N(t) \|_1 \;\leqslant\; C \| u_N(0) \|_1 ,$$

however as T increases C increases exponentially. We note that for this example with $\alpha = 1$, $\beta = \gamma = 0$, the analytic solution is

$$u(x,t) = f\!\left[ 2 \, \tan^{-1}(e^t \, \tan \tfrac{x}{2}) \right].$$

For all t the solution is bounded when f is bounded. However, for large t there is a steep gradient near $x = 0$. Hence, when one uses a fixed number of modes N and lets t increase, one eventually reaches a time for which the mesh can no longer resolve the gradient. Hence, the main problem is not one of stability but rather one of resolution.

One can show that the growth with time is not only a difficulty with the collocation method but the same difficulty occurs with the Galerkin method. Nevertheless, Galerkin methods are used for long term meteorological calculations using nonlinear equations where the eigenvalues change sign. Hence, one may conclude that in some sense (6.1) is a harder problem than the nonlinear Euler equations (with smooth solutions) since (6.1) develops a sharp gradient as t increases.

We also note that one can stabilize the pseudospectral algorithm for (6.1) by rewriting (6.1) as

$$u_t = \frac{(au)_x + au_x}{2} - \frac{a_x u}{2} , \qquad (6.5)$$

and discretizing by

$$\frac{du_N}{dt} = \frac{1}{2}\left[D(au) + aDu\right] - \frac{a_x u}{2} . \qquad (6.6)$$

The approximation is now clearly axisymmetric plus a lower order term and hence stable for all $a(x)$. Nevertheless computations indicate that there is no advantage to (6.6) over (6.2).

Returning to (6.1) Tal-Ezer [48] has also shown that the solution is stable when $a(x) = \sin 2x$. Computations show that the eigenvalues have a bounded (independent of $N$) real part for $a(x) = \sin kx$. Presently there is no proof that (6.1) is stable for any $a(x)$ using the Fourier pseudospectral methods. Nevertheless many computations indicate that the difficulty is one of growth in time and lack of resolution and not stability and convergence.

One can alleviate the growth in time by filtering the higher modes in the expansion of $u_N$. Specifically one modifies the collocation method by considering (Kreiss and Oliger [23] and Majda, McDonough, and Osher [31]).

$$\overline{u}_N = \sum_{k=-N}^{N} a_k \, \rho_k \, e^{ikx} ,$$

where

$$\rho_k = \begin{cases} 1 & |k| < k_0 \\ \\ e^{-\sigma\left(\frac{|k| - k_0}{N - k_0}\right)^4} & |k| > k_0 \end{cases} . \qquad (6.7)$$

One can consider this as a type of artificial viscosity in Fourier space which does not alter the spectral accuracy of the scheme.

For the Chebyshev method we begin with the model scalar homogeneous equation

$$u_t = u_x \qquad\qquad -a \leqslant x \leqslant 1$$
$$u(x,0) = f(x) \qquad\qquad u(1,t) = 0. \qquad\qquad (6.8)$$

We consider an expansion of $u$ in terms of Chebyshev polynomials

$$u_N(x,t) = \sum_{k=0}^{N} a_k(t)T_k(x), \qquad\qquad (6.9)$$

where the $a_k(t)$ are chosen so that the equation is exact at the collocation points. We consider two sets of collocation points, see (3.2), (3.24)

$$x_j = \cos \frac{\pi j}{N} \qquad\qquad j = 0, \cdots, N$$

and

$$y_j = \cos \frac{\pi j}{N+1} \qquad\qquad j = 0, \cdots, N.$$

Hence, we can consider $u_N$ to be a polynomial of degree $N$.

To prove stability we first choose an appropriate norm. A natural norm would be

$$\|u_N\|_\omega^2 = \frac{\pi}{N} \sum_{k=0}^{N} u_N^2(x_j,t).$$

When $u$ is a polynomial of degree $N$ or less this is equivalent to

$$\|u_N\|_\omega^2 = \int_{-1}^{1} \frac{u_N^2(x,t)}{\sqrt{1-x^2}} \, dx.$$

However, we have already seen that even for the partial differential equation one cannot find an estimate of the type

$$\|u(x,t)\| \leqslant C\|f(x)\|.$$

Hence, we shall instead consider the norm

$$\|u\|^2_{\omega_1} = \|(1+x)u\|^2_{\omega} = \frac{\pi}{N} \sum_{k=0}^{N} (1+x_j)u(x_j,t). \tag{6.10}$$

Again when $u \varepsilon P_{N-1}$, $\|u\|^2_{\omega_1} = \int_{-1}^{1} \frac{(1+x)u^2(x,t)dx}{\sqrt{1-x^2}}$ . For simplicity we shall

consider the collocation points $y_j$ and simply state the results for the $x_j$.

Since $u_N$ is a polynomial of degree $N$ which satisfies $u$ at the

points $y_j$, $u_N$ must satisfy the differential equation

$$\frac{\partial u_N}{\partial t} = \frac{\partial u_N}{\partial x} - \frac{T'_{N+1}(x)}{N+1} u_x(1,t) \qquad u(1,t) = 0, \tag{6.11}$$

$$u_N(x,0) = f(x)$$

before proceeding we state the following lemma which is an extension of one by

Rivlin [45].

## Lemma

Let $u = \sum_{k=0}^{\infty} a_k T_k(x)$, then

$$\frac{1}{\pi} \int_{-1}^{1} \frac{u \, dx}{\sqrt{1-x^2}} = \frac{1}{N} \sum_{k=0}^{N} \frac{u(x_k)}{c_k} - \sum_{\ell=1}^{\infty} a_{2\ell N} \qquad \begin{array}{l} C_0 = C_N = 2 \\ \\ C_j = 1 \quad \text{otherwise} \end{array} \tag{6.12}$$

in particular, if $u$ is a polynomial of degree $4N - 1$ or less

$$\frac{1}{\pi} \int_{-1}^{1} \frac{udx}{\sqrt{1-x^2}} = \frac{1}{N} \sum_{k=0}^{N} \frac{u(x_k)}{c_k} - a_{2N}. \tag{6.13}$$

30

We now multiply (6.11) by $(1+y_j)u$ and sum over $j$ to get

$$\frac{\pi}{2(N+1)} \frac{d}{dt} \sum_{j=0}^{N+1} (1+y_j)u^2(y_j,t) = \frac{\pi}{2(N+1)} \sum_{j=0}^{N+1} (1+y_j)\left[u^2(y_j)\right]_x, \qquad (6.14)$$

since the equation is exact at the points $y_j$, $j = 0, \cdots, N$ while at $j = N+1$, $1+y_j = 0$. Using (6.13) we replace the sum by an integral, and so

$$\frac{\pi}{2(N+1)} \frac{d}{dt} \sum_{j=0}^{N} (1+y_j)u^2(y_j,t) = \frac{1}{2} \int_{-1}^{1} \frac{(1+x)(u^2)_x}{\sqrt{1-x^2}}\, dx$$

$$= -\frac{1}{2} \int_{-1}^{1} \frac{u^2\, dx}{(1-x)\sqrt{1-x^2}} < 0, \qquad (6.15)$$

or using the lemma again

$$\frac{1}{2} \frac{d}{dt} \int_{-1}^{1} \frac{(1+x)u^2}{\sqrt{1-x^2}}\, dx = -\frac{1}{2} \int_{-1}^{1} \frac{u^2\, dx}{(1-x)\sqrt{1-x^2}} < 0. \qquad (6.16)$$

Hence,

$$\|u_N(t)\|_{\omega_1} < \|f\|_{\omega_1}.$$

If we now integrate both sides of (6.15) with respect to $t$ we have

$$\int_{0}^{T} \int_{-1}^{1} \frac{u_N^2\, dx\, dt}{(1-x)\sqrt{1-x^2}} = -\int_{-1}^{1} \frac{(1+x)u_N^2(x,T)}{\sqrt{1-x^2}}\, dx + \int_{-1}^{1} \frac{(1+x)\, f^2(x)}{\sqrt{1-x^2}}\, dx$$

but

$$\int_{-1}^{1} \frac{u_N^2\, dx}{\sqrt{1-x^2}} < 2 \int_{-1}^{1} \frac{u_N^2\, dx}{(1-x)\sqrt{1+x^2}}$$

and

$$\int_{-1}^{1} (1+x)\, \frac{f^2\, dx}{\sqrt{1-x^2}} < C \int_{-1}^{1} \frac{f^2(x)}{\sqrt{1-x^2}}$$

so

$$\int_0^T \int_{-1}^1 \frac{u_N^2 \; dx \; dt}{\sqrt{1-x^2}} \le C \int_{-1}^1 \frac{f^2(x)}{\sqrt{1-x^2}} \; dx.$$

Hence $u_N$ satisfies the following a priori estimate.

## Theorem

Let $u \; \varepsilon \; P_N$ be a solution of (6.8) using the collocation points $y_j$ then for some constant $C$

$$\| u_N(t) \|_{\omega_1} \le \bar{C} \| f \|_{\omega_1}$$

and (6.17)

$$\int_0^T \| u_N(t) \|_\omega^2 \; dt \le C \| f \|_\omega^2 .$$

One can also prove this theorem by using the matrix representation for $D_1$ that is appropriate for the points $y_j$.

We also have a similar theorem for the points $x_j$.

## Theorem

Let $u \; \varepsilon \; P_N$ be a solution of (6.8) using the collocation points $x_j$. Then

$$\frac{\pi}{2N} \sum_{j=0}^N (1+x_j)(1 - \frac{x_j}{2}) u^2(x_j,t) + \frac{\pi(2N-1)}{32N} \left[ 2a_N(t) - a_{N-1}(t) \right]^2$$

(6.18)

$$\le \frac{\pi}{2N} \sum_{j=0}^N (1 - x_j)(1 - \frac{x_j}{2}) f^2(x_j) + \frac{\pi(2N-1)}{32N} \left[ 2a_N(0) - a_{N-1}(0) \right]^2.$$

The left hand side of (6.18) is indeed a norm. The sum is strictly positive for nontrivial $u^2$ unless $u^2(x_j,t) = 0$ for $j = 0, \cdots, N-1$ and $u^2(x_N,t) \ne 0$ but in this case $u = C(1-x)T_N'$ and so $2a_N - a_{N-1} \ne 0$.

## 7. STABILITY ANALYSIS OF PSEUDOSPECTRAL CHEBYSHEV SCHEMES

### — PARABOLIC EQUATIONS

The analysis of the pseudospectral Fourier method is straightforward and is similar to the discussion of the previous section. For the Chebyshev method we consider the equation

$$u_t = u_{xx} \qquad (7.1a) \qquad -1 < x < 1$$

$$u(1,t) = u(-1,t) = 0 \qquad (7.1b)$$

$$u(x,0) = f(x). \qquad (7.1c)$$

We now only consider the collocation points $x_j$ (3.4). The proof that the Chebyshev method is stable for (7.1) for these $x_j$ is similar to that of the previous section. When (7.1b) is replaced by Neumann data we derive the same error equation as in (4.11). We then differentiate (4.11) to obtain an equation for $v = u_x$ with Dirichlet boundary conditions. The same energy estimates can be used for $v$. We thus have

### Theorem

Using the Chebyshev collocation method for (7.1a-c) with the points $x_j = \cos \frac{\pi j}{N}$, $j = 0, \cdots, N$, we have

$$\frac{\pi}{N} \sum_{j=0}^{N} u_N^2(x_j,t) < \frac{\pi}{N} \sum_{j=0}^{N} f^2(x_j). \qquad (7.2)$$

When (7.1b) is replaced by $u_x(1,t) = u_x(-1,t)$ we have

$$\frac{\pi}{N} \sum_{j=0}^{N} (1 - x_j^2)(\frac{\partial u_N}{\partial x})^2 + N^2 a_N^2(t) < \frac{\pi}{N} \sum_{j=0}^{N} (1 - x_j^2)(\frac{\partial f}{\partial x})^2 + N^2 a_N^2(0), \quad (7.3)$$

where as before $a_N$ is the highest coefficient of the expansion of $u_N$ in terms of $T_j$.

We now consider (7.1) with the boundary condition (7.1b) replaced by the more general case

$$\alpha u(1,t) + \beta u_x(1,t) = 0$$
$$\gamma u(-1,t) + \delta u_x(-1,t) = 0. \quad (7.1c)$$

In this case a stability proof is not yet known for the Chebyshev collocation method. However, one can explicitly find the eigenfunctions. Furthermore, if $\alpha$ and $\beta$ have the same sign while $\gamma$ and $\delta$ have opposite signs then the eigenvalues are real, negative and distinct [10].

## 8. TIME DISCRETIZATIONS AND ITERATIVE METHODS

In the preceding sections we have described the construction of spectral approximations to the spatial operator $L$ in (1.2). In this section we present some of the difficulties one faces in solving the time-dependent equation

$$\frac{\partial u_N}{\partial t} = P_N L P_N u_N . \quad (8.1)$$

There are three distinct situations that arise in practice. At times one is only interested in the steady state version of (8.1). For such problems one can use a temporarily inaccurate or even inconsistent method but one wishes to reach the steady state quickly. When the operator $L$ is an elliptic operator

34

one can solve the steady state equation directly using multigrid or conjugate gradient methods. An efficient method for hyperbolic equations will be presented later.

A second possibility is that the solution to (8.1) changes in time at a much slower scale then it changes in space. For these types of problems one may use a comparatively low order scheme to discretize the time. When the temporal changes are on the same scale as the spatial changes, which is typical of many wave propagation problems, it is not useful to use a low order time integration scheme.

We now present several recent developments in the construction of time integration formulas for the Fourier method. In one case we consider hyperbolic equations where we are only interested in the steady state. The second case occurs when the time and space dimensions are of equal importance.

We consider the periodic problem

$$\frac{\partial u}{\partial t} = f_x(u) + g_y(u) \qquad \qquad 0 \le x \le \pi. \qquad (8.2)$$

Let $D_{1x}$ denote the Fourier derivative operator in the x-direction while $D_{1y}$ represents the y Fourier derivative. Define

$$H_{2x} u_j^n = u_{j+1,k}^n - 2u_{j,k}^n + u_{j-1,k}^n. \qquad (8.3)$$

We solve (8.2) by a multi-step procedure. The first two steps is a modified Euler approximation to (8.2). We thus have

$$u_{j,k}^{n+1/2} = u_{j,k}^n + \frac{\Delta t}{2}\left(D_{1x}\, f_{j,k}^n + D_{1y}\, g_{j,k}^n\right)$$

and (8.4)

$$\bar{u}_{j,k} = u_{j,k}^n + \Delta t\left(D_{1x}\, f_{j,k}^{n+1/2} + D_{1,y}\, g_{j,k}^{n+1/2}\right).$$

It is readily verified that this part of the scheme is unconditionally unstable. We thus add a correction term which is similar to one suggested by Lerat [27]. Thus, the final step is

$$\left(I - \frac{\alpha^2}{4}\left(\frac{\Delta t}{\Delta x}\right)^2 H_{2x}\right)\left(I - \frac{\alpha^2}{4}\left(\frac{\Delta t}{\Delta y}\right)^2 H_{2y}\right)\left(u_{j,k}^{n+1} - u_{j,k}^n\right) = \bar{u}_{j,k} - u_{j,k}^n. \quad (8.5)$$

It is clear that once a steady state is reached that $u_{j,k}^{n+1} = \bar{u}_{j,k} = u_{j,k}^n$. It follows from (8.4) that $D_{1x}\, f^{n+1/2} + D_{1y}\, g^{n+1/2} = 0$ and so the steady state solution is independent of the time step. We note that this is not true if the standard Lax-Wendroff finite difference formula is used instead of (8.4). We next show that (8.4) - (8.5) is linearly unconditionally stable if $\alpha^2 > \frac{\pi^2}{2}\rho^2(A)$, $\alpha^2 > \frac{\pi^2}{2}\rho^2(B)$ where $A = \frac{\partial t}{\partial u}$ and $B = \frac{\partial f}{\partial u}$. We now consider the scalar two-dimensional equation

$$u_t = u_x + u_y.$$

Since the problem is periodic we can take the Fourier transform of (8.4) - (8.5): let $\hat{u}$ be the Fourier transform of $u$ and $(\xi, \eta)$ the Fourier variables, $\lambda = \frac{\Delta t}{\Delta x} = \frac{\Delta t}{\Delta y}$ we have

$$\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\xi}{2}\right)\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\eta}{2}\right)\left(\hat{u}^{n+1} - \hat{u}^n\right) = \left[i\lambda(\xi+\eta) - \frac{\lambda^2}{2}(\xi+\eta)^2\right]\hat{u}^n \quad (8.6)$$

for $-\pi < \xi, \eta < \pi$.

Hence, the amplification matrix is

$$G(\xi,\eta) = \frac{\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\xi}{2}\right)\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\eta}{2}\right) - \frac{\lambda^2}{2}\left(\xi + \eta^2\right) + i\lambda(\xi + \eta)}{\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\xi}{2}\right)\left(1 + \alpha^2 \lambda^2 \sin^2 \frac{\eta}{2}\right)} \quad .(8.7)$$

By algebra it follows that $|G(\xi,\eta)| \leqslant 1$ if and only if

$$\frac{1}{4}(\xi+\eta)^2 \leqslant \alpha^2\left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2}\right) + \alpha^4 \sin^2 \frac{\xi}{2} \sin^2 \frac{\eta}{2} . \qquad (8.8)$$

A sufficient condition for (8.8) to hold is that

$$\frac{1}{4}(\xi+\eta)^2 \leqslant \frac{\xi^2 + \eta^2}{2} \leqslant \alpha^2\left(\sin^2 \frac{\xi}{2} + \sin^2 \frac{\eta}{2}\right),$$

i.e.,

$$\xi^2 \leqslant 2\alpha^2 \sin^2 \frac{\xi}{2} \qquad \text{for} \quad -\pi \leqslant \xi \leqslant \pi.$$

It is straightforward that this inequality holds when $\alpha^2 \geqslant \frac{\pi^2}{2}$ .

We next consider the other extreme when the time and space variations are of the same order of magnitude. We again start with the scalar equation

$$u_t = u_x \qquad\qquad (8.9)$$
$$u(x,0) = f(x) \qquad u \text{ is periodic.}$$

We solve this using a Fourier method with $2N+1$ modes. Since we wish spectral accuracy we wish to be able to represent the waves $e^{\pi i k t}$, $k = -N, \cdots, 0, \cdots, N$. Specifically, we assume that the solution to (8.9) is $u(x,t) = e^{i\pi N(x+t)}$. Using leapfrog in time and letting $v$ be the Fourier transform to $u$, the spectral leapfrog approximation to (8.9) is

$$v_N^{n+1} - v_N^{n-1} = 2\pi i \Delta t N \, v_N^n \qquad (8.10)$$

where for stability

$$N \, \Delta \, t \, \leqslant \, 1/\pi. \qquad (8.11)$$

Solving (8.10) we find that

$$v_N(t) = e^{in\alpha} \qquad \text{where} \qquad \sin \alpha = \pi i \Delta t N,$$

while the exact solution to (8.9) satisfies $v(t) = e^{iNt}$. Comparing $v_N(t)$ and $v(t)$ Tal-Ezer [48] found that the phase error behaves like $(\Delta t)^2 N^3$. This shows that if we wish to resolve the high modes we require

$$(\Delta t)^2 \, \leqslant \, C(\Delta x)^3, \qquad (8.12)$$

which is more stringent than (8.11). Hence, the accuracy requirements demand a much smaller $\Delta t$ than is allowed by stability.

In other words if we wish to resolve $N$ modes with a leapfrog-Fourier method we need not $2N+1$ modes but many more modes and hence the Fourier method is not efficient. To resolve this difficulty it is necessary to use a higher order accurate time integration formula. Tal-Ezer [48] has presented a scheme which is efficient and has spectral accuracy in time as well as space. The solution to

$$u_t = P_N L P_N u \qquad (8.13)$$
$$u(0) = u_0$$

is given by

$$u = e^{P_N LP_N t} u_0. \tag{8.14}$$

We approximate the solution operator by

$$u_N = H_m(P_N LP_N t)u_0, \tag{8.15}$$

where

$$H_m(i\theta R) = \sum_{k=0}^{m} i^k J_k(R) T_k(\theta),$$

i.e., $i^k T_k(\theta)$ is a function of $i\theta$ which we call $V_k(i\theta)$ then

$$H_m(P_N LP_N t)u_0 = \sum_{k=0}^{m} J_k(R) V_k(P_N LP_N t)u_0,$$

where $T_k$ is the k-th Chebyshev polynomial, $J_k$ is the Bessel function of order k and R is larger than the difference of the largest and smallest eigenvalues of $P_N LP_N$. Tal-Ezer [48] has found efficient ways of implementing (8.15) which do not require any complex arithmetic. He also shows that it is more efficient to use 2N+1 modes coupled with (8.15) in time in order to resolve $e^{i\pi Nx}$ rather than using more modes and leapfrog in time.

## 9. COMPRESSIBLE FLOW

We now consider the application of spectral methods to the Euler equations, i.e., compressible inviscid fluid dynamics. The main feature of this system of equations is that they constitute a nonlinear conservative system of hyperbolic equations. Hence, the solutions will frequently include

shocked flows. One might suspect that global methods are not suitable for problems with discontinuities. However, we shall see that it is still possible to use spectral methods for such problems.

Majda, McDonough and Osher [31] have shown that for a linear system of equations, even with constant coefficients the existence of discontinuities reduces the accuracy of the Fourier method if one does not specially treat the initial data. They also show that if one truncates the initial data using a Galerkin procedure one retains the spectral accuracy. For equations with variable coefficients one must also filter the higher modes at every time step. These results imply that discontinuities may reduce the global accuracy using an $L^p$ norm. Lax [25] on the other hand has argued that although the formal accuracy is lost nevertheless one can recover a highly accurate solution. This occurs since the numerical solution contains many small scale oscillations. Although this destroys local accuracy one can remove the oscillations by an appropriate post-processor and recover the spectral accuracy. Thus, spectral methods have high order resolution, even in the presence of discontinuities. It may also be shown [14] that the spectral method automatically satisfies the conservation form and hence the Rankine-Hugoniot conditions. It is also shown in [14] that one can then extend the theorem of Lax and Wendroff [26] and prove that if the spectral method converges it will converge to a weak solution of the differential equation. There are indications that filter (6.7) suggested by Majda, et al. [31], introduces an entropy condition into the numerical solution.

At present there are three trends in dealing with shocks. The simplest approach is to use a finite difference artificial viscosity to reduce the oscillations caused by the shock. This method reduces the formal accuracy and gives smeared profiles of the shock. Nevertheless, reasonable results have

been obtained by Taylor, et al. [51] and Hussaini and Zang [20]. A different approach is to use shock fitting as recommended by Moretti. Salas, Zang, and Hussaini [47] have used this technique for a bow shock which was mapped onto the boundary of the domain. Other applications are discussed in [21] and [22]. A third possibility [14] is to truncate the high modes at each time step and then to locate the shock and to filter the solution on each side of the shock only at the final time. It is possible to use the structure of the spectral method to locate the shock. Thus, we compare the given numerical solution in Chebyshev space with a step function in order to locate the shock and estimate its strength. This gives a much better shock locater than is possible with finite difference methods. The shock locator is based on the fact that spectral methods give sharp discontinuities. Hence, any artificial viscosity that smears discontinuities will destroy the usefulness of the shock locator.

## REFERENCES

[1] C. CANUTO and A. QUARTERONI, Variational methods in the theoretical analysis of spectral approximation, Proc. of the Symposium on Spectra Method, SIAM CBMS Series, 1983.

[2] C. CANUTO and A. QUARTERONI, Approximation results for orthogonal polynomials in Sobolev spaces, Math. Comp., 38 (1982), pp. 67-86.

[3] C. CANUTO and A. QUARTERONI, Error estimates for spectral and pseudospectral approximations of hyperbolic equations, SIAM J. Numer. Anal., 19 (1982), pp. 629-642.

[4] C. CANUTO and A. QUARTERONI, Spectral and pseudospectral methods for parabolic problems with nonperiodic boundary conditions, Calcolo, 18 (1981), pp. 197-217.

[5] B. FORNBERG, A numerical study of two-dimensional turbulence, J. Comp. Phys., 25 (1977), pp. 1.

[6] B. FORNBERG, On a Fourier method for the integration of hyperbolic equations, SIAM J. Numer. Anal., 12 (1975), pp. 509.

[7] D. GOTTLIEB, M. Y. HUSSAINI, and R. VOIGT, Proc. of the Symposium on Spectral Methods, SIAM CBMS Series, 1983.

[8] D. GOTTLIEB, M. Y. HUSSAINI, and S. A. ORSZAG, Introduction to the Proc. of the Symposium on Spectral Methods, SIAM CBMS Series, 1983.

[9] D. GOTTLIEB, L. LUSTMAN, and C. L. STREETT, Spectral methods for two-dimensional shocks, Proc. of the Symposium on Spectral Methods, <u>SIAM CBMS Series</u>, 1983

[10] D. GOTTLIEB and L. LUSTMAN, The spectrum of the Chebyshev collocation operator for the heat equation, to appear in <u>SIAM J. Numer. Anal.</u>, ICASE Report No. 82-12, May 1982.

[11] D. GOTTLIEB, M. GUNZBURGER, and E. TURKEL, On numerical boundary treatment for hyperbolic systems, <u>SIAM J. Numer. Anal.</u>, 19 (1982), pp. 671-697.

[12] D. GOTTLIEB and L. LUSTMAN, The DuFort-Frankel Chebyshev method for parabolic initial boundary value problems, <u>Comput. & Fluids</u>, 11 (1983), pp. 107-120.

[13] D. GOTTLIEB, The stability of pseudospectral Chebyshev methods, <u>Math. Comp.</u>, 36 (1981), pp. 107-118.

[14] D. GOTTLIEB, L. LUSTMAN and S. A. ORSZAG, Spectral calculations of one-dimensional inviscid compressible flow, <u>SIAM J. Sci. Statis. Comput.</u>, 2 (1981), pp. 296-310.

[15] D. GOTTLIEB, S. A. ORSZAG and E. TURKEL, Stability of pseudospectral and finite difference methods for variable coefficient problems, <u>Math. Comp.</u>, 37 (1981), pp. 293-305.

[16]    D. GOTTLIEB and E. TURKEL, On time discretization for spectral methods, Stud. Appl. Math., 63 (1980), pp. 67–86.

[17]    D. GOTTLIEB and S. A. ORSZAG, Numerical Analysis of Spectral Methods: Theory and Applications, CBMS Regional Conference Series in Applied Mathematics 26, SIAM, 1977.

[18]    R. S. HIRSH, T. D. TAYLOR, M. M. NADWORNY, and J. L. KERR, Techniques for efficient implementation of pseudospectral methods and comparison, Proc. of the Symposium on Spectral Methods, SIAM CBMS Series, 1983.

[19]    R. S. HIRSH and T. D. TAYLOR, Recent developments at APL in the use of pseudospectral techniques, Proc. of EUROMECH Conference on Spectral Methods, Nice, France, 1982.

[20]    M. Y. HUSSAINI, M. D. SALAS and T. A. ZANG, Spectral methods for inviscid, compressible flows, Advances in Computational Transonics, (W. G. Habashi, ed.), Pineridge Press, Swansea, United Kingdom, 1983.

[21]    M. Y. HUSSAINI, D. A. KOPRIVA, M. D. SALAS and T. A. ZANG, Spectral methods for the Euler equations, Proc. of the 6th AIAA Computational Fluid Dynamics Conferernce, Danvers, MA, July 1983.

[22]    M. Y. HUSSAINI, C. L. STREETT and T. A. ZANG, Spectral methods for partial differential equations, ICASE Report No. 83–46, August 1983.

44

[23]   H. O. KREISS and J. OLIGER, Stability of the Fourier method, <u>SIAM J.</u>
       <u>Numer. Anal.</u>, 16 (1979), pp. 421-433.


[24]   H. O. KREISS and J. OLIGER, Comparison of accurate methods for the
       integration of hyperbolic equations, <u>Tellus,</u> 24 (1972), pp. 199.


[25]   P. D. LAX, Accuracy and resolution in the computation of solutions of
       linear and nonlinear equations, <u>Recent Advances in Numerical Analysis,</u>
       Proc. Symp. Mathematical Research Center, University of Wisconsin,
       Academic Press, 1978, pp. 107-117.


[26]   P. D. LAX and B. WENDROFF, Systems of conservation laws, <u>Comm. Pure</u>
       <u>Appl. Math.</u>, 13 (1960), pp. 217-237.


[27]   A. LERAT, J. SIDES, and V. DARU, An implicit finite-volume method for
       solving the Euler equations, Eighth Intl. Conf. Numer. Methods Fluid
       Dynamics, <u>Springer-Verlag Lecture Notes in Physics,</u> 170, (E. Krause,
       ed.) 1982, pp. 343-349.


[28]   Y. MADAY and A. QUARTERONI, Approximation of Burgers' equation by
       pseudospectral methods, to appear in <u>R.A.I.R.O. An. Num.</u>


[29]   Y. MADAY and A. QUARTERONI, Spectral and pseudospectral approximations
       of Navier-Stokes equations, <u>SIAM J. Numer. Anal.</u>, 19 (1982), pp. 761-
       780.

[30]   Y. MADAY and A. QUARTERONI, Legendre and Chebyshev spectral approximations of Burgers' equation, Numer. Math., 37 (1981), pp. 321-332.

[31]   A. MAJDA, J. McDONOUGH and S. OSHER, The Fourier method for nonsmooth initial data, Math. Comp., 32 (1978), pp. 1041-1081.

[32]·  B. MERCIER, Stabilité et convergence des méthodes spectrales polynômiales:   application a l'équation d'advection, R.A.I.R.O. An. Num., 16 (1982), pp. 67-100.

[33]   B. METIVET and Y. MORCHOISNE, Resolution of unsteady Navier-Stokes equations by multi-domin spectral method, Proc. EUROMECH Conf. on Spectral Methods, Nice, France, 1982.

[34]   B. METIVET and Y. MORCHOISNE, Multi-domain spectral technique for viscous flow calculations, Proc. 4th GAMM Conf. on Numerical Methods in Fluid Mechanics (H. Viviand, ed.) Vieweg, 1982, pp. 207-219.

[35]   Y. MORCHOISNE, Resolution of Navier-Stokes equations by a space-time pseudospectral method, La Rechereche Aerospatiale, No. 1979-5, pp. 293-306 (English translation).

[36]   S. A. ORSZAG, Spectral methods for problems in complex geometries, J. Comp. Phys., 37 (1980), pp. 70-92.

[37]   S. A. ORSZAG and D. GOTTLIEB, High resolution spectral calculations of invisicid compressible flows, approximation method for Navier-Stokes problems, Lecture Notes in Mathematics, 771, Springer-Verlag, Berlin, New York, 1980, pp. 381.

[38]   S. A. ORSZAG and M. ISRAELI, Numerical simulation of viscous incompressible flows, Annual Review of Fluid Mechanics, Annual Reviews, Inc., Palo Alot, CA, 6 (1974), pp. 281.

[39]   S. A. ORSZAG, Comparison of pseudospectral and spectral approximations, Stud. Appl. Math., 51 (1972), pp. 253-259.

[40]   S. A. ORSZAG, Numerical simulations of incompressible flows within simple boundaries: accuracy, J. Fluid Mech., 49 (1971), pp. 75-112.

[41]   S. OSHER, Smoothing for Spectral Methods, Proc. of the Symposium on Spectral Methods, SIAM CBMS Series, 1983.

[42]   J. E. PASCIAK, Spectral methods for a nonlinear initial value problem involving pseudo-differential operators, SIAM J. Numer. Anal., 19 (1982), pp. 142-154.

[43]   J. E. PASCIAK, Spectral and pseudospectral methods for advection equations, Math. Comp., 35 (1980), pp. 1081-1092.

[44]  A. QUARTERONI, Theoretical and computational aspects of spectral methods, Proc. 5th Intl. Conf. on Computing Methods in Applied Sciences and Engineering, Versailles, 1981, North-Holland, also Istituto di Analisi Numerica Pub. 306.

[45]  T. J. RIVLIN, The Chebyshev Polynomials, Wiley, New York, 1974.

[46]  L. SAKELL, Solution to the Euler equation of motion, pseudospectral techniques, 10th IMAC World Congress System, Simul. and Sci. Comput., 1982.

[47]  M. D. SALAS, T. A. ZANG and M. Y. HUSSAINI, Shock-fitted Euler solutions to shock-vortex interactions, Eighth Intl. Conf. Numer. Methods Fluid Dynamics, Springer-Verlag Lecture Notes in Physics 170, (E. Krause, ed.), 1982.

[48]  H. TAL-EZER, Ph.D. Thesis, Tel-Aviv University, 1983.

[49]  T. D. TAYLOR and J. W. MURDOCK, Application of spectral methods to the solution of the Navier-Stokes equation, Comput. Fluids, 9 (1981), pp. 255-262.

[50]  T. D. TAYLOR, R. S. HIRSH, and M. M. NADWORNY, Comparison of FFT, direct inversion and conjugate gradient methods for solution of flow equations by pseudospectral methods, Proc. 4th GAMM Conf. Numerical Methods, Paris, France, 1981.

[51]  T.  D.  TAYLOR,  R.  B.  MYERS  and  H.  A.  JEFFREY,  Pseudospectral calculations of shock waves, rarefaction waves and contact surfaces, Comput. Fluids, 9 (1981), pp. 469-473.


[52]  E.  TURKEL,  Numerical  methods  for  large-scale  time-dependent  partial differential equations, Computational Fluid Dynamics, 2, Hemisphere Publishing Corp., von Karman Institute, 1980, pp. 127-262.


[53]  E.  TURKEL,  Progress  in  computational  physics,  Comput.  Fluids,  11 (1983), pp. 121-144.


[54]  T.  A.  ZANG and M.  Y.  HUSSAINI,  Mixed  spectral/finite  difference approximations for slightly viscous flows, Lecture Notes in Physics 141, Springer-Verlag, 1980, pp. 461.

| 1. Report No. NASA CR-172241 | 2. Government Accession No. | 3. Recipient's Catalog No. |
|---|---|---|
| 4. Title and Subtitle Spectral Methods for Time Dependent Partial Differential Equations | | 5. Report Date October 1983 |
| | | 6. Performing Organization Code |
| 7. Author(s) David Gottlieb and Eli Turkel | | 8. Performing Organization Report No. 83-54 |
| | | 10. Work Unit No. |
| 9. Performing Organization Name and Address Institute for Computer Applications in Science and Engineering Mail Stop 132C, NASA Langley Research Center Hampton, VA 23665 | | 11. Contract or Grant No. NAS1-16394, NAS1-17130 |
| 12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, D.C. 20546 | | 13. Type of Report and Period Covered Contractor report |
| | | 14. Sponsoring Agency Code |

15. Supplementary Notes    Additional support: Air Force Office of Scientific Research under Contract Nos. AFOSR 77-3405 and AFOSR 83-0089 and U.S. Army Research and Standardization Group (Europe) Contract DAJA38-80-C-0032. Langley Technical Monitor: Robert H. Tolson, Final Report

16. Abstract

The theory of spectral methods for time dependent partial differential equations is reviewed. When the domain is periodic Fourier methods are presented while for nonperiodic problems both Chebyshev and Legendre methods are discussed. The theory is presented for both hyperbolic and parabolic systems using both Galerkin and collocation procedures. While most of the review considers problems with constant coefficients the extension to nonlinear problems is also discussed. Some results for problems with shocks are presented.

| 17. Key Words (Suggested by Author(s)) spectral methods differential equations Chebyshev methods | 18. Distribution Statement 64   Numerical Analysis  Unclassified-Unlimited |
|---|---|

| 19. Security Classif. (of this report) Unclassified | 20. Security Classif. (of this page) Unclassified | 21. No. of Pages 50 | 22. Price A03 |
|---|---|---|---|