Nonlinear Galerkin Method Using Chebyshev and Legendre Polynomials I. The One-Dimensional Case

Article *in* SIAM Journal on Numerical Analysis · February 1995

DOI: 10.1137/0732007 · Source: CiteSeer

2 authors:

Jie Shen
Purdue University

Roger Temam
Indiana University Bloomington

Proceeding of the International Conference on Nonlinear Evolution Partial Differential Equations, June 1993, Beijing, China

NONLINEAR GALERKIN METHOD USING LEGENDRE POLYNOMIALS

Jie Shen† & Roger Temam‡

Abstract. A nonlinear Galerkin method using Legendre polynomials is presented for solving linear elliptic and nonlinear dissipative evolution equations. The essential idea is to decompose the approximation space into a long-wavelength part and a short-wavelength part which are mutually orthogonal with respect to the principal elliptic operator of the equation.

1. Introduction. The exchange of energy between the long- and short-wavelength components of a flow is an important aspect of nonlinear phenomena. Although the short-wavelength component usually carries only a small fraction of the total energy, its effect through the nonlinear interaction with the long-wavelength component over a long time integration is not negligible and sometimes essential (cf. [**5**], [**4**]). However, it is computationally inefficient to allocate as much computing resources to compute the short-wavelength component of the flow carrying little energy as we do with the long-wavelength component of the flow which carries most of the energy. The nonlinear Galerkin method introduced in [**8**] was a first attempt to address this issue from a computational point of view. It has been applied to many different space discretizations (cf. [**9**], [**2**], [**10**]) and has proven to be computationally efficient (cf. [**3**], [**7**]).

We consider in this article the application of the nonlinear Galerkin method to spectral discretizations using Legendre polynomials. *Our objective is to derive simplified versions of the classical algorithms which produce better conditioned systems and a reduction in computing time without affecting the discretization error of the scheme under consideration.*

This work was supported in part by NSF grants DMS-9205300 and DMS-9024769 and DOE grant DE-FG02-92ER25120.

We now describe the nonlinear Galerkin Method in an abstract setting. Let H be a given Hilbert space with a scalar product $(\cdot,\cdot)$ and a norm $\|\cdot\|$. We shall consider a class of nonlinear evolution equations of the form:

$$\frac{du}{dt} + \nu A u + C u + B(u) = f,\ t > 0, \tag{1.1}$$

with $u(0) = u_0$. The operator A is a strictly positive linear unbounded self-adjoint operator with domain $D(A)$ dense in H. Hence, A^s ($s \in \mathbb{R}$) is well defined and we set $V = D(A^{\frac{1}{2}})$. B is a nonlinear operator such that $B(u) = B(u,u)$, where $B(\cdot,\cdot)$ is a bilinear form from $V \times V$ into V', C is a linear operator from V into H and there exists $c_1 > 0$ such that

$$((A + C)v, v) \geq c_1 \|v\|_V^2,\ \forall\, v \in V.$$

The variational formulation for (1.1) is to find $u(t) \in V$ such that

$$(\frac{du}{dt}, v) + \nu(Au, v) + (Cu, v) + (B(u), v) = (f, v),\ \forall\, v \in V \tag{1.2}$$

with $u(0) = u_0$. Let V_h, where h is the discretization parameter, be a finite dimensional approximation of V. The classical Galerkin method for (1.2) in V_h is to find $u_h(t) \in V_h$ such that

$$(\frac{du_h}{dt}, v) + \nu(Au_h, v) + (Cu_h, v) + (B(u_h), v) = (f, v),\ \forall\, v \in V_h, \tag{1.3}$$

with $u_h(0) = u_{0h}$, where u_{0h} is some projection of u_0 in V onto V_h.

The nonlinear Galerkin method stems from the recently developed theory on Inertial Manifolds and Approximate Inertial Manifolds. It is intended to provide an approximate interaction law between the long- and short-wavelength components. Essential to the method is a decomposition of the space $V_h = Y_{h_c} \oplus Z_h$, where $h < h_c \to 0$, Y_{h_c} consists of the long-wavelength elements and Z_h consists of the short-wavelength elements. With this decomposition, we can set $u_h = y + z$ for $y \in Y_{h_c}$ and $z \in Z_h$, and rewrite (1.3) as follows:
Find $y \in Y_{h_c}$ and $z \in Z_h$ such that $\forall\, v \in Y_{h_c}$ and $w \in Z_h$,

$$\frac{d}{dt}(y + z, v) + \nu(A(y + z), v) + (C(y + z), v) + (B(y + z), v) = (f, v), \tag{1.4}$$

$$\frac{d}{dt}(y + z, w) + \nu(A(y + z), w) + (C(y + z), w) + (B(y + z), w) = (f, w), \tag{1.5}$$

together with $y(0) = u_{0h_c}$, $z(0) = u_{0h} - u_{0h_c}$. Let us note that the short-wavelength component z usually carries only a small part of the total energy. Hence, some higher order (in z) terms can be dropped without affecting the accuracy. If in addition the decomposition is L^2-orthogonal, i.e.

$$(y, z) = 0,\ \forall y \in Y_{h_c}, z \in Z_h, \tag{A}$$

we can then approximate the system (1.4)-(1.5) by the following nonlinear Galerkin scheme:

Find $y(t) \in Y_{h_c}$, $z(t) \in Z_h$ such that $\forall v \in Y_{h_c},\ w \in Z_h$,

$$\frac{d}{dt}(y, v) + \nu(A(y+z), v) + (C(y+z), v) + (B(y+z), v) = (f, v), \tag{1.6}$$

$$\gamma \frac{d}{dt}(z, w) + \nu(A(y+z), w) + (C(y+z), w) + (B(y), w) = (f, w), \tag{1.7}$$

with $y(0) = u_{0h_c}$; γ is either 1 or 0; $z(0) = u_{0h} - u_{0h_c}$ if $\gamma = 1$.

We use $\gamma = 0$ in case $\dfrac{dz}{dt}$ is small compared to other terms.[1] Otherwise we take $\gamma = 1$.

Let us mention that the wavelet-like incremental unknowns method [**2**] and the two-grid finite element method [**10**] provide decompositions which are L^2-orthogonal.

The above system can be interpreted in two different aspects:

(i) Equation (1.7) defines a short-wavelength correction z in terms of the long-wavelength component y. Hence, (1.6)–(1.7) can be viewed as *a corrected Galerkin scheme in* Y_{h_c}.

(ii) Since (1.6)–(1.7) was obtained by neglecting some small terms in (1.4)–(1.5), we can also view (1.6)–(1.7) as *a simplified Galerkin scheme in* V_h.

The first interpretation leads us to expect that (1.6)–(1.7) gives a better approximation than the Galerkin scheme in Y_{h_c}, while the second interpretation leads us to expect that (1.6)–(1.7) provides the same accuracy as the Galerkin scheme in V_h.

The system (1.6)–(1.7) can be further simplified if the decomposition is also A-orthogonal, i.e.

$$(Ay, z) = 0,\ \forall y \in Y_{h_c}, z \in Z_h. \tag{B}$$

[1] This fact is rigorously established for some special decompositions (cf. [**4**], [**10**]).

In this case, we can approximate (1.4)–(1.5) by the following Nonlinear Galerkin scheme:

Find $y \in Y_{h_c}$, $z \in Z_h$ such that $\forall v \in Y_{h_c}$, $z \in Z_h$,

$$\frac{d}{dt}(y,v)+\nu(Ay,v)+(Cy,v)+(B(y+z),v)=(f,v), \tag{1.8}$$

$$\gamma\frac{d}{dt}(z,w)+\nu(Az,w)+(Cz,w)+(B(y),w)=(f,w), \tag{1.9}$$

with $y(0)=u_{0h_c}$; γ is either 1 or 0; $z(0)=u_{0h}-u_{0h_c}$ if $\gamma=1$.

The above scheme is certainly computationally more efficient than the scheme (1.6)–(1.7). The advantage of the nonlinear Galerkin scheme (1.8)–(1.9) over the classical Galerkin scheme (1.3) is two-fold: Firstly, the system (1.8)–(1.9) is greatly simplified compared to the Galerkin scheme (1.3). Secondly, the system (1.8)–(1.9) significantly reduces the stiffness of the system (1.3), since the stability or CFL (Courant-Friedrichs-Levy) condition for a full or partially explicit time discretization scheme for (1.6)–(1.7) only depends on the long-wavelength components (see for instance [**11**], [**14**]). Consequently, larger time steps can be used to greatly reduce the computing time, especially when one has to integrate over a long time interval.

However, the only available decomposition satisfying both (A) and (B) is based on the eigenfunctions of A. Unfortunately, this decomposition is only practical for problems with periodic boundary conditions. In order to apply scheme (1.8)–(1.9) to more general problems, we propose to construct decompositions which are A-orthogonal and "nearly" L^2-orthogonal (in a sense to be specified later), and we shall show that with this type of decompositions (1.8)–(1.9) is still as accurate as (1.3). In the next section, we construct several decompositions of this type with respect to different elliptic operators. Then in Section 3, we present an error estimate for the nonlinear Galerkin scheme.

2. Long- and short-wavelength decompositions. The following notations will be used hereafter. Let $I=(-1,1)$, and we denote by $H^s(I)$, $H_0^s(I)$ the usual Sobolev spaces with the norm $\|\cdot\|_s$. Let $L_n(x)$ be the n^{th}-order Legendre polynomial. We describe below our spectral discretization with Legendre polynomials.

For any $d \geq 1$ such that dm is an integer, we set

$$\begin{aligned} S_{dm} &= \ \text{span}\{L_0(x),\ L_1(x),\ \dots,\ L_{dm}(x)\}, \\ V_{dm} &= \{v \in S_{dm} : v(\pm 1)=0\}, \\ W_{dm} &= \{v \in S_{dm} : v(\pm 1)=v'(\pm 1)=0\}. \end{aligned}$$

S_{dm}, V_{dm} and W_{dm} are respectively finite dimensional approximations of $L^2(I)$, $H_0^1(I)$ and $H_0^2(I)$. We denote by π_{dm} the orthogonal projector from $L^2(I)$ onto S_{dm}.

2.1. 1-D case: decomposition of V_{dm} with respect to $A = -\frac{d^2}{dx^2}$. Let us write

$$\phi_k(x) = L_k(x) - L_{k+2}(x).$$

It is an easy matter to verify that

$$V_{dm} = \text{ span}\{\phi_0(x), \phi_1(x), \cdots, \phi_{dm-2}(x)\}. \tag{2.1}$$

Furthermore, using properties of the Legendre polynomials, we can show that

$$\int_I \phi_k'(x)\phi_j'(x)dx = 0, \ \forall\, k \neq j, \tag{2.2}$$

$$\int_I \phi_k(x)\phi_j(x)dx = 0, \ \forall\, j \neq k-2, k, k+2. \tag{2.3}$$

In analogy with the Fourier series, we consider a high (resp. low) order Legendre polynomial as a short (resp. long) wavelength component. Hence, a natural decomposition of V_{dm} is: $V_{dm} = V_m \oplus Z_{dm}$ where

$$Z_{dm} = \text{span}\{\phi_{m-1}(x), \phi_m(x), \cdots, \phi_{dm-2}(x)\}.$$

We observe immediately that $(Ay, z) = 0, \ \forall\, y \in V_m, z \in Z_{dm}$, and

$$(y, z) = (y, (I - \pi_{m-2})z), \ \forall\, y \in V_m, z \in Z_{dm}. \tag{2.4}$$

In other words, V_m and Z_{dm} are A-orthogonal and nearly L^2-orthogonal.

We note that the scheme (1.8)–(1.9) can also be applied to a linear steady problem. Let us consider for instance the second order linear steady equation:

$$\alpha u - u_{xx} = f \text{ in } I; \ u(\pm 1) = 0, \tag{2.5}$$

where α is a constant such that the operator $\alpha I - D_{xx}$ is positive in $H_0^1(I)$. The classical Galerkin approximation of (2.5) in V_{dm} is: Find $u_{dm} \in V_{dm}$ such that

$$\alpha(u_{dm}, v) + (u_{dm}', v') = (f, v), \ \forall\, v \in V_{dm}. \tag{2.6}$$

It can be easily shown [**1**] that

$$\|u_{dm} - u\|_1 \le C(s)(dm)^{1-s}\|f\|_{s-2},\ \forall\, s \ge 1.$$

The scheme (1.8)–(1.9) applied to (2.6) is:
Find $y_m \in V_m$ and $z_{dm} \in Z_{dm}$ such that

$$\alpha(y_m, v) + ({y_m}', v') = (f, v),\ \forall\, v \in V_m, \tag{2.7}$$

$$\alpha(z_{dm}, w) + ({z_{dm}}', w') = (f, w),\ \forall\, w \in Z_{dm}. \tag{2.8}$$

We note that $y_m + z_{dm} = u_{dm}$ if and only if $\alpha = 0$. However, we can prove the following results.

Lemma 2.1.

$$\|y_m + z_{dm} - u\|_1 \le C(s)\left(|\alpha| m^{-1-s} + (dm)^{1-s}\right)\|f\|_{s-2},\ \forall\, s \ge 1.$$

Remark 2.1. *We have approximately decomposed a large system* (2.6) *into two small systems* (2.7)*–*(2.8) *with the same accuracy as long as* $|\alpha| d^{s-1} \lessapprox m^2$*. We note that relation* (2.4) *is of crucial importance for this type of results.*

A proof of Lemma 2.1 with $\alpha = 1$ *is given in [***13***]. However the same proof carries over to the general case.*

2.2. 1-D case: decomposition of W_{dm} with respect to $A = \frac{d^4}{dx^4}$. Let us denote

$$\psi_k(x) = L_k(x) - \frac{2(2k+5)}{2k+7}L_{k+2}(x) + \frac{2k+3}{2k+7}L_{k+4}(x). \tag{2.9}$$

It is shown in [**12**] that

$$W_{dm} = \mathrm{span}\{\psi_0(x), \psi_1(x), \cdots, \psi_{dm-4}(x)\},$$

and

$$(\psi_i''(x), \psi_j''(x)) = 0,\ \forall\, i \ne j. \tag{2.10}$$

Hence a natural decomposition of W_{dm} is: $W_{dm} = W_m \oplus \tilde{Z}_{dm}$ where

$$\tilde{Z}_{dm} = \mathrm{span}\{\phi_{m-3}(x), \phi_{m-2}(x), \cdots, \phi_{dm-4}(x)\}.$$

Thanks to (2.9) and (2.10), we derive that $(Ay, z) = 0$ and

$$(y, z) = (y, (I - \pi_{m-4})z), \ \forall\, y \in W_m, z \in \tilde{Z}_{dm}. \tag{2.11}$$

Let us now apply the decomposition to the fourth order linear steady equation:

$$\alpha u - \beta u_{xx} + u_{xxxx} = f \text{ in } I, \ u(\pm 1) = u_x(\pm 1) = 0, \tag{2.12}$$

where α and β are two constants such that $\alpha I - \beta D_{xx} + D_{xxxx}$ is positive in $H_0^2(I)$. The Galerkin approximation of (2.12) in W_{dm} is:

Find $u_{dm} \in W_{dm}$ such that

$$\alpha(u_{dm}, v) + \beta(u'_{dm}, v') + (u''_{dm}, v'') = (f, v), \ \forall\, v \in W_{dm}. \tag{2.13}$$

and it can be shown [**1**] that

$$\|u_{dm} - u\|_2 \le C(s)(dm)^{2-s}\|f\|_{s-4}, \ \forall\, s \ge 2.$$

The scheme (1.8)–(1.9) applied to (2.12) is:

Find $y_m \in W_m$ and $z_{dm} \in \tilde{Z}_{dm}$ such that

$$\alpha(y_m, v) + \beta(y'_m, v') + ({y_m}'', v'') = (f, v), \ \forall\, v \in W_m, \tag{2.14}$$

$$\alpha(z_{dm}, w) + \beta(z'_{dm}, w') + ({z_{dm}}'', w'') = (f, w), \ \forall\, w \in \tilde{Z}_{dm}. \tag{2.15}$$

Equations (2.14)–(2.15) can be efficiently solved as indicated in [**12**]. The following lemma indicates that (2.14)–(2.15) is as accurate as (2.13) if $|\alpha| d^{s-2} \lessapprox m^4$ and $|\beta| d^{s-2} \lessapprox m^2$.

Lemma 2.2.

$$\|u - y_m - z_{dm}\|_2 \le C(s)(|\alpha| m^{-2-s} + |\beta| m^{-s} + (dm)^{2-s})\|f\|_{s-2}, \ \forall\, s \ge 2.$$

Sketch of the Proof. The proof is quite similar to the proof of Lemma 2.1 given in [**13**]. Let $\pi_{dm}^{2,0}$ be the orthogonal projector from $H_0^2(I)$ onto W_{dm}, i.e. $((u - \pi_{dm}^{2,0} u)'', v'') = 0$ for all $v \in W_{dm}$. It can be shown [**1**] that

$$\|u - \pi_{dm}^{2,0} u\|_\mu \le C(s)(dm)^{\mu - s}\|u\|_s, \ \forall\, 0 \le \mu \le 2, \ s \ge 2. \tag{2.16}$$

We derive from (2.12) that $\forall\, v \in W_m$, and $w \in \tilde{Z}_{dm}$,

$$\alpha(u, v+w) + \beta(u', v'+w') + ((\pi_{dm}^{2,0} u)'', v''+w'') = (f, v+w). \tag{2.17}$$

Let us denote $\xi = \pi_m^{2,0} u - y_m$ and $\eta = (\pi_{dm}^{2,0} - \pi_m^{2,0})u - z_{dm}$. We note that $\eta \in \tilde{Z}_{dm}$. Indeed, by definition of $\pi_m^{2,0}$ we find that

$$(\pi_{dm}^{2,0} u - \pi_m^{2,0} u, v'''') = 0,\ \forall\, v \in W_m.$$

Since the map $\frac{d^4}{dx^4}$: $W_m \to S_{m-4}$ is surjective, we infer that $(\pi_{dm}^{2,0} u - \pi_m^{2,0} u, v) = 0,\ \forall\, v \in S_{m-4}$. Therefore, $\pi_{dm}^{2,0} u - \pi_m^{2,0} u \in \tilde{Z}_{dm}$, which implies that $\eta \in \tilde{Z}_{dm}$.

Now subtracting (2.14) and (2.15) from (2.17), we obtain
$\forall\, v \in V_m$ and $w \in Z_{dm}$,

$$\begin{aligned} &\alpha(u - y_m, v) + \alpha(u - z_{dm}, w) + \beta((u-y_m)', v') \\ &\quad + \beta((u - z_{dm})', w') + ((\pi_{dm}^{2,0} u - y_m - z_{dm})'', (v+w)'') = 0. \end{aligned} \tag{2.18}$$

Since

$$\begin{aligned} \alpha(u - y_m, v) &= \alpha(u - \pi_m^{2,0} u, v) + \alpha(\xi, v), \\ \alpha(u - z_{dm}, w) &= \alpha(u - (\pi_{dm}^{2,0} - \pi_m^{2,0})u, w) + \alpha(\eta, w), \\ \beta((u - y_m)', v') &= \beta((u - \pi_m^{2,0} u)', v') + \beta(\xi', v'), \\ \beta((u - z_{dm})', w') &= \beta((u - (\pi_{dm}^{2,0} - \pi_m^{2,0})u)', w') + \beta(\eta', w'), \end{aligned}$$

setting $v = \xi$ and $w = \eta$ in (2.18), we find that

$$\begin{aligned} &\alpha(\|\xi\|^2 + \|\eta\|^2) + \beta(|\xi|_1^2 + |\eta|_1^2) + (|\xi|_2^2 + |\eta|_2^2) \\ &\qquad = -\,\alpha(u - \pi_m^{2,0} u, \xi) - \alpha(u - (\pi_{dm}^{2,0} - \pi_m^{2,0})u, \eta) \\ &\qquad\quad - \beta((u - \pi_m^{2,0} u)', \xi') - \beta((u - (\pi_{dm}^{2,0} - \pi_m^{2,0})u)', \eta'), \end{aligned} \tag{2.19}$$

where $|\cdot|_i$, $(i = 1, 2)$ is the semi-norm defined by $|v|_i = \sqrt{(D^i v, D^i v)}$.

The Lemma can then be established by bounding properly the four terms on the right-hand side, using various inequalities of the projection operators π_m and $\pi_m^{2,0}$. $\square$

2.3. 2-D periodic-nonperiodic case. It is in general very difficult, except in the pure periodic case, to construct a decomposition which is A-orthogonal in a multi-dimensional domain other than using the impractical eigenfunctions. We shall make a first attempt to construct such a decomposition for $A = -\Delta$ subject to the homogeneous Dirichlet boundary condition in one direction and to the periodic boundary condition in the other direction.

Let us write

$$H_p^n(0,2\pi) =$$
$$\left\{ v = \sum_{k=-\infty}^{+\infty} a_k e^{ikx} : \ \bar{a}_j = a_{-j}; \sum_{k=-\infty}^{+\infty} |k|^l |a_k| < +\infty, \ \forall\, l = 0, 1, \cdots, n \right\},$$

$$P_{dm} = \left\{ v = \sum_{|k| \le \frac{dm}{2}} a_k e^{ikx} : \bar{a}_j = a_{-j} \right\}.$$

Let $\Omega = [0, 2\pi] \times I$ and $\boldsymbol{m} = (m_1, m_2)$. We then set

$$\boldsymbol{H} = H_p^0(0,2\pi) \times L^2(I), \ \boldsymbol{V} = H_p^1(0,2\pi) \times H_0^1(I),$$

$$\boldsymbol{S_{dm}} = P_{dm_1} \times S_{dm_2}, \ \boldsymbol{V_{dm}} = P_{dm_1} \times V_{dm_2}.$$

We denote by $\Pi_{\boldsymbol{dm}}$ the orthogonal projector in $\boldsymbol{H}$ onto $\boldsymbol{S_{dm}}$.

We are looking for a decomposition $\boldsymbol{V_{dm}} = \boldsymbol{Y_m} \oplus \boldsymbol{Z_{dm}}$ which is A-orthogonal and nearly L^2-orthogonal. In general we can fix either $\boldsymbol{Y_m}$ or $\boldsymbol{Z_{dm}}$ and try to find its complement in $\boldsymbol{V_{dm}}$ with the desired properties. Since it is important that $\boldsymbol{Z_{dm}}$ only contains high-mode elements, we set

$$\boldsymbol{Z_{dm}} = \left\{ v = \sum_{\substack{\frac{m_1}{2} < |k| \le dm_1, \text{ or} \\ m_2+1 \le j \le dm_2 - 2}} a_{kj} e^{ikx} \phi_j(y) : \bar{a}_{kj} = a_{-kj} \right\}.$$

We note that in particular $(I - \Pi_{\boldsymbol{m}})z = z, \ \forall\, z \in \boldsymbol{Z_{dm}}$.

Thanks to (2.2) and (2.3), we observe immediately that for $|k| \le \frac{m_1}{2}$ and $0 \le j \le m_2 - 2$, we have

$$(A(e^{ikx}\phi_j(y)), w) = k^2 (e^{ikx}\phi_j(y), w) - (e^{ikx}\phi_j''(y), w) = 0, \ \forall\, w \in \boldsymbol{Z_{dm}},$$

where $(\cdot,\cdot)$ is the inner product in $\pi_m^{2,0} H$. Therefore, $e^{ikx}\phi_j(y) \in \boldsymbol{Y_m}$ for $|k| \le \frac{m_1}{2}$ and $0 \le j \le m_2 - 2$.

In order to obtain a complete basis for $\boldsymbol{Y_m}$, we only need to construct $\gamma_{k,j}(y)$ for $1 \le k \le \frac{m_1}{2}$ and $j = m_2 - 1, m_2$ such that

$$(A(e^{ikx}\gamma_{k,j}(y)), w) = 0, \ \forall\, w \in Z_{d\boldsymbol{m}}, \ \begin{cases} 1 \le k \le \frac{m_1}{2} \\ j = m_2 - 1, m_2 \end{cases}.$$

In fact, the above condition reduces to:

$$\begin{aligned} & k^2 \int_I \gamma_{k,j}(y)\phi_l(y)dy + \int_I \gamma'_{k,j}(y)\phi'_l(y)dy = 0, \\ & l = m_2+1, \cdots, dm_2 - 2, \ \begin{cases} 1 \le k \le \frac{m_1}{2} \\ j = m_2 - 1, m_2 \end{cases}. \end{aligned} \tag{2.20}$$

These relations lead us to search $\gamma_{k,j}(y)$ in the following form:

$$\gamma_{k,j}(y) = \phi_j(y) + \sum_{l=m_2+1}^{dm_2-2} \gamma_l^{(k,j)} \phi_l(y), \ \begin{cases} 1 \le k \le \frac{m_1}{2} \\ j = m_2 - 1, m_2 \end{cases}.$$

Again, thanks to (2.2) and (2.3), for each pair (k,j), relations (2.20) become a symmetric positive definite tridiagonal system which uniquely determines $\{\gamma_l^{(k,j)}\}_{l=m_2+1,\cdots,dm_2-2}$.

To summarize, we write

$$\begin{aligned} & \gamma_{k,j}(y) = \phi_j(y), \text{ if } j = 0, 1, \cdots, m_2 - 2 \text{ or } k = 0, \\ & \gamma_{-k,j}(y) = \gamma_{k,j}(y), \text{ if } k = 1, \cdots, \frac{m_1}{2}, \text{ and } j = m_2 - 1, m_2. \end{aligned}$$

Then setting

$$\boldsymbol{Y_m} = \left\{ v = \sum_{k=-\frac{m_1}{2}}^{\frac{m_1}{2}} \sum_{j=0}^{m_2} a_{kj} e^{ikx} \gamma_{k,j}(y) : \bar{a}_{kj} = a_{-kj} \right\},$$

we have by construction $\boldsymbol{V_{dm}} = \boldsymbol{Y_m} \oplus \boldsymbol{Z_{dm}}$ and

$$(Ay, z) = 0, \ (y, z) = (y, (I - \Pi_m)z), \ \forall\, y \in \boldsymbol{Y_m}, \ z \in \boldsymbol{Z_{dm}}.$$

Remark 2.2. *We can apply in particular the above decomposition to the Helmholtz equation: Find $y_{\boldsymbol{m}} \in \boldsymbol{Y_m}$, $z_{d\boldsymbol{m}} \in \boldsymbol{Z}_{d\boldsymbol{m}}$ such that*

$$\alpha(y_{\boldsymbol{m}}, v) + (\nabla y_{\boldsymbol{m}}, \nabla v) = (f, v), \ \forall\, v \in \boldsymbol{Y_m},$$
$$\alpha(z_{\boldsymbol{m}}, w) + (\nabla z_{\boldsymbol{m}}, \nabla v) = (f, w), \ \forall\, v \in \boldsymbol{Z}_{d\boldsymbol{m}}.$$

As in the 1-D case, it can be shown that the above scheme is as accurate as the Galerkin scheme in $\boldsymbol{V}_{d\boldsymbol{m}}$ provided $|\alpha| d^{s-1} \lessapprox m_2^2$, where s is the largest number such that $f \in H^{s-2}(\Omega)$.

The procedure can be directly applied to construct a proper decomposition with respect to the biharmonic operator. However, the extension to the two-dimensional case with Dirichlet boundary condition in both directions is more involved and will be considered in a future work.

3. Error estimates for the nonlinear Galerkin method. Let us consider the abstract nonlinear evolution equation (1.1). We set $b(u, v, w) = (B(u, v), w)$ and assume that the trilinear form b satisfies

$$b(u, v, v) = 0, \ \forall\, u, v \in V, \tag{3.1}$$

$$b(u, v, w) \le C \|u\|_V \|v\|_V \|w\|, \ \forall\, u, v, w \in V. \tag{3.2}$$

It is standard to show that the initial value problem (1.2) is well posed in the following sense: *for any $u_0 \in V$ and $f \in L^2(0, T; H)$, there exists a unique solution u of* (1.2) *such that $u \in C([0, T]; V)$.*

We can perform error analyses for the nonlinear Galerkin method based on all the decompositions presented in the last section. To fix the idea, we shall restrict ourselves to the decomposition in Section 2.1. In other words, we consider a 1-D problem with $A = -D_{xx}$, $H = L^2(I)$ and $V = H_0^1(I)$. A particular example is the 1-D Burgers' equation, for which we set

$$C = 0, \ B(u) = u u_x, \ B(u, v) = \frac{2}{3} u v_x + \frac{1}{3} v u_x.$$

One can readily check that the assumptions (3.1)–(3.2) are satisfied.

Using the notations in Section 2.1, the Galerkin approximation in V_{dm} to (1.2) in the above case is:

Find $u_{dm} \in V_{dm}$ such that $\forall\, v \in V_{dm}$,

$$\frac{d}{dt}(u_{dm}, v) + \nu(u_{dm}', v') + (C u_{dm}, v) + b(u_{dm}, u_{dm}, v) = (f, v), \tag{3.3}$$

with $u_{dm}(\cdot,0)=\pi_{dm}u_0(\cdot)$.

We propose to approximate (3.3) by the following nonlinear Galerkin scheme:

Find $y_m \in V_m$ and $z_{dm}\in Z_{dm}$ such that $\forall\, v\in V_m$ and $w\in Z_{dm}$,

$$\text{(3.4)}\quad \frac{d}{dt}(y_m,v)+\nu(y_m',v')+(Cy_m,v)+b(y_m+z_{dm},y_m+z_{dm},v)=(f,v),$$

$$\text{(3.5)}\quad \gamma\frac{d}{dt}(z_{dm},w)+\nu(z_{dm}',w')+(Cz_{dm},w)+b(y_m+z_{dm},y_m,w)=(f,w),$$

with $y_m(x,0)=u_{0m}(x)$; γ is either 1 or 0; $z_{dm}(x,0)=u_{0dm}(x)-u_{0m}(x)$ if $\gamma=1$; u_{0m} (resp. u_{0dm}) is the projection of u_0 in V onto V_m (resp. V_{dm}).

We have the following error estimates for the above scheme.

Theorem 4.1. *Under the assumptions* (3.1), (3.2) *and for m sufficiently large, we have*

$$\text{(3.6)}\quad \begin{aligned}&\Big(\int_0^T \|(u-y_m-z_{dm})(\rho)\|_1^2\,d\rho\Big)^{\frac12}\le\\ &\quad K\left((dm)^{1-s}\|u\|_{L^2(0,T;H^s)}+\gamma(dm)^{1-\alpha}\|u_t\|_{L^2(0,T;H^{\alpha-2})}\right)\\ &\quad +K\left(m^{-s}\|u\|_{L^2(0,T;H^s)}+(1-\gamma)m^{1-\alpha}\|u_t\|_{L^2(0,T;H^{\alpha-2})}\right),\end{aligned}$$

where K is a constant independent of d and m.

Remark 3.1.

(a) *The first two terms on the right hand side of the error estimate in* (3.6) *are inherited from the classical Galerkin approximation* (3.3), *while the last two terms are due to the nonlinear Galerkin treatment. We suggest to use $\gamma=1$ if f is time dependent and $\gamma=0$ if f is time independent. In the latter case, we can show (see [**6**]) that u is analytic in time with value in V. Using Cauchy's formula for analytic functions, we can prove $\|u_t\|_s\sim\|u\|_s$. Hence, by taking $\alpha=s+2$ in* (3.6), *the last term can be absorbed into the first term. Therefore, in both cases we find that as long as $d^{s-1}\lessapprox m$,* (3.4)–(3.5) *is as accurate as the classical Galerkin scheme* (3.3) *in V_{dm}.*

(b) *It is also possible to derive error estimates in $L^\infty(0,T;H^1(\Omega))$ norm. We shall not pursue in this direction since the main purpose of our analysis is to determine a quantitative guideline for the proper choice of d.*

(c) *A proof of the Theorem in the special case $C = 0$ is given in [**13**]. The case $C \neq 0$ can be proved similarly. Moreover, the same technique can be used to obtain error estimates for nonlinear Galerkin schemes using the decompositions in Sections 2.2 and 2.3.*

(d) *We note that compared to* (1.9)*, there is an additional term $b(z_{dm}, y_m, w)$ in* (3.5)*. This term is added due to a technical difficulty. It is believed that the same results would hold without this additional term.*

References

1. C. Bernardi and Y. Maday, *Approximations Spectrales de Problèmes aux Limites Elliptiques*, Springer-Verlag, Paris, 1992.
2. M. Chen and R. Temam, *Nonlinear Galerkin Method in the Finite Difference Case and Wavelet-like Incremental Unknowns*, Numer. Math. **64** (1993), 271–294 .
3. T. Dubois, F. Jauberteau and R. Temam, *Solution of the incompressible Navier-Stokes equations by the nonlinear Galerkin method*, J. Scient. Comput. **8** (1993), 167–194.
4. C. Foias, O. Manley and R. Temam, *Modeling of the interaction of small and large eddies in two dimensional turbulent flows*, Math. Model. and Num. Anal. **22** (1988).
5. C. Foias, G. R. Sell and R. Temam, *Inertial manifolds for nonlinear evolutionary equations*, J. Diff. Eqn. **73** (1988), 308–353.
6. C. Foias and R. Temam, *Some analytic and geometric properties of the evolution Navier-Stokes equations*, J. Math. Pure et Appl. **58** (1979), 339–368.
7. F. Jauberteau, C. Rosier and R. Temam, *The nonlinear Galerkin method in computational fluid dynamics*, Appl. Numer. Math. **6** (1989/90), 361–370.
8. M. Marion and R. Temam, *Nonlinear Galerkin Methods*, SIAM J. Numer. Anal. **26** (1989), 1139–1157.
9. ———, *Nonlinear Galerkin Methods: the finite element case*, Numer. Math. **57** (1990), 205–226.
10. M. Marion and J. Xu, *Error estimates on a new nonlinear Galerkin method based on two-grid finite elements*, To appear in SIAM J. Numer. Anal.
11. J. Shen, *Long time stability and convergence for fully discrete nonlinear Galerkin methods*, Appl. Anal. **38** (1990), 201–229.

12. ———, *Efficient spectral-Galerkin method I. Direct solvers for second- and fourth-order equations by using Legendre polynomials*, SIAM J. Sci. Comput. **15** (1994), 1489–1505.

13. J. Shen and R. Temam, *Nonlinear Galerkin methods using Chebyshev or Legendre polynomials I. One dimensional case*, SIAM J. Numer. Anal. **32** (1995), 215–234.

14. R. Temam, *Stability analysis of the nonlinear Galerkin method*, Math. Comp. **57** (1991), 477–505.

†Department of Mathematics, Penn State University
E-mail address: shen_j@math.psu.edu

‡The Institute for Scientific Computing and Applied Mathematics, Indiana University