A Mathematical Model Illustrating the Theory of Turbulence

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CONTENTS

I. Introduction

The application of methods of statistical analysis and statistical mechanics to the problem of turbulent fluid motion has attracted much attention in recent years. From the theoretical side we are faced with the necessity of investigating a complicated system of nonlinear equations, in order to find out enough about the properties of the solutions of these equations that insight can be obtained into the various patterns exhibited by the field and that data can be derived concerning the relative frequencies of these patterns, in the hope that in this way a basis may be found for the calculation of important values. The difficulties encountered are of a twofold nature: in part they are connected with the complicated geometrical character of the hydrodynamical equations (vectorial character of the velocity, condition imposed by the equation of continuity, properties of vortex motion); in part they are dependent upon the presence of nonlinear terms, containing derivatives of the first order of the velocity components, along with derivatives of the second order multiplied by the very small coefficient of viscosity. The latter feature in particular is responsible for a number of important characteristics of turbulence, among which are prominent those connected with the balance of energy and with the appearance of dissipation layers. These layers (boundary layers along the walls and similar phenomena in the interior of the field) play an important part in the energy exchange, as they represent the main regions where energy is dissipated.

These features, together with the properties of the spectrum of the turbulent motion (which is obtained when the field is analyzed into a series of elementary components), the transfer of energy through the spectrum, and the appearance of a practical limit to this spectrum (responsible for the finite value of the total dissipation), all can be elucidated with the aid of a system of mathematical equations much simpler than those of hydrodynamics. It is the object of the following pages to discuss these equations, which in a sense form a mathematical model of turbulence, and to indicate the bearing of the results obtained upon the hydrodynamical problem.[1]

II. The Equations Describing the Model System

Although the quantities occurring in our equations are abstract mathematical variables and parameters, we shall indicate them by names that show their analogies to the variables of hydrodynamics. Two dependent variables, $U(t)$ and $v(y,t)$, are introduced, representing velocities. The first one will be the analogue of the primary or mean motion in the case of a liquid flowing through a channel; in the model it is a function of the time only. The other variable v represents the secondary motion; when it differs from zero we shall say that there is *turbulence* in the system, even in a case where v should be independent of the time. The independent variable y occurring in v plays the part of the coordinate in the direction of the cross dimension of the channel. In the case with which we shall be primarily concerned the domain of y extends from 0 to b, and v

[1] References: (I) J. M. Burgers, Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, *Verhandel. Kon. Nederl. Akad. Wetenschappen Amsterdam, Afdeel. Natuurkunde (1st Sect.)*, **17**, No. 2, 1–53 (1939); (II) Application of a model system to illustrate some points of the statistical theory of free turbulence, *Proc. Acad. Sci. Amsterdam*, **43**, 2–12 (1940); (III) On the application of statistical mechanics to the theory of turbulent fluid motion. A hypothesis which can serve as a basis for a statistical treatment of some mathematical model systems, *Proc. Acad. Sci. Amsterdam*, **43**, 936–945, 1153–1159 (1940); (IV) Beschouwingen over de statistische theorie der turbulente stroming, *Nederl. Tijdschr. Natuurkunde*, **8**, 5–18 (1941).

These papers will be referred to as I, II, etc.

is subjected to the boundary condition that it must vanish at both ends.[2] Later on (in Sections VIII–IX) the case of an infinite domain for y will be considered.

The equations have the form:

$$b \frac{dU}{dt} = P - \frac{\nu U}{b} - \frac{1}{b} \int_0^b dy\, v^2 \tag{1}$$

$$\frac{\partial v}{\partial t} = \frac{U}{b} v + \nu \frac{\partial^2 v}{\partial y^2} - 2v \frac{\partial v}{\partial y}. \tag{2}$$

The quantities P and ν are constant parameters. P is the analogue of an exterior force acting upon the primary motion. The terms with ν represent frictional effects, in the case of the primary motion simply proportional to U/b, and in the case of the secondary motion proportional to the second derivative of v with respect to y. It is not necessary to use the same coefficient ν for both, but nothing is gained by introducing different coefficients.

The terms of the second degree in v, which in view of what has been said in section I form an essential feature of the system, have been chosen in such a way that an equation can be formed describing the balance of energy. We multiply (1) by U and (2) by v; the latter is integrated with respect to y from 0 to b, having regard to the boundary conditions, and added to the first one. Several terms cancel and we obtain:

$$\frac{d}{dt}\left[\frac{bU^2}{2} + \int_0^b dy \frac{v^2}{2}\right] = PU - \frac{\nu U^2}{b} - \nu \int_0^b dy \left(\frac{\partial v}{\partial y}\right)^2. \tag{3}$$

The quantity between [ ] in the left-hand member is interpreted as the kinetic energy of the motion; the equation shows that this energy increases in consequence of work PU being performed by the exterior force P upon the principal motion U, while at the same time there is dissipation of energy as a result of the frictional effects. The term Uv/b in equation (2) introduces a transmission of energy from the primary motion to the secondary motion; this transmission, being an internal feature of the system in consequence of a compensating term in equation (1), does not appear in the total balance of energy as given by (3). In the model this transmission has been made to depend upon the value of U itself, and

[2] In the original paper, referred to as (I) in footnote 1, the breadth b of the domain had been taken equal to unity. Further the minus sign introduced into Equation (4) had not been used, which results in a difference of sign in the ξ_n (and in the ζ_n of Section XIII).

not upon a gradient of the type dU/dy, as is the case with the equations of hydrodynamics. As will become evident from further developments the last term of equation (2) is responsible for an exchange of energy between the various components into which the secondary motion can be resolved; this likewise is an interior process, which does not appear in the total balance.

The equations can be brought into a different form by introducing a Fourier series for v. We put:

$$v = -\sum_{n=1}^{\infty} \xi_n \sin\left(\frac{\pi n y}{b}\right) \tag{4}$$

(the minus sign has been used in order to simplify some further expressions). The coefficients ξ_n in general will be functions of the time. This expression satisfies the boundary conditions for v. Equations (1) and (3) now take the forms:

$$b\frac{dU}{dt} = P - \frac{\nu U}{b} - \frac{1}{2}\sum \xi_n^2 \tag{5}$$

$$\frac{d\xi_n}{dt} = \left(\frac{U}{b} - \frac{\nu\pi^2 n^2}{b^2}\right)\xi_n + \frac{\pi n}{b}\left(\frac{1}{2}\sum_{k=1}^{n-1} \xi_k\xi_{n-k} - \sum_{k=1}^{\infty} \xi_k\xi_{n+k}\right). \tag{6}$$

The energy equation (3) at the same time becomes:

$$\frac{d}{dt}\left\{\frac{bU^2}{2} + \frac{b}{4}\sum \xi_n^2\right\} = PU - \frac{\nu U^2}{b} - \frac{\nu\pi^2}{2b}\sum n^2\xi_n^2. \tag{7}$$

It is seen that each term $(b/4)\xi_n^2$ can be considered as the energy associated with a component of the spectrum, while $(\nu\pi^2 n^2/2b)\xi_n^2$ is the dissipation associated with that component. Whereas in the energy equation and also in equation (5) the terms due to the various components of the spectrum appear in simple sums and thus are separated, the presence of the terms of the second degree in (6) insures that these components are not independent of each other, so that when one has been excited all the others of necessity will come into existence as well. In the language of hydrodynamics the terms of the second degree represent the mechanism by which small eddies are produced from large ones.

A slightly more general model, in which the secondary motion is represented by two variables, will be considered in Sections XIII and XIV.

III. Laminar Solution of the System (1), (2)

The equations (1) and (2) are rigorously fulfilled by the values:

$$U = \frac{Pb}{\nu}; \qquad v = 0. \tag{8}$$

In this solution turbulence is absent. To investigate its stability small values can be given to v, U being kept fixed. The discussion is effected most easily with the aid of equation (6), in which the terms of the second degree can be neglected for infinitely small values of v, leaving:

$$\frac{d\xi_n}{dt} = \left(\frac{U}{b} - \frac{\nu\pi^2 n^2}{b^2}\right)\xi_n. \tag{9}$$

When $U < \nu\pi^2/b$ the real part of the cofactor of ξ_n in the right-hand member is negative for every value of n; hence every ξ_n will be damped, so that the laminar solution is stable. On the other hand when $U > \nu\pi^2/b$, one or more of the ξ_n will increase exponentially; the laminar solution is unstable and a new solution will appear in which v is different from zero. As mentioned before, the presence of terms of the second degree in (6), which terms cannot be neglected for finite values of v, means that not only those ξ_n for which $U > \nu\pi^2 n^2/b$ but all ξ_n will become excited.

As no density factor has been introduced, the quantity Ub/ν plays the part of Reynolds number for the system. It will be denoted by Re; its critical value is: $Re_{crit} = \pi^2$.

IV. Stationary Turbulent Solutions of Equation (2)

When U is considered as a constant, equation (2) permits solutions for v that are independent of the time. With these solutions it is possible also to satisfy equation (1), provided the proper value is given to the exterior force P. The stationary solutions of equation (2) can be expressed in exact form by means of quadratures. They form two sets, related by the formula: $v_+(y) = -v_-(b - y)$, so that the two sets are antisymmetrical with respect to each other. In each set the solutions are characterized by a number m, which runs from unity to a maximum, determined by the largest integer contained in $\sqrt{Ub/\nu\pi^2} = \sqrt{Re/Re_{crit}}$. The solution with the index m has $m - 1$ zeros between $y = 0$ and $y = b$. There is a certain resemblance to the series of eigensolutions of a linear differential equation of the second order, satisfying the condition of vanishing at both boundaries; the part of the parameter (which in that case can take an infinite series of eigenvalues) in the case of equation (2) is assigned to the amplitude of the solution, which becomes smaller and smaller as m increases; the set of solutions in the case of (2) appears to be finite.

The full expressions of the stationary solutions, which have been given elsewhere,[3] are not necessary for our present purpose. Their most interesting property is that when Re is large and we restrict ourselves to low values of m, *e.g.*, $m = 1$ and $m = 2$, the domain $0 \leq y \leq b$ can be divided into (*a*) comparatively broad regions where the term $\nu(\partial^2 v/\partial y^2)$ can be neglected and equation (2) (with $\partial v/\partial t = 0$) approximately reduces to:

$$2v \frac{\partial v}{\partial y} - \frac{U}{b} v = 0, \tag{10}$$

and (*b*) extremely narrow regions in which the term Uv/b can be neglected in comparison with the other two, so that there remains:

$$\nu \frac{\partial^2 v}{\partial y^2} - 2v \frac{\partial v}{\partial y} = 0. \tag{11}$$

The solution of (10) is:

$$v = \frac{Uy}{2b} + C, \tag{12}$$

and that of (11):

$$v = -A \operatorname{tgh} \left\{ \frac{A(y - B)}{\nu} \right\}, \tag{13}$$

A, B, and C being constants. These constants must be determined in such a way that the solutions fit together without discontinuities in v, and that the boundary conditions at $y = 0$ and $y = b$ are satisfied. Important special cases can be represented with sufficient approximation by:

$$v = \frac{U}{2} \left(\frac{y}{b} - \operatorname{tgh} \frac{Uy}{2\nu} \right) \tag{14a}$$

$$v = \frac{U}{2} \left\{ \frac{y}{b} - 1 + \operatorname{tgh} \frac{U(b - y)}{2\nu} \right\} \tag{14b}$$

$$v = \frac{U}{4} \left\{ \frac{2y}{b} - 1 - \operatorname{tgh} \frac{Uy}{4\nu} + \operatorname{tgh} \frac{U(b - y)}{4\nu} \right\}. \tag{14c}$$

In the first case we have a region of steep change of v, comparable to a boundary layer, in the neighborhood of $y = 0$; in the second case such a region is found in the neighborhood of $y = b$; in the third case there are boundary layers along both walls. Solutions with regions of steep

[3] See (I), pp. 18–24.

change of v, *i.e.*, with dissipation layers, in the neighborhood of an interior point of the domain likewise can be obtained (for $m = 2$ and for a number of higher values); this will be seen also from the considerations of section VII.

V. Spectrum of the Stationary Turbulent Solutions

When the solutions (14a)–(14c) are developed into Fourier series, the following values are obtained for the coefficients:

$$\xi_n = \frac{\pi\nu}{b \sinh(\pi^2 n\nu/Ub)} \tag{15a}$$

$$\xi_n = \frac{(-1)^n \pi\nu}{b \sinh(\pi^2 n\nu/Ub)} \tag{15b}$$

$$\xi_n = \left\{ \begin{array}{ll} \dfrac{2\pi\nu}{b \sinh(2\pi^2 n\nu/Ub)} & (n \text{ even}) \\ 0 & (n \text{ odd}) \end{array} \right\}. \tag{15c}$$

In particular taking the first case (as this is the simplest and most typical one), it is found that for $n \ll Re$:

$$\xi_n = \frac{U}{\pi n}, \tag{16a}$$

whereas for $n > Re$:

$$\xi_n = \frac{2\pi\nu}{b} e^{-\pi^2 n\nu/Ub}. \tag{16b}$$

Hence in the spectrum the *energy per component* $\frac{1}{4}b\xi_n^2$ appears to be proportional to n^{-2} so long as $n \ll Re$, while this energy decreases exponentially when $n > Re$. It follows that the total energy can be calculated with sufficient accuracy from the approximate expression (16a), as in fact the total energy depends upon the first few components only. Its amount is $bU^2/24$; this amount is not influenced by the value of the viscosity. The same applies to the value of P, which (neglecting the term $\nu U/b$) is given by:

$$P = \frac{1}{2} \sum \xi_n^2 = \frac{U^2}{12}. \tag{17}$$

The latter result shows that when turbulence has set in, the "resistance" increases proportionally to the square of U and that this resistance is dependent upon the first few components of the turbulence only.

The *dissipation per component* (or, in other terms, per degree of freedom of the system) has a constant value $\frac{1}{2}\nu U^2/b$ as long as $n \ll Re$, while it decreases nearly exponentially when $n > Re$. As the number of degrees of freedom is unlimited the application of (16a) to all components would make the total dissipation infinite. The actual value of the total dissipation is dependent upon the extent of the spectrum before one arrives at the part to which applies formula (16b), which requires the number n to be of the order Re. The presence of a practical limit to the main part of the spectrum [as determined by (16a)] thus is of the utmost importance in fixing the total dissipation. It will be evident at the same time that the total dissipation embraces much more of the elementary components of the turbulence than does the total energy.

The magnitude of the total dissipation is calculated most easily from the integral $\nu \int_0^b dy(\partial v/\partial y)^2$, which occurs in the energy balance (3); for the case represented by formula (14a) it has the value $U^3/12$, corresponding to an effective number of components $N = Ub/6\nu = Re/6$.

Similar results can be obtained for the other cases. Formula (15c) shows that in certain cases a number of components of the spectrum will be zero.

The results obtained are an immediate consequence of the form of equation (2), in which a term of the second degree containing the product of v with its derivative of the first order is combined with a derivative of the second order multiplied by a very small constant. This will become still more clear from a discussion of the nonstationary solutions of equation (2).

VI. Additional Remarks Concerning the Spectrum of the Stationary Solutions

It is of interest to note that the expression (16a) gives an exact solution of the infinite system of equations:

$$0 = \frac{U\xi_n}{b} + \frac{\pi n}{b}\left(\frac{1}{2}\sum_{k=1}^{n-1} \xi_k\xi_{n-k} - \sum_{k=1}^{\infty} \xi_k\xi_{n+k}\right), \tag{18}$$

which is obtained from (6) (with $d\xi_n/dt = 0$) when ν is replaced by zero. The solution is not unique, as other solutions can be obtained in which the ξ_n periodically change sign, or in which, *e.g.*, the ξ_n of odd order are zero.

For large values of n the term with ν in equations (6) will be much more important than the term $U\xi_n/b$ (and presumably also than the term $d\xi_n/dt$ if this should not be rigorously zero). It is useful therefore to observe that the infinite system:

$$0 = -\frac{\nu\pi^2 n^2 \xi_n}{b^2} + \frac{\pi n}{b}\left(\frac{1}{2}\sum_{k=1}^{n-1} \xi_k\xi_{n-k} - \sum_{k=1}^{\infty} \xi_k\xi_{n+k}\right) \tag{19}$$

admits an asymptotic solution which can be obtained by assuming:

$$\xi_m = \beta\lambda^m \qquad \text{for} \qquad m > M \tag{20}$$

where M is some (large) number. Upon this assumption (19), for $n > 2M$, can be transformed into;

$$\beta\left(\frac{n}{2} - M - \frac{1}{2} - \frac{\lambda^{2M+2}}{1-\lambda^2}\right) = \frac{\nu\pi n}{b} - \sum_{k=1}^{M} \xi_k(\lambda^{-k} - \lambda^k),$$

which gives:

$$\beta = \frac{2\pi\nu}{b}. \tag{20a}$$

Substitution of this result leads to:

$$\frac{2\pi\nu}{b}\left(M + \frac{\lambda^{2M}}{1-\lambda^2}\right) = \sum_{k=1}^{M} \xi_k(\lambda^{-k} - \lambda^k) \tag{*}$$

(the term $\frac{1}{2}$ has been neglected in comparison with M, while λ^{2M+2} has been replaced by λ^{2M}, as λ evidently will differ very little from unity). We cannot come further with this equation alone and attention should be given to the circumstance that the infinite system (19) is equivalent to the differential equation (11), the general solution of which, as given by (13), cannot satisfy both boundary conditions. However, when we have an approximate expression for the ξ_n of low index numbers, this expression can be substituted into (*) and the latter can be used to determine λ, provided we find a suitable value for M. We may take as such that value of the number m for which the approximation (20) and the approximation used for small index numbers are nearest each other. For instance when we substitute the values of the ξ_n given by (16a) and determine M by the condition that the ratio $(U/\pi m)/(2\pi\nu\lambda^m/b)$ shall be a minimum (which requires $M = -1/\ln\lambda$), we find:

$$-\ln\lambda \cong \frac{1.01\pi^2\nu}{Ub},$$

which makes a good approximation to (16b).

In order to investigate the part played by the nonlinear terms of (18) and (19) in the transmission of energy, we multiply (6) by $b\xi_n/2$ and form the equation of energy for a single component of the spectrum:

$$\frac{d}{dt}\left(\frac{b}{4}\xi_n^2\right) = \frac{U\xi_n^2}{b} - \frac{\nu\pi^2 n^2 \xi_n^2}{2b} + \sigma_n' - \sigma_n''. \tag{21}$$

The terms of the right-hand member have the following meaning: the first one is the energy derived by the component ξ_n from the primary motion; the second one is the energy dissipation for that component; the third one:

$$\sigma_n' = \frac{\pi n}{4}\,\xi_n \sum_{k=1}^{n-1} \xi_k \xi_{n-k} \tag{22a}$$

represents the energy exchange with the components of index numbers below n; the fourth one:

$$\sigma_n'' = \frac{\pi n}{2}\,\xi_n \sum_{k=1}^{\infty} \xi_k \xi_{n+k} \tag{22b}$$

represents the energy exchange in which components with index numbers higher than n are involved.

The total amount of energy transmitted from the part of the spectrum for the index numbers 1, $\cdots$ n to the part beyond is given by the sum:

$$S_n = \sum_{m=1}^{n} (\sigma_m'' - \sigma_m') = \frac{\pi}{2} \sum_{k=n+1}^{\infty} \sum_{h=1}^{n} h \xi_h \xi_{k-h} \xi_k. \tag{22c}$$

The expressions for the exchange of energy all depend upon products of the third degree of the quantities ξ_n (which quantities themselves have the dimensions of a velocity).

When we exclude the first few values of n, in such a way that in the head of the spectrum the total energy to be derived from the primary motion $\frac{1}{2}U\Sigma\xi_n^2 = U^3/12(= PU)$ has already been taken up, the value of S_n will be equal to:

$$S_n = \frac{U^3}{12} - \left(\frac{\nu\pi^2}{2b}\right) \sum_{m=1}^{n} m^2 \xi_m^2. \tag{22d}$$

and for $n \ll Re/6$ will be practically independent of n.

VII. Nonstationary Solutions of Equation (2)

We keep to the case where U is treated as a constant. Based upon the results of section 4 we divide the domain for y into regions where the equation

$$\frac{\partial v}{\partial t} + 2v\frac{\partial v}{\partial y} - \frac{Uv}{b} = 0 \tag{23}$$

can be applied, and others where the approximation

$$\frac{\partial v}{\partial t} + 2v\frac{\partial v}{\partial y} - \nu\frac{\partial^2 v}{\partial y^2} = 0 \tag{24}$$

will be appropriate.

Equation (23) can be solved by means of its characteristics, which are determined by the equation: $dy/dt = 2v$. When the values of v given for $t = 0$ satisfy the relation $\partial v/\partial y > 0$, the solution for large values of t asymptotically approaches to: $v = \frac{1}{2}Uy/b + \text{constant}$. On the other hand when there are regions in which $\partial v/\partial y < 0$, the solution tends to generate discontinuities. When one should accept the presence of real discontinuities in the course of v, development into a Fourier series according to well-known theorems will lead to coefficients ultimately decreasing proportionally with n^{-1}. It is possible to calculate the total energy and the value of P with the aid of such a result; but the total dissipation will become infinite, as it has to be expected with $\partial v/\partial y$ assuming an infinite value in a discontinuity.

However, when a discontinuity threatens to appear, the approximation (23) no longer can be applied and we must use equation (24). A system of moving coordinates y', t' is introduced, defined by $y' = y - ct$; $t' = t$; equation (24) then transforms into:

$$\frac{\partial v}{\partial t'} + (2v - c)\frac{\partial v}{\partial y'} - \nu\frac{\partial^2 v}{\partial y'^2} = 0. \tag{24a}$$

We suppose that the term $\partial v/\partial t'$ will remain of the order unity when c is properly chosen, the other terms becoming of the order $1/\nu$ in the resulting solution. Hence we replace (24a) by

$$\nu\frac{\partial^2 v}{\partial y'^2} = (2v - c)\frac{\partial v}{\partial y'}, \tag{24b}$$

which gives

$$\nu\left(\frac{\partial v}{\partial y'}\right) = \left(v - \frac{1}{2}c\right)^2 - C_0,$$

C_0 being a constant. This equation must be valid through the region of steep change of v and on both sides of it, where the derivative $\partial v/\partial y'$ will return to values of the order unity so that $\nu(\partial v/\partial y')$ can be neglected. Using subscripts l and r to distinguish between points just to the left and just to the right of the discontinuity, we must have:

$$(v_l - \tfrac{1}{2}c)^2 = (v_r - \tfrac{1}{2}c)^2,$$

from which:

$$c = v_l + v_r. \tag{25}$$

The solution of (24b) now becomes:

$$v = \frac{v_l + v_r}{2} - \frac{v_l - v_r}{2} \operatorname{tgh} \frac{(v_l - v_r)(y' - y_0)}{2\nu}, \tag{26}$$

where y_0 is a second integration constant. In terms of the original coordinate y we have $y' - y_0 = y - ct - y_0$.

Summing up we see that in the case of large Reynolds numbers the nonlinear term in (2) can give rise to the appearance of discontinuities, but that the formation of a true discontinuity is prevented by the influence of the term $\nu(\partial^2 v/\partial y^2)$, in consequence of which every discontinuity is rounded off and a dissipation region is produced. The total dissipation in such a region has a finite value, for which is found $(v_l - v_r)^3/6$ (in a discontinuity always $v_l > v_r$). It is important to note that in a dissipation region the course of $v(y)$ is governed by the viscous term of the equation of motion (we may say that in the interior of this region the motion is laminar), but that nevertheless the total dissipation is independent of the viscosity and is of the third degree with respect to the velocities. The rounding off of the discontinuities also influences the asymptotic behavior of the coefficients in the Fourier development of v, which coefficients for $n > Re$ no longer are proportional to n^{-1}, but decrease exponentially in a way analogous to that described by (16b).

It will be seen that by this same process a boundary layer will be formed at $y = b$, when v is positive for values of y just smaller than b; and at $y = 0$ when v is negative for y immediately to the right of 0. Intermediate stationary dissipation regions can appear when $v_l + v_r$ takes the value 0. In a moving dissipation region v_l and v_r will be functions of the time, but we shall not enter into an investigation of their behavior.

VIII. Application of Equation (2) to an Infinite Domain

While equation (2) with the boundary conditions $v = 0$ at $y = 0$ and $y = b$ originally was meant to illustrate properties of the turbulence of a fluid moving under an exterior force through a channel, it may also be used without these boundary conditions to illustrate properties of free turbulence, by applying it to an open domain. In that case it is useful to take $U = 0$, in order to obtain results for free turbulence not activated by energy transmission from a primary motion. The equation governing the turbulence consequently becomes the same as (24):

$$\frac{\partial v}{\partial t} + 2v\frac{\partial v}{\partial y} - \nu\frac{\partial^2 v}{\partial y^2} = 0. \tag{24}$$

A typical solution of this equation, which can develop from various initial conditions, has the form:

$$v = \frac{V(\eta)}{\sqrt{t - t_0}}, \qquad \text{with} \qquad \eta = \frac{y - y_0}{\sqrt{t - t_0}}. \tag{27}$$

In the course of time this solution spreads out proportionally with the square root of time, while its amplitude decreases inversely proportionally to $\sqrt{t - t_0}$. As a function of the new variable η the quantity V must satisfy the equation:

$$\nu V'' - 2VV' + \tfrac{1}{2}\eta V' + \tfrac{1}{2}V = 0 \tag{28}$$

which can be integrated into:

$$\nu V' = V^2 - \tfrac{1}{2}\eta V. \tag{29}$$

The integration constant has been taken zero, in order to make possible solutions that for large (negative and positive) values of η decrease to zero. For infinitely small values of ν the solutions of (29) take an asymptotic form, which can be described by means of the following formulas:

$$\left.\begin{array}{c} V = 0 \qquad \text{for} \qquad \eta < 0 \\ V = \tfrac{1}{2}\eta \qquad \text{for} \qquad 0 < \eta < L \\ V \text{ decreases suddenly from } \tfrac{1}{2}L \text{ to } 0 \text{ for } \eta = L \\ V = 0 \qquad \text{for} \qquad L < \eta. \end{array}\right\} \tag{30}$$

Here L is a second integration constant, fixing the scale of the curve in η-measure. For small finite values of ν a correction to this asymptotic solution can be deduced [a graphical discussion of equation (29) gives much help in visualizing the resulting curve]; it is found that the true curve nowhere differs much from the broken line described by (30), while for values of η in the immediate neighborhood of L it can be approximated by the expression:

$$V \cong \frac{L}{4}\left\{1 - \operatorname{tgh}\frac{L(\eta - L)}{4\nu}\right\}. \tag{31}$$

Elsewhere it has been indicated in which way a solution of this type can arise from certain initial conditions.[4]

Returning to the y- and v-scales the linear dimension of the domain where v is appreciably different from zero, increases according to the formula:

$$l = L\sqrt{t - t_0} \tag{32a}$$

[4] See (II), pp. 8–9.

while the height of the front decreases and is equal to

$$v_{fr} = \frac{L}{2\sqrt{t - t_0}}. \tag{32b}$$

In this domain there is a momentum of amount:

$$\Omega = \int v\,dy = \tfrac{1}{4}L^2 \tag{32c}$$

which is independent of the time. It follows that L can be expressed by means of Ω: $L = 2\sqrt{\Omega}$. The kinetic energy has the value:

$$\frac{1}{2}\int v^2 dy = \frac{L^3}{24\sqrt{t - t_0}} = \frac{\Omega^{\frac{3}{2}}}{3\sqrt{t - t_0}} \tag{32d}$$

and thus decreases in the course of time; this is the consequence of the dissipation of energy in the front, which amounts to:

$$\frac{1}{6}\left(\frac{L}{2\sqrt{t - t_0}}\right)^3 = \frac{L^3}{48(t - t_0)^{\frac{3}{2}}} = \frac{\Omega^{\frac{3}{2}}}{6(t - t_0)^{\frac{3}{2}}}. \tag{32e}$$

The solution therefore represents the gradual broadening of an isolated "domain of coherence," in such a way that its momentum is preserved. It can be compared with the gradual increase of size of an eddy in the hydrodynamical case, where the same proportionality of the linear dimensions with $\sqrt{t - t_0}$ is found. It must be noted that a similar solution can be obtained in which both $y - y_0$ and v have changed sign; in that case the front moves to the left instead of to the right.

The steepness of the front is a function of its height, and decreases with decrease of the front height. There can be no steep front with a height of the order ν.

IX. Spatial Correlation in the Values of v for an Isolated Domain of Coherence

We denote by $I(r)$ the value of the correlation integral:

$$I(r) = \int_{-\infty}^{+\infty} dy_0 v(y)v(y + r) \tag{33}$$

which refers to the product of the values of v in two fixed points y and $y + r$, when the constant y_0 in the solution just found takes all values from $-\infty$ to $+\infty$, the value of l being kept unchanged. Evidently we can just as well write:

$$I(r) = \int_{-\infty}^{+\infty} dy\, v(y)v(y + r) \tag{33a}$$

where y_0 and r are kept fixed. A correlation coefficient is defined by

$$R(r) = \frac{I(r)}{I(0)}. \tag{34}$$

When r is comparable to l the calculation can be worked out by means of the asymptotic solution given in (30); we find:

$$R(r) = 1 - \frac{3}{2}\frac{r}{l} + \frac{1}{2}\frac{r^3}{l^3}. \tag{34a}$$

The value of $R(r)$ remains practically zero for values of r exceeding the length l of the domain. For very small values of r we better use the development:

$$\int dy\, v(y)v(y+r) = \int dy \left\{ v^2 + r \cdot v \frac{\partial v}{\partial y} + \frac{r^2}{2} \cdot v \frac{\partial^2 v}{\partial y^2} + \cdots \right\}$$
$$= \int dy\, v^2 - \frac{r^2}{2} \int dy \left(\frac{\partial v}{\partial y}\right)^2 + \cdots .$$

The most important contribution to the last integral comes from the steep front; making use of (31) we find:

$$R(r) = 1 - \frac{r^2}{8\nu t_1}. \tag{34b}$$

X. Application of Similarity Considerations

Following the lead given by Taylor and von Kármán with their investigations on the statistical properties of free turbulence, much attention in recent years is devoted to statistical problems, in particular in order to arrive at a theoretical expression for the correlation between velocity fluctuations in two points at a distance r apart from each other. A summary of some modern work on this subject has been given by G. K. Batchelor.[5] According to this summary the various theories proposed all start from the assumption of indefinitely high Reynolds numbers, and assume that the effect of viscosity on velocity correlations is negligible when r is large in comparison with the size of the smallest eddies; it is further supposed that the energy associated with a small range of wave numbers is received chiefly from wave numbers one order smaller (*i.e.*, eddies of larger size) and is passed on to larger wave numbers without appreciable loss through viscous dissipation; while finally the smallest existing eddies have laminar motion and are responsible for most of the

[5] G. K. Batchelor, *Nature* **158**, 883–884 (1946).

energy dissipation. It will be evident that all these features are present in our model. It is of interest therefore to consider to what extent the conclusions derived from these assumptions can be applied to the model.

In order to have a basis for the application of statistical considerations to the case of free turbulence, a supposition must be introduced concerning the way in which the field may have been generated. We can start with an arbitrary initial distribution of values for v at $t = 0$. It is to be expected that in course of time, by the generation and subsequent coalescence of discontinuities, a number of more or less clearly defined domains of coherence, of the type considered in Section VIII, will come into existence. The fronts of the domains with the larger velocities will overtake the other ones, and there will be a gradual coalescence of the domains into a smaller number. A picture of this process can be constructed without difficulty for a case in which the v-curve for $t = 0$ consists of segments where v is constant (eventually zero), with abrupt changes of v from one segment to the next one. The average size of the domains of coherence will be found to increase proportionally to $\sqrt{t - t_0}$.

Such a process could be repeated a large number of times, every time starting with an arbitrary distribution of the values of v. We then can take two fixed points, y and $y + r$, and ask for the average value of the product $v(y)v(y + r)$ referring to these points. In order to make this question definite, we either may compare the value of $v(y)v(y + r)$ each time with the value of $\{v(y)\}^2$, or something can be fixed about the initial conditions. We might require, *e.g.*, that the mean value of v^2, taken over regions of a definite length, initially shall always have the same value, and that the value of $v(y)v(y + r)$ is always observed after the same lapse of time since the process was started.

The dissipation of energy in every case is due to the circumstance that there is a continuous tendency to develop steep fronts, which are the main regions where viscosity is operative.

The introduction of an actual starting point in time, however, makes the picture less satisfactory as a description of the state to be found in an arbitrary field. In the statistical theory developed by A. N. Kolmogoroff the field, independently of its actual history, at any given instant is characterized by the mean dissipation per unit mass. This idea can be taken over in the model system.

At a given instant the following quantities describe important characteristics of the turbulent field: the mean dissipation of energy per unit length: ϵ; and the correlation function: $J(r) = \overline{v(y)v(y + r)}$, which latter embraces also the mean energy per unit length: $E = \frac{1}{2}J(0)$. Along with these we introduce an *average length l of the domains of coherence* and an *effective age T of the field.* These additional quantities can be related to the other two in the following way: on the one hand T must be of the order of magnitude of E/ϵ; on the other hand, as the mean velocity in the field is given by $\sqrt{2E}$, it should be of the order $l/\sqrt{E}$. Comparing these results we must have, with a numerical factor α:

$$\frac{E}{\epsilon} = \frac{\alpha l}{\sqrt{E}}. \tag{A}$$

A further relation can be found by observing that the average dissipation per unit length is given by the average value of $\nu(\partial v/\partial y)^2$, and thus must be of the order $\nu E/l^2$. In this way the viscosity is brought into the picture. It now becomes possible to eliminate the mean energy E; we obtain (with another numerical factor β):

$$E = \frac{\beta \epsilon l^2}{\nu}. \tag{B}$$

From (A) and (B) we deduce:

$$\left.\begin{aligned} E &= C_1 \nu^{\frac{1}{2}} \epsilon^{\frac{1}{2}} \\ l &= C_2 \nu^{\frac{3}{4}} \epsilon^{-\frac{1}{4}} \\ T &= C_3 \nu^{\frac{1}{2}} \epsilon^{-\frac{1}{2}} \end{aligned}\right\} \tag{35}$$

where C_1, C_2, C_3 again are numerical factors. Having obtained these results it is evident that the correlation function $J(r)$ must be of the form:

$$J(r) = \sqrt{\nu\epsilon} \cdot F\left(\frac{r\epsilon^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\right).$$

When r goes to zero the function F must reduce to a constant. When on the other hand the hypothesis is accepted that for values of r large in comparison with the size of the smallest domains of coherence, the part of $J(r)$ depending on r shall become independent of ν, it is necessary that F be of the form:

$$F = A_1 - A_2\left(\frac{r\epsilon^{\frac{1}{4}}}{\nu^{\frac{3}{4}}}\right)^{\frac{2}{3}},$$

where A_1 and A_2 are numerical factors. With $A_2/A_1 = \kappa$ the correlation coefficient $R(r)$ then becomes:

$$R(r) = 1 - \kappa \epsilon^{\frac{1}{6}} \nu^{-\frac{1}{2}} r^{\frac{2}{3}}. \tag{36}$$

The dependence upon $r^{\frac{2}{3}}$ is the same result as has been obtained by Kolmogoroff for hydrodynamical turbulence, apparently along similar lines of reasoning.

The reasoning is somewhat unsatisfactory in so far as it does not give a means for determining the numerical factors involved, nor does it enable us to find the relation between the scale of the eddies of coarser type and the size of the smallest eddies present (both quantities have the dimensions of l). The gross scale of the turbulence, for which we may take the quantity l itself, can be defined in such a way that the correlation is zero for $r \geq l$; hence we shall write:

$$R(r) = 1 - \left(\frac{r}{l}\right)^{\frac{2}{3}} \qquad \text{for} \qquad r < l \tag{36a}$$

with

$$l = \kappa^{-\frac{3}{2}}(\nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}}). \tag{36b}$$

XI. Continuation

According to a formula given by Taylor it is possible from the correlation coefficient to deduce an expression for the energy distribution in the relevant part of the spectrum of the turbulent motion.[6] In the present case, where the domain for y is infinite, we must replace the Fourier series of Sections V and VI by Fourier integrals; denoting the energy between the frequencies n and $n + dn$ by $E(n)dn$, Taylor's formula gives:

$$E(n) = \text{const.} \int_0^\infty dr\, R(r) \cos \frac{\pi n r}{l} \tag{37}$$

When we introduce the expression (36a) for $R(r)$ into this formula, taking $R(r) = 0$ for $r > l$, it can be applied to find in what way $E(n)$ depends upon n for wave numbers large compared with unity, and for which nevertheless l/n is still large in comparison with the size of the smallest eddies (for values of r of the order of that size (36a) should be replaced by a parabolic expression of the type given in (34); the application of (36a) consequently cannot give reliable results for very large n). The result obtained in this way is:[7]

$$E(n) \sim n^{-\frac{5}{3}}. \tag{38}$$

A result of similar nature has been obtained by L. Onsager,[8] and, according to Batchelor's paper, also by C. F. von Weizsäcker, in both cases from dimensional considerations taking as a datum the energy dissipation per unit volume. Onsager gives a proportionality with $n^{-\frac{11}{3}}$, but I suppose that in connection with the three-dimensional character of hydrodynamical turbulence a weight factor n^2 must be introduced. Both

[6] G. T. Taylor, *Proc. Roy. Soc.* (*London*) *A* **164**, p. 479, eq. (16) (1938).

[7] In order to deduce this result use can be made of certain integrals given in G. N. Watson, *Theory of Bessel Functions* (Cambridge 1944), p. 545, section 16.56, equations (1) and (2), together with the asymptotic expansion given on p. 550, equation (3).

[8] L. Onsager, *Phys. Rev.* (II) **68**, p. 286, 1945.

authors deduce the form (36a) of the function $R(r)$ from the expression for $E(n)$ by means of the conjugate formula to (37).

The physical basis for a deduction of the statistical behavior of the amplitudes ξ_n of the components into which the turbulent motion can be resolved should be found in the properties of the function S_n defined in Section VI, which represents the transmission of energy from the part of the spectrum extending up to a definite wave number n to the part beyond. In the present case the sums of form (22c) must be replaced by integrals. The average value $\bar{S}_n$ of S_n also in the case of free turbulence will be independent of n for the range of wave numbers considered above.

The expression for S_n is a sum of terms of odd degree in the ξ_n. As the ξ_n in a case starting from arbitrary initial conditions may have negative as well as positive values, it is necessary that triple correlations exist between them, in such a way that mean values of the type $\overline{\xi_h \xi_{k-h} \xi_k}$ will be different from zero. Onsager in his (very short) note points to the circumstance that the modulation of a given Fourier component will be mostly due to those other components that belong to wave numbers of comparable magnitude. It seems appropriate to suppose that a mean value $\overline{\xi_{p+\omega} \xi_{p-\omega} \xi_{2p}}$ will be largest when $\omega = 0$ and that it will decrease quickly with increasing values of ω. If we assume that the mean amplitude of a component is proportional to (wave number)$^\sigma$, an expression of the form:

$$\overline{\xi_{p+\omega} \xi_{p-\omega} \xi_{2p}} = A(2p)^{3\sigma} e^{-\beta\omega^2/2p} \tag{39}$$

will satisfy these requirements; it has been chosen in such a way that the range of the wave numbers of the components with which the component ξ_{2p} is in effective coherence is of a breadth proportional to $\sqrt{2p}$.[9] The factor A in (39) has the dimensions: $(velocity)^3$; in connection with (35) we must have $A = num.\ factor \cdot \nu^{\frac{3}{4}} \epsilon^{\frac{1}{4}} = num.\ factor \cdot \epsilon l$ (ϵ represents the average dissipation per unit of length). β must be a pure number, of such magnitude that, say, $\beta n/4 > 9$. Making use of (39) we obtain the following formula for $\bar{S}_n$:

$$\frac{2}{\pi} \bar{S}_n = \int_n^\infty dk \int_0^n dh\, h \overline{\xi_h \xi_{k-h} \xi_k} = \int_n^\infty dn \int_{-k/2}^{n-k/2} d\omega \left(\frac{k}{2} + \omega\right) \overline{\xi_{\frac{k}{2}+\omega} \xi_{\frac{k}{2}-\omega} \xi_k}$$
$$= A \int_n^\infty dk\, k^{3\sigma} \int_{-k/2}^{n-k/2} d\omega \left(\frac{k}{2} + \omega\right) e^{-\beta\omega^2/k}.$$

[9] Onsager states "that the subdivision of the energy must be a stepwise process, such that an n-fold increase of the wave number is reached by a number of steps of the order $\log n$." The idea involved in the formula given in the text is in accordance with this statement, as it assumes that the component ξ_{2p} mainly derives its energy from components with wave numbers nearly equal to p, so that the principal effect is a doubling of the wave numbers in each step.

So long as k satisfies the condition $n - k/2 > 3\sqrt{k/\beta}$, the integral with respect to ω practically gives $\pi^{\frac{1}{2}}k^{\frac{3}{2}}/2\beta^{\frac{1}{2}}$. When k increases and takes the value $2sn$, the range of integration for ω extends from $-sn$ to $-(s-1)n$, and the exponential function will not exceed $e^{-\beta(s-1)^2 n/2s}$, which quickly decreases to zero when $s > 4$. Hence we shall obtain an approximation to the value of $\bar{S}_n$ by restricting the range of integration for k from the lower limit n up to an upper limit $2sn$ (with a properly chosen value for s), if throughout this range we use the value $\pi^{\frac{1}{2}}k^{\frac{3}{2}}/2\beta^{\frac{1}{2}}$ as the result of the integration with respect to ω, so that:

$$\frac{2}{\pi}\bar{S}_n = \frac{A\sqrt{\pi}}{2\sqrt{\beta}}\int_n^{sn} dk\, k^{3\sigma+\frac{3}{2}}.$$

This expression will become independent of the value of n, when $3\sigma + \frac{3}{2} = -1$, which gives:

$$\sigma = -\tfrac{5}{6}. \tag{40}$$

In this way we again arrive at the expression (38) for $E(n)$. It is evident that this result is dependent upon the choice made in equation (39).

The argument is a tentative one, but it clearly shows the importance of triple correlations between the components of the spectrum.

XII. Statistical Treatment, Based upon the Energy Balance, of the System Considered in Sections II–VII

The result obtained in (40) differs from that found in Sections II–VII for the original system of equations, which referred to turbulent motion between two walls in the presence of a primary motion U differing from zero. In particular (40) is at variance with the property of equal dissipation per degree of freedom for wave numbers of average magnitude, which was obtained in the case of the original system and was intimately connected with the properties of the dissipation layers. It was deduced from an exact solution of the equations. Although this solution was independent of the time and consequently presents a simpler character than actual turbulence, the result cannot be discarded as unimportant. It was supported by the results of Section VII for nonstationary solutions; moreover in Sections XIII and XIV a system will be considered, with properties similar to those of the system (1) and (2) in every significant respect, but with turbulent solutions which cannot be independent of the time and thus come nearer to the character of actual turbulence.

There is a possibility that the statistical behavior of free turbulence is different from that of turbulence in a region of limited extent, subjected to boundary conditions at the walls and deriving its energy from a continuously present primary motion. This will be evident when we keep in mind that the presence of a primary velocity U introduces a quantity that did not occur in the similarity considerations of Section X, so that we can no longer rely upon the expressions (35). This destroys the basis upon which equation (36) had been deduced. We cannot be certain, therefore, that in the case of turbulence connected with a primary motion the correlation will be a linear function of r^{1}; neither can we be certain of the applicability of formula (38). It is not to be excluded that in this case the statistical properties of turbulence must be described by relations of the type found in Section V.

The present author made an attempt in treating the statistical problem of turbulence for the motion between fixed walls in the presence of a primary motion U, assuming the equation of energy in the form (7) to represent a key condition, to which was assigned a similar part as to the condition of constant energy in the case of conservative systems. Turbulence is considered as a sequence of states, the order of which is of no importance in the calculation of mean values. It must be required that every such sequence will satisfy the condition that the mean value of the right-hand member of (7) for the sequence should be zero. To every possible sequence a statistical weight is given, determined by the number of ways in which the various states of the sequence can be arranged, using an algorithm similar to that applied in ordinary statistical mechanics. In order to express the condition implied by equation (7), every state is resolved into its elementary components, so that the expression for the dissipation can be formed without difficulty. By means of this procedure mean values can be calculated, and the result is obtained that every component of the turbulence will have the same average dissipation. This is in accordance with the approximation represented by (16a), but, as was pointed out in connection with that equation, makes the total dissipation infinite.

Evidently the procedure applied left out an important feature connected with the part played by the nonlinear terms in the equations of motion, and in order to remedy the "violet catastrophe" threatening in this way a datum must be found representing the influence of these terms. Now the results of Sections V–VII have shown that in the case of turbulence associated with a primary motion the law of decrease of the ξ_n with increasing wave number changes in form as soon as the ξ_n have fallen below an amount of the order ν/b, in such a way that values below ν/b (other than zero) can be considered as relatively improbable. By defining a "threshold value" and requiring that the ξ_n shall never take values below this threshold except zero, an element can be introduced related to the introduction of the quantum hypothesis in the statistical theory of radiation. The simplest way of doing this is to require that the

ξ_n shall take only values that are integer multiples of $\delta\nu/b$, δ being a numerical factor of order unity, zero multiple of course being included. When the statistical calculations are worked out upon this basis,[10] the result can be expressed by means of θ-functions. From these functions approximations can be obtained both for moderate n and for very large n, which have the same form as those represented by formulas (16a) and (16b), so that the total dissipation becomes finite.

One conclusion will be evident from the results obtained: the situation that is found in the well-known statistical theory of conservative systems, where the equation of energy alone is sufficient for obtaining the statistical distribution and where no special notice has to be taken of the full equations of motion, is totally different from that found in the case of nonconservative systems. It may be taken for certain that this fact is connected with the failure of Liouville's theorem for nonconservative systems. The consequence is that in the statistical treatment of nonconservative systems special regard must be given to the influence of the nonlinear terms of the equations of motion, either (as was done in Section XI) by investigating the transmission of energy through the spectrum, or (as was done in the present section) by introducing some particular result deduced from the equations of motion—in this case referring to the nature of the dissipation regions.

It is of importance to observe that in the case of hydrodynamics relations can be found concerning the dissipation in vortex systems, which are closely connected to those obtained for the model system (see the last part of Section XV).

XIII. System with Two Components in the Secondary Motion

Although all important features could be elucidated with the aid of the system governed by equations (1) and (2), for certain purposes it is of interest to consider a slightly more general system, in which the secondary motion possesses two components: $v(t,y)$ and $w(t,y)$. This system is described by the equations:

$$b\frac{dU}{dt} = P - \frac{\nu U}{b} - \frac{1}{b}\int_0^b dy(v^2 + w^2) \tag{41}$$

$$\frac{\partial v}{\partial t} = \frac{U}{b}(v - w) + \nu\frac{\partial^2 v}{\partial y^2} - \frac{\partial}{\partial y}(v^2 - w^2) \tag{42a}$$

$$\frac{\partial w}{\partial t} = \frac{U}{b}(v + w) + \nu\frac{\partial^2 w}{\partial y^2} + \frac{\partial}{\partial y}(2vw). \tag{42b}$$

[10] See (III) and (IV).

The terms of the second degree have been adjusted in such a way that for the case of a limited domain $0 \leqq y \leqq b$, with boundary conditions $v = w = 0$ at both ends, an equation of energy can be formed as before:

$$\frac{d}{dt}\left\{\frac{bU^2}{2} + \int_0^b dy\,\frac{v^2 + w^2}{2}\right\} = PU - \frac{\nu U^2}{b} - \nu\int_0^b dy\left\{\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\}. \tag{43}$$

Moreover the following equation holds:

$$\int_0^b dy\left(w\frac{\partial v}{\partial t} - v\frac{\partial w}{\partial t}\right) = -\frac{U}{b}\int_0^b dy\,(v^2 + w^2) \tag{44}$$

which is obtained when (42a) is multiplied by w, (42b) by v, after which the latter is subtracted from the former one and the result is integrated with respect to y from 0 to b, having regard to the boundary conditions. Equation (44) shows that the system has no solution independent of the time unless v and w are zero for all values of y (provided $U \neq 0$); this is a consequence of the cyclic terms $-Uw$, $+Uv$ introduced into (42a), (42b) respectively.

By introducing a complex variable $\psi = v + iw$ (using the notation $\psi^* = v - iw$ for the conjugate quantity), equations (41)–(42b) can be contracted into:

$$b\frac{dU}{dt} = P - \frac{\nu U}{b} - \frac{1}{b}\int_0^b dy\,\psi\psi^* \tag{45}$$

$$\frac{\partial\psi}{\partial t} = (1 + i)\frac{U}{b}\psi + \nu\frac{\partial^2\psi}{\partial y^2} - \frac{\partial}{\partial y}(\psi^*)^2. \tag{46}$$

We restrict ourselves to the case where U is treated as a constant, which necessitates the replacing of equation (41) or (45) by its mean value with respect to the time:

$$P = \frac{\nu U}{b} + \frac{1}{b}\int_0^b dy\,\overline{\psi\psi^*}. \tag{47}$$

A Fourier development can be introduced by putting:

$$\psi = -\sum_{n=1}^{\infty}\zeta_n\sin\left(\frac{\pi n y}{b}\right) \tag{48}$$

where now the ζ_n are complex coefficients. Then equations (47) and (46) become:

$$P = \frac{\nu U}{b} + \frac{1}{2}\sum\overline{\zeta_n\zeta_n^*} \tag{49}$$

$$\frac{d\zeta_n}{dt} = \left\{(1+i)\frac{U}{b} - \frac{\nu\pi^2n^2}{b^2}\right\}\zeta_n + \frac{\pi n}{b}\left(\frac{1}{2}\sum_{k=1}^{n-1}\zeta_k{}^*\zeta^*{}_{n-k} - \sum_{k=1}^{\infty}\zeta_k{}^*\zeta^*{}_{n+k}\right), \tag{50}$$

while the equation of energy assumes the form:

$$\frac{d}{dt}\left\{\frac{bU^2}{2} + \frac{b}{4}\sum \zeta_n\zeta_n{}^*\right\} = PU - \frac{\nu U^2}{b} - \frac{\nu\pi^2}{2b}\sum n^2\zeta_n\zeta_n{}^*. \tag{51}$$

The system again has the nonturbulent solution

$$U = \frac{Pb}{\nu}; \qquad v = w = 0 \tag{52}$$

which is stable when $Re = Ub/\nu < \pi^2$ and unstable when $Re > \pi^2$.

XIV. Properties of the Turbulent Solutions

In consequence of (44) no stationary turbulent solutions exist. It seems probable that periodic solutions can be found, but so far no exact result has been obtained.

In treating the system for large values of Re we again distinguish between narrow boundary layers to which applies the equation:

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial y}(\psi^*)^2 - \nu\frac{\partial^2\psi}{\partial y^2} = 0 \tag{53}$$

and comparatively broad regions in which we can write:

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial y}(\psi^*)^2 - (1+i)\frac{U\psi}{b} = 0. \tag{54}$$

Equation (53), in which for boundary-layer problems the term $\partial\psi/\partial t$ proves to be unimportant, has solutions of the two types:

$$\psi_{\mathrm{I}} = A\tilde{\omega}_{\mathrm{I}}\,\mathrm{tgh}\left(\frac{Ay}{\nu}\right); \qquad \psi_{\mathrm{II}} = A\tilde{\omega}_{\mathrm{II}}\,\mathrm{tgh}\left\{A\left(\frac{b-y}{\nu}\right)\right\} \tag{55}$$

where:

$$\tilde{\omega}_{\mathrm{I}} = e^{\pi i/3}; \quad e^{\pi i}; \quad e^{5\pi i/3}; \quad \tilde{\omega}_{\mathrm{II}} = 1; \quad e^{2\pi i/3}; \quad e^{4\pi i/3}$$

[in the expressions (55) the coefficient A may be a function of the time].

Equation (54) can be transformed by writing

$$\psi^3 = \tfrac{1}{2}(\rho + i\sigma)^2.$$

After separation of real and imaginary parts we obtain the system:

$$\left.\begin{aligned}\frac{\partial\rho}{\partial t} &= -2z\frac{\partial\rho}{\partial y} + \frac{3}{2}\frac{U}{b}(\rho - \sigma)\\ \frac{\partial\sigma}{\partial t} &= +2z\frac{\partial\sigma}{\partial y} + \frac{3}{2}\frac{U}{b}(\rho + \sigma)\end{aligned}\right\} \tag{56}$$

where $z = \sqrt[3]{(\rho^2 + \sigma^2)/2}$. The boundary conditions that must be satisfied by ρ and σ at the ends of the interval are determined by the necessity of obtaining a proper fit with the solutions (55) without discontinuities in the value of ψ. This requires that $\rho = 0$ at $y = 0$ and that $\sigma = 0$ at $y = b$.

In equations (56) z is an (unknown) function both of y and of t. When z is replaced by its time mean value and thus is reduced to a function of y alone, a new variable can be substituted for y in such a way that a system of homogeneous linear equations with constant coefficients is obtained. A periodic solution of this simplified system, satisfying the boundary conditions for ρ and σ just mentioned, can be found; it can be used to obtain an approximate picture of a periodic solution of the system (56) and to make an estimate of the amplitude to be given to ρ and σ (again there is a connection between this amplitude and the eigenvalue of a parameter occurring in the linearized equation).[11]

When the motion starts from arbitrary initial conditions we find the same tendency toward the generation of dissipation regions as with equation (2). It is found that the dissipation in a region of steep change of v and w is given by:

$$\text{Real part of } \tfrac{1}{6}\{(v_l + iw_l) - (v_r + iw_r)\}^3.$$

This necessarily must be a positive quantity, which implies that the argument of $\{(v_l + iw_l) - (v_r + iw_r)\}$ can be situated only in certain definite sectors.

When equations (42a) and (42b), or, what comes to the same, equation (46) is applied to an unlimited domain, taking $U = 0$, a solution can be constructed similar to that given in Section VIII by writing

$$\psi = \frac{\Psi(\eta)}{\sqrt{t - t_0}}.$$

It is found that either

$$\arg\Psi = 0; \frac{2\pi}{3}; \frac{4\pi}{3}$$

[11] See (I), pp. 37–40. A minus sign ought to be introduced before A in the expression for σ, given in the second line of equation (18.13), p. 39.

the absolute value satisfying the equation:

$$\nu|\Psi|' = |\Psi|^2 - \tfrac{1}{2}\eta|\Psi|;$$

or

$$\arg \Psi = \frac{\pi}{3}; \pi; \frac{5\pi}{3}$$

the absolute value satisfying the equation:

$$\nu|\Psi|' = -|\Psi|^2 - \tfrac{1}{2}\eta|\Psi|.$$

The statistical method indicated in Section XII can be applied to the system (41)–(42b) (for the limited domain, with $U \neq 0$), as indeed was done in the original paper.[10]

XV. Concluding Remarks

The investigations described in the preceding pages can be considered in the first place from the mathematical point of view to be concerned with the properties of systems of quantities subjected to certain nonlinear differential equations.

As the quantities of interest we have taken the ξ_n (or ζ_n), *i.e.*, the amplitudes of the components that together constitute the spectrum of the system. The equations by which these amplitudes are connected are of the general type

$$\frac{d\xi_n}{dt} = \left(\frac{U}{b} - \frac{\nu\pi^2 n^2}{b^2}\right)\xi_n + f_n, \tag{57}$$

the first term of the right-hand member determining an exponential increase, the second term a damping, while a coupling between the various ξ_n is introduced through the f_n, which are quantities of the second degree in the ξ_n of such nature that

$$\sum_{n=1}^{\infty} \xi_n f_n = 0 \tag{58}$$

so that in a ξ_n-space the vector f is perpendicular to ξ. A direct investigation of the behavior of the ξ_n on the basis of these equations has not been given, but enough has been found out about them to bring into evidence highly interesting properties of the equations.

The most important question to be resolved is the mean value of $\Sigma\xi_n^2$ in function of U for large values of Ub/ν, which question involves the consideration of statistical problems.

Along with the system described by equations (6) other systems can be considered, *e.g.*, with a finite number of variables, or with more simple expressions for the f_n, such as $f_n = \xi_{n-1}\xi_n - \xi^2{}_{n+1}$. For problems connected with such systems reference must be made to Section 21 of the paper quoted under (I) in footnote 1. In all cases triple correlations between the ξ_n will be obtained.

In the case of the system (6), however, the main results deduced in the preceding pages were not obtained from the equations for the ξ_n, but from the partial differential equation (2) with which the particular system (6) was equivalent. The characteristic feature of (2) is the combination of the two terms $\nu(\partial^2 v/\partial y^2)$ and $2v(\partial v/\partial y)$, which gives rise to the appearance of the dissipation layers.

That this feature has its analogy in the equations of hydrodynamics, has been mentioned already. The considerations put forward in the present paper had the object of pointing out that these terms characterize the peculiar mechanism that is operative in producing turbulence and determine the statistical relations governing the transfer of energy. In view of its importance it is useful to look somewhat more closely into the way in which a similar feature appears in hydrodynamics. The group of terms

$$\frac{\partial v}{\partial t} + 2v\frac{\partial v}{\partial y} - \nu\frac{\partial^2 v}{\partial y^2}$$

of equation (2) will find its closest analogy in the terms

$$\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) - \nu\frac{\partial^2 u}{\partial x^2}$$

which are decisive in determining the appearance of shock waves in the supersonic motion of a gas. As shown by Rayleigh, and by Prandtl and others, the (theoretical) thickness of the transition layer in which kinetic energy of the mass motion is transformed into heat (internal energy) is of the order of magnitude ν/u. The ordinary problem of turbulence, however, refers to motion with velocities well below the velocity of sound, so that the moving fluid can be treated as incompressible. The condition imposed by the equation of continuity in that case prevents the appearance of shock waves, and the dissipation regions to be found in such liquids are of the nature of "quasi-slip" regions, of which boundary and vortex layers represent the typical examples. Now in all the well-known cases of boundary and vortex layers the thickness of the layer at high Reynolds numbers is found to be proportional to $\sqrt{\nu/u}$ instead of to ν/u. This implies a total dissipation (integrated over the thickness of the layer), per unit area of the layer, proportional to $\nu^{\frac{1}{2}}u^{\frac{5}{2}}$ instead of to u^3. If this result should be typical for all fluid motion, it would appear that the resistance in turbulence would be at most proportional to the $\frac{5}{2}$-power of the velocity.

Happily there is a case in which we obtain dissipation proportional to the third power of the velocity.[12] This is found when the motion is such that vortex lines are drawn out by the field. A typical example is arrived at by considering an axially symmetric field, having the velocity components (referred to cylindrical coordinates r, ϑ, z)

$$v_r = -Ar; \qquad v_\vartheta = u(t,r); \qquad v_z = 2Az,$$

where A is a constant (of the dimensions *velocity/length*); and the pressure:

$$p = -\frac{1}{2}\rho A^2(r^2 + 4z^2) + \rho \int dr \frac{u^2}{r}.$$

These expressions satisfy the equation of continuity and the equations of motion for the r- and z-directions. The equation of motion for the ϑ-direction has the form:

$$\frac{\partial u}{\partial t} - Ar\frac{\partial u}{\partial r} - Au = \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right).$$

This equation has a solution that is independent of the time, *viz.:*

$$u = \frac{C}{2\pi r}\left(1 - e^{-Ar^2/2\nu}\right) \tag{59}$$

where C represents the circulation around the vortex. The vorticity $\gamma = (\partial u/\partial r + u/r)$ is given by:

$$\gamma = \frac{AC}{2\pi\nu}e^{-Ar^2/2\nu} \tag{59a}$$

and the total dissipation per unit height in the z-direction becomes:

$$\rho\nu\int_0^\infty dr 2\pi r\gamma^2 = \frac{\rho A C^2}{4\pi} \tag{60}$$

which is of the third degree with respect to the velocities, while it is independent of ν.

I have not succeeded in constructing an analogous solution for the case of a vortex sheet; in that case one falls back upon a maximum value of the vorticity of the order $\nu^{-\frac{1}{2}}$, which is not sufficient for the

[12] See (II), pp. 11–12. That the drawing out of vortices must represent a fundamental process that controls the dissipation of energy in turbulent motion had been pointed out by G. I. TAYLOR; comp. *Proc. Roy. Soc.* (*London*) *A* **164**, p. 15, 1938.

purpose in view. This indicates that in hydrodynamical turbulence, at least in the case of the flow through a tube or channel, the fate of vortices extending in the direction of the motion is of great importance. At the same time it shows that the geometrical features of incompressible fluid flow introduce specific complications into the problems, which, however, stand apart from the basic dynamical relations to which attention has been given here.