"El saber de mis hijos hará mi grandeza"

# UNIVERSIDAD DE SONORA

## División de Ciencias Exactas y Naturales

**Programa de Posgrado en Matemáticas**

**título**

# T E S I S

Que para obtener el grado académico de:

**Maestro en Ciencias**
(Matemáticas)

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Hermosillo, Sonora, México, February 5, 2019

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# Chapter 1

# Introduction

## 1.1 Brief History of the Burger's Equation

The viscous Burgers equation was presented in 1940 and in 1950 Hopf and in 1951 Cole independently introduced the method that has come to be known as the Cole-Hopf transformation to solve the viscous Burgers equation. There is much conjecture as to the origin of the viscous Burger's equation as it had appeared in previously published literature though Burger made it famous by introducing it as a simple model of turbulence and presenting some results on a preliminary investigation of the properties of the equation. There is also conjecture regarding the origin of the Cole-Hopf transformation as Forsyth introduced a method and equation in 1906 in his work on differential equations that can be transformed into the viscous Burgers equation and the Cole-Hopf transformation. He did not discuss the equation, its applications or the transformation in any detail.

In the field of fluid dynamics, there exists an interesting problem called Stokes First Problem in which an infinite plate with a fluid on top of it is impulsively set into motion with a constant speed. This problem allows for the study of the propagation of a disturbance by viscous friction alone.

Some assumptions are made, namely: = constant (density), $\mu$ or $\nu$ = constant (viscosity), $P$ = constant (pressure). With $\nu > 0$ being the viscosity, $u$ the velocity field, t is time, y is the direction normal to the plate and $\alpha$ an arbitrary constant. This is the viscous Burgers equation. The effect of adding the viscosity term (as compared to the inviscid Burgers equation is to decrease the amplitude of $u$ in time and also to prevent multi-valued solutions from developing. In the case of Stokes First problem, $\alpha$ is 0 though there are other situations where it is $u > 0$. With $\alpha = 0$, there are simpler solutions to Stokes First problem though it can be solved by making use of the Cole-Hopf transform. The Cole-Hopf transform provides an important mechanism to solve equations of this form

Burgers' equation or Bateman-Burgers equation is a fundamental partial differential equation occurring in various areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow. The equation was first introduced by Harry Bateman in 1915 and later studied by Johannes Martinus Burgers in 1948.

# Chapter 2

# Preliminaries

## 2.1 Spectral Methods

The first spectral methods computations were simulations of homogeneous turbulence on periodic domains. For that type of problem, the natural choice of basis functions is the family of (periodic) trigonometric polynomials. In this chapter, we will discuss the behavior of these trigonometric polynomials when used to approximate smooth functions. We will consider the properties of both the continuous and discrete Fourier series, and come to an understanding of the factors determining the behavior of the approximating series.

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. If $\{e_k\}_k \in B$ is a countable orthonormal base of $\mathcal{H}$, then each $x$ of $\mathcal{H}$ element can be written as

$$x = \sum_{k \in B} \langle e_k, x\rangle e_k$$

This sum is known as the Fourier expansion of $x$ and the spectral methods are based on this type of series.

If we choose the base $B = span\{e^{inx} : |n| \leq \infty\}$ we obtain the Fourier series, then for $u(x) \in \mathcal{L}^2[0, 2\pi]$ the Fourier series $F[u]$ of the function $u$ is defined as follows

$$F[u] = \sum_{|n| \leq \infty} \hat{u}_n e^{inx} \tag{2.1}$$

Where $\hat{u}_n$ are known as the Fourier coefficients and are determined by:

$$\hat{u}_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{inx}dx = \begin{cases} \hat{a}_n & n = 0, \\ \frac{\hat{a}_n - i\hat{b}_n}{2} & n > 0, \\ \frac{\hat{a}_n + i\hat{b}_n}{2} & n < 0. \end{cases} \tag{2.2}$$

For our purposes, the relevant question is how well the Fourier series approaches a function. For this, two types of operators are defined, projection and interpolation operators.

### 2.1.1 Projection Operator

The operator $\mathcal{P}_N$ is defined as the truncated Fourier series, that is,

$$\mathcal{P}_N u(x) = \sum_{|n| \leq \frac{N}{2}} \hat{u}_n e^{inx} \tag{2.3}$$

Which is a projection in the space of finite dimension

$$\hat{B}_N = span\left\{ e^{inx} : |n| \leq \frac{N}{2} \right\}, \quad dim(\hat{B}_N) = N+1$$

**Example 1** *Consider the $C_p^\infty[0, 2\pi]$ function*

$$u(x) = \frac{3}{3 - 4cos(x)} \tag{2.4}$$

*Its expansion coefficients are*

$$\hat{u}_n = 2^{-|n|} \tag{2.5}$$

### 2.1.2 Differentiation of the continuous expansion

When approximating a function $u(x)$ by the finite Fourier series $\mathcal{P}_N u$, we can easily obtain the derivatives of $\mathcal{P}_N u$ by simply differentiating the basis functions.

If u is a sufficiently smooth function, then one can differentiate the sum

$$\mathcal{P}_N u(x) = \sum_{|n| \leq \frac{N}{2}} \hat{u}_n e^{inx} \tag{2.6}$$

term by term, to obtain

$$\frac{d^q}{dx^q} \mathcal{P}_N u(x) = \sum_{|n| \leq \frac{N}{2}} \hat{u}_n \frac{d^q}{dx^q} e^{inx} = \sum_{|n| \leq \frac{N}{2}} (in)^q \hat{u}_n e^{inx} \tag{2.7}$$

It follows that the projection and differentiation operators commute

$$\mathcal{P}_N \frac{d^q}{dx^q} u = \frac{d^q}{dx^q} \mathcal{P}_N u \tag{2.8}$$

This property implies that for any constant coefficient differention operator $\mathcal{L}$

$$\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N) u \tag{2.9}$$

known as the truncation error, vanishes. Thus, the Fourier approximation to the equation $u_t = \mathcal{L}u$ is exactly the projection of the analytic solution.

### 2.1.3 Approximation theory for smooth functions

When using the Fourier approximation to discretize the spatial part of the equation

$$u_t = \mathcal{L}u$$

where $\mathcal{L}$ is a differential operator, it is important that our approximation, both to $u$ and to $\mathcal{L}u$, be accurate. To establish consistency we need to consider not only the difference between $u$ and $\mathcal{P}_N u$, but also the distance between $\mathcal{L}u$ and $\mathcal{L}\mathcal{P}_N u$, measured in an appropriate norm. This is critical, because the actual rate of convergence of a stable scheme is determined by the truncation error

$$\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N)u$$

The truncation error is thus determined by the behavior of the Fourier approximations not only of the function, but of its derivatives as well. It is natural, therefore, to use the Sobolev q-norm denoted by $H_p^q[0, 2\pi]$, which measures the smoothness of the derivatives as well as the function,

$$\|u\|^2_{H_p^q[0,2\pi]} = \sum_{m=0}^{q} \int_0^{2\pi} |u^m(x)|^2 \, dx \tag{2.10}$$

The subscript p indicates the fact that all our functions are periodic, for which the Sobolev norm can be written in mode space as

$$\|u\|^2_{H_p^q[0,2\pi]} = 2\pi \sum_{m=0}^{q} \sum_{|n|\leq\infty} |n|^{2m} |\hat{u}_n|^2 = 2\pi \sum_{|n|\leq\infty} \left( \sum_{m=0}^{q} |n|^{2m} \right) |\hat{u}_n|^2 \tag{2.11}$$

where the interchange of the summation is allowed provided u(x) has sufficient smoothness, since for $n \neq 0$ and $q > \frac{1}{2}$

$$(1 + n^{2q}) \leq \sum_{m=0}^{q} n^{2m} \leq (q+1)(1 + 2^{2q}) \tag{2.12}$$

the norm $\| \cdot \|_{W_p^q[0,2\pi]}$ defined by

$$\|u\|_{W_p^q[0,2\pi]} = \left( \sum_{|n|\leq\infty} (1 + n^{2q}) |\hat{u}_n|^2 \right)^{1/2} \tag{2.13}$$

is equivalent to $\| \cdot \|_{H_p^q[0,2\pi]}$.

### 2.1.4 Results for the continuous expansion

Consider, first, the continuous Fourier series

$$\mathcal{P}_{2N} u(x) = \sum_{|n|\leq N} \hat{u}_n e^{inx} \tag{2.14}$$

We start with an $L^2$ estimate for the distance between u and its trigonometric approximation $\mathcal{P}_{2N}u$.

$$\|u-\mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq CN^{-q}\|u^{(q)}\|_{L^2[0,2\pi]} \tag{2.15}$$

$$\|u-\mathcal{P}_{2N}u\|^2_{L^2[0,2\pi]} = 2\pi \sum^{|n|>N} |\hat{u}_n|^2 \tag{2.16}$$

$$\begin{aligned}
\sum^{|n|>N} |\hat{u}_n|^2 &= \sum^{|n|>N} \frac{1}{N^{2q}n^{2q}}|\hat{u}_n|^2 && (2.17)\\
&\leq N^{-2q}\sum^{|n|>N} n^{2q}|\hat{u}_n|^2 && (2.18)\\
&\leq N^{-2q}\sum^{|n|\geq 0} n^{2q}|\hat{u}_n|^2 && (2.19)\\
&= \frac{1}{2\pi}N^{-2q}\|u^{(q)}\|^2_{L^2[0,2\pi]} && (2.20)
\end{aligned}$$

$$\|u^{(q)}\|_{L^2[0,2\pi]} = Cq!\|u\|_{L^2[0,2\pi]} \tag{2.21}$$

$$\|u-\mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq N^{-q}\|u^{(q)}\|_{L^2[0,2\pi]} \leq C\frac{q!}{N^q}\|u\|_{L^2[0,2\pi]} \tag{2.22}$$

$$\|u-\mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq\sim C\left(\frac{q}{N}\right)^q e^{-q}\|u\|_{L^2[0,2\pi]} \sim Ke^{-CN}\|u\|_{L^2[0,2\pi]} \tag{2.23}$$

$$\|u-\mathcal{P}_{2N}u\|_{W_p^q[0,2\pi]} \leq \frac{C}{N^{r-q}}\|u\|_{W_p^r[0,2\pi]} \tag{2.24}$$

$$\|u-\mathcal{P}_{2N}u\|^2_{W_p^q[0,2\pi]} = 2\pi\sum_{|n|>N}(1+|n|^{2q})|\hat{u}_n|^2 \tag{2.25}$$

$$\begin{aligned}
(1+|n|^{2q}) \leq (1+|n|)^{2q} = \frac{(1+|n|)^{2r}}{(1+|n|)^{2(r-q)}} &\leq \frac{(1+|n|)^{2r}}{N^{2(r-q)}} && (2.26)\\
&\leq (1+r)\frac{(1+n^{2r})}{N^{2(r-q)}} && (2.27)
\end{aligned}$$

$$\|u - \mathcal{P}_{2N}u\|^2_{W^q_p[0,2\pi]} \leq C \sum_{|n|>N} \frac{(1+n^{2r})}{N^{2(r-q)}} |\hat{u}_n|^2 \leq C \frac{\|u\|^2_{W^r_p[0,2\pi]}}{N^{2(r-q)}} \tag{2.28}$$

$$\mathcal{L}u = \sum_{j=1}^{s} a_j \frac{d^j u}{dx^j} \tag{2.29}$$

$$\|\mathcal{L}u - \mathcal{L}\mathcal{P}_N u\|_{W^q_p[0,2\pi]} \leq CN^{-(r-q-s)} \|u\|^2_{W^r_p[0,2\pi]} \tag{2.30}$$

$$\begin{aligned}
\|\mathcal{L}u - \mathcal{L}\mathcal{P}_N u\|_{W^q_p[0,2\pi]} &\leq \left\| \sum_{j=1}^{s} a_j \frac{d^j u}{dx^j} - \sum_{j=1}^{s} a_j \frac{d^j \mathcal{P}_N u}{dx^j} \right\|_{W^q_p[0,2\pi]} && (2.31)\\
&\leq \max_{0\leq j\leq s} |a_j| \left\| \sum_{j=1}^{s} \frac{d^j}{dx^j}(u - \mathcal{P}_N u) \right\|_{W^q_p[0,2\pi]} && (2.32)\\
&\leq \max_{0\leq j\leq s} |a_j| \sum_{j=1}^{s} \|(u - \mathcal{P}_N u)\|_{W^{q+s}_p[0,2\pi]} && (2.33)\\
& && (2.34)
\end{aligned}$$

### 2.1.5 Discrete trigonometric polynomials

$$x_j = \frac{2\pi}{N} j, \qquad j \in [0, ..., N-1]$$

$$\widetilde{u}_n = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-inx_j} \tag{2.35}$$

### 2.1.6 Theorem 2.5

The quadrature formula

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)dx = \frac{1}{N} \sum_{j=0}^{N-1} f(x_j) \tag{2.36}$$

is exact for any trigonometric polynomial $f(x) = e^{inx}$ , $|n| < N$
Given a function $f(x) = e^{inx}$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.37}$$

$$\frac{1}{N}\sum_{j=0}^{N-1} f(x_j) = \frac{1}{N}\sum_{j=0}^{N-1} e^{in(2\pi j/N)} \tag{2.38}$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} q^j \tag{2.39}$$

### 2.1.7 Interpolation Operator

We define the set of equispaced points, consisting of an even number $N$ of points $x_j \in [0, 2\pi)$

$$x_j = \frac{2\pi}{N}j, \qquad j \in [0, ..., N-1]$$

Then the discrete Fourier transform, which will be defined as an interpolation operator, is written as

$$\mathcal{I}_N u(x) = \sum_{|n|\leq \frac{N}{2}} \widetilde{u}_n e^{inx} \tag{2.40}$$

Where the coefficients $\widetilde{u}_n$ are the approximate Fourier coefficients $\hat{u}_n$ using the trapezoid rule.

$$\widetilde{u}_n = \frac{1}{N\widetilde{C}_n}\sum_{j=0}^{N-1} u(x_j)e^{inj}, \quad \widetilde{C}_n = \begin{cases} 2 & \text{si} \quad |n| = \frac{N}{2} \\ \\ 1 & \text{si} \quad |n| < 0 \end{cases}$$

Then for any periodic function such that $u(x) \in \mathcal{C}_p^0[0, 2\pi]$ the interpolation operator $\mathcal{I}_N$ complies with the following

$$\mathcal{I}_N u(x_j) = u(x_j), \quad \forall x_j = \frac{2\pi j}{N}, \quad j = 0, ..., N-1$$

### 2.1.8 Definitions

To fulfill our purpose we need to understand the following definitions.

Let's define the following problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \mathcal{L}u(x,t), && x \in \mathcal{D}, \quad t \geq 0 \\ u(x,0) &= g(x), && x \in \mathcal{D}, \quad t = 0 \\ \mathcal{B}u(x,t) &= 0, && x \in \delta\mathcal{D}, \quad t > 0 \end{aligned}$$

Donde $\mathcal{L}$ es independiente del tiempo y el espacio. Asumamos que el operador $\mathcal{B}[\mathcal{D}]$ esta incluido en el operador $\mathcal{L}$.

Entonces el problema esta bien definido si, para cada $g \in C_0^r$ y para cada tiempo $T_0 > 0$ existe una unica solucion $u(x,t)$ tal que

$$\|u(t)\| \leq Ce^{\alpha t}\|g\|_{\mathcal{H}^p(\mathcal{D})} \quad 0 \leq t \leq T_0$$

Para $p \leq r$ y algunas constantes positivas $C$ y $\alpha$. Para $p = 0$ se dice que esta fuertemente definido, es decir, para la norma $\mathcal{L}^2$

- Convergencia. Una aproximacion es convergente si

$$\|\mathcal{P}_N u(t) - u_N(t)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.41}$$

Para toda $t \in [0,T]$, $u(0) \in B$, y $u_N(0) \in B_N$

- Consistencia. Una aproximacion es consistente si

$$\|\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N)u)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.42}$$

$$\|\mathcal{P}_N u(0) - u_N(0)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.43}$$

Para toda $u(0) \in B$ y $u_N(0) \in B_N$

- Estabilidad. Una aproximacion es estable si

$$\|e^{\mathcal{L}_N t}\|_{\mathcal{L}^2_w(\mathcal{D})} \leq C(t), \quad \forall N \tag{2.44}$$

con la norma asociada al operador norma

$$\|e^{\mathcal{L}_N t}\|_{\mathcal{L}^2_w(\mathcal{D})} = \sup_{u \in B} \frac{\|e^{\mathcal{L}_N t}u\|_{\mathcal{L}^2_w(\mathcal{D})}}{\|u\|_{\mathcal{L}^2_w(\mathcal{D})}}$$

y $C(t)$ es independiente de $N$ y acotada para cualquier $t \in [0,T]$.

- **Convergencia Espectral.**

Si la suma de los cuadrados de los coeficientes de Fourier es acotada, es decir,

$$\sum_{|n| \leq \infty} |\hat{u}_N|^2 < \infty$$

Entonces la serie truncada converge en la norma $L^2$

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

Si, mas aun, la suma de los valores absolutos de los coeficientes de Fourier es acotada, es decir,

$$\sum_{|n|\leq\infty} |\hat{u}_N| < \infty$$

Entonces la serie truncada converge uniformemente

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

El hecho que la suma truncada converge implica que el error es dominado por la cola de la serie, es decir,

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} = 2\pi \sum_{|n|>\frac{N}{2}} |\hat{u}_N|^2$$

y

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \leq \sum_{|n|>\frac{N}{2}} |\hat{u}_N|$$

Sea $u(x)$ tal que su derivada $u'(x) \in L^2[0,\pi]$, entonces para $n \neq 0$ tenemos que,

$$\begin{aligned} 2\pi\hat{u}_N &= \int_0^{2\pi} u(x)e^{inx}dx \\ &= -\frac{1}{in}(u(2\pi) - u(0)) - \frac{1}{in}\int_0^{2\pi} u'(x)e^{inx}dx \end{aligned}$$

Por lo tanto,

$$|\hat{u}_N| \propto \frac{1}{n}$$

Ahora, si para $u(x)$, las $(m-1)$ derivadas, sus extensiones periodicas son todas continuas y si la m-esima derivada $u^{(m)} \in L^2[0,2\pi]$, entonces $\forall n \neq 0$ , haciendo el mismo procedimiento anterior, tenemos que los coeficientes de Fourier $\hat{u}_N$ de $u(x)$ decaen como

$$|\hat{u}_N| \propto \left(\frac{1}{n}\right)^m$$

Esto se conoce como convergencia espectral, es decir entre mas suave la funcion, la serie converge mas rapido.

### 2.1.9 Stability of the Fourier-collocation method for hyperbolic problems

$$(f_N, g_N)_N = \frac{1}{N+1}\sum_{j=0}^{N} f_N(x_j)\bar{g}_N(x_j)$$

$$\|f_N\|_N^2 = (f_N, f_N)_N$$

$$(f_N, g_N)_N = \frac{1}{2\pi}\int_0^{2\pi} f_N \bar{g}_N dx, \|f_N\|_{L^2[0,2\pi]} = \|f_N\|_N^2$$

$$(f_N, g_N)_N = \frac{1}{N}\sum_{j=0}^{N-1} f_N(x_j)\widetilde{g}_N(x_j)dx, \|f_N\|_N^2 = (f_N, f_N)_N$$

$$K^{-1}\|f_N\|_{L^2[0,2\pi]}^2 \leq \|f_N\|_N^2 \leq K\|f_N\|_{L^2[0,2\pi]}^2$$

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0, \tag{2.45}$$

$$u(x,0) = g(x) \tag{2.46}$$

$$\sum_{j=0}^{N-1} u_N^2(x_j,t) \leq \sum_{j=0}^{N-1} u_N^2(x_j,0)$$

$$\left|\frac{\partial u_N}{\partial t}\right|_{x_j} + a(x_j)\left|\frac{\partial u_N}{\partial x}\right|_{x_j} = 0 \tag{2.47}$$

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = -\sum_{j=0}^{N-1} u_N^2(x_j,t)\left|\frac{\partial u_N}{\partial x}\right|_{x_j}$$

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = -\frac{N}{2\pi}\int_0^{2\pi} u_N(x)\frac{\partial u_N}{\partial x}dx = 0$$

$$\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,0)$$

$$\begin{aligned}\frac{1}{k}\sum_{j=0}^{N-1} u_N^2(x_j,t) \leq \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) &= \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,0)\\ &\leq k\sum_{j=0}^{N-1} u_N^2(x_j,0)\end{aligned}$$

$$\|u_N(x,t)\|_N \leq k\|u_N(x,0)\|_N$$

$$\frac{du_N(t)}{dt} + ADu_N(t) = 0 \tag{2.48}$$

$$u(t) = e^{-ADt}u(0)$$

$$\|e^{-ADt}\| \leq K(t)$$
$$e^{-ADt}e^{-(AD)^T t} \leq K^2(t)$$

$$\|e^{-ADt}\| \leq k$$

### 2.1.10 Stability for parabolic equations

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial^2 u}{\partial x^2} \tag{2.49}$$
$$u(x,0) = g(x) \tag{2.50}$$

$$\|u_N(t)\|_N \leq \sqrt{\frac{\max b(x)}{\min b(x)}}\|u_N(0)\|_N$$

$$\frac{du(t)}{dt} = BD^{(2)}u(t)$$

$$D^{(2)} \equiv D \cdot D$$

$$u^T B^{-1}\frac{d}{dt}u = u^T D^{(2)}u = u^T DDu \tag{2.51}$$
$$= (D^T u)^T (Du) = -(Du)^T (Du) \leq 0 \tag{2.52}$$

$$\frac{d}{dt}u^T B^{-1}u \leq 0$$

$$\frac{1}{\max b(x)}\|u_N(t)\|_N^2 \leq u^t(t)B^{-1}u(t) \leq u^T(0)B^{-1}u(0) \leq \frac{1}{\min b(x)}\|u_N(0)\|_N^2$$

## 2.2 Fourier Spectral Methods

### 2.2.1 Metodo de Fourier-Galerkin

Sea $u(x,t) \in \mathcal{L}^2[0, 2\pi]$ y definamos el siguiente problema.

$$\frac{\partial u(x,t)}{\partial t} = \mathcal{L}u(x,t), \quad x \in [0, 2\pi], \quad t \geq 0$$
$$u(x,0) = g(x), \qquad x \in [0, 2\pi], \quad t = 0$$

Debemos encontrar las funciones $u_N(x,t)$ del espacio $\hat{B_N}$, es decir,

$$u_N(x,t) = \sum_{|n|\leq \frac{N}{2}} a_n(t)e^{inx}$$

Donde los coeficientes $a_n(t)$ se determinan del residuo $R_N$

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} - \mathcal{L}u_N(x,t)$$

Si expresamos a $R(t)$ como una serie de fourier obtenemos

$$R(x,t) = \sum_{|n|\leq \infty} \hat{R}_n(t)e^{inx}$$

Donde

$$\hat{R}_n(t) = \frac{1}{2\pi}\int_{-1}^{1} R_N(x,t)e^{inx}dx = 0, \quad \forall |n| \leq \frac{N}{2}$$

Lo cual equivale a $N+1$ ecuaciones diferenciales ordinarias, lo cual nos permite determinar los coeficientes $a_n(t)$ con las siguientes condiciones iniciales

$$u_N(x,0) = \sum_{|n|\leq \frac{N}{2}} a_n(0)e^{inx}, \quad a_n(0) = \frac{1}{2\pi}\int_{-1}^{1} g(x)e^{inx}dx$$

### 2.2.2 Operador semi-acotado

Un caso especial de un problema bien definido es cuando $\mathcal{L}$ es semi-acotado, es decir, $\mathcal{L} + \mathcal{L}^* \leq \alpha I$ para alguna constante $\alpha$.

Por ejemplo, dada la solucion $u(x,t)$ que esta en un espacio de Hilbert $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, suponiendo que el producto interior es el de $L^2[0,2\pi]$, la cual satisface el siguiente problema de valor de frontera con su valor inicial adecuado.

$$\frac{\partial u}{\partial t} = \mathcal{L}u \tag{2.53}$$

Mostraremos que este problema esta bien definido si $\mathcal{L}$ es semi-acotado. Empezaremos estimando la derivada de la norma como sigue

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= \frac{d}{dt}(u,u) = (\frac{du}{dt},u) + (u,\frac{du}{dt})\\ &= (\mathcal{L}u,u) + (u,\mathcal{L}u) = (u,\mathcal{L}^*u) + (u,\mathcal{L}u)\\ &= (u,(\mathcal{L}+\mathcal{L}^*)u).\end{aligned}$$

Como $\mathcal{L} + \mathcal{L}^* \leq \alpha I$, tenemos que $\frac{d}{dt}\|u\|^2 \leq \alpha\|u\|^2$, lo cual equivale a $\frac{d}{dt}\|u\| \leq \alpha\|u\|$, y por lo tanto la norma es acotada, es decir, resolviendo la ecuacion obtenemos

$$\|u(t)\| \leq e^{\alpha t}\|u(0)\|$$

Y por lo tanto el problema esta bien definido.

Ahora veamos el siguiente ejemplo de un operador semiacotado y consideremos el siguiente problema

$$\mathcal{L} = a(x)\frac{\partial}{\partial x}$$

Donde $a(x)$ es una funcion real periodica con derivada acotada. Primero encontraremos a $\mathcal{L}^*$ como sigue

$$\begin{aligned}
(\mathcal{L}u, v)_{L^2[0,2\pi]} &= \int_0^{2\pi} a(x)\frac{\partial u}{\partial x}\bar{v}dx \\
&= -\int_0^{2\pi} u\frac{\partial}{\partial x}a(x)\bar{v}dx \\
&= \left(u, \left[a(x)\frac{\partial}{\partial x} + \frac{da(x)}{dx}\right]v\right)_{L^2[0,2\pi]}
\end{aligned}$$

Entonces,

$$\mathcal{L}^* = -\frac{\partial}{\partial x}a(x)I = -a(x)\frac{\partial}{\partial x} - a'(x)I$$

De esto ultimo obtenemos que,

$$\mathcal{L} + \mathcal{L}^* = a'(x)I$$

Como $a'(x)$ acotada, es decir, $|a'(x)| \leq 2\alpha$ para algun $\alpha$, tenemos que,

$$\mathcal{L} + \mathcal{L}^* \leq 2\alpha I$$

Dado el problema $\frac{\partial u}{\partial t} = \mathcal{L}u$, donde $\mathcal{L}$ es un operador semi-acotado en el producto escalar usual de $\mathcal{L}^2[0, 2\pi]$, entonces el metodo de Fourier-Galerkin es estable.

*Demostracion.* Primero, Vamos a mostrar que $\mathcal{P}_N = \mathcal{P}_N^*$. Comencemos con la simple observacion que.

$$(u, \mathcal{P}_N v) = (\mathcal{P}_N u, \mathcal{P}_N v) + ((I - \mathcal{P}_N)u, \mathcal{P}_N v).$$

$$\frac{\partial u_N}{\partial t} = \mathcal{P}_N\mathcal{L}\mathcal{P}_N u_N = \mathcal{L}_N u_N$$

Entonces,

$$\begin{aligned}\mathcal{L}_N + \mathcal{L}_N^* &= \mathcal{P}_N \mathcal{L} \mathcal{P}_N + \mathcal{P}_N \mathcal{L}^* \mathcal{P}_N \\ &= \mathcal{P}_N (\mathcal{L} + \mathcal{L}^*) \mathcal{P}_N \leq 2\alpha \mathcal{P}_N\end{aligned}$$

Siguiente el procedimiento anterior tenemos que

$$\|u_N(t)\| \leq e^{\alpha t} \|u_N(0)\|$$

Lo cual significa que es estable y por lo tanto convergente

### 2.2.3 Metodo de Fourier-Colocacion

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{2\pi}{N} j, \quad j \in [0, ..., N-1]$$

Para $u(x,t) \in \mathcal{L}^2[0, 2\pi]$ debemos encontrar las funciones

$$u_N \in \widetilde{B}_N = span\left\{ \left( cos(nx),\;\; 0 \leq n \leq \frac{N}{2} \right) \cup \left( sin(nx),\;\; 1 \leq n \leq \frac{N}{2} - 1 \right) \right\}$$

Las cuales tienen la forma

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}$$

Si las expresamos en terminos de polinomios

$$u_N(x,t) = \sum_{0}^{N-1} u_N(x_j, t) g_j(x)$$

Donde $g_j(x)$ son los polinomios de Lagrange y ademas cumple que $g_j(x_i) = \delta_{ij}$, entonces el residuo

$$R_N(x,t) = \frac{\partial u_n(x,t)}{\partial t} - \mathcal{L} u_N(x,t)$$

Y debera cumplir

$$R_N(x_j, t) = 0, \quad \forall j \in [0, ..., N-1]$$

De modo que la ecuacion satisface

$$\frac{\partial u_N(x,t)}{\partial t} - \mathcal{I}_N \mathcal{L} u_N(x,t) = 0$$

# Chapter 3

# Numerical Solutions of the Burger's Equation

Sea $u(x,t) \in \mathcal{L}^2[0,2\pi]$ y definamos el siguiente problema.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in [0,1], \quad t \geq 0 \tag{3.1}$$

With an initial condition

$$u(x,0) = g(x) \tag{3.2}$$

## 3.1 Fourier Galerkin

The approximate function $u_N(x,t)$ is represented as the truncated Fourier Series in $\hat{B_N}$,

$$u_N(x,t) = \sum_{|n|\leq \frac{N}{2}} a_n(t)e^{inx}$$

where $a_n(t)$ are determinated of the residue $R_N$

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} - \mathcal{L}u_N(x,t)$$

Si expresamos a $R(t)$ como una serie de fourier obtenemos

$$R(x,t) = \sum_{|n|\leq\infty} \hat{R}_n(t)e^{inx}$$

Donde

$$\hat{R}_n(t) = \frac{1}{2\pi}\int_{-1}^{1} R_N(x,t)e^{inx}dx = 0, \quad \forall |n| \leq \frac{N}{2}$$

Lo cual equivale a $N+1$ ecuaciones diferenciales ordinarias, lo cual nos permite determinar los coeficientes $a_n(t)$ con las siguientes condiciones iniciales

$$u_N(x,0) = \sum_{|n|\leq \frac{N}{2}} a_n(0)e^{inx}, \quad a_n(0) = \frac{1}{2\pi}\int_{-1}^{1} g(x)e^{inx}dx$$

## 3.2 Fourier Collocation

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{2\pi}{N} j, \quad j \in [0, ..., N-1]$$

Para $u(x,t) \in \mathcal{L}^2[0, 2\pi]$ debemos encontrar las funciones

$$u_N \in \widetilde{B}_N = span\left\{ \left( cos(nx), \;\; 0 \leq n \leq \frac{N}{2} \right) \cup \left( sin(nx), \;\; 1 \leq n \leq \frac{N}{2} - 1 \right) \right\}$$

Las cuales tienen la forma

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}$$

Si las expresamos en terminos de polinomios

$$u_N(x,t) = \sum_{0}^{N-1} u_N(x_j, t) g_j(x)$$

Donde $g_j(x)$ son los polinomios de Lagrange y ademas cumple que $g_j(x_i) = \delta_{ij}$, entonces el residuo

$$R_N(x,t) = \frac{\partial u_n(x,t)}{\partial t} - \mathcal{L} u_N(x,t)$$

Y debera cumplir

$$R_N(x_j, t) = 0, \quad \forall j \in [0, ..., N-1]$$

De modo que la ecuacion satisface

$$\frac{\partial u_N(x,t)}{\partial t} - \mathcal{I}_N \mathcal{L} u_N(x,t) = 0$$

## 3.3 Chebyshev Collocation

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{cos(\pi j}{N}, \quad j \in [0, ..., N-1]$$

$$\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \left| \frac{\partial^2 u^N}{\partial x^2} \right|_{x=x_j} = 0 \tag{3.3}$$

$$u^N(-1,t) = u_L(t), \;\; u^N(1,t) = u_R(t),$$
$$u^N(x_j, 0) = u_0(x_j), \qquad j = 0, \ldots N.$$

Let $u(t) = (u^N(x_0,t), \ldots, u^N(x_N,t))^T$. Then can be written as, for all $t > 0$,

$$\left(\frac{du}{dt} + u \boxtimes D_N u - \nu D_N^2 u\right) = 0 \tag{3.4}$$

## 3.4 Chebyshev Tau

$$u(-1,t) = u_L(t), \;\; u(1,t) = u_R(t), \tag{3.5}$$

$$u_N(x,t) = \sum_{k=0}^{N} \hat{u}_k(t) T_k(x) \tag{3.6}$$

$$\int_{-1}^{1} \left(\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}\right)(x) T_k(x)(1-x^2)^{-1/2} dx = 0, \tag{3.7}$$

$$k = 0, \ldots, N-2 \tag{3.8}$$

$$u^N(-1,t) = u_L(t), \;\; u^N(1,t) = u_R(t), \tag{3.9}$$

$$\frac{\partial \hat{u}_k}{\partial t} + \left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\hat{}} - \nu \hat{u}_k^{(2)} = 0, \;\; k = 0, 1, \ldots, N-2, \tag{3.10}$$

$$\left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\hat{}} = \frac{2}{\pi c_k} \int_{-1}^{1} \left(u^N \frac{\partial u^N}{\partial x}\right)(x) T_k(x)(1-x^2)^{-1/2} dx \tag{3.11}$$

$$\sum_{k=0}^{N} \hat{u}_k = u_R, \sum_{k=0}^{N} (-1)^k \hat{u}_k = u_L \tag{3.12}$$

$$\hat{u}_k(0) = \frac{2}{\pi c_k} \int_{-1}^{1} u_0(x) T_k(x)(1-x^2)^{-1/2} dx, \;\; k = 0, \ldots, N \tag{3.13}$$

$$(uv)_k^{\hat{}} = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) v(x) T_k(x)(1-x^2)^{-1/2} dx \tag{3.14}$$

$$(uv)_k^{\hat{}} = \frac{1}{2} \sum_{p+q=k} \hat{u}_p \hat{v}_q + \sum_{|p-q|=k} \hat{u}_p \hat{v}_q \tag{3.15}$$

# Chapter 4

# Numerical Solution of the Stochastic Burgers equation

## 4.1 Numerical Approximation

Sea $\mathcal{H} = \mathcal{L}^2(0,1)$.

$$\begin{aligned} dX(t,\xi) &= \left[\nu \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + \frac{1}{2}\frac{\partial (X^2(t,\xi))}{\partial \xi}\right] dt + dW_t(t,\xi), \;\; \xi \in [0,1] \\ X(t,0) &= X(t,1) = 0, \;\; t > 0 \\ X(0,\xi) &= x(\xi), \;\; x \in \mathcal{H} \end{aligned}$$

Si definimos $A = \nu \partial_\xi^2$, $B = \frac{1}{2}\partial_\xi(X^2)$

$$\begin{aligned} dX &= [AX + B(X)]dt + dW_t \\ X(0) &= x, \;\; x \in \mathcal{H} \end{aligned}$$

Si $u(t,x) = \mathbb{E}[u_0(X_t^x)]$, y $u \in \mathbb{H} = L^2(\mathcal{H}, \mu)$ satisface

$$\frac{\partial u}{\partial t} = \frac{1}{2} Tr(QD^2 u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \; x \in D(A)$$

Definamos el siguiente conjunto de indices

$$\mathcal{J} = \{\alpha = (\alpha_i, i \geq 1) | \alpha_i \in \mathbb{N} \cup 0, |\alpha| := \sum_{i=0}^{\infty} \alpha_i < \infty\}$$

Funcionales de Hermite

$$H_n(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), h \in \mathcal{H}_0, n \in \mathcal{J}$$

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

Si para M, N fijos tomamos un subconjuto de $\mathcal{J}$

$$\mathcal{J}^{M,N} = \{\alpha = (\alpha_i, 1 \leq i \leq M) | \alpha_i \in \{0,1,...,N\}\}$$

Descomposición dada por la expansión de Wiener-Chaos (Serie de Fourier-Hermite).

$$u_N(t,x) = \sum_{n \in \mathcal{J}^{M,N}} u_n(t) H_n(x),\ x \in \mathcal{H},\ t \in [0,T]$$

Sustituyendo en la ecuación y haciendo unos ajustes

$$\dot{u}_{m_i}(t) = -u_{m_i}(t)\lambda_{m_i} + \sum_{j=1}^{M} u_{n_j}(t) C_{n_j,m_i},\ 1 \leq i \leq M$$

Donde

$$C_{n,m} = \int_{\mathcal{H}} \langle B(x), D_x H_n(x) \rangle_{\mathcal{H}} H_m(x) \mu(dx)$$

Si escribimos la solución como sigue

$$U^M(t) = \begin{pmatrix} u_{m_1}(t) & u_{m_2}(t) & \dots & u_{m_M}(t) \end{pmatrix}^T$$

$$\dot{U}^M(t) = \begin{pmatrix} \dot{u}_{m_1}(t) & \dot{u}_{m_2}(t) & \dots & \dot{u}_{m_M}(t) \end{pmatrix}^T$$

$$\dot{U}^M(t) = AU^M(t)$$

Donde la matriz $A$ es:

$$\begin{pmatrix} -\lambda_1 + C_{1,1} & C_{2,1} & \dots & C_{M-1,1} & C_{M,1} \\ C_{1,2} & -\lambda_2 + C_{2,2} & \dots & C_{M-1,2} & C_{M,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,M-1} & C_{2,M-1} & \dots & -\lambda_{M-1} + C_{M-1,M-1} & C_{M,M-1} \\ C_{1,M} & C_{2,M} & \dots & C_{M-1,M} & -\lambda_M + C_{M,M} \end{pmatrix}$$

Donde $\lambda_i = \lambda_{mi}$ y $C_{i,j} = C_{n_i,m_j}$ para $1 \leq i,j \leq M$.

Entonces la solucion del sistema es

$$U^M(t) = \sum_{j=1}^{M} c_i V_i e^{\eta_i t}$$

Donde las constantes $c_i$ se obtienen evaluando en $t = 0$

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{M-1}(t) \\ u_M(t) \end{pmatrix} = \begin{pmatrix} V1 & V2 & \dots & V_{M-1} & V_M \end{pmatrix} \begin{pmatrix} c_1 e^{\eta_1 t} \\ c_2 e^{\eta_2 t} \\ \vdots \\ c_{M-1} e^{\eta_{M-1} t} \\ c_M e^{\eta_M t} \end{pmatrix}$$

**Condiciones Iniciales:** Definamos los puntos $z_i$, $i = 0, 1, \dots, p$, tales que $z_0 = a$ y $z_p = b$.

$$u(0,x) = \mathbb{E}[u_0^{z_i}(X_0^x)] = X^x(0,z_i) = x(z_i)$$

Por otra parte tenemos

$$u(0,x) = \sum_{n \in \mathcal{J}^{M,N}} u_n(0) H_n(x)$$

Multiplicando por $H_m(x)$ e integrando en el espacio $\mathcal{L}^2(\mathcal{H}, \mu)$

$$u_m(0) = \int_{\mathcal{H}} x(z_i) H_m(x) \mu(dx)$$

Entonces la solución del sistema

$$\dot{U}^M(t) = AU^M(t)$$

Se resuelve con las condiciones iniciales

$$\begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_{M-1}(0) \\ u_M(0) \end{pmatrix} = \begin{pmatrix} V1 & V2 & \dots & V_{M-1} & V_M \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{M-1} \\ c_M \end{pmatrix}$$

**Continuidad con respecto las condiciones iniciales:**
Sean $x, y \in \mathcal{H}$ condiciones iniciales distinas y $\mathbb{H} = L^2(\mathcal{H}, \mu)$, con $\mu = \frac{1}{\sqrt{2\pi}} e^{\frac{x^2}{2}} dx$. Dadas las aproximaciones de la solución,

$$\psi_t^x = \sum_{n \in \mathcal{J}} u_n^x(t) H_n(x), \quad \psi_t^y = \sum_{n \in \mathcal{J}} u_n^y(t) H_n(y)$$

Estimamos $\psi_t^x - \psi_t^y$

$$\|\psi_t^x - \psi_t^y\|_{(L^2(\mathcal{H},\mu))^2}$$

# Chapter 5

# Numerical Results

## 5.1 Simulations

# Bibliography

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