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CONVERGENCE OF SPECTRAL METHODS FOR BURGERS' EQUATION*

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Abstract. In this paper, a general framework is presented for analyzing numerical methods for the evolutionary equations that admit semigroup formulations. This framework is then applied to spectral and pseudospectral methods for the Burgers' equation, using trigonometric, Chebyshev, and Legendre polynomials. Optimal order of convergence is obtained, which implies the spectral accuracy of these methods.

Key words. spectral method, Burgers' equation, semigroup

AMS(MOS) subject classifications. 65M10, 65M99

1. Introduction. In this paper we first present a general framework for analyzing numerical methods for the evolutionary equations that admit semigroup formulations. This framework is then applied to spectral and pseudospectral methods for Burgers' equation, using trigonometric, Chebyshev, and Legendre polynomials. Optimal order of convergence under suitable norms is proved for these methods in the sense that the error in the numerical solution is of the same order as the error in the initial approximation. As a consequence, we know that these methods achieve spectral accuracy if the exact solution to the differential equation is smooth enough.

There are many papers concerning the analysis of spectral methods. For linear evolutionary problems, we mention the important work of Kreiss and Oliger [9], Gottlieb and Orszag [7], and Majda, McDonough, and Osher [15]. For nonlinear problems in incompressible flow, we mention the work of Canuto, Maday, Quarteroni, and others which lays the foundation of this subject. Fundamental ingredients of their work include a basic approximation theory for the spectral expansions and interpolations in the setting of Sobolev spaces [1], [3], [17] and the variational techniques for problems of elliptic type, including the steady-state Navier–Stokes equation [1], [12]–[14]. The abstract framework of Brezzi, Rappaz, and Raviart [2] plays an essential role in the analysis of the nonlinear problems [2]. On the other hand, relatively little has been done for the full time-dependent Navier–Stokes equation. A review of the available results is contained in [5] and will not be repeated here. The purpose of this paper is to show that the ideas in [5, § 4] can be developed into a general framework which can be useful for other problems. We will show this by applying this framework to various types of spectral and pseudospectral methods for Burgers' equation.

Our fundamental tools are the semigroup formulation and the variation of constants formula. In § 2 we set up the problem and state our main theorem (Theorem 2.1). In §§ 3–5 we consider the application of this framework to the numerical solutions of Burgers' equation by Fourier, Legendre, and Chebyshev methods, respectively. These methods admit semigroup formulations if we suitably define the underlying spaces and the infinitesimal generators. Our approach provides an alternative to the conventional one of using energy estimates.

2. A general framework for the approximation of evolutionary equations.

2.1. Linear second-order parabolic equations. We consider a general second-order parabolic problem represented by the following abstract evolutionary equation (higher-

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order equations can be treated similarly):

$$\text{(1)} \qquad \frac{du}{dt} + Au = f(t), \qquad u(0) = a$$

in the space $X = L^2_\omega(\Omega)$. We have in mind the application to spectral methods, so we take general weighted Sobolev spaces with weight $\omega(x) > 0$. As usual, we define

$$(\varphi, \psi)_\omega = \int_\Omega \varphi(x)\psi(x)\omega(x)\,dx \quad \text{for } \varphi, \psi \in L^2_\omega(\Omega),$$

$$\|\varphi\|^2_{0,\omega} = (\varphi, \varphi)_\omega,$$

and for $k \in \mathbf{Z}$,

$$H^k_\omega(\Omega) = \{u \in L^2_\omega(\Omega),\ D^\alpha u \in L^2_\omega(\Omega) \text{ for } |\alpha| \leqq k\}$$

with the norm

$$\|u\|_{k,\omega} = \left(\sum_{|\alpha| \leqq k} \|D^\alpha u\|^2_{0,\omega} \right)^{\frac{1}{2}},$$

$$H^k_{0,\omega}(\Omega) = \text{the closure of } C^\infty_0(\Omega) \text{ in } H^k_\omega(\Omega).$$

For a noninteger $s \in \mathbf{R}^+$, H^s_ω and $H^s_{0,\omega}$ are defined by interpolation [19]. $H^{-s}_\omega(\Omega) =$ dual of $H^s_{0,\omega}(\Omega)$ in the L^2_ω-inner product. For $v(t) \in C\ ([0, T], H^\sigma_\omega(\Omega))$, we denote

$$|||v|||_{\sigma,\omega} = \sup_{0 \leqq t \leqq T} \|v(t)\|_{\sigma,\omega}.$$

The operator A in (1) is defined via a bilinear form $a(\cdot,\cdot): H^1_{0,\omega}(\Omega) \times H^1_{0,\omega}(\Omega) \to \mathcal{R}$ satisfying

$$\text{(A1)} \qquad a(u, u) \geqq C_0 \|u\|^2_{1,\omega} \quad \text{for } u \in H^1_{0,\omega}(\Omega)$$

$$\text{(A2)} \qquad |a(u, v)| \leqq C_1 \|u\|_{1,\omega} \|v\|_{1,\omega} \quad \text{for } u, v \in H^1_{0,\omega}(\Omega).$$

For $u \in \mathcal{D}(A) = H^2_\omega \cap H^1_{0,\omega}(\Omega)$, we define Au by $Au \in X$, and $(Au, v)_\omega = a(u, v)$ for any $v \in H^1_{0,\omega}(\Omega)$.

These conditions are satisfied in general by a parabolic equation. From these conditions we immediately get the following lemma.

LEMMA 2.1. *Suppose* (A1) *and* (A2) *are satisfied. Then* $-A$ *is the infinitesimal generator of an analytic semigroup in* X, *denoted by* $\{e^{-tA}\}_{t \geqq 0}$.

For a proof of this lemma, see Kato [8].

Now the usual estimates for analytic semigroups hold for $\{e^{-tA}\}_{t \geqq 0}$, namely, for $\alpha \geqq 0$, $t > 0$,

$$\text{(2)} \qquad \|A^\alpha e^{-tA}\| \leqq C_\alpha t^{-\alpha}.$$

Here and in the following, the operator norms are taken in the space of linear operators from X to X. We also have the following estimate on the resolvent of A, $R(\lambda, A) = (\lambda I - A)^{-1}$. Namely, there is a constant $\delta \in (\pi/2, \pi)$, such that $\sigma(A) \supset \Sigma_\delta = \{\xi \in \mathbf{C}, |\arg \xi| < \delta\}$, where $\sigma(A)$ is the resolvent set of A, and for $0 \leqq \alpha \leqq 1$, $\lambda \in \Sigma_\delta$,

$$\text{(3)} \qquad \|A^\alpha R(\lambda, A)\| \leqq \frac{C_\alpha}{|\lambda|^{1-\alpha}}.$$

Let $\{V_N\}$ be a sequence of finite-dimensional subspaces of $H^1_{0,\omega}(\Omega)$. Let $a_N(\cdot,\cdot)$, $V_N \times V_N \to \mathbf{R}$ be a discretization of the bilinear form $a(\cdot,\cdot)$ satisfying

(A1′) $$a_N(\varphi, \varphi) \geqq C_0' \|\varphi\|^2_{1,\omega} \quad \text{for } \varphi \in V_N,$$

(A2′) $$|a_N(\varphi, \psi)| \leqq C_1' \|\varphi\|_{1,\omega} \|\psi\|_{1,\omega} \quad \text{for } \varphi, \psi \in V_N.$$

We consider the following approximation scheme for (1):

$$u_N(t) : [0, T] \to V_N,$$

(4) $$\frac{du_N}{dt} + A_N u_N = f_N(t),$$

$$u_N(0) = a_N,$$

where A_N is defined by $A_N : V_N \to V_N$

$$(A_N v, \varphi)_\omega = a_N(v, \varphi) \quad \text{for any } v, \varphi \in V_N,$$

and $f_N(t) \in C([0, T], V_N)$ is an approximation of $f(t)$.

Conditions (A1′) and (A2′) guarantee that the numerical scheme (4) has some smoothing properties as shown in the following lemma.

LEMMA 2.2. *There exists a constant $\delta \in (\pi/2, \pi)$, such that $\sigma(A_N) \supset \Sigma_\delta = \{\xi \in \mathbf{C}, |\arg \xi| < \delta\}$. Moreover, we have the following estimates for $0 \leqq \alpha \leqq 1$, $\lambda \in \Sigma_\delta$,*

(5) $$\|A_N^\alpha e^{-A_N t} \varphi\|_{0,\omega} \leqq \frac{C_\alpha}{t^\alpha} \|\varphi\|_{0,\omega}, \qquad \varphi \in V_N,$$

(5a) $$\|A_N^\alpha R(\lambda, A_N)\| \leqq \frac{C_\alpha}{|\lambda|^{1-\alpha}},$$

where the C_α's are constants independent of N and φ.

This lemma is proved in [11].

In the following we will use C to denote generic constants which may have different values in different locations. The subscript ω will be omitted if $\omega(x) \equiv 1$. As a standard convention, we will view a function $\varphi(t, x)$ as a function of t with values in another function space (in x). This function will be denoted by $\varphi(t)$.

2.2. Nonlinear parabolic equations. In this subsection, we will consider the nonlinear evolutionary equations in X

(6) $$\frac{du}{dt} + Au + F(t, u) = 0, \qquad u(0) = a,$$

where A is the operator considered in the previous subsection. We will assume that this system has a unique solution that satisfies some regularity requirements specified below.

We consider an approximation scheme in the following form:

$$u_N(t) : [0, T] \to V_N,$$

(7) $$\frac{du_N}{dt} + A_N u_N + F_N(t, u_N) = 0,$$

$$u_N(0) = P_N a \in V_N,$$

where $F_N(t, v):[0, T]\times V_N \to V_N$ is the approximation of the nonlinear term and P_N is an operator (a projection operator in some sense) chosen in each application,

$$P_N : X = L^2_\omega(\Omega) \to V_N.$$

Using the variation of constants formula, we can write (6) and (7) in integral forms

$$u(t) = e^{-At}a - \int_0^t e^{-A(t-s)}F(s, u(s))\,ds, \tag{8}$$

$$u_N(t) = e^{-A_Nt}P_Na - \int_0^t e^{-A_N(t-s)}F_N(s, u_N(s))\,ds. \tag{8a}$$

Here (8) makes sense under some smoothness conditions for u and F.

The next lemma is the version of Gronwall's inequality we are going to use. Many inequalities of this type can be found in [10]. The version to be used here is proved in [16].

LEMMA 2.3. *Let T, α, β, ν be positive constants. $0<\nu<1$. Then for any continuous functions $f:[0, T]\to[0,\infty)$ satisfying*

$$f(t) \leqq \alpha + \beta \int_0^t (t-s)^{-\nu} f(s)\,ds, \qquad 0 \leqq t \leqq T,$$

we have

$$f(t) \leqq C\alpha \exp\{C\beta^{1/(1-\nu)}t\} \quad \textit{for } 0 \leqq t \leqq T,$$

with a positive constant C that depends only on ν.

We can now state the main results in this section.

THEOREM 2.1. *Let $0 \leqq \alpha, \beta \leqq 1$, $0 \leqq t \leqq T$. Assume that the semidiscrete approximation (7) satisfies*

$$\begin{aligned}\varepsilon_N = \sup_{0\leqq t,\tau\leqq T} \Big\{ &\|A_N^\alpha(P_N e^{-A(t-\tau)}u(\tau) - e^{-A_N(t-\tau)}P_Nu(\tau))\|_{0,\omega} \\ &+ \left\| \int_\tau^t A_N^\alpha\{P_N e^{-A(t-s)}F(s, u(s)) - e^{-A_N(t-\tau)}F_N(s, P_Nu(s))\}\,ds \right\|_{0,\omega} \Big\} \\ &\to 0 \quad \textit{as } N\to+\infty;\end{aligned} \tag{X}$$

$$\|A_N^{\alpha-\beta}\{F_N(s,v)-F_N(s,\varphi)\}\|_{0,\omega} \leqq L(\|A_N^\alpha\|_{0,\omega}, \|A_N^\alpha\varphi\|_{0,\omega})\|A_N^\alpha(v-\varphi)\|_{0,\omega} \quad \textit{for } 0\leqq s\leqq T,\ v, \varphi\in V_N, \tag{Y}$$

where $L(\cdot,\cdot)$ is an increasing function with respect to each argument and $u(t)$ is the true solution to (1);

$$|||A_N^\alpha P_Nu(t)|||_{0,\omega} \leqq K. \tag{Z}$$

Then there exist constants $N_0(T)$, $K_0(T)$, such that for $N>N_0(T)$, (7) has a unique solution $u_N(T):[0, T]\to V_N$. Moreover, we have for $0\leqq t\leqq T$

$$\|A_N^\alpha u_N(t)\|_{0,\omega} \leqq K+1, \tag{9}$$

$$\|A_N^\alpha(u_N(t)-P_Nu(t))\|_{0,\omega} \leqq K_0(T)\varepsilon_N. \tag{9a}$$

Remark 2.1. Although it looks awkward, assumption (X) merely says that our numerical method should in some sense converge for the linear problems (discussed in the previous section) and the approximation for the nonlinear term should be consistent. Assumption (Y) plays the role of a stability condition. Notice that it is a very weak form of the stability assumption that corresponds to the local Lipschitz condition on F needed for the existence of a local solution of (6) [10]. In the application of this result to be presented in §§ 3–5, we will indicate a procedure for systematically checking these assumptions.

Remark 2.2. Theorem 2.1 actually gives a way of extending approximation results for the linear problems to the nonlinear ones. It is in the spirit of Bressi, Rappaz, and Raviart's [2] framework for steady equations.

We omit the proof of Theorem 2.1 here since it is quite lengthy. The proof makes use of classical ideas of fixed point methods, similar to the proof of existence of local solutions to nonlinear evolutionary equations. The interested reader can check [6] for details.

3. Fourier collocation method for Burgers' equation. The equation we are going to study in this section is a standard model in fluid mechanics, namely, the Burgers' equation:

$$\begin{aligned} &u_t - u_{xx} + \tfrac{1}{2}(u^2)_x = f, \quad t>0, \quad x \in I = (-\pi, \pi),\\ &u(x,0) = a(x), \\ &\text{periodic boundary condition,} \end{aligned} \tag{10}$$

where $f(t, x)$ is periodic in x with period I.

For $m \geqq 0$, define

$$H_p^m = \left\{ u = \sum_{k \in \mathbf{Z}} C_k e^{ikx}, \bar{C}_k = C_{-k}, \quad \|u\|_m^2 = \sum_{k \in \mathbf{Z}} (1+|k|)^{2m} |C_k|^2 < +\infty \right\}. \tag{11}$$

For $m < 0$, define

$$H_p^m = \text{the dual of } H_p^{-m}.$$

For $u \in H_p^2 = D(A)$ define $Au = -u_{xx} + u \in H_p^0$. It is easy to see that A can be extended as a positive definite linear selfadjoint operator on H_p^0. Therefore, we can define the powers of A, A^α, for $\alpha \in \mathbf{R}$. Furthermore, we have $D(A^\alpha) = H_p^{2\alpha}$ and there exist constants C_1 and C_2, such that

$$C_1 \|u\|_{2\alpha} \leqq \|A^\alpha u\|_0 \leqq C_2 \|u\|_{2\alpha} \quad \text{for any } u \in D(A^\alpha). \tag{12}$$

Now (10) can be put in the dynamical form (6) by taking $X = H_p^0$, $\omega(x) = 1$, and $F(t, u) = \frac{1}{2}(u^2)_x - u - f(t)$.

S_N is the space of trigonometric polynomials of degree N

$$S_N = \left\{ u = \sum_{|k| \leqq N} C_k e^{ikx} \right\}.$$

For a continuous periodic function $\varphi(x)$ on I, we define its Fourier interpolant $P_c\varphi(x)$ by

$$P_c\varphi(x) \in S_N, \qquad P_c\varphi(x_j) = \varphi(x_j),$$

where $x_j = 2\pi j/(2N+1)$ for $|j| \leqq N$. More precisely,

$$P_c\varphi(x) = \sum_{|k| \leqq N} C_k e^{ikx} \tag{13}$$

where $C_k = h \sum_{|j| \leqq N} \varphi(x_j) e^{-ix_j k}$, $h = 2\pi/(2N+1)$.

The approximation property of P_c is given in the following lemma.

LEMMA 3.1 [3]. *There exists a constant C, such that for $u \in H_p^m$, $m > \frac{1}{2}$, $0 \leqq \mu \leqq m$, we have*

$$\|u - P_c u\|_\mu \leqq CN^{\mu - m}\|u\|_m. \tag{14}$$

We are going to consider the Fourier collocation approximation of (10) fomulated as the following problem:

$$\begin{aligned} &\text{Find } u_N(x, t) \in C([0, T], S_N), \text{ s.t. for } |j| \leqq N, \\ &\frac{\partial u_N}{\partial t}(x_j) - \frac{\partial^2 u_N}{\partial x^2}(x_j) + u_N(x_j)\frac{\partial u_N}{\partial x}(x_j) = f(x_j), \\ &u_N(x_j, 0) = a(x_j). \end{aligned} \tag{15}$$

Equation (15) can be equivalently written as

$$\frac{\partial u_N}{\partial t} - \frac{\partial^2 u_N}{\partial x^2} + \frac{1}{2} P_c(u_N^2)_x = P_c f(t).$$

This can be put into the dynamical form (7) by taking $A_N = A$, $F_N(t, v) = \frac{1}{2}P_c(v_x^2) - v - P_c f(t)$ (denoted also by $F_c(t, v)$) for $v \in S_N$.

Before proving the main results of this section, we state a lemma which will be used in the theorem below.

LEMMA 3.2. $\|P_c \varphi\|_0 \leqq 3\|\varphi\|_0$ *for* $\varphi \in S_{2N}$, *where* P_c *is the Fourier interpolation operator in* S_N.

This lemma is a simple consequence of the aliasing expansion [18].

Now we can prove our main estimates for the Fourier collocation method.

THEOREM 3.1. *Assume for* $\sigma \geqq 2$, $u(t) \in C([0, T], H_p^\sigma)$, $f(t) \in C([0, T], H_p^{\sigma-1})$. *Then there exist* $K_0(T)$, $N_0(T)$, *such that for* $N > N_0(T)$, (15) *has a unique solution* $u_N(t) \in C[(0, T], S_N)$.

Furthermore, we have

$$|||u_N(t)|||_1 \leqq |||u(t)|||_1 + 1, \tag{16}$$

$$|||u_N(t) - u(t)|||_1 \leqq \frac{K_0(T)}{N^{\sigma-1}}(|||u(t)|||_\sigma^2 + \|a\|_\sigma + |||f(t)|||_{\sigma-1}). \tag{16a}$$

Proof. First, we note that with very little change, the framework developed in the previous section carries over to the case of periodic boundary conditions. It is easy to see that the basic assumptions (A1) and (A2) are satisfied by the operator A. Next we observe that the restriction of A to S_N gives A_N. Therefore, for a proper version of the conditions (X), (Y), and (Z), we can replace A_N by A, and $P_N u(t)$ by $u(t)$. This allows us to obtain more accurate estimates. Now let us estimate ε_N with $\alpha = \frac{1}{2}$.

From (14), we have

$$\|A^{\frac{1}{2}} e^{-A(t-\tau)}(u(\tau) - P_c u(\tau))\|_0 \leqq C\|u(\tau) - P_c u(\tau)\|_1 \leqq \frac{C}{N^{\sigma-1}}\|u(\tau)\|_\sigma.$$

Applying (14) to $(u^2)_x$ and using the fact that $\|u^2\|_\sigma \leqq C\|u\|_\sigma^2$, we get

$$\begin{aligned} &\left\| \int_\tau^t A^{\frac{1}{2}} e^{-A(t-s)}\{F(s, u(s)) - F_c(s, u(s))\}\, ds \right\|_0 \\ &\leqq C \int_\tau^t (t-s)^{-\frac{1}{2}}\{\|(u^2)_x - P_c(u^2)_x\|_0 + \|f - P_c f\|_0\}\, ds \\ &\leqq \frac{C}{N^{\sigma-1}}(|||u(s)|||_\sigma^2 + |||f(s)|||_{\sigma-1}). \end{aligned}$$

Hence, we have

$$\varepsilon_N \leqq \frac{C}{N^{\sigma-1}} (|||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}). \tag{17}$$

Next we check condition (Y) with $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$. For $v, \varphi \in S_N$, from Lemma 3.2, we have

$$\begin{aligned}\|F_c(s, \varphi) - F_c(s, v)\|_0 &\leqq \tfrac{1}{2}\|P_c(\varphi_x^2 - v_x^2)\|_0 + \|v-\varphi\|_0 \\ &\leqq C(\|\varphi_x^2 - v_x^2\|_0 + \|v - \varphi\|_0) \\ &\leqq C(\|\varphi + v\|_1 \|\varphi - v\|_1 + \|\varphi - v\|_0) \\ &\leqq C(1 + \|A^{\frac{1}{2}}\varphi\|_0 + \|A^{\frac{1}{2}} v\|_0)\|A^{\frac{1}{2}}(v-\varphi)\|_0.\end{aligned}$$

Hence (Y) is satisfied. It is easy to see that (Z) is satisfied with $\alpha = \frac{1}{2}$. Now (16) and (16a) follows from (17), (9), and (9a). □

Remark 3.1. Another version of the Fourier collocation method appeared in the literature [13] uses the following formulation

$$\begin{aligned}&\text{Find } u_N(t):[0, T] \to S_N, \\ &\frac{\partial u_N}{\partial t} - \frac{\partial^2 u_N}{\partial x^2} + \frac{1}{2}(P_c u_N^2)_x = P_c f(t), \\ &u_N(0) = P_c a.\end{aligned} \tag{18}$$

With this formulation, the above argument carries through. An optimal error estimate follows directly from Theorem 2.1. Furthermore, for this formulation, it is straightforward to prove an additional optimal L^2-estimate under the assumptions in Theorem 3.1:

$$\|u(t) - u_N(t)\|_0 \leqq \frac{K_0(T)}{N^\sigma} (|||u(t)|||_\sigma^2 + \|a\|_\sigma + |||f(t)|||_{\sigma-1}) \tag{19}$$

for $N > N_0(T)$, where $N_0(T)$ and $K_0(T)$ are constants independent of N.

4. Legendre methods for Burgers' equation. Spectral and pseudospectral methods using Legendre polynomials are potentially another type of methods having spectral accuracy. They are good candidates for problems with general boundary conditions. The disadvantage of these methods, compared with Fourier methods or Chebyshev methods, is that we have no fast transform devices for them. The effect of this, however, could be minimized with the advance of parallel computers.

4.1. Legendre–Galerkin method for Burgers' equation. The equation we will deal with is the following:

$$\begin{aligned}&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}(u^2)_x = f(t, x), \quad t > 0, \quad x \in [-1, 1] = I, \\ &u(-1, t) = u(1, t) = 0, \\ &u(x, 0) = a(x).\end{aligned} \tag{20}$$

Since the weight function $\omega(x) = 1$, we will use the conventional Sobolev spaces. For $u, v \in H_0^1(I)$, define

$$a(u, v) = \int_I u_x v_x \, dx, \tag{21}$$

and for $u \in D(A) = H^2 \cap H_0^1(I)$, define

$$Au = -u_{xx}. \tag{22}$$

It is clear that assumptions (A1) and (A2) are satisfied. A can be extended as a selfadjoint, positive definite operator on $L^2(I)$. Therefore, its fractional powers are well defined. Furthermore, we have for $u \in D(A^\alpha)$, $\alpha \in \mathbf{R}$,

$$\frac{1}{C}\|u\|_{2\alpha} \leqq \|A^\alpha u\|_0 \leqq C\|u\|_{2\alpha}, \tag{23}$$

where C is a constant depending only on α. As a consequence of (2) and (23), we have, for $\sigma \geqq 0$, $v \in D(A^{\sigma/2})$,

$$\|e^{-At}v\|_\sigma \leqq C\|v\|_\sigma. \tag{24}$$

Now (20) can be written in the dynamical form (6) with $F(t, u) = \frac{1}{2}(u^2)_x - f(t)$. Next, we turn to the Legendre–Galerkin approximation of (23). Let

$S_N = \{\varphi, \varphi$ is a polynomial on $[-1, 1]$ of degree $\leqq N\}$,

$V_N = S_N \cap H_0^1(I)$,

$P_N : L^2(I) \to V_N$ is the L^2-projection operator for $v \in L^2(I)$,

$(v - P_N v, \varphi) = 0$ for $\varphi \in V_N$.

The Legendre–Galerkin method can be formulated as the following problem:

Find $u_N(t) : [0, T] \to V_N$, s.t. for every $\varphi \in V_N$,

$$\left(\frac{\partial u_N}{\partial t} - \frac{\partial^2 u_N}{\partial x^2}, \varphi\right) + \frac{1}{2}((u_N^2)_x, \varphi) = (f, \varphi), \tag{25}$$

$u_N(0) = P_N a$.

For $v, \varphi \in V_N$, let $a_N(v, \varphi) = -\int_I (\partial^2 v/\partial x^2)\varphi\, dx = a(v, \varphi)$. Then assumptions (A1′) and (A2′) are trivially satisfied. Furthermore, our numerical scheme can be put into the form of (7), with $F_N(t, u) = P_N(\frac{1}{2}(v_x^2) - f(t))$ for $v \in V_N$, A_N defined via $a_N(\cdot, \cdot)$ as in § 2.

Before going into the framework set up by Theorem 2.1, let us first establish some properties of the operator A and A_N.

LEMMA 4.1. (a) *For $0 \leqq \alpha \leqq \frac{1}{2}$, there is a constant C, such that for any $\varphi \in V_N$,*

$$\|A^\alpha \varphi\|_0 \leqq C\|A_N^\alpha \varphi\|_0. \tag{26}$$

(b) *For $0 \leqq \alpha \leqq 1$, there is a constant C, such that for any $\varphi \in V_N$,*

$$\|A_N^\alpha \varphi\|_0 \leqq C\|A^\alpha \varphi\|_0. \tag{26a}$$

Proof. (a) Let $\varphi \in V_N$, then

$$(A^{\frac{1}{2}}\varphi, A^{\frac{1}{2}}\varphi) = (A\varphi, \varphi) = a(\varphi, \varphi) = (A_N\varphi, \varphi) = (A_N^{\frac{1}{2}}\varphi, A_N^{\frac{1}{2}}\varphi).$$

Hence

$$\|A^{\frac{1}{2}}\varphi\|_0 \leqq \|A_N^{\frac{1}{2}}\varphi\|_0.$$

Obviously (26) holds for $\alpha=0$. For $0<\alpha<\frac{1}{2}$, (26) follows from Heinz-Kato inequality [16].

The proof of (26a) is similar. □

LEMMA 4.2. *The following estimates hold*:

(a) *For* $v\in H^\sigma\cap H_0^1(I)$, $0\leqq\mu\leqq 1$, $\mu\leqq\sigma$,

$$\|v-P_Nv\|_\mu\leqq CN^{\frac{3}{2}\mu-\sigma}\|v\|_\sigma. \tag{27}$$

(b) *For* $g\in H^{\sigma-2}$, $0\leqq\mu\leqq 1\leqq\sigma$,

$$\|(P_NA^{-1}-A_N^{-1}P_N)g\|_\mu\leqq CN^{\frac{3}{2}\mu-\sigma}\|g\|_{\sigma-2}. \tag{27a}$$

(c) *For* $g\in H^{-1}(I)$,

$$\|A_N^{-\frac{1}{2}}P_Ng\|_0\leqq C\|A^{-\frac{1}{2}}g\|_0\leqq C\|g\|_{-1}. \tag{27b}$$

(d) *For* $\varphi\in V_N$, $0\leqq\alpha\leqq\frac{1}{2}$.

$$\|A_N^{-\alpha}\varphi\|_0\leqq C\|A^{-\alpha}\varphi\|_0. \tag{27c}$$

Proof. Inequality (27) is proved in [3]. Inequality (27a) follows from (27) and Theorem 1.6 in [12]. If we let $v_N=A_N^{-1}P_Ng$, $v=A^{-1}g$, then it is easy to see that $a(v_N,v_N)\leqq a(v,v)$. This proves the first half of (27b), and the second half is even more obvious. Inequality (27c) follows from (27b) and the Heinz-Kato inequality. □

LEMMA 4.3. *Let* $\alpha=\frac{1}{2}$. *Assume that for* $\sigma\geqq 2$, $u(t)\in C([0,T],H^\sigma\cap H_0^1)$, $f(t)\in C([0,T],H^{\sigma-1})$. *Then there is a constant* C, *such that*

$$\varepsilon_N\leqq CN^{\frac{3}{2}-\sigma}(|||u(t)|||_\sigma^2+|||u(t)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{28}$$

Proof. We first recall a resolvent identity (see [11])

$$P_NR(\lambda,A)-R(\lambda,A_N)P_N=A_NR(\lambda,A_N)(P_NA^{-1}-A_N^{-1}P_N)AR(\lambda,A).$$

From the Dunford integral formula, we get

$$\begin{aligned}P_Ne^{-tA}a-e^{-tA_N}P_Na&=\int_\Gamma e^{\lambda t}[P_NR(\lambda,A)-R(\lambda,A_N)P_N]a\,d\lambda\\&=\int_\Gamma e^{\lambda t}A_NR(\lambda,A_N)[P_NA^{-1}-A_N^{-1}P_N]AR(\lambda,A)a\,d\lambda,\end{aligned} \tag{29}$$

where the path of integration in the complex plane is shown in Fig. 1.

Therefore, from (27) and (27a) we obtain

$$\begin{aligned}\|A_N^{\frac{1}{2}}(P_Ne^{-At}a-e^{-A_Nt}P_Na)\|_0&\leqq C\int_\Gamma|e^{\lambda t}|\,\|A_NR_N\|\,\|(P_NA^{-1}-A_N^{-1}P_N)ARa\|_1\,d\lambda\\&\leqq CN^{\frac{3}{2}-\sigma}\int_\Gamma|e^{\lambda t}|\,\|R(\lambda,a)a\|_\sigma\,d\lambda\\&\leqq CN^{\frac{3}{2}-\sigma}\int_\Gamma|e^{\lambda t}|\,\|R(\lambda,A)\|\,\|a\|_\sigma\,d\lambda\\&\leqq CN^{\frac{3}{2}-\sigma}\int_\Gamma\frac{|e^{\lambda t}|}{|\lambda|}\,d\lambda\cdot\|a\|_\sigma\leqq CN^{\frac{3}{2}-\sigma}\|a\|_\sigma,\end{aligned} \tag{27d}$$

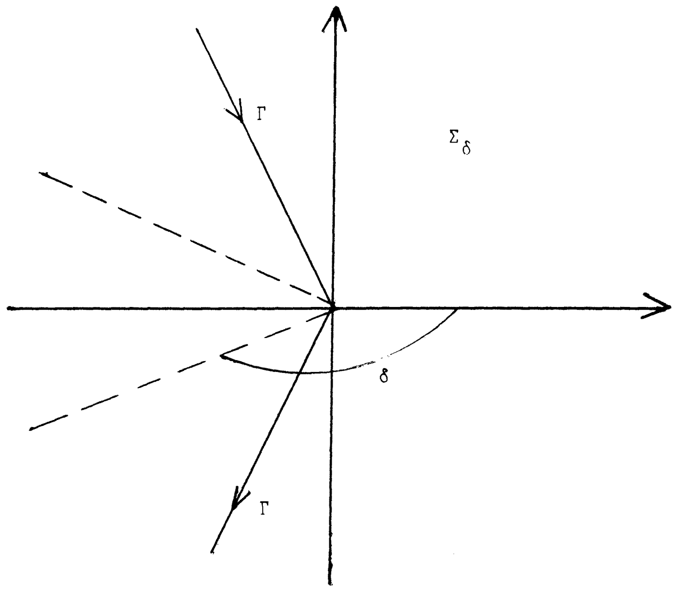

FIG. 1. *The path* Γ *for the contour integral in* (29). Γ *should be in the left half plane but contained in* Σ_δ.

where R_N and R denote $R(\lambda, A_N)$ and $R(\lambda, A)$, respectively, and we have used (3) and (5a). We also get

$$\|A_N^{\frac12}(P_N e^{-At}a-e^{-A_Nt}P_Na)\|_0 \leqq CN^{\frac32-\sigma}\int_\Gamma |e^{\lambda t}|\,\|A^{\frac12}R(\lambda, A)\|\,\|a\|_{\sigma-1}\,d\lambda$$
$$\leqq CN^{\frac32-\sigma}\|a\|_{\sigma-1}\int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|^{\frac12}}\,d\lambda \leqq Ct^{-\frac12}N^{\frac32-\sigma}\|a\|_{\sigma-1}. \tag{27e}$$

From these we get

$$\|A_N^{\frac12}(P_N e^{-A(t-\tau)}u(\tau)-e^{-A_N(t-\tau)}P_Nu(\tau))\|_0 \leqq CN^{\frac32-\alpha}\|u(\tau)\|_\sigma. \tag{30}$$

$$\|A_N^{\frac12}(P_N e^{-A(t-s)}(u^2(s))_x-e^{-A_N(t-s)}P_N(u^2(s))_x)\|_0$$
$$\leqq C(t-s)^{-\frac12}N^{\frac32-\sigma}\|u^2(s)_x\|_{\sigma-1}\leqq C(t-s)^{-\frac12}N^{\frac32-\sigma}\|u(s)\|_\sigma^2. \tag{31}$$

On the other hand, we have

$$\|A_N^{\frac12}\, e^{-A_N(t-s)}P_N\{u^2(s)_x-(P_Nu(s))_x^2\}\|_0$$
$$\leqq C(t-s)^{-\frac12}\|u^2(s)_x-(P_Nu(s))_x^2\|_0$$
$$\leqq C(t-s)^{-\frac12}\|u^2(s)-(P_Nu(s))^2\|_1 \tag{32}$$
$$\leqq C(t-s)^{-\frac12}\|u(s)+P_Nu(s)\|_1\|u(s)-P_Nu(s)\|_1$$
$$\leqq C(t-s)^{-\frac12}N^{\frac32-\sigma}\|u(s)\|_2\|u(s)\|_\sigma.$$

Here we have used the fact that $\|P_Nu(s)\|_1\leqq C\|u(s)\|_2$. Similarly, we get

$$\|A_N^{\frac12}(P_N e^{-A(t-s)}f(s)-e^{-A_N(t-s)}P_Nf(s))\|_0\leqq CN^{\frac32-\sigma}(t-s)^{-\frac12}\|f(s)\|_{\sigma-1}. \tag{33}$$

Combining (30)-(33), we get (28). □

Our next lemma takes care of the condition (Y) in Theorem 3.1.

LEMMA 4.4. *Condition* (Y) *in Theorem* 1.1 *is satisfied with* $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$.

Proof. Let $v, \varphi \in V_N$. From (23) and (26) we get,

$$\begin{aligned}\|A_N^{\alpha-\beta}(F_N(s, v)-F_N(s,\varphi))\|_0 &\leqq C\|P_N(v_x^2-\varphi_x^2)\|_0\\ &\leqq C\|v_x^2-\varphi_x^2\|_0 \leqq C\|v^2-\varphi^2\|_1\\ &\leqq C\|v+\varphi\|_1\|v-\varphi\|_1\\ &\leqq C(\|A_N^{\frac{1}{2}}v\|_0+\|A_N^{\frac{1}{2}}\varphi\|_0)\|A_N^{\frac{1}{2}}(v-\varphi)\|_0.\end{aligned}$$ □

Now we can prove our main result for Legendre-Galerkin method.

THEOREM 4.1. *Assume that for* $\sigma \geqq 2$, $u(t)\in C([0,T], H^\sigma \cap H_0^1)$, $f(t)\in C([0,T], H^{\sigma-1})$. *Then there exist constants* $N_0(T)$ *and* $K_0(T)$, *such that for* $N > N_0(T)$, *the Legendre-Galerkin approximation* $u_N(t)$ *exists for* $0 \leqq t \leqq T$. *Furthermore, we have, for* $0 \leqq t \leqq T$,

$$\|u(t)-u_N(t)\|_1 \leqq K_0(T)N^{\frac{3}{2}-\sigma}(|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{34}$$

Proof. First we note that condition (Z) is also satisfied because

$$\|A_N^{\frac{1}{2}}P_N u(t)\|_0 \leqq C\|P_N u(t)\|_1 \leqq C\|u(t)\|_2.$$

Now we can use Theorem 2.1 to obtain the first half of Theorem 4.1 together with an estimate

$$\|P_N u(t)-u_N(t)\|_1 \leqq CN^{\frac{3}{2}-\sigma}(|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{35}$$

Inequality (34) is a direct consequence of (35) and (27). □

THEOREM 4.2. *Under the same assumptions as in Theorem* 4.1, *we have the following estimate for* $N > N_0(T)$, $0 \leqq t \leqq T$:

$$\|u(t)-u_N(t)\|_0 \leqq CN^{-\sigma}(|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{36}$$

Proof. Let $e_N(t)=u(t)-u_N(t)$. Using variation of constants formula, we get

$$\begin{aligned}e_N(t) &= (e^{-At}a-e^{-A_Nt}P_Na)-\frac{1}{2}\int_0^t \{e^{-A(t-s)}u_x^2-e^{-A_N(t-s)}P_N(u_x^2)\}\,ds\\ &\quad -\frac{1}{2}\int_0^t e^{-A_N(t-s)}P_N\{u_x^2-(u_N^2)_x\}\,ds\\ &\quad +\int_0^t (e^{-A(t-s)}f(s)-e^{-A_N(t-s)}P_Nf(s))\,ds\\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

From (27) and (29), we obtain

$$\begin{aligned}\|P_N e^{-At}a-e^{-A_Nt}P_Na\|_0 &\leqq CN^{-\sigma}\int_\Gamma |e^{\lambda t}|\,\|R(\lambda, A)a\|_\sigma\,d\lambda\\ &\leqq CN^{-\sigma}\int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|}\|a\|_\sigma\,d\lambda \leqq CN^{-\sigma}\|a\|_\sigma,\end{aligned}$$

$$\|P_N e^{-At}a-e^{-A_Nt}P_Na\|_0 \leqq Ct^{-\frac{1}{2}}N^{-\sigma}\|a\|_{\sigma-1}.$$

Therefore,

$$\|I_1\|_0 = \|e^{-At}a - e^{-A_Nt}P_Na\|_0 \leqq CN^{-\sigma}\|a\|_\sigma,$$

$$\|e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_N(u_x^2)\|_0 \leqq C(t-s)^{-\frac{1}{2}}N^{-\sigma}\|u_x^2\|_{\sigma-1},$$

$$\leqq C(t-s)^{-\frac{1}{2}}N^{-\sigma}\|u\|_\sigma^2,$$

$$\|I_2\|_0 \leqq CN^{-\sigma}\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_\sigma^2\, ds \leqq CN^{-\sigma}|||u(s)|||_\sigma^2.$$

Similarly, $\|I_4\|_0 \leqq CN^{-\sigma}|||f(s)|||_{\sigma-1}$.

Notice that from (27b) we have

$$\|A_N^{-\frac{1}{2}}P_N(u^2-u_N^2)_x\|_0 \leqq C\|(u^2-u_N^2)_x\|_{-1} \leqq C\|u^2-u_N^2\|_0.$$

Hence,

$$\|I_3\|_0 \leqq C\int_0^t (t-s)^{-\frac{1}{2}}\|u^2-u_N^2\|_0\, ds$$

$$\leqq C\int_0^t (t-s)^{-\frac{1}{2}}\|u+u_N\|_{L^\infty}\|u-u_N\|_0\, ds$$

$$\leqq L\int_0^t (t-s)^{-\frac{1}{2}}\|e_N(s)\|_0\, ds,$$

where L depends on $|||u(s)|||_2$. Combining these results, we get

$$\|e_N(t)\|_0 \leqq CN^{-\sigma}(|||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}) + L\int_0^t (t-s)^{-\frac{1}{2}}\|e_N(s)\|_0\, ds.$$

Using Lemma 2.3, we obtain (36). □

4.2. Legendre collocation method. Now we consider the pseudospectral method using Legendre polynomials, which is computationally more efficient than the previous method. In the pseudospectral version, collocation is used instead of projection. For continuous functions $\varphi(x)$, $\psi(x)$, we define the interpolation operator $P_c: C^0(I)\to S_N$ by

$$P_c\varphi(x_j) = \varphi(x_j), \qquad 0 \leqq j \leqq N,$$

and a discrete inner product $(\cdot,\cdot)_N$,

$$(\varphi, \psi)_N = \sum_{j=0}^N \omega_j\varphi(x_j)\psi(x_j),$$

where $\{(x_j, \omega_j)\}_{j=0}^N$ is the set of nodes and weights of the Gauss–Lobatto integration formula relative to the weight function $\omega(x)=1$. It is well known that for $\varphi\in S_{2N-1}$,

$$\int_I \varphi(x)\, dx = \sum_{j=0}^N \varphi(x_j)\omega_j. \tag{37}$$

From (46), we have for $\varphi\cdot\psi\in S_{2N-1}$,

$$(\varphi, \psi)_N = (\varphi, \psi).$$

Furthermore, P_c is uniquely determined by

$$(\varphi, \psi)_N = (P_c\varphi, \psi)_N \quad \text{for any } \varphi, \psi \in C(I).$$

The Legendre collocation approximation for (20) is given by the following problem:

$$\text{Find } u_N(t):[0, T]\to V_N, \text{ s.t. for } 1 \leqq j \leqq N-1,$$

$$\frac{\partial u_N}{\partial t}(x_j, t) - \frac{\partial^2 u_N}{\partial x^2}(x_j, t) + \frac{1}{2}(P_c u_N^2)_x(x_j, t) = f(x_j, t), \tag{38}$$

$$u_N(x_j, 0) = a(x_j), \qquad 0 \leqq j \leqq N.$$

Or equivalently,

$$\begin{aligned} &\left(\frac{\partial u_N}{\partial t}, \varphi\right)_N - \left(\frac{\partial^2 u_N}{\partial x^2}, \varphi\right)_N + \frac{1}{2}((P_c u_N^2)_x, \varphi)_N = (f, \varphi)_N \quad \text{for any } \varphi \in V_N,\\ &u_N(0) = P_c a. \end{aligned} \tag{39}$$

For $v \in V_N$, define $A_N v \in V_N$ by

$$(A_N v, \varphi)_N = a_N(v, \varphi) = -(v_{xx}, \varphi)_N \quad \text{for any } \varphi \in V_N.$$

From (37) we know that

$$a_N(v, \varphi) = a(v, \varphi) \quad \text{for } v, \varphi \in V_N.$$

So conditions (A1′) and (A2′) are satisfied, and we can use the framework in § 2.

Now (39) can be written in the dynamical form (7) with $F_c(t, \varphi) = \frac{1}{2}(P_c\varphi^2)_x - P_c f(t)$ for $\varphi \in V_N$.

The analysis of the Legendre collocation method is parallel to that of the Legendre-Galerkin method, as we will see. Lemma 4.1 carries over, whereas Lemma 4.2 must be changed as follows.

LEMMA 4.5. *The following estimates hold*:

(a) *For* $0 \leqq \mu \leqq \sigma$, $\sigma > \frac{1}{2}$, $v \in H^\sigma$,

$$\|v - P_c v\|_\mu \leqq CN^{2\mu + \frac{1}{2} - \sigma}\|v\|_\sigma. \tag{40}$$

(b) *For* $\sigma \geqq 2$, $g \in H^{\sigma-2}$

$$\|(P_c A^{-1} - A_N^{-1} P_c)g\|_1 \leqq CN^{\frac{5}{2} - \sigma}\|g\|_{\sigma-2}. \tag{40a}$$

Proof. Inequality (40) was proved in [2]. To prove (40a), let $A^{-1}g = v$, $A_N^{-1}P_c g = v_N$, i.e., $Av = g$, $A_N v_N = P_c g$. Then for any $\varphi \in V_N$, we have

$$a(v, \varphi) = (g, \varphi), \qquad a(v_N, \varphi) = (g, \varphi)_N.$$

Taking $\varphi = \chi - V_N$, where $\chi \in V_N$ satisfies $\|v - \chi\|_1 = \inf_{\varphi \in V_N} \|v - \varphi\|_1$, then

$$a(v - v_N, v - v_N) = a(v - v_N, v - \chi) + (g, \chi - v_N) - (g, \chi - v_N)_N. \tag{41}$$

Using the following result from [11]: for $\varphi \in V_N$

$$|(g, \varphi) - (g, \varphi)_N| \leqq C\|\varphi\|_0(\|g - P_c g\|_0 + \|g - P_{N-1}g\|),$$

we obtain,

$$\begin{aligned} |(g, \chi - v_N) - (g, \chi - v_N)_N| &\leqq C\|\chi - v_N\|_0(\|g - P_c g\|_0 + \|g - P_{N-1}g\|_0)\\ &\leqq C\|v - v_N\|_1(\|g - P_c g\|_0 + \|g - P_{N-1}g\|_0). \end{aligned}$$

Going back to (41), we get (using Poincare's inequality)

$$\beta\|v-v_N\|_1 \leqq a(v-v_N, v-v_N)$$
$$\leqq C\|v-v_N\|_1(\|v-\chi\|_1+\|g-P_cg\|_0+\|g-P_{N-1}g\|_0)$$

or

$$\|v-v_N\|_1 \leqq C(\|v-\chi\|_1+\|g-P_cg\|_0+\|g-P_{N-1}g\|_0)$$
$$\leqq CN^{1-\sigma}\|v\|_\sigma+CN^{\frac{5}{2}-\sigma}\|g\|_{\sigma-2} \leqq CN^{\frac{5}{2}-\sigma}\|g\|_{\sigma-2}. \qquad \square$$

LEMMA 4.6. *Assume that for* $\sigma>\frac{1}{2}$, $a\in H^\sigma$, *we have*

$$\|P_c\, e^{-At}a-e^{-A_Nt}P_ca\|_1 \leqq CN^{\frac{5}{2}-\sigma}\|a\|_\sigma, \tag{42}$$

$$\|P_c\, e^{-At}a-e^{-A_Nt}P_ca\|_1 \leqq CN^{\frac{5}{2}-\sigma}t^{-\frac{1}{2}}\|a\|_{\sigma-1}. \tag{42a}$$

Proof. From (26) and (29), (40a), we get

$$\|P_c\, e^{-At}a-e^{-A_Nt}P_ca\|_1 \leqq C\|A_N^{\frac{1}{2}}(P_c\, e^{-At}a-e^{-A_Nt}P_ca)\|_0$$
$$\leqq C\int_\Gamma |e^{\lambda t}|\,\|A_NR_N\|\,\|(P_cA^{-1}-A_N^{-1}P_c)ARa\|_1\,d\lambda$$
$$\leqq CN^{\frac{5}{2}-\sigma}\int_\Gamma |e^{\lambda t}|\,\|R(\lambda,A)a\|_\sigma\,d\lambda \leqq CN^{\frac{5}{2}-\sigma}\|a\|_\sigma.$$

The proof of (42a) is similar. $\square$

LEMMA 4.7. *Assume that for* $\sigma>\frac{5}{2}$, $u(t)\in C([0,T], H^\sigma\cap H_0^1), f(t)\in C([0,T], H^{\sigma-1})$. *Then we have*

$$\varepsilon_N \leqq CN^{\frac{5}{4}-(\sigma/2)}|||u(s)|||_\sigma^2+CN^{\frac{5}{2}-\sigma}\{|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}\}. \tag{43}$$

Proof. From (42), we get

$$\|A_N^{\frac{1}{2}}(P_c\, e^{-A(t-\tau)}u(t)-e^{-A_N(t-\tau)}P_cu(\tau))\|_0 \leqq CN^{\frac{5}{2}-\sigma}\|u(\tau)\|_\sigma.$$

The second part in ε_N is more complicated. First, let us note that from (26), we get easily, for $0\leqq\alpha\leqq\frac{1}{2}$,

$$\|A_N^{-\alpha}\varphi\|_0 \leqq C\|A^{-\alpha}\varphi\|_0 \quad \text{for } \varphi\in V_N. \tag{26$'$}$$

As a consequence, we have, for $\varphi\in V_N$,

$$\|A_N^{-\frac{3}{8}}\varphi_x\|_0 \leqq C\|A^{-\frac{3}{8}}\varphi_x\|_0 \leqq C\|\varphi_x\|_{-\frac{3}{4}} \leqq C\|\varphi\|_{\frac{1}{4}}.$$

Let $\sigma'=(\sigma/2)-\frac{1}{4}$, then

$$\|A_N^{\frac{1}{2}}\, e^{-A_N}(t-s)\{P_c(P_cu)^2-(P_cu)^2\}_x\|_0$$
$$\leqq \|A_N^{\frac{7}{8}}\, e^{-A_N(t-s)}\|\,\|A_N^{-\frac{3}{8}}\{P_c(P_cu)^2-(P_cu)^2\}_x\|_0$$
$$\leqq C(t-s)^{-\frac{7}{8}}\|P_c(P_cu)^2-(P_cu)^2\|_{\frac{1}{4}}$$
$$\leqq C(t-s)^{-\frac{7}{8}}N^{1-\sigma'}\|P_cu\|_{\sigma'}^2 \leqq C(t-s)^{-\frac{7}{8}}N^{\frac{5}{4}-(\sigma/2)}\|u\|_\sigma^2.$$

We also have

$$\|A_N^{\frac{1}{2}}\, e^{-A_N(t-s)}\{(P_cu)_x^2-P_c(u_x^2)\}\|_0 \leqq C(t-s)^{-\frac{1}{2}}\|(P_cu)_x^2-P_c(u_x^2)\|_0$$
$$\leqq C(t-s)^{-\frac{1}{2}}(\|u_x^2-(P_cu)_x^2\|_0+\|(u^2)_x-P_c(u_x^2)\|_0)$$
$$\leqq C(t-s)^{-\frac{1}{2}}(\|u^2-(P_cu)^2\|_1+\|(u^2)_x-P_c(u_x^2)\|_0)$$
$$\leqq C(t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\|u\|_\sigma^2.$$

Finally, from (42a), we get

$$\|A_N^{\frac{1}{2}}\{P_c\, e^{-A(t-s)}(u_x^2)-e^{-A_N(t-s)}P_c(u_x^2)\}\|_0 \leqq c(t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\|u(s)\|_\sigma^2,$$

$$\|A_N^{\frac{1}{2}}\{P_c\, e^{-A(t-s)}f(s)-e^{-A_N(t-s)}P_c f(s)\}\|_0 \leqq CN^{\frac{5}{2}-\sigma}(t-s)^{-\frac{1}{2}}\|f(s)\|_{\sigma-1}.$$

Now (43) follows readily from these estimates.

LEMMA 4.8. *Condition* (Y) *in Theorem* 2.1 *is satisfied with* $\alpha=\frac{1}{2}$, $\beta=\frac{7}{8}$.

Proof. Let $v, \varphi \in V_N$;

$$\begin{aligned}\|A_N^{\alpha-\beta}\{F_c(s,v)-F_c(s,\varphi)\}\|_0 &= \tfrac{1}{2}\|A_N^{-\frac{3}{8}}\{P_c v^2-P_c\varphi^2\}_x\|_0\\ &\leqq C\|P_c(v^2-\varphi^2)\|_{\frac{1}{4}} \leqq C\|v^2-\varphi^2\|_1\\ &\leqq C\|v+\varphi\|_1\|v-\varphi\|_1\\ &\leqq C(\|A_N^{\frac{1}{2}}v\|_0+\|A_N^{\frac{1}{2}}\varphi\|_0)\|A_N^{\frac{1}{2}}(v-\varphi)\|_0.\end{aligned}$$

□

THEOREM 4.3. *Assume for* $\sigma>\frac{5}{2}$, $u(t)\in C([0,T], H^\sigma\cap H_0^1)$, $f(t)\in C([0,T], H^{\sigma-1})$. *Then there exist constants* $N_0(T)$, $K_0(T)$ *such that for* $N>N_0(T)$, *the Legendre collocation approximation* $u_N(t)$ *exists on* $[0,T]$, *and the following estimate holds*:

$$\|u(t)-u_N(t)\|_1 \leqq K_0(T)N^{\frac{5}{4}-(\sigma/2)}(|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{44}$$

Proof. Inequality (44) is a direct consequence of Theorem 2.1, (43), and (40). The only assumption in Theorem 2.1 to be checked is (Z). This is easy to accomplish because

$$\|A_N^{\frac{1}{2}}P_c u(t)\|_0 \leqq C\|P_c u(t)\|_1 \leqq C\|u(t)\|_\sigma.$$

□

However, (44) is not a satisfactory result because it is much worse than (40). It is improved in the next theorem by comparing $u_N(t)$ directly with $u(t)$.

THEOREM 4.4. *Under the assumptions of Theorem* 4.3, *we have for* $N>N_0(T)$,

$$\|u(t)-u_N(t)\|_1 \leqq CN^{\frac{5}{2}-\sigma}(|||u(s)|||_\sigma^2+|||u(s)|||_\sigma+|||f(s)|||_{\sigma-1}). \tag{45}$$

Proof. Let $e_N(t)=u(t)-u_N(t)$; then

$$\begin{aligned}e_N(t) &= e^{-At}a-e^{-A_N t}P_c a-\frac{1}{2}\int_0^t \{e^{-A(t-s)}u_x^2-e^{-A_N(t-s)}P_c(u_x^2)\}\,ds\\ &\quad -\frac{1}{2}\int_0^t e^{-A_N(t-s)}\{P_c(u_x^2)-(P_c u_N^2)_x\}\,ds\\ &\quad +\int_0^t \{e^{-A(t-s)}f(s)-e^{-A_N(t-s)}P_c f(s)\}\,ds\\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

From (40) and (42), we have

$$\|I_1\|_1 \leqq CN^{\frac{5}{2}-\sigma}\|a\|_\sigma,$$

$$\|I_2\|_1 \leqq C\int_0^t (t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\|u_x^2\|_{\sigma-1}\,ds \leqq CN^{\frac{5}{2}-\sigma}|||u(s)|||_\sigma^2,$$

$$\|I_4\|_1 \leqq C\int_0^t (t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\|f(s)\|_{\sigma-1}\,ds \leqq CN^{\frac{5}{2}-\sigma}|||f(s)|||_{\sigma-1}.$$

In Lemma 4.7, we proved

$$\|A_N^{\frac{1}{2}}\, e^{-A_N(t-s)}\{(P_c u^2)_x-P_c(u_x^2)\}\|_0 \leqq C(t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\|u\|_\sigma^2.$$

On the other hand, we also have

$$\begin{aligned}\left\| \int_0^t A_N^{\frac{1}{2}}\, e^{-A_N(t-s)}\{P_c u^2 - P_c u_N^2\}_x\, ds \right\|_0 &\leqq C \int_0^t (t-s)^{-\frac{7}{8}} \| A_N^{-\frac{3}{8}}(P_c u^2 - P_c u_N^2)_x \|_0\, ds \\ &\leqq C \int_0^t (t-s)^{-\frac{7}{8}} \| P_c(u^2 - u_N^2)\|_{\frac{1}{4}}\, ds \\ &\leqq C \int_0^t (t-s)^{-\frac{7}{8}} \| u^2 - u_N^2 \|_1\, ds \\ &\leqq C \int_0^t (t-s)^{-\frac{7}{8}} \| u + u_N \|_1 \| e_N(s) \|_1\, ds \\ &\leqq CL(u,f) \int_0^t (t-s)^{-\frac{7}{8}} \| e_N(s)\|_1\, ds,\end{aligned}$$

where $L(u,f) = |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}$ from Theorem 4.3.

Combining these results, we get

$$\| e_N(t) \|_1 \leqq CN^{\frac{5}{2}-\sigma} L(u,f) + CL(u,f) \int_0^t (t-s)^{-\frac{7}{8}} \| e_N(s) \|_1\, ds. \tag{46}$$

Now (45) follows from (46) and Lemma 2.3. □

5. Chebyshev methods for Burgers' equation. The ideas in Chebyshev methods are very similar to those of the Legendre methods, except that instead of Legendre polynomials, Chebyshev polynomials are used as the base functions. With Chebyshev polynomials, fast Fourier transform can be invoked to reduce the operation count. The analysis of Chebyshev methods gives rise to new difficulties because a singular weight function is used in the integration formula. Nevertheless, the arguments to be used is parallel to the ones for Legendre methods. Therefore, we will skip some of the details and emphasize their differences.

5.1. Chebyshev–Galerkin method. Define the Chebyshev weight function as

$$\omega(x) = (1-x^2)^{-\frac{1}{2}} \quad \text{on } I = (-1, 1). \tag{47}$$

For $u, v \in H^1_{0,\omega}(I)$, define

$$a(u, v) = \int_1 u_x (v\omega)_x\, dx. \tag{48}$$

Some basic properties of this bilinear form were proved by Canuto and Quarteroni [4].

LEMMA 5.1. *There exist positive constants $\beta_1, \beta_2, \beta_3$, such that for any $v \in H^1_{0,\omega}(I)$, we have*

$$\| v \|_{0,\omega} \leqq \beta_1 \| v_x \|_{0,\omega}, \tag{49}$$

$$a(v, v) \geqq \beta_2 \| v \|^2_{1,\omega}, \tag{49a}$$

$$a(u, v) \leqq \beta_3 \| u \|_{1,\omega} \| v \|_{1,\omega}. \tag{49b}$$

For $u \in D(A) = H^2_\omega \cap H^1_{0,\omega}(I)$, define

$$Au = -u_{xx}.$$

We see that $(Au, v)_\omega = a(u, v)$, for $u \in D(A)$, $v \in H^1_{0,\omega}$. Therefore the basic assumptions (A1) and (A2) are satisfied because of Lemma 5.1, and we conclude that A generates an analytic semigroup, the fractional powers of A are well defined.

LEMMA 5.2. (a) *For* $\alpha > \frac{1}{2}$, $H^\alpha_\omega(I)$ *is continuously imbedded in* $L^\infty(I)$.

(b) *For* $\alpha > 1$, $H^\alpha_\omega(I)$ *is a Banach algebra, i.e.,*

$$\|uv\|_{\alpha,\omega} \leqq C\|u\|_{\alpha,\omega}\|v\|_{\alpha,\omega} \quad \textit{for } u, v \in H^\alpha_\omega(I). \tag{50}$$

(c) *For* $\alpha \geqq 0$, *there is a constant* C, *such that for* $u \in D(A^\alpha)$,

$$\frac{1}{C}\|u\|_{2\alpha,\omega} \leqq \|A^\alpha u\|_{0,\omega} \leqq C\|u\|_{2\alpha,\omega}. \tag{51}$$

For a proof of these results, see [12].

Next we turn to the Chebyshev–Galerkin method. The projection operator P_N is now defined by

$$P_N : L^2_\omega(I) \to V_N,$$

$$(v - P_N v, \varphi)_\omega = 0 \quad \text{for } v \in L^2_\omega(I), \quad \varphi \in V_N.$$

For $v, \varphi \in V_N$, let $a_N(v, \varphi) = -\int v_{xx}\varphi\omega\, dx = a(v, \varphi)$. With this setup, the formulation of the Chebyshev–Galerkin method is the same as that of the Legendre–Galerkin method. Assumptions (A1′) and (A2′) are satisfied.

Now we proceed to the approximation properties of this method. Similar to Lemma 4.2 and (27d), (27e), we have the following lemma (see also [3], [4]).

LEMMA 5.3. *The following estimates hold*:

(a) *For* $0 \leqq \mu \leqq 1 \leqq \sigma$, $v \in H^\sigma_\omega \cap H^1_{0,\omega}(I)$,

$$\|v - P_N v\|_{\mu,\omega} \leqq CN^{\frac{3}{2}\mu - \sigma}\|v\|_{\sigma,\omega}. \tag{52}$$

(b) *For* $0 \leqq \mu \leqq 1 \leqq \sigma$, $g \in H^{\sigma-2}_\omega(I)$,

$$\|(P_N A^{-1} - A_N^{-1} P_N)g\|_{\mu,\omega} \leqq CN^{\frac{3}{2}\mu - \sigma}\|g\|_{\sigma-2}. \tag{52a}$$

(c) *For* $\sigma \geqq 2$, $a \in H^\sigma_\omega \cap H^1_{0,\omega}(I)$,

$$\|A_N^{\frac{1}{3}}(P_N e^{-At} - e^{-A_N t}P_N)a\|_{0,\omega} \leqq CN^{1-\sigma}\|a\|_{\sigma,\omega}, \tag{52b}$$

$$\|A_N^{\frac{1}{3}}(P_N e^{-At} - e^{-A_N t}P_N)a\|_{0,\omega} \leqq CN^{1-\sigma}t^{-\frac{1}{2}}\|a\|_{\sigma-1,\omega}. \tag{52c}$$

For a proof of (52), see [3]. The proof of (52a–c) is similar to that of (27a), (27d), and (27e), respectively, and is therefore omitted.

Because the bilinear form $a(\cdot\,,\cdot)$ is nonsymmetric, estimates of the type (26) and (26′) become more delicate. Nevertheless, we can prove the following.

LEMMA 5.4. *For* $0 \leqq \alpha < \frac{1}{2}$, *there exists a constant* C, *such that for* $\varphi \in V_N$,

$$\|A^\alpha \varphi\|_{0,\omega} \leqq C\|A_N^\alpha \varphi\|_{0,\omega}, \tag{53}$$

$$\|A_N^{-\alpha}\varphi\|_{0,\omega} \leqq C\|A^{-\alpha}\varphi\|_{0,\omega}. \tag{53a}$$

Proof. We will use the setup in § 4 of [8]. We can view A_N as a regularly accretive operator on the space V_N. Its real part H_N is a selfadjoint operator on V_N defined via a symmetric bilinear form $b(\cdot\,,\cdot)$ by, $H_N : V_N \to V_N$,

$$(H_N v, \varphi)_{0,\omega} = b(v, \varphi) = \tfrac{1}{2}(a(v, \varphi) + a(\varphi, v)) \quad \text{for } v, \varphi \in V_N.$$

Then from Theorem 3.1 of [8], we get for $0 \leqq \alpha < \frac{1}{2}$, $\varphi \in V_N$,

$$\|H_N^\alpha \varphi\|_{0,\omega} \leqq \left(1 - \tan\frac{\pi\alpha}{2}\right)^{-1}\|A_N^\alpha \varphi\|_{0,\omega}. \tag{54}$$

On the other hand, from (49) and (51), we get

$$\|H_N^{\frac{1}{2}}\varphi\|_{0,\omega}^2=(H_N\varphi,\varphi)_\omega=a(\varphi,\varphi)\geqq\beta_2\|\varphi\|_{1,\omega}^2\geqq\tilde{\beta}\|A^{\frac{1}{2}}\varphi\|_{0,\omega}^2.$$

Therefore, we have

$$\|A^{\frac{1}{2}}\varphi\|_{0,\omega}\leqq C\|H_N^{\frac{1}{2}}\varphi\|_{0,\omega}.$$

Now (53) follows from Heinz-Kato inequality, together with (54).

To prove (53a), let us first remark that, parallel to (53), we also have, for $0\leqq\alpha<\frac{1}{2}$,

$$\|(A^*)^\alpha\varphi\|_{0,\omega}\leqq C\|(A_N^*)^\alpha\varphi\|_{0,\omega}\quad\text{for }\varphi\in V_N,$$

where A^* and A_N^* are the adjoint operators of A and A_N, respectively.

For $g\in V_N$, let $v=(A_N^*)^{-\alpha}g\in V_N$, $g_1=(A^*)^\alpha v$,

$$\begin{aligned}\|A_N^{-\alpha}\varphi\|_{0,\omega}&=\sup_{g\in V_N}\frac{(A_N^{-\alpha}\varphi,g)_\omega}{\|g\|_{0,\omega}}=\sup_{g\in V_N}\frac{(\varphi,(A_N^*)^{-\alpha}g)_\omega}{\|g\|_{0,\omega}}=\sup\frac{(\varphi,v)}{\|(A_N^*)^\alpha v\|}\\&\leqq C\sup\frac{(\varphi,v)_\omega}{\|(A^*)^\alpha v\|_{0,\omega}}=C\sup_{g_1\in L_\omega^2}\frac{(A^{-\alpha}\varphi,g_1)_\omega}{\|g_1\|_{0,\omega}}\\&\leqq C\|A^{-\alpha}\varphi\|_{0,\omega}.\end{aligned}$$ □

LEMMA 5.5. *Let $\alpha=\frac{1}{3}$. Assume for $\alpha\geqq 2$, $u(t)\in C([0,T],H_\omega^\sigma\cap H_{0,\omega}^1(I))$, $f(t)\in C([0,T],H_\omega^{\sigma-1}(I))$; then there exists a constant C, such that*

$$\varepsilon_N\leqq CN^{1-\sigma}(|||u(s)|||_{\sigma,\omega}^2+|||u(s)|||_{\sigma,\omega}+|||f(s)|||_{\sigma-1,\omega})\quad\text{for }0\leqq t\leqq T. \tag{55}$$

We omit the proof of this lemma since it is very similar to that given for Lemma 3.3.

LEMMA 5.6. *Condition* (Y) *is satisfied with $\alpha=\frac{1}{3}$, $\beta=\frac{5}{6}$.*

Proof. First we remark that similar to (27b), we have for $g\in H_\omega^{-1}(I)$,

$$\|A_N^{-\frac{1}{2}}P_Ng\|_{0,\omega}\leqq C\|g\|_{-1,\omega}.$$

Let $v,\varphi\in V_N$,

$$\begin{aligned}\|A_N^{-\frac{1}{2}}\{F_N(s,v)-F_N(s,\varphi)\}\|_{0,\omega}&\leqq\|A_N^{-\frac{1}{2}}P_N(v_x^2-\varphi_x^2)\|_{0,\omega}\\&\leqq C\|v^2-\varphi^2\|_{0,\omega}\leqq C\|v+\varphi\|_{L^\infty}\|v-\varphi\|_{0,\omega}\\&\leqq\|CA^{\frac{1}{3}}(v+\varphi)\|_{0,\omega}\|v-\varphi\|_{0,\omega}\\&\leqq C(\|A_N^{\frac{1}{3}}v\|_{0,\omega}+\|A_N^{\frac{1}{3}}\varphi\|_{0,\omega})\|A_N^{\frac{1}{3}}(v-\varphi)\|_{0,\omega}.\end{aligned}$$ □

Now we are ready to state our main results on Chebyshev-Galerkin method. The proofs are the same as those of Theorems 4.1 and 4.2.

THEOREM 5.1. *Assume that for $\sigma\geqq 2$, $u(t)\in C([0,T],H_\omega^\sigma\cap H_{0,\omega}^1(I))$, $f(t)\in C([0,T],H_\omega^\sigma\cap H_{0,\omega}^1(I))$. Then there exist constants $N_0(T)$, $K_0(T)$, such that for $N>N_0(T)$, the Chebyshev-Galerkin approximation $u_N(t)$ of $u(t)$ exists on $[0,T]$. Furthermore, we have for $0\leqq t\leqq T$*

$$\|u(t)-u_N(t)\|_{\frac{2}{3},\omega}\leqq K_0(T)N^{1-\sigma}(|||u(s)|||_{\sigma,\omega}^2+|||u(s)|||_{\sigma,\omega}+|||f(s)|||_{\sigma-1,\omega}). \tag{56}$$

THEOREM 5.2. *Under the same assumptions as in Theorem* 4.1, *we also have for $N>N_0(T)$, $0\leqq t\leqq T$,*

$$\|u(t)-u_N(t)\|_{0,\omega}\leqq CN^{-\sigma}(|||u(s)|||_{\sigma,\omega}^2+|||u(s)|||_{\sigma,\omega}+|||f(s)|||_{\sigma-1,\omega}). \tag{57}$$

5.2. Chebyshev collocation method. For a continuous function $\varphi(x)$, we define its Chebyshev interpolant $P_c\varphi(x) \in S_N$ by

$$P_c : C(\bar{I}) \to S_N,$$

$$P_c\varphi(x_j) = \varphi(x_j), \qquad 0 \leqq j \leqq N,$$

where $x_j = \cos(\pi j/N)$ $(0 \leqq j \leqq N)$ are the collocation points for the Chebyshev method. The weights in the corresponding Gauss–Lobatto integration formula are denoted by ω_j. We define a discrete inner product on $C(\bar{I})$ by (for $\varphi(x), \psi(x) \in C(\bar{I})$),

$$(\varphi, \psi)_{N,\omega} = \sum_{j=0}^{N} \varphi(x_j)\psi(x_j)\omega_j. \tag{58}$$

The interpolation operator P_c and the discrete inner product enjoy the same properties as those for the Legendre method, as we enumerated in § 4.2. Also the Chebyshev collocation approximation of (20) and the operator A_N can be defined in a similar fashion, e.g., we have for $v, \varphi \in V_N$,

$$(A_N v, \varphi)_{N,\omega} = a_N(v, \varphi) = -(v_{xx}, \varphi)_{N,\omega} = -(v_{xx}, \varphi)_\omega = a(v, \varphi).$$

Conditions (A1′) and (A2′) are satisfied.

Now we proceed to the analysis of this method. Similar to Lemma 4.5, we have Lemma 5.7.

LEMMA 5.7. *The following estimates hold*:

(a) *For* $0 \leqq \mu \leqq \sigma$, $\sigma > \frac{1}{2}$, $v \in H_\omega^\sigma(I)$.

$$\|v - P_c v\|_{\mu,\omega} \leqq CN^{2\mu-\sigma}\|v\|_{\sigma,\omega}. \tag{59}$$

(b) *For* $\sigma \geqq 2$, $g \in H_\omega^{\sigma-2}(I)$.

$$\|(P_c A^{-1} - A_N^{-1} P_c)g\|_1 \leqq CN^{2-\sigma}\|g\|_{\sigma-2}, \tag{59a}$$

$$\|(P_c A^{-1} - A_N^{-1} P_c)g\|_0 \leqq CN^{1-\sigma}\|g\|_{\sigma-2}. \tag{59b}$$

Equation (58) is proved in [3]. The argument for (40a) also proves (59a). Inequality (59b) follows directly from (59a) and Nitsche's trick.

Similar to Lemma 4.4, we have the following estimates for the linear problem.

LEMMA 5.8. *Assume that for* $\sigma > \frac{1}{2}$, $a \in H_\omega^\sigma \cap H_{0,\omega}^1(I)$, *then we have*

$$\|P_c\, e^{-At}a - e^{-A_N t}P_c a\|_{1,\omega} \leqq CN^{2-\sigma}\|a\|_{\sigma,\omega}, \tag{60}$$

$$\|P_c\, e^{-At}a - e^{-A_N t}P_c a\|_{1,\omega} \leqq CN^{2-\sigma}t^{-\frac{1}{2}}\|a\|_{\sigma-1,\omega}. \tag{60a}$$

LEMMA 5.9. *Assume that for* $\sigma > 2$, $u(t) \in C([0, T], H_\omega^\sigma \cap H_{0,\omega}^1(I))$, $f(t) \in C([0, T], H_\omega^{\sigma-1}(I))$. *Then we have, for* $\alpha = \frac{1}{3}$,

$$\varepsilon_N \leqq N^{\frac{2}{3}-(\sigma/2)}(|||u(s)|||_{\sigma,\omega}^2 + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}). \tag{61}$$

Proof. From (60), we get

$$\|A_N^{\frac{1}{3}}(P_c\, e^{-A(t-\tau)}u(\tau) - e^{-A_N(t-\tau)}P_c u(\tau))\|_{0,\omega} \leqq CN^{2-\sigma}\|u(\tau)\|_{\sigma,\omega}.$$

The second part of ε_N is dealt with according to the following identity,

$$(P_c(P_c u)^2)_x = P_c(u_x^2) + \{(P_c u)_x^2 - P_c(u_x^2)\} + \{P_c(P_c u)^2 - (P_c u)^2\}_x. \tag{62}$$

From (60), we know

$$\|A_N^{\frac{1}{3}}(P_c\, e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_c u_x^2)\|_{0,\omega} \leqq CN^{2-\sigma}(t-s)^{-\frac{1}{2}}\|u(s)\|_{\sigma,\omega}^2.$$

We also have, similar to the situation in Lemma 4.7,

$$\begin{aligned}\|A_N^{\frac{1}{3}} e^{-A_N(t-s)}\{(P_c u)_x^2 - P_c(u_x^2)\}\|_{0,\omega} &\leqq C(t-s)^{-\frac{1}{3}}\|(P_c u)_x^2 - P_c(u_x^2)\|_{0,\omega} \\ &\leqq C(t-s)^{-\frac{1}{3}}(\|u_x^2 - (P_c u)_x^2\|_{0,\omega} + \|u_x^2 - P_c(u_x^2)\|_{0,\omega}) \\ &\leqq C(t-s)^{-\frac{1}{3}} N^{2-\sigma}\|u\|_{\sigma,\omega}^2.\end{aligned}$$

To estimate the last term in (62), we first note that Lemma 5.2 still holds (the same proof goes through) for our new A_N defined via the discrete inner product. Therefore, for $\varphi \in V_N$, we have

$$\|A_N^{-\frac{1}{3}}\varphi_x\|_{0,\omega} \leqq C\|A^{-\frac{1}{3}}\varphi_x\|_{0,\omega} \leqq C\|\varphi_x\|_{-\frac{2}{3},\omega} \leqq C\|\varphi\|_{\frac{1}{3},\omega}.$$

Let $\sigma' = \sigma/2$, then

$$\begin{aligned}\|A_N^{\frac{1}{3}} e^{-A_N(t-s)}\{P_c(P_c u)^2 - (P_c u)^2\}_x\|_{0,\omega} &\leqq \|A_N^{\frac{2}{3}} e^{-A_N(t-s)}\| \, \|A_N^{-\frac{1}{3}}\{P_c(P_c u)^2 - (P_c u)^2\}_x\|_{0,\omega} \\ &\leqq C(t-s)^{-\frac{2}{3}}\|P_c(P_c u)^2 - (P_c u)^2\|_{\frac{1}{3},\omega} \\ &\leqq C(t-s)^{-\frac{2}{3}} N^{\frac{2}{3}-\sigma'}\|P_c u\|_{\sigma',\omega}^2 \\ &\leqq C(t-s)^{-\frac{2}{3}} N^{\frac{2}{3}-\sigma'}\|u\|_{\sigma,\omega}^2.\end{aligned}$$

Combining these estimates and the estimate

$$\|A_N^{\frac{1}{3}}(P_c e^{-A_N(t-s)} f(s) - e^{-A_N(t-s)} P_c f(s))\|_{0,\omega} \leqq C N^{2-\sigma}(t-s)^{-\frac{1}{2}}\|f(s)\|_{\sigma,\omega}$$

we get (61). □

Next, we check condition (Y) in Theorem 2.1.

LEMMA 5.10. *Condition* (Y) *in Theorem* 2.1 *is satisfied with* $\alpha = \frac{1}{3}$, $\beta = \frac{5}{6}$.

Proof. We will make use of the following inequality which can be shown directly by using the intrinsic norms for $H_\omega^s(I)$, $\frac{1}{2} < s < 1$ (see [19]).

If $f, g \in H_\omega^s(I)$, $s > \frac{1}{2}$, then $f \cdot g \in H_\omega^s(I)$ and

$$\|fg\|_{s,\omega} \leqq C(\|f\|_{L^\infty}\|g\|_{s,\omega} + \|g\|_{L^\infty}\|f\|_{s,\omega}). \tag{63}$$

Now let $v, \varphi \in V_N$

$$\begin{aligned}\|A_N^{\alpha-\beta}(F_c(s, v) - F_c(s, \varphi))\|_{0,\omega} &\leqq C\|A_N^{-\frac{1}{3}}(P_c v^2 - P_c \varphi^2)_x\|_{0,\omega} \\ &\leqq C\|P_c(v^2 - \varphi^2)\|_{\frac{1}{3},\omega} \leqq C\|v^2 - \varphi^2\|_{\frac{2}{3},\omega} \\ &\leqq C(\|v+\varphi\|_{L^\infty}\|v-\varphi\|_{\frac{2}{3},\omega} + \|v-\varphi\|_{L^\infty}\|v+\varphi\|_{\frac{2}{3},\omega}) \\ &\leqq C\|v+\varphi\|_{\frac{2}{3},\omega}\|v-\varphi\|_{\frac{2}{3},\omega} \\ &\leqq C(\|A_N^{\frac{1}{3}} v\|_{0,\omega} + \|A_N^{\frac{1}{3}}\varphi\|_{0,\omega})\|A_N^{\frac{1}{3}}(v-\varphi)\|_{0,\omega}.\end{aligned}$$ □

Now we can state our main results on Chebyshev collocation method. We omit the proofs because they are the same as those of Theorems 4.3 and 4.4.

THEOREM 5.3. *Assume for* $\sigma > 2$, $u(t) \in C([0, T], H_\omega^\sigma \cap H_{0,\omega}^1(I))$, $f(t) \in C([0, T], H_\omega^{\sigma-1}(I))$. *Then there exist constants* $N_0(T)$, $K_0(T)$, *such that for* $N > N_0(T)$, *the Chebyshev collocation approximation* $u_N(t)$ *of* $u(t)$ *exists on* $[0, T]$, *and the following estimate holds for* $0 \leqq t \leqq T$:

$$\|u(t) - u_N(t)\|_{\frac{2}{3},\omega} \leqq K_0(T) N^{\frac{2}{3}-(\sigma/2)}(|||u(s)|||_{\sigma,\omega}^2 + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}). \tag{64}$$

THEOREM 5.4. *Under the same assumptions as in Theorem* 4.3, *we have for* $N > N_0(T)$,

$$\|u(t)-u_N(t)\|_{1,\omega} \leqq CN^{2-\sigma}(|||u(s)|||^2_{\sigma,\omega}+|||u(s)|||_{\sigma,\omega}+|||f(s)|||_{\sigma-1,\omega}). \tag{65}$$

In conclusion, we remark that the approach presented here applies to much more general situations such as the Burgers' equation with Neumann boundary condition. However, it is essential that the semigroup be regularizing (here we required that the semigroup be analytic). Therefore this approach does not apply to hyperbolic equations. In particular, the estimates presented in this paper deteriorate when the viscosity coefficient goes to zero. In the limit of zero viscosity, discontinuities form spontaneously in the solution, even with smooth initial data. Numerical experience indicates that the methods analyzed in this paper do not converge for discontinuous solutions, unless some kind of smoothing is done [15].

It is also interesting to analyze the full discretized problem. This can also be done in the present framework. The result will be some kind of approximation to the semigroup in the spirit of Trotter's product formula. Sharp estimates can be obtained by combining standard techniques from numerical solutions of ODEs and the results presented above.

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