NoDEA 1 (1994) 389-402 1021-9722/94/040389-14 $ 1.50+0.20

Stochastic Burgers' equation

Giuseppe DA PRATO
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 6, 56126 Pisa, Italy

Arnaud DEBUSSCHE
CNRS et Université Paris Sud, 91405 Orsay, France

Roger TEMAM
Université Paris Sud, 91405 Orsay, France

Abstract

We consider a Burgers' equation perturbed by white noise. We prove the existence and uniqueness of the global solution as well as the existence of an invariant measure for the corresponding transition semigroup.

1 Introduction

It is well known that the Burgers' equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity.

However the situation is totally different when the force is a random one. Several authors have indeed suggested to use the stochastic Burgers' equation as a simple model for turbulence, [1],[2],[4],[6]. The equation has also been proposed in [7] to study the dynamics of interfaces.

Here we consider the Burgers' equation with a random force which is a space–time white noise (or Brownian sheet)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x}(u^2(t,x)) + \frac{\partial^2 \widetilde{W}}{\partial t \partial x}. \tag{1.1}$$

We recall that $\widetilde{W}(t,x),\ t \geq 0,\ x \in \mathbf{R}$ is a zero mean Gaussian process whose covariance function is given by

$$\mathbf{E}[\widetilde{W}(t,x)\widetilde{W}(s,y)] = (t \wedge s)(x \wedge y),\ t,s \geq 0,\ x,y \in \mathbf{R}.$$

Alternatively we can consider a cylindrical Wiener process W by setting

$$W(t) = \frac{\partial \widetilde{W}}{\partial x} = \sum_{h=1}^{\infty} \beta_h e_h, \tag{1.2}$$

where $\{e_h\}$ is an orthonormal basis of $L^2(0,1)$ and $\{\beta_h\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The series (1.2) defining W does not converge in $L^2(0,1)$ but it is convergent in any Hilbert space U such that the embedding

$$L^2(0,1) \subset U$$

is Hilbert–Schmidt (see [8], Chapter 9).

In the following we shall write (1.1) as follows:

$$du(t,x) = \left(\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2} \frac{\partial}{\partial x}(u^2(t,x)) \right) dt + dW, \; x \in [0,1], t > 0, \tag{1.3}$$

where W is defined by (1.2).

Equation (1.3) is supplemented with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{1.4}$$

and the initial condition

$$u(x,0) = u_0(x), \; x \in [0,1]. \tag{1.5}$$

Our aim in this paper is to prove that problem (1.3), with boundary and initial conditions (1.4), (1.5) has a unique global solution. To the best of our knowledge this is the first existence result for the stochastic Burgers' equation forced by a cylindrical Wiener process.

In §2, we set the notations, introduce the stochastic convolution and prove local existence in time. Then in §3 we derive an a–priori estimate which yields global existence. Finally §4 is devoted to the existence of an invariant measure. The method consists in proving that the sequence of laws $\{\mathcal{L}(u_\lambda(0))\}_{\lambda\geq 0}$, where u_λ is the solution to (1.3)–(1.4) with the initial condition at time $-\lambda$, $u(x,-\lambda) = 0$, is tight so that a suitable subsequence is weakly convergent to the required invariant measure. For this purpose, we need to derive bounds uniform with respect to time on $u(t)$ in different spaces. Classical techniques do not work here and we use an argument similar to that in [5].

2 Local existence in time

We define the unbounded self-adjoint operator A on $L^2(0,1)$ by

$$Au = \frac{\partial^2}{\partial x^2} u,$$

for u on the domain

$$D(A) = \{u \in H^2(0,1): \quad u(0) = u(1) = 0\}.$$

We denote by e^{tA}, $t \geq 0$ the semigroup on $L^2(0,1)$ generated by A. It is well known that e^{tA}, $t \geq 0$, has a natural extension, that we still denote by e^{tA}, $t \geq 0$, as a contraction semigroup on $L^p(0,1)$ for any $p \geq 1$. Finally we denote by $\{e_k\}$ the complete orthonormal system on $L^2(0,1)$ which diagonalizes A and by $\{\lambda_k\}$ the corresponding eigenvalues. We have

$$e_k(x) = \sqrt{\frac{2}{\pi}} \sin k\pi x, \ k = 1, 2, ...$$

and

$$\lambda_k = -\pi^2 k^2, \ k = 1, 2, ...$$

Now we rewrite (1.3),(1.4), (1.5) as the abstract differential stochastic equation

$$\begin{cases} du = \left(Au + \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(u^2\right)\right) dt + dW, \\ \\ u(0) = u_0. \end{cases} \tag{2.1}$$

We recall that the solution to the linear problem

$$\begin{cases} du = Audt + dW, \\ \\ u(0) = u_0 \end{cases} \tag{2.2}$$

is unique and given by the so–called stochastic convolution

$$W_A(t) = \int_0^t e^{(t-s)A} dW(s). \tag{2.3}$$

It can be shown that W_A is a Gaussian process and it is mean–square continuous with values in $L^2(0,1)$. Moreover W_A has a version which is, a.s. for $\omega \in \Omega$, α–Hölder continuous with respect to (t,x) for any $\alpha \in [0, 1/4[$ (see [8], §5.5, Theorem 5.20 and Example 5.21).

We set

$$v(t) = u(t) - W_A(t), \ t \geq 0,$$

then u satisfies (2.1) if and only if v is a solution of

$$\begin{cases} \dfrac{dv}{dt} = Av + \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(v + W_A\right)^2, \\ \\ v(0) = u_0. \end{cases} \tag{2.4}$$

From now on we will study equation (2.4) a. s. $\omega \in \Omega$ and consider for the moment that W_A is an α–Hölder continuous function with respect to (t, x) for any $\alpha \in [0, 1/4[$. We will return to the stochastic point of view (and to equation (2.1)) at the end of §3.

Let us write (2.4) as

$$v(t) = e^{tA}u_0 + \frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2\, ds; \tag{2.5}$$

then if v satisfies (2.5) we say that it is a mild solution of (2.4).

We are going to solve equation (2.5) by a fixed point argument in the space $C([0, T^*]; L^p(0, 1))$ for $p > 1$ and for some $T^* > 0$. We set

$$\Sigma_p(m, T^*) = \{v \in C([0, T^*]; L^p(0, 1)) : |v(t)|_{L^p(0,1)} \le m,\ \forall t \in [0, T^*]\},$$

and consider an initial datum u_0 $\mathcal{F}_0$–measurable and belonging to $L^p(0, 1)$, $\omega \in \Omega$ a. s. We will see, in the proof of the Lemma 2.1 below that if $z(t)$ is, a bounded function from $[0, T]$ into $L^p(0, 1)$, then, for $t > 0$, the function $e^{tA}\dfrac{\partial}{\partial x}z^2$ is also in $L^p(0, 1)$. Hence the integral in (2.5) is convergent in $L^p(0, 1)$ a.s. Thus (2.5) has a meaning as an equality in $L^p(0, 1)$.

Lemma 2.1 *For any $p \ge 2$ and $m > |u_0|_{L^p(0,1)}$, there exists a stopping time $T^* > 0$ such that (2.5) has a unique solution in $\Sigma_p(m, T^*)$.*

Proof – Take any v in $\Sigma_p(m, T^*)$ and define $z = Gv$ by

$$z(t) = e^{tA}u_0 + \frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2\, ds.$$

Then

$$\begin{aligned} |z(t)|_{L^p(0,1)} &\le |e^{tA}u_0|_{L^p(0,1)} \\ &+ \frac{1}{2}\int_0^t \left| e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2 \right|_{L^p(0,1)} ds. \end{aligned}$$

As we noticed before, e^{tA}, $t \ge 0$ is a contraction semigroup on $L^p(0, 1)$ which has a regularizing effect and, for any $s_1 \le s_2$ in $\mathbf{R}$, and $r \ge 1$, e^{tA} maps $W^{s_1,r}(0, 1)$ into $W^{s_2,r}(0, 1)$, for all $t > 0$. Moreover the following estimate holds

$$|e^{tA}z|_{W^{s_2,r}(0,1)} \le C_1\left(t^{\frac{s_1 - s_2}{2}} + 1\right)|z|_{W^{s_1,r}(0,1)} \text{ for all } z \in W^{s_1,r}(0, 1). \tag{2.6}$$

The constant C_1 depends only on s_1, s_2 and r, see for instance [9, Lemma 3, Part I].

Using the Sobolev embedding theorem we have

$$\left| e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2 \right|_{L^p(0,1)} \le C_2\left| e^{(t-s)A}\frac{\partial}{\partial x}(v + W_A)^2 \right|_{W^{\frac{1}{p},\frac{p}{2}}(0,1)},$$

and, thanks to (2.6) with $s_1 = -1$, $s_2 = 1/p$, $r = p/2$

$$\left| e^{(t-s)A} \frac{\partial}{\partial x}(v + W_A)^2 \right|_{L^p(0,1)}$$

$$\leq C_1 C_2 \left(t^{-\frac{1}{2}-\frac{1}{2p}} + 1 \right) \left| \frac{\partial}{\partial x}(v + W_A)^2 \right|_{W^{-1,\frac{p}{2}}(0,1)}$$

$$\leq C_1 C_2 \left(t^{-\frac{1}{2}-\frac{1}{2p}} + 1 \right) \left| (v + W_A)^2 \right|_{L^{\frac{p}{2}}(0,1)}.$$

Therefore

$$\begin{aligned} |z(t)|_{L^p(0,1)} &\leq |u_0|_{L^p(0,1)} \\ &+ \frac{1}{2} C_1 C_2 \int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}} + 1 \right) \left| (v + W_A)^2 \right|_{L^{p/2}(0,1)} ds \\ &\leq |u_0|_{L^p(0,1)} + \frac{1}{2} C_1 C_2 (m + \mu_p)^2 \int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}} + 1 \right) ds \\ &\leq |u_0|_{L^p(0,1)} + \frac{1}{2} C_1 C_2 (m + \mu_p)^2 \left(\frac{p-1}{2p} t^{\frac{1}{2}-\frac{1}{2p}} + t \right), \end{aligned}$$

where

$$\mu_p = \sup_{t \in [0,T]} |W_A(t)|_{L^p(0,1)}.$$

Hence $|z(t)|_{L^p(0,1)} \leq m$ for all $t \in [0, T^*]$ provided

$$|u_0|_{L^p(0,1)} + \frac{1}{2} C_1 C_2 (m + \mu_p)^2 \left(\frac{p-1}{2p} (T^*)^{\frac{1}{2}-\frac{1}{2p}} + T^* \right) \leq m. \tag{2.7}$$

It is clear that for any $m > |u_0|_{L^p(0,1)}$ there exists a T^* satisfying (2.7).

Now consider $v_1, v_2 \in \Sigma_p(m, T^*)$ and set $z_i = G v_i$, $i = 1, 2$ and $z = z_1 - z_2$. Then

$$z(t) = \frac{1}{2} \int_0^t e^{(t-s)A} \frac{\partial}{\partial x} \left[(v_1 + W_A)^2 - (v_2 + W_A)^2 \right] ds,$$

and we derive as above

$$|z(t)|_{L^p(0,1)}$$

$$\leq \frac{1}{2} C_1 C_2 \int_0^t \left((t-s)^{-\frac{1}{2}-\frac{1}{2p}} + 1 \right) \left| (v_1 + W_A)^2 - (v_2 + W_A)^2 \right|_{L^{p/2}(0,1)} ds.$$

Using Hölder's inequality, we have

$$\left| (v_1 + W_A)^2 - (v_2 + W_A)^2 \right|_{L^{p/2}(0,1)} = |(v_1 + v_2 + 2W_A)(v_1 - v_2)|_{L^{p/2}(0,1)}$$

$$\leq |v_1 + v_2 + 2W_A|_{L^p(0,1)} |v_1 - v_2|_{L^p(0,1)} \leq 2(m + \mu_p) |v_1 - v_2|_{L^p(0,1)},$$

hence

$$|z(t)|_{L^p(0,1)} \leq C_1C_2(m+\mu_p)\left(\frac{p-1}{2p}(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*\right).$$

We take T^* such that

$$C_1C_2(m+\mu_p)(\left(\frac{1}{2}-\frac{1}{2p}\right)(T^*)^{\frac{1}{2}-\frac{1}{2p}}+T^*)<1$$

and (2.7) holds so that G is a strict contraction on $\Sigma_p(m,T^*)$. ∎

Remark 2.1 As mentioned before all the previous results are valid a.s. for $\omega \in \Omega$; in particular μ_p and T^* depend on ω. In the next section we will show that $T^* = T$ a.s. for $\omega \in \Omega$ and hence remove the dependence on ω for the time interval on which the solution exists.

3 Global existence

We are still considering equation (2.5) as a deterministic one, working a. s. for $\omega \in \Omega$.

Lemma 3.1 *If $v \in C([0,T];L^p(0,1))$ satisfies* (2.5) *then*

$$|v(t)|_{L^p(0,1)} \leq C_3\left(\mu_\infty^2+|u_0|_{L^p(0,1)}\right)e^{(2p\mu_\infty+1)t}.$$

Proof Let $\{u_0^n\}$ be a sequence in $C^\infty(0,1)$ such that

$$u_0^n \to u_0, \text{ in } L^p(0,1),$$

and let $\{W^n\}$ be a sequence of regular processes such that

$$W_A^n(t)=\int_0^t e^{(t-s)A}dW^n(s) \to W_A(t)$$

in $C([0,1]\times[0,1])$ a. s. for $\omega \in \Omega$.

Let v^n be the solution of

$$v^n(t)=e^{tA}u_0^n+\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left(v^n+W_A^n\right)^2 ds$$

given by Lemma 2.1. It is easy to see that v^n does exist on an interval of time $[0,T_n]$ such that $T_n \to T^*$ a. s. and that v^n converges to v in $C([0,T^*];L^p(0,1))$ a. s. Moreover v^n is regular a.s. and satisfies

$$\frac{\partial v^n}{\partial t}-\frac{\partial^2 v^n}{\partial x^2}-\frac{1}{2}\frac{\partial}{\partial x}\left(v^n+W_A^n\right)^2=0. \tag{3.1}$$

Multiplying (3.1) by $|v^n|^{p-2}v^n$ and integrating over $[0,1]$ we find

$$\begin{aligned}&\frac{1}{p}\frac{\partial}{\partial t}|v^n|^p_{L^p(0,1)}+(p-1)\int_0^1|v^n|^{p-2}\left(\frac{\partial}{\partial x}v^n\right)^2dx\\&-\frac{1}{2}\int_0^1\left(\frac{\partial}{\partial x}(v^n+W_A^n)^2\right)|v^n|^{p-2}v^ndx=0.\end{aligned}\tag{3.2}$$

We integrate by parts the last integral

$$\begin{aligned}&\int_0^1\left(\frac{\partial}{\partial x}(v^n+W_A^n)^2\right)|v^n|^{p-2}v^ndx\\&=-(p-1)\int_0^1\left(|v^n+W_A^n|^2\right)|v^n|^{p-2}\frac{\partial}{\partial x}v^ndx\\&=-(p-1)\int_0^1|v^n|^p\frac{\partial}{\partial x}v^ndx\\&-2(p-1)\int_0^1W_A^nv_n|v^n|^{p-2}\frac{\partial}{\partial x}v^ndx\\&-(p-1)\int_0^1(W_A^n)^2|v^n|^{p-2}\frac{\partial}{\partial x}v^ndx.\end{aligned}$$

The first term is zero, indeed

$$\int_0^1|v^n|^p\frac{\partial}{\partial x}v^ndx=\frac{1}{p+1}\int_0^1\frac{\partial}{\partial x}\left(|v^n|^pv^n\right)dx=0,$$

thanks to the boundary conditions.

We bound the second term as follows

$$\begin{aligned}&2(p-1)\left|\int_0^1W_A^nv_n|v^n|^{p-2}\frac{\partial}{\partial x}v^ndx\right|\\&\le 2(p-1)|W_A^n|_{L^\infty(0,1)}|v_n|^{p/2}_{L^p(0,1)}\left(\int_0^1|v^n|^{p-2}\left(\frac{\partial}{\partial x}v^n\right)^2dx\right)^{1/2}\\&\le 2(p-1)\mu^2_{n,\infty}|v_n|^p_{L^p(0,1)}+\frac{p-1}{2}\int_0^1|v^n|^{p-2}\left(\frac{\partial}{\partial x}v^n\right)^2dx,\end{aligned}$$

with $\mu_{n,\infty}=\sup_{t\in[0,T]}|W_A^n(t)|_{L^\infty(0,1)}$ for a. e. $\omega\in\Omega$.

For the third term we write

$$2(p-1)\left|\int_0^1 (W_A^n)^2 |v^n|^{p-2} \frac{\partial}{\partial x} v^n dx\right|$$

$$\leq 2(p-1)\mu_{n,\infty}^2 |v_n|_{L^p(0,1)}^{\frac{p-2}{2}} \left(\int_0^1 |v^n|^{p-2}\left(\frac{\partial}{\partial x}v^n\right)^2 dx\right)^{1/2}$$

$$\leq \frac{p-1}{2}\int_0^1 |v^n|^{p-2}\left(\frac{\partial}{\partial x}v^n\right)^2 dx$$

$$+\frac{4(p-1)}{p}\mu_{n,\infty}^{2p} + \frac{2(p-1)(p-2)}{p}|v_n|_{L^p(0,1)}^p.$$

Going back to (3.2) we obtain

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}|v_n|_{L^p(0,1)}^p &\leq \left(2(p-1)\mu_{n,\infty} + \frac{2(p-1)(p-2)}{p}\right)|v_n|_{L^p(0,1)}^p \\ &+ \frac{4(p-1)}{p}\mu_{n,\infty}^{2p} \\ &\leq 2p(\mu_{n,\infty}+1)|v_n|_{L^p(0,1)}^p + 4\mu_{n,\infty}^{2p}\end{aligned}$$

and, thanks to Gronwall's lemma

$$|v_n|_{L^p(0,1)}^p \leq \left(\frac{2\mu_{n,\infty}^{2p}}{p(\mu_{n,\infty}+1)} + |u_0^n|_{L^p(0,1)}^p\right) e^{2p^2(\mu_{n,\infty}+1)t}.$$

Taking the limit as $n \to \infty$, we see that a.s.

$$|v|_{L^p(0,1)}^p \leq \left(\frac{2\mu_\infty^{2p}}{p(\mu_\infty+1)} + |u_0|_{L^p(0,1)}^p\right) e^{2p^2(\mu_\infty+1)t}.$$

It follows

$$|v|_{L^p(0,1)} \leq \left(\frac{2\mu_\infty^{2p}}{p(\mu_\infty+1)} + |u_0|_{L^p(0,1)}^p\right)^{1/p} e^{2p(\mu_\infty+1)t}$$

$$\leq C_3\left(\mu_\infty^2 + |u_0|_{L^p(0,1)}\right) e^{2p(\mu_\infty+1)t},$$

for a constant C_3. ∎

It is now easy to derive from Lemma 2.1 and Lemma 3.1,

Theorem 3.1 *Let u_0 be given which is $\mathcal{F}_0$–measurable and such that for some $p \geq 2$, $u_0 \in L^p(0,1)$ a.s. Then there exists a unique mild solution of equation* (2.1)*, which belongs a. s. to $C([0,T]; L^p(0,1))$.*

4 Invariant measures

We now want to prove that there exists an invariant measure for the equation

$$du = \left(Au + \frac{1}{2} \frac{\partial}{\partial x}(u^2) \right) dt + dW(t). \tag{4.1}$$

As usual in this context, we extend the cylindrical Wiener process $W(t)$ to all $\mathbf{R}$ by setting

$$W(t) = V(-t), \ t \leq 0,$$

where $V(t), t \geq 0$ is another cylindrical H–valued Wiener process independent of $W(t), t \geq 0$.

For each $\lambda \geq 0$ let us consider the solution u_λ of the problem

$$\left\{ \begin{array}{lcl} du_\lambda & = & \left(Au_\lambda + \frac{1}{2} \frac{\partial}{\partial x}(u_\lambda^2) \right) dt + dW(t) \\ \\ u_\lambda(-\lambda) & = & 0. \end{array} \right. \tag{4.2}$$

Such a solution does exist by an obvious generalization of Theorem 3.1. It is classical that an invariant measure exists provided the family of laws $\{\mathcal{L}(u_\lambda(0))\}_{\lambda \geq 0}$ is tight. Also, by the compactness of the embedding of $H^\sigma(0,1)$ into $L^2(0,1)$ for all $\sigma > 0$ it is enough to prove that $\{u_\lambda(0)\}_{\lambda \geq 0}$ is bounded in probability in $H^\sigma(0,1)$ for some $\sigma > 0$.

We remark that the estimate obtained in Lemma 3.1 is useless here, since it gives a bound on solutions that grows indefinitely when $t \to \infty$.

Similarly as in [3], §9 we introduce a modified stochastic convolution. For any $\alpha > 0$ we define

$$W_A^\alpha(t) = \int_{-\infty}^t e^{(t-s)(A-\alpha)} dW(s). \tag{4.3}$$

W_A^α is the mild solution of the linear equation

$$\left\{ \begin{array}{lcl} dz & = & (Az - \alpha z)dt + dW(t) \\ \\ z(0) & = & z_0, \end{array} \right. \tag{4.4}$$

where

$$z_0 = \int_{-\infty}^0 e^{-s(A-\alpha)} dW(s).$$

It is well known, and easy to check, that W_A^α is a stationary process. Define

$$v_\lambda^\alpha(t) = u_\lambda(t) - W_A^\alpha(t), \ t \geq -\lambda.$$

Then v_λ^α is the mild solution to the semilinear equation

$$\begin{cases} \dfrac{dv_\lambda^\alpha(t)}{dt} &= \left(Av_\lambda^\alpha(t) + \dfrac{1}{2}\dfrac{\partial}{\partial x}\left(v_\lambda^\alpha(t) + W_A^\alpha\right)^2\right) + \alpha W_A^\alpha(t), \\ v_\lambda^\alpha(-\lambda) &= -W_A^\alpha(-\lambda). \end{cases} \tag{4.5}$$

Since W_A^α is bounded in probability in $H^{1/4}(0,1)$, it is enough to prove that $v_\lambda^\alpha(0)$ is also bounded in probability in $H^{1/4}(0,1)$. As a first step, we will prove boundedness in $L^2(0,1)$. The key in deriving this estimate is the fact that, when α is large enough, then $|W_A^\alpha(t)|_{L^2(0,1)}$ is small in mean square. More precisely, we have

Lemma 4.1 *For any $\varepsilon > 0$ and $\sigma \in [0, 1/4[$, there exist α depending on ε and σ such that*

$$\mathbf{E}\left(|(-A)^\sigma W_A^\alpha(t)|^2_{L^2(0,1)}\right) < \varepsilon$$

for all $t \in \mathbf{R}$.

Proof — Since

$$W(t) = \sum_{k=1}^\infty \beta_k(t) e_k,$$

we have

$$W_A^\alpha(t) = \sum_{k=1}^\infty \left(\int_{-\infty}^t e^{(t-s)(A-\alpha)} d\beta_k(s)\right) e_k,$$

and, by definition of the stochastic integral,

$$\begin{aligned} \mathbf{E}\left(|(-A)^\sigma W_A^\alpha(t)|^2_{L^2(0,1)}\right) &= \sum_{k=1}^\infty \int_{-\infty}^t (\pi k)^{4\sigma} e^{-2(\pi^2k^2+\alpha)(t-s)} ds \\ &= \sum_{k=1}^\infty \frac{(\pi k)^{4\sigma}}{2(\pi^2 k^2 + \alpha)}. \end{aligned}$$

Let N be such that

$$\sum_{k=N+1}^\infty (\pi k)^{4\sigma - 2} < \varepsilon,$$

and α be such that

$$\sum_{k=1}^N \frac{(\pi k)^{4\sigma}}{\pi^2 k^2 + \alpha} < \varepsilon/2$$

Then

$$\mathbf{E}\left(|(-A)^\sigma W_A^\alpha(t)|^2_{L^2(0,1)}\right) \le \varepsilon. \ \blacksquare$$

Let us now state the main result of this section.

Theorem 4.1 *There exists an invariant measure for problem* (4.1).

Proof —

We first choose $\alpha > 0$ such that, for all $t \in \mathbf{R}$

$$\mathbf{E}\left[|W_A^\alpha(t)|_{L^4(0,1)}^{8/3}\right] \leq \frac{\pi^2}{8C_6}, \tag{4.6}$$

where C_6 is an absolute constant defined hereafter. This is possible since by Hölder's inequality

$$\mathbf{E}\left[|W_A^\alpha(t)|_{L^4(0,1)}^{8/3}\right] \leq \left(\mathbf{E}\left[|W_A^\alpha(t)|_{L^\infty(0,1)}^{4}\right]\right)^{1/3} \left(\mathbf{E}\left[|W_A^\alpha(t)|_{L^2(0,1)}^{2}\right]\right)^{2/3},$$

and the first term on the right hand side is bounded by [8, pag. 336] whereas the second one can be made as small as we want by Lemma 4.1.

Let us now consider $v_\lambda^\alpha(t)$ defined above, multiply (4.5) by $v_\lambda^\alpha(t)$ and integrate over $]0,1[$. Two integrations by parts yield

$$\frac{1}{2}\frac{d}{dt}|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + |(-A)^{1/2}v_\lambda^\alpha(t)|_{L^2(0,1)}^2$$

$$+\frac{1}{2}\int_0^1 (v_\lambda^\alpha(t) + W_A^\alpha(t))^2 \frac{\partial v_\lambda^\alpha(t)}{\partial x}dx = \alpha \int_0^1 W_A^\alpha(t)v_\lambda^\alpha(t)dx.$$

The computation, as well as those below, is not rigourous, but the results may be justified by an argument similar to the one in the proof of Lemma 3.1

It is clear that

$$\int_0^1 (v_\lambda^\alpha(t))^2 \frac{\partial v_\lambda^\alpha(t)}{\partial x}dx = 0,$$

therefore

$$\frac{1}{2}\frac{d}{dt}|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + |(-A)^{1/2}v_\lambda^\alpha(t)|_{L^2(0,1)}^2$$

$$= -\int_0^1 v_\lambda^\alpha(t)W_A^\alpha(t)\frac{\partial v_\lambda^\alpha(t)}{\partial x}dx - \frac{1}{2}\int_0^1 (W_A^\alpha(t))^2 \frac{\partial v_\lambda^\alpha(t)}{\partial x}dx$$

$$+\alpha\int_0^1 W_A^\alpha(t)v_\lambda^\alpha(t)dx$$

$$\leq |W_A^\alpha(t)|_{L^4(0,1)}\,|v_\lambda^\alpha(t)|_{L^4(0,1)}\left|\frac{\partial v_\lambda^\alpha(t)}{\partial x}\right|_{L^2(0,1)} + \frac{1}{2}\,|W_A^\alpha(t)|_{L^4(0,1)}^2\left|\frac{\partial v_\lambda^\alpha(t)}{\partial x}\right|_{L^2(0,1)}$$

$$+\alpha\,|W_A^\alpha(t)|_{L^2(0,1)}\,|v_\lambda^\alpha(t)|_{L^2(0,1)}.$$

Since, by the Sobolev embedding theorem:

$$|v_\lambda^\alpha(t)|_{L^4(0,1)} \le C_4\,|v_\lambda^\alpha(t)|_{H^{1/4}(0,1)} \le C_5\,|v_\lambda^\alpha(t)|_{L^2(0,1)}^{3/4}\,|v_\lambda^\alpha(t)|_{H^1(0,1)}^{1/4},$$

we have

$$\frac{1}{2}\frac{d}{dt}|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + |(-A)^{1/2}v_\lambda^\alpha(t)|_{L^2(0,1)}^2$$

$$\le C_5\,|W_A^\alpha(t)|_{L^4(0,1)}\,|v_\lambda^\alpha(t)|_{L^2(0,1)}^{3/4}\left|\frac{\partial v_\lambda^\alpha(t)}{\partial x}\right|_{L^2(0,1)}^{5/4} + \frac{1}{2}\,|W_A^\alpha(t)|_{L^4(0,1)}^2\left|\frac{\partial v_\lambda^\alpha(t)}{\partial x}\right|_{L^2(0,1)}^2$$

$$+\alpha\,|W_A^\alpha(t)|_{L^2(0,1)}\,|v_\lambda^\alpha(t)|_{L^2(0,1)}.$$

$$\le C_6\,|W_A^\alpha(t)|_{L^4(0,1)}^{8/3}\,|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + \frac{1}{2}\,|W_A^\alpha(t)|_{L^4(0,1)}^4$$

$$+\frac{1}{4}\left|\frac{\partial v_\lambda^\alpha(t)}{\partial x}\right|_{L^2(0,1)}^2 + \frac{\alpha^2}{\pi^2}\,|W_A^\alpha(t)|_{L^4(0,1)}^2 + \frac{\pi^2}{4}\,|v^\alpha(t)|_{L^2(0,1)}^2.$$

Recalling Poincaré's inequality we have

$$\frac{d}{dt}|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + \pi^2|v_\lambda^\alpha(t)|_{L^2(0,1)}^2$$

$$\le 2C_6\,|W_A^\alpha(t)|_{L^4(0,1)}^{8/3}\,|v_\lambda^\alpha(t)|_{L^2(0,1)}^2 + |W_A^\alpha(t)|_{L^4(0,1)}^4$$

$$+\frac{2\alpha^2}{\pi^2}\,|W_A^\alpha(t)|_{L^2(0,1)}^2.$$

It follows that

$$|v_\lambda^\alpha(0)|_{L^2(0,1)}^2 \le e^{-\pi^2\lambda + 2C_6\int_{-\lambda}^0 |W_A^\alpha(s)|_{L^4(0,1)}^{8/3}ds}\,|W_A^\alpha(-\lambda)|_{L^2(0,1)}^2$$

$$+\int_{-\lambda}^0\left(|W_A^\alpha(s)|_{L^4(0,1)}^4 + \frac{2\alpha^2}{\pi^2}\,|W_A^\alpha(s)|_{L^2(0,1)}^2\right)e^{\pi^2 s + 2C_6\int_s^0 |W_A^\alpha(\sigma)|_{L^4(0,1)}^{8/3}d\sigma}ds. \tag{4.7}$$

We now follow an argument from F. FLANDOLI, [3], [5]. By stationarity of the process W_A^α and the ergodic theorem, we know that

$$\frac{1}{\lambda}\int_{-\lambda}^0 |W_A^\alpha(s)|_{L^4(0,1)}^{8/3}\,ds \to \mathbf{E}\left(|W_A^\alpha(0)|_{L^4(0,1)}^{8/3}\right),\ a.e.$$

as $\lambda \to \infty$. We deduce from (4.6) that there exists a real random variable $\lambda_0(\omega)$ such that

$$e^{-\pi^2\lambda + 2C_6\int_{-\lambda}^0 |W_A^\alpha(s)|_{L^4(0,1)}^{8/3}ds} \le e^{-\frac{\pi^2}{2}\lambda},\quad \text{for } \lambda \ge \lambda_0, a.s.$$

Now, since $|W_A^\alpha(s)|^2_{L^2(0,1)}$ and $|W_A^\alpha(s)|^4_{L^4(0,1)}$ have at most polynomial growth, when $s \to -\infty$, for almost every $\omega \in \Omega$, it is clear that the right hand side of (4.7) is bounded almost surely. In other words, there exists a random variable $R_1(\omega)$ such that

$$|v_\lambda^\alpha(0)|_{L^2(0,1)} \leq R_1(\omega), \ a.s.$$

Let us write that $v_\lambda^\alpha(0)$ is a mild solution of (4.5) :

$$v_\lambda^\alpha(0) = -e^{\lambda A} W_A^\alpha(-\lambda) + \frac{1}{2}\int_{-\lambda}^0 e^{-sA}\frac{\partial}{\partial x}\left(v_\lambda^\alpha(s) + W_A^\alpha(s)\right)^2 ds + \alpha \int_{-\lambda}^0 e^{-sA} W_A^\alpha ds.$$

We deduce

$$\begin{aligned}\left|(-A)^{1/8} v_\lambda^\alpha(0)\right|_{L^2(0,1)} &\leq \left|e^{\lambda A}(-A)^{1/8} W_A^\alpha(-\lambda)\right|_{L^2(0,1)} \\ &+\frac{1}{2}\int_{-\lambda}^0 \left|(-A)^{1/8} e^{-sA}\frac{\partial}{\partial x}\left(v_\lambda^\alpha(s) + W_A^\alpha(s)\right)^2\right|_{L^2(0,1)} ds \\ &+\alpha\int_{-\lambda}^0 \left|(-A)^{1/8} e^{-sA} W_A^\alpha(s)\right|_{L^2(0,1)} ds.\end{aligned}$$

As in the proof of Lemma 2.1 we have

$$\begin{aligned}&\left|(-A)^{1/8} e^{-sA}\frac{\partial}{\partial x}\left(v_\lambda^\alpha(s) + W_A^\alpha(s))\right)^2\right|_{L^2(0,1)} \\ &\leq C_7\left(t^{-7/8}+1\right) e^{\pi^2 s}|v_\lambda^\alpha(s) + W_A^\alpha(s)|^2_{L^2(0,1)},\end{aligned}$$

and

$$\left|(-A)^{1/8} e^{-sA} W_A^\alpha(s)\right|_{L^2(0,1)} \leq e^{\pi^2 s}\left|(-A)^{1/8} W_A^\alpha(s)\right|_{L^2(0,1)},$$

for any $s \geq -\lambda$. Thus, using the polynomial growth of $\left|(-A)^{1/8} W_A^\alpha(s)\right|_{L^2(0,1)}$ when $s \to -\infty$ for almost all $\omega \in \Omega$, we deduce the existence of a random variable $R_2(\omega)$ such that

$$|(-A)^{1/8} v_\lambda^\alpha(0)|_{L^2(0,1)} \leq R_2(\omega), \ a.s.;$$

hence $\{v_\lambda^\alpha(0)\}_{\lambda\geq 0}$, and therefore $\{u_\lambda(0)\}_{\lambda \geq 0}$, is almost surely bounded in $H^{1/4}(0,1)$. Since almost sure boundedness implies boundedness in probability, and the embedding $H^{1/4}(0,1) \subset L^2(0,1)$ is compact, the family of laws $\{\mathcal{L}(u_\lambda(0))\}_{\lambda t \geq 0}$ is tight and there exists an invariant measure. ∎

Acknowledgments. We want to thank F. FLANDOLI, for useful discussions about the existence of invariant measure.

References

[1] D. H. CHAMBERS, R. J. ADRIAN, P. MOIN, D. S. STEWART, H. J. SUNG, Karhunen–Loéve expansion of Burgers' model of turbulence, *Phys. Fluids* **31**, (9), 2573–2582 (1988)

[2] H. CHOI, R. TEMAM, P. MOIN, J. KIM, Feedback control for unsteady flow and its application to Burgers equation, Center for Turbulence Research, Stanford University, CTR Manuscript 131. To appear on J. Fluid Mechanics (1992)

[3] H. CRAUEL, F. FLANDOLI, Attractors for Random Dynamical Systems, Preprint n. 148 Scuola Normale Superiore di Pisa (1992)

[4] DAH–TENG JENG Forced Model Equation for Turbulence, *The Physics of Fluids* **12**, 10, 2006–2010 (1969)

[5] F. FLANDOLI, Dissipativity and invariant measures for stochastic Navier–Stokes equations Preprint n. 24 Scuola Normale Superiore di Pisa (1993)

[6] I. HOSOKAWA, K. YAMAMOTO, Turbolence in the randomly forced one dimensional Burgers flow, *J. Stat. Phys.* **13**, 245 (1975)

[7] M. KARDAR, M. PARISI, J. C. ZHANG Dynamical scaling of growing interfaces, *Phys. Rev. Lett.* **56**, 889 (1986)

[8] G. DA PRATO, J. ZABCZYK, Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, Cambridge University Press (1992)

[9] F. ROTHE (1984), Global Solutions of Reaction–Diffusion Systems, *Lecture Notes in Mathematics*, **1072**, Springer–Verlag (1984)

Received September 12, 1993