NoDEA 1 (1994) 389-402 1021-9722/94/040389-14 $ 1.50+0.20

Stochastic Burgers' equation

Giuseppe DA PRATO
Scuola Normale Superiore di Pisa
Piazza dei Cavalieri 6, 56126 Pisa, Italy

Arnaud DEBUSSCHE
CNRS et Université Paris Sud, 91405 Orsay, France

Roger TEMAM
Université Paris Sud, 91405 Orsay, France

Abstract

We consider a Burgers' equation perturbed by white noise. We prove the existence and uniqueness of the global solution as well as the existence of an invariant measure for the corresponding transition semigroup.

1 Introduction

It is well known that the Burgers' equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity.

However the situation is totally different when the force is a random one. Several authors have indeed suggested to use the stochastic Burgers' equation as a simple model for turbulence, [1],[2],[4],[6]. The equation has also been proposed in [7] to study the dynamics of interfaces.

Here we consider the Burgers' equation with a random force which is a space–time white noise (or Brownian sheet)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x}(u^2(t,x)) + \frac{\partial^2 \widetilde{W}}{\partial t \partial x}. \tag{1.1}$$

We recall that $\widetilde{W}(t,x),\ t \geq 0,\ x \in \mathbf{R}$ is a zero mean Gaussian process whose covariance function is given by

$$\mathbf{E}[\widetilde{W}(t,x)\widetilde{W}(s,y)] = (t \wedge s)(x \wedge y),\ t,s \geq 0,\ x,y \in \mathbf{R}.$$

Alternatively we can consider a cylindrical Wiener process W by setting

$$W(t) = \frac{\partial \widetilde{W}}{\partial x} = \sum_{h=1}^{\infty} \beta_h e_h, \tag{1.2}$$

where $\{e_h\}$ is an orthonormal basis of $L^2(0,1)$ and $\{\beta_h\}$ is a sequence of mutually independent real Brownian motions in a fixed probability space $(\Omega, \mathcal{F}, \mathbf{P})$ adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The series (1.2) defining W does not converge in $L^2(0,1)$ but it is convergent in any Hilbert space U such that the embedding

$$L^2(0,1) \subset U$$

is Hilbert–Schmidt (see [8], Chapter 9).

In the following we shall write (1.1) as follows:

$$du(t,x) = \left(\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x}(u^2(t,x)) \right) dt + dW, \; x \in [0,1], t > 0, \tag{1.3}$$

where W is defined by (1.2).

Equation (1.3) is supplemented with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \tag{1.4}$$

and the initial condition

$$u(x,0) = u_0(x), \; x \in [0,1]. \tag{1.5}$$

Our aim in this paper is to prove that problem (1.3), with boundary and initial conditions (1.4), (1.5) has a unique global solution. To the best of our knowledge this is the first existence result for the stochastic Burgers' equation forced by a cylindrical Wiener process.

In §2, we set the notations, introduce the stochastic convolution and prove local existence in time. Then in §3 we derive an a–priori estimate which yields global existence. Finally §4 is devoted to the existence of an invariant measure. The method consists in proving that the sequence of laws $\{\mathcal{L}(u_\lambda(0))\}_{\lambda\geq 0}$, where u_λ is the solution to (1.3)–(1.4) with the initial condition at time $-\lambda$, $u(x,-\lambda) = 0$, is tight so that a suitable subsequence is weakly convergent to the required invariant measure. For this purpose, we need to derive bounds uniform with respect to time on $u(t)$ in different spaces. Classical techniques do not work here and we use an argument similar to that in [5].

2 Local existence in time

We define the unbounded self-adjoint operator A on $L^2(0,1)$ by

$$Au = \frac{\partial^2}{\partial x^2} u,$$