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Statistical mechanics of the Burgers model of turbulence

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The velocity field of the Burgers one-dimensional model of turbulence at extremely large Reynolds numbers is expressed as a train of random triangular shock waves. For describing this field statistically the distributions of the intensity and the interval of the shock fronts are defined. The equations governing the distributions are derived taking into account the laws of motion of the shock fronts, and the self-preserving solutions are obtained. The number of shock fronts is found to decrease with time t as $t^{-\alpha}$, where α $(0 \leqslant \alpha < 1)$ is the rate of collision, and consequently the mean interval increases as t^{α}. The distribution of the intensity is shown to be the exponential distribution. The distribution of the interval varies with α, but it is proved that the maximum entropy is attained by the exponential distribution which corresponds to $\alpha = \frac{1}{2}$. For $\alpha = \frac{1}{2}$, the turbulent energy is shown to decay with time as t^{-1}, in good agreement with the numerical result of Crow & Canavan (1970).

1. Introduction

Turbulence is in its broadest sense a random motion of a continuous medium. The turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$ is a random function of the co-ordinate $\mathbf{x}$ and the time t, but at the same time it must satisfy a deterministic equation of motion and, possibly, subsidiary equations. When the governing equations are nonlinear as is the case in the hydrodynamical turbulence, they impose a relationship between different degrees of freedom which compose the turbulent field. If we expand the velocity field in a three-dimensional Fourier series,

$$\mathbf{u}(\mathbf{x}, t) = \sum_{\mathbf{k}} \mathbf{v}(\mathbf{k}, t) \exp [i\mathbf{k}.\mathbf{x}],$$

assuming the spatial periodicity of the field, the relevant degrees of freedom are the Fourier components $\mathbf{v}(\mathbf{k}, t)$, and then the relationship arises between $\mathbf{v}(\mathbf{k}, t)$ and $\mathbf{v}(\mathbf{k}', t)$ for $\mathbf{k} \neq \pm \mathbf{k}'$.

In hydrodynamical turbulence the nonlinear term is proportional to the Reynolds number. Almost all analytical theories of turbulence proposed so far express the results in ascending powers of the Reynolds number, and in this sense they are all regular perturbation methods around the state of vanishing Reynolds number, where every Fourier component behaves independently of the others. So long as the Reynolds number is small the relation between different Fourier components is so weak that the interaction may adequately be dealt with by means of a regular perturbation. On the other hand, at large Reynolds numbers, with which most cases of practical importance are concerned, the nonlinear terms are dominant and strong interaction between different Fourier components prevails. In this situation any regular perturbation methods must become either invalid or very inefficient. The occurrence of the negative energy spectrum (Ogura 1962, 1963) as a consequence of the approximation of zero fourth-order cumulant (Proudman & Reid 1954; Tatsumi 1955, 1957) is a typical example. Even if a more elaborate perturbation technique could bring about physically tolerable results there is no guarantee for them to be correct at large Reynolds numbers since the terms neglected cannot be expected to have a small effect. It is well known in the case of laminar flows that the Navier–Stokes equation forms a singular perturbation problem in the limit of infinite Reynolds number and presumably the same would be the case also for turbulence.

A more effective way of dealing with this system of strong interaction or turbulence at large Reynolds numbers may be to look for a new set of degrees of freedom for which the mutual interaction is minimized at large Reynolds numbers. There exists no general rule for finding such a set of degrees of freedom, but the experience of the singular perturbation in laminar flows suggests taking up as the new degrees of freedom the asymptotic solutions of the governing equation at extremely large Reynolds numbers. In turbulence in an incompressible fluid, for instance, they would be furnished by the vortex tubes and sheets, and in the Burgers model of turbulence they may be triangular shock waves. Unlike the Fourier components, these nonlinear solutions have discontinuous structure which is characteristic of the velocity field at large Reynolds numbers, and in this sense they already embody a substantial part of the nonlinear interaction of Fourier components. Of course there still exists a residual part of the nonlinear interaction, that is the interaction between these nonlinear solutions. In dealing with this type of interaction, however, we are greatly helped by the fact that the interaction takes the form of 'collision' of discrete 'particles' and its effect can be evaluated by means of kinetic-theory methods.

In the present paper we shall deal with the Burgers model of turbulence. The reason for taking up this model first is that the mechanics of the asymptotic solutions, or the triangular shock waves, has already been investigated in great detail by Burgers (1950) and it has been shown that there exist simple laws of motion for these shock waves. A preliminary approach along these lines was undertaken by one of the authors (Tatsumi 1969), who expanded the velocity field in a series of periodic functions of saw-tooth profile. This work, however, suffered seriously from the incompleteness of the above function series, and the desired purpose does not seem to have been achieved by this expansion. In the present work no series expansion is proposed, but instead the turbulent velocity field is expressed as a random sequence of triangular shock waves. The distribution functions of the intensity and the interval of shock fronts are defined, and the equations for the distribution functions are constructed taking into account the laws of motion of shock waves. By assuming the self-preservation of solutions in time we can solve the equations and obtain the distribution functions and the change in the number of shock fronts with time. The law of decay of turbulent energy is derived from this information.†

It may be obvious that this scheme of dealing with Burgers turbulence cannot be extended straightforwardly to the real turbulence in an incompressible fluid. In incompressible turbulence the asymptotic solutions for infinite Reynolds number are not shock waves but discrete vortex tubes and sheets surrounded by an irrotational velocity field, and the laws of motion of these vortices are necessarily different from those for Burgers' shock waves. Nevertheless, the basic idea of expressing the turbulent field in terms of a set of weakly interacting degrees of freedom at large Reynolds numbers is equally applicable to the incompressible turbulence. In this sense the present work on Burgers turbulence may also be taken as a primary step towards the study of the incompressible turbulence.

Lastly it should be noted that Burgers turbulence gives a realistic model of the weakly supersonic turbulence in a compressible fluid. The Burgers equation, being an approximate equation of motion for slightly supersonic flows, describes the formation and decay of weak shock waves. Thus Burgers turbulence expresses the real supersonic turbulence composed of a random array of shock waves. In view of the existence of many important phenomena in both nature and in the laboratory closely related to supersonic turbulence, it may be needless to say that Burgers turbulence is of considerable interest in its own right.

2. Train of shock waves

The Burgers equation of motion is written in non-dimensional form as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{1}{R}\frac{\partial^2 u}{\partial x^2}, \tag{2.1}$$

where $u = u(x,t)$ denotes the velocity, x the space co-ordinate and t the time, and $R = u_0 l_0/\nu$ is the Reynolds number defined with reference to a representative velocity u_0, the length l_0 of the turbulent field, and the kinetic viscosity ν.

Equation (2.1) has the peculiar feature that negative slopes $(\partial u/\partial x < 0)$ of the solution are steepened in time until they build up into discontinuous steps while positive slopes $(\partial u/\partial x > 0)$ are all diminished to zero, so that at large Reynolds numbers all solutions eventually take the form of a train of triangular shock waves. Since we are dealing with Burgers turbulence at large Reynolds

† After this paper was submitted for publication the authors were informed by Prof. J. M. Burgers that he also had done work dealing with the same problem (Burgers 1972). In his paper Burgers attempts to derive all the information about the positions and the amplitudes of the shock waves from statistical treatment of the initial velocity fields, which are assumed to be random with respect to the space co-ordinate. In the present work, on the other hand, this information is looked for as the solutions of a truncated hierarchy of equations for the distribution functions. Thus, in spite of the similarity in the basic formulation of the problem the ways of approach in Burgers' and the present work are quite different, and it may be interesting to compare the consequences of these theories when they come out in full.

numbers we shall examine in more detail the asymptotic structure of the velocity field at large Reynolds numbers.

It was shown by Hopf (1950) and Cole (1951) that the transformation

$$u(x,t) = -\frac{2}{R}\frac{\partial}{\partial x}\log\theta(x,t), \tag{2.2}$$

or

$$\theta(x,t) = \exp\left[-\tfrac{1}{2}R\int^{x} u(x',t)\,dx'\right], \tag{2.3}$$

converts (2.1) into an equation of heat conduction:

$$\frac{\partial\theta}{\partial t} = \frac{1}{R}\frac{\partial^2\theta}{\partial x^2}. \tag{2.4}$$

The general solution of (2.4) is expressed as

$$\theta(x,t) = \left(\frac{R}{4\pi t}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\theta(x',0)\exp\left[-\frac{R}{4t}(x-x')^2\right]dx', \tag{2.5}$$

where $\theta(x,0)$ denotes the initial state. The solution (2.5) may also be written as

$$\theta(x,t) = \left(\frac{R}{4\pi t}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\exp\left[-\tfrac{1}{2}RU(x')-\frac{R}{4t}(x-x')^2\right]dx', \tag{2.6}$$

where

$$U(x,t) = \int^{x} u(x',0)\,dx'. \tag{2.7}$$

Now let us examine the integrand of (2.6). $U(x)$ takes minimum values at those zeros of $u(x,0)$ at which $du(x,0)/dx > 0$, and we denote these zeros by $x = x_j$ say, j being an integer and $x_j < x_{j+1}$. It may be a sensible restriction for the initial state that both the intervals of zeros $x_{j+1}-x_j$ and the gradients $du(x,0)/dx$ at the x_j are all finite. Then for sufficiently large values of R, the function $\exp[-\frac{1}{2}RU(x')]$ vanishes almost everywhere except for the neighbourhood of the x_j, where it has separate peaks of the width of $O[1/(RU''(x_j))^{\frac{1}{2}}]$, $U''(x)$ being $d^2U(x)/dx^2$. On the other hand, the function $\exp[-R(x-x')^2/4t]$ has a peak of the width of $O((t/R)^{\frac{1}{2}})$ around the point x. Thus, under the condition that $R \gg 1$ we can express the integral of (2.6) as

$$\theta(x,t) = \left(\frac{R}{4\pi t}\right)^{\frac{1}{2}}\sum_j\int_{x_j-\epsilon}^{x_j+\epsilon}\exp\left[-\tfrac{1}{2}RU(x')-\frac{R}{4t}(x-x')^2\right]dx',$$

where ϵ is a small but finite positive number. Further, provided that $t \gg 1$ the above integrals can be asymptotically evaluated as

$$\theta(x,t) \approx \frac{1}{t^{\frac{1}{2}}}\sum_j\frac{1}{[U''(x_j)]^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}R\left\{U(x_j)+\frac{1}{2t}(x-x_j)^2\right\}\right]. \tag{2.8}$$

Here we introduce the functions

$$A(x,x_j) = \frac{1}{R}\log U''(x_j)+U(x_j)+\frac{1}{2t}(x-x_j)^2. \tag{2.9}$$

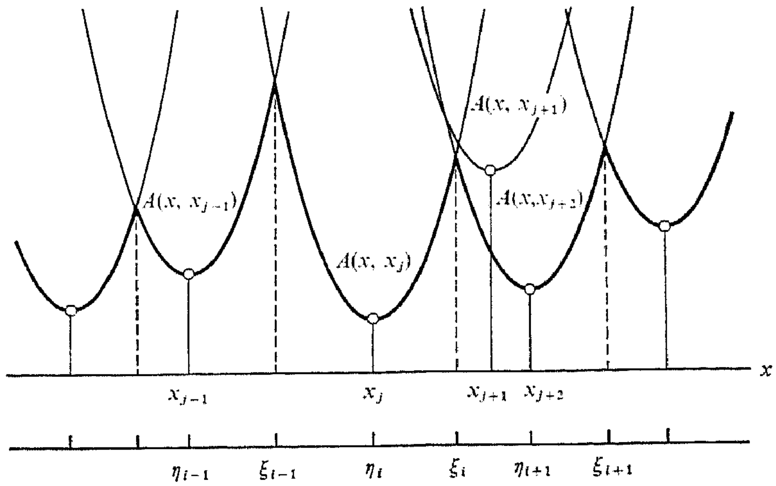

FIGURE 1. The functions $A(x, x_j)$.

As a function of x each $A(x,x_j)$ has a minimum at x_j. For a given x, on the other hand, there exists some $A(x,x_j)$, $A(x,x_l)$ say, which gives a lowest value. Then if we let x vary from $-\infty$ to ∞ we obtain a sequence of such $A(x,x_l)$, where $\{x_l\}$ forms a subset of $\{x_j\}$. Let us redefine l as l_i, where i runs over all integers, and put $x_{l_i} = \eta_i$. As stated above, some $A(x,\eta_i)$ takes the lowest value among all the $A(x,\eta_i)$ over a certain range of x, and we denote this range by $\xi_{i-1} < x < \xi_i$. At the point ξ_i obviously $A(\xi_i,\eta_i) = A(\xi_i,\eta_{i+1})$, or from (2.9),

$$\frac{1}{[U''(\eta_i)]^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}RU(\eta_i)+\frac{1}{2t}(\xi_i-\eta_i)^2\right] = \frac{1}{[U''(\eta_{i+1})]^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}RU(\eta_{i+1})+\frac{1}{2t}(\xi_i-\eta_{i+1})^2\right]. \quad (2.10)$$

Thus for extremely large R the right-hand side of (2.8) is dominated by the term with $j = l$ and we can neglect all other terms with $j \neq l$. More precisely, for the range $\frac{1}{2}(\xi_{i-1}+\xi_i) < x < \frac{1}{2}(\xi_i+\xi_{i+1})$, we have to retain only the two terms corresponding to η_i and η_{i+1}:

$$\theta(x,t) = \frac{1}{t^{\frac{1}{2}}}\left\{\frac{1}{[U''(\eta_i)]^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}R\left\{U(\eta_i)+\frac{1}{2t}(x-\eta_i)^2\right\}\right] + \frac{1}{[U''(\eta_{i+1})]^{\frac{1}{2}}}\exp\left[-\tfrac{1}{2}R\left\{U(\eta_{i+1})+\frac{1}{2t}(x-\eta_{i+1})^2\right\}\right]\right\}. \quad (2.11)$$

Substituting (2.11) into (2.2) and making use of (2.10), we obtain the following asymptotic expression of $u(x,t)$ which is valid in the region

$$\tfrac{1}{2}(\xi_{i-1}+\xi_i) < x < \tfrac{1}{2}(\xi_i+\xi_{i+1})$$

and for $R \gg 1$, $t \gg 1$:

$$u(x,t) = \frac{1}{t}[x-\tfrac{1}{2}(\eta_i+\eta_{i+1})]-\frac{1}{2t}(\eta_{i+1}-\eta_i)\tanh\left[\frac{R}{4t}(\eta_{i+1}-\eta_i)(x-\xi_i)\right]. \quad (2.12)$$

Further, in the limit of $R/t \to \infty$, the expression (2.12) reduces to

$$u(x,t) = \begin{cases} (1/t)(x-\eta_i) & \text{at} \quad \tfrac{1}{2}(\xi_{i-1}+\xi_i) < x < \xi_i, \\ (1/t)(x-\eta_{i+1}) & \text{at} \quad \xi_i < x < \tfrac{1}{2}(\xi_i+\xi_{i+1}). \end{cases} \tag{2.13}$$

The expressions (2.12) and (2.13) show that at very large Reynolds numbers and times such that $1 \ll t \ll R$ the velocity field is represented by a sequence of vertical shock fronts, each connected with a linear slope of different length but of common gradient $\beta = 1/t$. This velocity field, or the train of shock waves, is the subject of study in this paper.

3. Mechanics of shock fronts

In this section we shall consider the mechanics of the system of triangular shock waves which has been derived in §2. Let us assume for convenience that the velocity field is spatially periodic,

$$u(x,t) = u(x+L,t), \tag{3.1}$$

and restrict ourselves to the basic domain $0 < x < L$, in which N shock waves are included. As is easily seen from (2.13) the co-ordinates of the shock fronts are given by ξ_i $(i = 1,2,\ldots,N)$ and those of the intersections of the slopes with the x axis by η_i $(i = 1,2,\ldots,N)$. Then the state of the system at a given time is completely determined by specifying the set of ξ_i and η_i. Define the following variables:

$$\lambda_i = \xi_{i+1} - \xi_i, \tag{3.2}$$

$$\mu_i = \eta_{i+1} - \eta_i, \tag{3.3}$$

where λ_i represents the interval of two consecutive shock fronts located at ξ_i and ξ_{i+1}, and $\beta\mu_i$ the intensity of the shock front at ξ_i measured by the height of the step.

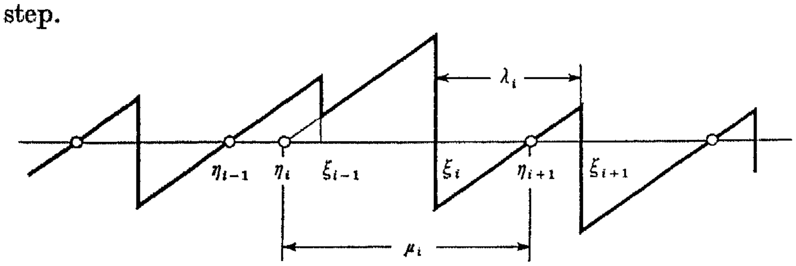

FIGURE 2. Turbulent velocity field as a train of shock waves.

The laws of motion which govern the system of shock waves at infinite Reynolds number were investigated in great detail by Burgers (1950) and they can be summarized as follows.

(i) The gradient β of the slopes decreases with time as $\beta = 1/t$.

(ii) The intersections η_i and their intervals μ_i are invariant in time,

$$d\eta_i/dt = d\mu_i/dt = 0, \tag{3.4}$$

except at the instants of collision.

(iii) The shock fronts move with constant velocity according to the equation

$$d\xi_i/dt = (1/t)\left[\xi_i - \tfrac{1}{2}(\eta_i + \eta_{i+1})\right], \tag{3.5}$$

and hence the interval of the shock fronts changes in time as

$$d\lambda_i/dt = (1/t)\left[\lambda_i - \tfrac{1}{2}(\mu_i + \mu_{i+1})\right]. \tag{3.6}$$

(iv) When two shock fronts of intensity $\beta\mu_i$ and $\beta\mu_{i+1}$ come into collision they coalesce to form a single front of intensity $\beta(\mu_i + \mu_{i+1})$.

(v) As a consequence of the above collision the intersection η_{i+1} disappears from the system and hence the number N of shock waves decreases by one.

The above laws of motion may more clearly be stated in terms of the 'particle' representation of the system. We can regard each shock front as a particle located at ξ_i, having mass μ_i, velocity $c_i = d\xi_i/dt$ and momentum $\mu_i c_i$. Laws (i), (ii) and (iii) state that the mass μ_i and the velocity c_i are invariant in time except at the instants of collision, while laws (iii), (iv) and (v) guarantee the conservation of the mass μ_i and the momentum $\mu_i c_i$ at every collision. All the collisions are entirely inelastic owing to law (iv). In the following we shall use the above terminology, which describes the shock waves as if they were cohesive particles.

4. Equations of distribution functions

Now let us consider an ensemble of the states possible for the system of N particles and define various probability distributions. Denote the probability distribution density of the mass μ by $f(\mu, t)$, and that of the interval λ by $g(\lambda, t)$. Moreover, define the joint probability distribution density of n successive intervals, $\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)}$, sandwiched alternately by $n+1$ successive masses, $\mu^{(0)}, \mu^{(1)}, \ldots, \mu^{(n)}$, by $g_n(\lambda^{(1)}, \ldots, \lambda^{(n)}; \mu^{(0)}, \ldots, \mu^{(n)}; t)$. Then it follows from the definition that

$$\int_0^\infty f(\mu, t)\, d\mu = 1, \tag{4.1}$$

$$\int_0^\infty g(\lambda, t)\, d\lambda = 1, \tag{4.2}$$

$$\int_0^\infty d\mu \int_0^\infty g_1(\lambda; \mu, \mu'; t)\, d\mu' = g(\lambda, t). \tag{4.3}$$

We shall derive the equations which govern these distribution functions from the mechanics of the particle system. First, consider the change in time of the number $N(t) f(\mu, t)\, \delta\mu$ of masses which are included in the range $(\mu, \mu + \delta\mu)$. The number of pairs of masses μ' and μ'' which are about to collide with each other at time t is $N(t)\, g_1(0; \mu', \mu''; t)\, \delta\mu'\, \delta\mu''$ and the relative speed with which they approach is $(\mu' + \mu'')/2t$, so that the number of such collisions per unit time is given by $N(t)\, g_1(0; \mu', \mu''; t)\, \delta\mu'\, \delta\mu''(\mu' + \mu'')/2t$. Since the net change of the number of the mass μ is equal to the difference between the production and the destruction of the mass μ due to collision, we have the following equation:

$$\begin{aligned}\frac{d}{dt}[N(t) f(\mu, t)\, \delta\mu] = N(t)\, \delta\mu \Bigg[&\int_0^\mu g_1(0; \mu', \mu - \mu'; t) \frac{\mu}{2t}\, d\mu' \\ &- \int_0^\infty \{g_1(0; \mu, \mu'; t) + g_1(0; \mu', \mu; t)\} \frac{\mu + \mu'}{2t}\, d\mu' \Bigg],\end{aligned} \tag{4.4}$$

where d/dt denotes total differentiation. The first term of the right-hand side of (4.4) represents the production of mass μ and the second term the destruction due to collision.

Since $d\mu/dt = 0$ according to (3.4), integration of (4.4) with respect to μ leads to the following equation for $N(t)$:

$$\frac{d}{dt}N(t) = -\frac{N(t)}{2t}\int_0^\infty d\mu \int_0^\infty (\mu+\mu')\, g_1(0;\mu,\mu';t)\, d\mu'. \tag{4.5}$$

On substitution of (4.5) into (4.4) we obtain the following equation for $f(\mu,t)$:

$$\begin{aligned}\frac{d}{dt}f(\mu,t) &= \frac{1}{2t}f(\mu,t)\int_0^\infty d\mu' \int_0^\infty (\mu'+\mu'')\, g_1(0;\mu',\mu'';t)\, d\mu'' \\ &\quad +\frac{\mu}{2t}\int_0^\mu g_1(0;\mu',\mu-\mu';t)\, d\mu' - \frac{1}{2t}\int_0^\infty (\mu+\mu')\,[g_1(0;\mu,\mu';t)+g_1(0;\mu',\mu;t)]\, d\mu'.\end{aligned} \tag{4.6}$$

In a similar manner we can derive the equation for $g_n(\lambda^{(1)},\ldots,\lambda^{(n)};\mu^{(0)},\ldots,\mu^{(n)};t)$ as follows:

$$\begin{aligned}&d[N(t)\, g_n(\lambda^{(1)},\ldots,\lambda^{(n)};\mu^{(0)},\ldots,\mu^{(n)};t)\,\delta\lambda^{(1)}\ldots\delta\lambda^{(n)}]/dt \\ &= N(t)\,\delta\lambda^{(1)}\ldots\delta\lambda^{(n)}\Bigg[\frac{\mu^{(0)}}{2t}\int_0^{\mu^{(0)}} g_{n+1}(0,\lambda^{(1)},\ldots,\lambda^{(n)};\mu,\mu^{(0)}-\mu,\ldots,\mu^{(n)};t)\, d\mu + \frac{1}{2t}\sum_{i=1}^{n-1}\mu^{(i)} \\ &\quad \times \int_0^{\mu^{(i)}} g_{n+1}(\lambda^{(1)},\ldots,\lambda^{(i)},0,\lambda^{(i+1)},\ldots,\lambda^{(n)};\mu^{(0)},\ldots,\mu^{(i-1)},\mu,\mu^{(i)}-\mu,\mu^{(i+1)},\ldots,\mu^{(n)};t)\, d\mu \\ &\quad +\frac{\mu^{(n)}}{2t}\int_0^{\mu^{(n)}} g_{n+1}(\lambda^{(1)},\ldots,\lambda^{(n)},0;\mu^{(0)},\ldots,\mu,\mu^{(n)}-\mu;t)\, d\mu \\ &\quad -\frac{1}{2t}\int_0^\infty (\mu+\mu^{(0)})\, g_{n+1}(0,\lambda^{(1)},\ldots,\lambda^{(n)};\mu,\mu^{(0)},\ldots,\mu^{(n)};t)\, d\mu \\ &\quad -\frac{1}{2t}\int_0^\infty (\mu^{(n)}+\mu)\, g_{n+1}(\lambda^{(1)},\ldots,\lambda^{(n)},0;\mu^{(0)},\ldots,\mu^{(n)},\mu;t)\, d\mu\Bigg].\end{aligned} \tag{4.7}$$

If we take the lowest order equation of (4.7) and integrate it with respect to $\mu^{(0)}$ and $\mu^{(1)}$ we obtain

$$\frac{d}{dt}\left[N(t)\,\delta\lambda\int_0^\infty d\mu\int_0^\infty g_1(\lambda;\mu,\mu';t)\, d\mu'\right] = 0, \tag{4.8}$$

or, taking into account (4.3),

$$d[N(t)\, g(\lambda,t)\,\delta\lambda]/dt = 0. \tag{4.9}$$

Equation (4.9) shows that the number $N(t)\, g(\lambda,t)\,\delta\lambda$ of intervals included in the range $(\lambda,\lambda+\delta\lambda)$ does not change on collision and is conserved in time. Since $d\delta\lambda/dt = \delta\lambda/t$ according to (3.6), equation (4.8) may be written as

$$\begin{aligned}&\int_0^\infty d\mu\int_0^\infty \Bigg[\frac{\partial}{\partial t}+\frac{1}{t}\left(\lambda-\frac{\mu+\mu'}{2}\right)\frac{\partial}{\partial\lambda}+\frac{1}{t} \\ &\quad -\frac{1}{2t}\int_0^\infty d\mu''\int_0^\infty (\mu''+\mu''')\, g_1(0;\mu'',\mu''';t)\, d\mu'''\Bigg] g_1(\lambda;\mu,\mu';t)\, d\mu' = 0,\end{aligned} \tag{4.10}$$

where use has been made of (4.5).

If we proceed to higher order equations of (4.7), we shall obtain a hierarchy of equations for g_n ($n \geqslant 2$), which, however, always contains one more unknown than equation. This unclosedness of the hierarchy is due to the effect of particle collisions and hence is nothing but a reflexion of the nonlinearity of the turbulent field.

In dealing with the infinite sequence of equations (4.7) we shall retain only the first equation, or, more precisely, its integrated form (4.10). This is a truncation of the infinite set of equations (4.7) and may remind us of the similar truncation processes in the conventional perturbation expansion around the non-interacting state. There are, however, essential differences between the present and conventional truncation processes. First of all, as was mentioned in § 1, the present truncation is concerned only with the residual part of the nonlinear interaction, whereas the conventional truncation is applied to the whole nonlinear interaction. The second difference lies in the nature of the present truncation, where the equation for $g_1(\lambda;\mu,\mu';t)$ is replaced by an integrated equation (4.10). As will be shown below, (4.10) allows a group of self-preserving solutions. Since the exact solution must be included in all possible solutions of (4.10) it is quite probable that the exact solution is either akin to or identical with one of these self-preserving solutions. Lastly, it should be emphasized that even a truncated form of (4.7) gives some probability distributions as solutions, and therefore there is no danger of its leading to unphysical consequences, which sometimes occur in the conventional perturbation processes.

5. Self-preserving solutions

Now we proceed to solving the set of equations (4.5), (4.6), (4.3) and (4.10) for $N(t), f(\mu,t), g(\lambda,t)$ and $g_1(\lambda;\mu,\mu';t)$ respectively. We look for solutions within the self-preserving distributions which change in time only through the change of a characteristic length, $l(t)$ say. The self-preserving forms of $f(\mu,t)$, $g(\lambda,t)$ and $g_1(\lambda;\mu,\mu';t)$ may be expressed as follows:

$$f(\mu,t) = \frac{1}{l(t)} f\left(\frac{\mu}{l(t)}\right), \tag{5.1}$$

$$g(\lambda,t) = \frac{1}{l(t)} g\left(\frac{\lambda}{l(t)}\right), \tag{5.2}$$

$$g_1(\lambda;\mu,\mu';t) = \frac{1}{l(t)^3} g_1\left(\frac{\lambda}{l(t)}, \frac{\mu}{l(t)}, \frac{\mu'}{l(t)}\right). \tag{5.3}$$

It immediately follows from (5.1) and (5.2) that

$$\left.\begin{aligned} \langle\mu\rangle &= \int_0^\infty \mu f(\mu,t)\,d\mu = l(t), \\ \langle\lambda\rangle &= \int_0^\infty \lambda g(\lambda,t)\,d\lambda = l(t), \end{aligned}\right\} \tag{5.4}$$

where $\langle\ \rangle$ denotes the probability mean. Thus $l(t)$ is equivalent to the mean mass

$\langle\mu\rangle$ and also the mean interval $\langle\lambda\rangle$. It also follows from (5.3) that

$$\int_0^\infty d\mu \int_0^\infty (\mu+\mu')\, g_1(0;\mu,\mu';t)\, d\mu' = \text{constant in time} \\ = 2\alpha, \text{ say.} \tag{5.5}$$

As we have seen in the argument which led to (4.4), α/t is equivalent to the rate of collision per unit time.

On substitution from (5.5), equation (4.5) is reduced to

$$dN(t)/dt = -(\alpha/t)\, N(t), \tag{5.6}$$

whose solution is given by

$$N(t) = N_0 (t_0/t)^\alpha, \tag{5.7}$$

where N_0 stands for $N(t_0)$ at an initial time t_0.

Integrating (4.6) multiplied by μ with respect to μ gives

$$dl(t)/dt = (\alpha/t)\, l(t), \tag{5.8}$$

which has the solution

$$l(t) = l_0 (t/t_0)^\alpha, \tag{5.9}$$

where $l_0 = l(t_0)$. Obviously (5.7) and (5.9) satisfy the consistency condition

$$N(t)\, l(t) = N_0 l_0 = L, \tag{5.10}$$

showing that the number of particles N decreases monotonically in time while the mean interval l increases monotonically keeping $N(t)\, l(t)$ invariant.

So far we have introduced no assumption except for restricting ourselves to the self-preserving solutions, but in order to proceed further we have to introduce some assumption concerning the joint distribution $g_1(\lambda;\mu,\mu';t)$ since (4.10) governs g_1 only in the mean. First we assume that the distributions of the masses μ and μ' are independent of each other. Then g_1 may be expressed as

$$g_1(\lambda;\mu,\mu';t) = f(\mu,t) f(\mu',t)\, h_1(\lambda|\mu,\mu';t), \tag{5.11}$$

where $h_1(\lambda|\mu,\mu';t)$ represents the conditional probability density of λ for a given pair of μ and μ'. Next we shall assume that the turbulent velocity fields have a statistically similar structure so that the conditional distribution of λ for given μ and μ' depends only upon ratios λ/μ and λ/μ'. Since λ changes in time according to (3.6) the above assumption requires that h_1 be a function of $\lambda/(\mu+\mu')$ alone. Thus, we let

$$h_1(\lambda|\mu,\mu';t) = \frac{2\alpha}{\mu+\mu'}\, h\left(\frac{\lambda}{\mu+\mu'}\right). \tag{5.12}$$

It follows from the definition that

$$1 = \int_0^\infty h_1(\lambda|\mu,\mu';t)\, d\lambda = 2\alpha \int_0^\infty h(\lambda)\, d\lambda, \tag{5.13}$$

and from (5.5) and (5.11),

$$2\alpha = \int_0^\infty d\mu \int_0^\infty (\mu+\mu') f(\mu,t) f(\mu',t)\, h_1(0|\mu,\mu';t)\, d\mu' \\ = 2\alpha h(0) \int_0^\infty f(\mu,t)\, d\mu \int_0^\infty f(\mu',t)\, d\mu',$$

so that, in view of (4.1),

$$h(0) = 1. \tag{5.14}$$

On substitution from (5.11), (5.12) and (5.14), equation (4.6) becomes an equation for $f(\mu, t)$:

$$\frac{d}{dt} f(\mu, t) = -\frac{\alpha}{t} f(\mu, t) + \frac{\alpha}{t} \int_0^\mu f(\mu', t) f(\mu - \mu', t)\, d\mu'. \tag{5.15}$$

Under the restriction to a similar solution of the type (5.1), equation (5.15) is reduced to

$$-z \frac{df(z)}{dz} = \int_0^z f(z') f(z - z')\, dz', \tag{5.16}$$

where $z = \mu / l(t)$.

Equation (5.16) is solved by introducing the Laplace transform of $f(z)$,

$$F(p) = \int_0^\infty f(z)\, e^{-pz}\, dz. \tag{5.17}$$

Then the following equation for $F(p)$ is deduced from (5.16):

$$d[pF(p)]/dp = F(p)^2, \tag{5.18}$$

whose general solution is given by

$$F(p) = 1/(1 + Cp), \tag{5.19}$$

where C is an integration constant. Since, from (5.4) and (5.1),

$$\begin{aligned} 1 &= \frac{1}{l(t)} \int_0^\infty \mu f(\mu, t)\, d\mu = \int_0^\infty z f(z)\, dz \\ &= -\frac{\partial}{\partial p} F(p)\big|_{p=0} = C, \end{aligned} \tag{5.20}$$

(5.19) becomes

$$F(p) = 1/(1 + p). \tag{5.21}$$

The inverse Laplace transformation of (5.21) gives

$$f(z) = e^{-z}. \tag{5.22}$$

Then on substituting (5.22) into (5.1) we obtain the following distribution for the mass μ:

$$f(\mu, t) = \frac{1}{l(t)} \exp\left[-\frac{\mu}{l(t)}\right]. \tag{5.23}$$

The solution (5.23) does not involve the collision rate and represents the exponential distribution, which is equivalent to the limiting distribution of the intervals between the nearest neighbours of N particles distributed uniformly and independently of each other on a line of the length L as $L \to \infty$, $N \to \infty$, with $L/N = 1$.

Next we shall obtain $g(\lambda, t)$ by solving (4.10), which is reduced on substitution from (5.23) to

$$\begin{aligned} \int_0^\infty d\mu \int_0^\infty &\left[\frac{\partial}{\partial t} + \frac{1}{t}\left(\lambda - \frac{\mu + \mu'}{2}\right) \frac{\partial}{\partial \lambda} + \frac{1}{t}(1 - \alpha)\right] \\ &\times \frac{1}{l(t)^2} \exp\left[-\frac{\mu + \mu'}{l(t)}\right] \frac{2\alpha}{\mu + \mu'} h\left(\frac{\lambda}{\mu + \mu'}\right) d\mu' = 0. \end{aligned} \tag{5.24}$$

Through a transformation of variables (5.24) becomes

$$\int_0^\infty e^{-\lambda/z}\left[(1-2\alpha)h(z)+\{(1-\alpha)z-\tfrac{1}{2}\}\frac{dh(z)}{dz}\right]\frac{dz}{z^2}=0.$$

In order that this equation is satisfied for an arbitrary λ the integrand must vanish identically:

$$(1-2\alpha)h(z)+\{(1-\alpha)z-\tfrac{1}{2}\}dh(z)/dz = 0. \tag{5.25}$$

The solution of (5.25) which satisfies the condition (5.14) is readily expressed as follows.

When $0 \leqslant \alpha < 1$,

$$h(z) = \begin{cases} \{1-2(1-\alpha)z\}^{(2\alpha-1)/(1-\alpha)} & \text{for} \quad z < 1/2(1-\alpha), \\ 0 & \text{for} \quad z > 1/2(1-\alpha). \end{cases} \tag{5.26}$$

When $\alpha = 1$,

$$h(z) = e^{-2z}. \tag{5.27}$$

When $\alpha > 1$,

$$h(z) = \{1-2(1-\alpha)z\}^{(2\alpha-1)/(1-\alpha)}. \tag{5.28}$$

Using (5.11), (5.12) and (5.23) the joint distribution $g_1(\lambda;\mu,\mu';t)$ is expressed in terms of the function $h(z)$ as follows:

$$g_1(\lambda;\mu,\mu';t) = \frac{2\alpha}{l(t)^2(\mu+\mu')}\exp\left[-\frac{\mu+\mu'}{l(t)}\right]h\left(\frac{\lambda}{\mu+\mu'}\right). \tag{5.29}$$

Then substitution of (5.29) into (4.3) leads to the distribution $g(\lambda,t)$:

$$g(\lambda,t) = \frac{2\alpha}{l(t)^2}\int_0^\infty \exp\left[-\frac{z}{l(t)}\right]h\left(\frac{\lambda}{z}\right)dz. \tag{5.30}$$

Unlike the distribution (5.23) for $f(\mu,t)$, the distribution (5.30) changes with the collision rate α. The distribution curves for various α are calculated by substituting (5.26)–(5.28) into (5.30) and are shown graphically in figure 3.

Now we shall determine the value of α. It is easily proved by integrating (2.1) that the integral scale of the turbulent velocity field defined by

$$J = \int_0^\infty \langle u(x,t)\,u(x+r,t)\rangle\,dr \tag{5.31}$$

is invariant in time (see Burgers 1950, p. 249; Saffman 1968, p. 542):

$$dJ/dt = 0. \tag{5.32}$$

On the other hand, the dimensional analysis and the assumption of self-preservation made in this section lead to the relation

$$J \propto l(t)^3 t^{-2} \propto t^{3\alpha-2}, \tag{5.33}$$

where use has been made of (5.9). Thus, in order that (5.32) and (5.33) be compatible we must have $\alpha = \frac{2}{3}$. In this case (5.30) becomes

$$g(\lambda,t) = \frac{4}{3l(t)}\left\{\exp\left[-\frac{2\lambda}{3l(t)}\right]+\frac{2\lambda}{3l(t)}Ei\left[-\frac{2\lambda}{3l(t)}\right]\right\}, \tag{5.34}$$

where

$$Ei(-z) = -\int_z^\infty \frac{e^{-t}}{t}\,dt.$$

The above argument, however, does not apply to the case of $J = 0$. If the turbulent velocity field $u(x,t)$ is taken as a stationary random process in which $u(x,t)$ and $u(x',t)$ for $x \neq x'$ are independent of each other, the non-zero value of J is an unavoidable consequence. However, in real turbulence, where the values of $u(x,t)$ for different x's are related to each other, the vanishing value of J is quite possible and in fact a limiting process leads to $J = 0$. In this case α is not determined by the dimensional analysis and we have to introduce a new principle for this purpose.

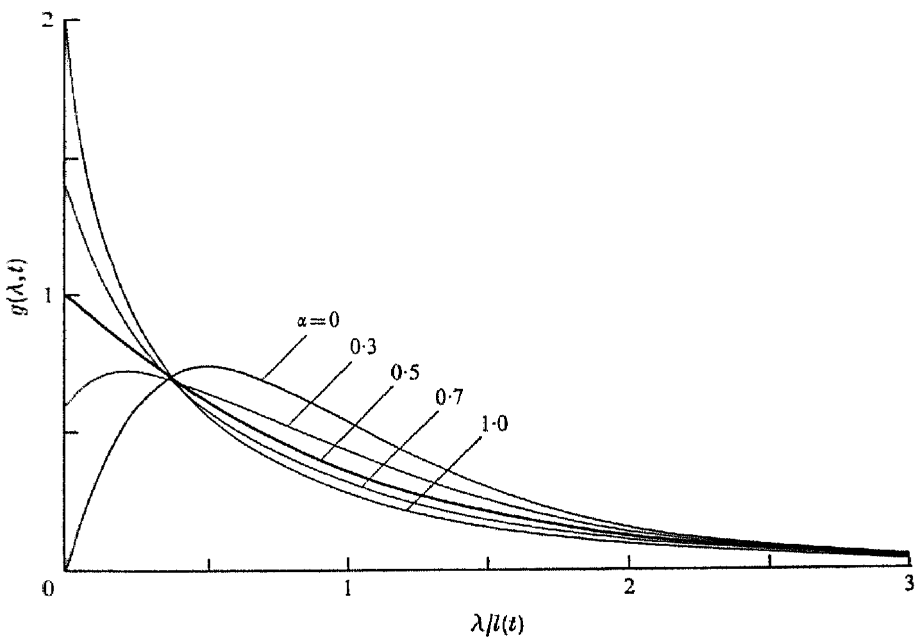

FIGURE 3. Distributions of the intervals.

Let us define the entropy of the distribution $g(\lambda,t)$ by

$$S = -\int_0^\infty g(\lambda,t)\log g(\lambda,t)\,d\lambda. \tag{5.35}$$

The entropy S differs for different distributions of $g(\lambda,t)$, and at any time the distribution which is realized in nature would be the one associated with the maximum entropy. Such a distribution is easily found by making use of Lagrange's method of undetermined multipliers under the conditions (4.2) and (5.4) and is shown to be

$$g(\lambda,t) = \frac{1}{l(t)}\exp\left[-\frac{\lambda}{l(t)}\right], \tag{5.36}$$

which is the exponential distribution of the same form as (5.23). The distribution (5.36) is nothing but a special case of (5.30) for $\alpha = \frac{1}{2}$, and therefore $\alpha = \frac{1}{2}$ may be said to correspond to the state of maximum randomness which would be realized if no subsidiary condition were imposed. This may also be confirmed by figure 4, which shows that the value of S calculated numerically for the distributions (5.30) actually takes a maximum at $\alpha = \frac{1}{2}$.

It should be noted that the exponential distribution (5.36) possesses the maximum entropy among not only the distributions of the type of (5.30) but all possible distributions. Thus we shall take the collision rate $\alpha = \frac{1}{2}$ and the exponential distribution (5.36) for $g(\lambda, t)$ as corresponding to the state which is most likely to occur in the real process of turbulence.

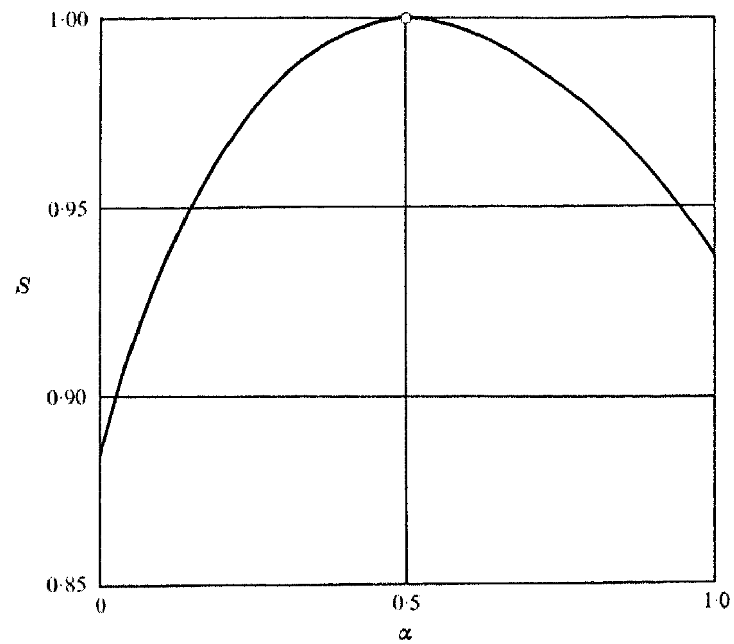

FIGURE 4. Entropy of the distribution of the interval.

6. Decay of turbulent energy

Now, having obtained the distribution function $f(\mu, t)$, we can derive the law of decay of turbulent energy per unit length defined by

$$\mathscr{E}(t) = \lim_{L\to\infty} \frac{1}{2L} \int_0^{\infty} u(x,t)^2\, dx. \tag{6.1}$$

The equation for $\mathscr{E}(t)$ is derived by integrating (2.1) multiplied by $u(x,t)$ and taking into account the periodicity condition (3.1):

$$\begin{aligned} \frac{d\mathscr{E}(t)}{dt} &= \lim_{L\to\infty} \frac{1}{2L} \frac{d}{dt} \int_0^{L} u(x,t)^2\, dx \\ &= -\lim_{L\to\infty} \frac{1}{RL} \int_0^{L} \left(\frac{\partial u}{\partial x}\right)^2 dx. \end{aligned} \tag{6.2}$$

Investigation of the asymptotic expression (2.12) for $u(x,t)$ for $1 \ll t$, $1 \ll R$ shows that the dominant contribution to $\partial u/\partial x$ comes from the shock fronts, contributions from other regions being of minor order, $O(1/R)$, where $R = u_0 l_0/\nu$ and $u_0 = l_0/t_0$. Thus,

$$\frac{\partial u}{\partial x} \approx -\frac{R\mu_i^2}{8t^2}\cosh^{-2}\left[\frac{R\mu_i}{4t}(x-\xi_i)\right] \quad \text{at} \quad \xi_i - \epsilon < x < \xi_i + \epsilon, \tag{6.3}$$

where ϵ is a small but finite positive number. Substitution of the expression (6.3) into the right-hand side of (6.2) leads to

$$\begin{aligned}
\frac{1}{RL}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx &= \frac{1}{RL}\sum_{i=1}^{N}\int_{\xi_i-\epsilon}^{\xi_i+\epsilon}\left(\frac{\partial u}{\partial x}\right)^2 dx \\
&= \frac{1}{RL}\sum_{i=1}^{N}\frac{R^2\mu_i^4}{8^2t^4}\int_{\xi_i-\epsilon}^{\xi_i+\epsilon}\cosh^{-4}\left[\frac{R\mu_i}{4t}(x-\xi_i)\right]dx \\
&= \frac{1}{16Lt^3}\int_{-\infty}^{\infty}\cosh^{-4}s\,ds\sum_{i=1}^{N}\mu_i^3 \\
&= \frac{1}{12Lt^3}\sum_{i=1}^{N}\mu_i^3.
\end{aligned}$$

In the limit of $L \to \infty$, $N \to \infty$, the law of large numbers guarantees the equivalency of the arithmetic mean to the probability mean, and hence for $1 \ll t \ll R$,

$$\begin{aligned}
\frac{1}{N}\sum_{i=1}^{N}\mu_i^3 &= \langle\mu^3\rangle \\
&= 6l(t)^3,
\end{aligned}$$

where use has been made of the distribution (5.23).

Thus, (6.2) becomes

$$\begin{aligned}
\frac{d\mathscr{E}(t)}{dt} &= -\frac{l(t)^2}{2t^3} \\
&= -\frac{l_0^2}{2t_0^2}t^{2\alpha-3},
\end{aligned} \tag{6.4}$$

where (5.9) has been taken into account. Integration of (6.4) immediately gives the following law of energy decay:

$$\mathscr{E}(t) = \frac{l_0^2}{4(1-\alpha)\,t_0^2}t^{2(\alpha-1)}. \tag{6.5}$$

In order that this result be physically meaningful ($\mathscr{E} > 0$) the value of α must be in the range $0 \leqslant \alpha < 1$.

It has been shown in §5 that $\alpha = \frac{2}{3}$ is a unique value for the case of $J \neq 0$, and $\alpha = \frac{1}{2}$ corresponds to the decay process most likely to occur in the case of $J = 0$. For $\alpha = \frac{2}{3}$, (6.5) becomes

$$\mathscr{E}(t) = \frac{3l_0^2}{4t_0^2}t^{-\frac{2}{3}}, \tag{6.6}$$

and for $\alpha = \frac{1}{2}$,

$$\mathscr{E}(t) = \frac{l_0^2}{2t_0^2}t^{-1}, \tag{6.7}$$

which gives an inverse power law of energy decay. A numerical calculation of the energy of Burgers turbulence was carried out by Crow & Canavan (1970) for various values of the Reynolds number and the limiting law of energy decay

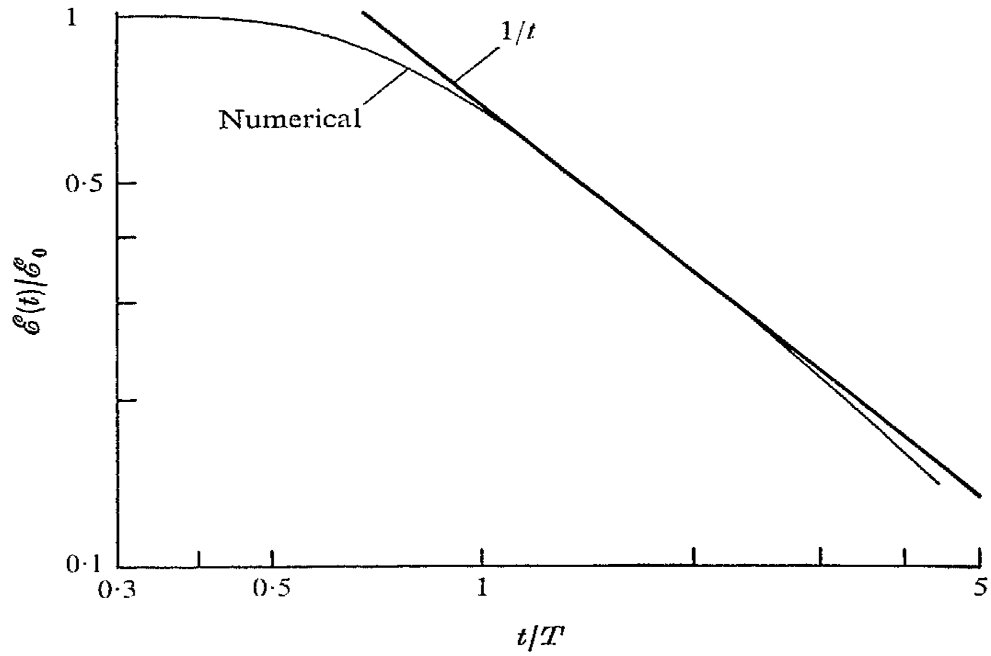

FIGURE 5. Decay of turbulent energy.

for infinite Reynolds number was derived through extrapolation. The curve of energy decay thus obtained is depicted in figure 5 together with the inverse power law due to (6.7). The agreement of the law (6.7) with the numerical curve is striking in the range of time $1 < t/T < 3$, where $T = 1/(k_0\langle u^2\rangle_0^{\frac{1}{2}})$, $\langle u^2\rangle_0$ and k_0 being the initial energy and the wavenumber at which the initial spectrum has the maximum respectively. This agreement seems to provide an actual basis for the choice $\alpha = \frac{1}{2}$. The discrepancy of the curves before this time range, $t/T < 1$, should be attributed to the fact that triangular shock waves have not yet been fully developed in the numerical experiment, whereas the deviation after this period, $t/T > 3$, is due to the fact that the condition $1 \ll t \ll R$ is no longer fulfilled in the numerical calculation.

Lastly it should be noted that in the present approach the turbulent energy has been derived directly from the distribution of the lowest order without resort to any other statistical averages. This situation is essentially different from that in the conventional theories of turbulence dealing with the mean velocity products or their Fourier transforms, where the energy had to be derived as the limit of the velocity correlation $\langle u(x,t)\,u(x+r,t)\rangle$ as $r \to 0$ or equivalently as an integral of the energy spectrum. However, this by no means implies that the velocity correlation, the energy spectrum and other statistical averages are of no interest in the present approach. Indeed a theory of turbulence will not be satisfactory without information of these statistical quantities. To work out these quantities requires solving the equations of the joint distributions of higher order, g_n ($n \geqslant 2$), and this will be dealt with in a separate paper.

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