SIAM J. NUMER. ANAL.
Vol. 14, No. 2, April 1977

SOME CONVERGENCE ESTIMATES FOR SEMIDISCRETE GALERKIN TYPE APPROXIMATIONS FOR PARABOLIC EQUATIONS*

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Abstract. In this paper we derive convergence estimates for certain semidiscrete methods used in the approximation of solutions of initial boundary value problems with homogeneous Dirichlet boundary conditions for parabolic equations. These methods contain the ordinary Galerkin method based on approximating subspaces with functions vanishing on the boundary of the basic domain, and also some methods without such restrictions. The results include L_2 estimates, maximum norm estimates, interior estimates for difference quotients and superconvergence estimates. Some proofs depend on known results for the associated elliptic problem. Several of these estimates are derived for positive time under weak assumptions on the initial data.

1. Introduction. We shall consider the initial boundary value problem

$$
\begin{aligned}
&D_t u = -Lu = \sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a_{jk}\frac{\partial u}{\partial x_k}\right) - a_0 u \quad \text{in } \Omega \times (0, T_0],\\
(1.1)\qquad &u(x,t)=0 \quad \text{on } \partial\Omega \times (0, T_0],\\
&u(x,0) = v(x).
\end{aligned}
$$

Here Ω is a bounded domain in R^N with sufficiently smooth boundary $\partial\Omega$, a_{jk} and a_0 are sufficiently smooth functions which are independent of t, the matrix (a_{jk}) is symmetric and uniformly positive definite and a_0 is nonnegative on $\bar{\Omega}$.

The exact solution of (1.1) can be represented as

$$
(1.2)\qquad u(x,t) = \sum_{j=1}^{\infty} \beta_j e^{-\lambda_j t}\varphi_j(x), \quad \text{with } \beta_j = (v, \varphi_j)
$$

where $(\cdot\,,\cdot)$ denotes the inner product in $L_2 = L_2(\Omega)$ and where $\{\lambda_j\}_1^\infty$ and $\{\varphi_j\}_1^\infty$ are the eigenvalues (in nondecreasing order) and (L_2 orthonormal) eigenfunctions of the associated elliptic problem

$$
Lw = \lambda w \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.
$$

The eigenvalues are positive and tend to infinity with j.

For $s \geqq 0$ we let $\dot{H}^s = \dot{H}^s(\Omega)$ be the space of w in L_2 for which

$$
\|w\|_s = \left(\sum_{j=1}^{\infty} \lambda_j^s \beta_j^2\right)^{1/2} < \infty, \qquad \beta_j = (w, \varphi_j).
$$

For $s = 0$ we shall often write $\|\cdot\| = \|\cdot\|_0$. It can be shown ([7, Lemma 2.2]) that for s a nonnegative integer, $\dot{H}^s$ consists of the functions w in $H^s = W_2^s(\Omega)$ which satisfy the boundary conditions $L^j w = 0$ on $\partial\Omega$ for $j < s/2$, and that the $\|\cdot\|_s$ norm is equivalent to the usual norm in H^s. In particular, $\dot{H}^\infty = \bigcap_{s>0} \dot{H}^s$ consists of all $w \in C^\infty(\bar{\Omega})$ for which $L^j w = 0$ on $\partial\Omega$ for all $j \geqq 0$.

* Received by the editors December 18, 1975, and in revised form May 4, 1976.
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For $s>0$ we also define $\dot{H}^{-s}$ to be the dual of $\dot{H}^{s}$ with respect to the inner product in L_2; hence for $w \in L_2$,

$$\|w\|_{-s} = \left(\sum_{j=1}^{\infty} \lambda_j^{-s} \beta_j^2 \right)^{1/2}, \qquad \beta_j = (w, \varphi_j).$$

Note that since $\varphi_j \in \dot{H}^{\infty}$ the solution formula (1.2) is also meaningful for $v \in \dot{H}^{-s}$, $s>0$.

As a consequence of the presence of the exponential factors in (1.2) it follows that the solution of (1.1) is in $\dot{H}^{\infty}$ for positive t, even for example with v only in L_2. From the definition of the $\dot{H}^s$ norms, we have for $p \geqq q$,

$$\|u(t)\|_p \leqq Ct^{-(p-q)/2} \|v\|_q. \tag{1.3}$$

For A a linear operator from $\dot{H}^q$ to $\dot{H}^p$ we shall sometimes use the notation

$$\|A\|_{p,q} = \sup_{\substack{w \in \dot{H}^q \\ w \neq 0}} \frac{\|Aw\|_p}{\|w\|_q}.$$

Letting $E(t)$ denote the solution operator of (1.1), so that $u(t) = E(t)v$, we may then express (1.3) in the form

$$\|E(t)\|_{p,q} \leqq Ct^{-(p-q)/2} \quad \text{for } p \geqq q. \tag{1.4}$$

We now introduce the solution operator T of the elliptic problem

$$Lw = f \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega,$$

as $w = Tf$. This operator can be represented by its eigenfunction expansion

$$Tf = \sum_j u_j \beta_j \varphi_j \quad \text{where } \beta_j = (f, \varphi_j), \quad u_j = \lambda_j^{-1},$$

with φ_j and λ_j as above, and it follows at once that T is a bounded operator from $\dot{H}^s$ into $\dot{H}^{s+2}$ for s nonnegative. In terms of T we may write the parabolic problem as

$$D_t T u + u = 0, \qquad u(0) = v. \tag{1.5}$$

For the purpose of approximating the solution of this problem, let $\{S_h\}$ (h small and positive) be a family of finite dimensional subspaces of L_2. We shall assume that we are given a corresponding family of operators $T_h: L_2 \to S_h$ which approximate T, and then consider the semidiscrete problem to find $u_h(t) \in S_h$ for $t \geqq 0$ such that

$$D_t T_h u_h(t) + u_h(t) = 0, \qquad u_h(0) = v_h \in S_h, \tag{1.6}$$

where v_h is a suitable approximation to v.

We shall make the following assumptions about the family $\{T_h\}$:

(i) T_h is selfadjoint, positive semidefinite on L_2 and positive definite on S_h.

(iia) There is a positive integer $r \geqq 2$ and a constant C such that

$$\|T_h - T\|_{0,q} \leqq Ch^{q+2} \quad \text{for } 0 \leqq q \leqq r-2.$$

In certain cases it will be necessary to replace the L_2 error bound in (iia) by error bounds in negative norms:

(iib) There is a positive integer $r>2$ and a constant C such that

$$\|T_h - T\|_{-p,q} \leqq Ch^{p+q+2} \quad \text{for } 0 \leqq p, q \leqq r-2.$$

By subtracting (1.5) from (1.6) we obtain for the error $e_h = e_h(t) = u_h(t) - u(t)$ the equation

$$D_t T_h e_h + e_h = \rho \equiv (T - T_h) D_t u, \tag{1.7}$$

which will be the starting point of some proofs given below.

An example of a family of operators satisfying such assumptions is given by the following: Suppose that $S_h \subset \dot{H}^1$ so that in particular the elements of S_h vanish on $\partial\Omega$. Assume further that S_h has the approximation property

$$\inf_{\chi \in S_h} \{\|w - \chi\| + h\|w - \chi\|_1\} \leqq Ch^s \|w\|_s, \quad \text{for } 1 \leqq s \leqq r. \tag{1.8}$$

Introducing the bilinear form

$$A(\varphi, \psi) = \int_\Omega \left(\sum_{j,k} a_{jk} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_k} + a_0 \varphi \psi \right) dx,$$

we may define $w_h = T_h f$ by

$$A(w_h, \chi) = (f, \chi), \quad \text{for } \chi \in S_h.$$

Then $\{T_h\}$ satisfies the above assumptions. The semidiscrete differential equation can now be written in the form

$$(D_t u_h(t), \chi) + A(u_h(t), \chi) = 0, \quad \text{for } \chi \in S_h, \quad t > 0. \tag{1.9}$$

More details about this method and other examples where the functions in S_h are not required to vanish on $\partial\Omega$ are given in § 8.

Instead of Dirichlet boundary conditions on $\partial\Omega \times (0, T_0]$ in (1.1) we could have considered, for instance, homogeneous Neumann type boundary conditions. Assuming in such a case that a_0 is positive on $\bar{\Omega}$, the eigenvalues of the corresponding elliptic problem are again positive so that the spaces $\dot{H}^s$ may be defined in the analogous way. The smoothing property (1.3) still holds and we may again introduce T and T_h and consider the semidiscrete approximation (1.6) of (1.5). The analysis below also covers this case.

In addition to the norms $\|\cdot\|_s$ defined above we shall use the norms in several other Banach spaces of functions on Ω and subsets Ω_1 of $\bar{\Omega}$. The function space will then in general appear as a subscript, with the convention that if the basic domain of the space is all of Ω, then it is omitted. For the norm in $W_\infty^s(\Omega_1)$ we simply write $|\cdot|_{s,\Omega_1}$, again with Ω_1 omitted if $\Omega_1 = \Omega$. The norms in $L_2(\Omega_1)$ and $L_\infty(\Omega_1)$ are denoted by $\|\cdot\|_{\Omega_1}$ and $|\cdot|_{\Omega_1}$ ($\|\cdot\|$ and $|\cdot|$ if $\Omega_1 = \Omega$).

Error estimates for semidiscrete Galerkin problems, containing the above example, have been given by the energy method in e.g. Price and Varga [18], Douglas and Dupont [10], Fix and Nassif [12], Wheeler [23] and Dupont [11]. In § 2 we prove such estimates in the present situation, thus in particular generalizing known results to cases in which the elements of S_h are not required to vanish on $\partial\Omega$ (cf. also Bramble and Thomée [7] in the completely discrete case).

The results of § 2, although derived under essentially as weak regularity assumptions as possible, require that the initial data v are at least in $\dot{H}^r$ in order to show optimal order convergence. In the special case that $S_h \subset \dot{H}^2$, Helfrich [13] was able to show that for t bounded away from zero the convergence is of optimal order if v is only in L_2 (cf. also Babuška–Aziz [2, Chap. 11] and in one dimension, Thomée [21][1]). In § 3 we prove such results in the present framework, and we also briefly consider the case when the data v is less smooth than in L_2.

In the results described so far, the error is measured in L_2. In § 4 we show how it is possible to derive maximum norm estimates, provided L_2 estimates are known and that maximum norm error bounds are known for the stationary problem, see also Thomée [21], Wheeler [24] and Wahlbin [22] in the one-dimensional case.

In the next three sections we derive certain interior error estimates in the maximum norm under the assumption that the semidiscrete solution satisfies the equation (1.9) for such $\chi \in S_h$ which vanish outside some subset of Ω. In § 5 the error in the interior is estimated in the maximum norm under the assumption that the analogous results hold for the elliptic problem and that L_2 error bounds are available. In §§ 6 and 7 we assume that the elements of S_h are translation invariant in the interior of Ω in a way to be made precise below. In § 6 we then prove, using results for the elliptic case due to Nitsche and Schatz [17] and Bramble, Nitsche and Schatz [4] that arbitrary derivatives of u can be approximated to optimal order in the interior of Ω by difference quotients of u_h if $u - u_h$ is of optimal order in L_2. Finally, in § 7, it is shown that an averaging process described in the elliptic case by Bramble and Schatz [6] can be used to derive superconvergent $O(h^{2r-2})$ approximations of the solution provided negative norm error estimates of this order are available.

In each of these last four sections the results described above are combined with the results of § 3 to show that all of these high order convergence results hold under weak assumptions on the initial data.

2. Error estimates L_2 for smooth initial data. In this section we prove a new optimal error estimate for the semidiscrete problem (1.6), which is valid uniformly for $0 \leqq t \leqq T_0$. This estimate is derived under as weak regularity assumptions as possible on the initial data, and is proved using only the approximation assumption (iia). The result is stronger than that of Dupont [11], in which he made essential use of negative norm estimates. Dupont's results exclude, for example, the ordinary Galerkin method with piecewise linear functions, a case which is included in Theorem 2.1 below.

[1] Cf. also J. J. Blair, *Approximate solution of elliptic and parabolic boundary value problems*, Thesis, University of California, Berkeley, 1970.

Let $u=u(t)$, $u_h=u_h(t)$, v and v_h be as in § 1, and assume that v_h is chosen in such a way that for some fixed s, $0\leqq s\leqq r$,

(2.1) $$\|v_h-v\|\leqq Ch^s\|v\|_s.$$

An example of such a choice is $v_h=P_0v$, where P_0 is the L_2 projection onto S_h. For, using (iia),

$$\|P_0v-v\|=\inf_{\chi\in S_h}\|\chi-v\|=\inf_{\chi\in S_h}\|\chi-TLv\|\leqq\|(T_h-T)Lv\|$$
$$\leqq Ch^r\|Lv\|_{r-2}=Ch^r\|v\|_r$$

and since $\|P_0v-v\|\leqq 2\|v\|$, the inequality (2.1) for any s, $0\leqq s\leqq r$, follows by interpolation.

Our result for $e_h=e_h(t)=u_h-u$ is then:

THEOREM 2.1. *Assume that* (i), (iia) *and* (2.1) *hold for some fixed* s, $0\leqq s\leqq r$. *Then there is a constant* C *such that*

$$\|e_h(t)\|\leqq Ch^s\|v\|_s \quad \text{for } 0\leqq t\leqq T_0.$$

Proof. We first show that it suffices to consider the case in which the initial data are taken as P_0v and $s=r$. For, let $\tilde u_h$ denote the solution of (1.6) with $\tilde u_h(0)=P_0v$, and assume for the moment that we have shown

(2.2) $$\|(\tilde u_h-u)(t)\|\leqq Ch^r\|v\|_r \quad \text{for } 0\leqq t\leqq T_0.$$

By (1.6) and (i) we have

$$\frac{d}{dt}\|\tilde u_h\|^2=-2(T_hD_t\tilde u_h, D_t\tilde u_h)\leqq 0$$

and hence

(2.3) $$\|\tilde u_h(t)\|\leqq\|v\|.$$

Thus using (1.4),

$$\|(u-\tilde u_h)(t)\|\leqq 2\|v\|$$

and it follows by interpolation that

(2.4) $$\|(u-\tilde u_h)(t)\|\leqq Ch^s\|v\|_s, \qquad 0\leqq s\leqq r, \quad 0\leqq t\leqq T_0.$$

As in (2.3) we have, using also (2.1) and its counterpart for the L_2 projection,

$$\|(u_h-\tilde u_h)(t)\|\leqq\|v_h-P_0v\|\leqq\|v_h-v\|+\|v-P_0v\|\leqq Ch^s\|v\|_s.$$

From this and (2.4) the theorem follows.

It remains to prove (2.2). In the rest of the proof we set $e_h=\tilde u_h-u$. Recall now the error equation

(2.5) $$T_hD_te_h+e_h=\rho\equiv(T-T_h)D_tu.$$

Clearly

$$(T_hD_te_h, D_te_h)+\frac{1}{2}\frac{d}{dt}\|e_h\|^2=(\rho, D_te_h)$$

and hence, using (i),

$$\int_0^t\tau\frac{d}{d\tau}\|e_h\|^2\,d\tau\leqq 2\int_0^t\tau(\rho, D_\tau e_h)\,d\tau.$$

Integrating by parts we obtain

$$t\|e_h\|^2 \leqq \int_0^t \|e_h\|^2\,d\tau + 2t(\rho, e_h) - 2\int_0^t (\rho, e_h)\,d\tau - 2\int_0^t \tau(D_\tau\rho, e_h)\,d\tau.$$

Using the Cauchy–Schwarz inequality it follows that

$$\|e_h(t)\|^2 \leqq C\left\{\frac{1}{t}\int_0^t \|e_h\|^2\,d\tau + \|\rho(t)\|^2 + \frac{1}{t}\int_0^t \|\rho\|^2\,d\tau + \frac{1}{t}\int_0^t \tau^2\|D_\tau\rho\|^2\,d\tau\right\}. \tag{2.6}$$

We shall next show that

$$\int_0^t \|e_h\|^2\,d\tau \leqq \int_0^t \|\rho\|^2\,d\tau. \tag{2.7}$$

For, from (2.5) we obtain

$$\frac{1}{2}\frac{d}{dt}(T_he_h, e_h) + \|e_h\|^2 = (\rho, e_h) \leqq \tfrac{1}{2}\|\rho\|^2 + \tfrac{1}{2}\|e_h\|^2.$$

Integrating, and noting that $(T_he_h(0), e_h(0)) = (T_he_h(0), P_0v - v) = 0$, (2.7) follows.

From (2.6) and (2.7) we have

$$\|e_h(t)\|^2 \leqq C\left\{\|\rho(t)\|^2 + \frac{1}{t}\int_p^t \|\rho\|^2\,d\tau + \int_0^t \tau\|D_\tau\rho\|^2\,d\tau\right\}. \tag{2.8}$$

Finally, we shall estimate the right hand side of (2.8). By (iia) and (1.4),

$$\begin{aligned}\|\rho(t)\|^2 + \frac{1}{t}\int_0^t \|\rho\|^2\,d\tau &\leqq 2\sup_{0\leqq\tau\leqq t}\|\rho(\tau)\|^2 \\ &= 2\sup_{0\leqq\tau\leqq t}\|(T_h - T)D_\tau u(\tau)\|^2 \leqq Ch^{2r}\|v\|_r^2.\end{aligned} \tag{2.9}$$

For the last term in (2.8), we have by (iia),

$$\int_0^t \tau\|D_\tau\rho\|^2\,d\tau \leqq Ch^{2r}\int_0^t \tau\|u(\tau)\|_{r+2}^2\,d\tau. \tag{2.10}$$

From the definition of $\|\cdot\|_{r+2}$ and with $\beta_j = (v, \varphi_j)$ we obtain

$$\begin{aligned}\int_0^t \tau\|u(t)\|_{r+2}^2\,d\tau &= \sum_{j=1}^\infty \lambda_j^{r+2}\beta_j^2\int_0^t \tau e^{-2\lambda_j\tau}\,d\tau \\ &\leqq \sum_{j=1}^\infty \lambda_j^r\beta_j^2\int_0^t \eta e^{-2\eta}\,d\eta = \tfrac{1}{4}\|v\|_r^2.\end{aligned} \tag{2.11}$$

By use of (2.9), (2.10) and (2.11) in (2.8), the desired inequality (2.2) follows. This completes the proof of the theorem.

3. Error estimates by spectral representation. We recall that the solution of (1.5) can be written by means of the eigenfunctions and eigenvalues of T in the form

$$u(t)=\sum_j e^{-t/\mu_j}\beta_j\varphi_j(x), \qquad \beta_j=(v,\varphi_j).$$

Introducing the resolvent $R_z(T)=(z-T)^{-1}$, we shall use the fact that the L_2-projection onto the eigenspace corresponding to the eigenvalue μ can be expressed as

$$\sum_{\mu_j=\mu}\beta_j\varphi_j(x)=\frac{1}{2\pi i}\int_{\gamma_\mu} R_z(T)v\,dz$$

where γ_μ is a curve in the complex plane enclosing only this eigenvalue. It follows that

$$\sum_1^J e^{-t/\mu_j}\beta_j\varphi_j=\frac{1}{2\pi i}\int_{\Gamma_J} e^{-t/z}R_z(T)v\,dz,$$

where Γ_J is a curve enclosing only the eigenvalues $\mu_1,\cdots,\mu_J$ (provided $\mu_{J+1}\neq\mu_J$). In the limit we have

$$u(t)=E(t)v=\frac{1}{2\pi i}\int_{\Gamma_a} e^{-t/z}R_z(T)v\,dz. \tag{3.1}$$

Here and below we shall take Γ_a to be the positively oriented curve defined by $|\arg z|=\pi/4$, $|z|<a$ or $|\arg z|<\pi/4$, $|z|=a$, with a sufficiently large.

We notice that since

$$R_z(T)v=\sum_j\frac{1}{z-\mu_j}\beta_j\varphi_j,$$

it easily follows that

$$\|R_z(T)\|_{s,s}\leqq\frac{1}{|\mathrm{Im}\,z|}\leqq\frac{\sqrt{2}}{|z|}, \tag{3.2}$$

and

$$\|R_z(T)\|_{s,s+2}\leqq\sqrt{2} \tag{3.3}$$

for $|\arg z|=\pi/4$.

Correspondingly, let the eigenvalues and eigenfunctions of T_h be $\{\mu_{j,h}\}$ and $\{\varphi_{j,h}\}$. Since S_h is finite-dimensional, only finitely many $\mu_{j,h}$ are positive, and since T_h is positive definite on S_h, S_h is the linear span of the corresponding eigenfunctions $\{\varphi_{j,h}\}^{J_h}$. Hence, if $v_h\in S_h$, the solution of (1.6) is

$$u_h(t)=\sum_1^{J_h} e^{-t/\mu_{j,h}}\beta_{j,h}\varphi_{j,h}, \qquad \beta_{j,h}=(v_h,\varphi_{j,h}).$$

In particular, if $v\in L_2$ and $v_h=P_0v$, we have $\beta_{j,h}=(v,\varphi_{j,h})$ and we may conclude as above that

$$u_h(t)=E_h(t)v=\frac{1}{2\pi i}\int_{\Gamma_a} e^{-t/z}R_z(T_h)v\,dz. \tag{3.4}$$

Since T_h is selfadjoint on L_2 we have again

$$\|R_z(T_h)\|_{0,0} \leqq \frac{1}{|\operatorname{Im} z|} \leqq \frac{\sqrt{2}}{|z|} \tag{3.5}$$

for $|\arg z| = \pi/4$.

We shall now use the spectral representations to derive estimates for

$$e_h(t) = F_h(t)v = E_h(t)v - E(t)v = u_h(t) - u(t).$$

By (3.1) and (3.4) we have taking the limit as $a \to \infty$

$$F_h(t) = \frac{1}{2\pi i} \int_\Gamma e^{-t/z}(R_z(T_h) - R_z(T))\, dz \tag{3.6}$$

where Γ is the positively oriented curve defined by $|\arg z| = \pi/4$. The limiting process is justified by the following lemma, which also allows us to obtain estimates for the error $e_h(t)$.

LEMMA 3.1. *Assume that T_h satisfies* (i) *and* (iia) (*or* (iib)). *Then for $p = 0$, $0 \leqq q \leqq r-2$ ($0 \leqq p, q \leqq r-2$)*

$$\|R_z(T_h) - R_z(T)\|_{-p,q} \leqq Ch^{p+q+2}/|z|^2, \quad \text{for } z \in \Gamma.$$

Proof. As a preliminary step we prove that if T_h satisfies (i), (iia), then

$$\|R_z(T_h) - R_z(T)\|_{0,s} \leqq Ch^s/|z|, \quad \text{for } 0 \leqq s \leqq r, \quad z \in \Gamma. \tag{3.7}$$

To see this we write

$$R_z(T_h) - R_z(T) = R_z(T_h)(T_h - T)R_z(T),$$

and obtain hence by (3.5), (iia) and (3.3),

$$\begin{aligned}\|R_z(T_h) - R_z(T)\|_{0,r} &\leqq \|R_z(T_h)\|_{0,0}\|T_h - T\|_{0,r-2}\|R_z(T)\|_{r-2,r} \\ &\leqq Ch^r/|z|,\end{aligned}$$

which is the desired result for $s = r$. Also,

$$\|R_z(T_h) - R_z(T)\|_{0,0} \leqq \|R_z(T_h)\|_{0,0} + \|R_z(T)\|_{0,0} \leqq C/|z|.$$

Together these two estimates prove (3.7) by interpolation.

In order to prove the lemma we now write

$$\begin{aligned}R_z(T_h) - R_z(T) = {}& R_z(T)(T_h - T)R_z(T) \\ & + R_z(T)(T_h - T)(R_z(T_h) - R_z(T)).\end{aligned}$$

We hence obtain, using (3.2) and (3.7),

$$\begin{aligned}\|R_z(T_h) - R_z(T)\|_{-p,q} \leqq {}& \|R_z(T)\|_{-p,-p}\|T_h - T\|_{-p,q}\|R_z(T)\|_{q,q} \\ & + \|R_z(T)\|_{-p,-p}\|T - T_h\|_{-p,0}\|R_z(T_h) - R_z(T)\|_{0,q} \\ \leqq {}& Ch^{p+q+2}/|z|^2,\end{aligned}$$

which completes the proof of the lemma.

We can now derive the following convergence result in which, for later use, we also estimate time derivatives of the error $e_h(t) = u_h(t) - u(t)$.

THEOREM 3.1. *Assume that* (i) *and* (iia) (*or* (iib)) *hold and that* $v_h = P_0 v$. *Then for* $p=0$, $0 \leqq q \leqq r-2$ $(0 \leqq p,\ q \leqq r-2)$ *there is a constant* C *such that for* $0 \leqq t \leqq T_0$,

$$\|D_t^j e_h(t)\|_{-p} \leqq Ch^{p+q+2} t^{-1-j} \|v\|_q.$$

Proof. We have by (3.6),

$$D_t^j F_h(t) = \frac{(-1)^j}{2\pi i} \int_\Gamma z^{-j} e^{-t/z} (R_z(T_h) - R_z(T))\, dz,$$

and hence by Lemma 3.1,

$$\begin{aligned} \|D_t^j F_h(t)\|_{-p,q} &\leqq Ch^{p+q+2} \int_\Gamma |z|^{-j-2} e^{-ct/|z|} d|z| \\ &= Ch^{p+q+2} \int_0^\infty \sigma^j e^{-t\sigma}\, d\sigma = Ch^{p+q+2} t^{-1-j}, \end{aligned}$$

which proves the theorem.

In particular the above result shows a convergence rate of $O(h^2)$ in L_2 for data in L_2 and t bounded away from zero. In the next theorem we shall sharpen this result to $O(h^r)$.

THEOREM 3.2. *Assume that* (i) *and* (iia) *hold and that* $v_h = P_0 v$. *Then there is a constant* C *such that for* $0 \leqq t \leqq T_0$,

$$\|D_t^j e_h(t)\| \leqq Ch^r t^{-r/2-j} \|v\|.$$

Proof. The case $h^2 t^{-1} > 1$ follows immediately from Theorem 3.1. Hence it is sufficient to take $h^2 t^{-1} \leqq 1$. We first consider the case $j=0$. From the definition of F_h it easily follows that

$$F_h(t) = F_h(t/2)E(t/2) + E(t/2)F_h(t/2) - F_h(t/2)^2. \tag{3.8}$$

By (1.4) and Theorem 3.1,

$$\|F_h(t/2)E(t/2)\|_{0,0} \leqq \|F_h(t/2)\|_{0,r-2} \|E(t/2)\|_{r-2,0} \leqq Ch^r t^{-r/2}.$$

Since the second term on the right in (3.8) is the adjoint of the first, its norm is the same and since by Theorem 3.1,

$$\|F_h(t/2)^2\|_{0,0} \leqq \|F_h(t/2)\|_{0,0}^2 \leqq Ch^2 t^{-1} \|F_h(t/2)\|_{0,0},$$

we conclude

$$\|F_h(t)\|_{0,0} \leqq Ch^r t^{-r/2} + Ch^2 t^{-1} \|F_h(t/2)\|_{0,0}.$$

By repeated application we obtain

$$\|F_h(t)\|_{0,0} \leqq Ch^r t^{-r/2} + C(h^2 t^{-1})^s \|F_h(t/2^s)\|_{0,0}, \qquad s = 1, 2, \cdots,$$

from which the result follows if $s \geqq r/2$.

For $j > 0$ the result is proved inductively by differentiating (3.8) using Theorem 3.1 and using instead of (1.4) the inequality

$$\|D_t^l E(t)\|_{r-2,0} \leqq Ct^{-(r-2)/2-l}.$$

Consider now the discrete problem (1.6) with initial data $v_h \in S_h$ other than $P_0 v$. We then have:

THEOREM 3.3. *Assume that* (i) *and* (iia) *hold. Then for* $k \geqq 0$ *there is a constant* C *such that for* $0 < t \leqq T_0$,

$$\|D_t^j e_h(t)\| \leqq Ch^r t^{-r/2-j}\|v_h\| + Ct^{-k/2-j}\|v - v_h\|_{-k}.$$

Proof. Let $\tilde{u}$ be the solution of the continuous problem with initial data v_h. Then by Theorem 3.2,

$$\|D_t^j(\tilde{u} - u_h)\| \leqq Ch^r t^{-r/2-j}\|v_h\|.$$

On the other hand, since $u - \tilde{u}$ is a solution of the continuous problem with initial data $v - v_h$ we have

$$\|D_t^j(u - \tilde{u})\| \leqq C\|D_t^j E(t)\|_{0,-k} \cdot \|v - v_h\|_{-k}.$$

The result now follows by (1.4) and the triangle inequality.

In particular, if v_h is chosen to be bounded in L_2 and so that it approximates v to order $O(h^r)$ in some negative norm, the error is $O(h^r)$ in L_2 for the t positive. This is satisfied, for instance, if (iib) (or (iia) if $r = 2$) holds and $v_h = P_1 v = T_h L v$ (the "elliptic projection" of v), since then

$$\|v_h - v\|_{-(r-2)} = \|(T_h - T)Lv\|_{-(r-2)} \leqq Ch^r\|Lv\| = Ch^r\|v\|_2$$

and

$$\|v_h\| \leqq \|TLv\| + \|(T_h - T)Lv\| \leqq \|v\| + Ch^2\|Lv\| \leqq C\|v\|_2.$$

Finally, we shall briefly consider the case when $v \in \dot{H}^{-m}$ for some $m > 0$. We assume then that we have at our disposal a family of smoothing operators J_ε which operate on v such that $J_\varepsilon v \in L_e(\Omega)$ and such that with a fixed $k > m$ and a constant C independent of ε,

$$\|J_\varepsilon v\| \leqq C\varepsilon^{-m}, \tag{3.9}$$

$$\|J_\varepsilon v - v\|_{-k} \leqq C\varepsilon^{k-m}. \tag{3.10}$$

If v has compact support in Ω, such a family of operators can be constructed for any k by means of convolution with an appropriate kernel, for example the kernel appearing in § 7 below.

We then have the following theorem which shows that if k is large enough we obtain almost optimal rate of convergence for positive t.

THEOREM 3.4. *Assume that* (i), (iia), (3.9) *and* (3.10) *hold,* $v \in \dot{H}^{-m}$ *for* $m > 0$ *and* $v_h = P_0 J_{h^{r/k}} v$. *Then for* $t_0 > 0$ *there exists a constant* C *such that for* $t_0 \leqq t \leqq T_0$,

$$\|D_t^j e_h(t)\| \leqq Ch^{r(1-m/k)}.$$

Proof. Let $\tilde{u}$ be the solution of the continuous problem with data $J_{h^{r/k}} v$. We have in the same way as in the proof of Theorem 3.3, using (3.9) and (3.10),

$$\begin{aligned}\|D_t^j e_h\| &\leqq \|D_t^j(u_h - \tilde{u})\| + \|D_t^j(\tilde{u} - u)\| \\ &\leqq C(t)h^r\|J_{h^{r/k}} v\| + \|D_t^j E(t)\|_{0,-k}\|J_{h^{r/k}} v - v\|_{-k} \\ &\leqq C(t)h^{r(1-m/k)}\end{aligned}$$

which completes the proof.

Let us remark that if (iib) holds and instead of (3.9) one has $J_\varepsilon v \in \dot{H}^{r+1}(\Omega)$ with $\|J_\varepsilon v\|_{r+1} \leqq C\varepsilon^{-(m+r+1)}$, then if $k \geqq m + r(m+r+1)$ and v_h is taken as $T_h L J_{\bar\varepsilon} v$, $\bar\varepsilon = h^{1/(m+r+1)}$, one can prove that $\|e_h(t)\| \leqq C(t)h^r$.

4. Global estimates in the maximum norm. In this section we shall show that if suitable error estimates in the maximum norm are available for the associated elliptic problem, then the question of error estimates in the maximum norm for the parabolic problem can be reduced to obtaining error estimates in L_2. Using the L_2 estimates derived above, we may then obtain maximum norm estimates in terms of data.

We shall assume throughout this section that the family $\{T_h\}$ has the property that there is a function $\gamma(h)$ and a constant C such that for sufficiently small h,

$$\text{(i}')\qquad |T_h w| \leqq C|Tw|_1, \qquad \|T_h w\| \leqq C\|Tw\|_1,$$

$$\text{(ii}')\qquad |(T_h - T)w| \leqq \gamma(h)|Tw|_r.$$

Letting as usual $e_h = e_h(t) = u_h - u$ we then have the following result.

THEOREM 4.1. *Assume that* (i′) *and* (ii′) *hold. Then there exist an integer J depending only on N and a constant C such that*

$$|e_h(t)| \leqq C\{\gamma(h)|u(t)|_{r+2J-2} + \|D_t^J e_h(t)\|\}.$$

In the proof we shall use the following inequality.

LEMMA 4.1. *Let $2 \leqq p \leqq \infty$ and $0 < 1/q - 1/p < 1/N$. Then there exists a constant C such that*

$$\|T_h w\|_{L_p} \leqq C\|w\|_{L_q}.$$

Proof. From (i′) we obtain by use of Sobolev's inequality and elliptic regularity

$$|T_h w| \leqq C|Tw|_1 \leqq C\|Tw\|_{W_s^2} \leqq C\|w\|_{L_s} \quad \text{for } \frac{1}{s} < \frac{1}{N},$$

and similarly that

$$\|T_h w\| \leqq C\|Tw\|_1 \leqq C\|Tw\|_{W_s^2} \leqq C\|w\|_{L_s} \quad \text{for } \frac{1}{s} - \frac{1}{2} < \frac{1}{N}.$$

Thus the desired result holds for $p = 2$ and $p = \infty$, and hence by interpolation in the general case.

We can now give the

Proof of Theorem 4.1. We have by (1.7) for $j \geqq 0$,

$$D_t^j e_h = D_t^j \rho - T_h D_t^{j+1} e_h \quad \text{with } D_t^j \rho = (T - T_h) D_t^{j+1} u.$$

Hence (ii′) and Lemma 4.1 give for $1/p_{j+1} - 1/p_j < 1/N$,

$$\|D_t^j e_h\|_{L_{p_j}} \leqq C\{\gamma(h)|D_t^j u|_r + \|D_t^{j+1} e_h\|_{L_{p_{j+1}}}\}. \tag{4.1}$$

The result therefore follows by repeated application of (4.1), using a suitable decreasing sequence of p's with $p_0 = \infty$, $p_J = 2$.

Theorem 4.1 can be combined with the results of § 2 or § 3. We have for instance:

THEOREM 4.2. *Assume that* (i), (i′), (iia) *and* (ii′) *hold, and let* $v_h = P_0 v$. *Then for any* $t_0 > 0$ *there is a constant* C *such that for* $t_0 \leqq t \leqq T_0$,

$$|e_h(t)| \leqq C\{\gamma(h) + h^r\} \|v\|.$$

Proof. This follows immediately from Theorems 4.1 and 3.2, since by Sobolev's inequality and (1.4), (with $N_0 = [N/2] + 1$)

$$|u(t)|_{r+2J-2} \leqq C\|u(t)\|_{r+2J-2+N_0} \leqq Ct^{-(r+2J-2+N_0)/2}\|v\|.$$

5. Interior maximum norm estimates. In this and the following two sections we shall assume that for some domain Ω_0 with $\Omega_0 \subset\subset \Omega$, the solution u_h of the semidiscrete problem satisfies the interior equations

(5.1) $$(D_t u_h, \chi) + A(u_h, \chi) = 0 \quad \text{for } \chi \in \mathring{S}_h(\Omega_0), \quad t > 0.$$

Here

$$A(\varphi, \psi) = \int_\Omega \left(\sum_{j,k} a_{jk} \frac{\partial \varphi}{\partial x_j} \frac{\partial \psi}{\partial x_k} + a_0 \varphi \psi \right) dx$$

and

(5.2) $$\mathring{S}_h(\Omega_0) = \{\chi \in S_h : \operatorname{supp} \chi \subset \Omega_0\}.$$

In the present section we shall suppose that certain interior maximum norm estimates are known for the corresponding elliptic problem

(5.3) $$A(\tilde{e}_h, \chi) = 0, \quad \text{for } \chi \in \mathring{S}_h(\Omega_0),$$

where $\tilde{e}_h = w_h - w$, with $w_h \in S_h$. More precisely, we shall assume that there is a function $\gamma(h)$ and for each pair of sets Ω_1, Ω_2 with $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ a constant C such that if (5.3) is satisfied, then (for sufficiently small h, depending on Ω_1 and Ω_2)

(i″) $$|w_h|_{\Omega_2} \leqq C\{|w|_{1,\Omega_1} + \|w_h\|_{\Omega_1}\},$$

(ii″) $$|\tilde{e}_h|_{\Omega_2} \leqq C\{\gamma(h)|w|_{r,\Omega_1} + \|\tilde{e}_h\|_{\Omega_1}\}.$$

Setting as usual $e_h = e_h(t) = u_h(t) - u(t)$ we then have the following interior analogue of Theorem 4.1:

THEOREM 5.1. *Assume that* (i″), (ii″) *and* (5.1) *hold. Then there exists an integer* J *depending only on* N *and for each* $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ *a constant* C *such that*

$$|e_h(t)|_{\Omega_2} \leqq C\left\{ \gamma(h)|u(t)|_{r+2J-2,\Omega_1} + \sum_{j=0}^{J} \|D_t^j e_h(t)\|_{\Omega_1} \right\}.$$

The proof will be an easy consequence of the following lemma.

LEMMA 5.1. *Let* $2 \leqq q < p \leqq \infty$ *with* $1/q - 1/p < 1/N$ *and assume that* $\tilde{e}_h = w_h - w$, *with* $w_h \in S_h$ *satisfies*

(5.4) $$A(\tilde{e}_h, \chi) = (f_h, \chi), \quad \textit{for } \chi \in \mathring{S}_h(\Omega_0).$$

Then for $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ there exists a constant C such that

$$\|\tilde{e}_h\|_{L_p(\Omega_2)} \leqq C\{\gamma(h)|w|_{r,\Omega_1} + \|\tilde{e}_h\|_{\Omega_1} + \|f_h\|_{L_q(\Omega_1)}\}.$$

Proof. Without loss of generality we may assume that Ω_1 is smooth. We introduce the solution $\psi \in \dot{H}^1(\Omega_1)$ of the continuous Dirichlet problem

$$A(\psi, \varphi) = (f_h, \varphi), \quad \text{for } \varphi \in \dot{H}^1(\Omega_1),$$

and its approximate solution $\psi_h \in \mathring{S}_h(\Omega_1)$ defined by

$$A(\psi - \psi_h, \chi) = 0, \quad \text{for } \chi \in \mathring{S}_h(\Omega_1).$$

The equation (5.4) may then be written

$$A(\tilde{e}_h - \psi_h, \chi) = 0, \quad \text{for } \chi \in \mathring{S}_h(\Omega_1),$$

so that by (ii″),

$$|\tilde{e}_h - \psi_h|_{\Omega_2} \leqq C\{\gamma(h)|w|_{r,\Omega_1} + \|\tilde{e}_h - \psi_h\|_{\Omega_1}\}.$$

Since

$$\|\tilde{e}_h\|_{L_p(\Omega_2)} \leqq \|\tilde{e}_h - \psi_h\|_{L_p(\Omega_2)} + \|\psi_h\|_{L_p(\Omega_2)},$$

$$\|\tilde{e}_h - \psi_h\|_{L_p(\Omega_2)} \leqq C|\tilde{e}_h - \psi_h|_{\Omega_2}$$

and

$$\|\psi_h\|_{\Omega_1} \leqq C\|\psi_h\|_{H^1(\Omega_1)} \leqq C\|f_h\|_{\Omega_1} \leqq C\|f_h\|_{L_q(\Omega_1)},$$

the result will follow if we can prove that

$$\|\psi_h\|_{L_p(\Omega_2)} \leqq C\|f_h\|_{L_q(\Omega_1)} \quad \text{for } 2 \leqq p \leqq \infty, \quad 0 < 1/q - 1/p < \frac{1}{N}. \tag{5.5}$$

Now since

$$\|\psi_h\|_{\Omega_1} \leqq C\|\psi_h\|_{H^1(\Omega_1)} \leqq C\|\psi\|_{H^1(\Omega_1)} \leqq C|\psi|_{1,\Omega_1},$$

we obtain by (i″), Sobolev's inequality and elliptic regularity

$$|\psi_h|_{\Omega_2} \leqq C|\psi|_{1,\Omega_1} \leqq C\|\psi\|_{W_s^2(\Omega_1)} \leqq C\|f_h\|_{L_s(\Omega_s)} \quad \text{for } \frac{1}{s} < \frac{1}{N}.$$

Similarly,

$$\|\psi_h\|_{\Omega_2} \leqq \|\psi_h\|_{H^1(\Omega_2)} \leqq C\|\psi\|_{H^1(\Omega_1)} \leqq C\|\psi\|_{W_s^2(\Omega_1)} \leqq C\|f_h\|_{L_s(\Omega_1)}$$

$$\text{for } \frac{1}{s} - \frac{1}{2} < \frac{1}{N}.$$

The inequality (5.5) now follows by interpolation.

We can now give the

Proof of Theorem 5.1. From (5.1) it follows that for each $j \geqq 0$

$$A(D_t^j e_h, \chi) = -(D_t^{j+1} e_h, \chi), \quad \text{for } \chi \in \mathring{S}_h(\Omega_0),$$

so that by Lemma 5.1, for $\Omega_2 \subset\subset \Omega_j' \subset\subset \Omega_j'' \subset\subset \Omega_1$, $1/p_{j+1} - 1/p_j < 1/N$,

$$\|D_t^j e_h\|_{L_{p_j}(\Omega_j')} \leqq C\{\gamma(h)\,|D_t^j u|_{r,\Omega_j''} + \|D_t^j e_h\|_{\Omega_j''} + \|D_t^{j+1} e_h\|_{L_{p_{j+1}}(\Omega_j'')}\}. \tag{5.6}$$

The result therefore follows by repeated applications of (5.6) for $j = 0, 1, \cdots$, a suitable decreasing finite sequence, $\{p_j\}$, and an expanding sequence of domains between Ω_2 and Ω_1.

As an immediate consequence of Theorem 5.1 we now obtain, using (1.4) and Theorem 3.2:

THEOREM 5.2. *Assume that* (i), (i″), (iia), (ii″) *and* (5.1) *hold, and let* $v_h = P_0 v$. *Then for any* $t_0 > 0$ *and* $\Omega_1 \subset\subset \Omega_0$ *there is a constant* C *such that for* $t_0 \leqq t \leqq T_0$,

$$|e_h(t)|_{\Omega_1} \leqq C\{\gamma(h) + h^r\}\|v\|.$$

6. Interior estimates for difference quotients. In this and the next section we shall obtain error estimates on some interior subdomain Ω_0 of Ω, where we shall now require that when restricted to Ω_0, the functions in the subspace are piecewise polynomials on a uniform partition. More precisely, let Q be a bounded domain in R^N which is partitioned into disjoint open sets π_j, $j = 1, \cdots, l$. For $\nu \in Z^N$, let Q^ν and π_j^ν denote the translations of Q and π_j by ν. We shall assume that the Q^ν are disjoint and that their closures cover R^N. Let $\psi_1, \cdots, \psi_k$ be continuous functions with compact supports which reduce to polynomials on the sets π_j^ν. We shall assume that the linear span of the restrictions to each π_j^ν of all functions of the form $\psi_l(x - \alpha)$, $l = 1, \cdots, k$, $\alpha \in Z^N$, contains all polynomials of degree less than r.

Let $S_h(\Omega_0)$ be the restrictions to Ω_0 of functions in S_h. We shall assume that $\chi \in S_h(\Omega_0)$ if and only if it can be written in the form

$$\chi(x) = \sum_{j,\alpha} a_{j\alpha}\psi_j(h^{-1}x - \alpha), \qquad x \in \Omega_0.$$

In order to be able to quote interior estimates for the elliptic problem we shall make two additional assumptions which concern the local approximability properties of the family $\{S_h\}$. For l a nonnegative integer we denote (with a slight abuse of notation) by $\|\cdot\|_{l,\Omega_0}$ the norm in $H^l(\Omega_0) = W_2^l(\Omega_0)$. We shall assume that for $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ there exists a constant C such that for h sufficiently small the following holds: If $1 \leqq l \leqq r$ and if $w \in \dot{H}^l(\Omega_1)$ and vanishes outside Ω_2, then with $\mathring{S}_h(\Omega_1)$ defined by (5.2),

$$\inf_{\eta \in \mathring{S}_h(\Omega_1)} \|w - \eta\|_{1,\Omega_2} \leqq Ch^{l-1}\|w\|_{l,\Omega_1}.$$

If $\omega \in C_0^\infty(\Omega_2)$, then

$$\inf_{\eta \in \mathring{S}_h(\Omega_1)} \|\omega\chi - \eta\|_{1,\Omega_2} \leqq Ch\|\chi\|_{1,\Omega_1}, \quad \text{for } \chi \in S_h(\Omega_1).$$

For brevity we shall say that $\{S_h\}$ is r-regular on Ω_0 if it satisfies the above assumptions.

Examples of families of subspaces satisfying the above hypotheses on Ω_0 are furnished by the restrictions to Ω_0 of the tensor products of one dimensional splines on a uniform mesh [4, Appendix], the plane triangular elements of Bramble and Zlámal [9], [17] (provided the triangulation is uniform on Ω_0) and the restrictions to Ω_0 of continuous piecewise linear functions defined on a uniform partition by N-dimensional simplices.

For technical reasons in proving maximum norm estimates we shall introduce the piecewise polynomial space $\tilde{S}_h(\Omega_0)$ consisting of exactly the same elements as those of $S_h(\Omega_0)$ when restricted to any $h\pi_j^\nu \cap \Omega_0$. The space $\tilde{S}_h(\Omega_0)$ may be described as being obtained from $S_h(\Omega_0)$ by removing all continuity requirements across the boundaries of the partitions. Clearly $\tilde{S}_h(\Omega_0) \supset S_h(\Omega_0)$.

We shall need a finite difference Sobolev type inequality and some approximation properties for $\tilde{S}_h(\Omega_0)$. For this purpose let ∂_h^α denote the forward difference quotient

$$\partial_h^\alpha = \partial_{h,1}^{\alpha_1} \cdots \partial_{h,N}^{\alpha_N} \quad \text{with } \partial_{h,j} w(x) = h^{-1}(w(x+he_j) - w(x)),$$

where e_j is the unit vector in the jth direction in R^N. From $m \geqq 0$ an integer we introduce the norms

$$|w|_{m,\Omega_0,h} = \sum_{|\alpha| \leqq m} |\partial_h^\alpha w|_{\Omega_0}$$

and

$$\|w\|_{m,\Omega_0,h} = \left(\sum_{|\alpha| \leqq m} \|\partial_h^\alpha w\|_{\Omega_0}^2 \right)^{1/2}.$$

Here and below we use the notation $N_0 = [N/2] + 1$.

LEMMA 6.1. *Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$. Then there exists a constant C such that for h sufficiently small*

$$|\chi|_{\Omega_2} \leqq C \|\chi\|_{N_0,\Omega_1,h}, \quad \text{for } \chi \in \tilde{S}_h(\Omega_0).$$

LEMMA 6.2. *Let $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ and let m be a nonnegative integer. Then there is a constant C and a function $\eta \in \tilde{S}_h(\Omega_0)$ such that for h sufficiently small*

$$|w - \eta|_{m,\Omega_2,h} \leqq Ch^r |w|_{r+m,\Omega_1}$$

and

$$\|w - \eta\|_{m,\Omega_2,h} \leqq Ch^r \|w\|_{r+m,\Omega_1}.$$

The proofs may be found in [4].

We shall assume now as in § 5 that the solution u_h of the discrete problem satisfies the interior equations

$$(6.1) \qquad (D_t u_h, \chi) + A(u_h, \chi) = 0, \quad \text{for } \chi \in \mathring{S}_h(\Omega_0), \quad t > 0.$$

We say that the finite difference operator

$$(6.2) \qquad Q_h w(x) = \sum_{\gamma, |\beta| = m} q_{\gamma,\beta} \partial_h^\beta w(x - \gamma h)$$

approximates the derivative $D_x^\alpha w$ with accuracy r if for $\Omega_2 \subset\subset \Omega_1$

$$|Q_h w - D_x^\alpha w|_{\Omega_2} \leqq Ch^r |w|_{r+m,\Omega_1}. \tag{6.3}$$

The following is then the main result of this section.

THEOREM 6.1. *Assume that* $\{S_h\}$ *is r-regular on* Ω_0 *and that* (6.1) *holds. Let* $j \geqq 0$ *and let* Q_h *approximate* D_x^α *with accuracy r. Then for any* $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ *there is a constant C such that*

$$\begin{aligned} &|Q_h D_t^j u_h(t) - D_x^\alpha D_t^j u(t)|_{\Omega_2} \\ &\quad \leqq C\{h^r \|u(t)\|_{r+|\alpha|+2j+N_0,\Omega_1} + \sum_{l \leqq j+1+|\alpha|/2} \|D_t^l e_h(t)\|_{\Omega_1}\}. \end{aligned} \tag{6.4}$$

For the proof we shall need the following result for the elliptic problem from [17, Thm. 5.2]. In addition to the norm in $H^p(\Omega_0)$ we employ for $w \in L_2$ the weaker norm

$$\|w\|_{-p,\Omega_1} = \sup_{\varphi \in \mathring{C}_0^\infty(\Omega_1)} \frac{(w, \varphi)}{\|\varphi\|_{p,\Omega_1}},$$

and more generally for a linear functional f_h on $\mathring{H}^p(\Omega_1)$,

$$\|f_h\|_{-p,\Omega_1} = \sup_{\varphi \in \mathring{C}_0^\infty(\Omega_1)} \frac{f_h(\varphi)}{\|\varphi\|_{p,\Omega_1}}.$$

LEMMA 6.3. *Assume that* $\{S_h\}$ *is r-regular on* Ω_0 *and let* $\tilde{e}_h = w_h - w$ *with* $w_h \in S_h$, $w \in H^r(\Omega_0)$ *satisfy*

$$A(\tilde{e}_h, \chi) = f_h(\chi), \quad \text{for } \chi \in \mathring{S}_h(\Omega_0),$$

where f_h *is a linear functional on* $\mathring{H}^1(\Omega_1)$. *Then for* $p \geqq 0$ *and* $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ *there is a constant C such that*

$$\|\tilde{e}_h\|_{\Omega_2} + h\|\tilde{e}_h\|_{1,\Omega_2} \leqq C\{h^r\|w\|_{r,\Omega_1} + \|\tilde{e}_h\|_{-p,\Omega_1} + h\|f_h\|_{-1,\Omega_1} + \|f_h\|_{-2,\Omega_1}\}.$$

We can now prove the following technical lemma for the difference quotients of $e_h(t)$.

LEMMA 6.4. *Assume that* $\{S_h\}$ *is r-regular on* Ω_0. *Let* $j \geqq 0$ *and* α *be arbitrary. Then for any* $\Omega_2 \subset\subset \Omega_1 \subset\subset\subset\subset \Omega_0$ *there is a constant C such that*

$$\begin{aligned} &\|\partial_h^\alpha D_t^j e_h(t)\|_{\Omega_2} + h\|\partial_h^\alpha D_t^j e_h(t)\|_{1,\Omega_2} \\ &\quad \leqq C\{h^r\|u(t)\|_{r+|\alpha|+2j,\Omega_1} + \sum_{l \leqq j+1+|\alpha|/2} \|D_t^l e_h(t)\|_{\Omega_1}\}. \end{aligned} \tag{6.5}$$

Proof. We shall prove this by induction over $|\alpha|$. Considering first $\alpha = 0$, we notice that since the coefficients of L are independent of t we have

$$A(D_t^j e_h, \chi) = -(D_t^{j+1} e_h, \chi), \quad \text{for } \chi \in \mathring{S}_h(\Omega_0).$$

We may now apply Lemma 6.3 with $p = 0$ to $D_t^j e_h = D_t^j u_h - D_t^j u$. Since

$$\|D_t^j u\|_{r,\Omega_1} \leqq C\|u\|_{r+2j,\Omega_1}$$

and

$$h\|D_t^{j+1}e_h\|_{-1,\Omega_1}+\|D_t^{j+1}e_h\|_{-2,\Omega_1}\leqq C\|D_t^{j+1}e_h\|_{\Omega_1},$$

(6.5) follows for $\alpha=0$.

Now assume that (6.5) has been proved for difference quotients of orders $<|\alpha|$ and let $\Omega_2\subset\subset\Omega_2'\subset\subset\Omega_2''\subset\subset\Omega_1$. By a simple calculation, using the discrete Leibniz formula

$$\partial_h^\alpha(vw)=\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}T_h^\beta\partial_h^{\alpha-\beta}v\,\partial_h^\beta w,$$

we obtain

$$A(\partial_h^\alpha D_t^j e_h(t),\chi)=(\partial_h^\alpha D_t^{j+1}e_h(t),\chi)+f_h^{(\alpha,j)}(\chi), \tag{6.6}$$

with

$$\begin{aligned} f_h^{(\alpha,j)}(\varphi)=-\int\sum_{\beta<\alpha}\binom{\alpha}{\alpha-\beta}\Big\{&\sum_{j,k}\partial_h^{\alpha-\beta}a_{jk}T_h^{\alpha-\beta}\partial_h^\beta\frac{\partial}{\partial x_k}D_t^je_h\frac{\partial\varphi}{\partial x_j}\\ &+\partial_h^{\alpha-\beta}a_0T_h^{\alpha-\beta}\partial_h^\beta D_t^je_h\cdot\varphi\Big\}\,dx. \end{aligned}\tag{6.7}$$

This time we shall apply Lemma 6.3 with $p=|\alpha|$ to $\partial_h^\alpha D_t^je_h$. We have

$$\|\partial_h^\alpha D_t^ju\|_{r,\Omega_2'}\leqq C\|u\|_{r+|\alpha|+2j,\Omega_1},$$

and

$$\|\partial_h^\alpha D_t^je_h\|_{-|\alpha|,\Omega_2'}\leqq C\|D_t^je_h\|_{\Omega_1}.$$

Furthermore, if $|\alpha|=1$,

$$h\|\partial_h^\alpha D_t^{j+1}e_h\|_{-1,\Omega_2'}+\|\partial_h^\alpha D_t^{j+1}e_h\|_{-2,\Omega_2'}\leqq C\|D_t^{j+1}e_h\|_{\Omega_1}, \tag{6.8}$$

and if $|\alpha|\geqq 2$,

$$\begin{aligned} &h\|\partial_h^\alpha D_t^{j+1}e_h\|_{-1,\Omega_2'}+\|\partial_h^\alpha D_t^{j+1}e_h\|_{-2,\Omega_2'}\\ &\qquad\leqq C\sum_{|\beta|\leqq|\alpha|-2}\{h\|\partial_h^\beta D_t^{j+1}e_h\|_{1,\Omega_2''}+\|\partial_h^\beta D_t^{j+1}e_h\|_{\Omega_2''}\}, \end{aligned}$$

which is bounded by the right hand side of (6.5) by the induction assumption. Finally, we find by (6.7),

$$\begin{aligned} &h\|f_h^{(\alpha,j)}\|_{-1,\Omega_2'}+\|f_h^{(\alpha,j)}\|_{-2,\Omega_2'}\\ &\qquad\leqq C\sum_{|\beta|<|\alpha|}\{h\|\partial_h^\beta D_t^je_h\|_{1,\Omega_2''}+\|\partial_h^\beta D_t^je_h\|_{\Omega_2''}\}, \end{aligned}\tag{6.9}$$

which is again bounded in the appropriate way by the induction assumption. This completes the proof of the lemma.

The following lemma is a maximum norm analogue of the previous lemma.

LEMMA 6.5. *Under the assumptions of Lemma* 6.4 *there is a constant* C *such that*

$$|D_t^j e_h^{(t)}|_{m,\Omega_2,h} \leqq C\left\{h^r \|u(t)\|_{r+m+2j+N_0,\Omega_1} + \sum_{l \leqq j+1+(m+N_0)/2} \|D_t^l e_h(t)\|_{\Omega_1}\right\}.$$

Proof. We have by the triangle inequality for arbitrary χ in the space $\tilde{S}_h(\Omega_0)$ introduced above,

$$|D_t^j e_h|_{m,\Omega_2,h} \leqq |D_t^j u_h - \chi|_{m,\Omega_2,h} + |\chi - D_t^j u|_{m,\Omega_2,h}.$$

We obtain by Lemma 6.1 for $\Omega_2 \subset\subset \Omega_2' \subset\subset \Omega_1$,

$$\begin{aligned}|D_t^j u_h - \chi|_{m,\Omega_2,h} &\leqq \|D_t^j u_h - \chi\|_{m+N_0,\Omega_2,h}\\ &\leqq \|D_t^j e_h\|_{m+N_0,\Omega_2',h} + \|D_t^j u - \chi\|_{m+N_0,\Omega_2',h}.\end{aligned}$$

Hence

$$\begin{aligned}|D_t^j e_h|_{m,\Omega_2,h} \leqq &\inf_{\chi \in \tilde{S}_h(\Omega_0)} \{|D_t^j u - \chi|_{m,\Omega_2',h} + \|D_t^j u - \chi\|_{m+N_0,\Omega_2',h}\}\\ &+ \|D_t^j e_h\|_{m+N_0,\Omega_2',h}.\end{aligned}$$

The first term is now estimated by Lemma 6.2 and the second by Lemma 6.4.

We can now give the

Proof of Theorem 6.1. We have by the triangle inequality, (6.2) and (6.3)

$$\begin{aligned}|Q_h D_t^j u_h - D_x^\alpha D_t^j u|_{\Omega_2} &\leqq |(Q_h - D_x^\alpha) D_t^j u|_{\Omega_2} + |Q_h D_t^j e_h|_{\Omega_2}\\ &\leqq C\{h^r |D_t^j u|_{r+|\alpha|,\Omega_1} + \sum_{|\beta| \leqq |\alpha|} |\partial_h^\beta D_t^j e_h|_{\Omega_1}\}.\end{aligned}$$

The first term on the right hand side is now bounded by the first term on the right in (6.4), by Sobolev's inequality and the differential equation, and by Lemma 6.5 the second term is bounded by the right hand side of (6.4).

Combining Theorems 6.1 and 3.2 we obtain at once the following corollary:

THEOREM 6.2. *Assume that* (i), (iia) *and* (6.1) *hold*, $\{S_h\}$ *is* r*-regular on* Ω_0, *and let* $v_h = P_0 v$. *Furthermore let* $j \geqq 0$ *and* Q_h *approximate* D_x^α *with accuracy* r. *Then for any* $t_0 > 0$ *and* $\Omega_1 \subset\subset \Omega_0$ *there is a constant* C *such that for* $t_0 \leqq t \leqq T_0$,

$$|Q_h D_t^j u_h(t) - D_x^\alpha D_t^j u(t)|_{\Omega_1} \leqq C h^r \|v\|.$$

7. Interior superconvergence estimates. In the case of an elliptic problem it was shown by Bramble and Schatz [6] that if a certain averaging operator is applied to the solution of certain Galerkin equations, then in the interior of the domain the result approximates the exact solution to the order $O(h^{2r-2})$. We shall now prove that a similar result holds for the parabolic problem we are treating.

The construction of the averaging operator is based on the following lemma from [6]. Here we denote by ψ the N-dimensional analogue of the B-spline of order r-2, that is, the convolution $\chi * \cdots * \chi$ with r-2 factors where χ is the characteristic function of $[-\frac{1}{2}, \frac{1}{2}]^N$.

LEMMA 7.1. *There exists a function K_h of the form*

$$K_h(x) = h^{-N} \sum_\alpha k_\alpha \psi(h^{-1}x - \alpha)$$

with $k_\alpha = 0$ when $|\alpha_j| \geqq r-1$ such that for $\Omega_2 \subset\subset \Omega_1$,

$$|w - K_h * w|_{\Omega_2} \leqq Ch^{2r-2}|w|_{2r-2,\Omega_1}, \tag{7.1}$$

$$|K_h * w|_{\Omega_2} \leqq C\Big\{\sum_{|\alpha| \leqq N_0+r-2} \|\partial_h^\alpha w\|_{-(r-2),\Omega_1} + h^{r-2} \sum_{|\alpha| \leqq r-2} |\partial_h^\alpha w|_{\Omega_1}\Big\}. \tag{7.2}$$

The main result in this section is the following:

THEOREM 7.1. *Assume that $\{S_h\}$ is r-regular on Ω_0, that* (6.1) *holds and let K_h be as in Lemma* 7.1. *Then for each $t > 0$ and $\Omega_2 \subset\subset \Omega_1 \subset\subset \Omega_0$ there is a constant C such that*

$$\begin{aligned}|K_h * u_h(t) - u(t)|_{\Omega_2} \leqq C\Big\{& h^{2r-2}\|u(t)\|_{2r-2+N_0,\Omega_1} \\ &+ \sum_{l \leqq r/2} (h^{r-2}\|D_t^l e_h(t)\|_{\Omega_1} + \|D_t^l e_h(t)\|_{-(r-2),\Omega_1}\Big\}.\end{aligned} \tag{7.3}$$

The proof will depend on estimates for difference quotients of e_h in § 6 and negative norm estimates of the same difference quotients to be obtained below. We first have

LEMMA 7.2. *Under the assumptions of Lemma* 6.3,

$$\begin{aligned}\|\tilde{e}_h\|_{-(r-2),\Omega_0} \leqq C\{& h^{r-1}\|\tilde{e}_h\|_{1,\Omega_1} + \|\tilde{e}_h\|_{-p,\Omega_1} \\ &+ h^{r-1}\|f_h\|_{-1,\Omega_1} + \|f_h\|_{-r,\Omega_1}\}.\end{aligned}$$

Proof. This is an immediate consequence of Lemma 4.2 of [17].

LEMMA 7.3. *Under the assumptions of Lemma* 6.4 *we have*

$$\begin{aligned}\|\partial_h^\alpha D_t^j e_h\|_{-(r-2),\Omega_2} \leqq C\{& h^{2r-2}\|u\|_{r+|\alpha|+2j,\Omega_1} \\ &+ \sum_{l \leqq j+1+|\alpha|/2} (h^{r-2}\|D_t^l e_h\|_{\Omega_1} + \|D_t^l e_h\|_{-(r-2),\Omega_1})\}.\end{aligned} \tag{7.4}$$

Proof. We shall prove this by induction over $|\alpha|$. Since the case $\alpha = 0$ is trivial we may now assume that (7.4) has been proved for difference quotients of orders less than $|\alpha|$. Let $\Omega_2 \subset\subset \Omega_2' \subset\subset \Omega_2'' \subset\subset \Omega_1$. We shall then apply Lemma 7.2 with $p = r-2+|\alpha|$ to $\partial_h^\alpha D_t^j e_h$ which function again satisfies (6.6), (6.7). By Lemma 6.4 we obtain

$$\begin{aligned}& h^{r-1}\|\partial_h^\alpha D_t^j e_h\|_{1,\Omega_2'} \\ &\leqq C\Big\{h^{2r-2}\|u\|_{r+|\alpha|+2j,\Omega_1} + h^{r-2} \sum_{l \leqq j+1+|\alpha|/2} \|D_t^l e_h\|_{\Omega_1}\Big\}\end{aligned}$$

and

$$\|\partial_h^\alpha D_t^j e_h\|_{-(r-2+|\alpha|),\Omega_2'} \leqq C\|D_t^j e_h\|_{-(r-2),\Omega_1}.$$

From (6.8), (6.9) and (6.5) we have

$$h^{r-1}\{\|\partial_h^\alpha D_t^{j+1} e_h\|_{-1,\Omega_2'}+\|f_h^{(\alpha,j)}\|_{-1,\Omega_2'}\}$$

$$\leqq C\{h^{2r-2}\|u\|_{r+|\alpha|+2j,\Omega_1}+h^{r-2}\sum_{l\leqq j+1+|\alpha|/2}\|D_t^l e_h\|_{\Omega_1}\}.$$

Clearly, if $|\alpha|=1$,

$$\|\partial_h^\alpha D_t^{j+1} e_h\|_{-r,\Omega_2'}\leqq C\|D_t^{j+1} e_h\|_{-(r-2),\Omega_1}$$

and if $|\alpha|\geqq 2$,

$$\|\partial_h^\alpha D_t^j e_h\|_{-r,\Omega_2'}\leqq C\sum_{|\beta|\leqq|\alpha|-2}\|\partial_h^\beta D_t^{j+1} e_h\|_{-(r-2),\Omega_2''}$$

which is bounded in the appropriate way by the induction assumption. The same holds for

$$\|f_h^{(\alpha,j)}\|_{-r,\Omega_2'}\leqq C\sum_{|\beta|<|\alpha|}\|\partial_h^\beta D_t^j e_h\|_{-(r-2),\Omega_2''}.$$

Together these estimates complete the proof of the lemma.

We can now give the

Proof of Theorem 7.1. We may write

$$|K_h * u_h-u|_{\Omega_2}\leqq|K_h * u-u|_{\Omega_2}+|K_h * e_h|_{\Omega_2}.$$

Here by (7.1),

$$|K_h * u-u|_{\Omega_2}\leqq Ch^{2r-2}|u|_{2r-2,\Omega_1}\leqq Ch^{2r-2}\|u\|_{2r-2+N_0,\Omega_1}.$$

By (7.2) we obtain

$$|K_h * e_h|_{\Omega_2}\leqq C\left\{\sum_{|\alpha|\leqq r-2+N_0}\|\partial_h^\alpha e_h\|_{-(r-2),\Omega_1}+h^{r-2}\sum_{|\alpha|\leqq r-2}|\partial_h^\alpha e_h|_{\Omega_1}\right\}$$

which is estimated by the right side of (7.3), by Lemmas 6.5 and 7.3. This completes the proof.

Combining Theorems 7.1 and 3.1 we obtain this time:

THEOREM 7.2. *Assume that* (i), (iib) *and* (6.1) *hold and that* $\{S_h\}$ *is r-regular on* Ω_0. *Let* $v_h=P_0v$ *and* K_h *be as in Lemma* 7.1. *Then for each* $t_0>0$ *and* $\Omega_1\subset\subset\Omega_0$ *there is a constant* C *such that for* $t_0\leqq t\leqq T_0$,

$$|K_h * u_h(t)-u(t)|_{\Omega_1}\leqq Ch^{2r-2}\|v\|_{r-2}.$$

8. Examples. We shall consider briefly some examples of families $\{T_h\}$ which approximate the solution operator T of an elliptic problem, and discuss the validity of the assumptions made above on such a family, and hence of the conclusions concerning the semidiscretization of the corresponding parabolic problem. In addition to the original papers quoted below we refer to Bramble and Osborn [3] for more details.

In each of these examples, the interior equations (5.1) are satisfied, and hence under the appropriate assumptions, the results of §§ 5, 6 and 7 are applicable. In particular, situations where assumptions of the forms (i″) and (ii″) of § 5 hold, with $\gamma(h)=Ch^r$ or $\gamma(h)=Ch^r \log 1/h$ are described in e.g. [5], [8] and [19].

8.1. The ordinary Galerkin method. This is the example mentioned in the introduction with $\{S_h\}$ satisfying (1.8) and $S_h \subset \dot{H}^1$ and with $T_h\colon L_2 \to S_h$ defined by

$$A(T_hf, \chi) = (f, \chi) \quad \text{for } \chi \in S_h. \tag{8.1}$$

It follows from (8.1) that

$$(T_hf, g) = A(T_hf, T_hg) = (f, T_hg) \quad \text{for } f, g \in L_2,$$

so that T_h is symmetric and positive semidefinite on L_2. For $f_h \in S_h$ and $T_hf_h = 0$ we conclude from (8.1) that $f_h = 0$ so that T_h is positive definite on S_h. This proves that condition (i) is satisfied for $\{T_h\}$.

As is well-known we have for $w = Tf$, $w_h = T_hf$,

$$\|w_h - w\|_{H^{-p}} \leqq Ch^{p+q}\|w\|_{H^q} \quad \text{for } 0 \leqq p \leqq r-2, \quad 2 \leqq q \leqq r. \tag{8.2}$$

Hence, using the fact that $\dot{H}^{q+2} \subset H^{q+2}$ and elliptic regularity, we obtain for $\partial\Omega$ smooth enough,

$$\|T_hf - Tf\|_{H^{-p}} \leqq Ch^{p+q+2}\|Tf\|_{q+2} \leqq Ch^{p+q+2}\|f\|_q$$
$$\text{for } 0 \leqq p, q \leqq r-2.$$

Since the norm on the left majorizes the $\|\cdot\|_{-p}$ norm, we conclude that (iia) holds, and if $r>2$, also (iib).

As a consequence, all the corresponding results of §§ 2, 3, 6 and 7 apply for the semidiscrete problem (1.9).

In [16], Nitsche considered the case $L = -\Delta$, and $r > 2$ with S_h based on a quasi-uniform triangulation (see [16] for the exact hypotheses). He proved that with $w = Tf$, $w_h = T_hf$,

$$|w_h| + h|w_h|_1 \leqq C\{|w| + h|w|_1\}, \qquad |w_h - w| \leqq Ch^r|w|_r.$$

Since obviously

$$\|w_h\| \leqq C\|w_h\|_1 \leqq C\|w\|_1,$$

it follows at once that (i′) and (ii′) are satisfied with $\gamma(h) = Ch^r$, so that also the results of § 4 apply.

Consider now the case of Neumann type boundary conditions mentioned in the introduction. Letting now S_h be contained in H^1 (without boundary conditions) and assume that $\{S_h\}$ instead of (1.8) satisfies

$$\inf_{\chi \in S_h} \{\|w-\chi\| + h\|w-\chi\|_{H^1}\} \leqq Ch^s\|w\|_{H^s} \quad \text{for } 1 \leqq s \leqq r. \tag{8.3}$$

The operator T_h can be defined again by (8.1) and it follows as above from known results about the Neumann problem that (i), (iia) and, if $r > 2$, (iib) hold.

Scott [20] considered the case $L=-\Delta+1$, $N=2$, and quasi-uniform triangulation (cf. [20] for details) and proved that

$$|w_h-w|\leqq \inf_{\chi\in S_h}|w-\chi|_1\begin{cases} Ch & \text{for } r>2,\\ Ch\log\dfrac{1}{h} & \text{for } r=2.\end{cases}$$

Under the asumptions of [20], (i′) and (ii′) follow with $\gamma(h)=Ch^r$ for $r>2$, $\gamma(h)=Ch^2\log 1/h$ for $r=2$.

In the remaining examples we shall consider Dirichlet boundary conditions and methods which do not require that the elements of S_h vanish on $\partial\Omega$.

8.2. Two methods of Nitsche. In [14], Nitsche introduced the bilinear form

$$B_h(\varphi,\psi)=A(\varphi,\psi)-\left\langle\varphi,\frac{\partial\psi}{\partial\nu}\right\rangle-\left\langle\frac{\partial\varphi}{\partial\nu},\psi\right\rangle+\beta h^{-1}\langle\varphi,\psi\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L_2(\partial\Omega)$, $\partial/\partial\nu$ the conormal derivative on $\partial\Omega$ and β a positive constant. It can be shown that if $S_h\subset H^1$ with the restrictions to $\partial\Omega$ in $H^1(\partial\Omega)$, and if $\{S_h\}$ satisfies

$$\inf_{\chi\in S_h}\{\|w-\chi\|+h\|w-\chi\|_{H^1}+h^{1/2}\|w-\chi\|_{L_2(\partial\Omega)}+h^{3/2}\|w-\chi\|_{H^1(\partial\Omega)}\}\leqq Ch^s\|w\|_{H^s},\quad 2\leqq s\leqq r, \tag{8.4}$$

and in addition an "inverse" assumption

$$\left\|\frac{\partial\chi}{\partial\nu}\right\|_{L_2(\partial\Omega)}\leqq Ch^{-1/2}\|\chi\|_{H^1}\quad\text{for }\chi\in S_h, \tag{8.5}$$

then B_h is positive definite on S_h for β suitably chosen. Defining the approximate solution operator T_h of the Dirichlet problem by

$$B_h(T_hf,\chi)=(f,\chi)\quad\text{for }\chi\in S_h, \tag{8.6}$$

(i) follows at once, and since again (8.2) is known, (iia) and (for $r>2$) (iib) follow as above.

In [15], Nitsche showed that if in addition to (8.4) and (8.5) we assume that for a sufficiently small constant C_0,

$$\|\chi\|_{L_2(\partial\Omega)}\leqq C_0h^{1/2}\|\chi\|_{H^1}\quad\text{for }\chi\in S_h,$$

(so that the functions in S_h are small on $\partial\Omega$) then B_h is positive definite on S_h even with $\beta=0$. Defining again T_h by (8.6) the conclusions are as above.

The semidiscrete parabolic equation in both cases takes the form

$$(D_tu_h,\chi)+B_h(u_h,\chi)=0\quad\text{for }\chi\in S_h.$$

8.3. The Lagrange multiplier method of Babuška. Let $\{\mathcal{S}_h\}$ be a family of subspaces of H^1 which satisfy (8.3) and let $\{\mathcal{S}'_h\}$ be a family of subspaces of $H^1(\partial\Omega)$ such that

$$\inf_{\chi'\in\mathcal{S}'_h}\{h^{-1/2}\|w'-\chi'\|_{H^{-1/2}(\partial\Omega)}+h^{1/2}\|w'-\chi'\|_{H^{1/2}(\partial\Omega)}\}$$
$$\leqq Ch^s\|w'\|_{H^s(\partial\Omega)} \quad \text{for } \tfrac{1}{2}\leqq s\leqq r-\tfrac{3}{2}, \quad w'\in H^s(\partial\Omega).$$

Assume also the inverse property

$$\|\chi'\|_{H^1(\partial\Omega)}\leqq Ch^{-1}\|\chi'\|_{L_2(\partial\Omega)} \quad \text{for } \chi'\in\mathcal{S}'_h.$$

With δ a sufficiently small positive number we may then define a family $\{S_h\}$ by

$$S_h=\{\chi\in\mathcal{S}_{\delta h}, \langle\chi,\chi'\rangle=0, \forall\chi'\in\mathcal{S}'_h\}.$$

It can be shown that the bilinear form $A(\varphi,\psi)$ is positive definite on S_h and we may define $T_h: L_2\to S_h$ by

$$A(T_hf,\chi)=(f,\chi) \quad \text{for } \chi\in S_h.$$

In the original paper by Babuška [1], this method was formulated in terms of Lagrange multipliers. The condition (i) is satisfied as above. Since again (8.2) holds, we have as before (iia) and (for $r>2$) (iib).

The semidiscrete parabolic equation can be written

$$(D_tu_h,\chi)+A(u_h,\chi)=0 \quad \text{for } \chi\in S_h.$$

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