J. Differential Equations 255 (2013) 572–592

Contents lists available at SciVerse ScienceDirect

Journal of Differential Equations

www.elsevier.com/locate/jde

Well-posedness of stochastic partial differential equations with Lyapunov condition ☆

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ARTICLE INFO

Article history:
Received 21 October 2011
Revised 14 April 2013
Available online 23 April 2013

MSC:
60H15
37L30
34D45

Keywords:
Local monotonicity
Lyapunov condition
Navier–Stokes equation
Mean curvature flow
p-Laplace equation
Fast diffusion equation

ABSTRACT

In this paper we show the existence and uniqueness of strong solutions for a large class of SPDE where the coefficients satisfy the local monotonicity and Lyapunov condition (one-sided linear growth condition). Moreover, some new invariance result and stronger regularity estimate are also established for the solutions. As examples, the main result is applied to stochastic tamed 3D Navier–Stokes equations, stochastic generalized curve shortening flow, singular stochastic p-Laplace equations, stochastic fast diffusion equations, stochastic Burgers type equations and stochastic reaction–diffusion equations.

1. Introduction

The main aim of this work is to prove the existence and uniqueness of strong solutions for a large class of stochastic partial differential equations (SPDE) using the variational approach. The variational framework has been used intensively for studying PDE and SPDE where the coefficients satisfying the classical monotonicity and coercivity conditions. In the case of deterministic equations, the theory of monotone operators started from the substantial work of Minty [25,26], then it was studied

☆ Supported in part by NSFC (No. 11201234), NSF of Jiangsu Higher Education Institutions (No. 12KJB110014), A Project Funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions and the DFG through the SFB-701 and IRTG 1132.

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0022-0396/$ – see front matter
http://dx.doi.org/10.1016/j.jde.2013.04.021

systematically by Browder [5,6] in order to obtain the existence of solutions for quasilinear elliptic and parabolic partial differential equations. We refer to the monographs [4,19,36,40] for more extensive exposition and references. Concerning the stochastic equations, it was first investigated in the seminal works of Pardoux [27] and Krylov and Rozovskii [18], where they adapted the monotonicity tricks to prove the existence and uniqueness of solutions for a class of semilinear and quasilinear SPDE. Later on, this result has been further generalized by many authors, see e.g. Gyöngy [14], Ren et al. [29], Röckner and Wang [30], Zhang [41] (and the references therein). Recently, this framework has been substantially extended by Röckner and the author in [23] for more general class of SPDE with coefficients satisfying the coercivity and local monotonicity conditions, hence many more fundamental examples such as stochastic Burgers type equations and stochastic 2D Navier–Stokes equations can be included into this framework now (see [1,7,12,22,24] for more examples).

However, the standard coercivity condition has been assumed in all the literatures mentioned above, which excludes some interesting examples. In this paper we will show the existence and uniqueness of strong solutions for a class of SPDE where the coefficients satisfy a specific type Lyapunov condition (we call it "one-sided linear growth" here) instead of the classical coercivity condition. Based on [23], we also use the local monotonicity condition here to replace the standard monotonicity condition. One motivating example is the stochastic tamed 3D Navier–Stokes equation, which is a regularized version of classical stochastic 3D Navier–Stokes equation and has been investigated by Röckner and Zhang [31,32] (see also [33,34]). One interesting feature of this equation is that one cannot find one appropriate Gelfand triple such that both the (local) monotonicity and coercivity conditions hold at the same time. Therefore, here we use a modified (coercivity) condition to overcome this difficulty. This Lyapunov type condition (see (H3) below) is inspired by the recent work of Es-sarhir and von Renesse [8](see also [9]), where they introduce this type of condition to study the stochastic curve shortening flow in the plane.

The main result (see Theorem 1.1) generalize the recent existence and uniqueness theorem in [8] (see Remark 1.2), hence it can be applied to the equation of stochastic curve shortening flow with some locally monotone perturbations in the drift (see Section 3.2). In fact, we apply the main result to a general divergence form of SPDE in Section 3.2, which also covers the example of singular stochastic p-Laplace equations. Comparing with the classical result (cf. [18,28]) on singular stochastic p-Laplace equations, we also obtain a new invariance result in a smaller state space for the solutions of a class of SPDE and also establish a much stronger regularity estimate for the solution, which could be used to substantially improve the recent result obtained in [21]. This line of investigation will be further continued in future works. Here we need to emphasize that we only consider nuclear noise in our framework instead of white noise, the reason is that we want to use the variational approach to study a very general class of SPDE and we want to establish stronger regularity estimates for the solution. The main result of this paper can be also applied to stochastic tamed 3D Navier–Stokes equations to establish the existence and uniqueness result. Unlike in [31,32,34], here we can use a different Gelfand triple to obtain the existence and uniqueness of solutions for stochastic tamed 3D Navier–Stokes equations and the proof is significantly simplified comparing with some previous works. Moreover, we should remark that the main result is also applicable to stochastic fast diffusion equations, stochastic Burgers type equations and stochastic reaction–diffusion equations having any (odd) degree polynomial perturbation with negative leading coefficient in the drift, which avoid the standard linear growth condition assumed in [18,27,28] and certain polynomial growth condition in [23]. We refer to Section 3 for more details.

Now we introduce the variational framework in details. Let $(H, \langle\cdot,\cdot\rangle_H)$ be a separable Hilbert space and identified with its dual space H^* by the Riesz isomorphism, and let $(V, \langle\cdot,\cdot\rangle_V)$ be a Hilbert space such that it is continuously and densely embedded into H. Then we have the following Gelfand triple

$$V \subset H \equiv H^* \subset V^*,$$

where V^* is the dual space of V (*w.r.t.* $\langle\cdot,\cdot\rangle_H$).

If ${}_{V^*}\langle\cdot,\cdot\rangle_V$ denotes the dualization between V and V^*, then it follows that

$${}_{V^*}\langle u, v\rangle_V = \langle u, v\rangle_H, \quad u \in H,\ v \in V.$$

Let $\{W_t\}_{t\geqslant 0}$ be a cylindrical Wiener process on a separable Hilbert space $(U, \langle\cdot,\cdot\rangle_U)$ *w.r.t.* a complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ and $(L_2(U;V), \|\cdot\|_2)$ denote the space of all Hilbert–Schmidt operators from U to V. We consider the following stochastic evolution equation

$$\mathrm{d}X_t = A(t, X_t)\,\mathrm{d}t + B(t, X_t)\,\mathrm{d}W_t, \tag{1.1}$$

where for some fixed time T,

$$A : [0,T] \times V \times \Omega \to V^*, \qquad B : [0,T] \times V \times \Omega \to L_2(U;V)$$

are progressively measurable, i.e. for every $t \in [0,T]$, these maps restricted to $[0,t] \times V \times \Omega$ are $\mathcal{B}([0,t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$-measurable (where $\mathcal{B}$ denotes the corresponding Borel σ-algebra).

We need to assume a further assumption concerning the Gelfand triple:

(H0) There exists an orthogonal set $\{e_1, e_2, \ldots\}$ in $(V, \langle\cdot,\cdot\rangle_V)$ such that it constitute an orthonormal basis of $(H, \langle\cdot,\cdot\rangle_H)$.

The precise conditions on the coefficients of (1.1) can be formulated as follows:

Suppose there exist constants $\alpha \geqslant 2$, K and a positive adapted process f such that the following conditions hold for all $v, v_1, v_2 \in V$ and $(t,\omega) \in [0,T] \times \Omega$.

(H1) (Hemicontinuity) The map $s \mapsto {}_{V^*}\langle A(t, v_1 + s v_2), v\rangle_V$ is continuous on $\mathbb{R}$.

(H2) (Local monotonicity) There exists a locally bounded measurable function $\rho : V \to [0, +\infty)$ such that

$$2{}_{V^*}\big\langle A(t,v_1) - A(t,v_2), v_1 - v_2\big\rangle_V + \big\|B(t,v_1) - B(t,v_2)\big\|^2_{L_2(U;H)} \leqslant \big(K + \rho(v_2)\big)\|v_1 - v_2\|^2_H.$$

(H3) (One-sided linear growth) For any $n \in \mathbb{N}$, the operator A maps $H^n := \mathrm{span}\{e_1, \ldots, e_n\}$ into V such that

$$\big\langle A(t,v), v\big\rangle_V \leqslant f_t + K\|v\|^2_V, \quad v \in H^n.$$

(H4) (Growth)

$$\big\|A(t,v)\big\|_{V^*} \leqslant f_t^{\frac{\alpha-1}{\alpha}} + K\|v\|_V^{\alpha-1},$$
$$\big\|B(t,v)\big\|_2^2 \leqslant K\big(f_t + \|v\|^2_V\big),$$
$$\rho(v) \leqslant K\big(1 + \|v\|^\alpha_V\big).$$

Remark 1.1. Recall that $\|\cdot\|_2$ denotes the norm $\|\cdot\|_{L_2(U;V)}$, hence there exists a constant C such that

$$\|\cdot\|_{L_2(U;H)} \leqslant C\|\cdot\|_2.$$

Definition 1.1 *(Solution of SPDE).* A continuous H-valued $(\mathcal{F}_t)$-adapted process $\{X_t\}_{t\in[0,T]}$ is called a solution of (1.1), if for its $\mathrm{d}t \times \mathbb{P}$-equivalent class $\bar{X}$ we have

$$\bar{X} \in L^\alpha\big([0,T] \times \Omega, \mathrm{d}t \times \mathbb{P}; V\big)$$

and $\mathbb{P}$-*a.s.*,

$$X_t = X_0 + \int_0^t A(s, \bar{X}_s)\,\mathrm{d}s + \int_0^t B(s, \bar{X}_s)\,\mathrm{d}W_s, \quad t \in [0,T].$$

Now we can state the main result of this work.

Theorem 1.1. *Suppose* (H0)–(H5) *hold for* $f \in L_H^{\frac{\alpha}{2}}([0,T] \times \Omega; dt \times \mathbb{P})$. *Then for any* $X_0 \in L^\alpha(\Omega, \mathcal{F}_0; \mathbb{P}; V)$, (1.1) *has a unique solution* $\{X_t\}_{t\in[0,T]}$ *and it satisfies*

$$X \in L^\alpha\big(\Omega; L^\infty\big([0,T]; V\big)\big) \cap L^\alpha\big(\Omega; C\big([0,T]; H\big)\big).$$

In particular, we have

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_H^\alpha + \mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_V^\alpha < \infty. \tag{1.2}$$

Moreover, if $A(t,\cdot)(\omega)$, $B(t,\cdot)(\omega)$ *are independent of* $t \in [0,T]$ *and* $\omega \in \Omega$, *then the solution* $\{X_t\}_{t\in[0,T]}$ *of* (1.1) *is a Markov process.*

Remark 1.2. (1) If (H3) is replaced by the following classical coercivity condition ($\delta > 0$):

$${}_{V^*}\big\langle A(t,v), v\big\rangle_V \leqslant -\delta\|v\|_V^\alpha + f_t + K\|v\|_H^2, \quad v \in V, \tag{1.3}$$

then the existence and uniqueness of strong solutions has been recently established in [23] ((H0) can be avoided in this case). Comparing with (1.2), in [23] one can only show the solution satisfies the following regularity estimate:

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|X_t\|_H^\alpha + \int_0^T \|X_t\|_V^\alpha \, dt\right) < \infty.$$

(2) Similar type of regularity estimate as (1.2) has been established in [21] for a class of SPDE with additive noise using some Yosida approximation techniques (see also [11,30]). Using similar techniques as in [21], under some conditions, it is possible to prove the right continuity of solution in V and obtain the following stronger estimate:

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_V^\alpha < \infty.$$

This property and some further applications will be investigated in future work.

(3) The role of classical coercivity condition is to obtain some a priori estimate of the solution *w.r.t.* V-norm (cf. [18,23,28]) by applying Itô's formula to $\|\cdot\|_H^2$ (in this case $\|\cdot\|_H^2$ is a good Lyapunov function). However, for some SPDEs (see Section 3 for concrete examples) the coercivity condition (1.3) fails to hold, however, one can show the one-sided linear growth condition (H3) (we refer to the recent work of Brzeźniak et al. [2] for a similar type condition posed in the setting of SPDE in M-type 2 Banach spaces and Brzeźniak and Peszat [3] for similar techniques used for the case $\alpha > 2$). In this case $\|\cdot\|_V^2$ turns out to be a good Lyapunov function, one can get some analog a priori estimate from (H3) by using Itô's formula to $\|\cdot\|_V^\alpha$.

Another remark is that in [22](see also [24]) we introduced a more general type of local monotonicity condition (see (3.2) in Section 3) than (H2), which links to the important concept of pseudo monotone operator introduced by Brézis.

(4) This theorem also generalizes the recent result of Es-sarhir and von Renesse in [8] (Theorem 2.3), where they proved a similar result in the case of $\rho \equiv 0$ in (H2) and $\alpha = 2$ in (H4). Note that $\alpha = 2$ in (H4) implies that A has at most linear growth, which excludes some interesting examples. Therefore, the result in [8] cannot be applied to the examples of stochastic tamed 3D Navier–Stokes equations, stochastic Burgers type equations and stochastic reaction–diffusion equations (see Section 3 for the details).

The rest of the paper is organized as follows: the proof of the main theorem is given in the next section; in Section 3 some concrete examples of SPDE will be investigated as the application of the main result. Throughout this paper C always denote some generic constant which may change from line to line.

2. Proof of Theorem 1.1

The proof combines the arguments used in [8] and [23]. Based on Galerkin approximation, we will use the one-sided linear growth condition (H3) to obtain some a priori estimates for the approximated solutions by applying Itô's formula to $\|\cdot\|_V^\alpha$, then we will show the limit of the approximated solutions solves the original equation (1.1) using the local monotonicity techniques in [23].

We first consider the standard Galerkin approximation to (1.1). Recall that

$$\{e_1, e_2, \ldots\} \subset V$$

is an orthonormal basis of H and $H_n = \mathrm{span}\{e_1, \ldots, e_n\}$. Let $P_n : V^* \to H_n$ be defined by

$$P_n y = \sum_{i=1}^{n} {}_{V^*}\langle y, e_i\rangle_V e_i, \quad y \in V^*.$$

Hence $P_n|_H$ is the orthogonal projection onto H_n in H and we have

$${}_{V^*}\big\langle P_n A(t,u), v\big\rangle_V = \big\langle P_n A(t,u), v\big\rangle_H = {}_{V^*}\big\langle A(t,u), v\big\rangle_V, \quad u \in V,\ v \in H_n.$$

By (H0) we know that for $v \in H_n$ we have

$$\langle e_i, v\rangle_V = 0, \quad i \geqslant n+1.$$

By (H3) we know that for $u \in H_n$ we have $A(t,u) \in V$ and

$$\sum_{i=1}^{\infty} {}_{V^*}\big\langle A(t,u), e_i\big\rangle_V e_i = \sum_{i=1}^{\infty} \big\langle A(t,u), e_i\big\rangle_H e_i = A(t,u).$$

Then for any $u, v \in H_n$ we have

$$\begin{aligned}\big\langle P_n A(t,u), v\big\rangle_V &= \left\langle \sum_{i=1}^{n} {}_{V^*}\big\langle A(t,u), e_i\big\rangle_V e_i, v\right\rangle_V = \left\langle \sum_{i=1}^{\infty} {}_{V^*}\big\langle A(t,u), e_i\big\rangle_V e_i, v\right\rangle_V \\ &= \big\langle A(t,u), v\big\rangle_V. \end{aligned} \tag{2.1}$$

Let $\{g_1, g_2, \ldots\}$ be an orthonormal basis of U and

$$W_t^{(n)} := \sum_{i=1}^{n} \langle W_t, g_i\rangle_U g_i = \tilde{P}_n W_t,$$

where $\tilde{P}_n$ is the orthogonal projection onto $\mathrm{span}\{g_1, \ldots, g_n\}$ in U.

Then for each $n \in \mathbb{N}$ we consider the following (finite-dimensional) stochastic equation on H_n:

$$\mathrm{d}X_t^{(n)} = P_n A\big(t, X_t^{(n)}\big)\,\mathrm{d}t + P_n B\big(t, X_t^{(n)}\big)\,\mathrm{d}W_t^{(n)}, \qquad X_0^{(n)} = P_n X_0. \tag{2.2}$$

According to the classical result for the solvability of SDE in finite-dimensional space (cf. e.g. [17,18]), (H0)–(H4) imply that (2.2) has a unique strong solution.

For convenience we use the following notations in the proof:

$$K = L^\alpha\big([0,T]\times\Omega, \mathrm{d}t\times\mathbb{P}; V\big),$$
$$K^* = L^{\frac{\alpha}{\alpha-1}}\big([0,T]\times\Omega, \mathrm{d}t\times\mathbb{P}; V^*\big),$$
$$J = L^2\big([0,T]\times\Omega, \mathrm{d}t\times\mathbb{P}; L_2(U;H)\big).$$

Now we construct the solution of (1.1). As preparation we first need to get some a priori estimates for $X^{(n)}$.

Lemma 2.1. *Under the assumptions in Theorem 1.1, there exists $C>0$ such that for all $n\in\mathbb{N}$ we have*

$$\mathbb{E}\sup_{t\in[0,T]}\big\|X_t^{(n)}\big\|_V^\alpha \leqslant C\left(\mathbb{E}\|X_0\|_V^\alpha + \mathbb{E}\int_0^T f_t^{\alpha/2}\,\mathrm{d}t\right). \tag{2.3}$$

In particular, there exists $C>0$ such that for all $n\in\mathbb{N}$,

$$\big\|X^{(n)}\big\|_K + \big\|A\big(\cdot, X^{(n)}\big)\big\|_{K^*} + \mathbb{E}\sup_{t\in[0,T]}\big\|X_t^{(n)}\big\|_H^\alpha \leqslant C\left(\mathbb{E}\|X_0\|_V^\alpha + \mathbb{E}\int_0^T \big(f_t + f_t^{\alpha/2}\big)\,\mathrm{d}t\right).$$

Proof. By Itô's formula, (2.1), (H4) and (H3) we have for any $p\geqslant 2$,

$$\begin{aligned}
\big\|X_t^{(n)}\big\|_V^p &= \big\|X_0^{(n)}\big\|_V^p + p(p-2)\int_0^t \big\|X_s^{(n)}\big\|_V^{p-4}\big\|\big(P_nB\big(s,X_s^{(n)}\big)\tilde{P}_n\big)^*X_s^{(n)}\big\|_V^2\,\mathrm{d}s \\
&\quad + \frac{p}{2}\int_0^t \big\|X_s^{(n)}\big\|_V^{p-2}\big(2\big\langle A\big(s,X_s^{(n)}\big), X_s^{(n)}\big\rangle_V + \big\|P_nB\big(s,X_s^{(n)}\big)\tilde{P}_n\big\|_2^2\big)\,\mathrm{d}s \\
&\quad + p\int_0^t \big\|X_s^{(n)}\big\|_V^{p-2}\big\langle X_s^{(n)}, P_nB\big(s,X_s^{(n)}\big)\,\mathrm{d}W_s^{(n)}\big\rangle_V \\
&\leqslant \|X_0\|_V^p + C\int_0^t \big(\big\|X_s^{(n)}\big\|_V^p + f_s\cdot\big\|X_s^{(n)}\big\|_V^{p-2}\big)\,\mathrm{d}s \\
&\quad + p\int_0^t \big\|X_s^{(n)}\big\|_V^{p-2}\big\langle X_s^{(n)}, P_nB\big(s,X_s^{(n)}\big)\,\mathrm{d}W_s^{(n)}\big\rangle_V \\
&\leqslant \|X_0\|_V^p + C\int_0^t \big(\big\|X_s^{(n)}\big\|_V^p + f_s^{p/2}\big)\,\mathrm{d}s + p\int_0^t \big\|X_s^{(n)}\big\|_V^{p-2}\big\langle X_s^{(n)}, P_nB\big(s,X_s^{(n)}\big)\,\mathrm{d}W_s^{(n)}\big\rangle_V, \\
&\quad t\in[0,T],
\end{aligned} \tag{2.4}$$

where constant C is independent of n.

For any given n we define the stopping time

$$\tau_R^{(n)} = \inf\{t \in [0,T]: \|X_t^{(n)}\|_V > R\} \wedge T, \quad R > 0,$$

where we take $\inf\emptyset = \infty$ as convention.

It is obvious that

$$\lim_{R\to\infty} \tau_R^{(n)} = T, \quad \mathbb{P} - a.s., \ n \in \mathbb{N}.$$

Then by the Burkholder–Davis–Gundy inequality and (H4) we have

$$\begin{aligned}
&\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r \|X_s^{(n)}\|_V^{p-2}\langle X_s^{(n)}, P_n B(s, X_s^{(n)})\,\mathrm{d}W_s^{(n)}\rangle_V\right| \\
&\leqslant 3\mathbb{E}\left(\int_0^t \|X_s^{(n)}\|_V^{2p-2}\|B(s, X_s^{(n)})\|_2^2\,\mathrm{d}s\right)^{1/2} \\
&\leqslant 3\mathbb{E}\left(\sup_0 s\in[0,t]\|X_s^{(n)}\|_V^{2p-2}\cdot C\int_0^t \left(\|X_s^{(n)}\|_V^2 + f_s\right)\mathrm{d}s\right)^{1/2} \\
&\leqslant 3\mathbb{E}\left[\varepsilon\sup_{s\in[0,t]}\|X_s^{(n)}\|_V^p + C_\varepsilon\left(\int_0^t \left(\|X_s^{(n)}\|_V^2 + f_s\right)\mathrm{d}s\right)^{p/2}\right] \\
&\leqslant 3\varepsilon\mathbb{E}\sup_{s\in[0,t]}\|X_s^{(n)}\|_V^p + 3\cdot(2T)^{p/2-1}C_\varepsilon\mathbb{E}\int_0^t \left(\|X_s^{(n)}\|_V^p + f_s^{p/2}\right)\mathrm{d}s, \quad t\in[0,\tau_R^{(n)}],
\end{aligned} \tag{2.5}$$

where $\varepsilon > 0$ is a small constant and C_ε comes from Young's inequality.

Then by (2.4), (2.5) and Gronwall's lemma we have

$$\mathbb{E}\sup_{t\in[0,\tau_R^{(n)}]}\|X_t^{(n)}\|_V^p \leqslant C\left(\mathbb{E}\|X_0\|_V^p + \mathbb{E}\int_0^T f_s^{p/2}\,\mathrm{d}s\right), \quad n \geqslant 1,$$

where C is independent of n.

Let $p = \alpha$ and $R \to \infty$, then (2.3) follows from the monotone convergence theorem. □

The rest of the proof is similar to the argument in [23], we include it here for completeness.

Proof of Theorem 1.1. (1) Existence: by Lemma 2.1 there exists a subsequence $n_k \to \infty$ such that

(i) $X^{(n_k)} \to \bar{X}$ weakly in K and weakly star in $L^\alpha(\Omega; L^\infty([0,T]; V))$.
(ii) $Y^{(n_k)} := A(\cdot, X^{(n_k)}) \to Y$ weakly in K^*.
(iii) $Z^{(n_k)} := P_{n_k}B(\cdot, X^{(n_k)}) \to Z$ weakly in J and hence

$$\int_0^\cdot P_{n_k}B(s, X_s^{(n_k)})\,\mathrm{d}W_s^{(n_k)} \to \int_0^\cdot Z_s\,\mathrm{d}W_s$$

weakly in $L^\infty([0,T], \mathrm{d}t; L^2(\Omega, \mathbb{P}; H))$.

Now we define the following process

$$X_t := X_0 + \int_0^t Y_s \, \mathrm{d}s + \int_0^t Z_s \, \mathrm{d}W_s, \quad t \in [0, T], \tag{2.6}$$

then it is easy to show that $X = \bar{X}$, $\mathrm{d}t \times \mathbb{P}$*-a.e.*

By [28, Theorem 4.2.5] (or [18]) and Lemma 2.1 we know that X is an H-valued continuous $(\mathcal{F}_t)$-adapted process and satisfies

$$\mathbb{E} \sup_{t \in [0,T]} \|X_t\|_H^\alpha + \mathbb{E} \operatorname*{ess\,sup}_{t \in [0,T]} \|X_t\|_V^\alpha < \infty.$$

Therefore, for the existence of solutions to (1.1) it is sufficient to show that

$$A(\cdot, \bar{X}) = Y, \qquad B(\cdot, \bar{X}) = Z, \qquad \mathrm{d}t \times \mathbb{P} - \text{a.e.}$$

Define

$$\mathcal{M} = \left\{ \phi \colon \phi \text{ is } V\text{-valued } (\mathcal{F}_t)\text{-adapted process such that } \mathbb{E} \int_0^T \rho(\phi_s) \, \mathrm{d}s < \infty \right\}.$$

For $\phi \in K \cap \mathcal{M} \cap L^\alpha(\Omega; L^\infty([0, T]; H))$, using Itô's formula we have (cf. e.g. the proofs of Lemma 3.3 and Theorem 4.1 in [35])

$$\begin{aligned}
&\mathbb{E}\big(e^{-\int_0^t (K+\rho(\phi_s))\,\mathrm{d}s} \big\|X_t^{(n_k)}\big\|_H^2\big) - \mathbb{E}\big(\big\|X_0^{(n_k)}\big\|_H^2\big) \\
&\quad = \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r} \big(2_{V^*}\big\langle P_{n_k} A\big(s, X_s^{(n_k)}\big), X_s^{(n_k)}\big\rangle_V \\
&\qquad + \big\|P_{n_k} B\big(s, X_s^{(n_k)}\big)\tilde{P}_{n_k}\big\|_{L_2(U;H)}^2 - \big(K + \rho(\phi_s)\big)\big\|X_s^{(n_k)}\big\|_H^2\big)\,\mathrm{d}s\Bigg] \\
&\quad \leqslant \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r} \big(2_{V^*}\big\langle A\big(s, X_s^{(n_k)}\big), X_s^{(n_k)}\big\rangle_V \\
&\qquad + \big\|B\big(s, X_s^{(n_k)}\big)\big\|_{L_2(U;H)}^2 - \big(K + \rho(\phi_s)\big)\big\|X_s^{(n_k)}\big\|_H^2\big)\,\mathrm{d}s\Bigg] \\
&\quad = \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r} \big(2_{V^*}\big\langle A\big(s, X_s^{(n_k)}\big) - A(s, \phi_s), X_s^{(n_k)} - \phi_s\big\rangle_V \\
&\qquad + \big\|B\big(s, X_s^{(n_k)}\big) - B(s, \phi_s)\big\|_{L_2(U;H)}^2 - \big(K + \rho(\phi_s)\big)\big\|X_s^{(n_k)} - \phi_s\big\|_H^2\big)\,\mathrm{d}s\Bigg] \\
&\qquad + \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r} \big(2_{V^*}\big\langle A\big(s, X_s^{(n_k)}\big) - A(s, \phi_s), \phi_s\big\rangle_V
\end{aligned}$$

$$+ 2{}_{V^*}\big\langle A(s,\phi_s), X_s^{(n_k)}\big\rangle_V - \big\|B(s,\phi_s)\big\|^2_{L_2(U;H)} + 2\big\langle B\big(s, X_s^{(n_k)}\big), B(s,\phi_s)\big\rangle_{L_2(U,H)}$$
$$- 2\big(K+\rho(\phi_s)\big)\big\langle X_s^{(n_k)}, \phi_s\big\rangle_H + \big(K+\rho(\phi_s)\big)\|\phi_s\|_H^2\big)\,\mathrm{d}s\Bigg]. \tag{2.7}$$

Let $k \to \infty$, by (H2) and the lower semicontinuity (cf. [28, (4.2.27)] for details) we have for every nonnegative $\psi \in L^\infty([0,T];\mathrm{d}t)$,

$$\mathbb{E}\Bigg[\int_0^T \psi_t\big(e^{-\int_0^t (K+\rho(\phi_s))\,\mathrm{d}s}\|X_t\|_H^2 - \|X_0\|_H^2\big)\,\mathrm{d}t\Bigg]$$
$$\leqslant \liminf_{k\to\infty}\mathbb{E}\Bigg[\int_0^T \psi_t\big(e^{-\int_0^t (K+\rho(\phi_s))\,\mathrm{d}s}\big\|X_t^{(n_k)}\big\|_H^2 - \big\|X_0^{(n_k)}\big\|_H^2\big)\,\mathrm{d}t\Bigg]$$
$$\leqslant \mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r}\big(2{}_{V^*}\big\langle Y_s - A(s,\phi_s), \phi_s\big\rangle_V$$
$$+ 2{}_{V^*}\big\langle A(s,\phi_s), \bar{X}_s\big\rangle_V - \big\|B(s,\phi_s)\big\|^2_{L_2(U;H)} + 2\big\langle Z_s, B(s,\phi_s)\big\rangle_{L_2(U,H)}$$
$$- 2\big(K+\rho(\phi_s)\big)\langle X_s, \phi_s\rangle_H + \big(K+\rho(\phi_s)\big)\|\phi_s\|_H^2\big)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]. \tag{2.8}$$

By Itô's formula we have for $\phi \in K \cap \mathcal{M} \cap L^\alpha(\Omega; L^\infty([0,T];H))$,

$$\mathbb{E}\big(e^{-\int_0^t (K+\rho(\phi_s))\,\mathrm{d}s}\|X_t\|_H^2\big) - \mathbb{E}\big(\|X_0\|_H^2\big)$$
$$= \mathbb{E}\Bigg[\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r}\big(2{}_{V^*}\langle Y_s, \bar{X}_s\rangle_V + \|Z_s\|^2_{L_2(U;H)} - \big(K+\rho(\phi_s)\big)\|X_s\|_H^2\big)\,\mathrm{d}s\Bigg]. \tag{2.9}$$

Combining (2.9) with (2.8) we obtain that

$$0 \geqslant \mathbb{E}\Bigg[\int_0^T \psi_t\Bigg(\int_0^t e^{-\int_0^s (K+\rho(\phi_r))\,\mathrm{d}r}\big(2{}_{V^*}\big\langle Y_s - A(s,\phi_s), \bar{X}_s - \phi_s\big\rangle_V$$
$$+ \big\|B(s,\phi_s) - Z_s\big\|^2_{L_2(U;H)} - \big(K+\rho(\phi_s)\big)\|X_s - \phi_s\|_H^2\big)\,\mathrm{d}s\Bigg)\,\mathrm{d}t\Bigg]. \tag{2.10}$$

Note that Lemma 2.1 and (H4) imply that

$$\bar{X} \in K \cap \mathcal{M} \cap L^\alpha\big(\Omega; L^\infty\big([0,T];H\big)\big),$$

then by taking $\phi = \bar{X}$ we obtain that $Z = B(\cdot, \bar{X})$.

Moreover, if we first take $\phi = \bar{X} - \varepsilon\tilde{\phi}v$ for $\tilde{\phi} \in L^\infty([0,T] \times \Omega; \mathrm{d}t \times \mathbb{P}; \mathbb{R})$ and $v \in V$, then divide it by ε and let $\varepsilon \to 0$ we have

$$0 \geqslant \mathbb{E}\left[\int_0^T \psi_t \left(\int_0^t e^{-\int_0^s (K+\rho(\bar{X}_r))\,\mathrm{d}r} \tilde{\phi}_s\, {}_{V^*}\langle Y_s - A(s,\bar{X}_s), v\rangle_V \,\mathrm{d}s\right) \mathrm{d}t\right]. \tag{2.11}$$

Then $Y = A(\cdot, \bar{X})$ follows from the arbitrariness of ψ and $\tilde{\phi}$.

Therefore, $\bar{X}$ is a solution of (1.1).

(2) Uniqueness: suppose X_t and Y_t are the solutions of (1.1) with initial conditions X_0 and Y_0 respectively, i.e.

$$X_t = X_0 + \int_0^t A(s,X_s)\,\mathrm{d}s + \int_0^t B(s,X_s)\,\mathrm{d}W_s, \quad t \in [0,T],$$

$$Y_t = Y_0 + \int_0^t A(s,Y_s)\,\mathrm{d}s + \int_0^t B(s,Y_s)\,\mathrm{d}W_s, \quad t \in [0,T]. \tag{2.12}$$

Then by Itô's formula and (H2) we have (cf. e.g. [35])

$$\begin{aligned} &e^{-\int_0^t (K+\rho(Y_s))\,\mathrm{d}s} \|X_t - Y_t\|_H^2 \\ &\quad \leqslant \|X_0 - Y_0\|_H^2 + 2\int_0^t e^{-\int_0^s (K+\rho(Y_r))\,\mathrm{d}r} \langle X_s - Y_s, B(s,X_s)\,\mathrm{d}W_s - B(s,Y_s)\,\mathrm{d}W_s\rangle_H, \quad t \in [0,T]. \end{aligned}$$

By a standard localization argument we have

$$\mathbb{E}\left[e^{-\int_0^t (K+\rho(Y_s))\,\mathrm{d}s} \|X_t - Y_t\|_H^2\right] \leqslant \mathbb{E}\|X_0 - Y_0\|_H^2, \quad t \in [0,T].$$

If $X_0 = Y_0$, $\mathbb{P}$-*a.s.*, then

$$\mathbb{E}\left[e^{-\int_0^t (K+\rho(Y_s))\,\mathrm{d}s} \|X_t - Y_t\|_H^2\right] = 0, \quad t \in [0,T].$$

Since (H4) and Lemma 2.1 imply that

$$\int_0^t \big(K + \rho(Y_s)\big)\,\mathrm{d}s < \infty, \quad \mathbb{P}\text{-}a.s.,\ t \in [0,T],$$

we have

$$X_t = Y_t, \quad \mathbb{P}\text{-}a.s.,\ t \in [0,T].$$

Therefore, the pathwise uniqueness follows from the path continuity of X and Y in H.

(3) Markov property: the proof of Markov property is same as in [28, Proposition 4.3.5] (see also [18, Theorem II.2.4]), hence we omit the details here. □

3. Application to examples

3.1. Stochastic tamed 3D Navier–Stokes equation

Let Λ be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary and $C_0^\infty(\Lambda, \mathbb{R}^3)$ denote the set of all smooth functions from Λ to $\mathbb{R}^3$ with compact support. For $p \geqslant 1$, let $(L^p(\Lambda, \mathbb{R}^3), \|\cdot\|_{L^p})$ be the vector valued L^p-space. For any integer $m > 0$, let $W_0^{m,2}$ denote the standard Sobolev space on Λ with values in $\mathbb{R}^3$, i.e. the closure of $C_0^\infty(\Lambda, \mathbb{R}^3)$ with respect to the following norm:

$$\|u\|_{W_0^{m,2}}^2 = \left(\sum_{0 \leqslant |\alpha| \leqslant m} \int_\Lambda |D^\alpha u|^2 \, dx \right)^2.$$

For the reader's convenience, we recall the following Gagliardo–Nirenberg interpolation inequality, which is used very often in the study of PDE theory.

Proposition 3.1. *If $q \in [1, \infty]$ such that*

$$\frac{1}{q} = \frac{1}{2} - \frac{m\gamma}{3}, \quad 0 \leqslant \gamma \leqslant 1,$$

then there exists a constant $C_{m,q} > 0$ such that for any $u \in W_0^{m,2}$,

$$\|u\|_{L^q} \leqslant C_{m,q} \|u\|_{W_0^{m,2}}^\gamma \|u\|_{L^2}^{1-\gamma}. \tag{3.1}$$

Now we define

$$H^0 := \{ u \in L^2(\Lambda, \mathbb{R}^3) \colon div(u) = 0 \}, \qquad H^m := \{ u \in W_0^{m,2} \colon div(u) = 0 \}.$$

The norm of $W_0^{m,2}$ restricted to H^m will be denoted by $\|\cdot\|_{H^m}$. Note that H^0 is a closed linear subspace of the Hilbert space $L^2(\Lambda, \mathbb{R}^3)$. In the literature it is well known that one can use the following Gelfand triple

$$V := H^1 \subseteq H := H^0 \subseteq V^*, \tag{$\star$}$$

to analyze the Navier–Stokes equation and it works very well in 2D case even with general stochastic perturbations (cf. [1,23,39] and the references therein). However, as pointed out in [23,22], the growth condition in [23] (which is different with (H4) here) fails to hold on this Gelfand triple for the 3D Navier–Stokes equation.

Motivated by the recent works on (stochastic) tamed 3D Navier–Stokes equation (cf. [31–33]), one can verify the growth condition in [23,22] if we use the following Gelfand triple:

$$V := H^2 \subseteq H := H^1 \subseteq V^*. \tag{$\star\star$}$$

But there exist two problems for working with the Gelfand triple ($\star\star$). The first one is that the classical coercivity condition (see (1.3)) does not hold anymore for 3D Navier–Stokes equation under this Gelfand triple. One possibility to verify the classical coercivity condition is to add a dissipative control term (see g_N in (3.3) below) to (stochastic) 3D Navier–Stokes equation.

The second problem is that, instead of (H2), we can only verify the following type local monotonicity condition for 3D Navier–Stokes equation under the triple $(\star\star)$ (cf. [22]):

$$ {}_{V^*}\langle A(t,v_1)-A(t,v_2), v_1-v_2\rangle_V \leqslant \big(K+\eta(v_1)+\rho(v_2)\big)\|v_1-v_2\|_H^2, \tag{3.2}$$

where $\eta,\rho: V\to[0,+\infty)$ are some locally bounded measurable functions. The additional term η (depend on another variable v_1) will cause many technical difficulties in the proof of existence of solutions, we refer to [22] where the well-posedness of a class of PDE has been established under this type of local monotonicity condition.

Therefore, the motivation of using (H2) and (H3) here is that one can combine the advantages of working with these two Gelfand triples such that both the local monotonicity condition and one-sided linear growth condition hold on one Gelfand triple. More precisely, we will use the Gelfand triple $(\star)$ for stochastic tamed 3D Navier–Stokes equation in order to verify the local monotonicity condition (H2) (instead of (3.2)); then we can verify the one-sided linear growth condition (H3) under this triple, which is formally equivalent to the classical coercivity condition (1.3) under the Gelfand triple $(\star\star)$; finally, we can show that the growth condition (H4) also holds for stochastic tamed 3D Navier–Stokes equation on the triple $(\star)$.

For all the examples in below, $\{W_t\}_{t\geqslant 0}$ denotes a cylindrical Wiener process on a separable Hilbert space $(U,\langle\cdot,\cdot\rangle_U)$ *w.r.t.* a complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$.

The first example is a tamed version of stochastic 3D Navier–Stokes equation, which has been recently investigated in a series of works of Röckner et al [31–34]. The classical 3D Navier–Stokes equations (i.e. $g_N=0$, $B=0$ in (3.3)) is a standard model to describe the evolution of velocity fields of an incompressible fluid (cf. [10,20,39]), the uniqueness and regularity of weak solutions are still open problems up to now. The stochastic tamed 3D Navier–Stokes equation is a regularized version of the classical stochastic 3D Navier–Stokes equation and it can be formulated as follows:

$$\begin{aligned} &\mathrm{d}X_t=\big[\nu\Delta X_t-(X_t\cdot\nabla)X_t+\nabla p(t)-g_N\big(|X_t|^2\big)X_t\big]\,\mathrm{d}t+B(X_t)\,\mathrm{d}W_t,\\ &div(X_t)=0,\qquad X_0=x_0,\\ &X_t|_{\partial\Lambda}=0, \end{aligned}\tag{3.3}$$

where $\nu>0$ is the viscosity constant, p is the (unknown) pressure and the taming function $g_N:\mathbb{R}_+\to\mathbb{R}_+$ is smooth and satisfies for some $N>0$,

$$\begin{cases} g_N(r)=0, & \text{if}\, r\leqslant N,\\ g_N(r)=(r-N)/\nu, & \text{if}\, r\geqslant N+1,\\ 0\leqslant g_N'(r)\leqslant C, & r\geqslant 0. \end{cases}$$

The main feature of (3.3) is that if there is a bounded smooth solution to the classical (stochastic) 3D Navier–Stokes equation, then this smooth solution must also satisfy this tamed equation for some large enough N.

Let $\mathcal{P}$ be the orthogonal (Helmhotz–Leray) projection from $L^2(\Lambda,\mathbb{R}^3)$ to H^0 (cf. [20,39]). It's well known that $\mathcal{P}$ is continuous (cf. [13,37]). For any $u\in H^0$ and $v\in L^2(\Lambda,\mathbb{R}^3)$ we have

$$\langle u,v\rangle_{H^0}:=\langle u,\mathcal{P}v\rangle_{H^0}=\langle u,v\rangle_{L^2}.$$

We consider the following Gelfand triple:

$$V:=H^1\subseteq H:=H^0\subseteq V^*=\big(H^1\big)^*,$$

then it is well known that the following operators

$$A : W^{2,2}\big(\Lambda, \mathbb{R}^3\big) \cap V \to H, \qquad Au = \nu \mathcal{P} \Delta u,$$
$$F : \mathcal{D}_{\mathcal{F}} \subset H \times V \to H, \qquad F(u,v) = -\mathcal{P}\big[(u \cdot \nabla) v\big], \qquad F(u) := F(u,u)$$

can be extended to the following well defined operators:

$$A : V \to V^*, \qquad F : V \times V \to V^*.$$

Moreover, we have

$$_{V^*}\big\langle F(u,v), w\big\rangle_V = -_{V^*}\big\langle F(u,w), v\big\rangle_V, \qquad _{V^*}\big\langle F(u,v), v\big\rangle_V = 0, \quad u, v, w \in V. \tag{3.4}$$

Without loss of generality we may assume $\nu = 1$. Now we show the existence and uniqueness of solutions to (3.3).

Example 3.2. Suppose $x_0 \in L^4(\Omega, \mathcal{F}_0, \mathbb{P}; H^1)$ and $B : V \to (L_2(U;V); \|\cdot\|_2)$ satisfies that

$$\begin{aligned} &\big\|B(v)\big\|_2^2 \leqslant C\big(1 + \|v\|_V^2\big), \quad v \in V, \\ &\big\|B(v_1) - B(v_2)\big\|_{L_2(U;H)}^2 \leqslant C\|v_1 - v_2\|_H^2, \quad v_1, v_2 \in V. \end{aligned} \tag{3.5}$$

Then (3.3) has a unique solution $X \in L^4(\Omega, \mathbb{P}, L^\infty([0,T]; H^1)) \cap L^4(\Omega, \mathbb{P}, C([0,T]; H^0))$. In particular, we have

$$\mathbb{E} \sup_{t \in [0,T]} \|X_t\|_H^4 + \mathbb{E} \operatorname*{ess\,sup}_{t \in [0,T]} \|X_t\|_V^4 < \infty.$$

Proof. It is well known that (3.3) can be rewritten into the following variational form:

$$\mathrm{d}X_t = \big[AX_t + F(X_t) - \mathcal{P}\big(g_N\big(|X_t|^2\big)X_t\big)\big]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0.$$

It is easy to see that all eigenvectors $\{e_i, i = 1, 2, \ldots\} \subset H^2$ of A constitute an orthonormal basis of H^0 and an orthogonal set in H^1, i.e. (H0) holds.

By Hölder's inequality we have the following estimate:

$$\|\psi\|_{L^3(\Lambda;\mathbb{R}^3)} \leqslant \|\psi\|_{L^2(\Lambda;\mathbb{R}^3)}^{1/2} \|\psi\|_{L^6(\Lambda;\mathbb{R}^3)}^{1/2}, \quad \psi \in L^6\big(\Lambda; \mathbb{R}^3\big).$$

Note that $W_0^{1,2}(\Lambda; \mathbb{R}^3) \subseteq L^6(\Lambda; \mathbb{R}^3)$, then by (3.4) one can show that

$$\begin{aligned} _{V^*}\big\langle F(u) - F(v), u - v\big\rangle_V &= -_{V^*}\big\langle F(u - v), v\big\rangle_V \\ &\leqslant C\|u - v\|_V \|u - v\|_{L^3(\Lambda;\mathbb{R}^3)} \|v\|_{L^6(\Lambda;\mathbb{R}^3)} \\ &\leqslant C\|u - v\|_V^{3/2} \|u - v\|_H^{1/2} \|v\|_{L^6(\Lambda;\mathbb{R}^3)} \\ &\leqslant \frac{1}{2}\|u - v\|_V^2 + C\|v\|_{L^6(\Lambda;\mathbb{R}^3)}^4 \|u - v\|_H^2, \quad u, v \in V. \end{aligned}$$

Hence we have the following estimate (recall that $\nu = 1$):

$$_{V^*}\big\langle Au + F(u) - Av - F(v), u - v\big\rangle_V \leqslant -\frac{1}{2}\|u - v\|_V^2 + C\big(1 + \|v\|_{L^6(\Lambda;\mathbb{R}^3)}^4\big)\|u - v\|_H^2.$$

By the definition of g_N and (3.1) we have

$$\begin{aligned}
&-{}_{V^*}\big\langle \mathcal{P}\big(g_N\big(|u|^2\big)u\big) - \mathcal{P}\big(g_N\big(|v|^2\big)v\big), u-v\big\rangle_V \\
&\quad = -\big\langle g_N\big(|v|^2\big)(u-v), u-v\big\rangle_H + \big\langle \big(g_N\big(|v|^2\big) - g_N\big(|u|^2\big)\big)u, u-v\big\rangle_H \\
&\quad \leqslant \int\limits_{\{|u|>|v|\}} \big(g_N\big(|v|^2\big) - g_N\big(|u|^2\big)\big)\big(|u|^2 - u\cdot v\big)\,dx \\
&\qquad + \int\limits_{\{|u|\leqslant|v|\}} \big(g_N\big(|v|^2\big) - g_N\big(|u|^2\big)\big)\big(|u|^2 - u\cdot v\big)\,dx \\
&\quad \leqslant C \int\limits_{\{|u|\leqslant|v|\}} \big||v|^2 - |u|^2\big|\cdot|u|\cdot|u-v|\,dx \\
&\quad \leqslant C \int\limits_{\{|u|\leqslant|v|\}} |u|^2\cdot|u-v|^2\,dx \\
&\quad \leqslant C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)}\|u-v\|^2_{L^3(\Lambda;\mathbb{R}^3)} \\
&\quad \leqslant C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)}\|u-v\|_H\|u-v\|_V \\
&\quad \leqslant \frac{1}{4}\|u-v\|_V^2 + C\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)}\|u-v\|_H^2, \quad u,v\in V.
\end{aligned}$$

Hence (H2) holds with $\rho(v) = C\|v\|^4_{L^6(\Lambda;\mathbb{R}^3)}$.

We recall the following estimate for $v\in \operatorname{span}\{e_1, e_1, \ldots, e_n\}$ (cf. [31, Lemma 2.3]):

$$\begin{aligned}
\langle Av, v\rangle_V &= \big\langle \mathcal{P}\Delta v, (I-\Delta)v\big\rangle_H \leqslant -\|v\|^2_{H^2} + \|v\|_V^2, \\
\big\langle F(v), v\big\rangle_V &= -\big\langle \mathcal{P}\big[(v\cdot\nabla)v\big], (I-\Delta)v\big\rangle_H \leqslant \frac{1}{2}\|v\|^2_{H^2} + \frac{1}{2}\big\||v|\cdot|\nabla v|\big\|_H^2, \\
-\big\langle \mathcal{P}\big(g_N\big(|v|^2\big)v\big), v\big\rangle_V &= -\big\langle \mathcal{P}\big(g_N\big(|v|^2\big)v\big), (I-\Delta)v\big\rangle_H \leqslant -\big\||v|\cdot|\nabla v|\big\|_H^2 + CN\|v\|_V^2.
\end{aligned} \tag{3.6}$$

Then it is easy to verify (H3) as follows:

$$\big\langle Av + F(v) - \mathcal{P}\big(g_N\big(|v|^2\big)v\big), v\big\rangle_V \leqslant -\frac{1}{2}\|v\|^2_{H^2} + C(N+1)\|v\|_V^2, \quad v\in \operatorname{span}\{e_1, e_1, \ldots, e_n\}.$$

Concerning the growth condition, we have that

$$\big\|F(v)\big\|_{V^*} \leqslant C\|v\|^2_{L^4(\Lambda;\mathbb{R}^3)} \leqslant C\|v\|_V^2, \quad v\in V.$$

By (3.1) we have

$$\big\|g_N\big(|v|^2\big)v\big\|^2_{V^*} \leqslant C\|v\|^2_{L^6(\Lambda;\mathbb{R}^3)} \leqslant C\|v\|_V^2, \quad v\in V.$$

Hence we know that (H4) holds with $\alpha = 4$.

Then the existence and uniqueness of solutions to (3.3) follows from Theorem 1.1. □

Remark 3.1. (1) In [31], the existence of solutions was obtained by Yamada–Watanabe theorem, i.e. they proved the existence of martingale solutions and pathwise uniqueness. Here we construct the solution directly by the Galerkin approximation and local monotonicity arguments.

(2) Another main difference is that in [31,34] the authors used the Gelfand triple ($\star\star$) to analyze the equation. This is very crucial in their works because the following inequality (cf. [16]) plays a very important role in their proofs:

$$\sup_x \left|u(x)\right|^2 \leqslant C \|\Delta u\|_{H^0} \|\nabla u\|_{H^0}. \tag{3.7}$$

However, in this work we use the different Gelfand triple (i.e. ($\star$)) to study the tamed equation (3.3) and we obtain the existence and uniqueness of solutions with better regularity estimate in H^1 by applying Theorem 1.1.

3.2. Stochastic (generalized) curve shortening flow and singular stochastic p-Laplace equations

The study of the motion by mean curvature of curves and surfaces attracts more and more attentions in recent years. It not only connects to many interesting mathematical theories such as nonlinear PDEs, geometric measure theory, asymptotic analysis and singular perturbations, but also has important applications in image processing and materials science etc (cf. [38,42]). The incorporation of stochastic perturbations has also been widely used in these models, where the noise can come from the thermal fluctuations, impurities and the atomistic processes describing the surface motions. However, the mathematical theory for the study of those stochastic models are quite incomplete (cf. [8] and the references therein).

The second example here is the equation of stochastic curve shortening flow, which has been investigated recently by Es-sarhir, von Renesse and Stannat in [8,9]. The deterministic part is a simplified model in geometric PDE theory which describes the motion by mean curvature of embedded surfaces (in the present model the surface is just some curve in the 2-dimensional plane), we refer to [8] for more detailed exposition on the model. The random forcing was introduced to refine the model by taking the influence of thermal noise into account. The stochastic curve shortening flow (cf. [8,9]) is formulated in the following form:

$$\mathrm{d}X_t = \frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2}\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t,$$

where ∂_x, ∂_x^2 denote the first and second (spatial) derivative.

Based on the crucial observation

$$\frac{\partial_x^2 X_t}{1 + (\partial_x X_t)^2} = \partial_x\big(\arctan(\partial_x X_t)\big),$$

this equation has been investigated in [8,9] using the variational framework with following Gelfand triple:

$$V := W_0^{1,2}\big([0,1]\big) \subseteq H := L^2\big([0,1]\big) \subseteq V^* = W^{-1,2}\big([0,1]\big).$$

Now we consider the following form of SPDE, which covers a large class of stochastic evolution equations such as stochastic curve shortening flow (with some nonlinear perturbations), stochastic p-Laplace equations and stochastic reaction–diffusion equations. For simplicity we only formulate the result for 1-dimensional underlying domain $[0,1]$ here.

$$\mathrm{d}X_t = \big[\partial_x\big(f(\partial_x X_t)\big) + g(X_t)\big]\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0. \tag{3.8}$$

Example 3.3. Suppose that functions $f, g \in C^1(\mathbb{R})$ and there exist constants $C, p \geqslant 2$ such that

$$\begin{aligned}
&f'(x) \geqslant 0, \qquad \big|f(x)\big| \leqslant C\big(1+|x|\big), \quad x \in \mathbb{R},\\
&g'(x) \leqslant C, \qquad \big|g(x)\big| \leqslant C\big(1+|x|^{p-1}\big), \quad x \in \mathbb{R},\\
&\big(g(x)-g(y)\big)(x-y) \leqslant C\big(1+|y|^p\big)|x-y|^2, \quad x, y \in \mathbb{R},
\end{aligned} \tag{3.9}$$

and $B : V \to (L_2(U;V); \|\cdot\|_2)$ satisfies that

$$\begin{aligned}
&\big\|B(v)\big\|_2^2 \leqslant C\big(1+\|v\|_V^2\big), \quad v \in V,\\
&\big\|B(v_1)-B(v_2)\big\|_{L_2(U;H)}^2 \leqslant C\|v_1-v_2\|_H^2, \quad v_1, v_2 \in V.
\end{aligned}$$

Then for $x_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; V)$, (3.8) has a unique solution $X \in L^p(\Omega, \mathbb{P}, L^\infty([0,T];V)) \cap L^p(\Omega, \mathbb{P}, C([0,T];H))$. In particular, we have

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_{L^2}^p + \mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_{W^{1,2}}^p < \infty.$$

Proof. We consider the following Gelfand triple:

$$V := W_0^{1,2}\big([0,1]\big) \subseteq H := L^2\big([0,1]\big) \subseteq V^* = W^{-1,2}\big([0,1]\big).$$

(H0) holds since all eigenvectors $\{e_i, i = 1, 2, \dots\}$ of the Laplace operator constitute an orthonormal basis of H and an orthogonal set in V.

By the assumptions on f we have

$$\begin{gathered}
\big\langle \partial_x\big(f(\partial_x v)\big), v\big\rangle_V = -\int_0^1 f'(\partial_x v)\big(\partial_x^2 v\big)^2 \,\mathrm{d}x \leqslant 0, \quad v \in H_n \subseteq V,\\
\big\|\partial_x\big(f(\partial_x v)\big)\big\|_{V^*} \leqslant \big\|\big(f(\partial_x v)\big)\big\|_H \leqslant C\big(1+\|v\|_V\big), \quad v \in V,\\
{}_{V^*}\big\langle \partial_x\big(f(\partial_x u)\big) - \partial_x\big(f(\partial_x u)\big), u-v\big\rangle_V = -\int_0^1 \big(f(\partial_x u) - f(\partial_x u)\big)(\partial_x u - \partial_x v)\,\mathrm{d}x \leqslant 0, \quad u, v \in V.
\end{gathered}$$

Then it is easy to show that (H1)–(H4) hold for $\partial_x(f(\partial_x v))$ (with $\rho \equiv 0$ and $\alpha = 2$).

Hence now it is enough to show that (H1)–(H4) hold for the term g in the drift.

By the continuity of g and dominated convergence theorem it is easy to show that (H1) holds for F.

By (3.9) and Sobolev's inequality we have

$$\begin{aligned}
{}_{V^*}\big\langle g(u) - g(v), u-v\big\rangle_V &= \int_0^1 \big(g(u)-g(v)\big)(u-v)\,\mathrm{d}x\\
&\leqslant C\big(1+\|v\|_{L^\infty}^p\big)\int_0^1 |u-v|^2\,\mathrm{d}x\\
&\leqslant C\big(1+\|v\|_V^p\big)\|u-v\|_H^2, \quad u, v \in V,
\end{aligned}$$

i.e. (H2) holds with $\rho(v) = \|v\|_V^p$.

(H3) also holds since (3.9) implies that

$$\langle g(v), v\rangle_V = -\langle g(v), \partial_x^2 v\rangle_H = \int_0^1 g'(v)(\partial_x v)^2\,\mathrm{d}x \leqslant C\|v\|_V^2, \quad v \in H_n \subseteq V.$$

(H4) with $\alpha = p$ follows from the following estimate:

$$\|g(v)\|_{V^*} \leqslant C\|g(v)\|_{L^1} \leqslant C\big(1 + \|v\|_{L^\infty}^{p-1}\big) \leqslant C\big(1 + \|v\|_V^{p-1}\big), \quad v \in V.$$

Therefore, the conclusion follows from Theorem 1.1. □

Remark 3.2. (1) If we take $f(x) = \arctan x$ and $g(x) \equiv 0$, then (3.8) reduces back to the model of stochastic curve shortening flow (cf. [8]). Although we use the same Gelfand triple as in [8], the main result (Theorem 2.3) in [8] cannot be applied to (3.8) because the perturbation term g is only locally monotone and superlinear (i.e. $\rho(u) = C\|u\|_V^p$ in (H2) and $\alpha \geqslant 2$ in (H4)).

(2) The simple example of g satisfying (3.9) is any polynomial of odd degree with negative leading coefficients. Hence (3.8) also covers stochastic reaction–diffusion equations (i.e. $f(x) = x$).

(3) If $f(x) = |x|^{p-2}x (1 < p \leqslant 2)$, then (3.8) covers the singular stochastic p-Laplace equations. The classical variational method (cf. [28]) was based on the following Gelfand triple

$$V := W_0^{1,p}\big([0,1]\big) \subseteq H := L^2\big([0,1]\big) \subseteq V^* = W^{-1,q}\big([0,1]\big)$$

and the solution of (3.8) has the following estimate:

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_{L^2}^2 + \mathbb{E}\int_0^T \|X_t\|_{W^{1,p}}^p dt < \infty.$$

However, using another Gelfand triple

$$V := W_0^{1,2}\big([0,1]\big) \subseteq H := L^2\big([0,1]\big) \subseteq V^* = W^{-1,2}\big([0,1]\big)$$

we obtain a new invariance result of the solution, i.e. the solution X_t of singular stochastic p-Laplace equation (3.8) take values in $W_0^{1,2}$ (which is smaller subspace of $W_0^{1,p}$) if the initial value X_0 does so. Moreover, we also prove that the solution has the following much stronger regularity estimate:

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_{W^{1,2}}^2 < \infty.$$

Another advantage of using this new Gelfand triple is that we do not have any restriction on p (i.e. $1 < p \leqslant 2$) if we extend the example to the multi-dimensional case (i.e. to replace $[0,1]$ by some bounded open domain Λ in $\mathbb{R}^d$), but the classical result usually need some assumption on p due to the Sobolev embedding of $W_0^{1,p}(\Lambda) \subseteq L^2(\Lambda)$ (i.e. $\frac{2d}{d+2} \leqslant p \leqslant 2$).

(4) If $f(x) = |x|^{p-2}x (p > 2)$, then (3.8) reduces to the degenerate stochastic p-Laplace equations and the result above cannot be applied to this case. Some invariance result in $W^{1,2}$ of the solution for this type of equation was established in [21].

3.3. Stochastic fast diffusion equations

Let Λ be a bounded open domain in $\mathbb{R}^d$ with smooth boundary and Δ be the standard Laplace operator with Dirichlet boundary condition. We consider the following stochastic fast diffusion equations with general multiplicative noise (cf. [18,28,29]):

$$\mathrm{d}X_t = \big(\Delta\Psi(X_t) + \gamma X_t\big)\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0, \tag{3.10}$$

where $\gamma \in \mathbb{R}$ and $\Psi : \mathbb{R} \to \mathbb{R}$ is measurable. In particular, if $\gamma = 0, B = 0$ and $\Psi(s) = s^r := |s|^{r-1}s$ for some $r \in (0,1)$, then (3.10) reduces back to the classical fast diffusion equations.

Using the Gelfand triple

$$V := L^2(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq V^* = \big(L^2(\Lambda)\big)^*,$$

we obtain the following new invariance result and regularity estimate for the solution of (3.10), which improves the recent result obtained in [21].

Example 3.4. Suppose that $\Psi \in C^1(\mathbb{R})$ and there exists a constant $C > 0$ such that

$$\Psi'(x) \geqslant 0, \qquad \big|\Psi(x)\big| \leqslant C\big(1 + |x|\big), \quad x \in \mathbb{R},$$

and $B : V \to (L_2(U;V); \|\cdot\|_2)$ satisfies that

$$\begin{aligned}
&\big\|B(v)\big\|_2^2 \leqslant C\big(1 + \|v\|_V^2\big), \quad v \in V,\\
&\big\|B(v_1) - B(v_2)\big\|_{L_2(U;H)}^2 \leqslant C\|v_1 - v_2\|_H^2, \quad v_1, v_2 \in V.
\end{aligned}$$

Then for any $x_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; V)$, (3.10) has a unique solution $X \in L^2(\Omega, \mathbb{P}, L^\infty([0,T]; V)) \cap L^2(\Omega, \mathbb{P}, C([0,T]; H))$. In particular, we have

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_{L^2}^2 < \infty.$$

Proof. According to the classical result for (3.10) (cf. [28, Example 4.1.11]), here we only need to verify the one-sided linear growth condition (H3) for (3.10). In fact, we have

$$\big\langle \Delta\Psi(v) + \gamma v, v\big\rangle_V = -\int_\Lambda \Psi'(v)|\nabla v|^2\,\mathrm{d}x + \gamma\|v\|_V^2 \leqslant \gamma\|v\|_V^2, \quad v \in H^n.$$

Therefore, the assertions follow directly from Theorem 1.1. □

Remark 3.3. (1) If $\Psi(x) = |x|^{r-1}x (0 < r < 1)$, then (3.10) covers the stochastic fast diffusion equations. The classical variational method (cf. [28]) use the following Gelfand triple

$$V := L^{r+1}(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq V^* = \big(L^{r+1}(\Lambda)\big)^*$$

to show the solution of (3.10) satisfies the following estimate:

$$\mathbb{E}\sup_{t\in[0,T]} \|X_t\|_{W^{-1,2}}^2 + \mathbb{E}\int_0^T \|X_t\|_{L^{r+1}}^{r+1}\,\mathrm{d}t < \infty.$$

Here by using a different Gelfand triple

$$V := L^2(\Lambda) \subseteq H := W^{-1,2}(\Lambda) \subseteq V^* = \big(L^2(\Lambda)\big)^*$$

we obtain a new invariance result of the solution, i.e. the solution X_t of stochastic fast diffusion equation (3.10) take values in $L^2(\Lambda)$ if the initial value X_0 does so. Moreover, the solution also satisfies the following stronger regularity estimate:

$$\mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_{L^2}^2 < \infty.$$

Another improvement is that we do not need to assume the standard assumption on r here, while the classical result requires that $\frac{d-2}{d+2} \leqslant r < 1$ (if $d \geqslant 3$) due to the Sobolev embedding of $L^{r+1}(\Lambda) \subseteq W^{-1,2}(\Lambda)$ (cf. [28, Remark 4.1.15]).

(2) If $\Psi(x) = |x|^{r-1}x$ $(r > 1)$, then (3.10) is the stochastic porous medium equations (cf. [1,12,28]). Some L^2-invariance result of the solution for this type of equation with additive type noise was established in [21,30].

3.4. Stochastic Burgers type and reaction–diffusion equations

The last example is a semilinear type SPDE which is formulated as follows:

$$\mathrm{d}X_t = \big(\partial_x^2 X_t + f(X_t)\partial_x X_t + g(X_t)\big)\,\mathrm{d}t + B(X_t)\,\mathrm{d}W_t, \quad X_0 = x_0. \tag{3.11}$$

If we take $f = 0$ and $g(x) = \sum_{i=0}^{2n+1} a_i x^i$ with $a_{2n+1} < 0$ (for some fixed $n \in \mathbb{N}$), then (3.11) reduces to the classical stochastic reaction–diffusion equations. If $g = 0$, then (3.11) covers the stochastic Burgers type equations (see [23, Remark 3.1]), which have been extensively used in the study of turbulent fluid motion (cf. [20,39]).

The existence and uniqueness results of (3.11) driven by space–time White noise has been obtained by Gyöngy in [15] under a different framework, in which both f and g were assumed to have linear growth. In this work we will use different assumptions (i.e. f is bounded, g has polynomial growth and the noise is regular in space), then we can obtain the existence and uniqueness of strong solutions for (3.11) with better regularity estimates (see (1.2)).

Similarly as in [23, Example 3.2], we consider the following Gelfand triple for (3.11):

$$V := W_0^{1,2}\big([0,1]\big) \subseteq H := L^2\big([0,1]\big) \subseteq V^* = W^{-1,2}\big([0,1]\big).$$

However, unlike in [23] where $n = 1$ is assumed for g, here by Theorem 1.1 we can obtain the existence and uniqueness of strong solutions for (3.11) with any odd degree polynomial g having negative leading coefficients.

Example 3.5. Suppose that f is a bounded Lipschitz function on $\mathbb{R}$ and $g \in C^1(\mathbb{R})$ and there exists constants $C, p \geqslant 2$ such that

$$\begin{aligned}
&\big(g(x) - g(y)\big)(x - y) \leqslant C\big(1 + |y|^p\big)|x - y|^2, \quad x, y \in \mathbb{R},\\
&\big|g(x)\big| \leqslant C\big(1 + |x|^{p-1}\big), \quad x \in \mathbb{R},\\
&g'(x) \leqslant C, \quad x \in \mathbb{R},
\end{aligned}$$

and $B : V \to (L_2(U; V); \|\cdot\|_2)$ satisfies that

$$\|B(v)\|_2^2 \leqslant C\left(1+\|v\|_V^2\right), \quad v \in V,$$
$$\|B(v_1)-B(v_2)\|_{L_2(U;H)}^2 \leqslant C\|v_1-v_2\|_H^2, \quad v_1, v_2 \in V.$$

Then for $x_0 \in L^p(\Omega, \mathcal{F}_0, \mathbb{P}; V)$, (3.11) has a unique solution $X \in L^p(\Omega, \mathbb{P}, L^\infty([0,T]; V)) \cap L^p(\Omega, \mathbb{P}, C([0,T]; H))$. In particular, we have

$$\mathbb{E} \sup_{t\in[0,T]} \|X_t\|_{L^2}^p + \mathbb{E} \operatorname*{ess\,sup}_{t\in[0,T]} \|X_t\|_{W^{1,2}}^p < \infty.$$

Proof. Combining with the result in the previous example, here we only need to show (H1)–(H4) hold for the term $\partial_x^2 + f(\cdot)\partial_x$.

According to the result showed in [23, Example 3.2], (H1), (H2) and (H4) hold.

Since f is bounded, by Hölder's inequality and Young's inequality we have

$$\begin{aligned}
\langle \partial_x^2 v + f(v)\partial_x v, v\rangle_V &= -\langle \partial_x^2 v + f(v)\partial_x v, \partial_x^2 v\rangle_H \\
&= -\|\partial_x^2 v\|_{L^2}^2 - \int_0^1 f(v)\partial_x v \partial_x^2 v \, dx \\
&\leqslant -\|\partial_x^2 v\|_{L^2}^2 + C\|\partial_x^2 v\|_{L^2}\|v\|_V \\
&\leqslant -\frac{1}{2}\|\partial_x^2 v\|_{L^2}^2 + C\|v\|_V^2, \quad v \in H_n \subseteq V,
\end{aligned}$$

i.e. (H3) also holds.

Therefore, the assertion follows from Theorem 1.1. □

Remark 3.4. (1) We should remark that the existence and uniqueness of strong solutions for stochastic reaction–diffusion equations (i.e. $f = 0$ in (3.11)) has been established by Zhang in [41] and Gess in [11] using different methods. However, the regularity estimate (1.2) *w.r.t.* V-norm seems new for (3.11).

(2) Similarly, one can also extend the above result from the underlying domain $[0, 1]$ to more general high dimensional domain $\Lambda \subseteq \mathbb{R}^d$ (see e.g. [23, Example 3.2]).

Acknowledgments

The author would like to thank the referees for many helpful comments and suggestions.

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