"El saber de mis hijos hará mi grandeza"

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T E S I S

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Chapter 1

Introduction

1.1 Brief History of the Burger's Equation

The major challenges in the field of complex systems is a thorough under-standing of the phenomenon of turbulence. Direct numerical simulations (DNS) have substantially contributed to our understanding of the disordered flow phenomena inevitably arising at high Reynolds numbers. However, a successful theory of turbulence is still lacking which should allow to predict features of technologically important phenomena like turbulent mixing, turbulent convection, and turbulent combustion on the basis of the fundamental fluid dynamical equations. This is due to the fact that already the evolution equation for the simplest fluids, which are the so-called Newtonian incompressible fluids, have to take into account nonlinear as well as nonlocal properties.

Consider the Navier Stokes equations.

$$\begin{cases} \nabla \cdot v = 0, \\ (\rho v)_t + (\nabla \cdot \rho v)v + \nabla p - \mu \nabla^2 v - \rho F = 0. \end{cases} \tag{1.1}$$

It is well known that when ρ consider to be the density, p the pression, v the velocity and μ the viscosity of a fluid, this equations describe the dynamics of a divergence free incompressible ($\rho_t = 0$) flow where F represents the gravitational effects.

The Burgers equation or the Bateman-Burgers equation is a differential partial equation that appears in several areas of applied mathematics, such as fluid mechanics, nonlinear acoustics, gas dynamics, traffic flow. The equation was first introduced by Harry Bateman in 1915 and later studied by Johannes Martinus Burgers in 1948.

Burgers equations appear as a simplification of the Navier-Stokes equation (1.1) by just dropping the pressure term. Hence it is usually thought as a toy model, namely, a tool that is used to understand some of the inside behavior of the general problem.

In contrast to equation (1.1), this equation can be investigated in one spatial dimension

Simplification in (1.1) of the x componet of the velocity vector, which we will call v^x, gives

$$\rho\frac{\partial v^x}{\partial t}+\rho v^x\frac{\partial v^x}{\partial x}+\rho v^y\frac{\partial v^x}{\partial y}+\rho v^z\frac{\partial v^x}{\partial z}+\frac{\partial p}{\partial x}-\mu\left(\frac{\partial^2 v^x}{\partial x^2}+\frac{\partial^2 v^x}{\partial y^2}+\frac{\partial^2 v^x}{\partial z^2}\right)-\rho F_x=0.$$

Considering a $1D$ problem with no pressure gradient, the above equation reduces to

$$\rho\frac{\partial v^x}{\partial t}+\rho v^x\frac{\partial v^x}{\partial x}-\mu\frac{\partial^2 v^x}{\partial x^2}-\rho F_x=0 \tag{1.2}$$

If we use now the traditional variable v rather than v^x and take ϵ to be the kinematic viscosity, i.e, $\epsilon = \frac{\mu}{\rho}$, then the last equation becomes just the viscid Burgers equation

$$\frac{\partial u(x,t)}{\partial t}+u(x,t)\frac{\partial u(x,t)}{\partial x}-\epsilon\frac{\partial^2 u(x,t)}{\partial x^2}-F(x,t)=0 \tag{1.3}$$

Some assumptions are made, namely: $\rho =$ constant (density), $\mu =$ constant (viscosity), $P =$ constant (pressure).

The Burgers equation is nonlinear and one expects to find phenomena similar to turbulence. However, the homogeneous Burgers equation lacks the most important property attributed to turbulence: The solutions do not exhibit chaotic features like sensitivity with respect to initial conditions. This can explicitly shown using the Hopf-Cole transformation which transforms Burgers equation into a linear parabolic equation. From the numerical point of view, however, this is of importance since it allows one to compare numerically obtained solutions of the nonlinear equation with the exact one. This comparison is important to investigate the quality of the applied numerical schemes.

1.1.1 Exact solution of the Burgers equation by the Cole-Hopf transformation.

Hopf (1950) and Cole (1951) introduced the method that has come to be known as the Cole-Hopf transformation to solve the viscous Burgers equation.

Consider now the initial value problem for the viscid Burgers equation

$$\begin{cases} u_t+uu_x=\epsilon u_{xx} & x\in\mathbb{R},\ \ t>0,\ \ \epsilon>0 \\ u(x,0)=u_0(x) & x\in\mathbb{R}, \end{cases} \tag{1.4}$$

The Cole-Hopf transformation is defined by

$$u = -2\epsilon \frac{\varphi_x}{\varphi} \tag{1.5}$$

Operating in (1.4) we find that

$$u_t = \frac{2\epsilon(\varphi_t \varphi_x - \varphi\varphi_{xt})}{\varphi^2} \ , \ uu_x = \frac{4\epsilon^2 \varphi_x(\varphi\varphi_{xx} - \varphi^2)}{\varphi^3} \tag{1.6}$$

and

$$\epsilon u_{xx} = -\frac{2\epsilon^2(2\varphi_x^3 - 3\varphi\varphi_{xx}\varphi_x + \varphi^2\varphi_{xxx})}{\varphi^3} \tag{1.7}$$

Substituting this expressions into (1.4),

$$\begin{aligned} \frac{2\epsilon(-\varphi\varphi_{xt} + \varphi_x(\varphi_t - \epsilon\varphi_{xx}) + \epsilon\varphi\varphi_{xxx})}{\varphi^2} &\iff -\varphi\varphi_{xt} + \varphi_x(\varphi_t - \epsilon\varphi_{xx}) + \epsilon\varphi\varphi_{xxx} = 0 \\ &\iff \varphi_x(\varphi_t - \epsilon\varphi_{xx}) = \varphi(\varphi_{xt} - \epsilon\varphi_{xxx}) \\ &\iff \varphi_x(\varphi_t - \epsilon\varphi_{xx}) = \varphi(\varphi_t - \epsilon\varphi_{xx})_x \end{aligned}$$

Therefore, if φ solves the heat equation $\varphi_t - \epsilon\varphi_{xx} = 0$, $x \in \mathbb{R}$, then $u(x,t)$ given by transformation (1.5) solves the viscid Burgers equation.

To completely transform the problem (1.3) we still have to work with the initial condition function. To do this, note that (1.5) can be written as

$$u = -2\epsilon(\log\varphi)_x \tag{1.8}$$

Hence

$$\varphi(x,t) = e^{-\int \frac{u(x,t)}{2\epsilon} dx} \tag{1.9}$$

It is clear from (1.5) that multiplying φ by a constant does not affect u, so we can write the last equation as

$$\varphi(x,t) = e^{-\int_0^x \frac{u(y,t)}{2\epsilon} dy} \tag{1.10}$$

The initial condition on (1.4) must therefore be transformed by using (1.10) into

$$\varphi(x,0) = \varphi_0(x) = e^{-\int_0^x \frac{u_0(y)}{2\epsilon}dy} \tag{1.11}$$

In summary, we have reduced the problem (1.4) to this one

$$\begin{cases} \varphi_t - \epsilon\varphi_{xx} = 0 & x \in \mathbb{R}, t > 0, \epsilon > 0 \\ \varphi(x,0) = \varphi_0(x) = e^{-\int_0^x \frac{u_0(y)}{2\epsilon}dy} & x \in \mathbb{R}, \end{cases} \tag{1.12}$$

Heat equation. The general solution of the initial value problem for the heat equation is well known and can be handled by a variety of methods. Taking the Fourier transform with respect to x for both heat equation and the initial condition $\varphi_0(x)$ in problem (1.12) we obtain the first order ODE

$$\begin{cases} \hat{\varphi}_t = \xi^2\epsilon\hat{\varphi} & \xi \in \mathbb{R}, t > 0, \epsilon > 0 \\ \hat{\varphi}(\xi,0) = \hat{\varphi}_0(\xi) & \xi \in \mathbb{R}, \end{cases} \tag{1.13}$$

where $\hat{\varphi}(\xi,t) = \displaystyle\int_{-\infty}^{\infty} \varphi(x,t)e^{i\xi x}dx$. The solution for this problem is

$$\hat{\varphi}(\xi,t) = \hat{\varphi}_0(\xi)e^{\xi^2\epsilon t}$$

To recover $\varphi(x,t)$ we have to use the inverse Fourier transformation F^{-1} , namely,

$$\varphi(x,t) = F^{-1}(\hat{\varphi}(\xi,t)) = F^{-1}(\hat{\varphi}_0 e^{\xi^2\epsilon t}) = \varphi_0(x) * F{-1}(e^{\xi^2\epsilon t})$$

where $*$ denotes the convolution product.

On the other hand

$$F^{-1}(e^{\xi^2\epsilon t}) = \frac{1}{2\sqrt{\pi\epsilon t}}e^{-\frac{x^2}{4\epsilon t}}$$

so the initial value problem (1.12) has the analytic solution

$$\varphi(x,t) = \frac{1}{2\sqrt{\pi\epsilon t}}\int_{-\infty}^{\infty} \varphi_0(\xi)e^{-\frac{(x-\xi)^2}{4\epsilon t}}d\xi$$

Finally, from (1.5), we obtain the analytic solution for the problem (1.4)

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t}\varphi_0(\xi)e^{-\frac{(x-\xi)^2}{4\epsilon t}}d\xi}{\int_{-\infty}^{\infty} \varphi_0(\xi)e^{-\frac{(x-\xi)^2}{4\epsilon t}}d\xi} \tag{1.14}$$

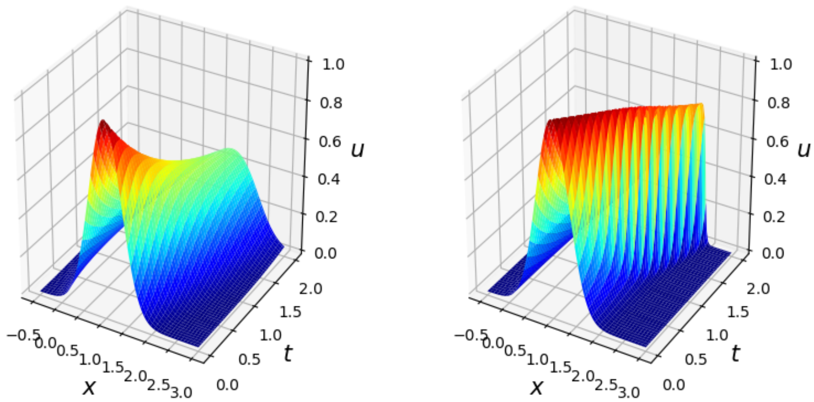

Figure 1.1: Exact solution for $u_0(x) = e^{-(2(x-1))^2}$ with $\epsilon = 0.1$ and $\epsilon = 0.01$ respectively.

1.2 Stochastic Burger's equation

It is well known that the Burgers' equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right hand side all solutions converge to a unique stationary solution as time goes to infinity. However the situation is totally different when the force is a random one. Several authors have indeed suggested to use the stochastic Burgers' equation as a simple model for turbulence. The equation has also been proposed to study the dynamics of interfaces. Here we consider the Burgers' equation with a random force which is a space-time white noise (or Brownian sheet).

$$dX(t,\xi) = \left[\nu \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + \frac{1}{2}\frac{\partial (X^2(t,\xi))}{\partial \xi}\right] dt + dW_t(t,\xi) \tag{1.15}$$

1.2.1 Elements of Probability

Definition 2.2.1 (One-dimensional Brownian motion) A one-dimensional continuous time stochastic process $W(t)$ is called a standard Brownian motion if

- $W(t)$ is almost surely continuous in t,
- $W(t)$ has independent increments,
- $W(t) - W(s)$ obeys the normal distribution with mean zero and variance $t - s$.
- $W(0) = 0$

It can be readily shown that $W(t)$ is Gaussian process. We then call $\dot{W}(t) = \frac{d}{dt}W$, formally the first-order derivative of $W(t)$ in time, white noise.

The Brownian motion and white noise can also be defined in terms of orthogonal expansions. Suppose that $\{m_k(t)\}_{k\geq 1}$ is a complete orthonormal system in $L^2([0,T])$. The Brownian motion $W(t)$, $t \in [0,T]$ can be defined by

$$W(t) = \sum_{0}^{\infty} \xi_i \int_0^t m_i(s)ds, t \in [0,T], \tag{1.16}$$

here ξ_is are mutually independent standard Gaussian random variables. It can be checked that the Gaussian process defined by (2.2.1) is indeed a standard Brownian motion by Definition 2.2.1. Correspondingly, the white noise is defined by

$$W(t) = \sum_{0}^{\infty} \xi_i m_i(t), t \in [0,T], \tag{1.17}$$

Chapter 2

Spectral Methods

2.1 Brief History of the Spectral Methods

Spectral methods involve seeking representing the solution to a differential equation in terms of a series truncated of known, smooth functions of the independent variables. They have recently emerged as a viable alternative to finite difference and finite element methods for the numerical solution of partial differential equations. The key recent advance was the development of fast transform methods for the efficient implementation of spectral equations. Spectral methods have proved particularly useful in numerical fluid dynamics where large spectral hydrodynamics codes are now regularly used to study turbulence and transition, numerical weather prediction, and ocean dynamics and others problems where high accuracy is desired for complicated solutions. We will discuss the formulation and analysis of spectral methods. It turns out that several features of this analysis involve interesting extensions of the classical numerical analysis of initial value problems.

The origin of the terminology spectral is not entirely clear but probably arises from the original use of Fourier sines and cosines as basis functions (Gottlieb and Orszag (1977)) especially in connection with a time series analysis and the fundamental frequencies of a process, namely the spectrum.

Spectral methods are a class of spatial discretizations for differential equations. The key components for their formulation are the trial functions (also called the expansion or approximating functions) and the test functions (also known as weight functions). The trial functions, which are linear combinations of suitable trial basis functions, are used to provide the approximate representation of the solution. The trial functions are used as the basis functions for a truncated series expansion of the solution. The test functions are used to ensure that the differential equation is satisfied as closely as possible by the truncated series expansion. This is achieved by minimizing the residual, i.e., the error in the differential equation produced by using the truncated expansion instead of the exact solution, with respect to a suitable norm. An equivalent requirement is that the residual satisfy a suitable orthogonality condition with respect to each of the test functions. The residual accounts for the differential equation and sometimes the boundary conditions, either explicitly or implicitly. For this reason they may be viewed as a special case of the method of weighted residuals (Finlayson and Scriven (1966)). An equivalent requirement is that the residual satisfy a suitable orthogonality condition with respect to each of the test functions.

The choice of trial functions is one of the features which distinguish spectral methods from finite-element and finite-difference methods. The trial functions for spectral methods are infinitely differentiable global functions. In the case of finite element methods, the domain is divided into small elements, and a trial function is specified in each element. The trial functions are thus local in character, and well suited for handling complex geometries. The finite-difference trial functions are likewise local. Between the three most commonly used spectral schemes, namely, the Galerkin, collocation, and tau versions. In the Galerkin approach, the test functions are the same as the trial functions. They are, therefore, infinitely smooth functions which individually satisfy the boundary conditions. The differential equation is enforced by requiring that the integral of the residual times each test function be zero. In the collocation approach the test functions are translated Dirac delta functions centered at special, so-called collocation points. This approach requires the differential equation to be satisfied exactly at the collocation points. Spectral tau methods are similar to Galerkin methods in the way that the differential equation is enforced. However, none of the test functions need satisfy the boundary conditions. Hence, a supplementary set of equations is used to apply the boundary conditions.

The first spectral methods computations were simulations of homogeneous turbulence on periodic domains. For that type of problem, the natural choice of basis functions is the family of (periodic) trigonometric polynomials. In this chapter, we will discuss the behavior of these trigonometric polynomials when used to approximate smooth functions. We will consider the properties of both the continuous and discrete Fourier series, and come to an understanding of the factors determining the behavior of the approximating series.

2.2 Spectral Methods

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. If $\{e_k\}_{k\in I}$ is a countable orthonormal base of $\mathcal{H}$, then each x of $\mathcal{H}$ element can be written as

$$x = \sum_{k\in I} \langle e_k, x\rangle e_k$$

This sum is known as the Fourier expansion of x and the spectral methods are based on this type of series.

If we choose the base $B = span\{e^{inx} : |n| \leq \infty\}$ we obtain the Fourier series, then for $u(x) \in L^2[0, 2\pi]$ the Fourier series $F[u]$ of the function u is defined as follows

$$F[u] = \sum_{|n| \leq \infty} \hat{u}_n e^{inx} \tag{2.1}$$

Which is known as the classical continuous series of trigonometric polynomials, where $\hat{u}_n$ are known as the Fourier coefficients and are determined by:

$$\hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{inx} dx = \begin{cases} \hat{a}_n & n = 0, \\ \frac{\hat{a}_n - i\hat{b}_n}{2} & n > 0, \\ \frac{\hat{a}_n + i\hat{b}_n}{2} & n < 0. \end{cases} \tag{2.2}$$

For our purposes, the relevant question is how well the Fourier series approaches a function. For this, two types of operators are defined, projection and interpolation operators.

2.2.1 Projection Operator

The operator $\mathcal{P}_N$ is defined as the truncated Fourier series, that is,

$$\mathcal{P}_N u(x) = \sum_{|n| \leq \frac{N}{2}} \hat{u}_n e^{inx} \tag{2.3}$$

Which is a projection in the space of finite dimension

$$\hat{B}_N = span \left\{ e^{inx} : |n| \leq \frac{N}{2} \right\}, \quad dim(\hat{B}_N) = N + 1$$

Theorem.

If the sum of squares of the Fourier cofficients is bounded

$$\sum_{|n| \leq \infty} |\hat{u}_N|^2 < \infty$$

Then the truncated series converges in the L^2 norm

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

If, moreover, the sum of the absolute values of the Fourier coefficients is bounded

$$\sum_{|n| \leq \infty} |\hat{u}_N| < \infty$$

Then the truncated series converges uniformly

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

The fact that the truncated sum convergences implies that the error is dominated by the tail of the series, i.e.,

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} = 2\pi \sum_{|n|>\frac{N}{2}} |\hat{u}_N|^2$$

and

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \leq \sum_{|n|>\frac{N}{2}} |\hat{u}_N|$$

Thus, the error committed by replacing $u(x)$ with its N th-order Fourier series depends solely on how fast the expansion coefficients of $u(x)$ decay. This, in turn, depends on the regularity of $u(x)$ in $[0, 2\pi]$ and the periodicity of the function and its derivatives.

To appreciate this, let $u(x)$ such that its derivative $u'(x) \in L^2[0, \pi]$, then for $n \neq 0$ we have to

$$\begin{aligned}
2\pi\hat{u}_N &= \int_0^{2\pi} u(x)e^{inx}dx \\
&= -\frac{1}{in}(u(2\pi) - u(0)) - \frac{1}{in}\int_0^{2\pi} u'(x)e^{inx}dx
\end{aligned}$$

Therefore

$$|\hat{u}_N| \propto \frac{1}{n}$$

Now, if a function $u(x)$, its first $(m-1)$ derivatives, and their periodic extensions are all continuous and if the mth derivative $u^{(m)} \in L^2[0, 2\pi]$, then $\forall n \neq 0$, repeating the above procedure, we have that the Fourier coefficients $\hat{u}_N$ of $u(x)$ decay as

$$|\hat{u}_N| \propto \left(\frac{1}{n}\right)^m$$

This is known as spectral convergence, which means that the smoother the function, the series converges faster.

Example 1 *Consider the $C_p^\infty[0, 2\pi]$ function*

$$u(x) = \frac{3}{3 - 4\cos(x)} \tag{2.4}$$

Its expansion coefficients are

$$\hat{u}_n = 2^{-|n|} \tag{2.5}$$

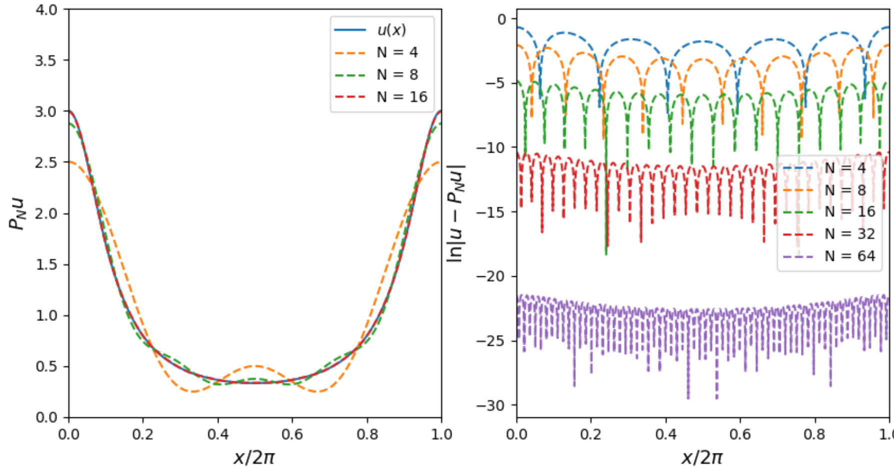

Figure 2.1: (a) Continuous Fourier series approximation of Example 1 for increasing resolution. (b) Pointwise error of approximation for increasing resolution.

Example 2 *The expansion coefficients of the function*

$$u(x) = \sin(\frac{x}{2}) \tag{2.6}$$

are given by

$$\hat{u}_n = \frac{2}{\pi}\frac{1}{(1-4n^2)} \tag{2.7}$$

Differentiation of the continuous expansion

When approximating a function $u(x)$ by the finite Fourier series $\mathcal{P}_N u$, we can easily obtain the derivatives of $\mathcal{P}_N u$ by simply differentiating the basis functions.

If u is a sufficiently smooth function, then one can differentiate the sum

$$\mathcal{P}_N u(x) = \sum_{|n|\leq\frac{N}{2}} \hat{u}_n e^{inx} \tag{2.8}$$

term by term, to obtain

$$\frac{d^q}{dx^q}\mathcal{P}_N u(x) = \sum_{|n|\leq\frac{N}{2}} \hat{u}_n \frac{d^q}{dx^q} e^{inx} = \sum_{|n|\leq\frac{N}{2}} (in)^q \hat{u}_n e^{inx} \tag{2.9}$$

It follows that the projection and differentiation operators commute

$$\mathcal{P}_N \frac{d^q}{dx^q} u = \frac{d^q}{dx^q}\mathcal{P}_N u \tag{2.10}$$

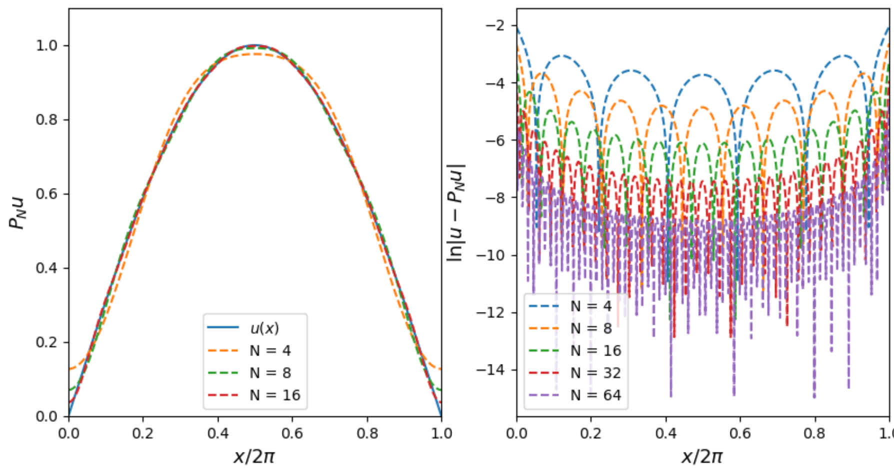

Figure 2.2: (a) Continuous Fourier series approximation of Example 2 for increasing resolution. (b) Pointwise error of approximation for increasing resolution.

This property implies that for any constant coefficient differention operator $\mathcal{L}$,

$$\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N)u \tag{2.11}$$

known as the truncation error, vanishes. Thus, the Fourier approximation to the equation $u_t = \mathcal{L}u$ is exactly the projection of the analytic solution.

Approximation theory for smooth functions

When using the Fourier approximation to discretize the spatial part of the equation

$$u_t = \mathcal{L}u$$

where $\mathcal{L}$ is a differential operator, it is important that our approximation, both to u and to $\mathcal{L}u$, be accurate. To establish consistency we need to consider not only the difference between u and $\mathcal{P}_N u$, but also the distance between $\mathcal{L}u$ and $\mathcal{L}\mathcal{P}_N u$, measured in an appropriate norm. This is critical, because the actual rate of convergence of a stable scheme is determined by the truncation error

$$\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N)u$$

The truncation error is thus determined by the behavior of the Fourier approximations not only of the function, but of its derivatives as well. It is natural, therefore, to use the Sobolev q-norm denoted by $H_p^q[0, 2\pi]$, which measures the smoothness of the derivatives as well as the function,

$$\|u\|^2_{H_p^q[0,2\pi]} = \sum_{m=0}^{q} \int_0^{2\pi} |u^m(x)|^2 \, dx \tag{2.12}$$

The subscript p indicates the fact that all our functions are periodic, for which the Sobolev norm can be written in mode space as

$$\|u\|^2_{H^q_p[0,2\pi]} = 2\pi \sum_{m=0}^{q} \sum_{|n|\leq\infty} |n|^{2m}|\hat{u}_n|^2 = 2\pi \sum_{|n|\leq\infty} \left(\sum_{m=0}^{q} |n|^{2m} \right) |\hat{u}_n|^2 \tag{2.13}$$

where the interchange of the summation is allowed provided $u(x)$ has sufficient smoothness. For example for $u(x) \in C^q_p[0, 2\pi]$, $n \neq 0$ and $q > \frac{1}{2}$

$$(1 + n^{2q}) \leq \sum_{m=0}^{q} n^{2m} \leq (q + 1)(1 + 2^{2q}) \tag{2.14}$$

the norm $\| \cdot \|_{W^q_p[0,2\pi]}$ defined by

$$\|u\|_{W^q_p[0,2\pi]} = \left(\sum_{|n|\leq\infty} (1 + n^{2q})|\hat{u}_n|^2 \right)^{1/2} \tag{2.15}$$

is equivalent to $\| \cdot \|_{H^q_p[0,2\pi]}$.

Results for the continuous expansion

Consider, first, the continuous Fourier series

$$\mathcal{P}_{2N} u(x) = \sum_{|n|\leq N} \hat{u}_n e^{inx} \tag{2.16}$$

We start with an L^2 estimate for the distance between u and its trigonometric approximation $\mathcal{P}_{2N}u$.

Theorem. For any $u(x) \in \mathcal{H}^r_p[0, 2\pi]$, there exists a positive constant C, independent of N , such that

$$\|u - \mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq CN^{-q}\|u^{(q)}\|_{L^2[0,2\pi]} \tag{2.17}$$

provided $0 \leq q \leq r$.

Proof: By Parsevals identity,

$$\|u - \mathcal{P}_{2N}u\|^2_{L^2[0,2\pi]} = 2\pi \sum_{|n|>N} |\hat{u}_n|^2 \tag{2.18}$$

We rewrite this summation

$$\sum_{|n|>N} |\hat{u}_n|^2 = \sum_{|n|>N} \frac{1}{N^{2q}n^{2q}}|\hat{u}_n|^2 \tag{2.19}$$

$$\leq N^{-2q} \sum_{|n|>N} n^{2q}|\hat{u}_n|^2 \tag{2.20}$$

$$\leq N^{-2q} \sum_{|n|\geq 0} n^{2q}|\hat{u}_n|^2 \tag{2.21}$$

$$= \frac{1}{2\pi}N^{-2q}\|u^{(q)}\|^2_{L^2[0,2\pi]} \tag{2.22}$$

Putting this all together and taking the square root, we obtain our result.

Note that the smoother the function, the larger the value of q, and therefore, the better the approximation. This is in contrast to finite difference or finite element approximations, where the rate of convergence is fixed, regardless of the smoothness of the function. This rate of convergence is referred to in the literature as spectral convergence.

If $u(x)$ is analytic then

$$\|u^{(q)}\|_{L^2[0,2\pi]} = Cq!\|u\|_{L^2[0,2\pi]} \tag{2.23}$$

and so

$$\|u - \mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq N^{-q}\|u^{(q)}\|_{L^2[0,2\pi]} \leq C\frac{q!}{N^q}\|u\|_{L^2[0,2\pi]} \tag{2.24}$$

Using Stirlings formula, $q! \sim q^q e^{-q}$, and assuming that $q \propto N$, we obtain

$$\|u - \mathcal{P}_{2N}u\|_{L^2[0,2\pi]} \leq\sim C\left(\frac{q}{N}\right)^q e^{-q}\|u\|_{L^2[0,2\pi]} \sim Ke^{-cN}\|u\|_{L^2[0,2\pi]} \tag{2.25}$$

Thus, for an analytic function, spectral convergence is, in fact, exponential convergence.

Theorem. For any real r and any real q where $0 \leq q \leq r$, if $u(x) \in W_p^r[0,2\pi]$, then there exists a positive constant C, independent of N , such that

$$\|u - \mathcal{P}_{2N}u\|_{W_p^q[0,2\pi]} \leq \frac{C}{N^{r-q}}\|u\|_{W_p^r[0,2\pi]} \tag{2.26}$$

Proof: Parsevals identity yields

$$\|u - \mathcal{P}_{2N}u\|^2_{W_p^q[0,2\pi]} = 2\pi\sum_{|n|>N}(1+|n|^{2q})|\hat{u}_n|^2 \tag{2.27}$$

Since $|n| + 1 \geq N$, we obtain

$$(1+|n|^{2q}) \leq (1+|n|)^{2q} = \frac{(1+|n|)^{2r}}{(1+|n|)^{2(r-q)}} \leq \frac{(1+|n|)^{2r}}{N^{2(r-q)}} \tag{2.28}$$

$$\leq (1+r)\frac{(1+n^{2r})}{N^{2(r-q)}} \tag{2.29}$$

for any $0 \leq q \leq r$.

This immediately yields

$$\|u - \mathcal{P}_{2N}u\|^2_{W_p^q[0,2\pi]} \leq C\sum_{|n|>N}\frac{(1+n^{2r})}{N^{2(r-q)}}|\hat{u}_n|^2 \leq C\frac{\|u\|^2_{W_p^r[0,2\pi]}}{N^{2(r-q)}} \tag{2.30}$$

Theorem. Let $\mathcal{L}$ be a constant coefficient differential operator

$$\mathcal{L}u = \sum_{j=1}^{s} a_j \frac{d^j u}{dx^j} \tag{2.31}$$

For any real r and any real q where $0 \leq q + s \leq r$, if $u(x) \in W_p^r[0, 2\pi]$, then there exists a positive constant C, independent of N , such that

$$\|\mathcal{L}u - \mathcal{L}\mathcal{P}_N u\|_{W_p^q[0,2\pi]} \leq CN^{-(r-q-s)} \|u\|^2_{W_p^r[0,2\pi]} \tag{2.32}$$

Proof: Using the definition of $\mathcal{L}$,

$$\|\mathcal{L}u - \mathcal{L}\mathcal{P}_N u\|_{W_p^q[0,2\pi]} \leq \left\| \sum_{j=1}^{s} a_j \frac{d^j u}{dx^j} - \sum_{j=1}^{s} a_j \frac{d^j \mathcal{P}_N u}{dx^j} \right\|_{W_p^q[0,2\pi]} \tag{2.33}$$

$$\leq \max_{0 \leq j \leq s} |a_j| \left\| \sum_{j=1}^{s} \frac{d^j}{dx^j} (u - \mathcal{P}_N u) \right\|_{W_p^q[0,2\pi]} \tag{2.34}$$

$$\leq \max_{0 \leq j \leq s} |a_j| \sum_{j=1}^{s} \|(u - \mathcal{P}_N u)\|_{W_p^{q+s}[0,2\pi]} \tag{2.35}$$

$$\tag{2.36}$$

This last term is bounded in Theorem ??, and the result immediately follows.

2.2.2 Interpolation Operator

The continuous Fourier series method requires the evaluation of the coefficients

$$\hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(x) e^{-inx} dx \tag{2.37}$$

In general, these integrals cannot be computed analytically, and one resorts to the approximation of the Fourier integrals by using quadrature formulas, yielding the discrete Fourier coefficients. Quadrature formulas differ based on the exact position of the grid points, and the choice of an even or odd number of grid points results in slightly different schemes.

The even expansion

Define an equidistant grid, consisting of an even number N of gridpoints $x_j \in [0, 2\pi)$, defined by

$$x_j = \frac{2\pi}{N} j, \qquad j \in [0, ..., N-1]$$

The trapezoidal rule yields the discrete Fourier coefficients $\widetilde{u}_n$, which approximate the continuous Fourier coefficients $\hat{u}_n$,

$$\widetilde{u}_n = \frac{1}{N} \sum_{j=0}^{N-1} u(x_j) e^{-inx_j} \tag{2.38}$$

As the following theorem shows, the trapezoidal quadrature rule is a very natural approximation when trigonometric polynomials are involved.

Theorem. The quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{N}\sum_{j=0}^{N-1} f(x_j) \tag{2.39}$$

is exact for any trigonometric polynomial $f(x) = e^{inx}$, $|n| < N$.

Proof: Given a function $f(x) = e^{inx}$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.40}$$

On the other hand,

$$\frac{1}{N}\sum_{j=0}^{N-1} f(x_j) = \frac{1}{N}\sum_{j=0}^{N-1} e^{in(2\pi j/N)} \tag{2.41}$$

$$= \frac{1}{N}\sum_{j=0}^{N-1} q^j \tag{2.42}$$

where $q = e^{i\frac{2\pi n}{N}}$. If n is an integer multiple of N , i.e., $n = mN$, then $\frac{1}{N}\sum_{j=0}^{N-1} f(x_j) = 1$. Otherwise, $\frac{1}{N}\sum_{j=0}^{N-1} f(x_j) = \frac{q^N - 1}{q-1} = 0$. Thus, the quadrature formula is exact for any function of the form $f(x) = e^{inx}$, $|n| < N$. The quadrature formula is exact for $f(x) \in \hat{B}_{2N-2}$ where $\hat{B}_N$ is the space of trigonometric polynomials of order N,

$$\hat{B}_N = span\{e^{inx}| \ |n| \le N/2\}.$$

Using the trapezoid rule, the discrete Fourier coefficients become

$$\widetilde{u}_n = \frac{1}{N\widetilde{c}_n}\sum_{j=0}^{N-1} u(x_j)e^{-inx_j} \tag{2.43}$$

where we introduce the coefficients

$$\widetilde{c}_n = \begin{cases} 2 \ \ \text{si} & |n| = N/2 \\ \\ 1 \ \ \text{si}|n| < N/2 \end{cases}$$

for ease of notation. These relations define a new projection of u

$$\mathcal{I}_N u(x) = \sum_{|n|\leq \frac{N}{2}} \widetilde{u}_n e^{inx} \tag{2.44}$$

This is the complex discrete Fourier transform, based on an even number of quadrature points. Note that

$$\widetilde{u}_{-N/2} = \widetilde{u}_{N/2},$$

so that we have exactly N independent Fourier coefficients, corresponding to the N quadrature points. As a consequence, $\mathcal{I}_N \sin(\frac{N}{2}x) = 0$ so that the function $\sin(\frac{N}{2}x)$ is not represented in the expansion, Equation (2.9).

Onto which finite dimensional space does $\mathcal{I}_N$ project? Certainly, the space does not include $\sin(\frac{N}{2}x)$, so it is not $\hat{B}_N$. The correct space is

$$\widetilde{B}_N = span\left\{\left(cos(nx),\ \ 0 \leq n \leq \frac{N}{2}\right) \cup \left(sin(nx),\ \ 1 \leq n \leq \frac{N}{2} - 1\right)\right\}$$

which has dimension $\dim(\widetilde{B}_N) = N$.

Theorem. Let the discrete Fourier transform be defined by Equations (2.8)-(2.9). For any periodic function, $C_p^0[0, 2\pi]$, we have

$$\mathcal{I}_N u(x_j) = u(x_j), \quad \forall x_j = \frac{2\pi j}{N}, \quad j = 0, ..., N-1$$

Proof: Substituting Equation (2.8) into Equation (2.9) we obtain

$$\mathcal{I}_N u(x) = \sum_{|n|\leq \frac{N}{2}} \left(\frac{1}{N\widetilde{c}_n} \sum_{j=0}^{N-1} u(x_j) e^{-inx_j}\right) e^{inx} \tag{2.45}$$

Exchanging the order of summation yields

$$\mathcal{I}_N u(x) = \sum_{j=0}^{N-1} u(x_j) g_j(x), \tag{2.46}$$

where

$$\begin{aligned} g_j(x) &= \sum_{|n|\leq \frac{N}{2}} \frac{1}{N\widetilde{c}_n} e^{in(x-x_j)} \\ &= \frac{1}{N} \sin\left[N\frac{x-x_j}{2}\right] \cot\left[\frac{x-x_j}{2}\right] \end{aligned}$$

by summing as a geometric series. It is easily verified that $g_j(x_i) = \delta_{ij}$

We still need to show that $g_j(x) \in \widetilde{B}_N$. Clearly, $g_j(x) \in \hat{B}_N$ as $g_j(x)$ is a polynomial of degree $\leq N/2$. However, since

$$\frac{1}{2}e^{-i\frac{N}{2}x_j} = \frac{1}{2}e^{i\frac{N}{2}x_j} = \frac{(-1)^j}{2},$$

and, by convention $\widetilde{u}_{-N/2} = \widetilde{u}_{N/2}$, we do not get any contribution from the term $\sin(\frac{N}{2}x)$, hence $g_j(x) \in \widetilde{B}_N$.

The odd expansion

we define a grid with an odd number of grid points,

$$x_j = \frac{2\pi}{N+1}j, \quad j \in [0, ..., N]$$

and use the trapezoidal rule

$$\widetilde{u}_n = \frac{1}{N+1}\sum_{j=0}^{N} u(x_j)e^{-inx_j} \tag{2.47}$$

to obtain the interpolation operator

$$\mathcal{J}_N u(x) = \sum_{|n|\leq\frac{N}{2}} \widetilde{u}_n e^{inx} \tag{2.48}$$

Again, the quadrature formula is highly accurate:

Theorem. The quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{N+1}\sum_{j=0}^{N} f(x_j) \tag{2.49}$$

is exact for any $f(x) = e^{inx}$, $|n| < N$, i.e., for all $f(x) \in \widetilde{B}_{2N}$

The scheme may also be expressed through the use of a Lagrange interpolation polynomial,

$$\mathcal{J}_N u(x) = \sum_{j=0}^{N} u(x_j)h_j(x)$$

where

$$h_j(x) = \frac{1}{N+1}\frac{\sin(\frac{N+1}{2}(x-x_j))}{\sin(\frac{x-x_j}{2})} \tag{2.50}$$

One easily shows that $h_j(x_l) = \delta_{jl}$ and that $h_j(x) \in \hat{B}_N$.

Differentiation of the discrete expansions

To implement the Fouriercollocation method, we require the derivatives of the discrete approximation. Once again, we consider the case of an even number of grid points. In the following subsections, we assume that our function u and all its derivatives are continuous and periodic on $[0, 2\pi]$.

Using expansion coefficients Given the values of the function $u(x)$ at the points x_j , differentiating the basis functions in the interpolant yields

$$\frac{d}{dx}\mathcal{I}_N u(x) = \sum_{|n|\leq N/2} in\widetilde{u}_n e^{inx}, \tag{2.51}$$

where

$$\widetilde{u}_n = \frac{1}{N\widetilde{c}_n}\sum_{j=0}^{N-1} u(x_j)e^{-inx_j},$$

are the coefficients of the interpolant $\mathcal{I}_N u(x)$ given in Equations (2.8)(2.9). Higher order derivatives can be obtained simply by further differentiating the basis functions.

The procedure for differentiating using expansion coefficients can be described as follows: first, we transform the point values $u(x_j)$ in physical space into the coefficients $\widetilde{u}_n$ in mode space. We then differentiate in mode space by multiplying $\widetilde{u}_n$ by in, and return to physical space.

The matrix method. The use of the Lagrange interpolation polynomials yields

$$\mathcal{I}_N u(x) = \sum_{j=0}^{N-1} u(x_j)g_j(x)$$

where

$$g_j(x) = \frac{1}{N}\sin\left[N\frac{x-x_j}{2}\right]\cot\left[\frac{x-x_j}{2}\right]$$

An approximation to the derivative of $u(x)$ at the points, x_i, is then obtained by differentiating the interpolation directly,

$$\frac{d}{dx}\mathcal{I}_N(x)\Big|_{x_l} = \sum_{j=0}^{N-1} u(x_j)\frac{d}{dx}g_j(x)\Big|_{x_l} = \sum_{j=0}^{N-1} D_{lj}u(x_j)$$

The entries of the differentiation matrix are given by

$$D_{ij} = \frac{d}{dx}g_j(x)\Big|_{x_i} = \begin{cases} \frac{(-1)^{i+j}}{2}\cot\left[\frac{x_i-x_j}{2}\right] & i \neq j \\ 0 & i = j \end{cases} \tag{2.52}$$

It is readily verified that D is circulant and skew-symmetric.

The approximation of higher derivatives follows the exact same route as taken for the first order derivative. The entries of the second order differentiation matrix $D^{(2)}$, based on an even number of grid points, are

$$D_{ij}^{(2)} = \frac{d^2}{dx^2} g_j(x)\Big|_{x_i} = \begin{cases} -\frac{(-1)^{i+j}}{2}\left[\sin\left[\frac{x_i - x_j}{2}\right]\right]^{-1} & i \neq j \\ -\frac{N^2+2}{12} & i = j \end{cases} \tag{2.53}$$

For the sake of completeness, we list the entries of the differentiation matrix $\widetilde{D}$ for the interpolation based on an odd number of points,

$$\widetilde{D}_{ij} = \begin{cases} -\frac{(-1)^{i+j}}{2}\left[\sin\left[\frac{x_i - x_j}{2}\right]\right]^{-2} & i \neq j \\ 0 & i = j \end{cases} \tag{2.54}$$

which is the limit of the finite difference schemes as the order increases. Once again $\widetilde{D}$ is a circulant, skew-symmetric matrix. As mentioned above, in this case we have

$$\widetilde{D}^{(q)} = \mathcal{J}_N \frac{d^q}{dx^q} \mathcal{J}_N = \widetilde{D}^q$$

for all values of q.

In this method, we do not go through the mode space at all. The differentiation matrix takes us from physical space to physical space, and the act of differentiation is hidden in the matrix itself.

Results for the discrete expansion

The approximation theory for the discrete expansion yields essentially the same results as for the continuous expansion, though with more effort. The proofs for the discrete expansion are based on the convergence results for the continuous approximation, and as the fact that the Fourier coefficients of the discrete approximation are close to those of the continuous approximation.
Recall that the interpolation operator associated with an even number of grid points is given by

$$\mathcal{I}_{2N} u = \sum_{|n|<N} \widetilde{u}_n e^{inx},$$

with expansion coefficients

$$\widetilde{u}_n = \frac{1}{2N\widetilde{c}_n} \sum_{j=0}^{2N-1} u(x_j) e^{-inx_j}, \;\; x_j = \frac{\pi}{N} j.$$

Rather than deriving the estimates of the approximation error directly, we shall use the results obtained in the previous section and then estimate the difference between the two different expansions, which we recognize as the aliasing error.
The relationship between the discrete expansion coefficients $\widetilde{u}_n$, and the continuous expansion coefficients $\hat{u}_n$, is given in the following lemma.

Lemma Consider $u(x) \in W_p[0, 2\pi]$, where $q > 1/2$. For $|n| \leq N$ we have

$$\widetilde{c}_n \widetilde{u}_n = \hat{u}_n + \sum_{\substack{|m| \leq \infty \\ m \neq 0}} \hat{u}_{n+2Nm}$$

Proof: Substituting the continuous Fourier expansion into the discrete expansion yields

$$\widetilde{c}_n \widetilde{u}_n = \frac{1}{2N} \sum_{j=0}^{2N-1} \sum_{|l| \leq \infty} \hat{u}_l e^{i(l-n)x_j}$$

To interchange the two summations we must ensure uniform convergence, i.e., $\sum_{|l| \leq \infty} |\hat{u}_l| < \infty$. This is satisfied, since

$$\begin{aligned} \sum_{|l| \leq \infty} |\hat{u}_l| &= \sum_{|l| \leq \infty} (1 + |l|)^q \frac{|\hat{u}_l|}{(1 + |l|)^q} \\ &\leq \left(2q \sum_{|l| \leq \infty} (1 + l^{2q}) |\hat{u}_l|^2 \right)^{1/2} \left(\sum_{|l| \leq \infty} (1 + |l|)^{-2q} \right)^{1/2} \end{aligned}$$

where the last expression follows from the CauchySchwarz inequality. As $u(x) \in W_p[0, 2\pi]$ the first part is clearly bounded. Furthermore, the second term converges provided $q > 1/2$, hence ensuring boundedness.
Interchanging the order of summation and using orthogonality of the exponential function at the grid yields the desired result.

As before, we first consider the behavior of the approximation in the L^2-norm. We will first show that the bound on the aliasing error, $\mathcal{A}_N$, in Equation (2.16) is of the same order as the truncation error. The error caused by truncating the continuous expansion is essentially the same as the error produced by using the discrete coefficients rather than the continuous coefficients.

Lemma For any $u(x) \in W_p^r[0, 2\pi]$, where $r > 1/2$, the aliasing error

$$||\mathcal{A}_N||_{L^2[0,2\pi]} = \left(\sum_{|n| < \infty} |\widetilde{c}_n \widetilde{u}_n - \hat{u}_n|^2 \right)^{1/2} \leq C N^{-r} ||u^r||_{L^2[0,2\pi]}$$

Proof: From Lemma 2.14 we have

$$|\widetilde{c}_n \widetilde{u}_n - \hat{u}_n|^2 = \left| \sum_{\substack{|m|\leq\infty \\ m\neq 0}} \hat{u}_{n+2Nm} \right|^2$$

To estimate this, we first note that

$$\begin{aligned} \left| \sum_{\substack{|m|\leq\infty \\ m\neq 0}} \hat{u}_{n+2Nm} \right|^2 &= \left| \sum_{\substack{|m|\leq\infty \\ m\neq 0}} |n+2Nm|^r \hat{u}_{n+2Nm} \frac{1}{|n+2Nm|^r} \right|^2 \\ &\leq \left(\sum_{\substack{|m|\leq\infty \\ m\neq 0}} |n+2Nm|^{2r} |\hat{u}_{n+2Nm}|^2 \right) \left(\sum_{\substack{|m|\leq\infty \\ m\neq 0}} \frac{1}{|n+2Nm|^{2r}} \right) \end{aligned}$$

using the CauchySchwartz inequality.
Since $|n| \leq N$, bounding of the second term is ensured by

$$\sum_{\substack{|m|\leq\infty \\ m\neq 0}} \frac{1}{|n+2Nm|^{2r}} \leq \frac{2}{N^{2r}} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^{2r}} = C_1 N^{-2r},$$

provided $r > 1/2$. Here, the constant C_1 is a consequence of the fact that the power series converges, and it is independent of N.
Summing over n, we have

$$\begin{aligned} \sum_{|n|\leq N} \left| \sum_{\substack{|m|\leq\infty \\ m\neq 0}} \hat{u}_{n+2Nm} \right|^2 &\leq \sum_{|n|\leq N} C_1 N^{-2r} \sum_{\substack{|m|\leq\infty \\ m\neq 0}} |n+2Nm|^{2r} |\hat{u}_{n+2Nm}|^2 \\ &\leq C_2 N^{-2r} ||u^{(r)}||^2_{L^2[0,2\pi]} \end{aligned}$$

We are now in a position to state the error estimate for the discrete approximation.

Theorem For any $u(x) \in W_p^r[0, 2\pi]$ with $r > 1/2$, there exists a positive constant C, independent of N , such that

$$||u - \mathcal{I}_{2N} u||_{L^2[0,2\pi]} \leq C N^{-r} ||u^{(r)}||_{L^2[0,2\pi]}$$

Proof: Lets write the difference between the function and its discrete approximation

$$\begin{aligned} ||u - \mathcal{I}_{2N} u||_{L^2[0,2\pi]} &= ||(\mathcal{P}_{2N} - \mathcal{L}_{2N})u + u - \mathcal{P}_{2N} u||^2_{L^2[0,2\pi]} \\ &\leq ||(\mathcal{P}_{2N} - \mathcal{L}_{2N})u||^2_{L^2[0,2\pi]} + ||u - \mathcal{P}_{2N} u||^2_{L^2[0,2\pi]} \end{aligned}$$

Thus, the error has two components. The first one, which is the difference between the continuous and discrete expansion coefficients, is the aliasing error, which is bounded in Lemma 2.15. The second, which is the tail of the series, is the truncation error, which is bounded by the result of Theorem 2.10. The desired result follows from these error bounds.

Theorem 2.16 confirms that the approximation errors of the continuous expansion and the discrete expansion are of the same order, as long as $u(x)$ has at least half a derivative. Furthermore, the rate of convergence depends, in both cases, only on the smoothness of the function being approximated.
The above results are in the L^2 norm. We can obtain essentially the same information about the derivatives, using the Sobolev norms. First, we need to obtain a Sobolev norm bound on the aliasing error.

Lemma Let $u(x) \in W_p^r[0, 2\pi]$, where $r > 1/2$. For any real q, for which $0 \le q \le r$, the aliasing error

$$||\mathcal{A}_N||_{W_p^q[0,2\pi]} = \left(\sum_{-\infty}^{\infty} |\widetilde{c}_n \widetilde{u}_n - \hat{u}_n|^2 \right)^{1/2} \le CN^{-(r-q)} ||u||_{W_p^r[0,2\pi]}$$

Proof:

$$\left| \sum_{\substack{|m| \le \infty \\ m \ne 0}} \hat{u}_{n+2Nm} \right|^2 = \left| \sum_{\substack{|m| \le \infty \\ m \ne 0}} (1 + |n + 2Nm|)^r \hat{u}_{n+2Nm} \frac{1}{(1 + |n + 2Nm|)^2} \right|^2 ,$$

such that

$$\left| \sum_{\substack{|m| \le \infty \\ m \ne 0}} \hat{u}_{n+2Nm} \right|^2 \le \left(\sum_{\substack{|m| \le \infty \\ m \ne 0}} (1 + |n + 2Nm|)^{2r} |\hat{u}_{n+2Nm}|^2 \right) \times \left(\sum_{\substack{|m| \le \infty \\ m \ne 0}} \frac{1}{(1 + |n + 2Nm|)^{2r}} \right)$$

The second factor is, as before, bounded by

$$\sum_{\substack{|m| \le \infty \\ m \ne 0}} \frac{1}{(1 + |n + 2Nm|)^{2r}} \le \frac{2}{N^{2r}} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^{2r}} = C_1 N^{-2r}$$

provided $r > 1/2$ and $|n| \leq N$.
Also, since $(1+|n|)^{2q} \leq C_2 N^{2q}$ for $|n| \leq N$ we recover

$$\begin{aligned}
\sum_{|n|\leq N} (1+|n|)^{2q} \left| \sum_{\substack{|m|\leq\infty \\ m\neq 0}} \hat{u}_{n+2Nm} \right|^2 \\
&\leq C_1 C_2 N^{-2(r-q)} \sum_{\substack{|m|\leq\infty \\ m\neq 0}} (1+|n+2Nm|)^{2r} |\hat{u}_{n+2Nm}|^2 \\
&\leq C_3 N^{-2(r-q)} ||u||^2_{W_p^r[0,2\pi]}
\end{aligned}$$

With this bound on the aliasing error, and the truncation error bounded by Theorem 2.11, we are now prepared to state.

Theorem. Let $u(x) \in W_p^r[0,2\pi]$, where $r > 1/2$. Then for any real q for which $0 \leq q \leq r$, there exists a positive constant, C, independent of N , such that

$$||u - \mathcal{I}_{2N} u||_{W_p^q[0,2\pi]} \leq C N^{-(r-q)} ||u||_{W_p^r[0,2\pi]}$$

The proof follows closely that of Theorem 2.16. As in the case of the continuous expansion, we use this result to bound the truncation error.

Theorem 2.19 Let $\mathcal{L}$ be a constant coefficient differential operator

$$\mathcal{L}u = \sum_{j=1}^{s} a_j \frac{d^j u}{dx^j}$$

then there exists a positive constant, C, independent of N such that

$$||\mathcal{L}u - \mathcal{L}\mathcal{I}_N u||_{W_p^q[0,2\pi]} \leq C N^{-(r-q-s)} ||u||_{W_p^r[0,2\pi]}$$

2.2.3 Definitions

To fulfill our purpose we need to understand the following definitions.

Let's define the following problem

$$\begin{aligned}
\frac{\partial u(x,t)}{\partial t} &= \mathcal{L}u(x,t), \quad x \in \mathcal{D}, \quad t \geq 0 \\
u(x,0) &= g(x), \quad x \in \mathcal{D}, \quad t = 0 \\
\mathcal{B}u(x,t) &= 0, \quad x \in \delta\mathcal{D}, \quad t > 0
\end{aligned}$$

Donde $\mathcal{L}$ es independiente del tiempo y el espacio. Asumamos que el operador $\mathcal{B}[\mathcal{D}]$ esta incluido en el operador $\mathcal{L}$.

Entonces el problema esta bien definido si, para cada $g \in C_0^r$ y para cada tiempo $T_0 > 0$ existe una unica solucion $u(x,t)$ tal que

$$\|u(t)\| \leq Ce^{\alpha t}\|g\|_{\mathcal{H}^p(\mathcal{D})} \quad 0 \leq t \leq T_0$$

Para $p \leq r$ y algunas constantes positivas C y α. Para $p = 0$ se dice que esta fuertemente definido, es decir, para la norma $\mathcal{L}^2$

- Convergencia. Una aproximacion es convergente si

  $$\|\mathcal{P}_N u(t) - u_N(t)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.55}$$

  Para toda $t \in [0,T]$, $\quad u(0) \in B$, $\quad$ y $\quad u_N(0) \in B_N$

- Consistencia. Una aproximacion es consistente si

  $$\|\mathcal{P}_N \mathcal{L}(I - \mathcal{P}_N)u)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.56}$$

  $$\|\mathcal{P}_N u(0) - u_N(0)\|_{\mathcal{L}^2_w(\mathcal{D})} \to 0 \quad \text{cuando} \quad N \to \infty \tag{2.57}$$

  Para toda $u(0) \in B$ y $u_N(0) \in B_N$

- Estabilidad. Una aproximacion es estable si

  $$\|e^{\mathcal{L}_N t}\|_{\mathcal{L}^2_w(\mathcal{D})} \leq C(t), \quad \forall N \tag{2.58}$$

  con la norma asociada al operador norma

  $$\|e^{\mathcal{L}_N t}\|_{\mathcal{L}^2_w(\mathcal{D})} = \sup_{u \in B} \frac{\|e^{\mathcal{L}_N t} u\|_{\mathcal{L}^2_w(\mathcal{D})}}{\|u\|_{\mathcal{L}^2_w(\mathcal{D})}}$$

  y $C(t)$ es independiente de N y acotada para cualquier $t \in [0,T]$.

- **Convergencia Espectral.**

  Si la suma de los cuadrados de los coeficientes de Fourier es acotada, es decir,

  $$\sum_{|n| \leq \infty} |\hat{u}_N|^2 < \infty$$

  Entonces la serie truncada converge en la norma L^2

  $$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

Si, mas aun, la suma de los valores absolutos de los coeficientes de Fourier es acotada, es decir,

$$\sum_{|n|\leq\infty} |\hat{u}_N| < \infty$$

Entonces la serie truncada converge uniformemente

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \to 0 \quad \text{cuando} \quad N \to \infty$$

El hecho que la suma truncada converge implica que el error es dominado por la cola de la serie, es decir,

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^2[0,2\pi]} = 2\pi \sum_{|n|>\frac{N}{2}} |\hat{u}_N|^2$$

y

$$\|u - \mathcal{P}_N u\|_{\mathcal{L}^\infty[0,2\pi]} \leq \sum_{|n|>\frac{N}{2}} |\hat{u}_N|$$

Sea $u(x)$ tal que su derivada $u'(x) \in L^2[0,\pi]$, entonces para $n \neq 0$ tenemos que,

$$\begin{aligned} 2\pi\hat{u}_N &= \int_0^{2\pi} u(x)e^{inx}dx \\ &= -\frac{1}{in}(u(2\pi) - u(0)) - \frac{1}{in}\int_0^{2\pi} u'(x)e^{inx}dx \end{aligned}$$

Por lo tanto,

$$|\hat{u}_N| \propto \frac{1}{n}$$

Ahora, si para $u(x)$, las $(m-1)$ derivadas, sus extensiones periodicas son todas continuas y si la m-esima derivada $u^{(m)} \in L^2[0,2\pi]$, entonces $\forall n \neq 0$, haciendo el mismo procedimiento anterior, tenemos que los coeficientes de Fourier $\hat{u}_N$ de $u(x)$ decaen como

$$|\hat{u}_N| \propto \left(\frac{1}{n}\right)^m$$

Esto se conoce como convergencia espectral, es decir entre mas suave la funcion, la serie converge mas rapido.

2.2.4 Stability of the Fourier-collocation method for hyperbolic problems

$$(f_N, g_N)_N = \frac{1}{N+1}\sum_{j=0}^{N} f_N(x_j)\bar{g}_N(x_j)$$

$$\|f_N\|_N^2 = (f_N, f_N)_N$$

$$(f_N, g_N)_N = \frac{1}{2\pi}\int_0^{2\pi} f_N \bar{g}_N dx, \|f_N\|_{L^2[0,2\pi]} = \|f_N\|_N^2$$

$$(f_N, g_N)_N = \frac{1}{N}\sum_{j=0}^{N-1} f_N(x_j)\widetilde{g}_N(x_j)dx, \|f_N\|_N^2 = (f_N, f_N)_N$$

$$K^{-1}\|f_N\|^2_{L^2[0,2\pi]} \leq \|f_N\|_N^2 \leq K\|f_N\|^2_{L^2[0,2\pi]}$$

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0, \tag{2.59}$$

$$u(x,0) = g(x) \tag{2.60}$$

$$\sum_{j=0}^{N-1} u_N^2(x_j,t) \leq \sum_{j=0}^{N-1} u_N^2(x_j,0)$$

$$\left|\frac{\partial u_N}{\partial t}\right|_{x_j} + a(x_j)\left|\frac{\partial u_N}{\partial x}\right|_{x_j} = 0 \tag{2.61}$$

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = -\sum_{j=0}^{N-1} u_N^2(x_j,t)\left|\frac{\partial u_N}{\partial x}\right|_{x_j}$$

$$\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = -\frac{N}{2\pi}\int_0^{2\pi} u_N(x)\frac{\partial u_N}{\partial x}dx = 0$$

$$\sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) = \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,0)$$

$$\begin{aligned}\frac{1}{k}\sum_{j=0}^{N-1} u_N^2(x_j,t) \leq \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,t) &= \sum_{j=0}^{N-1}\frac{1}{a(x_j)}u_N^2(x_j,0) \\ &\leq k\sum_{j=0}^{N-1} u_N^2(x_j,0)\end{aligned}$$

$$\|u_N(x,t)\|_N \leq k\|u_N(x,0)\|_N$$

$$\frac{du_N(t)}{dt} + ADu_N(t) = 0 \tag{2.62}$$

$$u(t) = e^{-ADt}u(0)$$

$$\|e^{-ADt}\| \leq K(t)$$
$$e^{-ADt}e^{-(AD)^T t} \leq K^2(t)$$

$$\|e^{-ADt}\| \leq k$$

2.2.5 Stability for parabolic equations

$$\frac{\partial u}{\partial t} = b(x)\frac{\partial^2 u}{\partial x^2} \tag{2.63}$$
$$u(x,0) = g(x) \tag{2.64}$$

$$\|u_N(t)\|_N \leq \sqrt{\frac{\max b(x)}{\min b(x)}}\|u_N(0)\|_N$$
$$\frac{du(t)}{dt} = BD^{(2)}u(t)$$

$$D^{(2)} \equiv D \cdot D$$

$$u^T B^{-1}\frac{d}{dt}u = u^T D^{(2)}u = u^T DDu \tag{2.65}$$
$$= (D^T u)^T (Du) = -(Du)^T (Du) \leq 0 \tag{2.66}$$

$$\frac{d}{dt}u^T B^{-1}u \leq 0$$

$$\frac{1}{\max b(x)}\|u_N(t)\|_N^2 \leq u^t(t)B^{-1}u(t) \leq u^T(0)B^{-1}u(0) \leq \frac{1}{\min b(x)}\|u_N(0)\|_N^2$$

The problems to be studied here are mixed initial-boundary value problems of the form

$$\frac{\partial u(x,t)}{\partial t} = L(x,t)u(x,t) + f(x,t), \quad x \in \mathcal{D}, \quad t \geqq 0, \tag{2.67}$$
$$B(x)u(x,t) = 0, \quad x \in \partial\mathcal{D}, \quad t > 0, \tag{2.68}$$
$$u(x,0) = g(x), \quad x \in \mathcal{D}, \tag{2.69}$$

where $\mathcal{D}$ is a spatial domain with boundary $\partial\mathcal{D}$, $L(x,t)$ is a linear (spatial) differential operator and $B(x)$ is a linear (time-independent) boundary operator. Here we write (2.55)-(2.57) for a single dependent variable u and a single space coordinate x with the understanding that much of the following analysis generalizes to systems of equations in higher space dimensions.

It is assumed that, for each t, $u(x,t)$ is an element of a Hilbert space $\mathcal{H}$ with inner product $(\cdot,\cdot)$ and norm $||\cdot||$. For each $t>0$, the solution $u(t)$ belongs to the subspace $\mathcal{B}$ of $\mathcal{H}$ consisting of all functions $u \in \mathcal{H}$ satisfying $Bu = 0$ on $\partial\mathcal{D}$. We do not require that $u(x,0) = g(x) \in \mathcal{B}$ but only that $u(x,0) \in \mathcal{H}$. The operator L is typically an unbounded differential operator whose domain is dense in, but smaller than, $\mathcal{H}$.

If the problem (2.55)-(2.7) is well posed, the evolution operator is a bounded linear operator from $\mathcal{H}$ to $\mathcal{B}$. Boundedness implies that the domain of the evolution operator can be extended in a standard way from the domain of L to the whole space $\mathcal{H}$. For notational convenience we shall assume henceforth that L is time independent so that the evolution operator is $e^{(Lt)}$. In this case the formal solution of (2.55)-(2.57) is

$$u(t) = e^{Lt}u(0) + \int_0^t e^{L(t-s)}f(s)ds \tag{2.70}$$

This formal solution is justified under the conditions that $f(t)$, $Lf(t)$ and $L^2 f(t)$ exist and are continuous functions of t in the norm $||\cdot||$ for all $t \geqq 0$

The semi-discrete approximations to (2.55) to be studied here are of the form

$$\frac{\partial u_N(x,t)}{\partial t} = L_N u_N(x,t) + f_N(x,t) \tag{2.71}$$

where, for each t, $u_N(x,t)$ belongs to an N-dimensional subspace $\mathcal{B}_N$ of $\mathcal{B}$, and L_N is a linear operator from $\mathcal{H}$ to $\mathcal{B}_N$ of the form

$$L_N = P_N L P_N \tag{2.72}$$

Here P_N is a projection operator of $\mathcal{H}$ onto $\mathcal{B}_N$ and $f_N = P_N f$. We shall assume that $\mathcal{B}_N \subset \mathcal{B}_M$ when $N < M$. To be definite, we shall also assume the initial conditions for the approximate equations (2.59) to be $u(0) = P_N u(0)$, where $u(0) = g(x)$ is the initial condition (2.55).

The fundamental problem of the numerical analysis of initial value problems is to find conditions under which $u_N(x,t)$ converges to $u(x,t)$ as $N \to \infty$ for some time interval $0 \leqq t \leqq T$ and to estimate the error $||u - u_N||$.

The classical Lax-Richtmyer equivalence theorem relating the above definitions states that "a consistent approximation to a well-posed linear problem is stable if and only if it is convergent."

Proof of the equivalence theorem. To show that stability implies convergence we use (2.55) and (2.59) to obtain

$$\frac{\partial(u-u_N)}{\partial t}=L_N(u-u_N)+Lu-L_Nu+f-f_N$$

Thus,

$$u(t)-u_N(t)=e^{L_Nt}[u(0)-u_N(0)] \tag{2.73}$$
$$+\int_0^t e^{L_N(t-s)}[Lu(s)-L_Nu(s)+f(s)-f_N(s)]ds \tag{2.74}$$

Using (2.30) and (2.61) and the triangle inequality we obtain the estimate

$$||u(t)-u_N(t)||\leqq K(t)||u(0)-u_N(0)|| \tag{2.75}$$
$$+\int_0^t K(t-s)[||Lu(s)-L_Nu(s)||+||f(s)-f_N(s)||]ds \tag{2.76}$$

Thus, if $u(t)$ belongs to the dense subspace of $\mathcal{H}$ satisfying (2.31) and if $f(t)$ belongs to the dense subspace of $\mathcal{H}$ satisfying $||f-P_Nf||\to 0$ as $N\to\infty$, then $||u(t)-u_N||\to 0$ as $N\to\infty$. Since all solutions $u(t)$ of (2.55) can be approximated arbitrarily well by functions satisfying (2.32), the proof that stability implies convergence is completed.

Conversely, to show that convergence implies stability, we first observe that, for any $u\in\mathcal{H}$, $||e^{L_Nt}u||$ is bounded for all N and each fixed t. In fact, convergence implies

$$0\leqq |||e^{L_Nt}u||-||e^{Lt}u|||\leqq ||e^{L_Nt}u-e^{Lt}u||\to 0,\qquad N\to\infty,$$

while well-posedness requires that $||e^{Lt}u||$ is finite. However, $\max||e^{L_Nt}u||$ may depend on u and on t, so stability is not yet proved. To complete the proof we use the fact that $\mathcal{H}$ is a Hilbert space. The principle of uniform boundedness (Richtmyer and Morton (1967)) implies that if $||e^{L_Nt}u||$ is bounded as $N\to\infty$ for each t and $u\in\mathcal{H}$, then $||e^{L_Nt}||$ is bounded as $N\to\infty$ for each t. This proves stability and completes the proof of the equivalence theorem.
Using the equivalence theorem, the study of the convergence of discrete approximations to the solutions of initial-value problems is reduced to the study of the stability of the discrete approximations, assuming the approximations are consistent. Thus, the development of conditions for the stability of families of finite-dimensional operators L_N is of primary interest in numerical analysis.

2.3 Fourier Spectral Methods

2.3.1 Metodo de Fourier-Galerkin

Sea $u(x,t) \in \mathcal{L}^2[0,2\pi]$ y definamos el siguiente problema.

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} &= \mathcal{L}u(x,t), \quad x \in [0,2\pi], \quad t \geq 0 \\ u(x,0) &= g(x), \qquad x \in [0,2\pi], \quad t = 0 \end{aligned}$$

Debemos encontrar las funciones $u_N(x,t)$ del espacio $\hat{B_N}$, es decir,

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}$$

Donde los coeficientes $a_n(t)$ se determinan del residuo R_N

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} - \mathcal{L}u_N(x,t)$$

Si expresamos a $R(t)$ como una serie de fourier obtenemos

$$R(x,t) = \sum_{|n| \leq \infty} \hat{R}_n(t) e^{inx}$$

Donde

$$\hat{R}_n(t) = \frac{1}{2\pi} \int_{-1}^{1} R_N(x,t) e^{inx} dx = 0, \quad \forall |n| \leq \frac{N}{2}$$

Lo cual equivale a $N+1$ ecuaciones diferenciales ordinarias, lo cual nos permite determinar los coeficientes $a_n(t)$ con las siguientes condiciones iniciales

$$u_N(x,0) = \sum_{|n| \leq \frac{N}{2}} a_n(0) e^{inx}, \quad a_n(0) = \frac{1}{2\pi} \int_{-1}^{1} g(x) e^{inx} dx$$

2.3.2 Operador semi-acotado

Un caso especial de un problema bien definido es cuando $\mathcal{L}$ es semi-acotado, es decir, $\mathcal{L} + \mathcal{L}^* \leq \alpha I$ para alguna constante α.

Por ejemplo, dada la solucion $u(x,t)$ que esta en un espacio de Hilbert $(\mathcal{H}, \langle \cdot, \cdot \rangle)$, suponiendo que el producto interior es el de $L^2[0,2\pi]$, la cual satisface el siguiente problema de valor de frontera con su valor inicial adecuado.

$$\frac{\partial u}{\partial t} = \mathcal{L}u \tag{2.77}$$

Mostraremos que este problema esta bien definido si $\mathcal{L}$ es semi-acotado. Empezaremos estimando la derivada de la norma como sigue

$$\begin{aligned}\frac{d}{dt}\|u\|^2 &= \frac{d}{dt}(u,u) = (\frac{du}{dt},u) + (u,\frac{du}{dt}) \\ &= (\mathcal{L}u,u) + (u,\mathcal{L}u) = (u,\mathcal{L}^*u) + (u,\mathcal{L}u) \\ &= (u,(\mathcal{L}+\mathcal{L}^*)u).\end{aligned}$$

Como $\mathcal{L} + \mathcal{L}^* \leq \alpha I$, tenemos que $\frac{d}{dt}\|u\|^2 \leq \alpha\|u\|^2$, lo cual equivale a $\frac{d}{dt}\|u\| \leq \alpha\|u\|$, y por lo tanto la norma es acotada, es decir, resolviendo la ecuacion obtenemos

$$\|u(t)\| \leq e^{\alpha t}\|u(0)\|$$

Y por lo tanto el problema esta bien definido.

Ahora veamos el siguiente ejemplo de un operador semiacotado y consideremos el siguiente problema

$$\mathcal{L} = a(x)\frac{\partial}{\partial x}$$

Donde $a(x)$ es una funcion real periodica con derivada acotada. Primero encontraremos a $\mathcal{L}^*$ como sigue

$$\begin{aligned}(\mathcal{L}u,v)_{L^2[0,2\pi]} &= \int_0^{2\pi} a(x)\frac{\partial u}{\partial x}\bar{v}dx \\ &= -\int_0^{2\pi} u\frac{\partial}{\partial x}a(x)\bar{v}dx \\ &= \left(u, \left[a(x)\frac{\partial}{\partial x} + \frac{da(x)}{dx}\right] v\right)_{L^2[0,2\pi]}\end{aligned}$$

Entonces,

$$\mathcal{L}^* = -\frac{\partial}{\partial x}a(x)I = -a(x)\frac{\partial}{\partial x} - a'(x)I$$

De esto ultimo obtenemos que,

$$\mathcal{L} + \mathcal{L}^* = a'(x)I$$

Como $a'(x)$ acotada, es decir, $|a'(x)| \leq 2\alpha$ para algun α, tenemos que,

$$\mathcal{L} + \mathcal{L}^* \leq 2\alpha I$$

Dado el problema $\frac{\partial u}{\partial t} = \mathcal{L}u$, donde $\mathcal{L}$ es un operador semi-acotado en el producto escalar usual de $\mathcal{L}^2[0,2\pi]$, entonces el metodo de Fourier-Galerkin es estable.

Demostracion. Primero, Vamos a mostrar que $\mathcal{P}_N = \mathcal{P}_N^*$. Comencemos con la simple observacion que.

$$(u, \mathcal{P}_N v) = (\mathcal{P}_N u, \mathcal{P}_N v) + ((I - \mathcal{P}_N)u, \mathcal{P}_N v).$$

$$\frac{\partial u_N}{\partial t} = \mathcal{P}_N \mathcal{L} \mathcal{P}_N u_N = \mathcal{L}_N u_N$$

Entonces,

$$\begin{aligned} \mathcal{L}_N + \mathcal{L}_N^* &= \mathcal{P}_N \mathcal{L} \mathcal{P}_N + \mathcal{P}_N \mathcal{L}^* \mathcal{P}_N \\ &= \mathcal{P}_N (\mathcal{L} + \mathcal{L}^*) \mathcal{P}_N \leq 2\alpha \mathcal{P}_N \end{aligned}$$

Siguiente el procedimiento anterior tenemos que

$$\|u_N(t)\| \leq e^{\alpha t} \|u_N(0)\|$$

Lo cual significa que es estable y por lo tanto convergente

2.3.3 Metodo de Fourier-Colocacion

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{2\pi}{N} j, \quad j \in [0, ..., N-1]$$

Para $u(x,t) \in \mathcal{L}^2[0, 2\pi]$ debemos encontrar las funciones

$$u_N \in \widetilde{B}_N = span \left\{ \left(cos(nx), \;\; 0 \leq n \leq \frac{N}{2} \right) \cup \left(sin(nx), \;\; 1 \leq n \leq \frac{N}{2} - 1 \right) \right\}$$

Las cuales tienen la forma

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}$$

Si las expresamos en terminos de polinomios

$$u_N(x,t) = \sum_{0}^{N-1} u_N(x_j, t) g_j(x)$$

Donde $g_j(x)$ son los polinomios de Lagrange y ademas cumple que $g_j(x_i) = \delta_{ij}$, entonces el residuo

$$R_N(x,t) = \frac{\partial u_n(x,t)}{\partial t} - \mathcal{L} u_N(x,t)$$

Y debera cumplir

$$R_N(x_j, t) = 0, \quad \forall j \in [0, ..., N-1]$$

De modo que la ecuacion satisface

$$\frac{\partial u_N(x,t)}{\partial t} - \mathcal{I}_N \mathcal{L} u_N(x,t) = 0$$

Chapter 3

Numerical Solutions of the Burger's Equation

Sea $u(x,t) \in \mathcal{L}^2[0,2\pi]$ y definamos el siguiente problema.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \;\; x \in [0,1], \;\; t \geq 0 \tag{3.1}$$

With an initial condition

$$u(x,0) = g(x) \tag{3.2}$$

3.1 Fourier Galerkin

The approximate function $u_N(x,t)$ is represented as the truncated Fourier Series in $\hat{B_N}$,

$$u_N(x,t) = \sum_{|n|\leq \frac{N}{2}} a_n(t)e^{inx}$$

where $a_n(t)$ are determinated of the residue R_N

$$R_N(x,t) = \frac{\partial u_N(x,t)}{\partial t} - \mathcal{L}u_N(x,t)$$

Si expresamos a $R(t)$ como una serie de fourier obtenemos

$$R(x,t) = \sum_{|n|\leq\infty} \hat{R}_n(t)e^{inx}$$

Donde

$$\hat{R}_n(t) = \frac{1}{2\pi}\int_{-1}^{1} R_N(x,t)e^{inx}dx = 0, \quad \forall |n| \leq \frac{N}{2}$$

Lo cual equivale a $N+1$ ecuaciones diferenciales ordinarias, lo cual nos permite determinar los coeficientes $a_n(t)$ con las siguientes condiciones iniciales

$$u_N(x,0) = \sum_{|n|\leq \frac{N}{2}} a_n(0)e^{inx}, \quad a_n(0) = \frac{1}{2\pi}\int_{-1}^{1} g(x)e^{inx}dx$$

3.2 Fourier Collocation

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{2\pi}{N} j, \quad j \in [0, ..., N-1]$$

Para $u(x,t) \in \mathcal{L}^2[0, 2\pi]$ debemos encontrar las funciones

$$u_N \in \widetilde{B}_N = span \left\{ \left(cos(nx), \ 0 \leq n \leq \frac{N}{2} \right) \cup \left(sin(nx), \ 1 \leq n \leq \frac{N}{2} - 1 \right) \right\}$$

Las cuales tienen la forma

$$u_N(x,t) = \sum_{|n| \leq \frac{N}{2}} a_n(t) e^{inx}$$

Si las expresamos en terminos de polinomios

$$u_N(x,t) = \sum_{0}^{N-1} u_N(x_j, t) g_j(x)$$

Donde $g_j(x)$ son los polinomios de Lagrange y ademas cumple que $g_j(x_i) = \delta_{ij}$, entonces el residuo

$$R_N(x,t) = \frac{\partial u_n(x,t)}{\partial t} - \mathcal{L} u_N(x,t)$$

Y debera cumplir

$$R_N(x_j, t) = 0, \quad \forall j \in [0, ..., N-1]$$

De modo que la ecuacion satisface

$$\frac{\partial u_N(x,t)}{\partial t} - \mathcal{I}_N \mathcal{L} u_N(x,t) = 0$$

3.3 Chebyshev Collocation

Consideremos de nuevo el sistema anterior y definamos el siguiente conjunto de puntos en el espacio como sigue.

$$x_j = \frac{cos(\pi j}{N}, \quad j \in [0, ..., N-1]$$

$$\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \left| \frac{\partial^2 u^N}{\partial x^2} \right|_{x=x_j} = 0 \qquad (3.3)$$

$$u^N(-1,t) = u_L(t), \ \ u^N(1,t) = u_R(t),$$
$$u^N(x_j, 0) = u_0(x_j), \qquad j = 0, \ldots N.$$

Let $u(t) = (u^N(x_0,t), \ldots, u^N(x_N,t))^T$. Then can be written as, for all $t > 0$,

$$\left(\frac{du}{dt} + u \boxtimes D_N u - \nu D_N^2 u\right) = 0 \tag{3.4}$$

3.4 Chebyshev Tau

$$u(-1,t) = u_L(t), \;\; u(1,t) = u_R(t), \tag{3.5}$$

$$u_N(x,t) = \sum_{k=0}^{N} \hat{u}_k(t) T_k(x) \tag{3.6}$$

$$\int_{-1}^{1} \left(\frac{\partial u^N}{\partial t} + u^N \frac{\partial u^N}{\partial x} - \nu \frac{\partial^2 u^N}{\partial x^2}\right)(x) T_k(x)(1-x^2)^{-1/2} dx = 0, \tag{3.7}$$

$$k = 0, \ldots, N-2 \tag{3.8}$$

$$u^N(-1,t) = u_L(t), \;\; u^N(1,t) = u_R(t), \tag{3.9}$$

$$\frac{\partial \hat{u}_k}{\partial t} + \left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\hat{}} - \nu \hat{u}_k^{(2)} = 0, \;\; k = 0, 1, \ldots, N-2, \tag{3.10}$$

$$\left(u^N \frac{\partial u^N}{\partial x}\right)_k^{\hat{}} = \frac{2}{\pi c_k} \int_{-1}^{1} \left(u^N \frac{\partial u^N}{\partial x}\right)(x) T_k(x)(1-x^2)^{-1/2} dx \tag{3.11}$$

$$\sum_{k=0}^{N} \hat{u}_k = u_R, \sum_{k=0}^{N} (-1)^k \hat{u}_k = u_L \tag{3.12}$$

$$\hat{u}_k(0) = \frac{2}{\pi c_k} \int_{-1}^{1} u_0(x) T_k(x)(1-x^2)^{-1/2} dx, \;\; k = 0, \ldots, N \tag{3.13}$$

$$(uv)_k^{\hat{}} = \frac{2}{\pi c_k} \int_{-1}^{1} u(x) v(x) T_k(x)(1-x^2)^{-1/2} dx \tag{3.14}$$

$$(uv)_k^{\hat{}} = \frac{1}{2} \sum_{p+q=k} \hat{u}_p \hat{v}_q + \sum_{|p-q|=k} \hat{u}_p \hat{v}_q \tag{3.15}$$

Chapter 4

Numerical Solution of the Stochastic Burgers equation

4.1 Numerical Approximation

Sea $\mathcal{H} = \mathcal{L}^2(0,1)$.

$$
\begin{aligned}
dX(t,\xi) &= \left[\nu \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + \frac{1}{2}\frac{\partial (X^2(t,\xi))}{\partial \xi}\right] dt + dW_t(t,\xi), \;\; \xi \in [0,1] \\
X(t,0) &= X(t,1) = 0, \;\; t > 0 \\
X(0,\xi) &= x(\xi), \;\; x \in \mathcal{H}
\end{aligned}
$$

Si definimos $A = \nu \partial_\xi^2$, $B = \frac{1}{2}\partial_\xi (X^2)$

$$
\begin{aligned}
dX &= [AX + B(X)]dt + dW_t \\
X(0) &= x, \;\; x \in \mathcal{H}
\end{aligned}
$$

Si $u(t,x) = \mathbb{E}[u_0(X_t^x)]$, y $u \in \mathbb{H} = L^2(\mathcal{H},\mu)$ satisface

$$
\frac{\partial u}{\partial t} = \frac{1}{2} Tr(QD^2 u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \; x \in D(A)
$$

Definamos el siguiente conjunto de indices

$$
\mathcal{J} = \{\alpha = (\alpha_i, i \geq 1) | \alpha_i \in \mathbb{N} \cup 0, |\alpha| := \sum_{i=0}^{\infty} \alpha_i < \infty\}
$$

Funcionales de Hermite

$$
H_n(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), h \in \mathcal{H}_0, n \in \mathcal{J}
$$

$$
P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}
$$

Si para M, N fijos tomamos un subconjuto de $\mathcal{J}$

$$
\mathcal{J}^{M,N} = \{\alpha = (\alpha_i, 1 \leq i \leq M) | \alpha_i \in \{0,1,...,N\}\}
$$

Descomposición dada por la expansión de Wiener-Chaos (Serie de Fourier-Hermite).

$$u_N(t,x) = \sum_{n\in\mathcal{J}^{M,N}} u_n(t)H_n(x),\ x\in\mathcal{H},\ t\in[0,T]$$

Sustituyendo en la ecuación y haciendo unos ajustes

$$\dot{u}_{m_i}(t) = -u_{m_i}(t)\lambda_{m_i} + \sum_{j=1}^{M} u_{n_j}(t)C_{n_j,m_i},\ 1\le i\le M$$

Donde

$$C_{n,m} = \int_{\mathcal{H}} \langle B(x), D_x H_n(x)\rangle_{\mathcal{H}} H_m(x)\mu(dx)$$

Si escribimos la solución como sigue

$$U^M(t) = \begin{pmatrix} u_{m_1}(t) & u_{m_2}(t) & \dots & u_{m_M}(t)\end{pmatrix}^T$$

$$\dot{U}^M(t) = \begin{pmatrix} \dot{u}_{m_1}(t) & \dot{u}_{m_2}(t) & \dots & \dot{u}_{m_M}(t)\end{pmatrix}^T$$

$$\dot{U}^M(t) = AU^M(t)$$

Donde la matriz A es:

$$\begin{pmatrix}
-\lambda_1 + C_{1,1} & C_{2,1} & \dots & C_{M-1,1} & C_{M,1}\\
C_{1,2} & -\lambda_2 + C_{2,2} & \dots & C_{M-1,2} & C_{M,2}\\
\vdots & \vdots & \ddots & \vdots & \vdots\\
C_{1,M-1} & C_{2,M-1} & \dots & -\lambda_{M-1} + C_{M-1,M-1} & C_{M,M-1}\\
C_{1,M} & C_{2,M} & \dots & C_{M-1,M} & -\lambda_M + C_{M,M}
\end{pmatrix}$$

Donde $\lambda_i = \lambda_{mi}$ y $C_{i,j} = C_{n_i,m_j}$ para $1\le i,j\le M$.

Entonces la solucion del sistema es

$$U^M(t) = \sum_{j=1}^{M} c_i V_i e^{\eta_i t}$$

Donde las constantes c_i se obtienen evaluando en $t=0$

$$\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_{M-1}(t)\\ u_M(t)\end{pmatrix} = \begin{pmatrix} V1 & V2 & \dots & V_{M-1} & V_M\end{pmatrix}\begin{pmatrix} c_1e^{\eta_1 t}\\ c_2e^{\eta_2 t}\\ \vdots\\ c_{M-1}e^{\eta_{M-1}t}\\ c_Me^{\eta_M t}\end{pmatrix}$$

Condiciones Iniciales: Definamos los puntos z_i, $i=0,1,\dots,p$, tales que $z_0=a$ y $z_p=b$.

$$u(0,x) = \mathbb{E}[u_0^{z_i}(X_0^x)] = X^x(0,z_i) = x(z_i)$$

Por otra parte tenemos

$$u(0,x) = \sum_{n\in\mathcal{J}^{M,N}} u_n(0)H_n(x)$$

Multiplicando por $H_m(x)$ e integrando en el espacio $\mathcal{L}^2(\mathcal{H},\mu)$

$$u_m(0) = \int_{\mathcal{H}} x(z_i)H_m(x)\mu(dx)$$

Entonces la solución del sistema

$$\dot{U}^M(t) = AU^M(t)$$

Se resuelve con las condiciones iniciales

$$\begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_{M-1}(0) \\ u_M(0) \end{pmatrix} = \begin{pmatrix} V1 & V2 & \dots & V_{M-1} & V_M \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{M-1} \\ c_M \end{pmatrix}$$

Continuidad con respecto las condiciones iniciales:
Sean $x, y \in \mathcal{H}$ condiciones iniciales distinas y $\mathbb{H} = L^2(\mathcal{H},\mu)$, con $\mu = \frac{1}{\sqrt{2\pi}}e^{\frac{x^2}{2}}dx$. Dadas las aproximaciones de la solución,

$$\psi_t^x = \sum_{n\in\mathcal{J}} u_n^x(t)H_n(x), \quad \psi_t^y = \sum_{n\in\mathcal{J}} u_n^y(t)H_n(y)$$

Estimamos $\psi_t^x - \psi_t^y$

$$\|\psi_t^x - \psi_t^y\|_{(L^2(\mathcal{H},\mu))^2}$$

Chapter 5

Numerical Results

5.1 Simulations

Bibliography

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