Hopf-Cole Transformation

Tai-Ping Liu

Academia Sinica, Taiwan
Stanford University

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Brown University

- Hopf, Eberhard The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Comm. Pure Appl. Math.* 3, (1950), 201-230.
- Julian D. Cole On a quasi-linear parabolic equation occurring in aerodynamics. *Quart. Appl. Math.* 9, (1951), 225-236.

Hopf-Cole transformation:

$u_t + uu_x = \kappa u_{xx}$ Burgers equation,
$\Rightarrow \; B_t + \frac{(B_x)^2}{2} = \kappa B_{xx}, \; B_x = u,$ Hamilton-Jacobi equation,
Introduce Hopf-Cole relation $B(x,t) = -2\kappa \log[\phi(x,t)]$,
$\Rightarrow \; \phi_t = \kappa \phi_{xx},$ Heat equation.

Hopf-Cole Transformation

Solution formula for initial value problem for Burgers equation:

$$u(x,t) = u(x,t,\kappa) = \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0-}^{y} u(z,0)dz} dy}{\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0-}^{y} u(z,0)dz} dy}.$$

Hopf-Cole Transformation

Burgers equation:

- Bateman proposed it for considering shock profile: Bateman, H. Some recent researches on the motion of fluids, *Monthly Weather Review* 43, (1915), 163-170.
- Burgers proposed it for the study of turbulence: Burgers, J. M. Application of a model system to illustrate some points of the statistical theory of free turbulence. Nederl. Akad. Wetensch., Proc. 43, (1940), 2-12.
- Cole derived the Burgers equation from gas dynamics:

$$\frac{\partial w}{\partial t} + \beta \frac{\partial w}{\partial x} = \frac{4}{3}\nu^* \frac{\partial^2 w}{\partial x^2}.$$

"for $w =$ excess of flow velocity over a sonic velocity, where $\beta = (\gamma + 1)/2,\ \nu^* =$ the kinematic viscosity at sonic condition"

Hopf-Cole Transformation:

- Hopf: ”The reduction of (1) to the heat equation was known to me since the end of 1946. However, it was not until 1949 that I became sufficiently acquainted with the recent development of fluid dynamics to be convinced that a theory of (1) could serve as an instructive introduction into some of the mathematical problems involved.”
- Friedrichs, K. O. Formation and decay of shock waves. Communications on Appl. Math. 1, (1948). 211245.
- Hopf was inspired by the works of Burgers on turbulence and Friedrichs’ theory of N-waves.
- Forsyth, A.R. Theory of Differential Equations, Vol. VI, Cambridge University Press (1906), Page 102, Ex. 3.
- The Hopf-Cole transformation is embedded in this exercise in Forsyth’s book.

Hopf-Cole Transformation

Hopf:

- Solution formula for the inviscid Burgers equation $u_t + (u^2/2)_x = 0$ in the zero dissipation limit $\kappa \to 0+$:

$$\begin{aligned} &F(x,y,t) = \tfrac{(x-y)^2}{2t} + \textstyle\int_{0_-}^{y} u(z,0)dz, \\ &\min_y F(x,y,t) = F(\xi,t), \\ &\lim_{\kappa\to 0_+} u(x,t,\kappa) = u(\xi,0). \end{aligned}$$

- Metastable states:

$$\lim_{t\to\infty}\lim_{\kappa\to 0+} u(x,t,\kappa) \neq \lim_{\kappa\to 0+}\lim_{t\to\infty} u(x,t,\kappa).$$

- Modern theory of hyperbolic conservation laws.

Hopf-Cole Transformation

Hopf:

$$b_t + (\frac{b^2}{2})_x = \kappa b_{xx},\ b(x,0) = A\delta(x),\ \text{Burgers kernel}$$

By Hopf-Cole transformation,

$$b(x,t;A) = \frac{\frac{\sqrt{\kappa}}{\sqrt{t}}(e^{\frac{A}{2\kappa}} - 1)e^{-\frac{x^2}{4\kappa t}}}{\sqrt{\pi} + \int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty}(e^{\frac{A}{2\kappa}} - 1)e^{-y^2}dy}.$$

For initial data with finite mass, a solution of the Burgers equation tends to Burgers kernel:

$$\int_{-\infty}^{\infty}|u(x,t) - b(x,t;A)|dx = O(1)t^{-\frac{1}{2}},\ \text{as } t \to \infty,\ A = \int_{-\infty}^{\infty} u(x,0)dx.$$

Hopf-Cole Transformation

Hopf:
On the other hand, for inviscid Burgers equation, the solution tends to N-waves:

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t;p,q)| = O(1)t^{-\frac{1}{2}},$$
$$p = \min_x \int_{-\infty}^{x} u(x,0)dx,\ \ q = \max_x \int_{x}^{\infty} u(x,0)dx,\ \text{two time invariants}$$
$$N(x,t;p,q) = \begin{cases} \frac{x}{t}, & \text{for } -\sqrt{-2pt} < x < \sqrt{2qt}; \\ 0, & \text{otherwise.} \end{cases}$$

$$\lim_{t\to\infty} \lim_{\kappa\to 0+} u(x,t,\kappa) = N-\text{waves, two time invariants;}$$

$$\neq \lim_{\kappa\to 0+} \lim_{t\to\infty} u(x,t,\kappa) \text{ one time invariant.}$$

N-waves represent metastable states for the Burgers solutions.

Hopf-Cole Transformation

Outside of gas dynamics:

- Miura, R. M. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Mathematical Phys.* 9 (1968), 1202-1204.
- $$V_t - 6VV_x + V_{xxx} = 0, \text{ KdV } \Rightarrow \phi_t - 6\phi^2\phi_x + \phi_{xxx} = 0, \text{ Modified}$$
  $$V = \phi^2 \pm \phi_x, \text{ Miura transformation.}$$
- "It is rare and surprising to find a transformation between two simple nonlinear partial differential equations of independent interest. One is reminded of the Hopf-Cole transformation of quadratically nonlinear Burgers equation into the heat conduction (diffusion) equation. A number of investigators (including us) have attempted unsuccessfully to find a similar simple linearizing transformation for the KdV equation, but a complicated one will be given in VI."

Hopf-Cole Transformation

Outside of gas dynamics:

- Kardar, M.; Parisi, G.; Zhang, Y.-C. Dynamic Scaling of Growing Interfaces *Phys. Rev. Lett.* Vol. 56, Iss. 9 -3 March (1986), 889-892.
- Evolution of the profile of a growing interface: the Hamilton-Jacobi equation plus a noise η:

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\boldsymbol{x}, t).$$

- Hopf-Cole transform $\Rightarrow$ linear equation with a source:

$$\begin{cases} \frac{\partial W}{\partial t} = \nu \nabla^2 W + \frac{\lambda}{2\nu}\eta(\boldsymbol{x}, t) W, \\ W(\boldsymbol{x}, t) = e^{\frac{\lambda}{2\nu} h(\boldsymbol{x}, t)}. \end{cases}$$

- New scaling, distinct from deterministic dissipation equations comes up due to the noise.
- "We thus have an intriguing connection between evolutions of a hydrodynamic and a growth pattern!"

Hopf-Cole Transformation

- Scalar convex hyperbolic conservation law, $f''(u) \neq 0$,

$$u_t + f(u)_x = 0 \;\Rightarrow\; \lambda_t + \lambda\lambda_x = 0, \text{ inviscid Burgers }, \lambda = f'(u).$$

- System of hyperbolic conservation laws, e.g. Euler equations in gas dynamics

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0,\ \boldsymbol{u} \in \mathbb{R}^n,\ \boldsymbol{f}'(\boldsymbol{u})\boldsymbol{r}_j(\boldsymbol{u}) = \lambda_j(\boldsymbol{u})\boldsymbol{r}_j(\boldsymbol{u}),$$
$$\boldsymbol{l}_j(\boldsymbol{u})\boldsymbol{f}'(\boldsymbol{u}) = \lambda_j(\boldsymbol{u})\boldsymbol{l}_j(\boldsymbol{u}),\ \boldsymbol{l}_j(\boldsymbol{u}) \cdot \boldsymbol{r}_k(\boldsymbol{u}) = \delta_{jk},\ j, k = 1, 2, \cdots, n.$$

- "Convexity": $\nabla_{\boldsymbol{u}}\lambda_j(\boldsymbol{u}) \cdot \boldsymbol{r}_j(\boldsymbol{u}) \neq 0$ "genuine nonlinear" field, e.g. acoustic waves.
- j-simple waves: $\boldsymbol{u}(x, t)$ moves along integral curve of $\boldsymbol{r}_j(\boldsymbol{u})$.

$$\lambda_t + \lambda\lambda_x = 0,\ \lambda(x, t) = \lambda_j(\boldsymbol{u}(x, t)) \;\Rightarrow\; \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0.$$

Hopf-Cole Transformation

- Viscous conservation laws, e.g. Compressible Navier-Stokes equations

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\mathbb{B}(\boldsymbol{u}, \boldsymbol{\nu})\boldsymbol{u}_x)_x.$$

- Dissipation parameters, e.g. $\boldsymbol{\nu} = (\mu, \kappa)$ viscosity and heat conductivity.
- The Burgers equation is used for construction of approximate j-simple waves for each genuinely nonlinear field

$$\lambda_t + \lambda\lambda_x = \kappa\lambda_{xx},\ \lambda(x,t) = \lambda_j(\boldsymbol{u}(x,t)).$$

- Burgers dissipation parameter κ is the diagonal element of the viscosity matrix $\mathbb{B}$ in the characteristic coordinates of the hyperbolic part:

$$\kappa = \boldsymbol{l}_j(\boldsymbol{u})\mathbb{B}(\boldsymbol{u}, \boldsymbol{\nu})\boldsymbol{r}_j(\boldsymbol{u}).$$

Hopf-Cole Transformation

- $u_t + f(u)_x = 0$ hyperbolic conservation laws
  Solutions of finite mass tends to N-waves at the rate of $t^{-1/4}$ in $L_1(x)$ as consequence of pointwise estimate.
  Liu, T.-P. Pointwise convergence to N-waves for solutions of hyperbolic conservation laws. *Bull. Inst. Math. Acad. Sinica* 15, (1987), no. 1, 1-17.
- $u_t + f(u)_x = (\mathbb{B}(u, \nu)u_x)_x$, viscous conservation laws.
  Solutions of finite mass tends to Burgers and heat kernels also at the rate of $t^{-1/4}$ in $L_1(x)$ and as consequence of pointwise estimate..
  Liu, T.-P.; Zeng, Y. Large time behavior of solutions for general quasilinear hyperbolic-parabolic systems of conservation laws. Mem. Amer. Math. Soc. 125 (1997), no. 599, viii+120 pp.
- Open problem: Metastability.

Hopf-Cole Transformation

- Hopf-Cole transformation is used for finding exact expression of Burgers Nonlinear waves.
- The Burgers nonlinear waves is used for construction of approximate nonlinear waves for system of conservation laws.
- The linearized Hopf-Cole transformation is used for the explicit construction of Green's function for Burgers equation linearized around a nonlinear wave.
- The construction is Green's function for systems is based on the Burgers Green's function.
- This is essential for the study of shock, initial layers for system of conservation laws.

Hopf-Cole Transformation

Burgers shock formation, $\lambda_0 > 0$, using Hopf-Cole:

$$\begin{cases} (u_S)_t + u_S(u_S)_x = \kappa (b_S)_{xx}, \\ u_S(x,0) = \begin{cases} \lambda_0, & \text{for } x < 0, \\ -\lambda_0, & \text{for } x > 0, \end{cases} \end{cases}$$

$$u_s(x,t) = -\lambda_0 \frac{Erfc(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{\kappa}} Erfc(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}{Erfc(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{\kappa}} Erfc(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}.$$

The thickness T_0 of the initial layer to form Burgers shock profile b_S, the time when the error function *Erfc* approaches $\sqrt{\pi}$,

$$b_S(x) = \lim_{t\to\infty} u_S(x,t) = -\lambda_0 \tanh(\frac{\lambda_0 x}{2\kappa}).$$

$$\frac{\lambda_0 T_0}{\sqrt{4\kappa T_0}} = O(1), \text{ or } T_0 = O(1)\frac{\kappa}{(\lambda_0)^2}.$$

Hopf-Cole Transformation

Burgers rarefaction wave

$$\begin{cases} (h_R)_t + (\frac{(h_R)^2}{2})_x = 0, \\ h_R(x,0) = \begin{cases} -\lambda_0, & \text{for } x < 0, \\ \lambda_0, & \text{for } x > 0; \end{cases} \end{cases}$$

$$b_R(x,t) = \lambda_0 \frac{e^{\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}{e^{\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) + e^{-\frac{\lambda_0 x}{2\kappa}} Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}.$$

Within the hyperbolic rarefaction wave region, $x \in (-\lambda_0 t + M\sqrt{4\kappa t}, \lambda_0 t - M\sqrt{4\kappa t})$, and after initial layer time, the difference of the Burgers rarefaction wave b_R and the inviscid rarefaction wave x/t:

$$b_R(x,t) - \frac{x}{t} = O(1)[\frac{1}{|x-\lambda_0 t|} + \frac{1}{|x+\lambda_0 t|}], \ t > O(1)\frac{\kappa}{(\lambda_0)^2}.$$

Linear Hopf-Cole transformation

Burgers equation linearized around a given solution $\bar{u}(x,t)$:

$$\bar{u}_t + (\frac{\bar{u}^2}{2})_x = \kappa \bar{u}_{xx}$$
$$\bar{U}_x = \bar{u},\ \bar{U}(x,t) = -2\kappa \log[\bar{\phi}(x,t)],$$
$$v_t + (\bar{u}v)_x = \kappa v_{xx},$$ Burgers equation linearized around $\bar{u}(x,t)$.

Linearize the Hopf-Cole relation $V + \bar{U} = -2\kappa \log[\bar{\phi} + \zeta]$:

$$V = -2\kappa \frac{\zeta}{\bar{\phi}},$$ linearized Hopf-Cole relation, $\Rightarrow$

$\zeta_t = \kappa \zeta_{xx}$ and the solution representation to the solution of the linearized Burgers equation:

$$v(x,t) = \frac{\partial}{\partial x} \frac{\int_{-\infty}^{\infty} [\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y,0) V(y,0)] dy}{\bar{\phi}(x,t)}.$$

Green's function for shock profile

Green's function $G_S(x,t;x_0,t-t_0)$ for the shock profile $b_S(x)$ using linearized Hopf-Cole:

$$(G_S)_t + b_S(G_S)_x = \kappa(G_S)_{xx},\ G_S(x,0) = \delta(x-x_0);$$

$$G_S(x,t;x_0) = \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-x_0)^2}{4\kappa t}} \frac{e^{\frac{\lambda_0 x_0}{2\kappa}} + e^{-\frac{\lambda_0 x_0}{2\kappa}}}{e^{-\frac{\lambda_0 x}{2\kappa}} + e^{-\frac{\lambda_0 x}{2\kappa}}} e^{\frac{(\lambda_0)^2 t}{4\kappa}}.$$

The Green's function as weighted combination of the heat kernel with speeds $\pm\lambda_0$:

$$G_S(x,t;x_0) = \frac{1+e^{-\frac{\lambda_0|x_0|}{\kappa}}}{1+e^{-\frac{\lambda_0|x|}{\kappa}}} \begin{cases} H(x+\lambda_0 t, t),\ \text{for } x>0,\ x_0>0; \\ e^{-\frac{\lambda_0|x|}{\kappa}} H(x+\lambda_0 t, t),\ \text{for } x<0,\ x_0>0; \\ H(x-\lambda_0 t, t),\ \text{for } x<0,\ x_0<0; \\ e^{-\frac{\lambda_0|x|}{\kappa}} H(x-\lambda_0 t, t),\ \text{for } x>0,\ x_0<0. \end{cases}$$

Hopf-Cole Transformation

Green's function for rarefaction waves:

$$G_R(x,t;x_0,t_0)=e^{-\frac{[x-x_0-(\lambda_0(t-t_0)]^2}{4\kappa(t-t_0)}}\frac{Erfc(\frac{-x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}})+Erfc(\frac{x_0+\lambda_0 t_0}{\sqrt{4\kappa t_0}})}{Erfc(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}})+Erfc(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}.$$

The propagation of waves is around the zero line of the exponential, along inviscid characteristics $x=x_0+\lambda_0(t-t_0)$. The essential support of the information is in the region given by

$$\frac{t(x_0-t_0x/t)^2}{4\kappa t_0(t-t_0)}=O(1),\ \text{or}\ |x-\frac{t}{t_0}x_0|=O(1)\sqrt{\kappa(t-t_0)\frac{t}{t_0}},$$

varying from sub-linear, dissipatve scale $\sqrt{t-t_0}$ for $t-t_0$ small, to linear, hyperbolic scale t for $t-t_0$ large.

Hopf-Cole Transformation

Open problem: Riemann problem

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\mathbb{B}(\boldsymbol{u}, \boldsymbol{\nu})\boldsymbol{u}_x)_x,$$

$$\boldsymbol{u}(x,0) = \begin{cases} \boldsymbol{u}_l, & x < 0, \\ \boldsymbol{u}_r, & x > 0. \end{cases}$$

$t \to \infty \;\Rightarrow \boldsymbol{\nu} \to 0,$ zero dissipation limit,

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\mathbb{B}(\boldsymbol{u}, \boldsymbol{\nu})\boldsymbol{u}_x)_x \;\Rightarrow \boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0.$$

- Hoff, D.; Liu, T.-P. The inviscid limit for the Navier-Stokes equations of compressible, isentropic flow with shock data. *Indiana Univ. Math.* J. 38 (1989), no. 4, 861-915. single shock, zero mass, using Hopf-Cole for initial layer
- Bianchini, S.; Bressan, A. Vanishing viscosity solutions of nonlinear hyperbolic systems. Ann. of Math. (2) 161 (2005), no. 1, 223-342. general initial values, artificial viscosity, generalized Glimm.

Boltzmann equation

$$\partial_t \mathrm{f}(\boldsymbol{x}, t, \xi) + \xi \cdot \partial_{\boldsymbol{x}} \mathrm{f}(\boldsymbol{x}, t, \xi) = \frac{1}{k} Q(\mathrm{f}, \mathrm{f})(\boldsymbol{x}, t, \xi)$$

Open problem: Riemann problem

$$\mathrm{f}(\boldsymbol{x}, t, \xi) = \begin{cases} M_l(\xi), & x < 0, \\ M_r(\xi), & x < 0. \end{cases}$$

$t \to \infty \ \Rightarrow k \to 0,$ zero mean free path,

Boltzmann solutions $\Rightarrow$ Euler solutions,

- Yu, S.-H. Initial and shock layers for Boltzmann equation. *Arch. Ration. Mech. Anal.* 211 (2014), no. 1, 1-60. single shock, nonzero mass, use Boltzmann Green's function, Hopf-Cole, etc.