SHOCK CAPTURING BY THE SPECTRAL VISCOSITY METHOD[1]

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ABSTRACT

A main disadvantage of using spectral methods for nonlinear conservation laws lies in the formation of Gibbs phenomenon, once spontaneous shock discontinuities appear in the solution. The global nature of spectral methods then pollutes the unstable Gibbs oscillations overall the computational domain, and the lack of entropy dissipation prevents convergences in these cases.

In this paper, we discuss the Spectral Viscosity method, which is based on high frequency-dependent vanishing viscosity regularization of the classical spectral methods. We show that this method enforces the convergence of nonlinear spectral approximations without sacrificing their overall spectral accuracy.

[1]Research was supported in part by NASA Contract No. NAS1-18107 while the author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665. Additional support was provided by U. S.-Israel BSF Grant No. 85-00346.

1. INTRODUCTION

We consider the 2π-periodic one-dimensional system of conservation laws

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.1a}$$

with prescribed initial conditions, $u(x, t = 0) = u_0(x)$. We recall, [4], that solutions of (1.1a) may develop spontaneous jump discontinuities (shock waves) and hence the class of weak solutions must be admitted. Moreover, since there are many possible weak solutions, the system (1.1a) is augmented with an entropy condition which requires

$$\frac{\partial U(u)}{\partial t} + \frac{\partial F(u)}{\partial x} \leq 0. \tag{1.1b}$$

Here, $U(u)$ and $F(u) \equiv \int^u < U'(w), f'(w)dw >$ is any entropy function and the corresponding entropy flux pair associated with (1.1a), so that a strict inequality in (1.1b) reflects the existence of physically relevant shock waves in the entropy solution of (1.1a), (1.1b).

We want to solve the 2π-periodic initial-value problem, (1.1a), (1.1b), by (pseudo-) spectral methods. To this end, we use an N-trigonometric polynomial

$$u_N(x, t) = \sum_{k=-N}^{N} \hat{u}_k(t) e^{ikx},$$

in order to approximate the spectral or ψdospectral projection of the exact solution, $u(x, t)$. Let $S_N u$ and $\psi_N u$ denote, respectively, the spectral-Fourier and the ψdospectral-Fourier projections of $u(x, t)$, and let $P_N u$ stands for either one of these two projections. Then, starting with $u_N(x, 0) = P_N u_0(x)$, the standard Fourier method, [1], lets $u_N(x, t)$ evolves at a later time according to the $(2N + 1)$-dimensional approximate model

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[P_N f(u_N)] = 0. \tag{1.2}$$

We can rewrite this approximation in the equivalent form

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[f(u_N)] = \frac{\partial}{\partial x}[Eu_N], \quad Eu_N = (I - P_N)f(u_N). \tag{1.3}$$

The expression inside the right brackets is the local error (due to spatial discretization). The approximation (1.2) - or equivalently (1.3), is consistent with the conservation law (1.1) in the sense that its local error does not exceed

$$\|Eu_N = (I - P_N)f(u_N)\| \leq Const_1 \cdot \frac{1}{N}\|u_N\|_{H^1}. \tag{1.4}$$

In fact, the usual spectral estimates, [10], tell us that the approximation (1.2) is arbitrarily high-order (or spectrally) accurate approximation of the conservation law (1.1a). Namely, for *any* $s \geq 1$ there exists a constant, $Const_s$, such that

$$\|Eu_N\| \leq Const_s \cdot N^{-s}\|u_N\|_{H^s}. \tag{1.5}$$

The consistency condition (1.4) guarantees that as $N \uparrow \infty$, the approximation (1.2) approaches the conservation law (1.1a). Nevertheless, its approximate solution, $u_N(x,t)$, need not approximate the exact entropy solution, $u(x,t)$. The following example shows what could go wrong.

EXAMPLE [12, Section 2]. The ψdospectral approximation of the periodic scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial (e^u)}{\partial x} = 0, \tag{1.6}$$

reads

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[\psi_N e^{u_N}] = 0. \tag{1.7}$$

Let us sum (1.7) against $\psi_N e^{u_N}$ at the $2N+1$ equidistant collocation points, $x_\nu = \nu \Delta x$, $\Delta x \equiv \frac{2\pi}{2N+1}$

$$\frac{d}{dt} \sum_{\nu=0}^{2N} e^{u_N(x_\nu,t)} \Delta x = - \sum_{\nu=0}^{2N} \psi_N e^{u_N} \frac{\partial}{\partial x} [\psi_N e^{u_N}]_{|x=x_\nu} \cdot \Delta x.$$

Since the trapezoidal rule agrees with exact integration of the $2N$-trigonometric polynomial on the right, we obtain

$$\frac{d}{dt} \sum_{\nu=0}^{2N} e^{u_N(x_\nu,t)} \Delta x = - \int_0^{2\pi} \frac{1}{2} \frac{2}{\partial x} [(\psi_N e^{u_N})^2] dx = 0.$$

Thus, we find that the total amount of exponential entropy is conserved in time

$$\int_0^{2\pi} \psi_N e^{u_N(x,t)} dx = \int_0^{2\pi} \psi_N e^{u_N(x,0)} dx. \tag{1.8}$$

Now assume that $u_N(x,t)$ converges (possibly weakly) to the discontinuous entropy solution, $\overline{u}(x,t)$, of (1.6). This means that $\psi_N e^{u_N(x,t)}$ tends (weakly, at least) to $e^{\overline{u}(x,t)}$. Consequently, (1.8) would imply the global entropy conservation of $\int_0^{2\pi} e^{\overline{u}(x,t)} dx$ in time, which would be incompatible with the entropy condition (1.1b) if $\overline{u}(x,t)$ should contain shock discontinuities. This contradiction shows that $u_N(x,t)$ *cannot* converge to the (discontinuous) entropy solution of the conservation law (1.6).

The last example demonstrates a consistent (and in fact, spectrally accurate) approximation of the conservation law (1.1a). Nevertheless, convergence fails in this case due to the (exponential) entropy conservation expressed in (1.8). In other words, the essential ingredient behind the failure of convergence demonstrated above is the lack of *entropy dissipation* which is inconsistent with the augmenting entropy condition (1.1b).

One of the main disadvantages of using spectral methods for nonlinear conservation laws lies in the formation of Gibbs phenomena, once spontaneous shock discontinuities appear in the solution. The global nature of spectral methods then pollutes the unstable Gibbs oscillations overall the computational domain and the lack of entropy dissipation prevents the convergence of spectral approximations in these cases. The Spectral Viscosity (SV) method,

proposed in [11], attempts to stabilize the Gibbs oscillations, and consequently to guarantee the convergence of spectral methods, without sacrificing their overall spectral accuracy. This is achieved by appending a given spectral method with a spectrally small amount of vanishing viscosity which is activated only for modes with high wavenumbers. In this paper, we shall focus our attention on semi-discrete approximations to periodic conservation laws. We shall use this periodic framework to demonstrate the stability and convergence behavior of the Spectral Viscosity method.

2. THE SPECTRAL VISCOSITY METHOD

It is well-known, [4], that the entropy solution of (1.1) is the one identified with the small viscosity limit of the regularized problem

$$\frac{\partial}{\partial t}[u_\varepsilon] + \frac{\partial}{\partial x}[f(u_\varepsilon)] = \varepsilon \frac{\partial}{\partial x}[Q_\varepsilon \frac{\partial u_\varepsilon}{\partial x}], \quad \varepsilon \downarrow 0. \tag{2.1}$$

With the vanishing viscosity method, [7], we replace the exact derivatives in (2.1) by their discrete counterpart. The viscous regularization on the right of (2.1) is responsible for the entropy dissipation of the resulting approximation. It depends on the viscosity coefficient, Q_ε, which is yet to be determined, and on the viscosity amplitude which is of the order of the vanishing grid size, $\epsilon \sim \Delta x$. The modern high-resolution finite-difference methods, e.g., [3], employ essentially nonlinear viscosity coefficients, Q_ε, satisfying a.e. $\varepsilon Q_\varepsilon \sim (\Delta x)^s$, which yield s-order accurate approximations of (1.1). Convergence is then established by compactness estimates together with entropy dissipation arguments, e.g., [8].

In order to respect spectral accuracy, however, a more delicate viscous regularization is required. To this end, the Spectral Viscosity (SV) method makes use of viscous regularization of (1.1) which takes the form

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[P_N f(u_N)] = \varepsilon_N \frac{\partial}{\partial x}[Q_N * \frac{\partial u_N}{\partial x}]. \tag{2.2}$$

Here $Q_N(x,t)$ is a (possibly nonlinear) viscosity kernel which is activated only on high modes, say with wavenumbers $|k| \geq m_N$,

$$Q_N(x,t) = \sum_{m_N \leq |k| \leq N} \hat{Q}_k(t) e^{ikx}. \tag{2.3}$$

This kind of spectral viscosity can be efficiently implemented in the Fourier rather than in the physical space, i.e.,

$$\varepsilon_N \frac{\partial}{\partial x}[Q_N * \frac{\partial u_N}{\partial x}] \equiv -\varepsilon_N \sum_{m_N \leq |k| \leq N} k^2 \hat{Q}_k(t) \hat{u}_k(t) e^{ikx}. \tag{2.4}$$

It depends on two free parameters: the viscosity amplitude, $\varepsilon = \varepsilon_N$, and the effective size of the inviscid spectrum, $m = m_N$. The spectral viscosity should be small enough so that it retains the spectral accuracy of the overall approximation as $m_N \uparrow \infty$, and the same time, it should be sufficiently large to enforce the correct amount of entropy dissipation that is missing otherwise (with $\varepsilon_N = 0$).

The SV method was introduced in [11] where compensated compactness arguments were used to show convergence in the special case of the scalar Burgers' equation. The convergence proof of the SV method was then extended in [5], [9], [12], to include general scalar and certain 2×2 systems of conservation laws. In the next sections, we outline the practical considerations involved in the implementation of the SV method, and we give a bird's eye view of the above mentioned convergence results. Finally, we provide numerical evidence which shows how post-processing of the SV solutions enables to recover with spectral accuracy the pointwise values of the exact entropy solution.

3. CONVERGENCE OF THE SV METHOD

The modified equation associated with the SV method (2.2) takes the form

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[f(u_N)] = \frac{\partial}{\partial x}[Eu_N]. \tag{3.1a}$$

The local error in this case, Eu_N, consists of two contributions: the discretization error we had before in (1.3), and the additional viscosity error introduced on the right of (2.2), i.e.,

$$Eu_N = (I - P_N)f(u_N) + \varepsilon_N Q_N * \frac{\partial u_N}{\partial x}. \tag{3.1b}$$

Integrating (3.1) against u_N over the 2π-period we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^{2\pi} u_N^2(x,t)dx = -(\frac{\partial u_N}{\partial x}, Eu_N). \tag{3.2}$$

Thus, the quantity on the right represents the amount of (quadratic) entropy dissipation rate. The counterexample discussed in Section 1 tells us that the control of such quantity is *necessary* for convergence. This brings us to

DEFINITION. [12, Section 3]. The approximation (3.1a) is consistent with the entropy condition, if there exist constants, Const. > 0 and $\varepsilon_N \downarrow 0$ such that

$$-(\frac{\partial u_N}{\partial x}, Eu_N) \le -Const.\varepsilon_N \|\frac{\partial u_N}{\partial x}\|^2 + \|u_N\|^2, \quad \varepsilon_N \ge \frac{1}{N}. \tag{3.3}$$

Using compensated compactness arguments, the main results of [12, Section 6] asserts that this kind of consistency together with L^∞-stability imply convergence. We shall use this framework to prove the convergence of the SV method.

The special form of the local error, Eu_N, in (3.1b) reveals that the entropy consistency requirement (3.3) is fulfilled, if a sufficiently large amount of SV regularization, $\varepsilon_N Q_N$, is present. Indeed, taking into account the a'priori estimate (1.4) and using Parseval's relation, we conclude that entropy consistency is achieved with SV kernels satisfying,

$$\hat{Q}_k(t) \ge Const. - \frac{1}{\varepsilon_N k^2}. \tag{3.4}$$

Can we use such a viscosity kernels without sacrificing spectral accuracy? using a vanishing viscosity amplitude of order, say, $\varepsilon_N \overset{>}{\sim} \frac{1}{N}$, we find by (3.4), that the viscosity coefficients, $\hat{Q}_k(t)$, should be activated only for high modes with wavenumbers $|k| \geq m_N$, where

$$m_N \sim N^\beta, \quad \beta < \frac{1}{2}. \tag{3.5a}$$

The resulting SV method then takes the form

$$\frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[P_N f(u_N)] = -\frac{1}{N} \sum_{m_N \leq |k| \leq N} k^2 \hat{Q}_k(t) \hat{u}_k(t) e^{ikx}. \tag{3.5b}$$

It differs from the standard Fourier method (1.2) by the spectrally small amount of viscosity added on the right of (3.5b). Indeed, the contribution of this kind of viscosity to the local error in (3.1b), does not exceed

$$\|\varepsilon_N Q_N * \frac{\partial u_N}{\partial x}\| \leq Const_s \cdot m_N^{-s} \|u\|_{H^s},$$

which retains the (formal) overall spectral accuracy of the Fourier method

$$\|Eu_N\| \leq Const_s \cdot N^{-\beta s} \|u_N\|_{H^s}. \tag{3.6}$$

Thus, by augmenting the Fourier method with a spectrally small amount of viscosity, we are able to enforce a sufficient amount of entropy dissipation so that the entropy consistency requirement (3.3) is met. Moreover, in the next section we show that this kind of spectral viscosity guarantees that our approximation, $u_N(x,t)$, remains uniformly bounded

$$\|u_N(x,t)\|_{L^\infty} \leq M(t). \tag{3.7}$$

Consequently, the (entropy) consistency and L^∞-stability imply that the SV method converges for a wide class of scalar conservation laws as well as certain 2×2 conservative systems, [9], [12].

4. DECAY AND CONVERGENCE RATES

In order to gain a better insight into the role of spectral viscosity, it is instructive to study the decay rate of the Fourier coefficients, $\hat{u}_k(t)$ and $\hat{f}_k(t) \equiv \hat{f}_k(u_N(t))$. This program was carried out in [5] for the special case of Burgers' equation, and in [9] for the general scalar problem. An iterative argument outlined in [5], [9], shows that the presence of SV separates the computed spectrum into three different regions.

1. Wavenumbers in the "inviscid" region $|k| \leq m_N$. The corresponding amplitudes, $\hat{u}_k(t)$ and $\hat{f}_k(t)$ are then governed by the underlying nonlinear conservation law.

2. Wavenumbers outside the "inviscid" region, $m_N \leq |k| < N$. Here we find that due to nonlinear interaction with the spectral viscosity, the corresponding amplitude dissipate at rate

   $$|\hat{u}_k(t)| + |\hat{f}_k(t)| \leq Const_s \cdot \left(\frac{N}{\epsilon_N k^2}\right)^s + e^{-Nt}, \quad m_N \leq |k| \leq N. \tag{4.1a}$$

   In particular, we have here a transition to a third distinctive region, namely

3. Wavenumbers located at the highest portion of the spectrum, say $\frac{1}{2}N \le |k| \le N$. Then, by (4.1a), the corresponding amplitudes are *negligibly* small, i.e., with vanishing viscosity amplitude of order $\epsilon_N \overset{>}{\sim} N^{-2\beta}$, we have

$$(4.1b) \qquad |\hat{u}_k(t)| + |\hat{f}_k(t)| \le Const_s \cdot N^{-(1-2\beta)s} + e^{-Nt}, \quad \beta < \frac{1}{2}, \quad \frac{1}{2}N \le |k| \le N.$$

We conclude that the presence of spectral viscosity enforces a spectral decay of the discretization error

$$(4.2) \qquad \|(I - P_N)f(u_N)\| \le Const_s \cdot N^{-(1-2\beta)s} + e^{-Nt}, \quad \beta < \frac{1}{2}.$$

We observe that the spectral decay of the discretization error is valid *independently whether the underlying solution is smooth or not*, [5, Section 3], [9, Section 3]. This shows that the SV solution is *essentially* governed by the equation

$$(4.3) \qquad \frac{\partial}{\partial t}[u_N] + \frac{\partial}{\partial x}[f(u_N)] = \varepsilon_N \frac{\partial}{\partial x}[Q_N * \frac{\partial u_N}{\partial x}].$$

This equation is closely related to the standard viscous regularization (2.1). We note, however, that unlike (2.1), the viscosity regularization in (4.3) is nonlocal due to the finite support of the convoluted kernel $Q_N(x,t)$.

One can use now the viscous equation (4.3) in order to obtain a uniform bound on the SV solution. Integrated against $pu_N^{p-1}(x,t)$ over the 2π- period, equation (4.3) yields

$$(4.4) \qquad \frac{d}{dt}\|u_N\|_{L^p} \le \varepsilon_N \|\frac{\partial}{\partial x}[I - Q_N]\frac{\partial u_N}{\partial x}\|_{L^p}.$$

Standard trigonometric estimates can be used to upper bound the RHS of (4.4),

$$(4.5) \qquad \varepsilon_N \|\frac{\partial}{\partial x}[I - Q_N]\frac{\partial u_N}{\partial x}\|_{L^p} \le Const.\|u_N\|_{L^p}.$$

Indeed, the inequality (4.5) is just an L^p-version of the entropy consistency estimate stated in (3.3), consult [12, Section 5]. Hence, by combining (4.4), (4.5) and carefully iterating on the L^p-norms of $u_N(x,t)$, we derive the L^∞-bound of $u_N(x,t)$ promised earlier.

Furthermore, Schochet [9] used the viscous equation (4.3) in order to conclude an almost optimal L_1-convergence rate of the SV method, namely

$$(4.6) \qquad \|u_N - u\|_{L^1} \le Const.N^{-\beta}, \quad \beta < \frac{1}{2}.$$

Figures 4.1 and 4.2 compare the behavior of the spectral-Fourier method with and without spectral viscosity in the case of Burgers' equation, which is subject to initial condition, $u_0(x) = \sin x$. The resulting ODE system for the Fourier coefficients

$$(4.7) \qquad \frac{d}{dt}\hat{u}_k(t) + \frac{1}{2}ik \sum_{p+q=k} \hat{u}_p(t)\hat{u}_q(t) = -\varepsilon k^2 \hat{Q}_k \hat{u}_k(t), \quad |k| \le N,$$

was integrated up to time $t = 1.5$, using the fourth-order Runge-Kutta method. The method was complemented with viscosity coefficients $\hat{Q}_k \equiv 1$ only for wavenumbers $|k| \geq m_N$ where $m_N \sim 2N^{\frac{1}{2}}$. Figure 4.1 shows that the SV solution convergence strongly (but not uniformly) to the exact entropy solution, in sharp contrast to the oscillatory behavior of the viscosity-free Fourier method in Figure 4.2. Improved results were obtained in Figure 4.3, by using C^∞ viscosity coefficients, $\hat{Q}_k$, connecting wavenumbers in the inviscid region, $|k| < m_N$, and the highest wavenumbers, $|k| \sim N$. This kind of *smoothly* varying SV prevents the propagation of the Gibbs phenomenon into the whole computational domain that can be noticed in Figure 4.2, consult [11]. Moreover, Figure 4.4 shows this SV solution after it was post-processed by the spectrally accurate smoothing procedure discussed in [2]. Finally, in Table 4.5, we quote from [6], the pointwise errors of the post-processed SV solution for the Burgers' equation. The results indicate the spectral convergence rate obtained by the post-processed SV solution in the shock-free zones of the entropy solution.

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Figure 4.1. The SV method for Burgers' equation.

Figure 4.2. The spectral Fourier method for Burgers' equation.

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Figure 4.3 Figure 4.4

The pseudo-SV method with smooth viscosity kernel before and after post-processing.

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	$\lvert u_N(x,t) - u(x,t)\rvert$ at $t = 1.5$	
$x =$	$N = 16$	$N = 32$
.314	$5.7 \cdot 10^{-3}$	$2.3 \cdot 10^{-4}$
.628	$3.9 \cdot 10^{-3}$	$2.8 \cdot 10^{-4}$
.942	$7.2 \cdot 10^{-3}$	$2.7 \cdot 10^{-4}$
1.25	$1.2 \cdot 10^{-2}$	$2.8 \cdot 10^{-4}$
1.57	$1.4 \cdot 10^{-2}$	$2.8 \cdot 10^{-4}$
1.88	$1.7 \cdot 10^{-2}$	$3.0 \cdot 10^{-4}$
2.19	$2.0 \cdot 10^{-2}$	$3.6 \cdot 10^{-4}$
2.51	$2.1 \cdot 10^{-2}$	$4.3 \cdot 10^{-4}$

Table 4.5

Pointwise errors of the post-processed pseudo-SV approximation compared with the exact entropy solution of Burgers' equation.

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