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Abstract Evolution Equations

In this chapter we reconsider partial differential equations in which there is a distinguished variable t, usually time in physical problems. We might think of such equations as ordinary differential equations in Banach spaces. Consider, for example, the heat equation in a bounded domain Ω. If we let $A = \Delta$, then the equation can be written

$$\frac{du}{dt} = Au, \tag{E}$$

where $u(t)$ is a vector-valued function (Chapter 8). As we will see, if the temperature u vanishes on the boundary of Ω, one reasonable choice of the "state space" is $H_0^1(\Omega)$.

There are several ways of giving a mathematical theory based on this idea. One, which might be described as an "energy-based theory," is given in Section 2. Here we consider a class of equations that are *parabolic*. We will not make use of a general definition of parabolicity in this book. The equations $u_t = Au$ where A is an elliptic operator are the basic examples for us, and we may simply think of these examples when using the term. (In a similar way, the equations $u_{tt} = Au$ where A is an elliptic operator are examples of a class of equations called *hyperbolic*; see also Section 4 of Chapter 1. We consider a large class of hyperbolic equations and systems that are of great practical significance in Chapter 7.)

It turns out that a natural class of initial data for (E) is $L^2(\Omega)$. As the heat equation is instantaneously smoothing, we can expect $u(t)$ for $t > 0$ to be in a more regular class of functions. On the other hand, if we want to think of solutions of (E) as evolving in some fixed Sobolev space, there are difficulties as $A(H^s(\Omega))$ is not contained in $H^s(\Omega)$ for any $s > 0$. The theory in Section 2 deals with this problem in a way closely related to the theory given for elliptic equations in Chapter 5. In special cases the time variable can be "separated" and eigenfunctions of A can be used. This will be illustrated in the special case of the heat equation in Section 1.

P. Bassanini et al., *Theory and Applications of Partial Differential Equations*

1. Solution of the Heat Equation by Eigenfunction Expansions

We will show in this section how the formal separation of variables method that was introduced in Section 5 of Chapter 3 can be set in the framework of Sobolev spaces. Essential use will be made of the variational theory of eigenvalues and eigenfunctions for $-\Delta$ that was developed in Section 5 of Chapter 4.

The problem considered will be

$$\frac{du}{dt} = Au + f, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where $A = \Delta$. We will consider only the boundary condition $u = 0$ on $\partial\Omega$, and this will be imposed weakly by assuming that $u \in H_0^1(\Omega)$. As $\Delta u \in H^{-1}(\Omega)$ for $u \in H_0^1(\Omega)$, we will assume $u'(t) = du/dt$ and $f(t)$ are elements of $H^{-1}(\Omega)$ for each t. This leaves the question of how to measure the time dependence of solutions. Many choices are possible, but here we will assume $u \in L^2(0, T, H_0^1(\Omega))$, $u' \in L^2(0, T, H^{-1}(\Omega))$ and suppose $f \in L^2(0, T, H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$ for the data. This yields a Hilbert space structure that will be useful in Section 2.

Suppose that $\{z_k\}$ are the eigenfunctions corresponding to the eigenvalues $\{\lambda_k\}$ of $-\Delta$. We need the following result conserning expansion of functions in Sobolev spaces.

Theorem 1.1 A series $\sum_{n=1}^{\infty} b_n z_n$ represents an element v of $H_0^1(\Omega)$, $L^2(\Omega)$, and $H^{-1}(\Omega)$ if and only if the series

$$\sum_{n=1}^{\infty} \lambda_n |b_n|^2, \qquad \sum_{n=1}^{\infty} |b_n|^2, \qquad \sum_{n=1}^{\infty} |b_n|^2/\lambda_n$$

converge, respectively.

Proof. As $\|z_n\|_{L^2} = 1$, $\|z_n/\sqrt{\lambda_n}\| = 1$, the result for the first two series follows from Parseval's equality (Chapter 8, Exercise 1.13) in L^2 and $H_0^1(\Omega)$ and from the relations

$$\left\| \sum_{n=N}^{N+p} b_n z_n \right\|_{L^2}^2 = \sum_{n=N}^{N+p} |b_n|^2, \qquad \left\| \sum_{n=N}^{N+p} b_n z_n \right\|^2 = \sum_{n=N}^{N+p} \lambda_n |b_n|^2$$

with $b_n = (v, z_n)$. [We recall that $\|\cdot\|$ is the norm in $H_0^1(\Omega)$ defined by the Dirichlet integral.] If $\varphi \in H_0^1(\Omega)$ and $\varphi^{(N)} := \sum_{n=1}^{N} \varphi_n z_n$, where $\varphi_n = (z_n, \varphi)$, and

$F \in H^{-1}(\Omega)$, set $F(z_n) := b_n$. Then, letting $v^{(N)} = \sum_{n=1}^{N} b_n z_n$,

$$F(\varphi) = \sum_{n=1}^{N} \varphi_n b_n = (v^{(N)}, \varphi).$$

As $z_n \in L^2(\Omega)$, $v^{(N)} \in L^2(\Omega)$. The Schwarz inequality yields

$$|F(\varphi)|^2 \leq \sum_{n=1}^{N} \frac{|b_n|^2}{\lambda_n} \sum_{n=1}^{N} \lambda_n \varphi_n^2 \leq \|\varphi\|^2 \sum_{n=1}^{N} \frac{|b_n|^2}{\lambda_n}.$$

Letting $N \to \infty$ we see that

$$v = \sum_{n=1}^{\infty} b_n z_n \in H^{-1}(\Omega) \quad \text{and} \quad \|v\|_{H^{-1}} \leq \left(\sum_{n=1}^{\infty} |b_n|^2/\lambda_n \right)^{1/2}$$

if the third series converges. On the other hand, the function

$$\phi := \sum_{n=1}^{N} b_n z_n/\lambda_n$$

is an element of $H_0^1(\Omega)$ and has norm $\|\phi\|^2 = \sum_{n=1}^{N} |b_n|^2/\lambda_n = |F(\phi)|$. Hence, $\|v\|_{H^{-1}} \geq |F(\phi)|/\|\phi\| = \|\phi\|$. It follows that if $v \in H^{-1}(\Omega)$, the series $\sum_{n=1}^{\infty} |b_n|^2/\lambda_n$ converges, and

$$\|v\|_{H^{-1}}^2 = \sum_{n=1}^{\infty} |b_n|^2/\lambda_n.$$

This completes the proof.

For $g \in L^2(0, T, \mathscr{H})$ where $\mathscr{H}$ is a Hilbert space with orthonormal basis $\{z_k\}$, an application of the dominated convergence theorem implies that

$$g = \sum_{k=1}^{\infty} g_k(t) z_k$$

in $L^2(0, T, \mathscr{H})$ where $g_k(t) = (g(t), z_k)_{\mathscr{H}}$, $0 \leq t \leq T$. Suppose that we write

$$f = \sum_{k=1}^{\infty} f_k(t) z_k, \qquad u_0 = \sum_{k=1}^{\infty} a_k z_k$$

in our problem [the series will then converge in $H^{-1}(\Omega)$ for a.e. t and the second in $L^2(\Omega)$]. We seek a solution in the form

$$u = \sum_{k=1}^{\infty} u_k(t) z_k \tag{3}$$

with the series convergent in $L_2(0, T, H_0^1(\Omega))$. Assuming (3) we obtain, formally,

$$u_k'(t) + \lambda_k u_k(t) = f_k(t), \qquad 0 < t < T,$$
$$u_k(0) = a_k,$$

which are solved by setting

$$u_k(t) = e^{-\lambda_k t} a_k + \int_0^t e^{-\lambda_k (t-s)} f_k(s) ds := v_k(t) + w_k(t). \tag{4}$$

We need to investigate the convergence of the series (3) with $u_k(t)$ given by (4). The series $v := \sum_{k=1}^{\infty} v_k(t) z_k = \sum_{k=1}^{\infty} e^{-\lambda_k t} a_k z_k$ is easily seen to be convergent in $L^2(0, T, H_0^1(\Omega))$ as

$$\int_0^T \|v(t)\|^2 dt = \sum_{k=1}^{\infty} \lambda_k |a_k|^2 \int_0^T e^{-2\lambda_k t} dt \leq \sum_{k=1}^{\infty} \frac{|a_k|^2}{2} (1 - e^{-2\lambda_k T}) \leq C \|u_0\|^2_{L^2(\Omega)}, \tag{5}$$

where $C = C(T)$. Moreover, $v'(t) \in C((0, T], H_0^1(\Omega))$ because of the exponential factors.

Let $w_N(t) = \sum_{k=1}^{N} w_k(t) z_k$. Then

$$\int_0^T \|w_N(t)\|^2 dt = \sum_{k=1}^{N} \lambda_k \int_0^T \left[\int_0^t e^{-\lambda_k (t-s)} f_k(s) ds \right]^2 dt.$$

We recall the following proposition from real analysis.

Proposition 1.1. Suppose $g \in L^1(0, T)$, $h \in L^2(0, T)$, and $k(t) = \int_0^t g(t-s) h(s) ds$. Then

$$\|k\|_{L^2} \leq \|g\|_{L^1} \|h\|_{L^2},$$

where $L^p = L^p(0, T) (p = 1, 2)$.

The proof is a direct application of Hölder's inequality. (A general theorem implying this is given in Chapter 9 of Ref. 1.) Letting $g(s) = e^{-\lambda_k s}$, $h(s) = f_k(s)$, we have

$$\int_0^T \|w_N(t)\|^2 dt \le \sum_{k=1}^N \lambda_k \left[\int_0^t e^{-\lambda_k t} dt\right]^2 \int_0^T |f_k(s)|^2 ds$$
$$= \sum_{k=1}^N (1 - e^{1-e^{-\lambda_k T}})^2 \lambda_k^{-1} \int_0^T |f_k(s)|^2 ds.$$

If we define $f_N(t) = \sum_{k=1}^N f_k(t) z_k$, this inequality and the previous theorem imply

$$\int_0^t \|w_N(t)\|^2 dt \le C \int_0^T \|f_N(t)\|^2_{H^{-1}(\Omega)} dt, \tag{6}$$

where, again, $C = C(T)$. We can deduce from (6) that $w = \sum_{k=1}^\infty w_k(t) z_k$ is convergent in $L^2(0, T, H_0^1(\Omega))$. As $f_k(t) \in L^2(0, T)$, we can differentiate the integral in the definition of $w_k(t)$. An argument analogous to the above then shows that the series obtained by letting $N \to \infty$ in $w'_N(t)$ converges in $L^2(0, T, H^{-1}(\Omega))$ and the limit is $w'(t)$. Hence, $u'(t) \in L^2(0, T, H^{-1}(\Omega))$. Finally, we can replace w_N and f_N in (6) by w and f, respectively, and (5) and (6) imply that

$$\int_0^T \|u(t)\|^2 dt \le C\left[\|u_0\|^2_{L^2(\Omega)} + \int_0^T \|f(t)\|^2_{H^{-1}(\Omega)} dt\right],$$

where C depends only on T. This inequality shows uniqueness and continuous dependence in the appropriate norms. We see then that the formal eigenfunction expansion method provides a unique solution in the class of functions considered. In this special case this may be considered an alternative to the theory of Section 2, and motivates the choice of function spaces used there.

We remark that this method can be used for any equation $u_t = Au$ for which an appropriate set of eigenfunctions of A are available.

2. Parabolic Evolution Equations

Here consider a class of parabolic problems of the form $u' = A(t)u$ where the operators $A(t)$ are elliptic for each t. In order to introduce the basic ideas with the minimum technical complications, we consider first the initial value problem for the heat equation,

$$\begin{aligned} u_t - u_{xx} &= f, \qquad (x,t) \in \mathbb{R} \times (0,T), \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}, \end{aligned} \tag{IVP}$$

in this setting. The operator $-\partial^2/\partial x^2$ will be thought of as a bounded linear operator mapping $H^1(\mathbb{R}) = H^1$ into $H^{-1}(\mathbb{R}) = H^{-1}$ and this extends in an obvious way to an operator mapping $L^2(0,T,H^1)$ into $L^2(0,T,H^{-1})$. From the notation $W(0,T)$ in Chapter 8,

$$W(0,T) = \{u \in L^2(0,T,H^1) : u_t \in L^2(0,T,H^{-1})\},$$

and we recall the fact that this is a Hilbert space, with the norm

$$\|u\| = \left(\int_0^T (\|u(t)\|_1^2 + \|u_t(t)\|_{-1}^2)dt\right)^{1/2}$$

in which $C^\infty([0,T],H^1)$ is dense. Further, as stated in Proposition 4.2 of Chapter 8, if $C([0,T],L^2)$ is given the norm $\sup\|u(t)\|_0$, then the natural injection of $C^\infty([0,T],H^1)$ into $C([0,T],L^2)$ can be extended to a continuous injection of $W(0,T)$ into $C([0,T],L^2)$. In particular, functions in $W(0,T)$ may be thought of as being continuous functions on $[0,T]$ taking values in L^2, and functions in $W(0,T)$ have initial values that are taken on continuously *in the L^2 sense*, i.e.,

$$\lim_{t\to 0} \|u(t) - u(0)\|_0 = 0,$$

We can now state the basic existence theorem for (IVP) in this context.

Theorem 2.1. For every $f \in L^2(0,T,H^{-1})$ and $u_0 \in L^2 = L^2(\mathbb{R})$, there is exactly one solution $u \in W(0,T)$ of (IVP), with $u(0) = u_0$ in the L^2 sense. ■

The proof is accomplished using the following proposition.

Proposition 2.1. Let $\mathscr{E}$ be a Hilbert space and $\mathscr{H}$ a subspace of $\mathscr{E}$. Let $a(u,v)$ be a continuous bilinear form defined on $\mathscr{E} \times \mathscr{H}$ such that:

i. For fixed $h, w \to a(w,h)$ is a continuous linear functional on $\mathscr{E}$.
ii. $a(h,h) \geq c\|h\|_{\mathscr{E}}^2$ for $h \in \mathscr{H}$, where $\|\cdot\|_{\mathscr{E}}$ denotes the norm in $\mathscr{E}$.

Then there is a bounded linear operator G from the dual $\mathscr{E}'$ of $\mathscr{E}$ into $\mathscr{E}$ with $\|G\| \leq c^{-1}$ such that $\lambda \in \mathscr{E}'$ implies

$$a(G\lambda, h) = \lambda(h) \qquad \text{for all } h \in \mathscr{H}.$$

Proof. For each $h \in \mathscr{H}$, (i) implies that there is an $Rh \in \mathscr{E}$ such that $a(w, h = (w, Rh)_{\mathscr{E}}$ for all $w \in \mathscr{E}$, where $(,)_{\mathscr{E}}$ denotes the inner product on $\mathscr{E}$. Further, (ii) implies that $\|h\|_{\mathscr{E}} \leq c^{-1}\|Rh\|_{\mathscr{E}}$, hence $R : \mathscr{H} \to \mathscr{E}$ is to a one-to-one linear transformation, and the inverse transformation $Rh \to h$ defined on $\underline{R(\mathscr{H})}$ (given the norm of $\mathscr{E}$) is continuous. If we extend this transformation to $\overline{R(\mathscr{H})}$ by continuity, and set it to zero on the orthogonal complement of $R(\mathscr{H})$, we obtain an operator $G^* \in B(\mathscr{E})$ with $\|G^*\| \leq c^{-1}$. Let G_1 be the adjoint of G^*. Then

$$a(G_1 w, h) = (G_1 w, Rh)_{\mathscr{E}} = (w, G^* Rh)_{\mathscr{E}} = (w, h)_{\mathscr{E}}.$$

If J is the usual isomorphism of $\mathscr{E}'$ onto $\mathscr{E}$, we can set $G = G_1 J$. □

Proof of Theorem 2.1. We let $\mathscr{E}$ be $L^2(0, T, H^1) \times L^2$, and

$$\mathscr{H} = \{(u, v) \in \mathscr{E} : u \in W(0, T), v = u(0), u(T) = 0\}.$$

The norm in $\mathscr{E}$ is given by

$$\|(u, v)\|^2 = \int_0^T \|u(t)\|_1^2 \, dt + \|v\|_0^2.$$

We define

$$a((w, w_0), (h, h_0)) = \int_0^T \left(-\langle w, h_t \rangle + \int_{\mathbb{R}} w_x h_x \, dx \right) dt,$$

where $\langle \ , \ \rangle$ denotes the natural duality between H^1 and H^{-1}. We observe that, for $u \in W(0, T)$,

$$2 \int_0^T \langle u, u_t \rangle \, dt = \int_{\mathbb{R}} (|u(x, T)|^2 - |u(x, 0)|^2) dx.$$

For $u \in C^\infty([0, T], H^1)$ this is immediate as the integrand on the left-hand side is just $d\|u\|_0^2/dt$. The density of $C^\infty([0, T], H^1)$ in $W(0, T)$ and Proposition 4.2 of Chapter 8 imply this identity in general. Now

$$\begin{aligned} a((h, h_0), (h, h_0)) &= \int_0^T (-\langle h, h_t\rangle + \|h(t)\|_1^2)dt \\ &= \frac{1}{2}\int_{\mathbb{R}} |h(x, 0)|^2\, dx - \frac{1}{2}\int_{\mathbb{R}} |h(x, T)|^2\, dx + \int_0^T \|h(t)\|_1^2\, dt \\ &\geq \tfrac{1}{2}\|(h, h_0)\|^2, \end{aligned}$$

as $h(x, T) = 0$ and $c = 1/2$. We see that the hypotheses of Proposition 2.1 are satisfied. We let $\lambda \in \mathscr{E}'$ be given by

$$\lambda((h, h_0)) = \int_0^T \langle f, h\rangle dt + \int_{\mathbb{R}} u_0 h_0\, dx.$$

There is, then, a unique $(v, v_0) \in \mathscr{E}$ such that

$$\int_0^T \left(\langle -v, h_t\rangle + \int_{\mathbb{R}} v_x h_x\, dx\right) dt = \int_0^T \langle f, h\rangle dt + \int_{\mathbb{R}} u_0 h(x, 0) dx \tag{7}$$

for all $h \in \mathscr{H}$. We choose first $h \in C_0^\infty([0, T], H^1)$. As $\int_0^T (\langle v, h_t\rangle + \langle v_t, h\rangle)\, dt = 0$ then, and as $\int_{\mathbb{R}} v_x h_x\, dx = -\int_{\mathbb{R}} v_{xx} h\, dx$ in the distributional sense for each t, we have

$$\int_0^T \langle v_t - v_{xx} - f, h\rangle dt = 0.$$

It follows that $v_t - v_{xx} - f = 0$ is an element of H^{-1}. If we now choose a general $h \in \mathscr{H}$ and use

$$\int_0^T \langle -v, h_t\rangle dt = \int_0^T \langle v_t, h\rangle dt + \int_{\mathbb{R}} v(x, 0) h(x, 0) dx,$$

we have

$$\int_{\mathbb{R}} (v(x, 0) - u_0(x)) h(x, 0) dx = 0.$$

As $h(x, 0)$ can be an arbitrary element of L_2, the initial condition is satisfied. We have shown the existence of a solution in $L^2(0, T, H^1)$. The conclusion that

$u \in W(0, T)$ follows from the differential equation. The uniqueness question is dealt with by considering

$$b(v, w) := \int_0^T \left(\langle v_t, w \rangle + \int_{\mathbb{R}} v_x w_x \, dx \right) dt$$

on $W(0, T) \times W(0, T)$. In general, we have the identity

$$b(u, u) + \frac{1}{2} \int_{\mathbb{R}} |u(x, 0)|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}} |u(x, T)|^2 \, dx + \int_0^T \|u(t)\|_1^2 \, dt. \tag{8}$$

Suppose that u_1, u_2 are two solutions and $u = u_1 - u_2$. Then

$$b(u, u) = \int_0^T \langle u_t - u_{xx}, u \rangle dt = 0.$$

As $u(x, 0) = 0$, the above identity (8) now implies that

$$\int_0^T \|u(t)\|_1 \, dt = 0.$$

The inequality $\|u_{xx}\|_{-1} \leq \|u\|_1$ and the differential equation then yield $\|u\| = 0$ in $W(0, T)$. □

We remark that the norm of the functional λ defined in the proof is given by

$$\|\lambda\|^2 = \int_0^T \|f(t)\|_{-1}^2 \, dt + \|u_0\|_0^2.$$

It follows from the estimate $\|G\| \leq c^{-1} = 2$ in Proposition 2.1 that

$$\int_0^T \|u(t)\|_1^2 \, dt + \|u_0\|_0^2 \leq 2 \left(\int_0^T \|f(t)\|_{-1}^2 \, dt + \|u_0\|_0^2 \right).$$

This shows that the solution depends continuously on the data in the sense that the mapping $(u_0, f) \to u$ from $L^2 \times L^2(0, T, H^{-1})$ to $L^2(0, T, H^1)$ is continuous.

For later work we need to establish an additional identity for the solutions guaranteed by Theorem 2.1.

Theorem 2.2. If $f \in L^2(0, T, H^{-1})$ is of the form $f = f_0 + (f_1)_x$ with $f_0, f_1 \in L^2(0, T, L^2)$, then, for a.e. t,

$$\langle u_t, z\rangle + \int_{\mathbb{R}} u_x z_x \, dx = \int_{\mathbb{R}} (f_0 z - f_1 z_x) dx$$

for all $z \in H^1$.

Proof. Choosing $v = u$ and $h \in C_0^\infty([0, T], H^1)$ as $\varphi(t)z$, where $\varphi \in C_0^\infty([0, T])$, in the identity (7) of the proof of Theorem 2.1, we find

$$\int_0^T \varphi(t)\left(\langle u_t, z\rangle + \int_{\mathbb{R}} u_x z_x \, dx - \langle f, z\rangle\right) dt = 0,$$

and, as φ is arbitrary,

$$\langle u_t, z\rangle + \int_{\mathbb{R}} u_x z_x \, dx - \langle f, z\rangle = 0$$

for a.e. t. Observing that

$$\langle f, z\rangle = \int_{\mathbb{R}} (f_0 z - f_1 z_x) dx,$$

the result follows. □

We also need the following result, which improves the properties of the solution if the data are improved.

Theorem 2.3. If $f \in L^2(0, T, L^2)$ and $u_0 \in H^1$, then $u \in L^2(0, T, H^2)$ and $u_t \in L^2(0, T, L^2)$.

Proof. Formally, $v = u_x$ is a solution of $v_t - v_{xx} = f_x$ with $v(0) = u_{0x}$ and $u = \int_{-\infty}^x v \, dx$. Applying Theorem 2.1 to v implies the result. □

We can also apply this method to the initial–boundary value problem on a finite interval. Suppose for simplicity that we are considering the problem

$$\begin{aligned} u_t - u_{xx} &= f, \qquad (x, t) \in (0, 1) \times (0, T), \\ u(x, 0) &= u_0(x), \qquad x \in (0, 1), \end{aligned} \tag{IBVP}$$

with the boundary conditions $u(0, t) = u(1, t) = 0, t > 0$. We will impose the boundary conditions in a weak sense by simply requiring that $u \in H_0^1(0, 1)$ for fixed t. More precisely, let

$$W_0(0, T) = \{u \in L^2(0, T, H_0^1(I)) : u_t \in L^2(0, T, H^{-1}(I))\}, \qquad I = (0, 1).$$

Then we have

Theorem 2.4. For every $f \in L^2(0, T, H^{-1}(I))$ and $u_0 \in L^2(I)$, there is exactly one solution $u \in W_0(0, T)$ of (IBVP).

In the proof we apply Proposition 2.1 with $\mathscr{E} = L^2(0, T, H_0^1(I)) \times L^2(I)$ and $\mathscr{H}$ the subspace of pairs (v, v_0) with $v(0) = v_0$ and $v(T) = 0$. The rest is almost identical to the proof of Theorem 2.1. The details are left as an exercise.

Suppose now that Ω is a bounded domain in $\mathbb{R}^n$. We will show how Theorem 2.1 extends to a general second-order parabolic initial boundary value problem:

a. $u_t = A(t)u + f, (\mathbf{x}, t) \in \Omega \times (0, T)$,
b. $u(\mathbf{x}, 0) = u_0(\mathbf{x}), \mathbf{x} \in \Omega$,
c. $u(\mathbf{x}, t) = g(\mathbf{x}, t), (\mathbf{x}, t) \in \partial\Omega \times [0, T]$,

where

$$A(t)u = \sum_{i,j=1}^{n} (a_{ij}(\mathbf{x}, t)u_{x_i})x_j$$

and the functions $a_{ij} \in L^\infty(\Omega \times [0, T]), a_{ij} = a_{ji}$, satisfy

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq c_0|\xi|^2$$

for all $\xi \in \mathbb{R}^n$. It is straightforward to show that $u \to A(t)u$ defines a bounded linear operator $L^2(0, T, H^1(\Omega)) \to L^2(0, T, H^{-1}(\Omega))$. We will assume that g is the trace of a function, again denoted by g, in $H^1(\Omega)$ for almost every t, and the boundary condition (c) will be imposed in the form $u - g \in H_0^1(\Omega)$ for almost every t.

In the definition of $W(0, T)$ we replace H^1 and H^{-1} by $H^{-1}(\Omega)$ and $H^{-1}(\Omega)$, and we define

$$W_0(0, T) = \{u \in W(0, T) : w \in H_0^1(\Omega), t - \text{a.e.}\}.$$

The boundary condition can then be stated: $u - g \in W_0(0, T)$. The proofs of Propositions 4.1 and 4.2 of Chapter 8 go over to this situation unchanged.

We will use a technical assumption about the (lifting of the) boundary values g:

$$\|g(\cdot, t) - g(\cdot, 0)\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to 0. \tag{g}$$

We can state the following theorem.

Theorem 2.5. Under the stated hypotheses on $A(t)$, for every $f \in L^2(0, T, H^{-1}(\Omega))$, $u_0 \in L^2(\Omega)$ and $g \in L^2(0, T, H^1(\Omega))$ satisfying (g) and $g_t \in L^2(0, T, H^{-1}(\Omega))$, there is a unique solution $u \in L^2(0, T, H^1(\Omega))$ of (a), (b), (c). The initial and boundary conditions are interpreted in the form

b′. $\|u(\cdot, t) - u_0\|_{L^2(\Omega)} \to 0$ as $t \to 0$,
c′. $u - g \in W_0(0, T)$.

Proof. The proof begins by transforming to homogeneous boundary values using the substitution $U = u - g$. The hypotheses made on g are crucial here. We now define

$$a((w, w_0), (h, h_0)) = \int_0^T \left(-\langle w, h_t \rangle + \int_\Omega \sum_{i,j=1}^n a_{ij} w_{x_i} h_{x_j} \, d\mathbf{x} \right) dt$$

and

$$b(v, w) = \int_0^T \left(\langle v_t, w \rangle + \int_\Omega \sum_{i,j=1}^n a_{ij} v_{x_i} w_{x_j} \, d\mathbf{x} \right) dt.$$

The proof is carried out now exactly as for Theorem 2.1. □

We remark that these ideas can be generalized to an abstract situation where H^1, L^2, and H^{-1} are replaced by Hilbert spaces $V \subset H \subset V'$, V dense in H, and a quadratic form $q(t, u, v) = -(A(t)u, v)$ defined on $\mathbb{R} \times V \times V$. If

$$q(t, u, u) \geq c_1 \|u\|_V^2 - c_2 \|u\|_H^2, \qquad c_1, c_2 > 0, \tag{G}$$

then an existence theorem can be given for

$$\frac{du}{dt} = A(t)u + f(t), \qquad u(0) = u_0.$$

This allows for the possibility of considering higher-order operators $A(t)$ and systems of equations.

3. Nonlinear Initial Value Problem

We now consider an initial value problem for a nonlinear equation that will have an important application in Chapter 7. The equation is

$$u_t + f(u)_x = \mu u_{xx} \tag{9}$$

($u, x \in \mathbb{R}, t > 0$) with initial condition

$$u(x, 0) = u_0(x) \tag{10}$$

($x \in \mathbb{R}$). The parameter μ is positive and a crucial aspect of the results obtained is control of estimates on u with respect to μ.

Theorem 3.1. Assume that $f \in C^1$ satisfies the uniform Lipschitz condition

$$|f(u) - f(v)| \leq M|u - v| \tag{UL}$$

and $u_0 \in L^2 \equiv L^2(\mathbb{R})$. Then there is a unique solution $u \in W(0, T)$ of (9), (10).

Proof. (See Ref. 2). The substitution $v = e^{-\lambda t}u, \lambda > 0$, transforms (9) into

$$v_t - \mu v_{xx} + \lambda v = -(e^{-\lambda t} f(e^{\lambda t} v))_x \equiv g_{1x}.$$

For $v \in L^2(0, T, L^2)$, let $g = g_{1x}$ be the right-hand side of this equation. Without loss of generality $f(0) = 0$, and $|g_1| \leq M|v|$ implies that $g_1 \in L^2(0, T, L^2)$, $g \in L^2(0, T, H^{-1})$:

$$\int_0^T \|g\|_{-1}^2 \, dt \equiv \int_0^T \|g_{1x}\|_{-1}^2 \, dt \leq \int_0^T \|g_1\|_0^2 \, dt \leq M^2 \int_0^T \|v\|_0^2 \, dt < \infty.$$

Theorem 2.1 implies that there is a unique solution $w \in W(0, T)$ of

$$w_t - \mu w_{xx} + \lambda w = g, \qquad w(x, 0) = u_0(x),$$

and by Theorem 2.2,

$$\langle w_t, z\rangle + \int_{\mathbb{R}} (\mu w_x z_x + \lambda wx) = -\int_{\mathbb{R}} g_1 z_x \, dx \tag{11}$$

for all $z \in H^1$ and for a.e. t. We claim that the mapping F_λ that takes v to w is a contraction on $L^2(0, T, L^2)$ for λ sufficiently large. For $w_i = F_\lambda(v_i)$, $i = 1, 2$, and $w = w_1 - w_2$, (11) implies

$$\langle w_t, z\rangle + \int_{\mathbb{R}} (\mu w_x z_x + \lambda wz)dx = e^{-\lambda t} \int_{\mathbb{R}} (f(e^{\lambda t} v_1) - f(e^{\lambda t} v_2))z_x \, dx$$

for $z \in H^1$, for a.e. t. Setting $z = w(\cdot, t)$ and integrating from 0 to t, we obtain

$$\frac{1}{2}\|w(t)\|_0^2 + \int_0^t (\mu\|w\|_1^2 + \lambda\|w\|_0^2 \, d\tau \leq M \int_0^t \int_{\mathbb{R}} |v_1 - v_2\|w_x| dx d\tau \tag{12}$$

The Cauchy–Schwarz inequality and $ab \leq (\varepsilon/2)a^2 + (1/2\varepsilon)b^2$ imply

$$\int_{\mathbb{R}} |v_1 - v_2\|w_x| dx \leq \frac{1}{2\varepsilon}\|v_1 - v_2\|^2 + \frac{\varepsilon}{2}\|w\|_1^2.$$

Using this on the right-hand side of (12) with $\varepsilon = 2\mu/M$ in conjunction with Gronwall's lemma yields

$$\lambda \int_0^t \|w\|_0^2 \, d\tau \leq \frac{M^2}{4\mu} \int_0^t \|v_1 - v_2\|_0^2 \, d\tau.$$

Letting t approach T,

$$\|w_1 - w_2\| \leq \frac{M^2}{4\mu\lambda}\|_1 - v_2\|,$$

where $\|\cdot\|$ is the norm in $L^2(0, T, L^2)$. □

We need the following proposition, which contains results from Propositions 3.10 and 4.3 and Exercise 3.16 of Chapter 8.

Proposition 3.1.

a. If G satisfies a Lipschitz condition and G' is continuous except at a finite number of points, then $G \circ v \in H^1$ for $v \in H^1$.

b. $|v|_x = \text{sgn}(v)v_x$ and $v \to |v|$ is continuous on H^1.
c. If $v \in W(0, T)$, $v_+ = \max\{v, 0\} \in L^2(0, T, H^1) \cap C([0, T], L^2)$ and

$$2\int_{t_1}^{t_2} \langle v_t, v_+\rangle dt = \|v_+(t_2)\|_0^2 - \|v_+(t_1)\|_0^2.$$

Theorem 3.2. If $f \in C^1$ and $u_0 \in L^2 \cap L^\infty$, there is a unique solution of (9), (10) in $W(0, T) \cap L^\infty(\mathbb{R} \times [0, T])$ and $\|u(t)\|_\infty \leq \|u_0\|_\infty$ for a.e. t.

Proof. We need to show that the assumption (UL) can be dispensed with. Let $\Psi(u) = \Psi(|u|) \in C^\infty(\mathbb{R})$ with

$$\Psi = \begin{cases} 1, & |u| \leq \|u_0\|_\infty, \\ 0, & |u| \geq \|u_0\|_\infty + 1, \end{cases}$$

and $\tilde{f}(u) = \Psi(u)f(u)$. Then $\tilde{f}$ satisfies (UL) and Theorem 3.1 implies the existence of a solution of (9), (10) with f replaced by $\tilde{f}$, which we will denote by u. Let $v = u - \|u_0\|_\infty$. We will show $v_+ = 0$. First, observe that $v_+ \in H^1$ for a.e. t. As $v_t - \mu v_{xx} = -\tilde{f}'(u)v_x$ and

$$\langle v_t, z\rangle + \mu \int_{\mathbb{R}} v_x z_x \, dx = -\int_{\mathbb{R}} \tilde{f}'(u)v_x z \, dx$$

for all $z \in H^1$, if we take $z = v_+$, and integrate from 0 to t [noting Proposition 3.1, (c) and $v_+(0) = 0$],

$$\frac{1}{2}\|v_+(t)\|_0^2 + \mu \int_0^t \int_{\mathbb{R}} v_x v_{+x} \, dx d\tau = -\int_0^t \int_{\mathbb{R}} \tilde{f}'(u)v_x v_+ \, dx d\tau.$$

Proposition 3.1, (b) implies $\int_{\mathbb{R}} v_x v_{+x} \, dx = \int_{\mathbb{R}} (v_{+x})^2 \, dx$. Applying the Cauchy–Schwarz inequality and $ab \leq \varepsilon a^2 + (1/4\varepsilon)b^2$ we obtain

$$\frac{1}{2}\|v_+(t)\|_0^2 + (\mu - \varepsilon M)\int_0^t \|v_+\|_1^2 d\tau \leq \frac{M}{4\varepsilon}\int_0^t \|v_+\|_0^2 \, d\tau.$$

Setting $\varepsilon = \mu/M$,

$$\|v_+(t)\|_0^2 \leq \frac{M^2}{2\mu}\int_0^t \|v_+\|_0^2 \, d\tau,$$

and Gronwall's inequality implies that $\|v_+(t)\| = 0$ t—a.e. Similarly, we can show $(-u - \|u_0\|_\infty)_+ = 0$ t—a.e. and $\|u\|_\infty \leq \|u_0\|_\infty$ follows. As $\tilde{f}(u) = f(u)$ for $|u| \leq \|u_0\|_\infty$, u is a solution of the original problem.

To prove uniqueness, suppose u_1 and u_2 are two solutions, and truncate f outside $\max\{\|u_i\|_{L^\infty(\mathbb{R}\times[0,T])}\}$ as above. □

We now derive (in Theorem 3.3) some a priori estimates that will be required in Chapter 7 in order to deal with the limit $\mu \to 0$ in (9). We need a sequence of lemmas.

Lemma 3.1. (a) If $f \in C^1$ and $u_0 \in H^1$, then the solution u of (9), (10) obtained in Theorem 3.2 has the further properties $u \in L^2(0, T, H^2) \cap C([0, T], H^1)$, $u_t \in L^2(0, T, L^2)$.

(b) If, in addition, $f \in C^2$ and $u_0 \in H^2$, then $u \in L^2(0, T, H^3) \cap C([0, T], H^2)$, and $u_t \in L^2(0, T, H^1) \cap C([0, T], L^2)$.

Proof. (a) As $-f'(u)u_x \in L^2(0, T, L^2)$, Theorem 2.3 implies that $u \in L^2(0, T, H^2)$ and $u_t \in L^2(0, T, L^2)$. In addition, the proof of that theorem shows that $u_x \in C([0, T], L^2)$.

(b) Setting $v = u_t$, we have

$$v_t - \mu v_{xx} = -(f(u)_t)_x \equiv -(f'(u)u_t)_x,$$

and $f'(u)u_t \in L^2(0, T, L^2)$ [*part*(a)], $v(x, 0) = \mu u_{0xx} - (f(u_0))_x \in L^2$ imply that $v \in L^2(0, T, H^1) \cap C([0, T], L^2)$. We note that $u \in L^2(0, T, H^2)$ implies that $u_{xx} \in L^2([0, T] \times \mathbb{R})$ and $u_t \in L^2(0, T, H^1)$ implies that $u_{xt} \in L^2([0, T] \times \mathbb{R})$; hence, $u_x \in H^1([0, T] \times \mathbb{R})$ and the Sobolev embedding theorem (Section 3 of Chapter 8) says that $u_x^2 \in L^2([0, T] \times \mathbb{R})$. Differentiating (9) with respect to x yields

$$(u_x)_t - \mu(u_x)_{xx} = -f''(u)u_x^2 + f'(u)u_{xx}.$$

The right-hand side is in $L^2([0, T] \times \mathbb{R})$ [$f''(u)$ and $f'(u)$ are in $L^\infty([0, T] \times \mathbb{R})$] and $u_{0x} \in H^1$, so $u_x \in L^2(0, T, H^2) \cap C([0, T], H^1)$ as above, i.e., $u \in L^2(0, T, H^3) \cap C([0, T], H^2)$. □

Lemma 3.2. We can choose a polynomial $p(\rho)$ such that

$$\chi(\rho) := \begin{cases} 1, & \rho \leq 1/2, \\ p(\rho), & 1/2 \leq \rho \leq 1, \\ e^{-p}, & 1 \leq \rho, \end{cases}$$

is in $C^2([0,\infty))$ and $\chi(\rho) \geq 0$ for $\rho \geq 0$. Further, if $\Psi_R(x) = \chi(|x|/R)$, then

$$|\Psi_{R_x}| \leq C\Psi_R/R, \qquad |\Psi_{Rxx}| \leq C\Psi_R/R^2$$

for a constant C independent of R.

Proof. There are many choices for $p(\rho)$. After finding such a p we can choose C such that $|\chi'(\rho), |\chi''(\rho)| \leq C\chi(\rho)$ on $[\frac{1}{2}, 1]$, and hence everywhere. As $\Psi_R'(x) = \chi'(|x|/R)\ \mathrm{sgn}(x)/R$, $\Psi_R''(x) = \chi''(|x|/R)/R^2$, the rest is immediate. □

Lemma 3.3 If v, $\psi \in H^2$ and $\psi \geq 0$ then

$$\int_{\mathbb{R}} v_{xx}\ \mathrm{sgn}(v)\psi dx \leq \int_{\mathbb{R}} |v|\psi_{xx} dx.$$

Proof. Consider the piecewise linear function $s_\theta(x)$ approximating $\mathrm{sgn}(x)$,

$$s_\theta = \begin{cases} -1, & x < -\theta, \\ x/\theta, & -\theta \leq x \leq \theta, \\ 1, & x > \theta. \end{cases}$$

As $s_\theta'(x) \geq 0$ (for a.e. x) we have

$$\begin{aligned}\int_{\mathbb{R}} v_{xx}s_\theta(v(x))\psi dx &= -\int_{\mathbb{R}} v_x\psi_x s_\theta(v(x))dx - \int_{\mathbb{R}} v_x s_\theta'(v(x))v_x\psi dx \\ &\leq -\int_{\mathbb{R}} v_x\psi_x s_\theta(v(x))dx.\end{aligned}$$

Letting $\theta \to 0$ and using Lebesgue's dominated convergence theorem and Proposition 3.1, (b),

$$\int_{\mathbb{R}} v_{xx}\ \mathrm{sgn}(v)\psi\ dx \leq -\int_{\mathbb{R}} v_x\psi_x\ \mathrm{sgn}(v)dx = -\int_{\mathbb{R}} |v|_x\psi_x\ dx = \int_{\mathbb{R}} |v|\psi_{xx}\ dx. \quad \square$$

Theorem 3.3 If $f \in C^2$ and $u_0 \in H^2 \cap L^1$, and $M = \sup\{|f'(u)| : |u| \leq \|u_0\|_\infty\}$, then the solution u of (9), (10) satisfies

i. $\int_{\mathbb{R}} |u_x(x,t)|\, dx \leq \int_{\mathbb{R}} |u_{0x}| dx$,
ii. $\int_{\mathbb{R}} |u_t(x,t)|\, dx \leq M \int_{\mathbb{R}} |u_{0x}|\, dx + \mu \int_{\mathbb{R}} |u_{0xx}| dx$,
iii. $\int_{\mathbb{R}} |u(x,t)|\, dx \leq \int_{\mathbb{R}} |u_0|\, dx + MT|_{\mathbb{R}}|u_{0x}| dx$.

Proof. If we differentiate (9) with respect to x and multiply by $\operatorname{sgn}(u_x)$, we obtain, using Proposition 3.1, (b),

$$\begin{aligned}
0 &= \operatorname{sgn}(u_x)(u_{xt} + f''(u)u_x^2 + f'(u)u_{xx} - \mu u_{xxx}) \\
&= (|x_x|)_t + f''(u)u_x|u_x| + f'(u)|u_x|_x - \mu u_{xxx} \operatorname{sgn}(u_x) \\
&= (|u_x|)_t + (f'(u)|u_x|)_x - \mu u_{xxx} \operatorname{sgn}(u_x).
\end{aligned}$$

If this equation is multiplied by Ψ_R and integrated over $\mathbb{R}$, we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |u_x|\Psi_R\,dx &= \int_{\mathbb{R}} f'(u)|u_x|\Psi_{Rx}\,dx + \mu\int_{\mathbb{R}} u_{xxx} \operatorname{sgn}(u_x)\Psi_r\,dx \\
&\le \frac{CM}{R}\int_{\mathbb{R}} |u_x|\Psi_{\mathbb{R}}\,dx + \mu\int_{\mathbb{R}} |u_x|\Psi_{Rxx}\,dx \\
&\le \frac{C}{R}\left(M + \frac{\mu}{R}\right)\int_{\mathbb{R}} |u_x|\Psi_{\mathbb{R}}\,dx,
\end{aligned}$$

where Lemmas 3.2 and 3.3 have been used. If we assume $R \ge 1$, and let $F(t) := \int_{\mathbb{R}} |u_x|\Psi_{\mathbb{R}}\,dx$, this becomes

$$0 \le F(t) \le F(0) + \frac{K}{R}\int_0^t F(s)ds$$

($K > 0$). Gronwall's inequality implies

$$F(t) \le F(0)e^{Kt/R},$$

and if we let $R \to \infty$ and use Lebesgue's dominated convergence theorem, conclusion (i) follows.

In a similar way, we can differentiate (0) with respect to t and multiply by $\operatorname{sgn}(u_t)\Psi_{\mathbb{R}}(x)$ to get

$$0 = (|u_t|_t + (f'(u)|u_t|)_x - \mu u_{txx} \operatorname{sgn}(u_t))\Psi_R$$

and

$$\frac{d}{dt}\int_{\mathbb{R}} |u_t|\Psi_{\mathbb{R}}\,dx \le \frac{C}{R}\left(M + \frac{\mu}{R}\right)\int_{\mathbb{R}} |u_t|\Psi_{\mathbb{R}}\,dx.$$

This implies

$$\int_{\mathbb{R}} |u_t(x,t)|dx \le \int_{\mathbb{R}} |u_t(x,0)|dx,$$

as before. As $u_t(x,0) = -f'(u_0)u_{0x} + \mu u_{0xx}$, conclusion (ii) follows.
Finally, multiplication of (9) by sgn(u) yields

$$\begin{aligned}\frac{d}{dt}\int \mathbb{R}|u|\Psi_R\,dx &= -\int_{\mathbb{R}} f'(u)u_x \operatorname{sgn}(u)\Psi_R\,dx + \mu\int_{\mathbb{R}} u_{xx}\operatorname{sgn}(u)\Psi_R\,dx\\ &\le M\int_{\mathbb{R}} |u_x|\Psi_R\,dx + \mu\frac{C}{R^2}\int_{\mathbb{R}} |u|\Psi_R\,dx\end{aligned}$$

Integrating and letting $R \to \infty$,

$$\int_{\mathbb{R}} |u(x,t)|dx \le \int_{\mathbb{R}} |u_0(x)|dx + M\int_0^t\int_{\mathbb{R}} |u_x(x,\tau)|dx\,d\tau,$$

and conclusion (iii) follows from conclusion (i). □

We have illustrated several important techniques in the above. In Theorem 3.1 a fixed point argument is used to obtain a solution. In Theorem 3.2 and a priori estimate is used to extend the scope of the existence theorem. In Lemma 3.1 additional regularity is obtained using the differential equation and added hypotheses on the data. In Theorem 3.3 integral norm estimates of the solution are obtained by differentiating the equation and using appropriate "test functions."

Exercises

1.1 Prove that the initial conditions are taken on in the sense that

$$\|u(t) - u_0\|_{L^2(\Omega)} \to 0 \qquad \text{as } t \to 0.$$

1.2. Consider the initial value problem

$$u_{tt} = Au,$$

$$u(0) = u_0, \qquad u_t(0) = u_1,$$

where $A = \Delta$, $u_0 \in H_0^1(\Omega)$, $u_1 \in L^2(\Omega)$. Assume a solution of the form (3), and show that there is a unique $u \in C([0,T], H_0^1(\Omega))$ with $u_t \in C([0,T],$

$L^2(\Omega))$ and $u_{tt} \in C([0,T], H^{-1}(\Omega))$. *Hint*: First show $\|u(t)\| \le \|u_0\| + C\|u_1\|_{L^2(\Omega)}$, and then consider $\sum_{n=1}^{\infty}\lambda_n|u_n(t) - u_n(s)|^2$.

2.1. Prove Theorem 2.4.

2.2. Give the details in proving Theorem 2.5.

2.3. In Theorem 2.5 we can add lower-order terms and obtain the same result. In particular, we can replace $A(t)$ by $A(t) + B(t)$ where $B(t) = \sum_i b_i u_{x_i} + cu$ and $b_i, c \in L^\infty(\Omega \times [0,T])$. The transformation to homogeneous boundary conditions now becomes $U = e^{-\tau t}(u - g)$ where τ is to be chosen. The equation for U is $U_t + (\tau + A(t) + B(t))U = F$, $F = e^{-\tau t}(f - g_t - (A+B)g)$, and we are led to consider the quadratic form

$$q(t,u,v;\tau) = \tau\int_\Omega uv d\mathbf{x} + \int_\Omega \left(\sum_{i,j} a_{ij}u_{x_i}v_{x_j} + \sum_i b_i u_{x_i} v + cuv\right)d\mathbf{x}$$

on $H^1(\Omega) \times H^1(\Omega)$. Show that τ can be chosen so that

$$q(t,u,u;\tau) \ge C\|u\|^2_{H^1(\Omega)}$$

and carry out the details of the extension of Theorem 2.5.

2.4. Let V, H, V' be a triplet of Hilbert spaces with $V \subset H \subset V'$, V dense in H. We may use the Riesz map J (Section 1 of Chapter 8) to identify H and H': This means that for every $u \in H$ we identify u and Ju, writing

$${}_{H'}\langle u, v\rangle_H = (u,v)_H$$

for every $v \in H$. This implies the inclusion $H = H' \subset V'$, i.e., ${}_{V'}\langle u, v\rangle_V = (u,v)_H$ for every $u \in H$, $v \in V$. One might think of using the Riesz theorem (Section 1 of Chapter 8) to enforce the further identification $V = V'$ (remember that two infinite-dimensional sets A, B with A a proper subset of B may be isomorphic). Show that this further identification is not allowed unless $V = H$ with equal norms. *Hint*: ${}_{V'}\langle u, v\rangle_V = (u,v)_V$ for every $u, v \in V$ implies $(u,v)_V = (u,v)_H$ and as V is dense in H, $V = H$.

3.1. (Gronwall's inequality). If f is continuous and nonnegative with

$$f(t) \le C + K\int_0^t f(s)ds, \qquad C, K \ge 0,$$

on $0 \le t \le a$, then $f(t) \le Ce^{Kt}$ there. *Hint*: Let $F(t) = C + K\int_0^t f(s)ds$ and show $F(t) \le Ce^{Kt}$.

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