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Boundary Conditions for Nonlinear Hyperbolic Systems of Conservation Laws

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For nonlinear hyperbolic systems of conservation laws, the initial-boundary value problem is studied. Two formulations of boundary conditions are proposed: an entropy boundary inequality is derived thanks to the viscosity method, and a second formulation is based on the Riemann problem. These two formulations are equivalent for linear systems and scalar nonlinear equations. For nonlinear systems, the second formulation leads to well-posed problems. Nonlinear local structure is studied. The p-system and the isentropic Euler equations are detailed. © 1988 Academic Press, Inc.

0. INTRODUCTION

We consider nonlinear hyperbolic systems of conservation laws in one space dimension,

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, \qquad u = u(x, t) \in \mathcal{U} \subset \mathbb{R}^n, \ t > 0, \tag{0.1}$$

with a Cauchy datum u_0 for $t = 0$,

$$u(x, 0) = u_0(x). \tag{0.2}$$

The flux function f is regular on the set of the states $\mathcal{U} \subset \mathbb{R}^n$ with values in $\mathbb{R}^n$.

The purely Cauchy problem with $x \in \mathbb{R}$ is well known: the system (0.1)–(0.2) does not admit (in the large) any classical solution, even if the datum u_0 is regular. Hence, it is necessary to consider weak solutions (in the sense of distributions). However, in this class uniqueness is lost, and to select the physical meaningful solutions, an entropy condition has to be added. Two classical methods (at least) can be used to derive such a condition: the vanishing viscosity method or the Lax compatibility relations for wave velocities (Lax [13–15], Smoller [23]).

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In this paper, we are interested in the initial-boundary value problem associated to (0.1)–(0.2). The unknown u is defined for $x > 0$, $t > 0$ and, at the boundary $x = 0$, a datum $\bar{u}_0(t)$ is prescribed. With a strong Dirichlet boundary condition

$$u(0, t) = \bar{u}_0(t), \qquad t > 0$$

the associated problem is not well-posed: generally, there is neither existence nor uniqueness (refer to Leroux [18]). In fact, for the case where the domain boundary is not characteristic, the initial-boundary value problem has been studied by different authors (Liu [19], Nishida and Smoller [20]) for particular systems of gas dynamics. (Refer also to Goodmann [7] for general nonlinear systems.)

We propose and study in this paper two general approaches to formulating the boundary condition (at $x = 0$). In Section 1, the vanishing viscosity method is applied and leads to a *boundary entropy inequality*, which generalizes the previous result of Bardos, Leroux, and Nedelec [3] (see also Le Floch and Nedelec [16, 17]). From this inequality, a first set $\mathscr{E}(\bar{u}_0(t))$ of admissible values at the boundary is defined for each $t > 0$. The first boundary condition is written

$$u(0, t) \in \mathscr{E}(\bar{u}_0(t)), \qquad t > 0. \tag{B.1}$$

In Section 2, the *Riemann problem* is used and leads to the definition of a second set of admissible values at the boundary, denoted by $\mathscr{V}(\bar{u}_0(t))$ (Section 2.1). The second boundary condition is

$$u(0, t) \in \mathscr{V}(\bar{u}_0(t)), \qquad t > 0. \tag{B.2}$$

Condition (B.2) is a natural way to take into account a boundary datum when the Godunov scheme is numerically implemented (Godunov [6]).

Concerning strictly hyperbolic linear systems and (not necessarily convex) scalar conservation laws, we prove that the two sets $\mathscr{E}$ and $\mathscr{V}$ of admissible values are equal and lead to well-posed problems (Sections 1.2, 1.3, 2.2, and 2.3).

We make the conjecture that for general systems of conservation laws, these two sets are equal and therefore that the two formulations of boundary conditions are equivalent. The major difficulty is the description of the sets $\mathscr{E}$ because for this we need knowledge of "entropy functions" for the system (0.1). (Section 1.4).

The second formulation allows an explicit description of the boundary condition. Therefore, the nonlinear local structure of the sets $\mathscr{V}$ for a general system is described in Section 2.4. Then, a complete study of the p-system (Section 2.5) and the isentropic Euler equations of gas dynamics

(Section 2.6) are presented. In Section 3, our conjecture is illustrated by a 2×2 nonlinear system whose characteristic fields are both linearly degenerate.

Notations. Except for the scalar case where we consider nonconvex equations, our hypotheses and notations are the following: the eigenvalues of the Jacobian matrix $A(u) = Df(u)$ are real and distinct,

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u),$$

and $r_1(u)$, $r_2(u), ..., r_n(u)$ is a basis of corresponding eigenvectors. The i-characteristic fields are supposed either genuinely nonlinear $(\nabla\lambda_i(u) \cdot r_i(u) \equiv 1)$ or linearly degenerate $(\nabla\lambda_i(u) \cdot r_i(u) \equiv 0)$.

1. Approach by the Method of Viscosity

1.1. *First Formulation of the Boundary Condition*

A classical method which leads to an entropy condition for selecting the physical solution of the system (0.1) is the viscosity method. The perturbation of (0.1) by a viscosity term $\varepsilon(D(u)\,u_x)_x$ (with $\varepsilon > 0$, and $D(u)$ a positive matrix) leads to a well-posed parabolic problem (refer to Ladyzenskaya and Uralceva [12]),

$$\frac{\partial}{\partial t} u^\varepsilon + \frac{\partial}{\partial x} f(u^\varepsilon) = \varepsilon \frac{\partial}{\partial x}\left(D(u^\varepsilon)\frac{\partial u^\varepsilon}{\partial x}\right), \qquad x > 0, t > 0 \tag{1.1}$$

$$u^\varepsilon(x, 0) = u_0^\varepsilon(x), \qquad x > 0 \tag{1.2}$$

$$u^\varepsilon(0, t) = \bar{u}_0^\varepsilon(t), \qquad t > 0, \tag{1.3}$$

where u_0^ε and $\bar{u}_0^\varepsilon$ are regular approximations of the data u_0 and $\bar{u}_0$.

Classically, a D-pair of entropy-flux is defined as a pair (η, q) of functions $\mathscr{U} \to \mathbb{R}$ such that

$$q'(u) = \eta'(u)\,A(u), \qquad u \in \mathscr{U}$$

$$\eta''(u)\,D(u) \geqslant 0, \qquad u \in \mathscr{U}\ (D\text{-convexity}),$$

and if (u^ε) remains bounded in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^n)$, and converges in L^1_{loc} norm (as ε tends to zero) to a limit function u then in the sense of distributions this limit satisfies the entropy inequality in the domain $\{(x, t), x > 0, t > 0\}$,

$$\frac{\partial}{\partial t}\eta(u) + \frac{\partial}{\partial x} q(u) \leqslant 0, \tag{1.4}$$

for any D-pair of entropy-flux (refer to Kruzkov [11], Lax [15], Smoller [23]). Such an entropy weak solution u satisfies a Rankine–Hugoniot inequality along any smooth curve of discontinuity $x = \varphi(t)$;

$$q(u_+) - q(u_-) - \varphi'(t)(\eta(u_+) - \eta(u_-)) \leqslant 0, \tag{1.5}$$

with $u_\pm = u(\varphi(t) \pm 0, t)$.

The previous entropy inequality (1.4) is concerned only with the values of the limit function $u(\cdot, \cdot)$ *into* the domain $\{(x, t)/x > 0, t > 0\}$. Now, we establish a supplementary condition: a *boundary entropy inequality* valid along the boundary. Making the same assumptions as previously on the sequence (u^ε), we get:

THEOREM 1.1. *For each $t > 0$, the boundary value $u(0+, t)$ of a limit-function u of the sequence u^ε is linked to the boundary limit-value $\bar{u}_0(t)$ of the functions u^ε by the boundary entropy inequality*

$$q(u(0+, t)) - q(\bar{u}_0(t)) - \eta'(\bar{u}_0(t)) \cdot \{f(u(0+, t)) - f(\bar{u}_0(t))\} \leqslant 0 \tag{1.6}$$

for any D-pair of entropy-flux (η, q).

The inequality (1.6) was derived previously in the scalar case with Kruzkov entropies by Bardos, Leroux, and Nedelec [3]. Here, we only show that this inequality remains valid (at least formally) for a *system* of conservation laws with an arbitrary viscosity term. In the derivation of inequality (1.6) we establish a preliminary result (Lemma 1.1) which clearly underlines a phenomenon of *boundary-layer*: the function u^ε remains bounded for $\varepsilon \to 0$, but its derivative $\partial u^\varepsilon/\partial x$ is not necessarily bounded. This fact explains that discontinuities can appear for the limit function u. Refer also, for scalar equations, to Howes [9].

Like Kruzkov [11] and Bardos, Leroux, and Nedelec [3], let us introduce a family of functions $\{\rho_\delta\}$ $(\delta > 0)$ such that

$$\begin{aligned} &\rho_\delta \in \mathscr{C}^2(\mathbb{R}^+; [0, 1]), \qquad \rho_\delta(x) = 0, \forall x > \delta \\ &\rho_\delta(0) = 1 \qquad \text{and} \qquad |\rho'_\delta(\cdot)| \leqslant c/\delta \end{aligned} \tag{1.7}$$

with a constant $c > 0$ independent of δ.

LEMMA 1.1.

$$\lim_{\varepsilon \to 0} \left(\varepsilon \cdot D(\bar{u}_0(\cdot)) \cdot \frac{\partial u^\varepsilon}{\partial x}(0, \cdot) \right) = f(\bar{u}_0(\cdot)) - f(u(0+, t)),$$

in the sense of distributions.

Proof of Lemma 1.1. Consider $G \in \mathscr{C}^2(\mathbb{R}^+; \mathbb{R}^n)$ with compact support in $]0, \infty[$ and the Green formula

$$\int_0^\infty G(t) \cdot \int_0^\delta \frac{\partial}{\partial x}\left(D(u^\varepsilon) \cdot \frac{\partial}{\partial x} u^\varepsilon\right) \cdot \rho_\delta(x) \cdot dx \cdot dt$$

$$= -\int_0^\infty G(t) \cdot D(\bar{u}_0) \cdot \frac{\partial u^\varepsilon}{\partial x} dx - \int_0^\infty \int_0^\delta G(t) \cdot D(u^\varepsilon) \cdot \frac{\partial}{\partial x} u^\varepsilon \cdot \rho'_\delta(x)\, dx \cdot dt.$$

We take the limit in this formula first for $\varepsilon \to 0$, then for $\delta \to 0$. With Eqs. (1.1), we deduce

$$A(\delta) \equiv \lim_{\varepsilon \to 0} \int_0^\infty G(t) \cdot D(\bar{u}_0(t)) \cdot \frac{\partial}{\partial x} u^\varepsilon(0, t)\, dt$$

$$= \lim_{\varepsilon \to 0} -\int_0^\infty \int_0^\delta G \cdot (u_t^\varepsilon + f(u^\varepsilon)_x) \cdot \rho_\delta\, dx\, dt$$

because for each $\delta > 0$ we have

$$\left| \varepsilon \int_0^\infty \int_0^\delta G \cdot D(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x} \rho'_\delta\, dx\, dt \right| \leqslant \varepsilon.\text{cst. } \|u^\varepsilon\|_{W^{1,1}_{\text{loc}}} \to 0.$$

Integrating by parts, we obtain

$$A(\delta) = \lim_{\varepsilon \to 0} \left\{ + \int_0^\infty \int_0^\delta G'(t) \cdot u^\varepsilon \cdot \rho_\delta\, dx\, dt + \int_0^\infty \int_0^\delta G(t) f(u^\varepsilon) \cdot \rho'_\delta\, dx\, dt \right.$$

$$\left. + \int_0^\infty G(t) f(\bar{u}_0(t))\, dt \right\}$$

$$= \int_0^\infty \int_0^\delta G'(t)\, u \rho_\delta\, dx\, dt + \int_0^\infty \int_0^\delta G(t) f(u) \cdot \rho'_\delta\, dx\, dt$$

$$+ \int_0^\infty G(t) f(\bar{u}_0(t))\, dt.$$

Then, when δ tends to zero, for $t > 0$ we use

$$\lim_{\delta \to 0} \int_0^\delta f(u(x, t)) \cdot \rho'_\delta(x)\, dx = -f(u(0, t)),$$

and it results

$$\lim_{\delta \to 0} A(\delta) = \int_0^\infty G(t) \cdot \{f(\bar{u}_0(t)) - f(u(0, t))\}\, dt,$$

which proves the lemma. ∎

Proof of Theorem 1.1. Now, multiplying Eq. (1.1) by $\eta'(u(x,t))\,\psi(t)\,\rho_\delta(x)$, with an arbitrary function $\psi \in \mathscr{C}^2(0,\infty;\mathbb{R})$ with compact support, we get

$$\begin{aligned}&\int_0^\infty \int_0^\delta \{\eta(u^\varepsilon)_t + q(u^\varepsilon)_x\} \cdot \psi(t)\,\rho_\delta(x)\,dx\,dt \\ &\quad - \varepsilon \int_0^\infty \int_0^\delta \eta'(u^\varepsilon) \cdot \frac{\partial}{\partial x}\left(D(u^\varepsilon) \cdot \frac{\partial}{\partial x} u^\varepsilon\right) \cdot \psi(t) \cdot \rho_\delta(x)\,dx\,dt = 0.\end{aligned} \tag{1.8}$$

The first term in (1.8) is

$$\begin{aligned}&-\int_0^\infty \int_0^\delta \eta(u^\varepsilon)\,\psi'(t)\,\rho_\delta(x)\,dx\,dt - \int_0^\infty \int_0^\delta q(u^\varepsilon) \cdot \psi(t) \cdot \rho'_\delta(x)\,dx\,dt \\ &\quad -\int_0^\infty q(\bar{u}_0(t)) \cdot \psi(t)\,dt,\end{aligned}$$

which tends to

$$\begin{aligned}&-\int_0^\infty \int_0^\delta \eta(u)\,\psi'(t)\,\rho_\delta(x)\,dx\,dt - \int_0^\infty \int_0^\delta q(u)\,\psi(t)\,\rho'_\delta(x)\,dx\,dt \\ &\quad -\int_0^\infty q(\bar{u}_0)\,\psi\,dt,\end{aligned}$$

when $\varepsilon \to 0$. Then for $\delta \to 0$, it becomes

$$\int_0^\infty \{-q(\bar{u}_0(t)) + q(u(0,t))\}\,\psi(t)\,dt. \tag{1.9}$$

The second term in (1.8) equals

$$\begin{aligned}&-\varepsilon \int_0^\infty \int_0^\delta \frac{\partial}{\partial x}\left(\eta'(u^\varepsilon) \cdot D(u^\varepsilon) \cdot \frac{\partial}{\partial x} u^\varepsilon\right) \psi(t) \cdot \rho_\delta(x)\,dx\,dt \\ &\quad + \varepsilon \int_0^\infty \int_0^\delta \eta''(u^\varepsilon) \cdot \frac{\partial u^\varepsilon}{\partial x} \cdot D(u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x}\,\psi(t)\,\rho_\delta(x)\,dx\,dt.\end{aligned} \tag{1.10}$$

The first integral in (1.10) can be written as

$$\begin{aligned}&+\varepsilon \int_0^\infty \int_0^\delta \eta'(u^\varepsilon)\,D(u^\varepsilon) \cdot \frac{\partial u^\varepsilon}{\partial x}\,\psi(t) \cdot \rho'_\delta(x)\,dx\,dt \\ &\quad + \varepsilon \int_0^\infty \eta'(\bar{u}_0) \cdot D(\bar{u}_0) \cdot \frac{\partial u^\varepsilon}{\partial x}(0,t)\,\psi\,dt,\end{aligned}$$

which tends ($\varepsilon \to 0$) to

$$+\int_0^{\infty} \eta'(\bar{u}_0(t)) \cdot \{f(\bar{u}_0(t)) - f(u(0, t))\} \cdot \psi(t)\, dt; \tag{1.11}$$

applying Lemma 1.1 and the inequality (valid for each $\delta > 0$),

$$\left| -\varepsilon \int_0^{\infty} \int_0^{\delta} \eta'(u^{\varepsilon})\, D(u^{\varepsilon}) \frac{\partial u^{\varepsilon}}{\partial x} \psi(t) \cdot \rho'_{\delta}(x)\, dx\, dt \right| \leqslant \varepsilon.\text{cst. } \|u^{\varepsilon}\|_{W^{1,1}_{\text{loc}}}.$$

The terms (1.9) and (1.10) correspond exactly to the inequality (1.6). To conclude, note that the second integral in (1.10) is positive because of the D-convexity property of the entropy η. ∎

Remark 1.1. (1) The inequality (1.6) is an algebraic one. For an entropy-flux pair such that $\eta'(\bar{u}_0(t)) = 0$ ($t > 0$ fixed), it becomes

$$q(u(0+, t)) \leqslant q(\bar{u}_0(t)), \tag{1.12}$$

which expresses the nonincreasing of entropy-flux across the boundary. For the p-system and with the entropy associated with the energy, this inequality (1.12) was derived previously by Benabdallah [4].

(2) The inequality (1.12) is equivalent (formally) with the inequality (1.5) when the discontinuity curve in (1.5) is a vertical line ($\varphi'(t) = 0$).

(3) Note that the derivation of boundary inequality (1.6) does not need a hypothesis concerning the characteristic fields.

In the following, we are interested in specifying the *admissible states at the boundary.*

Definition 1.1. For each state $\bar{u}_0$ in $\mathcal{U}$, the set $\mathcal{E}(\bar{u}_0)$ of the admissible values at the boundary is defined as all the states u in $\mathcal{U}$ such that

$$q(u) - q(\bar{u}_0) - \eta'(\bar{u}_0) \cdot \{f(u) - f(\bar{u}_0)\} \leqslant 0 \tag{1.13}$$

for each D-pair (η, q) of entropy-flux.

Moreover we propose to formulate the boundary condition for the problem (0.1)–(0.2) as

$$u(0+, t) \in \mathcal{E}(\bar{u}_0(t)), \qquad t > 0 \tag{B.1}$$

for an arbitrary boundary datum $\bar{u}_0 \colon \mathbb{R}_+ \to \mathcal{U} \subset \mathbb{R}^n$. Note that $\bar{u}_0(t)$ itself belongs to $\mathcal{E}(\bar{u}_0(t))$. But generally the set $\mathcal{E}(\bar{u}_0(t))$ is not reduced to this point, and the condition (B.1) is an extension of the usual Dirichlet boundary condition.

We hope that in general, this formulation (B.1) will lead to a well-posed problem. For a linear system we recover the well-known formulation (Section 1.2). Concerning scalar conservation laws, Bardos, Leroux, and Nedelec [3] proved that (0.1)–(0.2)–(B.1) admits one and only one entropy solution. In Section 1.3, we give a complete description of the sets $\mathscr{E}(\bar{u}_0)$ for nonconvex scalar equations; many entropies are needed for this study. For a general system, the difficulty is to find suitable different entropies.

1.2. *Strictly Hyperbolic Linear Systems*

The flux function is now given by $f(u) = A \cdot u$, with a constant matrix A whose eigenvalues are real and distinct. If we limit ourselves to a diffusion matrix $D = I$, the entropy-flux pairs (Lemma 1.2) are well known for such systems, and we are able to specify the sets $\mathscr{E}(\bar{u}_0)$. Let p in $\{1, ..., n\}$ be the index of the greatest nonpositive eigenvalue of A,

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p \leqslant 0 < \lambda_{p+1} < \cdots < \lambda_n.$$

PROPOSITION 1.1. *For $\bar{u}_0$ in $\mathbb{R}^n$, the set $\mathscr{E}(\bar{u}_0)$ is the affine space containing $\bar{u}_0$ and generated by the p first eigenvectors of the matrix A,*

$$\mathscr{E}(\bar{u}_0) = \left\{ \bar{u}_0 + \sum_{i=1}^{p} \alpha_i \cdot r_i / (\alpha_i, ..., \alpha_p) \in \mathbb{R}^p \right\}.$$

This statement is well known: for a linear hyperbolic system, the values given at the boundary are the components on the basis of the eigenvectors corresponding to the *incoming characteristics*; and this formulation of the boundary condition leads to a well-posed problem (Kreiss [10], Gustafsson, Kreiss, and Sundström [8]).

Let us begin with the description of the entropy-flux pairs for a linear *strictly* hyperbolic system.

LEMMA 1.2. *All the pairs of I-entropy-flux are given by*

$$\eta(u) = \sum_{i=1}^{n} \varphi_i(\alpha_i(u))$$

$$q(u) = q_0 + \sum_{i=1}^{n} \lambda_i \varphi_i(\alpha_i(u)),$$

where u is decomposed as $u = \sum_{i=1}^{n} \alpha_i(u)\, r_i$ for n arbitrary convex functions $\varphi_1, \varphi_2, ..., \varphi_n$ and a constant $q_0 \in \mathbb{R}^n$.

Proof of Lemma 1.2. It can be shown (Bardos [2]) that a convex function η is an entropy if and only if $\forall u \in \mathbb{R}^n$, $\eta''(u) \cdot A$ is a symmetric matrix. But taking $P = (r_1, r_2, ..., r_n)$ and $\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_n)$, we have

$$\Lambda = P^{-1} \cdot A.P.$$

Thus using $\eta''(u) \cdot P \cdot \Lambda \cdot P^{-1} = {}^t(\eta''(u) \cdot P \cdot \Lambda \cdot P^{-1})$ and the symmetry of the matrix $\eta''(u)$,

$$\eta''(u) \cdot P \cdot \Lambda \cdot P^{-1} = {}^tP^{-1} \cdot \Lambda \cdot {}^tP \cdot \eta''(u),$$

we get the relation $({}^tP \cdot \eta''(u) \cdot P) \cdot \Lambda = \Lambda({}^tP \cdot \eta''(u) \cdot P)$.

The matrix ${}^tP \cdot \eta''(u) \cdot P$ commutes with the diagonal matrix Λ whose elements are all *distinct*. Therefore, it is also a diagonal matrix. The expressions of η and q are direct consequences of this fact. ∎

Proof of Proposition 1.1. First, we prove that if $u = \bar{u}_0 + \sum_{i=1}^n \alpha_i r_i$ with at least one index $i_0 > p$ such that $\alpha_{i_0} \neq 0$, then u does not belong to $\mathscr{E}(\bar{u}_0)$. Let us take the pair of entropy-flux,

$$\eta\left(\bar{u}_0 + \sum_{i=1}^n \beta_i \cdot r_i\right) = (\beta_{i_0})^2$$

$$q\left(\bar{u}_0 + \sum_{i=1}^n \beta_i \cdot r_i\right) = \lambda_{i_0} \cdot (\beta_{i_0})^2.$$

We obtain

$$q(u) - q(\bar{u}_0) - \eta'(\bar{u}_0) \cdot \{f(u) - f(\bar{u}_0)\} = \lambda_{i_0} \cdot (\alpha_{i_0})^2 > 0,$$

which is not consistent with (1.13).

Second, let us consider a state $u = \bar{u}_0 + \sum_{i=1}^p \alpha_i \cdot r_i$. For any entropy-flux pair (η, q), by Lemma 1.2 we get

$$\eta\left(\bar{u}_0 + \sum_{i=1}^p \alpha_i \cdot r_i\right) = \eta(\bar{u}_0) + \eta'(\bar{u}_0) \cdot \sum_{i=1}^p \alpha_i \cdot r_i + \sum_{i=1}^p \varphi_i(\alpha_i)$$

$$q\left(\bar{u}_0 + \sum_{i=1}^p \alpha_i \cdot r_i\right) = q(\bar{u}_0) + q'(\bar{u}_0) \cdot \sum_{i=1}^p \alpha_i \cdot r_i + \sum_{i=1}^p \lambda_i \varphi_i(\alpha_i),$$

with convex functions φ_i such that

$$\varphi_i(0) = \varphi_i'(0) = 0 \qquad (\text{thus } \varphi_i \geqslant 0).$$

The result is

$$q(u) - q(\bar{u}_0) - \eta'(\bar{u}_0) \cdot \{f(u) - f(\bar{u}_0)\} = \sum_{i=1}^p \lambda_i \varphi_i(\alpha_i) \leqslant 0,$$

which is exactly (1.13). ∎

1.3. *Study of Nonconvex Scalar Conservation Laws*

For a scalar conservation law ($n=1$), every convex function is an entropy. Hence we can entirely characterize the set $\mathscr{E}(\bar{u}_0)$ for each state $\bar{u}_0$ in $\mathbb{R}$. Note that the flux function f is not necessarily convex; that is, the (unique) characteristic field is not globally genuinely nonlinear or linearly degenerated.

Using the Kruzkov entropies (refer to Kruzkov [11] and [18–25])

$$\eta(u)=|u-k|, \qquad q(u)=\operatorname{sgn}(u-k)(f(u)-f(k)), k\in\mathbb{R}$$

with $\operatorname{sgn} v=1$ if $v>0$, -1 if $v<0$, we note that the boundary entropy inequalities (1.13) are equivalent to

$$\begin{aligned}\operatorname{sgn}(u-k)(f(u)-f(k))-\operatorname{sgn}(\bar{u}_0-k)(f(\bar{u}_0)-f(k))&\\ -\operatorname{sgn}(\bar{u}_0-k)(f(u)-f(\bar{u}_0))\leqslant 0,& \qquad k\in\mathbb{R}.\end{aligned}$$

Thus we get

$$(\operatorname{sgn}(u-k)-\operatorname{sgn}(\bar{u}_0-k)(f(u)-f(k))\leqslant 0, \qquad k\in\mathbb{R}.$$

The latter inequality holds trivially if k is not between u and $\bar{u}_0$. Therefore we have proved the following property:

PROPOSITION 1.2. *The set $\mathscr{E}(\bar{u}_0)$ of admissible states u is characterized by the inequalities*

$$\frac{f(u)-f(k)}{u-k}\leqslant 0 \tag{1.14}$$

for every k between u and $\bar{u}_0$.

Geometrically, this last condition means that the slope of any chord joining the point $(u,f(u))$ to $(k,f(k))$ is negative for every real k between u and $\bar{u}_0$. To make explicit (1.14) we introduce a piecewise regular function g which depends on the state $\bar{u}_0$, g: $(\inf_{u\leqslant\bar{u}_0}f(u), \sup_{u\geqslant\bar{u}_0}f(u))\to\mathbb{R}$ defined by

$$g(v)=\begin{cases}\inf_h h(v), & \text{if } \quad v\leqslant f(\bar{u}_0)\\ \\ \sup_h h(v), & \text{if } \quad v\geqslant f(\bar{u}_0),\end{cases} \tag{1.15}$$

where h are functions satifying the constraints

$$h{:}f(\mathbb{R})\to\mathbb{R}, \qquad f(h(v))\equiv v, \qquad \forall v\in f(\mathbb{R}) \tag{1.16a}$$

$$(v-f(\bar{u}_0))(h(v)-\bar{u}_0)\leqslant 0, \qquad \forall v\in f(\mathbb{R}). \tag{1.16b}$$

In the plane $\{u, f\}$, if we look at the graph of the function f from the point $(\bar{u}_0, f(\bar{u}_0))$ to the left (resp. the right) and above $f(\bar{u}_0)$ (resp. below $f(\bar{u}_0)$) and if we eliminate hidden lines, we obtain the graph of g^{-1} (see Fig. 1.1). Moreover we can verify that g is a nonincreasing function.

PROPOSITION 1.3. *Let $\mathscr{D}$ be the set of discontinuity points of the function g. We get*

$$\mathscr{E}(\bar{u}_0) = \operatorname{Im} g \cup f^{-1}(\mathscr{D}).$$

In the particular case where the flux function is strictly convex tending towards infinity at infinity, denote by u_* the state minimizing f and for $\bar{u}_0 \neq u_*$ let $\bar{u}_1$ be the solution $u \neq \bar{u}_0$ of the equation $f(u) = f(\bar{u}_0)$. The previous result is simpler.

PROPOSITION 1.4 (convex case).

$$\mathscr{E}(\bar{u}_0) = \begin{cases}]-\infty, \bar{u}_1] \cup \{\bar{u}_0\}, & \text{if } \bar{u}_0 > u_* \\ ]-\infty, u_*], & \text{if } \bar{u}_0 \leqslant u_*. \end{cases}$$

This result was obtained previously by Le Floch and Nedelec in [16, 17]. For the (interesting) case of an incoming value $\bar{u}_0 > u_*$, see Fig. 1.2.

Proof of Propositions 1.3 *and* 1.4. To fix the ideas, suppose that $u < \bar{u}_0$.

(a) If $f(u) < f(\bar{u}_0)$ then on the one hand (1.14) is not valid with $k = \bar{u}_0$, and thus $u \notin \mathscr{E}(\bar{u}_0)$; on the other hand the condition (1.16b) is also wrong with $v = f(u)$, and thus $u \notin \operatorname{Im} g$ and clearly $u \notin f^{-1}(\mathscr{D})$ (see (1.15)).

(b) If $f(u) \geqslant f(\bar{u}_0)$, we set $I = \{u' \in]u, \bar{u}_0], f(u') > f(u)\}$.

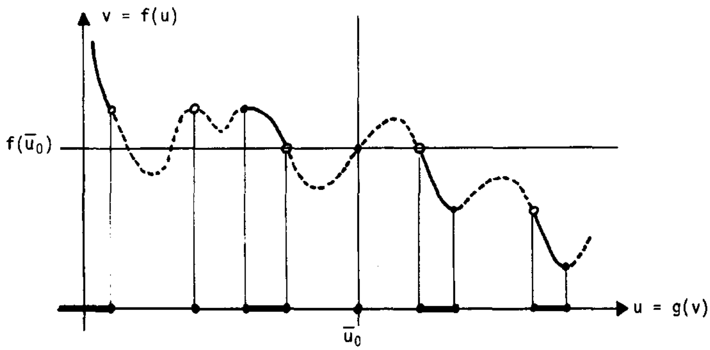

FIGURE 1.1

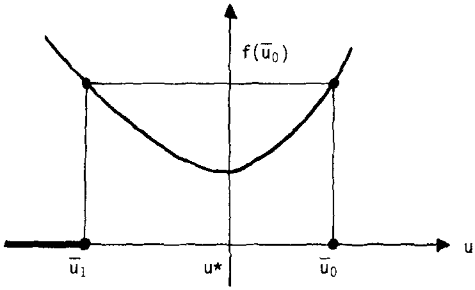

FIGURE 1.2

• Either card $I=0$ and on the one hand (1.14) is clearly satisfied. Define $J=\{u' \in]u, \bar{u}_0], f(u')=f(u)\}$. When card $J=0$, we necessarily get $g(f(u))=u$ and thus $u \in \operatorname{Im} g$. When card $J \geqslant 1$, $f(u)$ is a point of discontinuity of g and $u \in f^{-1}(\mathscr{D})$.

• Or card $I \geqslant 1$ and we have $g(f(u))=\sup I>u$. So $u \notin \operatorname{Im} g \cup f^{-1}(\mathscr{D})$. But (1.14) is not satisfied with k into the interval limited by two values of the set I which contains at least two points because of the continuity of f.

Henceforth the assertion for a general flux function is proved. Concerning the convex case, it is sufficient to note that $g: [f(\bar{u}_0), +\infty[\to]-\infty, \bar{u}_1]$ is continuous except eventually at $f(\bar{u}_0)$. Moreover $f^{-1}(f(\bar{u}_0))=\{\bar{u}_1, \bar{u}_0\}$. ∎

Remark 1.2. When the function f is strictly convex it is sufficient to take into account only *one entropy* $\eta(u)=|u-u_0|$ in the inequality (1.13) to recover the entire set $\mathscr{E}(\bar{u}_0)$. Moreover, every other Kruzkov entropy is not sufficient.

1.4. *Local Nonlinear Structure for General Systems*

We now consider the local nonlinear structure of $\mathscr{E}(\bar{u}_0)$ with $\bar{u}_0 \in \mathscr{U}$, assuming that the state $\bar{u}_0$ has no null eigenvalues

$$\lambda_1(\bar{u}_0)<\cdots<\lambda_p(\bar{u}_0)<0<\lambda_{p+1}(\bar{u}_0)<\cdots<\lambda_n(\bar{u}_0). \tag{1.17}$$

The proof of Lemma 1.2 in this case shows that for a pair of entropy-flux (η, q), the Hessian matrix of η at point $\bar{u}_0$ admits the diagonal form $\operatorname{diag}(\psi_1(\eta), \psi_2(\eta), ..., \psi_n(\eta))$ in the basis of eigenvectors $r_1(\bar{u}_0)$, $r_2(\bar{u}_0), ..., r_n(\bar{u}_0)$, with n nonnegative reals $\psi_1(\eta), \psi_2(\eta), ..., \psi_n(\eta)$.

For u in the neighbourhood of $\bar{u}_0$,

$$u-\bar{u}_0=\sum_{i=1}^{n} \alpha_i \cdot r_i(\bar{u}_0),$$

the inequality (1.13) can be developed by means of the variables $\alpha_1, \alpha_2, ..., \alpha_n$,

$$\sum_{i=1}^{n} \psi_i(\eta) \cdot \lambda_i(\bar{u}_0) \cdot \alpha_i^2 + R_3(\alpha, \eta) \leqslant 0, \tag{1.18}$$

with a rest $R_3(\alpha, \eta)$ that depends on the choice of the entropy η and is at least of order 3 with respect to α. Unfortunately, a geometrical description of $\mathscr{E}(\bar{u}_0)$ is not easily deductible from inequality (1.18). We hope that this set $\mathscr{E}(\bar{u}_0)$ is locally a manifold in the neighbourhood of $\bar{u}_0$. Making the hypothesis that this manifold admits a tangent space and moreover that there exists a suitable entropy η such that

$$\psi_1 = \psi_2 = \cdots = \psi_p = 0, \qquad \psi_{p+1} > 0, ..., \psi_n > 0,$$

the inequality (1.18) shows that the p first $\alpha_1, \alpha_2, ..., \alpha_p$ are zero if u now belongs in this tangent space. Inversely, a state $u = \bar{u}_0 + \sum_{p+1}^{n} \alpha_i \cdot r_i(\bar{u}_0)$, for (α_i) sufficiently small, satisfies the inequality (1.18) for any entropy η.

2. Approach by the Riemann Problem

2.1. *Second Formulation of the Boundary Condition*

A classical and well-understood problem associated with the system (0.1) is the Riemann problem corresponding to an initial datum composed of two constant states (refer to Lax [13–15]). Hence, in this section, we limit ourselves to a model problem, assuming the initial datum $u_0(\cdot)$ and the boundary datum $\bar{u}_0(\cdot)$ to be constant; and in the following, we formulate the boundary condition thanks to the notion of Riemann problem.

We first recall well-known results concerning the *Riemann problem*

$$R(u_{\mathrm{L}}, u_{\mathrm{R}}): \begin{cases} \dfrac{\partial}{\partial t} w + \dfrac{\partial}{\partial x} f(w) = 0, & w(x, t) \in \mathscr{U}, x \in \mathbb{R}, t > 0 \\ w(x, 0) = \begin{cases} u_{\mathrm{L}}, & x < 0 \\ u_{\mathrm{R}}, & x > 0, \end{cases} \end{cases}$$

with u_{L} and u_{R} in $\mathscr{U}$. Classically, we define particular solutions of system (0.1), which depend only on the variable $x/t = \xi$; they are called elementary waves: shock waves, rarefaction waves, and contact discontinuities. In the space of states $\mathscr{U} \subset \mathbb{R}^n$, the waves issuing from a given state u_* are denoted by $\mathscr{S}_i(u_*)$, $\mathscr{R}_i(u_*)$, and $\mathscr{C}_i(u_*)$ respectively; and the i-curves are denoted by $\mathscr{W}_i(u_*)$, $1 \leqslant i \leqslant n$. In neighbourhood of the state u_*, the tangent direction of each curve $\mathscr{W}_i(u_*)$, $1 \leqslant i \leqslant n$, is given by the eigenvector $r_i(u_*)$. Moreover, the curves are globally in the $\mathscr{C}^2$ class. For details concerning the complete definition and the different properties of the i-curves, we refer to Smoller [23].

Assuming the vicinity of the states u_L and u_R, the Riemann problem $R(u_L, u_R)$ admits one and only one entropy weak solution composed of $(n+1)$ constant states separated by n elementary waves (Lax [14]) written as

$$w = w\left(\xi = \frac{x}{t};\ u_L, u_R\right).$$

In the following, the set $\mathscr{U}$ is chosen such that the Riemann problem is always well-posed in this sense.

From the notion of Riemann problem, we now introduce a second set of *admissible states at the boundary*:

DEFINITION 2.1. For each state $\bar{u}_0$ in $\mathscr{U}$, the set $\mathscr{V}(\bar{u}_0)$ of admissible values is defined as

$$\mathscr{V}(\bar{u}_0) = \{w(0+;\bar{u}_0, u_R) \mid u_R \in \mathscr{U}\}.$$

Hence, we propose *a second formulation of the boundary condition* for the problem (0.1)–(0.2),

$$u(0+, t) \in \mathscr{V}(\bar{u}_0(t)),\ t > 0, \tag{B.2}$$

for a boundary datum $\bar{u}_0 \colon \mathbb{R}_+ \to \mathscr{U} \subset \mathbb{R}^n$. Note that, in particular, $\bar{u}_0(t)$ belongs to $\mathscr{V}(\bar{u}_0(t))$. But generally, $\mathscr{V}(\bar{u}_0(t))$ is not reduced to this point as we noted for the set $\mathscr{E}(\bar{u}_0(t))$ in Section 1.

First, we prove that the problem (0.1)–(0.2)–(B.2), with constant data u_0 and $\bar{u}_0$, is *well-posed.*

THEOREM 2.1. *Let $\bar{u}_0$ and u_0 be data in $\mathscr{U}$. The problem* (0.1)–(0.2)–(B.2) *admits one and only one solution in the class of functions which consist of constant states separated by at most n elementary waves. (The set $\mathscr{U}$ is chosen such that any Riemann problem $R(\cdot, \cdot)$ with data in $\mathscr{U}$ is well-posed.)*

Let us begin with a preliminary result which shows that two solutions of Riemann problems can be mixed together.

LEMMA 2.1. *Let $\xi_* \in \mathbb{R}$, $u_* \in \mathscr{U}$. Let $w_L(\xi)$ and $w_R(\xi)$, $\xi \in \mathbb{R}$, be two solutions of Riemann problems such that*

$$w_R(\xi_* + 0) = w_L(\xi_* + 0) = u_*.$$

Let $w(\xi)$, $\xi \in \mathbb{R}$, *be given by*

$$w(\xi)=\begin{cases} w_L(\xi), & \xi<\xi_* \\ w_R(\xi), & \xi>\xi_*. \end{cases}$$

Then, $w(\cdot)$ *is an entropy weak solution of one Riemann problem.*

Proof of Lemma 2.1. We give the proof for $\xi_*=0$. Let i (resp. j) be the index of the greatest nonpositive (resp. smallest nonnegative) wave of w_L (resp. w_R). Let $\bar{\sigma}_i$ (resp. $\underline{\sigma}_j$) be the greatest (resp. smallest) velocity of the i-wave (resp. the j-wave) for w_L (resp. for w_R). Using Lax inequalities, we get

$$\lambda_i(u_*) \leqslant \bar{\sigma}_i \leqslant 0 \leqslant \underline{\sigma}_j \leqslant \lambda_j(u_*). \tag{2.1}$$

Hence, either $\lambda_i(u_*)<\lambda_j(u_*)$, implying $i<j$, or $\lambda_i(u_*)=\lambda_j(u_*)=0$, implying $i=j$ and that u_* is a state in an i-rarefaction wave. ∎

Proof of Theorem 2.1. First, we prove the existence. The solution is immediately constructed via a classical Riemann problem. Consider the restriction of the Riemann problem with the two data $\bar{u}_0$ on the left and u_0 on the right:

$$u(\xi)=w(\xi;\bar{u}_0,u_0), \qquad \xi\in\mathbb{R}^+. \tag{2.2}$$

Clearly, it satisfies (0.1)–(0.2) and (B.2).

Second, concerning the uniqueness, note that a solution $u(\cdot)$ of (0.1)–(0.2)–(B.2) necessarily satisfies

$$u(0+)\in\mathscr{V}(\bar{u}_0)$$

on the one hand, and

$$u(0+)\in\mathscr{W}(u_0)$$

on the other hand. The set $\mathscr{W}(u_0)$ is defined by

$$\mathscr{W}(u_0)=\{w(0+;u_L,u_0)\,|\,u_L\in\mathscr{U}\}.$$

Recall that we consider only solutions consisting of at most $(n+1)$ constant states separated by elementary waves.

We now prove that $\mathscr{V}(\bar{u}_0)\cap\mathscr{W}(u_0)$ is reduced to *one point*. It is clear that there is at least one point; and if u_1^* and u_2^* are in $\mathscr{V}(\bar{u}_0)\cap\mathscr{W}(u_0)$ then there exist z_1 and z_2 solutions of the Riemann problem $R(\bar{u}_0,u_0)$ constructed as follows: u_i^* belongs to $\mathscr{V}(\bar{u}_0)$ (resp. $\mathscr{W}(u_0)$) and thus there exists $u_{R,i}$ (resp. $u_{L,i}$) such that $z_i=w(x/t;\bar{u}_0,u_{R,i})$ if $x/t<0$, and $z_i=w(x/t;u_{L,i},u_0)$ if $x/t>0$, $i=1,2$.

Acccording to Lemma 2.1, we verify that the two functions z_1 and z_2 are effectively weak entropy solutions of the Riemann problem $R(\bar{u}_0, u_0)$. Hence, $z_1 = z_2$ and in particular $u_1^* = u_2^*$. Thus, we deduce that

$$u(0+) = w(0+; \bar{u}_0, u_0) \equiv u_* .$$

We now prove that the solution (2.2) of (0.1)–(0.2)–(B.2) is the *only way* to join u_* (at $\xi = 0+$) to u_0 (at $\xi = +\infty$). Let $z(\xi)$, $\xi > 0$, be another entropy construction. We extend it for $\xi < 0$ by means of the Riemann problem $R(\bar{u}_0, u_0)$:

$$z(\xi) = w(\xi; \bar{u}_0, u_0), \qquad \forall \xi < 0.$$

Then, by Lemma 2.1, $z = w$ as previously. ∎

Next, we obtain a complete description of the sets $\mathscr{V}$ in two cases: linear strictly hyperbolic systems and nonconvex scalar conservation laws (Sections 2.2 and 2.3). Then, we analyze the local structure of $\mathscr{V}$ for a general system of conservation laws (Section 2.4). Finally, we study the p-system and the isentropic Euler equations of gas dynamics thoroughly (Sections 2.5 and 2.6).

2.2. *Study of Strictly Hyperbolic Linear Systems*

When the flux f is linear, $f(u) = A \cdot u$, we consider the index p such that

$$\lambda_1 < \cdots < \lambda_p \leqslant 0 < \lambda_{p+1} < \cdots < \lambda_n,$$

and we obtain the following result.

PROPOSITION 2.1. *For each state $\bar{u}_0$ in $\mathbb{R}^n$, the set $\mathscr{V}(\bar{u}_0)$ is the affine manifold containing $\bar{u}_0$ and generated by the p eigenvectors $r_1, ..., r_p$. That is, $\mathscr{V}(\bar{u}_0) = \mathscr{E}(\bar{u}_0)$.*

This proposition is an immediate consequence of the resolution of a Riemann problem $R(u_L, u_R)$ for a linear system. Taking

$$u_R - u_L = \sum_{i=1}^{n} \alpha_i \cdot r_i, \tag{2.3}$$

the solution $w(\xi; u_L, u_R)$ is given by $(0 \leqslant i \leqslant n)$

$$w(\xi) = u_L + \sum_{j=1}^{i} \alpha_j \cdot r_j \qquad \text{for} \quad \lambda_i < \xi < \lambda_{i+1}, \tag{2.4}$$

with $\lambda_0 = -\infty$ and $\lambda_{n+1} = +\infty$.

The set $\mathscr{V}(\bar{u}_0)$ is exactly equal to the set $\mathscr{E}(\bar{u}_0)$ introduced in Section 1. Hence, we recover the well-known formulation (Kreiss [10]).

2.3. *Study of Nonconvex Scalar Conservation Laws*

In this section, we take $n=1$ and an arbitrary (nonconvex) $\mathscr{C}^1$-class flux function $f\colon \mathbb{R} \to \mathbb{R}$. Hence, the hypotheses are more general than the usual hypotheses for a system: the (unique) field is not globally genuinely nonlinear or linearly degenerated. But, following Leroux [18] and Ballou [1], the resolution of the Riemann problem is explicit. The solution $w(\xi; u_{\mathrm{L}}, u_R)$ of a Riemann problem $R(u_{\mathrm{L}}, u_{\mathrm{R}})$ is composed of the two constant states u_{L} and u_{R} separated by a succession of rarefaction waves and shock waves. These waves are obtained by means of the convex (resp. concave) hull of the flux f on the interval $[u_{\mathrm{L}}, u_{\mathrm{R}}]$ if $u_{\mathrm{L}} < u_{\mathrm{R}}$ (resp. $[u_{\mathrm{R}}, u_{\mathrm{L}}]$ if $u_{\mathrm{R}} < u_{\mathrm{L}}$). For details, see [18].

In Section 1.3, we introduced a set $\mathscr{E}(\bar{u}_0)$ for each $\bar{u}_0 \in \mathbb{R}$. Now, we prove that the two sets $\mathscr{V}(\bar{u}_0)$ and $\mathscr{E}(\bar{u}_0)$ are equal.

PROPOSITION 2.2. $\forall \bar{u}_0 \in \mathbb{R}, \mathscr{V}(\bar{u}_0) = \mathscr{E}(\bar{u}_0)$.

LEMMA 2.2. *Let $h\colon [0,1] \to \mathbb{R}$. Consider $\tilde{h}\colon [0,1] \to \mathbb{R}$ its convex hull. Let u_* be the greatest point realizing the minimum of $\tilde{h}$,*

$$u_* = \sup\{u \mid \forall v, \tilde{h}(u) \leqslant \tilde{h}(v)\}.$$

Then we have

$$\tilde{h}(u_*) = h(u_*).$$

Refer to Rockafellar's classical text on convex functions [21].

Proof of Proposition 2.2. Recall that the set $\mathscr{E}(\bar{u}_0)$ is described in Section 1.3 by means of a function g defined from the state $\bar{u}_0 \in \mathbb{R}$ (see (1.15)–(1.16)),

$$\mathscr{E}(\bar{u}_0) = \operatorname{Im} g \cup f^{-1}(\mathscr{D}).$$

First, let u_* be in $\mathscr{E}(\bar{u}_0)$. Then geometrically the Riemann problem $R(\bar{u}_0; u_*)$ contains only waves with nonpositive velocities (Fig. 1.1),

$$w(\xi; \bar{u}_0, u_*) = u_*, \qquad \forall \xi > 0.$$

Thus, $u_* = w(0+; \bar{u}_0, u_*)$. So, u_* belongs to $\mathscr{V}(\bar{u}_0)$.

Second, let u_* be in $\mathscr{V}(\bar{u}_0)$: $u_* = w(0+; \bar{u}_0, u_R)$ with $u_R > \bar{u}_0$ to fix the ideas. Using the explicit resolution of $R(\bar{u}_0, u_R)$, the state u_* is exactly the greatest point realizing the minimum value of the convex hull $\tilde{f}$ of f on $[\bar{u}_0, u_R]$. For k in $[\bar{u}_0, u_*]$, we clearly have

$$\tilde{f}(u_*) \leqslant \tilde{f}(k) \leqslant f(k).$$

Moreover, by Lemma 2.2, we have $f(u_*) = \tilde{f}(u_*)$. Thus, we get $f(u_*) \leqslant f(k)$. So (1.14) holds, which is equivalent to $u_* \in \mathscr{E}(\bar{u}_0)$ (Proposition 1.2). ∎

2.4. *Local Nonlinear Structure for a General System*

We turn back to the general situation of Section 2.1. Recall that by hypothesis, the set of states $\mathscr{U} \subset \mathbb{R}^n$ is chosen such that the Riemann problem with data in $\mathscr{U}$ is well-posed. The following theorem describes the local nonlinear structure of $\mathscr{V}(\bar{u}_0)$, when all the eigenvalues of the state $\bar{u}_0 \in \mathscr{U}$ are not equal to zero or when the null eigenvalue corresponds to a genuinely nonlinear field.

THEOREM 2.2. *For any state $\bar{u}_0$ in $\mathscr{U}$, let p be the index of the greatest nonpositive eigenvalue at the point $\bar{u}_0$,*

$$\lambda_1(\bar{u}_0) < \cdots < \lambda_p(\bar{u}_0) \leqslant 0 < \lambda_{p+1}(\bar{u}_0) < \cdots < \lambda_n(\bar{u}_0). \tag{2.5}$$

(1) *If $\lambda_p(\bar{u}_0) < 0$, then the set $\mathscr{V}(\bar{u}_0)$ is, locally in the neighbourhood of $\bar{u}_0$, a manifold $\mathscr{V}_p(\bar{u}_0)$ with dimension p whose tangent vector space is generated by the p first eigenvectors $r_1(\bar{u}_0)$, $r_2(\bar{u}_0)$, ..., $r_p(\bar{u}_0)$.*

(2) If $\lambda_p(\bar{u}_0) = 0$ *and* if the p-field is genuinely nonlinear, *then the set $\mathscr{V}(\bar{u}_0)$ is locally around $\bar{u}_0$ the union of $\mathscr{V}'_p(\bar{u}_0)$ and $\mathscr{V}''_p(\bar{u}_0)$ with dimension p and a manifold $\mathscr{V}_{p-1}(\bar{u}_0)$ defined inductively as*

$$\begin{aligned} &\mathscr{V}_1(\bar{u}_0) = \mathscr{W}_1(\bar{u}_0) \\ &\mathscr{V}_k(\bar{u}_0) = \{u \mid \exists u' \in \mathscr{V}_{k-1}(\bar{u}_0), u \in \mathscr{W}_k(u')\} \end{aligned} \tag{2.6}$$

for $k = 2, ..., n$; and

$$\mathscr{V}'_p(\bar{u}_0) = \{u \mid \exists u' \in \mathscr{V}_{p-1}(\bar{u}_0), u \in \mathscr{S}_p(u') \quad \text{and} \quad \sigma_p(u', u) \leqslant 0\} \tag{2.7a}$$

$$\mathscr{V}''_p(\bar{u}_0) = \{u \mid \exists u' \in \mathscr{V}_{p-1}(\bar{u}_0), u \in \mathscr{R}_p(u') \quad \text{and} \quad \lambda_p(u) \leqslant 0\} \tag{2.7b}$$

($\sigma_p(u', u)$ is the speed of the p-shock between u' and u). (See Fig. 2.1.)

Proof of Theorem 2.2. (1) *When* $\lambda_p(\bar{u}_0) < 0$, we restrict ourselves to a neighbourhood $\mathscr{N}$ of the state $\bar{u}_0$ such that

$$\lambda_1(u) < \cdots < \lambda_p(u) < 0 < \lambda_{p+1}(u) < \cdots < \lambda_n(u). \tag{2.8}$$

The set $\mathscr{N}$ can be parameterized by a neighbourhood $\mathscr{N}_0$ of zero in $\mathbb{R}^n$,

$$\mathscr{N}_0 \ni \varepsilon = (\varepsilon_1, ..., \varepsilon_n) \mapsto u = \phi(\varepsilon_1, ..., \varepsilon_n) \in \mathscr{N},$$

with

$$\phi(\varepsilon_1, ..., \varepsilon_n) = W_n(\varepsilon_n; W_{n-1}(\varepsilon_{n-1}; \cdots); W_1(\varepsilon_1; \bar{u}_0)) \cdots),$$

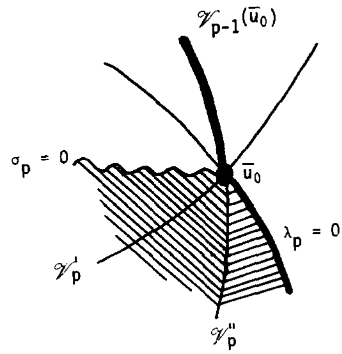

FIGURE 2.1

where $\varepsilon_i \mapsto W_i(\varepsilon_i; v)$ is a parameterization of the i-curve $\mathscr{W}_i(v)$ issuing from the state v. If the i-characteristic field is genuinely nonlinear, the part $\varepsilon_i > 0$ (resp. $\varepsilon_i < 0$) corresponds to the curve $\mathscr{R}_i(v)$ (resp. $\mathscr{S}_i(v)$). We have the properties (Lax [15])

$$\begin{aligned} \phi(0) &= \bar{u}_0 \\ D\phi(0) &= (r_1(\bar{u}_0), ..., r_n(\bar{u}_0)). \end{aligned} \tag{2.9}$$

According to (2.8), all the velocities of the i-waves are negative for $i \leqslant p$ and positive for $i > p$. Thus we get (see Fig. 2.2)

$$w(0+; \bar{u}_0, \phi(\varepsilon_1, ..., \varepsilon_n)) = \phi(\varepsilon_1, ..., \varepsilon_p, 0, ..., 0), \tag{2.10}$$

which implies that

$$\mathscr{V}(\bar{u}_0) \cap \mathscr{N} = \mathscr{V}_p(\bar{u}_0) \cap \mathscr{N}.$$

The local structure of $\mathscr{V}(\bar{u}_0)$ results from (2.9).

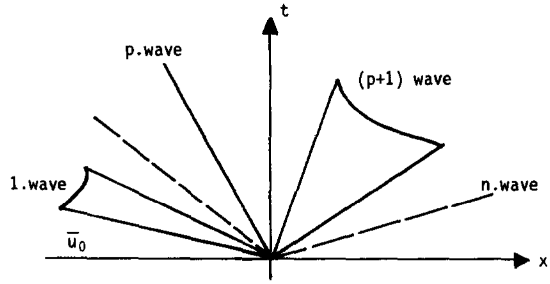

FIGURE 2.2

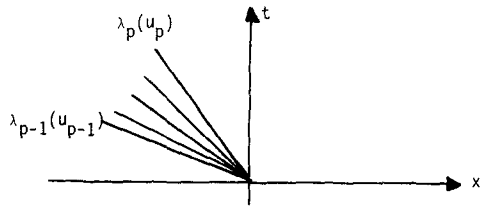

FIGURE 2.3

(2) *When* $\lambda_p(\bar{u}_0)=0$ *and the p-field is genuinely nonlinear*, the set $\mathscr{V}(\bar{u}_0)\cap\mathscr{N}$ contains $\mathscr{V}_{p-1}(\bar{u}_0)\cap\mathscr{N}$, as previously. Moreover, take $u_{p-1}=\phi(\varepsilon_1, ..., \varepsilon_{p-1}, 0, ..., 0)$. The state $w(0+;\bar{u}_0, \phi(\varepsilon_1, ..., \varepsilon_n))$ depends only on the relative location between the p-wave issued of u_{p-1} and the axis $\xi=x/t=0$. We distinguish between two cases:

(a) *Either the p-wave is a rarefaction*, that is, $\varepsilon_p \geqslant 0$.

Necessarily, the points of $\mathscr{V}(\bar{u}_0)$ which are not in $\mathscr{V}_{p-1}(\bar{u}_0)$ are obtained for states u_{p-1} such that $\lambda_p(\phi(\varepsilon_1, ..., \varepsilon_{p-1}, 0, ..., 0))\leqslant 0$.

- Either $u_p=\phi(\varepsilon_1, ..., \varepsilon_p, 0, ..., 0)$ satisfies $\lambda_p(u_p)\leqslant 0$ (Fig. 2.3); then (2.10) holds and u_p describes a part of $\mathscr{V}_p(\bar{u}_0)$.
- Or u_p satisfies $\lambda_p(u_p)>0$ and $w(0+;\bar{u}_0, \phi(\varepsilon_1, ...,\varepsilon_n))$ is a state where λ_p is equal to 0 (Fig. 2.4).

The tangent spaces limiting $\mathscr{V}(\bar{u}_0)$ in case (a) consist of the space generated by $r_1(\bar{u}_0), ..., r_{p-1}(\bar{u}_0)$ and the intersection between the space generated by $r_1(\bar{u}_0), ..., r_p(\bar{u}_0)$ and the hyperplan tangent to the manifold $\{v \mid \lambda_p(v)=0\}$. Locally, the position of $\mathscr{V}(\bar{u}_0)$ is given by the two constraints

$$\varepsilon_p \geqslant 0, \qquad \lambda_p(\phi(\varepsilon_1, ..., \varepsilon_p, 0, ..., 0))\leqslant 0. \tag{2.11}$$

(b) *Or the p-wave is a shock wave*, that is, $\varepsilon_p<0$.

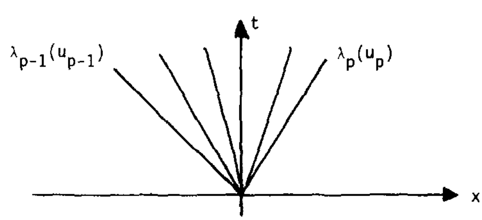

FIGURE 2.4

The "new" states of $\mathscr{V}(\bar{u}_0)$ are obtained for a nonpositive velocity $\sigma_p(u_{p-1}, u_p)$ of the p-shock. (see Fig. 2.5). The tangent spaces limiting $\mathscr{V}(\bar{u}_0)$ in case (b) are the same as previously except that the set $\{v \mid \lambda_p(v)=0\}$ becomes $\{u_p=\phi(\varepsilon_1, ..., \varepsilon_p, 0, ..., 0)/\sigma_p(u_{p-1}, u_p)=0\}$. Thus, $\mathscr{V}(\bar{u}_0)$ satisfies the two constraints

$$\varepsilon_p \leqslant 0, \qquad \sigma_p(u_{p-1}, u_p) \leqslant 0. \tag{2.12}$$

We can determine the equations of limiting spaces. On the one hand, we get

$$\lambda_p(\phi(\varepsilon_1, ..., \varepsilon_p, 0, ..., 0)) = \sum_{j=1}^{p-1} \nabla\lambda_p(\bar{u}_0) \cdot r_j(\bar{u}_0) \cdot \varepsilon_j + \varepsilon_p + O(\varepsilon^2),$$

by (2.9). And the equation is

$$\varepsilon_p + \sum_{j=1}^{p-1} \nabla\lambda_p(\bar{u}_0) \cdot r_j(\bar{u}_0) \cdot \varepsilon_j = 0. \tag{2.13}$$

On the other hand, we get

$$\begin{aligned}\sigma_p(u_{p-1}, u_p) &= \frac{1}{2}(\lambda_p(u_{p-1}) + \lambda_p(u_p)) + O(\varepsilon^2)\\ &= \frac{1}{2}\varepsilon_p + \sum_{j=1}^{p-1} \nabla\lambda_p(\bar{u}_0) \cdot r_j(\bar{u}_0) \cdot \varepsilon_j + O(\varepsilon^2).\end{aligned}$$

And the corresponding equation is

$$\frac{\varepsilon_p}{2} + \sum_{j=1}^{p-1} \nabla\lambda_p(\bar{u}_0) \cdot r_j(\bar{u}_0) \cdot \varepsilon_j = 0. \quad \blacksquare \tag{2.14}$$

Remark 2.1. (1) Note that (2.13) and (2.14) give the equations of tangent spaces of the limiting manifolds of $\mathscr{V}'_p$ and $\mathscr{V}''_p$: they generally differ.

(2) The case $\lambda_p(\bar{u}_0) = 0$, with the p-field linearly degenerated, is more complicated.

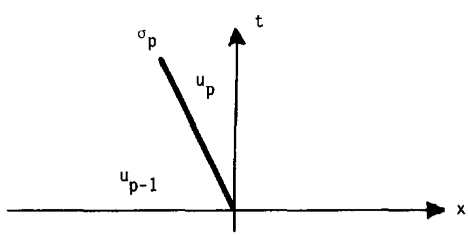

FIGURE 2.5

2.5. Study of the p-system

Let us now consider the example of the p-system of gas dynamics (v, specific volume; u, velocity),

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ u \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} -u \\ p(v) \end{pmatrix} = 0, \qquad v > 0, u \in \mathbb{R}, \tag{2.15}$$

with a pressure p satisfying $p > 0$, $p' < 0$, and $p'' > 0$. The eigenvalues are $\lambda_1 = -\lambda_2 = \sqrt{-p'(v)}$. Because those eigenvalues always keep their own sign,

$$\lambda_1(v, u) < 0 < \lambda_2(v, u), \qquad \forall (v, u), \tag{2.16}$$

a 1-wave (resp. a 2-wave) always has a negative (resp. positive) speed(s). Immediately we get:

PROPOSITION 2.3. $\forall (\bar{v}_0, \bar{u}_0)$, $\mathscr{V}(\bar{v}_0, \bar{u}_0) = \mathscr{W}_1(\bar{v}_0, \bar{u}_0)$.

In this example, the set $\mathscr{V}(\bar{v}_0, \bar{u}_0)$ is globally a manifold of dimension one.

2.6. Isentropic Euler System of Gas Dynamics

The modelling of the isentropic evolution of a polytropic perfect gas leads to the Euler (2×2)-system. The unknowns are the density $\rho(x, t) > 0$ and the momentum $q(x, t)$,

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x} q = 0 \tag{2.17a}$$

$$\frac{\partial}{\partial t} q + \frac{\partial}{\partial x}\left(\frac{q^2}{\rho} + p(\rho)\right) = 0. \tag{2.17b}$$

The pressure $p(\rho)$ is given by the equation of state

$$p(\rho) = k \cdot \rho^\gamma, \qquad k > 0, 1 \leqslant \gamma < 3. \tag{2.18}$$

Taking

$$U = \begin{pmatrix} \rho \\ q \end{pmatrix} \qquad \text{and} \qquad F(U) = \begin{pmatrix} q \\ q^2/\rho + p(\rho) \end{pmatrix},$$

we get

$$A(U) \equiv DF(U) = \begin{pmatrix} 0 & 1 \\ -u^2 + c^2 & 2u \end{pmatrix},$$

where $u = q/\rho$ is the velocity and $c = \sqrt{p'(\rho)}$ is the sound velocity. The eigenvalues of the matrix $A(U)$ are

$$\lambda_1(U) = u - c < \lambda_2(U) = u + c.$$

Two corresponding eigenvectors are

$$r_1(U)=\begin{pmatrix}1\\ u-c\end{pmatrix}, \qquad r_2(U)=\begin{pmatrix}1\\ u+c\end{pmatrix}.$$

The two characteristic fields are both genuinely nonlinear.

The rarefaction curves containing $\bar{U}_0={}^t(\bar{\rho}_0, \bar{q}_0)$ (whose corresponding speed and sound velocity are denoted by $\bar{u}_0$ and $\bar{c}_0$) admit the equations

$$\mathscr{R}_1(\bar{U}_0)\colon q=\rho\left(\bar{u}_0+\frac{2}{\gamma-1}\bar{c}_0-\frac{2}{\gamma-1}c(\rho)\right), \qquad \rho<\bar{\rho}_0 \tag{2.19.a}$$

$$\mathscr{R}_2(\bar{U}_0)\colon q=\rho\left(\bar{u}_0-\frac{2\bar{c}_0}{\gamma-1}+\frac{2}{\gamma-1}c(\rho)\right), \qquad \rho>\bar{\rho}_0. \tag{2.19.b}$$

The shock curves are computed by means of the Rankine–Hugoniot relations,

$$\mathscr{S}_1(\bar{U}_0)\colon q=\rho\cdots\left(\bar{u}_0-\sqrt{\left(p(\rho)-\bar{p}_0\right)\left(\frac{1}{\bar{\rho}_0}-\frac{1}{\rho}\right)}\right), \qquad \rho>\bar{\rho}_0 \tag{2.20.a}$$

$$\mathscr{S}_2(\bar{U}_0)\colon q=\rho\left(\bar{u}_0-\sqrt{\left(p(\rho)-\bar{p}_0\right)\left(\frac{1}{\bar{\rho}_0}-\frac{1}{\rho}\right)}\right), \qquad \rho<\bar{\rho}_0. \tag{2.20.b}$$

Those curves are represented in Fig. 2.6. We have also drawn the curves $\{\lambda_1(\rho, q)=0\}$, $\{\lambda_2(\rho, q)=0\}$, which limit the three open disjoined sets:

$$\mathscr{A}\colon 0<\lambda_1(\rho, q)<\lambda_2(\rho, q) \tag{2.21.a}$$

$$\mathscr{B}\colon \lambda_1(\rho, q)<0<\lambda_2(\rho, q) \tag{2.21.b}$$

$$\mathscr{C}\colon \lambda_1(\rho, q)<\lambda_2(\rho, q)<0. \tag{2.21.c}$$

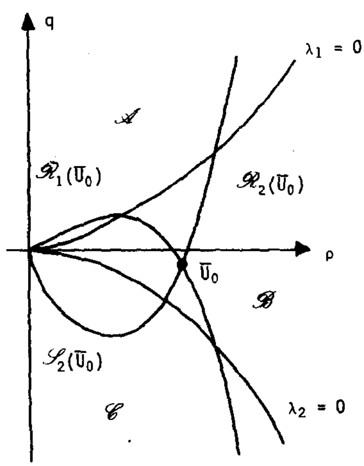

FIGURE 2.6

We now compute the sets $\mathscr{V}(\bar{u}_0)$. Since several cases can occur (Figs. 2.7a–2.7d) the details of the construction of Fig. 2.7b are now given.

Computation of the Set $\mathscr{V}(\bar{U}_0)$. If we consider $\bar{U}_0$ in $\mathscr{B}$, let U be in $\mathscr{V}(\bar{u}_0)$. Then the Riemann problem $R(\bar{U}_0, U)$ has all its waves in the part $\xi \leqslant 0$ of the (x, t) plane. We distinguish between two cases, with respect to the number of waves in this Riemann problem.

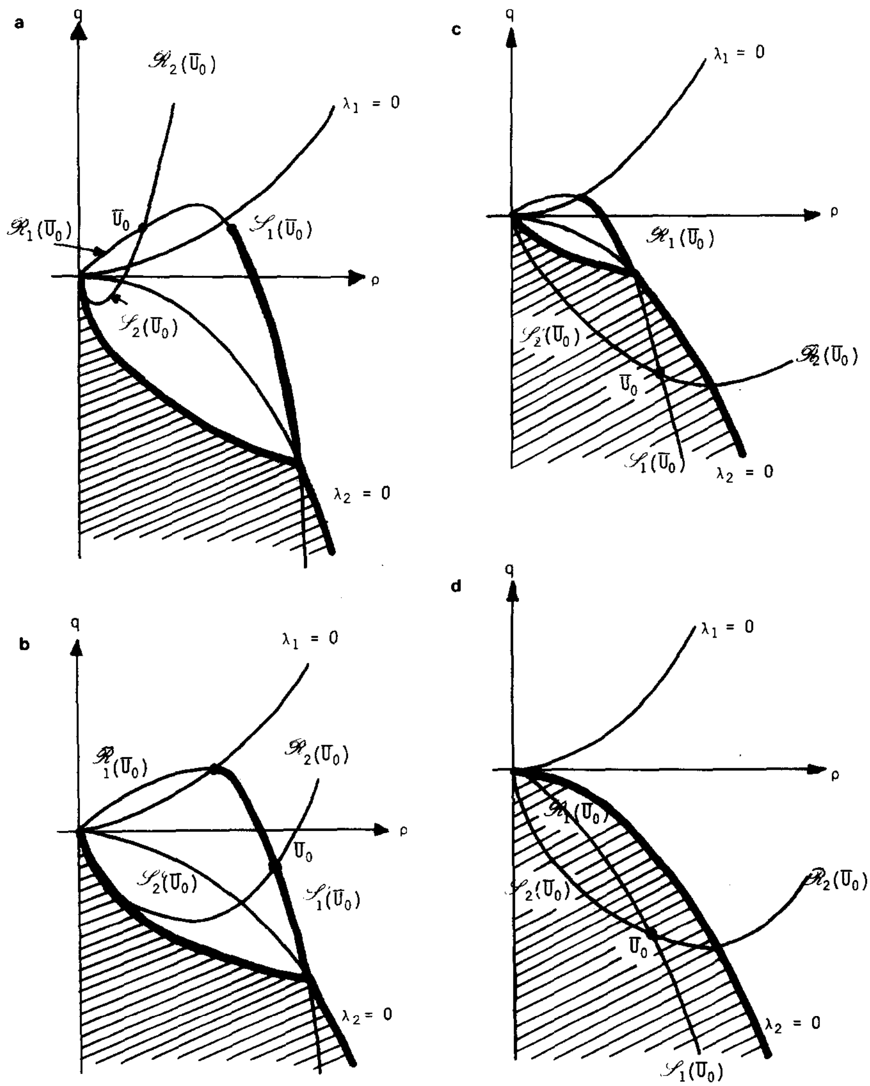

FIGURE 2.7

(1) $U \in \mathscr{W}_1(\bar{U}_0) \cup \mathscr{W}_2(\bar{U}_0) \equiv \mathscr{S}_1(\bar{U}_0) \cup \mathscr{R}_1(\bar{U}_0) \cup \mathscr{S}_2(\bar{U}_0) \cup \mathscr{R}_2(\bar{U}_0)$.

— All the values in $\mathscr{R}_1(\bar{U}_0)$ are admissible because the speed $\sigma_1(\bar{U}_0, U)$ of the 1-shock is always negative.

— The admissible part of $\mathscr{S}_1(\bar{U}_0)$ is exactly $\mathscr{R}_1(\bar{U}_0) \cap \text{adh } (\mathscr{B})$ because if $U \in \mathscr{R}_1(\bar{U}_0) \cap \mathscr{A}$, $\lambda_1(U) > 0$, and U is not the stationary value $w(0+; \bar{u}_0, u)$ of the corresponding Riemann problem.

— When U belongs to $\mathscr{S}_2(\bar{U}_0)$, the shock speed $\sigma_2(\bar{U}_0, U)$ must be negative; we obtain a branch of $\mathscr{S}_2(\bar{U}_0)$ which is not in the vicinity of $\bar{U}_0$.

— No value in $\mathscr{R}_2(\bar{U}_0)$ is admissible because if $U \in \mathscr{R}_2(\bar{U}_0)$, $\lambda_2(U) > \lambda_2(\bar{U}_0) > 0$ (cf. Fig. 2.8)

(2) *The Riemann problem $R(\bar{U}_0, U)$ contains an intermediate state U^*.* Necessarily, U^* belongs to a 1-wave in the set described in case (1). (The 1-wave must have only negative velocities.) We distinguish essentially between two cases:

(i) $U^* \in \mathscr{W}_1(\bar{U}_0) \cap \mathscr{C}$. The speed $\sigma_2(U^*, U)$ of a 2-shock-wave issuing from U^* is always negative; hence each U in $\mathscr{S}_2(U^*)$ is admissible. Furthermore a 2-rarefaction issuing from U^* has only negative velocities if and only if U remains in adh $\mathscr{C}$. These two cases are displayed in Fig. 2.9.

(ii) $U^* \in \mathscr{W}_1(\bar{U}_0) \cap \text{adh } \mathscr{B}$. The speed $\sigma_2(U^*, U)$ of a 2-shock issuing from U^* is negative if and only if the momentum q^* of U^* is negative *and* q (momentum of U) is greater than q^*. This leads to the shaded set in Fig. 2.10, limited on the one hand by the curve $\mathscr{S}_2(V_0)$

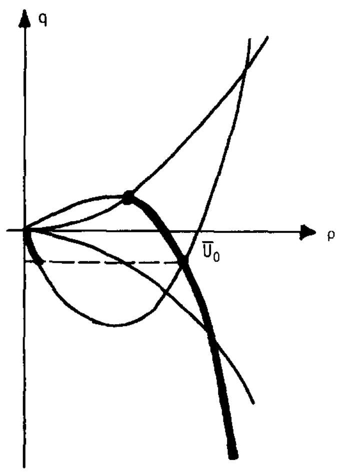

FIGURE 2.8

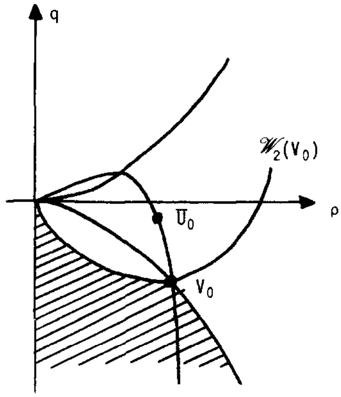

FIGURE 2.9

$(V_0 = \mathscr{W}_1(\bar{U}_0) \cap \{\lambda_2 = 0\})$ and on the other hand by a curve $\mathscr{J}(\bar{U}_0)$ which consists of all the states U^{**} issuing from the states U^* (on $\mathscr{W}_1(\bar{U}_0) \cap \{-c^* \leqslant u^* \leqslant 0\}$) by a 2-shock wave with zero velocity. The curve $\mathscr{J}(\bar{u}_0)$ is parameterized by ρ^* with the implicit equations

$$\begin{gathered} q^{**} = \rho^* u^* \\ \sqrt{\left(p(\rho^{**}) - p(\rho^*)\right) \Big/ \left(\frac{1}{\rho^*} - \frac{1}{\rho^{**}}\right)} = -\rho^* u^* \end{gathered} \tag{2.22}$$

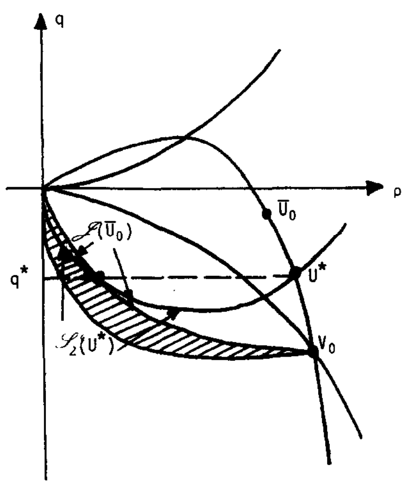

FIGURE 2.10

with

$$u^* = \begin{cases} \bar{u}_0 - \sqrt{\left(p(\rho^*) - p(\bar{\rho}_0)\right) \Big/ \left(\dfrac{1}{\bar{\rho}_0} - \dfrac{1}{\rho^*}\right)} & \text{if} \quad \rho^* > \bar{\rho}_0 \\ \bar{u}_0 + \dfrac{2\bar{c}_0}{\gamma - 1} - \dfrac{2c(\rho^*)}{\gamma - 1} & \text{if} \quad \rho^* \leqslant \bar{\rho}_0 \end{cases} \tag{2.23}$$

and $-c(\rho^*) \leqslant u^* \leqslant 0$. The 2-rarefaction $\mathscr{R}_2(U^*)$ is not admissible, as in case (1) ▮

Figures 2.7a, 2.7c, and 2.7d are obtained by similar computations. We now give a qualitative description of those figures:

- *Figure* 2.7a. $\bar{U}_0 \in \mathscr{A}$ ($\bar{u}_0 > \bar{c}_0$: *supersonic inlet*). We remark that $\mathscr{V}(\bar{U}_0) \cap \mathscr{A} = \{\bar{U}_0\}$, which is compatible with Theorem 2.2. Note that $\mathscr{V}(\bar{U}_0) \cap \mathscr{B}$ is a curve and that $\mathscr{V}(\bar{U}_0) \cap \mathscr{C}$ is locally of dimension 2.

- *Figure* 2.7b. $\bar{U}_0 \in \mathscr{B}(-\bar{c}_0 \leqslant \bar{U}_0 \leqslant \bar{c}_0$: *subsonic inlet or outlet*). $\mathscr{V}(\bar{U}_0) \cap \mathscr{A}$ is void, and we have seen previously that $\mathscr{V}(\bar{U}_0) \cap \mathscr{B}$ is a branch of the curve $\mathscr{W}_1(\bar{U}_0)$.

- *Figures* 2.7c, 2.7d. $\bar{U}_0 \in \mathscr{C}$ ($\bar{u}_0 < -\bar{c}_0$: *supersonic outlet*). $\mathscr{V}(\bar{U}_0) \cap \mathscr{A}$ is void and two subcases can occur:

— *Figure* 2.7c. $-(2/(\gamma-1))\,\bar{c}_0 < \bar{u}_0 < -\bar{c}_0$. $\mathscr{V}(\bar{U}_0) \cap \mathscr{B}$ is a curve and $\mathscr{V}(\bar{U}_0) \cap \mathscr{C}$ is as shown previously.

— *Figure* 2.7d. $\bar{u}_0 < -(2/(\gamma-1))\,\bar{c}_0$. $\mathscr{V}(\bar{U}_0)$ is exactly the adherence of $\mathscr{C}$.

The interest in this study lies in the very strong nonlinear behaviour of the boundary condition. (Compare with previous works of Veuillot and Viviand [24], for example.)

3. Conjecture, Study of an Example

In this paper, we have proposed two approaches to the formulation of the boundary condition for nonlinear systems of conservation laws. These two formulations are equivalent and lead to well-posed initial-boundary value problems for linear strictly hyperbolic systems and (nonconvex) scalar conservation laws. Concerning general nonlinear systems, the boundary entropy inequality (first formulation) is difficult to use by lack of entropy-flux pairs. But, with Riemann problems (second formulation) many interesting physical examples can be treated explicitly.

We conjecture that the two approaches lead to the same boundary condition, that is, $\mathscr{V} = \mathscr{E}$.

To conclude, we verify this conjecture for a nonlinear (2×2)-system whose characteristic fields are both linearly degenerated (considered by Serre [22]),

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} &= 0 \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x}\left(\frac{v(v+1)}{u}\right) &= 0. \end{aligned} \tag{3.1}$$

This system is strictly hyperbolic in the domain $\{(u, v) \mid u > 0\}$. The eigenvalues are $\lambda_1(u, v) = v/u < (v+1)/u = \lambda_2(u, v)$.

We know [22] all the entropy-flux pairs

$$\begin{aligned} \eta(u, v) &= u \cdot \left(a\left(\frac{v}{u}\right) + b\left(\frac{v+1}{u}\right)\right) \\ q(u, v) &= (v+1) \cdot a\left(\frac{v}{u}\right) + v \cdot b\left(\frac{v+1}{u}\right), \end{aligned} \tag{3.2}$$

where a and b are convex functions. For a state $(\bar{u}_0, \bar{v}_0)$ fixed, the boundary entropy inequality (1.13) can be written

$$(v+1) \cdot a\left(\frac{v}{u}\right) + v \cdot b\left(\frac{v+1}{u}\right) \leqslant 0, \tag{3.3}$$

with $a'(\bar{v}_0/\bar{u}_0) = b'((\bar{v}_0 + 1)/\bar{u}_0) = 0$ (without restriction). From (3.3), we deduce easily the set $\mathscr{E}(\bar{u}_0, \bar{v}_0)$.

PROPOSITION 3.1. *Let* $\mathscr{A} = \{(u, v) \mid v \leqslant -1, u > 0\}$.

- *If* $\bar{v}_0 \leqslant 0$,

$$\mathscr{E}(\bar{u}_0, \bar{v}_0) = \mathscr{A} \cup \{(\lambda \cdot \bar{u}_0, \lambda \cdot \bar{v}_0) \mid \lambda > 0\}.$$

- *If* $\bar{v}_0 > 0$

$$\mathscr{E}(\bar{u}_0, \bar{v}_0) = \mathscr{A} \cup \{(\bar{u}_0, \bar{v}_0)\}.$$

Now, following [22], we must reduce the domain of study for the unknowns, say $\mathscr{U}_{\alpha,\beta}$ with $\alpha < \beta$,

$$\mathscr{U}_{\alpha\beta} = \{(u, v) \mid u > 0, \lambda_1(u, v) \leqslant \alpha < \beta \leqslant \lambda_2(u, v)\},$$

to be able to solve the Riemann problem with data in $\mathscr{U}_{\alpha\beta}$. A straightforward computation leads to the equivalence of the two boundary conditions:

PROPOSITION 3.2. *For the system* (3.1) *with unknown in* $\mathscr{U}_{\alpha,\beta}$, *we get*

$$\mathscr{V}(\bar{u}_0, \bar{v}_0) = \mathscr{E}(\bar{u}_0, \bar{v}_0) \cap \mathscr{U}_{\alpha,\beta}.$$

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