Giuseppe Da Prato

Introduction to Stochastic Analysis and Malliavin Calculus

EDIZIONI DELLA NORMALE

13

APPUNTI

LECTURE NOTES

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Contents

Introduction

This book collects some lecture notes from a one year course delivered in the past twenty years in Scuola Normale on differential stochastic equations and Malliavin calculus.

Basic elements on Gaussian measures in infinite dimensional separable Hilbert spaces and Gaussian random variables are gathered in Chapters 1 and 2. Moreover, in the last part of Chapter 2 we introduce the *white noise function* W which will play an important role later in the definition of Brownian motion. Given a non degenerate Gaussian measure $\mu = N_Q$ of mean 0 and covariance Q in H, we consider the subspace $Q^{1/2}(H)$, called the *Cameron–Martin space* of μ, which is dense in H but different from H. Then we define

$$W : Q^{1/2}(H) \subset H \to L^2(H, \mu),\ \ f \to W_f,$$

with

$$W_f(x) = \langle x, Q^{-1/2} f\rangle, \quad x \in H.$$

It is easy to see that W is an isometry of H into $L^2(H, \mu)$ so that it is uniquely extendible to the whole H.

A basic property of W is the following. Given n vectors of H, $f_1, \ldots, f_n$, the random variable with values in $\mathbb{R}^n$, $(W_{f_1}, \ldots, W_{f_n})$ is Gaussian with mean 0 and covariance operator given by the matrix,

$$(Q_{f_1,\ldots,f_n})_{i,j} = \langle f_i, f_j\rangle, \quad i, j = 1, \ldots, n.$$

Consequently W transforms a sequence $f_1, \ldots, f_n$ of orthogonal elements of H into a sequence $W_{f_1}, \ldots, W_{f_n}$ of independent random variables.

Chapter 3 deals with the Malliavin derivative $M = Q^{1/2}D$ (where D denotes the gradient), defined on a space of smooth functions (exponential functions). Using a classical integration by parts formula, we show that M is closable in $L^2(H, \mu)$. The domain of its closure, still denoted by M, is the *Malliavin-Sobolev space* $D^{1,2}(H, \mu)$. The space $D^{1,2}(H, \mu)$

is larger than the usual Sobolev space $W^{1,2}(H,\mu)$, an important property with far reaching consequences being that it includes the white noise function W_f for any $f \in H$.

In the second part of this chapter we study several properties of the adjoint operator M^*, also called *Gaussian divergence* or *Skorohod* operator.

Chapter 4 is devoted to a construction of the Brownian motion. We consider the space $H = L^2(0,T)$, where $T > 0$ is fixed, and take any non degenerate Gaussian measure $\mu = N_Q$ in H. Then we define

$$B(t) = W_{\mathbb{1}_{[0,t]}}, \quad t \geq 0,$$

where $\mathbb{1}_{[0,t]}$ is the characteristic function of $[0,t]$. Using the aforementioned property of the white noise function, one checks easily the following properties of $B(t)$. For each $t \geq 0$, $B(t)$ is a Gaussian, real, random variable $N_{0,t}$. Moreover, if $0 < t_1 < \cdots < t_n$, the random variables

$$B(t_1),\ B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1}),$$

are mutually independent. In other words, $B(t)$ is a process with *independent increments*. Finally, we show that $B(t)$ is a continuous process using the factorization method, see [6].

Next, we prove that the trajectories of $B(t)$ are not of bounded variation with probability one. So we cannot define the integral

$$I(f) := \int_0^T f(t)dB(t)$$

(where $f : [0,T] \to \mathbb{R}$ is for instance continuous) in the sense of Stieltjes. However, we show that the Riemannnian sums of f are convengent in $L^2(H,\mu)$ to a random variable, which we call the *Wiener integral* of f. Moreover, we identify $I(f)$ with the white noise W_f.

Chapter 5 is devoted to some sharp properties of the Brownian motion as: Markov and strong Markov properties and reflexion principle. We also introduce the transition semigroup

$$P_t\varphi(x) = \mathbb{E}[\varphi(B(t) + x)], \quad t \geq 0,\ x \in \mathbb{R},$$

for any real, bounded and Borel function φ. We show its connection with the heat equation both in the real line and in the half-line. Then we consider Dirichlet, Neuman and Ventzell problems in the half-line.

The chapter ends with a final section on some remarkable properties of the set of zeros of the Brownian motion.

In Chapter 6 we consider a Brownian motion $B(t), t \geq 0$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and introduce the Itô integral,

$$\int_0^T F(t)dB(t)$$

for suitable processes $F(t),\ t \in [0, T]$ (not necessarily deterministic as in the case of Wiener's integral). The basic requirement on F is that it is *adapted* to B, that is $F(t)$ is measurable with respect to the σ-algebra $\mathscr{F}_t$ generated by $\{B(s) :\ s \in [0, t]\}$ called the *natural filtration of* B. For the sake of simplicity we shall work with the natural filtration of B in all the book, except in a section of Chapter 12, where a larger filtration is considered.

We first define Itô's integral for *elementary processes*, that is adapted processes of the form

$$F = \sum_{i=1}^{n} F_{t_{i-1}} \mathbb{1}_{[t_{i-1},t]},$$

setting

$$\int_0^T F(t)dB(t) = \sum_{i=1}^{n} F_{t_{i-1}}(B(t_i) - B(t_{i-1}))$$

Then we prove the identity,

$$\mathbb{E}\left(\left|\int_0^T F(t)dB(t)\right|^2\right) = \int_0^T \mathbb{E}(|F(t)|^2)dt.$$

(where $\mathbb{E}$ represents the expectation). This identity allows us to extend the Itô integral to the closure of the space of elementary processes in $L^2((0, T) \times \Omega)$. The elements of this closure are called (square integrable) *predictable* processes.

Then we consider Itô's integrals with stopping times and for almost surely square integrable functions.

Finally we study the mapping $t \to \int_0^t F(s)dB(s)$, showing that it is a continuous *martingale*.

Chapter 7 is devoted to Itô's formula. Given a real function φ continuous with its first and second derivatives, we show that

$$\varphi(B(t) + x) = \varphi(x) + \frac{1}{2} \int_0^t \varphi''(B(s) + x)ds + \int_0^t \varphi'(B(s) + x)dB(s),$$

or in differential forms

$$d\varphi(B(t)+x) = \varphi'(B(t)+x)dB(t) + \frac{1}{2}\,\varphi''(B(t)+x)dt.$$

Note that this formula (called *Itô's formula*) contains an extra term with respect the usual calculus rules.

Then we discuss several generalizations of Itô's formula to functions with more variables and for more general processes.

In Chapter 8 we study a *stochastic differential equation* of the form

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s) \tag{1}$$

written also in differential form as

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), \quad t \in [s, T], \\ \\ X(s) = x, \end{cases}$$

where $B(t)$ is an r-dimensional Brownian motion, $b : [0, T] \times \mathbb{R}^d$, $(t,x) \to b(t,x)$ and $\sigma : [0, T] \times \mathbb{R}^d \to L(\mathbb{R}^d; \mathbb{R}^m)$, $(t, x) \to \sigma(t, x)$ are continuous in (t, x) and Lipschitz continuous in x uniformly in time.

We prove existence and uniqueness of a solution $X(t, s, x) \in \mathbb{R}^d$ of (1) and the continuous dependence in mean square of $X(t, s, x)$ with respect to t, s, x.

Moreover, under suitable additional regularity properties of b and σ, we prove the *backward Itô's formula* (which is used to get Itô's differentiability of $X(t, s, x)$ with respect to s) and moreover that $X(t, s, x)$ is twice differentiable with respect to x.

In Chapter 9 we introduce the *transition evolution operator*

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad s \le t \le T,\ x \in \mathbb{R}^d,$$

where φ is any real, Borel, bounded function. We prove that

$$P_{s,r}P_{r,t} = P_{s,t}, \quad s \le r \le t \le T.$$

We note that when the coefficients b and σ are independent of time (autonomous case) we have

$$P_{s,t} = P_{0,t-s}.$$

In this case $P_t := P_{0,t}$ fulfills the semigroup law, that is $P_{t+s} = P_t P_s$.

We continue the chapter by studying the connection between the differential stochastic equation (1) and the parabolic problem

$$\begin{cases} v_s(s,x) + \frac{1}{2}(\mathscr{L}(s)v(s,\cdot)(x) = 0, \quad s \in [0,T], \\ \\ v(T,x) = \varphi(x),\ x \in \mathbb{R}^d, \end{cases} \tag{2}$$

where $\mathscr{L}(s)$ is the *Kolmogorov operator*

$$\mathscr{L}(s)\varphi(x) = \frac{1}{2}\ \mathrm{Tr}\ [\sigma(s,x)\sigma^*(s,x)D_x^2\varphi(x)] + \langle b(s,x), D_x\varphi(x)\rangle \tag{3}$$

and Tr means the trace. More precisely we show, using the backward Itô formula, that for any function φ uniformly continuous and bounded together with its first and second derivatives, the function $v(s,x) = (P_{s,T})\varphi(x)$ solves problem (3). Then we prove the uniqueness of the solution by using Itô's formula.

When the operator $\mathscr{L}(s)$ is elliptic (non degenerate) we prove that $P_{s,t}\varphi$ is differentiable for $t > s$ with respect to x for any Borel bounded function φ (*the Bismut–Elworthy–Li* formula).

In Chapter 10 we consider parabolic equations with a potential term and with a modified drift proving the *Feynman-Kac* and the *Girsanov* formulae. We also introduce and discuss the *Girsanov* transform which allow to solve (1) in a weak sense when b and σ are merely continuous.

In Chapter 11 we consider the Malliavin derivative (which was introduced in Chapter 3)

$$M : D^{1,2}(H,\mu) \subset L^2(H,\mu) \to L^2(H,\mu; H),$$

when the Hilbert space H coincides with $L^2(0,1)$ and $\mu = N_Q$ is a given non degenerate Gaussian measure on H and the real Brownian motion $B(t),\ t \in [0,1]$, in $(H, \mathscr{B}(H))$ defined in Chapter 4,

$$B(t)(x) = W_{\mathbb{1}_{[0,t]}}(x), \quad t \in [0,1],\ x \in L^2(0,1).$$

In the first part of this chapter we shall study some properties of the Malliavin derivatives of some interesting random variables in $(H, \mathscr{B}(H), \mu)$, in particular of functions of W_f, $f \in H$, and of Itô' integrals. We show also that a process F in $C_B([0,1]; L^2(\Omega))$ belongs to the domain of M^* (the adjoint of M) and that M^*F is the Itô' integral, $\int_0^1 F(s)dB(s)$.

In the second part we shall consider a simple stochastic differential equation

$$dX(t) = b(X(t))dt + \sqrt{C}dB(t), \quad X(0) = \xi \in \mathbb{R}, \tag{2}$$

where $b \in C_b^3(\mathbb{R})$ and $C \in L(\mathbb{R}^d)$ is symmetric nonnegative, and compute the Malliavin derivative of the solution $X(t,\xi)$.

We prove, when $\det C > 0$, two important formulae $DP_t\varphi$ and $P_tD\varphi$ which are valid for any φ namely continuous. As an application we prove the existence of a density for the law of $X(t,\xi)$.

Finally, we prove the *Clark-Ocone* formula.

Chapter 12 is devoted to asymptotic properties of the transition semigroup P_t for a stochastic differential equation. For the sake of simplicity we have limited ourselves to stochastic differential equations) with additive noise of the form (2).

We study several properties of the *transition semigroup* $P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))]$ defined for all real functions φ that are Borel and bounded and of the law of $X(t,x)$ which we denote by $\Pi_{t,x}$. We show that P_t is Feller, that is it acts on $C_b(\mathbb{R}^d)$. We prove the *Chapman–Kolmogorov* equation for the transition probabilities. Some important properties of P_t as *irreducibility*, *regularity* and *strong Feller* properties are studied together with the *Hasminskii Theorem*.

The main concept of the chapter is that of *invariant measure* which is basic for studying the asymptotic behaviour of P_t. Given an invariant measure μ, we prove that if η is the law of μ then $X(\cdot,\eta)$ (suitably defined) is a stationary process.

Then we show that if μ is an invariant measure there exists the limit of time averages

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T P_t\varphi\,dt =: S\varphi \quad \text{in } L^2(\mathbb{R}^d,\mu),$$

where S is a projection operator on $L^2(H,\mu)$ ($S^2 = S$) (*von Neumann Theorem*). A particular attention will be payed to the important situation where

$$S\varphi = \int_{\mathbb{R}^d}\varphi\,d\mu, \quad \forall\,\varphi\in L^2(\mathbb{R}^d,\mu).$$

In this case μ is said to be *ergodic*.

General results about existence and uniqueness of invariant measures including the *Krylov–Bogoliubov Theorem* are proved and applied to the stochastic differential equation above. Fore more results on these topics see the monograph [6].

Finally we give some short informations about the densities of invariant measures. For instance we show, using Malliavin calculus, that if det $C > 0$ then any invariant measure has a density with respect to the Lebesgue measure in $\mathbb{R}^d$. For more general equations, the reader can look at the classical monograph by R. Z. Has'minskii, [12].

Also, some appendices are devoted to some classical topics used in the text : conditional expectation, π and λ-systems, martingales, fixed points depending on a parameter and ergodic averages.

We conclude by highlighting some differences of this third edition with respect to the edition of 2011. Besides several small changes, corrections of misprints and some mistakes and improvements, here are listed the main differences

- The two first chapters are been unified in Chapter 1 which, however, includes several more details.
- In Chapter 2 only Malliavin spaces are taken in consideration. The classical Sobolev spaces which are never used in this course have been dropped.
- In Chapter 5 a new section has been added, concerning the structure of the set of zeros of the Brownian motion.
- A novelty in Chapter 6 is the extension by localization of Itô's integrals to almost surely square integrable process.
- Chapter 7 on Itô's formula has been completely rewritten with different and, we think, simpler proofs. It includes now Ito's formula for general C^2 (not necessarily bounded) functions and the Burkhölder inequality.
- The two old chapters on Malliavin calculus have been unified in one, which, however, contains more details.
- Chapter 12 has been completely rewritten and much new material added.

I am strongly indebted with my students, their questions allowed me to make several improvements and correct misprints and mistakes and with Luciano Tubaro for continuous discussions and suggestions on different topics of this course.

Pisa, December 15, 2013 Giuseppe Da Prato

Chapter 1
Gaussian measures in Hilbert spaces

This chapter is devoted to the basic concept of *Gaussian measure* in a separable Hilbert spaces H.

We shall use the following notations.

• $\mathbb{N}$ denotes the set of all natural numbers: $1, 2, \ldots$

• $\langle \cdot, \cdot \rangle$ is the inner product and $|\cdot|$ the norm in H.

• $L(H)$ is the Banach algebra of all linear bounded operators $T : H \to H$ endowed with the norm

$$\|T\| = \sup_{x \in H,\ |x|=1} |Tx|.$$

A linear operator $T \in \mathscr{L}(H)$ is said to be *symmetric* if $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in H$, *positive* if $\langle Tx, x \rangle \geq 0$ for all $x \in H$. We shall denote by $L^+(H)$ the set of all symmetric and positive operators of $L(H)$.

• $\mathscr{B}(H)$ is the σ-algebra of all Borel sets of H, that is the smallest σ-algebra including all open (or closed) subsets of H.

• For any $I \in \mathscr{B}(H)$, $\mathbb{1}_I$ denotes the characteristic function of I

$$\mathbb{1}_I = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{if } x \notin I. \end{cases}$$

• $\mathscr{P}(H)$ is the set of all Borel probability measures on $(H, \mathscr{B}(H))$.

In Section 1.1 we recall the definitions of moments, mean, covariance and Fourier transform of a probability measure $\mu \in \mathscr{P}(H)$.

In Section 1.2 we define Gaussian measures in H starting from the case where H is finite dimensional and compute some important Gaussian integrals.

Section 1.3 is devoted to some complement about trace class operators. It can be skipped in a first reading.

1.1. Moments, mean and covariance

Let $k \in \mathbb{N}$; we say that the k^{th} moment of μ is finite if

$$\int_H |x|^k \mu(dx) < \infty.$$

Assume that the first moment of μ is finite. Then the linear functional $F : H \to \mathbb{R}$ defined as

$$F(x) = \int_H \langle x, y \rangle \, \mu(dy), \quad \forall \, x \in H,$$

is continuous because

$$|F(x)| \le |x| \int_H |y| \, \mu(dy), \quad \forall \, x \in H.$$

By the Riesz representation theorem there exists a unique element $m \in H$ such that

$$\langle m, x \rangle = \int_H \langle x, y \rangle \, \mu(dy), \quad \forall \, x \in H. \tag{1.1}$$

m is called the *mean* of μ.

Assume now that the second moment of μ is finite

$$\int_H |x|^2 \mu(dx) < \infty.$$

Then, obviously, the first moment is finite as well, so that the mean m of μ is well definite. Moreover, the bilinear form $G : H \times H \to \mathbb{R}$

$$G(x, y) = \int_H \langle x, z - m \rangle \langle y, z - m \rangle \mu(dz), \quad \forall \, x, y \in H,$$

is continuous because

$$|G(x, y)| \le |x| \, |y| \int_H |z - m|^2 \, \mu(dz), \quad \forall \, x, y \in H.$$

Again, by the Riesz representation theorem there exists a unique operator $Q \in L(H)$ such that

$$\langle Qx, y \rangle = \int_H \langle x, z - m \rangle \, \langle y, z - m \rangle \, \mu(dz), \quad \forall \, x, y \in H. \tag{1.2}$$

Q is called the *covariance* of μ.

The following result gives basic properties of Q.

Proposition 1.1. *Assume that the second moment of μ is finite. Let m and Q be the mean and the covariance of μ respectively. Then*

(i) *Q is symmetric and positive.*

(ii) *For any orthonormal basis (e_k) in H we have*

$$\sum_{k=1}^{\infty}\langle Qe_k, e_k\rangle = \int_H |x-m|^2\mu(dx) < +\infty. \tag{1.3}$$

(iii) *Q is compact* [(1)].

Proof. Symmetry and positivity of Q are clear; let us prove (1.3). By (1.2) we have

$$\langle Qe_k, e_k\rangle = \int_H |\langle x-m, e_k\rangle|^2\mu(dx), \quad \forall\, k \in \mathbb{N}.$$

Summing up on k and using the monotone convergence theorem and the Parseval identity, yields

$$\sum_{k=1}^{\infty}\langle Qe_k, e_k\rangle = \sum_{k=1}^{\infty}\int_H |\langle x-m, e_k\rangle|^2\mu(dx)$$

$$= \int_H \sum_{k=1}^{\infty}|\langle x-m, e_k\rangle|^2\mu(dx) = \int_H |x-m|^2\mu(dx) < \infty.$$

We show finally (iii). Set

$$Q_n x := \sum_{j=1}^{n}\langle Qx, e_j\rangle e_j, \quad \forall\, x \in H,\ n \in \mathbb{N}.$$

Since Q_n is a finite rank operator (its range is included in the linear span of $e_1, \ldots, e_n$), to show that Q is compact it is enough to prove that $\|Q - Q_n\| \to 0$ as $n \to \infty$, see *e.g.* [18, Section 83]. Now, since Q is symmetric and positive, there exists a unique $T \in L^+(H)$ such that $T^2 = Q$; we set $T = \sqrt{Q}$ and call $\sqrt{Q}$ the square root of Q. (see *e.g.* [18, page 365].)

(1) Equivalently, Q is *completely continuous*.

Next write

$$|Qx - Q_n x|^2 = \sum_{j=n+1}^{\infty} |\langle Qx, e_j\rangle|^2 = \sum_{j=n+1}^{\infty} |\langle \sqrt{Q}x, \sqrt{Q}e_j\rangle|^2$$

$$\leq |x|^2\ \|Q\| \sum_{j=n+1}^{\infty} |\sqrt{Q}e_j|^2 = |x|^2\ \|Q\| \sum_{j=n+1}^{\infty} \langle Qe_j, e_j\rangle.$$

Therefore

$$\|Q - Q_n\|^2 \leq \|Q\| \sum_{j=n+1}^{\infty} \langle Qe_j, e_j\rangle \overset{n\to\infty}{\to} 0,$$

thank's to (1.3). □

We set

$$\mathrm{Tr}\ Q = \sum_{j=1}^{\infty}\langle Qe_j, e_j\rangle.$$

Tr Q is called the *trace* of Q. Clearly Tr Q is independent on the choice of the orthonormal basis (e_j). We shall denote by $L_1^+(H)$ the subset of $L^+(H)$ of those operators having finite trace.

For a definition of not necessarily symmetric trace class operators, see Section 1.4 below.

Remark 1.2. Assume that the second moment of μ is finite and let Q be the covariance of μ. Since Q is symmetric and compact, there exists a complete orthonormal basis (e_k) in H and a sequence of nonnegative numbers (λ_k) such that

$$Qe_k = \lambda_k e_k, \quad \forall\ k \in \mathbb{N}. \tag{1.4}$$

See *e.g.* [18, Section 93]. Obviously by (1.3) it follows that

$$\mathrm{Tr}\ Q = \sum_{k=1}^{\infty} \lambda_k. \tag{1.5}$$

Exercise 1.3. Assume that the second moment of μ is finite and let m and Q be its mean and covariance respectively. Prove that

$$\int_H |x|^2 \mu(dx) = \mathrm{Tr}\ Q + |m|^2. \tag{1.6}$$

Finally, the *Fourier transform* of μ is defined by

$$\widehat{\mu}(h) = \int_H e^{i\langle h,x\rangle}\mu(dx), \quad \forall\, h \in H.$$

Remark 1.4. One can show that $\widehat{\mu}$ uniquely determines μ, see [1, Theorem 6.35] when $H = \mathbb{R}$ and [17] in the general case.

1.2. Gaussian measures

A probability measure μ on $(H, \mathscr{B}(H))$ is called *Gaussian* if there exist $m \in H$ and $Q \in L_1^+(H)$ such that

$$\widehat{\mu}(h) = e^{i\langle m,h\rangle} e^{-\frac{1}{2}\langle Qh,h\rangle}, \quad \forall\, h \in H.$$

In this case we shall set $\mu = N_{m,Q}$ and if $m = 0$ $\mu = N_Q$. If Ker $Q = \{x \in H : \ Qx = 0\} = \{0\}$, μ is called *non degenerate*.

We are going to show that for any pair (m, Q) with $m \in H$ and $Q \in L_1^+(H)$, there exists a unique Gaussian measure $N_{m,Q}$. Moreover, all moments of μ are finite, m is the mean and Q the covariance of μ.

We shall first consider the case when H has finite dimension.

1.2.1. Gaussian measures in $\mathbb{R}$

Here we take $H = \mathbb{R}$.

Proposition 1.5. *The following statements hold.*

(i) *Let $m \in \mathbb{R}$. Then $\delta_m = N_{m,0}$, where δ_m is the Dirac measure at m. Moreover m is the mean of δ_m and* 0 *its covariance.*

(ii) *Let $\lambda > 0$, $m \in \mathbb{R}$. Let μ be defined by*

$$\mu(dx) = \frac{1}{\sqrt{2\pi\lambda}}\, e^{-\frac{(x-m)^2}{2\lambda}}\, dx. \tag{1.7}$$

Then $\mu = N_{m,\lambda}$. Moreover m is the mean and λ the covariance of $N_{m,\lambda}$.

Proof. (i) Let $\lambda = 0$. Then the Fourier transform of δ_m is given by

$$\int_{\mathbb{R}} e^{ixh}\delta_m(dx) = e^{imh},$$

so that $\delta_m = N_{m,0}$.

(ii) Let now $\lambda > 0$ and let μ be defined by (1.7). μ is a probability measure since

$$\mu(\mathbb{R}) = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{+\infty} e^{-\frac{(x-m)^2}{2\lambda}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1.$$

It is now elementary to check that

$$\int_{\mathbb{R}} e^{ixh} \mu(dx) = e^{imh-\frac{1}{2}\lambda h^2}, \quad \forall\, h \in \mathbb{R},$$

and

$$\int_{\mathbb{R}} x\mu(dx) = m, \quad \int_{\mathbb{R}} (x-m)^2 \mu(dx) = \lambda. \qquad \square$$

Remark 1.6. If $\lambda > 0$, $N_{m,\lambda}$ is absolutely continuous with respect to the Lebesgue measure $\lambda_1(dx) = dx$.

Exercise 1.7. Prove that for any $\lambda > 0$ and any $k \in \mathbb{N}$ it results

$$\int_{\mathbb{R}} x^{2k} N_\lambda(dx) = \frac{(2k)!}{2^k\, k!}\, \lambda^k, \tag{1.8}$$

so that all moments of N_λ are finite.

Exercise 1.8. Prove that for any $\lambda > 0$ and $\alpha \in \mathbb{R}$

$$\int_{\mathbb{R}} e^{\alpha x^2} N_\lambda(dx) = \begin{cases} (1-2\lambda\alpha)^{-1/2}, & \text{if } \alpha < \dfrac{1}{2\lambda} \\ +\infty \quad \text{if } \alpha \geq \dfrac{1}{2\lambda}. \end{cases} \tag{1.9}$$

1.2.2. Gaussian measures in $\mathbb{R}^n$

We take here $H = \mathbb{R}^n$ with $n \in \mathbb{N}$ and $n > 1$. Let $m \in \mathbb{R}^n$ and $Q \in L^+(\mathbb{R}^n)$. Since Q is symmetric there exists an orthonormal basis $(e_1, ..., e_n)$ in $\mathbb{R}^n$ and nonnegative numbers $(\lambda_1, ..., \lambda_n)$ such that

$$Qe_h = \lambda_h e_h, \quad h = 1, ..., n. \tag{1.10}$$

Set

$$m_h = \langle m, e_h \rangle, \quad h = 1, ..., n. \tag{1.11}$$

Consider the product measure in $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$, see *e.g.* [1],

$$\mu := \bigtimes_{h=1}^{n} N_{m_h,\lambda_h} \tag{1.12}$$

Proposition 1.9. *We have* $\mu = N_{m,Q}$. *Moreover the second moment of* $N_{m,Q}$ *is finite,* m *coincides with the mean and* Q *with the covariance of* μ.

Proof. For any $h \in \mathbb{R}^n$ we have,

$$\begin{aligned}\widehat{\mu}(h) &= \int_{\mathbb{R}^n} e^{x_1h_1+\cdots+x_nh_n} N_{m_1,\lambda_1}(dx_1)\cdots N_{m_n,\lambda_n}(dx_n)\\ &= \prod_{j=1}^n e^{im_jh_j-\frac{1}{2}\lambda_j h_j^2} = e^{i\langle m,h\rangle-\frac{1}{2}\langle Qh,h\rangle}.\end{aligned} \tag{1.13}$$

Therefore $\mu = N_{m,Q}$. The other assertions are left to the reader. □

Exercise 1.10. Let $m \in \mathbb{N}$ and $Q \in L_1^+(\mathbb{R}^n)$ such that $\det Q > 0$. Show that $N_{m,Q}$ is absolutely continuous with respect to the Lebesgue measure λ_n in $\mathbb{R}^n$ and

$$\frac{dN_{m,Q}}{d\lambda_n}(x) = (2\pi \det Q)^{-1/2} e^{-\frac{1}{2}\langle Q^{-1}x,x\rangle}, \quad \forall\, x \in \mathbb{R}^n. \tag{1.14}$$

1.2.3. Gaussian measures in Hilbert spaces

Here H is an infinite dimensional separable Hilbert. Let $m \in H$, $Q \in L_1^+(H)$. We are going to show that there exists a Gaussian measure $N_{m,Q}$ in H. As we have seen, since Q is compact, there is a complete orthonormal basis (e_k) in H and a sequence of non-negative numbers (λ_k) such that $Qe_k = \lambda_k e_k$ for all $k \in \mathbb{N}$. For any $x \in H$ we set $x_k = \langle x, e_k\rangle,\ k \in \mathbb{N}$.

Let us consider the natural isomorphism γ between H and the Hilbert space ℓ^2 of all sequences (x_k) of real numbers such that

$$\sum_{k=1}^{\infty} |x_k|^2 < \infty,$$

defined by

$$H \to \ell^2,\ x \mapsto \gamma(x) = (x_k).$$

We shall construct the required Gaussian measures starting from the countable product of measures μ, see *e.g.* [1, Section 6.3],

$$\mu = \bigtimes_{k=1}^{\infty} N_{m_k,\lambda_k} \tag{1.15}$$

Notice, however, that μ is defined in $\mathbb{R}^\infty$, the space of all sequences of real numbers endowed with the product topology.

Theorem 1.11. *The restriction of the measure* μ*, defined by* (1.15)*, to* $(\ell^2, \mathscr{B}(\ell^2))$ *is a Gaussian measure* $N_{m,Q}$ *in* ℓ^2*, where* $m = (m_k)$ *and* $Qx = (\lambda_k x_k)$*. Moreover,* m *is the mean and* Q *the covariance of* $N_{m,Q}$*.*

Proof. Step 1. μ is concentrated in ℓ^2, that is $\mu(\ell^2) = 1$.

We first note that ℓ^2 is a Borel subset of $\mathbb{R}^\infty$. Then, by the monotone convergence theorem we have

$$\begin{aligned}\int_{\mathbb{R}^\infty} |x|^2_{\ell^2}\,\mu(dx) &= \int_{\mathbb{R}^\infty} \sum_{k=1}^\infty x_k^2\,\mu(dx) = \sum_{k=1}^\infty \int_{\mathbb{R}} x_k^2\, N_{m_k,\lambda_k}(dx_k)\\ &= \sum_{k=1}^\infty (\lambda_k + m_k^2) = \mathrm{Tr}\ Q + |m|^2.\end{aligned} \tag{1.16}$$

Therefore

$$\mu\left(\left\{x \in \mathbb{R}^\infty :\ |x|^2_{\ell^2} < \infty\right\}\right) = 1.$$

This proves that μ is concentrated on ℓ^2.

Step 2. $\mu = N_{m,Q}$.

For any $n \in \mathbb{N}$ define

$$P_n x := \sum_{h=1}^n \langle x, e_k\rangle e_k, \quad \forall\, x \in H. \tag{1.17}$$

It is well known that $\lim_{n\to\infty} P_n x = x$ for all $x \in H$. Now for any $h \in H$ we have by the dominated convergence theorem and taking into account (1.15),

$$\begin{aligned}\int_H e^{i\langle x,h\rangle}\mu(dx) &= \lim_{n\to\infty} \int_H e^{i\langle P_n x,h\rangle}\mu(dx)\\ &= \lim_{n\to\infty} \prod_{k=1}^n \int_{\mathbb{R}} e^{ix_k h_k} N_{m_k,\lambda_k}(dx_k)\\ &= \lim_{n\to\infty} \prod_{k=1}^n e^{ia_k h_k - \frac{1}{2}\,\lambda_k h_k^2} = \lim_{n\to\infty} e^{i\langle P_n m,h\rangle} e^{-\frac{1}{2}\,\langle P_n Qh,h\rangle}\\ &= e^{i\langle m,h\rangle} e^{-\frac{1}{2}\,\langle Qh,h\rangle}.\end{aligned}$$

So, $\mu = N_{m,Q}$ as required.

Step 3. The second moment of μ is finite, moreover m is the mean and Q the covariance of μ.

By (1.16) it follows that

$$\int_H |x|^2 \mu(dx) = \text{ Tr } Q + |m|^2,$$

so that the second moment of μ is finite. The other assertions follow easily, they are left to the reader. □

1.2.4. Computation of some Gaussian integral

We are here concerned with the Gaussian measure $\mu = N_Q$ defined by (1.15) with $m = 0$ for simplicity. With a possible rearrangement of the eigenvalues (λ_k) of Q, we can assume that the sequence (λ_k) is non increasing.

We start by computing

$$\int_H e^{\alpha|x|^2} \mu(dx),$$

where $\alpha \in \mathbb{R}$. Let $n \in \mathbb{N}$, then, recalling that μ is the product measure of N_{λ_n}, we have

$$\int_H e^{\alpha|P_n x|^2} \mu(dx) = \int_H e^{\alpha|P_n x|^2} N_{Q_n}(dx),$$

where $Q_n = QP_n$. On the other hand, we have

$$\int_H e^{\alpha|P_n x|^2} N_{Q_n}(dx) = \prod_{h=1}^{n} \int_{\mathbb{R}} e^{\alpha\xi^2} N_{\lambda_n}(d\xi). \tag{1.18}$$

Now, by (1.9), we obtain (recall that the sequence (λ_k) is not increasing)

$$\int_H e^{\alpha|P_n x|^2} N_{Q_n}(dx) = \begin{cases} \displaystyle\prod_{h=1}^{n} (1 - 2\lambda_h \alpha)^{-1/2}, & \text{if } \alpha < \dfrac{1}{2\lambda_1} \\ +\infty \quad \text{if } \alpha \geq \frac{1}{2\lambda_1}. \end{cases} \tag{1.19}$$

Exercise 1.12. Show that if $\alpha < \frac{1}{2\lambda_1}$ there exists finite and positive the limit

$$\prod_{h=1}^{\infty} (1 - 2\lambda_h \alpha) := \lim_{n\to\infty} \prod_{h=1}^{n} (1 - 2\lambda_h \alpha).$$

Hint. Consider the series

$$\sum_{h=1}^{\infty} \log(1 - 2\lambda_h \alpha).$$

and use that $\sum_{h=1}^{\infty} \lambda_h < \infty$.

When $\alpha < \frac{1}{2\lambda_1}$ we set

$$\det(1 - 2\alpha Q) = \prod_{h=1}^{\infty}(1 - 2\lambda_h \alpha).$$

Proposition 1.13. *We have*

$$\int_H e^{\alpha|x|^2} N_Q(dx) = \begin{cases} [\det(1 - 2\alpha Q)]^{-1/2}, & \text{if } \alpha < \dfrac{1}{2\lambda_1} \\ \\ +\infty \quad \text{if } \alpha \geq \dfrac{1}{2\lambda_1}. \end{cases} \tag{1.20}$$

Proof. By (1.19) we have

$$\int_H e^{\alpha|P_n x|^2} \mu(dx) = \int_H e^{\alpha|P_n x|^2} N_{Q_n}(dx)] = \prod_{h=1}^{n}(1 - 2\lambda_h \alpha)^{-1/2}. \tag{1.21}$$

Now the conclusion follows letting $n \to \infty$ and using the monotone convergence theorem. □

Exercise 1.14. Let $T \in L_1^+(H)$ and let (e_k) be an orthonormal basis such that

$$T e_k = t_k e_k, \quad \forall\, k \in \mathbb{N},$$

for some $t_k \geq 0$. Define

$$\det(1 - 2\epsilon T) := \prod_{k=1}^{\infty}(1 - 2\epsilon t_k), \quad \epsilon \in (0, \|T\|/2).$$

Check the following formula

$$\frac{d}{d\epsilon} \det(1 - 2\epsilon T) = -2\mathrm{Tr}\, [T(1 - 2\epsilon T)^{-1}] \det(1 - 2\epsilon T). \tag{1.22}$$

Example 1.15. Let us compute

$$J_2 := \int_H |x|^4 \mu(dx).$$

Set

$$F(\varepsilon) = \int_H e^{\varepsilon|x|^2} \mu(dx), \quad \varepsilon > 0.$$

Then we have $J_2 = F''(0)$. On the other hand, by Proposition 1.13 it follows that

$$F(\varepsilon) = [\det(1 - 2\epsilon Q)]^{-1/2}$$

for $\epsilon < \frac{1}{2\lambda_1}$. Consequently, recalling (1.22), we get

$$F'(\epsilon) = \text{Tr}\,[Q(1-2\epsilon Q)^{-1}]F(\epsilon)$$

and

$$F''(\epsilon) = \text{Tr}\,[Q(1-2\epsilon Q)^{-1}|F'(\epsilon) + 2\text{Tr}\,[Q^2(1-2\epsilon Q)^{-2}]F(\epsilon).$$

Consequently

$$\int_H |x|^4 \mu(dx) = F''(0) = (\text{Tr}\ Q)^2 + 2\text{Tr}\ (Q^2). \tag{1.23}$$

Exercise 1.16. Compute

$$J_m := \int_H |x|^{2m} \mu(dx) < \infty,$$

for any $m \in \mathbb{N}$.

We prove finally

Proposition 1.17. *We have*

$$\int_H e^{\langle h,x\rangle} \mu(dx) = e^{\langle m,h\rangle} e^{\frac{1}{2}\langle Qh,h\rangle}, \qquad h \in H. \tag{1.24}$$

Proof. For any $\epsilon > 0$ we have

$$e^{\langle h,x\rangle} \le e^{|x|\,|h|} \le e^{\epsilon|x|^2}\, e^{\frac{1}{\epsilon}|h|^2}.$$

Choosing $\epsilon < \frac{1}{2\lambda_1}$ we have, by the dominated convergence theorem, that

$$\begin{aligned}\int_H e^{\langle h,x\rangle} \mu(dx) &= \lim_{n\to\infty} \int_H e^{\langle h,P_n x\rangle} \mu(dx) = \lim_{n\to\infty} \int_H e^{\langle h,P_n x\rangle} \mu_n(dx) \\ &= \lim_{n\to\infty} e^{\langle P_n m,h\rangle} e^{\frac{1}{2}\langle P_n Qh,h\rangle} = e^{\langle m,h\rangle} e^{\frac{1}{2}\langle Qh,h\rangle}.\end{aligned}$$

□

Exercise 1.18. Let $\epsilon, \alpha > 0$. Show that

$$\int_H e^{\alpha|x|^{2+\epsilon}}\, \mu(dx) = \infty. \tag{1.25}$$

Exercise 1.19. Assume that the dimension of H is greater than 2. Show that

$$\int_H \frac{1}{|x|^2}\, \mu(dx) < \infty. \tag{1.26}$$

Hint. Note that for all $n \in \mathbb{N}$

$$\int_H \frac{1}{|x|^2}\, \mu(dx) \le \int_H \frac{1}{|P_n x|^2}\, \mu(dx) = \int_H \frac{1}{|P_n x|^2}\, \mu_n(dx).$$

1.3. Trace class operators (*)

An operator $T \in L(H)$ (not necessarily symmetric) is said to be of *trace class* or *nuclear* if there exist two sequences $(a_j),\ (b_j)$ of elements of H such that

$$\sum_{j=1}^{\infty} |a_j|\,|b_j| < \infty \tag{1.27}$$

and

$$Tx = \sum_{j=1}^{\infty} a_j \langle x, b_j\rangle, \quad \forall x \in H. \tag{1.28}$$

Obviously the series in (1.28) is norm convergent [(2)].

The set of all trace class operators T is a Banach space with the norm

$$\|T\|_1 := \inf\left\{\sum_{j=1}^{\infty} |a_j|\,|b_j| :\ Tx = \sum_{j=1}^{\infty} a_j\langle x, b_j\rangle, \quad \forall x \in H\right\}.$$

Proposition 1.20. *Assume that $T \in L(H)$ is of trace class, then*

(i) T *is compact.*

(ii) *For any orthonormal basis (e_k) in H the series*

$$\sum_{j=1}^{\infty} \langle Te_j, e_j\rangle \tag{1.29}$$

is absolutely convergent. Moreover

$$\sum_{j=1}^{\infty} \langle Te_j, e_j\rangle = \sum_{j=1}^{\infty} \langle a_j, b_j\rangle. \tag{1.30}$$

We set

$$\mathrm{Tr}\, T := \sum_{j=1}^{\infty} \langle Te_j, e_j\rangle$$

and call Tr T *the* trace *of* T.

Proof. Assume that T is given by (1.28) and that (1.27) holds.

Let us show compactness of T. For any $n \in \mathbb{N}$ set

$$T_n x := \sum_{j=1}^{n} a_j \langle x, b_j\rangle.$$

(2) That is $\sum_{j=1}^{\infty} |a_j|\,|\langle x, b_j\rangle| < \infty$.

Since T_n has finite rank it is enough to show that $\|T - T_n\| \to 0$ as $n \to \infty$ (as noticed in the proof of Proposition 1.1). We have in fact

$$|Tx - T_n x| \le \sum_{j=n+1}^{\infty} |a_j|\,|\langle x, b_j\rangle| \le |x| \sum_{j=n+1}^{\infty} |a_j|\,|b_j|.$$

Therefore

$$\|T - T_n\| \le \sum_{j=n+1}^{\infty} |a_j|\,|b_j|$$

and the conclusion follows from (1.27).

Now we fix an orthonormal basis (e_k) in H and show that the series (1.29) is absolutely convergent. We have in fact

$$\begin{aligned}\sum_{k=1}^{\infty} |\langle Te_k, e_k\rangle| &\le \sum_{j,k=1}^{\infty} |\langle a_j, e_k\rangle|\,|\langle b_j, e_k\rangle| \\ &\le \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |\langle a_j, e_k\rangle|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} |\langle b_j, e_k\rangle|\right)^{1/2} \\ &= \sum_{j=1}^{\infty} |a_j|\,|b_j| < \infty.\end{aligned}$$

Finally, we have

$$\sum_{k=1}^{\infty}\langle Te_k, e_k\rangle = \sum_{j,k=1}^{\infty} \langle a_j, e_k\rangle\, \langle b_j, e_k\rangle$$

$$\sum_{j=1}^{\infty}\left\langle b_j, \sum_{k=1}^{\infty}\langle a_j, e_k\rangle e_k\right\rangle = \sum_{j=1}^{\infty}\langle a_j, b_j\rangle.$$

The conclusion follows. □

Chapter 2
Gaussian random variables

This chapter is devoted to definitions and main properties of *Gaussian random variables*.

In Sections 2.1 and 2.3 we recall some basic tools from probability as the *law of a random variable*, with the change of variables formula, and the *independence* of random variables.

In Sections 2.2 and 2.4 we study Gaussian random variables in a separable Hilbert space H.

Finally, Section 2.5 is devoted to the important concept of *white noise*.

2.1. Law of a random variable

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, E a Polish space[1] and $\mathscr{B}(E)$ the σ-algebra of all Borel subsets of E. An E-valued *random variable* in $(\Omega, \mathscr{F}, \mathbb{P})$ is a mapping $X : \Omega \to E$ such that

$$I \in \mathscr{B}(E) \Rightarrow X^{-1}(I) \in \mathscr{F}.$$

The *law* of X is the probability measure $X_{\#}\mathbb{P}$ on $(E, \mathscr{B}(E))$ defined by

$$(X_{\#}\mathbb{P})(I) = \mathbb{P}(X^{-1}(I)), \quad \forall\, I \in \mathscr{B}(E).$$

The following *change of variables* theorem is basic.

Theorem 2.1. *Let $X : \Omega \to H$ be an E-valued random variable. Then for any $\varphi\colon E \to \mathbb{R}$ bounded and Borel, we have*

$$\int_{\Omega} \varphi(X(\omega))\mathbb{P}(d\omega) = \int_{E} \varphi(x)(X_{\#}\mathbb{P})(dx). \tag{2.1}$$

[1] That is a complete separable metric space.

Proof. Let first $\varphi = 1\!\!1_I$ where $I \in \mathscr{B}(\mathbb{R})$. Then

$$\varphi(X(\omega)) = 1\!\!1_{X^{-1}(I)}(\omega), \quad \omega \in \Omega,$$

so that

$$\int_\Omega \varphi(X(\omega))\mathbb{P}(d\omega) = \mathbb{P}(X^{-1}(I)).$$

On the other hand,

$$\int_H \varphi(x)(X_\#\mathbb{P})(dx) = \mathbb{P}(X^{-1}(I))$$

and (2.1) follows. Consequently, (2.1) holds for all simple functions φ. Since φ is a monotonic limit of simple functions, the conclusion follows, see *e.g.* [1]. □

Exercise 2.2. Let $n \in \mathbb{N}$ and let $X_1, \ldots, X_n$ be real random variables in $(\Omega, \mathscr{F}, \mathbb{P})$. Show that the mapping $X\colon \Omega \to \mathbb{R}^n$,

$$X(\omega) = (X_1(\omega), \ldots, X_n(\omega)), \quad \omega \in \Omega,$$

is an $\mathbb{R}^n$-valued random variable.

Take now $E = H$, where H is a separable Hilbert space. A random variable $X : \Omega \to H$ is called *Gaussian* if its law is Gaussian. If $X_\#\mathbb{P}$ is Gaussian with mean m and covariance Q we say that X is a *Gaussian random variable* $N_{m,Q}$. By (2.1) it follows that $X_\#\mathbb{P}$ is Gaussian $N_{m,Q}$ if and only if

$$\widehat{X_\#\mathbb{P}}(h) = \int_\Omega e^{i\langle X(\omega),h\rangle}\mathbb{P}(d\omega) = e^{i\langle m,h\rangle - \frac{1}{2}\langle Qh,h\rangle}, \quad \forall\, h \in H.$$

Exercise 2.3. Let $X : \Omega \to H$ be a Gaussian random variable $N_{m,Q}$ and let $z \in H$. Show that the real random variable

$$\omega \to \langle X(\omega), z\rangle,$$

is Gaussian $N_{\langle m,z\rangle,\langle Qz,z\rangle}$.

2.2. Random variables on a Gaussian Hilbert space

Let H be a separable Hilbert space and let μ be a Gaussian measure on H; we shall call (H, μ) a *Gaussian Hilbert space*.

Let K be another separable Hilbert space; we show that any linear and affine transformation from H into K is Gaussian.

Proposition 2.4. *Let $b \in K$, $A \in L(H, K)$* [(2)] *and $T(x) = Ax + b$, $x \in H$. Then T is a Gaussian random variable N_{Aa+b, AQA^*}, where A^* is the transpose of A.*

Proof. It is enough to show that

$$\widehat{T_\# \mu}(k) = e^{i\langle k, Aa+b\rangle} e^{-\frac{1}{2}\langle AQA^* k, k\rangle}, \quad \forall\, k \in K.$$

We have in fact

$$\begin{aligned}\widehat{T_\# \mu}(k) &= \int_K e^{i\langle k, y\rangle} T_\# \mu(dy) = \int_H e^{i\langle k, T(x)\rangle} \mu(dx) \\ &= \int_H e^{i\langle k, Ax+b\rangle} \mu(dx) = e^{i\langle k, b\rangle} \int_H e^{i\langle A^* k, x\rangle} \mu(dx) \\ &= e^{i\langle k, Aa+b\rangle} e^{-\frac{1}{2}\langle AQA^* k, k\rangle}, \quad k \in K.\end{aligned}$$ □

Corollary 2.5. *Let $\mu = N_Q$ be a Gaussian measure on H and let $z_1, \ldots, z_n \in H$. Let $A \colon H \to \mathbb{R}^n$ be defined as*

$$Ax = (\langle x, z_1\rangle, \ldots, \langle x, z_n\rangle), \quad x \in H.$$

Then A is an $\mathbb{R}^n$-valued Gaussian random variable N_{Q_A} and

$$(Q_A)_{i,j} = \langle Q z_i, z_j\rangle, \quad i, j = 1, \ldots, n.$$

Proof. By an elementary computation we see that the transpose $A^* \colon \mathbb{R}^n \to H$ of A is given by

$$A^* \xi = \sum_{i=1}^n \xi_i z_i, \quad \forall\, \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

Consequently we have

$$AQA^* \xi = A \sum_{i=1}^k \xi_i (Q z_i) = \sum_{i=1}^k \xi_i \left(\langle Q z_i, z_1\rangle, ..., \langle Q z_i, z_k\rangle\right),$$

and the conclusion follows from Proposition 2.4. □

[(2)] $L(H, K)$ is the space of all linear bounded mappings from H into K.

Exercise 2.6. Let $\mu = N_Q$, (e_k) an orthonormal basis which diagonalizes Q and set

$$P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \quad n \in \mathbb{N}.$$

Let moreover $\varphi \in L^2(H, \mu)$, $n \in \mathbb{N}$ and $\varphi_n(x) = \varphi(P_n x)$, $x \in H$. In general φ_n does not belong to $L^2(H, \mu)$. Show that setting

$$\widetilde{\varphi}_n(x) = \int_H \varphi(P_n x + (I - P_n)y)\mu(dy), \quad x \in H, \quad n \in \mathbb{N},$$

we have $\widetilde{\varphi}_n \in L^2(H, \mu)$ for any $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \widetilde{\varphi}_n = \varphi \quad \text{in } L^2(H, \mu).$$

Hint. Consider the linear tansformaton

$$T_n : H \times H \to H, \quad (x, y) \to T_n(x, y) = P_n x + (1 - P_n)y.$$

Show that $(T_n)_\#(\mu \times \mu) = \mu$ for all $n \in \mathbb{N}$.

Exercise 2.7 (Invariance by rotations). Let $\mu = N_Q$. Consider the linear map $\mathscr{R}$ from $H \times H$ into $H \times H$, defined by

$$\mathscr{R}(x, y) = (\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y), \tag{2.2}$$

where $\theta \in [0, 2\pi]$. Let $F : H \times H \to H \times H$ be bounded and Borel. Show that

$$\int_{H \times H} F(x, y)\mu(dx)\mu(dy) = \int_{H \times H} F(\mathscr{R}(x, y))\mu(dx)\mu(dy). \tag{2.3}$$

2.3. Independence

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

Definition 2.8. (i) We say that $A_1, \dots, A_n \in \mathscr{F}$ are *independent* if for any permutation $\sigma : i \to \sigma_i$ of $1, 2, \dots, n$ we have

$$\mathbb{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \prod_{j=1}^{n} \mathbb{P}(A_{\sigma_j}).$$

(ii) Let $X_1, \dots, X_n$ be E-valued random variables, where E is a Polish space. We say that $X_1, \dots, X_n$ are *independent* if for any $I_1, \dots I_n \in \mathscr{B}(E)$, the sets

$$\{X_1 \in I_1\}, \dots, \{X_1 \in I_n\}$$

are independent.

(iii) A sequence (X_j) of random variables is said independent if any finite subset of (X_j) is independent.

(iv) Let $\mathscr{G}_1, ..., \mathscr{G}_n$ be σ-algebras included in $\mathscr{F}$. They are called *independent* if for any choice

$$A_1 \in \mathscr{G}_1, ..., A_n \in \mathscr{G}_n,$$

sets $A_1, ..., A_n$ are independent.

Exercise 2.9. Let X and Y be real independent random variables and let $\varphi : \mathbb{R} \to \mathbb{R}$ Borel. Show that $\varphi(X)$ and Y are independent.

Let $X_1, ..., X_n$ be H-valued random variables, where H is a separable Hilbert space and denote by X the H^n-valued random variable $X := (X_1, ..., X_n)$. Let us consider the Fourier transforms of X_j, $j = 1, ..., n$, and X,

$$\widehat{X}_j(h) = \mathbb{E}[e^{i\langle h, X_j\rangle}], \quad h \in H, \ j = 1, ..., n,$$

and

$$\widehat{X}(h) = \mathbb{E}[e^{i\sum_{j=1}^n \langle h_j, X_j\rangle}], \quad (h_1, ..., h_n) \in H^n.$$

The following result is classical, see *e.g.* [8].

Proposition 2.10. *Let $X_1, ..., X_n$ be H-valued random variables and let $X := (X_1, ..., X_n)$. The following statements are equivalent*

(i) $X_1, ..., X_n$ *are independent.*

(ii) $\widehat{X}(h) = \prod_{j=1}^n \widehat{X}_j(h_j), \quad \forall\ (h_1, ..., h_n) \in H^n.$

(iii) $\mathbb{E}[\varphi_1(X_1)\cdots\varphi_n(X_n)] = \mathbb{E}[\varphi_1(X_1)]\cdots\mathbb{E}[\varphi_n(X_n)]$ *for any choice of functions $\varphi_1, ..., \varphi_n$ real, bounded and Borel.*

Exercise 2.11. For $k = 1, ..., n$ let X_k be a real Gaussian random variable N_{m_k,λ_k}. Assume that $(X_1, ..., X_n)$ are independent. Then $X := (X_1, ..., X_n)$ is an $\mathbb{R}^n$-valued Gaussian random variable with mean $(m_1, ..., m_n)$ and covariance diag $(\lambda_1, ..., \lambda_k)$.

Hint. Compute the Fourier transform of X.

2.4. Linear random variables on a Gaussian Hilbert space

Let H be a separable Hilbert space and let $\mu = N_Q$ be a Gaussian measure on H. Among random variables in H, of particular interest are linear mappings from H into $\mathbb{R}$. For any $v \in H$ we define

$$F_v(x) = \langle x, v\rangle, \quad \forall x \in H.$$

By Proposition 2.4 we know that F_v is a Gaussian random variable $N_{\langle Qv,v\rangle}$ and that given $v_1, ..., v_n \in H$, setting

$$F_{v_1,....,v_n}(x) = (F_{v_1}(x), ..., F_{v_n}(x)), \quad \forall\, x \in H,$$

$F_{v_1,....,v_n}$ is an $\mathbb{R}^n$-valued Gaussian random variable with mean 0 and covariance

$$(Q_{v_1,....,v_n})_{j,k} = \langle Qv_j, v_k\rangle, \quad j,k = 1, ..., n.$$

Proposition 2.12. *Let $v_1, ..., v_n \in H$. Then $F_{v_1}, ..., F_{v_n}$ are independent if and only if the matrix $(Q_{v_1,....,v_n})$ is diagonal, that is if and only if*

$$\langle Qv_j, v_k\rangle = 0, \quad \text{if } j \neq k.$$

Proof. We have

$$\widehat{F_{v_1,....,v_n}}(\xi_1, ..., \xi_n) = e^{-\frac{1}{2}\sum_{j,k=1}^{n}(Q_{v_1,....,v_n})_{j,k}\xi_j\xi_k},$$

and

$$\widehat{F_{v_j}}(\xi_j) = e^{-\frac{1}{2}\xi_j^2\langle Qv_j,v_j\rangle}, \quad j = 1, ..., n,$$

Therefore the conclusion of the proposition follows from Proposition 2.10. □

Example 2.13. Let μ be a Gaussian measure N_Q in H and let (e_k) be an orthonormal basis which diagonalizes Q. Set

$$x_k = \langle x, e_k\rangle, \quad k \in \mathbb{N}.$$

Then the random variables (x_k) are independent.

2.5. The white noise function

Let $\mu = N_Q$ with Ker $Q = \{0\}$. For any $f \in Q^{1/2}(H)$ set

$$W_f(x) = \langle Q^{-1/2}f, x\rangle, \quad \forall\, x \in H.$$

Remark 2.14. By Proposition 2.12, W_f enjoys the following remarkable property. Let $f_1, ..., f_n \in Q^{1/2}(H)$, then the random variables $W_{f_1}, ..., W_{f_n}$ are independent if and only if system $f_1, ..., f_n$ is orthogonal.

We are going to show that the mapping $f \to W_f$ can be extended to the whole H; this extension, however, will belong to $L^2(H, \mu)$. So, it will be an equivalence class of random variables rather a random variable.

2.5.1. Equivalence classes of random variables

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let H be a separable Hilbert space. We denote by $\mathscr{R}(H)$ the set of all H-valued random variables.

We say that $X, Y \in \mathscr{R}(H)$ are *equivalent* (and write $X \sim Y$) if

$$\mathbb{P}(\{\omega \in \Omega : \ X(\omega) = Y(\omega)\}) = 1.$$

One can check easily that $X \sim Y, \ \ X, Y \in \mathscr{R}(H)$ is an equivalence relation, so that the set $\mathscr{R}(H)$ is a disjoint union of equivalences classes.

Assume that $X \sim Y$, then for any $I \in \mathscr{B}(H)$ we have

$$\begin{aligned}\mathbb{P}(X \in I) &= \mathbb{P}(X \in I, X = Y) + \mathbb{P}(X \in I, X \neq Y)\\ &= \mathbb{P}(Y \in I, X = Y) \le \mathbb{P}(Y \in I).\end{aligned}$$

Similarly, $\mathbb{P}(Y \in I) \le \mathbb{P}(X \in I)$, so that $\mathbb{P}(X \in I) = \mathbb{P}(Y \in I)$ for all $I \in \mathscr{B}(H)$ and the laws of X and Y coincide. Consequently, all random variables belonging to a fixed equivalence class $\widetilde{X}$ have the same law, which is called the *law* of $\widetilde{X}$.

In the following we shall not distinguish between a random variable X and the equivalence class $\widetilde{X}$ including X, except when some confusion could arise.

By $L^p(\Omega, \mathscr{F}, \mathbb{P}; H), \ \ p \ge 1$, we mean the space of all equivalence class of random variables $X\colon \Omega \to H$ such that

$$\int_\Omega |X(\omega)|^p \, \mathbb{P}(d\omega) < +\infty.$$

$L^p(\Omega, \mathscr{F}, \mathbb{P}; H)$, endowed with the norm

$$\|X\|_{L^p(\Omega, \mathscr{F}, \mathbb{P}; H)} = \left(\int_\Omega |X(\omega)|^p \, \mathbb{P}(d\omega)\right)^{1/p},$$

is a Banach space. We shall write $L^p(\Omega, \mathscr{F}, \mathbb{P}; H) = L^p(\Omega, \mathbb{P}; H)$ for brevity.

Now we show that L^2-limits of sequences of Gaussian random variables is Gaussian.

Proposition 2.15. *Let (X_n) be a sequence of H-valued Gaussian random variables with laws (N_{m_n, Q_n}) convergent to X in $L^2(\Omega, \mathbb{P}; H)$. Let m and Q be the mean and the covariance respectively of the law of X. Then we have*

$$\lim_{n\to\infty} \langle m_n, h\rangle = \langle m, h\rangle, \quad \forall\, h \in H,$$

and

$$\lim_{n\to\infty}\langle Q_n h, h\rangle = \langle Qh, h\rangle, \quad \forall\, h \in H.$$

Moreover X is a Gaussian random variable $N_{m,Q}$.

Proof. Let $h \in H$. Since $X_n \to X$ in $L^2(\Omega, \mathbb{P}; H)$ we have

$$\lim_{n\to\infty}\langle m_n, h\rangle = \lim_{n\to\infty}\int_\Omega \langle X_n(\omega), h\rangle \mathbb{P}(d\omega) = \int_\Omega \langle X(\omega), h\rangle \mathbb{P}(d\omega) = \langle m, h\rangle$$

and

$$\begin{aligned}\lim_{n\to\infty}\langle Q_n h, h\rangle &= \lim_{n\to\infty}\int_\Omega \langle X_n(\omega) - m_n, h\rangle\ \langle X_n(\omega) - m_n, h\rangle \mathbb{P}(d\omega)\\ &= \int_\Omega \langle X(\omega) - m, h\rangle\ \langle X(\omega) - m, h\rangle \mathbb{P}(d\omega) = \langle Qh, h\rangle.\end{aligned}$$

To show that X is a Gaussian random variable with law $N_{m,Q}$ it is enough to prove that

$$\int_H e^{i\langle x,h\rangle} X_\#\mathbb{P}(dx) = e^{i\langle m,h\rangle} e^{-\frac{1}{2}\ \langle Qh,h\rangle}, \quad h \in H.$$

We have in fact

$$\begin{aligned}\int_H e^{i\langle x,h\rangle}(X_\#\mu)\mathbb{P}(dy) &= \int_\Omega e^{i\langle X(\omega),h\rangle}\mathbb{P}(d\omega)\\ &= \lim_{n\to\infty}\int_\Omega e^{i\langle X_n(\omega),h\rangle}\mathbb{P}(d\omega)\\ &= \lim_{n\to\infty} e^{i\langle m_n,h\rangle} e^{-\frac{1}{2}\ \langle Q_n h,h\rangle}\\ &= e^{i\langle m,k\rangle} e^{-\frac{1}{2}\ \langle Qh,h\rangle}.\end{aligned}$$

Therefore X is Gaussian and $X_\#\mu = N_{m,Q}$. □

2.5.2. Definition of the white noise function

Here we assume that the Hilbert space H is infinite dimensional and consider a non degenerate Gaussian measure $\mu = N_Q$ in H, that is such that Ker $(Q) = \{0\}$. Since Q is compact there exists an orthonormal basis (e_k) on H and a sequence of positive numbers (λ_k) such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

We notice that Q^{-1} is unbounded, since

$$Q^{-1}e_k = \frac{1}{\lambda_k}\, e_k, \quad k \in \mathbb{N}$$

and $\lambda_k \to 0$ as $k \to \infty$ because $\operatorname{Tr} Q < +\infty$. Consequently, recalling the closed graph theorem, we see that the range $Q(H)$ does not coincide with H. However, $Q(H)$ is dense in H as proved in the next lemma.

Lemma 2.16. *$Q(H)$ is a dense subspace of H.*

Proof. Let $x_0 \in H$ be orthogonal to $Q(H)$. Then we have

$$\langle Qx, x_0\rangle = \langle x, Qx_0\rangle = 0, \quad \forall\, x \in H,$$

which yields $Qx_0 = 0$, and so $x_0 = 0$ because $\operatorname{Ker}(Q) = \{0\}$. □

It is useful to consider the square root $Q^{1/2}$ of Q. Clearly

$$Q^{1/2}x = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\, \langle x, e_k\rangle e_k, \quad x \in H.$$

$Q^{1/2}(H)$ is called the *Cameron–Martin space* of the measure μ. Arguing as before we see that $Q^{1/2}(H)$ is a subspace of H different of H and dense in H.

Consider now the following mapping W,

$$\begin{cases} W\colon Q^{1/2}(H) \subset H \to L^2(H,\mu),\ z \mapsto W_z, \\ \\ W_z(x) = \langle x, Q^{-1/2}z\rangle, \quad x \in H. \end{cases}$$

W is an isomorphism of $Q^{1/2}(H)$ into $L^2(H,\mu)$ because for all $z_1, z_2 \in H$ we have

$$\int_H W_{z_1}(x)W_{z_2}(x)\mu(dx) = \langle QQ^{-1/2}z_1, Q^{-1/2}z_2\rangle\rangle = \langle z_1, z_2\rangle. \tag{2.4}$$

Consequently, since $Q^{1/2}(H)$ is dense in H, the mapping W can be uniquely extended to the whole H. Thus for any $f \in H$, W_f is a well defined element of $L^2(H,\mu)$. We shall often write "par abus de langage" [(3)]

$$W_f(x) = \langle Q^{-1/2}x, f\rangle.$$

(3) See Remark 2.21 below.

W_f is linear in the sense that for all $\alpha, \beta \in \mathbb{R}$ we have

$$W_f(\alpha x + \beta y) = \alpha W_f(x) + \beta W_f(y), \quad \text{for } \mu\text{-a.e. } x, y \in H.$$

W is called the *white noise* function.

Exercise 2.17. Let $f \in H$. Show that

$$W_f(x) = \sum_{h=1}^{\infty} \lambda_h^{-1/2} \langle x, e_h \rangle \langle f, e_h \rangle, \quad x \in H,$$

the series being convergent on $L^2(H, \mu)$.

Proposition 2.18. *Let $z \in H$. Then W_z is a real Gaussian random variable with mean 0 and covariance $|z|^2$.*

Proof. Set $\nu_z := (W_z)_\# \mu$; we have to prove that

$$\widehat{\nu_z}(\eta) = \int_{\mathbb{R}} e^{i\xi\eta} \nu_z(d\xi) = \int_H e^{i\eta W_z(x)} \mu(dx) = e^{-\frac{1}{2}\eta^2 |z|^2}.$$

Let $(z_n) \subset Q^{1/2}(H)$ such that $z_n \to z$ in H. Then, by the dominated convergence theorem, we have

$$\begin{aligned} \int_H e^{i\eta W_z(x)} \mu(dx) &= \lim_{n\to\infty} \int_H e^{i\eta \langle Q^{-1/2} z_n, x \rangle} \mu(dx) \\ &= \lim_{n\to\infty} e^{-\frac{1}{2}\eta^2 |z_n|^2} = e^{-\frac{1}{2}\eta^2 |z|^2}. \end{aligned}$$

So, the conclusion follows. □

In a similar way, proceeding as in the proof of Corollary 2.5 we can prove the following result.

Proposition 2.19. *Let $n \in \mathbb{N}$, $z_1, \dots, z_n \in H$. Then $(W_{z_1}, \dots, W_{z_n})$ is an n-dimensional Gaussian random variable with mean 0 and covariance operator $Q_{z_1,\dots,z_n}$ given by*

$$(Q_{z_1,\dots,z_n})_{h,k} = \langle z_h, z_k \rangle, \quad h, k = 1, \dots, n. \tag{2.5}$$

Now by Proposition 2.12 it follows that

Corollary 2.20. *The random variables $W_{z_1}, \dots, W_{z_n}$ are independent if and only if $z_1, \dots, z_n$ are mutually orthogonal.*

Remark 2.21. When $z \notin Q^{1/2}(H)$ it would be tempting to define the random variable W_z by setting,

$$W_z(x) = \langle Q^{-1/2}x, z\rangle, \quad \mu\text{-a.e..}$$

This definition, however, is meaningless because $\mu(Q^{1/2}(H)) = 0$, see the next proposition.

Proposition 2.22. *We have* $\mu(Q^{1/2}(H)) = 0$.

Proof. For any $n, k \in \mathbb{N}$ set

$$U_n = \left\{ y \in H : \sum_{h=1}^{\infty} \lambda_h^{-1} y_h^2 < n^2 \right\},$$

and

$$U_{n,k} = \left\{ y \in H : \sum_{h=1}^{2k} \lambda_h^{-1} y_h^2 < n^2 \right\}.$$

Clearly $U_n \uparrow Q^{1/2}(H)$ as $n \to \infty$, and for any $n \in \mathbb{N}$, $U_{n,k} \downarrow U_n$ as $k \to \infty$. So it is enough to show that

$$\mu(U_n) = \lim_{k\to\infty} \mu(U_{n,k}) = 0. \tag{2.6}$$

We have in fact

$$\mu(U_{n,k}) = \int_{\left\{y\in\mathbb{R}^{2k}:\sum_{h=1}^{2k}\lambda_h^{-1}y_h^2<n^2\right\}} \mathop{\times}_{h=1}^{2k} N_{\lambda_k}(dy_k),$$

which, setting $z_h = \lambda_h^{-1/2} y_h$ is equivalent to

$$\mu(U_{n,k}) = \int_{\{z\in\mathbb{R}^{2k}:|z|<n\}} N_{I_{2k}}(dz) = \omega_{2k} \int_0^n e^{-\frac{r^2}{2}} r^{2k-1} dr,$$

where I_{2k} is the identity in $\mathbb{R}^{2k}$ and ω_{2k} is the surface measure of the unitary ball in $\mathbb{R}^{2k}$. Therefore we have

$$\mu(U_{n,k}) = \frac{\mu(U_{n,k})}{\mu(H)} = \frac{\int_0^n e^{-\frac{r^2}{2}} r^{2k-1} dr}{\int_0^{+\infty} e^{-\frac{r^2}{2}} r^{2k-1} dr} = \frac{\int_0^{n^2/2} e^{-\rho} \rho^{k-1} d\rho}{\int_0^{+\infty} e^{-\rho} \rho^{k-1} d\rho}.$$

Let us compute $\mu(U_{n,k})$. We have

$$\mu(U_{n,k}) = \frac{\mu(U_{n,k})}{\mu(H)} = \frac{\int_0^n e^{-\frac{r^2}{2}} r^{2k-1} dr}{\int_0^{+\infty} e^{-\frac{r^2}{2}} r^{2k-1} dr} = \frac{\int_0^{n^2/2} e^{-\rho} \rho^{k-1} d\rho}{\int_0^{+\infty} e^{-\rho} \rho^{k-1} d\rho}.$$

Therefore

$$\mu(U_{n,k}) = \frac{1}{(k-1)!}\int_0^{n^2/2} e^{-\rho}\rho^{k-1}d\rho$$
$$\leq \frac{1}{(k-1)!}\int_0^{n^2/2} \rho^{k-1}d\rho = \frac{1}{k!}\left(\frac{n^2}{2}\right)^k,$$

and (2.6) follows. □

Exercise 2.23. Let $L \in L(H)$ be symmetric (not necessarily positive or compact). Prove that there is $\epsilon_0 > 0$ such that for any $\epsilon < \epsilon_0$ we have

$$\int_H e^{-\epsilon\langle Lx,x\rangle} N_Q(dx) = [\det(1+2\epsilon Q^{1/2}LQ^{1/2})]^{-1/2}. \tag{2.7}$$

Hint. Set $M = Q^{1/2}LQ^{1/2}$, notice that M is compact and denote by (f_k) an orthonormal basis of eigenvectors of M and by (β_k) the corresponding eigenvalues. Then show that

$$\langle Lx, x\rangle = \langle MQ^{-1/2}x, Q^{-1/2}x\rangle = \sum_{k=1}^n \beta_k|W_{f_k}(x)|^2.$$

Finally, use that the random variables (W_{f_k}) are independent. (Corollary 2.20).

Exercise 2.24. Compute

$$\int_H |\langle Lx, x\rangle|^{2m}\mu(dx), \quad m \in \mathbb{N}.$$

Exercise 2.25. Prove that the range of W, $\{W_f : f \in H\}$, coincides with the closure of the dual H^* of H in $L^2(H,\mu)$.

Chapter 3
The Malliavin derivative

Let H be an infinite dimensional separable Hilbert space and $\mu = N_Q$ a non degenerate Gaussian measure. For any $\varphi \in \mathscr{E}(H)$ (the space of all exponential functions, see Section 3.2) we define the *Malliavin derivative* of φ setting

$$M\varphi := Q^{1/2} D\varphi,$$

where D represents the gradient. The factor $Q^{1/2}$ in front of the gradient, has far reaching consequences that we shall explain later (see Remark 3.2 and Chapter 11).

After some preliminaries in Section 3.1, we show in Section 3.2, using a basic integration by parts formula, that M is a closable operator in $L^2(H, \mu)$. The domain of the closure of M (still denoted by M) is the Malliavin–Sobolev space $D^{1,2}(H, \mu)$.

Section 3.4 is devoted to differential calculus with the Malliavin derivative and finally, Section 3.5 to the adjoint M^* of M, called in the literature, *Skorohod integral* or *Gaussian divergence operator*.

3.1. Notations and preliminaries

We are given a non degenerate Gaussian measure $\mu = N_Q$ in an infinite dimensional separable Hilbert space H, we denote by (e_k) an orthonomal basis in H and by (λ_k) a sequence of positive numbers such that

$$Qe_k = \lambda_k e_k, \quad k \in \mathbb{N}.$$

For any $n \in \mathbb{N}$ we define

$$P_n x = \sum_{k=1}^{n} \langle x, e_k \rangle e_k, \quad x \in H. \tag{3.1}$$

We denote by $C_b(H)$ the space of all uniformly continuous and bounded mappings from H into $\mathbb{R}$. $C_b(H)$, endowed with the norm

$$\|\varphi\|_0 = \sup_{x \in H} |\varphi(x)|, \quad \varphi \in C_b(H),$$

is a Banach space. Moreover, for any $k \in \mathbb{N}$ we denote by $C_b^k(H)$ the space of those mappings from $C_b(H)$ which are uniformly continuous and bounded together with their Fréchet derivatives of order less or equal to k. $C_b^k(H)$, endowed with the norm

$$\|\varphi\|_k = \|\varphi\|_0 + \sum_{h=1}^{k} \sup_{x\in H} |D^h\varphi(x)|, \quad \varphi \in C_b^k(H),$$

is a Banach space ($D^h\varphi$ represents the Fréchet derivative of φ of order k).

For any $k \in \mathbb{N}$ we set $x_k = \langle x, e_k\rangle$, $x \in H$, and for any $\varphi \in C_b^1(H)$ we set $D_k\varphi = \langle D\varphi, e_k\rangle$.

We denote by $L^2(H,\mu)$ the Hilbert space of all equivalence classes of μ-square integrable mappings from H into $\mathbb{R}$ endowed with the inner product,

$$\langle \varphi, \psi\rangle_{L^2(H,\mu)} = \int_H \varphi\, \psi\, d\mu, \quad \forall\, \varphi, \psi \in L^2(H,\mu).$$

For any $\varphi \in L^2(H,\mu)$ we set

$$\|\varphi\|_{L^2(H,\mu)} = \left(\int_H |\varphi(x)|^2 \mu(dx)\right)^{1/2}.$$

Finally, we denote by $L^2(H,\mu; H)$ the space of all equivalence classes of μ-square integrable mappings $F\colon H \to H$, such that

$$\|F\|_{L^2(H,\mu;H)} := \left(\int_H |F(x)|^2 \mu(dx)\right)^{1/2} < +\infty.$$

$L^2(H,\mu; H)$, endowed with the inner product,

$$\langle F, G\rangle_{L^2(H,\mu;H)} = \int_H \langle F(x), G(x)\rangle \mu(dx), \quad F, G \in L^2(H,\mu; H),$$

is a Hilbert space.

Exercise 3.1. Let $F \in L^2(H,\mu; H)$ and set

$$F_k(x) = \langle F(x), e_k\rangle, \quad k \in \mathbb{N}.$$

Show that $F_k \in L^2(H,\mu)$ for all $k \in \mathbb{N}$ and

$$F(x) = \sum_{k=1}^{\infty} F_k(x) e_k, \quad \mu\text{-a.e.},$$

the series being convergent in $L^2(H,\mu; H)$.

3.2. Definition of Malliavin derivative

Let us consider the linear operator

$$M = Q^{1/2}D \colon \mathscr{E}(H) \subset L^2(H,\mu) \to L^2(H,\mu;H),\ \varphi \mapsto Q^{1/2}D\varphi,$$

where $\mathscr{E}(H)$ is the linear span of all real parts of functions of the form

$$\varepsilon_h(x) = e^{i\langle x,h\rangle}, \quad x \in H, \quad h \in H.$$

$\mathscr{E}(H)$ is called the space of *exponential functions*. We are going to show that M is closable; the domain of its closure (still denoted by M) will be called the *Malliavin-Sobolev* space $D^{1,2}(H,\mu)$.

Remark 3.2. The reason to deal with the operator $Q^{1/2}D$, rather than the simple gradient D, is that $Q^{1/2}DW_f = f$ is well defined for all $f \in H$, whereas DW_f is only defined for $f \in Q^{1/2}(H)$. This may seem a slight difference, in fact it is essential, as we shall see, in studying Brownian motion and stochastic differential equations.

Next subsection is devoted to some properties of $\mathscr{E}(H)$, needed in what follows.

3.2.1. Approximation by exponential functions

Proposition 3.3. *For all $\varphi \in C_b(H)$ there exists a two-index sequence $(\varphi_{n,k}) \subset \mathscr{E}(H)$ such that*

(i) $\|\varphi_{n,k}\|_0 \le \|\varphi\|_0, \quad \forall\, n,k \in \mathbb{N}$.

(ii) $\displaystyle\lim_{n\to\infty}\lim_{k\to\infty} \varphi_{n,k}(x) = \varphi(x), \quad \forall\, x \in H$.

Notice that we cannot replace $(\varphi_{n,k})$ with a one-index sequence by a diagonal extraction because the convergences considered here are only pointwise.

Proof. Fix $n \in \mathbb{N}$ and set $\varphi_n(x) = \varphi(P_n x),\ x \in H$. We identify $H_n := P_n(H)$ with $\mathbb{R}^n$. Let us take a sequence $(\psi_{n,k})_{k\in\mathbb{N}} \subset C_b(H_n)$ with the following properties,

(i) $\psi_{n,k}$ is periodic with period n in all its coordinates.

(ii) $\psi_{n,k}(x) = \varphi_n(x)$ for all $x \in [-n+1/2, n-1/2]^n$.

(iii) $\|\psi_{n,k}\|_0 \le \|\varphi_n\|_0 \le \|\varphi\|_0$.

Clearly $\lim_{k\to\infty} \psi_{n,k}(x) = \varphi_n(x)$ for all $x \in H$. Moreover, by using Fourier series, we can find a sequence $(\varphi_{n,k})$ in $\mathscr{E}(H)$ (“close” to $(\psi_{n,k})$)

fulfilling

(i) $\|\varphi_{n,k}\|_0 \le \|\varphi\|_0, \quad \forall\, k \in \mathbb{N}.$
(ii) $\lim_{k\to\infty} \varphi_{n,k}(x) = \varphi_n(x), \quad \forall\, x \in H.$

Now it is clear that the two-index sequence $(\varphi_{n,k})$ fulfills the required conditions. □

In a similar way one can prove the following generalization of Proposition 3.3.

Proposition 3.4. *For all $\varphi \in C_b^1(H)$ there exists a double sequence $(\varphi_{n,k}) \subset \mathscr{E}(H)$ such that*

$$\begin{aligned} &\text{(i)} && \lim_{n\to\infty}\lim_{k\to\infty} \varphi_{n,k}(x) = \varphi(x), \quad \forall\, x \in H,\\ &\text{(ii)} && \lim_{n\to\infty}\lim_{k\to\infty} D\varphi_{n,k}(x) = D\varphi(x), \quad \forall\, x \in H,\\ &\text{(iii)} && \|\varphi_{n,k}\|_0 + \|D\varphi_{n,k}\|_0 \le \|\varphi\|_0 + \|D\varphi\|_0, \quad \forall\, n,k \in \mathbb{N}. \end{aligned} \tag{3.2}$$

Corollary 3.5. *$\mathscr{E}(H)$ is dense in $L^2(H,\mu)$.*

Proof. By using Dynkin's Theorem (see Appendix A) we check easily that $C_b(H)$ is dense in $L^2(H,\mu)$. Then the conclusion follows from Proposition 3.3 and the dominated convergence theorem. □

It is also useful to consider a dense subspace of regular functions of $L^2(H,\mu;H)$. We denote by $\mathscr{E}(H;H)$ the linear span of all functions of the form

$$F(x) = \varepsilon_h(x)z, \quad h, z \in H.$$

The following result is straightforward, its simple proof is left to the reader.

Proposition 3.6. *$\mathscr{E}(H;H)$ is dense in $L^2(H,\mu;H)$.*

3.3. The space $D^{1,2}(H,\mu)$

We shall prove here that the operator $M = Q^{1/2}D$, defined on $\mathscr{E}(H)$, is closable; that is if (φ_n) is a sequence in $\mathscr{E}(H)$ such that

$$\varphi_n \to 0 \text{ in } L^2(H,\mu), \quad Q^{1/2}D\varphi_n \to F \text{ in } L^2(H,\mu;H),$$

we have $F = 0$. [1]

To show that $Q^{1/2}D$ is closable we need an integration by parts formula.

[1] Let X be a Banach space, $T : D(T) \subset X \to X$ a mapping. T is *closable* iff $(x_n) \subset D(T)$, $x_n \to 0$, $Tx_n \to y$ implies $y = 0$.

Lemma 3.7. *For any $\varphi, \psi \in \mathscr{E}(H)$, $k \in \mathbb{N}$ the following identity holds.*

$$\int_H D_k\varphi\, \psi\, d\mu = -\int_H \varphi\, D_k\psi\, d\mu + \frac{1}{\lambda_k}\int_H x_k\, \varphi\, \psi\, d\mu. \tag{3.3}$$

Proof. Since $\mathscr{E}(H)$ is dense in $L^2(H,\mu)$, it is enough to show (3.3) when $\varphi = \varepsilon_f,\ \psi = \varepsilon_g$ where $f, g \in H$. In this case for $k \in \mathbb{N}$ we have,

$$\int_H D_k\varphi\ \psi\ d\mu = \int_H i f_k\, \varepsilon_{f+g}\ d\mu = i f_k e^{-\frac{1}{2}\langle Q(f+g), f+g\rangle}, \tag{3.4}$$

$$\int_H \varphi D_k\psi\ d\mu = \int_H i g_k\, \varepsilon_{f+g}\ d\mu = i g_k e^{-\frac{1}{2}\langle Q(f+g), f+g\rangle}. \tag{3.5}$$

Now set

$$J := \int_H x_k\varphi\psi d\mu = \int_H x_k\varepsilon_{f+g}(x)\mu(dx). \tag{3.6}$$

To compute J set

$$G(t) := \int_H e^{i\langle f+g+te_k, x\rangle}\mu(dx) = e^{-\frac{1}{2}\langle Q(f+g+te_k), f+g+te_k\rangle},$$

so that

$$J = G'(0) = i\lambda_k(f_k + g_k)e^{-\frac{1}{2}\langle Q(f+g), f+g\rangle}. \tag{3.7}$$

Now summing up (3.4), (3.5) and (3.7), yields (3.3). □

Proposition 3.8. *Let $\varphi, \psi \in \mathscr{E}(H)$ and $z \in H$. Then the following identity holds.*

$$\int_H \langle M\varphi, z\rangle\, \psi\, d\mu = -\int_H \langle M\psi, z\rangle\, \varphi\, d\mu + \int_H W_z\, \varphi\, \psi\, d\mu. \tag{3.8}$$

Proof. Using Lemma 3.7 and Exercise 2.17 we find

$$\int_H \langle M\varphi, z\rangle\, \psi\ d\mu = \sum_{k=1}^{\infty} \lambda_k^{1/2} z_k \int_H D_k\varphi\, \psi\ d\mu$$

$$= -\sum_{k=1}^{\infty} \lambda_k^{1/2} z_k \int_H D_k\psi\, \varphi\, d\mu + \sum_{k=1}^{\infty}\int_H \lambda_k^{-1/2} z_k\, x_k\, \varphi\, \psi\, d\mu$$

so, the conclusion follows.

□

Exercise 3.9. Show that (3.8) holds for all $\varphi \in \mathscr{E}(H)$, $z \in H$ and $\psi \in D^{1,2}(H,\mu)$.

We are now ready to show

Proposition 3.10. *The mapping*

$$M\colon \mathscr{E}(H) \subset L^2(H,\mu) \to L^2(H,\mu;H),\ \varphi \mapsto M\varphi,$$

is closable.

Proof. Let $(\varphi_n) \subset \mathscr{E}(H)$ be such that

$$\varphi_n \to 0 \text{ in } L^2(H,\mu), \quad M\varphi_n \to F \text{ in } L^2(H,\mu;H).$$

We claim that $F = 0$, which will prove closability of M.

Let $\psi \in \mathscr{E}(H)$ and $z \in Q^{1/2}(H)$. Then by (3.8) it follows that

$$\int_H \langle M\varphi_n, z\rangle\, \psi\, d\mu = -\int_H \langle M\psi, z\rangle\, \varphi_n\, d\mu + \int_H W_z\, \varphi_n \psi\, d\mu.$$

Letting $n \to \infty$ yields

$$\int_H \langle F(x), z\rangle \psi(x)\mu(dx) = 0.$$

By the arbitrariness of ψ and Proposition 3.6 we obtain

$$\langle F(x), z\rangle = 0, \quad \forall\, z \in H,\ x\ \mu\text{-a.e. in } H,$$

which implies $F = 0$. □

We shall denote by M the closure of $Q^{1/2}D$ and by $D^{1,2}(H,\mu)$ its domain. It is easy to see that, $D^{1,2}(H,\mu)$, endowed with the scalar product

$$\langle \varphi, \psi\rangle_{D^{1,2}(H,\mu)} = \int_H (\varphi\psi + \langle M\varphi, M\psi\rangle) d\mu,$$

is a Hilbert space. We set

$$\|\varphi\|^2_{D^{1,2}(H,\mu)} = \int_H \varphi^2\, d\mu + \int_H |M\varphi|^2\, d\mu.$$

For all $\varphi \in D^{1,2}(H,\mu)$ we call $M\varphi$ the *Malliavin derivative* of φ.

Exercise 3.11. Prove that for any $k \in \mathbb{N}$ the linear operator D_k is closable and denote by D_k its closure. Prove morover that if $\varphi \in D^{1,2}(H,\mu)$ we have

$$\langle M\varphi, e_k\rangle = \lambda_k^{1/2} D_k\varphi, \quad k \in \mathbb{N},$$

$$M\varphi(x) = \sum_{k=1}^{\infty} \lambda_k^{1/2} D_k\varphi(x) e_k, \quad \mu\text{-a.s.}$$

and

$$\|\varphi\|^2_{D^{1,2}(H,\mu)} = \int_H \varphi^2\, d\mu + \sum_{k=1}^{\infty} \int_H \lambda_k |D_k\varphi(x)|^2 \mu(dx).$$

We prove now that functions from $C^1(H)$ belong to $D^{1,2}(H,\mu)$ and that $M\varphi = Q^{1/2}D\varphi$, provided they do not grow too fast. We shall assume that φ has a polynomial growth for brevity.

Proposition 3.12. *Let $\varphi\colon H \to \mathbb{R}$ be continuously differentiable. Assume that there exists $N \in \mathbb{N}\cup\{0\}$ and $K > 0$ such that*

$$|\varphi(x)| + |D\varphi(x)| \le K(1 + |x|^{2N}), \quad \forall\, x \in H.$$

Then $\varphi \in D^{1,2}(H,\mu)$ and $M\varphi(x) = Q^{1/2}D\varphi(x)$, μ-a.e.

Proof. Assume first that $N = 0$. By Proposition 3.4 there exists a sequence $(\varphi_{n,k}) \subset \mathscr{E}(H)$ such that (3.2) holds. Consequently, by the dominate convergence theorem it follows that

$$\lim_{n\to\infty}\lim_{k\to\infty} \varphi_{n,k} = \varphi \quad \text{in } L^2(H,\mu)$$

and

$$\lim_{n\to\infty}\lim_{k\to\infty} M\varphi_{n,k} = M\varphi \quad \text{in } L^2(H,\mu;H).$$

This implies that $\varphi \in D^{1,2}(H,\mu)$ and that $M\varphi = Q^{1/2}D\varphi$.

Let now $N \in \mathbb{N}$ and set

$$\varphi_n(x) = \frac{\varphi(x)}{1 + n^{-1}|x|^{2N}}, \quad x \in H.$$

Then $\varphi_n \to \varphi$ in $L^2(H,\mu)$ as $n \to \infty$ since

$$|\varphi(x) - \varphi_n(x)| \le \frac{1}{n}\,|\varphi(x)|\,\frac{|x|^{2N}}{1 + n^{-1}|x|^{2N}} \le \frac{K}{n}\,(1 + |x|^{2N})|x|^{2N}$$

and consequently, recalling Exercise 1.16,

$$\int_H |\varphi(x) - \varphi_n(x)|^2 \mu(dx) \le \frac{K^2}{n^2}\int_H (1 + |x|^{2N})^2|x|^{4N}\mu(dx) \le \frac{K_1}{n^2},$$

where $K_1 < \infty$. In a similar way one can check that $M\varphi_n \to M\varphi$ in $L^2(H,\mu)$ as $n \to \infty$ and so, the conclusion follows. □

Remark 3.13. Proposition 3.12 extends easily to functions φ of class C^1 such that $|\varphi(x)| \le Ce^{\epsilon|x|^2}$ with $\epsilon < \inf_{k\in\mathbb{N}} \lambda_k^{-1}$ (recall Proposition 1.13).

Exercise 3.14. Let $f \in H$. Show that $W_f \in D^{1,2}(H,\mu)$ and $MW_f = f$.

We end this section by proving an useful result

Proposition 3.15. *Let $\varphi \in D^{1,2}(H,\mu)$ and $z \in H$. Then $W_z\varphi \in L^2(H,\mu)$ and*

$$\int_H (W_z\varphi)^2 d\mu \le 2|z|^2 \int_H \varphi^2 d\mu + 4\int_H |\langle M\varphi, z\rangle|^2 d\mu. \tag{3.9}$$

Proof. Step 1. $\varphi \in \mathscr{E}(H)$.

First we notice that, recalling Exercise 3.9, we have

$$\int_H \langle M\varphi, z\rangle\ \psi\ d\mu = -\int_H \langle M\psi, z\rangle\ \varphi\ d\mu + \int_H W_z\ \varphi\ \psi\ d\mu, \tag{3.10}$$

for all $\varphi \in \mathscr{E}(H)$, $\psi \in D^{1,2}(H,\mu)$ and $z \in H$. Now we replace ψ by $W_z\varphi$, which belongs to $D^{1,2}(H,\mu)$ by Exercise 3.14 and we find

$$M(W_z\varphi) = z\varphi + W_z M\varphi = z\varphi + W_z Q^{1/2} D\varphi.$$

It follows that

$$\int_H (W_z\ \varphi)^2\ d\mu = 2\int_H \langle M\varphi, z\rangle\ W_z\ \varphi\ d\mu + \int_H |z|^2\varphi^2\ d\mu.$$

Consequently[2]

$$\int_H (W_z\ \varphi)^2 d\mu \le \frac{1}{2}\int_H (W_z\ \varphi)^2\ d\mu + 2\int_H |\langle M\varphi, z\rangle|^2 d\mu + \int_H |z|^2\varphi^2\ d\mu,$$

which yields

$$\int_H (W_z\ \varphi)^2\ d\mu \le 4\int_H |\langle M\varphi, z\rangle|^2 d\mu + 2\int_H |z|^2\varphi^2\ d\mu. \tag{3.11}$$

Step 2. $\varphi \in D^{1,2}(H,\mu)$.

Let $(\varphi_n) \subset \mathscr{E}(H)$ be a sequence such that

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{in } L^2(H,\mu), \qquad \lim_{n\to\infty} M\varphi_n = M\varphi \quad \text{in } L^2(H,\mu;H).$$

Then by (3.11) it follows that

$$\begin{aligned}\int_H [W_z(\varphi_m - \varphi_n)]^2\ d\mu &\le 4\int_H |\langle M(\varphi_m - \varphi_n), z\rangle|^2 d\mu \\ &\quad + 2\int_H |z|^2(\varphi_m - \varphi_n)^2\ d\mu.\end{aligned} \tag{3.12}$$

Therefore, $(W_z\varphi_n)$ is Cauchy in $L^2(H,\mu)$ and so, $W_z\varphi \in L^2(H,\mu)$ and (3.9) follows. □

[2] By the obvious inequality: $ab \le \frac{1}{2}\,(a^2+b^2)$.

3.4. Differential calculus with Malliavin derivative

Let us show the *chain rule*.

Proposition 3.16. *Let $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_N \in D^{1,2}(H, \mu)$ and $g \in C_b^1(\mathbb{R}^N)$. Then $g(\varphi) \in D^{1,2}(H, \mu)$ and we have*

$$M(g(\varphi)) = \sum_{k=1}^{N} D_k g(\varphi) M\varphi_k. \tag{3.13}$$

Proof. For $k = 1, \ldots, N$ there exists a sequence $(\varphi_{k,n}) \subset \mathscr{E}(H)$ such that

$$\lim_{n\to\infty} \varphi_{k,n} = \varphi_k \quad \text{in } L^2(H, \mu), \qquad \lim_{n\to\infty} M\varphi_{k,n} = M\varphi_k \quad \text{in } L^2(H, \mu; H).$$

On the other hand, $g(\varphi_{1,n}, \ldots, \varphi_{N,n}) \in D^{1,2}(H, \mu)$ and

$$Mg(\varphi_{1,n}, \ldots, \varphi_{N,n}) = \sum_{k=1}^{N} D_k g(\varphi_{1,n}, \ldots, \varphi_{N,n}) M\varphi_{n,k}.$$

Therefore $Mg(\varphi_{1,k}, \ldots, \varphi_{N,k}) \in L^2(H, \mu; H)$ and

$$\lim_{n\to\infty} Mg(\varphi_{1,n}, \ldots, \varphi_{N,n}) = \sum_{k=1}^{N} D_k g(\varphi_1, \ldots, \varphi_N) M\varphi_k \quad \text{in } L^2(H, \mu; H).$$

So, (3.13) holds. □

3.4.1. Lipschitz continuous functions

We denote by $\mathrm{Lip}(H)$ the set all functions $\varphi : H \to \mathbb{R}$ such that

$$[\varphi]_1 := \sup_{x,y\in H} \frac{|\varphi(x) - \varphi(y)|}{|x - y|} < +\infty.$$

We shall prove that any real Lipschitz continuous function on H belongs to $D^{1,2}(H, \mu)$. To this purpose we recall an analytic result.

Proposition 3.17. *Assume that E is a Hilbert space and that $T : D(T) \subset E \to E$ is a linear closed operator in E. Let $\phi \in E$ and let (ϕ_n) be a sequence in E such that $\phi_n \to \phi$ in E and $\sup_{n\in\mathbb{N}} |T\phi_n|_E < \infty$. Then $\phi \in D(T)$. Moreover, there exists a subsequence (ϕ_{n_k}) of (ϕ_n) such that*

$$T\phi_{n_k} \to T\phi \quad \textit{weakly as } k \to \infty.$$

Proof. Since E is reflexive there exists a subsequence (ϕ_{n_k}) of (ϕ_n) weakly convergent to some element $\psi \in E$. Thus we have

$$\phi_{n_k} \to \phi, \quad T\phi_{n_k} \to \psi \quad \text{weakly},$$

as $k \to \infty$. Since the graph of a closed operator is also closed in the weak topology of $E \times E$, see *e.g.* [21], it follows that $\phi \in D(T)$ and $T\phi = \psi$. □

Proposition 3.18. *We have* $\mathrm{Lip}(H) \subset D^{1,2}(H,\mu)$.

Proof. *Step* 1. We assume that H is N-dimensional, $N \in \mathbb{N}$.

For any $\varphi \in C_b(H)$ define,

$$\varphi_n(x) = \int_{\mathbb{R}^N} \varphi(x-y) N_{\frac{1}{n} I_N}(dy) = \left(\frac{n}{2\pi}\right)^{N/2} \int_{\mathbb{R}^N} e^{-\frac{n}{2}|x-y|^2} \varphi(y)dy.$$

Then

$$\lim_{n\to\infty} \varphi_n(x) = \varphi(x), \quad x \in H.$$

It is clear that φ_n is continuously differentiable and that, since

$$|\varphi_n(x) - \varphi_n(x_1)| \le [\varphi]_1 |x - x_1|, \quad x, x_1 \in H,$$

we have $\|D\varphi_n\|_0 \le [\varphi]_1$ for all $n \in \mathbb{N}$. Moreover, since

$$|\varphi(x)| \le |\varphi(0)| + [\varphi]_1|x|, \quad x \in H,$$

we have

$$|\varphi_n(x)| \le |\varphi(0)| + \left(\frac{n}{2\pi}\right)^{N/2} [\varphi]_1 \int_{\mathbb{R}^N} e^{-\frac{n}{2}|x-y|^2} |y| dy, \quad x \in H,$$

so that there exists $C_1 > 0$ such that

$$|\varphi_n(x)| \le C_1(1 + |x|), \quad x \in H.$$

Therefore, $(\varphi_n) \subset D^{1,2}(H,\mu)$ (by Proposition 3.12) and we have

$$\sup_{n\in\mathbb{N}} \left[\int_{\mathbb{R}^N} |\varphi_n|^2 d\mu + \int_{\mathbb{R}^N} |Q^{1/2} D\varphi_n|^2 d\mu \right] < \infty$$

and $\varphi \in D^{1,2}(H,\mu)$ by Proposition 3.18.

Step 2. H infinite dimensional.

For any $k \in K$ set $\varphi_k(x) = \varphi(P_k x)$. Then $\varphi_k \in D^{1,2}(H, \mu)$ for all $k \in H$ by the previous step. Since

$$\varphi_k(x) \to \varphi(x), \quad Q^{1/2} D\varphi_k(x) = P_k M\varphi(P_k x), \quad x \in H,$$

we have

$$\sup_{k\in\mathbb{N}} \left[\int_{\mathbb{R}^N} |\varphi_k|^2 d\mu + \int_{\mathbb{R}^N} |Q^{1/2} D\varphi_k|^2 d\mu \right] < \infty$$

and so, $\varphi \in D^{1,2}(H, \mu)$ again by Proposition 3.18. □

Exercise 3.19. Let $N \in \mathbb{N}, \varphi_1, ..., \varphi_n \in D^{1,2}(\mu)$ and $g \in \mathrm{Lip}(\mathbb{R}^n)$. Show that $g(\varphi) \in D^{1,2}(\mu)$.

3.5. The adjoint of M

In this section we are going to study the adjoint operator M^* of the Malliavin derivative M. By definition of adjoint, $F \in D(M^*)$ if and only if there exists a positive constant K_F such that

$$\left| \int_H \langle M\varphi, F \rangle \, d\mu \right|^2 \le K_F \int_H \varphi^2 \, d\mu, \quad \forall\, \varphi \in \mathscr{E}(H). \tag{3.14}$$

In this case there is an element in $L^2(H, \mu)$, denoted $M^*(F)$, such that

$$\int_H \langle M\varphi, F \rangle \, d\mu = \int_H \varphi \, M^*(F) \, d\mu, \quad \forall\, \varphi \in \mathscr{E}(H). \tag{3.15}$$

Proposition 3.20. *We have $\mathscr{E}(H; H) \subset D(M^*)$. Moreover, for any $F \in \mathscr{E}(H; H)$ it results*

$$\begin{aligned} M^*F(x) &= -\sum_{k=1}^{\infty} (\lambda_k)^{1/2} D_k F_k(x) + \sum_{k=1}^{\infty} (\lambda_k)^{-1/2} x_k F_k(x) \\ &= -\mathrm{div}\, [Q^{1/2} F(x)] + \langle Q^{-1/2} x, F(x) \rangle, \quad x \in H. \end{aligned} \tag{3.16}$$

Proof. Since $\mathscr{E}(H; H)$ is dense in $L^2(H, \mu; H)$, it is enough to show (3.16) for

$$F(x) = \varepsilon_h(x) z, \quad h, z \in H,$$

where $\varepsilon_h(x) = e^{i\langle h,x\rangle}$, $x \in H$. In this case, for any $\varphi \in \mathscr{E}(H)$ we have, taking into account (3.8)

$$\int_H \langle M\varphi(x), F(x)\rangle\, \mu(dx) = \int_H \langle M\varphi(x), z\rangle\, \varepsilon_h(x)\mu(dx)$$

$$= -\int_H \langle M\varepsilon_h(x), z\rangle\, \varphi(x)\, \mu(dx) + \int_H W_z(x)\, \varepsilon_h(x)\, \varphi(x)\, \mu(dx)$$

$$= -i\int_H \langle h, z\rangle\, \varepsilon_h(x)\varphi(x)\, \mu(dx) + \int_H W_z(x)\, \varepsilon_h(x)\, \varphi(x)\, \mu(dx).$$

Consequently $F \in D(M^*)$ and

$$M^*(F)(x) = -i\langle h, z\rangle\, \varepsilon_h(x) + W_z(x)\, \varepsilon_h(x),$$

which yields (3.16). □

In a similar way one can show the following result.

Proposition 3.21. *Let $\varphi \in D^{1,2}(H,\mu)$, $z \in Q^{1/2}(H)$ and let F be given by*

$$F(x) = \varphi(x)z. \tag{3.17}$$

Then F belongs to the domain of M^* and

$$M^*F(x) = -\langle M\varphi(x), z\rangle + \varphi(x)W_z, \quad x \in H. \tag{3.18}$$

Finally, we prove

Proposition 3.22. *Let $\psi \in D^{1,2}(H,\mu)$, $F \in D(M^*) \cap L^\infty(H,\mu; H)$ and $M^*(F) \in L^\infty(H,\mu)$. Then $\psi F \in D(M^*)$ and it results*

$$M^*(\psi F) = \psi M^*(F) - \langle M\psi, F\rangle. \tag{3.19}$$

Proof. Step 1. Identity (3.19) holds for all $\psi \in \mathscr{E}(H)$.

For any $\varphi \in \mathscr{E}(H)$ write

$$\begin{aligned}\int_H \langle M\varphi, \psi F\rangle\, d\mu &= \int_H \langle M(\varphi\psi), F\rangle\, d\mu - \int_H \varphi\langle M(\psi), F\rangle\, d\mu\\ &= \int_H \varphi(\psi M^*(F) - \langle M(\psi), F\rangle)\, d\mu.\end{aligned}$$

Since $\psi M^*(F) - \langle M(\psi), F\rangle \in L^2(H,\mu)$, identity (3.19) follows by the arbitrariness of φ.

Step 2. Conclusion.

Let $\psi \in D^{1,2}(H,\mu)$ and let $(\psi_n) \subset \mathscr{E}(H)$ such that

$$\lim_{n\to\infty} \psi_n = \psi \quad \text{in } L^2(H,\mu), \qquad \lim_{n\to\infty} M(\psi_n) = M\psi \quad \text{in } L^2(H,\mu; H).$$

Now by Step 2 we have

$$M^*(\psi_n F) = \psi_n M^*(F) - \langle M(\psi_n), F\rangle,$$

and the conclusion follows letting $n \to \infty$. □

Remark 3.23. More results on operators M and M^* (generally denoted by δ and called Skorohod integral) can be found in [16] and [20].

Chapter 4
Brownian Motion

This chapter is devoted to the definition and construction of a Brownian motion $B = B(t),\ t \in [0, T]$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Section 4.1 is devoted to generalities on stochastic processes, whereas Brownian motion is defined in Section 4.2.

The Wiener measure is defined as the law of a Brownian motion in the space $C([0, T])$ of continuous functions in Section 4.3.

Section 4.4 is devoted to the law of the Brownian motion in $L^2(0, T)$. This section can be skipped in a first lecture.

Section 4.5 is devoted to the quadratic variation of the Brownian motion. We show that its trajectories are not of bounded variation with $\mathbb{P}$-probability one.

Section 4.6 deals with the Wiener integral

$$\int_0^T f(t)dB(t),$$

of a (deterministic) function $f : [0, T] \to \mathbb{R}$ of square integrable. We show that it can be identified with the white noise function W_f introduced in Chapter 2.

Finally, in Section 4.7 we extend the previous results to multi dimensional Brownian motions.

4.1. Stochastic Processes

For any $T > 0$ we consider the product space $[0, T] \times \Omega$ endowed with the product σ-algebra $\mathscr{B}([0, T]) \times \mathscr{F}$. A real *stochastic process* X on $[0, T],\ T > 0$, is a measurable mapping

$$X : [0, T] \times \Omega \to \mathbb{R}, \qquad (t, \omega) \mapsto X(t, \omega),$$

such that $X^{-1}(I) \in \mathscr{B}([0,T]) \times \mathscr{F}$ for all $I \in \mathscr{B}(\mathbb{R})$. In the following X will represent a stochastic process. [(1)]

By the well known properties of product spaces it follows that the section $X(t,\cdot)$ is $\mathscr{F}$-measurable for all $t \in [0,T]$ and that the section $X(\cdot,\omega)$ is $\mathscr{B}([0,T])$-measurable for all $\omega \in \Omega$, see e. g. [1]. $X(\cdot,\omega)$ are called the *trajectories* of X.

As we did for random variables, it is useful to consider equivalence classes of stochastic processes. We say that two stochastic processes X, Y are *equivalent* if

$$\begin{aligned}\mathbb{P}(X(t,\cdot)=Y(t,\cdot))\\ =\mathbb{P}(\{\omega\in\Omega:\ (X(t,\omega)=Y(t,\omega)\})=1,\quad \forall\, t\in[0,T].\end{aligned} \tag{4.1}$$

It is easy to see that (4.1) defines an equivalence relation, so that the set of all stochastic processes splits in mutually disjoint equivalence classes. Any element of an equivalence class of processes $\widetilde{X}$ is called a *version* of $\widetilde{X}$.

Exercise 4.1. Let X and Y be equivalent stochastic processes and let $(t_1,\ldots,t_n) \in [0,T]^n$. Show that the n–dimensional random variables

$$(X(t_1),\ldots,X(t_n)),\quad (Y(t_1),\ldots,Y(t_n))$$

have the same law (see Section 2.4.1).

In the following when no confusion may arise, we will not distinguish between processes and equivalence classes of processes. Also we shall often write $X(t,\cdot) = X(t)$.

4.1.1. Continuous stochastic processes

A stochastic process X on $[0,T]$ is said to be *continuous* if $X(\cdot,\omega)$ is continuous on $[0,T]$ for $\mathbb{P}$-almost all $\omega \in A$.

X is said to be *p-mean continuous*, $p \geq 1$, if

$$\mathbb{E}(|X(t,\cdot)|^p) < \infty,\quad \forall\, t \in [0,T]$$

and the mapping

$$[0,T] \to L^p(\Omega, ,\mathbb{P}),\quad t \to X(t,\cdot),$$

is continuous.

(1) We notice that in the literature often the measurability of X is not required.

An equivalence class $\widetilde{X}$ of stochastic processes is said to be *continuous* if it possesses a continuous version.

An equivalence class of processes $\widetilde{X}$ is said to be *p-mean continuous* if all its versions are *p-mean continuous*. It is easy to check that the set of all p-mean continuous equivalence class of processes, endowed with the norm

$$\|X\| := \left(\sup_{t\in[0,T]} \mathbb{E}(|X(t,\cdot)|^p) \right)^{1/p}, \tag{4.2}$$

is a Banach space, denoted $C([0,T]; L^p(\Omega, \mathscr{F}, \mathbb{P}))$.

A stochastic process X in $[0,T]$ is called *Gaussian* if for any $n \in \mathbb{N}$, $0 < t_1 < ... < t_n \le T$, the $\mathbb{R}^n$-valued random variable $(X(t_1), ..., X(t_n))$ is Gaussian.

4.1.2. Brownian motion

Definition 4.2. A real *Brownian motion* on $[0,T]$ is a real stochastic process B such that

(i) $B(0,\cdot) = 0$ and if $0 \le s < t \le T$, $B(t,\cdot) - B(s,\cdot)$ is a real Gaussian random variable with law N_{t-s}.

(ii) For all $0 < t_1 < ... < t_n \le T$ the random variables,

$$B(t_1,\cdot),\ \ B(t_2,\cdot) - B(t_1,\cdot), \cdots, B(t_n,\cdot) - B(t_{n-1},\cdot)$$

are independent. We say that B is a process with *independent increments*.

(iii) B is continuous.

An equivalence class of stochastic processes $\widetilde{B}$ is called a Brownian motion if it possesses a version which is a Brownian motion.

We show now that any Brownian motion is a Gaussian process.

Proposition 4.3. *Let $B(t)$, $t\in[0,T]$, be a Brownian motion on $(\Omega,\mathscr{F},\mathbb{P})$. Then B is a Gaussian process. Moreover, if $n \in \mathbb{N}$, $0 < t_1 < ... < t_n \le T$, the $\mathbb{R}^n$-valued random variable $(B(t_1), ..., B(t_n))$ is Gaussian with mean* 0 *and covariance operator*

$$Q_{t_1,...,t_n} = SD_{t_1,...,t_n}S^*, \tag{4.3}$$

where

$$D_{t_1,...,t_n} = \text{diag }(t_1, t_2 - t_1, ..., t_n - t_{n-1}), \tag{4.4}$$

$S \in L(\mathbb{R}^n)$ is defined by,

$$S(x_1, ..., x_n) = (x_1, x_1 + x_2, ..., x_1 + \cdots + x_n) \tag{4.5}$$

and S^* is the adjoint of S. Finally, if $A \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\begin{aligned}\mathbb{P}((B(t_1), ..., B(t_n) \in A)) = & \frac{1}{\sqrt{(2\pi)^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})}} \\ & \times \int_A e^{-\frac{\xi_1^2}{2t_1} - \frac{(\xi_2-\xi_1)^2}{2(t_2-t_1)} - \cdots - \frac{(\xi_n-\xi_{n-1})^2}{2(t_n-t_{n-1})}} \, d\xi_1 \cdots d\xi_n.\end{aligned} \tag{4.6}$$

Proof. Let $0 < t_1 < ... < t_n$, set

$$X := (B(t_1), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1}))$$

and

$$Z := (B(t_1), ..., B(t_n)).$$

Since random variables $B(t_1)$, $B(t_2) - B(t_1)$, ..., $B(t_n) - B(t_{n-1})$ are independent, by Exercise 2.11 it follows that X is an n-dimensional Gaussian random variable of mean 0 and covariance

$$Q_X = \text{diag } (t_1, t_2 - t_1, ..., t_n - t_{n-1}) =: D_{t_1,...,t_n}.$$

Now, consider the linear mapping $S \in L(\mathbb{R}^n)$ defined by (4.5). It is clear that $Z = S(X)$. Therefore by Proposition 2.4 Z is Gaussian and its covariance operator is given by (4.3). Let us finally show (4.6). First notice that det $S = 1$ and that

$$S^{-1}\xi = (\xi_1, \xi_2 - \xi_1, ..., \xi_n - \xi_{n-1}).$$

Since the law of $(B(t_1), ..., B(t_n))$ is $N_{SD_{t_1,...,t_n}S^*}$, we have

$$\begin{aligned}&\mathbb{P}((B(t_1), ..., B(t_n) \in A) \\ &= \frac{1}{\sqrt{(2\pi)^n \det[SD_{t_1,...,t_n}S^*]}} \int_A e^{-\frac{1}{2}\langle (SD_{t_1,...,t_n}S^*)^{-1}\xi, \xi\rangle} \, d\xi_1 \cdots d\xi_n.\end{aligned}$$

But

$$\begin{aligned}\langle (SD_{t_1,...,t_n}S^*)^{-1}\xi, \xi\rangle &= \langle (S^*)^{-1}D^{-1}_{t_1,...,t_n}S^{-1}\xi, \xi\rangle = \langle D^{-1}_{t_1,...,t_n}S^{-1}\xi, S^{-1}\xi\rangle \\ &= \frac{\xi_1^2}{t_1} + \frac{(\xi_2 - \xi_1)^2}{(t_2 - t_1)} + \cdots + \frac{(\xi_n - \xi_{n-1})^2}{(t_n - t_{n-1})}.\end{aligned}$$

Finally, taking into account that det $S = 1$ and

$$\det\,[SD_{t_1,\dots,t_n}S^*] = t_1(t_2 - t_1)\cdots(t_n - t_{n-1}),$$

we obtain (4.6). □

Exercise 4.4. (i) Let $s, t \in [0, T]$. Show that

$$\mathbb{E}[B(s)B(t)] = \min\{s, t\}.$$

(ii) Let $s \in [0, T]$. Show that the random variables $B(s) - sB(T)$ and $B(T)$ are independent.

Hint. For (ii) use that $(B(s) - sB(T),\, B(T))$ is a two-dimensional Gaussian random variable.

4.2. Construction of a Brownian motion

We are going to construct a Brownian motion on $[0, T]$, $T > 0$, on the probability space $(H, \mathscr{B}(H), \mu)$, where $H = L^2(0, T)$ and $\mu = N_Q$, Q being an arbitrary (but fixed) non degenerate Gaussian measure in H.

We define

$$B(t) = W_{\mathbb{1}_{[0,t]}}, \quad t \in [0, T], \tag{4.7}$$

where

$$\mathbb{1}_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0, t],\\[1ex] 0 & \text{otherwise.}\end{cases}$$

Here W is the white noise function introduced in Chapter 2.

Theorem 4.5. *Let $B(t)$, $t \geq 0$, the equivalence class of processes defined by* (4.7)*; then B is a Brownian motion.*

Moreover B belongs to $C([0, T]; L^{2m}(H, \mu))$ for any $m \in \mathbb{N}$.

Proof. Clearly, for any $t \geq 0$, $B(t)$ is a Gaussian random variable N_t and for any $t > s \geq 0$, $B(t) - B(s) = W_{\mathbb{1}_{(s,t]}}$ is a Gaussian random variable N_{t-s}. So, B fulfills (i). Let us prove (ii). Since the system of elements of H,

$$(\mathbb{1}_{[0,t_1]}, \mathbb{1}_{(t_1,t_2]}, \dots, \mathbb{1}_{(t_{n-1},t_n]}),$$

is orthogonal, it follows from Corollary 2.20 that the random variables

$$B(t_1),\; B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1})$$

are independent. Thus (ii) follows. To prove the last statement we notice that, since $B(t) - B(s)$ is a Gaussian random variable N_{t-s} for every $t > s$, we have

$$\mathbb{E}(|B(t) - B(s)|^{2m}) = \int_{\mathbb{R}} |\xi|^{2m} N_{t-s}(d\xi) = \frac{(2m)!}{2^m m!} \, |t - s|^m.$$

Therefore $B \in C([0, T]; L^{2m}(H, \mu))$.

It remains to show that B is continuous. For this we shall use the *factorization* method based on the following elementary identity

$$\int_s^t (t - \sigma)^{\alpha-1}(\sigma - s)^{-\alpha} d\sigma = \frac{\pi}{\sin \pi\alpha}, \quad 0 \le s \le \sigma \le t, \tag{4.8}$$

where $\alpha \in (0, 1)$. To check (4.8) it is enough to set $\sigma = r(t - s) + s$ so that (4.8) becomes

$$\int_0^1 (1 - r)^{\alpha-1} r^{-\alpha} dr = \frac{\pi}{\sin \pi\alpha}.$$

By (4.8) it is easy to check that

$$1\!\!1_{[0,t]}(s) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t - \sigma)^{\alpha-1} g_\sigma(s) d\sigma, \quad s \ge 0, \tag{4.9}$$

where

$$g_\sigma(s) = 1\!\!1_{[0,\sigma]}(s)(\sigma - s)^{-\alpha}, \quad s \ge 0. \tag{4.10}$$

Moreover, notice that $g_\sigma \in L^2(0, T)$ and $|g_\sigma|^2 = \frac{\sigma^{1-2\alpha}}{1-2\alpha}$, provided $\alpha \in (0, 1/2)$. So, we shall assume that $\alpha \in (0, 1/2)$ from now on. Recalling that the mapping

$$H \to L^2(H, \mu), \; f \mapsto W_f,$$

is continuous, we obtain the following representation formula for B,

$$B(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t - \sigma)^{\alpha-1} W_{g_\sigma} d\sigma. \tag{4.11}$$

We show now that the mapping $\sigma \to W_{g_\sigma}(x) \in L^{2m}(0, T)$ for μ-almost all $x \in H$. In fact, taking in account that W_{g_σ} is a real Gaussian random variable with law $N_{\frac{\sigma^{1-2\alpha}}{1-2\alpha}}$, we have

$$\int_H |W_{g_\sigma}(x)|^{2m} \mu(dx) = \frac{(2m)!}{2^m m!}(1 - 2\alpha)^{-m} \sigma^{m(1-2\alpha)}. \tag{4.12}$$

Since $\alpha < 1/2$ we have, by the Fubini theorem,

$$\begin{aligned} &\int_0^T \left[\int_H |W_{g_\sigma}(x)|^{2m} \mu(dx)\right] d\sigma \\ &= \int_H \left[\int_0^T |W_{g_\sigma}(x)|^{2m} d\sigma\right] \mu(dx) < +\infty. \end{aligned} \tag{4.13}$$

Therefore $\sigma \to W_{g_\sigma}(x) \in L^{2m}(0, T)$ μ–a.e. and the conclusion follows from the analytic Lemma 4.6 below. □

Lemma 4.6. *Let* $m > 1, \alpha \in (1/(2m), 1), T > 0,$ *and* $f \in L^{2m}(0, T; H)$. *Set*

$$F(t) = \int_0^t (t-\sigma)^{\alpha-1} f(\sigma) d\sigma, \quad t \in [0, T].$$

Then $F \in C([0, T]; H)$.

Proof. By Hölder's inequality we have (notice that $2m\alpha - 1 > 0$),

$$|F(t)| \le \left(\int_0^t (t-\sigma)^{(\alpha-1)\frac{2m}{2m-1}} d\sigma\right)^{\frac{2m-1}{2m}} |f|_{L^{2m}(0,T;H)}. \tag{4.14}$$

Therefore $F \in L^\infty(0, T; H)$. It remains to show the continuity of F. Continuity at 0 follows from (4.14). Let us prove that F is continuous on $[\frac{t_0}{2}, T]$ for any $t_0 \in (0, T]$. Let us set for $\varepsilon < \frac{t_0}{2}$,

$$F_\varepsilon(t) = \int_0^{t-\varepsilon} (t-\sigma)^{\alpha-1} f(\sigma) d\sigma, \; t \in [0, T].$$

F_ε is obviously continuous on $[\frac{t_0}{2}, T]$. Moreover, using once again Hölder's inequality, we find

$$|F(t) - F_\varepsilon(t)| \le M \left(\frac{2m-1}{2m\alpha - 1}\right)^{\frac{2m-1}{2m}} \varepsilon^{\alpha - \frac{1}{2m}} |f|_{L^{2m}(0,T;H)}.$$

Thus $\lim_{\varepsilon \to 0} F_\varepsilon(t) = F(t)$, uniformly on $[\frac{t_0}{2}, T]$, and F is continuous as required. □

Exercise 4.7. Prove that any Brownian motion B possesses a version having Hölder continuous trajectories with any exponent $\beta < 1/2$.

Exercise 4.8. Let B be a Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Prove that the following are Brownian motions.

(i) $B_1(t) = B(t+h) - B(h), \; t \ge 0$, where $h > 0$ is given.
(ii) $B_2(t) = \alpha B(\alpha^{-2} t), \; t \ge 0$, where $\alpha > 0$ is given.
(iii) $B_3(t) = t B(1/t), \; t > 0, \; B_3(0) = 0$.
(iv) $B_4(t) = -B(t), \; t \ge 0$.

4.3. The Wiener measure

Let us consider a Brownian motion $B(t)$, $t \in [0, T]$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We are going to study the law of $B(\cdot)$ on $C_0([0, T])$.

$C([0, T])$ is the Banach space of all real continuous functions in $[0, T]$ endowed with the sup norm and $C_0([0, T])$ is the subspace of $C([0, T])$ of functions vanishing at 0. We set $C_T := C_0([0, T])$.

We already know that $\mathbb{P}$-a.s. $B(\cdot)$ belongs to $C_0([0, T])$. We are going to show that the mapping

$$\gamma : \Omega \to C_T, \quad \omega \mapsto B(\cdot, \omega),$$

is measurable. For this it is convenient to introduce the family $\mathscr{C}$ of all *cylindrical subsets* of C_T, that is all sets of the form

$$C_{t_1,t_2,\dots,t_n;A} := \{\alpha \in C_T : \ (\alpha(t_1), \dots, \alpha(t_n)) \in A\}.$$

where $0 < t_1 < \cdots < t_n \le T$ and $A \in \mathscr{B}(\mathbb{R}^n)$. Note that

$$C_{t_1,t_2,\dots,t_n;A} = C_{t_1,t_2,\dots,t_n,t_{n+1},\dots,t_{n+k};A\times\mathbb{R}^k}, \quad k, n \in \mathbb{N}.$$

Using this identity one can easily see that $\mathscr{C}$ is an algebra of sets. Moreover, the σ-algebra generated by $\mathscr{C}$ coincides with $\mathscr{B}(C_T)$ since any ball of C_T is a countable intersection of cylindrical sets, as easily seen.

Proposition 4.9. *The mapping*

$$\gamma : \Omega \to C_T, \quad \omega \mapsto B(\cdot, \omega), \tag{4.15}$$

is measurable.

Proof. Let $n \in \mathbb{N}, 0 < t_1 < \cdots < t_n$ and $A \in \mathscr{B}(\mathbb{R}^n)$. Then

$$\gamma^{-1}(C_{t_1,t_2,\dots,t_n;A}) = \{\omega \in \Omega : \ (B(t_1), \dots, B(t_n) \in A\} \in \mathscr{F}.$$

Since cylindrical sets generate the σ-algebra $\mathscr{B}(C_T)$, the conclusion follows. $\square$

The law $\mathbb{Q}$ of the mapping γ (defined by (4.15)) is called the *Wiener measure* on $(C_T, \mathscr{B}(C_T))$. Therefore we have.

$$\int_\Omega \varphi(B(\cdot, \omega))\, \mathbb{P}(d\omega) = \int_{C_T} \varphi(\alpha)\, \mathbb{Q}(d\alpha), \tag{4.16}$$

for any $\varphi : C_T \to \mathbb{R}$ bounded and Borel.

Remark 4.10. The Wiener measure is explicit on cylindrical sets. In fact, by (4.16) we have

$$\mathbb{Q}(C_{t_1,t_2,\dots,t_n;A}) = \mathbb{P}((B(t_1), \dots, B(t_n)) \in A)$$

so that, by formula (4.6)

$$\mathbb{Q}(C_{t_1,t_2,\dots,t_n;A}) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \times \int_A e^{-\frac{\xi_1^2}{2t_1}-\frac{(\xi_2-\xi_1)^2}{2(t_2-t_1)}-\dots-\frac{(\xi_n-\xi_{n-1})^2}{2(t_n-t_{n-1})}}\, d\xi_1\cdots d\xi_n. \tag{4.17}$$

A Borel mapping $\psi : C_T \to \mathbb{R}$ is said to be *cylindrical* if there exists $n \in \mathbb{N}$, $0 < t_1 < \cdots t_n \le T$ and a Borel function $g : \mathbb{R}^n \to \mathbb{R}$, such that

$$\psi(\alpha) = g(\alpha(t_1), \dots, \alpha(t_n)), \quad \alpha \in C_T. \tag{4.18}$$

Exercise 4.11. Let ψ be given by (4.18) and assume that g is bounded. Show that

$$\int_{C_T} \psi(\alpha)\mathbb{Q}(d\alpha) = \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \times \int_{\mathbb{R}^n} e^{-\frac{\xi_1^2}{2t_1}-\frac{(\xi_2-\xi_1)^2}{2(t_2-t_1)}-\dots-\frac{(\xi_n-\xi_{n-1})^2}{2(t_n-t_{n-1})}}\, g(\xi_1, \dots, \xi_n)\, d\xi_1\cdots d\xi_n. \tag{4.19}$$

4.3.1. The standard Brownian motion

Let us define the following stochastic process in $(C_T, \mathscr{B}(C_T), \mathbb{Q})$,

$$Z(t)(\alpha) = \alpha(t), \quad \alpha \in C_T, \quad t \ge 0.$$

Proposition 4.12. *Z is a Brownian motion in $(C_T, \mathscr{B}(C_T), \mathbb{Q})$, called the* standard Brownian motion.

Proof. It is obvious that $Z(\cdot)$ is continuous. Let us now show that $Z(t) - Z(s)$ is a Gaussian random variable N_{t-s} for any $t > s > 0$. We have in fact by (4.16),

$$\int_{C_T} e^{ih(Z(t)-Z(s))} d\mathbb{Q} = \int_{C_T} e^{i(\alpha(t)-\alpha(s))h}\mathbb{Q}(d\alpha)$$

$$= \int_{\Omega} e^{i(B(t,\omega)-B(s,\omega))h}\mathbb{P}(d\omega)$$

$$= \mathbb{E}[e^{i(B(t)-B(s))}] = e^{-\frac{1}{2}(t-s)h^2}, \quad h \in \mathbb{R}.$$

Let us show finally that $Z(t),\ t \geq 0$, has independent increments. Let $0 = t_1 < t_2 < \cdots < t_n \leq T$. Then, recalling that B has independent increments, we have for $h_1, ..., h_n \in \mathbb{R}$,

$$\int_{C_T} e^{ih_1 Z(t_1)+ih_2(Z(t_2)-Z(t_1))+\cdots+ih_n(Z(t_n)-Z(t_{n-1}))} d\mathbb{Q}$$

$$= \int_{\Omega} e^{ih_1 B(t_1)+ih_2(B(t_2)-B(t_1))+\cdots+ih_n(B(t_n)-B(t_{n-1}))} d\mathbb{P}$$

$$= \int_{\Omega} e^{ih_1 B(t_1)} d\mathbb{P} \int_{\Omega} e^{ih_2(B(t_2)-B(t_1))} d\mathbb{P} \times \cdots \times \int_{\Omega} e^{ih_n(B(t_n)-B(t_{n-1}))} d\mathbb{P}$$

$$= \int_{C_T} e^{ih_1 Z(t_1)} d\mathbb{Q} \int_{C_T} e^{ih_2(Z(t_2)-Z(t_1))} d\mathbb{Q} \times \cdots \times \int_{C_T} e^{ih_n(Z(t_n)-Z(t_{n-1}))} d\mathbb{Q}.$$

The conclusion follows. □

4.4. Law of the Brownian motion in $L^2(0, T)^{(*)}$

Let B be the Brownian motion defined by (4.7) in $(H, \mathscr{B}(H), \mu)$. Let us consider the mapping,

$$B : H \to L^2(0, T),\ x \mapsto B(x),$$

where $B(x)(t) = B_t(x),\ x \in H,\ t \in [0, T]$ and denote by $\nu := B_{\#}\mu$ the law of B. ν is a probability measure on $(L^2(0, T), \mathscr{B}(L^2(0, T)))$. [2]

Proposition 4.13. *ν is a non degenerate Gaussian measure on $L^2(0, T)$ with mean 0 and covariance operator $\mathscr{Q}$ defined by*

$$(\mathscr{Q}h)(t) = \int_0^T \min\{t, s\} h(s) ds, \quad h \in L^2(0, T),\ t \in [0, T]. \quad (4.20)$$

Proof. We have to show that

$$\mathbb{E}[e^{i\langle h,B\rangle_{L^2(0,T)}}] = \mathbb{E}[e^{i\int_0^T h(s)B(s)ds}] = e^{-\frac{1}{2}\langle \mathscr{Q}h,h\rangle_{L^2(0,T)}}$$

Given $h \in L^2(0, T)$ set

$$k(t) = \int_t^T h(s) ds.$$

[2] Since $B(\cdot)(x)$ is continuous for almost all $x \in H$, the restriction of ν to $C_0([0, T])$ coincides with the Wiener measure introduced before.

Then, taking into account (4.38), we have

$$\int_0^T h(t)B(t)dt = \int_H k(t)dB(t).$$

Therefore by Proposition 4.22 the law of $\langle h, B\rangle_{L^2(0,T)}$ is the Gaussian measure with mean 0 and covariance

$$\int_0^T k^2(t)dt.$$

Therefore

$$\mathbb{E}[e^{i\int_0^T h(s)B(s)ds}] = e^{-\frac{1}{2}\int_0^T k^2(t)dt}$$

and one can easily check that

$$\int_0^T k^2(t)dt = \int_0^T (\mathscr{Q}h)(t)h(t)dt.$$

Let us finally show that ν is non degenerate. Let $h \in L^2(0,T)$ be such that $\mathscr{Q}h = 0$. Then we have

$$\int_0^t sh(s)ds + t\int_t^T h(s)ds = 0, \quad t \in [0,T]. \tag{4.21}$$

Differentiating (4.21) with respect to t yields

$$th(t) + \int_t^T h(s)ds - th(t) = 0,$$

so that

$$\int_t^T h(s)ds = 0, \quad \forall\, t \in [0,T].$$

Therefore $h = 0$. So, $\mathscr{Q}$ is not degenerate. □

Exercise 4.14. Let $\mathscr{A} = \mathscr{Q}^{-1}$. Show that

$$\begin{cases} \mathscr{A}h(t) = -h''(t), \ \forall\, h \in D(\mathscr{A}) \\ \\ D(\mathscr{A}) = \{h \in H^2(0,T) : \ h(0) = h'(T) = 0\}. \end{cases}$$

4.5. Quadratic variation of the Brownian motion

We are concerned with a real Brownian motion $B(t),\ t \in [0, T]$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We are going to show that the *quadratic variation* of B is $\mathbb{P}$-a.e. finite (see below the definition of quadratic variation). As a consequence, we shall show that almost all trajectories of B are not of bounded variation.

We need some notation. A *partition* σ of $[0, T]$ is a set $\sigma = \{t_0, t_1, ..., t_n\}$ of points of $[0, T]$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$. For any partition $\sigma = \{t_0, t_1, ..., t_n\}$ we set

$$|\sigma| = \max_{1\le i\le n} |t_i - t_{i-1}|.$$

We denote by $\Sigma(0, T)$ the set of all partitions of $[0, T]$, endowed with the partial ordering

$$\sigma_1 \le \sigma_2 \quad \text{if and only if } |\sigma_1| \le |\sigma_2|.$$

For any $\sigma = \{t_0, t_1, ..., t_n\} \in \Sigma$ we set

$$J_\sigma = \sum_{k=1}^{n} |B(t_k) - B(t_{k-1}|^2.$$

Theorem 4.15. *We have*

$$\lim_{|\sigma|\to 0} J_\sigma = T \quad \text{in } \ L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

We express the theorem by saying that the quadratic variation of B is T.

Proof. Note first that

$$\mathbb{E}(J_\sigma) = \sum_{k=1}^{n} \mathbb{E}(|B(t_k) - B(t_{k-1}|^2) = \sum_{k=1}^{n}(t_k - t_{k-1}) = T,$$

because $B_{t_k} - B_{t_{k-1}}$ is a real Gaussian random variable with law $N_{t_k - t_{k-1}}$.

Now let us compute $\mathbb{E}(|J_\sigma - T|^2)$. Write

$$\mathbb{E}(|J_\sigma - T|^2) = \mathbb{E}(J_\sigma^2) - 2T\mathbb{E}(J_\sigma) + T^2 = \mathbb{E}(J_\sigma^2) - T^2. \tag{4.22}$$

Moreover

$$\begin{aligned}\mathbb{E}|J_\sigma|^2 &= \mathbb{E}\left|\sum_{k=1}^{n} |B(t_k) - B(t_{k-1})|^2\right|^2 \\ &= \mathbb{E}\sum_{k=1}^{n} |B(t_k) - B(t_{k-1})|^4 \\ &\quad + 2\sum_{h<k=1}^{n} \mathbb{E}|B(t_h) - B(t_{h-1})|^2|B(t_k) - B(t_{k-1})|^2.\end{aligned}$$

But we have

$$\mathbb{E}\sum_{k=1}^{n}|B(t_k)-B(t_{k-1})|^4 = 3\sum_{k=1}^{n}(t_k-t_{k-1})^2, \tag{4.23}$$

and, since $B(t_h)-B(t_{h-1})$ and $B(t_k)-B(t_{k-1})$ are independent, we have

$$\sum_{h<k=1}^{n}\mathbb{E}|B(t_h)-B(t_{h-1})|^2|B(t_k)-B(t_{k-1})|^2 = \sum_{h<k=1}^{n}(t_h-t_{h-1})(t_k-t_{k-1}). \tag{4.24}$$

Therefore

$$\begin{aligned}
\mathbb{E}|J_\sigma|^2 &= 3\sum_{k=1}^{n}(t_k-t_{k-1})^2 + 2\sum_{h<k=1}^{n}(t_h-t_{h-1})(t_k-t_{k-1})\\
&= 2\sum_{k=1}^{n}(t_k-t_{k-1})^2 + \left(\sum_{k=1}^{n}(t_k-t_{k-1})\right)^2.\\
&= 2\sum_{k=1}^{n}(t_k-t_{k-1})^2 + T^2.
\end{aligned} \tag{4.25}$$

Now, substituting (4.25) on (4.22), we obtain

$$\mathbb{E}\left(|J_\sigma - T|^2\right) = 2\sum_{k=1}^{n}(t_k-t_{k-1})^2 \to 0,$$

as $|\sigma|\to 0$. □

4.5.1. Irregularity of trajectories of B

We consider here the Wiener measure and the standard Brownian motion defined in Section 4.3.1. Then for any $\alpha \in C_T$ and any $\sigma = \{t_0, t_1, \ldots, t_n\} \in \Sigma$, we set

$$J_\sigma(\alpha) = \sum_{k=1}^{n}|\alpha(t_k)-\alpha(t_{k-1}|^2.$$

By Theorem 4.15 it follows that

$$\lim_{|\sigma|\to 0} J_\sigma = T \quad \text{in } L^2(C_T, \mathscr{B}(C_T), \mathbb{Q}).$$

Consequently there exists a sequence $(\sigma_N)\subset\Sigma$ and a Borel subset $\Lambda\subset C_T$ such that $\mathbb{Q}(\Lambda)=1$ and

$$\lim_{N\to\infty} J_{\sigma_N}(\alpha) = T, \quad \forall\, \alpha\in\Lambda. \tag{4.26}$$

Obviously Λ depends on the particular sequence (σ_N) chosen, which we consider fixed from now on.

Proposition 4.16. *We have* $\Lambda \cap BV(0,T) = \varnothing$.

By $BV(0,T)$ we denote the set of all real function α on $[0,T]$ such that

$$\|\alpha\|_{TV} := \sup_{\sigma\in\Sigma} \sum_{j=1}^{m} |\alpha_{t_j} - \alpha_{t_{j-1}}| < \infty, \tag{4.27}$$

where $\sigma = \{t_0, t_1, ..., t_n\} \in \Sigma(0,T)$.

Proof. Assume by contradiction that there exists $\alpha \in \Lambda \cap BV(0,T)$. Then for any partition $\sigma = \{t_0, t_1, ..., t_n\} \in \Sigma(0,T)$ we have

$$\sum_{j=1}^{m} |\alpha_{t_j} - \alpha_{t_{j-1}}| \le \|\alpha\|_{TV} < \infty. \tag{4.28}$$

Now, given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$t, s \in [0,T],\ \ |t-s| < \delta_\epsilon \Rightarrow |\alpha(t) - \alpha(s)| < \epsilon \tag{4.29}$$

and $N_\epsilon \in \mathbb{N}$ such that

$$N > N_\epsilon \Rightarrow |\sigma_N| < \delta_\epsilon. \tag{4.30}$$

Let $N > N_\epsilon$ and $\sigma_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$. Then we have

$$J_{\sigma_N}(\alpha) = \sum_{k=1}^{N} |\alpha(t_k) - \alpha(t_{k-1})|^2 \le \epsilon \sum_{k=1}^{N} |\alpha(t_k) - \alpha(t_{k-1})| \le \epsilon \|\alpha\|_{TV}.$$

This contradicts (4.26) by the arbitrariness of ϵ. □

Remark 4.17. We note an unexpected property of the set Λ. Take $\alpha \in \Lambda$, so that α is not identically 0. We know that α fulfills (4.26). But then $\kappa\alpha$ does not fulfill (4.26) for any $\kappa \neq 1$ and so, does not belong to Λ.

4.6. Wiener integral

We are still concerned with a real Brownian motion $B(t)$, $t \in [0,T]$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Given a continuous function $f : [0,T] \to \mathbb{R}$ we aim to define the stochastic integral

$$\int_0^T f(s) dB(s).$$

The first idea would be to consider the Stieltjes integral

$$\int_0^T f(s) dB(s, \omega),$$

for all $\omega \in \Omega$ such that $B(\cdot, \omega) \in BV(0, T)$. But we know from Theorem 4.15 that the set of such ω has probability zero. Therefore we shall define the integral as the limit in $L^2(\Omega, \mathscr{F}, \mathbb{P})$ as $|\sigma| \to 0$ of the following Riemannian sums [(3)]

$$I_\sigma = \sum_{j=1}^{n} f(t_{j-1})(B(t_j) - B(t_{j-1})),$$

where $\sigma = \{t_1, ..., t_n\} \in \Sigma$. To this purpose we first prove some basic identities.

Lemma 4.18. *Let $\sigma \in \Sigma$. Then we have*

$$\mathbb{E}(I_\sigma) = 0 \tag{4.31}$$

and

$$\mathbb{E}\left(I_\sigma^2\right) = \sum_{j=1}^{n} |f(t_{j-1})|^2 (t_j - t_{j-1}). \tag{4.32}$$

Proof. Identity (4.31) is obvious. Let us prove (4.32). We have

$$\begin{aligned} \mathbb{E}(|I_\sigma|^2) &= \mathbb{E}\left(\sum_{j=1}^{n} |f(t_{j-1})|^2 [B(t_j) - B(t_{j-1})]^2\right) \\ &+ 2\mathbb{E}\left(\sum_{j<k} f(t_{j-1}) f(t_{k-1})[B(t_j) - B(t_{j-1})][B(t_k) - B(t_{k-1})]\right). \end{aligned} \tag{4.33}$$

Now the conclusion follows taking into account that the law of $B(t_j) - B(t_{j-1})$ is $N_{t_{j-1}-t_j}$ and that $B(t_j) - B(t_{j-1})$ is independent of $B(t_k) - B(t_{k-1})$ for $k \neq j$. □

We can now prove the following result.

Theorem 4.19. *For any $f \in C([0, T])$ there exists the limit*

$$\lim_{|\sigma| \to 0} I_\sigma =: \int_0^T f(s) dB(s), \tag{4.34}$$

(3) It is not essential here, unless than in the definition of Itô's integral (see Chapter 7) to take $f(t_{j-1})$ in the integral sums, one could take instead $f(t_j^*)$ where $t_j^* \in [t_{j-1}, t_j]$.

in $L^2(\Omega, \mathscr{F}, \mathbb{P})$. Moreover we have

$$\mathbb{E}\left(\int_0^T f(s)dB(s)\right) = 0, \tag{4.35}$$

and

$$\mathbb{E}\left|\int_0^T f(s)dB(s)\right|^2 = \int_0^T |f(s)|^2 ds. \tag{4.36}$$

$\int_0^T f(s)dB(s)$ the *Wiener integral* of f with respect to B.

Proof. For any $\varepsilon > 0$ let $\delta_\varepsilon > 0$ be such that

$$|t-s| < \delta_\varepsilon \ \Rightarrow |f(t) - f(s)|^2 < \varepsilon. \tag{4.37}$$

Let moreover $\sigma, \eta \in \Sigma$ be such that

$$|\sigma| < \delta_\varepsilon, \ |\eta| < \delta_\varepsilon,$$

and

$$\sigma = \{t_0, t_1, \cdots, t_n\},$$

$$\eta = \{s_0, s_1, \cdots, s_m\}.$$

We denote by $\sigma \cup \eta$ the partition

$$\sigma \cup \eta = \{r_0, r_1, \cdots, r_N\},$$

where

$$\{r_0, ..., r_N\} = \{t_0, t_1, ..., t_n\} \cup \{s_0, s_1, ..., s_m\}.$$

We have

$$I_\sigma(f) - I_\eta(f) = \sum_{j=1}^N (f_1(r_{j-1}) - f_2(r_{j-1}))(B(r_j) - B(r_{j-1})),$$

where, for any $k = 0, ..., N-1$,

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in \ \{t_0, t_1, ..., t_n\} \\ \\ f(t_l) & \text{if } x \in \ (t_l, t_{l+1}), \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{if } x \in \ \{s_0, s_1, ..., s_m\} \\ \\ f(s_l) & \text{if } x \in \ (s_l, s_{l+1}). \end{cases}$$

Since $r_j - r_{j-1} \le \delta_\varepsilon$, by Lemma 4.18 we have

$$\mathbb{E}(|I_\sigma(f) - I_\eta(f)|^2) = \sum_{j=1}^{N} (f_1(r_{j-1}) - f_2(r_{j-1}))^2 (r_j - r_{j-1}) \le \varepsilon.$$

So, (I_σ) is Cauchy and there exists the limit in (4.34) follows.

We have finally

$$\mathbb{E}\left(\int_0^T f(s)dB(s)\right) = \lim_{|\sigma|\to 0} \mathbb{E}(I_\sigma) = 0,$$

and

$$\begin{aligned}\mathbb{E}\left|\int_0^T f(s)dB(s)\right|^2 &= \lim_{|\sigma|\to 0} \mathbb{E}|I_\sigma|^2 \\ &= \lim_{|\sigma|\to 0} \sum_{j=1}^{n} |f(t_{j-1})|^2 (t_j - t_{j-1}) \\ &= \int_0^T |f(s)|^2 ds. \qquad \square\end{aligned}$$

We now extend the definition of Wiener integral to functions from $L^2(0,T)$. We notice that the linear mapping

$$C([0,T]) \subset L^2(0,T) \to L^2(\Omega, \mathscr{F}, \mathbb{P}), \ \ f \mapsto \int_0^T f(s)dB(s),$$

is continuous since

$$\mathbb{E}\left|\int_0^T f(s)dB(s)\right|^2 = \int_0^T |f(s)|^2 ds.$$

Therefore we may extend by density the Wiener integral to any $f \in L^2(0,T)$.

The random variable $\int_0^T f(s)dB(s)$, which belongs to $L^2(\Omega, \mathscr{F}, \mathbb{P})$, is called the *Wiener integral* of f in $[0,T]$.

We define in an obvious way the Wiener integral $\int_a^b f(s)dB(s)$ in any interval $[a,b] \subset [0,T]$.

Exercise 4.20. Let $f \in L^2(0,T)$ such that $\int_0^T f(s)dB(s) = 0$. Show that $f = 0$ almost everywhere.

We show finally, that if $f \in C^1([0,T])$ then the Wiener integral of f can be written as a Riemann integral through an integration by parts formula.

Proposition 4.21. *If $f \in C^1([0,T])$ we have*

$$\int_0^T f(s)dB(s) = f(T)B(T) - \int_0^T f'(s)B(s)ds, \qquad \mathbb{P}\text{-a.e.} \tag{4.38}$$

Proof. Let $\sigma = \{t_0, t_1, \cdots, t_n\} \in \Sigma(0,T)$. Then we have

$$\begin{aligned}
I_\sigma(f) &= \sum_{k=1}^n f(t_{k-1})(B(t_k) - B(t_{k-1})) \\
&= \sum_{k=1}^n (f(t_k)B(t_k) - f(t_{k-1})B(t_{k-1})) \\
&\quad - \sum_{k=1}^n (f(t_k) - f(t_{k-1}))B(t_k) \\
&= f(T)B(T) - \sum_{k=1}^n (f(t_k) - f(t_{k-1}))B(t_k) \\
&= f(T)B(T) - \sum_{k=1}^n f'(t_k^*)B(t_k)(t_k - t_{k-1}),
\end{aligned}$$

where t_k^* are suitable numbers in the interval $[t_{k-1}, t_k]$, $k = 1, ..., n$. It follows that

$$\lim_{|\sigma|\to 0} I_\sigma(f) = f(T)B(T) - \int_0^T f'(s)dB(s)ds, \quad \mathbb{P}\text{-a.s.} \qquad \square$$

Proposition 4.22. *Let $f \in L^2(0,T)$, $X = \int_0^T f(s)dB(s)$. Then X is a real Gaussian random variable N_q with $q = \int_0^T |f(s)|^2 ds$.*

Proof. It is enough to prove the result for $f \in C([0,T])$. For any $\sigma \in \Sigma$ set

$$X_\sigma = \sum_{i=1}^n f(t_{i-1})(B(t_i) - B(t_{i-1})).$$

Since random variables

$$B(t_1),\ B(t_2) - B(t_1), \cdots, B(t_n) - B(t_{n-1}),$$

are independent, X_σ is a real Gaussian random variable N_{q_σ} with

$$q_\sigma = \sum_{i=1}^n f^2(t_{i-1})(t_i - t_{i-1}).$$

Now the conclusion follows from Proposition 2.15 letting $|\sigma|$ tend to zero. $\square$

4.6.1. Identification of the Wiener integral with a white noise function

Here we take $\Omega = L^2(0, T)$, $\mathscr{F} = \mathscr{B}(H)$ and $\mathbb{P} = N_Q$ where $Q \in L_1^+(H)$. We consider the Brownian motion defined by (4.7).

Proposition 4.23. *Let $f \in L^2(0, T)$. Then we have*

$$W_f = \int_0^T f(s)dB(s). \tag{4.39}$$

Proof. For any $\sigma = \{0 = t_0 < t_1 < \cdots < t_n = T\} \in \Sigma(0, T)$ set

$$f_\sigma : = \sum_{k=1}^n f(t_{k-1})\mathbb{1}_{(t_{k-1},t_k]}.$$

Then we have

$$\begin{aligned} W_{f_\sigma} &= \sum_{k=1}^n f(t_{k-1})W_{\mathbb{1}_{(t_{k-1},t_k]}} = \sum_{k=1}^n f(t_{k-1})(W_{t_k} - W_{t_{k-1}}) \\ &= W_{\sum_{k=1}^n f(t_{k-1})\mathbb{1}_{(t_{k-1},t_k]}}. \end{aligned} \tag{4.40}$$

Since

$$f = \lim_{|\sigma|\to 0} f_\sigma \quad \text{in } L^2(0, T),$$

and W is a continuous mapping from $H = L^2(0, T)$ into $L^2(H, \mu)$, identity (4.39) follows letting $|\sigma| \to 0$. □

4.7. Multidimensional Brownian motions

Definition 4.24. Let $r \in \mathbb{N}$. An *r–dimensional Brownian motion* on $[0, T]$ is an r–dimensional stochastic process B such that

(i) $B(0) = 0$ and if $0 \le s < t \le T$, $B(t) - B(s)$ is an r-dimensional Gaussian random variable with law $N_{(t-s)I}$, where I is the identity operator in $\mathbb{R}^r$.

(ii) For all $0 < t_1 < ... < t_n \le T$ the random variables,

$$B(t_1),\ B(t_2) - B(t_1), \cdots , B(t_n) - B(t_{n-1})$$

are independent.

(iii) B is continuous.

An equivalence class of stochastic processes $\widetilde{B}$ is called a Brownian motion if it possesses a version which is a Brownian motion.

Exercise 4.25. Let $B = (B_1, ..., B_n)$ be an r–dimensional Brownian motion. Show that $B_1, ..., B_n$ are independent real Brownian motions.

Remark 4.26. Let $B(t) = (B_1(t), ..., B_r(t))$ be an r-dimensional Brownian motion. Then, by the independence of $B_1, ..., B_r$ it follows that

$$\mathbb{E}[B_i(t)B_j(t)] = 0 \quad \text{if } i \neq j,\ i, j = 1, ..., r. \tag{4.41}$$

Moreover,

$$\mathbb{E}\left[|B(t) - B(s)|^2\right] = r(t - s), \tag{4.42}$$

because

$$\mathbb{E}\left[|B(t) - B(s)|^2\right] = \sum_{k=1}^{r} \mathbb{E}\left[|B_k(t) - B_k(s)|^2\right] = r(t - s).$$

Exercise 4.27. Prove that for $0 \le s < t$ we have

$$\mathbb{E}\left[|B(t) - B(s)|^4\right] = (2r + r^2)(t - s)^2. \tag{4.43}$$

Exercise 4.28. Denote by $(e_1, ..., e_r)$ the canonical basis in $\mathbb{R}^r$. Let $H := L^2(0, T; \mathbb{R}^r)$, choose an operator $Q \in L_1^+(H)$ such that $\operatorname{Ker} Q = \{0\}$ and set $\mu = N_Q$. Define

$$B_i(t) = W_{e_i \mathbb{1}_{[0,t]}}, \quad t \ge 0,\ i = 1, ..., r.$$

Show that $B(t) = (B_1(t), ..., B_r(t))$ is an r–dimensional Brownian motion in $(H, \mathscr{B}(H), \mu)$.

4.7.1. Multidimensional Wiener integral

Let $r, d \in \mathbb{N}$ and $T > 0$. Let moreover $B(t) = (B_1(t), ..., B_r(t))$ be an r–dimensional Brownian motion in $[0, T]$ and let F belong to $L^2(0, T; L(\mathbb{R}^r; \mathbb{R}^d))$. We denote by $F_{h,k}(t)$, $h = 1, ..., d$, $k = 1, .., r$, the matrix elements of $F(t)$. We define the integral

$$X := \int_0^T F(s)dB(s), \quad t \in [0, T],$$

setting $X = (X_1, ..., X_d)$ where

$$X_h = \sum_{k=1}^{d} \int_0^T F_{h,k}(t)dB_k(t), \quad t \in [0, T],\ h = 1, ..., d,\ k = 1, ..., r. \tag{4.44}$$

We call $\int_0^T F(s)dB(s)$ the *Wiener integral* of F with respect to B.

If $d = 1$ we shall write

$$\int_0^T F(s)dB(s) = \int_0^T \langle F(s), dB(s)\rangle.$$

Theorem 4.29. *Let $F \in L^2(0,T;L(\mathbb{R}^r;\mathbb{R}^d))$. Then we have*

$$\mathbb{E}\left(\int_0^T F(s)dB(s)\right) = 0, \tag{4.45}$$

and

$$\mathbb{E}\left|\int_0^T F(s)dB(s)\right|^2 = \int_0^T \mathrm{Tr}\,[F(s)F^*(s)]ds. \tag{4.46}$$

Finally, X is a d-dimensional Gaussian random variable $N_{\int_0^T F^*(s)F(s)ds}$.

Proof. (4.45) is clear, let us show (4.46). Since B_1, ..., B_r are independent, we have

$$\mathbb{E}(|X_h|^2) = \sum_{h=1}^d \int_0^T |F_{h,k}(t)|^2 dt,$$

from which, summing up over h, yields

$$\mathbb{E}|X|^2 = \sum_{h=1}^d \sum_{k=1}^r \int_0^T |F_{h,k}(t)|^2 dt = \int_0^T \mathrm{Tr}\,[F(t)F^*(t)]\,dt.$$

For the last statement, one can show by a direct computation, that

$$\mathbb{E}e^{i\langle X,z\rangle} = e^{-\frac{1}{2}\langle \int_0^T F^*(s)F(s)ds\, z,z\rangle}, \quad \forall\, z \in \mathbb{R}^d. \qquad \square$$

The following result can be proved as Proposition 4.23.

Proposition 4.30. *Let $f \in L^2(0,T;\mathbb{R}^r)$. Then we have*

$$W_f = \int_0^T \langle f(s), dB(s)\rangle. \tag{4.47}$$

Chapter 5
Markov property of Brownian motion

This chapter deals with the Markov and strong Markov properties of the Brownian motion and to several their interesting consequences.

Sections 5.1 and 5.2 include some needed preliminaries on filtrations and stopping times. We prove the useful result that if τ is a stopping time, then $B(t+\tau) - B(\tau)$ is still a Brownian motion. Markov and strong Markov properties are proved in Section 5.3 together with some of their consequences.

In Section 5.4 we give several applications to the Cauchy problem for the heat equation on $[0, +\infty)$ equipped with a Dirichlet, Neumann or Ventzell boundary condition at 0.

Finally, Section 5.5 is devoted to some fine properties of the zeros of the Brownian motion.

5.1. Filtrations

We are given a real Brownian motion B on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let us introduce the *natural filtration* of B. For any $n \in \mathbb{N}$, $0 \le t_1 < \cdots < t_n$, $A \in \mathscr{B}(\mathbb{R}^n)$ we consider the cylindrical set

$$C_{t_1,\cdots,t_n;A} = \{\omega \in \Omega : \ (B(t_1)(\omega), ..., B(t_n)(\omega) \in A\}.$$

Then for any $t \ge 0$ we denote by $\mathscr{C}_t$ the family of all cylindrical sets $C_{t_1,\cdots,t_n;A}$ with $t_n \le t$ and by $\mathscr{F}_t$ the σ-algebra generated by $\mathscr{C}_t$.

It is clear that $(\mathscr{F}_t)_{t\ge 0}$ is an increasing family of σ–algebras and that $\mathscr{F}_0 = \{\varnothing, \Omega\}$. $(\mathscr{F}_t)_{t\ge 0}$ is called the *natural filtration* of B. One also says that $\mathscr{F}_t$ contains the *story* of B up to t.

The following useful lemma shows that independence of $B(t) - B(s)$ by $B(s)$ extends to independence of $B(t) - B(s)$ by any $\mathscr{F}_s$-measurable random variable.

Lemma 5.1. *Let $t > s > 0$, and let X be a real random variable $\mathscr{F}_s$–measurable. Then $B(t) - B(s)$ and X are independent.*

Proof. We have to show that for any $I \in \mathscr{B}(\mathbb{R})$, and any $A \in \mathscr{F}_s$ it results

$$\mathbb{P}[(B(t) - B(s) \in I) \cap A] = \mathbb{P}(B(t) - B(s) \in I)\,\mathbb{P}(A). \tag{5.1}$$

By a straightforward application of Dynkin's theorem (see Appendix A), it is enough to prove that

$$\mathbb{P}[(B(t) - B(s) \in I) \cap C_{s_1,\dots,s_n;J_n}] = \mathbb{P}(B(t) - B(s) \in I)\,\mathbb{P}(C_{s_1,\dots,s_n;J_n}), \tag{5.2}$$

for any cylindrical set of $\mathscr{F}_s$, $C_{s_1,\dots,s_n;J_n}$, with $0 < s_1 < \cdots < s_n \le s,\ J_n \in \mathscr{B}(\mathbb{R}^n)$.

Now we recall that by Proposition 4.3 the $\mathbb{R}^{n+2}$-valued random variable

$$(B(s_1), \dots, B(s_n), B(s), B(t))$$

is Gaussian and possesses a density $\rho(\eta_1, \dots, \eta_n, \xi_1, \xi_2)$ with respect to the Lebesgue measure in $\mathbb{R}^{n+2}$ given by

$$\begin{aligned}
&\rho((\eta_1, \dots, \eta_n, \xi_1, \xi_2))\\
&= (2\pi)^{-(n+2)/2}(s_n, s_{n-1} - s_n \dots, s_2 - s_1, s - s_1, t - s)^{-1/2}\\
&\times e^{-\frac{\eta_1^2}{2s_n} - \frac{(\eta_2-\eta_1)^2}{2(s_{n-1}-s_n)} - \dots - \frac{(\eta_n-\eta_{n-1})^2}{2(s_{n-2}-s_{n-1})} - \frac{(\xi_1-\eta_n)^2}{2(s-s_n)} - \frac{(\xi_1-\xi_2)^2}{2(t-s)}}\, d\eta_1 \cdots d\eta_n\, d\xi_1\, d\xi_2.
\end{aligned} \tag{5.3}$$

Now, checking (5.2) is a simple computation which is left to the reader. □

5.1.1. Some properties of the filtration $(\mathscr{F}_t)$

We first notice that, as easily checked, for all $t > 0$ $\mathscr{F}_t$ coincides with the σ-algebra generated by all cylindrical sets of the form

$$C_{t_1,\dots,t_n;A_n}, \quad 0 < t_1 < \cdots < t_n < t,\ A_n \in \mathscr{B}(\mathbb{R}^n),\ n \in \mathbb{N}.$$

We say that filtration $(\mathscr{F}_t)_{t\ge 0}$ is *left continuous*.

For any $t \ge 0$ we define the *right limit* at t $\mathscr{F}_{t^+}$ of $(\mathscr{F}_t)_{t\ge 0}$ setting

$$\mathscr{F}_{t^+} : = \bigcap_{\epsilon>0} \mathscr{F}_{t+\epsilon}.$$

Exercise 5.2. Consider the set

$$D := \{\omega \in \Omega : \ B(\cdot)(\omega) \text{ is differentiable at } 0\}.$$

Show that D is non empty and belong to $\mathscr{F}_{0^+}$.

Remark 5.3. From the previous exercise it follows that the filtration $(\mathscr{F}_t)_{t\geq 0}$ is not right continuous for all $t \geq 0$, that is $\mathscr{F}_{t^+} \neq \mathscr{F}_t$.

Let us prove now the *Blumenthal one-zero law*.

Proposition 5.4. *Assume that* $A \in \mathscr{F}_{0^+}$. *Then either* $\mathbb{P}(A) = 1$ *or* $\mathbb{P}(A) = 0$.

Proof. Let $A \in \mathscr{F}_{0^+}$. Denote by $\mathscr{G}$ the σ-algebra generated by all sets of the form

$$\begin{aligned} D_{t_1,\dots,t_n,h;I} &= \{\omega \in \Omega : \ (B(t_1+h)(\omega) - B(h)(\omega), \\ &\qquad \dots\dots, B(t_n+h)(\omega) - B(h)(\omega)) \in I\} \end{aligned}$$

where $n \in \mathbb{N}, 0 < t_1 < \cdots < t_n, h > 0, I \in \mathscr{B}(\mathbb{R}^n)$. It is clear that A is independent of $\mathscr{G}$, since it belongs to all $\mathscr{F}_t$ (in particular to $\mathscr{F}_{t_1}$) and B has independent increments. Then we have

$$\mathbb{P}(A \cap G) = \mathbb{P}(A)\mathbb{P}(G), \quad \forall\, G \in \mathscr{G}. \tag{5.4}$$

We claim that $\mathscr{G} = \mathscr{F}$. To prove the claim, we notice that any cylindrical set $C_{t_1,\dots,t_n,h;I}$ belongs to $\mathscr{G}$, as easily seen. Now, since $\mathscr{G} = \mathscr{F}$ we can set in (5.4) $G = A$, so that $\mathbb{P}^2(A) = \mathbb{P}(A)$ which yields $\mathbb{P}(A)$ equal to zero or one. □

Exercise 5.5. For any $t \geq 0$ denote by $\overline{\mathscr{F}_t}$ the σ-algebra generated by $\mathscr{F}_t$ and all null sets of C_0 [(1)]. Show that the filtration $(\overline{\mathscr{F}_t})_{t\geq 0}$ is both right and left continuous.

5.2. Stopping times

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probabilty space. A *filtration* on $(\Omega, \mathscr{F}, \mathbb{P})$ is a family $(\mathscr{F}_t)_{t\geq 0}$ of σ-algebras included on $\mathscr{F}$ such that

$$t < s \Rightarrow \mathscr{F}_t \subset \mathscr{F}_s.$$

An example of filtration is the natural filtration of the Brownian motion seen before.

A random variable τ in $(\Omega, \mathscr{F}, \mathbb{P})$ with values in $[0, +\infty]$ is called a *stopping time* with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ if

$$\{\tau \leq t\} \in \mathscr{F}_t \quad \forall\, t \geq 0.$$

(1) A null set is a subset of C_0 having exterior measure 0.

If τ is stopping time, then $\{\tau < t\}$ and $\{\tau = t\}$ belong to $\mathscr{F}_t$ for all $t \geq 0$. In fact

$$\{\tau < t\} = \bigcup_{k=1}^{\infty} \left\{\tau \leq t - \frac{1}{k}\right\} \in \mathscr{F}_t.$$

To each stopping time τ we associate the σ-algebra

$$\mathscr{F}_\tau: = \{A \in \mathscr{F}: \ A \cap \{\tau \leq t\} \in \mathscr{F}_t, \quad \forall\, t \geq 0\}. \tag{5.5}$$

$\mathscr{F}_\tau$ is called the σ-algebra of the events *preceding* τ. τ is $\mathscr{F}_\tau$-measurable; in fact, if $A = \{\tau \leq s\}$ we have

$$A \cap \{\tau \leq t\} = \{\tau \leq t \wedge s\} \in \mathscr{F}_{t \wedge s} \subset \mathscr{F}_t.$$

In other words we have

$$\mathscr{F}_\tau \supset \sigma(\tau),$$

where $\sigma(\tau)$ is the σ-algebra generated by τ.

Remark 5.6. Let τ be a random variable with values in $[0, +\infty]$ such that

$$\{\tau < t\} \in \mathscr{F}_t, \quad \text{for all } t \geq 0.$$

Then τ is a stopping time with respect to the filtration $(\mathscr{F}_{t^+})_{t \geq 0}$. In fact

$$\{\tau \leq t\} = \bigcap_{k=1}^{\infty} \left\{\tau \leq t + \frac{1}{k}\right\} \in \mathscr{F}_{t^+}.$$

5.2.1. Basic examples of stopping times

Let $B(t)$, $t \geq 0$, be a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$ and let $(\mathscr{F}_t)_{t \geq 0}$ its natural filtration. Let $a \in \mathbb{R}$ and set

$$\tau_a = \begin{cases} \inf\{t \geq 0: \ B(t) = a\}, \ \text{if } \exists\, t \geq 0 \text{ such that } B(t) = a, \\ \\ +\infty \text{ oherwise.} \end{cases} \tag{5.6}$$

τ_a is called the *first moment* where $B(\cdot)$ reaches a. τ_a is a stopping time with respect to the filtration $(\mathscr{F}_t)_{t \geq 0}$. In fact

$$\begin{aligned} \{\tau_a > t\} &= \bigcap_{s \in [0,t]} \{B(s) < a\} \\ &= \bigcup_{k \in \mathbb{N}} \bigcap_{s \in [0,t]} \left\{B(s) \leq a - \frac{1}{k}\right\} \\ &= \bigcup_{k \in \mathbb{N}} \bigcap_{s \in [0,t] \cap \mathbb{Q}} \left\{B(s) \leq a - \frac{1}{k}\right\} \in \mathscr{F}_t, \end{aligned}$$

where $\mathbb{Q}$ is the set of all rational numbers, so that

$$\{\tau_a \le t\} \in \mathscr{F}_t.$$

Exercise 5.7. Let $a > 0$. Show that

$$\{\tau_a > t\} = \{\omega \in \Omega : \; B(s) < a, \; \forall \, s \in [0, t]\} \tag{5.7}$$

and

$$\{\tau_a \ge t\} = \{\omega \in \Omega : \; B(s) < a, \; \forall \, s \in [0, t)\}. \tag{5.8}$$

Let now

$$\tau = \inf\{t \ge 0 : \; B(t) > a\}.$$

Then we have

$$\{\tau \ge t\} = \bigcap_{s \in [0,t]} \{B(s) \le a\} = \bigcap_{s \in [0,t] \cap \mathbb{Q}} \{B(s) \le a\} \in \mathscr{F}_t.$$

Consequently, by Remark 5.6, τ is a stopping time with respect to the filtration $(\mathscr{F}_{t+})_{t \ge 0}$.

5.2.2. Properties of stopping times

Exercise 5.8. Assume that a nonnegative random variable τ is discrete: $\tau(\Omega) = (\mu_k)_{k \in \mathbb{N}}$ where μ_k is an increasing sequence of positive numbers. Show that τ is a stopping time if and only if

$$\{\tau = \mu_k\} \in \mathscr{F}_{\mu_k}, \quad \forall \, k \in \mathbb{N}.$$

In this case $\mathscr{F}_\tau$ is the σ-algebra

$$\mathscr{F}_\tau : \; = \{A \in \mathscr{F} : \; A \cap \{\tau = \mu_k\} \in \mathscr{F}_{\mu_k}, \quad \forall \, k \in \mathbb{N}\}.$$

Proposition 5.9. *Let τ be a stopping time. Then there exists a sequence (τ_n) of discrete stopping times such that*

$$\tau_n(\omega) \downarrow \tau(\omega), \quad \forall \, \omega \in \Omega, \tag{5.9}$$

as $n \to \infty$ and $\mathscr{F}_{\tau_n} \supset \mathscr{F}_\tau$ for all $n \in \mathbb{N}$.

Proof. Define for any $n \in \mathbb{N}$ and any $\omega \in \Omega$

$$\begin{aligned} \tau_n(\omega) &= \frac{k}{2^n} \quad \text{if} \quad \frac{k-1}{2^n} \le \tau(\omega) < \frac{k}{2^n}, \quad k = 1, 2, ..., 2^n, \\ \tau_n(\omega) &= 1 + \frac{k}{2^n} \;\; \text{if} \;\; 1 + \frac{k-1}{2^n} \le \tau(\omega) < 1 + \frac{k}{2^n}, \quad k = 1, 2, ..., 2^n \end{aligned} \tag{5.10}$$

and so on. It is clear that the sequence (τ_n) is non increasing and that τ_n is a discrete stopping time. In fact, if for instance $t = \frac{k}{2^n}$ with $k \in \mathbb{N}$ we have

$$\{\tau_n = t\} = \left\{\frac{k-1}{2^n} \le \tau < \frac{k}{2^n}\right\} \in \mathscr{F}_t. \tag{5.11}$$

Finally, let $A \in \mathscr{F}_\tau$, that is

$$A \cap \{\tau \le t\} \in \mathscr{F}_t, \quad \forall\, t \ge 0.$$

Then we have

$$A \cap \left\{\tau_n = \frac{k}{2^n}\right\} = A \cap \left\{\frac{k-1}{2^n} \le \tau < \frac{k}{2^n}\right\} \in \mathscr{F}_{\frac{k}{2^n}}, \quad \forall\, k \in \mathbb{N},$$

$$\begin{aligned} &A \cap \left\{\tau_n = 1 + \frac{k}{2^n}\right\} \\ &\quad = A \cap \left\{1 + \frac{k-1}{2^n} \le \tau < 1 + \frac{k}{2^n}\right\} \in \mathscr{F}_{\frac{k}{2^n}}, \quad \forall\, k \in \mathbb{N}, \end{aligned}$$

and so on. Therefore $A \in \mathscr{F}_{\tau_n}$. □

Proposition 5.10. *Let τ be a stopping time and set*

$$B_\tau(\omega) = B(\tau(\omega), \omega), \quad \omega \in \Omega.$$

Then B_τ is $\mathscr{F}_\tau$-measurable.

Proof. Assume first that τ is of the form,

$$\tau(\Omega) = \{t_1, ..., t_n\}, \quad 0 < t_1 < \cdots < t_n$$

and set $A_k = \{\tau = t_k\}, k = 1, ..., n$. Then we have

$$B_\tau(\omega) = B(t_k)(\omega), \quad \forall\, \omega \in A_k,\ k = 1, ..., n.$$

Let $I \in \mathscr{B}(\mathbb{R})$. Then

$$\{B_\tau \in I\} = \bigcup_{k\in\mathbb{N}} [\{B_\tau \in I\} \cap A_k] = \bigcup_{k\in\mathbb{N}} [\{B_{t_k} \in I\} \cap A_k].$$

Consequently, for any $h = 1, ..., n$ we have

$$\begin{aligned} \{B_\tau \in I\} \cap \{\tau = t_h\} &= \bigcup_{k\in\mathbb{N}} [\{B_{t_k} \in I\} \cap A_k] \cap A_h \\ &= \{B_{t_h} \in I\} \cap A_h \in \mathscr{F}_{t_h}. \end{aligned}$$

So, B_τ is $\mathscr{F}_\tau$-measurable. Let now τ be arbitrary, let τ_n be defined by (5.10) and set

$$B_{\tau_n}(\omega) = B(\tau_n(\omega), \omega), \quad \mathbb{P}\text{-a.s.},$$

since B is continuous we have

$$\lim_{n\to\infty} B_{\tau_n}(\omega) = B_\tau(\omega), \quad \mathbb{P}\text{-a.s.}.$$

Fix $t \geq 0$. We have to show that

$$\{B_\tau \in I\} \cap \{\tau \leq t\} \in \mathscr{F}_t \quad \text{for all } I \in \mathscr{B}(\mathbb{R}). \tag{5.12}$$

Therefore, it is enough to show that

$$\{B(\tau_n(\omega), \omega) \in I\} \cap \{\tau_n \leq t\} \in \mathscr{F}_t \quad \text{for all } I \in \mathscr{B}(\mathbb{R}). \tag{5.13}$$

Let $t \in [\frac{k}{2^n}, \frac{k+1}{2^n})$. Then we have

$$\begin{aligned}\{B(\tau_n(\omega),\omega) \in I\} \cap \{\tau_n \leq t\} &= \bigcup_{h=1}^{k} \{B(\tau_n(\omega), \omega) \in I\} \cap \{\tau_n = h2^{-n}\} \\ &= \bigcup_{h=1}^{k} \{B(h2^{-n}, \omega) \in I\} \cap \{\tau_n = h2^{-n}\} \in \mathscr{F}_t.\end{aligned}$$

So, (5.12) and then (5.13) follow. □

5.2.3. The Brownian motion $B(t+\tau) - B(\tau)$

Proposition 5.11. *Let τ be a stopping time. Then*

$$C(t) := B(t+\tau) - B(\tau), \quad t \geq 0,$$

is a Brownian motion.

Proof. Let us first prove that the law of $C(t)$ is N_t, the Gaussian measure in $\mathbb{R}$ with mean 0 and covariance t. For this it is enough to show that for any $\alpha \in \mathbb{R}$ we have

$$\mathbb{E}\left(e^{i\alpha C(t)}\right) = \mathbb{E}\left(e^{i\alpha(B(t+\tau)-B(\tau))}\right) = e^{-\frac{1}{2}\alpha^2 t}, \quad \alpha \in \mathbb{R}. \tag{5.14}$$

Assume first that τ is discrete, $\tau(\Omega) = (t_k)$ and set

$$A_i = \{\tau = t_i\} \in \mathscr{F}_{t_i}, \quad i \in \mathbb{N}.$$

Then we have

$$\mathbb{E}\left(e^{i\alpha(B(t+\tau)-B(\tau))}\right) = \sum_{i=1}^{\infty}\int_{A_i} e^{i\alpha(B(t+t_i)-B(t_i))}d\mathbb{P}$$
$$= \sum_{i=1}^{\infty}\mathbb{E}\left(\mathbb{1}_{A_i}e^{i\alpha(B(t+t_i)-B(t_i))}\right).$$

Since $\mathbb{1}_{A_i}$ and $B(t+t_i)-B(t_i)$ are independent, it follows that

$$\mathbb{E}\left(e^{i\alpha(B(t+\tau)-B(\tau))}\right) = \sum_{i=1}^{\infty}\mathbb{P}(A_i)\mathbb{E}\left(e^{i\alpha(B(t+t_i)-B(t_i))}\right) = e^{-\frac{1}{2}\alpha^2 t}$$

and so (5.14) is proved.

Let now τ be general and let (τ_n) be the sequence of finite stoppping times defined by (5.10). We have just proved that

$$\mathbb{E}\left(e^{i\alpha(B(t+\tau_n)-B(\tau_n))}\right) = e^{-\frac{1}{2}\alpha^2 t}, \quad \alpha\in\mathbb{R}.$$

Now (5.14) follows letting n tend to infinity. By (5.14) it follows that $C(t)$ is a Gaussian random variable N_t. Proceeding similarly one can prove that the law of $C(t)-C(s)$ with $t>s>0$ is N_{t-s} and that $C(t)$ has independent increments. Continuity of $C(\cdot)$ is obvious. □

5.3. Markov semigroups and Markov property

We consider a Brownian motion $B(t),\ t\ge 0$, on a probabilty space $(\Omega,\mathscr{F},\mathbb{P})$ and denote by $(\mathscr{F}_t)_{t\ge 0}$ the natural filtration of B.

By $B_b(H)$ we mean the space of all mappings $\varphi: H\to\mathbb{R}$ which are bounded and Borel and by $C_b(H)$ the subspace of $B_b(H)$ of uniformly continuous and bounded mappings.

5.3.1. Transition semigroup

Let us consider the stochastic process $B(t)+x,\ t\ge 0$, where $x\in\mathbb{R}$ and $t\ge 0$. Given $\varphi\in C_b(\mathbb{R})$ we want to study the evolution in time of $\varphi(B(t)+x)$. To this purpose, let us define the *transition semigroup*

$$P_t\varphi(x) = \mathbb{E}[\varphi(B(t)+x)], \quad t\ge 0,\ x\in\mathbb{R},\ \varphi\in B_b(\mathbb{R}). \tag{5.15}$$

Notice the following properties of P_t.

(i) $P_0\varphi(x)=\varphi(x), \quad \forall\, x\in\mathbb{R},\ \varphi\in B_b(\mathbb{R})$.
(ii) $P_t\mathbb{1}=\mathbb{1}, \forall\, t\ge 0$.

(iii) P_t conserves the positivity, that is if $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, we have $P_t\varphi(x) \geq 0$ for all $x \in \mathbb{R}$.

We say that P_t is a *Markov semigroup*.

Let us now find an explicit expression of P_t. Let $t > 0$; since the law of $B(t) + x$ is $N_{x,t}$, we have

$$\begin{aligned} P_t\varphi(x) &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2t}(x-y)^2}\varphi(y)dy \\ &= \int_{-\infty}^{+\infty} g_t(x-y)\varphi(y)dy, \end{aligned} \tag{5.16}$$

where

$$g_t(z) := \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{z^2}{2t}}, \quad z \in \mathbb{R}. \tag{5.17}$$

Proposition 5.12. *Let $\varphi \in B_b(\mathbb{R})$ and set $u(t,x) = P_t\varphi(x)$, $t \geq 0$, $x \in \mathbb{R}$. Then u is infinitely many times differentiable on $(0,+\infty) \times \mathbb{R}$ and it fulfills the heat equation*

$$u_t(t,x) = \frac{1}{2}\, u_{x,x}(t,x), \quad \forall\, (t,x) \in (0,+\infty) \times \mathbb{R}. \tag{5.18}$$

Moreover, if $\varphi \in C_b(\mathbb{R})$, u is continuous on $[0,+\infty) \times \mathbb{R}$ and it results

$$u(0,x) = \varphi(x), \quad \forall\, x \in \mathbb{R}. \tag{5.19}$$

Proof. If $t > 0$ we see by (5.16) that u is of C^∞ class on $(0,+\infty) \times \mathbb{R}$ and (5.18) follows by a straightforward verification.

Let us show the continuity of u on $[0,+\infty) \times \mathbb{R}$ when $\varphi \in C_b(\mathbb{R})$. Since

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2t}(x-y)^2} dy = 1,$$

we have for $t > 0$ and $x \in \mathbb{R}$

$$u(t,x) - \varphi(x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2t}(x-y)^2}(\varphi(y) - \varphi(x))dy. \tag{5.20}$$

By the change of variable $y - x = \sqrt{t}z$ we deduce by (5.20) that

$$|u(t,x) - \varphi(x)| \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}z^2}|\varphi(x+\sqrt{t}z) - \varphi(x)|dz. \tag{5.21}$$

Since φ is uniformly continuous we have

$$\lim_{t\to 0} u(t,x) = \varphi(x), \quad \text{uniformly on } x. \qquad \square$$

5.3.2. Markov property

Proposition 5.13. *Let $t > s \geq 0$ and $\varphi \in B_b(H)$. Then we have*

$$\mathbb{E}[\varphi(B(t)+x)|\mathscr{F}_s] = (P_{t-s}\varphi)(B(s)+x), \tag{5.22}$$

where for any $r \geq 0$, P_r is the transition semigroup defined by (5.15).

Proof. To compute $\mathbb{E}[\varphi(B(t)+x)|\mathscr{F}_s]$ we shall use Proposition B.8. Write

$$\varphi(B(t)+x) = \varphi(B(s)+x+(B(t)-B(s)) = \varphi(X+Y),$$

where $X = B(s)+x$, $Y = B(t)-B(s)$ and notice that X is $\mathscr{F}_s$–measurable whereas Y is independent of $\mathscr{F}_s$. By Proposition B.8 we deduce

$$\mathbb{E}[\varphi(B(t)+x)|\mathscr{F}_s] = h(B(s)+x),$$

where

$$\begin{aligned} h(r) &= \mathbb{E}[\varphi(r+B(t)-B(s))] \\ &= \frac{1}{\sqrt{2\pi(t-s)}}\int_{\mathbb{R}} e^{-\frac{(x-r)^2}{2(t-s)}}\varphi(x)dx \\ &= (P_{t-s}\varphi)(r). \end{aligned}$$

□

As an application of the Markov property we prove

Proposition 5.14. *P_t fulfills the semigroup law*

$$P_{t+s} = P_t P_s, \quad \forall\, s, t > 0. \tag{5.23}$$

Proof. Let $t > s > 0$, $\varphi \in B_b(\mathbb{R})$. Taking into account (5.22) we have

$$\begin{aligned} P_t P_s\varphi(x) &= \mathbb{E}[P_s\varphi(X(t,x))] = \mathbb{E}\{\mathbb{E}[\varphi(X(t+s,x))|\mathscr{F}_s]\} \\ &= \mathbb{E}[\varphi(X(t+s,x))] = P_{t+s}\varphi(x). \end{aligned}$$

□

5.3.3. Strong Markov property

We want now to extend identity (5.22) to the conditional expectation of $\varphi(B(t)+x)$ with respect to $\mathscr{F}_\tau$ where τ is a stopping time.

Proposition 5.15. *Let τ be a stopping time with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$. For all $t \geq \tau$ and $\varphi \in B_b(\mathbb{R})$ we have*

$$\mathbb{E}[\varphi(B(t)+x)|\mathscr{F}_\tau] = (P_{t-\tau}\varphi)(B(\tau)+x), \tag{5.24}$$

where $\mathscr{F}_\tau$ is defined in (5.5).

Property (5.24) is called the *strong Markov* property of the process $B(\cdot)+x$.

Proof. We set $x = 0$ for simplicity. Then (5.24) is equivalent to

$$\int_A \varphi(B(t))d\mathbb{P} = \int_A (P_{t-\tau}\varphi)(B(\tau))d\mathbb{P}, \quad \forall\ A \in \mathscr{F}_\tau. \tag{5.25}$$

Assume first that τ is discrete, $\tau(\Omega) = \{t_i\}$. If $A \in \mathscr{F}_\tau$, we have, recalling (5.22),

$$\begin{aligned}\int_A (P_{t-\tau}\varphi)(B(\tau))d\mathbb{P} &= \sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} (P_{t-\tau}\varphi)(B(\tau))d\mathbb{P}\\ &= \sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} (P_{t-t_i}\varphi)(B(t_i))d\mathbb{P} \qquad (5.26)\\ &= \sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} \mathbb{E}[\varphi(B(t)|\mathscr{F}_{t_i}]d\mathbb{P}.\end{aligned}$$

On the other hand, by the definition of $\mathscr{F}_\tau$ we have

$$A\cap\{\tau=t_i\} \in \mathscr{F}_{t_i}, \quad i \in \mathbb{N},$$

so that

$$\sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} \mathbb{E}[\varphi(B(t)|\mathscr{F}_{t_i}]d\mathbb{P} = \sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} \varphi(B(t))d\mathbb{P}.$$

Now by (5.26) it follows that

$$\int_A (P_{t-\tau}\varphi)(B(\tau))d\mathbb{P} = \sum_{i=1}^\infty \int_{A\cap\{\tau=t_i\}} \varphi(B(t))d\mathbb{P} = \int_A \varphi(B(t))d\mathbb{P}.$$

Therefore, (5.25) is proved.

Let now τ be an arbitrary stopping time and let $(\tau_n) \downarrow \tau$ be a sequence of discrete stopping times such that (see Proposition 5.9)

$$\mathscr{F}_\tau \subset \mathscr{F}_{\tau_n} \quad \forall\, n \in \mathbb{N}.$$

Let $A \in \mathscr{F}_\tau$. Then by (5.25) it follows that

$$\int_A \varphi(B(t))d\mathbb{P} = \int_A (P_{t-\tau_n}\varphi)(B(\tau_n))d\mathbb{P} \quad \forall\, A \in \mathscr{F}_\tau.$$

Since

$$|(P_{t-\tau_n}\varphi)(B(\tau_n))| \le \sup_{y\in\mathbb{R}} |\varphi(y)|,$$

the conclusion follows letting $n \to \infty$ and using the dominated convergence theorem. □

5.3.4. Some key random variables related to B

In this section we are concerned with the following random variables:

- The *first time where B reaches a*:

$$\tau_a = \inf\{t \ge 0 : \ B(t) = a\}, \quad a \in \mathbb{R}.$$

- The *maximum function*:

$$M(t) = \max_{s\in[0,t]} B(s), \quad t \ge 0.$$

- The *minimum function*:

$$m(t) = \min_{s\in[0,t]} B(s), \quad t \ge 0.$$

Obviously we have $\tau_0 = M(0) = m(0) = 0$.

Let us notice two obvious but useful identities.

$$\{\tau_a \le t\} = \{M(t) \ge a\}, \quad t \ge 0,\ a \ge 0 \tag{5.27}$$

and

$$\{\tau_a \le t\} = \{m(t) \le a\}, \quad t \ge 0,\ a \le 0. \tag{5.28}$$

To determine the laws of τ_a and $M(t)$ we need another useful identity.

Lemma 5.16. *Let $a \ge 0$ and $t \ge 0$. Then we have*

$$\mathbb{P}(B(t) \le a, M(t) \ge a) = \mathbb{P}(B(t) \ge a). \tag{5.29}$$

Proof. We start with an intuitive but not rigorous proof. Assume that $M(t) \geq a$, then for symmetry it is natural to assume that

$$\mathbb{P}(M(t) \geq a,\, B(t) \leq a) = \mathbb{P}(M(t) \geq a,\, B(t) \geq a).$$

Since obviously $\mathbb{P}(M(t) \geq a,\, B(t) \geq a) = \mathbb{P}(B(t) \geq a)$, (5.29) follows.

Now we present a rigorous proof, which requires the strong Markov property of B. Taking into account (5.27) we have

$$\begin{aligned} \mathbb{P}(B(t) \leq a,\, M(t) \geq a) &= \mathbb{P}(B(t) \leq a,\, \tau_a \leq t) \\ &= \int_{\{\tau_a \leq t\}} \mathbb{1}_{(-\infty,a]}(B(t))d\mathbb{P} \\ &= \int_{\{\tau_a \leq t\}} \mathbb{E}[\mathbb{1}_{(-\infty,a]}(B(t))|\mathscr{F}_{\tau_a}]d\mathbb{P}, \end{aligned} \tag{5.30}$$

since $\{\tau_a \leq t\} \in \mathscr{F}_{\tau_a}$.

Now by (5.30) and the strong Markov property it follows that

$$\begin{aligned} \mathbb{P}(B(t) \leq a,\, M(t) \geq a) &= \int_{\{\tau_a \leq t\}} [P_{t-\tau_a}\mathbb{1}_{(-\infty,a]}(B(\tau_a))]d\mathbb{P} \\ &= \int_{\{\tau_a \leq t\}} [P_{t-\tau_a}\mathbb{1}_{(-\infty,a]}(a)]d\mathbb{P}. \end{aligned} \tag{5.31}$$

Finally, using the elementary identity

$$P_s\mathbb{1}_{(-\infty,a]}(a) = P_s\mathbb{1}_{[a,+\infty)}(a),$$

we deduce from (5.31) that

$$\begin{aligned} &\mathbb{P}(B(t) \leq a,\, M(t) \geq a) \\ &= \int_{\{\tau_a \leq t\}} [P_{t-\tau_a}\mathbb{1}_{(-\infty,a]}(a)]d\mathbb{P} = \int_{\{\tau_a \leq t\}} [P_{t-\tau_a}\mathbb{1}_{[a,+\infty)}(a)]d\mathbb{P} \\ &= \mathbb{P}(B(t) \geq a,\, \tau_a \leq t) = \mathbb{P}(B(t) \geq a), \end{aligned}$$

as claimed. □

Proposition 5.17 (Reflection principle). *For all* $a \geq 0$ *we have*

$$\mathbb{P}(M(t) \geq a) = 2\mathbb{P}(B(t) \geq a) = \mathbb{P}(|B(t)| \geq a), \tag{5.32}$$

Proof. Write

$$\mathbb{P}(M(t) \geq a) = \mathbb{P}(M(t) \geq a,\, B(t) \leq a) + \mathbb{P}(M(t) \geq a,\, B(t) \geq a).$$

Now, by Lemma 5.16 we have $\mathbb{P}(M(t) \geq a,\, B(t) \leq a) = \mathbb{P}(B(t) \geq a)$. Moreover, it is clear that $\mathbb{P}(M(t) \geq a,\, B(t) \geq a) = \mathbb{P}(B(t) \geq a)$ so, the conclusion follows. □

By Proposition 5.17 we can easily deduce the expressions of the laws of $M(t)$ and τ_a for all $a \in \mathbb{R}$.

Corollary 5.18 (Law of $M(t)$). *For all $t \geq 0$ we have*

$$(M(t)_{\#}\mathbb{P})(d\xi) = \frac{2}{\sqrt{2\pi t}}\, e^{-\frac{\xi^2}{2t}}\, \mathbb{1}_{[0,+\infty)}(\xi) d\xi. \tag{5.33}$$

Proof. By Proposition 5.17 we have for any $\xi \geq 0$

$$\mathbb{P}(M(t) \geq \xi) = 2\mathbb{P}(B(t) \geq \xi) = \frac{2}{\sqrt{2\pi t}} \int_{\xi}^{+\infty} e^{-\frac{\eta^2}{2t}} d\eta.$$

Consequently, $M(t)_{\#}\mathbb{P}$ has a density given by

$$-\frac{d}{d\xi}\mathbb{P}(M(t) \geq \xi) = \frac{2}{\sqrt{2\pi t}}\, e^{-\frac{\xi^2}{2t}}\, \mathbb{1}_{[0,+\infty)}(\xi).$$

□

Remark 5.19. From Corollary 5.18 it follows that the laws of $M(t)$ and $|B(t)|$ coincide; notice, however, that the laws of $M(\cdot)$ and $|B(\cdot)|$ in $C([0,T])$ do not coincide for any $T > 0$ because $M(\cdot)$ is increasing whereas $B(\cdot)$ is not.

Corollary 5.20 (Law of τ_a). *Let $a \geq 0$ and $t \geq 0$. Then we have*

$$((T_a)_{\#}\mathbb{P})(dt) = \frac{a}{\sqrt{2\pi t^3}}\, e^{-\frac{a^2}{2t}} dt. \tag{5.34}$$

Proof. By (5.27) and Proposition 5.17 we have

$$\mathbb{P}(T_a \leq t) = \mathbb{P}(M(t) \geq a) = \frac{2}{\sqrt{2\pi t}} \int_a^{+\infty} e^{-\frac{\xi^2}{2t}} d\xi = \frac{2}{\sqrt{2\pi}} \int_{at^{-1/2}}^{+\infty} e^{-\frac{\eta^2}{2}} d\xi.$$

It follows that

$$\frac{d}{dt}\mathbb{P}(T_a \leq t) = \frac{a}{\sqrt{2\pi t^3}}\, e^{-\frac{a^2}{2t}}$$

which yields (5.34). □

The following series of results can be proved as before, we leave the proofs as an exercise to the reader.

Lemma 5.21. *Let $a \le 0$ and $t \ge 0$. Then we have*

$$\mathbb{P}(B(t) \ge a, m(t) \le a) = \mathbb{P}(B(t) \le a). \tag{5.35}$$

Proposition 5.22 (Reflection principle). *For all $a \le 0$ we have*

$$\mathbb{P}(m(t) \le a) = 2\mathbb{P}(B(t) \le a). \tag{5.36}$$

Corollary 5.23 (Law of $m(t)$). *For all $t \ge 0$ we have*

$$(m(t)_{\#}\mathbb{P})(d\xi) = -\frac{2}{\sqrt{2\pi t}}\, e^{-\frac{\xi^2}{2t}} \mathbb{1}_{(-\infty,0]}(\xi)d\xi. \tag{5.37}$$

Corollary 5.24 (Law of τ_a). *Let $a \in \mathbb{R}$ and $t \ge 0$. Then we have*

$$((\tau_a)_{\#}\mathbb{P})(dt) = \frac{|a|}{\sqrt{2\pi t^3}}\, e^{-\frac{a^2}{2t}} dt. \tag{5.38}$$

Exercise 5.25. Let $a > 0$ and let

$$\tau_a' = \inf\{t \ge 0 : \ B(t) > a\}. \tag{5.39}$$

As we have seen τ_a' is a stopping time with respect to the σ–algebra $\mathscr{F}_{t^+}$. Show that the laws of τ_a' and τ_a coincide.

Hint. Notice that for any $t > 0$ we have

$$\mathbb{P}(\tau_a \le t) \le \mathbb{P}(\tau_a' \le t) \le \mathbb{P}(\tau_{a+\epsilon} \le t)$$

Then recall (5.34) and let $\epsilon \to 0$.

5.4. Application to partial differential equations

5.4.1. The Cauchy–Dirichlet problem

Let $x \ge 0$ and consider the process

$$X(t,x) = (B(t) + x) \quad \text{if } t \le [0, \tau_{-x}].$$

Notice that this process is only defined in the stochastic interval $[0, \tau_{-x}]$; after τ_{-x} it disappears. We say that the process $B(\cdot) + x$ is *killed* at 0.

Let $\varphi \in B_b([0, +\infty))$ and set

$$\begin{aligned} U_t\varphi(x) := u(t,x) &:= \int_{\{t \le \tau_{-x}\}} \varphi(B(t) + x) d\mathbb{P} \\ &= \mathbb{E}[\varphi(B(t) + x)\mathbb{1}_{t \le \tau_{-x}}], \quad x \ge 0,\ t \ge 0. \end{aligned} \tag{5.40}$$

Remark 5.26. Let $t > 0$ and $x = 0$, since $\tau_0 = 0$ we have

$$\{t \le \tau_{-x}\} = \{t \le 0\} = \varnothing.$$

So, u fulfills a Dirichlet condition at 0,

$$u(t, 0) = 0, \quad \forall\, t > 0. \tag{5.41}$$

Let us find a representation formula for $u(t, x)$.

Proposition 5.27. *Let $\varphi \in B_b(0, +\infty)$ and set $u(t, x) = U_t\varphi(x)$. Then we have*

$$u(t, x) = \int_0^{+\infty} [g_t(x - y) - g_t(x + y)]\varphi(y)dy, \quad x \ge 0,\ t \ge 0, \tag{5.42}$$

where

$$g_t(\xi) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{\xi^2}{2t}}, \quad \xi \ge 0,\ t > 0. \tag{5.43}$$

Proof. Let us first extend φ to $\mathbb{R}$ setting

$$\varphi(x) = 0, \quad x < 0.$$

Then write

$$\begin{aligned} u(t, x) &= \int_{\{t \le \tau_{-x}\}} \varphi(B(t) + x)d\mathbb{P} \\ &= \int_\Omega \varphi(B(t) + x)d\mathbb{P} - \int_{\{t > \tau_{-x}\}} \varphi(B(t) + x)d\mathbb{P} \\ &= P_t\varphi(x) - \mathbb{E}[\varphi(B(t) + x)1\!\!1_{t > \tau_{-x}}]. \end{aligned} \tag{5.44}$$

Now $P_t\varphi(x)$ coincides with the first term on the right hand side of (5.42), so it remains to show that

$$\int_0^{+\infty} g_t(x + y)\varphi(y)dy = \mathbb{E}\left[\varphi(B(t) + x)1\!\!1_{t > \tau_{-x}}\right] =: J. \tag{5.45}$$

Using (B.3) and Proposition B.6 we see that

$$J = \mathbb{E}\left[\varphi(B(t) + x)1\!\!1_{t > \tau_{-x}}\right] = \mathbb{E}\left[1\!\!1_{t > \tau_{-x}}\mathbb{E}[\varphi(B(t) + x)|\mathscr{F}_{\tau_{-x}}]\right]$$

Now, by the strong Markov property it follows that,

$$J = \mathbb{E}\left[\mathbb{1}_{t>\tau_{-x}} P_{t-\tau_{-x}}\varphi(B(\tau_{-x})+x)\right]$$

$$= \mathbb{E}\left[\mathbb{1}_{t>\tau_{-x}} P_{t-\tau_{-x}}\varphi(0)\right] =: \mathbb{E}[\psi(\tau_{-x})],$$

where

$$\psi(\lambda) = \mathbb{1}_{t>\lambda}\,\frac{1}{\sqrt{2\pi(t-\lambda)}}\int_{\mathbb{R}} e^{-\frac{y^2}{2(t-\lambda)}}\varphi(y)dy$$

$$= \mathbb{1}_{t>\lambda}\int_{\mathbb{R}} g_{t-\lambda}(y)dy, \quad \lambda > 0.$$

Recalling that the law of τ_{-x} is given by

$$((\tau_{-x})_{\#}\mathbb{P})(d\lambda) = \frac{x}{\sqrt{2\pi\lambda^3}}\,e^{-\frac{x^2}{2\lambda}}d\lambda. \tag{5.46}$$

we deduce

$$J = \int_{\mathbb{R}}\psi(\lambda)((\tau_{-x})_{\#}\mathbb{P})(d\lambda) = \int_{\mathbb{R}}\psi(\lambda)\frac{x}{\sqrt{2\pi\lambda^3}}\,e^{-\frac{x^2}{2\lambda}}d\lambda$$

$$= \int_0^t\left[\int_{\mathbb{R}} g_{t-s}(y)\varphi(y)dy\right]\frac{x}{\sqrt{2\pi s^3}}\,e^{-\frac{x^2}{2s}}ds$$

$$= -\frac{\partial}{\partial x}\int_0^t\left[\int_{\mathbb{R}} g_{t-s}(y)\varphi(y)dy\right]g_s(x)ds$$

$$= -\frac{\partial}{\partial x}\int_{\mathbb{R}} G_{x,y}\varphi(y)dy,$$

where[(2)]

$$G_{x,y} = \int_0^t g_{t-s}(y)g_s(x)ds = \frac{1}{2}\,\mathrm{Erfc}\left(\frac{|x|+|y|}{\sqrt{2t}}\right).$$

Since, for $x > 0$,

$$\frac{\partial}{\partial x}G_{x,y} = \frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(x+|y|)^2}{2t}} = -g_t(x+|y|),$$

[(2)] We recall that Erfc $(a) = \frac{2}{\sqrt{\pi}}\int_a^{+\infty} e^{-r^2}dr$.

we have

$$J = -\frac{\partial}{\partial x} \int_{\mathbb{R}} G_{x,y}\varphi(y)dy$$

$$= \int_{\mathbb{R}} g_t(x+|y|)\varphi(y)dy = \int_0^{+\infty} g_t(x+y)\varphi(y)dy,$$

and (5.44) is proved. □

Next result shows that $u(t,x) = U_t\varphi(x)$ is the solution to a Cauchy–Dirichlet problem. The corresponding proof, similar to that of Proposition 5.12, is left to the reader.

Proposition 5.28. *Let* $\varphi \in B_b([0,+\infty))$ *and* $U_t\varphi(x) = u(t,x)$*. Then* u *is infinitely many times differentiable in* $(0,+\infty)\times[0,+\infty)$ *and it results*

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\, u_{xx}(t,x), & x \geq 0,\ t > 0 \\ \\ u(t,0) = 0, & t > 0. \end{cases} \tag{5.47}$$

If moreover $\varphi \in C_b([0,+\infty))$ *then* u *is continuous in* $[0,+\infty)\times[0,+\infty)$ *and fulfills*

$$u(0,x) = \varphi(x), \quad x \geq 0. \tag{5.48}$$

Remark 5.29. U_t is not a Markov semigroup because

$$U_t\mathbb{1}(x) = \mathbb{P}(t \leq \tau_{-x}) < 1.$$

5.4.2. The Cauchy–Neumann problem

We are here concerned with the process

$$Y(t,x) = |B(t)+x|, \quad x \geq 0,\ t \geq 0.$$

$Y(\cdot,0)$ is called the *Brownian motion reflected at* 0.

For any $\varphi \in B_b([0,+\infty))$ we set

$$Q_t\varphi(x) = \mathbb{E}[\varphi(|B(t)+x|)] = (2\pi t)^{-1/2}\int_{\mathbb{R}} e^{-\frac{|x-y|^2}{2t}}\varphi(|y|)dy.$$

Therefore

$$Q_t\varphi(x) = (2\pi t)^{-1/2}\int_0^{+\infty} e^{-\frac{|x-y|^2}{2t}}\varphi(y)dy$$

$$+(2\pi t)^{-1/2}\int_{-\infty}^0 e^{-\frac{|x-y|^2}{2t}}\varphi(-y)dy.$$

Replacing in the last integral y with $-y$, we see that

$$u(t,x) := Q_t\varphi(x) = \int_0^{+\infty} [g_t(x-y) + g_t(x+y)]\varphi(y)dy, \qquad (5.49)$$

where g_t is defined by (5.43). Notice that for any $t > 0$, $u(t,x)$ fulfills a *Neumann* condition at 0,

$$u_x(t,x) = 0, \quad x \geq 0, \ t > 0.$$

Next result shows that $u(t,x) = Q_t\varphi(x)$ is the solution to a Cauchy–Neumann problem; even here we leave the proof to the reader.

Proposition 5.30. *Let* $\varphi \in B_b([0,+\infty))$ *and* $Q_t\varphi(x) = u(t,x)$. *Then* u *is infinitely many times differentiable in* $(0,+\infty)\times[0,+\infty)$ *and it results*

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\, u_{xx}(t,x), & x \geq 0, \ t > 0 \\ \\ u_x(t,0) = 0, & t > 0. \end{cases} \qquad (5.50)$$

If moreover $\varphi \in C_b([0,+\infty))$ *then* u *is continuous in* $[0,+\infty)\times[0,+\infty)$ *and fulfills*

$$u(0,x) = \varphi(x), \quad x \geq 0. \qquad (5.51)$$

Remark 5.31. U_t is a Markov semigroup because

$$U_t \mathbb{1} = \mathbb{1}.$$

5.4.3. The Cauchy–Ventzell problem

Let us consider the stochastic process,

$$Z(t,x) = \begin{cases} B(t) + x, & \text{if } t \geq \tau_{-x} \\ \\ 0, & \text{if } t \geq \tau_{-x}. \end{cases}$$

Note that when $Z(t,x)$ reaches 0, it remains at 0 for ever. We say that $Z(t,x)$ is *absorbed* at 0.

The corresponding transition semigroup is given by

$$V_t\varphi(x) = \int_{\{t\leq\tau_{-x}\}} \varphi(B(t)+x)d\mathbb{P} + \int_{\{t>\tau_{-x}\}} \varphi(0)d\mathbb{P}.$$

Therefore

$$V_t\varphi(x) = U_t\varphi(x) + \varphi(0)\,\mathbb{P}(\tau_{-x} \leq t),$$

where U_t is defined by (5.40). So, recalling (5.38), we have

$$V_t\varphi(x) = \int_0^{+\infty} [g_t(x-y) - g_t(x+y)]\varphi(y)dy + \varphi(0)\int_{-\infty}^{x} \frac{x}{\sqrt{2\pi t^3}}\, e^{-\frac{x^2}{2t}}dt.$$

Proposition 5.32. *Let $\varphi \in B_b([0,+\infty))$ and $V_t\varphi(x) = u(t,x)$. Then u is infinitely many times differentiable in $(0,+\infty)\times[0,+\infty)$ and it results*

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\, u_{xx}(t,x), \quad x \ge 0,\ t>0 \\ \\ u_{x,x}(t,0) = 0, \quad t>0. \end{cases} \tag{5.52}$$

If moreover $\varphi \in C_b([0,+\infty))$ then u is continuous in $[0,+\infty)\times[0,+\infty)$ and fulfills

$$u(0,x) = \varphi(x), \quad x \ge 0. \tag{5.53}$$

Remark 5.33. We notice that

$$V_t \mathbb{1} = \mathbb{1},$$

and so, V_t is a Markov semigroup.

5.5. The set of zeros of the Brownian motion

We are going to study the set

$$\Lambda(\omega) = \{t \in [0,+\infty):\ B(t,\omega) = 0\}, \quad \omega \in \Omega.$$

We first show that, $\mathbb{P}$–a.s., 0 is not isolated in $\Lambda(\omega)$ and so, it is an accumulation point of zeros of B. To prove this fact, it is useful to consider the random variable

$$\zeta_s = \inf\{t > s:\ B(t) = 0\}, \quad s \ge 0. \tag{5.54}$$

ζ_s is the infimum of all zeros of $B(t)$ after s. Notice that obviously ζ_0 is different of τ_0 (the first moment where $B(t) = 0$), because $\tau_0 = 0$; however, $\mathbb{P}(\zeta_0 > 0) = 0$ as the following proposition shows.

Proposition 5.34. *We have* $\mathbb{P}(\zeta_0 > 0) = 0$.

By the proposition it follows that 0 is not an isolated point of $\Lambda(\omega)$, $\mathbb{P}$-a.s.

Proof. It is enough to show that $\mathbb{P}(\zeta_0 > \lambda) = 0$ for any $\lambda > 0$. In fact, for any $\epsilon > 0$, we have

$$\begin{aligned} \{\zeta_0 > \lambda\} &= \{B(s) > 0,\ \forall\, s \in (0,\lambda]\} \cup \{B(s) < 0,\ \forall\, s \in (0,\lambda]\} \\ &\subset \{m(\lambda) \ge -\epsilon\} \cup \{M(\lambda) \le \epsilon\} \end{aligned}$$

But, taking into account (5.33) and (5.37), we find

$$\mathbb{P}(\zeta_0 > \lambda) \le \mathbb{P}(m(\lambda) > -\epsilon) + \mathbb{P}(M(\lambda) \le \epsilon)$$

$$= \frac{2}{\sqrt{2\pi\lambda}} \int_{-\epsilon}^{0} e^{-\frac{\xi^2}{2\epsilon}} d\xi + \frac{2}{\sqrt{2\pi\lambda}} \int_{0}^{\epsilon} e^{-\frac{\xi^2}{2\epsilon}} d\xi \overset{\epsilon\to 0}{\to} 0.$$

So, $\mathbb{P}(\zeta_0 > \lambda) = 0$ as claimed. □

We show now that the Lebesgue measure λ_1 of $\Lambda(\omega)$ is zero for almost all $\omega \in \Omega$.

Proposition 5.35. *We have* $\lambda_1(\Lambda(\omega)) = 0, \quad \mathbb{P}$-a.s..

Proof. Define

$$\Sigma := \{(t, \omega) \in [0, +\infty) \times \Omega : \; B(t, \omega) = 0\},$$

and consider the sections of Σ,

$$\Sigma^t := \{\omega \in \Omega : \; B(t, \omega) = 0\}, \; t \in [0, +\infty)\}$$

and

$$\Sigma_\omega := \{t \in [0, +\infty) : \; B(t, \omega) = 0\} = \Lambda(\omega).$$

Notice that for any $t > 0$ we have

$$\mathbb{P}(\Sigma^t) = \mathbb{P}(B(t) = 0) = 0.$$

Now, by the Fubini Theorem we have

$$\mathbb{P}(\Sigma) = \int_\Omega \lambda_1(\Lambda(\omega))\mathbb{P}(d\omega) = \int_0^\infty \mathbb{P}(\Sigma^t)dt = 0.$$

Therefore

$$\lambda_1(\Lambda(\omega)) = 0, \quad \mathbb{P}\text{-a.e.}$$ □

We show finally that the set $\Lambda(\omega)$ has no isolated points $\mathbb{P}$-a.s., equivalently, that for any interval (s_1, s_2) of $[0, \infty)$ the probability that B has exactly one zero in the interval (s_1, s_2) is zero. For this we need a lemma

Lemma 5.36. *For all* $t \ge 0$ *we have*

$$\zeta_{\zeta_t(\omega)}(\omega) = \zeta_t(\omega), \quad \mathbb{P}\text{-a.s.}$$

Proof. Recall that

$$\zeta_t(\omega) = \inf\{s > t : \ B(s)(\omega) = 0\}$$

so that

$$\zeta_{\zeta_t(\omega)}(\omega) = \inf\{s > \zeta_t(\omega) : \ B(s)(\omega) = 0\}.$$

We can obviously write

$$\zeta_{\zeta_t(\omega)}(\omega) = \zeta_t(\omega) + \inf\{s > 0 : \ B(s + \zeta_t)(\omega) - B(\zeta_t)(\omega) = 0\}$$

Since, by Proposition 5.11, $B(\cdot + \zeta_t) - B(\zeta_t)$ is a Brownian motion we have

$$\inf\{s > 0 : \ B(s + \zeta_t)(\omega) - B(\zeta_t)(\omega) = 0\} = 0, \quad \mathbb{P}\text{-a.s..}$$

Therefore

$$\zeta_{\zeta_t(\omega)}(\omega) = \zeta_t(\omega), \quad \mathbb{P}\text{-a.s.,}$$

as required. □

We now prove

Proposition 5.37. *For any* $0 \le s_1 < s_2$ *the probability* p *that* B *has exactly one zero in the interval* (s_1, s_2) *is zero.*

Proof. We have

$$\begin{aligned} p &= \mathbb{P}(\zeta_{s_1} \in (s_1, s_2), \ \zeta_{\zeta_{s_1}} > s_2) \\ &\le \mathbb{P}(\zeta_{s_1} < s_2, \ \zeta_{\zeta_{s_1}} > s_2). \end{aligned}$$

Since by Lemma 5.36 $\zeta_{\zeta_{s_1}}$ is almost surely equal to ζ_{s_1}, we have $p = 0$. □

Remark 5.38. Summarizing, we have proved that for $\mathbb{P}$–almost all $\omega \in \Omega$ the set $\Lambda(\omega)$ of all zeros of $B(\cdot, \omega)$ has Lebesgue measure zero, is closed and do not possess isolated points; so, it is uncountable, see [19]. So, $\Lambda(\omega)$ behaves as the Cantor set. For other interesting properties of the Brownian motion see *e.g.* [13].

Chapter 6
Itô's integral

We are given a real Brownian motion $B(t),\ t \ge 0$, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and we denote by $(\mathscr{F}_t)$ its natural filtration. For any $T > 0$ we are going to define a stochastic integral

$$\int_0^T F(s)dB(s),$$

called *Itô's integral*, where F is generally not deterministic as in the Wiener integral considered before, but instead it is a stochastic process fulfilling suitable conditions, roughly speaking such that $F(t)$ is $\mathscr{F}_t$-measurable for all $t \in [0, T]$ (we say that F is *adapted* to the filtration $(\mathscr{F}_t)$).

First we define in Section 6.1 the Itô integral for elementary processes, that is processes which take on only a finite number of values and are adapted to the filtration $(\mathscr{F}_t)$. Then in Section 6.2 we extend the integral to a more general class of processes, called *predictable* and such that

$$\int_0^T \mathbb{E}[|F(s)|^2]ds < \infty. \tag{6.0}$$

In Section 6.3 we consider the Itô integral

$$I(t) := \int_0^t F(s)dB(s), \quad t \in [0, T],$$

as a function of t, proving that it is a martingale. Then using an important martingale inequality for martingales, proved in Appendix C, we show that $I(\cdot)$ is a continuous process.

In Section 6.4 we define the Itô integral with a stopping time τ replacing t.

In Section 6.5 we show how define the Itô integral under a weaker condition than (6.0), namely when

$$\int_0^T |F(s)]^2 ds < \infty, \quad \mathbb{P}\text{–a.e..}$$

Finally, Section 6.6 is devoted to extend the obtained results to multi-dimensional Itô integrals.

Remark 6.1. In some circumstances[(1)] it is useful to consider on $(\Omega, \mathscr{F}, \mathbb{P})$ a filtration $(\mathscr{G}_t)_{t\geq 0}$ larger than the natural filtration of B with the property that $B(t+h) - B(t)$ is independent of $\mathscr{G}_t$ for any $t \geq 0,\ h > 0$. In this case we say that the Brownian motion B is *non anticipating* with respect to $(\mathscr{G}_t)_{t\geq 0}$.

It is easy to check that the definition of Itô's integral can be done using a non anticipating Brownian motion. All results proved in this chapter will remain essentially the same.

6.1. Itô's integral for elementary processes

A real *elementary process* $F(t)$, $t \in [0, T]$, is a real process on $(\Omega, \mathscr{F}, \mathbb{P})$ of the form

$$F(t) = \sum_{i=1}^{n} F_{i-1} \mathbb{1}_{[t_{i-1}, t_i)}(t), \quad t \in [0, T], \tag{6.1}$$

where $n \in \mathbb{N},\ 0 = t_0 < t_1 < \cdots < t_n = T$ and, for $i = 0, 1, ..., n-1$, F_i is an $\mathscr{F}_{t_i}$-measurable real random variable. Thus F is adapted.

We denote by $\mathscr{E}^2_B(0, T)$ the set of all elementary processes of the form (6.1) such that

$$F_i \in L^2(\Omega, \mathscr{F}_{t_i}, \mathbb{P}), \quad i = 0, 1, ..., n-1. \tag{6.2}$$

Then for any $F \in \mathscr{E}^2_B(0, T)$ we define the *Itô integral* of F setting

$$I(F) := \int_0^T F(s)dB(s) = \sum_{i=1}^{n} F_{i-1}(B(t_i) - B(t_{i-1})). \tag{6.3}$$

(1) See Chapter 12.

6.1.1. Basic identities

Proposition 6.2. *For each $F \in \mathscr{E}_B^2(0, T)$ we have*

$$\mathbb{E}\left(\int_0^T F(s)dB(s)\right) = 0 \tag{6.4}$$

$$\mathbb{E}\left[\left(\int_0^T F(s)dB(s)\right)^2\right] = \int_0^T \mathbb{E}(|F(s)|^2)ds. \tag{6.5}$$

Proof. To prove (6.4) write

$$\mathbb{E}[I(F)] = \sum_{j=1}^n \mathbb{E}[F(t_{j-1})(B(t_j) - B(t_{j-1}))].$$

Since $F(t_{j-1})$ is $\mathscr{F}_{t_{j-1}}$ measurable, it is independent of $B(t_j) - B(t_{j-1})$ by Lemma 5.1. Therefore we have

$$\mathbb{E}[I(F)] = \sum_{j=1}^n \mathbb{E}[F(t_{j-1})]\mathbb{E}[B(t_j) - B(t_{j-1})] = 0$$

and (6.4) follows.

Let us prove (6.5). Write

$$\mathbb{E}[|I(F)|^2] = \mathbb{E}\left[\sum_{j=1}^n |F(t_{j-1})|^2[B(t_j) - B(t_{j-1})]^2\right]$$

$$+2\mathbb{E}\left[\sum_{j<k} F(t_{j-1})F(t_{k-1})[B(t_j) - B(t_{j-1})]\ [B(t_k) - B(t_{k-1})]\right].$$

Notice now that for $j < k$ the random variable

$$F(t_{j-1})F(t_{k-1})[B(t_j) - B(t_{j-1})],$$

is $\mathscr{F}_{t_{k-1}}$–measurable and consequently is independent of $B(t_k) - B(t_{k-1})$. Therefore we have

$$\mathbb{E}\left[F(t_{j-1})F(t_{k-1})[B(t_j) - B(t_{j-1})][B(t_k) - B(t_{k-1})]\right]$$

$$= \mathbb{E}\left[F(t_{j-1})F(t_{k-1})[B(t_j) - B(t_{j-1})]\right]\ \mathbb{E}[B(t_k) - B(t_{k-1})] = 0.$$

It follows that

$$\begin{aligned}\mathbb{E}[|I(F)|^2] &= \sum_{j=1}^{n} \mathbb{E}[|F(t_{j-1})|^2]\mathbb{E}[(B(t_j) - B(t_{j-1})^2]\\ &= \sum_{j=1}^{n} \mathbb{E}[|F(t_{j-1})|^2](t_j - t_{j-1}),\end{aligned}$$

which coincides with (6.5). □

Corollary 6.3. For any $F, G \in \mathscr{E}^2_B(0, T)$ we have

$$\mathbb{E}\left(\int_0^T F(s)dB(s)\int_0^T G(s)dB(s)\right) = \int_0^T \mathbb{E}[F(s)G(s)]ds. \tag{6.6}$$

Proof. The conclusion follows easily from (6.5) using the obvious identity

$$ab = \frac{1}{2}\,(a+b)^2 - \frac{1}{2}\,a^2 - \frac{1}{2}\,b^2, \quad a, b \in \mathbb{R}.$$ □

6.2. Itô's integral for predictable processes

The basic identity (6.5) allows to extend the definition of Itô's integral to more general processes, called *predictable*. Let us first introduce the *predictable σ-algebra* of subsets of $[0, T] \times \Omega$. A *predictable rectangle* is a subset R of $\mathscr{B}([0, T]) \times \mathscr{F}$ of the form

$$R = [s, t) \times A, \quad 0 \le s \le t \le T, \quad A \in \mathscr{F}_s.$$

We shall denote by $\mathscr{R}$ the family of all predictable rectangles and by $\mathscr{P}$ the σ-algebra generated by $\mathscr{R}$. A random variable in the probability space

$$([0, T] \times \Omega, \mathscr{P}, \lambda_1 \times \mathbb{P})$$

(where λ_1 denotes the Lebesgue measure on $[0, T]$) is called a *predictable* process in $[0, T]$. Clearly, elementary processes are predictable.

We are going to extend the definition of Itô's integral to all processes belonging to the space $L^2([0, T] \times \Omega, \mathscr{P}, \lambda_1 \times \mathbb{P}))$ which we shall denote by $L^2_B([0, T]; L^2(\Omega))$ for short.

Proposition 6.4. *The space $\mathscr{E}^2_B(0,T)$ of all elementary processes is dense in $L^2_B([0, T]; L^2(\Omega))$.*

Proof. Since any element of $L_B^2([0,T];L^2(\Omega))$ can be approximated by a monotonic sequence of simple functions, it is enough to show that

$$\mathbb{1}_A \in \overline{\mathscr{E}_B^2(0,T)}, \quad \forall\, A \in \mathscr{P}, \tag{6.7}$$

where $\overline{\mathscr{E}_B^2(0,T)}$ is the closure of $\mathscr{E}_B^2([0,T])$ in $L_B^2([0,T];L^2(\Omega))$.

To prove (6.7) we shall use the Dynkin Theorem A.1 recalled in Appendix A. We first note that the set $\mathscr{R}$ of all predictable rectangles is a π-system. Then we set

$$\mathscr{D} = \{A \in \mathscr{P}:\ \mathbb{1}_A \in \overline{\mathscr{E}_B^2(0,T)}\}.$$

We claim that $\mathscr{D}$ is a λ-system, *i.e.* that it fulfills (A.1). Properties (A.1)-(i)-(ii) are clear, let us show (A.1)-(iii). Let $(A_n) \subset \mathscr{D}$ be mutually disjoint sets and set

$$\phi_n = \sum_{k=1}^n \mathbb{1}_{A_k}.$$

Then, by the monotone convergence theorem,

$$\lim_{n\to\infty} \phi_n \to \phi = \mathbb{1}_A \quad \text{in } L_B^2([0,T];L^2(\Omega)),$$

where $A = \bigcup_{k=1}^\infty A_k$. So, $A \in \mathscr{D}$ and (A.1)-(iii) is fulfilled as well. The conclusion follows now from Theorem A.1. □

We are now ready to extend Itô's integral to processes from $L_B^2([0,T];L^2(\Omega))$. Let $F \in L_B^2([0,T];L^2(\Omega))$ and let $(F_n) \subset \mathscr{E}_B^2(0,T)$ be convergent to F in $L_B^2([0,T];L^2(\Omega))$. In view of (6.5), we have

$$\mathbb{E}[|I(F_n) - I(F_m)|^2] = \int_0^T \mathbb{E}[|F_n(t) - F_m(t)|^2]dt, \quad \forall\, m,n \in \mathbb{N}.$$

Consequently the sequence $(I(F_n))$ is Cauchy in $L^2(\Omega,\mathscr{F}_T,\mathbb{P})$. Now we define

$$\int_0^T F(t)dB(t) = \lim_{n\to\infty} \int_0^T F_n(t)dB(t) \quad \text{in } L^2(\Omega,\mathscr{F}_T,\mathbb{P}).$$

It is easy to see that this definition does not depend on the choice of the sequence (F_n). Moreover, by Proposition 6.2 it follows that

$$\mathbb{E}\left(\int_0^T F(s)dB(s)\right) = 0 \tag{6.8}$$

$$\mathbb{E}\left[\left(\int_0^T F(s)dB(s)\right)^2\right] = \int_0^T \mathbb{E}(|F(s)|^2)ds. \tag{6.9}$$

Also, from (6.6) it follows that if $F, G \in L^2_B([0,T]; L^2(\Omega))$ we have

$$\mathbb{E}\left(\int_0^T F(s)dB(s) \int_0^T G(s)dB(s)\right) = \int_0^T \mathbb{E}[F(s)G(s)]ds. \quad (6.10)$$

We also define in an obvious way the Itô integral $\int_a^b F(s)dB(s)$ in any interval $[a,b] \subset [0,T]$. We set

$$\int_a^a F(s)dB(s) = 0, \quad \int_b^a F(s)dB(s) = -\int_a^b F(s)dB(s).$$

Then we have

$$\mathbb{E}\left[\left(\int_a^b F(s)dB(s)\right)^2\right] = \int_a^b (\mathbb{E}|F(s)|^2)ds, \quad a < b.$$

Moreover, for any $a, b, c \in [0,T]$ one can check easily that

$$\int_a^c F(s)dB(s) = \int_a^b F(s)dB(s) + \int_b^c F(s)dB(s).$$

Exercise 6.5. Let $F \in L^2_B([0,T]; L^2(\Omega))$, $[a,b] \subset [0,T]$ and let $X \in L^\infty(\Omega, \mathscr{F}_a, \mathbb{P})$. Show that $XF \in L^2_B([a,b]; L^2(\Omega))$ and

$$\int_a^b X\, F(r)dB(r) = X \int_a^b F(r)dB(r). \quad (6.11)$$

Exercise 6.6. Let $F \in L^2_B([0,T]; L^2(\Omega))$ such that

$$\int_0^T F(t)dB(t) = 0. \quad (6.12)$$

Show that $F = 0$, $\mathbb{P}$-a.s..

Exercise 6.7. Let $F \in L^2_B([0,T]; L^2(\Omega))$. Show that

$$\mathbb{E}\left[B(t)\int_0^T F(s)dB(s)\right] = \int_0^t \mathbb{E}[F(s)]ds. \quad (6.13)$$

Hint. Use that

$$B(t) = \int_0^T \mathbb{1}_{[0,t]}(s)dB(s)$$

and (6.10).

We prove now that any mean continuous predictable process is the limit of Riemmanian sums.

6.2.1. Mean square continuous predictable processes

We denote by $\mathscr{C}_B([0,T];\mathscr{L}^2(\Omega))$ the linear space of all processes X such that

(i) $X(t,\cdot) \in \mathscr{L}^2(\Omega, \mathscr{F}_t, \mathbb{P})$ for all $t \in [0,T]$.

(ii) $\|X\|_{C_B} : = \left(\sup_{t\in[0,T]} \mathbb{E}(|X(t)|^2) \right)^{1/2} < +\infty.$

(iii) X is mean square continuous, that is for all $t_0 \in [0,T]$ we have

$$\lim_{t\to t_0} \mathbb{E}(|X(t) - X(t_0)|^2) = 0.$$

Exercise 6.8. Show that all processes from $\mathscr{C}_B([0,T];\mathscr{L}^2(\Omega))$ are predictable.

Let us consider the following equivalence relation r_1:

$$X \sim Y \Longleftrightarrow \|X - Y\|_{C_B} = 0.$$

Obviously $X \sim Y$ if and only if, for all $t \in [0,T]$, we have $X(t,\omega) = Y(t,\omega)$ for $\mathbb{P}$–almost all $\omega \in \Omega$

Finally, we shall denote by $C_B([0,T];L^2(\Omega))$ the quotient space of $\mathscr{C}_B([0,T];\mathscr{L}^2(\Omega))$ with respect to the equivalence relation r_1. It is easy to check that $C_B([0,T];L^2(\Omega))$, endowed with the norm $\|X\|_{C_B}$, is a Banach space.

Exercise 6.9. Show that $C_B([0,T];L^2(\Omega))$ is dense in $L^2_B([0,T];L^2(\Omega))$.

Now let $F \in C_B([0,T];L^2(\Omega))$. We want to show that $I(F)$ is the limit of Riemanian sums. We first notice that, since the mapping

$$F : [0,T] \mapsto L^2(\Omega),\ t \mapsto F(t),$$

is uniformly continuous, for any $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$s,t \in [0,T],\ |t-s| < \delta_\epsilon \Rightarrow \mathbb{E}(|F(t) - F(s)|^2) < \epsilon. \tag{6.14}$$

Now for any decomposition $\varsigma = \{t_0, t_1, \cdots, t_n\} \in \Sigma(0,T)$ consider the elementary process

$$F_\varsigma := \sum_{j=1}^n F(t_{j-1})\mathbb{1}_{[t_{j-1},t_j)}.$$

Then

$$I_\varsigma(F) = \int_0^T F_\varsigma(s)dB(s) = \sum_{j=1}^n F(t_{j-1})(B(t_j) - B(t_{j-1})).$$

Clearly $F_\varsigma \in \mathscr{E}^2_B(0,T)$ and, using (6.14) one can check easily that

$$\lim_{|\varsigma|\to 0} F_\varsigma = F, \quad \text{in } L^2_B([0,T]; L^2(\Omega)). \tag{6.15}$$

Consequently we have

$$\lim_{|\varsigma|\to 0} I_\varsigma(F) = \int_0^T F(s)dB(s) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{6.16}$$

Example 6.10. Let us prove that

$$\int_0^T B(t)dB(t) = \frac{1}{2}(B^2(T) - T). \tag{6.17}$$

Let $\varsigma = \{t_0, t_1, ..., t_n\} \in \Sigma(0,T)$ and set

$$I_\varsigma(B) = \sum_{k=1}^n B(t_{k-1})(B(t_k) - B(t_{k-1}).$$

Since

$$B(t_{k-1})(B(t_k)-B(t_{k-1})) = \frac{1}{2}\ B^2(t_k) - \frac{1}{2}\ B^2(t_{k-1}) - \frac{1}{2}\ (B(t_k)-B(t_{k-1}))^2,$$

we have

$$I_\varsigma(B) = \frac{1}{2}\ B^2(T) - \frac{1}{2}\ \sum_{k=1}^n (B(t_k) - B(t_{k-1}))^2.$$

Recalling that the quadratic variation of B is T (Theorem 4.15), we deduce that

$$\int_0^T B(t)dB(t) = \lim_{|\varsigma|\to 0} I_\varsigma(B) = \frac{1}{2}\ (B^2(T) - T).$$

Exercise 6.11. Prove that

$$\lim_{|\varsigma|\to 0} \sum_{k=1}^n B(t_k)(B(t_k) - B(t_{k-1})) = \frac{1}{2}\ (B^2(T) + T) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}),$$

and

$$\lim_{|\varsigma|\to 0} \sum_{k=1}^n B\left(\tfrac{t_k + t_{k-1}}{2}\right)(B(t_k) - B(t_{k-1})) = \frac{1}{2}\ B^2(T) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

Therefore the definition of the Itô integral depends on the particular form of the chosen integral sums.

6.2.2. A local property of Itô' integrals

Proposition 6.12. *Let $A \in \mathscr{F}$ such that $\mathbb{P}(A) > 0$ and let $F, G \in L^2_B(0, T; L^2(\Omega))$ such that for all $t \in [0, T]$ we have*

$$F(t) = G(t), \quad \mathbb{P}\text{-}a.s.\ on\ \ A. \tag{6.18}$$

Then we have

$$I(F) = I(G), \quad \mathbb{P}\text{-}a.s.\ on\ A, \tag{6.19}$$

where

$$I(F) = \int_0^T F(s)ds, \quad I(G) = \int_0^T G(s)ds.$$

Proof. The statement is obvious when F and G are elementary. Assume now that $F, G \in C_B([0, T]; L^2(\Omega))$. Then for any decomposition $\varsigma = \{t_0, t_1, \cdots, t_n\}$ of $[0, T]$ consider the elementary processes

$$F_\varsigma := \sum_{j=1}^n F(t_{j-1}) 1\!\!1_{[t_{j-1}, t_j)}$$

and

$$G_\varsigma := \sum_{j=1}^n G(t_{j-1}) 1\!\!1_{[t_{j-1}, t_j)}.$$

Clearly, we have

$$I(F_\varsigma) = I(G_\varsigma), \quad \mathbb{P}\text{-a.s. on}\ \ A.$$

Since

$$I(F_\varsigma) \to I(F), \quad I(G_\varsigma) \to I(G), \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}),$$

there is a sequence of decompositions of $[0, T]$, (ς_n) such that $|\varsigma_n| \to 0$ and

$$I(F_\varsigma) \to I(F), \quad I(G_\varsigma) \to I(G), \quad \mathbb{P}\text{-a.s.}.$$

Consequently

$$I(F) = I(G), \quad \mathbb{P}\text{-a.s. on } A.$$

So, the proposition is proved when $F, G \in C_B([0, T]; L^2(\Omega))$. The general case follows recalling that $C_B([0, T]; L^2(\Omega))$ is dense in $L^2(0, T); L^2(\Omega))$. $\square$

6.3. The Itô integral as a stochastic process

Let $F \in L^2_B(0,T;L^2(\Omega))$ and set

$$X(t) = \int_0^t F(s)dB(s), \quad t \in [0,T].$$

In this section we are going to study $X(t)$ as a function of t. We first notice that the process $X(t),\ t \ge 0$, has not independent increments. However, it is worth noticing that its increments are orthogonal (in the sense of $L^2(\Omega,\mathscr{F},\mathbb{P})$) as the following result shows.

Proposition 6.13. *Let* $0 \le t_1 \le t_2 \le t_3 \le t_4 \le T$. *Then we have*

$$\mathbb{E}[(X(t_2)-X(t_1))(X(t_4)-X(t_3))] = 0$$

Proof. In fact, taking into account (6.10), we have

$$\begin{aligned}
&\mathbb{E}[(X(t_2)-X(t_1))(X(t_4)-X(t_3))]\\
&= \mathbb{E}\left(\int_{t_1}^{t_2} F(s)dB(s)\int_{t_3}^{t_4} F(s)dB(s)\right)\\
&= \mathbb{E}\left(\int_0^T \mathbb{1}_{[t_1,t_2]}F(s)dB(s)\int_0^T \mathbb{1}_{[t_3,t_4]}F(s)dB(s)\right)\\
&= \int_0^T \mathbb{1}_{[t_1,t_2]}\mathbb{1}_{[t_3,t_4]}\mathbb{E}(F^2(s))ds = 0.
\end{aligned}$$

□

we are going to study continuity properties of X. First we show that X is mean square continuous, then that it is a martingale (see Appendix C), finally, that it is a continuous process.

Proposition 6.14. *Assume that* $F \in L^2_B(0,T;L^2(\Omega))$ *and let*

$$X(t) = \int_0^t F(s)dB(s), \quad t \in [0,T].$$

Then $X \in C_B([0,T];L^2(\Omega))$.

Proof. We already know that $X(t) \in L^2(\Omega,\mathscr{F}_t,\mathbb{P})$ for any $t \in [0,T]$. Moreover, for any $t, t_0 \in [0,T]$ we have

$$\mathbb{E}(|X(t)-X(t_0)|^2) = \int_{t_0}^t \mathbb{E}(|F(r)|^2)dr,$$

so that

$$\lim_{t\to t_0}\mathbb{E}(|X(t)-X(t_0)|^2)=0.$$

The conclusion follows. □

Proposition 6.15. $X(t),\ t\in[0,T]$, *is an* $\mathscr{F}_t$*–martingale.*

Proof. Let $t>s$. Since

$$X(t)-X(s)=\int_s^t F(r)dB(r),$$

we have

$$\mathbb{E}[X(t)|\mathscr{F}_s]=X(s)+\mathbb{E}\left[\int_s^t F(r)dB(r)\Big|\mathscr{F}_s\right].$$

So, we have to prove that

$$\mathbb{E}\left[\int_s^t F(r)dB(r)\Big|\mathscr{F}_s\right]=0. \tag{6.20}$$

This is not obvious because in general $F(r)$ contains the "story" of the Brownian motion from 0 to r. On the other hand, by the very definition of conditional expectation, we have

$$\int_A\mathbb{E}\left[\int_s^t F(r)dB(r)\Big|\mathscr{F}_s\right]d\mathbb{P}=\int_A\int_s^t F(r)dB(r)d\mathbb{P},\quad\forall\ A\in\mathscr{F}_s,$$

equivalently

$$\mathbb{E}\left\{\zeta\left[\int_s^t F(r)dB(r)\Big|\mathscr{F}_s\right]\right\}=\mathbb{E}\left[\zeta\int_s^t F(r)dB(r)\right]\quad\forall\ \zeta\in L^\infty(\Omega,\mathscr{F}_s,\mathbb{P}).$$

But

$$E\left[\zeta\int_s^t F(r)dB(r)\right]=E\left[\int_s^t \zeta F(r)dB(r)\right]=0,$$

in view of of (6.11). Therefore (6.20) follows. □

We show now that

$$X(t)=\int_0^t F(s)dB(s),\quad t\in[0,T],$$

is continuous. For this it is convenient to introduce a suitable space of continuous processes.

We shall denote by $\mathscr{L}^2_B(\Omega; C([0,T]))$ the linear space of all processes X such that

(i) $X(\cdot,\omega)$ is continuous for $\mathbb{P}$-almost all $\omega \in \Omega$.
(ii) $X(t,\cdot)$ is $\mathscr{F}_t$–measurable for all $t \in [0,T]$.
(iii) $\|X\|_{L^2} := \left[\mathbb{E}\left(\sup_{t\in[0,T]} |X(t)|^2\right)\right]^{1/2} < +\infty.$

We consider the following equivalence relation r_2:

$$X \sim Y \Longleftrightarrow \|X - Y\|_{L^2} = 0,$$

that is $X \sim Y$ if and only if $X(\cdot,\omega) = Y(\cdot,\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Finally, we denote by $L^2_B(\Omega; C([0,T]))$ the quotient space of $\mathscr{L}^2_B(\Omega; C([0,T]))$ with respect to the equivalence relation r_2. One can show by standard arguments that $L^2_B(\Omega; C([0,T]))$ is complete.

Remark 6.16. There is a natural inclusion

$$\mathscr{L}^2_B(\Omega; C([0,T])) \subset C_B([0,T]; \mathscr{L}^2(\Omega)).$$

Therefore any process in $L^2_B(\Omega; C([0,T]))$ is predictable (see Exercise 6.5).

Remark 6.17. If X is r_2–equivalent to Y, then it is also r_1-equivalent to Y. Consequently, there is a natural inclusion

$$L^2_B(\Omega; C([0,T])) \subset C_B([0,T]; L^2(\Omega)).$$

We are now ready to prove the continuity of X.

Theorem 6.18. *Let* $F \in L^2_B(0,T; L^2(\Omega))$ *and let*

$$X(t) = \int_0^t F(s)dB(s), \quad t \in [0,T].$$

Then $X \in L^2_B(\Omega; C([0,T]))$ *and we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |X(t)|^2\right] \le 4\int_0^T \mathbb{E}|F(s)|^2 ds. \tag{6.21}$$

Proof. Let $(F_n) \subset \mathscr{E}^2_B(0,T)$ be such that

$$F_n \to F \quad \text{in } L^2_B(0,T; L^2(\Omega))$$

and set

$$X_n(t) = \int_0^t F_n(s)dB(s), \quad n \in \mathbb{N},\ t \in [0, T].$$

Since $B(t)$ is continuous it follows easily that $X_n(t)$ is continuous as well for all $n \in \mathbb{N}$. Taking into account Proposition 6.15 we see that $X(t),\ t \in [0, T]$, is a continuous $\mathscr{F}_t$–martingale. Then by Corollary C.6 it follows that for any $n, m \in \mathbb{N}$

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X_n(t) - X_m(t)|^2\right) \le 4\mathbb{E}(|X_n(T) - X_m(T)|^2)$$

$$= 4\mathbb{E}\left(\int_0^T |F_n(s) - F_m(s)|^2 ds\right).$$

Consequently (X_n) is Cauchy in $L^2_B(\Omega; C([0, T]))$ and its limit coincides (up to a version) with X. □

6.4. Itô's integral with stopping times

Let $F \in L^2_B(0, T; L^2(\Omega))$ and set

$$X(t) = \int_0^t F(s)dB(s), \quad t \in [0, T].$$

Let moreover $\tau \le T$ be a stopping time with respect to the filtration $(\mathscr{F}_t)$. We recall that this means

$$\{\tau \le t\} \text{ is } \mathscr{F}_t\text{-measurable}, \quad \forall\, t > 0.$$

Define

$$\int_0^\tau F(s)dB(s) =: X(\tau), \tag{6.22}$$

that is

$$X(\tau)(\omega) = X(\tau(\omega), \omega), \quad \omega \in \Omega.$$

Arguing as in Proposition 5.10 we see that $X(\tau)$ is $\mathscr{F}_\tau$–measurable. We are going to show that

$$\int_0^\tau F(s)dB(s) = \int_0^T \mathbb{1}_{[0,\tau]}(s)F(s)dB(s). \tag{6.23}$$

Notice that the integral on the right hand side of (6.23) is easier to handle than the one in (6.22). In particular, from (6.23) it follows at once that

$$\mathbb{E}\int_0^\tau F(s)dB(s) = 0. \tag{6.24}$$

In order to prove (6.23), we first need to show that $1\!\!1_{[0,\tau]}$ is a predictable process. This is provided by the following lemma.

Lemma 6.19. *Let $\tau : \Omega \to (0, +\infty)$ be a stopping time with respect to the filtration $(\mathscr{F}_t)$. Then the process*

$$h(t) := 1\!\!1_{[0,\tau]}(t), \quad t \geq 0,$$

is predictable.

Proof. In view of Proposition 5.9 it is enough to assume that τ is discrete, say

$$\tau(\Omega) = (t_k),\ t_k > 0, \text{ with } (t_k) \text{ increasing}. \tag{6.25}$$

Set

$$\Omega_h := \{\tau = t_h\}, \quad h \in \mathbb{N}.$$

Then if $t \in [0, t_1]$ we have $t \leq \tau$ so that

$$h(t, \omega) = 1, \quad \forall\, t \in [0, t_1],\ \forall\, \omega \in \Omega.$$

If $t \in (t_1, t_2]$ we have $t \leq \tau$ iff $\omega \notin \Omega_1$ so that

$$h(t, \omega) = \begin{cases} 1, & \forall\, \omega \in (\Omega_1)^c \\ \\ 0, & \forall\, \omega \in \Omega_1 \end{cases}$$

and so on. In general for $t \in (t_k, t_{k+1}]$ we have

$$h(t, \omega) = \begin{cases} 1, & \forall\, \omega \in (\Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_k)^c \\ \\ 0, & \forall\, \omega \in \Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_k. \end{cases}$$

In conclusion

$$h = 1\!\!1_{[0,t_1]\times\Omega} + 1\!\!1_{(t_1,t_2]\times(\Omega_1)^c} + \cdots + 1\!\!1_{(t_k,t_{k+1}]\times(\Omega_1\cup\Omega_1\cup\cdots\cup\Omega_k)^c} + \cdots \tag{6.26}$$

Since τ is a stopping time,

$$(\Omega_1 \cup \Omega_1 \cup \cdots \cup \Omega_k)^c \in \mathscr{F}_{t_k}$$

for each $k \in \mathbb{N}$ and therefore the process h is predictable. $\square$

Proposition 6.20. *Let $F \in L^2_B(0, T; L^2(\Omega))$ and let $\tau : \Omega \to (0, +\infty)$ be a stopping time with respect to the filtration $(\mathscr{F}_t)$. Then we have*

$$\int_0^\tau F(s)dB(s) = \int_0^T 1\!\!1_{[0,\tau]}(s)F(s)dB(s). \tag{6.27}$$

Proof. Again, it is enough to prove the result when τ is discrete; assume that τ is given by (6.25). Let us check (6.27). By (6.26), we have

$$\begin{aligned}\int_0^T \mathbb{1}_{[0,\tau]}(s)F(s)dB(s) &= \int_0^{t_1} F(s)dB(s) + \mathbb{1}_{(\Omega_1)^c}\int_{t_1}^{t_2} F(s)dB(s)\\ &\quad + \mathbb{1}_{(\Omega_1\cup\Omega_2\cup\cdots\cup\Omega_{n-1})^c}\int_{t_{n-1}}^{t_n} F(s)dB(s) + \cdots\end{aligned} \tag{6.28}$$

Consequently if $\omega \in \Omega_1$ we have $\tau(\omega) = t_1$ and

$$\int_0^T \mathbb{1}_{[0,\tau]}(s)F(s)dB(s) = \int_0^{t_1} F(s)dB(s) = \int_0^{\tau} F(s)dB(s).$$

If $\omega \in \Omega_2$ we have $\tau(\omega) = t_2$ and

$$\begin{aligned}\int_0^T \mathbb{1}_{[0,\tau]}(s)F(s)dB(s) &= \int_0^{t_1} F(s)dB(s) + \int_{t_1}^{t_2} \mathbb{1}_{(\Omega_1)^c}F(s)dB(s)\\ &= \int_0^{t_2} F(s)dB(s) = \int_0^{\tau} F(s)dB(s),\end{aligned}$$

and so on. So. (6.27), follows. □

6.5. Itô's integral for almost surely square integrable functions

We denote by $B_B(\Omega; L^2([0,T]))$ the linear space of all predictable processes F such that

$$\mathbb{P}\left(\int_0^T |F(s)|^2 ds < \infty\right) = 1. \tag{6.29}$$

In other words $B_B(\Omega; L^2([0,T]))$ is the set of all predictable processes F such that

$$\int_0^T |F(s,\omega)|^2 ds < \infty, \quad \mathbb{P}\text{-a.e..}$$

Let $F \in B_B(\Omega; L^2([0,T]))$. For any $n \in \mathbb{N}$ denote by τ_n the first time where $\int_0^t |F(s)|^2 ds = n$ if such a time does exist and T otherwise. In other terms

$$\tau_n = \begin{cases} \inf\left\{t \geq 0 : \int_0^t |F(s)|^2 ds = n\right\} \\ \text{if } \exists\, t \in [0, T] \text{ such that } \int_0^t |F(s)|^2 ds = n, \\ T \quad \text{if } \int_0^t |F(s)|^2 ds < n, \ \forall\, t \in [0, T]. \end{cases} \tag{6.30}$$

Then the process $\mathbb{1}_{[0,\tau_n]}F$ belongs to $L^2_B(0, T; L^2(\Omega))$ because

$$\int_0^T \mathbb{1}_{[0,\tau_n]}(s)|F(s)|^2 ds \leq n.$$

We set

$$\int_0^{\tau_n} F(s)dB(s) := \int_0^T \mathbb{1}_{[0,\tau_n]}(s)|F(s)|^2 ds.$$

Let now $\omega \in \Omega$ such that

$$\int_0^T |F(s, \omega)|^2 ds = k < \infty.$$

Then

$$\tau_n(\omega) = T, \quad \forall\, n \geq k + 1. \tag{6.31}$$

Since for $\mathbb{P}$-almost all $\omega \in \Omega$ the integral $\int_0^T |F(s, \omega)|^2 ds$ is finite, we can define

$$\int_0^T F(s)dB(s) := \lim_{n\to\infty} \int_0^{\tau_n} F(s)dB(s). \tag{6.32}$$

Notice that the limit above does exist, since by (6.31) it follows that the integral $\int_0^{\tau_n} F(s)dB(s)$ is definitively constant. So, we have defined the Itô integral for almost all $\omega \in \Omega$.

6.6. Multidimensional Itô's integrals

Let $r \in \mathbb{N}$ be fixed and consider an r-dimensional Brownian motion

$$B(t) = (B_1(t), ..., B_r(t)), \quad t \geq 0,$$

on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let us consider the cylindrical sets

$$C_{t_1,t_2,...,t_n;A} := \{\omega \in \Omega : \ (B(t_1)(\omega), ..., B(t_n)\omega) \in A\},$$

where $n \in \mathbb{N}, 0 < t_1 < \cdots < t_n$ and $A \in \mathscr{B}(\mathbb{R}^{nr})$.

We denote by $\mathscr{C}$ the σ-algebra generated by all cylindrical sets. Moreover, for any $t \geq 0$ we denote by $\mathscr{C}_t$ the family of all cylindrical sets $C_{t_1,t_2,\ldots,t_n;A}$ such that $t_n \leq t$ and by $\mathscr{F}_t$ the σ-algebra generated by $\mathscr{C}_t$.

$(\mathscr{F}_t)_{t\geq 0}$ is an increasing family of σ–algebras and $\mathscr{F}_0 = \{\varnothing, \Omega\}$. $(\mathscr{F}_t)_{t\geq 0}$ is called the *natural filtration* of B.

The following lemma can be proved as Lemma 5.1.

Lemma 6.21. *Let $t > s > 0$, and let X be a real random variable $\mathscr{F}_s$–measurable. Then $B(t) - B(s)$ and X are independent.*

Let us consider the predictable σ–algebra $\mathscr{P}$ defined as before. A random variable in the probability space

$$([0, T] \times \Omega, \mathscr{P}, \lambda_1 \times \mathbb{P})$$

(where λ_1 denotes the Lebesgue measure on $[0, T]$) is called a *predictable* process in $[0, T]$.

Now, given $d \in \mathbb{N}$, we shall define the Itô integral for predictable processes with values in $L(\mathbb{R}^r, \mathbb{R}^d)$ belonging to

$$L^2([0, T] \times \Omega, \mathscr{P}, \lambda_1 \times \mathbb{P}; L(\mathbb{R}^r, \mathbb{R}^d)) =: L^2_B(0, T; L^2(\Omega; L(\mathbb{R}^r, \mathbb{R}^d)).$$

First we need a lemma.

Lemma 6.22. *Let $f \in L^2_{B_i}(0, T; L^2(\Omega))$ and $g \in L^2_{B_j}(0, T; L^2(\Omega))$. Then we have*

$$\begin{aligned}&\mathbb{E}\left(\int_0^T f(s)dB_i(s)\int_0^T g(s)dB_j(s)\right)\\ &= \delta_{i,j}\int_0^T \mathbb{E}[f(s)g(s)]ds, \quad i, j = 1, \ldots, r.\end{aligned} \tag{6.33}$$

Proof. It is enough to show the assertion when f and g are elementary,

$$f = \sum_{k=1}^n f_{k-1}\mathbb{1}_{[t_{k-1},t_k)}, \quad g = \sum_{k=1}^n g_{k-1}\mathbb{1}_{[t_{k-1},t_k)},$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and f_k, g_k are $\mathscr{F}_{t_k}$-measurable, $k = 1, \ldots, n$. In this case we have

$$\int_0^T f(s)dB_i(s) = \sum_{k=1}^n f_{k-1}(B_i(t_k) - B_i(t_{k-1}))$$

and

$$\int_0^T g(s)dB_j(s) = \sum_{k=1}^n g_{k-1}(B_j(t_k) - B_j(t_{k-1})).$$

Now, if $i \neq j$ we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T f(s)dB_i(s) \int_0^T g(s)dB_j(s)\right) \\
&= \mathbb{E} \sum_{h,k=1}^n f_{h-1}g_{k-1}(B_i(t_h) - B_i(t_{h-1}))(B_j(t_k) - B_j(t_{k-1})).
\end{aligned} \tag{6.34}$$

It is easy to see that if $h \neq k$ it holds

$$\mathbb{E}[f_{h-1}g_{k-1}(B_i(t_h) - B_i(t_{h-1}))(B_j(t_k) - B_j(t_{k-1}))] = 0.$$

Let now $h = k$ and set

$$L_h := f_{h-1}g_{h-1}(B_i(t_h) - B_i(t_{h-1}))(B_j(t_h) - B_j(t_{h-1})).$$

Then $B(t_h) - B(t_{h-1})$ is independent of $f_{h-1}g_{h-1}$ and consequently any Borel function of $B(t_h) - B(t_{h-1})$ is independent of $f_{h-1}g_{h-1}$. Now

$$(B_i(t_h) - B_i(t_{h-1}))(B_j(t_h) - B_j(t_{h-1})) = \psi_{i,j}(B(t_h) - B(t_{h-1})),$$

where

$$\psi_{i,j}(x) = x_i x_j, \quad i, j = 1, ..., r.$$

This implies that $\mathbb{E}[L_h] = 0$. So, we have

$$\mathbb{E}\left(\int_0^T f(s)dB_i(s) \int_0^T g(s)dB_j(s)\right) = 0.$$

Finally, if $i = j$ we have

$$\mathbb{E}\left(\int_0^T f(s)dB_i(s) \int_0^T g(s)dB_i(s)\right) = \mathbb{E} \int_0^T f(s)g(s)ds.$$

The proof is complete. □

Let now $F \in L^2_B(0, T; L^2(\Omega; L(\mathbb{R}^r, \mathbb{R}^d))$. We define the Itô integral of F as the d-dimensional process

$$\left(\int_0^T F(t)dB(t)\right)_i = \sum_{j=1}^r \int_0^T F_{i,j}(t)dB_j(t), \quad i = 1, ..., d.$$

Proposition 6.23. *Let $F \in L^2_B(0,T;L^2(\Omega;L(\mathbb{R}^r,\mathbb{R}^d)))$. Then we have*

$$\mathbb{E}\left|\int_0^T F(t)dB(t)\right|^2 = \int_0^T \mathbb{E}[\mathrm{Tr}(F(t)F^*(t))]dt, \tag{6.35}$$

where

$$\mathrm{Tr}(F(t)F^*(t)) = \sum_{i=1}^d \sum_{j=1}^r F_{i,j}^2(t)$$

and $F^*(t)$ is the adjoint of $F(t)$ [(2)].

Proof. Set $I(F) = \int_0^T F(t)dB(t)$. Then we have

$$(I(F))_i = \sum_{j=1}^r \int_0^T F_{i,j}(t)dB_j(t), \quad i = 1, ..., d.$$

It follows that

$$\mathbb{E}|I(F)|^2 = \sum_{i=1}^d \mathbb{E}\left[\sum_{j=1}^r \int_0^T F_{i,j}(t)dB_j(t)\right]^2$$

and, taking into account (6.33),

$$\mathbb{E}|I(F)|^2 = \sum_{i=1}^d \sum_{j=1}^r \int_0^T \mathbb{E}[F_{i,j}(t)^2]dt,$$

which yields (6.35). □

Example 6.24. Assume that $d = 1$, then $F(t) \in L(\mathbb{R}^r, \mathbb{R})$, so that it can be identified with a vector setting

$$F(t)x = \langle F(t), x\rangle_{\mathbb{R}^r} = \sum_{h=1}^r F_h(t)x_h.$$

So, we write the Itô integral as

$$I(F) = \int_0^T \langle F(t), dB(t)\rangle.$$

(2) For each t, $F^*(t)$ is the mapping from $\mathbb{R}^d$ into $\mathbb{R}^r$ defined by $\langle x, F^*(t)y\rangle_{\mathbb{R}^d} = \langle F(t)x, y\rangle_{\mathbb{R}^r}$.

Moreover $F^*(t) \in L(\mathbb{R}, \mathbb{R}^r)$ is given by

$$F^*(t)\alpha = (F_1(t), ..., F_r(t))\alpha.$$

Then

$$F(t)F^*(t) = \sum_{h=1}^{n} F_h(t)^2 = |F(t)|^2$$

and (6.35) becomes

$$\mathbb{E}\left|\int_0^T \langle F(t), dB(t)\rangle\right|^2 = \int_0^T \mathbb{E}|F(t)|^2 dt. \tag{6.36}$$

Chapter 7
Itô's formula

Let B be a real Brownian motion on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let $(\mathscr{F}_t)_{t\geq 0}$ its natural filtration. We are given a function $\varphi \in C_b^2(\mathbb{R})$ [(1)] and a real process X on $[0, T]$ of the form

$$X(t) = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s), \quad t \in [0, T], \tag{7.1}$$

where $b,\ \sigma \in L_B^2(0, T; L^2(\Omega))$ and $x \in \mathbb{R}$. We call X a *Itô's process*.

The main result of this chapter is the following Itô's formula proved in Section 7.1,

$$\begin{aligned}\varphi(X(t)) = \varphi(x) &+ \int_0^t \varphi'(X(s))b(s)\\ &+ \int_0^t \varphi'(X(s))\sigma(s)dB(s)\\ &+\frac{1}{2}\int_0^t \varphi''(X(s))\sigma^2(s)ds, \quad t \in [0, T].\end{aligned} \tag{7.2}$$

We start with some preliminary on quadratic sums (generalizing the quadratic variation of the Brownian motion) given is Subsection 7.1.1. Then we prove (7.2) for *elementary processes* and finally for *predictable* ones.

Section 7.2 is devoted to a useful generalization of (7.2) to the case when φ is still C^2 but not bounded.

Finally, in Section 7.3 we generalize the previous results to multidimensional processes.

(1) Space $C_b^2(\mathbb{R})$ was defined in Section 3.1.

7.1. Itô's formula for one-dimensional processes

7.1.1. Preliminaries on quadratic sums

Let us consider a process $F \in C_B([0,T]; L^2(\Omega))$ and set

$$I(F) := \int_0^t F(s)dB(s), \quad t \in [0,T].$$

For any decomposition $\varsigma = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ of $[0,T]$ set

$$I_\varsigma(F) := \sum_{k=1}^n F(t_{k-1})(B(t_k) - B(t_{k-1})).$$

We already know that

$$\lim_{|\varsigma|\to 0} I_\varsigma(F) = \int_0^T F(s)dB(s) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

Now we want to study the limit as $\varsigma \to 0$ of the following quadratic sums

$$J_\varsigma(F) = \sum_{k=1}^n F(t_{k-1})(B(t_k) - B(t_{k-1}))^2.$$

When $F(t) \equiv 1$ we know from Theorem 4.15 that

$$J_\varsigma(F) \to T \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

The following proposition gives a generalization of this result.

Proposition 7.1. *We have*

$$\lim_{|\varsigma|\to 0} J_\varsigma(F) = \int_0^T F(s)ds \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.3}$$

Proof. It is enough to show that

$$\lim_{|\varsigma|\to 0} \mathbb{E}\left[\left(J_\varsigma(F) - \sum_{k=1}^n F(t_{k-1})(t_k - t_{k-1})\right)^2\right] = 0,$$

because obviously

$$\lim_{|\varsigma|\to 0} \sum_{k=1}^n F(t_{k-1})(t_k - t_{k-1}) = \int_0^T F(s)ds \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).$$

We have in fact

$$
\begin{aligned}
&\mathbb{E}\left[\left(J_{\varsigma}(F)-\sum_{k=1}^{n} F(t_{k-1})(t_k-t_{k-1})\right)^2\right]\\
&=\mathbb{E}\left[\left(\sum_{k=1}^{n} F(t_{k-1})\left((B(t_k)-B(t_{k-1}))^2-(t_k-t_{k-1})\right)\right)^2\right]\\
&=\sum_{k=1}^{n}\mathbb{E}\left(|F(t_{k-1})|^2\left[(B(t_k)-B(t_{k-1}))^2-(t_k-t_{k-1})\right]^2\right)\\
&\quad +2\sum_{j<k=1}^{n}\mathbb{E}\Big(F(t_{j-1})[(B(t_j)-B(t_{j-1}))^2-(t_j-t_{j-1})]\\
&\quad \times F(t_{k-1})[(B(t_k)-B(t_{k-1}))^2-(t_k-t_{k-1})]\Big)
\end{aligned}
$$

Now the last sum vanishes because B has independent increments, so that

$$
\begin{aligned}
&\mathbb{E}\left[\left(J_{\varsigma}(F)-\sum_{k=1}^{n} F(t_{k-1})(t_k-t_{k-1})\right)^2\right]\\
&=\sum_{k=1}^{n}\mathbb{E}\left(|F(t_{k-1})|^2\left[|B(t_k)-B(t_{k-1})|^2-(t_k-t_{k-1})\right]^2\right) \qquad (7.4)\\
&=\sum_{k=1}^{n}\mathbb{E}|F(t_{k-1})|^2\,\mathbb{E}\left(\left[|B(t_k)-B(t_{k-1})|^2-(t_k-t_{k-1})\right]^2\right),
\end{aligned}
$$

because $F(t_{k-1})$ and $B(t_k)-B(t_{k-1})$ are independent.

Next, taking into account that

$$
\mathbb{E}[|B(t_k)-B(t_{k-1})|^2]=(t_k-t_{k-1}),
$$

$$
\mathbb{E}[|B(t_k)-B(t_{k-1})|^4]=3(t_k-t_{k-1})^2,
$$

by (7.4) we have

$$
\begin{aligned}
&\mathbb{E}\left[\left(J_{\varsigma}(F)-\sum_{k=1}^{n} X(t_{k-1})(t_k-t_{k-1})\right)^2\right]\\
&=2\sum_{k=1}^{n}\mathbb{E}[|F(t_{k-1})|^2](t_k-t_{k-1})^2\\
&\le 2|\varsigma|\sum_{k=1}^{n}\mathbb{E}[|F(t_{k-1})|^2(t_k-t_{k-1})]\to 0,
\end{aligned}
$$

as $|\varsigma|\to 0$. The conclusion follows. $\square$

Proposition 7.1 can be intuitively interpreted by saying that $dB(t)$ behaves formally as $\sqrt{dt}$.

7.1.2. Proof of Itô's formula

Let us consider the Itô process X given by (7.1).

Remark 7.2. X belongs to $C_B([0,T]; L^2(\Omega))$ by Proposition 6.14 and to $L^2_B(\Omega; C([0,T]))$ by Theorem 6.18.

Let us prove the following *Itô's formula*

Theorem 7.3. *Let $\varphi \in C^2_b(\mathbb{R})$ and let X be defined by* (7.1) *Then we have*

$$\begin{aligned}\varphi(X(t)) = \varphi(x) + \int_0^t \varphi'(X(s))b(s)ds + \int_0^t \varphi'(X(s))\sigma(s)dB(s)\\ +\frac{1}{2}\int_0^t \varphi''(X(s))\sigma^2(s)ds, \quad t\in[0,T].\end{aligned} \tag{7.5}$$

Remark 7.4. Identity (7.5) holds both in $C_B([0,T]; L^2(\Omega))$ and in $L_B(\Omega; C([0,T]))$ (see Remark 7.2).

We shall also write identities (7.1) and (7.2) in the following differential forms

$$dX(t) = b(t)dt + \sigma(t)dB(t), \tag{7.6}$$

and

$$\begin{aligned}d\varphi(X(t)) = \varphi'(X(t))b(t)dt + \varphi'(X(t))\sigma(t)dB(t)\\ +\frac{1}{2}\varphi''(X(t))\sigma^2(t)dt,\end{aligned} \tag{7.7}$$

respectively.

We shall first prove Theorem 7.3 when b and σ are constant.

7.1.2.1. The case when b and σ are constant. We assume here that $b = b_0 \in \mathbb{R}$ and $\sigma = \sigma_0 \in \mathbb{R}$. Then Theorem 7.3 reduces to.

Lemma 7.5. *Let $X(t) = x + b_0 t + \sigma_0 B(t)$, $t\in[0,T]$, and let $\varphi \in C^2_b(\mathbb{R})$. Then we have*

$$\begin{aligned}\varphi(X(t)) = \varphi(x) + b_0\int_0^t \varphi'(X(s))ds + \sigma_0\int_0^t \varphi'(X(s))dB(s)\\ +\frac{\sigma_0^2}{2}\int_0^t \varphi''(X(s))ds.\end{aligned} \tag{7.8}$$

Proof. We first notice that it is enough to prove (7.8) when $\varphi \in C_b^3(\mathbb{R})$ because $C_b^3(\mathbb{R})$ is dense in $C_b^2(\mathbb{R})$. So, we shall assume that $\varphi \in C_b^3(\mathbb{R})$ in the following.

Fix $T > 0$ and let $\varsigma = \{0 = s_0 < s_1 < \cdots < s_n = T\}$ be a decomposition of $[0, T]$. Write

$$\varphi(X(T)) - \varphi(x) = \sum_{h=1}^{n}[\varphi(X(s_h)) - \varphi(X(s_{h-1}))]. \tag{7.9}$$

By Taylor's formula we have

$$\begin{aligned}\varphi(X(s_h)) - \varphi(X(s_{h-1})) &= \varphi'(X(s_{h-1}))(X(s_h) - X(s_{h-1})) \\ &\quad + \frac{1}{2}\varphi''(X(s_{h-1}))(X(s_h) - X(s_{h-1}))^2 + R_h,\end{aligned} \tag{7.10}$$

where

$$R_h = \int_0^1 (1-\xi)(\varphi''(Y_h(\xi)) - \varphi''(X(s_{h-1}))(X(s_h) - X(s_{h-1}))^2 d\xi \tag{7.11}$$

and

$$Y_h(\xi) = (1-\xi)X(s_{h-1}) + \xi X(s_h), \quad \xi \in [0, 1]. \tag{7.12}$$

Now we write (7.9) as

$$\begin{aligned}\varphi(X(T)) - \varphi(x) &= \sum_{h=1}^{n} \varphi'(X(s_{h-1}))(X(s_h) - X(s_{h-1})) \\ &\quad + \frac{1}{2}\sum_{h=1}^{n} \varphi''(X(s_{h-1}))(X(s_h) - X(s_{h-1}))^2 \\ &\quad + \sum_{h=1}^{n} R_h \\ &=: A_\varsigma + B_\varsigma + C_\varsigma.\end{aligned} \tag{7.13}$$

Since

$$X(s_h) - X(s_{h-1}) = b_0(s_h - s_{h-1}) + \sigma_0(B(s_h) - B(s_{h-1})) \tag{7.14}$$

we have

$$\begin{aligned}A_\varsigma &= b_0 \sum_{h=1}^{n} \varphi'(X(s_{h-1}))(s_h - s_{h-1}) \\ &\quad + \sigma_0 \sum_{h=1}^{n} \varphi'(X(s_{h-1}))(B(s_h) - B(s_{h-1})),\end{aligned} \tag{7.15}$$

so that

$$\lim_{|\varsigma|\to 0} A_\varsigma = b_0 \int_0^T \varphi'(X(s))ds + \sigma_0 \int_0^T \varphi'(X(s))dB(s) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.16}$$

Moreover

$$\begin{aligned} B_\varsigma &= \frac{\sigma_0^2}{2} \sum_{h=1}^n \varphi''(X(s_{h-1}))(B(s_h) - B(s_{h-1}))^2 \\ &\quad + b_0\sigma_0 \sum_{h=1}^n \varphi''(X(s_{h-1}))(s_h - s_{h-1}))(B(s_h) - B(s_{h-1})) \\ &\quad + \frac{b_0^2}{2} \sum_{h=1}^n \varphi''(X(s_{h-1}))(s_h - s_{h-1})^2 \\ &=: B_{\varsigma,1} + B_{\varsigma,2} + B_{\varsigma,3}. \end{aligned} \tag{7.17}$$

By Proposition 7.1 it follows that

$$\lim_{|\varsigma|\to 0} B_{\varsigma,1} = \frac{\sigma_0^2}{2} \int_0^T \varphi''(X(s))ds \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.18}$$

We claim that

$$\lim_{|\varsigma|\to 0} B_{\varsigma,2} = 0, \quad \lim_{|\varsigma|\to 0} B_{\varsigma,3} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.19}$$

We have in fact

$$B_{\varsigma,2} = b_0\sigma_0 \sum_{h=1}^n \varphi''(X(s_{h-1}))(s_h - s_{h-1})(B(s_h) - B(s_{h-1})),$$

and so

$$\begin{aligned} \mathbb{E}|B_{\varsigma,2}| &\le |b_0|\,|\sigma_0|\,\|\varphi\|_2 \sum_{h=1}^n (s_h - s_{h-1})\mathbb{E}|B(s_h) - B(s_{h-1})| \\ &\le |b_0|\,|\sigma_0|\,\|\varphi\|_2 \sum_{h=1}^n (s_h - s_{h-1}) \left(\mathbb{E}|B(s_h) - B(s_{h-1})|^2\right)^{1/2} \\ &\le |b_0|\,|\sigma_0|\,\|\varphi\|_2 \sum_{h=1}^n (s_h - s_{h-1})^{3/2} \le |b_0|\,|\sigma_0|\,\|\varphi\|_2 |\varsigma|^{1/2} T \end{aligned}$$

and so, the first identity in (7.19) is proved; the second one follows at once.

We finally claim that

$$\lim_{|\varsigma|\to 0} C_\varsigma = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}), \tag{7.20}$$

which, together with (7.16), (7.18) and(7.19), will conclude the proof.

Write

$$\begin{aligned}
C_\varsigma = & \frac{\sigma_0^2}{2} \sum_{h=1}^n \int_0^1 (1-\xi)(\varphi''(Y_h(\xi)) \\
& - \varphi''(X(s_{h-1}))(B(s_h) - B(s_{h-1}))^2 d\xi \\
& + b_0\sigma_0 \sum_{h=1}^n \int_0^1 (1-\xi)(\varphi''(Y_h(\xi)) \\
& - \varphi''(X(s_{h-1}))(s_h - s_{h-1})(B(s_h) - B(s_{h-1}))d\xi \\
& + \frac{b_0^2}{2} \sum_{h=1}^n \int_0^1 (1-\xi)(\varphi''(Y_h(\xi)) - \varphi''(X(s_{h-1}))(s_h - s_{h-1})^2 d\xi \\
=: & \, C_{\varsigma,1} + C_{\varsigma,2} + C_{\varsigma,3}.
\end{aligned}$$

Consequently, using again (7.14)

$$\begin{aligned}
|C_{\varsigma,1}| \le & \frac{\sigma_0^2}{2} \|\varphi\|_3 \sum_{h=1}^n \int_0^1 (1-\xi)|Y_h(\xi) - X(s_{h-1})| \, (B(s_h) - B(s_{h-1}))^2 d\xi \\
= & \frac{\sigma_0^2}{2} \|\varphi\|_3 \sum_{h=1}^n \int_0^1 \xi(1-\xi)|X(s_h) - X(s_{h-1})| \, (B(s_h) - B(s_{h-1}))^2 d\xi \\
\le & \frac{|b_0|\sigma_0^2}{2} \|\varphi\|_3 \sum_{h=1}^n (s_h - s_{h-1}) \, (B(s_h) - B(s_{h-1}))^2 \\
& + \frac{|\sigma_0|^3}{2} \|\varphi\|_3 \sum_{h=1}^n (B(s_h) - B(s_{h-1}))^3 \\
=: & \, C_{\varsigma,1,1} + C_{\varsigma,1,2}.
\end{aligned}$$

Now

$$\mathbb{E}|C_{\varsigma,1,1}| \le \frac{1}{2}\,|b_0|\sigma_0^2\,\|\varphi\|_3\sum_{h=1}^n(s_h-s_{h-1})\mathbb{E}(B(s_h)-B(s_{h-1}))^2$$

$$= \frac{1}{2}\,|b_0|\sigma_0^2\,\|\varphi\|_3\sum_{h=1}^n(s_h-s_{h-1})^2 \le |\varsigma|T.$$

and then

$$\lim_{|\varsigma|\to 0} C_{\varsigma,1,1} = 0 \quad \text{in } L^1(\Omega,\mathscr{F},\mathbb{P}). \tag{7.21}$$

Moreover

$$\mathbb{E}|C_{\varsigma,1,2}| \le \frac{1}{2}\,|\sigma_0|^3\,\|\varphi\|_3\sum_{h=1}^n\mathbb{E}(B(s_h)-B(s_{h-1}))^3$$

$$\le \frac{1}{2}\,|\sigma_0|^3\,\|\varphi\|_3\sum_{h=1}^n[\mathbb{E}(B(s_h)-B(s_{h-1}))^6]^{1/2}$$

$$= \frac{1}{2}\,\sqrt{15}\,|\sigma_0|^3\,\|\varphi\|_3\sum_{h=1}^n(s_h-s_{h-1})^{3/2}$$

$$\le \frac{1}{2}\,\sqrt{15}\,|\sigma_0|^3\,\|\varphi\|_3\,|\varsigma|\,T.$$

Therefore

$$\lim_{|\varsigma|\to 0} C_{\varsigma,1,2} = 0 \quad \text{in } L^1(\Omega,\mathscr{F},\mathbb{P}). \tag{7.22}$$

As regards $C_{\varsigma,2}$ we have

$$\mathbb{E}|C_{\varsigma,2}| \le 2|b_0|\,|\sigma_0|\,\|\varphi\|_2\sum_{h=1}^n(s_h-s_{h-1})\mathbb{E}|B(s_h)-B(s_{h-1})|$$

$$\le |b_0|\,|\sigma_0|\,\|\varphi\|_2\sum_{h=1}^n(s_h-s_{h-1})^{3/2}$$

$$\le |b_0|\,|\sigma_0|\,\|\varphi\|_2\,|\varsigma|^{1/2}\,T.$$

Therefore

$$\lim_{|\varsigma|\to 0} C_{\varsigma,2} = 0 \quad \text{in } L^1(\Omega,\mathscr{F},\mathbb{P}). \tag{7.23}$$

Finally, as regards $C_{\varsigma,3}$ we have

$$|C_{\varsigma,3}| \le \sigma_0^2 \|\varphi\|_2 \sum_{h=1}^{n} (s_h - s_{h-1})^2 \le \sigma_0^2 \|\varphi\|_2 \ \varsigma \ T.$$

Therefore

$$\lim_{|\varsigma|\to 0} C_{\varsigma,3} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.24}$$

So (7.20) holds and the proof is complete. □

7.1.2.2. Proof of Theorem 7.3.

Proof. *Step* 1. Assume that $b, \sigma \in \mathscr{E}_B^2(0, T)$.

Then the proof is an easy consequence of Lemma 7.1 which we leave the reader as an exercise.

Step 2. Conclusion.

Let $(b_n) \subset \mathscr{E}_B^2(0, T)$ and $(\sigma_n) \subset \mathscr{E}_B^2(0, T)$ be such that

$$\begin{cases} b_n \to b \quad \text{in } L_B^2(0, T; L^2(\Omega)) \\ \\ \sigma_n \to \sigma \quad \text{in } L_B^2(0, T; L^2(\Omega)) \end{cases}$$

and set

$$X_n(T) := x + \int_0^T b_n(s)ds + \int_0^T \sigma_n(s)dB(s). \tag{7.25}$$

Then

$$\lim_{n\to\infty} X_n(T) = X(T) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.26}$$

Moreover, by Step 1 we have

$$\begin{aligned} \varphi(X_n(T)) = \varphi(x) &+ \int_0^T \varphi'(X_n(s))b_n(s)ds \\ &+ \int_0^T \varphi'(X_n(s))\sigma_n(s)dB(s) \\ &+ \frac{1}{2}\int_0^T \varphi''(X_n(s))\sigma_n^2(s)ds. \end{aligned} \tag{7.27}$$

Now the conclusion follows letting $n \to \infty$ and using the dominated convergence theorem. □

Taking expectation on both sides of (7.2), yields the following important formula.

Proposition 7.6.

$$
\begin{aligned}
\mathbb{E}[\varphi(X(t))] = \varphi(x) + \mathbb{E}\int_0^t \varphi'(X(s))b(s)ds \\
+\frac{1}{2}\,\mathbb{E}\int_0^t \varphi''(X(s))\sigma^2(s)ds.
\end{aligned}
\tag{7.28}
$$

We can extend Itô's formula to functions $u(t, X(t))$ by a straightforward generalization of Theorem 7.3 whose proof is left to the reader.

Theorem 7.7. *Let $u : [0, T] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded together with its partial derivatives u_t, u_x, u_{xx} and let X be defined by* (7.1). *Then we have*

$$
\begin{aligned}
u(t, X(t)) = u(0, x) + \int_0^t u_t(s, X(s))ds \\
+ \int_0^t u_x(s, X(s))b(s)ds \\
+ \int_0^t u_x(s, X(s))\sigma(s)dB(s) \\
+\frac{1}{2}\int_0^t u_{xx}(s, X(s))\sigma^2(s)ds, \quad t \in [0, T].
\end{aligned}
\tag{7.29}
$$

7.2. The Itô formula for unbounded functions

We assume here that φ is of class C^2 but not necessarily belonging to $C_b^2(\mathbb{R})$. In this case we can show that Itô's formula (7.2) is still valid. However, the various terms which appears in the formula are not $\mathbb{P}$-integrable in general and we cannot conclude that the expectation of the stochastic integral in the formula vanishes.

Theorem 7.8. *Let*

$$
X(t) = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s), \quad t \in [0, T], \tag{7.30}
$$

where $b,\ \sigma \in L_B^2(0, T; L^2(\Omega))$ and $x \in \mathbb{R}$. Then for any $\varphi \in C^2(\mathbb{R})$ we *have*

$$\begin{aligned}\varphi(X(t)) &= \varphi(x) + \int_0^t \varphi'(X(s))b(s)ds \\ &\quad + \int_0^t \varphi'(X(s))\sigma(s)dB(s) \\ &\quad + \frac{1}{2}\int_0^t \varphi''(X(s))\sigma^2(s)ds, \quad t \in [0,T], \ \mathbb{P}\text{-a.s.}\end{aligned} \tag{7.31}$$

Remark 7.9. All terms in formula (7.31) are meaningful. In fact, first notice that since X is a continuous process then $\varphi'(X(\cdot))$ and $\varphi''(X(\cdot))$ are $\mathbb{P}$-a.s. continuous and bounded. So, $\varphi'(X(\cdot))b(\cdot)$ and $\varphi''(X(\cdot))\sigma^2(\cdot)$ are re $\mathbb{P}$-a.s. integrable.

For as the stochastic integral $\int_0^t \varphi'(X(s))\sigma(s)dB(s)$ is concerned, it can be defined proceeding as in Section 6.5 because $\varphi'(X(\cdot))\sigma(\cdot)$ is square integrable with probability 1.

Let us now prove Theorem 7.8. We shall proceed by localization.

Proof. For any $R > 0$ consider a function $\varphi_R \in C_b^2(\mathbb{R})$ such that

$$\varphi_R(x) = \begin{cases} \varphi(x) & \text{if } |x| \le R, \\ \\ 0 & \text{if } |x| \ge R+1. \end{cases}$$

Then, applying Itô's formula (7.2) to $\varphi_R(X(t))$ yields

$$\begin{aligned}\varphi_R(X(t)) &= \varphi_R(x) + \int_0^t \varphi_R'(X(s))b(s)ds \\ &\quad + \int_0^t \varphi_R'(X(s))\sigma(s)dB(s) \\ &\quad + \frac{1}{2}\int_0^t \varphi_R''(X(s))\sigma^2(s)ds.\end{aligned} \tag{7.32}$$

Let now τ_R be the stopping time

$$\tau_R = \begin{cases} \inf\{t \in [0,T] : \ |X(t)| = R\} & \text{if } \sup\limits_{t \in [0,T]} |X(t)| \ge R, \\ \\ T \quad \text{if } \sup\limits_{t\in[0,T]} |X(t)| < R. \end{cases} \tag{7.33}$$

We claim that

$$\lim_{R\to\infty} \mathbb{P}(\tau_R = T) = 1. \tag{7.34}$$

In fact,

$$\{\tau_R = T\} \supset \{|X(t)| < R,\ \forall\, t \in [0, T]\} = \left\{ \sup_{t\in[0,T]} |X(t)| < R \right\}.$$

Now recalling (6.21) we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |X(t)| &\le |x| + \int_0^T \mathbb{E}|b(s)|ds + 2\left(\int_0^t \mathbb{E}|\sigma(s)|^2 ds\right)^{1/2} \\ &\le |x| + C_T.\end{aligned} \tag{7.35}$$

Therefore

$$\mathbb{P}(\tau_R = T) \ge \mathbb{P}\left\{ \sup_{t\in[0,T]} |X(t)| < R \right\} = 1 - \mathbb{P}\left\{ \sup_{t\in[0,T]} |X(t)| \ge R \right\}$$

$$\ge 1 - \frac{1}{R}\, \mathbb{E} \sup_{t\in[0,T]} |X(t)| \ge 1 - \frac{|x| + C_T}{R}$$

and (7.34) follows. As a consequence we have

$$\lim_{R\to\infty} \tau_R = T \quad \text{in probability.} \tag{7.36}$$

Now, for any $|x| \le R$ we have

$$\begin{aligned}\varphi(X(\tau_R)) = \varphi(x) &+ \int_0^{\tau_R} \varphi'(X(s))b(s)ds \\ &+ \int_0^{\tau_R} \varphi'(X(s))\sigma(s)dB(s) \\ &+ \frac{1}{2}\int_0^{\tau_R} \varphi''(X(s))\sigma^2(s)ds.\end{aligned} \tag{7.37}$$

Finally the conclusion of the theorem follows letting $R \to \infty$ and taking into account (7.36). □

7.2.1. Taking expectation in Itô's formula

When $\varphi \in C_b^2(\mathbb{R})$ taking expectation on both sides of Itô's formula, yields the useful identity (7.28). Assuming now that $\varphi \in C^2(\mathbb{R})$, we show that identity (7.28) can still be proved under suitable additional assumptions.

Proposition 7.10. *Let $T > 0$ and assume that*

$$\mathbb{E} \int_0^T |\varphi'(X(s))|\,|b(s)|ds < \infty, \quad \mathbb{E} \int_0^T |\varphi''(X(s))|\,|\sigma^2(s)|ds < \infty. \tag{7.38}$$

Then $\varphi(X(T)) \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and we have

$$\begin{aligned}\mathbb{E}[\varphi(X(T))] = \varphi(x) + \mathbb{E} \int_0^T \varphi'(X(s))b(s)ds \\ + \frac{1}{2}\,\mathbb{E} \int_0^T \varphi''(X(s))\sigma^2(s)ds.\end{aligned} \tag{7.39}$$

Proof. We argue as in the proof of Theorem 7.8 and arrive to (7.37), which we writes as

$$\begin{aligned}\varphi(X(\tau_R)) = \varphi(x) + \int_0^T \mathbb{1}_{[0,\tau_R]}(s)\varphi'(X(s))b(s)ds \\ + \int_0^T \mathbb{1}_{[0,\tau_R]}(s)\varphi'(X(s))\sigma(s)dB(s) \\ + \frac{1}{2}\int_0^T \mathbb{1}_{[0,\tau_R]}(s)\varphi''(X(s))\sigma^2(s)ds.\end{aligned}$$

Taking expectation in both sides, the stochastic integral disappears, and we obtain

$$\begin{aligned}\mathbb{E}[\varphi(X(\tau_R))] = \varphi(x) + \mathbb{E}\int_0^T \mathbb{1}_{[0,\tau_R]}(s)\varphi'(X(s))b(s)ds \\ + \frac{1}{2}\,\mathbb{E}\int_0^T \mathbb{1}_{[0,\tau_R]}(s)\varphi''(X(s))\sigma^2(s)ds.\end{aligned}$$

Finally, by assumption (7.38) and by the dominated convergence theorem we see there exists the limit as $R \to \infty$ in both sides of the identity above and so, we arrive at the conclusion. $\square$

Remark 7.11. Notice that in Proposition 7.10 we are not assuming that

$$\int_0^T \mathbb{E}|\varphi'(X(s))\sigma(s)|^2 ds < \infty.$$

7.2.2. Examples and applications

We take here an Itô's process of the form

$$X(t) = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s), \quad t \in [0, T], \tag{7.40}$$

where $b,\ \sigma \in L^2_B(0, T; L^2(\Omega))$ and $x \in \mathbb{R}$.

Example 7.12. Let $\varphi(x) = x^2$. Then Itô's formula reads as follows

$$\begin{aligned} |X(t)|^2 = |x|^2 + 2\int_0^t X(s)b(s)ds + 2\int_0^t X(s)\sigma(s)dB(s) \\ + \int_0^t \sigma^2(s)ds, \quad t \in [0, T]. \end{aligned} \tag{7.41}$$

Notice that $X\sigma$ does not necessarily belong to $L^2_B(0, T; L^2(\Omega))$ so that we cannot take expectation in (7.41). However, since

$$\begin{cases} \mathbb{E}\displaystyle\int_0^T |X(s)|\ |b(s)|ds \le \left(\mathbb{E}\int_0^T |X(s)|^2 ds\right)^{1/2} \left(\mathbb{E}\int_0^T |b(s)|^2 ds\right)^{1/2} < \infty, \\ \mathbb{E}\displaystyle\int_0^T |\sigma(s)|^2 ds < \infty, \end{cases}$$

we can apply Proposition 7.10 and we obtain

$$\mathbb{E}[|X(t)|^2] = |x|^2 + 2\mathbb{E}\int_0^t X(s)b(s)ds + \mathbb{E}\int_0^t \sigma^2(s)ds.$$

Example 7.13. Let $\varphi(x) = x^4$, $X(t) = \int_0^t \sigma(s)dB(s)$ and let first assume that σ is bounded. Then Itô's formula reads as follows

$$X^4(t) = 4\int_0^t X^3(s)\sigma(s)dB(s) + 6\int_0^t X^2(s)\sigma^2(s)ds, \quad t \in [0, T]. \tag{7.42}$$

Again, we can apply Proposition 7.10 because

$$\mathbb{E}\int_0^t X^2(s)\sigma^2(s)ds \le \|\sigma\|_0^2\ \mathbb{E}\int_0^t |X^2(s)|ds < \infty$$

and we obtain

$$\mathbb{E}[|X(t)|^4] = 6\mathbb{E}\left[\int_0^t |X(s)|^2\ |\sigma(s)|^2 ds\right].$$

It follows that

$$\mathbb{E}[|X(t)|^4] \le 3\mathbb{E}\int_0^t |X(s)|^4 ds + 3\mathbb{E}\int_0^t |\sigma(s)|^4 ds.$$

By the Gronwall Lemma we find

$$\mathbb{E}[|X(t)|^4] \le 3\mathbb{E}\int_0^t e^{3(t-s)}|\sigma(s)|^4 ds \le 3e^{3T}\mathbb{E}\int_0^t |\sigma(s)|^4 ds. \qquad (7.43)$$

Finally, we can extend by density the validity of (7.43) to all $\sigma \in L^4_B(0,T,L^{2m}(\Omega))$. In fact, choose a sequence $(\sigma_n) \subset L^\infty_B(0,T;L^\infty(\Omega))$ such that

$$\lim_{n\to\infty} \sigma_n = \sigma \quad \text{in } L^4_B(0,T;L^\infty(\Omega))$$

and set $X_n(t) = \int_0^t \sigma_n(s)dB(s)$. Then by (7.43) it follows that for any $m, n \in \mathbb{N}$,

$$\mathbb{E}[|X_n(t - X_m(t))|^4] \le 3e^{3T}\mathbb{E}\int_0^t |\sigma_n(s) - \sigma_m(s)|^4 ds.$$

So, (X_n) is Cauchy in $C_B([0,T];L^4(\Omega))$ and the conclusion follows.

The useful estimate (7.43) can be generalized for σ belonging to $L^{2m}_B(0,T,L^4(\Omega))$, $m \in \mathbb{N}$, as the following proposition shows.

Proposition 7.14. *Assume that* $\sigma \in L^{2m}_B([0,T];L^{2m}(\Omega))$, $m \in \mathbb{N}$, *and set*

$$X(t) = \int_0^t \sigma(s)dB(s), \quad t \in [0,T].$$

Then $X \in C_B([0,T];L^{2m}(\Omega))$ *and there exists a constant* $C_{m,T}$, *independent of* σ, *such that*

$$\mathbb{E}[|X(t)|^{2m}] \le C_{m,T}\int_0^t \mathbb{E}\left[|\sigma(s)|^{2m}\right] ds. \qquad (7.44)$$

Proof. We already know that the result holds for $m = 1, 2$. Let now $m = 3$. It is enough to show that

$$\mathbb{E}[|X(t)|^6] \le C_{3,T}\int_0^t \mathbb{E}\left[|\sigma(s)|^6\right] ds, \qquad (7.45)$$

when σ is bounded. By Itô's formula with $\varphi(x) = x^6$ we have

$$|X(t)|^6 = 6\int_0^t X^5(s)\sigma(s)dB(s) + 15\int_0^t X^4(s)\sigma^2(s)ds, \quad t \in [0,T]. \tag{7.46}$$

Again, we can apply Proposition 7.10 because

$$\mathbb{E}\int_0^t X^4(s)\sigma^2(s)ds \le \|\sigma\|_0^2\, \mathbb{E}\int_0^t |X^4(s)|ds < \infty$$

by (7.43). Thus we have

$$\mathbb{E}|X(t)|^6 = 15\int_0^t X^4(s)\sigma^2(s)ds, \quad t \in [0,T]. \tag{7.47}$$

Now we apply the Young inequality

$$ab \le \frac{1}{p}\, a^p + \frac{1}{q}\, a^q, \quad a > 0,\ b > 0,\ \frac{1}{p} + \frac{1}{q} = 1$$

with $p = 3/2,\ q = 3,\ a = X^4(s),\ b = \sigma^2(s)$ and obtain by (7.47)

$$\mathbb{E}|X(t)|^6 \le 10\int_0^t \mathbb{E}|X^6(s)|ds + 5\int_0^t \mathbb{E}|\sigma^6(s)|ds, \quad t \in [0,T]. \tag{7.48}$$

Then (7.45) follows from Gronwall's lemma. For the general case we proceed by iteration. □

As a last application we prove now an inequality due to Burkhölder.

Theorem 7.15. *Let* $\sigma \in L^2_B(0,T;L^2(\Omega))$ *and set* $X(t) = \int_0^t \sigma(s)dB(s)$. *Then there exists* $c_m > 0$ *such that*

$$\mathbb{E}\left(\sup_{s\in[0,T]} |\sigma(s)|^{2m}\right) \le c_m \mathbb{E}\left[\left(\int_0^T |\sigma(s)|^2\, ds\right)^m\right]. \tag{7.49}$$

Proof. By Itô's formula applied to $f(x) = |x|^{2m},\ x \in \mathbb{R}$, we obtain (again by Proposition 7.10)

$$\mathbb{E}|X(t)|^{2m} = m(2m-1)\mathbb{E}\left(\int_0^t |X(s)|^{2m-2}\, |\sigma(s)|^2 ds\right)$$

$$\le m(2m-1)\mathbb{E}\left(\sup_{s\in[0,t]} |X(s)|^{2m-2} \int_0^t |\sigma(s)|^2 ds\right).$$

By Hölder's inequality it follows that

$$\mathbb{E}|X(t)|^{2m} \leq m(2m-1)\left[\mathbb{E}\left(\sup_{s\in[0,t]}|X(s)|^{2m}\right)\right]^{\frac{m-1}{m}}\left[\mathbb{E}\left[\left(\int_0^t|\sigma(s)|^2ds\right)^m\right]\right]^{\frac{1}{m}}.$$

Using (C.5), yields

$$\mathbb{E}\left(\sup_{s\in[0,T]}|X(s)|^{2m}\right) \leq c_m\mathbb{E}|X(T)|^{2m}$$

$$\leq \left[\mathbb{E}\left(\sup_{s\in[0,T]}|X(s)|^{2m}\right)\right]^{\frac{m-1}{m}}\left[\mathbb{E}\left[\left(\int_0^T|F(s)|^2ds\right)^m\right]\right]^{\frac{1}{m}}.$$

So

$$\mathbb{E}\left(\sup_{s\in[0,t]}|X(s)|^{2m}\right) \leq c_m^2\mathbb{E}\left[\left(\int_0^t|F(s)|^2ds\right)^m\right]. \qquad \square$$

7.3. Itô's formula for multidimensional processes

We are given an r-dimensional Brownian motion $B = (B_1, ..., B_r)$ on a probability space $(\Omega,\mathscr{F},\mathbb{P})$ whose natural filtration we denote by $(\mathscr{F}_t)_{t\geq 0}$, and a stochastic process X on $[0, T]$ of the form

$$X(t) := x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s), \tag{7.50}$$

where $x \in \mathbb{R}^d$, $b \in L^2_B(0, T; L^2(\Omega; L(\mathbb{R}^d)))$ and $\sigma \in L^2_B(0, T; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d)))$. By Theorem 6.18 we know that X has a continuous version. We call X an $\mathbb{R}^d$-valued *Itô's process*.

Before stating and proving Itô's formula, let us give some notations.

Spaces $C_b(\mathbb{R}^d)$ and $C_b^k(\mathbb{R}^d)$ were defined in Section 3.1. It is well known that $C_b^k(\mathbb{R}^d)$ is dense in $C_b^{k-1}(\mathbb{R}^d)$ for all $k \in \mathbb{N}$.

Our aim now is to prove the Itô's formula below. The proof is very similar to that of Theorem 7.3 except some straightforward algebraic technicalities. However, due to the importance of this result, we shall present a complete proof.

Theorem 7.16. *Let* $\varphi \in C_b^2(\mathbb{R}^d)$ *and let* X *given by* (7.50). *Then we have*

$$
\begin{aligned}
\varphi(X(t)) = \varphi(x) &+ \int_0^t \langle D_x\varphi(X(s)), b(s)\rangle ds \\
&+ \int_0^t \langle D_x\varphi(X(s)), \sigma(s)dB(s)\rangle \qquad (7.51)\\
&+\frac{1}{2}\int_0^t \mathrm{Tr}\,[D_x^2\varphi(X(s))\sigma(s)\sigma^*(s)]ds, \quad \mathbb{P}\text{-a.s.}.
\end{aligned}
$$

We shall often write equations (7.50) and (7.51) in the following (equivalent) differential forms

$$dX(t) = b(t)dt + \sigma(t)dB(t), \tag{7.52}$$

and

$$
\begin{aligned}
d\varphi(X(t)) = \langle\varphi(X(t)), b(t)\rangle dt &+ \langle D\varphi(X(t)), \sigma(t)dB(t)\rangle \\
&+\frac{1}{2}\mathrm{Tr}\,[D^2\varphi''(X(t))\sigma(t)\sigma^*(t)]dt,
\end{aligned} \tag{7.53}
$$

respectively.

As we did in the one dimensional case, we shall first prove Theorem 7.16 when b and σ are constant.

7.3.1. The case when b and σ are constant

We need a lemma, which generalizes Proposition 7.1.

Lemma 7.17. *Let* $F \in C_B([0,T]; L^2(\Omega))$, $\varsigma = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ *a decomposition of* $[0,T]$ *and* $i, j \in \{1, ..., r\}$. *Then we have* [(2)]

$$
\begin{aligned}
&\lim_{|\varsigma|\to 0}\sum_{k=1}^n F(t_{k-1})(B_i(t_k) - B_i(t_{k-1}))(B_j(t_k) - B_j(t_{k-1})) \\
&= \delta_{i,j}\int_0^T F(s)ds \quad in \;\; L^2(\Omega, \mathscr{F}, \mathbb{P}).
\end{aligned} \tag{7.54}
$$

(2) $\delta_{i,j}$ is the Kronecker symbol.

Proof. If $i = j$ the conclusion follows from Proposition 7.1. Let $i \neq j$ and set

$$I_{i,j}^{\varsigma} = \sum_{k=1}^{n} F(t_{k-1})(B_i(t_k) - B_i(t_{k-1}))(B_j(t_k) - B_j(t_{k-1})).$$

Then we have

$$\begin{aligned}\mathbb{E}[(I_{i,j}^{\varsigma})^2] &= \mathbb{E} \sum_{h,k=1}^{n} F(t_{h-1})F(t_{k-1})(B_i(t_h) - B_i(t_{h-1}))(B_j(t_h) - B_j(t_{h-1})) \\ &\quad \times (B_i(t_k) - B_i(t_{k-1}))(B_j(t_k) - B_j(t_{k-1})) \\ &= \mathbb{E} \sum_{h=1}^{n} F^2(t_{h-1})(B_i(t_h) - B_i(t_{h-1}))^2(B_j(t_h) - B_j(t_{h-1}))^2 \\ &= \sum_{h=1}^{n} \mathbb{E}(F^2(t_{h-1}))(t_h - t_{h-1})^2 \to 0,\end{aligned}$$

as $|\sigma| \to 0$. □

We now assume that $b = b_0 \in \mathbb{R}^d$ and $\sigma = \sigma_0 \in L(\mathbb{R}^r; \mathbb{R}^d)$. Then Theorem 7.16 reduces to.

Lemma 7.18. *Let* $\varphi \in C_b^2(\mathbb{R}^d)$ *and let* $X(t) = x + b_0 t + \sigma_0 B(t),\ t \in [0, T]$. *Then we have*

$$\begin{aligned}\varphi(X(t)) = \varphi(x) &+ \int_0^t \langle D\varphi(X(s)), b_0\rangle\, ds \\ &+ \int_0^t \langle D\varphi(X(s)), \sigma_0 dB(s)\rangle \qquad (7.55) \\ &+ \int_0^t \mathrm{Tr}\,[D^2\varphi(X(s))\sigma_0\sigma_0^*]ds.\end{aligned}$$

Proof. It is enough to prove (7.55) when $\varphi \in C_b^3(\mathbb{R}^d)$ because $C_b^3(\mathbb{R}^d)$ is dense in $C_b^2(\mathbb{R}^d)$.

Let $\varsigma = \{0 = s_0 < s_1 < \cdots < s_n = T\}$ be a decomposition of $[0, T]$. Write

$$\varphi(X(T)) - \varphi(x) = \sum_{h=1}^{n} [\varphi(X(s_h)) - \varphi(X(s_{h-1}))]. \qquad (7.56)$$

By Taylor's formula with the integral remainder we have

$$\begin{aligned}\varphi(X(s_h)) - \varphi(X(s_{h-1})) &= \langle D\varphi(X(s_{h-1})), X(s_h) - X(s_{h-1})\rangle \\ &+\frac{1}{2}\,\langle D^2\varphi(X(s_{h-1}))(X(s_h)-X(s_{h-1})), (X(s_h)-X(s_{h-1}))\rangle + R_h,\end{aligned} \tag{7.57}$$

where

$$\begin{aligned}R_h = \int_0^1 (1-\xi)\langle (D^2\varphi(Y_h(\xi)) - D^2\varphi(X(s_{h-1})))\\ \times(X(s_h) - X(s_{h-1})), (X(s_h) - X(s_{h-1}))\rangle\, d\xi\end{aligned} \tag{7.58}$$

and

$$Y_h(\xi) = (1-\xi)X(s_{h-1}) + \xi X(s_h), \quad \xi \in [0,1]. \tag{7.59}$$

Now we write (7.56) as

$$\begin{aligned}\varphi(X(T)) - \varphi(x) &= \sum_{h=1}^{n} \langle D\varphi(X(s_{h-1})), X(s_h)) - X(s_{h-1})\rangle \\ &+\frac{1}{2}\sum_{h=1}^{n}\langle D^2\varphi(X(s_{h-1}))(X(s_h) - X(s_{h-1})), (X(s_h) - X(s_{h-1}))\rangle \\ &+\sum_{h=1}^{n} R_h =: A_\varsigma + B_\varsigma + C_\varsigma.\end{aligned} \tag{7.60}$$

Taking into account that

$$X(s_h) - X(s_{h-1}) = b_0(s_h - s_{h-1}) + \sigma_0(B(s_h) - B(s_{h-1})),$$

we have

$$\begin{aligned}A_\varsigma &= \sum_{h=1}^{n}\langle D\varphi(X(s_{h-1})), b_0\rangle(s_h - s_{h-1}) \\ &+\sum_{h=1}^{n}\langle D\varphi(X(s_{h-1})), \sigma_0(B(s_h) - B(s_{h-1})\rangle,\end{aligned} \tag{7.61}$$

so that

$$\begin{aligned}\lim_{|\varsigma|\to 0} A_\varsigma &= \int_0^T \langle D\varphi(X(s)), b_0\rangle ds \\ &+ \int_0^T \langle \varphi(X(s)), \sigma_0 dB(s)\rangle \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}).\end{aligned} \tag{7.62}$$

Moreover

$$
\begin{aligned}
B_\varsigma &= \frac{1}{2}\sum_{h=1}^{n}\langle D^2\varphi(X(s_{h-1}))\sigma_0(B(s_h)-B(s_{h-1})),\sigma_0(B(s_h)-B(s_{h-1}))\rangle\\
&\quad+\sum_{h=1}^{n}\langle D^2\varphi(X(s_{h-1}))b_0,\sigma_0(B(s_h)-B(s_{h-1}))\rangle(s_h-s_{h-1})\\
&\quad+\frac{1}{2}\sum_{h=1}^{n}\langle D^2\varphi(X(s_{h-1}))b_0,b_0\rangle(s_h-s_{h-1})^2\\
&=: B_{\varsigma,1}+B_{\varsigma,2}+B_{\varsigma,3}.
\end{aligned}
\tag{7.63}
$$

We claim that

$$
\lim_{|\varsigma|\to 0} B_{\varsigma,1} = \frac{1}{2}\int_0^T \mathrm{Tr}\,[D^2\varphi(B(s))\sigma_0\sigma_0^*]ds \quad \text{in } L^2(\Omega,\mathscr{F},\mathbb{P}). \tag{7.64}
$$

We have in fact

$$
\begin{aligned}
B_{\varsigma,1} &= \frac{1}{2}\sum_{h=1}^{n}\langle D^2\varphi(X(s_{h-1}))\sigma_0(B(s_h)-B(s_{h-1})),\sigma_0(B(s_h)-B(s_{h-1}))\rangle\\
&=\frac{1}{2}\sum_{h=1}^{n}\sum_{\alpha,\beta=1}^{d}D_\alpha D_\beta\varphi(X(s_{h-1}))\\
&\quad\times(\sigma_0(B(s_h)-B(s_{h-1})))_\alpha(\sigma_0(B(s_h)-B(s_{h-1})))_\beta\\
&=\frac{1}{2}\sum_{h=1}^{n}\sum_{\alpha,\beta,\gamma,\delta}D_\alpha D_\beta\varphi(X(s_{h-1}))(\sigma_0)_{\alpha,\gamma}(B_\gamma(s_h)-B_\gamma(s_{h-1}))\\
&\quad\times(\sigma_0)_{\beta,\delta}(B_\delta(s_h)-B_\delta(s_{h-1}))
\end{aligned}
\tag{7.65}
$$

Taking into account Lemma 7.18 it follows that

$$
\begin{aligned}
&\lim_{|\varsigma|\to 0} B_{\varsigma,1}\\
&=\frac{1}{2}\lim_{|\varsigma|\to 0}\sum_{h=1}^{n}\sum_{\alpha,\beta,\gamma}D_\alpha D_\beta\varphi(X(s_{h-1}))(\sigma_0)_{\alpha,\gamma}(B_\gamma(s_h)-B_\gamma(s_{h-1}))^2(\sigma_0)_{\beta,\gamma}\\
&=\frac{1}{2}\int_0^T\sum_{\alpha,\beta,\gamma}D_\alpha D_\beta\varphi(X(s)(\sigma_0)_{\alpha,\gamma}(\sigma_0)_{\beta,\gamma}ds\\
&=\frac{1}{2}\int_0^T \mathrm{Tr}\,[D^2\varphi(B(s))\sigma_0\sigma_0^*]ds.
\end{aligned}
\tag{7.66}
$$

So, the claim is proved.

We now prove that

$$\lim_{|\varsigma|\to 0} B_{\varsigma,2} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.67}$$

We have in fact

$$B_{\varsigma,2} = \sum_{h=1}^{n} \langle D^2\varphi(X(s_{h-1}))b_0, \sigma_0(B(s_h) - B(s_{h-1}))\rangle (s_h - s_{h-1}),$$

and so

$$\begin{aligned}
\mathbb{E}|B_{\varsigma,2}| &\le |b_0|\, \|\varphi\|_2 \sum_{h=1}^{n}(s_h - s_{h-1}) \sum_{h=1}^{n} \mathbb{E}|\sigma_0(B(s_h) - B(s_{h-1}))| \\
&\le |b_0| \left(\mathrm{Tr}\,[\sigma_0\sigma_0^*]\right)^{1/2} \|\varphi\|_2 \sum_{h=1}^{n}(s_h - s_{h-1})^{3/2} \\
&\le |b_0| \left(\mathrm{Tr}\,[\sigma_0\sigma_0^*]\right)^{1/2} \|\varphi\|_2 |\varsigma|^{1/2} T
\end{aligned}$$

and (7.67) follows.

The proof that

$$\lim_{|\varsigma|\to 0} B_{\varsigma,3} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}), \tag{7.68}$$

follows at once. We finally show that

$$\lim_{|\varsigma|\to 0} C_{\varsigma} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}), \tag{7.69}$$

which, together with (7.64), (7.67) and (7.68), will conclude the proof.

Write

$$\begin{aligned}
C_\varsigma = {} & \frac{1}{2}\sum_{h=1}^{n} \int_0^1 (1-\xi)\langle (D^2\varphi(Y_h(\xi)) - D^2\varphi(X(s_{h-1}))\sigma_0(B(s_h) - B(s_{h-1})), \\
& \qquad \sigma_0(B(s_h) - B(s_{h-1}))\rangle d\xi \\
& + \sum_{h=1}^{n} \int_0^1 (1-\xi)\langle (\varphi''(Y_h(\xi)) - \varphi''(X(s_{h-1}))b_0, \sigma_0(B(s_h) - B(s_{h-1}))\rangle \\
& \times (s_h - s_{h-1}) d\xi \\
& + \frac{1}{2}\sum_{h=1}^{n} \int_0^1 (1-\xi)\langle (\varphi''(Y_h(\xi)) - \varphi''(X(s_{h-1}))b_0, b_0\rangle (s_h - s_{h-1})^2 d\xi \\
& =: C_{\varsigma,1} + C_{\varsigma,2} + C_{\varsigma,3}.
\end{aligned}$$

Consequently

$$
\begin{aligned}
|C_{\varsigma,1}| &\leq \frac{1}{2}\,\|\varphi\|_3 \sum_{h=1}^{n} \int_0^1 (1-\xi)|Y_h(\xi)) - X(s_{h-1})|\;|\sigma_0(B(s_h) - B(s_{h-1}))|^2 d\xi \\
&= \frac{1}{2}\,\|\varphi\|_3 \sum_{h=1}^{n} \int_0^1 (1-\xi)^2 |X(s_h) - X(s_{h-1})|\;|\sigma_0(B(s_h) - B(s_{h-1}))|^2 d\xi \\
&\leq \frac{|b_0|}{2}\,\|\varphi\|_3 \sum_{h=1}^{n} (s_h - s_{h-1})\;|\sigma_0(B(s_h) - B(s_{h-1}))|^2 \\
&\quad + \frac{1}{2}\,\|\varphi\|_3 \sum_{h=1}^{n} |\sigma_0(B(s_h) - B(s_{h-1}))|^3 \\
&=: C_{\varsigma,1,1} + C_{\varsigma,1,2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}|C_{\varsigma,1,1}| &\leq \frac{1}{2}\;|b_0|\;\mathrm{Tr}\,[\sigma_0\sigma_0^*]\;\|\varphi\|_3 \sum_{h=1}^{n} (s_h - s_{h-1})^2 \\
&\leq \frac{1}{2}\;|b_0|\;\mathrm{Tr}\,[\sigma_0\sigma_0^*]\;\|\varphi\|_3 \sum_{h=1}^{n} (s_h - s_{h-1})^2 \leq \frac{1}{2}\;b_0\sigma_0^2\;\|\varphi\|_3|\varsigma|T.
\end{aligned}
$$

and obviously

$$
\lim_{|\varsigma|\to 0} C_{\varsigma,1,1} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.70}
$$

Moreover

$$
\mathbb{E}|C_{\varsigma,1,2}| \leq \frac{1}{2}\;\left(\mathrm{Tr}\,[\sigma_0\sigma_0^*]\right)^{3/2}\;\|\varphi\|_3 \sum_{h=1}^{n} (s_h) - s_{h-1})^{3/2}.
$$

Therefore

$$
\lim_{|\varsigma|\to 0} C_{\varsigma,1,2} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.71}
$$

As regards $C_{\varsigma,2}$ we have

$$
\mathbb{E}|C_{\varsigma,2}| \leq 2|b_0|\;|\sigma_0|\;\|\varphi\|_2 \sum_{h=1}^{n} (s_h - s_{h-1})\mathbb{E}|B(s_h) - B(s_{h-1})|
$$

$$
\leq |b_0|\;|\sigma_0|\;\|\varphi\|_2 \sum_{h=1}^{n} (s_h - s_{h-1})^{3/2} \leq |b_0|\;|\sigma_0|\;\|\varphi\|_2\;|\varsigma|\;T.
$$

Therefore

$$
\lim_{|\varsigma|\to 0} C_{\varsigma,2} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.72}
$$

Finally, as regards $C_{\varsigma,3}$ we have

$$|C_{\varsigma,3}| \leq \text{Tr}\,[\sigma_0\sigma_0^*]\|\varphi\|_2 \sum_{h=1}^{n}(s_h - s_{h-1})^2 \leq \sigma_0^2\|\varphi\|_2\, \varsigma\, T.$$

Therefore

$$\lim_{|\varsigma|\to 0} C_{\varsigma,3} = 0 \quad \text{in } L^1(\Omega, \mathscr{F}, \mathbb{P}). \tag{7.73}$$

So (7.69) holds and the proof is complete. □

7.3.2. Proof of Theorem 7.16

Proof. Step 1. Assume that $b \in \mathscr{E}^1_B(0, T; L(\mathbb{R}^d))$ and $\sigma \in \mathscr{E}^2_B(0, T; L(\mathbb{R}^r; \mathbb{R}^d))$.

Then the proof is a straightforward consequence of Lemma 7.18.

Step 2. Conclusion.

Let $(b_n) \subset \mathscr{E}^1_B(0, T; L(\mathbb{R}^r; \mathbb{R}^d))$ and $(\sigma_n) \subset \mathscr{E}^2_B(0, T; L(\mathbb{R}^r; \mathbb{R}^d))$ such that

$$\begin{cases} b_n \to b & \text{in } L^1_b(0, T; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d))) \\[2ex] \sigma_n \to \sigma & \text{in } L^2_b(0, T; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d))) \end{cases}$$

and set

$$X_n(t) := x + \int_0^t b_n(s)ds + \int_0^t \sigma_n(s)dB(s). \tag{7.74}$$

Then by Step 1 we have

$$\begin{aligned} \varphi(X_n(t)) &= \varphi(x) + \int_0^t \langle D\varphi(X_n(s)), b_n(s)\rangle ds \\ &\quad + \int_0^t \langle D\varphi(X_n(s)), \sigma_n(s)dB(s)\rangle \\ &\quad + \frac{1}{2}\int_0^t \text{Tr}\,[D^2\varphi(X_n(s))\sigma_n(s)\sigma_n^*(s)]ds. \end{aligned} \tag{7.75}$$

Now the conclusion follows letting $n \to \infty$. □

Taking expectation on both sides of (7.51), yields the following important formula.

Proposition 7.19.

$$\begin{aligned}\mathbb{E}[\varphi(X(t))] = \varphi(x) + \mathbb{E}\int_0^t \langle D\varphi(X(s)), b(s)\rangle ds \\ +\frac{1}{2}\,\mathbb{E}\int_0^t \mathrm{Tr}\,[D^2\varphi(X_n(s))\sigma_n(s)\sigma_n^*(s)]ds.\end{aligned} \tag{7.76}$$

Example 7.20. Let

$$dX(t) = b(t)dt + \sum_{i=1}^r \sigma_i(t)dB_i(t),$$

where $b, \sigma_1, ..., \sigma_r \in L^2_B([0,T]; L^2(\Omega;\mathbb{R}^d))$, and $B = (B_1, ..., B_r)$, is an r-valued Brownian motion. Then given $\varphi \in C^2_b(\mathbb{R}^d)$, the Itô formula reads as follows

$$\begin{aligned}d\varphi(X(t)) = \langle D\varphi(X(t)), b(t)\rangle dt \\ +\sum_{i=1}^r \langle D\varphi(X(t)), \sigma_i(t)\rangle dB_i(t) \\ +\frac{1}{2}\sum_{i=1}^r \langle D^2\varphi(X(t))\sigma_i(t), \sigma_i(t)\rangle dt.\end{aligned} \tag{7.77}$$

In fact let $\sigma(t) \in L(\mathbb{R}^r;\mathbb{R}^d)$ be defined as

$$\sigma(t)h = \sum_{i=1}^r \sigma_i(t)\langle h, e_i\rangle \quad h \in \mathbb{R}^d,$$

where $(e_1, ..., e_r)$ is an orthonormal basis in $\mathbb{R}^r$. Then

$$\langle \sigma(t)h, y\rangle = \left\langle h, \sum_{i=1}^r \langle \sigma_i(t), y\rangle e_i\right\rangle, \quad \forall\, y \in \mathbb{R}^d$$

and $\sigma^*(t) \in L(\mathbb{R}^d;\mathbb{R}^r)$ is given by

$$\sigma^*(t)y = \sum_{i=1}^r \langle \sigma_i(t), y\rangle e_i, \quad \forall\, y \in \mathbb{R}^d$$

so that

$$\begin{aligned}\sigma(t)\sigma^*(t)y &= \sum_{i=1}^r \langle \sigma_i(t), y\rangle \sigma(t)e_i = \sum_{i,j=1}^r \langle \sigma_i(t), y\rangle \sigma_j(t)\langle e_i, e_j\rangle \\ &= \sum_{i=1}^r \langle \sigma_i(t), y\rangle \sigma_i(t).\end{aligned}$$

It follows that

$$\text{Tr}\,[D^2\varphi\sigma(t)\sigma^*(t)] = \sum_{i=1}^{d}\langle\varphi_{x,x}\langle\sigma_i(t),\sigma_i(t)$$

and (7.77) follows.

Example 7.21. Let

$$dX_i(t) = b_i(t)dt + f_i(t)dB(t), \quad i = 1, 2,$$

where $b_i,\ f_i, \in L^2_B(0,T); L^2(\Omega))$ and and $B(t),\ t \geq 0$, is a real Brownian motion. Then we have

$$d(X_1(t)X_2(t)) = dX_1(t)\,X_2(t) + X_1(t)\,dX_2(t) + f_1(t)f_2(t)dt. \quad (7.78)$$

Set in fact

$$dX(t) = b(t)dt + f(t)dB(t)$$

where $X(t) = (X_1(t), X_2(t))$, $f(t) = (f_1(t), f_2(t))$ and

$$\varphi(x) = x_1x_2, \quad x = (x_1, x_2) \in \mathbb{R}^2.$$

Then

$$D\varphi(x) = (x_2, x_1)$$

and

$$D^2\varphi(x) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So

$$\varphi_{xx}(X(t))(f(t), f(t)) = 2f_1(t)f_2(t)$$

and the conclusion follows from (7.77).

We conclude this section with some generalization. First we consider the case when φ is unbounded. Then the following result can be proved as Theorem 7.8.

Theorem 7.22. *Let X be given by* (7.50). *Then for any $\varphi \in C^2(\mathbb{R}^d)$ we have*

$$\begin{aligned}\varphi(X(t)) \;=&\; \varphi(x) + \int_0^t \langle D\varphi(X(s)), b(s)\rangle ds \\ &+ \int_0^t \langle D\varphi(X(s)), \sigma(s)\rangle dB(s) \\ &+ \frac{1}{2}\int_0^t \text{Tr}\,[D^2\varphi(X(s))\sigma(s)\sigma^*(s)]ds, \quad t \in [0,T],\ \mathbb{P}\text{-a.s.}\end{aligned} \quad (7.79)$$

We finally extends Itô's formula to functions $u(t, X(t))$, the proof is left to the reader.

Theorem 7.23. *Let* $u : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *be continuous and bounded together with its partial derivatives* u_t, u_x, u_{xx} *and let* X *be defined by* (7.50). *Then we have*

$$\begin{aligned} u(t,X(t)) = {} & u(0, x) + \int_0^t u_t(s, X(s))ds \\ & + \int_0^t \langle D_x u(s, X(s)), b(s)\rangle ds \\ & + \int_0^t \langle D_x u(s, X(s)), \sigma(s) dB(s)\rangle \\ & + \frac{1}{2}\int_0^t \mathrm{Tr}\,[D_x^2 u(s, X(s))\sigma(s)\sigma^*(s)]ds, \quad t \in [0, T]. \end{aligned} \tag{7.80}$$

Chapter 8
Stochastic differential equations

We are given two positive integers r, d and an r-dimensional Brownian motion $B(t),\ \ t \geq 0$, in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ whose natural filtration we denote by $(\mathscr{F}_t)_{t\geq 0}$. We are concerned with the following integral equation,

$$X(t) = \eta + \int_s^t b(r, X(r))dr + \int_s^t \sigma(r, X(r))dB(r), \tag{8.1}$$

where $T > 0$, $s \in [0, T)$, $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$, $b\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to L(\mathbb{R}^r, \mathbb{R}^d)$. b is called the *drift* and σ the *diffusion coefficient* of the equation.

We shall write (8.1) in the following differential form

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) \\[2ex] X(s) = \eta. \end{cases} \tag{8.2}$$

By a *solution* of equation (8.1) in the interval $[s, T]$ we mean a function $X \in C_B([s, T]; L^2(\Omega; \mathbb{R}^d))$ that fulfills identity (8.1).

In Section 8.1 we shall prove existence and uniqueness of a solution $X = X(t, s, x)$ of (8.1). Section 8.2 is devoted to continuous dependence of $X(t, s, x)$ from t, s, x, and Section 8.3 to x-differentiability of $X(t, s, x)$.

Finally, in Section 8.4 we show that $X(t, s, x)$ is also Itô's differentiable with respect to s (in a suitable sense). To prove this result we introduce a backward Itô's integral.

We shall use throughout the basic isometric identity

$$\mathbb{E}\left|\int_a^b G(t)dB(t)\right|^2 = \int_a^b \mathbb{E}\left[\mathrm{Tr}\ (G(t)G^*(t))\right] dt,$$

for all $G \in L^2_B([0,T];L^2(\Omega,L(\mathbb{R}^r,\mathbb{R}^d)))$ and $0 \le a < b \le T$. This suggests to endow $L(\mathbb{R}^r,\mathbb{R}^d)$ with the *Hilbert–Schmidt* norm, setting

$$\|S\|_{HS} :\ = \left[\mathrm{Tr}(SS^*)\right]^{1/2}, \quad \forall\, S \in L(\mathbb{R}^r,\mathbb{R}^d)$$

and to write

$$\mathbb{E}\left|\int_a^b G(t)dB(t)\right|^2 = \int_a^b \mathbb{E}\left(\|G(t)\|^2_{HS}\right)dt. \tag{8.3}$$

8.1. Existence and uniqueness

The standard assumptions implying well-posedness of problem (8.1) are the following

Hypothesis 8.1.

(i) b and σ are continuous on $[0,T]\times\mathbb{R}^d$.

(ii) There exists $M > 0$ such that for all $t \in [0,T]$ and all $x, y \in \mathbb{R}^d$, we have

$$|b(t,x)-b(t,y)|^2 + \|\sigma(t,x)-\sigma(t,y)\|^2_{HS} \le M^2|x-y|^2 \tag{8.4}$$

and

$$|b(t,x)|^2 + \|\sigma(t,x)\|^2_{HS} \le M^2(1+|x|^2). \tag{8.5}$$

Notice that (8.5) is a consequence of (8.4) (by possibly changing the constant M).

Theorem 8.2. *Assume that Hypothesis* 8.1 *holds and let* $s \in [0,T)$*,* $\eta \in L^2(\Omega,\mathscr{F}_s,\mathbb{P};\mathbb{R}^d)$. *Then problem* (8.1) *has a unique solution*

$$X \in C_B([s,T];L^2(\Omega;\mathbb{R}^d)).$$

Proof. We shall solve (8.1) by a fixed point argument in the space

$$C_B := C_B([s,T];L^2(\Omega;\mathbb{R}^d)).$$

first when T is sufficiently close to s and then in general.

Define

$$\gamma_1(X)(t) := \int_s^t b(r,X(r))dr, \quad X \in C_B,\ t \in [s,T],$$

$$\gamma_2(X)(t) := \int_s^t \sigma(r,X(r))dB(r), \quad X \in C_B,\ t \in [s,T]$$

and set

$$\gamma(X) := \eta + \gamma_1(X) + \gamma_2(X), \qquad X \in C_B.$$

$X \in C_B$ is a solution of equation (8.1) if and only if it is a fixed point X of γ,

$$X = \eta + \gamma_1(X) + \gamma_2(X) = \gamma(X). \tag{8.6}$$

We are going to show that γ maps C_B into itself, then that it is Lipschitz and finally that it is a contraction provided $T - s$ be sufficiently small. At the end of the proof we show how to get rid of this restriction.

Step 1. γ maps C_B into itself.

As regards γ_1 we have, using Hölder's inequality and taking into account (8.5),

$$\begin{aligned} |\gamma_1(X)(t)|^2 &\le (t-s)\int_s^t |b(r, X(r))|^2 dr \\ &\le M^2(t-s)\int_s^t (1+|X(r)|^2)dr \\ &\le M^2(t-s)^2(1+\|X\|_{C_B}^2). \end{aligned}$$

Since $\gamma_1(X)(t)$ is $\mathscr{F}_t$–measurable for all $t \in [s, T]$, γ_1 maps C_B into itself and

$$\|\gamma_1(X)\|_{C_B} \le M(T-s)(1+\|X\|_{C_B}).$$

As regards γ_2 we have, taking into account (8.5),

$$\begin{aligned} \mathbb{E}|\gamma_2(X)(t)|^2 &= \int_s^t \mathbb{E}(\|\sigma(r, X(r))\|_{HS}^2)dr \\ &\le M^2\int_s^t (1+|X(r)|^2)dr \le M^2(t-s)(1+\|X\|_{C_B}^2) \end{aligned}$$

So, Step 1 is proved.

Step 2. γ is Lipschitz continuous.

Let $X, Y \in C_B$. Using again the Hölder inequality and taking into account (8.4), yields

$$\begin{aligned} |\gamma_1(X)(t) - \gamma_1(Y)(t)|^2 &\le (t-s)\int_s^t |b(r, X(r)) - b(r, Y(r))|^2 dr \\ &\le (t-s)M^2\int_s^t |X(r) - Y(r)|^2 du \\ &\le (t-s)^2 M^2\|X-Y\|_{C_B}^2. \end{aligned}$$

Consequently

$$\|\gamma_1(X) - \gamma_1(Y)\|_{C_B} \le M\ (T-s)\ \|X-Y\|_{C_B}, \quad \forall\ X, Y \in C_B \tag{8.7}$$

Furthermore, taking into account (8.4),

$$\mathbb{E}|\gamma_2(X)(t)-\gamma_2(Y)(t)|^2 = \int_s^t \mathbb{E}\|\sigma(r,X(r))-\sigma(r,Y(r))\|_{HS}^2 dr$$
$$\leq M^2(t-s)\|X-Y\|_{C_B}^2,$$

and so,

$$\|\gamma_2(X)-\gamma_2(Y)\|_{C_B} \leq M\sqrt{T-s}\,\|X-Y\|_{C_B}, \quad \forall\, X,Y\in C_B. \quad (8.8)$$

By (8.7) and (8.8) it follows that

$$\|\gamma(X)-\gamma(Y)\|_{C_B} \leq M(T-s+\sqrt{T-s}\,)\|X-Y|\|_{C_B},$$

for all $X, Y \in C_B$. Now if s is such that

$$M\left(T-s+\sqrt{T-s}\,\right) \leq 1/2, \quad (8.9)$$

γ is a $1/2$–contraction on C_B, and so, it possesses a unique fixed point. If (8.9) does not hold we choose $T_1 \in (s, T]$ such that

$$M\left(T_1-s+\sqrt{T_1-s}\,\right) \leq 1/2.$$

Then by the previous argument there is a unique solution to (8.2) on $[s, T_1]$. Now we repeat the proof with T_1 replacing s and in a finite number of steps we arrive to show existence of a solution of (7.3). Uniqueness follows from a straightforward argument left to the reader. □

Remark 8.3. By Theorem 8.2 it follows that there exists a version of the solution $X(\cdot, s, \eta)$ which belongs to $L^2_B(\Omega, C([s,T])$ and so it is a continuous process. Consequently we can say that identity (7.3) holds $\mathbb{P}$-a.s.

Let us prove an important property of $X(\cdot, s, \eta)$.

Proposition 8.4. *Assume that Hypothesis* 8.1 *holds and let*

$$\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d).$$

Then

$$X(t,s,\eta) = X(t,r,X(r,s,\eta)), \quad 0 \leq s \leq r \leq t \leq T. \quad (8.10)$$

Proof. Define $Z(t) = X(t, s, \eta),\ t \in [s, T]$. Then Z is a solution of the problem

$$\begin{cases} dZ(t) = b(t, Z(t))dt + \sigma(t, Z(t))dB(t), \\ \\ Z(r) = X(r, s, \eta). \end{cases}$$

By the uniqueness part of Theorem 8.2 it follows that

$$Z(t) = X(t, s, \eta) = X(t, r, X(r, s, \eta)),$$

as required. □

In the following we shall denote by $X(\cdot, s, \eta)$ the solution of problem (8.2). We shall use greek letters for stochastic initial data and latin letters for deterministic ones.

Remark 8.5. The solution $X(t, s, \eta)$ of problem (8.2) can be obtained as a limit of successive approximations. More precisely, define $X_0(t, s, \eta) = \eta$ and for any $N \in \mathbb{N}$,

$$X_{N+1}(t, s, \eta) = \eta + \int_s^t b(r, X_N(r, s, \eta))dr + \int_s^t \sigma(r, X_N(r, s, \eta))dB(r). \tag{8.11}$$

Then by the contraction principle we have

$$\lim_{N\to\infty} X_N(\cdot, s, \eta) = X(\cdot, s, \eta) \quad \text{in } C_B([s, T]; L^2(\Omega; \mathbb{R}^d)), \tag{8.12}$$

provided $T - s$ is sufficiently small. One can get rid of this assumption by a standard argument.

Let us prove an useful relationship between $X(t, s, \eta)$, and $X(t, s, x)$, where $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Proposition 8.6. *Assume that Hypothesis* 8.1 *holds and that*

$$\eta = \sum_{k=1}^n x_k \mathbb{1}_{A_k}, \tag{8.13}$$

where $x_1, \ldots, x_n \in \mathbb{R}^d$, *and* $A_1, \ldots, A_n$ *are mutually disjoints sets in* $\mathscr{F}_s$ *such that*

$$\Omega = \bigcup_{k=1}^n A_k.$$

Then we have

$$X(t, s, \eta) = \sum_{k=1}^n X(t, s, x_k)\mathbb{1}_{A_k}. \tag{8.14}$$

Proof. Let X_N be defined by (8.11). We claim that

$$X_N(t,s,\eta) = \sum_{k=1}^{n} \mathbb{1}_{A_k} X_N(t,s,x_k), \quad \forall\, N \in \mathbb{N}. \tag{8.15}$$

Once (8.15) is proved, the conclusion follows letting N tend to infinity. Now we proceed by recurrence. Equality (8.15) is clear for $N = 0$. Assume that it holds for a given $N \in \mathbb{N}$,

$$X_N(t,s,\eta) = X_N(t,s,x_k) \quad \text{in} \;\; A_k,\;\; k = 1,\dots,n.$$

Then we have

$$b(r, X_N(r,s,\eta)) = b(r, X_N(r,s,x_k)) \quad \text{in} \;\; A_k,\;\; k = 1,\dots,n,$$

$$\sigma(r, X_N(r,s,\eta)) = \sigma(r, X_N(r,s,x_k)) \quad \text{in} \;\; A_k,\;\; k = 1,\dots,n,$$

so that

$$b(r, X_N(r,s,\eta)) = \sum_{k=1}^{n} \mathbb{1}_{A_k} b(r, X_N(r,s,x_k)),$$

$$\sigma(r, X_N(r,s,\eta)) = \sum_{k=1}^{n} \mathbb{1}_{A_k} \sigma(r, X_N(r,s,x_k)).$$

Consequently

$$\begin{aligned} X_{N+1}(t,s,\eta) &= \sum_{k=1}^{n} \mathbb{1}_{A_k} \Big(X_0(t,s,x_k) + \int_s^t b(r, X_N(r,s,x_k)du \\ &\qquad\qquad + \int_s^t \sigma(r, X_N(r,s,x_k))dB(r) \Big) \\ &= \sum_{k=1}^{n} \mathbb{1}_{A_k} X_{N+1}(t,s,x_k) \end{aligned}$$

and (8.15) holds for $N+1$. So, the conclusion follows. □

8.1.1. Existence and uniqueness in $L^{2m}(\Omega)$

One can study equation (8.2) on the space

$$C_B([s,t]; L^{2m}(\Omega;\mathbb{R}^d)), \quad m \in \mathbb{N}$$

and prove the result

Theorem 8.7. *Assume that Hypothesis* 8.1 *holds and let* $s \in [0, T)$, $m \in \mathbb{N}$, $\eta \in L^{2m}(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$. *Then problem* (8.1) *has a unique solution*

$$X \in C_B([s, T]; L^{2m}(\Omega; \mathbb{R}^d)).$$

Proof. One can repeat the proof of Theorem 8.2 but using Proposition 7.14. □

8.1.2. Examples

Example 8.8 (Linear equations perturbed by noise). Consider the stochastic differential equation

$$\begin{cases} dX = AXdt + CdB(t), \\ \\ X(0) = x, \end{cases} \tag{8.16}$$

where $A \in L(\mathbb{R}^d)$, $C \in L(\mathbb{R}^r; \mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Clearly Theorem 8.2 applies so that (8.16) has a unique solution X,

$$X(t) = x + A \int_0^t X(s)ds + CB(t), \quad t \geq 0. \tag{8.17}$$

Setting

$$Y(t) = \int_0^t X(s)ds, \quad t \geq 0,$$

Y fulfills the deterministic equation (with random coefficients)

$$Y'(t) = AY(t) + x + CB(t), \quad Y(0) = 0, \quad t \geq 0,$$

which can be solved by the method of variation of constants, giving

$$Y(t) = \int_0^t e^{(t-s)A}(x + CB(s))ds, \quad t \geq 0.$$

By substituting $Y(t)$ in (8.17) yields

$$X(t) = A \int_0^t e^{(t-s)A}(x + CB(s))ds + x + CB(t).$$

Recalling the integration by parts formula (Proposition 4.21),

$$\int_0^t e^{(t-s)A}CdB(s) = CB(t) + A \int_0^t e^{(t-s)A}CB(s)ds,$$

we find finally

$$X(t) = e^{tA}x + \int_0^t e^{(t-s)A}CdB(s). \tag{8.18}$$

Example 8.9 (Stochastic linear equations). Let $r = d = 1$ and consider the stochastic differential equation

$$\begin{cases} dX = aXdt + cXdB(t), \\ \\ X(0) = x, \end{cases} \tag{8.19}$$

where $a, c, x \in \mathbb{R}$. Again Theorem 8.2 applies. We want to show that the solution of (8.19) is given by

$$X(t) = e^{t\left(a - \frac{1}{2}\, c^2\right)} e^{cB(t)} x, \quad t \geq 0. \tag{8.20}$$

Let us check in fact that $X(t)$ given by (8.20) solves (8.19). Write $X(t) = e^{F(t)}x$ where $F(t) = t\left(a - \frac{1}{2}\, c^2\right) + cB(t)$. Then we have

$$dF(t) = \left(a - \frac{1}{2}\, c^2\right) dt + cdB(t)$$

and, by Itô's formula,

$$\begin{aligned} dX(t) &= e^{F(t)} x dF(t) + \tfrac{1}{2}\, c^2 e^{F(t)} x dt \\ &= e^{F(t)} \left(a - \tfrac{1}{2}\, c^2\right) x dt + c e^{F(t)} x dB(t) + \tfrac{1}{2}\, c^2 e^{F(t)} x dt \\ &= aX(t)dt + cXdB(t). \end{aligned}$$

Exercise 8.10. Let $r = 1$ and consider the differential stochastic equation

$$\begin{cases} dX = AXdt + CXdB(t), \\ \\ X(0) = x, \end{cases} \tag{8.21}$$

where $A, C \in L(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $AC = CA$. Show that the solution of (8.21) is given by

$$X(t) = e^{t(A - C^2/2)} e^{CB(t)} x. \tag{8.22}$$

8.1.3. Equations with random coefficients

In some situations (see for instance Section 8.2) we shall deal with stochastic differential equations having random coefficients,

$$\begin{aligned} X(t, \omega) = \eta(\omega) &+ \int_s^t b(r, X(r, \omega), \omega) dr \\ &+ \int_s^t \sigma(r, X(r, \omega), \omega) dB(r). \end{aligned} \tag{8.23}$$

Here $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{R}^d)$, $b\colon [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ and $\sigma : [0,T] \times L(\mathbb{R}^r, \mathbb{R}^d) \times \Omega \to \mathbb{R}^d$ are such that:

Hypothesis 8.11. (i) There exists $M > 0$ such that for all $t \in [0,T]$, $x, y \in \mathbb{R}^d$, $\omega \in \Omega$

$$|b(t,x,\omega)-b(t,y,\omega)|^2+\|\sigma(t,x,\omega)-\sigma(t,y,\omega)\|_{HS}^2 \le M^2|x-y|^2 \quad (8.24)$$

and

$$|b(t,x,\omega)|^2 + \|\sigma(t,x,\omega)\|_{HS}^2 \le M^2(1+|x|^2). \quad (8.25)$$

(ii) For all $Y \in C_B([0,T]; L^2(\Omega, \mathbb{R}^d))$ we have

$$U \in C_B([0,T]; L^2(\Omega, \mathbb{R}^d)), \quad V \in C_B([0,T]; L^2(\Omega, L(\mathbb{R}^r, \mathbb{R}^d))),$$

where

$$U(t,\omega)=b(t,Y(t,\omega),\omega)),\ \ V(t,\omega)=\sigma(t,Y(t,\omega),\omega)),\ \ t\in[0,T],\ \omega\in\Omega.$$

The following result can be proved as Theorem 8.2.

Theorem 8.12. *Assume that Hypothesis* 8.11 *holds. Let* $s \in [0,T)$ *and* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{R}^d)$. *Then problem* (8.23) *has a unique solution*

$$X \in C_B([s,T]; L^2(\Omega; \mathbb{R}^d)).$$

Example 8.13. Consider the stochastic differential equation on $\mathbb{R}$

$$\begin{cases} dX(t) = X(t)\langle F(t), dB(t)\rangle, & t \in [0,T], \\ \\ X(0) = x, \end{cases} \quad (8.26)$$

where B is a d-dimensional Brownian motion and $F \in C_B([0,T]; L^2(\Omega; \mathbb{R}^d))$ is bounded. It is easy to check that Theorem 8.12 applies and so there exists a solution X of (8.26). Let us show that X is given by the formula

$$X(t)=\exp\left\{-\frac{1}{2}\int_0^t |F(s)|^2 ds+\int_0^t \langle F(s), dB(s)\rangle\right\} x, \quad t\ge 0. \quad (8.27)$$

For this we check that $X(t)$ given by (8.27) solves (8.26). Write $X(t) = e^{H(t)}x$ where

$$H(t) = -\frac{1}{2}\int_0^t |F(s)|^2 ds + \int_0^t \langle F(s), dB(s)\rangle.$$

Then

$$dH(t) = -\frac{1}{2}\,|F(t)|^2 dt + \langle F(t), dB(t)\rangle, \quad t \geq 0.$$

Now by Itô's formula we find

$$\begin{aligned} dX(t) &= e^{H(t)} dH(t)x + \frac{1}{2}\, e^{H(t)}|F(t)|^2 x dt \\ &= e^{H(t)}\langle F(t), dB(t)\rangle x = X(t)\langle F(t), dB(t)\rangle x, \quad t \geq 0. \end{aligned}$$

So, X fulfills (8.26) as claimed.

8.2. Continuous dependence of $X(t, s, \eta)$ from (t, s, η)

We assume here that Hypothesis 8.1 holds. We are going to prove that the solution $X(t, s, \eta)$ to (8.2) is Hölder continuous on t, s and Lipschitz continuous on η in *mean square*. We first need a lemma.

Lemma 8.14. *Assume that Hypothesis* 8.1 *holds. Then for all* $s \in [0, T]$ *and all* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$ *we have*

$$\mathbb{E}\left(|X(t,s,\eta)|^2\right) \leq 3[\mathbb{E}(|\eta|^2) + M^2((T-s)^2 + (T-s)]e^{3M^2(T-s+1)}. \tag{8.28}$$

Proof. Writing $X(t, s, \eta) = X(t)$ for short, we have

$$\begin{aligned} \mathbb{E}\left(|X(t)|^2\right) \;\leq\; & 3\mathbb{E}(|\eta|^2) + 3\mathbb{E}\left(\left|\int_s^t b(r, X(r))dr\right|^2\right) \\ & + 3\int_s^t \mathbb{E}(\|\sigma(r, X(r))\|_{HS}^2)dr. \end{aligned}$$

By the Hölder inequality and Hypothesis 8.1(ii) we deduce that

$$\begin{aligned} \mathbb{E}\left(|X(t)|^2\right) \;\leq\; & 3\mathbb{E}(|\eta|^2) + 3M^2(t-s)\int_s^t (1 + \mathbb{E}(|X(r)|^2))dr \\ & + 3M^2\int_s^t (1 + \mathbb{E}(|X(r)|^2))dr. \end{aligned}$$

Consequently

$$\begin{aligned} \mathbb{E}\left(|X(t)|^2\right) \;\leq\; & 3\mathbb{E}(|\eta|^2) + 3M^2((T-s)^2 + (T-s)) \\ & + 3M^2((T-s) + 1)\int_s^t \mathbb{E}(|X(r)|^2)dr. \end{aligned}$$

Now the conclusion follows from the Gronwall lemma. $\square$

We are now ready to study the regularity of $X(t,s,\eta)$ with respect to t,s,η. We note that, by Lemma 8.14, there exists a constant $C_{T,\eta}$ such that

$$\mathbb{E}\left(|X(t,s,\eta)|^2\right) \le C_{T,\eta}, \quad 0 \le s < t \le T. \tag{8.29}$$

As regards the regularity of $X(t,s,\eta)$ with respect to t we prove

Proposition 8.15. *Assume that Hypothesis* 8.1 *holds. Let* $0 \le s \le t_1 < t \le T$ *and* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{R}^d)$. *Then we have*

$$\mathbb{E}\left(|X(t,s,\eta) - X(t_1,s,\eta)|^2\right) \le 2M^2((t-t_1)^2+t-t_1)(1+C^2_{T,\eta}). \tag{8.30}$$

Proof. Write

$$\begin{aligned}\mathbb{E}\left(|X(t,s,\eta) - X(t_1,s,\eta)|^2\right) &\le 2M^2(t-t_1)\int_{t_1}^t (1+\mathbb{E}(|X(r,s,\eta)|^2)dr\\ &+2M^2\int_{t_1}^t (1+\mathbb{E}(|X(r,s,\eta)|^2))dr.\end{aligned}$$

Now the conclusion follows taking into account (8.29). □

As regards the regularity of $X(t,s,\eta)$ with respect to η we prove

Proposition 8.16. *Assume that Hypothesis* 8.1 *holds, let* $0 \le s < t \le T$ *and* $\eta, \zeta \in L^2(\Omega, \mathscr{F}_s, \mathbb{R}^d)$. *Then*

$$\mathbb{E}\left(|X(t,s,\eta) - X(t,s,\zeta)|^2\right) \le 3e^{3M^2(T-s+1)(t-s)}\mathbb{E}(|\eta-\zeta|^2). \tag{8.31}$$

Proof. Write

$$\begin{aligned}|X(t,s,\eta) - X(t,s,\zeta)|^2 &\le 3|\eta-\zeta|^2\\ &+3\left|\int_s^t (b(r,X(r,s,\eta)) - b(r,X(r,s,\zeta))du\right|^2\\ &+3\left|\int_s^t (\sigma(r,X(r,s,\eta)) - \sigma(r,X(r,s,\zeta))dB(r)\right|^2.\end{aligned}$$

Taking expectation and using (8.4) we obtain

$$\begin{aligned}\mathbb{E}\left(|X(t,s,\eta) - X(t,s,\zeta)|^2\right) &\le 3\mathbb{E}(|\eta-\zeta|^2)\\ &+3M^2(T-s+1\int_s^t \mathbb{E}\left(|X(r,s,\eta) - X(r,s,\zeta)|^2\right)dr\end{aligned}$$

and the conclusion follows from the Gronwall lemma. □

We finally study the regularity of $X(t, s, \eta)$ with respect to s.

Proposition 8.17. *Assume that Hypothesis* 8.1 *holds, let* $0 < s < s_1 < t \le T$ *and* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$. *Then there exists a constant* $C'_{T,\eta} > 0$ *such that*

$$\mathbb{E}\left(|X(t, s, \eta) - X(t, s_1, \eta)|^2\right) \le C'_{T,\eta}|s - s_1|. \tag{8.32}$$

Proof. Taking into account (8.10), we write

$$X(t, s, \eta) - X(t, s_1, \eta) = X(t, s_1, X(s_1, s, \eta)) - X(t, s_1, \eta).$$

By (8.31) there exists $C_T > 0$ such that

$$\begin{aligned}\mathbb{E}\left(|X(t, s, \eta) - X(t, s_1, \eta)|^2\right) &\le C_T^2\, \mathbb{E}\left(|X(s_1, s, \eta) - \eta|^2\right) \\ &= C_T^2\, \mathbb{E}\left(|X(s_1, s, \eta) - X(s, s, \eta)|^2\right).\end{aligned}$$

The conclusion follows now from (8.30). □

8.3. Differentiability of $X(t, s, x)$ with respect to x

In this section we assume, besides Hypothesis 8.1, that

Hypothesis 8.18.

(i) The partial derivatives $D_x b$, $D_x^2 b$, $D_x\sigma$ and $D_x^2\sigma$ exist and are continuous on $[0, T] \times \mathbb{R}^d$.
(ii) There is $M > 0$ such that

$$\|D_x b(t, \cdot)\|_0 + \|D_{xx} b(t, \cdot)\|_0 + \|D_x\sigma(t, \cdot)\|_0 + \|D_{xx}\sigma(t, \cdot)\|_0 \le M. \tag{8.33}$$

We set as before

$$C_B := C_B([s, T]) = C_B([s, T]; L^2(\Omega; \mathbb{R}^d)).$$

8.3.1. Existence of $X_x(t, s, x)$

Theorem 8.19. *Assume that Hypotheses* 8.1 *and* 8.18 *hold. Then for any* $s \in [0, T]$ *the mapping*

$$\mathbb{R}^d \to C_B, \quad x \to X(\cdot, s, x),$$

is $\mathbb{P}$*-a.s. differentiable with respect to* x *in any direction* $h \in \mathbb{R}^d$. *Moreover, setting*

$$X_x(t, s, x) \cdot h = \eta^h(t, s, x), \quad x, h \in \mathbb{R}^d, \tag{8.34}$$

$\eta^h(\cdot) = \eta^h(\cdot, s, x)$ *is the solution to the following stochastic differential equation with random coefficients,*

$$\begin{cases} d\eta^h(t) = b_x(t, X(t)) \cdot \eta^h(t)dt + \sigma_x(t, X(t))(\eta^h(t), dB(t)) \\ \eta^h(s) \quad = h. \end{cases} \tag{8.35}$$

Proof. Note that the coefficients of equation (8.35) fulfill Hypothesis 8.11, so it possesses a unique solution by Theorem 8.12.

To prove the theorem we use Theorem D.4 from Appendix D (with $\Lambda = \mathbb{R}^d$ and $E = C_B = C_B([s, T_1])$). Let us define a mapping

$$F : \mathbb{R}^d \times C_B \to C_B,$$

setting

$$[F(x,X)](t) : = x + \int_s^t b(r, X(r))dr + \int_s^t \sigma(r, X(r))dB(r), \; t \in [s, T_1], \tag{8.36}$$

where $T_1 > s$ is chosen such that

$$\|F(x,X_1) - F(x,X_2)\|_{C_B} \leq \frac{1}{2} \|X_1 - X_2\|_{C_B} \text{ for all } X_1, X_2 \in C_B, \; x \in \mathbb{R}^d. \tag{8.37}$$

Then F fulfills Hypothesis D.1 so that there exists a unique $X(x) \in C_B$ such that

$$F(x, X(x)) = X(x), \quad x \in \mathbb{R}^d,$$

which depends continuously on x. $X(x)$ coincides with the solution $X(\cdot, s, x)$ of (8.3).

Let us check that Hypothesis D.3 is fulfilled as well. Taking into account that $F(x, X)$ is differentiable in x and $F_x(x, X) = I$, Hypothesis D.3-(i) reduces to:

(i). For any $x \in \mathbb{R}^d$, $X \in C_B$ there exist a linear continuous operator $F_X(x, X) \in L(C_B)$ such that

$$\lim_{h \to 0} \frac{1}{h} (F(x + hy, X + hY) - F(x, X)) = y + F_X(x, X) \cdot Y.$$

This follows from (8.36) with

$$\begin{aligned}[F_X(x, X) \cdot Y](t) : &= \int_s^t b_x(r, X(r)) \cdot Y(r)dr \\ &+ \int_s^t \sigma_x(r, X(r)) \cdot Y(r)dB(r), \quad t \in [s, T_1].\end{aligned}$$

As regards Hypothesis D.3-(ii), it reduces to:

(ii). For any $x, y \in \mathbb{R}^d$, $X, Y \in C_B$ the function

$$[0,1] \to C_B, \quad \xi \to F((1-\xi)x + \xi y, (1-\xi)X + \xi Y),$$

is continuously differentiable and

$$\frac{d}{d\xi} F((1-\xi)x + \xi y, (1-\xi)X + \xi Y) = x - y$$

$$+F_X((1-\xi)x + \xi y, (1-\xi)X + \xi Y) \cdot (Y - X), \quad \forall\, \xi \in [0,1]. \tag{8.38}$$

Also this identity can be easily checked. So, the conclusion follows from Theorem D.4. □

The following estimate of $|\eta^h(t,x)|$ will be used in the sequel.

Proposition 8.20. *For any $h \in \mathbb{R}^d$ and any $T > 0$ there is $C_T > 0$ such that*

$$\mathbb{E}|\eta^h(t,x)|^2 \le C_T t^{-1} |h|^2, \quad \forall\, t \in [0,T]. \tag{8.39}$$

Proof. By (8.35) we have

$$\mathbb{E}|\eta^h(t,x)|^2 \;\le\; 3|h|^2 + 3M^2 t\, \mathbb{E} \int_0^t |\eta^h(s,x)|^2 ds$$

$$+3M^2 \mathbb{E} \int_0^t |\eta^h(s,x)|^2 ds.$$

Now the conclusion follows from Gronwall's lemma. □

8.3.2. Existence of $X_{xx}(t,s,x)$

We now prove the existence of the second derivative of $X(t,s,x)$ with respect to x.

Theorem 8.21. *Assume that Hypotheses* 8.1 *and* 8.18 *hold. Then the mapping*

$$\mathbb{R}^d \to C_B, \quad x \to X(\cdot, s, x),$$

is $\mathbb{P}$-a.s. twice differentiable with respect to x in any couple of directions (h,k) in $\mathbb{R}^d$. Moreover, setting

$$X_{xx}(t,s,x)(h,k) = \zeta^{h,k}(t,s,x), \quad x, h \in \mathbb{R}^d, \tag{8.40}$$

$\zeta^{h,k}(t,s,x)$ is the solution to the stochastic differential equation (with random coefficients)

$$\begin{cases} d\,\zeta^{h,k}(t,s,x) = b_x(t,X(t,s,x))\cdot\zeta^{h,k}(t,s,x)dt \\ \qquad\qquad +b_{xx}(t,X(t,s,x))(\eta^h(t,s,x),\eta^k(t,s,x))dt \\ \qquad\qquad +\sigma_x(t,X(t,s,x))(\zeta^{h,k}(t,s,x),dB(s)) \\ \qquad\qquad +\sigma_{xx}(t,X(t,s,x))\cdot(\eta^h(t,s,x),\eta^k(t,s,x),dB(t)) \\ \zeta^{h,k}(s,s,x) \quad = 0. \end{cases} \tag{8.41}$$

We shall take $n = r = 1$ for simplicity. We first prove a lemma.

Lemma 8.22. *Let $\eta(\cdot,s,x) \in C_B([s,T];L^2(\Omega))$ be the solution of the equation*

$$\begin{aligned} \eta(t,s,x) = 1 + \int_s^t b_x(r,X(r,s,x))\eta(r,s,x)dr \\ + \int_s^t \sigma_x(r,X(r,s,x))\eta(r,s,x)dB(r). \end{aligned} \tag{8.42}$$

Then $\eta(\cdot,s,x) \in C_B([s,T];L^4(\Omega))$ and there exists $C > 0$ such that

$$\mathbb{E}|\eta(\cdot,s,x)|^4 \le C, \quad \forall\, s \in [0,T),\ x \in \mathbb{R}^d. \tag{8.43}$$

Proof. First we notice that equation (8.43) has a solution $\eta(\cdot,s,x)$ belonging to $C_B([s,t];L^4(\Omega,\mathscr{F}_s,\mathbb{P};\mathbb{R}^d))$ in view of Theorem 8.7.

Now write

$$\begin{aligned} |\eta(t,s,x)|^4 \le 27 + 27\left|\int_s^t b_x(r,X(r,s,x))\eta(r,s,x)dr\right|^4 \\ +27\left|\int_s^t \sigma_x(r,X(r,s,x))\eta(r,s,x)dB(r)\right|^4. \end{aligned}$$

By using (8.33) and the Hölder inequality we see that there exists a constant C_1 such that

$$\begin{aligned} |\eta(t,s,x)|^4 \le 27 + C_1\int_s^t |\eta(r,s,x)|^4 dr \\ +C_1\left|\int_s^t \sigma_x(r,X(r,s,x))\eta(r,s,x)dB(r)\right|^4. \end{aligned}$$

Now, taking expectation on both sides of this inequality and using Proposition 7.14, we find that

$$\mathbb{E}|\eta(t,s,x)|^4 \le C_2(1+\mathbb{E}\int_s^t |\eta(r,s,x)|^4 dr), \quad 0 \le s \le t \le T, \ x \in \mathbb{R},$$

where C_2 is another constant. The conclusion follows from the Gronwall lemma. □

We can now prove Theorem 8.21.

Proof. Let us choose T_1 as in (8.37) and set $C_B = C_B([s, T_1])$ as before. By Theorem 8.19 we know that $X(t,s,x)$ is differentiable with respect to x and that its derivative $\eta(\cdot,s,x) = X_x(\cdot,s,x)$ belongs to C_B and fulfills equation (8.42). For any $x \in \mathbb{R}$ we define a linear bounded operator $T(x)$ from C_B into C_B setting for all $t \in [s, T_1]$,

$$\begin{aligned}(T(x)Z)(t) = &-\int_s^t b_x(r, X(r,s,x))Z(r)dr \\ &-\int_s^t \sigma_x(r, X(r,s,x))Z(r)dB(r).\end{aligned} \tag{8.44}$$

Notice that, since $\eta(\cdot,s,x) \in C_B([s,T]; L^4(\Omega))$, $T(x)Z$ is differentiable with respect to x for any $Z \in C_B([s,T]; L^4(\Omega))$ and it results

$$\begin{aligned}(T'(x)Z)(t) = &-\int_s^t b_{xx}(r, X(r,s,x))Z(r)\eta(\cdot,r,x)dr \\ &-\int_s^t \sigma_{xx}(r, X(r,s,x))Z(r)\eta(\cdot,r,x)dB(r).\end{aligned} \tag{8.45}$$

Now we write equation (8.42) as

$$\eta(\cdot,s,x) = 1 + T(x)\eta(\cdot,s,x) \tag{8.46}$$

By (8.37) it follows that

$$\|T(x)\|_{L(C_B)} \le 1/2, \quad \forall\, x \in \mathbb{R}.$$

Thus the solution of (8.46) is given by

$$\eta(\cdot,s,x) = (1-T(x))^{-1}(\mathbb{1}), \tag{8.47}$$

where $\mathbb{1}$ denotes the function identically equal to 1, From this identity it is easy to show the existence of $\eta_x(\cdot,s,x) := \zeta(\cdot,s,x)$. We have in fact, by a straightforward computation

$$\eta_x(\cdot,s,x) = (1-T(x))^{-1}(T'(x)\eta(\cdot,s,x)), \tag{8.48}$$

where

$$
\begin{aligned}
T'(x)\eta(\cdot,s,x)(t) = & \int_s^t b_{xx}(r,X(r,s,x))\eta^2(\cdot,s,x)dr \\
& + \int_s^t \sigma_{xx}(r,X(r,s,x))\eta^2(\cdot,s,x)dB(r).
\end{aligned}
\tag{8.49}
$$

Now by (8.48) it follows that

$$
\begin{aligned}
\eta_x(t,s,x) - T(x)\eta_x(\cdot,s,x)(t) = & \int_s^t b_{xx}(r,X(r,s,x))\eta^2(\cdot,s,x)dr \\
& + \int_s^t \sigma_{xx}(r,X(r,s,x))\eta^2(\cdot,s,x)dB(r),
\end{aligned}
$$

and the conclusion follows. □

8.4. Itô differentiability of $X(t,s,x)$ with respect to s.

It is useful first to recall some results from the deterministic case.

8.4.1. The deterministic case

Let us consider the problem

$$
\begin{cases}
X'(t) = b(t,X(t)), \quad t \in [s,T], \\
\\
X(s) = s,
\end{cases}
\tag{8.50}
$$

under Hypotheses 8.1 and 8.18 (but with $\sigma = 0$). Denote by $X(t,s,x)$ the solution of (8.50), it is well known that it is C^1 in all variables. To compute $X_s(t,s,x)$ write

$$
X(t,s,x) = X(t,r,X(r,s,x)), \quad t \geq r \geq s. \tag{8.51}
$$

Differentiating (8.51) with respect to r yields

$$
0 = X_s(t,r,X(r,s,x)) + X_x(t,r,X(r,s,x))X_t(r,s,x).
$$

Setting $r = s$ we find

$$
X_s(t,s,x) = -X_x(t,s,x)b(s,x), \quad 0 \leq s \leq t \leq T, \tag{8.52}
$$

which is equivalent to

$$
X(t,s,x) = x + \int_s^t X_x(t,r,x)b(r,x)dr, \quad 0 \leq s \leq t \leq T. \tag{8.53}
$$

In the next subsection we are going to generalize this formula for the solution $X(t,s,x)$ of the stochastic differential equation (8.3).

8.4.2. The stochastic case

Notice that the process $s \mapsto X(t,s,x)$ is not $\mathscr{F}_s$-measurable. It happens, however, that for any $s \in [0,T]$, $X(t,s,x)$ is measurable with respect to the σ–algebra $\mathscr{F}_s^+$ generated by all sets of the form

$$\{(B(s_1) - B(s), ..., B(s_n) - B(s)) \in A\},$$

where $n \in \mathbb{N}$, $0 \le s \le s_1 < ... < s_n \le T$ and $A \in \mathscr{B}(\mathbb{R}^n)$. The family $(\mathscr{F}_s^+)_{s\in[0,T]}$ is called the *future filtration* of B.

Proposition 8.23. *Assume that Hypotheses* 8.1 *holds. Let $x \in \mathbb{R}^d$, $s \in [0,T]$. Then $X(t,s,x)$ is $\mathscr{F}_s^+$-measurable.*

Proof. Let $X_N(t,s,x)$ be defined by (8.13), $N \in \mathbb{N}$. Let us starting by showing that $X_1(t,s,x)$ is $\mathscr{F}_s^+$–measurable. We have in fact

$$X_1(t,s,x) = x + \int_s^t b(u,x)du + \int_s^t \sigma(u,x)dB(u).$$

Since

$$\int_s^t \sigma(u,x)dB(u) = \lim_{|\eta|\to 0} \sum_{k=1}^n \sigma(s_{k-1})(B(s_k) - B(s_{k-1})),$$

where $\eta = \{s = s_0 < s_1 < \cdots < s_n = t\}$, then $X_1(t,s,x)$ is $\mathscr{F}_s^+$-measurable. In fact

$$(B(s_1) - B(s)) \text{ is } \mathscr{F}_s^+\text{-measurable},$$

$$(B(s_2) - B(s_1)) = (B(s_2) - B(s)) - (B(s_1) - B(s)) \text{ is } \mathscr{F}_s^+\text{-measurable},$$

and so on. We end the proof by recurrence on N. □

Now we introduce the *backward Itô integral* for a process which is adapted to the future filtration. For this we need the following result which can be proved as Lemma 5.1.

Lemma 8.24. *Let $t_1 < t_2 \le s$, and let $\varphi \in L^2(\Omega, \mathscr{F}_s^+, \mathbb{P})$. Then $B(t_2) - B(t_1)$ and φ are independent.*

We define $C_{B^+}([0,T]; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d)))$ by a straightforward generalization of the space $C_B([0,T]; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d)))$ defined before.

The elements of $C_{B^+}([0,T]; L^2(\Omega; L(\mathbb{R}^r; \mathbb{R}^d))$ are called *stochastic processes adapted to the future filtration* $(\mathscr{F}_t^+)$ *and continuous in quadratic mean.*

Let $F \in C_{B^+}([0,T]; L^2(\Omega; L(\mathbb{R}^r;\mathbb{R}^d)))$. For any $\varsigma \in \Sigma$ with $\varsigma = \{0 = s_0 < s_1 < \cdots < s_n = T\}$ we set

$$I_\varsigma(F) = \sum_{k=1}^n F(t_k)(B(t_k) - B(t_{k-1}))$$

The proof of next theorem is completely similar to that of equation (6.15).

Theorem 8.25. *For any* $F \in C_{B^+}([0,T]; L^2(\Omega; L(\mathbb{R}^r;\mathbb{R}^d)))$ *there exists the limit*

$$\lim_{|\varsigma|\to 0} I_\varsigma(F) =: \oint_0^T F(s)dB(s), \tag{8.54}$$

in $L^2(\Omega)$. *Moreover we have*

$$\mathbb{E}\oint_0^T F(s)dB(s) = 0, \tag{8.55}$$

and

$$\mathbb{E}\left|\oint_0^T F(s)dB(s)\right|^2 = \int_0^T \mathbb{E}\left[\|F(s)\|^2_{HS}\right]\, ds. \tag{8.56}$$

$\oint_0^T F(s)dB(s)$ is called the *backward Itô integral* of the function F in $[0,T]$.

Exercise 8.26. Let $t > s$. Prove that

$$\oint_s^t B(r)dB(r) = \frac{1}{2}\,(B(t)^2 - B(s)^2 + (t-s)).$$

8.4.3. Backward Itô's formula

Theorem 8.27. *Assume that Hypotheses* 8.1 *and* 8.18 *hold with* $d = r = 1$. *Then we have*

$$\begin{aligned} X(t,s,x) - x = &\int_s^t X_x(t,r,x)b(r,x)dr \\ &+ \frac{1}{2}\int_s^t X_{xx}(t,r,x)\sigma^2(r,x)dr \\ &+ \oint_s^t X_x(t,r,x)\sigma(r,x)dB(r). \end{aligned} \tag{8.57}$$

Proof. For any $\eta \in \Sigma(s,t)$ we set

$$|\eta| = \max_{k=1,\dots,n} (t_k - t_{k-1}).$$

If $\eta \in \Sigma(s,t)$ we have

$$\begin{aligned} X(t,s,x) - x &= -\sum_{k=1}^{n} [X(t,s_k,x) - X(t,s_{k-1},x)] \\ &\quad - \sum_{k=1}^{n} [X(t,s_k,x) - X(t,s_k,X(s_k,s_{k-1},x))] \\ &= -\sum_{k=1}^{n} X_x(t,s_k,x)(x - X(s_k,s_{k-1},x)) \\ &\quad - \frac{1}{2}\sum_{k=1}^{n} X_{xx}(t,s_k,x)(x - X(s_k,s_{k-1},x))^2 + o(|\eta|). \end{aligned} \tag{8.58}$$

Arguing as in the proof of Itô's formula one can show, after some tedious but straighforward computations, that

$$\lim_{|\eta|\to 0} o(|\eta|) = 0, \ \ \mathbb{P}\text{-a.s..}$$

On the other hand we have

$$\begin{aligned} X(s_k,s_{k-1},x) - x &= \int_{s_{k-1}}^{s_k} b(r, X(r,s_{k-1},x))dr \\ &\quad + \int_{s_{k-1}}^{s_k} \sigma(r, X(r,s_{k-1},x))dB(r) \\ &= b(s_k,x)(s_k - s_{k-1}) + \sigma(s_k,x)(B(s_k) - B(s_{k-1})) \\ &\quad + o(s_k - s_{k-1}). \end{aligned} \tag{8.59}$$

(Notice that, since b is deterministic, one can replace in (8.59) $b(s_k,x)$ with $b(\xi_k,x)$ where ξ_k is any point in $[s_{k-1},s_k]$.) Substituting (8.59) in

(8.58) we find that

$$
\begin{aligned}
X(t,s,x) - x &= \sum_{k=1}^{n} X_x(t,s_k,x) b(s_k,x)(s_k - s_{k-1}) \\
&+ \sum_{k=1}^{n} X_x(t,s_k,x)\sigma(s_k,x)(B(s_k) - B(s_{k-1})) \\
&+ \frac{1}{2}\sum_{k=1}^{n} X_{xx}(t,s_k,x)\sigma^2(s_k,x)(B(s_k) - B(s_{k-1}))^2 \\
&+ I_1(\eta) + I_2(\eta) + I_3(\eta) + o_1(|\eta|).
\end{aligned}
\tag{8.60}
$$

Obviously

$$\lim_{|\eta|\to 0} I_1(\eta) = \int_s^t X_x(r,x) b(r,x) dr.$$

As regards $I_2(\eta)$, we note that it is an integral sum corresponding to the backward Itô integral since $X_x(t,s_k,x)$ is $\mathscr{F}_{s_k}^{+}$ measurable by Proposition 8.23. Therefore we have

$$\lim_{|\eta|\to 0} I_2(\eta) = \oint_s^t X_x(r,x)\sigma(r,x) dB(r).$$

The other terms $I_3(\eta)$ and $o_1(|\eta|)$ can be handled as in the proof of Itô's formula. □

In a similar way one can prove

Theorem 8.28. *Assume that Hypotheses* 8.1 *and* 8.18 *hold. Then we have*

$$
\begin{aligned}
X(t,s,x) - x &= \int_s^t X_x(t,r,x)\cdot b(r,x) dr \\
&+ \frac{1}{2}\int_s^t \mathrm{TR}\,[X_{xx}(t,r,x)(\sigma(r,x),\sigma(r,x))] dr \\
&+ \oint_s^t X_x(t,r,x)(\sigma(r,x), dB(r))\rangle,
\end{aligned}
\tag{8.61}
$$

where the vector trace TR is defined by

$$\mathrm{TR}\,[X_{xx}(t,r,x)(\sigma(r,x),\sigma(r,x))] = \sum_{k=1}^{d} X_{xx}(t,r,x)(\sigma(r,x)e_k, \sigma(r,x)e_k)$$

and (e_k) *is any orthonormal basis in* $\mathbb{R}^d$.

Also we can prove the following *backward Itô formula*.

Theorem 8.29. *Let* $\varphi \in C_b^2(\mathbb{R}^d)$. *Then for any* $0 \le s < t \le T$, *we have*

$$\begin{aligned}\varphi(X(t,s,x)) - \varphi(x) = &\int_s^t \langle D_x[\varphi(X(t,r,x))], b(r,x)\rangle dr \\ &+ \frac{1}{2}\int_s^t \mathrm{TR}\ \{D_x^2[\varphi(X(t,r,x))]\sigma(r,x)\sigma^*(r,x)\} dr \\ &+ \oint_s^t \langle D_x[\varphi(X(t,r,x))], \sigma(r,x) dB(r).\end{aligned} \tag{8.62}$$

Chapter 9
Relationship between stochastic and parabolic equations

We are still concerned with the stochastic evolution equation

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), & 0 \le s < t \le T, \\ X(s) = x \in \mathbb{R}^d \end{cases} \tag{9.1}$$

where $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{R}^d \to L(\mathbb{R}^r; \mathbb{R}^d)$ fulfill Hypothesis 8.1 whereas B is an r-dimensional Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ (we shall denote by $(\mathscr{F}_t)_{t\ge 0}$ its natural filtration). By Theorem 8.2 we know that problem (9.1) has a unique solution $X(\cdot, s, x)$.

Let us define the *transition evolution operator* $P_{s,t}$, setting for any $\varphi \in B_b(\mathbb{R}^d)$ [(1)]

$$P_{s,t}\varphi(x) = \mathbb{E}[\varphi(X(t, s, x))], \quad x \in \mathbb{R}^d, \ 0 \le s \le t \le T. \tag{9.2}$$

More informations about $P_{s,t}$, in particular that $P_{s,t}$ acts on $B_b(\mathbb{R}^d)$, can be found in Chapter 12.

The goal of this chapter is to show that, given $\varphi \in C_b^2(\mathbb{R}^d)$ and setting

$$z(s, x) = \mathbb{E}[\varphi(X(s, x))], \quad s \in [0, T], \ x \in \mathbb{R}^d,$$

then z is the solution of the following parabolic equation.

$$\begin{cases} z_s(s, x) + (\mathscr{L}(s)z(s, \cdot))(x) = 0, & 0 \le s < T, \\ z(T, x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{9.3}$$

(1) We recall that $B_b(\mathbb{R}^d)$ is the space of all real, bounded and Borel mappings on $\mathbb{R}^d$.

where $\mathscr{L}(s)$ is for any $s \in [0, T]$ the *Kolmogorov operator*

$$
\begin{aligned}
(\mathscr{L}(s)\varphi)(x) = {} & \frac{1}{2}\,\mathrm{Tr}\,[\varphi_{xx}(x)\sigma(s,x)\sigma^*(s,x)] \\
& + \langle b(s,x), \varphi_x(x)\rangle, \quad \varphi \in C_b^2(\mathbb{R}^d).
\end{aligned} \tag{9.4}
$$

This fact has already been proved in Chapter 5 in a very particular case $(r, d = 1, b = 0, \sigma = 1)$.

We start in Section 9.1 by proving the *Markov* property of $X(\cdot, s, x)$ from which it follows that

$$
P_{s,t} = P_{s,r}P_{r,t}, \quad 0 \le s \le r \le t \le T. \tag{9.5}
$$

We shall consider, for pedagogical reasons, the deterministic case $(\sigma \equiv 0)$ first (in Section 9.2), the general stochastic will be treated in Section 9.3.

Section 9.4 will be devoted to examples. Finally, in Section 9.5 we shall prove an important smoothing property of the transition semigroup $P_{s,t}$ when for all $x \in \mathbb{R}^d$ $\sigma(x)$ is invertible and there exists $K > 0$ such that $\|(\sigma(x))^{-1}\| \le K$ for all $x \in \mathbb{R}^d$ (this is a uniform ellipticity condition for $\mathscr{L}(s)$).

In this case for any $\varphi \in C_b(\mathbb{R}^d)$ there exists a unique continuous function z in $[s, T] \times \mathbb{R}^d$ such that $z(T, \cdot) = \varphi$, which is C^2 on $[s, T) \times \mathbb{R}^d$ and solves equation (9.3).

We notice that the proof of all results from this chapter do not require any knowledge of PDE's theory, but they are purely probabilistic.

9.1. Markov property

We want to extend the Markov property (already proved in Chapter 5 in the particular case when $X(t, s, x) = B(t) - B(s) + x$), to the solution $X(t, s, \eta)$ of equation

$$
\begin{cases}
dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), \\
\\
X(s) = \eta
\end{cases} \tag{9.6}
$$

where $0 \le s < t \le T$, $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$.

Lemma 9.1. *For all $\varphi \in B_b(\mathbb{R}^d)$ and for all $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$ we have*

$$
\mathbb{E}[\varphi(X(t, s, \eta))|\mathscr{F}_s] = P_{s,t}\varphi(\eta), \quad 0 \le s < t \le T. \tag{9.7}
$$

and

$$
\mathbb{E}[\varphi(X(t, s, \eta))] = \mathbb{E}[P_{s,t}\varphi(\eta)], \quad 0 \le s < t \le T. \tag{9.8}
$$

Proof. Identity (9.8) follows by (9.7) taking expectation. So, we have only to prove (9.7). It is enough to take η of the form

$$\eta = \sum_{k=1}^{n} x_k \mathbb{1}_{A_k},$$

where $x_1, \ldots, x_n \in \mathbb{R}^d$ and A_1, ..., A_n are mutually disjoints sets in $\mathscr{F}_s$ such that

$$\Omega = \bigcup_{k=1}^{n} A_k.$$

In this case from Proposition 8.6 it follows that

$$X(t, s, \eta) = \sum_{k=1}^{n} X(t, s, x_k) \mathbb{1}_{A_k}, \quad 0 \le s \le t \le T.$$

Consequently

$$\varphi(X(t, s, \eta)) = \sum_{k=1}^{n} \varphi(X(t, s, x_k)) \mathbb{1}_{A_k}, \quad 0 \le s \le t \le T,$$

which implies

$$\mathbb{E}[\varphi(X(t, s, \eta))|\mathscr{F}_s] = \sum_{k=1}^{n} \mathbb{E}[\varphi(X(t, s, x_k)) \mathbb{1}_{A_k} |\mathscr{F}_s].$$

But, since $\mathbb{1}_{A_k}$ is $\mathscr{F}_s$-measurable and $\varphi(X(t, s, x_k))$ is independent of $\mathscr{F}_s$, we have by Propositions B.4 and B.6 that

$$\begin{aligned}\mathbb{E}[\varphi(X(t, s, x_k)) \mathbb{1}_{A_k} |\mathscr{F}_s] &= \mathbb{1}_{A_k} \mathbb{E}[\varphi(X(t, s, x_k))|\mathscr{F}_s] \\ &= \mathbb{E}[\varphi(X(t, s, x_k))] \mathbb{1}_{A_k} \\ &= P_{s,t}\varphi(x_k) \mathbb{1}_{A_k}.\end{aligned}$$

In conclusion

$$\mathbb{E}[\varphi(X(t, s, \eta))|\mathscr{F}_s] = \sum_{k=1}^{n} P_{s,t}\varphi(x_k) \mathbb{1}_{A_k} = P_{s,t}\varphi(\eta). \qquad \square$$

Corollary 9.2. *Let* $0 \le s < r < t \le T$ *and let* $\varphi \in B_b(\mathbb{R}^d)$. *Then we have*

$$P_{s,t}\varphi(x) = \mathbb{E}[P_{r,t}\varphi(X(r, s, x))]. \tag{9.9}$$

Moreover,

$$P_{s,t}\varphi = P_{s,r} P_{r,t}\varphi. \tag{9.10}$$

Proof. In fact by (9.7) we have

$$\begin{aligned}\mathbb{E}[P_{r,t}\varphi(X(r,s,x))] &= \mathbb{E}[\varphi(X(t,r,X(r,s,x))] = \mathbb{E}[\varphi(X(t,s,x))]\\ &= P_{s,t}\varphi(x),\end{aligned}$$

so that (9.9) follows. Finally, since

$$\mathbb{E}[P_{r,t}\varphi(X(r,s,x))] = P_{s,r}[P_{r,t}\varphi(x)],$$

equation (9.10) follows as well. □

Theorem 9.3 (Markov property). *Let* $0 \leq s < r < t \leq T$ *and let* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$. *Then for all* $\varphi \in B_b(\mathbb{R}^d)$ *we have*

$$\mathbb{E}[\varphi(X(t,s,\eta))|\mathscr{F}_r] = P_{r,t}\varphi(X(r,s,\eta)). \tag{9.11}$$

Proof. Set $\zeta = X(r,s,\eta)$. Then by Lemma 9.1 we have, using (8.10),

$$\begin{aligned}\mathbb{E}[\varphi(X(t,s,\eta))|\mathscr{F}_r] &= \mathbb{E}[\varphi(X(t,r,X(r,s,\eta))|\mathscr{F}_s]\\ &= \mathbb{E}[\varphi(X(t,r,\zeta)|\mathscr{F}_r] = P_{t,r}\varphi(\zeta)\end{aligned}$$

and the conclusion follows. □

Proceeding as in the proof of Proposition 5.15 we can show the *strong Markov property*.

Proposition 9.4. *Let* τ *be a stopping time with respect to the filtration* $(\mathscr{F}_t)$. *Let* $0 < s < \tau < t \leq T$ *and* $\eta \in L^2(\Omega, \mathscr{F}_s, \mathbb{P}; \mathbb{R}^d)$. *Then for all* $\varphi \in C_b(\mathbb{R}^d)$ *we have*

$$\mathbb{E}[\varphi(X(t,s,\eta))|\mathscr{F}_\tau] = P_{\tau,t}\varphi(X(\tau,s,\eta)).$$

9.2. The deterministic case

We consider here the problem

$$\begin{cases} X'(t) = b(t, X(t)), \quad t \in [s,T],\\ X(s) = x \in \mathbb{R}^d, \end{cases} \tag{9.12}$$

where $s \in [0,T)$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ fulfills Hypothesis 8.1 (with $\sigma = 0$). We assume moreover that b possesses a partial derivative b_x which is continuous and bounded on $[0,T] \times \mathbb{R}^d$.

As well known, under these assumptions problem (9.12) has a unique solution $X(\cdot) = X(\cdot, s, x) \in C^1([s,T];\mathbb{R}^d)$. Moreover by (8.52) we have

$$X_s(t,s,x) + X_x(t,s,x)\cdot b(s,x) = 0, \quad s,t,u \in [0,T],\ x \in \mathbb{R}^d. \tag{9.13}$$

In this case the transition evolution operator $P_{s,t}$ reduces to

$$P_{s,t}\varphi(x) = \varphi(X(t,s,x)), \quad x \in \mathbb{R}^d, \quad s,t \in [0,T],\ \varphi \in B_b(\mathbb{R}^d). \tag{9.14}$$

From the identity

$$X(t,s,x) = X(t,r,X(r,s,x)), \quad s,t,r \in [0,T],\ x \in \mathbb{R}^d, \tag{9.15}$$

it follows that

$$P_{s,t} = P_{s,r}P_{r,t}, \quad s,t,r \in [0,T]. \tag{9.16}$$

Proposition 9.5. *For any $\varphi \in C_b^1(\mathbb{R}^d)$ we have*

$$\frac{d}{dt}\, P_{s,t}\varphi = P_{s,t}\ \mathscr{L}(t)\varphi, \quad t,s \in [0,T] \tag{9.17}$$

and

$$\frac{d}{ds}\, P_{s,t}\varphi = -\mathscr{L}(s)P_{s,t}\varphi, \quad t,s \in [0,T], \tag{9.18}$$

where

$$\mathscr{L}(t)\varphi(x) = \langle b(t,x), \varphi_x(x)\rangle, \quad \varphi \in C_b^1(\mathbb{R}^d), \quad x \in \mathbb{R}^d,\ t \in [0,T]. \tag{9.19}$$

Proof. Write

$$\frac{d}{dt}\, P_{s,t}\varphi(x) = \frac{d}{dt}\,\varphi(X(t,s,x)) = \langle b(t,X(t,s,x)), \varphi_x(X(t,s,x))\rangle.$$

Since

$$P_{s,t}\mathscr{L}(t)\varphi(x) = \langle b(t,X(t,s,x)), \varphi_x(X(t,s,x))\rangle,$$

(9.17) follows.

Let us prove (9.18). Taking into account (9.13) we have

$$\begin{aligned}\frac{d}{ds}\, P_{s,t}\varphi(x) &= \frac{d}{ds}\,\varphi(X(t,s,x))\\ &= -\langle \varphi_x(X(t,s,x)), X_x(t,s,x)\cdot b(s,x)\rangle\end{aligned} \tag{9.20}$$

and

$$\begin{aligned}\mathscr{L}(s)P_{s,t}\varphi(x) &= \mathscr{L}(s)\varphi(X(t,s,x))\\ &= \langle D_x\varphi(X(t,s,x)), b(s,x)\rangle\\ &= \langle \varphi_x(X(t,s,x)), X_x(t,s,x)b(s,x)\rangle\end{aligned} \tag{9.21}$$

The conclusion follows comparing (9.20) with (9.21). $\square$

Let us now consider the following partial differential equation (the *transport equation*)

$$\begin{cases} z_s(s,x) + \langle b(s,x), z_x(s,x)\rangle = 0, & s \in [0,T],\\[2mm] z(T,x) = \varphi(x), \end{cases} \tag{9.22}$$

where $\varphi \in C_b^1(\mathbb{R}^d)$ and $T > 0$.

Theorem 9.6. *Let $\varphi \in C_b^1(\mathbb{R}^d)$. Then problem* (9.22) *has a unique solution z of class C^1 given by*

$$z(s,x) = P_{s,T}\varphi(x) = \varphi(X(T,s,x)), \quad s \in [0,T],\ x \in \mathbb{R}^d. \tag{9.23}$$

Proof. *Existence*. It is enough to notice that z, given by (9.23), is a solution of (9.22) in view of (9.18).

Uniqueness. If z is a C^1 solution of problem (9.22) we have

$$\begin{aligned} \frac{d}{ds}[z(s,X(s,u,x))] &= z_s(s,X(s,u,x)) \\ &\quad + \langle z_x(s,X(s,u,x)), X_t(s,u,x)\rangle \\ &= z_s(s,X(s,u,x)) \\ &\quad + \langle z_x(s,X(s,u,x)), b(s,X(s,u,x))\rangle = 0. \end{aligned}$$

Therefore $z(s,X(s,u,x))$ is constant in s. Setting $s=T$ and $s=u$ we find that $z(T,X(T,u,x)) = z(u,X(u,u,x))$ which implies $\varphi(X(T,u,x)) = z(u,x)$ as required. □

9.2.1. The autonomous case

We assume here that $b(t,x) = b(x)$ and consider the problem

$$\begin{cases} X'(t) = b(X(t)), & t \in \mathbb{R},\\[2mm] X(0) = x \in \mathbb{R}^d, \end{cases} \tag{9.24}$$

whose solution we denote by $X(\cdot,x)$. In this case it is easy to check that we have $P_{s,t} = P_{0,t-s}$ for any $t,s \in \mathbb{R}$.

Define

$$P_t\varphi(x) = \varphi(X(t,x)), \quad \varphi \in C_b(\mathbb{R}^d),\ t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{9.25}$$

so that by (9.16) the *semigroup* law

$$P_{t+s} = P_t P_s, \quad t, s \geq 0, \tag{9.26}$$

follows. P_t is called the *transition semigroup* associated with (9.24). By Proposition 9.5 we deduce

Proposition 9.7. *For any $\varphi \in C_b^1(\mathbb{R}^d)$ we have*

$$\frac{d}{dt} P_t\varphi = P_t\mathscr{L}\varphi = \mathscr{L}P_t\varphi, \quad t \geq 0, \tag{9.27}$$

where

$$\mathscr{L}\varphi(x) = \langle b(x), \varphi_x(x)\rangle, \quad \varphi \in C_b^1(\mathbb{R}^d),\ x \in \mathbb{R}^d. \tag{9.28}$$

Proof. The conclusion follows from the identities

$$\frac{d}{dt} P_t\varphi = \frac{d}{dt} P_{0,t}\varphi = P_{0,t}\mathscr{L}\varphi = P_t\mathscr{L}\varphi$$

and

$$\frac{d}{dt} P_t\varphi = \frac{d}{dt} P_{-t,0}\varphi = \mathscr{L}P_{-t,0} = \mathscr{L}P_t\varphi.$$

□

Finally, by Theorem 9.6 we have

Theorem 9.8. *Assume that $\varphi \in C_b^1(\mathbb{R})$. Then problem*

$$\begin{cases} u_t(t,x) = \langle b(x), u_x(t,x)\rangle, & t \geq 0,\ x \in \mathbb{R}^d \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{9.29}$$

has a unique solution of class C^1 given by

$$u(t,x) = P_t\varphi(x), \quad t \geq 0,\ x \in \mathbb{R}^d. \tag{9.30}$$

9.3. Stochastic case

We are here concerned with the stochastic evolution equation (9.6). Let us consider the Kolmogorov operator (9.4) and prove two basic identities which generalize (9.17) and (9.18) respectively. The first identity is proved by the following proposition.

Proposition 9.9. *Assume that Hypothesis* 8.1 *holds and let* $\varphi \in C_b^2(\mathbb{R}^d)$. *Then* $P_{s,t}\varphi$ *is differentiable with respect to* t *and we have*

$$\frac{d}{dt}\, P_{s,t}\varphi = P_{s,t}\mathscr{L}(t)\varphi, \quad t \geq 0, \tag{9.31}$$

where for all $t \in [0, T]$,

$$\mathscr{L}(t)\varphi = \frac{1}{2}\, \mathrm{Tr}\, [D_x^2\varphi(\sigma\sigma^*)(t, x)] + \langle b(t, x), D_x\varphi\rangle, \quad \varphi \in C_b^2(H). \tag{9.32}$$

Proof. By Itô's formula we have

$$\begin{aligned} d_t\varphi(X(t, s, x)) = {} & (\mathscr{L}(t)\varphi)(X(t, s, x)) \\ & + \langle \varphi_x(X(t, s, x)), \sigma(t, X(t, s, x))dB(t)\rangle. \end{aligned}$$

Integrating with respect to t between s and t and taking expectation, yields

$$\mathbb{E}[\varphi(X(t, s, x))] = \varphi(x) + \int_s^t \mathbb{E}[(\mathscr{L}(r)\varphi)(X(r, s, x))]dr,$$

that is

$$P_{s,t}\varphi(x) = \varphi(x) + \int_s^t P_{s,t}(\mathscr{L}(r)\varphi)(x)dr,$$

which coincides with (9.31). □

The second basic identity is proved by the following proposition.

Proposition 9.10. *Assume that Hypotheses* 8.1 *and* 8.18 *hold and let* $\varphi \in C_b^2(\mathbb{R}^d)$. *Then* $P_{s,t}\varphi$ *is differentiable with respect to* s *and we have*

$$\frac{d}{ds}\, P_{s,t}\varphi = -\mathscr{L}(s)P_{s,t}\varphi, \quad t \geq 0. \tag{9.33}$$

Proof. Taking expectation in the backward Itô formula (8.62), we find

$$P_{s,t}\varphi(x) - \varphi(x) = \int_s^t \mathscr{L}(r)P_{s,r}\varphi(x)dr,$$

which yields (9.33). □

9.3.1. Parabolic equations

We consider here the parabolic equation

$$\begin{cases} z_s(s,x) + (\mathscr{L}(s)z(s,\cdot))(x) = 0, \quad 0 \le s \le T, \\ \\ z(T,x) = \varphi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{9.34}$$

We say that a function $z\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a solution to (9.34) if z is continuous and bounded together with its partial derivatives $z_t,\ z_x,\ z_{xx}$ and if it fulfills (9.34).

Theorem 9.11. *Assume that Hypotheses* 8.1 *and* 8.18 *hold and let* $\varphi \in C_b^2(\mathbb{R}^d)$. *Then there exists a unique solution* z *of problem* (9.34), *given by*

$$z(s,x) = \mathbb{E}[\varphi(X(T,s,x))], \quad 0 \le s \le T. \tag{9.35}$$

Proof. Existence. By (9.33) it follows that the function

$$z(s,x) = P_{s,T}\varphi(x), \quad s \in [0,T],\ x \in \mathbb{R}^d,$$

fulfills (9.34).

Uniqueness. Let z be a solution to (9.34), and let $0 \le u \le s \le T$. Let us compute the Itô differential (with respect to s) of $z(s, X(s,u,x))$. We have

$$\begin{aligned} d_s z(s,X(s,u,x)) &= z_s(s,X(s,u,x))ds + (L(s)z(s,X(s,u,\cdot))(x)ds \\ &\quad + \langle z_x(s,X(s,u,x)), \sigma(s,X(s,u,x))dB(s)\rangle \\ &= \langle z_x(s,X(s,u,x)), \sigma(s,X(s,u,x))dB(s)\rangle. \end{aligned}$$

since z fulfills (9.34). Integrating in s between u and T yields

$$z(T,X(T,u,x)) - z(u,X(u,u,x)) = \varphi(X(t,u,x)) - z(u,x)$$

$$= \int_u^t z_x(s,X(s,u,x))\sigma(s,X(s,u,x))dB(s).$$

Now, taking expectation we find $z(u,x) = \mathbb{E}[\varphi(X(t,u,x))]$. □

Remark 9.12. The reader can check that the uniqueness part of the proof of Theorem 9.11 only requires that b and σ are continuous.

Remark 9.13. Notice that the proof of Theorem 9.11 does not require any knowledge of PDEs theories. In particular, the parabolic equation (9.35) can be degenerate.

Corollary 9.14. *Let us consider the problem*

$$\begin{cases} dY(t) = b(t, Y(t))dt + \sigma(t, Y(t))dB_1(t), \quad t \geq s, \\ \\ Y(s) = x \in \mathbb{R}^d, \end{cases} \tag{9.36}$$

where b and σ fulfill Hypotheses 8.1 and 8.18, whereas $B_1(t)$ is another Brownian motion. Let $Y(\cdot, s, x)$ be the solution of (9.36). *Then the laws of $X(t, s, x)$ and $Y(t, s, x)$ coincide for all $x \in \mathbb{R}^d$, $s \leq t$.*

Proof. Let $\varphi \in C_b^2(\mathbb{R}^d)$, $t > 0$ and set

$$u(s, x) = \mathbb{E}[\varphi(X(t, s, x))], \quad v(s, x) = \mathbb{E}[\varphi(Y(t, s, x))].$$

Then by Theorem 9.11 it follows that $u = v$. This implies the conclusion by the arbitrariness of φ. □

9.3.2. Autonomous case

We assume here that b and σ are independent of t

$$b(t, x) = b(x), \quad \sigma(t, x) = \sigma(x), \quad x \in \mathbb{R}^d.$$

Then we set $\mathscr{L}(s) =: \mathscr{L}$ where

$$\mathscr{L}\varphi(x) = \frac{1}{2} \operatorname{Tr} [\varphi_{xx}(x)\sigma(x)\sigma^*(x)] + \langle b(x), \varphi_x(x)\rangle, \quad \varphi \in C_b^2(\mathbb{R}^d).$$

Proposition 9.15. *Let $X(t, s, x)$ be the solution of the stochastic evolution equation*

$$\begin{cases} dX(t) = b(X(t))dt + \sigma(X(t))dB(t) \\ \\ X(s) = x \in \mathbb{R}^d. \end{cases} \tag{9.37}$$

Then for any $a > 0$ the laws of $X(t, s, x)$ and $X(t+a, s+a, x)$ coincide.

Proof. Setting $Y(t) = X(t + a, s + a, x)$, we have

$$X(t + a, s + a, x) = x + \int_{s+a}^{t+a} b(X(r, s + a, x))dr + \int_{s+a}^{t+a} \sigma(X(r, s + a, x))dB(r).$$

By the change of variables $r - a = \rho$ we obtain

$$Y(t) = x + \int_s^t b(Y(\rho))d\rho + \int_s^t \sigma(Y(\rho))d[B(\rho + a) - B(a)].$$

Setting $B_1(t) = B(t + a) - B(a)$ we see that $Y(t)$ fulfills equation (9.37) but with the Brownian motion $B_1(t)$ replacing $B(t)$. Now the conclusion follows from Corollary 9.14. □

By Proposition 9.15 it follows that, setting

$$P_t = P_{0,t}, \quad t \geq 0,$$

we have

$$P_{t+s} = P_t P_s, \quad t, s \geq 0, \quad P_0 = 1. \tag{9.38}$$

Moreover, setting

$$v(s, x) = u(t, t - s, x), \quad t \geq 0, \ s \in [0, t], \ x \in \mathbb{R}^d,$$

problem (9.34) becomes

$$\begin{cases} v_s(s, x) = \mathscr{L} v(s, x), & s \in [0, t], \ x \in \mathbb{R}^d, \\ v(0, x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{9.39}$$

We define the *transition semigroup* $P_t, \ t \geq 0$, setting [(2)]

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t, x))], \quad \forall\, \varphi \in B_b(\mathbb{R}^d). \tag{9.40}$$

In view of (9.38), $P_t, \ t \geq 0$, is a semigroup of linear bounded operators on $B_b(\mathbb{R}^d)$ (but is not strongly continuous).

Now by Theorem 9.11 we find the result

Theorem 9.16. *Assume that Hypotheses* 8.1 *and* 8.18 *hold with b and σ independent of t. Then for any $\varphi \in C_b^2(\mathbb{R})$, problem* (9.39) *has a unique solution given by*

$$v(s, x) = P_{t-s,t}\ \varphi(x) = P_t\ \varphi(x), \quad t \geq 0, \ s \in [0, t], \ x \in \mathbb{R}^d. \tag{9.41}$$

(2) We recall that $B_b(\mathbb{R}^d)$ is the Banach space of all bounded and Borel real functions on $\mathbb{R}^d$ endowed with the sup norm $\|\varphi\|_0 = \sup_{x \in \mathbb{R}^d} |\varphi(x)|$.

9.4. Examples

Example 9.17. Consider the parabolic equation in $\mathbb{R}^d$,

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[Qu_{xx}(t,x)] + \langle Ax, u_x(t,x)\rangle, \\ \\ u(0,x) = \varphi(x), \end{cases} \tag{9.42}$$

where $A, Q \in L(\mathbb{R}^d)$, Q is symmetric and $\langle Qx, x\rangle \geq 0$ for all $x \in \mathbb{R}^d$.

The corresponding stochastic differential equation is

$$\begin{cases} dX(t) = AX(t)dt + \sqrt{Q}\,dB(t), \\ \\ X(0) = x, \end{cases} \tag{9.43}$$

where B is a d-dimensional Brownian motion in some probability space $(\Omega, \mathscr{F}, \mathbb{P})$. The solution of (9.43) is given by the variation of constants formula (recall Example 8.8)

$$X(t,x) = e^{tA}x + \int_0^t e^{(t-s)A}\sqrt{Q}\,dB(s), \quad t \geq 0. \tag{9.44}$$

Therefore the law of $X(t,x)$ is given by (see Theorem 4.29)

$$X(t,x)_{\#}\mathbb{P} = N_{e^{tA}x, Q_t}, \tag{9.45}$$

where

$$Q_t = \int_0^t e^{sA}Qe^{sA^*}ds, \quad t \geq 0, \tag{9.46}$$

and A^* is the adjoint of A. Consequently, the transition semigroup P_t looks like

$$P_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)N_{e^{tA}x, Q_t}(dy) \tag{9.47}$$

and so, if $\varphi \in C_b^2(\mathbb{R}^d)$, the solution of (9.42) is given by

$$u(t,x) = P_t\varphi(x), \quad t \geq 0,\ x \in H.$$

If, in particular, $\det Q_t > 0$ we have

$$u(t,x) = (2\pi)^{-d/2}[\det Q_t]^{-1/2}\int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Q_t^{-1}(y-e^{tA}x),(y-e^{tA}x)\rangle}\varphi(y)dy. \tag{9.48}$$

Notice that there are interesting situations where $\det Q = 0$ whereas $\det Q_t > 0$ for all $t > 0$. In these cases formula (9.48) hold.

Exercise 9.18. Consider the Kolmogorov operator in $\mathbb{R}^2$

$$\mathscr{L}\varphi = \frac{1}{2}\,\varphi_{x_1,x_1} + x_2\varphi_{x_1}, \quad x = (x_1, x_2).$$

Show that $\det Q_t > 0$ for all $t > 0$.

Example 9.19. Consider the parabolic equation in $\mathbb{R}$

$$\begin{cases} u_t(t,x) = \frac{1}{2}\,qx^2 u_{xx}(t,x) + axu_x(t,x)\rangle \\ \\ u(0,x) = \varphi(x), \end{cases} \tag{9.49}$$

where $q \geq 0$ and $a \in \mathbb{R}$. The corresponding stochastic differential equation is

$$\begin{cases} dX(t) = aX(t)dt + \sqrt{q}\;X(t)dB(t), \\ \\ X(0) = x, \end{cases} \tag{9.50}$$

where B is a real Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

The solution of (9.50) is given by

$$X(t,x) = e^{(a-q/2)t+\sqrt{q}\;B(t)}x. \tag{9.51}$$

Therefore

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))] = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2t}}\;\varphi(e^{(a-q/2)t+\sqrt{q}\;y}x)dy. \tag{9.52}$$

9.5. Smoothing properties of $P_{s,t}\varphi$

Here we assume, besides Hypothesis 8.1 that

Hypothesis 9.20. For all $x \in \mathbb{R}^d$ $\sigma(x)$ is invertible and there exists $K > 0$ such that $\|(\sigma(x))^{-1}\| \leq K$ for all $x \in \mathbb{R}^d$.

We are going to show that $P_{s,t}\varphi$ is smooth for any $\varphi \in C_b(\mathbb{R}^d)$ and any $t \in (s, T]$ (Bismut–Elworthy-Li formula).[(3)]

Given $\varphi \in C_b(\mathbb{R}^d)$, let us first estimate the difference

$$\mathbb{E}[\varphi(X(t,s,x))] - \varphi(X(t,s,x)) = P_{s,t}\varphi(x) - \varphi(X(t,s,x)).$$

(3) In fact one can show smoothing for any $\varphi \in B_b(\mathbb{R}^d)$, see Theorem 12.42 below.

Lemma 9.21. *Assume that Hypothesis* 8.1 *holds and let* $0 \le s \le t \le T$. *Then for any* $\varphi \in C_b^1(\mathbb{R}^d)$ *we have*

$$P_{s,t}\varphi(x) = \varphi(X(t,s,x)) - \int_s^t \langle (D_x P_{r,t}\varphi)(X(r,s,x)), \sigma(r, X(r,s,x))dB(r)\rangle. \tag{9.53}$$

Proof. Fix $t \ge 0$ and consider the solution $u(s,x) = P_{s,t}\varphi(x)$ to the problem

$$\begin{cases} u_s(s,x) + \mathscr{L}(s)u(s,x) = 0, & 0 \le s < t, \\ u(t,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases}$$

By the Itô formula we have

$$d_r u(r, X(r,s,x)) = \langle u_x(r, X(r,s,x)), \sigma(r, X(r,s,x))dB(r)\rangle.$$

Integrating with respect to r in $[s,t]$, yields

$$\varphi(X(t,s,x)) - u(s,x) = \int_s^t \langle u_x(r, X(r,s,x)), \sigma(r, X(r,s,x))dB(r)\rangle.$$

which coincides with (9.53). □

Now we are ready to prove the *Bismut–Elworthy–Li formula*.

Proposition 9.22. *Assume that Hypotheses* 8.1 *and* 9.20 *hold. Let* $\varphi \in C_b(\mathbb{R}^d)$. *Then for all* $0 \le s < t \le T$, $P_{s,t}\varphi$ *is differentiable in all directions* $h \in \mathbb{R}^d$ *and we have*

$$\begin{aligned}&\left\langle D_x P_{s,t}\varphi(x), h\right\rangle \\ &= \frac{1}{t-s}\mathbb{E}\left[\varphi(X(t,s,x)\int_s^t \langle \sigma^{-1}(r,X(r,s,x)) \cdot X_x(r,s,x)h, dB(r)\rangle\right].\end{aligned} \tag{9.54}$$

Proof. Step 1. We assume that $\varphi \in C_b^1(\mathbb{R}^d)$.

Multiplying both sides of (9.53) by

$$\int_r^t \langle \sigma^{-1}(r, X(t,r,x)) \cdot X_x(t,r,x)h, dB(r)\rangle$$

and taking expectation yields

$$\begin{aligned}
&\mathbb{E}\left[\varphi(X(t,s,x))\int_s^t \langle\sigma^{-1}(r,X(r,s,x))\cdot X_x(r,s,x)h, dB(r)\rangle\right]\\
&= \mathbb{E}\int_s^t \big\langle D_x(P_{r,t}\varphi)(X(r,s,x)), X_x(r,s,x)h\big\rangle dr\\
&= \mathbb{E}\int_s^t \big\langle D_x P_{r,t}\varphi(X(r,s,x)), h\big\rangle dr\\
&= \left\langle D_x\int_s^t \mathbb{E}[P_{r,t}\varphi(X(r,s,x))]ds, h\right\rangle.
\end{aligned} \tag{9.55}$$

Now by the Markov property (9.9) we deduce

$$\mathbb{E}[P_{r,t}\varphi(X(r,s,x))] = P_{s,r}P_{r,t}\varphi(x) = P_{s,t}\varphi(x),$$

so that

$$\begin{aligned}
\mathbb{E}\left[\varphi(X(t,s,x))\int_s^t \langle\sigma^{-1}(r,X(r,s,x))\cdot X_x(r,s,x)h, dB(r)\rangle\right]&\\
= (t-s)\big\langle D_x P_{s,t}\varphi(x), h\big\rangle&
\end{aligned}$$

which proves (9.54) when $\varphi \in C_b^1(\mathbb{R}^d)$.

Step 2. Conclusion.

Since $C_b^1(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, given $\varphi \in C_b(\mathbb{R}^d)$ there exists a sequence $(\varphi_n) \subset C_b^2(\mathbb{R}^d)$ convergent to φ in $C_b(\mathbb{R}^d)$. By Step 1 we have for all $h \in \mathbb{R}^d$

$$\begin{aligned}
&\big\langle D_x P_{s,t}\varphi_n(x), h\big\rangle\\
&= \frac{1}{t-s}\,\mathbb{E}\left[\varphi_n(X(t,s,x)\int_s^t \langle\sigma^{-1}(r,X(r,s,x))\cdot X_x(r,s,x)h, dB(r)\rangle\right].
\end{aligned}$$

It follows that

$$\begin{cases}
\lim_{n\to\infty} P_{s,t}\varphi_n(x) = P_{s,t}\varphi(x),\\[2ex]
\lim_{n\to\infty} \big\langle D_x P_{s,t}\varphi_n(x), h\big\rangle\\[2ex]
= \dfrac{1}{t-s}\,\mathbb{E}\left[\varphi(X(t,s,x)\displaystyle\int_s^t \langle\sigma^{-1}(r,X(r,s,x))\cdot X_x(r,s,x)h, dB(r)\rangle\right],
\end{cases}$$

convergences being uniform in $\mathbb{R}^d$. So, the conclusion follows by a well known elementary argument. □

Corollary 9.23. *Under the assumptions of Proposition* 9.22 *there exists a positive constant* K_1 *such that for all* $0 \le s < t \le T$ *we have*

$$\left|D_x P_{s,t}\varphi(x)\right| \le K_1 \frac{\|\varphi\|_0}{\sqrt{t-s}}, \quad \forall\, x \in \mathbb{R}^d. \tag{9.56}$$

Proof. By (9.54), using the Hölder inequality, we obtain

$$\begin{aligned}\left|\left\langle D_x P_{s,t}\varphi(x), h\right\rangle\right|^2 &\le \frac{\|\varphi\|_0^2}{(t-s)^2}\mathbb{E}\int_s^t |\sigma^{-1}(r, X(r,s,x)) \cdot X_x(r,s,x)h|^2 dr\\ &\le \frac{\|\varphi\|_0^2}{(t-s)^2}\, K^2\, \mathbb{E}\int_s^t |\eta^h(r,s,x)|^2 dr,\end{aligned}$$

where $\eta^h(r,s,x) = X_x(r,s,x)$. Now the conclusion follows easily from (8.39) and the arbitrariness of h. □

We now consider the second derivative of $P_{s,t}\varphi$ and prove

Proposition 9.24. *Assume that Hypotheses* 8.1, 8.18 *and* 9.20 *hold. Let* $\varphi \in C_b(\mathbb{R}^d)$. *Then for all* $0 \le s < t \le T$, $P_{s,t}\varphi$ *is twice differentiable in all directions* $h \in \mathbb{R}^d$ *and we have*

$$\begin{aligned}&\left\langle D_x^2 P_{s,t}\varphi(x)\cdot h, h\right\rangle\\ &= \frac{2}{t-s}\mathbb{E}\left[\langle D_x\psi(X(t,s,x),h\rangle \int_u^t \langle \sigma^{-1}(r,X(r,s,x)) \cdot X_x(r,s,x)h,\, dB(r)\rangle\right]\\ &+ \frac{4}{(t-s)^2}\,\mathbb{E}\Big[\psi(X(t,s,x) \int_u^t \Big\langle D_x\sigma^{-1}(r, X(r,s,x)) \cdot X_x(r,s,x)h,\\ &\qquad\qquad X_x(r,s,x)h,\, dB(r)\Big\rangle\Big]\\ &+ \frac{4}{(t-s)^2}\mathbb{E}\left[\psi(X(t,s,x) \int_u^t \langle \sigma^{-1}(r,X(r,s,x)) \cdot (X_{xx}(r,s,x)h,h), dB(r)\rangle\right].\end{aligned} \tag{9.57}$$

Proof. Step 1. We assume that $\varphi \in C_b^2(\mathbb{R}^d)$.

Set $u = \frac{t-s}{2}$, and write $P_{s,t}\varphi = P_{s,u}\psi$ where $\psi := P_{u,t}\varphi$. Then by (9.54) we find

$$\left\langle D_x P_{s,t}\varphi(x), h\right\rangle = \left\langle D_x P_{s,u}\psi(x), h\right\rangle$$

$$= \frac{1}{s-u}\,\mathbb{E}\left[\psi(X(u,s,x))\int_s^u \langle\sigma^{-1}(r, X(r,s,x)) \cdot X_x(r,s,x)h, dB(r)\rangle\right].$$

Now we can differentiate again in the direction of h, obtaining

$$\begin{aligned}
&\left\langle D_x^2 P_{s,t}\varphi(x)\cdot h, h\right\rangle \\
&= \frac{1}{u-s}\,\mathbb{E}\left[\langle D_x\psi(X(u,s,x)), h\rangle \int_s^u \langle\sigma^{-1}(r, X(r,s,x))\right. \\
&\qquad\left. \cdot X_x(r,s,x)h, dB(r)\rangle\right] \\
&\quad + \frac{1}{u-s}\,\mathbb{E}\left[\psi(X(u,s,x))\int_s^u \langle D_x\sigma^{-1}(r, X(r,s,x))\right. \\
&\qquad\left. \cdot X_x(r,s,x)h, X_x(r,s,x)h, dB(r)\rangle\right] \\
&\quad + \frac{1}{u-s}\,\mathbb{E}\left[\psi(X(u,s,x))\int_s^u \langle\sigma^{-1}(r, X(r,s,x))\right. \\
&\qquad\left. \cdot(X_{xx}(r,s,x)h, h), dB(r)\rangle\right] \\
&=: J_1 + J_2 + J_3.
\end{aligned} \tag{9.58}$$

As regards J_1 we have, using again (9.54)

$$\begin{aligned}
\left\langle D_x^2 P_{s,t}\varphi(x)\cdot h, h\right\rangle &= \frac{1}{(u-s)(t-u)}\,\mathbb{E}\Big[\mathbb{E}\Big(\varphi(X(t,u,y)) \\
&\times \int_u^t \langle\sigma^{-1}(r, X(r,u,y))\cdot(X_{xx}(r,u,y)h, h), dB(r)\rangle\Big)_{y=X(u,s,x)} \\
&\times \int_u^t \langle\sigma^{-1}(r, X(r,s,x))\cdot X_x(r,s,x)h, dB(r)\rangle\Big].
\end{aligned} \tag{9.59}$$

Then (9.57) follows.

Step 2. Conclusion.

Similar to the proof of Step 2 from Proposition 9.22. □

Exercise 9.25. Prove that there exists $K_2 > 0$ such that for all $0 \le s < t \le T$ we have

$$\left|D_x^2 P_{s,t}\varphi(x)\right| \le K_2\,\frac{\|\varphi\|_0}{t-s}. \tag{9.60}$$

Now we go back to the parabolic problem (9.34).

Definition 9.26. A *classical solution* of problem (9.34) is a continuous function $z : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ such that

(i) z possesses continuous partial derivatives z_s, z_x, z_{xx} on $[0,T)\times\mathbb{R}^d$.

(ii) We have

$$D_s z(s,x) + (\mathscr{L}(s)z(s,\cdot))(x) = 0, \quad \forall\, 0 \le s < T.$$

(iii) $z(T,x) = \varphi(x), \quad \forall\, x \in \mathbb{R}^d$.

Theorem 9.27. *Assume that Hypotheses* 8.1, 8.18 *and* 9.20 *hold. Then for any* $\varphi \in C_b(\mathbb{R}^d)$ *problem* (9.34) *has a unique classical solution given by* $z(s,x) = P_{s,T}\varphi(x)$.

Proof. Step 1. Existence.

Since $C_b^2(\mathbb{R}^d)$ is dense on $C_b(\mathbb{R}^d)$ there exists a sequence $(\varphi_n) \subset C_b^2(\mathbb{R}^d)$ convergent to φ in $C_b(\mathbb{R}^d)$. Let z_n be the solution of the problem

$$\begin{cases} D_s z_n(s,x) + (\mathscr{L}(s)z_n(s,\cdot))(x) = 0, \quad \forall\, 0 \le s < T.\\[2mm] z_n(t,x) = \varphi_n(x), \quad x \in \mathbb{R}, \end{cases} \tag{9.61}$$

which has a unique solution $z_n(s,x) = P_{s,T}\varphi_n(x)$ thanks to Theorem 9.11.

Now

$$\lim_{n\to\infty} z_n(s,x) = P_{s,T}\varphi(x) =: z(s,x), \text{ uniformly in } t \text{ and } x \text{ in } [s,T]\times\mathbb{R}^d. \tag{9.62}$$

Moreover, for all $m, n \in \mathbb{N}$ we have by (9.56)

$$|D_x z_m(s,x) - D_x z_n(s,x)| \le K_1\,\frac{\|\varphi_m - \varphi_n\|_0}{\sqrt{t-s}}, \quad \forall\, x \in \mathbb{R}^d,\ s \in [0,T). \tag{9.63}$$

and by (9.60)

$$|D_x^2 z_m(s,x) - D_x^2 z_n(s,x)| \le K_2 \, \frac{\|\varphi_m - \varphi_n\|_0}{t-s}, \quad \forall\, x \in \mathbb{R}^d,\ s \in [0,T). \tag{9.64}$$

Consequently by (9.61) there exists $K_3 > 0$ such that

$$|D_s z_m(s,x) - D_s z_n(s,x)| \le K_3 \, \frac{\|\varphi_m - \varphi_n\|_0}{t-s}, \quad \forall\, x \in \mathbb{R}^d,\ s \in [0,T). \tag{9.65}$$

By (9.63)-(9.65) it follows by classical arguments that sequence $(z_m(s,x))$ is uniformly convergent to a continuous function $z(s,x)$ in $[s,T] \times \mathbb{R}^d$ as $m \to \infty$. Moreover, sequences $(D_x z_m(s,x))$ and $(D_x^2 z_m(s,x))$ are uniformly convergent to $D_x z(t,x)$ and $D_x^2 z(t,x)$ respectively in any compact subset of $[s,T) \times \mathbb{R}^d$. As a consequence, $(D_t z_m(s,x))$ is uniformly convergent to $D_t z(t,x)$ in any compact subset of $[s,T) \times \mathbb{R}^d$ and finally, $z(s,x)$ is a classical solution to (9.34) as claimed.

Step 2. Uniqueness.

Let $v(s,x)$ be a classical solution of problem (9.34), $\eta \in (0,T)$, and set $\varphi_\eta(x) = v(T-\eta, x)$. Then $v(s,x)$ is clearly a solution of the following problem

$$\begin{cases} D_s v(s,x) + (\mathscr{L}(s)v(s,\cdot))(x) = 0, & 0 \le s < T-\eta \\ v(T-\eta, x) = \varphi_\eta(x), & x \in \mathbb{R}^d. \end{cases} \tag{9.66}$$

By the uniqueness part of Theorem 9.8 it follows that

$$v(s,x) = P_{s,T-\eta}\varphi_\eta(x), \quad x \in \mathbb{R}^d,\ s \in [0, T-\eta]. \tag{9.67}$$

As $\eta \to 0$, yields

$$\varphi_\eta(x) \to \varphi(x),$$

since v is a classical solution. Finally, letting η tend to 0 in (9.67) we find $v(s,x) = P_{s,t}\varphi(x)$, for every $s \in [0,T)$, $x \in \mathbb{R}^d$. $\square$

Chapter 10
Formulae of Feynman–Kac and Girsanov

We are here concerned with the stochastic differential equation

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), \\ \\ X(s) = x, \quad x \in \mathbb{R}^d, \end{cases} \tag{10.1}$$

under Hypotheses 8.1 and 8.18.

As before B is an r-dimensional Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\mathscr{F}_t)_{t\geq 0}$ denotes its natural filtration.

We recall that the corresponding Kolmogorov equation reads as follows

$$\begin{cases} u_s(s, x) + (\mathscr{L}(s)u(s, \cdot))(x) = 0, \quad 0 \leq s < T, \\ \\ u(T, x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{10.2}$$

where

$$(\mathscr{L}(s)\varphi)(x) = \frac{1}{2} \operatorname{Tr} [a(s, x)\varphi_{xx}(x)] + \langle b(s, x), \varphi_x(x)\rangle, \quad \varphi \in C_b^2(\mathbb{R}^d), \tag{10.3}$$

and $a(s, x) = \sigma(s, x)\sigma^*(s, x)$ for any $x \in \mathbb{R}^d$, $s \in [0, T]$.

In Section 10.1 we shall consider a perturbation of equation (10.2) by a potential term of the form $V(s, x)u(s, x)$ and shall find a representation formula for the solution of the perturbed problem, called the *Feynman–Kac formula*.

In Section 10.2 we shall deal with a more general perturbation of (10.2) also acting on the drift. We find again a representation formula called the *Girsanov formula*.

Finally, in Section 10.3 we define the *Girsanov transform*, based on a suitable change of probability. This is a very useful tool that allow solving (in a weak sense that it will be explained) some stochastic differential equation with only continuous coefficients.

We shall denote as before by $\Sigma(s,T)$ the set of all decompositions of the interval $[s,T]$, $\eta = \{s_0, s_1, \ldots, s_n = T\}$ where $s_0 = s < s_1 < \cdots < s_n = T$ and set

$$|\eta| = \max_{k=1,\ldots,n} (t_k - t_{k-1}).$$

$\Sigma(s,T)$ is endowed with the usual partial ordering.

10.1. The Feynman–Kac formula

We are given a function $V\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ such that

Hypothesis 10.1. V is continuous and bounded in $[0,T] \times \mathbb{R}^d$ together with its first and second partial derivatives with respect to x.

Let us consider a Kolmogorov equation with a *potential* term V

$$\begin{cases} v_s(s,x) + (\mathscr{L}(s)v(s,\cdot))(x) + V(s,x)v(s,x) = 0, & 0 \le s < T, \\ \\ v(T,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{10.4}$$

Theorem 10.2. *Assume that Hypotheses* 8.1, 8.18 *and* 10.1 *hold. Then for any* $\varphi \in C_b^2(\mathbb{R}^d)$ *problem* (10.4) *has a unique solution given by the formula*

$$v(s,x) = \mathbb{E}\left[\varphi(X(T,s,x))e^{\int_s^T V(u,X(u,s,x))du}\right], \quad (s,x) \in [0,T] \times \mathbb{R}^d. \tag{10.5}$$

Identity (10.5) is called the *Feynman–Kac formula*.

Proof. Existence. Let us consider for any $x \in \mathbb{R}^d$ the stochastic process

$$F(s,x) := \varphi(X(T,s,x))e^{\int_s^T V(u,X(u,s,x))du}, \quad s \in [0,T].$$

Then for any $\eta = \{s = s_0 < s_1 < \cdots < s_n = T\} \in \Sigma(s,T)$ write

$$F(s,x) - \varphi(x) = -\sum_{k=1}^n [F(s_k,x) - F(s_{k-1},x)],$$

which is equivalent to,

$$\begin{aligned} F(s,x) - \varphi(x) &= -\sum_{k=1}^n [F(s_k,x) - F(s_k, X(s_k, s_{k-1}, x))] \\ &\quad - \sum_{k=1}^n [F(s_k, X(s_k, s_{k-1}, x)) - F(s_{k-1}, x)]. \end{aligned}$$

Since

$$
\begin{aligned}
&F(s_k, X(s_k, s_{k-1}, x)) - F(s_{k-1}, x) \\
&= \varphi(X(T, s_k, X(s_k, s_{k-1}, x)))e^{\int_{s_k}^T V(u, X(u, s_k, X(s_k, s_{k-1}, x)))du} \\
&\quad -\varphi(X(T, s_{k-1}, x))e^{\int_{s_{k-1}}^T V(u, X(u, s_{k-1}, x))du} \\
&= \varphi(X(T, s_{k-1}, x))e^{\int_{s_k}^T V(u, X(u, s_{k-1}, x))du} \\
&\quad -\varphi(X(T, s_{k-1}, x))e^{\int_{s_{k-1}}^T V(u, X(u, s_{k-1}, x))du}
\end{aligned}
$$

we have

$$
\begin{aligned}
&F(s_k, X(s_k, s_{k-1}, x)) - F(s_{k-1}, x) \\
&= \varphi(X(T, s_{k-1}, x)) \left(e^{\int_{s_k}^T V(u, X(u, s_{k-1}, x))du} - e^{\int_{s_{k-1}}^T V(u, X(u, s_{k-1}, x))du} \right) \\
&= F(s_{k-1}, x) \left(e^{-\int_{s_{k-1}}^{s_k} V(u, X(u, s_{k-1}, x))du} - 1 \right)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
F(s, x) - \varphi(x) \;=\; &-\sum_{k=1}^{n} [F(s_k, x) - F(s_k, X(s_k, s_{k-1}, x))] \\
&-\sum_{k=1}^{n} F(s_{k-1}, x) \left(e^{-\int_{s_{k-1}}^{s_k} V(u, X(u, s_{k-1}, x))du} - 1 \right).
\end{aligned}
$$

Using Taylor's formula we write

$$
\begin{aligned}
&F(s, x) - \varphi(x) = \sum_{k=1}^{n} \langle D_x F(s_k, x), X(s_k, s_{k-1}, x) - x \rangle \\
&+\frac{1}{2} \sum_{k=1}^{n} \langle D_x^2 F(s_k, x)(X(s_k, s_{k-1}, x) - x), X(s_k, s_{k-1}, x) - x \rangle \\
&+\sum_{k=1}^{n} F(s_{k-1}, x) \left(1 - e^{-\int_{s_{k-1}}^{s_k} V(u, X(u, s_{k-1}, x))du} \right) + o(|\eta|).
\end{aligned}
$$

By proceeding as in the proof of the backward Itô's formula (see Section 8.4.3) we can show that $\lim_{|\eta|\to 0} o(|\eta|) = 0$. Now, by approximating $X(s_k, s_{k-1}, x) - x$ with

$$b(s_k, x)(s_k - s_{k-1}) + \sigma(s_k, x)(B(s_k) - B(s_{k-1})),$$

we find that

$$F(s,x) - \varphi(x) = \sum_{k=1}^{n} \langle D_x F(s_k, x), b(s_k, x)\rangle (s_k - s_{k-1})$$

$$+\sum_{k=1}^{n} \langle D_x F(s_k, x), \sigma(s_k, x)(B(s_k) - B(s_{k-1}))\rangle$$

$$+\frac{1}{2}\sum_{k=1}^{n} \langle D_x^2 F(s_k,x)\sigma(s_k,x)(B(s_k)-B(s_{k-1})), \sigma(s_k,x)(B(s_k)-B(s_{k-1}))\rangle$$

$$+\sum_{k=1}^{n} F(s_{k-1}, x)V(s_{k-1}, x)(s_k - s_{k-1}) + o_1(|\eta|),$$

where $\lim_{|\eta|\to 0} o_1(|\eta|) = 0$. Letting $|\eta| \to 0$ we obtain the identity

$$F(s,x)-\varphi(x) = \int_s^T \Big[\langle b(u,x), D_x F(u,x)\rangle + \frac{1}{2}\,\mathrm{Tr}\,[a(u,x)D_x^2 F(u,x)\Big]$$

$$+V(u,x)F(u,x)\Big]du + \oint_s^T \langle D_x F(u,x), \sigma(u,x)dB(u)\rangle.$$

Taking expectation of both sides, setting

$$v(s,x) = \mathbb{E}[F(s,x)], \quad s \in [0,T],\ x \in \mathbb{R}^d,$$

and taking into account the identities,

$$\mathbb{E}[D_x F(u,x)] = D_x \mathbb{E}[F(u,x)], \quad \mathbb{E}[D_x^2 F(u,x)] = D_x^2\, \mathbb{E}[F(u,x)],$$

we find that

$$v(s,x) - \varphi(x)$$

$$= \int_s^T \Big[\frac{1}{2}\mathrm{Tr}\,[a(u,x)v_{xx}(u,x)] + \langle b(u,x), v_x(u,x)\rangle + V(u,x)v(u,x)\Big]du.$$

Therefore v fulfills (10.4).

Uniqueness. Let v be a solution to (10.4). By using Itô's formula we find that

$$d_u v(u, X(u,s,x)) e^{\int_s^u V(\rho, X(\rho,s,x))d\rho}$$

$$= e^{\int_s^u V(\rho, X(\rho,s,x))d\rho} \langle v_x(u, X(u,s,x)), \sigma(u, X(u,s,x)) dB(u)\rangle.$$

Integrating with respect to u between s and T and taking expectation, we see that v is given by formula (10.5). □

10.1.1. Autonomous case

Assume that b, σ and V are independent of t and consider the problem

$$\begin{cases} v_t(t,x) = \dfrac{1}{2}\,\mathrm{Tr}\,[a(x)v_{xx}(t,x)] + \langle v_x(s,x), b(x)\rangle \\ \qquad\qquad + V(x)v(t,x), \quad t \geq 0, \\ v(0,x) = \varphi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{10.6}$$

Then by Theorem 10.2 we obtain the following result,

Theorem 10.3. *Assume that Hypotheses* 8.1, 8.18 *and* 10.1 *hold and that* b, σ *and* V *are independent of* t. *Then problem* (10.6) *has a unique solution given by the formula*

$$v(t,x) = \mathbb{E}\left[\varphi(X(t,x)) e^{\int_0^t V(X(s,x))ds}\right]. \tag{10.7}$$

Exercise 10.4. Consider the following heat equation with a potential V

$$\begin{cases} v_t(t,x) = \dfrac{1}{2}\,\Delta v(t,x) + V(x)v(t,x), \quad t \geq 0, \\ v(0,x) = \varphi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{10.8}$$

Show that

$$v(t,x) = \mathbb{E}\left[\varphi(x + B(t)) e^{\int_0^t V(x+B(s))ds}\right], \quad (t,x) \in [0,T] \times \mathbb{R}^d. \tag{10.9}$$

10.1.2. Differentiability of the Feynman–Kac transition evolution operator

Assume Hypotheses 8.1, 8.18 and 10.1. Then for any $\varphi \in B_b(H)$ we define

$$P^V_{s,t}\varphi(x) = \mathbb{E}\left[\varphi(X(t,s,x))e^{\int_s^t V(r,X(r,s,x))dr}\right], \quad 0 \le s < t \le T, \; x \in \mathbb{R}^d. \tag{10.10}$$

Let us prove a generalization of Lemma 9.1 and Corollary 9.2.

Proposition 10.5. *Let $\varphi \in C_b(\mathbb{R}^d)$, $0 \le r < t \le T$ and $\eta \in L^2(\Omega, \mathscr{F}_r, \mathbb{P})$. Then we have*

$$\mathbb{E}\left[\varphi(X(t,r,\eta))e^{\int_r^t V(u,X(u,r,\eta))du}|\mathscr{F}_r\right] = P^V_{r,t}(\eta). \tag{10.11}$$

Proof. It is enough to take η of the form

$$\eta = \sum_{i=1}^n x_i \mathbb{1}_{A_i},$$

where $A_1, ..., A_n$ are disjoint subsets of $\mathscr{F}_r$. Then we have

$$\varphi(X(t,r,\eta))e^{\int_r^t V(u,X(u,r,\eta))du} = \sum_{i=1}^n \varphi(X(t,r,x_i))e^{\int_r^t V(u,X(u,r,x_i))du} \; \mathbb{1}_{A_i}$$

$$= \sum_{i=1}^n \mathbb{E}\left[\varphi(X(t,r,x_i))e^{\int_r^t V(u,X(u,r,x_i))du}\right] e^{\int_r^s V(u,X(u,r,x_i))du} \; \mathbb{1}_{A_i}.$$

Since

$$\mathbb{E}\left[\varphi(X(t,r,x_i))e^{\int_r^t V(u,X(u,r,x_i))du} \; \mathbb{1}_{A_i}|\mathscr{F}_r\right] = P^V_{r,t}\varphi(X(t,r,x_i)) \; \mathbb{1}_{A_i},$$

the conclusion follows. □

Corollary 10.6. *Let $\varphi \in C_b(\mathbb{R}^d)$ and $0 \le s < r < t \le T$. Then we have*

$$\mathbb{E}\left[\varphi(X(t,s,x))e^{\int_r^t V(u,X(u,s,x))du}|\mathscr{F}_r\right] = P^V_{r,t}\varphi(X(r,s,x)). \tag{10.12}$$

and

$$P^V_{s,r}P^V_{r,t}\varphi(x) = \mathbb{E}\left[P^V_{r,t}\varphi(X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du}\right] = P^V_{s,t}\varphi(x). \tag{10.13}$$

Proof. Identity (10.12) follows setting in (10.11) $\eta = X(r,s,x)$. Let us prove (10.13). The first identity is clear, to show the second one multiply both sides of (10.12) by $I := e^{\int_s^r V(u,X(u,s,x))du}$. Since I is $\mathscr{F}_r$-measurable we find

$$\mathbb{E}\left[\varphi(X(t,s,x))e^{\int_s^t V(u,X(u,s,x))du}|\mathscr{F}_r\right] = P_{r,t}^V\varphi(X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du}.$$

Finally, taking expectation, yields (10.13). □

We call

$$P_{s,t}^V, \quad 0 \le s \le t \le T,$$

the *Feynman–Kac transition evolution operator*.

Now we assume in addition that the uniform ellipticity Hypothesis 9.20 is fulfilled. We want to prove a kind of Bismut–Elworthy-Li formula for $P_{s,t}^V\varphi$, see [7].

Theorem 10.7. *Assume that Hypotheses* 8.1*,* 8.18*,* 10.1 *and* 9.20 *are fulfilled. For all $\varphi \in C_b(H)$ $P_{s,t}^V\varphi$ is differentiable with respect to x in any direction $h \in \mathbb{R}^d$ and we have*

$$\begin{aligned}\left\langle D_x P_{s,t}^V\varphi(x), h\right\rangle &= \frac{1}{t-s}\,\mathbb{E}\Big[\varphi(X(t,s,x))e^{\int_s^t V(u,X(u,s,x))du} \\ &\qquad \times \int_s^t \langle \sigma(r,X(r,s,x))^{-1}X_x(r,s,x)h, dB(r)\rangle\Big] \\ &+\mathbb{E}\left[P_{s,t}^V\varphi(X(t,s,x))\int_s^t \frac{t-u}{t-s}\langle V_x(u,X(u,s,x)), X_x(u,s,x)h\rangle du\right].\end{aligned} \tag{10.14}$$

Proof. Step 1. We assume that $\varphi \in C_b^2(H)$.

Let $0 \le s < r < t \le T$ and set $v(r,x) = P_{r,t}^V\varphi(x)$. By Itô's formula we have

$$\begin{aligned}&d_r v(r,X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du} \\ &= \langle v_x(r,X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du}, \sigma(r,X(r,s,x))dB(r)\rangle.\end{aligned}$$

Integrating in r between s and t, yields

$$\begin{aligned}&\varphi(X(t,s,x))e^{\int_s^t V(u,X(u,s,x))du} \\ &= v(s,x) + \int_s^t \langle v_x(r,X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du}, \sigma(r,X(r,s,x))dB(r)\rangle.\end{aligned}$$

Now fix $h \in H$, multiply the identity above by

$$\int_s^t \langle \sigma^{-1}(r, X(r,s,x))X_x(r,s,x)h, dB(r)\rangle$$

and take expectation. Setting

$$\begin{aligned} J := \mathbb{E}\Big[&\varphi(X(t,s,x))e^{\int_s^t V(u,X(u,s,x))du} \\ &\times \int_s^t \langle \sigma^{-1}(r, X(r,s,x))X_x(r,s,x)h, dB(r)\rangle\Big], \end{aligned}$$

we obtain

$$\begin{aligned} J &= \mathbb{E}\int_s^t e^{\int_s^r V(u,X(u,s,x))du}\langle v_x(r, X(r,s,x)), X_x(r,s,x)h\rangle dr \\ &= \mathbb{E}\int_s^t e^{\int_s^r V(u,X(u,s,x))du}\left\langle D_x P_{r,t}^V(X(r,s,x)), h\right\rangle dr \\ &= \mathbb{E}\int_s^t \left\langle D_x\left[e^{\int_s^r V(u,X(u,s,x))du} P_{r,t}^V\varphi(X(r,s,x))\right], h\right\rangle dr \\ &-\mathbb{E}\int_s^t P_{r,t}^V\varphi(X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du} \\ &\times \int_s^r \langle V_x(u, X(u,s,x)), X_x(u,s,x)h\rangle du dr \\ &=: J_1 + J_2. \end{aligned} \tag{10.15}$$

Now, thanks to (10.13), we have

$$\begin{aligned} J_1 &= \mathbb{E}\int_s^t \left\langle D_x\left[e^{\int_s^r V(u,X(u,s,x))du} P_{r,t}^V\varphi(X(r,s,x))\right], h\right\rangle dr \\ &= \left\langle D_x\int_s^t \mathbb{E}\left[e^{\int_s^r V(u,X(u,s,x))du} P_{r,t}^V\varphi(X(r,s,x))\right]dr, h\right\rangle \\ &= \left\langle D_x\int_s^t P_{s,t}^V\varphi(x)dr, h\right\rangle = (t-s)\left\langle D_x P_{s,t}^V\varphi(x), h\right\rangle. \end{aligned}$$

Moreover, using (10.12) and Exercise B.9, we get

$$
\begin{aligned}
J_2 &= -\mathbb{E}\int_s^t P^V_{r,t}\varphi(X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du} \\
&\quad\times \int_s^r \langle V_x(u,X(u,s,x))du, X_x(u,s,x)h\rangle dr \\
&= -\mathbb{E}\Big[\int_s^t \mathbb{E}\Big[P^V_{r,t}\varphi(X(r,s,x))e^{\int_s^r V(u,X(u,s,x))du}|\mathscr{F}_r\Big] \\
&\quad\times \int_s^r \langle V_x(u,X(u,s,x))du, X_x(u,s,x)h\rangle dr\Big] \\
&= -\mathbb{E}\Big[\int_s^t P^V_{s,t}\varphi(X(t,s,x))\int_s^r \langle V_x(u,X(u,s,x)), X_x(u,s,x)h\rangle du dr\Big] \\
&= \mathbb{E}\Big[P^V_{s,t}\varphi(X(t,s,x))\int_s^t (t-u)\langle V_x(u,X(u,s,x)), X_x(u,s,x)h\rangle du\Big].
\end{aligned}
$$

The conclusion follows.

Step 2. Conclusion.

Since $C_b^1(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, given $\varphi \in C_b(\mathbb{R}^d)$ there exists a sequence $(\varphi_n) \subset C_b^2(\mathbb{R}^d)$ convergent to φ in $C_b(\mathbb{R}^d)$. So, it is enough to apply step 1 to each φ_n and then to let $n \to \infty$.

□

10.2. The Girsanov formula

In this section we consider a more general perturbation of equation (10.2), namely

$$
\begin{cases}
z_s(s,x) + (\mathscr{L}(s)z(s,\cdot)(x)) + \langle z_x(s,x), f(s,x)\rangle \\
\qquad\qquad + \Big[V(s,x) + \frac{1}{2}\,|\sigma(s,x)^{-1}f(s,x)|^2\Big] z(s,x) = 0, \quad 0 \le s < T, \\
z(T,x) = \varphi(x), \quad x \in \mathbb{R}^d,
\end{cases}
\tag{10.16}
$$

where f is a suitable mapping from $[0,T]\times\mathbb{R}^d$ into $\mathbb{R}^d$ and $\varphi \in C_b^2(\mathbb{R}^d)$.

Let us prove the following result.

Theorem 10.8. *Assume that Hypotheses* 8.1*,* 8.18 *and* 10.1 *hold. Let moreover* $f\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ *be such that*

(i) $f(s,x)$ *belongs the range of* $\sigma(s,x)$ *for all* $s\in[0,T]$ *and all* $x\in\mathbb{R}^d$.

(ii) $\sigma^{-1}f$ *is continuous and bounded in* $[0,T]\times\mathbb{R}^d$ *together with its first and second partial derivatives with respect to* x.

Then problem (10.16) *has a unique solution given by*

$$\begin{aligned}&z(s,x)\\ &=\mathbb{E}\left[\varphi(X(T,s,x))e^{\int_s^T V(u,X(u,s,x))du+\int_s^T \langle\sigma(u,X(u,s,x))^{-1}f(u,X(u,s,x)),dB(u)\rangle}\right].\end{aligned}\tag{10.17}$$

Formula (10.17) is called the *Girsanov formula*.

Proof. Existence. Set

$$F(t,x)=\sigma(t,x)^{-1}f(t,x),\quad (t,x)\in[0,T]\times\mathbb{R}^d$$

and

$$G(s,x)=\varphi(X(T,s,x))e^{\int_s^T V(u,X(u,s,x))du+\int_s^T F(u,X(u,s,x))dB(u)},\qquad (s,x)\in[0,T]\times\mathbb{R}^d.$$

Then for any $\eta=\{s_0=s<s_1<\cdots<s_n=T\}\in\Sigma(s,T)$ write

$$G(s,x)-\varphi(x)=-\sum_{k=1}^n[G(s_k,x)-G(s_{k-1},x)],$$

so that

$$\begin{aligned}G(s,x)-\varphi(x)&=-\sum_{k=1}^n[G(s_k,x)-G(s_k,X(s_k,s_{k-1},x))]\\ &\quad-\sum_{k=1}^n[G(s_k,X(s_k,s_{k-1},x))-G(s_{k-1},x)].\end{aligned}$$

Write now

$$\begin{aligned}G(s,x)-\varphi(x)&=-\sum_{k=1}^n[G(s_k,x)-G(s_k,X(s_k,s_{k-1},x))]\\ &-\sum_{k=1}^n\varphi(X(t,s_{k-1},x))\Big(e^{\int_{s_k}^T V(u,X(u,s_{k-1},x))du+\int_{s_k}^T F(u,X(u,s_{k-1},x))dB(u)}\\ &\qquad -e^{\int_{s_{k-1}}^T V(u,X(u,s_{k-1},x))du+\int_{s_{k-1}}^T F(u,X(u,s_{k-1},x))dB(u)}\Big).\end{aligned}$$

Using Taylor's formula we find that

$$G(s,x)-\varphi(x)=\sum_{k=1}^{n}\langle D_xG(s_k,x),X(s_k,s_{k-1},x)-x\rangle$$

$$+\frac{1}{2}\sum_{k=1}^{n}\langle D_x^2G(s_k,x)(X(s_k,s_{k-1},x)-x),X(s_k,s_{k-1},x)-x\rangle$$

$$+\sum_{k=1}^{n}G(s_{k-1},x)\left(1-e^{-\int_{s_{k-1}}^{s_k}V(u,X(u,s_{k-1},x))du-\int_{s_{k-1}}^{s_k}\langle F(u,X(u,s_{k-1},x)),dB(u)\rangle}\right)$$

$$+o_1(|\eta|),$$

where $\lim_{|\eta|\to 0}o_1(|\eta|)=0$. Next, approximating $X(s_k,s_{k-1},x)-x$ by

$$b(s_k,x)(s_k-s_{k-1})+\sigma(s_k,x)(B(s_k)-B(s_{k-1})),$$

we find that

$$G(s,x)-\varphi(x)=\sum_{k=1}^{6}J_k(\eta)+o(|\eta|),$$

where $\lim_{|\eta|\to 0}o(|\eta|)=0$ and

$$J_1(\eta)=\sum_{k=1}^{n}\langle D_xG(s_k,x),b(s_k,x)\rangle(s_k-s_{k-1}),$$

$$J_2(\eta)=\sum_{k=1}^{n}\langle D_xG(s_k,x),\sigma(s_k,x)(B(s_k)-B(s_{k-1}))\rangle,$$

$$J_3(\eta)=\frac{1}{2}\sum_{k=1}^{n}\langle D_x^2G(s_k,x)\sigma(s_k,x)(B(s_k)-B(s_{k-1})),$$

$$\sigma(s_k,x)(B(s_k)-B(s_{k-1}))\rangle,$$

$$J_4(\eta)=\sum_{k=1}^{n}G(s_{k-1},x)V(s_{k-1},x)(s_k-s_{k-1}),$$

$$J_5(\eta)=\sum_{k=1}^{n}G(s_{k-1},x)\langle F(s_{k-1},x),B(s_k)-B(s_{k-1})\rangle,$$

$$J_6(\eta)=-\frac{1}{2}\sum_{k=1}^{n}G(s_{k-1},x)[\langle F(s_{k-1},x),B(s_k)-B(s_{k-1})\rangle]^2.$$

It follows that

$$\lim_{|\eta|\to 0} J_1(\eta) = \int_s^T \langle D_x G(r,x), b(r,x)\rangle dr,$$

$$\lim_{|\eta|\to 0} J_2(\eta) = \oint_s^T \langle D_x G(r,x), \sigma(r,x) dB(r)\rangle,$$

$$\lim_{|\eta|\to 0} J_3(\eta) = \frac{1}{2} \int_s^T \mathrm{Tr}\,[a(r,x)D_x^2 G(r,x)]dr,$$

$$\lim_{|\eta|\to 0} J_4(\eta) = \int_s^T G(r,x)V(\rho,x)dr,$$

$$\lim_{|\eta|\to 0} J_6(\eta) = -\frac{1}{2} \int_s^T G(r,x)|F(r,x)|^2 dr.$$

We notice that one cannot pass immediately to the limit as $|\eta| \to 0$ in $J_5(\eta)$; in fact the term $G(s_{k-1},x)$ contains the "story" of the Brownian motion between s_{k-1} and T and so this integral sum does not correspond neither to a backward nor to a forward Itô integral.

For this reason, we transform $J_5(\eta)$ as follows

$$\begin{aligned} J_5(\eta) &= \sum_{k=1}^n G(s_k,x)\langle F(s_k,x), B(s_k)-B(s_{k-1})\rangle \\ &\quad + \sum_{k=1}^n [G(s_{k-1},x)-G(s_k,x)]\langle F(s_k,x), B(s_k)-B(s_{k-1})\rangle. \end{aligned}$$

Then we write

$$\begin{aligned} J_5(\eta) = &\sum_{k=1}^n G(s_k,x)\langle F(s_k,x), B(s_k)-B(s_{k-1})\rangle \\ &+ \sum_{k=1}^n [G(s_k, X(s_k,s_{k-1},x)) - G(s_k,x)] \\ &\qquad \times \langle F(s_k,x), B(s_k)-B(s_{k-1})\rangle \\ &+ \sum_{k=1}^n [G(s_{k-1}) - G(s_k, X(s_k,s_{k-1},x))] \\ &\qquad \times (\langle F(s_k,x), B(s_k)-B(s_{k-1})\rangle)^2 + o(|\eta|). \end{aligned}$$

We have used that

$$
\begin{aligned}
&\sum_{k=1}^{n}[G(s_k, X(s_k, s_{k-1}, x)) - G(s_{k-1}]\langle F(s_k, x), B(s_k) - B(s_{k-1})\rangle \\
&= \sum_{k=1}^{n} G(s_{k-1}) \left(e^{\int_{s_{k-1}}^{T} V(u, X(u, s_{k-1}, x)) du + \int_{s_{k-1}}^{T} F(u, X(u, s_{k-1}, x)) dB(u)} - 1 \right) \\
&\qquad \times \langle F(s_k, x), B(s_k) - B(s_{k-1})\rangle.
\end{aligned}
$$

Consequently, using again Taylor's formula,

$$
\begin{aligned}
J_5(\eta) = {} & \sum_{k=1}^{n} G(s_k, x)\langle F(s_k, x), B(s_k) - B(s_{k-1})\rangle \\
& + \sum_{k=1}^{n} \langle D_x G(s_k, x), \sigma(s_k, x)(B(s_k) - B(s_{k-1}))\rangle \\
& \quad \times \langle F(s_k, x), B(s_k) - B(s_{k-1})\rangle \\
& + \sum_{k=1}^{n} G(s_{k-1})[\langle F(s_k, x), B(s_k) - B(s_{k-1})\rangle]^2 + o(|\eta|)
\end{aligned}
$$

where $\lim_{|\eta|\to 0} o_3(|\eta|) = 0$.

It follows that

$$
\begin{aligned}
\lim_{|\eta|\to 0} J_5(\eta) = {} & \oint_s^T G(r, x)\langle F(r, x), dB(r)\rangle \\
& + \int_s^T \langle D_x G(r, x), \sigma(r, x) F(r, x)\rangle dr \\
& + \int_s^T G(r, x)|F(r, x)|^2 dr.
\end{aligned}
$$

In conclusion we have proved the identity

$$
\begin{aligned}
& G(s, x) - \varphi(x) = \int_s^T \Big[\langle [b(u, x) + \sigma(u, x) F(u, x)], D_x G(u, x)\rangle \\
& + \frac{1}{2} \operatorname{Tr} [a(u, x) D_x^2 G(u, x)] + [V(u, x) + \frac{1}{2} \, |F(u, x)|^2] G(u, x)\Big] du \\
& + \oint_s^T [\langle D_x G(u, x), \sigma(u, x) dB(u)\rangle + G(u, x)\langle F(u, x), dB(u)\rangle].
\end{aligned}
$$

Now taking expectation and recalling that $\sigma F = f$ we see that

$$z(s,x) = \mathbb{E}[G(s,x)].$$

fulfills (10.16).

Uniqueness. It can be proved by arguing as in the proof of Theorem 10.2. $\square$

10.2.1. The autonomous case

We assume here that $b(t,x) = b(x)$, $\sigma(t,x) = \sigma(x)$, $f(t,x) = f(x)$, $V(t,x) = V(x)$. We denote by $X(t,x)$ the solution of (10.6) and consider the problem

$$\begin{cases} z_t(t,x) = \mathscr{L} z(t,x) \\ \qquad\qquad + \left[V(x) + \frac{1}{2}\,|\sigma(x)^{-1} f(x)|^2\right] z(t,x), & 0 \le t \le T, \\ \\ z(0,x) = \varphi(x), & x \in \mathbb{R}^d, \end{cases} \tag{10.18}$$

where $\varphi \in C_b^2(\mathbb{R}^d)$.

By Theorem 10.8 we find

Theorem 10.9. *Assume that Hypotheses* 8.1, 8.18 *and* 10.1 *hold with* b, σ *and* V *independent of* t. *Let moreover* $f: \mathbb{R}^d \to \mathbb{R}^d$ *be such that*

(i) $f(x)$ *belongs the range of* $\sigma(x)$ *for all* $x \in \mathbb{R}^d$.
(ii) $\sigma^{-1} f$ *is continuous and bounded in* $[0,T] \times \mathbb{R}^d$ *together with its first and second partial derivatives with respect to* x.

Then problem (10.18) *has a unique solution given by the formula*

$$z(t,x) = \mathbb{E}\left[\varphi(X(t,x)) e^{\int_0^t V(X(u,x))du + \int_0^t \langle (\sigma^{-1} f)(X(u,x)), dB(u)\rangle}\right]. \tag{10.19}$$

10.2.2. Change of drift

Consider the following problem

$$\begin{cases} u_t(t,x) = \frac{1}{2}\,\mathrm{Tr}\,[\sigma(x)\sigma^*(x) u_{xx}(t,x)] + \langle u_x(t,x), b(x)\rangle, \\ \\ u(0,x) = \varphi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{10.20}$$

where b and σ fulfill Hypotheses 8.1, 8.18 and 10.1 and $\varphi \in C_b(\mathbb{R}^d)$.

Then by Theorem 9.27, problem (10.20) has a unique classical solution given by

$$u(t,x) = \mathbb{E}[\varphi(X(t,x))].$$

In some case it can be useful to express the solution $u(t,x)$ in terms of the solution of another stochastic differential equation with a different drift b',

$$dZ = b'(Z)dt + \sigma(Z)dB(t), \quad Z(0) = x. \tag{10.21}$$

In fact, using the Girsanov formula, we can write (provided $\sigma^{-1}(x)(b(x) - b'(x))$ be meaningful)

$$\begin{cases} u_t(t,x) \;=\; \frac{1}{2}\,\mathrm{Tr}\,[a(x)u_{xx}(t,x)] + \langle u_x(t,x), b'(x) + \sigma(x)F(x)\rangle \\ \qquad\qquad + \left(V + \frac{1}{2}\,|F(x)|^2\right) u(t,x) = 0, \\ u(0,x) \;=\; \varphi(x), \quad x \in \mathbb{R}, \end{cases} \tag{10.22}$$

choosing

$$F(x) = \sigma^{-1}(x)(b(x) - b'(x)), \quad V = -\frac{1}{2}\,|\sigma^{-1}(x)(b(x) - b'(x))|^2.$$

In this way we obtain the formula

$$u(t,x) = \mathbb{E}\left[\varphi(Z(t,x))\rho(t,x)\right], \tag{10.23}$$

where

$$\rho(t,x) = \exp\left\{\frac{1}{2}\int_0^t |\sigma^{-1}(Z(u,x))(b(Z(u,x)) - b'(Z(u,x)))|^2 du \right.$$
$$\left. + \int_0^t \langle\sigma^{-1}(Z(t,x))(b(Z(t,x)) - b'(Z(t,x))), dB(u)\rangle\right\}.$$

Example 10.10. Consider the Kolmogorov equation

$$\begin{cases} u_t(t,x) \;=\; \frac{1}{2}\,\mathrm{Tr}\,[u_{xx}(t,x)] + \langle u_x(t,x), Ax + F(x)\rangle, \\ u(0,x) \;=\; \varphi(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{10.24}$$

corresponding to the stochastic differential equation

$$\begin{cases} dX = (AX + F(X))dt + dB(t) \\ X(0) = x, \end{cases} \tag{10.25}$$

where $A \in L(\mathbb{R}^d)$ and $F : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz continuous. We want to express the solution of (10.24) in terms of the solution of the Ornstein–Uhlenbeck equation

$$\begin{cases} dZ = AZdt + dB(t) \\ \\ Z(0) = x, \end{cases} \tag{10.26}$$

which is given by

$$Z(t,x) = e^{tA}x + \int_0^t e^{(t-s)A} dB(s).$$

For this it is enough to use the Girsanov formula (10.23) with

$$f = F, \quad V = -\frac{1}{2}\,|F|^2.$$

We obtain

$$u(t,x) = \mathbb{E}[\varphi(Z(t,x))\rho(t,x)],$$

where

$$\rho(t,x) = e^{-\frac{1}{2}\int_0^t |F(Z(s,x))|^2 ds - \int_0^t \langle F(Z(s,x)), dB(s)\rangle}. \tag{10.27}$$

Let us consider the interesting case when F is a gradient, say $F(x) = -DU(x)$, where $U \in C^3(\mathbb{R}^d)$ and DU is Lipschitz. In this case equation (10.27) reads as follows

$$\rho(t,x) = e^{-\frac{1}{2}\int_0^t |DU(Z(s,x))|^2 ds + \int_0^t \langle DU(Z(s,x)), dB(s)\rangle}. \tag{10.28}$$

Now we can find a simpler formula for ρ which do not involve stochastic integrals. In fact, by Itô's formula we find

$$U(Z(t,x)) = U(x) + \int_0^t \mathscr{L}U(Z(s,x))ds + \int_0^t \langle DU(Z(t,x)), dB(s)\rangle, 2$$

where

$$\mathscr{L}U = \frac{1}{2}\,\Delta U + \langle Ax, D_x U\rangle.$$

We deduce

$$\int_0^t \langle DU(Z(t,x)), dB(s)\rangle = U(Z(t,x)) - U(x) - \int_0^t \mathscr{L}U(Z(s,x))ds.$$

Substituting in (10.28), yields finally

$$\rho(t,x) = e^{-\frac{1}{2}\int_0^t |DU(Z(s,x))|^2 ds + U(Z(t,x)) - U(x) - \int_0^t \mathscr{L}U(Z(s,x))ds}.$$

10.3. The Girsanov transform

In this section we take $r = d$, so that B is an $\mathbb{R}^d$-valued Brownian motion in $(\Omega, \mathscr{F}, \mathbb{P})$. We denote, as usual, by $(\mathscr{F}_t)_{t\geq 0}$ its natural filtration. We are given a bounded [1] predictable process α on $[0, T]$. Then we consider the solution $M(t),\ t \in [0, T]$, to the stochastic differential equation in $\mathbb{R}$

$$\begin{cases} dM(t) = M(t)\langle \alpha(t), dB(t)\rangle, & t \geq 0, \\ \\ M(0) = 1, \end{cases} \tag{10.29}$$

which is given by (see Example 8.13)

$$M(t) = e^{\int_0^t \langle \alpha(s), dB(s)\rangle - \frac{1}{2}\int_0^t |\alpha(s)|^2 ds}. \tag{10.30}$$

Since

$$M(t) = M(s) + \int_s^t M(r)\langle \alpha(r), dB(r)\rangle, \quad t \geq s,$$

we see that $M(t)$ is a martingale with respect to the filtration $(\mathscr{F}_t)_{t\geq 0}$ and that $\mathbb{E}[M(t)] = 1$ for all $t \in [0, T]$.

10.3.1. Change of probability

Fix $T > 0$ and introduce a measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ setting

$$\mathbb{Q}(A) = \int_A M(T) d\mathbb{P}, \quad \forall\ A \in \mathscr{F}. \tag{10.31}$$

$\mathbb{Q}$ is a probability measure (because $\mathbb{Q}(\Omega) = \mathbb{E}[M(T)] = 1$) which is equivalent to $\mathbb{P}$. Moreover

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = M(T).$$

We shall denote by $\mathbb{E}_{\mathbb{Q}}(X)$ the expectation of a random variable X with respect to $\mathbb{Q}$, *i.e.*

$$\mathbb{E}_{\mathbb{Q}}(X) = \int_\Omega X d\mathbb{Q} = \mathbb{E}[XM(T)]. \tag{10.32}$$

Set now

$$L(t) = B(t) - \int_0^t \alpha(s) ds, \quad t \in [0, T]. \tag{10.33}$$

We are going to show that L is a Brownian motion on $(\Omega, \mathscr{F}, \mathbb{Q})$. We start with a Lemma.

---

[1] In t and $\omega \in \Omega$.

Lemma 10.11. *Let $t \geq s \geq 0$, $h \in \mathbb{R}^d$. Then*

$$\mathbb{E}\left[e^{i\langle h,(L(t)-L(s))\rangle}M(t)\Big|\mathscr{F}_s\right] = e^{-\frac{|h|^2}{2}(t-s)}M(s). \tag{10.34}$$

Proof. Let us compute $d\left[e^{i\langle h,L(t)\rangle}M(t)\right]$ using Itô's formula. Taking into account that

$$\begin{cases} dL(t) = dB(t) - \alpha(t)dt, \\ \\ de^{i\langle h,L(t)\rangle} = ihe^{i\langle h,L(t)\rangle}dL(t) - \frac{1}{2}\,|h|^2e^{i\langle h,L(t)\rangle}dt, \\ \\ dM(t) = M(t)\langle\alpha(t), dB(t)\rangle, \end{cases}$$

we find

$$\begin{aligned} &d\left[e^{i\langle h,L(t)\rangle}M(t)\right] \\ &= d\left[e^{i\langle h,L(t)\rangle}\right]M(t) + e^{i\langle h,L(t)\rangle}dM(t) + iM(t)\langle h,\alpha(t)\rangle e^{i\langle h,L(t)\rangle}dt \\ &= -\frac{1}{2}M(t)|h|^2e^{i\langle h,L(t)\rangle}dt + e^{i\langle h,L(t)\rangle}M(t)\langle ih+\alpha(t), dB(t)\rangle, \end{aligned}$$

so that

$$d\left[e^{\frac{t}{2}|h|^2+i\langle h,L(t)\rangle}M(t)\right] = e^{\frac{t}{2}|h|^2+i\langle h,L(t)\rangle}M(t)\langle ih+\alpha(t), dB(t)\rangle.$$

which is equivalent for $t > s$ to

$$\begin{aligned} e^{\frac{t}{2}|h|^2+i\langle h,L(t)\rangle}M(t) &= e^{\frac{s}{2}|h|^2+i\langle h,L(s)\rangle}M(s) \\ &\quad + \int_s^t e^{i\langle h,L(r)\rangle}M(r)\langle ih+\alpha(r), dB(r)\rangle. \end{aligned}$$

Taking the conditional expectation with respect to $\mathscr{F}_s$ and taking into account that the Itô integral has orthogonal increments, yields

$$\mathbb{E}[e^{\frac{t}{2}|h|^2+i\langle h,L(t)\rangle}M(t)|\mathscr{F}_s] = e^{\frac{s}{2}|h|^2+i\langle h,L(s)\rangle}M(s), \tag{10.35}$$

which coincides with (10.34). □

Corollary 10.12. *Let $t \geq s \geq 0$, $h \in \mathbb{R}^d$. Then*

$$\mathbb{E}\left[e^{i\langle h,L(t)-L(s)\rangle}M(T)|\mathscr{F}_s\right] = e^{-\frac{|h|^2}{2}(t-s)}M(s), \tag{10.36}$$

Proof. Write

$$\mathbb{E}\left[e^{i\langle h,L(t)-L(s)\rangle}M(T)|\mathscr{F}_s\right] = e^{-i\langle h,L(s)\rangle}\mathbb{E}\left[e^{i\langle h,L(t)\rangle}M(T)\Big|\mathscr{F}_s\right]$$

$$= e^{-i\langle h,L(s)\rangle}\mathbb{E}\left[\mathbb{E}\left(e^{i\langle h,L(t)\rangle}M(T)|\mathscr{F}_t\right)\Big|\mathscr{F}_s\right]$$

$$= e^{-i\langle h,L(s)\rangle}\mathbb{E}\left[e^{i\langle h,L(t)\rangle}\mathbb{E}\left(M(T)|\mathscr{F}_t\right)\Big|\mathscr{F}_s\right]$$

$$= e^{-i\langle h,L(s)\rangle}\mathbb{E}\left[e^{i\langle h,L(t)\rangle}M(t)\Big|\mathscr{F}_s\right] = \mathbb{E}\left[e^{i\langle h,L(t)-L(s)\rangle}M(t)\Big|\mathscr{F}_s\right].$$

Now the conclusion follows from Lemma 10.11. □

Corollary 10.13. *Let $t > s > 0$, $h, k \in \mathbb{R}^d$. Then*

$$\mathbb{E}\left[e^{i\langle h,L(s)\rangle+i\langle k,L(t)-L(s)\rangle}M(T)\Big|\mathscr{F}_s\right] = e^{i\langle h,L(s)\rangle}M(s)e^{-\frac{|k|^2}{2}(t-s)} \tag{10.37}$$

Proof. Write

$$\mathbb{E}\left[e^{i\langle h,L(s)\rangle+i\langle k,L(t)-L(s)\rangle}M(T)\Big|\mathscr{F}_s\right]=e^{i\langle h,L(s)\rangle}\mathbb{E}\left[e^{i\langle k,L(t)-L(s)\rangle}M(T)\Big|\mathscr{F}_s\right].$$

Then the conclusion follows from Corollary 10.12. □

Theorem 10.14. *$L(t)$, $t \in [0,T]$, is an $\mathbb{R}^d$-valued Brownian motion on $(\Omega, \mathscr{F}, \mathbb{Q})$.*

Proof. First we notice that $L(\cdot)$ is a continuous process. Let us show that for all $t \geq s \geq 0$ and all $h, k \in \mathbb{R}^d$ we have

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[e^{i\langle h,L(s)\rangle+i\langle k(L(t)-L(s))\rangle}\right] &= \mathbb{E}\left[e^{i\langle h,L(s)\rangle+i\langle k(L(t)-L(s))\rangle}M(T)\right]\\ &= e^{-\frac{|h|^2+|k|^2}{2}(t-s)}.\end{aligned} \tag{10.38}$$

This will prove that $L(s)$ and $L(t)-L(s)$ are Gaussian (N_{It} and $N_{I(t-s)}$ respectively) and that they are independent. Then the conclusion will follow by a straightforward generalization of this argument.

In fact, taking expectation in (10.37), yields

$$\mathbb{E}\left[e^{i\langle h,L(s)\rangle+i\langle k,L(t)-L(s)\rangle}M(T)\right] = \mathbb{E}\left[e^{i\langle h,L(s)\rangle}M(s)\right]e^{-\frac{|k|^2}{2}(t-s)}$$

Finally, taking into account (10.34) (with s replacing t and 0 replacing s) we find (10.38).

□

10.3.2. Weak solutions

Let us consider the stochastic differential equation

$$\begin{cases} dX(t) = b(X(t))dt + dB(t), \\ \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{10.39}$$

where $b \in C_b(\mathbb{R}^d; \mathbb{R}^d)$ and B is an $\mathbb{R}^d$-valued Brownian motion. Notice that we cannot apply the standard existence and uniqueness result, because it requires b to be Lipschitz. However, we can prove the following

Theorem 10.15. *Let $b \in C_b(\mathbb{R}^d; \mathbb{R}^d)$. Fix $x \in \mathbb{R}^d$ and $T > 0$. Then there exists a probability measure $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ and a Brownian motion L on $(\Omega, \mathscr{F}, \mathbb{Q})$ such that the stochastic differential equation*

$$\begin{cases} dX(t) = b(X(t))dt + dL(t), \quad t \in [0, T], \\ \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{10.40}$$

has a solution.

Proof. Set

$$\alpha(t) = b(x + B(t)), \quad t \in [0, T],$$

so that α is bounded predictable process. Let M be the solution of (10.29), $\mathbb{Q}$ the probability defined by (10.31) and let L be the Brownian motion defined by (10.33).

Now set $X(t) = x + B(t),\ t \in [0, T]$. Then we have

$$\begin{aligned} L(t) &= B(t) - \int_0^t b(x + B(s))ds = B(t) - \int_0^t b(X(s))ds \\ &= X(t) - x - \int_0^t b(X(s))ds. \end{aligned}$$

This means that X is a solution to (10.40). □

Remark 10.16. By the previous theorem we know that equation (10.39) has a solution but in another probability space $(\Omega, \mathscr{F}, \mathbb{Q})$ and with another Brownian motion L. Notice that both $\mathbb{Q}$ and L depend on x and T. $X(t),\ t \in [0, T]$, is called a *weak solution* to (10.39).

Remark 10.17. Consider the transition semigroup

$$Q_t\varphi(x) = \mathbb{E}_{\mathbb{Q}}[\varphi(X(t))], \quad \varphi \in C_b(\mathbb{R}^d),\ t \in [0, T],\ x \in \mathbb{R}^d.$$

Then for any $t \in [0, T]$ we have

$$
\begin{aligned}
Q_t\varphi(x) &= \mathbb{E}[\varphi(x + B(t))M(T)] = \mathbb{E}[\varphi(x + B(t))M(T)|\mathscr{F}_t] \\
&= \mathbb{E}\{\mathbb{E}\{\varphi(x+B(t))\mathbb{E}[M(T)|\mathscr{F}_t]\} = \mathbb{E}[\varphi(x+B(t))M(t)]\} \\
&= \mathbb{E}\left[\varphi(x + B(t))e^{-\frac{1}{2}\int_0^t |b(x+B(s))|^2 ds + \int_0^t \langle b(x+B(s)), dB(s)\rangle}\right].
\end{aligned}
\tag{10.41}
$$

10.3.3. Another application

Let us consider the stochastic differential equation

$$
\begin{cases}
dX(t) = (AX(t) + b(X(t))dt + dB(t), \\
X(0) = x \in \mathbb{R}^d,
\end{cases}
\tag{10.42}
$$

where $A \in L(\mathbb{R}^d)$ and $b \in C_b(\mathbb{R}^d; \mathbb{R}^d)$. As before we cannot apply the standard existence and uniqueness result.

Theorem 10.18. *Let $A \in L(\mathbb{R}^d)$ and $b \in C_b(\mathbb{R}^d; \mathbb{R}^d)$ and let $x \in \mathbb{R}^d$. Then there exists a probability $\mathbb{Q}$ on $(\Omega, \mathscr{F})$ and a Brownian motion L on $(\Omega, \mathscr{F}, \mathbb{Q})$ such that the stochastic differential equation*

$$
\begin{cases}
dX(t) = (AX(t) + b(X(t))dt + dL(t), \\
X(0) = x \in \mathbb{R}^d,
\end{cases}
\tag{10.43}
$$

has a solution.

Proof. Set

$$Z(t) = e^{tA}x + W_A(t),$$

where

$$W_A(t) = \int_0^t e^{(t-s)A} dB(s).$$

Then

$$Z(t) = x + \int_0^t AZ(s)ds + B(t). \tag{10.44}$$

Now set

$$\alpha(t) = b(e^{tA}x + W_A(t)), \quad t \in [0, T],$$

so that $\alpha \in C_B(0, T; L^\infty(\Omega, \mathbb{R}^d))$. Let M be the solution of (10.29), $\mathbb{Q}$ the probability defined by (10.31) and let L be the Brownian motion defined by (10.33).

Then, taking into account (10.44), we have

$$\begin{aligned} L(t) &= B(t) - \int_0^t b(Z(s))ds \\ &= Z(t) - x - \int_0^t AZ(s)ds - \int_0^t b(Z(s))ds. \end{aligned}$$

This means that Z is a solution to (10.43). □

Remark 10.19. Consider the transition semigroup

$$Q_t\varphi(x) = \mathbb{E}_Q[\varphi(Z(t)))], \quad \varphi \in C_b(\mathbb{R}^d),\ t \in [0, T],\ x \in \mathbb{R}^d.$$

Then we have

$$\begin{aligned} Q_t\varphi(x) &= \mathbb{E}[\varphi(e^{tA}x + W_A(t))M(T)] \\ &= \mathbb{E}[\varphi(e^{tA}x + W_A(t))M(T)|\mathscr{F}_t] \\ &= \mathbb{E}\{\varphi(e^{tA}x + W_A(t))\mathbb{E}[M(T)|\mathscr{F}_t]\} = \mathbb{E}[\varphi(e^{tA}x + W_A(t))M(t)] \\ &= \mathbb{E}\left[\varphi(e^{tA}x + W_A(t))e^{-\frac{1}{2}\int_0^t |b(e^{sA}x+W_A(s))|^2ds+\int_0^t\langle b(e^{A}x+W_A(s)),dB(s)\rangle}\right]. \end{aligned} \tag{10.45}$$

Chapter 11
Malliavin calculus

By proceeding as in Section 4.6 we consider a probability space $(H, \mathscr{B}(H), \mu)$, where H is given by $L^2(0, 1; \mathbb{R}^d)$, $d \in \mathbb{N}$, and μ is an arbitrary non degenerate Gaussian measure on H of mean 0 and covariance $Q \in L(H)$. The probability space $(H, \mathscr{B}(H), \mu)$ will play the role of $(\Omega, \mathscr{F}, \mathbb{P})$ of the previous chapters, from which we shall use several notations. [(1)].

We consider the d-dimensional Brownian motion B in $(H, \mathscr{B}(H), \mu)$ defined by $B(t) = (B_1(t), ..., B_d(t))$ where

$$B_k(t) = W_{e_k \mathbb{1}_{[0,t]}}, \quad k = 1, ..., d, \ t \in [0, 1],$$

where $(e_1, ..., e_d)$ is the natural basis of $\mathbb{R}^d$ and for $f \in H$, W_f denotes the white noise function defined in Chapter 2. The natural filtration of B will be denoted by $(\mathscr{F}_t)$.

The main goal of this chapter is to study the Malliavin derivative of the solution of a stochastic differential equation. For the sake of simplicity, we shall limit ourselves to equations with constant diffusion coefficients,

$$\begin{cases} dX(t) = b(X(t))dt + \sqrt{C}\, dB(t), \\ \\ X(0) = \xi \in \mathbb{R}^d, \end{cases} \tag{11.1}$$

under the following assumptions

Hypothesis 11.1. (i) $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz, of class C^3 and with bounded derivatives of order lesser than 3.

(ii) $C : \mathbb{R}^d \to \mathbb{R}^d$ is linear, symmetric and nonnegative.

[(1)] A basic idea of the Malliavin calculus is to consider Ω as an explicit Banach or Hilbert space and to take derivatives with respect to $\omega \in \Omega$.

For more general results the reader can refer to the monographs [16] and [20].

After some preliminaries given in Section 11.1, we consider in Section 11.2 and 11.3 some properties of the Malliavin derivative and its adjoint operator (called Skorohod integral) in the space $L^2(0,1;\mathbb{R}^d)$. Section 11.4 deals with the Clark–Ocone formula; it can be skipped in a first lecture.

Section 11.5, devoted to the Malliavin derivative of the solution $X(t,\xi)$ of problem (11.1), is the core of the chapter. In particular, when det $C>0$, we prove two formulae for

$$D_\xi P_t f(\xi) = D_\xi \mathbb{E}[f(X(t,\xi)]$$

and

$$P_t D_\xi f(\xi) = \mathbb{E}[D_\xi f(X(t,\xi)],$$

which only involve f (and not its derivative). From the second formula it follows that the law of $X(t,\xi)$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$.

11.1. Preliminaries

We are here dealing with some properties of the white noise function. We recall that by Proposition 4.30 we have

$$W_f = \int_0^1 \langle f(t), dB(t)\rangle, \quad \forall\, f\in H. \tag{11.2}$$

We are going to show that, given $f\in H$ and $s\in[0,1]$, the conditional expectation of W_f with respect to $\mathscr{F}_s$ is still a white noise function. More precisely we prove

Lemma 11.2. *Let $f\in L^2(0,1;\mathbb{R}^d)$ and $s\in[0,1]$. Then we have*

$$\mathbb{E}[W_f|\mathscr{F}_s] = W_{\mathbb{1}_{[0,s]}f}. \tag{11.3}$$

Proof. Write

$$\mathbb{E}[W_f|\mathscr{F}_s] = \mathbb{E}\left[\int_0^s \langle f(r), dB(r)\rangle|\mathscr{F}_s\right] + \mathbb{E}\left[\int_s^1 \langle f(r), dB(r)\rangle|\mathscr{F}_s\right].$$

Now the first term is equal to $\int_0^s\langle f(r), dB(r)\rangle = W_{\mathbb{1}_{[0,s]}f}$ because it is $\mathscr{F}_s$-measurable, whereas the second one vanishes because is independent of $\mathscr{F}_s$. So, the conclusion follows. $\square$

Now we show that the conditional expectation of an exponential function from $\widetilde{\mathscr{E}}(H)$ with respect to $\mathscr{F}_s$ is still an exponential function.

By $\widetilde{\mathscr{E}}(H)$ we mean the linear span all real parts of functions e^{iW_f}, $f \in H$. Obviously $\widetilde{\mathscr{E}}(H) \supset \mathscr{E}(H)$ because for any $f \in H$ we have $\langle f, \cdot \rangle = W_{Q^{1/2}f}$. So, $\widetilde{\mathscr{E}}(H)$ is dense in $L^2(H, \mu)$. Similarly, $\widetilde{\mathscr{E}(H, H)} = \widetilde{\mathscr{E}}(H) \otimes H$ is dense in $L^2(H, \mu; H)$.

Proposition 11.3. *Let $f \in L^2(0, 1; \mathbb{R}^d)$ and $s \in [0, 1]$. Then we have*

$$\mathbb{E}[e^{iW_f}|\mathscr{F}_s] = e^{iW_{\mathbb{1}_{[0,s]}f}} e^{-\frac{1}{2}\int_s^1 |f(u)|^2 du}. \tag{11.4}$$

Proof. Set $X(t) = \int_0^t \langle f(u), dB(u) \rangle$. Then $dX(t) = \langle f(t), dB(t) \rangle$ and by Itô's formula we have

$$de^{iX(t)} = ie^{iX(t)}\langle f(t), dB(t) \rangle - \frac{1}{2}\, e^{iX(t)}|f(t)|^2 dt,$$

from which

$$e^{iX(t)} = e^{iX(s)} + i \int_s^t e^{iX(u)}\langle f(u), dB(u) \rangle - \frac{1}{2} \int_s^t e^{iX(u)}|f(u)|^2 du.$$

Taking conditional expectation with respect to $\mathscr{F}_s$ of both sides of this identity and setting $Y(t) = \mathbb{E}[e^{iX(t)}|\mathscr{F}_s]$ yields

$$Y(t) = Y(s) - \frac{1}{2} \int_s^t Y(u)|f(u)|^2\, du.$$

Now the conclusion follows by solving this integral equation. □

Remark 11.4. By Proposition 11.3 it follows that

$$\varphi \in \widetilde{\mathscr{E}}(H) \Rightarrow \mathbb{E}[\varphi|\mathscr{F}_t] \in \widetilde{\mathscr{E}}(H),$$

for all $t \in [0, 1]$.

We shall need the following simple result.

Proposition 11.5. *Let $\varphi \in L^2(H, \mathscr{F}_t, \mu)$ for some $t \in [0, 1]$. Then there exists a sequence $(\varphi_n) \subset \widetilde{\mathscr{E}}(H)$ such that $\varphi_n \to \varphi$ in $L^2(H, \mathscr{F}_t, \mu)$.*

Proof. First we choose a sequence $(\psi_n) \subset \widetilde{\mathscr{E}}(H)$ such that $\psi_n \to \varphi$ in $L^2(H, \mu)$. Then we set

$$\varphi_n = \mathbb{E}[\psi_n|\mathscr{F}_t].$$

Now by Remark 11.4 it follows that $(\varphi_n) \subset \widetilde{\mathscr{E}}(H)$, so the conclusion follows from the continuity of the conditional expectation. □

11.2. The Malliavin derivative

We shall consider the Malliavin derivative

$$M : D^{1,2}(H, \mu) \subset L^2(H, \mu) \to L^2(H, \mu; H)$$

defined in Chapter 3 (with $H = L^2(0, 1; \mathbb{R}^d)$).

Let $\varphi \in D^{1,2}(H,\mu)$. We know that M maps $D^{1,2}(H,\mu)$ into $L^2(H,\mu;H)$. Therefore for μ-a.e. $x \in H$, $(M\varphi)(x)$ is an element of $H = L^2(0, 1; \mathbb{R}^d)$; we shall set

$$M_\tau \varphi(x) = [(M\varphi)(x)](\tau), \quad \tau \in [0, 1], \ \mu\text{-a.e. } x \in H.$$

So, $M_\tau \varphi$ belongs to $L^2(H, \mu; \mathbb{R}^d)$; it is a random variable on $(H, \mathscr{B}(H))$ with values in $\mathbb{R}^d$. Consequently, the mapping

$$[0, 1] \to L^2(H, \mu; \mathbb{R}^r), \quad \tau \mapsto M_\tau \varphi$$

is a stochastic process with values in $\mathbb{R}^d$ more precisely an element of $L^2(0, 1; L^2(H, \mu; \mathbb{R}^d))$.

By definition M acts on scalar random variables, but it can be defined on d-random vectors (this will be needed later when dealing with Malliavin derivatives of stochastic flows). Assume that $Y = (Y_1, ..., Y_d)$ with $Y_1, ..., Y_d \in D^{1,2}(H, \mu)$; in this case we write $X \in D^{1,2}(H, \mu; \mathbb{R}^d)$. Then we set for each $\tau \in [0, 1]$

$$M_\tau Y = (M_\tau Y_1, ..., M_\tau Y_d),$$

and for any $h \in \mathbb{R}^d$,

$$M_\tau^h Y = \sum_{j=1}^d M_\tau Y_j \, h_j.$$

Notice that for each $\tau \in [0, 1]$ and any $h \in \mathbb{R}^d$, $M_\tau^h Y$ is an $\mathbb{R}^d$- random vector.

Example 11.6. Let $\varphi = W_f$ with $f \in L^2(0, 1; \mathbb{R}^d)$. Then $M\varphi = f$, so that

$$M_\tau \varphi = f(\tau), \quad \tau \in [0, 1].$$

Example 11.7. For any $\tau \in [0, 1]$ we have

$$M_\tau B(t) = (M_\tau B_1(t), ..., M_\tau B_d(t)) = 1\!\!1_{[0,t]}. \tag{11.5}$$

and for any $h \in \mathbb{R}^d$

$$M_\tau^h B(t) = h 1\!\!1_{[0,t]}. \tag{11.6}$$

Let us prove an important relation between the Malliavin derivative of a random variable and its conditional expectation with respect to $\mathscr{F}_t$, $t \in [0, 1]$.

Proposition 11.8. *Let* $\varphi \in D^{1,2}(H, \mu)$ *and* $t \in [0, 1]$. *Then* $\mathbb{E}[\varphi|\mathscr{F}_t] \in D^{1,2}(H, \mu)$ *and*

$$M_\tau(\mathbb{E}[\varphi|\mathscr{F}_t]) = \mathbb{E}[M_\tau\varphi|\mathscr{F}_t]\mathbb{1}_{[0,t]}(\tau), \quad \forall\, \tau \in [0, 1]. \tag{11.7}$$

Proof. Let us first prove (11.7) when $\varphi = e^{iW_f}$ with $f \in L^2(0, 1; \mathbb{R}^d)$. In this case we have by Proposition 11.3,

$$\mathbb{E}[e^{iW_f}|\mathscr{F}_t] = e^{iW_{\mathbb{1}_{[0,t]}f}} e^{-\frac{1}{2}\int_t^1 |f(r)|_d^2 dr}.$$

Therefore $\mathbb{E}[e^{iW_f}|\mathscr{F}_t] \in D^{1,2}(H, \mu)$ and, by the chain rule,

$$M_\tau(\mathbb{E}[e^{iW_f}|\mathscr{F}_t]) = i\,\mathbb{1}_{[0,t]}(\tau) f(\tau)\, e^{iW_{\mathbb{1}_{[0,t]}f}} e^{-\frac{1}{2}\int_t^1 |f(r)|_d^2 dr}.$$

On the other hand, $M_\tau(e^{iW_f}) = if(\tau)e^{iW_f}$, so that by (11.4)

$$\mathbb{E}[M_\tau(e^{iW_f})|\mathscr{F}_t]) = if(\tau)e^{iW_{\mathbb{1}_{[0,t]}}} e^{-\frac{1}{2}\int_t^1 |f(r)|_d^2 dr}$$

So, (11.7) is proved when $\varphi \in \widetilde{\mathscr{E}}(H)$.

Let now $\varphi \in D^{1,2}(H, \mu)$ and let $(\varphi_n) \subset \widetilde{\mathscr{E}}(H)$ be such that

$$\lim_{n\to\infty} \varphi_n = \varphi \quad \text{in } L^2(H, \mu), \qquad \lim_{n\to\infty} M\varphi_n = M\varphi \quad \text{in } L^2(H, \mu; H).$$

Then from the continuity of the conditional expectation it follows that

$$\lim_{n\to\infty} \mathbb{E}[\varphi_n|\mathscr{F}_t] = \mathbb{E}[\varphi|\mathscr{F}_t] \quad \text{in } L^2(H, \mu),$$

$$\lim_{n\to\infty} \mathbb{E}[M_\tau\varphi_n|\mathscr{F}_t] = \mathbb{E}[M_\tau\varphi|\mathscr{F}_t] \quad \text{in } L^2(H, \mu; H), \quad \tau \in [0, 1].$$

Since by (11.7)

$$M_\tau(\mathbb{E}[\varphi_n|\mathscr{F}_t]) = \mathbb{E}[M_\tau\varphi_n|\mathscr{F}_t]\mathbb{1}_{[0,t]},$$

we also have

$$\lim_{n\to\infty} M_\tau(\mathbb{E}[\varphi_n|\mathscr{F}_t]) = \mathbb{E}[M_\tau\varphi|\mathscr{F}_t]\mathbb{1}_{[0,t]} \quad \text{in } L^2(H, \mu; H).$$

Therefore formula (11.7) holds in general. $\square$

Corollary 11.9. *Assume that $\varphi \in D^{1,2}(H,\mu)$ is $\mathscr{F}_t$-measurable, $t \in [0,1]$. Then $M_\tau\varphi$ is $\mathscr{F}_t$-measurable as well. Moreover*

$$M_\tau\varphi = 0, \quad \forall\, \tau \in [t,1]. \tag{11.8}$$

Proof. Let $\varphi \in D^{1,2}(H,\mu)$ be $\mathscr{F}_t$-measurable. Then $\mathbb{E}[\varphi|\mathscr{F}_t] = \varphi$ and (11.7) reduces to

$$M_\tau\varphi = \mathbb{E}[M_\tau\varphi|\mathscr{F}_t]\mathbb{1}_{[0,t]}(\tau), \quad \forall\, \tau \in [0,1], \tag{11.9}$$

so that $M_\tau\varphi$ is $\mathscr{F}_t$-measurable as well. (11.9) is clear. □

11.3. The Skorohod integral

We are here concerned with the adjoint operator M^* of M,

$$M^* : D(M^*) \subset L^2(H,\mu;H) \to L^2(H,\mu).$$

For all $F \in D(M^*)$ the random variable $M^*(F)$ is called the *Skorohod integral* of F. By the definition of adjoint we have for all $\varphi \in D^{1,2}(H,\mu)$ and all $F \in D(M^*)$

$$\int_H \langle M\varphi(x), F(x)\rangle_H\, \mu(dx) = \int_H \varphi(x)\,(M^*F)(x)\,\mu(dx). \tag{11.10}$$

Recalling that $H = L^2(0,1;\mathbb{R}^d)$, this is equivalent to

$$\int_H \int_0^1 \langle M_\tau\varphi(x), F(x)(\tau)\rangle_{\mathbb{R}^r}\, d\tau\, d\mu = \int_H \varphi\, M^*(F)\, d\mu, \tag{11.11}$$

or also

$$\int_0^1 \mathbb{E}[\langle M_\tau\varphi, F(\cdot)(\tau)\rangle]\, d\tau = \mathbb{E}[\varphi\, M^*(F)]. \tag{11.12}$$

We recall that for any $F \in \mathscr{E}(H,H)$ (defined at the end of Section 3.2) we have

$$M^*F = -\text{div}\,[Q^{1/2}F] + \langle Q^{-1/2}x, F\rangle. \tag{11.13}$$

We shall use (11.13) in the following cases.

Case 1. $F(x) = \varphi(x)z$ where $\varphi \in D^{1,2}(H,\mu)$ and $z \in H$. In this case (11.13) becomes

$$M^*F = -\langle M\varphi, z\rangle + \varphi W_z. \tag{11.14}$$

This formula is proved in Proposition 3.21.

Case 2. $F \in D(M^*) \cap L^\infty(H,\mu;H)$, $M^*(F) \in L^\infty(H,\mu)$ and $\varphi \in D^{1,2}(H,\mu)$. Here by (11.13) we obtain

$$M^*(\varphi F) = \varphi M^*(F) + \langle M\varphi, F\rangle. \tag{11.15}$$

This formula is proved in Proposition 3.22.

The following important result links the Skorohod integral to Itô's integral.

Proposition 11.10. *Let* $F \in C_B([0,1]; L^2(H,\mu;\mathbb{R}^d))$. *Then* $F \in D(M^*)$ *and*

$$M^*F = \int_0^1 \langle F(s), dB(s)\rangle. \tag{11.16}$$

Proof. Let us first assume that $F(t) \in \widetilde{\mathscr{E}}(H)$ for all $t \in [0,1]$. Set

$$F(t) = \sum_{j=1}^d F_j(t)e_j, \quad t \in [0,1].$$

Let $\varsigma = \{0 = t_0 < \cdots < t_N = 1\}$ be a decomposition of $[0,1]$ and set

$$F_\sigma = \sum_{k=1}^N \sum_{j=1}^d F_j(t_{k-1})e_j \mathbb{1}_{(t_{k-1},t_k]}.$$

Then by (11.14) we have

$$M^*(F_\sigma) = -\sum_{k=1}^N \sum_{j=1}^d \int_0^1 M_\tau F_j(t_{k-1})\mathbb{1}_{(t_{k-1},t_k]}(\tau)d\tau$$

$$+\sum_{k=1}^N \langle F(t_{k-1}), (B(t_k) - B(t_{k-1}))\rangle.$$

The first term in the right hand side vanishes by (11.8), so that

$$M^*(F_\sigma) = \sum_{k=1}^n F(t_{k-1})(B(t_k) - B(t_{k-1})).$$

Letting $|\varsigma| \to 0$ yields (11.16). So, we have proved the result when $F(t) \in \mathscr{E}(H)$ for all $t \in [0,1]$. The general case follows by density. □

Finally, in view of Proposition 3.22 we have

Proposition 11.11. *Let $F \in C_B([0,1]; L^2(H,\mu;\mathbb{R}^d))$, $\varphi \in D^{1,2}(H,\mu) \cap L^\infty(H,\mu)$ and $M\varphi \in L^\infty(H,\mu;H)$. Then $\varphi F \in D(M^*)$ and*

$$M^*(\varphi F) = -\langle M\varphi, F\rangle + \varphi \int_0^1 \langle F(s), dB(s)\rangle. \tag{11.17}$$

Proof. Let us set for any $n \in \mathbb{N}$

$$F_n(t) = \begin{cases} F(t) & \text{if } |F(t)| \le n, \\ n & \text{otherwise.} \end{cases}$$

Then by Proposition 3.22 we have

$$M^*(\varphi F_n) = -\langle M\varphi, F_n\rangle + \varphi \int_0^1 \langle F_n(s), dB(s)\rangle.$$

Now the conclusion follows letting $n \to \infty$. □

11.4. The Clark-Ocone formula(*)

We are going to show that any random variable $\varphi \in D^{1,2}(H,\mu)$ is an Itô integral.

Proposition 11.12. *For all $\varphi \in D^{1,2}(H,\mu)$ we have*

$$\varphi = \mathbb{E}(\varphi) + \int_0^1 \mathbb{E}\langle [M_t\varphi|\mathscr{F}_t], dB(t)\rangle. \tag{11.18}$$

Proof. Let first $\varphi = e^{iW_f}$ where $f \in L^2(0,1;\mathbb{R}^d)$. Then we have

$$M_t\varphi = i e^{iW_f} f(t), \quad t \text{ a.e. in } [0,1],$$

so that, recalling Proposition 11.8, we see that (11.18) is equivalent to

$$e^{iW_f} = e^{-\frac{1}{2}|f|^2} + i \int_0^1 e^{iW_{1_{[0,t]}f}} e^{-\frac{1}{2}\int_t^1 |f(u)|^2 du} \langle f(t), dB(t)\rangle. \tag{11.19}$$

Set

$$Z(t) = e^{iW_{1_{[0,t]}f}} e^{\frac{1}{2}\int_0^t |f(u)|^2 du}, \quad t \in [0,1] \tag{11.20}$$

and

$$Y(t) = Z(t) e^{-\frac{1}{2}\int_0^t |f(u)|^2 du} = e^{iW_{1_{[0,t]}f}}, \quad t \in [0,1]. \tag{11.21}$$

Then Z is the solution of the integral equation

$$Z(t) = 1 + i \int_0^t Z(u) \langle f(u), dB(u)\rangle. \tag{11.22}$$

Putting in (11.22), $Z(t) = Y(t)e^{\frac{1}{2}\int_0^t |f(u)|^2du}$ we see that Y fulfills the equation

$$Y(t) = e^{-\frac{1}{2}\int_0^t |f(u)|^2du} + i\int_0^t Y(u)e^{-\frac{1}{2}\int_u^t |f(\rho)|^2d\rho}\langle f(u), dB(u)\rangle. \quad (11.23)$$

Since $Y(t) = e^{iW_{1_{[0,t]}f}}$, setting $t = 1$ in (11.23) yields (11.19). By (11.19) it follows that (11.18) holds for all exponential functions. Since the space $\widetilde{\mathscr{E}}(H)$ of exponential functions is a core for M the conclusion follows. $\square$

11.5. Malliavin derivative of the stochastic flow

We are here concerned with the stochastic differential equation (11.1) under Hypothesis 11.1. By Theorem 8.19 equation (11.1) has a unique solution $X(\cdot, \xi)$. Moreover $X(t, \xi)$ is differentiable with respect to ξ and for any $h \in \mathbb{R}^d$ its derivative $\eta^h(t, \xi) := X_\xi(t, \xi)h$ is the solution of the linear stochastic equation

$$\begin{cases} d\eta^h(t, \xi) = b'(X(t, \xi))\eta^h(t, \xi)dt, \\ \\ \eta^h(0, \xi) = h. \end{cases} \quad (11.24)$$

For all $\tau \in [0, 1]$ and $h \in \mathbb{R}^d$ let us consider the linear differential equation (with random coefficients) for a $d \times d$ matrix $U(\cdot) = U(\cdot, s, \xi)$,

$$\begin{cases} D_t U(t)h = b'(X(t, \xi)) \cdot U(t)h, \\ \\ U(s)h = h, \quad h \in \mathbb{R}^d, \end{cases} \quad (11.25)$$

which has a unique solution $U(t) = U(t, s, \xi)$. Notice that

$$\eta^h(t, x) = U(t, 0, x)h, \quad t \in [0, 1], \; h \in \mathbb{R}^d. \quad (11.26)$$

Moreover we have

$$U(t, s, \xi)U(s, r, \xi) = U(t, r, \xi), \quad \forall\, t, s, r \in [0, 1], \; \xi \in \mathbb{R}^d, \quad (11.27)$$

which implies in particular that

$$U(t, s, \xi)^{-1} = U(s, t, \xi), \quad \forall\, t, s \in [0, 1], \; \xi \in \mathbb{R}^d. \quad (11.28)$$

Let us prove for further use an estimate for $|U(t, s, \xi)h|$.

Proposition 11.13. *Assume that Hypothesis* 11.1 *is fulfilled. Then we have*

$$|U(t,s,\xi)h| \le e^{|t-s|\|b\|_1}\,|h|, \quad \forall\, t,s\in[0,1],\ h\in\mathbb{R}^d. \tag{11.29}$$

Proof. Multiplying scalarly (11.25) by $U(t)h$, yields

$$\frac{1}{2}\,D_t|U(t)h|^2 = \langle b'(X(t,\xi))\cdot U(t)h, U(t)h\rangle \le \|b\|_1\,|U(t)h|^2.$$

The conclusion follows by a standard comparison result. □

We prove now existence of the Malliavin derivative of $X(t,\xi)$.

Theorem 11.14. *Assume that Hypothesis* 11.1 *is fulfilled and let* $X(\cdot,\xi)$ *be the solution of* (11.1). *Then* $X(t,\xi)\in D^{1,2}(H,\mu;\mathbb{R}^d)$ *and for any* $h\in\mathbb{R}^d$ *we have*

$$M_\tau^h X(t,\xi) = U(t,\tau,\xi)\sqrt{C}\,h\,\mathbb{1}_{[0,t]}(\tau) \quad \forall\,\tau\in[0,1]. \tag{11.30}$$

Proof. By using the contraction principle depending on parameters, see Appendix D, it is not difficult to see that $X(t,\xi)\in D^{1,2}(H,\mu;\mathbb{R}^d)$ and that $M_\tau^h X(t,\xi)$ fulfills the equation obtained by applying M_τ^h to the identity

$$X(t,\xi) = \xi + \int_0^t b(X(s,\xi))ds + \sqrt{C}\,B(t).$$

So, using the chain rule, recalling (11.6) and that $M_\tau^h X(s,\xi) = 0$ for $t\ge\tau$ we find for any $h\in\mathbb{R}^d$,

$$M_\tau^h X(t,\xi) = \int_\tau^t b'(X(s,\xi))M_\tau^h X(s,\xi)ds + \sqrt{C}\ h\ \mathbb{1}_{[0,t](\tau)}.$$

So, (11.30) follows. □

Let us compute, for further use, the Malliavin derivative of $U(t,s,\xi)$.

Proposition 11.15. *Let* $0\le s\le t\le 1$ *and* $h,k\in\mathbb{R}^d$. *Then we have*

$$M_\tau^h(U(t,s,\xi)k)$$

$$= \int_{s\vee\tau}^t U(t,v,\xi)b''(X(v,\xi))(U(v,s,\xi)k, U(v,\tau,\xi)\sqrt{C}\ h)dv. \tag{11.31}$$

Moreover

$$|M_\tau^h(U(s,t,\xi)k)| \le \|C^{1/2}\|\,e^{3\|b\|_1}\ \|b\|_2\,|h|\,|k|, \quad (s,t)\in[0,1],\ h,k\in\mathbb{R}^d. \tag{11.32}$$

Proof. We first remark that

$$U(t,s,\xi)k = k + \int_s^t b'(X(v,\xi))U(v,s,\xi)k\,dv.$$

Consequently, using again a fixed point theorem depending on a parameter, and the chain rule we see that $U(t,s,\xi)k \in D^{1,2}(H,\mu;\mathbb{R}^d)$ and

$$M_\tau^h(U(t,s,\xi)k) = \int_s^t b'(X(v,\xi))M_\tau^h(U(v,s,\xi)k)\,dv$$

$$+\int_s^t b''(X(v,\xi))(U(v,s,\xi)k, M_\tau^h X(v,\xi))dv.$$

Therefore

$$\begin{aligned} M_\tau^h(U(t,s,\xi)k) &= \int_s^t b'(X(v,\xi))M_\tau^h(U(v,s,\xi)k)\,dv \\ &+\int_s^t b''(X(v,\xi))(U(v,s,\xi)k, U(v,\tau,\xi)\sqrt{C}\,h\mathbb{1}_{[0,v]}(\tau)dv. \end{aligned} \tag{11.33}$$

Consequently (11.31) follows.

It remains to show (11.32). We have in fact from (11.31), taking into account (11.29)

$$|M_\tau^h(U(t,s,\xi)k)| \le e^{3\|b\|_1}\,\|b\|_2\,|h|\,|k|,$$

as required. □

In a similar way we prove

Proposition 11.16. *Let* $0 \le s \le t \le 1$ *and* $h \in \mathbb{R}^r$, $k \in \mathbb{R}^d$. *Then we have*

$$\begin{aligned} &M_\tau^h(U(s,t,\xi)k) \\ &= -\int_{s\vee\tau}^t U(s,v,\xi)D^2b(X(v,\xi))(U(v,t,\xi)k,\sqrt{C}\,h)dv. \end{aligned} \tag{11.34}$$

Moreover

$$|M_\tau^h(U(s,t,\xi)k)| \le \|C^{1/2}\|\,e^{3\|b\|_1}\,\|b\|_2\,|h|\,|k|,\quad (s,t)\in[0,1],\ h,k\in\mathbb{R}^d. \tag{11.35}$$

11.5.1. Commutators between D and P_1

In this section we want to study the operators DP_1 and P_1D, showing that when $\det C > 0$, both act on $C_b(\mathbb{R}^d)$.

Then the formula obtained for DP_tf will coincide with the Bismut-Elworthy-Li formula (9.54), whereas the more tricky formula proved for P_1Df will imply the existence of a density for the law $\Pi_{1,\xi}$ of $X(1,\xi)$ with respect to the Lebesgue measure of $\mathbb{R}^d$(as we shall see in Section 11.5.4).

11.5.2. A formula for $DP_1\varphi$

Let us first consider $f \in C_b^1(\mathbb{R}^d)$. In this case P_1f is differentiable and for any $k \in \mathbb{R}^d$ we have

$$\begin{aligned}\langle DP_1f(\xi), k\rangle &= \mathbb{E}\langle Df(X(1,\xi)), X_\xi(1,\xi)k\rangle \\ &= \mathbb{E}\langle Df(X(1,\xi)), U(1,0,\xi)k\rangle, \quad \xi \in \mathbb{R}^d.\end{aligned} \tag{11.36}$$

The idea now is to eliminate $Df(X(1,\xi))$ from this formula using the Malliavin derivative of $f(X(1,\xi))$. In this way we shall obtain a formula for $D_\xi P_1 f$ which only involves f and not its derivative. This will imply, by a standard argument, that $P_1 f$ is differentiable for any $f \in C_b(\mathbb{R}^d)$.

Let $\tau \in [0,1]$ and $h \in \mathbb{R}^d$. Then, using the chain rule and recalling (11.30) we find for any $\xi \in \mathbb{R}^d$,

$$\begin{aligned}\langle M_\tau(f(X(1,\xi))), h\rangle &= \langle Df(X(1,\xi)), M_\tau^h X(1,\xi)\rangle \\ &= \langle Df(X(1,\xi)), U(1,\tau,\xi)h\rangle, \quad \tau \in [0,1].\end{aligned}$$

Therefore

$$M_\tau(f(X(1,\xi))) = U^*(1,\tau,\xi)Df(X(1,\xi)), \quad \tau \in [0,1]. \tag{11.37}$$

We shall distinguish two cases, according to $\det C > 0$ or $\det C = 0$.

11.5.2.1. The case when $\det C > 0$. Here we obtain another proof of the Bismut-Elworthy-Li formula.

Proposition 11.17. *Assume that Hypothesis* 11.1 *is fulfilled and let* $f \in C_b(\mathbb{R}^d)$. *Then* $P_1 f$ *is differentiable with respect to* ξ *and for any* $h \in \mathbb{R}^d$ *we have*

$$\left\langle D_\xi \ P_1 f(\xi), h\right\rangle = \mathbb{E}\left[f(X(1,\xi))\int_0^1 \langle C^{-1/2}U(\tau,0,\xi)h, dB(\tau)\rangle\right]. \tag{11.38}$$

Proof. We first take $f \in C_b^1(\mathbb{R}^d)$ and fix $\tau \in [0, 1]$. Then by (11.37) we deduce, using (11.28)

$$D_\xi f(X(1,\xi)) = U^*(\tau,1,\xi)G^{-1/2}M_\tau(f(X(1,\xi))), \quad \tau \in [0, 1]. \quad (11.39)$$

Substituting in (11.36) yields,

$$\begin{aligned}\langle D_\xi P_1 f(\xi),k\rangle &= \mathbb{E}\langle U^*(\tau,1,\xi)C^{-1/2}M_\tau(f(X(1,\xi))), U(1,0,\xi)k\rangle \\ &= \mathbb{E}\langle M_\tau(f(X(1,\xi))), C^{-1/2}U(\tau,1,\xi)U(1,0,\xi)h\rangle \quad (11.40) \\ &= \mathbb{E}\langle M_\tau(f(X(1,\xi))), C^{-1/2}k\rangle, \quad \tau \in [0, 1],\end{aligned}$$

having used (11.27). Consequently, integrating both sides of (11.40) with respect to τ over [0, 1], yields

$$\langle D_\xi \ P_1 f(\xi), h\rangle = \mathbb{E}\int_0^1 \left[\langle M_\tau[f(X(1,\xi))], C^{-1/2}U(\tau,0,\xi)k\rangle\right] d\tau.$$

Using the duality between M and M^* we finally obtain,

$$\langle D_\xi \ P_1 f(\xi), k\rangle = \mathbb{E}\left[f(X(1,\xi))M^*(G^{-1/2}U(\cdot,0,\xi)k)\right].$$

Since the process $C^{-1/2}U(\cdot, 0, \xi)k$ is adapted to $(\mathscr{F}_t)$, by Proposition 11.10 we obtain (11.38) for $f \in C_b^1(\mathbb{R}^r)$. Finally, since $C_b^1(\mathbb{R}^r)$ is dense on $C_b(\mathbb{R}^r)$, the conclusion follows by a standard argument. □

The proof of the following straightforward generalization is left to the reader as an exercise (Notice, however, that one has to replace $L^2(0,1;\mathbb{R}^d)$ by $L^2(0, t; \mathbb{R}^d)$ in the definition of H and consequently, one has to consider the Malliavin derivative on $L^2(0, t; \mathbb{R}^d)$.)

Proposition 11.18. *Assume that Hypothesis* 11.1 *is fulfilled, let* $f \in C_b(\mathbb{R}^d)$ *and* $t > 0$. *Then* $P_t f$ *is differentiable with respect to* ξ *and for any* $h \in \mathbb{R}^d$ *we have*

$$\langle D_\xi \ P_t f(\xi), h\rangle = \frac{1}{t}\,\mathbb{E}\left[f(X(t,\xi))\int_0^t \langle C^{-1/2}U(\tau,0,\xi)h, dB(\tau)\rangle\right]. \quad (11.41)$$

11.5.2.2. The case when $\det C = 0$. We take again $f \in C_b^1(\mathbb{R}^d)$. Since C is not invertible, we cannot recover $D_\xi f(X(1, \xi))$ from (11.37). Then we multiply both sides of (11.37) by

$$U(1, \tau, \xi)\sqrt{C}$$

and write

$$U(1,\tau,\xi)\sqrt{C}M_\tau(f(X(1,\xi)))$$
$$= U(1,\tau,\xi)CU^*(1,\tau,\xi)Df(X(1,\xi)), \quad \tau \in [0,1]$$

Now, integrating this identity with respect to τ over $[0,1]$, we find

$$\int_0^1 U(1,\tau,\xi)\sqrt{C}\,M_\tau(f(X(1,\xi)))d\tau = \mathscr{M}(1,\xi)Df(X(1,\xi)), \tag{11.42}$$

where the matrix

$$\mathscr{M}(1,\xi) := \int_0^1 U(1,\tau,\xi)CU^*(1,\tau,\xi)d\tau, \tag{11.43}$$

is called the *Malliavin matrix*.

In spite of the fact that $\det C = 0$ it may happens that

$$\det \mathscr{M}(1,\xi) > 0.$$

In this case by (11.42) we deduce

$$D_\xi f(X(1,\xi)) = \mathscr{M}(1,\xi)^{-1}\int_0^1 U(1,\tau,\xi)\sqrt{C}\,M_\tau(f(X(1,\xi)))d\tau.$$

Now by (11.36) we find for any $k \in \mathbb{R}^d$ that

$$\begin{aligned}&\left\langle D_\xi\ P_1 f(\xi), k\right\rangle\\ &= \int_0^1 \mathbb{E}\langle \mathscr{M}(1,\xi)^{-1}U(1,\tau,\xi)\sqrt{C}\,M_\tau(f(X(1,\xi))), U(1,0,\xi)k\rangle\,d\tau\\ &= \int_0^1 \mathbb{E}\langle M_\tau(f(X(1,\xi))), \sqrt{C}\,U^*(1,\tau,\xi)\mathscr{M}(1,\xi)^{-1}U(1,0,\xi)k\rangle\,d\tau.\end{aligned} \tag{11.44}$$

In conclusion, we obtain the result.

Proposition 11.19. *Assume, besides Hypothesis* 11.1 *that*

$$\sqrt{C}\,U^*(1,\cdot,\xi)\mathscr{M}(1,\xi)^{-1}U(1,0,\xi)k\rangle \in D(M^*), \quad \forall\, k \in \mathbb{R}^d \tag{11.45}$$

and let $f \in C^1_b(\mathbb{R}^d)$. Then we have

$$\begin{aligned}&\left\langle D_\xi\ P_1 f(\xi), k\right\rangle\\ &= \mathbb{E}[f(X(1,\xi))M^*(\sqrt{C}\,U^*(1,\cdot,\xi)\mathscr{M}(1,\xi)^{-1}U(1,0,\xi)k\rangle].\end{aligned} \tag{11.46}$$

Example 11.20. Assume that $b(x) = Ax$, with $A \in L(\mathbb{R}^d)$. Then we have

$$U(t,s,\xi) = e^{(t-s)A},$$

and

$$\mathscr{M}(1,\xi) = \int_0^t e^{sA} C e^{sA^*} ds =: Q_t.$$

Now if $\det Q_t > 0$, (11.46) becomes

$$\left\langle D_\xi P_1 f(\xi), k \right\rangle = \mathbb{E}\left[f(X(1,\xi)) \int_0^1 \langle \sqrt{C}\; e^{(1-\tau)A^*} Q_t^{-1} e^A k, dB(\tau)\rangle \right],$$
$$f \in C_b(\mathbb{R}^r).$$

Let us consider the particular case when $r = 2$ and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Then we have

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad e^{tA} C e^{tA^*} = \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}$$

It follows that

$$\int_0^t e^{sA} C e^{sA^*} ds = \int_0^t \begin{pmatrix} 1 & s \\ s & s^2 \end{pmatrix} ds = \begin{pmatrix} t & t^2/2 \\ t^2/2 & t^3/3 \end{pmatrix}.$$

Therefore $\det Q_t > 0$ and we can conclude that $P_1 f$ is differentiable (and also, iterating the previous procedure, that P_t is of class C^∞).
Notice that, setting $u(t,\xi) = P_t f(\xi)$, u is the solution of the Kolmogorov equation

$$\begin{cases} D_t u(t,\xi_1,\xi_2) = \dfrac{1}{2} D_{\xi_1}^2 u(t,\xi_1,\xi_2) + \xi_1 D_{\xi_2} u(t,\xi_1,\xi_2) =: \mathscr{L} u, \\ u(0,\xi) = f(\xi). \end{cases} \tag{11.47}$$

This proves that the Kolmogorov operator $\mathscr{L}$ is *hypoelliptic*.

11.5.3. A formula for $P_1 Df$

We limit ourselves to the case when $\det C > 0$.

Proposition 11.21. *Assume, besides Hypothesis* 11.1*, that*

$$C^{-1/2}U(\cdot, 1, \xi)h) \in D(M^*). \tag{11.48}$$

Then if $f \in C_b(\mathbb{R}^d)$ *we have*

$$P_1[\langle D_\xi f, h\rangle](\xi) = E[f(X(1,\xi))M^*(C^{-1/2}U(\cdot, 1, \xi)h\rangle)]. \tag{11.49}$$

Proof. It is enough to prove (11.49) when $f \in C_b^1(\mathbb{R}^d)$. In this case for any $h \in \mathbb{R}^d$ we have by (11.39),

$$\mathbb{E}[\langle D_\xi f(X(1,\xi)), h\rangle] = \mathbb{E}[\langle U^*(\tau, 1, \xi)C^{-1/2}M_\tau(f(X(1,\xi))), h\rangle],$$
$$\tau \in [0,1].$$

Integrating both sides of this identity with respect to τ over $[0,1]$, yields

$$\begin{aligned}\mathbb{E}[\langle D_\xi f(X(t,\xi)), h\rangle] &= \mathbb{E}\int_0^1 \langle M_\tau(f(X(1,\xi))), C^{-1/2}U(\tau, 1, \xi)h\rangle d\tau \\ &= \mathbb{E}[f(X(1,\xi))M^*(C^{-1/2}U(\cdot, 1, \xi)h\rangle)],\end{aligned} \tag{11.50}$$

by assumption (11.48). □

We are now going to show that (11.48) is fulfilled. This is not immediate because the process $U(\cdot, 1, \xi)k$ is not adapted to $(\mathscr{F}_t)$. For this reason we write,

$$C^{-1/2}U(\tau,1,\xi)h = C^{-1/2}U(\tau, 0, \xi)U(0,1,\xi)h, \quad \xi,\ k \in \mathbb{R}^d,\ \tau \in [0,1],$$

then we note that $U(\cdot, 0, \xi)$ is adapted to $(\mathscr{F}_t)$ and we apply Proposition 11.11.

For this we need a lemma.

Lemma 11.22. *Let* $Z(\tau) = (Z_{i,k}(\tau)),\ \tau \in [0,1]$, *be an adapted* $L(\mathbb{R}^d)$*-valued process such that*

$$Z_{i,k} \in C_B([0,1]; L^2(H, \mu; \mathbb{R}^d)), \quad \forall\, i,k = 1, ..., d.$$

Let $F \in D^{1,2}(H, \mu; \mathbb{R}^d)$ *be bounded together with* MF*. Then* $ZF \in D(M^*)$ *and we have*

$$\begin{aligned}M^*(ZF) &= \sum_{i,k=1}^d \int_0^1 \langle Z_{i,k}(\tau)e_i, dB(\tau)\rangle\, F_k e_i \\ &\quad + \sum_{i,k=1}^d \int_0^t \langle Z_{i,k}(\tau)e_i, M_\tau F_k\rangle d\tau.\end{aligned} \tag{11.51}$$

Proof. Write

$$(Z(\tau)F)_i = \sum_{k=1}^{d} Z_{i,k}(\tau)F_k.$$

Therefore

$$Z(\tau)F = \sum_{i=1}^{d}(Z(\tau)F)_i e_i = \sum_{i,k=1}^{d} Z_{i,k}(\tau)F_k e_i.$$

Moreover,

$$\begin{aligned} M^*((Z(\cdot)F)_i e_i) &= \sum_{k=1}^{d} M^*(Z_{i,k}(\cdot)F_k e_i) \\ &= \sum_{k=1}^{d} \int_0^1 \langle Z_{i,k}(\tau)e_i, dB(\tau)\rangle\, F_k e_i \\ &\quad + \sum_{k=1}^{d} \int_0^t \langle Z_{i,k}(\tau)e_i, M_\tau F_k\rangle d\tau \end{aligned}$$

Therefore by Proposition 11.11 it follows that

$$\begin{aligned} M^*(ZF) &= \sum_{i,k=1}^{d} \int_0^1 \langle Z_{i,k}(\tau)e_i, dB(\tau)\rangle\, F_k e_i \\ &\quad + \sum_{i,k=1}^{d} \int_0^t \langle Z_{i,k}(\tau)e_i, M_\tau F_k\rangle d\tau \end{aligned}$$ □

We are now ready to show that $C^{-1/2}U(\cdot, 1, \xi)k \in D(M^*)$ for all $k \in \mathbb{R}^d$ and to prove the announced formula for $P_1 D_\xi$.

Theorem 11.23. *Assume that Hypothesis* 11.1 *is fulfilled and that* $\det C > 0$. *Then for all* $f \in C_b(\mathbb{R}^d)$ *and* $h \in \mathbb{R}^d$ *we have*

$$\begin{aligned} &P_1[\langle D_\xi f, h\rangle](\xi) \\ &= \mathbb{E}[f(X(1,\xi)) \sum_{i,k=1}^{d} \int_0^1 \langle (C^{-1/2}U(\tau,0,\xi))_{i,k} e_i, dB(\tau)\rangle\, (U(0,1,\xi)h)_k e_i)] \\ &\quad + \mathbb{E}[f(X(1,\xi)) \sum_{i,k=1}^{d} \int_0^1 \langle (C^{-1/2}U(\tau,0,\xi))_{i,k} e_i, M_\tau (U(0,1,\xi)h)_k\rangle d\tau]. \end{aligned} \tag{11.52}$$

Moreover

$$|P_1[\langle D_\xi f, h\rangle](\xi)| \le \|f\|_0 d^2 \|C^{-1/2}\|(\, e^{2\|b\|_1} + \|b\|_2\, e^{4\|b\|_1}\,)|h|, \quad h \in H. \tag{11.53}$$

Proof. We start from (11.49) and write

$$C^{-1/2}U(\tau, 1, \xi)h = \sigma^{-1}(X(\cdot, \xi)U(\tau, 0, \xi)U(0, 1, \xi)h, \quad \xi,\ k \in \mathbb{R}^d,\ \ \tau \in [0, 1].$$

Next we apply (11.51) with $Z(\tau) = C^{-1/2}U(\tau,0,\xi)$ and $F = U(0,1,\xi)h$. This is possible in view of estimates (11.29) and (11.35). Then (11.52) follows.

Let us show finally (11.53). Write (11.52) as

$$P_1[\langle D_\xi f, h\rangle](\xi) =: J_1 + J_2.$$

Then, taking into account (11.29), yields

$$\begin{aligned}|J_1| &\le \|f\|_0\, e^{\|b\|_1} \sum_{i,k=1}^d \mathbb{E}\left|\int_0^1 \langle C^{-1/2}U(\tau, 0, \xi))_{i,k}e_i, dB(\tau)\rangle\right|\, |h| \\ &\le \|f\|_0\, e^{\|b\|_1} \sum_{i,k=1}^d \left(\mathbb{E}\int_0^1 |C^{-1/2}U(\tau, 0, \xi))_{i,k}e_i|^2\, d\tau\right)^{1/2} |h|\end{aligned}$$

Therefore, using again (11.29)

$$|J_1| \le \|f\|_0\, e^{2\|b\|_1} d^2 \|C^{-1/2}\|\ |h|. \tag{11.54}$$

As regards J_2 we have

$$|J_2| \le \|f\|_0 \sum_{i,k=1}^d e^{\|b\|_1}\ \|C^{-1/2}\| \int_0^t |M_\tau(U(0, 1, \xi)h)_k|d\tau,$$

from which, recalling (11.35),

$$|J_2| \le \|f\|_0 d^2\ \|b\|_2\, e^{4\|b\|_1}\ \|C^{-1/2}\|\ |h|. \tag{11.55}$$

Now the conclusion follows from (11.53) and (11.54). $\square$

11.5.4. Existence of the density of $X(t,\xi)$

In this section we prove that if $\det C > 0$ the law of the solution $X(t,\xi)$ of problem (11.24) is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$ which we denote by λ_d.

When $\det C = 0$ a similar result holds under suitable assumptions, see [16].

We shall use the following analytic lemma

Lemma 11.24. *Let m be a probability measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Assume that there is $C > 0$ such that*

$$\left|\int_{\mathbb{R}^d} \langle Df(\xi), h\rangle m(d\xi)\right| \le C|h|\|f\|_0, \qquad \forall\, f \in C_b^1(\mathbb{R}^d),\ h \in \mathbb{R}^d. \tag{11.56}$$

Then there exists $\rho \in L^1(\mathbb{R}^d)$ such that

$$m(dx) = \rho(x)dx, \quad x \in \mathbb{R}^d. \tag{11.57}$$

Proof. From (11.56) it follows that the distributional derivative of the measure m belongs to $L^1(\mathbb{R}^d)$. Therefore there exists a $BV(\mathbb{R}^d)$ function ρ such that

$$m(dx) = \rho(x)dx.$$

Moreover, $\rho \in L^{\frac{r}{r-1}}(\mathbb{R}^r)$ [2]. □

We are now ready to show

Proposition 11.25. *Assume that Hypothesis* 11.1 *is fulfilled and that* $\det C > 0$. *Then the law of $X(t,\xi)$ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^r$.*

Proof. By applying Theorem 11.23 we deduce that there is $M > 0$ such that

$$\mathbb{E}[|Df(X(1,\xi))|] \le M\|f\|_0,$$

so, the conclusion follows from Lemma 11.24. □

(2) For a proof based on the Fourier transform see [16].

Chapter 12
Asymptotic behaviour of transition semigroups

12.1. Introduction

For the sake of simplicity, we shall limit ourselves to stochastic differential equations with constant diffusion coefficients (additive noise) of the form

$$\begin{cases} dX(t) = b(X(t))dt + \sqrt{C}\, dB(t), \\ \\ X(0) = x \in \mathbb{R}^d, \end{cases} \tag{12.1}$$

under the following assumptions

Hypothesis 12.1.

(i) $b : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz, of class C^2 and with bounded second derivative.
(ii) $C : H \to H$ is linear, symmetric and nonnegative.
(iii) B is a d-dimensional Brownian motion in a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and $(\mathscr{F}_t)_{t\geq 0}$ is its natural filtration.

Several of the following concepts and results, however, can be easily generalized to equations with multiplicative noise, that is with a non constant diffusion coefficient σ.

Hypothesis 12.1 clearly implies Hypothesis 8.1, so that by Theorem 8.2 equation (12.1) has a unique solution $X(t) = X(t,x)$. We denote by $P_t,\ t \geq 0$, (or P_t for short) the *transition semigroup* P_t defined by (9.40) and by $\Pi_{t,x}$ the law of $X(t,x)$ so that

$$P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))] = \int_{\mathbb{R}^d} \varphi(y)\Pi_{t,x}(dy), \tag{12.2}$$

for all $t \geq 0,\ x \in \mathbb{R}^d,\ \varphi \in B_b(\mathbb{R}^d)$. $\Pi_{t,x}$ are called *transition probabilities* of $X(t,x),\ t \geq 0,\ x \in \mathbb{R}^d$.

In Section 12.2 we show that P_t acts both on $C_b(\mathbb{R}^d)$ (*Feller* property) and on $B_b(\mathbb{R}^d)$. [(1)] Then we prove the *Chapman–Kolmogorov* equation for the transition probabilities. Finally, we define the transposed transition semigroup $P_t^*, t \geq 0$, on the dual space $C_b^*(\mathbb{R}^d)$ of $C_b(\mathbb{R}^d)$.

In Section 12.3 we study some important properties of P_t as *irreducibility*, *regularity* and *strong Feller* properties and prove the *Hasminskii Theorem* which relates these concepts.

In Section 12.4 we introduce the important concept of *invariant measure* which is basic for studying the asymptotic behaviour of P_t. Given an invariant measure μ, we consider a filtration $(\mathscr{G}_t)_{t\geq 0}$ larger than $(\mathscr{F}_t)_{t\geq 0}$ such that B is non anticipating and there exists a random variable $\eta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ (see Remark 6.1) whose law is μ. We prove that $X(\cdot, \eta)$ is a stationary process.

Then we show that if μ is an invariant measure there exists the limit of time averages

$$\lim_{T\to+\infty} \frac{1}{T} \int_0^T P_t\varphi \, dt =: S\varphi \quad \text{in } L^2(\mathbb{R}^d, \mu),$$

where S is a projection operator on $L^2(H, \mu)$ ($S^2 = S$) (*von Neumann Theorem*). A particular attention will be payed to the important situation where

$$S\varphi = \int_{\mathbb{R}^d} \varphi \, d\mu, \quad \forall \varphi \in L^2(\mathbb{R}^d, \mu).$$

In this case μ is said to be *ergodic*.

Section 12.5 is devoted to general results about existence and uniqueness of invariant measures including the *Krylov–Bogoliubov Theorem*, whereas in Section 12.6 we consider applications to the stochastic differential equation (12.1). We note that results from Sections 12.3–12.6 hold for several Markov semigroups, see [6]

Finally, in Section 12.7 we give some short informations about the densities of invariant measures.

12.2. Feller property and Chapman–Kolmogorov equation

We start by proving continuity in time of $P_t\varphi$ where $\varphi \in C_b(H)$.

Proposition 12.2. *For any $\varphi \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ the mapping*

$$[0, +\infty) \to \mathbb{R}, \quad t \mapsto P_t\varphi(x),$$

is continuous.

(1) $P_t\varphi$ has been defined for all $\varphi \in B_b(\mathbb{R}^d)$ but the fact that it belongs to $B_b(\mathbb{R}^d)$ requires a proof, see Proposition 12.5 below.

Proof. Let us show continuity of $P_t\varphi(x)$ at $t_0 \geq 0$. Let first $\varphi \in C_b^1(\mathbb{R}^d)$. Then for all $t \geq 0$ write

$$|P_t\varphi(x) - P_{t_0}\varphi(x)| = |\mathbb{E}[\varphi(X(t,x)) - \varphi(X(t_0,x))]|$$

$$\leq \|\varphi\|_1 \left(\mathbb{E}[|X(t,x) - X(t_0,x)|^2\right)^{1/2}.$$

Choose $T > t$. Then by Proposition 8.12 there exists $C(T,x) > 0$ such that

$$\mathbb{E}[|X(t,x) - X(t_0,x)|^2] \leq C^2(T,x)|t - t_0|, \quad \forall\, t \in [0,T].$$

Consequently,

$$|P_t\varphi(x) - P_{t_0}\varphi(x)| \leq C(T,x)|t - t_0|^{1/2}, \quad \forall\, t \in [0,T],$$

and the result follows when $\varphi \in C_b^1(\mathbb{R}^d)$.

Let now $\varphi \in C_b(\mathbb{R}^d)$. Since $C_b^1(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, there exists a sequence $(\varphi_n) \subset C_b^1(\mathbb{R}^d)$ such that

$$\|\varphi - \varphi_n\|_0 \leq \frac{1}{n}, \quad \forall\, n \in \mathbb{N},$$

so that

$$\|P_t\varphi - P_t\varphi_n\|_0 \leq \frac{1}{n}, \quad \forall\, n \in \mathbb{N},\ t \geq 0.$$

Consequently, for any $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ we have

$$\begin{aligned}
&|P_t\varphi(x) - P_{t_0}\varphi(x)| \\
&\leq |P_t\varphi(x) - P_t\varphi_n(x)| + |P_t\varphi_n(x) - P_{t_0}\varphi_n(x)| + |P_{t_0}\varphi(x) - P_{t_0}\varphi_n(x)| \\
&\leq \frac{2}{n} + |P_t\varphi_n(x) - P_{t_0}\varphi_n(x)|.
\end{aligned}$$

Now the conclusion follows from the first part of the proof. □

Proposition 12.3. *P_t is Feller, that is it maps $C_b(\mathbb{R}^d)$ into itself for any $t \geq 0$.*

Proof. Let $t \geq 0$. Assume first that $\varphi \in C_b^1(\mathbb{R}^d)$. Let $x_0 \in \mathbb{R}^d$, then for any $x \in \mathbb{R}^d$ we have

$$|P_t\varphi(x) - P_t\varphi(x_0)| \leq \|\varphi\|_1 \mathbb{E}|X(t,x) - X(t,x_0)|.$$

By Proposition 8.16 it follows that there exists $C = C(t) > 0$ such that

$$\mathbb{E}|X(t,x) - X(t,x_0)| \le C|x - x_0|, \quad x \in \mathbb{R}^d,$$

so that $\lim_{x\to x_0} P_t\varphi(x) = P_t\varphi(x_0)$ and $P_t\varphi \in C_b(\mathbb{R}^d)$.

Let now $\varphi \in C_b(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$; let us show that $P_t\varphi$ is continuous at x_0. Since $C_b^1(\mathbb{R}^d)$ is dense in $C_b(\mathbb{R}^d)$, there exists a sequence $(\varphi_n) \subset C_b^1(\mathbb{R}^d)$ such that

$$\|\varphi - \varphi_n\|_0 \le \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Consequently, for any $n \in \mathbb{N}, x \in \mathbb{R}^d$ we have

$$\begin{aligned}
&|P_t\varphi(x) - P_t\varphi(x_0)| \\
&\le |P_t\varphi(x) - P_t\varphi_n(x)| + |P_t\varphi_n(x) - P_t\varphi_n(x_0)| + |P_t\varphi(x_0) - P_t\varphi_n(x_0)| \\
&\le \frac{2}{n} + |P_t\varphi_n(x) - P_t\varphi_n(x_0)|.
\end{aligned}$$

Fix now $\epsilon > 0$ and choose $n_\epsilon > \frac{1}{3\epsilon}$. Then

$$|P_t\varphi(x) - P_t\varphi(x_0)| \le \frac{2}{3}\,\epsilon\; + |P_t\varphi_{n_\epsilon}(x) - P_t\varphi_{n_\epsilon}(x_0)|.$$

Since, as we have just proved, $P_t\varphi_{n_\epsilon} \in C_b(\mathbb{R}^d)$, it follows that

$$\limsup_{x\to x_0} |P_t\varphi(x) - P_t\varphi(x_0)| \le \frac{2}{3}\,\epsilon,$$

which implies, by the arbitrariness of ϵ that

$$\lim_{x\to x_0} P_t\varphi(x) = P_t\varphi(x_0).$$

So, $P_t\varphi \in C_b(\mathbb{R}^d)$ as claimed. □

Exercise 12.4. Show that for all $\varphi \in C_b(\mathbb{R}^d)$ the mapping

$$[0,\infty) \times \mathbb{R}^d \to \mathbb{R}, \quad (t,x) \to P_t\varphi(x),$$

is continuous.

We show finally that P_t acts on $B_b(\mathbb{R}^d)$ for any $t > 0$.

Proposition 12.5. *$P_t\varphi \in B_b(\mathbb{R}^d)$ for all $\varphi \in B_b(\mathbb{R}^d)$ and all $t \geq 0$.*

Proof. Let first $\varphi = \mathbb{1}_C$, where C is a closed subset of $\mathbb{R}^d$. Define

$$\varphi_n(x) = \begin{cases} 1 & \text{if } x \in C, \\ 1 - nd(x, C) & \text{if } d(x, C) \leq \dfrac{1}{n} \\ 0 & \text{if } d(x, C) \geq \dfrac{1}{n}, \end{cases}$$

where

$$d(x, C) = \inf\{|x - c| : \ c \in C\}, \quad x \in \mathbb{R}^d,$$

denotes the distance of x from C. Then

$$\varphi_n(x) \downarrow \varphi(x), \quad \forall\, x \in \mathbb{R}^d, \quad \text{as } n \to \infty.$$

Consequently for all $t \geq 0$

$$P_t\varphi_n(x) \downarrow P_t\varphi(x), \quad \forall\, x \in \mathbb{R}^d, \quad \text{as } n \to \infty.$$

On the other hand, $P_t\varphi_n$ is continuous for all $n \in \mathbb{N}$ by Proposition 12.3. So, $P_t\varphi$ is measurable. Next let $\varphi = \mathbb{1}_I$, where I is a Borel subset of $\mathbb{R}^d$. Then we conclude that $P_t\varphi$ is measurable by using Dykin's Theorem. □

Exercise 12.6. Show that for all $\varphi \in B_b(\mathbb{R}^d)$ the mapping

$$[0, \infty) \times \mathbb{R}^d \to \mathbb{R}, \quad (t, x) \to P_t(x),$$

is Borel.

12.2.1. Transition probabilities

For any $x \in \mathbb{R}^d$ and any $t \geq 0$ we denote by $\Pi_{t,x}$ the law of $X(t, x)$,

$$\Pi_{t,x} = X(t, x)_{\#}\mathbb{P}, \quad x \in \mathbb{R}^d, \ t \geq 0.$$

We have $\Pi_{0,x} = \delta_x$ for all $x \in \mathbb{R}^d$ and moreover

$$P_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\Pi_{t,x}(dy), \quad \forall\, \varphi \in B_b(\mathbb{R}^d), \ t \geq 0, \ x \in \mathbb{R}^d. \quad (12.3)$$

In particular, for any $\Gamma \in \mathscr{B}(\mathbb{R}^d)$ we have

$$P_t\mathbb{1}_\Gamma(x) = \Pi_{t,x}(\Gamma). \quad (12.4)$$

$\Pi_{t,x}, \ t \geq 0, \ x \in \mathbb{R}^d$ are called the *transition probabilities* of $X(\cdot, \cdot)$ or of P_t.

Proposition 12.7. *Let $\Gamma \in \mathscr{B}(\mathbb{R}^d)$. Then the following statements hold.*

(i) *The mapping*

$$[0, \infty) \times \mathbb{R}^d \to \mathbb{R}, \quad (t, x) \mapsto \Pi_{t,x}(\Gamma),$$

is Borel.

(ii) *For any $t, s \geq 0$, $x \in \mathbb{R}^d$, the following Chapman–Kolmogorov equation holds*

$$\Pi_{t+s,x}(\Gamma) = \int_H \Pi_{s,y}(\Gamma)\Pi_{t,x}(dy). \tag{12.5}$$

Proof. Statement (i) is a consequence of Exercise 12.6 because $\Pi_{t,x}(\Gamma) = (P_t \mathbb{1}_\Gamma)(x)$.

To show (ii), write

$$\Pi_{t+s,x}(\Gamma) = (P_{t+s}\mathbb{1}_\Gamma)(x) = (P_t(P_s\mathbb{1}_\Gamma))(x)$$
$$= (P_t\Pi_{s,\cdot}(\Gamma))(x) = \int_{\mathbb{R}^d} \Pi_{s,y}(\Gamma)\Pi_{t,x}(dy). \qquad \square$$

We say that the mapping

$$[0, +\infty) \times \mathbb{R}^d \to \mathscr{P}(\mathbb{R}^d), \quad (t, x) \mapsto \Pi_{t,x}$$

is a *probability kernel*.

12.2.2. Joint law of $(X(t_1, x), ..., X(t_n, x))$

For any $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $t_1, ..., t_n$ nonnegative we denote by $\Pi_{t_1,...,t_n,x}$ the law of $(X(t_1, x), ..., X(t_n, x))$. Let us find an expression for $\Pi_{t_1,...,t_n,x}$ in terms of P_t. For the sake of simplicity we limit ourselves to $n = 2$.

Proposition 12.8. *Let $x \in \mathbb{R}^d$, $t > s > 0$, $\varphi, \psi \in B_b(H)$. Then*

$$\mathbb{E}[\varphi(X(t, x))\, \psi(X(s, x))] = [P_s(\varphi\, P_{t-s}\psi)](x). \tag{12.6}$$

Proof. By the Markov property and properties of conditional expectation we have

$$\mathbb{E}[\varphi(X(t, x))\, \psi(X(s, x))]$$
$$= \mathbb{E}[\mathbb{E}[\varphi(X(t, x))\, \psi(X(s, x))|\mathscr{F}_s]]$$
$$= \mathbb{E}[\psi(X(s, x))\mathbb{E}[\varphi(X(t, x))\,|\mathscr{F}_s]]$$
$$= \mathbb{E}[\psi(X(s, x))P_{t-s}\varphi(X(s, x))]$$
$$= [P_s(\varphi\, P_{t-s}\psi)](x). \qquad \square$$

Remark 12.9. *Let $x \in \mathbb{R}^d$, $0 < t < s$, $\varphi, \psi \in B_b(H)$. Then*

$$\mathbb{E}[\varphi(X(t,x))\,\psi(X(s,x))] = [P_t(\varphi\, P_{s-t}\psi)](x). \tag{12.7}$$

Exercise 12.10. Let $x \in \mathbb{R}^d$, $n \in \mathbb{N}$, $0 < t_1 < \cdots, t_n, \varphi_1, ..., \varphi_n \in B_b(H)$. Show that

$$\mathbb{E}[\varphi_1(X(t_1,x))...\varphi_n(X(t_n,x))] = [P_{t_1}(\varphi_1....(P_{t_n-t_{n-1}}\varphi_n)](x). \tag{12.8}$$

12.2.3. The transposed semigroup

We shall denote by $C_b^*(\mathbb{R}^d)$ the topological dual of $C_b(\mathbb{R}^d)$; it is endowed with the norm

$$\|F\| = \sup\{|F(x)|,\ x \in \mathbb{R}^d,\ |x| = 1\}, \quad \forall\, F \in C_b^*(\mathbb{R}^d).$$

The duality between $C_b(\mathbb{R}^d)$ and $C_b^*(\mathbb{R}^d)$ will be denoted by $\langle \cdot, \cdot \rangle$.

For any $\mu \in \mathscr{P}(\mathbb{R}^d)$ we define

$$F_\mu(\varphi) := \int_{\mathbb{R}^d} \varphi\, d\mu, \quad \varphi \in C_b(\mathbb{R}^d). \tag{12.9}$$

It is easy to see that $F_\mu \in C_b^*(\mathbb{R}^d)$ and $\|F_\mu\| = 1$.

Let $t > 0$. Since $P_t \in L(C_b(\mathbb{R}^d))$ (in view of Proposition 12.3) we can consider the transposed operator P_t^* of P_t, defined as

$$\langle \varphi, P_t^* F \rangle = \langle P_t\varphi, F \rangle, \quad \forall\, \varphi \in C_b(\mathbb{R}^d),\ F \in C_b^*(\mathbb{R}^d).$$

We shall identify $\mathscr{P}(\mathbb{R}^d)$ with a closed convex subset of $C_b^*(\mathbb{R}^d)$ through the isomorphism

$$\mu \in \mathscr{P}(\mathbb{R}^d) \to F_\mu \in C_b^*(\mathbb{R}^d).$$

Proposition 12.11. *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and let $t \geq 0$. Then $P_t^*\mu \in \mathscr{P}(\mathbb{R}^d)$.*

Proof. The element $P_t^*\mu$ of $C_b^*(\mathbb{R}^d)$ is defined by

$$\langle \varphi, P_t^*\mu \rangle = \langle P_t\varphi, \mu \rangle = \int_{\mathbb{R}^d} P_t\varphi(x)\mu(dx), \quad \forall\, \varphi \in C_b(\mathbb{R}^d).$$

Fix $t > 0$. For any $A \in \mathscr{B}(\mathbb{R}^d)$ set

$$\mu_t(A) = \int_{\mathbb{R}^d} P_t \mathbb{1}_A(x) = \int_{\mathbb{R}^d} \Pi_{t,x}(A)\mu(dx).$$

The integral above is meaningful thanks to Proposition 12.7(i). It is clear that μ_t is additive and that $\mu_t(\mathbb{R}^d) = 1$. Therefore, to show that μ_t is a probability measure it is enough to prove that

$$(A_n) \subset \mathscr{B}(\mathbb{R}^d),\ A_n \uparrow A \Rightarrow \mu_t(A_n) \uparrow \mu_t(A).$$

Since

$$\mu_t(A_n) = \int_{\mathbb{R}^d} \Pi_{t,x}(A_n)\mu(dx),$$

this follows from the dominated convergence theorem. □

12.3. Strong Feller, irreducibility and regularity

Definition 12.12. (i) We say that P_t is *Strong Feller* if it maps $B_b(\mathbb{R}^d)$ in $C_b(\mathbb{R}^d)$ for all $t > 0$.

(ii) P_t is called *irriducible* if

$$P_t(\mathbb{1}_{B(x_0,r)})(y) > 0, \quad \forall\, t > 0,\ r > 0,\ x_0, y \in \mathbb{R}^d,$$

where $B(x_0, r)$ denotes the open ball in $\mathbb{R}^d$ with center x_0 and radius r.

(iii) P_t is called *regular* if $\Pi_{t,x}$ is equivalent to $\Pi_{t,y}$ for all $t > 0, x, y \in \mathbb{R}^d$.

Let us first consider some consequences of irreducibility.

Proposition 12.13. *(i) P_t is irreducible if and only if for each $x_0 \in \mathbb{R}^d$ and each $r > 0$ one has*

$$P_t \mathbb{1}_{B(x_0,r)}(x) = \Pi_{t,x}(B(x_0, r)) > 0, \quad \forall\, x \in \mathbb{R}^d.$$

(ii) Assume that $\varphi \in C_b(\mathbb{R}^d)$ is nonnegative and not identically equal to 0. Then for all $t > 0$ and all $x \in \mathbb{R}^d$ we have $P_t\varphi(x) > 0$.

We express (ii) saying that P_t is *positivity improving*.

Proof. (i) is a simple consequence of Definition 12.12(ii), let us show (ii). Let $x_0 \in \mathbb{R}^d$ such that $\varphi(x_0) > 0$. Then there exists $r > 0$ such that $\varphi(x_0) > 0$ for all $x \in \overline{B(x_0, r)}$, the closure of $B(x_0, r)$. Therefore, setting

$$m = \min_{x \in B(x_0,r)} \varphi(x),$$

we have

$$\varphi(x) \geq m\, \mathbb{1}_{B(x_0,r)},$$

which implies, by the irreducibility of P_t,

$$P_t\varphi(x) \geq m\Pi_{t,x}(B(x_0, r)) > 0, \quad \forall\, x \in \mathbb{R}^d,\ t > 0,$$

as claimed. □

The following important result, *Hasminskii's Theorem*, connects strong Feller, irreducibility and regularity properties of P_t.

Theorem 12.14. *Assume that P_t is strong Feller and irreducible. Then it is regular.*

Proof. Let $t > 0$ and let $x_0, x_1 \in \mathbb{R}^d$ be fixed. We have to show that

$$\Pi_{t,x_0} \ll \Pi_{t,x_1}, \quad \forall\, x_0, x_1 \in \mathbb{R}^d. \tag{12.10}$$

To prove (12.10) it is enough to show the implication

$$E \in \mathscr{B}(\mathbb{R}^d),\ \Pi_{t,x_0}(E) > 0 \Longrightarrow \Pi_{t,x_1}(E) > 0, \quad \forall\, x_0, x_1 \in \mathbb{R}^d. \tag{12.11}$$

Assume that $\Pi_{t,x_0}(E) > 0$. Then for any $h \in (0,t)$ we have by the Chapman–Kolmogorov equation (12.5),

$$\Pi_{t,x_0}(E) = \int_{\mathbb{R}^d} \Pi_{t-h,y}(E)\Pi_{h,x_0}(dy) > 0.$$

It follows that there exists $y_0 \in \mathbb{R}^d$ such that $\Pi_{t-h,y_0}(E) > 0$. Since P_t is strong Feller, the function

$$y \mapsto \Pi_{t-h,y}(E) = P_{t-h}\mathbb{1}_E(y)$$

is continuous, so that there exists $r > 0$ such that

$$\Pi_{t-h,y}(B(y_0,r)) > 0.$$

Moreover, invoking again the Chapman–Kolmogorov equation, we have

$$\begin{aligned}\Pi_{t,x_1}(E) &= \int_{\mathbb{R}^d} \Pi_{t-h,y}(E)\Pi_{h,x_1}(dy)\\ &\geq \int_{B(y_0,r)} \Pi_{t-h,y}(E)\Pi_{h,x_1}(dy) > 0,\end{aligned}$$

because $\Pi_{h,x_1}(B(y_0,r)) > 0$ by the irreducibility of P_t and $\Pi_{t-h,y}(B(y_0,r)) > 0$. So, $\Pi_{t,x_1}(E) > 0$ and (12.11) is proved. □

12.4. Invariant measures

Definition 12.15. We say that $\mu \in \mathscr{P}(\mathbb{R}^d)$ [2] is *invariant* for P_t if

$$\int_{\mathbb{R}^d} P_t\varphi(x)\mu(dx) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \tag{12.12}$$

for all $\varphi \in C_b(\mathbb{R}^d)$ and all $t > 0$.

(2) Recall that $\mathscr{P}(\mathbb{R}^d)$ represents the set of all Borel probability measures on $\mathbb{R}^d$.

It is easy to see that if μ is invariant then identity (12.12) holds for all $\varphi \in B_b(\mathbb{R}^d)$.

Lemma 12.16. *A measure $\mu \in \mathscr{P}(\mathbb{R}^d)$ is invariant for P_t if and only if*

$$\mu(A) = \int_{\mathbb{R}^d} \Pi_{t,x}(A)\mu(dx), \quad \forall\, A \in \mathscr{B}(\mathbb{R}^d). \tag{12.13}$$

Proof. Assume that μ is invariant and that $A \in \mathscr{B}(\mathbb{R}^d)$. Then we have

$$\int_{\mathbb{R}^d} P_t \mathbb{1}_A \, d\mu = \int_{\mathbb{R}^d} \mathbb{1}_A \, d\mu,$$

which is equivalent to (12.13).

Assume conversely that (12.13) is fulfilled. Then for any simple function

$$\varphi := \sum_{k=1}^{n} c_k \mathbb{1}_{A_k},$$

where $A_1, \ldots, A_n \in \mathscr{B}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \varphi \, d\mu = \sum_{k=1}^{n} c_k \mu(A_k) = \int_{\mathbb{R}^d} P_t\varphi \, d\mu.$$

Since any function from $C_b(\mathbb{R}^d)$ is limit of simple functions, the conclusion follows. □

Exercise 12.17. Show that $\mu \in \mathscr{P}(\mathbb{R}^d)$ is invariant if and only if

$$P_t^* \mu = \mu, \quad \forall\, t \ge 0.$$

Hint. Recall (12.4).

12.4.1. Stationary processes

Definition 12.18. We say that a stochastic process $Z(t)$, $t \ge 0$ is *stationary* if for any $n \in \mathbb{N}$, $t_1, \ldots, t_n \ge 0$ and any $h > 0$ one has

$$\text{Law }(Z(t_1 + h), \ldots, Z(t_n + h)) = \text{Law }(Z(t_1), \ldots, Z(t_n)). \tag{12.14}$$

Property (12.14) can be interpreted by saying that informations available at time t coincide with that available at time $t + h$ for all $h > 0$.

Let μ be an invariant measure for P_t. Assume that there is a filtration $(\mathscr{G}_t)_{t\ge 0}$ in $(\Omega, \mathscr{F}, \mathbb{P})$ larger than $(\mathscr{F}_t)_{t\ge 0}$ and such that

(i) B is non anticipating with respect to $(\mathscr{G}_t)_{t\geq 0}$ [3]

(ii) The law of μ coincides with η.

We are going to show that $X(\cdot,\eta)$, the solution of (12.1) starting from $\eta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$, is a stationary process. For this we need some preliminaries.

For any random variable $\zeta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ we denote by $\Pi_{t,\zeta}$ the law of $X(t,\zeta)$ for all $t \geq 0$.

Proposition 12.19. *Assume that $\zeta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ and denote by ν the law of ζ. Then the following identities hold*

$$\int_{\mathbb{R}^d} f(x)\Pi_{t,\zeta}(dx) = \int_{\mathbb{R}^d} P_t f(x)\nu(dx), \quad \forall\, f \in B_b(\mathbb{R}^d). \tag{12.15}$$

and

$$\Pi_{t,\zeta}(I) = \int_{\mathbb{R}^d} \Pi_{t,x}(I)\nu(dx), \quad \forall\, I \in \mathscr{B}(\mathbb{R}^d), \tag{12.16}$$

Proof. It is enough to prove (12.15) because then (12.16) will follow setting in (12.15) $f = \mathbb{1}_I$.

To prove (12.15) write

$$\int_{\mathbb{R}^d} f(x)\Pi_{t,\zeta}(dx) = \mathbb{E}[f(X(t,\zeta)].$$

By the Markov property we deduce

$$\int_{\mathbb{R}^d} f(x)\Pi_{t,\zeta}(dx) = \mathbb{E}[(P_t f)(\zeta)] = \int_{\mathbb{R}^d} P_t f(x)\nu(dx).$$

□

Proposition 12.20. *Assume that $\mu \in \mathscr{B}(H)$ is invariant for P_t and that $\eta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ is the law is μ. Then we have*

$$\Pi_{t,\eta} = \mu. \tag{12.17}$$

Proof. Just note that by (12.15) (with η replacing ζ and μ replacing ν) it follows that, taking into account the invariance of μ

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\Pi_{t,\eta}(dx) &= \int_{\mathbb{R}^d} P_t f(x)\mu(dx)\\ &= \int_{\mathbb{R}^d} f(x)\mu(dx), \quad \forall\, f \in B_b(\mathbb{R}^d).\end{aligned}$$

[3] See Remark 6.1.

This implies

$$\Pi_{t,\eta} = \mu,$$

as claimed. □

Let us now consider the law of $(X(t,\eta), X(s,\eta))$, where $t, s \geq 0$ and $\eta \in \mathscr{G}_0$, which we denote by $\Pi_{t,s,\eta}$.

Proposition 12.21. *Assume that $\mu \in \mathscr{P}(H)$ is invariant for P_t and that $\eta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ is the law of μ. Then for any $t, s \geq 0$ and any $h > 0$ we have*

$$\Pi_{t+h,s+h,\eta} = \Pi_{t,s,\eta}. \tag{12.18}$$

Proof. We claim that

$$\int_{\mathbb{R}^2} \varphi(x)\psi(y)\Pi_{t,s,\eta}(dx, dy) = \int_{\mathbb{R}^d} P_s(\varphi P_{t-s}\psi)(x)\mu(dx). \tag{12.19}$$

By the claim and the invariance of μ we have

$$\int_{\mathbb{R}^2} \varphi(x)\psi(y)\Pi_{t,s,\eta}(dx, dy) = \int_{\mathbb{R}^d} \varphi(x)P_{t-s}\psi(x)\mu(dx),$$

from which (12.18) follows at once.

To prove the claim use the Markov property and write

$$\begin{aligned}\int_{\mathbb{R}^2} \varphi(x)\psi(y)\Pi_{t,s,\eta}(dx, dy) &= \mathbb{E}[\varphi(X(t,\eta))\psi(X(s,\eta))]\\[2ex] &= \mathbb{E}[\varphi(X(s,\eta))\mathbb{E}[\psi(X(t,\eta))|\mathscr{G}_s]]\\[2ex] &= \mathbb{E}[\varphi(X(s,\eta))P_{t-s}\psi(X(s,\eta))]\\[2ex] &= \mathbb{E}[g(X(s,\eta))],\end{aligned}$$

where

$$g(x) = \varphi(x)P_{t-s}\psi(x).$$

The proof is complete □

Now, given $t_1, ..., t_n \geq 0$ we consider in a similar way the law of

$$(X(t_1,\eta), ..., X(t_1,\eta))$$

which we denote by $\Pi_{t_1,...,t_n,\mu}$.

Exercise 12.22. Assume that μ is an invariant measure for P_t and that η is a $\mathscr{G}_0$-measurable with law equal to μ. Show that for any $h > 0$ one has

$$\Pi_{t_1+h,\dots,t_n+h,\eta} = \Pi_{t_1,\dots,t_n,\eta}. \tag{12.20}$$

Remark 12.23. By Exercise 12.22 it follows that if μ is an invariant measure for P_t and $\eta \in L^2(\Omega, \mathscr{G}_0, \mathbb{P})$ has law equal to μ, then $X(t,\eta)$, $t \geq 0$ is a stationary process.

12.4.2. Ergodicity

An important property of an invariant measure μ of P_t is that P_t can be uniquely extended to a strongly continuous semigroup of contractions in $L^P(H,\mu)$ for all $p \geq 1$.

Theorem 12.24. *Assume that μ is an invariant measure of P_t. Then for all $t \geq 0$, $p \geq 1$, P_t has a unique extension to a linear bounded operator on $L^p(\mathbb{R}^d,\mu)$ (which we still denote by P_t). Moreover*

$$\|P_t\varphi\|_{L^p(\mathbb{R}^d,\mu)} \leq \|\varphi\|_{L^p(\mathbb{R}^d,\mu)}, \quad \forall\, \varphi \in L^p(\mathbb{R}^d,\mu),\ t \geq 0. \tag{12.21}$$

Finally, semigroup P_t is strongly continuous on $L^p(\mathbb{R}^d,\mu)$.

Proof. Let first take $\varphi \in C_b(\mathbb{R}^d)$. By the Hölder inequality we have

$$|P_t\varphi(x)|^p \leq \int_{\mathbb{R}^d} |\varphi(y)|^p \Pi_{t,x}dy) = P_t(\varphi^p)(x), \quad x \in \mathbb{R}^d.$$

Now, integrating this inequality with respect to μ over $\mathbb{R}^d$, yields

$$\int_{\mathbb{R}^d} |P_t\varphi(x)|^p \mu(dx) \leq \int_{\mathbb{R}^d} P_t(|\varphi|^p)(x)\mu(dx) = \int_{\mathbb{R}^d} |\varphi(x)|^p \mu(dx),$$

in view of the invariance of μ. Since $C_b(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d,\mu)$, then P_t can be uniquely extended to a linear bounded operator in $L^p(\mathbb{R}^d,\mu)$ and (12.21) follows.

As regards the last statement, it is enough to show that

$$\lim_{t\to 0} \int_{\mathbb{R}^d} |P_t\varphi(x) - \varphi(x)|^p \mu(dx) = 0,$$

for all $\varphi \in L^p(H,\mu)$. This follows easily when $\varphi \in C_b(\mathbb{R}^d)$ and it follows for all $\varphi \in L^p(H,\mu)$ by density in view of (12.21). □

An important consequence of Theorem 12.24 is the *Von Neumann theorem* on the existence of the limit as $T \to +\infty$ of time averages $M(T)\varphi$ of $P_t\varphi$,

$$M(T)\varphi = \frac{1}{T}\int_0^T P_s\varphi ds, \quad \varphi \in L^2(\mathbb{R}^d, \mu),\ T > 0.$$

The proof is based on a classical result on convergence of averages of linear operators, recalled in Appendix E.

Let us denote by Σ the set

$$\Sigma = \{f \in L^2(\mathbb{R}^d, \mu) : P_t f = f,\ \mu\text{-a.e.}\ \forall\, t \geq 0\},$$

of all *stationary* points of P_t. Clearly Σ is a closed subspace of $L^2(\mathbb{R}^d, \mu)$ and $\mathbb{1}$ (the function identically equal to 1) belongs to Σ.

Theorem 12.25. *For all $\varphi \in L^2(\mathbb{R}^d, \mu)$ there exists the limit*

$$\lim_{T\to\infty} M(T)\varphi =: M_\infty\varphi \quad \text{in } L^2(\mathbb{R}^d, \mu). \tag{12.22}$$

Moreover $M_\infty \in L(L^2(\mathbb{R}^d, \mu))$, $M_\infty^2 = M_\infty$, $M_\infty(L^2(\mathbb{R}^d, \mu)) = \Sigma$ and

$$\int_{\mathbb{R}^d} M_\infty\varphi d\mu = \int_H \varphi d\mu. \tag{12.23}$$

Proof. For all $T > 0$ write

$$T = n_T + r_T, \quad n_T \in \mathbb{N} \cup \{0\},\ r_T \in [0, 1).$$

Let $\varphi \in L^2(H, \mu)$. Then we have

$$\begin{aligned}
M(T)\varphi &= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_k^{k+1} P_s\varphi ds + \frac{1}{T}\int_{n_T}^T P_s\varphi ds \\
&= \frac{1}{T}\sum_{k=0}^{n_T-1}\int_0^1 P_{s+k}\varphi ds + \frac{1}{T}\int_0^{r_T} P_{s+n(T)}\varphi ds \qquad (12.24)\\
&= \frac{n_T}{T}\,\frac{1}{n_T}\sum_{k=0}^{n_T-1}(P_1)^k M(1)\varphi + \frac{r_T}{T}(P_1)^{n_T} M(r_T)\varphi.
\end{aligned}$$

Since

$$\lim_{T\to\infty}\frac{n_T}{T} = 1,\ \lim_{T\to\infty}\frac{r_T}{T} = 0,$$

letting $n \to \infty$ in (12.24) and using Theorem E.1, we get (12.22).

We prove now that for all $t \geq 0$

$$M_\infty P_t = P_t M_\infty = M_\infty. \tag{12.25}$$

In fact, given $t \geq 0$ and $\varphi \in L^2(H, \mu)$ we have

$$\begin{aligned} M_\infty P_t \varphi &= \lim_{T\to\infty} \frac{1}{T} \int_0^T P_{t+s}\varphi ds = \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} P_s \varphi ds \\ &= \lim_{T\to\infty} \frac{1}{T} \left\{ \int_0^T P_s\varphi ds - \int_0^t P_s\varphi ds + \int_T^{T+t} P_s\varphi ds \right\} \\ &= M_\infty \varphi, \end{aligned}$$

and this yields (12.25).

By (12.25) it follows that $M^\infty f \in \Sigma$ for all $f \in L^2(\mathbb{R}^d, \mu)$, and moreover that

$$M_\infty M(T) = M(T) P_\infty = M_\infty,$$

that yields, letting $T \to \infty$, $M_\infty^2 = M_\infty$. Finally, let us prove (12.23). By the invariance of μ and Fubini's Thorem we have

$$\int_{\mathbb{R}^d} M(T)\varphi d\mu = \frac{1}{T} \int_0^T \left(\int_{\mathbb{R}^d} P_t \varphi d\mu \right) dt = \int_{\mathbb{R}^d} \varphi d\mu.$$

So, the conclusion follows letting $T \to \infty$. □

Definition 12.26. Let μ be an invariant measure of P_t.

(i) μ is called *ergodic* if for all $\varphi \in L^2(\mathbb{R}^d, \mu)$ it results

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T P_t \varphi dt = \int_{\mathbb{R}^d} \varphi(x)\mu(dx) \quad \text{in } L^2(\mathbb{R}^d, \mu), \tag{12.26}$$

(ii) μ is called *strongly mixing* if

$$\lim_{T\to\infty} P_t \varphi = \int_{\mathbb{R}^d} \varphi \, d\mu \quad \text{in } L^2(\mathbb{R}^d, \mu), \tag{12.27}$$

for all $\varphi \in L^2(\mathbb{R}^d, \mu)$.

Ergodicity is often interpreted by saying that the "time average" $\frac{1}{T} \int_0^T P_t \varphi dt$ converge as $T \to \infty$ to the "space" average $\int_{\mathbb{R}^d} \varphi(x)\mu(dx)$.

Proposition 12.27. *Let μ be an invariant measure of P_t. Then μ is ergodic if and only if the set Σ of all stationary points of P_t has dimension one.*

Proof. Let μ be ergodic. Then from (12.26) it follows that any element of Σ is constant, so that Σ is one-dimensional. Conversely, if Σ is one-dimensional there is a linear bounded functional F on $L^2(\mathbb{R}^d, \mu)$ such that

$$M_\infty \varphi = F(\varphi)\mathbb{1}.$$

By the Riesz representation theorem there exists $\varphi_0 \in L^2(\mathbb{R}^d, \mu)$ such that $F(\varphi) = \langle \varphi, \varphi_0 \rangle_{L^2(\mathbb{R}^d,\mu)}$. Integrating this equality over H with respect to μ and recalling (12.23), yields

$$\begin{aligned}\int_H M_\infty \varphi d\mu &= \int_{\mathbb{R}^d} \varphi d\mu = \langle \varphi, \mathbb{1} \rangle_{L^2(\mathbb{R}^d,\mu)} \\ &= \langle \varphi, \varphi_0 \rangle_{L^2(\mathbb{R}^d,\mu)}, \;\; \varphi \in L^2(\mathbb{R}^d, \mu).\end{aligned}$$

Therefore $\varphi_0 = \mathbb{1}$. □

12.4.3. Conditions for ergodicity

Definition 12.28. Let μ be an invariant measure of P_t. A Borel set $\Gamma \in \mathscr{B}(\mathbb{R}^d)$ is said to be *invariant* for P_t if its characteristic function $\mathbb{1}_\Gamma$ belongs to Σ. If $\mu(\Gamma)$ is equal to 0 or to 1, we say that Γ is *trivial*, otherwise that it is *non trivial*.

We are going to show that μ is ergodic if and only if all invariant sets of μ are trivial. For this it is important to realize that the set of all stationary points Σ is a *lattice*, as next proposition shows.

Proposition 12.29. *Assume that φ and ψ belong to Σ. Then*

(i) $|\varphi| \in \Sigma$.

(ii) $\varphi^+, \varphi^- \in \Sigma$ [4].

(iii) $\varphi \vee \psi,\; \varphi \wedge \psi \in \Sigma$ [5].

(iv) *For any $a \in \mathbb{R}$, function $\mathbb{1}_{\{\varphi > a\}}$ belongs to Σ.*

(4) $\varphi^+ = \max\{\varphi, 0\}$, $\varphi^- = \max\{-\varphi, 0\}$

(5) $\varphi \vee \psi = \max\{\varphi, \psi\}$, $\varphi \wedge \psi = \min\{\varphi, \psi\}$.

Proof. Let us prove (i). Let $t > 0$ and assume that $\varphi \in \Sigma$, so that $\varphi(x) = P_t\varphi(x)$ μ-a.s. Then we have

$$|\varphi(x)| = |P_t\varphi(x)| \le (P_t|\varphi|)(x), \quad \mu\text{-a.s.} \tag{12.28}$$

We claim that

$$|\varphi(x)| = (P_t|\varphi|)(x), \quad \mu\text{-a.s.}$$

Assume by contradiction that there is a Borel set $I \subset \mathbb{R}^d$ such that $\mu(I) > 0$ and

$$|\varphi(x)| < (P_t|\varphi|)(x), \quad \forall\, x \in I.$$

Then we have

$$\int_{\mathbb{R}^d} |\varphi(x)|\mu(dx) < \int_{\mathbb{R}^d} (P_t|\varphi|)(x)\mu(dx).$$

Since, by the invariance of μ,

$$\int_{\mathbb{R}^d} (P_t|\varphi|)(x)\mu(dx) = \int_{\mathbb{R}^d} |\varphi|(x)\mu(dx),$$

we find a contradiction.

Statements (ii) and (iii) follow from the obvious identities

$$\varphi^+ = \frac{1}{2}(\varphi + |\varphi|), \quad \varphi^- = \frac{1}{2}(\varphi - |\varphi|),$$

$$\varphi \vee \psi = (\varphi - \psi)^+ + \psi, \quad \varphi \wedge \psi = -(\varphi - \psi)^+ + \varphi.$$

Finally, let us prove (iv). It is enough to show that the set $\{\varphi > 0\}$ is invariant, or, equivalently, that $1\!\!1_{\{\varphi>0\}}$ belongs to Σ. We have in fact, as it is easily checked,

$$1\!\!1_{\{\varphi>0\}} = \lim_{n\to\infty} \varphi_n(x), \quad x \in H,$$

where $\varphi_n = (n\varphi^+) \wedge 1$, $n \in \mathbb{N}$, belongs to Σ by (ii) and (iii). Therefore $\{\varphi > 0\}$ is invariant. $\square$

We are now ready to prove the result.

Theorem 12.30. *Let μ be an invariant measure of P_t. Then μ is ergodic if and only if all their invariant sets are trivial.*

Proof. Let Γ be invariant for μ. Then if μ is ergodic $1\!\!1_\Gamma$ must be constant (otherwise $\dim \Sigma \ge 2$) and so Γ is trivial.

Assume conversely that the only invariant sets for μ are trivial and, by contradiction, that μ is not ergodic. Then there exists a non constant function $\varphi_0 \in \Sigma$. Therefore by Proposition 12.29(iv) for some $\lambda \in \mathbb{R}$ the invariant set $\{\varphi_0 > \lambda\}$ is not trivial. $\square$

12.4.4. The structure of the set of invariant measures

We denote by Ξ the set of all invariant measures of P_t. We first show

Theorem 12.31. *Assume that P_t possesses exactly one invariant measure μ. Then μ is ergodic.*

Proof. Assume by contradiction that μ is not ergodic. Then μ possesses a nontrivial invariant set Γ. We are going to prove that the measure μ_Γ defined as

$$\mu_\Gamma(A) = \frac{1}{\mu(\Gamma)}\, \mu(A \cap \Gamma), \quad \forall\, A \in \mathscr{B}(\mathbb{R}^d),$$

belongs to Ξ. This will produce a contradiction in view of Theorem 12.30.

We have to show (recall (12.13)) that

$$\mu_\Gamma(A) = \int_{\mathbb{R}^d} \Pi_{t,x}(A)\mu_\Gamma(dx), \quad \forall\, A \in \mathscr{B}(\mathbb{R}^d),$$

or, equivalently, that

$$\mu(A \cap \Gamma) = \int_\Gamma \Pi_{t,x}(A)\mu(dx), \quad \forall\, A \in \mathscr{B}(\mathbb{R}^d). \tag{12.29}$$

In fact, since Γ is invariant for μ, we have.

$$P_t \mathbb{1}_\Gamma(x) = \mathbb{1}_\Gamma(x), \quad P_t \mathbb{1}_{\Gamma^c}(x) = \mathbb{1}_{\Gamma^c}(x), \quad t \geq 0,\ x\ \mu\text{-a.e.},$$

where Γ^c is the complement of Γ. Since

$$\Pi_{t,x}(\Gamma) = \begin{cases} 1, & \text{if } x \in \Gamma \quad \mu\text{-a.e.} \\ 0, & \text{if } x \in \Gamma^c \quad \mu\text{-a.e.,} \end{cases}$$

we have

$$\begin{cases} \Pi_{t,x}(A \cap \Gamma^c) = 0, & \mu\text{-a.e. in } \Gamma \\[2ex] \Pi_{t,x}(A \cap \Gamma) = 0, & \mu\text{-a.e. in } \Gamma^c, \end{cases}$$

Therefore

$$\begin{aligned} &\int_{\mathbb{R}^d} \Pi_{t,x}(A)\mu(dx) \\ &= \int_\Gamma \Pi_{t,x}(A \cap \Gamma)\mu(dx) + \int_\Gamma \Pi_{t,x}(A \cap \Gamma^c)\mu(dx) \\ &= \int_\Gamma \Pi_{t,x}(A \cap \Gamma)\mu(dx) = \int_{\mathbb{R}^d} \Pi_{t,x}(A \cap \Gamma)\mu(dx) = \mu(A \cap \Gamma), \end{aligned}$$

and identity (12.29) follows. $\square$

12.4.4.1. Digression on extremal points. Let X be a Banach space and let K be a convex closed non empty subset of X. We say that a point $x_0 \in K$ is *extremal* if does not belong to any open segment included in K.

Theorem 12.32 (Krein–Milman). *The set K_e of all extremal points of K is non empty. Moreover, K coincides with the closure of all linear convex combinations of points from K_e.*

For a proof see *e.g.* [21, pages 362-363].

Let us now show that the set of all extremal points of Ξ, the set of of all invariant measures of P_t, is precisely the set of all its ergodic measures.

For this we need a lemma.

Lemma 12.33. *Let $\mu, \nu \in \Xi$ with μ ergodic and ν absolutely continuous with respect to μ. Then $\mu = \nu$.*

Proof. Let $\Gamma \in \mathscr{B}(\mathbb{R}^d)$. By the Von Neumann theorem there is a sequence $T_n \uparrow +\infty$ such that

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} P_t \mathbb{1}_\Gamma dt = \mu(\Gamma), \quad \mu\text{-a.e.} \tag{12.30}$$

Since $\nu \ll \mu$, identity (12.30) holds also ν-a.e. Now integrating (12.30) with respect to ν yields, taking into account the invariance of ν,

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} \left(\int_H P_t \mathbb{1}_\Gamma d\nu \right) dt = \nu(\Gamma), \quad \mu\text{-a.e.}$$

Consequently $\nu(\Gamma) = \mu(\Gamma)$ as required. □

Theorem 12.34. *The set of all invariant ergodic measures of P_t coincides with the set of all extremal points of Ξ.*

Proof. We claim that if μ is ergodic then it is an extremal point of Ξ, otherwise there exist $\mu_1, \mu_2 \in \Xi$ with $\mu_1 \neq \mu_2$, and $\alpha \in (0, 1)$ such that

$$\mu = \alpha\mu_1 + (1 - \alpha)\mu_2.$$

Then clearly $\mu_1 \ll \mu$ and $\mu_2 \ll \mu$. By Lemma 12.33 we find a contradiction. We finally prove that if μ is an extremal point of Ξ, then it is ergodic.

In fact, assume by contradiction that μ is not ergodic; then μ possesses a nontrivial invariant set Γ. Consequently, by Lemma 12.33, we have $\mu_\Gamma, \mu_{\Gamma^c} \in \Lambda$. Since

$$\mu = \mu(\Gamma)\mu_\Gamma + (1 - \mu(\Gamma))\mu_{\Gamma^c},$$

we find that μ is not extremal, again a contradiction. □

We finally show that different ergodic measures are mutually singular.

Proposition 12.35. *Assume that μ and ν are ergodic invariant measures such that $\mu \neq \nu$. Then μ and ν are singular.*

Proof. Let $\Gamma \in \mathscr{B}(\mathbb{R}^d)$ such that $\mu(\Gamma) \neq \nu(\Gamma)$. Then by the Von Neumann theorem there exists a sequence $T_n \uparrow +\infty$ and two Borel sets M and N such that $\mu(M) = 1$, $\nu(N) = 1$, and

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} (P_t 1\!\!1_\Gamma)(x)dt = \mu(\Gamma), \quad \forall\, x \in M,$$

$$\lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} (P_t 1\!\!1_\Gamma)(x)dt = \nu(\Gamma), \quad \forall\, x \in N.$$

Since $\mu(\Gamma) \neq \nu(\Gamma)$ this implies that $M \cap N = \varnothing$, and so μ and ν are singular. □

12.5. Existence and uniqueness of invariant measures

12.5.1. Weak convergence and tightness

Definition 12.36. (i) Let $(\mu_n) \subset \mathscr{P}(\mathbb{R}^d)$ and $\mu \in \mathscr{P}(\mathbb{R}^d)$. We say that (μ_n) is *weakly convergent* to μ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi\, d\mu_n = \int_{\mathbb{R}^d} \varphi\, d\mu, \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \tag{12.31}$$

(ii) Let $(\mu_t))_{t\geq 0} \subset \mathscr{P}(\mathbb{R}^d)$ and $\mu \in \mathscr{P}(\mathbb{R}^d)$. We say that $(\mu_t)_{t\geq 0}$ is *weakly convergent* to μ as $t \to \infty$ if

$$\lim_{t\to\infty} \int_{\mathbb{R}^d} \varphi\, d\mu_t = \int_{\mathbb{R}^d} \varphi\, d\mu, \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \tag{12.32}$$

(iii) A set $\Gamma \subset C_b^*(\mathbb{R}^d)$ is called *tight* if for any $\epsilon > 0$ there exists a compact set K_ϵ in $\mathbb{R}^d$ such that

$$\mu(K_\epsilon) \geq 1 - \epsilon, \quad \forall\, \mu \in \Gamma. \tag{12.33}$$

Theorem 12.37 (Prokhorov). *Assume that $\Gamma \subset \mathscr{P}(\mathbb{R}^d)$ is tight. Then there exists a sequence $(\mu_n) \subset \Gamma$ and $\mu \in \mathscr{P}(\mathbb{R}^d)$ such that (μ_n) is weakly convergent to μ.*

For a proof see *e.g.* [17].

12.5.2. Existence of invariant measure

A simple (but strong) condition implying existence and uniqueness of an invariant measure of P_t is provided by the following

Proposition 12.38. *Let $\mu \in \mathscr{P}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$ such that the sequence (Π_{t,x_0}) is weakly convergent to μ for some $x_0 \in \mathbb{R}^d$. Then μ is an invariant measure of P_t.*

Proof. By the assumption we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(y)\, \Pi_{t,x_0}(dy) = \int_{\mathbb{R}^d} \varphi(y)\, \mu(dy), \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \quad (12.34)$$

Now (12.34) is equivalent to

$$\lim_{t\to\infty} P_t\varphi(x_0) = \int_{\mathbb{R}^d} \varphi(y)\, \mu(dy), \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \quad (12.35)$$

Let $s \geq 0$ and $\varphi \in C_b(\mathbb{R}^d)$. By (12.35) with $P_s\varphi$ replacing φ, we find

$$\lim_{t\to\infty} P_{t+s}\varphi(x_0) = \int_{\mathbb{R}^d} P_s\varphi(y)\, \mu(dy)$$

$$= \lim_{t\to\infty} P_{t+s}\varphi(x_0) = \int_{\mathbb{R}^d} \varphi(y)\, \mu(dy),$$

so that μ is invariant. □

An important existence result is provided by the following *Krylov-Bogoliubov Theorem*. In its formulation we shall use the following notation. For any $x \in \mathbb{R}^d$ and $T > 0$ we define

$$\mu_{T,x}(A) = \frac{1}{T} \int_0^T \Pi_{t,x}(A)dt, \quad A \in \mathscr{B}(\mathbb{R}^d). \quad (12.36)$$

It is easy to check that $\mu_{T,x} \in \mathscr{P}(\mathbb{R}^d)$ for any $T > 0$ and any $x \in \mathbb{R}^d$.

Theorem 12.39. *Assume that the family of probability measures $(\mu_{T,x_0})_{T\geq 0}$ is tight for some $x_0 \in \mathbb{R}^d$. Then there exists an invariant measure μ of P_t.*

Proof. Since the family $(\mu_T)_{T\geq 0}$ is tight, by the Prokhorov theorem, there exists a sequence $T_n \uparrow \infty$ and a measure $\mu \in \mathscr{P}(\mathbb{R}^d)$ such that (μ_{T_k}) is weakly convergent to μ, that is

$$\lim_{k\to\infty} \int_{\mathbb{R}^d} \varphi(x)\mu_{T_k}(dx) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \quad (12.37)$$

Notice that by Fubini's theorem identity (12.37) is equivalent to

$$\lim_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} dt \int_{\mathbb{R}^d} \varphi(x)\Pi_{t,x_0}(dx) = \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \quad \forall\, \varphi \in C_b(\mathbb{R}^d) \tag{12.38}$$

and also (recalling (12.3)) to

$$\lim_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} P_t\varphi(x_0)dt = \int_{\mathbb{R}^d} \varphi(x)\mu(dx), \quad \forall\, \varphi \in C_b(\mathbb{R}^d). \tag{12.39}$$

Now we can show that μ is invariant for P_t. Fix in fact $s \geq 0$ and replace in (12.39) φ with $P_s\varphi$, we find

$$\begin{aligned}\int_{\mathbb{R}^d} P_s\varphi(x)\mu(dx) &= \lim_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} P_{t+s}\varphi(x_0)dt \\ &= \lim_{k\to\infty} \frac{1}{T_k} \int_s^{s+T_k} P_t\varphi(x_0)dt \\ &= \lim_{k\to\infty} \frac{1}{T_k} \int_0^{T_k} P_s\varphi(x_0)dt = \int_{\mathbb{R}^d} \varphi(x)\mu(dx). \qquad \square\end{aligned}$$

Note that in the last part of the proof of the theorem we have used the fact that P_t acts on $C_b(\mathbb{R}^d)$, in other words that it is Feller.

Remark 12.40. (i) Assume that the family of probability measures $(\Pi_{T,x_0})_{T\geq 0}$ is tight for some $x_0 \in \mathbb{R}^d$. Then obviously the family $(\mu_{T,x_0})_{T\geq 0}$ defined by (12.36) is tight as well, so that the assumption of Theorem 12.39 is fulfilled and we can apply the Krylov–Bogoliubov Theorem.

(ii) Assume that there exists $t_0 > 0$ and $x_0 \in \mathbb{R}^d$ such that the family of probability measures $(\mu_{T,t_0,x_0})_{T\geq t_0}$,

$$\mu_{T,t_0,x_0} = \frac{1}{T-t_0} \int_{t_0}^T \Pi_{s,x_0} ds$$

is tight. Then it is easy to see that Krylov–Bogoliubov Theorem generalizes and there exists an invariant measure μ of P_t.

12.5.3. Uniqueness of invariant measures

Proposition 12.41. *Assume that the transition semigroup P_t is regular and possesses an invariant measure μ. Then*

(i) *μ is equivalent to $\Pi_{t,x}$ for all $t > 0$ and all $x \in \mathbb{R}^d$.*

(ii) *μ is the unique invariant measure of P_t and so, it is* ergodic *(by Theorem* 12.31*).*

Recall that a sufficient condition for P_t to be regular is that it is irreducible and strong Feller (Theorem 12.14).

Proof. Since μ is invariant, by Lemma 12.16 we have

$$\mu(A) = \int_{\mathbb{R}^d} \Pi_{t,y}(A)\mu(dy), \tag{12.40}$$

for any $A \in \mathscr{B}(\mathbb{R}^d)$ and any $t > 0$.

Let now $x \in \mathbb{R}^d$. We claim that μ is equivalent to $\Pi_{t,x}$. In fact if for some $A \in \mathscr{B}(\mathbb{R}^d)$ we have $\Pi_{t,x}(A) = 0$ then $\Pi_{t,y}(A) = 0$ for all $y \in H$ since P_t is regular. Therefore $\mu(A) = 0$ by (12.40) and so $\mu \ll \Pi_{t,x}$.

Conversely, if $\mu(A) = 0$ we have, again by (12.40), $\Pi_{t,y}A) = 0$ for μ-almost all $y \in \mathbb{R}^d$. Again by the regularity of P_t we conclude that $\Pi_{t,y}(A) = 0$ for all $y \in H$. Therefore $\Pi_{t,x} \ll \mu$.

It remains to prove the uniqueness of μ. Assume by contradiction that there exist two ergodic measures μ and ν. Then we know by Proposition 12.35 that μ and ν are singular. Then there exist $A, B \in \mathscr{B}(\mathbb{R}^d)$ disjoint and such that μ is concentrated on A and ν is concentrated in B.

Since $\mu(A) = \nu(B) = 1$ we have, by the first part of the proof,

$$\Pi_{t,x}(A) = \Pi_{t,x}(B) = 1, \quad \forall\, t > 0,\ x \in \mathbb{R}^d,$$

which implies $\Pi_{t,x}(A \cup B) = 2$ for all $t > 0,\ x \in \mathbb{R}^d$, a contradiction. □

12.6. Invariant measures for differential stochastic equations

In this section we are going to show existence and uniqueness of invariant measures for equation (12.1) under Hypothesis 12.1. As regards existence, we shall apply the Krylov-Bogoliubov Theorem. To establish uniqueness, we shall limit ourselves to the case $\det C > 0$. In this case we show that the transition semigroup P_t is irreducible and strong Feller; this will imply that P_t is regular by the Hasminskii Theorem, so that the uniqueness of invariant measure will follow from Proposition 12.41. We start with the Strong Feller property.

12.6.1. Strong Feller property

Theorem 12.42. *Assume that* $\det C > 0$. *Then* P_t *is strong Feller.*

Proof. We first note that from (9.56) for any $T > 0$ there is $C_T > 0$ such that

$$\mathbb{E}|D_x X(t,x)h|^2 \le C_T |h|^2, \quad \forall\, t \in [0,T], \quad \forall\, h \in \mathbb{R}^d. \tag{12.41}$$

Then we conclude the proof in two steps.

Step 1. For all $\varphi \in C_b(\mathbb{R}^d)$ and all $x, y \in \mathbb{R}^d$ there is $C_{2,T} > 0$ such that

$$|P_t\varphi(x) - P_t\varphi(y)| \le C_{2,T}\, t^{-1/2}|x-y|, \quad t \in [0,T]. \tag{12.42}$$

By the Bismut–Elworthy–Li formula (Proposition 9.22) and the Hölder inequality we have in fact for any $h \in \mathbb{R}^d$

$$|\langle D_x P_t\varphi(x), h\rangle|^2 \le t^{-2} C_1^2 \|\varphi\|_0^2 \int_0^t \mathbb{E}|D_x X(s,x)h|^2 ds,$$

which in view of (12.41), yields

$$|\langle D_x P_t\varphi(x), h\rangle|^2 \le t^{-1} C_1^2 C_T |h|^2.$$

Now the conclusion follows from the arbitrariness of h and the mean value theorem.

Step 2. Conclusion.

Fix $t > 0$ and $x, y \in \mathbb{R}^d$. Then consider the linear functional (in fact a signed measure) in $C_b^*(\mathbb{R}^d)$ defined for all $\psi \in C_b(\mathbb{R}^d)$ as

$$F(\psi) = P_t\psi(x) - P_t\psi(y) = \Pi_{t,x}(\psi) - \Pi_{t,y}(\psi).$$

Then F is a signed measure, so that it acts on $B_b(\mathbb{R}^d)$. Finally, if $\varphi \in B_b(\mathbb{R}^d)$ we have

$$\begin{aligned}
P_t\varphi(x) - P_t\varphi(y) &= \int_{\mathbb{R}^d} \varphi(z)(\Pi_{t,x}(dz) - \Pi_{t,y}(dz)) \\
&\le \|\varphi\|_0 \|\Pi_{t,x} - \Pi_{t,y}\|_{C_b^*(\mathbb{R}^d)} \\
&\le \|\varphi\|_0 \sup_{\psi \in C_b(\mathbb{R}^d),\ \|\psi\|_0 \le 1} |F(\psi)| \\
&= \|\varphi\|_0 \sup_{\psi \in C_b(\mathbb{R}^d),\ \|\psi\|_0 \le 1} |P_t\psi(x) - P_t\psi(y)| \\
&\le C_{2,T}\, t^{-1/2}|x-y|,
\end{aligned}$$

in view of (12.42). This shows that $P_t\varphi$ is Lipschitz for any $t > 0$. □

Now we study irreducibility of P_t.

12.6.2. Irreducibility

We assume here that $\det C > 0$. To show that P_t is irreducible it is enough to show that for any $r > 0$ and any $x_0, x_1 \in \mathbb{R}^d$ we have

$$\mathbb{P}(|X(t, x_0) - x_1| \leq r) > 0. \tag{12.43}$$

We fix from now on r, x_0, x_1.

It is convenient to introduce the control system

$$\begin{cases} \xi'(t) = b(\xi(t)) + u(t), \quad t \geq 0, \\ \\ \xi(0) = x_0, \end{cases} \tag{12.44}$$

where $u \in C([0, T]; \mathbb{R}^d)$. Since b is Lipschitz, equation (12.44) has a unique solution $\xi \in C^1([0, T]; \mathbb{R}^d)$ which we denote by $\xi(t, x_0; u)$. We need a *controllability* result.

Proposition 12.43. *For any $x_0, \ x_1 \in \mathbb{R}^d$ and any $T > 0$ there exists $u \in C([0, T]; \mathbb{R}^d)$ such that $\xi(T, x_0; u) = x_1$.*

Proof. It is enough to set

$$\xi(t) = \frac{T - t}{T}\, x_0 + \frac{t}{T}\, x_1, \quad \forall\, t \in [0, T]$$

and

$$u(t) = \xi'(t) - b(\xi(t)), \quad t \in [0, T].$$

□

Theorem 12.44. *Assume that $\det C > 0$. Then P_t is irreducible.*

Proof. We compare the solution $X(t, x_0)$ of the stochastic equation

$$X(t, x_0) = x_0 + \int_0^t b(X(s, x_0))ds + B(t), \quad t \in [0, T], \tag{12.45}$$

with the solution of the controlled system

$$\xi(t, x_0; u) = x_0 + \int_0^t b(\xi(s, x_0; u))ds + \int_0^t u(s)ds, \ \ t \in [0, T], \tag{12.46}$$

where u is chosen such that $\xi(T, x_0; u) = x_1$. Le $M > 0$ be a Lipschitz constant of b. Then we deduce by (12.45) that

$$|X(t, x_0) - \xi(t, x_0; u)| \le M \int_0^t |X(s, x_0) - \xi(s, x_0; u)| ds$$
$$+ |B(t) - v(t)|,$$

where

$$v(t) = \int_0^t u(s) ds.$$

By the Gronwall lemma we deduce that

$$|X(t, x_0) - \xi(t, x_0; u)| \le \int_0^t e^{(t-s)L} |B(s) - v(s)| ds,$$

from which, setting $t = T$ and recalling that $\xi(T, x_0; u) = x_1$,

$$|X(T, x_0) - x_1| \le e^{TL} \int_0^T |B(s) - v(s)| ds.$$

Therefore

$$\mathbb{P}(|X(T, x_0) - x_1| \le r) \ge \mathbb{P}\left(\int_0^T |B(s) - v(s)| ds \le r e^{-TL}\right) > 0,$$

because the law of the Brownian motion in $L^2(0, T)$ is non degenerate (Proposition 4.13. Thus (12.43) is proved. □

12.6.3. Linear equations with additive noise

We are here concerned with the the following Ornstein–Uhlenbeck equation

$$\begin{cases} dX = AX dt + \sqrt{C}\, dB(t) \\ X(0) = x, \end{cases} \tag{12.47}$$

where $A \in L(\mathbb{R}^d)$. We know (Example 8.8) that the solution $X(t, x)$ is given by the variation of constants formula

$$X(t, x) = e^{tA} x + Z(t),$$

where the process

$$Z(t) = \int_0^t e^{(t-s)A} \sqrt{C}\, dB(s),$$

is called the *stochastic convolution*.

By (9.45) we know that the law of $X(t,x)$ is given by

$$\Pi_{t,x} = N_{e^{tA}x,Q_t}, \quad \forall\, t \ge 0,\ x \in \mathbb{R}^d,$$

where

$$Q_t = \int_0^t e^{sA} C e^{sA^*} ds.$$

Proposition 12.45. *Assume that A is of negative type. Then for all $x \in \mathbb{R}^d$ we have*

$$\lim_{t\to+\infty} \Pi_{t,x} = N_{Q_\infty} \quad \textit{weakly}, \tag{12.48}$$

where

$$Q_\infty = \int_0^\infty e^{sA} C e^{sA^*} ds.$$

Moreover $\mu = N_{Q_\infty}$ is the unique invariant measure of P_t.

Let us recall that A is said to be of negative type when all its eigenvalues have negative real parts. In this case there exists positive constants M, ω such that

$$\|e^{tA}\| \le M e^{-\omega t}, \quad \forall\, t \ge 0. \tag{12.49}$$

We shall assume (12.49) for simplicity. A more general result can be found in [5].

Proof. It is convenient to introduce the random variables

$$Y(t,x) = e^{tA}x + \int_0^t e^{sA}\sqrt{C}\, dB(s), \quad x \in \mathbb{R}^d,\ t \ge 0.$$

Proceeding as in the proof of (9.45) one can show that for each $t > 0, x \in \mathbb{R}^d$, the laws of $X(t,x)$ and $Y(t,x)$ coincide with $N_{e^{tA}x,Q_t}$ (obviously, the laws of $X(\cdot,x)$ and $Y(\cdot,x)$ in $C([0,T];\mathbb{R}^d)$, $T > 0$, do not coincide in general). Now we claim that

$$\lim_{t\to+\infty} Y(t,x) = \int_0^{+\infty} e^{sA}\sqrt{C}\, dB(s) \quad \text{in } L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R}^d).$$

Since, as easily checked, the law of $\int_0^{+\infty} e^{sA} dB(s)$ is precisely equal to N_{Q_∞}, this will imply (12.48).

We have in fact

$$\begin{aligned}
&\int_0^{+\infty} e^{sA}\sqrt{C}\,dB(s) - Y(t,x)\\
&= \int_0^{+\infty} e^{sA}\sqrt{C}\,dB(s) - \int_0^t e^{sA}\sqrt{C}\,dB(s) - e^{tA}x\\
&= \int_t^{+\infty} e^{sA}\sqrt{C}\,dB(s) - e^{tA}x.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbb{E}\left|\int_t^{+\infty} e^{sA}\sqrt{C}\,dB(s) - e^{tA}x\right|^2\\
&= \int_t^{+\infty} \operatorname{Tr}\,[e^{sA}Ce^{sA^*}]ds + |e^{tA}x|^2 \to 0, \quad \text{as } t \to +\infty.
\end{aligned}$$

Now by (12.48) it follows that μ is invariant in view of Proposition 12.38.

It remains to show uniqueness. Let $\nu \in \mathscr{P}(\mathbb{R}^d)$ be another invariant measure of P_t. Let $s \geq 0$ and $\varphi \in C_b(\mathbb{R}^d)$. By (12.48) it follows that

$$\lim_{t\to\infty} P_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\mu(dy), \quad \forall\, x \in \mathbb{R}^d.$$

Integrating with respect to ν over $\mathbb{R}^d$ and taking into account the invariance of ν, yields

$$\int_{\mathbb{R}^d} \varphi(y)\nu(dy) = \int_{\mathbb{R}^d} \varphi(y)\mu(dy), \quad \forall\, \varphi \in C_b(\mathbb{R}^d),$$

so that $\nu = \mu$. $\square$

Exercise 12.46. Let $A \in L(\mathbb{R}^d)$ be symmetric and of negative type and $G = I$. Show that P_t has unique invariant measure N_{Q_∞}, where

$$Q_\infty = -\frac{1}{2}\,A^{-1}.$$

12.6.4. Invariant measures for nonlinear equations

Let us still consider the stochastic differential equation (12.1) under Hypothesis 12.1. We shall assume moreover that there exists a function $V : \mathbb{R}^d \to \mathbb{R}$ of class C^2 and such that

Hypothesis 12.47. (i) $\lim_{|x|\to\infty} V(x) = +\infty.$

(ii) There exists $a > 0,\ b > 0$ such that

$$\mathscr{L}V(x) = \frac{1}{2}\,\mathrm{Tr}\,[CV''(x)] + \langle b(x), V'(x)\rangle \le b - cV(x), \quad \forall\, x \in \mathbb{R}^d.$$

We call V a *Lyapunov* function.

Let

$$K_\alpha := \{x \in \mathbb{R}^d :\ V(x) \le \alpha\}, \quad \alpha > 0.$$

It is easy to see that K_α is compact for any $\alpha > 0$ and moreover

$$K_\alpha \uparrow \mathbb{R}^d, \quad \text{as } \alpha \to +\infty.$$

Proposition 12.48. *Assume Hypotheses* 12.1 *and* 12.47. *Then the transition semigroup P_t possesses an invariant measure μ.*

Proof. By Itô's formula applied to the function $V(X(t,x))$, we obtain

$$\begin{aligned} V(X(t,x)) = {} & V(x) + \int_0^t (\mathscr{L}V)(X(s,x))ds \\ & + \int_0^t \langle V'(X(s,x)), \sqrt{C}\, dB(s)\rangle. \end{aligned}$$

Taking expectation, yields

$$\mathbb{E}[V(X(t,x))] = V(x) + \mathbb{E}\int_0^t (\mathscr{L}V)(X(s,x))ds,$$

from which, taking into account Hypothesis 12.47,

$$\frac{d}{dt}\,\mathbb{E}[V(X(t,x))] \le a - b\,\mathbb{E}[V(X(t,x))].$$

By an elementary comparison result we deduce

$$\mathbb{E}[V(X(t,x))] \le e^{-bt}V(x) + a\int_0^t e^{-b(t-s)}ds \le V(x) + \frac{b}{a}.$$

Now we can show that the family of probability measures $(\Pi_{t,x_0})_{t\ge 0}$ is tight for any $x_0 \in \mathbb{R}^d$. This will imply the existence of an invariant measure by the Krylov–Bogoliubov theorem (see Remark 12.40).

We have in fact

$$\begin{aligned}\Pi_{t,x_0}(K_\alpha^c) &= \int_{V>\alpha} \Pi_{t,x_0}(dy) \le \frac{1}{\alpha} \int_{\mathbb{R}^d} V(y)\Pi_{t,x_0}(dy)\\ &= \frac{1}{\alpha} P_t V(x_0) = \frac{1}{\alpha} \mathbb{E}[V(X(t,x_0))]\\ &\le \frac{1}{\alpha} \left(V(x_0) + \frac{b}{a}\right) < \epsilon,\end{aligned}$$

for α large enough. □

Remark 12.49. Assume that Hypotheses 12.1 and 12.47 are fulfilled and in addition that $\det C > 0$. Then P_t is irreducible and strong Feller by Propositions 12.42 and 12.44. So the invariant measure from Proposition 12.48 is unique and ergodic.

12.6.5. Gradient systems

We start from the Ornstein–Uhlenbeck equation

$$\begin{cases} dX = AXdt + dB(t) \\ X(0) = x, \end{cases} \tag{12.50}$$

where $A \in L(\mathbb{R}^d)$ is symmetric and of negative type. We call (12.50) a *free system*. We know by Exercise 12.46 that it possesses a unique invariant measure $\mu = N_Q$, $Q = -\frac{1}{2}\, A^{-1}$.

Let us consider a perturbation of equation (12.50) by the gradient of a potential U,

$$\begin{cases} dX = (AX - U'(X))dt + dB(t), \\ X(0) = x, \end{cases} \tag{12.51}$$

where $U : \mathbb{R}^d \to \mathbb{R}^+$ is of class C^3 with U' Lipschitz and U'', U''' bounded. By Theorem 8.2 equation (12.51) has a unique solution $X(t,x)$.

Proposition 12.50. *P_t has a unique invariant measure ν, given by*

$$\nu(dx) = Z^{-1} e^{-2U(x)} \mu(dx), \tag{12.52}$$

where μ is the Gaussian measure N_Q, and

$$Z = \int_{\mathbb{R}^d} e^{-2U(y)} \mu(dy).$$

By Theorem 12.31 μ is ergodic; it is called a *Gibbs measure* in statistical mechanics.

Proof. Since P_t is irreducible and strong Feller, the uniqueness of invariant measure is clear (Remark 12.49). To prove the existence we need some preliminary results. Let us consider the Kolmogorov operators

$$\mathscr{L}_0\varphi = \frac{1}{2}\,\Delta\varphi + \langle Ax, D\varphi\rangle, \quad \forall\, \varphi \in C_b^2(\mathbb{R}^d),$$

$$\mathscr{L}\varphi = \mathscr{L}_0\varphi - \langle U'(x), D\varphi\rangle, \quad \forall\, \varphi \in C_b^2(\mathbb{R}^d)$$

corresponding to the free and to the perturbed system respectively and proceed in two steps.

Step 1. The following identity holds

$$\int_{\mathbb{R}^d} \mathscr{L}_0\varphi\,\psi\,d\mu = -\frac{1}{2}\int_{\mathbb{R}^d}\langle D\varphi, D\psi\rangle d\mu, \quad \forall\,\varphi \in C_b^2(\mathbb{R}^d). \tag{12.53}$$

To prove (12.53), choose a basis $e_1, ..., e_n$ in $\mathbb{R}^d$ which diagonalizes A,

$$Ae_h = -\alpha_h e_h, \quad h = 1, ..., n.$$

Then

$$\begin{aligned} N_Q(dx) &= \; 2^d(2\pi)^{-d/2}[\alpha_1\cdots\alpha_d]^{1/2}\, e^{-\sum_{h=1}^n \alpha_h x_h^2} dx \\ &=: \; \rho(x)dx. \end{aligned}$$

Consequently, for $\varphi, \psi \in C_b^2(\mathbb{R}^d)$ we have

$$\frac{1}{2}\int_{\mathbb{R}^d}\Delta\varphi\,\psi d\mu = \frac{1}{2}\sum_{k=1}^n\int_{\mathbb{R}^d} D_{x_k}^2\varphi\,\psi\,\rho\,dx$$

$$= -\frac{1}{2}\sum_{k=1}^n\int_{\mathbb{R}^d} D_{x_k}\varphi\, D_{x_k}(\psi\,\rho\,)dx$$

$$= -\frac{1}{2}\sum_{k=1}^n\int_{\mathbb{R}^d} D_{x_k}\varphi\, D_{x_k}\psi\, d\mu + \sum_{k=1}^n\int_{\mathbb{R}^d} D_{x_k}\varphi\,\psi\,\alpha_k x_k\, d\mu$$

$$= -\frac{1}{2}\int_{\mathbb{R}^d}\langle D\varphi, D\psi\rangle d\mu - \int_{\mathbb{R}^d}\langle D\varphi, Ax\rangle\psi\, d\mu.$$

Step 2. Conclusion.

Taking into account Step 1 we write

$$\int_{\mathbb{R}^d} \mathscr{L}_0\varphi\, d\nu = Z^{-1}\int_{\mathbb{R}^d} \mathscr{L}_0\varphi\, e^{-2U}\, d\mu$$

$$= -\frac{1}{2}\, Z^{-1}\int_{\mathbb{R}^d} \langle D\varphi, D(e^{-2U})\rangle d\mu \;= \int_{\mathbb{R}^d} \langle D\varphi, DU\rangle d\nu.$$

Therefore

$$\int_{\mathbb{R}^d} \mathscr{L}\varphi\, d\nu = 0, \quad \forall\, \varphi \in C_b^2(\mathbb{R}^d). \tag{12.54}$$

Now by Theorem 9.16 for any $\varphi \in C_b^2(\mathbb{R}^d)$ and any $t \geq 0$ we have $P_t\varphi \in C_b^2(\mathbb{R}^d)$, so that

$$0 = \int_{\mathbb{R}^d} \mathscr{L}(P_t\varphi)\, d\nu = \int_{\mathbb{R}^d} \frac{d}{dt}\, P_t\varphi\, d\nu.$$

Integrating from 0 and t, yields

$$\int_{\mathbb{R}^d} P_t\varphi\, d\nu = \int_{\mathbb{R}^d} \varphi\, d\nu.$$

Therefore ν is invariant. □

Proposition 12.51. *Assume, besides the assumptions of Proposition* 12.48*, that U is convex and let ν be the invariant measure* (12.52)*. Then ν is strongly mixing, that is*

$$\lim_{t\to\infty} P_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)\nu(dy), \quad \forall\, x \in \mathbb{R}^d,\ \varphi \in L^2(\mathbb{R}^d, \nu). \tag{12.55}$$

Proof. Set first

$$\eta^h(t,x) = X_x(t,x)h, \quad t \geq 0,\ x, h \in \mathbb{R}^d,$$

then η^h is the solution to the problem

$$\begin{cases} \dfrac{d}{dt}\, \eta^h(t,x) = A\eta^h(t,x) - U''(X(t,x))\eta^h(t,x), \\[2ex] \eta^h(t,0) = h. \end{cases} \tag{12.56}$$

Multiplying scalarly both sides of (12.56) by $\eta^h(t,x)$ and recalling that $\langle Ax, x\rangle \le -\lambda|x|^2$ and U is convex, yields

$$\frac{1}{2}\,\frac{d}{dt}\,|\eta^h(t,x)|^2 = \langle (A - U''(X(t,x)))\eta^h(t,x), \eta^h(t,x)\rangle \le -\lambda|\eta^h(t,x)|^2.$$

This implies

$$|\eta^h(t,x)| \le e^{-\lambda t}|h|, \quad \forall\, h \in \mathbb{R}^d. \tag{12.57}$$

Let now $\varphi \in C_b^1(\mathbb{R}^d)$. Then by (12.57), since

$$\langle DP_t\varphi(x), h\rangle = \mathbb{E}[\langle D\varphi(X(t,x)), \eta^h(t,x)\rangle], \quad t \ge 0,\ x \in H,$$

we deduce that

$$|DP_t\varphi(x)| \le e^{-\lambda t}\|\varphi\|_1 \quad \forall\, x \in \mathbb{R}^d. \tag{12.58}$$

Now, taking into account the invariance of ν and (12.58), we have

$$\begin{aligned}\left|P_t\varphi(x) - \int_{\mathbb{R}^d}\varphi(y)\nu(dy)\right| &= \left|\int_{\mathbb{R}^d}(P_t\varphi(x) - P_t\varphi(y))\nu(dy)\right| \\ &\le e^{-\omega t}\|\varphi\|_1 \int_{\mathbb{R}^d}|x-y|\nu(dy) \\ &\le e^{-\omega t}\|\varphi\|_1 \left(|x| + \int_{\mathbb{R}^d}|y|\nu(dy)\right).\end{aligned}$$

Thus, (12.55) follows for $\varphi \in C_b^1(\mathbb{R}^d)$ since

$$\int_{\mathbb{R}^d}|x|d\nu = Z^{-1}\int_{\mathbb{R}^d}|x|e^{-2U(x)}d\mu < \infty,$$

because U being convex there exists $a, b > 0$ such that $U(x) \ge a - b|x|$. Now the conclusion follows because $C_b^1(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d, \mu)$. □

12.7. Some considerations on densities of invariant measures

We are still concerned with the stochastic differential equation (12.1), under Hypothesis 12.1.

12.7.1. A general result when $\det C > 0$

Proposition 12.52. *Assume that μ is an invariant measure for P_t and that $\det C > 0$. Then μ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$.*

Proof. Let $\varphi \in C_b^1(\mathbb{R}^d)$. Then by (11.53) for any $T > 0$ there exists a constant $C_T > 0$ such that

$$|\langle P_T D\varphi(x), h\rangle| \le C_T \|\varphi\|_0 \, |h|, \quad \forall\, x, h \in \mathbb{R}^d. \tag{12.59}$$

Taking into account the invariance of μ it follows that

$$\begin{aligned}\left|\int_H \langle D\varphi(x), h\rangle \, \mu(dx)\right| &= \left|\int_H P_T \langle D\varphi(x), h\rangle \, \mu(dx)\right| \\ &\le \int_H |\langle P_T D\varphi(x), h\rangle| \mu(dx) \\ &\le C_T \|\varphi\|_0 \, |h|, \quad \forall\, x \in \mathbb{R}^d.\end{aligned} \tag{12.60}$$

Now the conclusion follows from Lemma 11.24. □

12.7.2. A differential equation for the density

Assume that P_t has an invariant measure μ, so that

$$\int_{\mathbb{R}^d} P_t\varphi \, d\mu = \int_{\mathbb{R}^d} \varphi \, d\mu, \quad \forall\, \varphi \in C_b(H),\ t \ge 0. \tag{12.61}$$

Let $\varphi \in C_b^2(H)$ and $\mathscr{L}\varphi \in C_b(H)$. Then differentiating (12.61) with respect to t, applying Theorem 9.16 and setting $t = 0$, it follows that

$$\int_{\mathbb{R}^d} \mathscr{L}\varphi \, d\mu = 0. \tag{12.62}$$

We say that μ is *infinitesimally invariant*. The concept of infinitesimally invariant measure is obviously weaker than the one of invariant measure.

Assume now that μ is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$,

$$\mu(dx) = \rho(x)dx.$$

Then by (12.61) we find

$$\mathscr{L}^*\rho = 0,$$

where $\mathscr{L}^*$ is the formal adjoint of $\mathscr{L}$,

$$\mathscr{L}^*\psi := \frac{1}{2}\,\mathrm{div}\,[CD\psi - b\psi]\,, \quad \varphi \in C_b^2(\mathbb{R}^d). \tag{12.63}$$

So $\rho \in L^1(\mathbb{R}^d)$ is a weak solution of the equation

$$\mathrm{div}\,[CD\rho - 2b\rho] = 0. \tag{12.64}$$

A useful technique to find invariant measures consists in finding a regular solution of equation (12.64). However, once a solution ρ of equation (12.64) is found, it remains to check that $\mu(dx) = \rho(x)dx$ is invariant. For general results one can see the monograph [14].

Appendix A
The Dynkin Theorem

Let Ω be a non empty set. A non empty family $\mathscr{R}$ of parts of Ω is called a π*-system* if

$$A, B \in \mathscr{R} \Longrightarrow A \cap B \in \mathscr{R},$$

a λ*-system* if

$$\begin{cases} \text{(i)} \quad \Omega, \varnothing \in \mathscr{D}. \\ \\ \text{(ii)} \;\; A \in \mathscr{D} \Longrightarrow A^c \in \mathscr{D}. \\ \\ \text{(iii)} \; (A_i) \subset \mathscr{D} \textit{ mutually disjoint } \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathscr{D}. \end{cases} \tag{A.1}$$

Obviously any σ–algebra is a λ-system. Moreover, if $\mathscr{D}$ is a λ-system such that $A, B \in \mathscr{D} \Longrightarrow A \cap B \in \mathscr{D}$ then it is σ–algebra. In fact if (A_i) is a sequence in $\mathscr{D}$ of not necessarily disjoint sets we have

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup (A_2 \setminus A_1) \cup (A_3 \setminus A_2 \setminus A_1) \cup \cdots \in \mathscr{D}$$

and so $\bigcup_{i=1}^{\infty} A_i \in \mathscr{D}$ by (ii) and (iii).

Let us prove the following *Dynkin Theorem*.

Theorem A.1. *Let $\mathscr{R}$ be a π-system and let $\mathscr{D}$ be a λ-system including $\mathscr{R}$. Then we have $\sigma(\mathscr{R}) \subset \mathscr{D}$, where $\sigma(\mathscr{R})$ is the σ algebra generated by $\mathscr{R}$.*

Proof. Let $\mathscr{D}_0$ be the minimal λ-system including $\mathscr{R}$. We are going to show that $\mathscr{D}_0$ is a σ–algebra, which will imply the theorem. For this it is enough to show, as remarked before, that the following inclusion holds

$$A, B \in \mathscr{D}_0 \Longrightarrow A \cap B \in \mathscr{D}_0. \tag{A.2}$$

For any $B \in \mathscr{D}_0$ define

$$\mathscr{H}(B) := \{F \in \mathscr{D}_0 : \;\; B \cap F \in \mathscr{D}_0\}.$$

We claim that $\mathscr{H}(B)$ is a λ-system. In fact properties (i) and (iii) are clear. It remains to show that if $F \cap B \in \mathscr{D}_0$ then $F^c \cap B \in \mathscr{D}_0$ or, equivalently, that $F \cup B^c \in \mathscr{D}_0$. In fact, since $F \cup B^c = (F \setminus B^c) \cup B^c = (F \cap B) \cup B^c$ and $F \cap B$ and B^c are disjoint, we have that $F \cup B^c \in \mathscr{D}_0$ as required.

If we show that

$$\mathscr{H}(B) \supset \mathscr{R}, \quad \forall B \in \mathscr{D}_0 \tag{A.3}$$

then we conclude that $\mathscr{H}(B) = \mathscr{D}_0$ by the minimality of $\mathscr{D}_0$ and (A.2) is proved.

On the other hand, it is clear that if $R \in \mathscr{R}$ we have $\mathscr{R} \subset \mathscr{H}(R)$ since $\mathscr{R}$ is a π-system. Therefore $\mathscr{H}(R) = \mathscr{D}_0$ by the minimality of $\mathscr{D}_0$. Consequently, the following implication holds

$$R \in \mathscr{R},\ \ B \in \mathscr{D}_0 \Rightarrow R \cap B \in \mathscr{D}_0,$$

which yields $\mathscr{R} \subset \mathscr{H}(B)$ and (B.3) is fulfilled. $\square$

The following corollary is obvious.

Corollary A.2. *Let $\mathscr{R}$ be a π-system and let $\mathscr{D}$ be a λ-system included in $\sigma(\mathscr{R})$. Then we have $\sigma(\mathscr{R}) = \mathscr{D}$.*

Appendix B
Conditional expectation

B.1. Definition

We are given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a σ-algebra $\mathscr{G}$ included in $\mathscr{F}$. Let $X : \Omega \to \mathbb{R}$ be an integrable real random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ [(1)]. Then, for any $I \in \mathscr{B}(\mathbb{R})$ the set $X^{-1}(I)$ belongs to $\mathscr{F}$ but not necessarily to $\mathscr{G}$, so that X is not $\mathscr{G}$-measurable in general.

We want to find a random variable Y which is $\mathscr{G}$-measurable but such that Y is "as close to X as possible". So, following Kolmogorov, it is natural to look for a $\mathscr{G}$-measurable random variable Y such that

$$\int_G X d\mathbb{P} = \int_G Y d\mathbb{P} \quad \forall\, G \in \mathscr{G}. \tag{B.1}$$

Theorem B.1. *Let $X \in L^1(\Omega, \mathscr{G}, \mathbb{P})$ and let $\mathscr{G}$ be a σ-algebra included in $\mathscr{F}$. Then there exists a unique $Y \in L^1(\Omega, \mathscr{G}, \mathbb{P})$ such that* (B.1) *is fulfilled.*

Y is called the *conditional expectation of X given $\mathscr{G}$* and it is denoted by $\mathbb{E}[X|\mathscr{G}]$.

Proof. Let us consider the signed measure

$$\mu(G) = \int_G X d\mathbb{P}, \qquad G \in \mathscr{G}.$$

It is clear that μ is absolutely continuous with respect to $\mathbb{P}$ (more precisely with respect to the restriction of $\mathbb{P}$ to $\mathscr{G}$). Therefore, by the Radon-Nikodym Theorem there exists a unique $Y \in L^1(\Omega, \mathscr{G}, \mathbb{P})$ such that

$$\mu(G) = \int_G X d\mathbb{P} = \int_G Y d\mathbb{P}, \quad \forall\, G \in \mathscr{G}. \tag{B.2}$$

(1) In all this appendix by random variable we mean an equivalence class of random variables with respect to the usual equivalence relation.

$\square$

Remark B.2. We list here some properties of the conditional expectation, whose proof is obvious. To some other important properties will be devoted the next section.

(i) If X is $\mathscr{G}$–measurable, we have $\mathbb{E}[X|\mathscr{G}] = X$.

(ii) Setting $G = \Omega$ in (B.1) yields

$$\mathbb{E}[\mathbb{E}(X|\mathscr{G})] = \mathbb{E}(X). \tag{B.3}$$

(iii) The conditional expectation is linear,

$$\mathbb{E}(\alpha X + \beta Y|\mathscr{G}) = \alpha\mathbb{E}(X|\mathscr{G}) + \beta\mathbb{E}(Y|\mathscr{G}), \tag{B.4}$$

for all $\alpha, \beta \in \mathbb{R}$ and all $X, Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$.

(iii) If $X \geq 0$, $\mathbb{P}$-a.s., one has $\mathbb{E}(X|\mathscr{G}) \geq 0$, $\mathbb{P}$-a.s. From this one deduces the inequality

$$|\mathbb{E}(X|\mathscr{G})| \leq \mathbb{E}(|X|\ |\mathscr{G}). \tag{B.5}$$

Exercise B.3. Assume that $X \in L^2(\Omega, \mathscr{F}, \mathbb{P})$. Show that $\mathbb{E}(X|\mathscr{G})$ coincides with the orthogonal projection of X onto the closed subspace $L^2(\Omega, \mathscr{G}, \mathbb{P})$ of $L^2(\Omega, \mathscr{F}, \mathbb{P})$.

B.2. Other properties of conditional expectation

Proposition B.4. *Let $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G}$ be a σ-algebra included in $\mathscr{F}$. If X is independent of $\mathscr{G}$ we have*

$$\mathbb{E}(X|\mathscr{G}) = \mathbb{E}(X). \tag{B.6}$$

Proof. Let $A \in \mathscr{G}$. Then $\mathbb{1}_A$ and X are independent so that

$$\int_A X d\mathbb{P} = \int_\Omega \mathbb{1}_A\, X\, d\mathbb{P} = \mathbb{P}(A)\mathbb{E}(X) = \int_A \mathbb{E}(X)\, d\mathbb{P}.$$

Therefore

$$\mathbb{E}(X) = \mathbb{E}(X|\mathscr{G}).$$

$\square$

Proposition B.5. *Let $X \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ and let $\mathscr{G}$ be σ-algebra included in $\mathscr{F}$ and $\mathscr{H}$ a σ-algebra included in $\mathscr{G}$. Then we have*

$$\mathbb{E}(X|\mathscr{H}) = \mathbb{E}\left[\mathbb{E}(X|\mathscr{G})\big|\mathscr{H}\right]. \tag{B.7}$$

Proof. Let $A \in \mathscr{H}$. Then by definition of conditional expectation of X with respect to $\mathscr{H}$ we have

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}(X|\mathscr{H}) d\mathbb{P}. \tag{B.8}$$

Since $\mathscr{H} \subset \mathscr{G}$ we also have

$$\int_A X d\mathbb{P} = \int_A \mathbb{E}(X|\mathscr{G}) d\mathbb{P}. \tag{B.9}$$

Next by definition of conditional expectation of $\mathbb{E}(X|\mathscr{G})$ with respect to $\mathscr{H}$ we have

$$\int_A \mathbb{E}(X|\mathscr{G}) d\mathbb{P} = \int_A \mathbb{E}\left[\mathbb{E}(X|\mathscr{G})\big|\mathscr{H}\right] d\mathbb{P}. \tag{B.10}$$

So, comparing (B.8), (B.9) and (B.10) we see that

$$\int_A \mathbb{E}(X|\mathscr{H}) d\mathbb{P} = \int_A X d\mathbb{P} = \int_A \mathbb{E}\left[\mathbb{E}(X|\mathscr{G})\big|\mathscr{H}\right] d\mathbb{P}.$$

□

Proposition B.6. *Let $X, Y, XY \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Assume that X is $\mathscr{G}$-measurable. Then we have*

$$\mathbb{E}(XY|\mathscr{G}) = X\mathbb{E}(Y|\mathscr{G}). \tag{B.11}$$

Proof. It is enough to show (B.11) for $X = \mathbb{1}_A$ where $A \in \mathscr{G}$. Let now $G \in \mathscr{G}$, then since $G \cap A \in \mathscr{G}$ we have

$$\int_G \mathbb{E}(\mathbb{1}_A Y|\mathscr{G}) d\mathbb{P} = \int_G \mathbb{1}_A Y d\mathbb{P} = \int_{G\cap A} Y d\mathbb{P}$$

$$= \int_{G\cap A} \mathbb{E}(Y|\mathscr{G}) d\mathbb{P} = \int_G \mathbb{1}_A \mathbb{E}(Y|\mathscr{G}) d\mathbb{P},$$

for any $G \in \mathscr{G}$. □

Recalling Proposition B.4 we find.

Corollary B.7. *Let $X, Y, XY \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Assume that X is $\mathscr{G}$-measurable and that Y is independent of $\mathscr{G}$. Then we have*

$$\mathbb{E}(XY|\mathscr{G}) = X\mathbb{E}(Y). \tag{B.12}$$

Let us prove now a useful generalization of this Corollary.

Proposition B.8. *Let* $X, Y \in L^1(\Omega, \mathscr{F}, \mathbb{P})$ *and let* $\phi : \mathbb{R}^2 \to \mathbb{R}$ *be bounded and Borel. Assume that* X *is* $\mathscr{G}$*-measurable and* Y *is independent of* $\mathscr{G}$*. Then we have*

$$\mathbb{E}(\phi(X, Y)|\mathscr{G}) = h(X), \tag{B.13}$$

where

$$h(x) = \mathbb{E}[\phi(x, Y)], \quad x \in \mathbb{R}. \tag{B.14}$$

Proof. We have to show that

$$\int_G \phi(X, Y) d\mathbb{P} = \int_G h(X) d\mathbb{P}, \quad \forall\, G \in \mathscr{G}.$$

This is clearly equivalent to

$$\mathbb{E}(Z\phi(X, Y)) = \mathbb{E}(Zh(X)), \quad \forall Z \in L^1(\Omega, \mathscr{G}, \mathbb{P}). \tag{B.15}$$

Denote by μ the law of the random variable (X, Y, Z) with values in $\mathbb{R}^3$

$$\mu = (X, Y, Z)_\# \mathbb{P}.$$

So,

$$\mathbb{E}(Z\phi(X, Y)) = \int_{\mathbb{R}^3} z\phi(x, y)\mu(dx, dy, dz). \tag{B.16}$$

Since X and Z are $\mathscr{G}$-measurable and Y is independent of $\mathscr{G}$, the random variables (X, Z) and Y are independent so that

$$\mu(dx, dy, dz) = \nu(dx, dz)\lambda(dy),$$

where

$$\nu(dx, dz) = (X, Z)_\# \mathbb{P}(dx, dz), \quad \lambda(dy) = Y_\# \mathbb{P}(dy).$$

Therefore we can write (B.16) as

$$\mathbb{E}(Z\phi(X, Y)) = \int_{\mathbb{R}^3} z\phi(x, y)\nu(dx, dz)\lambda(dy).$$

Using the Fubini Theorem we get finally

$$\begin{aligned} \mathbb{E}(Z\phi(X, Y)) &= \int_{\mathbb{R}^2} z \left[\int_{\mathbb{R}} \phi(x, y)\lambda(dy) \right] \nu(dx, dz) \\ &= \int_{\mathbb{R}^2} zh(x)\nu(dx, dz) = \mathbb{E}(Zh(X)), \end{aligned}$$

as required. □

Exercise B.9. Let F, H, $FH \in L^1(\Omega, \mathscr{G}, \mathbb{P})$ and $Z = E(H|\mathscr{G})$. Prove that

$$\mathbb{E}(FH) = \mathbb{E}(FZ). \tag{B.17}$$

Exercise B.10. Let $g : \mathbb{R} \to \mathbb{R}$ be convex and let F, $g(F) \in L^1(\Omega, \mathscr{F}, \mathbb{P})$. Prove the *Jensen* inequality

$$\mathbb{E}(g(F)|\mathscr{G}) \geq g(\mathbb{E}(F|\mathscr{G})). \tag{B.18}$$

Appendix C
Martingales

C.1. Definitions

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $(\mathscr{F}_t) = (\mathscr{F}_t)_{t \geq 0}$ an increasing family of σ-algebras included in $\mathscr{F}$ and $(M(t))_{t \in [0,T]}$ with $M(t) \in L^1(\Omega, \mathscr{F}_t, \mathbb{P})$, $t \in [0, T]$, a stochastic process.

$M(t))$, $t \in [0, T]$, is said to be a $(\mathscr{F}_t)$-*martingale* if

$$\mathbb{E}[M(t)|\mathscr{F}_s] = M(s), \quad \forall\, 0 \leq s < t \leq T,$$

a $(\mathscr{F}_t)$-*submartingale* if

$$\mathbb{E}[M(t)|\mathscr{F}_s] \geq M(s), \quad \forall\, 0 \leq s < t \leq T,$$

a $(\mathscr{F}_t)$-*supermartingale* if

$$\mathbb{E}[M(t)|\mathscr{F}_s] \leq M(s), \quad \forall\, 0 \leq s < t \leq T.$$

Thus $(M(t))$, $t \in [0, T]$, is a $(\mathscr{F}_t)$-martingale if and only if

$$\int_A M(s)d\mathbb{P} = \int_A M(t)d\mathbb{P}, \quad \forall\, 0 \leq s < t \leq T,\ A \in \mathscr{F}_s,$$

a $(\mathscr{F}_t)$-submartingale if and only if

$$\int_A M(s)d\mathbb{P} \leq \int_A M(t)d\mathbb{P}, \quad \forall\, 0 \leq s < t \leq T,\ A \in \mathscr{F}_s,$$

and a $(\mathscr{F}_t)$-supermartingale if and only if

$$\int_A M(s)d\mathbb{P} \geq \int_A M(t)d\mathbb{P}, \quad \forall\, 0 \leq s < t \leq T,\ A \in \mathscr{F}_s.$$

Proposition C.1. *If* $M(t)$, $t \geq 0$, *is a* $(\mathscr{F}_t)$ *martingale then* $|M(t)|$, $t \geq 0$, *is a submartingale.*

Proof. Let $0 \le s < t \le T$, $A \in \mathscr{F}_s$. Set

$$A^+ = \{\omega \in A : \ M(s)(\omega) > 0\}, \quad A^- = \{\omega \in A : \ M(s)(\omega) \le 0\}.$$

Clearly A^+ and A^- belong to $\mathscr{F}_s$. Consequently we have

$$\int_A |M(s)| d\mathbb{P} = \int_{A^+} M(s) d\mathbb{P} - \int_{A^-} M(s) d\mathbb{P}$$

$$= \int_{A^+} M(t) d\mathbb{P} - \int_{A^-} M(t) d\mathbb{P} \le \int_A |M(t)| d\mathbb{P}.$$

This shows that $|M(t)|,\ t \ge 0$, is a submartingale. □

Example C.2. The Brownian motion B is a martingale with respect its natural filtration $(\mathscr{F}_t)$. In fact, let $t > s$ and $A \in \mathscr{F}_s$. Since $B(t) - B(s)$ and $\mathbb{1}_A$ are independent we have

$$\int_A (B(t) - B(s)) d\mathbb{P} = \mathbb{E}(\mathbb{1}_A (B(t) - B(s))) = 0,$$

so that

$$\int_A B(t) d\mathbb{P} = \int_A B(s) d\mathbb{P}, \quad \forall\ A \in \mathscr{F}_s,$$

which implies

$$\mathbb{E}[B(t)|\mathscr{F}_s) = B(s).$$

Exercise C.3. Using Jensen's inequality prove that any convex function of a martingale is a submartingale.

C.2. A basic inequality for martingales

Let $(M(t))$ be a $(\mathscr{F}_t)$-martingale. Let $0 < t_1 < t_2 < \dots < t_n \le T$ and set

$$S = \sup_{1 \le i \le n} |M(t_i)|.$$

We are going to prove an important estimate (due to Kolmogorov) of S in terms of $|M(t_n)|$.

Proposition C.4. *For all* $\lambda > 0$ *we have*

$$\mathbb{P}(S \ge \lambda) \le \frac{1}{\lambda} \int_{\{S \ge \lambda\}} |M(t_n)| d\mathbb{P}. \tag{C.1}$$

Proof. Set

$$\begin{aligned} &A_1 = \{|M(t_1)| \geq \lambda\}, \\ &A_2 = \{|M(t_1)| < \lambda,\ |M(t_2)| \geq \lambda\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &A_n = \{|M(t_1)| < \lambda,\ \ldots,\ |M(t_n)| \geq \lambda\}. \end{aligned}$$

Clearly, sets $A_1, \ldots, A_n$ are mutually disjoint. Moreover $A_i \in \mathscr{F}_{t_i},\ i = 1, \ldots, n$, and we have

$$\{S \geq \lambda\} = \bigcup_{i=1}^n A_i.$$

Let us estimate $\int_{\{S\geq\lambda\}} |M(t_n)| d\mathbb{P}$. We have obviously

$$\int_{A_n} |M(t_n)| d\mathbb{P} \geq \lambda \mathbb{P}(A_n).$$

Now we estimate $\int_{A_{n-1}} |X(t_n)| d\mathbb{P}$. We have, recalling that $|M(t)|$ is a sub–martingale,

$$\lambda \mathbb{P}(A_{n-1}) \leq \int_{A_{n-1}} |M(t_{n-1})| d\mathbb{P} \leq \int_{A_{n-1}} |M(t_n)| d\mathbb{P}.$$

Therefore

$$\int_{A_{n-1}} |M(t_n)| d\mathbb{P} \geq \lambda \mathbb{P}(A_{n-1}).$$

Proceeding in a similar way we obtain

$$\int_{A_k} |M(t_n)| d\mathbb{P} \geq \lambda \mathbb{P}(A_k), \quad k = 1, \ldots, n. \tag{C.2}$$

Summing up on k from 1 to n the conclusion follows. □

C.3. $2m$ integrable martingales

Let $m \geq 1$. We are given a $(\mathscr{F}_t)$-martingale $(M(t)),\ t \geq 0$, such that $M(t) \in L^{2m}(\Omega, \mathscr{F}, \mathbb{P})$ for all $t \in [0, T]$. We call M a *$2m$-integrable martingale*.

Let $0 < t_1 < t_2 < \ldots < t_n \leq T$ and set as before

$$S = \sup_{1\leq i\leq n} |M(t_i)|.$$

We are going to find an estimate of $\mathbb{E}[S^{2m}]$ in terms of $\mathbb{E}[M^{2m}(t_n)]$.

Proposition C.5. *We have*

$$\mathbb{E}\left(\sup_{1\le i\le n}|M(t_i)|^{2m}\right) \le \left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}(|M(t_n)|^{2m}). \tag{C.3}$$

Proof. Set

$$F(t)=\mathbb{P}(S>t),\quad t\ge 0.$$

By (C.1) we have

$$F(t)\le \frac{1}{t}\int_{\{S\ge t\}}|M(t_n)|d\mathbb{P}. \tag{C.4}$$

Consequently

$$\mathbb{E}(S^{2m})=\int_0^\infty \mathbb{P}(S^{2m}>t)dt=\int_0^\infty \mathbb{P}(S>t^{\frac{1}{2m}})dt=\int_0^\infty F(t^{\frac{1}{2m}})dt.$$

So, by (C.4) and the Fubini Theorem we have

$$\begin{aligned}
\mathbb{E}(S^{2m}) &\le \int_0^\infty \left[t^{-\frac{1}{2m}}\int_{\{S\ge t^{\frac{1}{2m}}\}}|M(t_n)|d\mathbb{P}\right]dt\\
&=\int_{[0,+\infty)\times\Omega}[t^{-\frac{1}{2m}}|M(t_n)|\mathbb{1}_{\{S\ge t^{\frac{1}{2m}}\}}\mathbb{P}(d\omega)dt\\
&=\int_\Omega |M(t_n)|\mathbb{P}(d\omega)\int_0^\infty t^{-\frac{1}{2m}}\ \mathbb{1}_{\{S\ge t^{\frac{1}{2m}}\}}dt\\
&=\int_\Omega |M(t_n)|\mathbb{P}(d\omega)\int_0^{S^{2m}} t^{-\frac{1}{2m}}dt\\
&=\frac{2m}{2m-1}\int_\Omega |M(t_n)|S^{2m-1}\mathbb{P}(d\omega)\\
&\le \frac{2m}{2m-1}\left(\int_\Omega |M(t_n)|^{2m}d\mathbb{P}\right)^{\frac{1}{2m}}\left(\int_\Omega S^{2m}d\mathbb{P}\right)^{\frac{2m-1}{2m}}.
\end{aligned}$$

Consequently

$$\mathbb{E}(S^{2m})\le\left(\frac{2m}{2m-1}\right)^{2m}\mathbb{E}(M(t_n)^{2m}),$$

as required. □

Corollary C.6. *Let $m \geq 1$ and let M be a $2m$-integrable continuous martingale. Then for any $T > 0$ we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M(t)|^{2m}\right] \leq \mathbb{E}[M^{2m}(T)]. \tag{C.5}$$

Proof. Let $0 < s_1 < s_2 < \cdots < s_m = T$. By Proposition $C.5$ it follows that

$$\mathbb{E}\left[\sup_{1\leq i\leq m} |M(s_i)|^{2m}\right] \leq \left(\frac{2m}{2m-1}\right)^{2m} \mathbb{E}\left[|M(T)|^{2m}\right].$$

Since M is continuous (C.5) follows, by the arbitrariness of the sequence $s_1, s_2, \ldots, s_m$. □

Appendix D
Fixed points depending on parameters

D.1. Introduction

Let Λ, E be Banach spaces (norms $|\cdot|$). We are given a continuous mapping

$$F : \Lambda \times E \to E,\ (\lambda, x) \mapsto F(\lambda, x)$$

and assume that

Hypothesis D.1. There exists $\kappa \in [0, 1)$ such that

$$|F(\lambda, x) - F(\lambda, y)| \le \kappa |x - y|, \quad \forall\, \lambda \in \Lambda,\ x, y \in E.$$

The following result (*contraction principle*) is classical.

Theorem D.2. (i). *There exists a unique continuous mapping*

$$x : \Lambda \to E, \quad \lambda \to x(\lambda),$$

such that

$$x(\lambda) = F(\lambda, x(\lambda)), \quad \forall\, \lambda \in \Lambda. \tag{D.1}$$

(ii). *If in addition F is of class C^1, then x is of class C^1 and*

$$x'(\lambda) = F_\lambda(\lambda, x(\lambda)) + F_x(\lambda, x(\lambda))x'(\lambda). \tag{D.2}$$

We want to generalize the second part of this result to mappings $F(\lambda, x)$ which are only differentiable in all directions of (λ, x) and for which the chain rule holds. To this purpose, we shall assume, besides Hypothesis D.1, that

Hypothesis D.3. (i) For any $\lambda \in \Lambda, x \in E$ there exist linear continuous operators $F_\lambda(\lambda, x) \in L(\Lambda, E)$ and $F_x(\lambda, x) \in L(E)$ such that

$$\lim_{h\to 0} \frac{1}{h}\,(F(\lambda + h\mu, x + hy) - F(\lambda, x))$$
$$= F_\lambda(\lambda, x)\cdot\mu + F_x(\lambda, x)\cdot y, \quad \forall\, \mu \in \Lambda,\ y \in E.$$

(ii) For any $\lambda, \mu \in \Lambda, x, y \in E$ the function

$$[0,1] \to E, \quad \xi \to F((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y),$$

is continuously differentiable and

$$\begin{aligned} &\frac{d}{d\xi} F((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y) \\ &= F_\lambda((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y) \cdot (\mu - \lambda) \\ &\quad + F_x((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y) \cdot (y - x), \ \forall \xi \in [0,1]. \end{aligned} \tag{D.3}$$

We notice that from (D.3) it follows that, for all $\lambda, \mu \in \Lambda, x, y \in E$,

$$\begin{aligned} &F(\mu, y) - F(\lambda, x) \\ &= \int_0^1 F_\lambda((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y) \cdot (\mu - \lambda) d\xi \\ &\quad + \int_0^1 F_x((1-\xi)\lambda + \xi\mu, (1-\xi)x + \xi y) \cdot (y - x) d\xi. \end{aligned} \tag{D.4}$$

D.2. The main result

Theorem D.4. *Assume that Hypotheses* D.1 *and* D.3 *hold and let* $x(\lambda)$ *be the solution of* (D.1). *Then* $x(\lambda)$ *is differentiable in any direction* $\mu \in \Lambda$ *and we have*

$$x'(\lambda) \cdot \mu = (1 - F_x(\lambda, x(\lambda)))^{-1} F_\lambda(\lambda, x(\lambda)) \cdot \mu, \tag{D.5}$$

equivalently

$$x'(\lambda) \cdot \mu = F_\lambda(\lambda, x(\lambda)) \cdot \mu + F_x(\lambda, x(\lambda))(x'(\lambda) \cdot \mu). \tag{D.6}$$

Proof. From (D.4) it follows that

$$\begin{aligned} &x(\lambda + h\mu) - x(\lambda) = F(\lambda + h\mu, x(\lambda + h\mu)) - F(\lambda, x(\lambda)) \\ &= h \int_0^1 F_\lambda(\lambda + \xi h\mu, x(\lambda) + \xi(x(\lambda + h\mu) - x(\lambda))) \cdot \mu d\xi \\ &\quad + \int_0^1 F_x(\lambda + \xi h\mu, x(\lambda) + \xi(x(\lambda + h\mu) - x(\lambda))) \\ &\qquad \cdot (x(\lambda + h\mu) - x(\lambda)) d\xi. \end{aligned} \tag{D.7}$$

Set now

$$
\begin{aligned}
&G(\lambda, x, \mu, h)z \\
&= Gz =: \int_0^1 F_x(\lambda + \xi h\mu, x(\lambda) + \xi(x(\lambda + h\mu) - x(\lambda))) \cdot z d\xi, \quad z \in E.
\end{aligned}
$$

Then $G \in L(E)$ and by Hypothesis D.1

$$
|Gz| \le \kappa |z|, \quad \forall\, z \in E.
$$

Then from equation (D.7) we have

$$
\begin{aligned}
&(1 - G(\lambda, x, \mu, h))(x(\lambda + h\mu) - x(\lambda)) \\
&\qquad = h \int_0^1 F_\lambda(\lambda + \xi h\mu, x(\lambda) + \xi(x(\lambda + h\mu) - x(\lambda))) \cdot \mu d\xi,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&\frac{1}{h}\, x(\lambda + h\mu) - x(\lambda)) = (1 - G(\lambda, x, \mu, h))^{-1} \\
&\qquad \times \int_0^1 F_\lambda(\lambda + \xi h\mu, x(\lambda) + \xi(x(\lambda + h\mu) - x(\lambda))) \cdot \mu d\xi.
\end{aligned}
$$

Therefore there exists the limit

$$
\begin{aligned}
x'(\lambda) \cdot \mu &= \lim_{h \to 0} \frac{1}{h}\, x(\lambda + h\mu) - x(\lambda)) \\
&= F_\lambda(\lambda, x(\lambda)) \cdot \mu + F_x(\lambda, x(\lambda))(x'(\lambda) \cdot \mu),
\end{aligned}
$$

and the conclusion follows. □

Appendix E
A basic ergodic theorem

We are given a linear bounded operator T on a Hilbert space E whose norm we denote by $\|\cdot\|$. We make the basic assumption that there exists a positive constant κ such that

$$\|T^n\| \leq \kappa, \quad \forall\, n \in \mathbb{N}.$$

This condition is for instance fulfilled if T is a contraction, that is if $\|T\| \leq 1$. We are interested in the asymptotic behavior of the discrete dynamical system $I, T, T^2, ..., T^n, ...$ Notice that if $Tx = x$ we have $x = Tx = T^2x = \cdots =$ so that obviously

$$\lim_{n\to\infty} T_n x = x.$$

In this case we say that x is a *stationary point*. So, the set of all stationary points is given by

$$\text{Ker}\,(1-T) := \{x \in E:\ (1-T)x = 0\}.$$

If $x \in E$ is an arbitrary element of E the sequence $(T^n x)$ does not possess a limit when $n \to \infty$ in general (take for instance $T = -I$). We are going to prove, however, the important fact that the limit of the means

$$M_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x, \quad n \in \mathbb{N},\ x \in E.$$

does exist for *all* $x \in E$.

Theorem E.1. *Assume that there exists $\kappa > 0$ such that $\|T^n\| < \kappa$ for all $n \in \mathbb{N}$. Then the following holds*

(i) *There exists the limit*

$$\lim_{n\to\infty} M_n x := M_\infty x, \quad \forall\, x \in E. \tag{E.1}$$

(ii) $M_\infty \in L(E)$ *is a projection operator,* i.e.

$$(M_\infty)^2 = M_\infty.$$

(iii) M_∞ *maps* E *onto the subspace of all stationary points of* T i.e.

$$M_\infty(E) = \text{Ker }(1 - T).$$

Proof. We start by proving the existence of the limit (E.1) for any $x \in E$.

Step 1. For all $x \in \text{Ker }(1 - T)$ we have

$$\lim_{n\to\infty} M_n x = x.$$

In fact, if $x \in \text{Ker }(1 - T)$ we have $Tx = x$ and consequently $M_n x = x$. Therefore the conclusion follows.

Step 2. For all $x \in (1 - T)(E)$ we have

$$\lim_{n\to\infty} M_n x = 0.$$

Assume that $x = (1 - T)y$. Then

$$M_n x = M_n(1 - T)y = \frac{1}{n}\,(1 - T^n)y, \quad n \in \mathbb{N}.$$

Recalling that $\|T^n\| \le \kappa$ it follows that

$$\lim_{n\to\infty} M_n x = 0.$$

Step 3. For all $x \in \overline{(1 - T)(E)}$ (the closure of $(1 - T)(E)$) we have

$$\lim_{n\to\infty} M_n x = 0.$$

Assume that $x \in \overline{(1 - T)(E)}$. Then there exists a sequence $(y_j) \subset E$ such that setting $x_j = x - (1 - T)y_j$, we have

$$\|x_j\| \le \frac{1}{j}, \quad \forall\ j \in \mathbb{N}.$$

It follows that

$$\|M_n x\| \le \|M_n x_j\| + \|M_n(1 - T)y_j\| \le \frac{\kappa}{j} + \|M_n(1 - T)y_j\|,$$

and the conclusion follows from Step 2.

Step 4. For any $x \in E$ there exists the limit

$$\lim_{n\to\infty} M_n x =: M_\infty x.$$

Let $x \in E$. Since $\|M_n x\| \le \kappa \|x\|$ and balls of E are weakly compact, there exists a sequence $(n_k) \subset \mathbb{N}$ and an element $y \in H$ such that

$$\lim_{k\to\infty} M_{n_k} x = y \quad \text{weakly} .$$

Therefore by the identity

$$M_{n_k}(1-T) = \frac{1}{n_k}\,(1-T^{n_k}),$$

we deduce that

$$\lim_{k\to\infty} M_{n_k}(1-T)x = 0 \quad \text{strongly} ,$$

which yields

$$\lim_{k\to\infty} M_{n_k} x = y = Ty \quad \text{weakly} .$$

This shows that $y \in \mathrm{Ker}\,(1-T)$.

Now we can show that $M_n x \to y$ strongly. First note that, since $y \in \mathrm{Ker}\,(1-T)$, we have $M_n y = y$, and so

$$M_n x = M_n y + M_n(x-y) = y + M_n(x-y).$$

We claim that $x - y \in \overline{(1-T)(E)}$, which will yield the conclusion because then by Step 3 we obtain $M_n(x-y) \to 0$ and so

$$\lim_{n\to\infty} M_n x = y.$$

To prove that $x - y \in \overline{(1-T)(E)}$, we write

$$x - y = \text{weak} \lim_{k\to\infty}(x - M_{n_k}x),$$

and show that $x - M_{n_k}x \in (1-T)(E)$. In fact

$$\begin{aligned} x - M_{n_k}x &= \frac{1}{n_k}\sum_{h=0}^{n_k-1}(1-T^h)x \\ &= \frac{1}{n_k}\,(1-T)\sum_{h=0}^{n_k-1}(1+T+\ldots+T^{h-1})x. \end{aligned}$$

Therefore the claims holds. We have proved that there exists the limit

$$M_\infty x := \lim_{n\to\infty} M_n x, \quad \forall\, x \in E.$$

Let us prove now (ii).

Since

$$(1-T)M_n = \frac{1}{n}\,(1-T^n),$$

letting $n \to \infty$ we find $(1-T)M_\infty = 0$, so that $M_\infty = TM_\infty$. This obviously implies $M_\infty = M_n M_\infty$ for all $n \in \mathbb{N}$ and so $M_\infty = (M_\infty)^2$, as required.

Let us prove finally (iii). Letting $n \to \infty$ in the identity

$$(1-T)M_n = \frac{1}{n}\,(1-T^n),$$

yields $(1-T)M_\infty = 0$ so that

$$M_\infty(E) \subset \mathrm{Ker}\,(1-T).$$

On the other hand, if $x \in \mathrm{Ker}\,(1-T)$ we have $x = M_\infty x$ so that $x \in M_\infty(E)$. Therefore

$$\mathrm{Ker}\,(1-T) \subset M_\infty(E).$$

The proof is complete. □

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