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Well-posedness of the stochastic KdV–Burgers equation

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Abstract

We are interested in rigorously proving the invariance of white noise under the flow of a stochastic KdV–Burgers equation. This paper establishes a result in this direction. After smoothing the additive noise (by a fractional spatial derivative), we establish (almost sure) local well-posedness of the stochastic KdV–Burgers equation with white noise as initial data. Next we observe that spatial white noise is invariant under the projection of this system to the first $N > 0$ modes of the trigonometric basis. Finally, we prove a global well-posedness result under an additional smoothing of the noise.

Keywords: Well-posedness; Stochastic PDEs; White noise invariance

1. Introduction

In this paper we study the stochastic Korteweg–de Vries (KdV)–Burgers equation

$$\begin{cases} du = (u_{xx} - u_{xxx} - (u^2)_x)dt + \phi\partial_x dW, & t \geq 0, x \in \mathbb{T}, \\ u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where ϕ is a bounded operator on $L^2(\mathbb{T})$, and $W(t,x)$ is a cylindrical white noise of the form

$$W(t,x) = \sum_{n\neq 0} B_n(t)e^{inx}. \tag{1.2}$$

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Here $(B_n(t))_{n\in\mathbb{N}}$ is a family of standard complex-valued Brownian motions mutually independent in a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and $B_{-n} = \overline{B_n}$, since we are interested in real-valued noise.

We consider (1.1) as a toy model for the stochastic Burgers equation

$$\begin{cases} du = \left(\frac{1}{2}u_{xx} - \frac{1}{2}(u^2)_x\right)dt + \partial_x dW, & t \geq 0, x \in \mathbb{T}, \\ u|_{t=0} = u_0. \end{cases} \tag{1.3}$$

Eq. (1.3) is a reformulation of the Kardar–Parisi–Zhang equation (KPZ). That is, letting $u = \partial_x h$, u satisfies (1.3) if and only if h satisfies KPZ, given by

$$dh = \left(\frac{1}{2}h_{xx} - \frac{1}{2}(h_x)^2\right)dt + dW, \quad t \geq 0, x \in \mathbb{T}. \tag{1.4}$$

Eq. (1.4) was introduced [20] to model the fluctuations (over long scales) of a growing interface. For example, $h(t, x)$ could describe the height of an interface between regions of opposite polarity inside a ferromagnet subject to an external magnetic field. Mathematical interest in (1.4) is motivated by evidence that second order dependence on the derivative $\partial_x h$ (over long scales) is universal – that is, independent of the microscopic dynamics – within a certain class of growth models [22]. This is verified mathematically in Bertini–Giacomin [1] for a specific growth model; they obtain (1.4) as the limit equation of a suitable particle system (up to the Cole–Hopf transformation).

By local well-posedness (LWP) of a stochastic PDE we mean pathwise LWP almost surely (a.s.). That is, for almost every fixed $\omega \in \Omega$, the corresponding PDE is locally well-posed. Similarly, global well-posedness (GWP) of a stochastic PDE will be defined as pathwise GWP a.s. As written, equations (1.3) and (1.4) are ill-posed. To make sense of well-posedness for (1.4), one strategy is to solve a stochastic heat equation which is equivalent to (1.4) under the Cole–Hopf transformation, as in [1]; this equivalence is modulo an infinite constant produced by the Itô formula. A more robust well-posedness for (1.4) under suitable renormalization (extending the Cole–Hopf solution) was recently established by Hairer [19] using the theory of rough paths. The approach taken in this paper is more classical in technique, and we are particularly interested in proving the invariance of white noise for (1.1) by adapting methods from the analysis of deterministic PDEs.

Besides the approach in [1] and [19], other studies of (1.4) (and (1.3)) have considered modified equations. For example, Da Prato–Debussche–Temam [13] considered (1.3) without the spatial derivative ∂_x applied the noise. That is, they smoothed the additive noise by one derivative in space. Then they established local (and global) well-posedness using a fixed point theorem. Our approach is closer to this strategy, but we modify the linear part of (1.3) instead. By including $-u_{xxx}$ in (1.3), we obtain (1.1) with $\phi = \mathrm{Id}$ (up to constants with no effect on our analysis). See also Da Prato–Debussche–Tubaro [14], where the spatial derivative ∂_x remains on the noise, but the nonlinearity is slightly smoothed, and well-posedness is established after renormalization.

With $\phi = \mathrm{Id}$, equations (1.1) and (1.3) share a physically significant property; both of these equations *formally* preserve mean zero spatial white noise. Mean zero spatial white noise is the unique probability measure μ on the space of mean zero distributions on $\mathbb{T}$ satisfying

$$\int e^{i\langle\lambda,u\rangle} d\mu(u) = e^{-\frac{1}{2}\|\lambda\|_2^2}, \tag{1.5}$$

for any mean zero smooth function λ on $\mathbb{T}$. From now on, we assume that the spatial mean is always zero, and omit the prefix "mean zero". Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of smooth functions in $L^2(\mathbb{T})$. Then white noise is represented as $u = \sum_{n=1}^{\infty} g_n e_n$, where $\{g_n\}_{n\in\mathbb{N}}$ is a sequence of independent Gaussian random variables with mean zero and variance one. In particular, white noise is supported in $H^s(\mathbb{T})$ for any $s < -\frac{1}{2}$.

It is important for us to identify that our analysis involves two distinct types of noise. In Eq. (1.1), the term $W(t, x)$ represents *space–time* white noise, as defined in (1.2). Separately, we have just defined *spatial* white noise, a probability measure supported on $H^s(\mathbb{T})$ for any $s < -\frac{1}{2}$, represented as $u = \sum_{n=1}^{\infty} g_n e_n$. That is, the stochastic PDE (1.1) is forced by space–time white noise, but we will consider spatial white noise as initial data.

Let us clarify what we mean by formal invariance of white noise for (1.1) and (1.3). The invariance of white noise has been proven rigorously [22,1] for the Cole–Hopf solution to (1.3). We interpret this invariance as "formal" for (1.3), because the Itô formula produces an infinite constant under the Cole–Hopf transformation. For (1.1), we observe (see Proposition 1.1 below) that spatial white noise is invariant under the projection of (1.1) to finitely many Fourier modes. The proof is based on decomposing this system into the truncated KdV equation, plus a rescaled Ornstein–Uhlenbeck process at each spatial frequency; each of these evolutions individually preserve white noise, and this extends to the super-position in the finite-dimensional setting.

For certain Hamiltonian PDEs (e.g. the nonlinear Schrödinger and generalized KdV equations), Bourgain [4] developed a strategy for rigorously proving the invariance of the Gibbs measure "$d\mu = e^{-H(u)} du$" under the flow. The main new idea implemented in [4] was to use the invariance of the Gibbs measure under the flow of a finite-dimensional system of approximating ODEs as a *substitute for a conservation law*, extending local solutions of the PDE (evolving from initial data in the support of the Gibbs measure) to global solutions, and subsequently proving the invariance of the Gibbs measure under the flow. That is, Bourgain found a way to use the (structure behind the) invariant measure in order to extend solutions globally-in-time. This strategy has now been successfully implemented for numerous PDEs and different invariant measures (see for example [5–8,12,25,27–29,34,35]).

There is a natural question: can we establish LWP of (1.1) in the support of white noise, then following the strategy developed in [4], use Proposition 1.1 to extend these solutions globally-in-time, and prove the invariance of white noise under the flow? This paper is designed as a preliminary step in this direction.

We are motivated to consider (1.1) instead of (1.3) (suitably renormalized) to implement this strategy for mathematical reasons; the combination of dispersion and dissipation in the linear part of (1.1) makes this equation amenable to Fourier restriction techniques (as developed by Bourgain [2]) to prove local well-posedness with low regularity initial data. Indeed, to ask the question of invariance of white noise, we first require local well-posedness for initial data in the support of white noise. In terms of Sobolev regularity, white noise is supported in $H^s(\mathbb{T})$ for any $s < -\frac{1}{2}$, and Dix [17] has established the failure of uniqueness below $H^{-\frac{1}{2}}(\mathbb{R})$ for the deterministic Burgers' equation (in the context of $x \in \mathbb{R}$), whereas Molinet–Vento [24] have shown that the deterministic KdV–Burgers equation is analytically globally well-posed in $H^s(\mathbb{T})$ for $s \geq -1$.

See Section 1.2 below for more details.

1.1. Results

The main result of this paper is LWP of (1.1) in the support of white noise with a rougher additive noise than in [13,16,26] (see Theorem 1.1 below). We exploit the combination of dispersion and dissipation in (the linear part of) (1.1) to (i) relax the conditions placed on the additive noise in some previous analyses of the stochastic Burgers equation [13] (dissipation only) and the stochastic KdV [16,26] (dispersion only), and (ii) allow for spatial white noise as initial data. This result is designed as a first step towards a rigorous proof of the invariance of white noise for (1.1).

We pause to introduce some notation. For our purposes, ϕ is assumed to be a Hilbert–Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ (written $\phi \in HS(L^2; H^s)$) for some $s \in \mathbb{R}$. The space $HS(L^2; H^s)$ is endowed with its natural norm

$$\|\phi\|_{HS(L^2;H^s)} = \left(\sum_{n\in\mathbb{N}} \|\phi e_n\|_{H^s}^2\right)^{1/2},$$

where $(e_n)_{n\in\mathbb{N}}$ is any complete orthonormal system in $L^2(\mathbb{T})$. For simplicity, we will further assume that ϕ is a convolution operator. That is, for any function f on $\mathbb{T}$, and any $n \in \mathbb{Z}$, we have

$$\widehat{(\phi f)}(n) = \phi_n \widehat{f}(n), \tag{1.6}$$

for some $\phi_n \in \mathbb{R}$. For example, in [13], they consider $\phi = \partial_x^{-1}$, which corresponds to $\phi_n = \frac{1}{in}$, for each n. Using the standard Fourier basis for $L^2(\mathbb{T})$, we find that for convolution operators ϕ, we have $\|\phi\|_{HS(L^2;H^s)} = (\sum_n \langle n\rangle^{2s}|\phi_n|^2)^{1/2} = \|\phi\|_{H^s}$ (with a slight abuse of notation). The point to take away is that by placing $\phi \in HS(L^2, H^s)$, we are smoothing the additive noise in (1.1) by $(s + \frac{1}{2})+$ spatial derivatives.

We proceed to state our main result.

Theorem 1.1 (*Local Well-posedness*)**.** *Given* $0 < \varepsilon < \frac{1}{16}$, *let* $s \geq -\frac{1}{2} - \varepsilon$. *Suppose* $\phi \in HS(L^2; H^{s+1-2\varepsilon})$ *of the form* (1.6). *Then* (1.1) *is LWP in* $H^s(\mathbb{T})$ *for mean zero data. That is, if* $u_0 \in H^s(\mathbb{T})$ *has mean zero, there exists a stopping time* $T_\omega > 0$ *and a unique process* $u \in C([0, T_\omega]; H^s(\mathbb{T}))$ *satisfying* (1.1) *on* $[0, T_\omega]$ *a.s.*

With $\varepsilon = \frac{1}{16}-$ and $s = -\frac{1}{2} - \varepsilon < -\frac{1}{2}$, we have $s + 1 - 2\varepsilon = \frac{5}{16}+$. Then $\phi = \partial_x^{-(\frac{13}{16}+)}$ satisfies $\phi \in HS(L^2, H^{\frac{5}{16}+})$, as required to apply Theorem 1.1. With this choice of ϕ, we have smoothed the noise in (1.1) by $\frac{13}{16}+$ spatial derivatives. In contrast, the space–time noise in [13,26] is smoothed by a full derivative in space (taking $\phi = \partial_x^{-1}$). Recall that spatial white noise is represented as $u = \sum_n g_n e_n \in H^s(\mathbb{T})$ a.s. for $s < -\frac{1}{2}$. It follows that Theorem 1.1 applies a.s. with initial data given by spatial white noise. We have arrived at a consequence of Theorem 1.1 worth mentioning.

Corollary 1.1. *Let* $\phi = \partial_x^{-(\frac{13}{16}+)}$, *then* (1.1) *is a.s. LWP with spatial white noise as initial data.*

Next we establish the invariance of white noise for (1.1) after projection to the first $N > 0$ modes of the trigonometric basis. This is a potential precursor to rigorously proving the invariance of white noise for (1.1). Given $N > 0$, let $\mathbb{P}_N$ denote the Dirichlet projection to $E_N = \operatorname{span}\{\sin(nx), \cos(nx) : 1 \leq n \leq N\}$. We consider the frequency truncated stochastic PDE

$$\begin{cases} du^N = \left(\frac{1}{2}u^N_{xx} - u^N_{xxx} - \mathbb{P}_N\left[\frac{1}{2}((u^N)^2)_x\right]\right)dt + \mathbb{P}_N\partial_x dW, & t \geq 0, x \in \mathbb{T} \\ u^N(0,x) = u_0^N(x) = \mathbb{P}_N(u_0(x)), \end{cases} \tag{1.7}$$

where $u^N = \mathbb{P}_N u^N$. For each $N > 0$, the system (1.7) is globally well-posed (see Proposition 3.1 in Section 3). Let $\Phi^N(t)$ denote the data-to-solution map for (1.7). Finite-dimensional spatial white noise μ_N is represented as $u^N = \sum_{0<|n|\le N} g_n e^{inx}$ (with $g_{-n} = \overline{g_n}$).

Proposition 1.1. *The flow of* (1.7) *preserves the spatial white noise* μ_N. *That is,* $(\Phi_t^N)^* \mu_N = \mu_N$ *(in distribution) for each* $t \ge 0$.

The proof of Proposition 1.1 is straightforward, and we refer to [32] for details. It is based on a decomposition of (3.1) into the (truncated) KdV equation, plus a rescaled Ornstein–Uhlenbeck process at each spatial frequency. That is, (3.1) is given by

$$du^N = \underbrace{-u^N_{xxx}dt + \frac{1}{2}\mathbb{P}_N\Big(\partial_x\big((u^N)^2\big)\Big)dt}_{\text{truncated KdV}} + \underbrace{u^N_{xx}dt + \mathbb{P}_N \partial_x dW}_{\text{OU-processes}}. \tag{1.8}$$

The truncated KdV preserves spatial white noise, and an Ornstein–Uhlenbeck process leaves the normal distribution invariant at each frequency. In the finite-dimensional setting, Proposition 1.1 is immediate from these observations. We remark that for the periodic KdV, the invariance of white noise has been proven rigorously; see Quastel–Valkó [31], Oh–Quastel–Valkó [30] and Oh [29]. We also remark that the strategy of [4] was used in [29].

Let us emphasize that we are interested in relaxing the smoothing hypothesis in the statement of Theorem 1.1. In particular, if we can prove LWP of (1.1) with $\phi = Id$, then following [4], we may be able to use Proposition 1.1 to extend this result to GWP, and to prove the invariance of spatial white noise under the flow.

Our final result is GWP of (1.1) with L^2-data and a smoothed noise.

Theorem 1.2 (*Global Well-posedness*)**.** *Let* $\phi \in HS(L^2; H^1)$. *Then* (1.1) *is GWP in* $L^2(\mathbb{T})$ *for mean zero data. That is, if* $u_0 \in L^2(\mathbb{T})$ *has mean zero, then for any* $T > 0$ *there is a unique process* $u \in C([0,T]; L^2(\mathbb{T}))$ *satisfying* (1.1) *on* $[0,T]$ *a.s.*

We proceed to provide more background, and to discuss some of the methods involved in the proofs of Theorems 1.1 and 1.2.

1.2. Background and method

Our analysis will take place in the $X^{s,b}$ space of functions of space–time adapted to the (linear) KdV–Burgers equation. As in [23], for $s, b \in \mathbb{R}$, the $X^{s,b}$ space is a weighted Sobolev space whose norm is given by

$$\|u\|_{X^{s,b}} = \left(\int_{\tau\in\mathbb{R}} \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} \langle i(\tau - n^3) + n^2\rangle^{2b} |\tilde{u}(n,\tau)|^2 d\tau\right)^{1/2}.$$

Recall that $\langle\cdot\rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. The time restricted $X_T^{s,b}$ space is defined with the norm

$$\|u\|_{X_T^{s,b}} = \inf\big\{\|v\|_{X^{s,b}} : v \in X^{s,b} \text{ and } v(t) \equiv u(t) \text{ for } t \in [0,T]\big\}.$$

We define the local-in-time version $X_I^{s,b} = X_{[a,b]}^{s,b}$ on an interval $I = [a,b]$ in an analogous way.

Bourgain [2] introduced the $X^{s,b}$ spaces, and used a fixed point argument to establish LWP in $L^2(\mathbb{T})$ for the periodic KdV. Global well-posedness followed from conservation of the $L^2(\mathbb{T})$-norm. Kenig–Ponce–Vega [21] improved this result to LWP in $H^{-\frac{1}{2}}(\mathbb{T})$ [21] (see also [9] where the corresponding global-in-time result is obtained with the I-method). These (local-in-time) results are based primarily on bilinear estimates in (sometimes modified) $X^{s,b}$ spaces. For example, in [21], the estimate

$$\|\partial_x(uv)\|_{X^{s,b-1}} \le \|u\|_{X^{s,b}}\|v\|_{X^{s,b}} \tag{1.9}$$

is established for $s \ge -\frac{1}{2}$, $b = \frac{1}{2}$. This estimate is then used to prove LWP of KdV with a fixed point argument.

It is also shown in [21] that (1.9) fails if $s < -\frac{1}{2}$, or $b < \frac{1}{2}$. That is, the bilinear $X^{s,b}$ estimate used for well-posedness of KdV requires spatial regularity $s \ge -\frac{1}{2}$ and temporal regularity $b \ge \frac{1}{2}$. In a related result, it is shown in [3] that the data-to-solution map $u_0 \in H^s(\mathbb{T}) \longmapsto u(t) \in C([0,T]; H^s(\mathbb{T}))$ for KdV is not C^4 for $s < -\frac{1}{2}$, and any $T > 0$. This indicates that there is no hope in using a fixed point argument (i.e. applying the contraction principle to an equivalent integral equation) to establish well-posedness of KdV in $H^s(\mathbb{T})$ for $s < -\frac{1}{2}$; the contraction principle dictates analyticity of the data-to-solution map. In a separate result, Dix [17] established ill-posedness of the deterministic Burgers equation

$$u_t = u_{xx} - \frac{1}{2}(u^2)_x, \quad t \ge 0, x \in \mathbb{R}, \tag{1.10}$$

in $H^s(\mathbb{R})$ for $s < -\frac{1}{2}$, due to a lack of uniqueness. We observe that, for both the deterministic KdV and the deterministic (non-periodic) Burgers equation, $s = -\frac{1}{2}$ is the optimal regularity for well-posedness in H^s via the fixed point method.

The barrier of $s = -\frac{1}{2}$ is not present in the analysis of the deterministic KdV–Burgers equation ((1.1) with $\phi = 0$). For example, Molinet–Ribaud [23], followed by Molinet–Vento [24], established GWP of the KdV–Burgers equation in $H^s(\mathbb{T})$ for $s \ge -1$ by using the fixed point method; the data-to-solution map is analytic. Moreover, in [23], an estimate of the form (1.9) is established for any $s > -1$, $b = \frac{1}{2}$ (and a similar estimate for $s = -1$ is obtained in [24]). In this way, the combination of dispersion and dissipation in the KdV–Burgers equation allows for an improved bilinear estimate of the form (1.9) (in particular, with spatial regularity $s < -\frac{1}{2}$), which yields superior well-posedness results.

Let us discuss function spaces which capture the regularity of space–time white noise. Spaces of this type were considered in De Bouard–Debussche–Tsutsumi [16], and Oh [26], to study the stochastic KdV (with additive noise), given by

$$du = (-u_{xxx} - (u^2)_x)dt + \phi dW, \quad t \ge 0, x \in \mathbb{T}. \tag{1.11}$$

The argument in [16] is based on the result of Roynette [33] on the endpoint regularity of Brownian motion in a Besov space; they prove a variant of the bilinear estimate (1.9) adapted to their Besov space setting, and establish LWP of (1.11) using the fixed point method. However, this modified bilinear estimate requires a slight regularization of the noise in space via the bounded operator ϕ, so that the smoothed noise has spatial regularity $s > -\frac{1}{2}$. In particular, they could not treat the case of space–time white noise, as in (1.11) (i.e. $\phi = Id$).

These developments are clarified in [26] with two observations. The first observation is that a certain modified Besov space, and the corresponding $X^{s,b}$-type space, capture the regularity of spatial and space–time white noise, respectively, for $s < -\frac{1}{2}$ and $b < \frac{1}{2}$. The second is that a priori estimates on the second iteration of the mild formulation of (1.11) (in these spaces) can be used to prove LWP of (1.11). The condition $b < \frac{1}{2}$ is dictated by the space–time white noise; recall that the bilinear $X^{s,b}$ estimate (1.9) for KdV requires $b \ge \frac{1}{2}$. This conflict creates the need for refined Besov-type spaces as in [16,26].

Notice that the additive noise in (1.11) lacks the spatial derivative ∂_x appearing in (1.3). That is, refined analyses of the stochastic KdV [16,26] can only treat noise which is a full derivative smoother than the noise in (1.3). Similarly, the noise considered in [13] is a full derivative smoother than the noise appearing in (1.3). In this paper, we find that the *combination of dispersion and dissipation* in (1.1) allows us to treat a rougher noise.

We will solve (1.1) in the mild formulation

$$u(t) = S(t)u_0 - \frac{1}{2}\int_0^t S(t-t')\partial_x(u^2(t'))dt' + \int_0^t S(t-t')\phi\partial_x dW(t'). \tag{1.12}$$

In Eq. (1.12), $S(t)u_0$ denotes the solution to the linear KdV–Burgers equation with initial data u_0 evaluated at time t, suitably extended to all $t \in \mathbb{R}$. That is, $\widehat{S(t)u_0}(n) = e^{-n^2|t|+in^3t}\widehat{u_0}(n)$ for any $t \in \mathbb{R}$, $n \in \mathbb{Z}$. To simplify notation, we let

$$\Phi(t) := \int_0^t S(t-t')\phi\partial_x dW(t').$$

The function $\Phi(t) = \Phi_\omega(t)$ is referred to as the stochastic convolution.

In the proof of Theorem 1.1, the combination of dispersion and dissipation in the KdV–Burgers propagator helps our analysis in the following way: we can establish a bilinear estimate of the form (1.9) (see Proposition 2.3 below) in the $X_T^{s,b}$ space adapted to the KdV–Burgers equation, with $s < -\frac{1}{2}$ and $b < \frac{1}{2}$ (recall that for the $X_T^{s,b}$ space adapted to the KdV equation, $s \geq -\frac{1}{2}$ and $b \geq \frac{1}{2}$ are required for this estimate to hold). Taking $s < -\frac{1}{2}$, we can treat white noise as initial data. With $b < \frac{1}{2}$, the $X_T^{s,b}$ space captures the regularity of space–time white noise. More precisely, the stochastic convolution $\Phi(t)$ is a.s. an element of $X_T^{s,b}$, under appropriate conditions on ϕ (see Proposition 2.4). Finally, the dissipative semigroup has a smoothing effect in space, and the conditions imposed on ϕ by Proposition 2.4 allow for a rougher space–time noise than is considered in [13,26] (we can relax the smoothing of the noise to $s + 2b + \frac{1}{2} = \frac{13}{16}+ < 1$ spatial derivatives, as witnessed in Corollary 1.1).

The remainder of this paper is organized as follows. In Section 2 we prove Theorem 1.1. We establish Theorem 1.2 in Section 3. In Section 4 we prove the bilinear estimate crucial to the proof of Theorem 1.1.

2. Local well-posedness

In this section we prove Theorem 1.1. For almost every fixed $\omega \in \Omega$, we will prove LWP using a fixed point argument in the space $X_{T_\omega}^{s,b}$ of space–time functions (adapted to the KdV–Burgers equation), for a stopping time $T_\omega > 0$, and suitable $s, b \in \mathbb{R}$.

2.1. Local estimates

The proof of Theorem 1.1 will require six key propositions and a lemma concerning the $X_T^{s,b}$ spaces. In this subsection we present these propositions and lemmata.

Proposition 2.1 (*Linear Estimate*)**.** *For any* $s \in \mathbb{R}$,

$$\|S(t)\phi\|_{X_T^{s,\frac{1}{2}}} \leq \|\phi\|_{H^s(\mathbb{T})},$$

for all $\phi \in H^s(\mathbb{T})$.

Proposition 2.2 (*Non-homogeneous Linear Estimate*)**.** *For any $s \in \mathbb{R}$, $\delta > 0$,*

$$\left\| \int_0^t S(t-t')v(t')dt' \right\|_{X_T^{s,\frac{1}{2}}} \le \|v\|_{X_T^{s,-\frac{1}{2}+\delta}}$$

for all $v \in X_T^{s,-\frac{1}{2}+\delta}$.

Propositions 2.1 and 2.2 are both established in [23, Propositions 2.1 and 2.3]. We will prove the following bilinear estimate.

Proposition 2.3 (*Bilinear Estimate*)**.** *For any $0 < \varepsilon < \frac{1}{16}$, if $s \ge -\frac{1}{2} - \varepsilon$, $0 < \gamma \le \varepsilon$, we have the estimate*

$$\|(uv)_x\|_{X_T^{s,-\frac{1}{2}+\gamma}} \le \|u\|_{X_T^{s,\frac{1}{2}-\varepsilon}} \|v\|_{X_T^{s,\frac{1}{2}-\varepsilon}}, \tag{2.1}$$

for all $u, v \in X_T^{s,\frac{1}{2}-\varepsilon}$ with spatial mean zero for all time.

The novelty of Proposition 2.3 is the use of temporal regularity $b = \frac{1}{2} - \varepsilon < \frac{1}{2}$ (on the right-hand side). Recall that for the $X^{s,b}$ space adapted to the KdV equation, this type of estimate requires $b \ge \frac{1}{2}$. This modification (taking $b < \frac{1}{2}$) relies on the combination of dispersion and dissipation in the KdV–Burgers propagator. Specifically, we will benefit from the algebraic identity identified of Bourgain for the KdV equation, $\max(\tau - n^3, \tau_1 - n_1^3, \tau_2 - n_2^3) \ge nn_1n_2$ (dispersion), but we will also exploit the lower bound on the KdV–Burgers weight $|i(\tau_j - n_j^3) + n_j^2| \ge n_j^2$ (dissipation) nearby the dispersive hypersurface $\tau_j = n_j^3$. The proof of Proposition 2.3 is found in Section 4.

To show that the stochastic convolution $\Phi(t)$ is a.s. an element of $X_T^{s,b}$, we compute:

Proposition 2.4 (*Stochastic Convolution Estimate*)**.** *Let $\phi \in HS(L^2; H^{s+2b})$ of the form (1.6). Given $b < \frac{1}{2}$ and $T > 0$, we have*

$$\mathbb{E}\Big(\|\Phi\|^2_{X_T^{s,b}}\Big) \lesssim T\|\phi\|^2_{H^{s+2b}}.$$

According to Proposition 2.4, if $\phi \in HS(L^2; H^{s+2b})$, then $\Phi(t) \in X_T^{s,b}$ a.s. Recall that placing $\phi \in HS(L^2; H^{s+2b})$ corresponds to smoothing the additive noise in (1.1) by $(s + 2b + \frac{1}{2})+$ spatial derivatives. The proof of Proposition 2.4 can be found in the next subsection.

The primary observation of this paper is that, with the combination of dispersion and dissipation, we can obtain the bilinear estimate needed for well-posedness of the KdV–Burgers equation (Proposition 2.3) in a function space ($X_{T_\omega}^{s,b}$) with the regularity of white noise in space ($s < -\frac{1}{2}$) and low regularity in time ($b < \frac{1}{2}$). Then, using Proposition 2.4, we can prove LWP of (1.1) with $\phi \in HS(L^2, H^{s+2b})$, and we have smoothed the additive noise by $(s+2b+\frac{1}{2})+ < 1$ spatial derivatives. That is, we can treat a rougher additive noise than in [13,26].

Remark 1. We remark that a sharp well-posedness result has been obtained for the deterministic KdV–Burgers equation in [24]. Indeed, the authors prove GWP in $H^{-1}(\mathbb{T})$, and ill-posedness in

$H^s(\mathbb{T})$ for $s < -1$. This suggests that, without significantly changing the technique of this paper, we could at best take $s = -1$, and produce LWP of (1.1) after smoothing the additive noise by $(2b - \frac{1}{2})+$ spatial derivatives. In particular we would need to lower $b < 1/4$ in order to treat the case of interest for proving the invariance of white noise: $\phi = Id$.

Using $b < \frac{1}{2}$ requires that we justify continuity of the solution with a separate argument (it is not automatic from the fixed point method). To establish continuity of the nonlinear part of the solution, we appeal to the following proposition of [23].

Proposition 2.5 (*Continuity of the Duhamel Map*)**.** *Let $s \in \mathbb{R}$ and $\gamma > 0$. For all $f \in X^{s,-\frac{1}{2}+\gamma}$,*

$$t \longmapsto \int_0^t S(t-t')f(t')dt' \in C(\mathbb{R}_+, H^{s+2\gamma}). \tag{2.2}$$

Moreover, if $\{f_n\}$ is a sequence with $f_n \to 0$ in $X^{s,-\frac{1}{2}+\gamma}$ as $n \to \infty$, then

$$\left\| \int_0^t S(t-t')f_n(t')dt' \right\|_{L^\infty(\mathbb{R}_+, H^{s+2\gamma})} \longrightarrow 0, \tag{2.3}$$

as $n \to \infty$.

For continuity of the stochastic convolution, we use a probabilistic argument to establish the following proposition.

Proposition 2.6 (*Continuity of the Stochastic Convolution*)**.** *Let $\phi \in HS(L^2, H^{s+2\alpha})$ of the form (1.6), for some $\alpha \in (0, 1/2)$. Then for any $T > 0$, $\Phi \in C([0, T]; H^s(\mathbb{T}))$ a.s.*

The proof of Proposition 2.6 can be found in the next subsection.

The proof of Theorem 1.1 will also require the following lemma concerning the $X_T^{s,b}$ spaces, which allows us to gain a small power of T by raising the temporal exponent b.

Lemma 2.7. *Let $0 < b < \frac{1}{2}$, $s \in \mathbb{R}$, then*

$$\|u\|_{X_T^{s,b}} \lesssim T^{(\frac{1}{2}-b)-} \|u\|_{X_T^{s,\frac{1}{2}}}.$$

The proof of Lemma 2.7 can be found in [10]. This proof relies on the following property of the $X_T^{s,b}$ spaces, for $b < \frac{1}{2}$, which will be exploited throughout this paper. For any $b < \frac{1}{2}$, letting $\chi_{[0,T]}$ denote the characteristic function of the interval $[0, T]$, we have

$$\|u\|_{X_T^{s,b}} \sim \|\chi_{[0,T]}u\|_{X^{s,b}}. \tag{2.4}$$

The justification of (2.4) follows by standard arguments. See for example [10,15].

The remainder of this section is organized as follows. First, in Section 2.2, we present the proofs of Propositions 2.4 and 2.6. Then, in Section 2.3, we will prove Theorem 1.1 using Propositions 2.1–2.6, and Lemma 2.7.

2.2. Stochastic convolution estimates

In this subsection we present the proofs of Propositions 2.4 and 2.6.

Proof of Proposition 2.4. Begin by observing that, for $b < \frac{1}{2}$, by applying (2.4) and using the definition of the $X_T^{s,b}$-norm, we have

$$\begin{aligned}\left\| \int_0^t S(t-t')\phi\partial_x dW(t') \right\|_{X_T^{s,b}} &\sim \left\| \chi_{[0,T]}(t) \int_0^t S(t-t')\phi\partial_x dW(t') \right\|_{X^{s,b}} \\ &= \left\| \chi_{[0,T]}(t) \int_0^t \chi_{[0,T]}(t') S(t-t')\phi\partial_x dW(t') \right\|_{X^{s,b}} \\ &\sim \left\| \int_0^t \chi_{[0,T]}(t') S(t-t')\phi\partial_x dW(t') \right\|_{X_T^{s,b}} \\ &\leq \left\| \chi_{[0,\infty)}(t) \int_0^t \chi_{[0,T]}(t') S(t-t')\phi\partial_x dW(t') \right\|_{X^{s,b}}. \quad (2.5)\end{aligned}$$

Then using the stochastic Fubini theorem, we compute that

$$\begin{aligned}&\left\| \chi_{[0,\infty)}(t) \int_0^t \chi_{[0,T]}(t') S(t-t')\phi\partial_x dW(t') \right\|_{X^{s,b}} \\ &\quad = \left\| |n\phi_n| \langle n\rangle^s \langle i\tilde{\tau} + n^2\rangle^b \int_0^T \left(\int_{t'}^\infty e^{-it\tilde{\tau}} e^{-n^2(t-t')} e^{-in^3 t'} dt \right) dB_n(t') \right\|_{L^2_{n,\tilde{\tau}}},\end{aligned}$$

where we have taken $\tilde{\tau} = \tau - n^3$. Now we find, for each $n \neq 0$,

$$\begin{aligned}\int_{t'}^\infty e^{-it\tilde{\tau}} e^{-n^2(t-t')} e^{-in^3 t'} dt &= e^{n^2 t'} e^{-in^3 t'} \left[\frac{1}{-i\tilde{\tau} - n^2} e^{-it\tilde{\tau}} e^{-n^2 t} \right]_{t=t'}^{t=\infty} \\ &= \frac{1}{i\tilde{\tau} + n^2} e^{-in^3 t'} e^{-it'\tilde{\tau}}.\end{aligned}$$

Then bringing the expectation inside, and applying the Itô isometry,

$$\mathbb{E}\left(\left\| \int_0^t S(t-t')\phi\partial_x dW(t') \right\|_{X_T^{s,b}}^2 \right) \lesssim T \sum_n \langle n\rangle^{2s} |n|^{4b} |\phi_n|^2 \int_{-\infty}^\infty \frac{1}{\langle\rho\rangle^{2-2b}} d\rho$$

$$\text{where } \rho = \frac{\tilde{\tau}}{n^2}, \sim T\|\phi\|_{H^{s+2b}}^2, \quad \text{for } b < \frac{1}{2}. \quad \square$$

Proof of Proposition 2.6. This proof follows the factorization method of Da Prato [11]. From the identity

$$\int_r^t (t-t')^{\alpha-1}(t'-r)^{-\alpha} dt' = \frac{\pi}{\sin \pi\alpha}$$

for $\alpha \in (0, 1/2)$, $0 \leq r \leq t' \leq t$. Letting

$$Y(t') = \int_0^{t'} S(t'-r)(t'-r)^{-\alpha}\phi\partial_x dW(r),$$

we find

$$\frac{\sin(\pi\alpha)}{\pi} \int_0^t S(t-t')(t-t')^{\alpha-1} Y(t') dt' = \int_0^t S(t-r)\phi\partial_x dW(r) = \Phi(t).$$

Next we use the following lemma from [11, Lemma 2.7]:

Lemma 2.8. *Let $T > 0$, $\alpha \in (0, 1/2)$, and $m > \frac{1}{2\alpha}$. For $f \in L^{2m}([0, T]; H^s(\mathbb{T}))$, let*

$$F(t) = \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')dt', \quad 0 \le t \le T.$$

Then $F \in C([0, T]; H^s(\mathbb{T}))$. Moreover $\exists\, C = C(m, T)$ such that

$$\|F(t)\|_{H^s(\mathbb{T})} \le C\|f\|_{L^{2m}([0,T];H^s(\mathbb{T}))}.$$

In [11], Lemma 2.8 is established for any semi-group $S(t)$ which is an isometry on H^s for each fixed $t \in [0, T]$. It is easily verified, however, that the proof in [11] requires only finiteness of the following expression:

$$\begin{aligned} M_T &= \sup_{t\in[0,T]} \|S(t)\|_{H^s\to H^s} = \sup_{t\in[0,T]} \sup_{\|f\|_{H^s}=1} \|S(t)f\|_{H^s} \\ &= \sup_{t\in[0,T]} \sup_{\|f\|_{H^s}=1} \Big(\sum_n \langle n\rangle^{2s} e^{-2n^2|t|} |\hat{f}(n)|^2\Big)^{1/2} \le 1. \end{aligned}$$

Therefore, Lemma 2.8 applies in our context. In view of Lemma 2.8, to prove Proposition 2.6 it suffices to show that, a.s.,

$$Y(t') \in L^{2m}([0, T]; H^s(\mathbb{T})).$$

Using the Itô isometry, we compute

$$\mathbb{E}\Big(|\widehat{Y(t')}(n)|^2\Big) = |\phi_n|^2 n^2 \int_0^{t'} e^{-2n^2(t'-r)}(t'-r)^{-2\alpha}dr, \tag{2.6}$$

and find

$$\int_0^{t'} e^{-2n^2(t'-r)}(t'-r)^{-2\alpha}dr \lesssim \frac{1}{n^{2(1-2\alpha)}}, \tag{2.7}$$

for $\alpha < \frac{1}{2}$. Combining (2.6) with (2.7), and applying the Minkowski integral inequality (with $2m > 2$), we have

$$\mathbb{E}\Big(\|Y(t')\|^{2m}_{H^s(\mathbb{T})}\Big) \lesssim \|\phi\|^{2m}_{H^{s+2\alpha}}.$$

Then

$$\mathbb{E}\Big(\int_0^T \|Y(t')\|^{2m}_{H^s(\mathbb{T})}dt'\Big) \lesssim \|\phi\|^{2m}_{H^{s+2\alpha}} T < \infty.$$

Thus $Y(t') \in L^{2m}([0, T]; H^s(\mathbb{T}))$ a.s., and the proof of Proposition 2.6 is complete. □

2.3. *Proof of local well-posedness*

Proof of Theorem 1.1. Fix ε, s and ϕ as in the statement of Theorem 1.1. Letting $z(t) := S(t)u_0$, we can rewrite (1.12) in terms of $v := u - z - \Phi$,

$$\begin{aligned} v &= -\frac{1}{2}\int_0^t S(t-t')\partial_x((v+z+\Phi)^2(t'))dt' \\ &=: \Gamma(v). \end{aligned} \tag{2.8}$$

This way, the solution u to (1.12) is the unique fixed point of $\tilde{\Gamma}(u) := z + \Phi + \Gamma(u - z - \Phi)$ on the unit ball in $X^{s,b}_{T_\omega}$ centered at $z + \Phi$ if and only if v is the unique fixed point of Γ on the unit ball in $X^{s,b}_{T_\omega}$ centered at 0. We define

$$T_\omega := \min\left\{T > 0 : 2CT^{\varepsilon-}\Big(\|u_0\|_{H^s(\mathbb{T})} + 2 + \|\chi_{[0,T]}\Phi\|_{X^{s,\frac{1}{2}-\varepsilon}}\Big)^2 \geq 1\right\}. \tag{2.9}$$

By Proposition 2.4,

$$\mathbb{E}(\|\Phi\|^2_{X_1^{s,\frac{1}{2}-\varepsilon}}) \lesssim \|\phi\|^2_{H^{s+1-2\varepsilon}} < \infty,$$

and we have, for any $0 < T < 1$,

$$\begin{aligned}\|\chi_{[0,T]}\Phi\|_{X^{s,\frac{1}{2}-\varepsilon}} &\lesssim \|\Phi\|_{X_1^{s,\frac{1}{2}-\varepsilon}}\\ &\leq C(\omega),\end{aligned}$$

a.s. In addition, since $b < \frac{1}{2}$, $\|\chi_{[0,T]}\Phi\|_{X^{s,\frac{1}{2}-\varepsilon}}$ is a.s. continuous with respect to T. It follows that $T_\omega > 0$ a.s. Since $\|\chi_{[0,T]}\Phi\|_{X^{s,b}}$ is $\mathcal{F}_T$-measurable, T_ω is a stopping time.

With these definitions in place, the proof that Γ is a contraction on the unit ball in $X^{s,b}_{T_\omega}$ follows from Propositions 2.1–2.6 and Lemma 2.7 using standard arguments. See for example [15,16], or Section 4.2.3 of [32] for details.

It remains to establish a.s. continuity of $u(t)$ in $H^s(\mathbb{T})$, and a.s. continuous dependence on the data. We begin by proving that $u \in C([0,T]; H^s(\mathbb{T}))$. Given that $u = z + v + \Phi$, it suffices to verify a.s. continuity of z, v and Φ with separate arguments. Continuity of $z(t) = S(t)u_0$ is trivial. Almost sure continuity of

$$v = \Gamma(v) = \int_0^t S(t-t')\partial_x((z + v + \Phi)^2(t'))dt'$$

follows from Proposition 2.5, and the following estimate:

$$\begin{aligned}\|\partial_x\big((z+v+\Phi)^2\big)\|_{X_T^{s,-\frac{1}{2}+\gamma}} &\lesssim \|v + z + \Phi\|^2_{X_T^{s,\frac{1}{2}-\varepsilon}}\\ &\lesssim (1 + \|u_0\|_{H^s(\mathbb{T})} + C(\omega))^2\\ &< \infty,\end{aligned} \tag{2.10}$$

a.s. In the statement (2.10) we have invoked Propositions 2.1, 2.3 and 2.4. Finally, almost sure continuity of Φ follows from Proposition 2.6, by witnessing that $\phi \in HS(L^2; H^{s+1-2\varepsilon}) \subset HS(L^2; H^{s+2\alpha})$, when $\alpha > 0$ is sufficiently small.

Having established that $u \in C([0,T]; H^s(\mathbb{T}))$ a.s., it remains to justify a.s. continuous dependence on the data. Suppose $\{u_0^n\}$ is a sequence in $H^s(\mathbb{T})$ converging to some u_0. From the dependence of the time of local existence $T_\omega > 0$ on the $H^s(\mathbb{T})$ norm of the initial data, it follows that for all n sufficiently large, the solutions u^n and u to (1.1) with initial data u_0^n and u_0, respectively, both exist on a time interval $[0, T_\omega]$, with $T_\omega > 0$ (independent of n). The fixed point method guarantees that the solution map $u_0 \in H^s(\mathbb{T}) \mapsto u \in X^{s,\frac{1}{2}-\varepsilon}_{T_\omega}$ is analytic. In particular, we have that $u_n \to u$ in $X_{T_\omega}^{s,\frac{1}{2}-\varepsilon}$. Letting $f_n = \partial_x\big((u^n-u)(u^n+u)\big)$, by Proposition 2.3, we have

$$\begin{aligned}\|f_n\|_{X_{T_\omega}^{s,-\frac{1}{2}+\gamma}} &= \|\partial_x\big((u^n-u)(u^n+u)\big)\|_{X_{T_\omega}^{s,-\frac{1}{2}+\gamma}} \\ &\lesssim \|u^n+u\|_{X_{T_\omega}^{s,\frac{1}{2}-\varepsilon}}\|u^n-u\|_{X_{T_\omega}^{s,\frac{1}{2}-\varepsilon}} \longrightarrow 0,\end{aligned}$$

as $n \to \infty$. Writing

$$u^n = S(t)u_0^n + \int_0^t S(t-t')\partial_x\big((u^n)^2\big)dt' + \Phi(t)$$

and

$$u = S(t)u_0 + \int_0^t S(t-t')\partial_x\big(u^2\big)dt' + \Phi(t),$$

we have by Proposition 2.5 that

$$\begin{aligned}u^n - u &= \int_0^t S(t-t')\partial_x\big((u^n-u)(u^n+u)\big)dt' \\ &= \int_0^t S(t-t')\partial_x f(t')dt' \longrightarrow 0,\end{aligned}$$

in $C([0,T_\omega]; H^s(\mathbb{T}))$ a.s. We conclude that the data-to-solution map for (1.1) is a.s. continuous.

Finally, we observe that the same argument can be used, for fixed $u_0 \in H^s(\mathbb{T})$, to verify that the map $\Phi \in X_{T_\omega}^{s,\frac{1}{2}-\varepsilon} \mapsto v = v(\Phi) \in C([0,T_\omega]; H^s(\mathbb{T}))$ satisfying (2.8) is a.s. continuous. In particular, this confirms that the map $\omega \in \Omega \mapsto v = v_\omega \in C([0,T_\omega]; H^s(\mathbb{T}))$ satisfying (2.8) is $\mathcal{F}_{T_\omega}$-measurable, as this is the composition of the measurable map $\omega \in \Omega \mapsto \Phi \in X_{T_\omega}^{s,\frac{1}{2}-\varepsilon}$ and the continuous map $\Phi \in X_{T_\omega}^{s,\frac{1}{2}-\varepsilon} \mapsto v = v(\Phi) \in C([0,T_\omega]; H^s(\mathbb{T}))$. Writing $u = z + v + \Phi$, we combine this observation with Proposition 2.6 to conclude that the map $\omega \in \Omega \mapsto u = u_\omega \in C([0,T_\omega]; H^s(\mathbb{T}))$ satisfying (1.12) is $\mathcal{F}_{T_\omega}$-measurable.

In conclusion, there is a stopping time $T_\omega > 0$ and a unique process $u \in C([0,T_\omega]; H^s(\mathbb{T}))$ satisfying (1.1) on $[0,T_\omega]$ a.s. The proof of Theorem 1.1 is complete. $\square$

3. Global well-posedness

This section is devoted to the proof of Theorem 1.2. We begin by establishing a priori bounds on the Galerkin approximation of (1.1). Given $N > 0$, let $\mathbb{P}_N$ denote the Dirichlet projection to $E_N = \operatorname{span}\{\cos(nx), \sin(nx) : 1 \le n \le N\}$. We consider the frequency truncated stochastic PDE

$$\begin{cases} du^N = \Big(u_{xx}^N - u_{xxx}^N - \dfrac{1}{2}\mathbb{P}_N\big[((u^N)^2)_x\big]\Big)dt + \phi^N\partial_x dW, \quad t \ge 0, x \in \mathbb{T} \\ u^N(0,x) = u_0^N(x) = \mathbb{P}_N(u_0(x)), \end{cases} \tag{3.1}$$

where $\phi^N = \mathbb{P}_N\phi$, and $u^N = \mathbb{P}_N u^N$. We will solve the Duhamel form of (3.1)

$$u^N = S(t)u_0^N - \frac{1}{2}\int_0^t S(t-t')\mathbb{P}_N\big((u^N)^2\big)_x dt' + \int_0^t S(t-t')\phi^N\partial_x dW(t'). \tag{3.2}$$

We will also take

$$\Phi^N(t) = \int_0^t S(t-t')\phi^N \partial_x dW(t')$$

to denote the frequency truncated stochastic convolution. In this subsection, we establish uniform bounds on the solution to (3.1). We begin with the following proposition.

Proposition 3.1. *Let $u_0 \in H^s(\mathbb{T})$, and $\phi \in HS(L^2(\mathbb{T}), H^s(\mathbb{T}))$ of the form (1.6). For every $N > 0$, and each $T > 0$, there is a.s. a unique solution $u^N(t)$ to (3.2) for all $t \in [0, T]$.*

Proposition 3.1 provides the existence of global-in-time solutions to the frequency truncated stochastic PDE (3.1). Because (3.1) is finite-dimensional, this result is (essentially) independent of any conditions placed on u_0 and ϕ; we have taken $u_0 \in H^s$ and $\phi \in HS(L^2, H^s)$ because this is sufficient for our purposes. The proof of Proposition 3.1 is somewhat lengthy so we avoid it here; an interested reader should consult Section 5.3.2 of [32].

Consider initial data $u_0 \in L^2(\mathbb{T})$, and additive noise smoothed by $\frac{3}{2}+$ spatial derivatives (that is, consider $\phi \in HS(L^2, H^1)$). We can apply Proposition 3.1 with $s = 0$, since $\phi \in HS(L^2, H^1) \subset HS(L^2, L^2)$. We conclude that there is a.s. a unique global-in-time solution $u^N(t)$ to (3.1). With these conditions on the data and noise, we can establish the following bound on the (expected) growth of the L^2-norm of the solution $u^N(t)$.

Proposition 3.2. *Let $u_0 \in L^2(\mathbb{T})$, and $\phi \in HS(L^2(\mathbb{T}), H^1(\mathbb{T}))$ of the form (1.6). The unique solution $u^N(t)$ to (3.1) satisfies*

$$\mathbb{E}\Big(\sup_{0\le t\le T} \|u^N(t)\|^2_{L^2_x} \Big) \le C, \tag{3.3}$$

where $C = C(T, \|u_0\|_{L^2_x}, \|\phi\|_{H^1})$.

The crucial point is that this bound is independent of $N > 0$. The proof of Proposition 3.2 uses standard Itô calculus and can be found in Section 4.3.1 of [32].

Proof of Theorem 1.2. Given $u_0 \in L^2(\mathbb{T})$, $\phi \in HS(L^2, H^1)$ of the form (1.6), for every $N > 0$ we form the cutoff functions $u_0^N = \mathbb{P}_N u_0$, $\phi^N = \mathbb{P}_N \phi$. Given $T > 0$, by Propositions 3.1 and 3.2, a unique solution $u^N(t)$ to (3.2) a.s. exists for $t \in [0, T]$ and satisfies (3.3). Hence, the sequence $\{u^N\}_{N\in\mathbb{N}}$ is bounded in $L^2(\Omega; L^\infty((0,T); L^2(\mathbb{T})))$, and we can extract a subsequence which converges weak-* to a limit $\tilde{u} \in L^2(\Omega; L^\infty((0,T); L^2(\mathbb{T})))$ satisfying (3.3). It remains to justify that $\tilde{u}$ satisfies (1.12) on $[0, T]$ a.s.

Letting $z^N(t) = S(t)u_0^N$, and $v^N = u^N - z^N - \Phi^N$, then for each N, v^N satisfies the truncated equation

$$\begin{aligned} v^N &= \int_0^t S(t-t')\mathbb{P}_N\Big(\partial_x\big((z^N + v^N + \Phi^N)^2\big)(t')\Big)dt' \\ &=: \Gamma^N(v^N). \end{aligned} \tag{3.4}$$

By repeating the proof of Theorem 1.1, we can show that Γ^N is a.s. a contraction on a ball of radius 1 in $X^{0,\frac{1}{2}-\varepsilon}_{\tilde{T}}$ for any $\tilde{T} > 0$ satisfying

$$2C\tilde{T}^{\varepsilon-}\Big(2 + \|u_0^N\|_{L^2_x} + \|\chi_{[0,\tilde{T}]}\Phi^N\|_{X^{0,\frac{1}{2}-\varepsilon}}\Big)^2 \le 1, \tag{3.5}$$

for a certain constant $C > 0$. Let

$$D(\omega) := \sup_{0 \le t \le T} \|\tilde{u}(t)\|_{L_x^2}^2.$$

Then since $\tilde{u}$ satisfies (3.3) on $[0, T]$, we have that $D(\omega) < \infty$ a.s. Consider $\tilde{T}_\omega > 0$ satisfying

$$2C\tilde{T}_\omega^{\varepsilon-}\Big(2 + \|u_0\|_{L_x^2} + D(\omega) + \|\chi_{[0,T]}\Phi\|_{X^{s,\frac{1}{2}-\varepsilon}}\Big)^2 \le 1. \tag{3.6}$$

Then for every $N > 0$, we have

$$\|u_0^N\|_{L_x^2} \le \|u_0\|_{L_x^2},$$

and

$$\|\chi_{[0,\tilde{T}_\omega]}\Phi^N\|_{X^{s,\frac{1}{2}-\varepsilon}} \le \|\chi_{[0,T]}\Phi\|_{X^{s,\frac{1}{2}-\varepsilon}}.$$

It follows that (3.5) is satisfied a.s. for every $N > 0$ with $\tilde{T} = \tilde{T}_\omega$. Furthermore, we have $\tilde{T}_\omega \le T_\omega$, where T_ω is the time of local existence for the full solution v coming from the proof of Theorem 1.1. We conclude that Γ^N and Γ are contractions (for every $N > 0$) in $X_{\tilde{T}_\omega}^{0,\frac{1}{2}-\varepsilon}$, where $\tilde{T}_\omega$ satisfies (3.6). In particular, a unique solution $v \in X_{\tilde{T}_\omega}^{0,\frac{1}{2}-\varepsilon}$ to (2.8) a.s. exists. Furthermore, for each $N > 0$, v^N and v are the unique fixed points of the contractions Γ^N and Γ, respectively.

Using the estimates of Section 2.1 we can show that $u^N \to u$ in $C([0, \tilde{T}_\omega]; L^2(\mathbb{T}))$, and conclude that $u = \tilde{u}$ for $t \in [0, \tilde{T}_\omega]$ a.s. (see Section 4.3.2 of [32] for details). This gives

$$\|u(\tilde{T}_\omega)\|_{L_x^2}^2 \le \sup_{0 \le t \le T} \|\tilde{u}(t)\|_{L_x^2}^2 = D(\omega). \tag{3.7}$$

By (3.6) we can iterate the argument above on $[\tilde{T}_\omega, 2\tilde{T}_\omega]$. Thus, we have $u = \tilde{u}$ on $[0, T]$ a.s., and the proof is complete. □

4. Bilinear estimate

Proof of Proposition 2.3. By (2.4), the estimate (2.1) follows from

$$\|\chi_{[0,T]}(uv)_x\|_{X^{s,-\frac{1}{2}+\gamma}} \lesssim \|\chi_{[0,T]}u\|_{X^{s,\frac{1}{2}-\varepsilon}} \|\chi_{[0,T]}v\|_{X^{s,\frac{1}{2}-\varepsilon}}.$$

Then since $\chi_{[0,T]}\partial_x(uv) = \partial_x\big((\chi_{[0,T]}u)(\chi_{[0,T]}v)\big)$, it suffices to show that

$$\|\partial_x\big((\chi_{[0,T]}u)(\chi_{[0,T]}v)\big)\|_{X^{s,-\frac{1}{2}+\gamma}} \lesssim \|\chi_{[0,T]}u\|_{X^{s,\frac{1}{2}-\varepsilon}} \|\chi_{[0,T]}v\|_{X^{s,\frac{1}{2}-\varepsilon}}. \tag{4.1}$$

Letting

$$f(n,\tau) = \langle n\rangle^s \langle i(\tau - n^3) + n^2\rangle^{\frac{1}{2}-\varepsilon} (\widehat{\chi_{[0,T]}u})(n,\tau)$$
$$g(n,\tau) = \langle n\rangle^s \langle i(\tau - n^3) + n^2\rangle^{\frac{1}{2}-\varepsilon} (\widehat{\chi_{[0,T]}v})(n,\tau),$$

the estimate (4.1) is equivalent to

$$\|\beta(f,g)\|_{L_{n,\tau}^2} \lesssim \|f\|_{L_{n,\tau}^2} \|g\|_{L_{n,\tau}^2}, \tag{4.2}$$

where

$$\beta(f,g)(n,\tau) := \sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s f(n_1,\tau_1) g(n_2,\tau_2)}{\langle n_1\rangle^s \langle n_2\rangle^s \langle\sigma_0\rangle^{\frac{1}{2}-\gamma} \langle\sigma_1\rangle^{\frac{1}{2}-\varepsilon} \langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_1$$

with $\sigma_i = i(\tau_i - n_i^3) + n_i^2$, for $i = 1, 2$, and $\sigma_0 = i(\tau - n^3) + n^2$.

We proceed to justify (4.2). In the analysis that follows, for a given $(n, n_1, n_2) \in \mathbb{Z}^3$, we order the magnitudes of the frequencies $|n|, |n_1|, |n_2|$ from largest to smallest, using capital letters and superscripts to denote the corresponding dyadic shell: i.e. $N^1 \geq N^2 \geq N^3$. We begin by performing a computation which will simplify subsequent estimates.

Lemma 4.1. *If $n = n_1 + n_2$ and $s \geq -\frac{1}{2} - \varepsilon$, then*

$$\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n n_1 n_2\rangle^{\frac{1}{2}-\varepsilon}} \lesssim |N^1|^{4\varepsilon}.$$

We will also require the following Calculus inequalities:

Lemma 4.2. *Let $0 < \delta_1 \leq \delta_2$ satisfy $\delta_1 + \delta_2 > 1$, and let $a \in \mathbb{R}$, then*

$$\int_{-\infty}^{\infty} \frac{d\theta}{\langle\theta\rangle^{\delta_1} \langle a-\theta\rangle^{\delta_2}} \lesssim \frac{1}{\langle a\rangle^{\alpha}},$$

where $\alpha = \delta_1 - (1-\delta_2)_+$. Recall that $(\lambda)_+ := \lambda$ if $\lambda > 0$, $= \varepsilon > 0$ if $\lambda = 0$, and $= 0$ if $\lambda < 0$.

Lemma 4.3. *Let $\delta > \frac{1}{2}$, $n \neq 0$, then*

$$\left\| \sum_{n_1 \neq 0, n_1 \neq n} \frac{1}{(1+|\mu - n_1(n-n_1)|)^{\delta}} \right\|_{L^\infty_{\mu,n}} < c < \infty.$$

Proofs of Lemmas 4.2 and 4.3 can be found in [18] and [21], respectively. We proceed with the proof of Lemma 4.1.

Proof of Lemma 4.1. We consider two cases depending on the relative sizes of n, n_1, n_2. Recall that as $n = n_1 + n_2$, we always have $N^1 \sim N^2$.

• **Case** 1: $|n| \sim N^1$.
Then

$$\begin{aligned}\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n n_1 n_2\rangle^{\frac{1}{2}-\varepsilon}} \sim \frac{|N^1|^{2\varepsilon}}{|N^3|^{s+\frac{1}{2}-\varepsilon}} &\leq |N^1|^{2\varepsilon}|N^3|^{2\varepsilon} \quad \text{since } s \geq -\frac{1}{2} - \varepsilon,\\ &\leq |N^1|^{4\varepsilon}.\end{aligned}$$

• **Case** 2: $|n| \sim N^3$, so that $|n_1| \sim |n_2| \sim N^1$.
Then

$$\begin{aligned}\frac{|n|\langle n\rangle^s}{\langle n_1\rangle^s \langle n_2\rangle^s \langle n n_1 n_2\rangle^{\frac{1}{2}-\varepsilon}} \sim \left|\frac{N^3}{N^1}\right|^{s+\frac{1}{2}} \frac{|N^3|^{\varepsilon}}{|N^1|^{s+\frac{1}{2}-2\varepsilon}} &\leq \left|\frac{N^3}{N^1}\right|^{s+\frac{1}{2}+\varepsilon} \frac{|N^1|^{4\varepsilon}}{|N^1|^{s+\frac{1}{2}+\varepsilon}}\\ &\leq |N^1|^{4\varepsilon}, \quad \text{since } s \geq -\frac{1}{2} - \varepsilon.\end{aligned}$$

This completes the proof of Lemma 4.1. □

We now turn to the proof of (4.2). In the estimates that follow, we will take $\gamma = \varepsilon$ for simplicity. For $i = 0, 1, 2$, let

$$A_i = \{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}^3 \times \mathbb{R}^3 : \max(|\sigma_0|, |\sigma_1|, |\sigma_2|) = |\sigma_i|\},$$

and let $\beta_i(f, g)$ denote the contribution to $\beta(f, g)$ coming from A_i. We separately estimate each $\beta_i(f, g)$, $i = 0, 1, 2$.

• **Case** 1: $\max(|\sigma_0|, |\sigma_1|, |\sigma_2|) = |\sigma_0|$.

From the algebraic relation $\max(|\sigma_0|, |\sigma_1|, |\sigma_2|) \geq |nn_1n_2|$, we have

$$\begin{aligned}
&\|\beta_0(f, g)\|_{L^2_{n,\tau}} \\
&\leq \left\| \sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s |f(n_1, \tau_1)||g(n_2, \tau_2)|}{\langle n_1\rangle^s \langle n_2\rangle^s \langle nn_1n_2\rangle^{\frac{1}{2}-\varepsilon} \langle \sigma_1\rangle^{\frac{1}{2}-\varepsilon} \langle \sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_1 \right\|_{L^2_{n,\tau}} \\
&\leq \left\| \sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|N^1|^{4\varepsilon} |f(n_1, \tau_1)||g(n_2, \tau_2)|}{\langle \sigma_1\rangle^{\frac{1}{2}-\varepsilon} \langle \sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_1 \right\|_{L^2_{n,\tau}} \qquad (4.3)
\end{aligned}$$

by Lemma 4.1. We will cancel the factor of $|N^1|^{4\varepsilon}$ by considering the relative sizes of n_1, n_2.

• **Case** 1.A: $|n_1| \sim N^1$.

First observe that

$$\begin{aligned}
&\int_{\tau=\tau_1+\tau_2} \frac{d\tau_1}{\langle \sigma_1\rangle^{1-6\varepsilon} \langle \sigma_2\rangle^{1-2\varepsilon}} \leq \int_{-\infty}^{\infty} \frac{d\tau_1}{\langle \tau_1 - n_1^3\rangle^{1-6\varepsilon} \langle \tau - \tau_1 - (n-n_1)^3\rangle^{1-2\varepsilon}} \\
&= \int_{-\infty}^{\infty} \frac{d\tau_1}{\langle \theta\rangle^{1-6\varepsilon} \langle a - \theta\rangle^{1-2\varepsilon}}, \quad \text{with } \theta = \tau_1 - n_1^3, a = \tau - n^3 + 3nn_1(n-n_1), \\
&\leq \frac{1}{\langle a\rangle^{1-8\varepsilon}}, \qquad (4.4)
\end{aligned}$$

by Lemma 4.2. Let

$$M_{n,\tau} := \Big(\sum_{n_1, n_1 \neq 0, n_1 \neq n} \frac{1}{\langle \tau - n^3 + 3nn_1(n-n_1)\rangle^{1-8\varepsilon}} \Big)^{\frac{1}{2}}.$$

For $n \neq 0$ we can take $\mu = \frac{1}{3}\big(\frac{\tau}{n} - n^2\big)$, and find

$$\frac{1}{\langle \tau - n^3 + 3nn_1(n-n_1)\rangle} \leq \frac{1}{\langle \mu - n_1(n-n_1)\rangle}.$$

This leads to

$$\begin{aligned}
\sup_{\substack{n,\tau \\ n\neq 0}} M_{n,\tau} &= \sup_{\substack{n,\tau \\ n\neq 0}} \Big(\sum_{\substack{n_1 \\ n_1\neq 0, n_1\neq n}} \frac{1}{\langle \tau - n^3 + 3nn_1(n-n_1)\rangle^{1-8\varepsilon}} \Big)^{\frac{1}{2}} \\
&\leq \sup_{\substack{n,\mu \\ n\neq 0}} \Big(\sum_{\substack{n_1 \\ n_1\neq 0, n_1\neq n}} \frac{1}{\langle \mu - n_1(n-n_1)\rangle^{1-8\varepsilon}} \Big)^{\frac{1}{2}} \\
&= C < \infty, \qquad (4.5)
\end{aligned}$$

by Lemma 4.3, if $\varepsilon < \frac{1}{16}$.

Returning to the proof of (4.2), notice that with $|\sigma_1| \geq n_1^2 \sim |N^1|^2$ we have $\frac{1}{\langle\sigma_1\rangle^{2\varepsilon}} \lesssim \frac{1}{|N^1|^{4\varepsilon}}$. Starting from (4.3), we invoke this inequality, then apply Cauchy–Schwarz and (4.4). Next we take out $\sup_{n,\tau} M_{n,\tau}$ and apply (4.5). This procedure gives

$$\begin{aligned}
\|\beta_0(f,g)\|_{L^2_{n,\tau}} &\lesssim \left\| \sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|f(n_1,\tau_1)||g(n_2,\tau_2)|}{\langle\sigma_1\rangle^{\frac{1}{2}-3\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_1 \right\|_{L^2_{n,\tau}} \\
&\leq \left\| \Big(\sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} |f(n_1,\tau_1)|^2 |g(n_2,\tau_2)|^2 d\tau_1 \Big)^{\frac{1}{2}} \right. \\
&\quad \left. \cdot \Big(\sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{d\tau_1}{\langle\sigma_1\rangle^{1-6\varepsilon}\langle\sigma_2\rangle^{1-2\varepsilon}} \Big)^{\frac{1}{2}} \right\|_{L^2_{n,\tau}} \\
&\lesssim \Big(\sup_{\substack{n,\tau \\ n\neq 0}} M_{n,\tau} \Big) \left\| \Big(\sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} |f(n_1,\tau_1)|^2 |g(n_2,\tau_2)|^2 d\tau_1 \Big)^{\frac{1}{2}} \right\|_{L^2_{n,\tau}} \\
&\lesssim \left\| \Big(\sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} |f(n_1,\tau_1)|^2 |g(n_2,\tau_2)|^2 d\tau_1 \Big)^{\frac{1}{2}} \right\|_{L^2_{n,\tau}} \\
&= \|f\|_{L^2_{n,\tau}} \|g\|_{L^2_{n,\tau}},
\end{aligned}$$

by Fubini's Theorem in the last line. This completes the analysis of Case 1.A.

• **Case** 1.B: $|n| \sim |n_2| \sim N^1$, $|n_1| \sim N^3$.
Then $|\sigma_2| \gtrsim |N^1|^2$, $\Rightarrow \frac{1}{\langle\sigma_2\rangle^{2\varepsilon}} \lesssim \frac{1}{|N^1|^{4\varepsilon}}$. With (4.3) this leads to

$$\|\beta_0(f,g)\|_{L^2_{n,\tau}} \lesssim \|f\|_{L^2_{n,\tau}} \|g\|_{L^2_{n,\tau}},$$

by the analysis done in Case 1.A. This completes Case 1. That is, the proof of (4.2) in the region A_0 is complete.

• **Case** 2: $|\sigma_1| = \max(|\sigma_0|, |\sigma_1|, |\sigma_2|)$.
Using $\langle\sigma_1\rangle \gtrsim \langle nn_1n_2\rangle \sim \langle n\rangle\langle n_1\rangle\langle n_2\rangle$ (recall that $n_i \neq 0$ as u and v are assumed to have mean zero for all time), the estimate

$$\|\beta_1(f,g)\|_{L^2_{n,\tau}} \lesssim \|f\|_{L^2_{n,\tau}} \|g\|_{L^2_{n,\tau}}$$

follows from

$$\|\eta(f,g)\|_{L^2_{n,\tau}} \lesssim \|f\|_{L^2_{n,\tau}} \|g\|_{L^2_{n,\tau}}$$

where

$$\eta(f,g)(n,\tau) := \sum_{\substack{n_1 \\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s f(n_1,\tau_1) g(n_2,\tau_2)}{\langle n_1\rangle^s \langle n_2\rangle^s \langle\sigma_0\rangle^{\frac{1}{2}-\varepsilon} (\langle n\rangle\langle n_1\rangle\langle n_2\rangle)^{\frac{1}{2}-\varepsilon} \langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_1.$$

Invoking duality, we choose to establish the equivalent estimate:

$$\|\tilde{\beta}(g,h)\|_{L^2_{n_1,\tau_1}} \lesssim \|g\|_{L^2_{n,\tau}}\|h\|_{L^2_{n,\tau}}, \tag{4.6}$$

where

$$\tilde{\beta}(g,h)(n_1,\tau_1) := \sum_{\substack{n\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s h(n,\tau) g(n_2,\tau_2)}{\langle n_1\rangle^s \langle n_2\rangle^s \langle\sigma_0\rangle^{\frac{1}{2}-\varepsilon}\langle nn_1n_2\rangle^{\frac{1}{2}-\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau.$$

By Lemma 4.1, we have

$$\|\tilde{\beta}(g,h)\|_{L^2_{n_1,\tau_1}} \le \left\| \sum_{\substack{n\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|N^1|^{4\varepsilon}|h(n,\tau)||g(n_2,\tau_2)|}{\langle\sigma_0\rangle^{\frac{1}{2}-\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau \right\|_{L^2_{n_1,\tau_1}}. \tag{4.7}$$

Again, we handle the factor of $|N^1|^{4\varepsilon}$ by considering the relative sizes of n_1, n_2.

• **Case** 2.A: $|n| \sim N^1$.
Then $|\sigma_0| \gtrsim |N^1|^2, \quad \Rightarrow \frac{1}{\langle\sigma_0\rangle^{2\varepsilon}} \lesssim \frac{1}{|N^1|^{4\varepsilon}}$. Combining this estimate with (4.7), and applying Cauchy–Schwarz, we find

$$\begin{aligned}\|\tilde{\beta}(g,h)\|_{L^2_{n_1,\tau_1}} &\le \left\| \sum_{\substack{n\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|h(n,\tau)||g(n_2,\tau_2)|}{\langle\sigma_0\rangle^{\frac{1}{2}-3\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau \right\|_{L^2_{n_1,\tau_1}}\\ &\le \|g\|_{L^2_{n,\tau}}\|h\|_{L^2_{n,\tau}},\end{aligned}$$

by the analysis done in Case 1.A.

• **Case** 2.B:, $|n_1| \sim |n_2| \sim N^1$, $|n| \sim N^3$.
Then $|\sigma_2| \gtrsim |N^1|^2, \quad \Rightarrow \frac{1}{\langle\sigma_2\rangle^{2\varepsilon}} \lesssim \frac{1}{|N^1|^{4\varepsilon}}$. Combined with (4.7) this leads to

$$\begin{aligned}\|\tilde{\beta}(g,h)\|_{L^2_{n_1,\tau_1}} &\lesssim \left\| \sum_{\substack{n_2\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|h(n,\tau)||g(n_2,\tau_2)|}{\langle\sigma_0\rangle^{\frac{1}{2}-\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-3\varepsilon}} d\tau_2 \right\|_{L^2_{n_1,\tau_1}}\\ &\lesssim \|g\|_{L^2_{n,\tau}}\|h\|_{L^2_{n,\tau}},\end{aligned}$$

by the analysis done in Case 1(A).

• **Case** 3: $|\sigma_2| = \max(|\sigma_0|, |\sigma_1|, |\sigma_2|)$.
Observe that we may rewrite (4.2) as

$$\|\beta(f,g)\|_{L^2_{n,\tau}} \lesssim \|f\|_{L^2_{n,\tau}}\|g\|_{L^2_{n,\tau}}, \tag{4.8}$$

with

$$\beta(f,g) := \sum_{\substack{n_2\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{|n|\langle n\rangle^s f(n_1,\tau_1) g(n_2,\tau_2)}{\langle n_1\rangle^s \langle n_2\rangle^s \langle\sigma_0\rangle^{\frac{1}{2}-\varepsilon}\langle\sigma_1\rangle^{\frac{1}{2}-\varepsilon}\langle\sigma_2\rangle^{\frac{1}{2}-\varepsilon}} d\tau_2.$$

Justifying (4.2) in the region A_2 is therefore equivalent to justifying (4.2) in the region A_1, and we are done by the analysis in Case 2. This completes Case 3, and the proof of Proposition 2.3. □

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