Time asymptotics for solutions of the Burgers equation with a periodic force

Article *in* Mathematische Zeitschrift · December 1999

DOI: 10.1007/PL00004778

Werner Kirsch
FernUniversität in Hagen

Math. Z. 232, 691–705 (1999)

Mathematische Zeitschrift

Time asymptotics for solutions of the Burgers equation with a periodic force

Werner Kirsch***, Almut Kutzelnigg***

Fakultät für Mathematik, Ruhr-Universität Bochum, D-44780 Bochum, Germany
(e-mail: werner@mathphys.ruhr-uni-bochum.de)

Received June 10, 1998

Abstract. We consider the Burgers equation with a periodic force $\frac{\partial}{\partial t}u + u \cdot \nabla u = \frac{1}{2}\Delta u + \nabla V(x)$ which presents a simplified model for turbulence. We are interested in the asymptotic behaviour of solutions for $t \to \infty$. This problem has been studied by Sinai who uses a probabilistic and very technical approach. Using methods from spectral theory we get similar results. This functional analytic approach gives an easier proof. For certain initial data (periodic or some random perturbations of those) we show time-convergence towards a deterministic periodic limit solution related to the ground state of a certain Schrödinger operator.

1 Introduction

In the present paper we deal with the Burgers equation with force ∇V

$$\frac{\partial}{\partial t}u + u \cdot \frac{\partial}{\partial x}u = \frac{1}{2}\Delta u + \nabla V(x), \qquad t \geq 0, \qquad x \in \mathbb{R}^d$$

with initial data $u(0,x) = u_0(x)$. The Burgers equation $\frac{\partial}{\partial t}u + u \cdot \frac{\partial}{\partial x}u = \frac{1}{2}\Delta u$ serves as a simplified model for the description of fluid motion. The solution u is interpreted as the velocity of the fluid. The Burgers equation shares its structure of diffusion term and nonlinearity with other differential equations of hydrodynamics, e.g. with the Navier Stokes equations. For a general review we refer to [1], [2], [6], [10], [17].

Here we consider an additional force ∇V which is the gradient of some "nice" potential V. By the so-called Cole-Hopf transformation this leads

* Sonderforschungsbereich 237

to the heat equation with a potential V. In this paper we are interested in periodic potentials. We might expect that the influence of the periodic force leads to asymptotically periodic solutions. In fact, under certain conditions we can prove convergence towards a periodic limit solution for a special class of (random) initial data.

This subject has been treated by Sinai [16]. In his paper he considers random initial data. He demonstrates almost-sure convergence of solutions towards a deterministic periodic limit solution. The assumptions on the initial data as well as the probabilistic approach are rather technical. This was the reason for us to look for an alternative approach.

We follow functional analytic methods. We write the heat equation with potential as

$$\frac{\partial}{\partial t}\varphi = -H\varphi \qquad \text{with} \qquad H = -\frac{1}{2}\Delta + V.$$

If the periodic potential is locally $\mathbf{L}^p$ (p depending on the dimension d) then H is a selfadjoint operator bounded from below. The time evolution for solutions is given by

$$\varphi(t,x) = e^{-tH}\varphi_0(x),$$

the action of the Schrödinger semigroup e^{-tH} on initial data φ_0. We may assume $\inf \sigma(H) = 0$.

We consider two kinds of initial data. For periodic initial $\mathbf{L}^2_{\mathrm{loc}}$-data we show convergence of φ towards a periodic limit solution, more precisely the generalized ground state of H. For this we use mainly methods from spectral theory. This is done in Sect. 5. In Sect. 6 we investigate random $\mathbf{L}^\infty$-deviations of initial data like those in Sect. 5. Using probabilistic arguments we show that those deviations do not contribute too much in the limit. We need some asymptotic results about the integral kernel $e^{-tH}(x,y)$. This will be done in Sect. 4. The main tool for investigations is the so-called direct integral decomposition for Schrödinger operators with periodic potentials.

This work is based on [11] where more details can be found.

2 Solving the Burgers equation with force

The so-called Cole-Hopf transformation which has been used by Cole [3] and Hopf [7] takes the Burgers equation over into the heat equation. The initial data can be easily obtained from each other. The same transformation works also for the Burgers equation with force

$$\frac{\partial}{\partial t}u + u\cdot\nabla u = \frac{1}{2}\Delta u + \nabla V(x), \qquad t \geq 0,\ x \in \mathbb{R}^d$$

where the force ∇V is the gradient of a "nice" potential V. By the Cole-Hopf transformation

$$u = -\nabla \ln \varphi = -\frac{\nabla \varphi}{\varphi} \qquad (\varphi > 0)$$

the equation is taken over into the heat equation with potential

$$\frac{\partial}{\partial t}\varphi = \frac{1}{2}\Delta\varphi - V(x)\varphi, \qquad t \geq 0,\ x \in \mathbb{R}^d.$$

Later on we want to use methods from functional analysis. For this reason we rewrite the last equation as

$$\frac{\partial}{\partial t}\varphi = -H\varphi$$

with the operator

$$H = -\frac{1}{2}\Delta + V.$$

The potential V should be at least so nice that the Schrödinger operator H is selfadjoint. For a periodic potential this is fulfilled by the Rellich-Kato theorem whenever V is locally $\mathbf{L}^p$ ($p = 2$ if $d \leq 3$, $p > 2$ if $d = 4$ and $p > \frac{d}{2}$ if $d \geq 5$) [12], [13]. Later on we will be more restrictive.

For such "nice" potentials the operator H is lower semi-bounded, and it makes sense to consider the Schrödinger semigroup e^{-tH} acting on initial data with suitable properties (e.g. boundedness) to obtain the solutions at time t as

$$\varphi(t,x) = e^{-tH}\varphi_0(x).$$

Without loss of generality we assume $\inf \sigma(H) = 0$. Adding a constant to the potential V changes solutions φ by an only time dependent factor which has no effect on solutions u for the Burgers equation.

The action of the operators e^{-tH} is given by

$$\varphi(t,x) = \int e^{-tH}(x,y)\,\varphi_0(y)\,\mathrm{d}y$$

where the (jointly continuous) integral kernel can be written explicitly by the Feynman Kac formula, see e.g. Kirsch [8], [14], [15]:

$$e^{-tH}(x,y) = p_t(x,y) \cdot \mathbb{E}_{0,y}^{t,x} \exp\left[-\int_0^t V(b_s)\,\mathrm{d}s\right].$$

Here $p_t(x,y) = \frac{1}{(2\pi t)^{d/2}} \exp[-\frac{\|x-y\|^2}{2t}]$ is the normal density, and the expectation $\mathbb{E}_{0,y}^{t,x}$ is taken with respect to the probability measure associated with the Brownian bridge starting at time 0 in y and ending at time t in x.

Except for some finiteness arguments based on the explicit representation we will *not* use the Feynman Kac formula. Instead we will make use of the periodicity of V which allows special techniques [5], [13]. The main tool will be the method of the so-called direct integral decomposition for Schrödinger operators with periodic potentials [13].

3 Statement of the Results

Now we will become more explicit and state our main result. Doing this, we will specialize the assumptions about the potential V and the initial data φ_0.

Assumptions 3.1 The (random) initial data u_0^ω resp. φ_0^ω are supposed to fulfil the following conditions:

(i) $\varphi_0^\omega > 0$.
(ii) $\mathbb{E}\varphi_0$ is $[0,1]^d$-periodic.
(iii) $\mathbb{E}\varphi_0|_{[0,1]^d} \in \mathbf{L}^2([0,1]^d)$.
(iv) $\varphi_0^\omega - \mathbb{E}\varphi_0 \in \mathbf{L}^\infty(\mathbb{R}^d)$ and $\sup_\omega \|\varphi_0^\omega - \mathbb{E}\varphi_0\|_\infty < \infty$.
(v) There is a decomposition $\varphi_0 = \sum_{j=1}^J \varphi_{0,j}$ such that

$$\left(\int_{l+[0,1]^d} \varphi_{0,j}(y) g(y)\,\mathrm{d}y\right)_{l\in\mathbb{Z}^d}$$

is an independent sequence of random variables for any j and functions g.
(vi) For $1 \le j \le J$ and $k \le 6$ we have $\sup_y \mathbb{E}|\varphi_{0,j}(y) - \mathbb{E}\varphi_{0,j}(y)|^k \le \mathrm{const}$.

Assumptions 3.2 The potential V is supposed to fulfil the following conditions:

a) (i) V is $[0,1]^d$-periodic.
(ii) $V|_{[0,1]^d} \in \mathbf{L}^p([0,1]^d)$ ($p = 2$ if $d \le 3$, $p > 2$ if $d = 4$ and $p > \frac{d}{2}$ if $d \ge 5$).
(iii) $\inf \sigma(H) = 0$ for $H := -\frac{1}{2}\Delta + V$.
(iv) Some further regularity depending on d which is certainly fulfilled if V is $\mathcal{C}^k$ for some $k > d$.

b) (v) $V \in \mathcal{C}^1(\mathbb{R}^d)$ with bounded partial second derivatives.

Especially this covers the following cases:

1. Deterministic initial data R, more precisely a positive $[0,1]^d$-periodic function $R \in \mathbf{L}^2_{\mathrm{loc}}(\mathbb{R}^d)$.

2. Stationary initial data

$$\varphi_0^\omega(x) := \sum_{l\in\mathbb{Z}^d} X_l(\omega)\varrho(x-l)$$

where the X_l are positive and bounded i.i.d. random variables and ϱ is positive and bounded on some finite set, e.g. on $[0,1[^d$.

3. Finite sums of 1. and 2.

Now we state the main result:

Theorem 3.3 *Let $u^\omega(t,x)$ be the solution of the Burgers equation with a periodic force with V as in the assumptions 3.2 for random initial data which fulfil the assumptions 3.1. Then we have for any compact set $K \subset \mathbb{R}^d$:*

$$\lim_{t\to\infty} \sup_{x\in K} \left| u^\omega(t,x) - \left(-\frac{\nabla\Psi_0(x)}{\Psi_0(x)} \right) \right| = 0 \qquad \text{almost-surely.}$$

Here Ψ_0 is the periodic continuation of the ground state ψ_0 of the operator $-\frac{1}{2}\Delta + V$ on $\mathbf{L}^2([0,1]^d)$ with periodic boundary conditions. Thus, the periodic limit solution does not depend on the initial data, but only on the potential V, more precisely only on the force ∇V.
In addition we have a better result for the case of deterministic (periodic) initial data. The regularity hypothesis for V can be weakened to the assumptions 3.2 a), and the convergence is globally uniform in x.

We will split this theorem into several propositions from which it can be deduced easily. The first proposition handles the case of deterministic initial data.

Proposition 3.4 *Let V be as in the assumptions 3.2 a), and let R be a $[0,1]^d$-periodic function with*

$$\varrho := R|_{[0,1]^d} \in \mathbf{L}^2([0,1]^d).$$

Let ψ_0 be the (strictly positive) normalized ground state of the operator $\tilde{H} := -\frac{1}{2}\Delta + V$ on $\mathbf{L}^2([0,1]^d)$ with periodic boundary conditions, and let Ψ_0 be its periodic continuation. Then for $t\to\infty$ we have

$$e^{-tH}R \quad \longrightarrow \quad c\cdot\Psi_0$$

in the sense of uniform convergence where
$c = \langle\psi_0,\varrho\rangle = \int_{[0,1[^d} \Psi_0(z)R(z)\,\mathrm{d}z\,.$

Under the same conditions and with the same notation we also have

$$\nabla\left(e^{-tH}R\right) \quad \longrightarrow \quad c\cdot\nabla\Psi_0$$

in the sense of uniform convergence.

The second proposition deals with random perturbations of periodic initial data. For this we do not need the periodicity of $\mathbb{E}\varphi_0$.

Proposition 3.5 *Let φ_0 be as in the assumptions 3.1 and let V be as in the assumptions 3.2. Define*

$$Z_t^x(\omega) := e^{-tH}(\varphi_0^\omega - \mathbb{E}\varphi_0)(x) = \int e^{-tH}(x,y) \cdot (\varphi_0^\omega(y) - \mathbb{E}\varphi_0(y))\,\mathrm{d}y\,.$$

Then we have for any compact set $K \subset \mathbb{R}^d$:

a) $\sup_{x\in K} |Z_t^x(\omega)| \overset{t\to\infty}{\longrightarrow} 0$ *almost-surely.*
b) $\sup_{x\in K} |\frac{\partial}{\partial x_i} Z_t^x(\omega)| \overset{t\to\infty}{\longrightarrow} 0$ *almost-surely for* $1 \le i \le d$.

Before we come to the proofs we have to prepare them by giving some results from spectral theory and investigating the integral kernel $e^{-tH}(x,y)$.

4 Asymptotic behaviour of the integral kernel $e^{-tH}(x,y)$

In this section we will prepare some results for the integral kernel $e^{-tH}(x,y)$.

Lemma 4.1 *Let V be as in the assumptions 3.2 a). Then we have*

$$\sup_{x,y} e^{-tH}(x,y) \le \text{const} \cdot \frac{1}{t^{d/2}}\,.$$

Remark 4.2 For $V = 0$ this is clear; for general V a priori we have only $e^{-tH}(x,y) \le p_t(x,y) \cdot e^{-t \inf V}$. Note that $\inf V < 0$ due to $\inf \sigma(H) = 0$.

Proof. Using the method of direct integral decomposition [13, Ch. XIII.16] we obtain for the integral kernel

$$e^{-tH}(x,y) = \int_{[0,1]^d} \mathrm{d}\theta\, e^{\mathrm{i}2\pi\theta\cdot([x]-[y])} e^{-tH^\theta}(\{x\},\{y\})$$

where $[x]$ resp. $\{x\}$ denotes the vector of the integer parts resp. fractional parts of x. The operators H^θ act on

$$\mathcal{D}(H^\theta) := \Bigg\{ f \in \mathbf{W}^{2,2}([0,1]^d) : f(x)\big|_{x_i=1} = e^{\mathrm{i}2\pi\theta_i}\, f(x)\big|_{x_i=0}\,, \\ \frac{\partial}{\partial x_i} f(x)\bigg|_{x_i=1} = e^{\mathrm{i}2\pi\theta_i}\, \frac{\partial}{\partial x_i} f(x)\bigg|_{x_i=0}, i = 1,\ldots,d \Bigg\}.$$

The operators H^θ have discrete spectrum. Let E_n^θ denote the n-th eigenvalue and ψ_n^θ the corresponding normalized eigenfunction (unique up to a factor of modulus 1). Then the spectral decomposition leads to the estimate

$$e^{-tH}(x,y) \le \int_{[0,1]^d} \mathrm{d}\theta \sum_{n=0}^{\infty} e^{-tE_n^\theta} \|\psi_n^\theta\|_\infty^2.$$

Using Sobolev inequalities and having in mind the boundary conditions we can estimate $\|\psi_n^\theta\|_\infty$ in terms of the energy E_n^θ. Let's first consider the case $d = 1$. We get

$$\|\psi_n^\theta\|_\infty \le d_1 \|(\psi_n^\theta)''\|_2 + d_2.$$

for appropriate constants d_1 and d_2. Using the eigenfunction property and keeping in mind that V is infinitesimally bounded w.r.t. $H_0 = -\frac{1}{2}\Delta$ we obtain

$$\|\psi_n^\theta\|_\infty^2 \le \tilde{q}(E_n^\theta)$$

for some polynomial $\tilde{q}$. For higher dimensions this is a bit more complicated. We need higher order derivatives to estimate the $\mathbf{L}^\infty$-norm of ψ_n^θ in terms of $\mathbf{L}^2$-norms. We find $\|\psi_n^\theta\|_\infty \le \tilde{d_1} \cdot \|\Delta^k \psi_n^\theta\|_2 + \tilde{d_2}$ for $k > \frac{d}{2}$. Here we have to use the eigenfunction property several times. Expressing and estimating the term by the energy E_n^θ there appear also derivatives of the potential V.

In any dimension we finally arrive at

$$e^{-tH}(x,y) \le \int_{[0,1]^d} \mathrm{d}\theta \sum_{n=0}^{\infty} e^{-tE_n^\theta} \tilde{q}(E_n^\theta)$$

for some polynomial $\tilde{q}$. First we look at the part coming from $n \ge 1$. All E_n^θ are bounded below by $E_1^{1/2} > 0$, and for n large enough we have

$$c_1 \cdot n^2 \le E_n^\theta \le c_2 \cdot n^2$$

for some constants c_1 and c_2 (and similar for $d > 1$). The last statement follows from the min-max principle because the spectrum of the free operator $-\frac{1}{2}\Delta^\theta$ is known as $\frac{1}{2}(\theta + \mathbb{Z})^2$.

Putting the estimates together we obtain exponential convergence for the sum $\sum_{n=1}^\infty e^{-tE_n^\theta} \tilde{q}(E_n^\theta)$ to 0 for $t \to \infty$.

For the ground state energies E_0^θ it is known [9] that they can be bounded from above and from below by some constant times $\|\theta\|^2$ if we shift the cube. Thus,

$$\int_{[0,1]^d} \mathrm{d}\theta\, e^{-tE_0^\theta} \le \int_{[-\frac{1}{2},\frac{1}{2}]^d} \mathrm{d}\theta\, e^{-tc\theta\cdot\theta} < \text{const} \cdot \frac{1}{t^{d/2}}$$

which is the behaviour we claimed in the lemma. □

We will need the analogue estimate for the derivatives:

Lemma 4.3 *Like in Lemma 4.1 we have for* $1 \le i \le d$

$$\sup_{x,y} \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right| \le \text{const} \frac{1}{t^{d/2}}.$$

Proof. The proof follows the line of the proof for Lemma 4.1, but at some points we have to look more closely. For $x_i \notin \mathbb{Z}$ we get

$$\frac{\partial}{\partial x_i} e^{-tH}(x,y) = \int_{[0,1]^d} \mathrm{d}\theta \, e^{\mathrm{i} 2\pi\theta([x]-[y])} \sum_{n=0}^{\infty} e^{-tE_n^\theta} \frac{\partial}{\partial x_i} (\psi_n^\theta)(\{x\}) \overline{\psi_n^\theta(\{y\})}$$

Again, we can prove a polynomial estimate for $\| \frac{\partial}{\partial x_i} (\psi_n^\theta) \|_\infty$ in terms of the energies E_n^θ. For $x_i \in \mathbb{Z}$ we use a shifted formula for the direct integral decomposition.

5 Proof of Proposition 3.4

Now we examine the operators e^{-tH} acting on periodic functions, i.e. the integral

$$e^{-tH} R(x) = \int e^{-tH}(x,y) R(y) \, \mathrm{d}y$$

where R is a $[0,1]^d$-periodic function.

We will split this into two main steps. In the first one we will show that H acts on a periodic function in the same way as the corresponding operator $\tilde{H}$ on $\mathbf{W}^{2,2}([0,1]^d)$ with periodic boundary conditions. The operator $e^{-t\tilde{H}}$ has discrete spectrum and acts asymptotically as projection onto the ground state.

Step 1:

Let $\mathcal{D}(U)$ be the space of $[0,1]^d$-periodic $\mathbf{L}^2_{\mathrm{loc}}$-functions with the scalar product

$$\langle f, g \rangle_{\mathcal{D}(U)} := \langle f|_{[0,1]^d}, g|_{[0,1]^d} \rangle = \int_{[0,1]^d} \overline{f(x)} \, g(x) \, \mathrm{d}x \, .$$

Let $\mathcal{D}_{[0,1]} := \mathbf{L}^2([0,1])$ with the usual scalar product. Then

$$U : \mathcal{D}(U) \ni G \mapsto G|_{[0,1]^d} \in \mathcal{D}_{[0,1]^d}$$

represents a unitary operator whose inverse acts as periodic continuation of $\mathbf{L}^2$-functions on $[0,1]^d$. For $H := -\frac{1}{2}\Delta + V$ on

$$\mathcal{D}(H) := \Bigg\{ f \in \mathcal{D}(U) : f|_{[0,1]^d} \in \mathbf{W}^{2,2}([0,1]^d), \; f(x)|_{x_i=1} = f(x)|_{x_i=0} \, , \\ \frac{\partial}{\partial x_i} f(x) \bigg|_{x_i=1} = \frac{\partial}{\partial x_i} f(x) \bigg|_{x_i=0} , i = 1, \ldots, d \Bigg\}$$

and $\tilde{H} := -\frac{1}{2}\Delta + V$ on

$$\mathcal{D}(\tilde{H}) := \Big\{ f \in \mathbf{W}^{2,2}([0,1]^d),\ f(x)|_{x_i=1} = f(x)|_{x_i=0}\,, \\ \frac{\partial}{\partial x_i} f(x)\Big|_{x_i=1} = \frac{\partial}{\partial x_i} f(x)\Big|_{x_i=0}, i = 1, \ldots, d \Big\}$$

having periodicity in mind we can show by direct calculation that for $f \in \mathcal{C}^\infty(\mathbb{R}^d) \cap \mathcal{D}(H)$ we have

$$UHf = \tilde{H}Uf.$$

Using a density argument and the spectral theorem we conclude

$$Ue^{-tH}f = e^{-t\tilde{H}}Uf \qquad \text{for } f \in \mathcal{D}(U).$$

Step 2:

Due to step 1 we only have to show

$$e^{-t\tilde{H}}\varrho \quad \longrightarrow \quad \psi_0 \cdot \langle \psi_0, \varrho \rangle$$

where ψ_0 is the ground state of $\tilde{H}$ which can be chosen strictly positive. Let $(E_n)_{n\in\mathbb{N}_0}$ be the eigenvalues of $\tilde{H}$ in increasing order and let ψ_n be the corresponding normalized eigenvectors. Then,

$$e^{-t\tilde{H}}\varrho = \psi_0\,\langle \psi_0, \varrho \rangle + \sum_{n=1}^{\infty} e^{-tE_n} \cdot \psi_n\,\langle \psi_n, \varrho \rangle.$$

The sum

$$\sum_{n=1}^{\infty} e^{-tE_n} \cdot \|\psi_n\|_\infty \cdot \|\psi_n\|_2 \cdot \|\varrho\|_2$$

converges to 0 exponentially in t as in the proof of Lemma 4.1.

Remark 5.1 The operator $e^{-t\tilde{H}}$ is positivity improving for all $t > 0$. Thus [13] ψ_0 can be chosen strictly positive, and E_0 is non-degenerate.

Remark 5.2 We state a consequence of the part of proposition 3.4 proven so far which gives a partial answer for our type of potentials to a question posed by Simon [15]. In general he knows

$$\|e^{-tH}\|_{\infty,\infty} \leq C\,(1+t)^{d/2}\,\|e^{-tH}\|_{2,2}$$

and asks for situations when the t-dependent factor can be dropped. This is indeed the case in our situation of periodic V: According to $\inf \sigma(H) = 0$ we have $\|e^{-tH}\mathbb{1}\|_{2,2} \equiv 1$ for any $t \geq 0$. From proposition 3.4 we also conclude

$$\sup_t \|e^{-tH}\|_{\infty,\infty} = \sup_t \|e^{-tH}\mathbb{1}\|_\infty < \infty.$$

We are not only interested in solutions φ but also in their x-derivatives. We can show

$$\nabla \left(e^{-tH} R\right) \quad \longrightarrow \quad c \cdot \nabla \Psi_0$$

by the same methods as before.

6 Proof of proposition 3.5

Proving proposition 3.5 we will proceed as follows:

First we will show:

Lemma 6.1 *For fixed $x \in \mathbb{R}^d$ and for any $\delta > 0$ we have*

a) For $\mathbb{N} \ni n \to \infty$ $Z^x_{n\cdot\delta}$ converges almost-surely to 0.
b) For $\mathbb{N} \ni n \to \infty$ $\frac{\partial}{\partial x_i} Z^x_{n\cdot\delta}$ converges almost-surely to 0 for $1 \le i \le d$.

Then by a uniform continuity argument we will conclude the statement of the proposition.

For the proof of the lemma we will use methods from probability theory.

Proof of the Lemma 6.1 a). The convergence statement can be reformulated into

$$\mathbb{P}(\limsup_{n\to\infty}\{|Z^x_{n\cdot\delta}| > \eta\}) = 0 \qquad \forall \eta > 0.$$

The last statement is a consequence of the “easy direction” of the Borel-Cantelli lemma if we can show

$$\sum_{n=1}^{\infty} \mathbb{P}\{|Z^x_{n\cdot\delta}| > \eta\} < \infty \qquad \forall \eta > 0.$$

By Chebyshev’s inequality it suffices to estimate appropriate moments of the Z^x_t; $\mathbb{E}|Z^x_t|^6 \le \mathrm{const} \cdot t^{-3/2}$ would be enough.

Now we write

$$Z^x_t = \sum_{l \in \mathbb{Z}^d} Y_l$$

with

$$Y_l := \int_{l+[0,1]^d} e^{-tH}(x,y) \left(\varphi_0(y) - \mathbb{E}\varphi_0(y)\right) \,\mathrm{d}y\,.$$

Without loss of generality we assume the Y_l to be independent. Otherwise we decompose φ_0 into a finite sum according to assumption (v).

It remains to show

$$\mathbb{E}|\sum_{l\in\mathbb{Z}^d} Y_l|^6 \le \mathrm{const} \cdot t^{-3/2}.$$

In general the 6th moment of a sum of independent random variables with mean value 0 can be calculated as

$$\mathbb{E}\left(\sum_l Y_l\right)^6 = \sum_l \mathbb{E}Y_l^6 + a_1 \sum_l \mathbb{E}Y_l^4 \sum_l \mathbb{E}Y_l^2 + a_2 \left(\sum_l \mathbb{E}Y_l^3\right)^2 + a_3 \left(\sum_l \mathbb{E}Y_l^2\right)^3$$

with some combinatorical factors a_1, a_2, a_3.

By Jensen's inequality and Fubini's theorem we have

$$\begin{aligned}\left|\sum_{l\in\mathbb{Z}^d} \mathbb{E}Y_l^k\right| &\leq \sum_{l\in\mathbb{Z}^d} \int_{l+[0,1]^d} \left(e^{-tH}(x,y)\right)^k \cdot \mathbb{E}|\varphi_0(y) - \mathbb{E}\varphi_0(y)|^k \,\mathrm{d}y \\ &\leq \int \left(e^{-tH}(x,y)\right)^k \,\mathrm{d}y \cdot \mathrm{const}\end{aligned}$$

by assumption (vi) about uniform boundedness of the higher moments.

If we can show

$$\int \left(e^{-tH}(x,y)\right)^k \,\mathrm{d}y \leq \mathrm{const} \cdot t^{-d\cdot\frac{k-1}{2}}.$$

for $k = 2, 3, 4, 6$ and large t then we are done because we obtain

$$\begin{aligned}\mathbb{E}|Z_t^x|^6 \leq \mathrm{const} \cdot \Big(& t^{-d\cdot 5/2} + t^{-d\cdot 3/2} \cdot t^{-d\cdot 1/2} \\ &+ (t^{-d\cdot 2/2})^2 + (t^{-d\cdot 1/2})^3\Big) \leq \mathrm{const} \cdot t^{-d\cdot 3/2}\end{aligned}$$

for large t. Now it becomes clear why we considered the sixth moment of Z_t^x. The fourth moment would have included a term $(t^{-d\cdot 1/2})^2$ which would have led to the non-summable harmonic series for dimension $d = 1$.

The integral can be estimated as

$$\begin{aligned}\int \left(e^{-tH}(x,y)\right)^k \,\mathrm{d}y &\leq \left(\sup_{x,y} e^{-tH}(x,y)\right)^{k-1} \cdot \int e^{-tH}(x,y)\,\mathrm{d}y \\ &\leq \mathrm{const} \cdot t^{-d\cdot\frac{k-1}{2}} \cdot \|e^{-tH}\mathbb{1}\|_\infty\end{aligned}$$

by the estimate for the integral kernel from Lemma 4.1. From remark 5.2 we know that $\|e^{-tH}\mathbb{1}\|_\infty$ remains bounded in t, and we achieve the stated result.

Proof of the Lemma 6.1 b). The proof for the statement about the derivative can almost be copied. Interchanging derivative and integral can be defended by the Feynman-Kac formula. The role of the integral kernel $e^{-tH}(x,y)$ is now played by the absolute value of the derivative $|\frac{\partial}{\partial x_i} e^{-tH}(x,y)|$. Again, we estimate

$$\int \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right|^k \, \mathrm{d}y \\ \le \sup_{x,y} \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right|^{k-1} \cdot \int \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right| \, \mathrm{d}y \, .$$

The first factor is less than $\mathrm{const} \cdot t^{-d \cdot \frac{k-1}{2}}$ for large t as before. For the boundedness of the second factor we have to look for a new argument. By the semigroup property we know for $t > 1$

$$\int \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right| \, \mathrm{d}y \le \int \left| \frac{\partial}{\partial x_i} e^{-H}(x,z) \right| \int e^{-(t-1)H}(z,y) \, \mathrm{d}y \, \mathrm{d}z \\ \le \sup_x \int \left| \frac{\partial}{\partial x_i} e^{-H}(x,z) \right| \, \mathrm{d}z \cdot \| e^{-(t-1)H} \mathbb{1} \|_\infty .$$

The second factor is bounded in t as we know from remark 5.2. But the z-integral does not depend on t and is finite. With this we finish the proof of Lemma 6.1. □

As next step we will prove the following lemma:

Lemma 6.2 *Let* $(\varphi^\omega)_{\omega \in \Omega} \subset \mathbf{L}^\infty(\mathbb{R}^d)$ *with* $\sup_\omega \|\varphi^\omega\|_\infty < \infty$ *and define* $Z_t^x(\omega) := e^{-tH} \varphi^\omega(x)$. *Then the following holds for large* t *uniformly in* ω *and in* x *resp.* t*:*

a) $Z_t^x(\omega)$ *and* $\frac{\partial}{\partial x_i} Z_t^x(\omega)$ *are uniformly continuous in* t.
b) $Z_t^x(\omega)$ *and* $\frac{\partial}{\partial x_i} Z_t^x(\omega)$ *are uniformly continuous in* x.

Proof. First we consider the uniform continuity of $Z_t^x(\omega)$ in t: We want to show

$$\sup_{t,x,\omega} |Z_{t+\tau}^x(\omega) - Z_t^x(\omega)| \quad \longrightarrow \quad 0 \qquad \text{for } 0 < \tau \to 0.$$

For the moment, let $(\varphi^\omega)_{\omega \in \Omega}$ be uniformly equicontinuous, i.e. we assume

$$\sup_{\omega,x} |\varphi^\omega(x+h) - \varphi^\omega(x)| \quad \longrightarrow \quad 0 \qquad \text{for } h \to 0$$

to be valid. Then

$$\sup_{t,x,\omega} |Z_{t+\tau}^x(\omega) - Z_t^x(\omega)| \le \sup_t \|e^{-tH}\|_{\infty,\infty} \cdot \sup_\omega \|(e^{-\tau H} - \mathbb{1})\varphi^\omega\|_\infty .$$

For the first factor we already know boundedness by remark 5.2. Investigating the second factor we compare to the case of the free operator.

Looking closely at the formula for the integral kernels we can estimate $\|e^{-\tau H} - e^{-\tau H_0}\|_{\infty,\infty}$ like $\sinh \tau \|V\|_\infty$ which tends to 0 for $\tau \to 0$. By straightforward arguments the part coming from the free operator can be discussed as tending to 0 because $e^{-\tau H_0}\varphi^\omega(x)$ is equicontinuous in $\tau = 0$ due to the uniform equicontinuity of (φ^ω). In the case of the smooth φ^ω we are done.

In general we write

$$Z^x_{t+\tau}(\omega) - Z^x_t(\omega) = e^{-(t-1)H}(e^{-\tau H} - \mathbb{1})e^{-H}\varphi^\omega(x).$$

The result follows because $(e^{-H}\varphi^\omega)_{\omega\in\Omega}$ is uniformly equicontinuous due to the uniform boundedness of the partial x-derivatives.

The proof for the uniform continuity of $\frac{\partial}{\partial x_i} Z^x_t(\omega)$ in t can again almost be copied. Instead of $\sup_t \|e^{-tH}\|_{\infty,\infty} = \sup_{t,x} \int e^{-tH}(x,y)\,\mathrm{d}y < \infty$ we use now

$$\sup_{t,x} \int \left| \frac{\partial}{\partial x_i} e^{-tH}(x,y) \right| \mathrm{d}y < \infty$$

which we have seen at the end of the proof of Lemma 6.1 b).

For the uniform continuity of $Z^x_t(\omega)$ and $\frac{\partial}{\partial x_i} Z^x_t(\omega)$ in x it suffices to show uniform boundedness for the first and second partial x-derivatives of $Z^x_t(\omega)$. But for $t \geq 1$,

$$\begin{aligned} &\sup_{t,x,\omega} \left| \frac{\partial}{\partial x_j} Z^x_t(\omega) \right| \\ &\leq \sup_x \int \left| \frac{\partial}{\partial x_j} e^{-H}(x,y) \right| \mathrm{d}y \cdot \sup_t \|e^{-(t-1)H}\mathbb{1}\|_\infty \cdot \sup_\omega \|\varphi^\omega\|_\infty < \infty \end{aligned}$$

and analoguously

$$\sup_{t,x,\omega} \left| \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} Z^x_t(\omega) \right| < \infty$$

because we also have $\sup_x \int \left| \frac{\partial^2}{\partial x_{j_1} \partial x_{j_2}} e^{-H}(x,y) \right| \mathrm{d}y < \infty$ as consequence of the boundedness of the second derivatives that we have assumed. So we have proven Lemma 6.2. □

We finish the proof of proposition 3.5 by an $\frac{\varepsilon}{3}$-argument using the continuity statements.

7 Proof of the theorem

From the propositions we obtain

$$\sup_{x\in K} |\varphi(t,x) - c \cdot \Psi_0(x)| \quad \longrightarrow \quad 0 \qquad \text{almost-surely}$$

and

$$\sup_{x\in K} |\nabla\varphi(t,x) - c\cdot\nabla\Psi_0(x)| \quad \longrightarrow \quad 0 \qquad \text{almost-surely.}$$

In fact we are not interested in solutions $\varphi(t,x)$ of the heat equation with potential but in solutions $u = -\frac{\nabla\varphi}{\varphi}$ of the Burgers equation with force. But we have

$$u - (-\frac{\nabla\Psi_0}{\Psi_0}) = -\frac{\nabla\varphi - c\nabla\Psi_0}{\varphi} + \frac{c\nabla\Psi_0(\varphi - c\Psi_0)}{\varphi c\Psi_0}.$$

Thanks to the properties of Ψ_0 ($\inf_x \Psi_0 > 0$, $\sup_x |\nabla\Psi_0(x)| < \infty$) and the locally uniform convergence $\varphi \to c\Psi_0$ which implies $\inf_{x\in K} |\varphi(t,x)| > 0$ for t large enough we obtain the locally uniform convergence

$$u \quad \longrightarrow \quad -\frac{\nabla\Psi_0}{\Psi_0} = -\nabla(\ln\Psi_0).$$

almost-surely which proves Theorem 3.3.

References

1. E. R. Benton, G. W. Platzman: A Table of Solutions of the One-Dimensional Burgers Equation. Quart. Appl. Math. **30**, 195-212 (1972)
2. J. M. Burgers: The Nonlinear Diffusion Equation. Dordrecht: Reidel 1974
3. J. D. Cole: On a Quasi-Linear Parabolic Equation Occuring in Aerodynamics. Quart. Appl. Math. **9**, 225-236 (1951)
4. H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon: Schrödinger Operators. Texts and Monographs in Physics. Berlin: Springer 1987
5. M. S. P. Eastham: The Spectral Theory of Periodic Differential Equations. Edinburgh: Scottish Academic Press 1973
6. H. Holden, B. Øksendal, J. Ubøe, T. Zhang: Stochastic Partial Differential Equations. Boston: Birkhäuser 1996
7. E. Hopf: The Partial Differential Equation $u_t + uu_x = \mu u_{xx}$. Commun. Pure Appl. Math. **3**, 201-230 (1950)
8. W. Kirsch: Random Schrödinger Operators. A Course. In: H. Holden, A. Jensen (eds.): Schrödinger Operators. Lecture Notes in Physics, Vol. 345, 264-370, Berlin: Springer 1989
9. W. Kirsch, B. Simon: Comparision Theorems for the Gap of Schrödinger Operators. J. Funct. Anal. **75**, 396-410 (1987)
10. H.-O. Kreiss, J. Lorenz: Initial-Boundary Value Problems and the Navier-Stokes Equations. New York: Academic Press 1989
11. A. Kutzelnigg: Zur Zeitasymptotik von Lösungen der Burgers-Gleichung mit periodischem Kraftterm. Dissertation Bochum 1997
12. M. Reed, B. Simon: Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness. New York: Academic Press 1975
13. M. Reed, B. Simon: Methods of Modern Mathematical Physics IV: Analysis of Operators. New York: Academic Press 1978

14. B. Simon: Functional Integration and Quantum Physics. New York: Academic Press 1979
15. B. Simon: Schrödinger Semigroups. Bull. Am. Math. Soc. **7**, 447-526 (1982)
16. Ya. G. Sinai: Two Results Concerning Asymptotic Behavior of Solutions of the Burgers Equation with Force. J. Stat. Phys. **64**, 1-12 (1991)
17. G. B. Whitham: Linear and Nonlinear Waves. New York: Wiley 1974