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Stochastic Burgers' equation driven by fractional Brownian motion

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ABSTRACT

In this paper, we consider the stochastic Burgers' equation driven by a genuine cylindrical fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. We first prove the regularities of the solution to the linear stochastic problem corresponding to the stochastic Burgers' equation. Then we obtain the local and global existence and uniqueness results for the stochastic Burgers' equation.

1. Introduction

Since pioneering work of Hurst [17,18] and Mandelbrot [20], stochastic calculus with respect to fractional Brownian motion (fBm) has led to various interesting mathematical applications. In particular, infinite-dimensional linear, bilinear and quasilinear fBm-driven evolution equations in which noise enters linearly have been studied recently in [7–10,12,22,23]. The stochastic heat equation with a multiplicative fractional white noise has been dealt with in [15,16]. A class of hyperbolic stochastic partial differential equation driven by a space time fractional noise has been studied in [11]. A stochastic wave equation with a fractional Gaussian noise has been considered in [3].

In the present paper, we consider the following stochastic initial–boundary Burgers' equation

$$du = (u_{xx} + uu_x)\,dt + dB^H(t), \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \quad x \in [0,1], \tag{1.3}$$

where $B^H = (B_t^H,\ t \geqslant 0)$ is a cylindrical fBm on a real and separable Hilbert space U with Hurst parameter $\frac{1}{4} < H < 1$.

It is well known that Burgers' equation is not a good model for turbulence. It does not display any chaos; even when a force is added to the right-hand side, any solution will converge to a unique stationary solution as time goes to infinity. However the situation is totally different when the force is random. Several authors have indeed suggested to study the stochastic Burgers' equation as a simple model for turbulence [19,14]; for more details, one may also refer to [2] and references therein. Notice that if $H = \frac{1}{2}$, the process B^H is a standard Brownian motion. The existence and regularity properties of the solution of the stochastic Burgers' equation with a standard Brownian motion have been obtained by several authors; see e.g. Da Prato et al. [4] and Gyongy and Nualart [13]. Since fBm is positively correlated for $H \in (1/2, 1)$ and negatively correlated for $H \in (0, 1/2)$, it is interesting to consider the Burgers equation with fBm type.

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Our aim in this paper is to obtain the global well-posedness of the solution for the problem (1.1), with boundary and initial conditions (1.2), (1.3). The key to the proof is the regularities of the solution to the stochastic linear problem corresponding to (1.1)–(1.3). With this result in hand, the local and global existence results of the solution can be proved following the standard way in [4].

We recall the following linear stochastic problem

$$Z\,dt = AZ\,dt + \Phi\,dB^H(t), \qquad Z(0) = 0, \tag{1.4}$$

where Φ is a bounded linear operator. Assume that A generates a strongly continuous semigroup $(S(t),\ t \geqslant 0)$. The solution to the linear problem (1.4), if it exists, is given by the stochastic convolution

$$Z(t) = \int_0^t S(t-s)\Phi\,dB^H(s). \tag{1.5}$$

The regularity of $Z(t)$ with $H \in (1/2, 1)$ has been proved in [10]. Under the conditions that Φ is a Hilbert–Schmidt operator, $(S(t), t \geqslant 0)$ is an analytic semigroup and $H \in (0, 1/2)$, it has been proved that $Z(t)$ has a Hölder continuous version with respect to t in [22]. A necessary and sufficient condition on A and Φ has been established for the existence and uniqueness of $Z(t)$ in [23].

In the present paper, let Φ be an identity operator, B^H a cylindrical fBm on a Hilbert space U and A a Laplacian operator in (1.4). In this case, the problem (1.4) corresponds to the linear problem of (1.1). We will prove the regularities of the process $Z(t)$ under the restriction $H > 1/4$, which is stated in Theorem 2.1.

When Φ is an identity operator, B^H is a genuine cylindrical fBm, i.e. B^H is fBm in time and white noise in space. The regularity of $Z(t)$ with respect to t depends only on the value of Hurst parameter H and the properties of the semigroup $(S(t))$ generated by Laplacian operator A. From (2.10) and (2.1)–(2.3) given below in Section 2, we know that the smaller the value of Hurst parameter H is, the worse the regularity of B^H is. The smooth properties of the semigroup $(S(t))$ can only assure the well-defined of $Z(t)$ as the Hurst parameter $H > \frac{1}{4}$ by carefully calculation in Section 3. This is the reason why we restrict Hurst parameter $H > \frac{1}{4}$ in this paper.

This paper is organized as follows. In Section 2, we give preliminaries: the Wiener integral with fBm in Section 2.1 and the linear stochastic evolution equation with fBm in Section 2.2. In Section 3, we prove the regularity properties of the stochastic convolution as Hurst parameter $H > \frac{1}{4}$. In the last section, we prove local and global existence results to the problem (1.1)–(1.3).

2. Preliminaries

2.1. The Wiener integral with fBm

Fix an interval $[0, T]$, and let $\beta^H(t)$, $t \in [0, T]$, be a fBm of Hurst parameter $H \in (0, 1)$ on the complete probability space $(\Omega, \mathcal{F}^H, \mathbb{P}^H)$ endowed with the natural filtration $(\mathcal{F}_t^H)_{t\in[0,T]}$ and the law $\mathbb{P}^H$ of β^H. This means by definition that $\beta^H(t)$ is a centered Gaussian process with covariance

$$R(t,s) = E\big(\beta^H(s)\beta^H(t)\big) = \frac{1}{2}\big(t^{2H} + s^{2H} - |t-s|^{2H}\big).$$

Notice that $\beta^{\frac{1}{2}}$ is a standard Brownian motion. Moreover β^H has the following Wiener integral representation [1]:

$$\beta^H(t) = \int_0^t K^H(t,s)\,dW(s). \tag{2.1}$$

$W = \{W(t)\colon\ t \in [0, T]\}$ is a Wiener process. $K^H(t,s)$ is a kernel. If $H \in (0, 1/2)$, the kernel $K^H(t,s)$ is given by

$$K^H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left(\frac{1}{2} - H\right)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - \left(\frac{s}{u}\right)^{\frac{1}{2}-H}\right)du, \tag{2.2}$$

where c_H is a constant given by

$$c_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

If $H \in (1/2, 1)$, the kernel $K^H(t,s)$ has a simpler expression

$$K^H(t,s) = c_H\left(H - \frac{1}{2}\right)s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}\,du. \tag{2.3}$$

From (2.2) we obtain

$$\frac{\partial K^H}{\partial t}(t,s) = c_H\left(H - \frac{1}{2}\right)(t-s)^{H-\frac{3}{2}}\left(\frac{s}{t}\right)^{\frac{1}{2}-H}. \tag{2.4}$$

In the sequel we will use the notation K instead of K^H for simplicity.

We will denote by $\mathscr{E}$ the linear space of step functions on $[0,T]$ of the form

$$\varphi(t) = \sum_{i=1}^{n-1} a_i 1_{(t_i,t_{i+1}]}(t), \tag{2.5}$$

where $0 = t_1 < t_2 < \cdots < t_n = T$, $n \in \mathbb{N}$, $a_i \in \mathbb{R}$ and by $\mathcal{H}$ the closure of $\mathscr{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R(t,s).$$

For $\varphi \in \mathscr{E}$ of the form (2.5) we define its Wiener integral with respect to the fractional Brownian motion as

$$\int_0^T \varphi_s\,d\beta^H(s) = \sum_{i=1}^{n} a_i\big(\beta^H(t_{i+1}) - \beta^H(t_i)\big).$$

Obviously, the mapping

$$\varphi = \sum_{i=1}^{n} a_i 1_{(t_i,t_{i+1}]}(t) \to \int_0^T \varphi_s\,d\beta^H(s)$$

is an isometry between $\mathscr{E}$ and the linear space $span\{\beta^H(t),\ t \in [0,T]\}$ viewed as a subspace of $L^2(\Omega)$ and it can be extended to an isometry between $\mathcal{H}$ and the first Wiener chaos of the fractional Brownian motion $\overline{span}^{L^2(\Omega)}\{\beta^H(t),\ t \in [0,T]\}$. The image on an element $\Phi \in \mathcal{H}$ by this isometry is called the Wiener integral of Φ with respect to β^H.

Let $K_\tau^* : \mathscr{E} \to L^2([0,T])$ be the linear map given by

$$\big(K_\tau^*\varphi\big)(s) = \varphi(s)K(\tau,s) + \int_s^\tau \big(\varphi(t) - \varphi(s)\big)\frac{\partial K}{\partial t}(t,s)\,dt. \tag{2.6}$$

We refer to [1] for the proof of the fact that K_τ^* is an isometry which can be extended to $\mathcal{H}$.

When $H > \frac{1}{2}$, the operator K_τ^* has a simpler expression

$$\big(K_\tau^*\varphi\big)(s) = \int_s^\tau \varphi(r)\frac{\partial K}{\partial r}(r,s)\,dr.$$

For any $t \in [0,T]$ we can define K_t^* similarly. Also, for φ in $\mathscr{E}$ and h in $L^2(0,T)$ the following duality holds

$$\int_s^T \big(K_T^*\varphi\big)(t)h(t)\,dt = \int_0^T \varphi(t)(Kh)(dt). \tag{2.7}$$

(2.7) certainly holds when φ belongs to $\mathcal{H}$.

As a consequence, we have the following relation between the Wiener integral with respect to fBm and the Itô integral with respect to the Wiener process

$$\int_0^T \varphi(s)\,d\beta^H(s) = \int_0^T \big(K_\tau^*\varphi\big)(s)\,dW(s), \tag{2.8}$$

which holds for every $\varphi \in \mathcal{H}$ if and only if $K_\tau^*\varphi \in L^2([0,T])$. For any $s,t \in [0,T]$, one can check the relation

$$K_T^*[\varphi 1_{[0,t]}](s) = K_T^*[\varphi](s)1_{[0,t]}(s).$$

Then if one defines the definite stochastic integral $\int_0^t \varphi(s)\,d\beta^H(s)$, as it should be, by $\int_0^t \varphi(s)1_{[0,t]}(s)\,d\beta^H(s)$, we obtain

$$\int_0^t \varphi(s)\,d\beta^H(s) = \int_0^t \big(K_\tau^*\varphi\big)(s)\,dW(s) \tag{2.9}$$

for every $t \in [0,T]$ and $\varphi 1_{[0,t]} \in \mathcal{H}$ if and only if $K_\tau^*\varphi \in L^2(0,T)$.

Note that in the general theory of Skorohod integration with respect to fBm with values in a Hilbert space V, a relation such as (2.9) requires careful justification of the existence of its right-hand side (see [21, Section 5.1]). But we will work only with Wiener integrals over Hilbert spaces; in this case we note that, if $u \in L^2([0,T];V)$ is a deterministic function, then relation (2.9) holds, the Wiener integral on the right-hand side being well defined in $L^2(\Omega;V)$ if K^*u belongs to $L^2([0,T]\times V)$.

2.2. Linear stochastic evolution equations with fBm

Generally, following the standard approach in [5] for $H=\frac{1}{2}$, we consider the case of a genuine cylindrical fBm on a real separable Hilbert U, i.e.

$$B^H(t) = \sum_{n=0}^{\infty} e_n\beta_n^H(t), \tag{2.10}$$

where $\beta_n^H(t)$ are real, independent fBm's, $(e_n)_{n\geqslant 0}$ form an orthonormal basis in U which assumed to be $L^2(0,1)$.

We now give the definition of the stochastic integral with respect to the cylindrical fBm B^H defined in (2.10). Let V be another real separable Hilbert space. Λ is a bounded operator from a Hilbert space U to V. The stochastic integral of Λ with respect to B^H is defined by, see for example [23],

$$\begin{aligned}\int_0^t \Lambda(s)\,dB^H(s) &= \sum_{j\in\mathbb{N}}\int_0^t \Lambda(s)e_j\,d\beta_j^H(s)\\ &= \sum_{j\in\mathbb{N}}\int_0^t \big(K_t^*\Lambda(\cdot)e_j\big)(s)\,d\beta_j(s),\end{aligned} \tag{2.11}$$

where $(\Lambda(t))_{t\in[0,T]}$ is such that

$$\sum_{j\in\mathbb{N}}\int_0^t \big\|\big(K_t^*\Lambda(\cdot)e_j\big)(s)\big\|_V^2\,ds < \infty.$$

We can check from (2.11) that $\Lambda(t)$ commutes with K_T^*.

We define the unbounded self-adjoint operator A on $L^2(0,1)$ by

$$Au = \frac{\partial^2}{\partial x^2}u,$$

for u on the domain

$$D(A) = \big\{u \in H^2(0,1)\colon\ u(0)=u(1)=0\big\}.$$

Denote by e^{tA}, $t>0$ the semigroup on $L^2(0,1)$ generated by A. Denote by $\{e_k\}$ the complete orthonormal system on $L^2(0,1)$ which diagonalizes A and by λ_k the corresponding eigenvalues. We have

$$e_k(x) = \sqrt{\frac{2}{\pi}}e^{ik\pi x}, \quad k=1,2,\ldots$$

and

$$\lambda_k = -\pi^2k^2, \quad k=1,2,\ldots.$$

The linear stochastic evolution equations corresponding to problem (1.1)–(1.3) is given by

$$du(t) = Au(t)\,dt + dB^H(t), \qquad u(0) = u_0. \tag{2.12}$$

The mild form of the linear problem (2.12), if it exists, can be written as

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}\,dB^H(s). \tag{2.13}$$

In the study of linear and nonlinear equations, just as the properties of the stochastic convolution deduced by standard Gaussian process [5], it is of great importance to establish the basic properties of the stochastic convolution

$$Z(t) = \int_0^t e^{(t-s)A}\,dB^H(s). \tag{2.14}$$

We state the basic properties of the stochastic convolution as follows.

Theorem 2.1. *The stochastic convolution $Z : t \mapsto \int_0^t e^{(t-s)A}\,dB^H(s)$ is well defined. It has a modification which is continuous with respect to $x \in [0,1]$, and $t \geqslant 0$ for any Hurst parameter $H > \frac{1}{4}$.*

Remark 1. This result for the fBm B^H corresponds to the one for the standard Brownian motion given in [5, Theorem 5.20].

Remark 2. As Hurst parameter $1 > H \geqslant \alpha > \frac{1}{4}$, we can get the Hölder continuous modification of $Z(t)$ with respect to t. However, the continuous modification of $Z(t)$ is enough for our purpose to prove the well-posedness of the solution to the problem (1.1)–(1.3).

3. The stochastic convolution

In this section we prove the regularities of the stochastic convolution $Z(t)$ which corresponds to the Ornstein–Uhlenbeck case when there is no nonlinearity.

Throughout the paper, the letters C and $C(H)$ denote some generic constants, where $C(H)$ depends on Hurst parameter H.

The following elementary two lemmas will provide the main tools to establish the basic properties of the process $Z(t)$.

Lemma 3.1. *Let*

$$B(a, A) = \int_0^1 e^{-2as}\left(\int_0^s \left(e^{ar} - 1\right) r^{A-1}\,dr\right)^2 ds,$$

where $a \geqslant 0$ and $A \in (-\frac{1}{2}, 0]$; then it holds

$$B(a, A) \leqslant K_A a^{-2A-1}$$

with K_A a positive constant depending only on A.

Lemma 3.2. *For arbitrary $\gamma \in [0,1]$ there exists $c_\gamma > 0$ such that*

$$\left|e^{-x} - e^{-y}\right| \leqslant c_\gamma |x - y|^\gamma, \tag{3.1}$$

for all $x \geqslant 0$, $y \geqslant 0$.

Proof of Theorem 2.1. We first prove that the stochastic convolution is well defined.

The Wiener type integral with respect to fBm and Itô integral can be connected by the linear operator K^*; see (2.8) and (2.11). The stochastic convolution $Z(t)$ is a Wiener type integral; then for t positive, we have

$$Z(t) = \int_0^t e^{(t-s)A}\,dB^H(s)$$

$$= \sum_{k \in \mathbb{N}} \int_0^t e^{(t-s)A} e_k \, d\beta_k^H(s)$$

$$= \sum_{k \in \mathbb{N}} \int_0^t \big(K_t^* e^{(t-\cdot)A} e_k\big)(\sigma) \, d\beta_k(s). \tag{3.2}$$

The transformation (3.2) makes the properties of Itô integral available; and noticing the expression of K^* (2.6), we get

$$\begin{aligned}
\mathbb{E}\|Z(t)\|_{L^2(0,1)}^2 &= \sum_{k \in \mathbb{N}} \int_0^t \big\|\big(K_t^* e^{(t-\cdot)A} e_k\big)(\sigma)\big\|_{L^2(0,1)}^2 \, d\sigma \\
&= \sum_{k \in \mathbb{N}} \int_0^t \left\| e^{(t-\sigma)A} e_k K(t,\sigma) + \int_\sigma^t \big(e^{(t-r)A} - e^{(t-\sigma)A}\big) e_k \frac{\partial K(r,\sigma)}{\partial r} \, dr \right\|_{L^2(0,1)}^2 d\sigma \\
&\leqslant 2 \sum_{k \in \mathbb{N}} \int_0^t \big\| e^{(t-\sigma)A} e_k K(t,\sigma)\big\|_{L^2(0,1)}^2 \, d\sigma \\
&\quad + 2 \sum_{k \in \mathbb{N}} \int_0^t \left\| \int_\sigma^t \big(e^{(t-r)A} - e^{(t-\sigma)A}\big) e_k \frac{\partial K(r,\sigma)}{\partial r} \, dr \right\|_{L^2(0,1)}^2 d\sigma.
\end{aligned} \tag{3.3}$$

Denote

$$I_1 := 2 \sum_{k \in \mathbb{N}} \int_0^t \big\| e^{(t-\sigma)A} e_k K(t,\sigma)\big\|_{L^2(0,1)}^2 \, d\sigma;$$

$$I_2 := 2 \sum_{k \in \mathbb{N}} \int_0^t \left\| \int_\sigma^t \big(e^{(t-r)A} - e^{(t-\sigma)A}\big) e_k \frac{\partial K(r,\sigma)}{\partial r} \, dr \right\|_{L^2(0,1)}^2 d\sigma.$$

With the help of the following inequality (see [6]),

$$K(t,s) \leqslant C(H)(t-s)^{H-\frac{1}{2}} s^{H-\frac{1}{2}}, \tag{3.4}$$

we obtain

$$\begin{aligned}
I_1 &\leqslant 2C(H) \sum_{k \in \mathbb{N}} \int_0^t e^{-2(t-\sigma)\pi^2 k^2} (t-\sigma)^{2H-1} \sigma^{2H-1} \, d\sigma \\
&= 2C(H) \sum_{k \in \mathbb{N}} \big(\pi^2 k^2\big)^{-2H} \int_0^{2\pi^2 k^2 t} e^{-v} v^{2H-1} \left(t - \frac{v}{2\pi^2 k^2}\right)^{2H-1} dv;
\end{aligned} \tag{3.5}$$

then

$$\begin{aligned}
I_1 &\leqslant 2C(H) \sum_{k \in \mathbb{N}} \big(\pi^2 k^2\big)^{-2H} \left\{ \left(\frac{t}{2}\right)^{2H-1} \int_0^\infty e^{-v} v^{2H-1} \, dv + \big(\pi^2 k^2 t\big)^{2H-1} \int_{\pi^2 k^2 t}^{2\pi^2 k^2 t} e^{-v} \left(t - \frac{v}{2\pi^2 k^2 t}\right)^{2H-1} dv \right\} \\
&\leqslant 2C(H) \sum_{k \in \mathbb{N}} \big(\pi^2 k^2\big)^{-2H} \left\{ C(t,H) + \big(\pi^2 k^2 t\big)^{2H-1} \int_0^{\pi^2 k^2 t} e^{-(2\pi^2 k^2 t - v')} \left(\frac{v'}{2\pi^2 k^2 t}\right)^{2H-1} dv' \right\} \\
&\leqslant 2C(H) \sum_{k \in \mathbb{N}} \big(\pi^2 k^2\big)^{-2H} \big\{ C(t,H) + C(t,H) e^{-\pi^2 k^2 t} \big(\pi^2 k^2 t\big)^{2H} \big\} \\
&\leqslant C(t,H),
\end{aligned} \tag{3.6}$$

where $C(t,H)$ depends only on t and $H \in (1/4, 1/2)$.

As for I_2, we have

$$I_2 = 2\sum_{k\in\mathbb{N}} \int_0^t \left\| \int_\sigma^t \left(e^{(t-r)A} - e^{(t-\sigma)A}\right) e_k \frac{\partial K(r,\sigma)}{\partial r}\, dr \right\|_{L^2}^2 d\sigma$$
$$\leqslant 2\sum_{k\in\mathbb{N}} \int_0^t \left| \int_\sigma^t \left(e^{-(t-r)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\right) \frac{\partial K(r,\sigma)}{\partial r}\, dr \right|^2 d\sigma. \tag{3.7}$$

In order to estimate I_2, we turn our attention to estimate the following integral

$$I_{21} =: \int_0^t \left| \int_\sigma^t \left(e^{-(t-r)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\right) \frac{\partial K(r,\sigma)}{\partial r}\, dr \right|^2 d\sigma. \tag{3.8}$$

Rewriting I_{21} as (2.2) shows that

$$I_{21} = \int_0^t \int_\sigma^t \left(e^{-(t-r_1)\pi^2k^2} - e^{(-t-\sigma)\pi^2k^2}\right) \frac{\partial K(r_1,\sigma)}{\partial r_1}\, dr_1 \int_\sigma^t \left(e^{-(t-r_2)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\right) \frac{\partial K(r_2,\sigma)}{\partial r_2}\, dr_2\, d\sigma.$$

Noticing (2.4), we have

$$I_{21} \leqslant C(H) \int_0^t \int_\sigma^t \int_\sigma^t \left(e^{-(t-r_1)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\right)\left(e^{-(t-r_2)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\right)(r_1-\sigma)^{H-\frac{3}{2}}(r_2-\sigma)^{H-\frac{3}{2}}\, dr_1\, dr_2\, d\sigma.$$

Changing the variables $t-\sigma = u$, $t-r_1 = v_1$, $t-r_2 = v_2$ and applying the Fubini theorem, we have

$$I_{21} \leqslant C(H) \int_0^t \int_0^u \int_0^u \left(e^{-v_1\pi^2k^2} - e^{-u\pi^2k^2}\right)\left(e^{-v_2\pi^2k^2} - e^{-u\pi^2k^2}\right)(u-v_1)^{H-\frac{3}{2}}(u-v_2)^{H-\frac{3}{2}}\, dv_1\, dv_2\, du$$
$$= C(H) \int_0^t \left(\int_0^u e^{-u\pi^2k^2}\left(e^{(u-v)\pi^2k^2} - 1\right)(u-v)^{H-\frac{3}{2}}\, dv \right)^2 du.$$

Changing the variable $u - v = r$ and going back to the initial variables, we have

$$I_{21} \leqslant C(H) \int_0^t e^{-2\sigma\pi^2k^2} \left(\int_0^\sigma \left(e^{r\pi^2k^2} - 1\right) r^{H-\frac{3}{2}} dr \right)^2 d\sigma$$
$$\leqslant C(H)\left(\pi^2k^2\right)^{-2H}.$$

As $H \in (1/4, 1/2)$, we get

$$I_2 \leqslant C(t,H). \tag{3.9}$$

All the above inequalities hold when $\frac{1}{4} < H < \frac{1}{2}$. Note that when $H > \frac{1}{2}$, the derivative of the kernel is integrable. We could obtain directly

$$\mathbb{E}\left\|Z(t)\right\|_{L^2(0,1)}^2 \leqslant C(t,H)t^{2H}.$$

When $H = 1/2$, B^H is standard Brownian motion; $Z(t)$ is well defined in [5]. Thus the stochastic convolution is well defined for $H \in (1/4, 1)$.

We now prove that for any positive $T > 0$, $Z(t)$ has a modification for $H \in (1/4, 1)$ in $C([0,T], L^2(0,1))$.

Without loss of generality, we suppose $0 < s < t$. We have

$$Z(t) - Z(s) = \int_0^t e^{(t-\sigma)A}\, dB^H(\sigma) - \int_0^s e^{(s-\sigma)A}\, dB^H(\sigma)$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{N}} \int_0^t e^{-(t-\sigma)\pi^2k^2} e_k \, d\beta^H(\sigma) - \sum_{k\in\mathbb{N}} \int_0^s e^{-(s-\sigma)\pi^2k^2} e_k \, d\beta^H(\sigma) \\
&= \sum_{k\in\mathbb{N}} \int_s^t K_t^*\big(e^{-(t-\cdot)\pi^2k^2} e_k\big)(\sigma) \, d\beta(\sigma) \\
&\quad + \sum_{k\in\mathbb{N}} \int_0^s \big(\big(K_t^* e^{-(t-\cdot)\pi^2k^2} e_k\big)(\sigma) - \big(K_s^* e^{-(s-\cdot)\pi^2k^2} e_k\big)(\sigma)\big) \, d\beta(\sigma).
\end{aligned} \tag{3.10}
$$

Denote the two parts of the right-hand side of the last equality by I_3, I_4 respectively.

We first estimate I_3. By the expression of K^* (2.6), we have

$$
\begin{aligned}
\mathbb{E}\|I_3\|_{L^2}^2 &= \sum_{k\in\mathbb{N}} \int_s^t \big\|K_t^*\big(e^{-(t-\cdot)\pi^2k^2} e_k\big)(\sigma)\big\|_{L^2}^2 \, d\sigma \\
&\leqslant \sum_{k\in\mathbb{N}} \int_s^t \left(\big|e^{-(t-\sigma)\pi^2k^2} K(t,\sigma)\big|^2 + \left| \int_\sigma^t \big(e^{-(t-r)\pi^2k^2} - e^{-(t-\sigma)\pi^2k^2}\big) \frac{\partial K(r,\sigma)}{\partial r} dr \right|^2 \right) d\sigma \\
&=: \sum_{k\in\mathbb{N}} (I_{31} + I_{32}).
\end{aligned} \tag{3.11}
$$

Noting the expression of K (2.2), we get

$$
\begin{aligned}
I_{31} &\leqslant 2 \int_s^t e^{-2(t-\sigma)\pi^2k^2} (t-\sigma)^{2H-1} \, d\sigma + 2 \int_s^t \left(\int_\sigma^t (r-\sigma)^{H-\frac{3}{2}} \left(1 - \left(\frac{\sigma}{r}\right)^{\frac{1}{2}-H}\right) dr \right)^2 d\sigma \\
&=: 2\big(I'_{31} + I'_{32}\big).
\end{aligned} \tag{3.12}
$$

Changing variable $2(t-\sigma)\pi^2k^2 = v$, we deduce that

$$
I'_{31} = \big(2\pi^2k^2\big)^{-2H} \int_0^{2\pi^2k^2(t-s)} e^{-v} v^{2H-1} \, dv. \tag{3.13}
$$

Choosing parameters $\varepsilon > 0$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{4} < H < \frac{1}{2}$ such that

$$
-1 + 2\varepsilon + \frac{1}{p'} > 0, \qquad 4(H-\varepsilon) + \frac{2}{p} > 1,
$$

it is obvious that there exist ε, p, p' satisfying the above inequalities. Using Lagrange mean value theorem and Hölder inequality, we get

$$
\begin{aligned}
I'_{32} &= \int_s^t e^{-2(t-\sigma)\pi^2k^2} \left(\int_\sigma^t (r-\sigma)^{H-\frac{3}{2}} \left(1 - \left(\frac{\sigma}{r}\right)^{\frac{1}{2}-H}\right) dr \right)^2 d\sigma \\
&\leqslant \int_s^t e^{-2(t-\sigma)\pi^2k^2} \left(\int_\sigma^t (r-\sigma)^{H-\frac{3}{2}} \left(1 - \left(\frac{\sigma}{r}\right)^{\frac{1}{2}-H}\right)^{\frac{1}{2}-\varepsilon} dr \right)^2 d\sigma \\
&\leqslant \int_s^t e^{-2(t-\sigma)\pi^2k^2} (t-\sigma)^{2(H-\varepsilon)} \sigma^{-1+2\varepsilon} \, d\sigma \\
&\leqslant \left(\int_s^t e^{-2p(t-\sigma)\pi^2k^2} (t-\sigma)^{2p(H-\varepsilon)} \, d\sigma \right)^{\frac{1}{p}} \left(\int_s^t \sigma^{(-1+2\varepsilon)p'} \, d\sigma \right)^{\frac{1}{p'}} \\
&\leqslant \big(2p\pi^2k^2\big)^{-2(H-\varepsilon)-\frac{1}{p}} (t-s)^{-1+2\varepsilon+\frac{1}{p'}} \left(\int_0^\infty e^{-v} v^{2p(H-\varepsilon)} \, dv \right)^{\frac{1}{p}}.
\end{aligned} \tag{3.14}
$$

Noticing Lemma 3.1, we have

$$I_{32}=\int_s^t\left|\int_\sigma^t\left(e^{-(t-r)\pi^2k^2}-e^{-(t-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right|^2 d\sigma$$
$$\leqslant C(H)\int_0^{t-s}e^{-2\sigma\pi^2k^2}\left(\int_0^\sigma\left(e^{r\pi^2k^2}-1\right)r^{H-\frac{3}{2}}\,dr\right)^2 d\sigma$$
$$\leqslant C(H)\left(\pi^2k^2\right)^{-2H}\int_0^{(t-s)\pi^2k^2}e^{-2x}\left(\int_0^\sigma\left(e^y-1\right)y^{H-\frac{3}{2}}\,dy\right)^2 dx.$$

As for I_4, we have

$$\mathbb{E}\|I_4\|_{L^2}^2=\sum_{k\in\mathbb{N}}\int_0^s\left\|\left(K_t^*e^{-(t-\cdot)\pi^2k^2}e_k\right)(\sigma)-\left(K_s^*e^{-(s-\cdot)\pi^2k^2}e_k\right)(\sigma)\right\|_{L^2}^2 d\sigma$$
$$=\sum_{k\in\mathbb{N}}\int_0^s\left|e^{-(t-\sigma)\pi^2k^2}K(t,\sigma)+\int_\sigma^t\left(e^{-(t-r)\pi^2k^2}-e^{-(t-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right.$$
$$\left.-e^{-(s-\sigma)\pi^2k^2}K(s,\sigma)+\int_\sigma^s\left(e^{-(s-r)\pi^2k^2}-e^{-(s-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right|^2 d\sigma.$$

Denote

$$I_{41}=\int_0^s\left|\left(e^{-(t-\sigma)\pi^2k^2}K(t,\sigma)-e^{-(s-\sigma)\pi^2k^2}K(s,\sigma)\right)\right|^2 d\sigma+\int_0^s\left|\int_s^t\left(e^{-(t-r)\pi^2k^2}-e^{-(t-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right|^2 d\sigma;$$
$$I_{42}=\int_0^s\left|\int_\sigma^s\left(e^{-(t-r)\pi^2k^2}-e^{-(t-\sigma)\pi^2k^2}+e^{-(s-r)\pi^2k^2}-e^{-(s-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right|^2 d\sigma. \tag{3.15}$$

The first term of I_{41} follows

$$\int_0^s\left|e^{-(t-\sigma)\pi^2k^2}K(t,\sigma)-e^{-(s-\sigma)\pi^2k^2}K(s,\sigma)\right|^2 d\sigma\leqslant\int_0^s e^{-(t-\sigma)\pi^2k^2}\left|K(t,\sigma)-K(s,\sigma)\right|^2 d\sigma$$
$$\leqslant e^{-(t-s)\pi^2k^2}\left(t^{2H}+s^{2H}\right); \tag{3.16}$$

and the second term of I_{41} follows

$$\int_0^s\left|\int_s^t\left(e^{-(t-r)\pi^2k^2}-e^{-(t-\sigma)\pi^2k^2}\right)\frac{\partial K(r,\sigma)}{\partial r}\,dr\right|^2 d\sigma$$
$$\leqslant C(H)\int_0^{t-s}e^{-2\sigma\pi^2k^2}\left(\int_0^\sigma\left(e^{r\pi^2k^2}-1\right)r^{H-\frac{3}{2}}\,dr\right)^2 d\sigma$$
$$\leqslant C(H)\left(\pi^2k^2\right)^{-2H}\int_0^{(t-s)\pi^2k^2}e^{-2x}\left(\int_0^\sigma\left(e^y-1\right)y^{H-\frac{3}{2}}\,dy\right)^2 dx. \tag{3.17}$$

As for I_{42}, noticing Lemma 3.2, for any $\gamma\in(0,1)$ we have

$$I_{42}\leqslant C(H)\left(e^{-\pi^2k^2(t-s)}-1\right)^2\int_0^s e^{-2\sigma\pi^2k^2}\left(\int_0^\sigma\left(e^{r\pi^2k^2}-1\right)r^{H-\frac{3}{2}}\,dr\right)^2 d\sigma$$

$$\leqslant C(H)\big(e^{-\pi^2k^2(t-s)}-1\big)^2\int_0^s e^{-2\sigma\pi^2k^2}\left(\int_0^\sigma \big(e^{r\pi^2k^2}-1\big)r^{H-\frac{3}{2}}\,dr\right)^2 d\sigma$$
$$\leqslant C(H)\big(e^{-(t-s)}-1\big)^2\big(\pi^2k^2\big)^{-2H}$$
$$\leqslant C(H,\gamma)\big(\pi^2k^2\big)^{-2H}|t-s|^{2\gamma}, \tag{3.18}$$

where $C(H,\gamma)$ depends only on H and γ.

Combining (3.13), (3.14), (3.16), (3.17), (3.18) with (3.10), for any positive $T>0$, Z has a modification in $C([0,T],L^2(0,1))$ for $1/4<H<1/2$. In the case $H>\frac{1}{2}$, the derivative of the kernel is integrable. Following the step given above, one can get the continuity of $Z(t)$. In the case $H=1/2$, the continuity of $Z(t)$ has been proved in [5].

For the continuous modification of Z with respect to x, we have

$$Z(y)-Z(x)=\sum_{k\in\mathbb{N}}\int_0^t K_t^*\big(e^{-(t-\cdot)\pi^2k^2}\big(e_k(y)-e_k(x)\big)\big)(\sigma)\,d\beta(\sigma). \tag{3.19}$$

Then

$$\mathbb{E}\big\|Z(y)-Z(x)\big\|^2_{L^2(0,1)}=\sum_{k\in\mathbb{N}}\int_0^t \big\|K_t^*\big(e^{-(t-\cdot)\pi^2k^2}\big(e_k(y)-e_k(x)\big)\big)(\sigma)\big\|^2_{L^2(0,1)}\,d\sigma$$
$$\leqslant C(H)\sum_{k\in\mathbb{N}}\big(\pi^2k^2\big)^{-2H}\big\|e_k(y)-e_k(x)\big\|^2_{L^2(0,1)}. \tag{3.20}$$

The continuous modification of Z with respect to x for $H\geqslant\frac{1}{2}$ is obvious.

Thus the proof of Theorem 2.1 is completed. $\square$

4. Global existence

In this section, we work pathwise and use a fixed point argument to prove local well-posedness results to the problem (1.1)–(1.3); then using a priori estimate, we prove global well-posedness results.

Set

$$v(t)=u(t)-Z(t),\quad t\geqslant 0;$$

then if u satisfies (1.1)–(1.3), v satisfies

$$\frac{dv}{dt}=Av+\frac{1}{2}\frac{\partial}{\partial_x}(v+Z)^2;$$
$$v(0)=u_0. \tag{4.1}$$

Let us write (4.1) as a mild form

$$v(t)=e^{tA}u_0+\frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\big(v+Z(s)\big)^2\,ds, \tag{4.2}$$

which is convenient to construct the local well-posedness of the problem (4.1).

From now on we will study Eq. (4.2) for a.s. $\omega\in\Omega$. We are going to solve Eq. (4.2) by a fixed point argument in the space $C([0,T^*];L^2(0,1))$ on each path, i.e. we fix ω and construct a solution to a deterministic equation. Set

$$B_R^{T^*}=\big\{v\in C\big(\big[0,T^*\big];L^2(0,1)\big):\ \big\|v(t)\big\|_{L^2(0,1)}\leqslant R,\ \forall t\in\big[0,T^*\big]\big\},$$

and consider the initial data that are $\mathcal{F}_0$-measurable and belonging to $L^2(0,1)$. Then we have the following lemma.

Lemma 4.1. *Let $\|u_0\|_{L^2(0,1)}<R$; then there exists a stopping time $T^*>0$ such that (4.2) has a unique solution in $B_R^{T^*}$.*

Proof. Take any $v\in B_R^{T^*}$ and define $v=\mathcal{T}v$ by

$$v(t)=\mathcal{T}v(t)=e^{tA}u_0+\frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\big(v+Z(s)\big)^2\,ds.$$

We need to show $\mathcal{T}: B_R^{T^*} \to B_R^{T^*}$ is a strict contraction on $B_R^{T^*}$.

$$\begin{aligned}\left\|\mathcal{T}v(t)\right\|_{L^2(0,1)} &\leqslant \left\|e^{tA}u_0\right\|_{L^2(0,1)} + \frac{1}{2}\int_0^t \left\|e^{(t-s)A}\frac{\partial}{\partial x}\left(v+Z(s)\right)^2\right\|_{L^2(0,1)} ds \\ &\leqslant \|u_0\|_{L^2(0,1)} + C\int_0^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(1-\frac{1}{2})}\left\|\left(v+Z(s)\right)^2\right\|_{L^2(0,1)} ds \\ &\leqslant \|u_0\|_{L^2(0,1)} + Ct^{\frac{1}{4}}\left(R^2+\mu^2\right),\end{aligned}$$

where

$$\mu = \sup_{s\in[0,t]}\left\|Z(s)\right\|_{L^2(0,1)}.$$

Hence $\|\mathcal{T}v(t)\|_{L^2(0,1)} \leqslant R$ for all $t \in [0, T^*]$ provided

$$\|u_0\|_{L^2(0,1)} + Ct^{\frac{1}{4}}\left(R^2+\mu^2\right) < R. \tag{4.3}$$

It is clear that for any $\|u_0\|_{L^2(0,1)} < R$ there exists a T^* satisfying (4.3).

Now consider $v_1, v_2 \in B_R^{T^*}$ and set $v_i = \mathcal{T}v_i$, $i = 1, 2$ and $v = v_1 - v_2$; then

$$\begin{aligned}v(t) &= \frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left(\left(v_1+Z(s)\right)^2 - \left(v_2+Z(s)\right)^2\right) ds \\ &\leqslant \frac{1}{2}\int_0^t (t-s)^{-\frac{3}{4}}\left(\left(v_1+Z(s)\right)^2 - \left(v_2+Z(s)\right)^2\right) ds.\end{aligned} \tag{4.4}$$

Using Hölder inequality, we have

$$\begin{aligned}\left\|\left(v_1+Z(t)\right)^2 - \left(v_2+Z(t)\right)^2\right\|_{L^2(0,1)} &\leqslant \left\|v_1+v_2+2Z(t)\right\|_{L^2(0,1)}\|v_1-v_2\|_{L^2(0,1)} \\ &\leqslant 2(R+\mu)\|v_1-v_2\|_{L^2(0,1)}.\end{aligned} \tag{4.5}$$

Hence

$$\left\|v(t)\right\|_{L^2(0,1)} \leqslant C_1C_2(R+\mu)\left(T^*\right)^{\frac{1}{4}}\|v_1-v_2\|_{L^2(0,1)}.$$

We take T^* such that

$$C_1C_2(R+\mu)\left(T^*\right)^{\frac{1}{4}} < 1,$$

and (4.3) holds so that $\mathcal{T}$ is a strict contraction on $B_R^{T^*}$.

Thus we complete the proof of this lemma. □

The results in Lemma 4.1 are proved with respect to a fixed ω. It is obvious that there exists $T^* = T^*(\omega)$ which depends on ω such that the results in Lemma 4.1 are valid for a.s. $\omega \in \Omega$. The following work will remove the dependence on ω for the time interval on which the solution exists.

Lemma 4.2. *If $v \in C([0, T]; L^2(0, 1))$ satisfies (4.1), then*

$$\left\|v(t)\right\|_{L^2(0,1)} \leqslant C\left(\mu_\infty^2 + \|u_0\|_{L^2(0,1)}\right)e^{2\mu_\infty^2 t}$$

for some constant $\mu_\infty > 0$.

Proof. Let $\{u_0^n\}_{n\in\mathbb{N}}$ be a sequence in $C^\infty(0, 1)$ such that $u_0^n \to u_0$. Let $\{(B^H)^n\}_{n\in\mathbb{N}}$ be a sequence of regular processes such that

$$Z^n(t) = \int_0^t e^{(t-s)A}\, d\left(B^H\right)^n(s) \to Z(t)$$

in $C([0, T] \times [0, 1])$.

Let v^n be the solution of

$$v^n(t) = e^{tA}u_0^n + \frac{1}{2}\int_0^t e^{(t-s)A}\frac{\partial}{\partial x}\left(v^n + Z^n\right)^2 ds.$$

It is easy to see that v^n does exist on an interval of time $[0, T_n]$ such that $T_n \to T^*$ and that v^n converges to v in $C([0, T^*]; L^2(0,1))$. Moreover v^n is regular and satisfies

$$\frac{dv^v}{dt} = Av^n + \frac{1}{2}\frac{\partial}{\partial x}\left(v^n + Z^n\right)^2, \qquad v^n(0) = u_0^n. \tag{4.6}$$

Multiplying the first equation of (4.6) by v^n and integrating over $(0,1)$ with respect to x, we find

$$\frac{\partial}{\partial t}\left\|v^n\right\|^2_{L^2(0,1)} + 2\left\|v^n\right\|^2_{H^1(0,1)} - \int_0^1 v^n\frac{\partial}{\partial x}\left(v^n + Z^n\right)^2 dx = 0. \tag{4.7}$$

We estimate the last term in Eq. (4.7) as follows:

$$\begin{aligned}\left|\int_0^1 v^n\frac{\partial}{\partial x}\left(v^n + Z^n\right)^2 dx\right| &\leqslant 2\left|\int_0^1 Z^n v^n\frac{\partial}{\partial x}v^n\,dx\right| + \left|\int_0^1 \left(Z^n\right)^2\frac{\partial}{\partial x}v^n\,dx\right| \\ &\leqslant 2\mu^4_{n,\infty} + 4\mu^2_{n,\infty}\left\|v^n\right\|^2_{L^2(0,1)} + \left\|v^n\right\|^2_{H^1(0,1)},\end{aligned} \tag{4.8}$$

where $\mu^2_{n,\infty} = \sup_{t\in[0,T]}|Z^n(t)|_{L^\infty(0,1)}$ for a.s. $\omega \in \Omega$.

Going back to (4.7) we obtain

$$\frac{\partial}{\partial t}\left\|v^n\right\|^2_{L^2(0,1)} \leqslant 2\mu^4_{n,\infty} + 4\mu^2_{n,\infty}\left\|v^n\right\|^2_{L^2(0,1)}.$$

Thanks to Gronwall's lemma, we have

$$\left\|v^n\right\|^2_{L^2(0,1)} \leqslant \left(\frac{\mu^2_{n,\infty}}{2} + \left\|u_0^n\right\|^2_{L^2(0,1)}\right)e^{4\mu^2_{n,\infty}t}.$$

Taking the limit as $n \to \infty$, we see that a.s.

$$\|v\|_{L^2(0,1)} \leqslant C\left(\mu^2_\infty + \|u_0\|_{L^2(0,1)}\right)e^{2\mu^2_\infty t}.$$

Thus we complete the proof of this lemma. □

From Lemma 4.1 and Lemma 4.2, we get the following result.

Theorem 4.3. *Let u_0 be given which is $\mathcal{F}_0$-measurable and such that $u_0 \in L^2(0,1)$ a.s. Then there exists a unique mild solution of Eq.* (1.1)–(1.3)*, which belongs to $C([0,T]; L^2(0,1))$ for a.s. $\omega \in \Omega$.*

References

[1] E. Alòs, O. Mazet, D. Nualart, Stochastic calculus with respect to Gaussian processes, Ann. Probab. 29 (2001) 766–801.
[2] J. Bec, K. Khanin, Burgers turbulence, Phys. Rep. 447 (2007) 1–66.
[3] P. Caithamer, The stochastic wave equation driven by fractional Brownian noise and temporally correlated smooth noise, Stoch. Dyn. 5 (1) (2005) 45–64.
[4] G. Da Prato, A. Debussche, R. Temam, Stochastic Burgers' equation, NoDEA 1 (1994) 389–402.
[5] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1992.
[6] L. Decreusefond, A.S. Ustunel, Stochastic analysis of the fractional Brownian motion, Potential Anal. 10 (1997) 177–214.
[7] T.E. Duncan, Some stochastic semilinear equations in Hilbert space with fractional Brownian motion, in: J. Menaldi, E. Rofman, A. Sulem (Eds.), Optimal Control and Partial Differential Equations, IOS Press, Amsterdam, 2000, pp. 241–247.
[8] T.E. Duncan, Y. Hu, B. Pasik-Duncan, Stochastic calculus for fractional Brownian motion, I: Theory, SIAM J. Control Optim. 38 (2) (2000) 582–612.
[9] T.E. Duncan, J. Jakubowski, B. Pasik-Duncan, Stochastic integration for fractional Brownian motion in a Hilbert space, Stoch. Dyn. 6 (2006) 53–75.
[10] T.E. Duncan, B. Maslowski, B. Pasik-Duncan, Fractional Brownian motion and stochastic equations in Hilbert spaces, Stoch. Dyn. 2 (2002) 225–250.
[11] M. Erraoui, D. Nualart, Y. Ouknine, Hyperbolic stochastic partial differential equations with additive fractional Brownian sheet, Stoch. Dyn. 3 (2003) 121–139.
[12] W. Grecksch, V.V. Ahn, A parabolic stochastic differential equation with fractional Brownian motion input, Statist. Probab. Lett. 41 (1999) 337–346.
[13] I. Gyongy, D. Nualart, On the stochastic Burgers' equation in the real line, Ann. Probab. 27 (2) (1999) 782–802.
[14] I. Hosokawa, K. Yamamoto, Turbulence in the randomly forced one dimensional Burgers flow, J. Stat. Phys. 13 (1975) 245–272.
[15] Y. Hu, Heat equation with fractional white noise potential, Appl. Math. Optim. 43 (2001) 221–243.

[16] Y. Hu, D. Nualart, Stochastic heat equation driven by fractional noise and local time, Probab. Theory Related Fields 143 (2009) 285–328.
[17] H.E. Hurst, Long-term storage capacity in reservoirs, Trans. Amer. Soc. Civil Eng. 116 (1951) 400–410.
[18] H.E. Hurst, Method of using long-term storage capacity in reservoirs, in: Proc. Inst. Civil Engineers, Part I, 1956, pp. 519–590.
[19] Dah-Teng Jeng, Forced model equation for turbulence, Phys. Fluids 12 (10) (1969) 2006–2010.
[20] B.B. Mandelbrot, The Fractal Geometry of Nature, Freeman, San Francisco, CA, 1983.
[21] D. Nualart, Malliavin Calculus and Related Topics, Springer-Verlag, 1995.
[22] B. Pasik-Duncan, T.E. Duncan, B. Maslowski, Linear stochastic equations in a Hilbert space with fractional Brownian motion, in: Control Theory Applications in Financial Engineering and Manufacturing, Springer-Verlag, 2006, pp. 201–222 (Chapter 11).
[23] S. Tindel, C.A. Tudor, F. Viens, Stochastic evolution equations with fractional Brownian motion, Probab. Theory Related Fields 127 (2003) 186–204.