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STABILITY ANALYSIS OF NONLINEAR GALERKIN & GALERKIN METHODS FOR NONLINEAR EVOLUTION EQUATIONS*

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Abstract. Our object in this article is to describe the Galerkin scheme and nonlinear Galerkin scheme for the approximation of nonlinear evolution equations, and to study the stability of these schemes. Spatial discretization can be performed by either Galerkin spectral method or nonlinear Galerkin spectral method; time discretization is done by Euler scheme which is explicit or implicit in the nonlinear terms. According to the stability analysis of the above schemes, the stability of nonlinear Galerkin method is better than that of Galerkin method.

0. Introduction

With the actual computing ability, one can imagine solving numerical problems which were unthinkable in the recent past. Much effort has been devoted in the past to the approximation of evolution equations on finite intervals of time or in the dynamically simple cases where the solutions converge to a steady state as time goes to infinity. However, in many physically relevant situations the solutions to a dissipative evolution equation do not converge to stable solutions. They rather remain time–dependent and converge to a compact attractor that encompasses the nonlinear dynamics. Hence, whether this is addressed implicitly or explicitly, the numerical integration of the evolution equations is then closely related to the approximation of the attractor that may be a complicated set, even a fractal.

Nonlinear Galerkin method is a new integration algorithm that stems from the dynamical systems. They are based on a differentiated treatment of the small and large eddies, and are particularly adapted to the integration of such equations on large intervals of time.

In the present article we address the question of both time and space discretizations. In particular, we will emphasize the study of the stability of such schemes. Compared with the Galerkin scheme, the stability of the nonlinear Galerkin scheme is better than that of the Galerkin scheme.

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1. Nonlinear Evolution Equations

We are given a Hilbert space H with a scalar product $(\cdot,\cdot)$ and a norm $|\cdot|$. Let A be a linear unbounded self–adjoint operator in H with domain $D(A)$ dense in H. We assume that A is positive closed and that A^{-1} is compact. We can then define the powers A^s of A for $s \in R$; the space $D(A^s)$ is a Hilbert space when endowed with the norm $|A^s \cdot|$. We set $V = D(A^{\frac{1}{2}})$ and denote by $||\cdot|| = |A^{\frac{1}{2}}\cdot|$ the norm on V.

The nonlinear evolution equation in V' that we will study has the form

$$\frac{du}{dt} + \lambda A u + B(u,u) = f, \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where $\lambda > 0$ is a viscosity parameter, $B(\cdot,\cdot)$ is a bilinear operator from $V \times V$ into $V', V' = D(A^{-\frac{1}{2}})$ is the dual space of V endowed with the equivalent norm

$$||f||_* = \sup_{v\in V} \frac{(f,v)}{||v||}, \quad \forall f \in V'.$$

Since A^{-1} is compact and self–adjoint, there exists an orthonormal basis $\{w_j\}$ of H consisting of eigenvectors of A:

$$Aw_j = \lambda_j w_j, \quad 0 < \lambda_1 \le \lambda_2 \le \cdots, \lambda_j \to \infty \text{ as } j \to \infty. \tag{1.3}$$

Let us denote by $a(\cdot,\cdot)$ and $b(\cdot,\cdot,\cdot)$ the bilinear and trilinear forms on V given by

$$a(u,v) = \lambda\langle Au, v\rangle, \quad \forall u,v \in V,$$

$$b(u,v,w) = \langle B(u,v), w\rangle, \quad \forall u,v,w \in V,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality of V and V'.

The following assumptions are valid:

$$b(u,v,w) = -b(u,w,v), \quad \forall u,v,w \in V, \tag{1.4}$$

$$|b(u,v,w)| \le c_0(|u|||u|||w|||w||)^{\frac{1}{2}}||v||, \quad \forall u,v,w \in V, \tag{1.5}$$

$$|a(u,v)| \le \lambda||u||\,||v||, \quad a(u,u) = \lambda||u||^2, \quad \forall u,v \in V. \tag{1.6}$$

Using the same methods as for these equations (see [6]), we can check that if $u_0 \in H$ and $f \in L^\infty(R^+;V')$, then the initial value problem (1.1)–(1.2) has a unique solution $u = u(t)$ defined for all $t > 0$ and such that

$$u \in L^\infty(R^+;H) \cap L^2(0,T;V), \quad \forall T > 0.$$

For integers m, n, we introduce the finite dimensional subspaces of H (or V):

$$H_m = \text{span}\{w_1, \cdots, w_m\}, \quad H_{mn} = H_{m+n} \setminus H_m,$$

$$H_{m+n} = \text{span}\{w_1, \cdots, w_m, \cdots, w_{m+n}\}.$$

By the definitions of H_m, H_{m+n}, H_{mn}, we have

$$\lambda_1|u|^2 \leq ||u||^2 \leq \lambda_k|u|^2, \quad \forall u \in H_k, \ k = m, m+n,$$

$$\lambda_{m+1}|u|^2 \leq ||u||^2 \leq \lambda_{m+n}|u|^2, \qquad \forall u \in H_{mn},$$

$$\begin{aligned} &|u+w|^2 = |u|^2 + |w|^2, \\ &||u+w||^2 = ||u||^2 + ||w||^2, \ \forall u \in H_m, \ w \in H_{mn}. \end{aligned} \tag{1.7}$$

2. Nonlinear Galerkin Method

The usual Galerkin method consists in finding $u_{m+n}(t) \in H_{m+n}$ such that

$$(\frac{du_{m+n}}{dt}, v_0) + a(u_{m+n}, v_0) + b(u_{m+n}, u_{m+n}, v_0) = (f, v_0), \quad \forall v_0 \in H_{m+n}, \tag{2.1}$$

$$u_{m+n}(0) = P_{m+n}u_0. \tag{2.2}$$

Taking into account the decompositions of u_{m+n} and v_0

$$u_{m+n} = y(t) + z(t), \quad y \in H_m, \ z \in H_{mn},$$

$$v_0 = v + w, \qquad v \in H_m, \ w \in H_{mn},$$

and neglecting some terms on z and w, we obtain the nonlinear Galerkin method, namely finding $(y, z) \in H_m \times H_{mn}$ such that

$$(\frac{dy}{dt}, v) + a(y, v) + b(y, y, v) + b(y, z, v) + b(z, y, v) = (f, v), \quad \forall v \in H_m, \tag{2.3}$$

$$(\frac{dz}{dt}, w) + a(z, w) + b(y, y, w) = (f, w), \quad \forall w \in H_{mn}, \tag{2.4}$$

$$y(0) = P_m u_0, \ z(0) = (P_{m+n} - P_m)u_0, \tag{2.5}$$

where P_m denotes the orthogonal projector of H (or V) onto $H_m, u_{m+n}(t), y(t)+z(t)$ are expected to be the approximation of $u(t)$.

Now, we consider the time discretization of (2.1)–(2.2) and (2.3)–(2.5) by the Euler scheme. Thus we obtain the following numerical schemes of (1.1)–(1.2).

Galerkin Explicit Scheme (GE Scheme): to find $u^k \in H_{m+n}$ such that

$$u^0 = P_{m+n}u_0, \tag{2.6}$$

$$\frac{1}{\Delta t}(u^{k+1} - u^k, v) + a(u^{k+1}, v) + b(u^k, u^k, v) = (f^{k+1}, v), \quad \forall v \in H_{m+n}. \tag{2.7}$$

Nonlinear Galerkin Explicit Scheme (NGE Scheme): to find $y^k \in H_m, z^k \in H_{mn}$ such that

$$y^0 = P_{m+n}u_0, z^0 = (P_{m+n} - P_m)u_0, \tag{2.8}$$

$$\begin{aligned}
&\frac{1}{\Delta t}(y^{k+1} - y^k, v) + \frac{1}{\Delta t}(z^{k+1} - z^k, w) \\
&\quad + a(y^{k+1} + z^{k+1}, v + w) + b(y^k, y^k, v + w) \\
&\quad + b(y^k, z^{k+1}, v) + b(z^{k+1}, y^k, v) \\
&= (f^{k+1}, v + w), \quad \forall v \in H_m, w \in H_{m+n},
\end{aligned} \tag{2.9}$$

where Δt is the time step, $t_k = k\Delta t$ and

$$f^{k+1} = \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} f(t)dt,$$

and $u^k, y^k + z^k$ are expected to be the approximations of $u(t_k), k = 0, 1, \cdots$.

Taking $v = u^{k+1}$ in (2.7) and using (1.4)–(1.7), we obtain

$$\begin{aligned}
&\frac{1}{\Delta t}(|u^{k+1}|^2 - |u^k|^2 + |u^{k+1} - u^k|^2) + \lambda||u^{k+1}||^2 \\
&\qquad + b(u^k, u^k, u^{k+1} - u^k) = (f^{k+1}, u^{k+1}),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
|b(u^k, u^k, u^{k+1} - u^k)| &\le |u^k|||u^k||||u^{k+1} - u^k|| \\
&\le \frac{\lambda}{4}||u^k||^2 + \frac{c_0^2}{\lambda}|u^k|^2||u^{k+1} - u^k||^2,
\end{aligned} \tag{2.11}$$

$$|(f^{k+1}, u^{k+1})| \le ||f^{k+1}||_*||u^{k+1}|| \le \frac{\lambda}{4}||u^{k+1}||^2 + \frac{1}{\lambda}||f^{k+1}||_*^2, \tag{2.12}$$

$$||u^{k+1} - u^k||^2 \le \lambda_{m+n}|u^{k+1} - u^k|^2, \tag{2.13}$$

$$\begin{aligned}
||f^{k+1}||_*^2 &= \frac{1}{\Delta t^2}||\int_{t_k}^{t_{k+1}} f(t)dt||_*^2 \\
&\le \frac{1}{\Delta t}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt \le \sup_{t\in R^+} ||f(t)||_*^2.
\end{aligned} \tag{2.14}$$

Combining (2.10) with (2.11)–(2.14), we obtain

$$\begin{aligned}|u^{k+1}|^2 - |u^k|^2 + (1 - 2\lambda_{m+n}c_0^2\Delta t\lambda^{-1}|u^k|^2)|u^{k+1} - u^k|^2 \\ + \lambda||u^{k+1}||^2\Delta t + \frac{\lambda}{2}(||u^{k+1}||^2 - ||u^k||^2)\Delta t \\ \leq \frac{2}{\lambda}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt.\end{aligned} \tag{2.15}$$

For NGE Scheme, we can obtain the similar estimates

$$\begin{aligned}|y^{k+1} + z^{k+1}|^2 - |y^k + z^k|^2 + (1 - 10\lambda_m c_0^2\Delta t\lambda^{-1}|y^k|^2)|y^{k+1} - y^k|^2 \\ + \lambda||y^{k+1} + z^{k+1}||^2\Delta t + \frac{\lambda}{2}(||y^{k+1}||^2 - ||y^k||^2)\Delta t \\ \leq \frac{2}{\lambda}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt.\end{aligned} \tag{2.16}$$

Moreover, we consider the implicit schemes of the Galerkin method and nonlinear Galerkin method, respectively.

Galerkin Implicit Scheme(GI Scheme)

$$u^0 = P_{m+n}u_0, \tag{2.17}$$

$$\begin{aligned}\frac{1}{\Delta t}(u^{k+1} - u^k, v) + a(u^{k+1}, v) + b(u^{k+1}, u^{k+1}, v) \\ =(f^{k+1}, v), \quad \forall v \in H_{m+n}.\end{aligned} \tag{2.18}$$

Nonlinear Galerkin Implicit Scheme (NGI Scheme)

$$y^0 = P_{m+n}u_0, \ z^0 = (P_{m+n} - P_m)u_0, \tag{2.19}$$

$$\begin{aligned}\frac{1}{\Delta t}(y^{k+1} - y^k, v) + \frac{1}{\Delta t}(z^{k+1} - z^k, w) + a(y^{k+1} + z^{k+1}, v + w) \\ + b(y^{k+1}, y^{k+1}, v + w) + b(y^{k+1}, z^{k+1}, v) + b(z^{k+1}, y^{k+1}, v) \\ =(f^{k+1}, v + w), \quad \forall v \in H_m, w \in H_{mn}.\end{aligned} \tag{2.20}$$

For GI Scheme and NGI Scheme we then obtain

$$|u^{k+1}|^2 - |u^k|^2 + \lambda||u^{k+1}||^2\Delta t \leq \frac{1}{\lambda}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt, \tag{2.21}$$

$$|y^{k+1} + z^{k+1}|^2 - |y^k + z^k|^2 + \lambda||y^{k+1} + z^{k+1}||^2\Delta t \leq \frac{1}{\lambda}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt. \tag{2.22}$$

3. Boundedness Analysis

Assume that $u_0 \in H$ and $f \in L^\infty(R^+; V')$, then we obtain the boundedness of solution for GE Scheme and NGE Scheme.

Theorem 3.1. *If Δt satisfies*

$$\Delta t \leq \alpha \lambda_{m+n}^{-1}, \qquad \alpha = \min\{\frac{2}{\lambda}, \frac{\lambda}{10} c_0^{-2} M^{-2}\}, \tag{3.1}$$

then the sequence $u^k, k \geq 0$, generated by the GE Scheme is bounded, namely

$$|u^J|^2 \leq 2|u_0|^2 + \lambda_1^{-1}(\frac{2}{\lambda})^2 f_\infty^2 = M^2, \qquad \forall J \geq 0, \tag{3.2}$$

$$\lambda \sum_{k=1}^{J} ||u^k||^2 \Delta t \leq 2|u_0|^2 + \frac{2}{\lambda} \int_0^{t_J} ||f(t)||_*^2 dt, \qquad \forall J \geq 1, \tag{3.3}$$

where

$$f_\infty = \sup_{t \in R^+} ||f(t)||_*.$$

Proof. We set

$$\eta^k = |u^k|^2 + \frac{\lambda}{2} \Delta t ||u^k||^2,$$

then (2.15) yields

$$\begin{aligned} &\eta^{k+1} - \eta^k + (1 - 2\lambda_{m+n} c_0^2 \Delta t |u^k|^2)|u^{k+1} - u^k|^2 \\ &\qquad + \lambda ||u^{k+1}||^2 \Delta t \leq \frac{2}{\lambda} f_\infty^2 \Delta t. \end{aligned} \tag{3.4}$$

Using again (1.7) and (3.1), one finds

$$\begin{aligned} ||u^{k+1}||^2 &\geq \frac{\lambda_1}{2}|u^{k+1}|^2 + \frac{1}{2}||u^{k+1}||^2 \geq \frac{\lambda_1}{2}(|u^{k+1}|^2 + \frac{1}{\lambda_1}||u^{k+1}||^2) \\ &\geq \frac{\lambda_1}{2}\eta^{k+1} \qquad (\text{due to} \frac{1}{\lambda_1} \geq \frac{\lambda}{2}\Delta t). \end{aligned} \tag{3.5}$$

So, (3.4) and (3.5) imply

$$\begin{aligned} &(1 + \frac{\lambda\lambda_1}{2}\Delta t)\eta^{k+1} + (1 - 2\lambda_{m+n} c_0^2 \Delta t \lambda^{-1} |u^k|^2)|u^{k+1} - u^k|^2 \\ \leq &\eta^k + \frac{2}{\lambda} f_\infty^2 \Delta t. \end{aligned} \tag{3.6}$$

We now prove (3.2) by induction. Since

$$|u^0|^2 = |P_{m+n} u_0|^2 \leq |u_0|^2 \leq M^2,$$

So (3.2) is true for $J = 0$. Assume that (3.2) holds for $J \le k$, we want to establish it for $J = k+1$.

According to the induction hypothesis, (3.1) yields

$$1 - 2\lambda_{m+n} c_0^2 \Delta t \lambda^{-1} |u^k|^2 \ge 0. \tag{3.7}$$

Thus, (3.6) implies

$$(1 + \frac{\lambda\lambda_1}{2}\Delta t)\eta^{k+1} \le \eta^k + \frac{2}{\lambda} f_\infty^2 \Delta t,$$

$$\eta^{k+1} \le (1 + \frac{\lambda\lambda_1}{2}\Delta t)^{-1}\eta^k + \frac{2}{\lambda}(1 + \frac{\lambda\lambda_1}{2}\Delta t)^{-1} f_\infty^2 \Delta t. \tag{3.8}$$

The analogous relations are also valid for the previous values of k, and thus

$$\begin{aligned}&(1 + \frac{\lambda\lambda_1}{2}\Delta t)^{-i}\eta^{k+1-i} \\ \le &(1 + \frac{\lambda\lambda_1}{2}\Delta t)^{-i-1}\eta^{k-i} + \frac{2}{\lambda}\Delta t (1 + \frac{\lambda\lambda_1}{2}\Delta t)^{-i-1} f_\infty^2.\end{aligned} \tag{3.9}$$

Summing (3.9) for $i = 0, 1, \cdots, k$, we obtain

$$\begin{aligned}\eta^{k+1} \le &\eta^0 + \lambda_1^{-1}(\frac{2}{\lambda})^2 f_\infty^2 \le (1 + \frac{\lambda}{2}\lambda_{m+n}\Delta t)|u^0|^2 + \lambda_1^{-1}(\frac{2}{\lambda})^2 f_\infty^2 \\ \le &2|u_0|^2 + \lambda_1^{-1}(\frac{2}{\lambda})^2 f_\infty^2 = M^2.\end{aligned} \tag{3.10}$$

The induction is complete, (3.2) is thus proven.

By applying (3.1) and (3.2), (2.15) gives

$$\begin{aligned}&|u^{k+1}|^2 - |u^k|^2 + \lambda ||u^{k+1}||^2 \Delta t + \frac{\lambda}{2}(||u^{k+1}||^2 - ||u^k||^2)\Delta t \\ \le &\frac{2}{\lambda}\int_{t_k}^{t_{k+1}} ||f(t)||_*^2 dt.\end{aligned} \tag{3.11}$$

Summing (3.11) for $k = 0, 1, \cdots, J-1$, we obtain

$$\begin{aligned}\lambda \sum_{k=1}^{J} ||u^k||^2 \Delta t &\le |u_0|^2 + \frac{\lambda}{2}||u^0||^2 \Delta t + \frac{2}{\lambda}\int_0^{t_J} ||f(t)||_*^2 dt \\ &\le 2|u_0|^2 + \frac{2}{\lambda}\int_0^{t_J} ||f(t)||_*^2 dt.\end{aligned} \tag{3.12}$$

So (3.3) is proved.

Theorem 3.2. *Assume*

$$\Delta t \le \alpha \lambda_m^{-1}, \tag{3.13}$$

then the sequence $(y^k, z^k), k \ge 0$ *generated by NGE Scheme* (2.9) *is bounded, namely*

$$|y^J + z^J|^2 \le M^2, \quad \forall J \ge 0, \tag{3.14}$$

$$\lambda \sum_{k=1}^{J} \|y^k + z^k\|^2 \Delta t \le 2|u_0|^2 + \frac{2}{\lambda} \int_0^{t_J} \|f(t)\|_*^2 dt, \quad \forall J \ge 1. \tag{3.15}$$

Proof. Setting

$$\xi^k = |y^k + z^k|^2 + \frac{\lambda}{2} \Delta t \|y^k\|^2,$$

then

$$\begin{aligned} \lambda \Delta t \|y^{k+1} + z^{k+1}\|^2 &\ge \frac{\lambda \lambda_1}{2} \Delta t (|y^{k+1} + z^{k+1}|^2 + \lambda_1^{-1} \|y^{k+1}\|^2) \\ &\ge \frac{\lambda \lambda_1}{2} \Delta t \xi^{k+1} \quad (\text{due to } \lambda_1^{-1} \ge \frac{\lambda}{2} \Delta t). \end{aligned} \tag{3.16}$$

So, (2.16) and (3.16) give

$$\begin{aligned} &\xi^{k+1} - \xi^k + \frac{\lambda \lambda_1}{2} \Delta t \xi^{k+1} + (1 - 10 \lambda_m c_0^2 \lambda^{-1} \Delta t |y^k|^2) |y^{y+1} - y^k|^2 \\ \le &\frac{2}{\lambda} \int_{t_k}^{t_{k+1}} \|f(t)\|_*^2 dt \le \frac{2}{\lambda} \Delta t f_\infty^2. \end{aligned} \tag{3.17}$$

Applying the previous method to (3.17), we can obtain (3.14)–(4.15). □

Moreover, by setting

$$\eta^k = |u^k|^2, \quad \xi^k = |y^k + z^k|^2,$$

we derive from (2.25)–(2.26) the following bounded theorem.

Theorem 3.3. *Assume* $\Delta t \le 2\lambda^{-1}\lambda_1^{-1}$, *then solution sequences* $u^k, (y^k, z^k), k \ge 0$, *generated respectively by GI Scheme and NGI Scheme, satisfy*

$$\begin{aligned} &|u^J|^2 \le |u_0|^2 + \lambda_1^{-1} \lambda^{-2} f_\infty^2, \quad \forall J \ge 0, \\ &\lambda \sum_{k=1}^{J} \|u^k\|^2 \Delta t \le |u_0|^2 + \lambda^{-1} \int_0^{t_J} \|f(t)\|_*^2 dt, \quad \forall J \ge 1, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &|y^J + z^J|^2 \le |u_0|^2 + \lambda_1^{-1} \lambda^{-2} f_\infty^2, \quad \forall J \ge 0, \\ &\lambda \sum_{k=1}^{J} \|y^k + z^k\|^2 \Delta t \le |u_0|^2 + \lambda^{-1} \int_0^{t_J} \|f(t)\|_*^2 dt, \quad \forall J \ge 1, \end{aligned} \tag{3.19}$$

namely, the sequences generated respectively by GI Scheme and NGI Scheme are absolutely bounded.

This proof is very similar to that of Theorem 3.1, thus omitted here.

4. Stability Analysis

Assume that $P_m u_0 + e^0, (P_{m+n} - P_m)u_0 + \varepsilon^0, f^k + \eta^k$ are the computer approximations of $P_m u_0, (P_{m+n} - P_m)u_0$ and f^k, respectively. Then $(y^k + e^k, z^k + \varepsilon^k), k \geq 0$, is the computer approximation of (y^k, z^k) and $u^k + \xi^k, k \geq 0$ is the computer approximation of u^k corresponding to $P_m u_0 + e^0, (P_{m+n} - P_m)u_0 + \varepsilon^0$ and $f^k + \eta^k$, respectively. Thus we can obtain the computer error equations corresponding to GE, NGE Schemes and GI, NGI Schemes, respectively.

For GE Scheme

$$\begin{aligned}&\frac{1}{\Delta t}(\xi^{k+1} - \xi^k, v) + a(\xi^{k+1}, v) + b(u^k, \xi^k, v)\\&\quad + b(\xi^k, u^k, v) + b(\xi^k, \xi^k, v) = (\eta^{k+1}, v), \quad \forall v \in H_{m+n}.\end{aligned} \tag{4.1}$$

For NGE Scheme

$$\begin{aligned}&\frac{1}{\Delta t}(e^{k+1} - e^k, v) + \frac{1}{\Delta t}(\varepsilon^{k+1} - \varepsilon^k, w) + a(e^{k+1} + \varepsilon^{k+1}, v + w)\\&\quad + b(y^k, e^k, v + w) + b(e^k, y^k, v + w) + b(e^k, e^k, v + w)\\&\quad + b(y^k, \varepsilon^{k+1}, v) + b(e^k, z^{k+1}, v) + b(e^k, \varepsilon^{k+1}, v)\\&\quad + b(z^{k+1}, e^k, v) + b(\varepsilon^{k+1}, y^k, v) + b(\varepsilon^{k+1}, e^k, v)\\&=(\eta^{k+1}, v + w), \quad \forall v \in H_m,\ w \in H_{mn}.\end{aligned} \tag{4.2}$$

For GI Scheme

$$\begin{aligned}&\frac{1}{\Delta t}(\xi^{k+1} - \xi^k, v) + a(\xi^{k+1}, v) + b(u^{k+1}, \xi^{k+1}, v) + b(\xi^{k+1}, u^{k+1}, v)\\&\qquad + b(\xi^{k+1}, \xi^{k+1}, v) = (\eta^{k+1}, v), \quad \forall v \in H_{m+n}.\end{aligned} \tag{4.3}$$

For NGI Scheme

$$\begin{aligned}&\frac{1}{\Delta t}(e^{k+1} - e^k, v) + \frac{1}{\Delta t}(\varepsilon^{k+1} - \varepsilon^k, w) + a(e^{k+1} + \varepsilon^{k+1}, v + w)\\&\quad + b(y^{k+1}, e^{k+1}, v + w) + b(e^{k+1}, y^{k+1}, v + w)\\&\quad + b(e^{k+1}, e^{k+1}, v + w) + b(y^{k+1}, \varepsilon^{k+1}, v) + b(e^{k+1}, z^{k+1}, v)\\&\quad + b(e^{k+1}, \varepsilon^{k+1}, v) + b(z^{k+1}, e^{k+1}, v) + b(\varepsilon^{k+1}, y^{k+1}, v)\\&\quad + b(\varepsilon^{k+1}, e^{k+1}, v) = (\eta^{k+1}, v + w), \quad \forall v \in H_m,\ w \in H_{mn}.\end{aligned} \tag{4.4}$$

Taking $v = \xi^{k+1}$ in (4.1),(4.3) and $v = e^{k+1}, w = \varepsilon^{k+1}$ in (4.2), (4.4), we then derive from (1.4) and (1.6) that for NGE Scheme

$$\frac{1}{2\Delta t}(|\xi^{k+1}|^2 - |\xi^k|^2 + |\xi^{k+1} - \xi^k|^2) + \lambda||\xi^{k+1}||^2 + b(u^k, \xi^k, \xi^{k+1})$$
$$+b(\xi^k, u^k, \xi^{k+1}) + b(\xi^k, \xi^k, \xi^{k+1} - \xi^k) = (\eta^{k+1}, \xi^{k+1}); \tag{4.5}$$

for NGE Scheme

$$\begin{aligned}&\frac{1}{2\Delta t}(|e^{k+1} + \varepsilon^{k+1}|^2 - |e^k + \varepsilon^k|^2 + |e^{k+1} - e^k|^2 + |\varepsilon^{k+1} - \varepsilon^k|^2)\\ &+ \lambda||e^{k+1} + \varepsilon^{k+1}||^2 + b(e^k, y^k, e^{k+1} + \varepsilon^{k+1}) + b(\varepsilon^{k+1}, y^k, e^{k+1})\\ &+ b(e^k, z^{k+1}, e^{k+1}) + b(y^k, e^k, e^{k+1} + \varepsilon^{k+1}) + b(y^k, \varepsilon^{k+1}, e^{k+1})\\ &+ b(z^{k+1}, e^k, e^{k+1}) + b(e^k, e^k, e^{k+1} - e^k) + b(e^k, \varepsilon^{k+1}, e^{k+1} - e^k)\\ &+ b(\varepsilon^{k+1}, e^k, e^{k+1} - e^k) = (\eta^{k+1}, e^{k+1} + \varepsilon^{k+1});\end{aligned} \tag{4.6}$$

for GI Scheme

$$\frac{1}{2\Delta t}(|\xi^{k+1}|^2 - |\xi^k|^2 + |\xi^{k+1} - \xi^k|^2) + \lambda||\xi^{k+1}||^2$$
$$+b(\xi^{k+1}, u^{k+1}, \xi^{k+1}) = (\eta^{k+1}, \xi^{k+1}); \tag{4.7}$$

for NGI Scheme

$$\begin{aligned}&\frac{1}{2\Delta t}(|e^{k+1} + \varepsilon^{k+1}|^2 - |e^k + \varepsilon^k|^2 + |e^{k+1} - e^k|^2 + |\varepsilon^{k+1} - \varepsilon^k|^2)\\ &+ \lambda||e^{k+1} + \varepsilon^{k+1}||^2 + b(e^{k+1}, y^{k+1} + z^{k+1}, e^{k+1})\\ &+ b(e^{k+1}, y^{k+1}, \varepsilon^{k+1}) + b(\varepsilon^{k+1}, y^{k+1}, e^{k+1})\\ =&(\eta^{k+1}, e^{k+1} + \varepsilon^{k+1}).\end{aligned} \tag{4.8}$$

According to (1.4)–(1.7), we obtain

$$\begin{aligned}&|b(u^k, \xi^k, \xi^{k+1}) + b(\xi^k, u^k, \xi^{k+1})|\\ \le&2c_0(|u^k|\ ||u^k||\ |\xi^k|\ ||\xi^k||)^{\frac{1}{2}}||\xi^{k+1}||\\ \le&\frac{\lambda}{4}||\xi^{k+1}||^2 + \frac{\lambda}{16}||\xi^k||^2 + (\frac{4}{\lambda})^3 c_0^4 |u^k|^2 ||u^k||^2 |\xi^k|^2,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&|b(\xi^k, \xi^k, \xi^{k+1} - \xi^k)| \le c_0|\xi^k|\ ||\xi^k||\ ||\xi^{k+1} - \xi^k||\\ \le&\frac{\lambda}{16}||\xi^k||^2 + \frac{\lambda}{4}c_0^4|\xi^k|^2||\xi^{k+1} - \xi^k||^2,\end{aligned} \tag{4.10}$$

$$||\xi^{k+1} - \xi^k||^2 \le \lambda_{m+n}|\xi^{k+1} - \xi^k|^2,$$

$$|(\eta^{k+1}, \xi^{k+1})| \le \frac{\lambda}{8}||\xi^{k+1}||^2 + \frac{2}{\lambda}||\eta^{k+1}||_*^2. \tag{4.11}$$

Combining (4.5) with (4.9)–(4.12), we obtain that for GE Scheme,

$$\begin{aligned}&|\xi^{k+1}|^2 - |\xi^k|^2 + (1 - 8\lambda^{-1}c_0^2\lambda_{m+n}\Delta t|\xi^k|^2)|\xi^{k+1} - \xi^k|^2\\&+ \lambda||\xi^{k+1}||^2\Delta t + \frac{\lambda}{4}\Delta t(||\xi^{k+1}||^2 - ||\xi^k||^2)\\\leq&\frac{4}{\lambda}||\eta^{k+1}||_*^2\Delta t + 2(\frac{4}{\lambda})^3c_0^4|u^k|^2||u^k||^2|\xi^k|^2\Delta t.\end{aligned} \tag{4.12}$$

Moreover, we will infer the similar estimates for NGE Scheme,

$$\begin{aligned}&|e^{k+1} + \varepsilon^{k+1}|^2 - |e^k + \varepsilon^k|^2 + (1 - \frac{40}{\lambda}c_0^2\lambda_m\Delta t|e^k|^2)|e^{k+1} - e^k|^2\\&\quad + \lambda||e^{k+1} + \varepsilon^{k+1}||^2\Delta t + \frac{3}{8}\lambda\Delta t(||e^{k+1}||^2 - ||e^k||^2)\\&\leq\frac{4}{\lambda}||\eta^{k+1}||_*^2\Delta t + \frac{8}{\lambda}(\frac{16}{\lambda})^2c_0^4(|y^k|^2||y^k||^2\\&\quad + |z^{k+1}|^2||z^{k+1}||^2)(|e^k|^2 + |\varepsilon^{k+1}|^2)\Delta t.\end{aligned} \tag{4.13}$$

Similarly, we also have for GI Scheme,

$$|\xi^{k+1}|^2 - |\xi^k|^2 + \lambda||\xi^{k+1}||\Delta t \leq \frac{2}{\lambda}||\eta^{k+1}||_*^2 + \frac{2}{\lambda}c_0^2||u^{k+1}||^2|\xi^{k+1}|^2\Delta t; \tag{4.14}$$

for NGI Scheme,

$$\begin{aligned}&|e^{k+1} + \varepsilon^{k+1}|^2 - |e^k + \varepsilon^k|^2 + \lambda||e^{k+1} + \varepsilon^{k+1}||^2\Delta t\\\leq& (\frac{2}{\lambda}||\eta^{k+1}||_*^2 + \frac{8}{\lambda}c_0^2||y^{k+1} + z^{k+1}||^2|e^{k+1} + \varepsilon^{k+1}|^2)\Delta t.\end{aligned} \tag{4.15}$$

To study the stability of these schemes, let us recall the following discrete Gronwall lemma (see [3]).

Lemma 4.1. *Let $a_n, b_n, c_n, d_n, n \geq 0$, and B be nonnegative numbers such that*

$$a_J + \Delta t\sum_{n=0}^{J} b_n \leq \Delta t\sum_{n=0}^{J} a_n d_n + \Delta t\sum_{n=0}^{J} c_n + B, \quad \forall J \geq 0. \tag{4.16}$$

Suppose that $\Delta t d_n < 1, n \geq 0$, and set $\sigma_n = (1 - \Delta t d_n)^{-1}$, then,

$$a_J + \Delta t\sum_{n=0}^{J} b_n \leq \exp(\Delta t\sum_{n=0}^{J}\sigma_n d_n)\{\Delta t\sum_{n=0}^{J} c_n + B\}, \quad \forall J \geq 0. \tag{4.17}$$

Theorem 4.1. *For any $J \geq 0$ assume that Δt satisfies*

$$\Delta t \leq \min\{\alpha, \frac{1}{4}(\frac{\lambda}{4})^3 c_0^{-4} M^{-4}, \frac{\lambda}{8} c_0^{-2} \theta_J^{-2}\} \lambda_{m+n}^{-1}. \tag{4.18}$$

Then GE Scheme is stable , namely

$$|\xi^J|^2 + \lambda \sum_{k=0}^{J} ||\xi^k||^2 \Delta t \leq \theta_J^2, \tag{4.19}$$

where $\eta^0 = 0$ and

$$\theta_J^2 = \exp(2\Delta t \sum_{k=0}^{J} d_k)\{\frac{4}{\lambda} \sum_{k=0}^{J} ||\eta^k||_*^2 \Delta t + 4|e^0 + \varepsilon^0|^2\},$$

$$d_k = 2(\frac{4}{\lambda})^3 c_0^4 |u^k|^2 ||u^k||^2.$$

Proof. We proceed (4.19) by induction. From (4.18) and $\xi^0 = e^k + \varepsilon^0$, we have

$$|\xi^0|^2 + \lambda ||\xi^0||^2 \Delta t \leq (1 + \lambda \lambda_{m+n} \Delta t_0) |\xi^0|^2 \leq 3|e^0 + \varepsilon^0|^2, \tag{4.20}$$

namely (4.19) holds for $J = 0$.

Assume that

$$(4.19) \text{ holds for} J = 0, 1, \cdots, k, \tag{4.21}$$

we want to prove that (4.19) holds for $J = k + 1$.

According to (4.18) and (4.21)

$$\begin{aligned} &1 - 8\lambda^{-1} c_0^2 \lambda_{m+n} \Delta t |\xi^k|^2 \\ \geq &1 - 8\lambda^{-1} c_0^2 \lambda_{m+n} \Delta t \theta_k^2 \\ \geq &1 - 8\lambda^{-1} c_0^2 \lambda_{m+n} \Delta t \theta_J^2 \geq 0. \end{aligned} \tag{4.22}$$

So, (4.13) and (4.23) give

$$\begin{aligned} &|\xi^{k+1}|^2 - |\xi^k|^2 + \lambda ||\xi^{k+1}||^2 \Delta t + \frac{\lambda}{4} \Delta t (||\xi^{k+1}||^2 - ||\xi^k||^2) \\ \leq &\frac{4}{\lambda} ||\eta^{k+1}||_*^2 \Delta t + d_k |\xi^k|^2 \Delta t. \end{aligned} \tag{4.23}$$

The analogous relations are also valid for the previous values of k, and thus

$$\begin{aligned} &|\xi^{r+1}|^2 - |\xi^r|^2 + \lambda ||\xi^{r+1}||^2 \Delta t + \frac{\lambda}{4} \Delta (||\xi^{r+1}||^2 - ||\xi^r||^2) \\ \leq &\frac{4}{\lambda} ||\eta^{r+1}||_*^2 \Delta t + d_r |\xi^r|^2 \Delta t. \end{aligned} \tag{4.24}$$

By summing (4.25) for $r = 0, 1, \cdots, k$, we obtain

$$
\begin{aligned}
&|\xi^{k+1}|^2 + \lambda \sum_{r=0}^{k+1} ||\xi^r||^2 \Delta t \\
\leq &(1 + \frac{5}{4}\lambda\lambda_{m+n}\Delta t_0)|\xi^0|^2 + \sum_{r=0}^{k+1} d_r |\xi^r|^2 \Delta t + \frac{4}{\lambda}\sum_{r=0}^{k+1} ||\eta^r||_*^2 \Delta t_r \\
\leq &\frac{7}{2}|e^0 + \varepsilon^0|^2 + \sum_{r=0}^{k+1} d_r |\xi^r|^2 \Delta t + \frac{4}{\lambda}\sum_{r=0}^{k+1} ||\eta^r||_*^2 \Delta t.
\end{aligned}
\tag{4.25}
$$

From (4.18), $\Delta t \leq \alpha\lambda_{m+n}^{-1}$. Thus, Theorem 3.1 gives

$$
|u^k|^2 \leq M^2, \qquad \forall k \geq 0. \tag{4.26}
$$

So, (4.18), (4.26) and (1.7) imply

$$
\begin{gathered}
d_r \Delta t \leq 2(\frac{4}{\lambda})^3 c_0^4 \lambda_{m+n} |u^r|^4 \Delta t \leq 2(\frac{4}{\lambda})^3 c_0^4 M^4 \lambda_{m+n} \Delta t \leq \frac{1}{2}, \\
\sigma_r = (1 - d_r \Delta t)^{-1} \leq 2.
\end{gathered}
\tag{4.27}
$$

Applying Lemma 4.1 to (4.25) with

$$
J = k+1,\ a_r = |\xi^r|^2,\ b_r = \lambda ||\xi^r||^2,\ c_r = \frac{4}{\lambda}||\eta^r||_*^2,\ B = \frac{7}{2}|e^0 + \varepsilon^0|^2,
$$

we obtain that (4.19) holds for $J = k+1$.

The induction is complete, (4.19) holds for any $J \geq 0$. □

Theorem 4.2. *If Δt satisfies*

$$
\Delta t \leq \min\{\alpha\lambda_m^{-1}, \frac{2}{\lambda}\lambda_{m+n}^{-1}, \frac{\lambda}{40}c_0^{-2}\theta_J^{-2}\lambda_m^{-1}, \frac{1}{4}(\frac{\lambda}{16})^3 c_0^{-4} M^{-4} \lambda_{m+n}^{-1}\}, \tag{4.28}
$$

then NGE Scheme is stable, namely

$$
|e^J + \varepsilon^J|^2 + \lambda \sum_{k=0}^{J} ||e^k + \varepsilon^k||^2 \Delta t \leq \theta_J^2, \tag{4.29}
$$

where

$$
\theta_J^2 = \exp(2\Delta t \sum_{k=0}^{J} d_r)\{\frac{4}{\lambda}\sum_{k=0}^{J} ||\eta^k||_*^2 \Delta t + 4|e^0 + \varepsilon^0|^2\},
$$

$$
\begin{aligned}
d_k = &\frac{8}{\lambda}(\frac{16}{\lambda})^2 c_0^4 (|y^k|^2 ||y^k||^2 + |y^{k-1}|^2 ||y^{k-1}||^2 \\
&+ |z^k|^2 ||z^k||^2 + |z^{k+1}|^2 ||z^{k+1}||^2).
\end{aligned}
$$

Proof. We prove (4.29) through induction. Obviously, by (4.28), there holds

$$|e^0+\varepsilon^0|^2+\lambda||e^0+\varepsilon^0||^2\Delta t \le 3|e^0+\varepsilon^0|^2, \tag{4.30}$$

namely (4.29) holds for $J=0$. Assume that (4.29) holds for $J=0,1,\cdots,k$, we want to prove that (4.29) holds for $J=k+1$.

According to the induction assumption and (4.28), we have

$$1-\frac{40}{\lambda}c_0^2\lambda_m\Delta t|e^k|_0^2 \ge 1-\frac{40}{\lambda}c_0^2\lambda_m\Delta t\theta_k^2 \ge 1-\frac{40}{\lambda}c_0^2\lambda_m\Delta t\theta_J^2 \ge 0. \tag{4.31}$$

So, (4.20) and (4.31) give

$$\begin{aligned}&|e^{k+1}+\varepsilon^{k+1}|^2-|e^k+\varepsilon^k|^2+\lambda||e^{k+1}+\varepsilon^{k+1}||^2\Delta t\\&+\frac{3}{8}\lambda\Delta t(||e^{k+1}||^2-||e^k||^2)\le\frac{4}{\lambda}||\eta^{k+1}||_*^2\Delta t\\&+\frac{8}{\lambda}(\frac{16}{\lambda})^2c_0^4(|y^k|^2||y^k||^2+|z^{k+1}|^2||z^{k+1}||^2)(|e^k|^2+|\varepsilon^{k+1}|^2)\Delta t.\end{aligned} \tag{4.32}$$

The analogous relations are valid for the previous values of k, and thus we derive from (4.32) that

$$\begin{aligned}&|e^{k+1}+\varepsilon^{k+1}|^2+\lambda\sum_{r=0}^{k+1}||e^r+\varepsilon^r||^2\Delta t\\\le&|e^0+\varepsilon^0|^2+\frac{11}{8}\lambda\Delta t||e^0+\varepsilon^0||^2+\frac{4}{\lambda}\sum_{r=0}^{k+1}||\eta^r||_*^2\Delta t+\sum_{r=0}^{k+1}d_r|e^r+\varepsilon^r|^2\Delta t.\end{aligned} \tag{4.33}$$

From (4.29) and Theorem 3.2,

$$\begin{aligned}d_r\Delta t=&\frac{8}{\lambda}(\frac{16}{\lambda})^2c_0^4(|y^r|^2||y^r||^2+|z^r|^2||z^r||^2\\&+|y^{r-1}|^2||y^{r-1}||^2+|z^{r+1}|^2||z^{r+1}||^2)\Delta t\\\le&(\frac{16}{\lambda})^3c_0^4M^4(\lambda_m+\lambda_{m+n})\Delta t\le\frac{1}{2},\end{aligned} \tag{4.34}$$

$$\sigma_r=(1-d_r\Delta t)^{-1}\le 2, \tag{4.35}$$

$$\frac{11}{8}\lambda\Delta t||e^0+\varepsilon^0||^2\le\frac{11}{8}\lambda\Delta t\lambda_{m+n}|e^0+\varepsilon^0|^2\le 3|e^0+\varepsilon^0|^2. \tag{4.36}$$

Applying Lemma 4.1, we get (4.29) from (4.33) and (4.29) holds for $J=k+1$.

Induction is complete, (4.29) holds for any $J\ge 0$. □

Finally, by the very similar method, we obtain the stability of GI Scheme and NGI Scheme.

Theorem 4.3. *If Δt satisfies*

$$\Delta t \leq \frac{\lambda}{16} c_0^{-2} \lambda_{m+n}^{-1}, \tag{4.37}$$

then the GI Scheme and NGI Scheme are stable, namely

$$|\xi^J|^2 + \lambda \sum_{k=0}^{J} ||\xi^J||^2 \Delta t \leq \theta_J^2, \tag{4.38}$$

$$|e^J + \varepsilon^J|^2 + \lambda \sum_{k=0}^{J} ||e^k + \varepsilon^k||^2 \Delta t \leq \theta_J^2, \tag{4.39}$$

where

$$\theta_J^2 = \exp(2\Delta t \sum_{k=0}^{J} d_r)\{\frac{1}{\lambda} \sum_{k=0}^{J} ||\eta^k||_*^2 \Delta t + (1 + (\frac{\lambda}{4c_0})^2)|e^0 + \varepsilon^0|^2\},$$

$$d_k = \frac{2}{\lambda} c_0^2 ||u^k||^2 \quad \textit{for GI Scheme,}$$

$$d_r = \frac{8}{\lambda} c_0^2 ||y^k + z^k||^2 \quad \textit{for GI Scheme.}$$

5. Conclusions

According to the definition of θ_J, we know that for any $J \geq 0$, $\theta_{J+1} \geq \theta_J$ and $\lim_{j \to \infty} \theta_J = +\infty$. So, there exists an integer number $J_0 > 0$ such that for any $J \geq J_0$,

$$\frac{\lambda}{8} c_0^{-2} \theta_J^{-2} \leq \min\{\alpha, \frac{1}{4}(\frac{\lambda}{4})^3 c_0^{-4} M^{-4}\}, \tag{5.1}$$

$$\frac{\lambda}{40} c_0^{-2} \theta_J^{-2} \lambda_m^{-1} \leq \min\{\alpha \lambda_m^{-1}, \frac{2}{\lambda} \lambda_{m+n}^{-1}, \frac{1}{4}(\frac{\lambda}{16})^3 c_0^{-4} M_0^{-4} \lambda_{m+n}^{-1}\}. \tag{5.2}$$

Thus, by (4.19) and (4.28), the stability condition of GE Scheme is

$$\Delta t \leq \frac{\lambda}{8} c_0^{-2} \theta_J^{-2} \lambda_{m+n}^{-1}, \tag{5.3}$$

and the stability condition of NGE Scheme is

$$\Delta t \leq \frac{\lambda}{40} c_0^{-2} \theta_J^{-2} \lambda_m^{-1}, \tag{5.4}$$

for any $J \geq J_0$. So, we find that the stability condition of NGE Scheme is better than that of GE Scheme.

Like the above discussions , we have

Theorem 5.1. *If Δt satisfies the stability conditions, then Δt satisfies the boundedness conditions for the above four schemes.*

Moreover, for the implicit scheme, the boundedness and stability of the nonlinear Galerkin method is the same as that of the Galerkin method; for the explicit scheme, the boundedness and stability of the nonlinear Galerkin method is better than that of the Galerkin method.

Thus, we conclude that the nonlinear Galerkin method is better than Galerkin method in the sense of the boundedness and stability.

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