The Partial Differential Equation
$u_t + uu_x = \mu_{xx}$*

By EBERHARD HOPF
Department of Mathematics, Indiana University

1. *Introduction*

In the last decades, mathematicians have become increasingly interested in problems connected with the behavior of the solutions of partial differential space-time systems in which the highest order terms occur linearly with small coefficients. These problems have originated from physical applications, mainly from modern fluid dynamics (compressible fluids of small viscosity μ and of small heat conductivity λ). Research in these fields has led to some general mathematical conjectures, such as the following two: The solution of the initial value problem (the solution is prescribed at $t = 0$) for the general equations of fluid flow tends in general, i.e. for "most" values of the space-time-coordinates, towards a limit function as $\mu \to 0$ and $\lambda \to 0$. The limit function is, in general, discontinuous and pieced together by solutions of the system in which those highest order coefficients have the value zero (ideal fluid with contact- and shock-discontinuities). These conjectures are probably valid in a much wider range of partial differential systems. The second one is restricted to non-linear systems, but it seems to point out a typical occurrence in this general case. Exact formulation and rigorous proof of these conjectures are still tasks for the future. These problems are closely tied up with the present or future theory of functional spaces. Continued study of special problems is still a commendable way towards greater insight into this matter.

Among the partial differential systems studied in these directions, we have never met one in which the totality of its solutions has been rigorously determined and in which those limit problems can thus be studied in all detail. In this paper we present such a complete solution for the case of the equation

$$u_t + uu_x = \mu u_{xx}, \qquad \mu > 0. \tag{1}$$

It was first introduced by J. M. Burgers[1] as the simplest model for the differ-

---

*Prepared under Navy Contract N6onr-180, Task Order No. 5, with Indiana University.

[1](a) Application of a model system to illustrate some points of the statistical theory of free turbulence, *Proc. Acad. Sci. Amsterdam*, Volume 43 (1940), pp. 2–12. (b) A mathematical model illustrating the theory of turbulence; *Advances in Applied Mechanics*, edited by R. v. Mises and T. v. Kármán, Volume 1, 1948, pp. 171–199, in particular pp. 182–184.

Burgers treats in (b) the equation $v_t + 2vv_y = \nu v_{yy}$ which goes over into equation (1) by ential equations of fluid flow. There is a close analogy between the left of (1) and the terms $u_{i,t} + u_\nu u_{i,\nu}$, and between the right of (1) and the viscosity terms $\mu u_{i,\nu\nu}$ of those equations. However, no additional dependent variables such as pressure, density or temperature appear in (1). Nonetheless Burgers observed analogies between certain solutions of (1) and certain one-dimensional flows of a compressible fluid. He had an intuitive picture of the limit case $\mu \to 0$ in the solutions of (1) and determined the origin and the law of propagation of a discontinuity. Like Burgers we study the boundary-free initial value problem: u given for all x at $t = 0$, u wanted for all x and all $t > 0$. The solution is achieved by an exact integration of (1).[2] Both problems, the behavior of the solutions as $t \to \infty$ while μ stays constant, and their behavior as $\mu \to 0$ while the initial values are kept fixed, are solved. The second problem turns out to be connected with the support lines of a plane (in general non-convex) curve. A fact brought out by the discussion is that the reversal of order in the successive limit passages $t \to \infty$, $\mu \to 0$ leads to different results. From a general point of view this is not surprising, but it is a reminder that, for the turbulence problem of hydrodynamics (behavior of the flow of a slightly viscous fluid as $t \to \infty$), the relevant order is the one stated. The limit function obtained if $\mu \to 0$ is in general discontinuous. Under the wide assumptions we have made about the initial values, the points x, t of discontinuity could be everywhere dense; but we show that they are always lined up along curves $x(t)$ that have certain differentiability properties (lines of discontinuity) and that the points of continuity on the limit surface $u = u(x, t)$ are lined up along characteristics of (1), $\mu = 0$. A line of discontinuity can only originate but never terminate as t increases. That the irreversibility is retained in the limit case $\mu \to 0$ of the solutions was already duly emphasized by Burgers (dissipation of energy in discontinuities). This is analogous to the apparent irreversibility of the general flow of an ideal fluid with its shock-discontinuities.

A question of general interest (considered in Section 8) is that of the natural functional equations which determine the limit functions of the solutions ($\mu \to 0$, $\lambda \to 0$ in fluid flow, $\mu \to 0$ in the solutions of (1)) directly and independently of that limit passage in the solutions of the higher order system. The answer to this question is a prerequisite to a mathematical theory of the general initial value problem of ideal fluid flow. We use the case of (1) to illustrate the answer. We put (1) into the following form (the double integrals are extended over the semiplane $t > 0$): The equation

---

the substitution $u = 2v$, $x = y$, $\mu = \nu$. We doubt that Burgers' equation fully illustrates the statistics of free turbulence. Kolmogoroff's idea about the probability distribution of the turbulent fluctuations in the small is essentially concerned with the velocity differences, not the velocities themselves. Equation (1) is too simple a model to display chance fluctuations of these differences.

[2]The reduction of (1) to the heat equation was known to me since the end of 1946. However, it was not until 1949 that I became sufficiently acquainted with the recent development of fluid dynamics to be convinced that a theory of (1) could serve as an instructive introduction into some of the mathematical problems involved.

$$\iint \left[uf_t + \frac{u^2}{2} f_x + \mu u f_{xx} \right] dx\, dt = 0$$

is to be satisfied by each function $f(x, t)$ of class C'' in $t > 0$ that vanishes outside some circle contained in $t > 0$. It can be shown that this problem has essentially the same solutions as (1) provided that $\mu > 0$. Nothing is thus gained by this transformation of the differential problem. In Section 8 we rigorously prove, however, that the limit functions u of the solutions of (1) obtained as $\mu \to 0$ satisfy the relation

$$\iint \left[uf_t + \frac{u^2}{2} f_x \right] dx\, dt = 0$$

(for each f as specified) which is simply the case $\mu = 0$ of the preceding relation. While the case $\mu = 0$ of the differential equation (1) cannot completely determine the limit functions, the indicated integral formulation of the problem grasps them. It is probable (though not proved by us) that, under very general assumptions about u, those limit functions are the only solutions of the problem in $t > 0$.

Analogous considerations must hold in a much wider class of such limit problems and certainly in the case of fluid dynamics. This concept of generalized solutions of differential systems is not new in itself. It has been successfully applied to other differential problems. But the modern concept of ideal fluid flow with all its discontinuities is perhaps the most striking example in which the classical differential description is insufficient and which demands the integral form of the fundamental equations for its complete and general description.

2. *The explicit solution for* $\mu > 0$

We introduce a new dependent variable $\varphi = \varphi(x, t)$ into Burgers' equation

$$u_t = \left(\mu u_x - \frac{u^2}{2} \right)_x \tag{2}$$

by means of the substitution

$$\varphi = \exp\left\{ -\frac{1}{2\mu} \int u\, \partial x \right\} \tag{3}$$

whose inverse is

$$u = -2\mu(\log \varphi)_x = -2\mu\, (\varphi_x/\varphi). \tag{4}$$

Then, (1) becomes

$$-2\mu(\log \varphi)_{xt} = -2\mu(\log \varphi)_{tx} = -2\mu\left(\frac{\varphi_t}{\varphi}\right)_x = -\left(2\mu^2\, \frac{\varphi_{xx}}{\varphi}\right)_x,$$

or upon integration with respect to x

$$\varphi_t = \mu\varphi_{xx} + C(t)\varphi,$$

where C is a suitable function of t only. If

$$\varphi \cdot \exp\left\{-\int C\,dt\right\}$$

is introduced as a new dependent variable φ one simply obtains the heat equation

$$\varphi_t = \mu\varphi_{xx}\,. \tag{5}$$

Precisely stated: If u solves (1) in an open rectangle R of the x, t-plane and if u, u_x, u_{xx} are continuous in R then there exists a positive function φ of the form (3) that solves the heat equation in R and for which φ, φ_x, φ_{xx}, φ_{xxx} are continuous in R. One easily shows that, conversely, every positive solution φ of (5) with the mentioned properties goes, by means of (4), over into a solution of (1) of the described general type. Let us call a function u that solves (1) in an x, t-domain D *a regular solution in D* if u, u_x, u_{xx} (and consequently u_t) are continuous in D.

Theorem 1. *Suppose that $u_0(x)$ is integrable in every finite x-interval and that*

$$\int_0^x u_0(\xi)\,d\xi = o(x^2) \tag{6}$$

for $|x|$ large. Then

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\left\{-\frac{1}{2\mu}F(x,y,t)\right\} dy}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\mu}F(x,y,t)\right\} dy}, \tag{7}$$

where

$$F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(\eta)\,d\eta, \tag{8}$$

is a regular solution of (1) *in the half plane $t > 0$ that satisfies the initial condition*

$$\int_0^x u(\xi,t)\,d\xi \to \int_0^a u_0(\xi)\,d\xi \qquad \text{as} \qquad x \to a,\ t \to 0, \tag{9}$$

for every a. If, in addition, $u_0(x)$ is continuous at $x = a$ then

$$u(x,t) \to u_0(a) \qquad \text{as} \qquad x \to a,\ t \to 0. \tag{10}$$

A solution of (1) *which is regular in some strip $0 < t < T$ and which satisfies* (9) *for each value of the number a necessarily coincides with* (7) *in the strip.*

The condition (6) on the initial values merely insures the existence of the solution for all $t > 0$. The weaker condition

$$\int_0^x u_0(\xi)\, d\xi = O(x^2)$$

entails the existence and regularity of the solution only in some finite strip $0 < t < T$. The example of the solution $u = x/(t - T)$ of (1) shows that this restriction of the conclusion is natural. The exact hydrodynamic equations for a homogeneous and incompressible fluid have, by the way, analogous solutions $u_i = a_{i\nu}x_\nu/(t - T)$, $a_{ik} = a_{ki}$, $a_{\nu\nu} = 0$. Note the generality and naturalness of the uniqueness statement. No restriction is imposed upon the behavior of $u(x, t)$ for $|x|$ large except, of course, in the initial case $t = 0$.

Proof of Theorem 1. The continuous function

$$\varphi_0(y) = \exp\left\{-\frac{1}{2\mu}\int_0^y u_0(\eta)\, d\eta\right\} \tag{11}$$

is, in virtue of (6), of the order

$$\varphi_0(y) = e^{o(x^2)}$$

for large $|x|$. The integral

$$\varphi(x, t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty} \varphi_0(y) \exp\left\{-\frac{(x-y)^2}{4\mu t}\right\} dy \tag{12}$$

therefore converges for all x and all $t > 0$. The formal derivatives of any order with respect to x, t are again absolutely convergent integrals for $t > 0$ which represent continuous functions of x, t. Therefore φ has, in $t > 0$, continuous partial derivatives of any order which are obtained by differentiation under the integral sign. It is well known that, for $t > 0$, $\varphi(x, t)$ solves (5) and satisfies the initial condition

$$\lim_{\substack{x=a\\ t=0}} \varphi(x, t) = \varphi_0(a)$$

for every a. For $t > 0$, φ is positive. The function (7)

$$u(x, t) = -2\mu\frac{\varphi_x(x, t)}{\varphi(x, t)}$$

solves, therefore, (1) in the half plane $t > 0$ and has continuous partial derivatives of all orders. It evidently satisfies (9) for each value of a.

Suppose, now, that in addition to (6) $u_0(x)$ is continuous at $x = a$. The formula

$$\varphi_x(x, t) = \frac{1}{\sqrt{4\pi\mu t}}\int_{-\infty}^{\infty} \varphi_0'(y) \exp\left\{-\frac{(x-y)^2}{4\mu t}\right\} dy \tag{13}$$

is valid for all x and all $t > 0$. It can, for instance, be obtained from (12) by differentiating under the integral sign and integrating by parts the differentiated integral. Integration by parts is legitimate under the assumption that $u_0(x)$ is integrable in every x-interval and that it satisfies (6). If, at $x = a$, $u_0(x)$ is continuous then $\varphi_0'(x)$ is continuous also. Hence

$$\lim_{\substack{x=a\\t=a}} \varphi_x(x, t) = \varphi_0'(a)$$

and therefore

$$u(x, t) = -2\mu \frac{\varphi_x(x, t)}{\varphi(x, t)} \to -2\mu \frac{\varphi_0'(a)}{\varphi_0(a)} = u_0(a)$$

as $x \to a$, $t \to 0$.

It remains to prove the uniqueness statement. Suppose that $u(x, t)$ is a regular solution of (1) in $0 < t < T$ and that it satisfies (9) for each value of a. We know that

$$u = -2\mu(\varphi_x/\varphi) \tag{14}$$

where $\varphi = \varphi(x, t)$,

$$\varphi(x, t) = \varphi(0, t)\psi(x, t), \qquad \psi(x, t) = \exp\left\{-\frac{1}{2\mu}\int_0^x u(\xi, t)\, d\xi\right\}, \tag{15}$$

is a positive and regular solution of (5) in this strip. $\varphi(0, t)$ is continuous and positive for $t > 0$. That it has a positive limit as $t \to 0$ is not obvious. This we prove in the following way. From (9) we infer that

$$\lim_{t=0} \psi(y, t) = \psi_0(y) = \exp\left\{-\frac{1}{2\mu}\int_0^y u_0(\xi)\, d\xi\right\} \tag{16}$$

holds uniformly in every finite y-interval. We apply a theorem on non-negative solutions of the heat equation, which was proved by D. V. Widder[3] in 1944. A solution of (5) which is positive and regular in a strip $\alpha < t < \beta$ and continuous on $t = \alpha$ is uniquely determined by the values on $t = \alpha$ and is, in the strip, represented by the classical integral formula. Applying this theorem to $\varphi(x, t + \epsilon)$ in the strip $0 < t < T - \epsilon$, $\epsilon > 0$, we find that

$$\varphi(0, t + \epsilon)\psi(x, t + \epsilon) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} \varphi(0, \epsilon)\psi(y, \epsilon) \exp\left\{-\frac{(x - y)^2}{4\mu t}\right\} dy \tag{17}$$

holds for $0 < t < T$. Let

$$\varphi_0 = \limsup_{\epsilon=0} \varphi(0, \epsilon), \qquad 0 \leqq \varphi_0 \leqq \infty,$$

[3] D. V. Widder, Positive temperatures on an infinite rod, *Trans. Am. Math. Soc.*, Volume 55 (1944), pp. 85–95.

and let ϵ_ν be a sequence of positive numbers satisfying

$$\epsilon_\nu \to 0, \qquad \varphi(0, \epsilon_\nu) \to \varphi_0 . \tag{18}$$

Let now $\epsilon = \epsilon_\nu$ and $\nu \to \infty$ in (17) while x, t are kept fixed. Since

$$\int_{-\infty}^{\infty} \geqq \int_{-A}^{A}$$

we infer from the uniformity of (16) in the interval $(-A, A)$, which we keep fixed, and from (18) that

$$\varphi(0, t)\psi(x, t) \geqq \frac{\varphi_0}{\sqrt{4\pi\mu t}} \int_{-A}^{A} \psi_0(y) \exp\left\{-\frac{(x-y)^2}{4\mu t}\right\} dy. \tag{19}$$

This shows that $0 \leqq \varphi_0 < \infty$. Inequality (19) holds for arbitrary A and the integral converges, in view of (6), also for $A = \infty$. Hence (19) is true for $A = \infty$. As $\psi_0(y)$ is continuous, the right hand side converges, for a fixed value of x, towards $\varphi_0\psi_0(x)$ as $t \to 0$. Hence, applying (16) to the left hand side of (19), we conclude that

$$[\liminf_{t=0} \varphi(0, t)]\psi_0(x) \geqq \varphi_0\psi_0(x).$$

Remembering the definition of φ_0, we see that

$$\lim_{t=0} \varphi(0, t) = \varphi_0$$

exists. According to (15) the non-negative solution $\varphi(x, t)$ of (5) is therefore continuous for $0 \leqq t < T$. The application of Widder's theorem to the full strip $0 < t < T$ is now permitted. The result is that $\varphi(x, t)$ is uniquely determined by the initial values $\varphi_0\psi_0(x) = \text{const.}\,\psi_0(x)$ where $\psi_0(x)$ is a given function. $\varphi(x, t) > 0$ now implies that $\varphi_0 > 0$. $\varphi(x, t)$ is, up to a constant factor, uniquely determined. $u(x, t)$ is, therefore, completely unique.

3. *Behavior of the solutions as $t \to +\infty$, $\mu > 0$*

In this section we consider only initial data $u_0(x)$ for which the integral

$$M = \int_{-\infty}^{\infty} u_0(x)\, dx$$

exists as a sum of two improper integrals: $\int_0^\infty + \int_{-\infty}^0$. Burgers calls it the moment (at time $t = 0$) of the velocity distribution. The existence of M means for the function $\varphi_0(x)$, defined by (4), that the limits $\varphi_0(-\infty)$ and $\varphi_0(\infty)$ both exist (they are > 0) and that

$$2\mu \log \frac{\varphi_0(-\infty)}{\varphi_0(\infty)} = M. \tag{20}$$

The solution $\varphi(x, t)$ of (5) with the initial values $\varphi_0(x)$ can be expressed by the equivalent formula

$$\varphi(\bar{x}\sqrt{2\mu t}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_0(\sqrt{2\mu t}\, y) \exp\left\{-\frac{(\bar{x}-y)^2}{2}\right\} dy \tag{21}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_0(\sqrt{2\mu t}\,(\bar{x}-y)) \exp\left\{-\frac{y^2}{2}\right\} dy.$$

If we let $\bar{x} \to \infty$ while t is kept fixed, the extreme right hand side evidently converges toward

$$\frac{\varphi_0(\infty)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\frac{y^2}{2}\right\} dy = \varphi_0(\infty).$$

An analogous result is found if $\bar{x} \to -\infty$. The limits $\varphi(-\infty, t)$, $\varphi(\infty, t)$ therefore exist and their values are independent of t. By virtue of (4) it follows: *The solution $u(x, t)$ of (1) with the initial values $u_0(x)$ in the sense of theorem 1 possesses a moment*

$$\int_{-\infty}^{\infty} u(x, t)\, dx$$

for every $t > 0$ if it possesses one for $t = 0$ (i.e. for $u = u_0$). The value of the moment is that assumed at $t = 0$. The moment is an "integral" of the differential equation (1).

It is easily seen what happens in the limit $t \to \infty$ as $\bar{x}$ and μ stay fixed. If the last integral in (21) is split up into

$$\int_{-\infty}^{\infty} = \int_{-\infty}^{\bar{x}-\epsilon} + \int_{\bar{x}+\epsilon}^{\infty} + \int_{\bar{x}-\epsilon}^{\bar{x}+\epsilon}, \qquad \epsilon > 0,$$

the φ_0-factor of the integrand behaves in the following way: In the first integral it tends uniformly toward $\varphi_0(\infty)$, in the second it tends uniformly to $\varphi_0(-\infty)$, while in the third term it is bounded by an upper bound for $\varphi_0(x)$. From this it easily follows that

$$\sqrt{2\pi} \lim_{t=\infty} \varphi(\bar{x}\sqrt{2\mu t}, t) \tag{22}$$
$$= \varphi_0(\infty) \int_{-\infty}^{\bar{x}} \exp\left\{-\frac{y^2}{2}\right\} dy + \varphi_0(-\infty) \int_{\bar{x}}^{\infty} \exp\left\{-\frac{y^2}{2}\right\} dy$$

holds uniformly in $\bar{x}$. The limit relation differentiated with respect to $\bar{x}$,

$$\lim_{t=\infty} \sqrt{4\pi\mu t}\, \varphi_x(\bar{x}\sqrt{2\mu t}, t) = [\varphi_0(\infty) - \varphi_0(-\infty)] \exp\left\{-\frac{\bar{x}^2}{2}\right\}, \tag{23}$$

also is readily shown to hold uniformly for all $\bar{x}$. One has only to differentiate the first integral in (21) and then to transform it by the substitution $y \to \bar{x} - y$.

The resulting integral is handled again like the second integral of (21). Since (22) is positively bounded from below, the limit

$$\lim_{t\to\infty} \sqrt{2\mu t}\, \frac{\varphi_x(\bar{x}\, \sqrt{2\mu t},\, t)}{\varphi(\bar{x}\, \sqrt{2\mu t},\, t)}$$

exists uniformly for all $\bar{x}$. This limit relation can again be differentiated any number of times, and the limit relations obtained hold uniformly with respect to $\bar{x}$ as is shown by perfectly analogous reasoning. According to (4) we have hereby proved the following:

Theorem 2. *Let $u(x, t)$ be a regular solution of* (1), $t > 0$, *of finite moment* $M = 2\mu K$. *Let*

$$\bar{x} = \frac{x}{\sqrt{2\mu t}}, \qquad \bar{u} = \sqrt{\frac{t}{2\mu}}\, u, \qquad \bar{u} = \bar{u}(\bar{x}\,, t), \tag{24}$$

and

$$G(\xi) = e^{-\frac{K}{2}} \int_{-\infty}^{\xi} e^{-\frac{y^2}{2}}\, dy + e^{\frac{K}{2}} \int_{\xi}^{\infty} e^{-\frac{y^2}{2}}\, dy. \tag{25}$$

Then the limit relation

$$\lim_{t\to\infty} \bar{u}(\bar{x},\, t) = -\frac{G'(\bar{x})}{G(\bar{x})} \tag{26}$$

holds uniformly with respect to $\bar{x}$. This relation may be differentiated any number of times with respect to $\bar{x}$ and the differentiated relation again holds uniformly with respect to $\bar{x}$.

The transformation (24), if coupled with

$$\bar{t} = \log t, \tag{24'}$$

carries Burgers' equation over into

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\, \frac{\partial \bar{u}}{\partial \bar{x}} = \frac{1}{2}\, \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + X \tag{27}$$

where

$$X = \frac{1}{2}\left(\bar{u} + \bar{x}\, \frac{\partial \bar{u}}{\partial \bar{x}}\right). \tag{27'}$$

The moment of a solution becomes

$$\int_{-\infty}^{\infty} \bar{u}(\bar{x},\, t)\, d\bar{x} = \frac{M}{2\mu} = K.$$

Theorem 2 simply asserts that any solution of (27), (27′) of finite moment K tends toward the time-independent solution of this equation with the same moment K. These stationary solutions form a family with exactly one member for every value of K.

Precisely the same transformation (24), (24') of the coordinates and velocities carriers the Navier-Stokes equations of a homogeneous ($\rho = 1$) and incompressible fluid occupying the entire space

$$\frac{\partial u_i}{\partial t} + u_\nu \frac{\partial u_i}{\partial x_\nu} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_\nu \partial x_\nu}, \qquad \frac{\partial u_\nu}{\partial x_\nu} = 0$$

over into

$$\frac{\partial \bar{u}_i}{\partial \bar{t}} + \bar{u}_\nu \frac{\partial \bar{u}_i}{\partial \bar{x}_\nu} = -\frac{\partial \bar{p}}{\partial \bar{x}_i} + \frac{1}{2} \frac{\partial^2 \bar{u}_i}{\partial \bar{x}_\nu \partial \bar{x}_\nu} + X_i, \qquad \frac{\partial \bar{u}_\nu}{\partial \bar{x}_\nu} = 0$$

where

$$X_i = \frac{1}{2}\left(\bar{u}_i + \bar{x}_\nu \frac{\partial \bar{u}_i}{\partial \bar{x}_\nu}\right).$$

The transformation is of a similar importance for the spreading of a turbulent disturbance. However, the hydrodynamic case presents additional complications with no analogue in Burgers' oversimplified case.

It is of interest to uncover the mathematical reason for the significance of the transformation. The transformed equation $\partial \bar{u}/\partial \bar{t} = \cdots$ has the property that the right hand side does not explicitly contain the time variable in common with the original equation $\partial u/\partial t = \cdots$. In general this is not the case if the equation is transformed by an arbitrary transformation

$$\bar{x}(x, t, u), \qquad \bar{t}(x, t, u), \qquad \bar{u}(x, t, u).$$

Why it is the case for the special transformation (24), (24') is made plain by the following observation which applies to the hydrodynamical case as well. The statement that the right hand side of $\partial \bar{u}/\partial \bar{t} = \cdots$ does not contain $\bar{t}$ explicitly is synonymous with the statement that the equation is invariant under the substitutions (indicated by an arrow)

$$\bar{t} \to \bar{t} + a. \tag{28}$$

Now (1) is carried into itself by the well-known group of substitutions

$$\begin{aligned} x &\to cx \\ t &\to c^2 t \\ u &\to c^{-1} u. \end{aligned} \tag{29}$$

If we introduce variables $\bar{x}$, $\bar{t}$, $\bar{u}$, then the transformed equation (1) must be invariant under the group (29) expressed in these new variables. The transformed equation will not contain $\bar{t}$ explicitly, if we choose $\bar{x}$, $\bar{t}$, $\bar{u}$ such that (29) has, in these new variables, the form of a translation in $\bar{t}$

$$\bar{x} \to \bar{x}$$

$$\bar{t} \to \bar{t} + a$$

$$\bar{u} \to \bar{u}.$$

This is obviously accomplished by choosing

$$\bar{x} = x/\sqrt{t}, \qquad \bar{u} = u\sqrt{t}, \qquad \bar{t} = \log t \tag{30}$$

which, essentially, is the transformation (24), (24′).

Using transformation (30) (which does not contain μ) rather than (24) our limit relation becomes

$$\lim_{t=\infty} \sqrt{t}\, u(\bar{x}\sqrt{t}, t) = -\sqrt{2\mu}\, \frac{G'(\bar{x}/\sqrt{2\mu})}{G(\bar{x}/\sqrt{2\mu})} = -2\mu \frac{d}{d\bar{x}} \log G(\bar{x}/\sqrt{2\mu}), \tag{31}$$

where $K = M/2\mu$ and where M is the moment of the solution $u(x, t)$ of (1). The right hand side of (31), as a time-independent solution of equation (1) transformed by (30), had been studied by Burgers.[4] He found that its limit as $\mu \to 0$, while M stays fixed, is the discontinuous function $\alpha f(\bar{x}/\alpha)$ where $\alpha \mid \alpha \mid = 2M$ and

$$f(\xi) = \begin{cases} 0, & \xi < 0, \\ \xi, & 0 < \xi < 1, \\ 0, & \xi > 1. \end{cases}$$

4. *Passage to the limit* $\mu \to 0$

We proceed to treat rigorously the behavior of the solutions $u(x, t; \mu)$ of (1) as $\mu \to 0$ while the initial values $u_0(x)$ are kept fixed. It will be shown that the limit function $u(x, t; 0)$ exists, but is in general discontinuous. Away from the discontinuities it fulfills essentially differential equation (1), $\mu = 0$,

$$u_t + uu_x = 0. \tag{32}$$

More precisely, the characteristic equations of (32)

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0 \tag{33}$$

(which, under somewhat stricter assumptions as to differentiability, are equivalent to (32)) are strictly fulfilled. A simple formula is derived that expresses $u(x, t; 0)$ in terms of $u_0(x)$ and that comprises, in particular, the results which Burgers obtained by intuitive reasoning.

[4] See Burgers, footnote 1(b), p. 183.

The formula makes use of certain properties of the function $F(x, y, t)$, $t > 0$, of (8) which we have to derive first. The properties refer to F as a function of y (which is the variable of integration in (7)) with x, t being fixed. F is continuous in y and has, according to (6), the property

$$\frac{F}{y^2} \to \frac{1}{2t} > 0, \qquad |y| \to \infty. \tag{34}$$

Hence F attains its smallest value for one or several values of y, the smallest and the largest of which are denoted by y_* and y^*, respectively,

$$y_*(x, t) \leqq y^*(x, t).$$

Lemma 1. *The functions* y_* *and* y^* *have the properties*

(a) $y^*(x, t) \leqq y_*(x', t)$ if $x < x'$,

(b) $y_*(x - 0, t) = y_*(x, t), \quad y^*(x + 0, t) = y^*(x, t)$,

(c) $y_*(+\infty, t) = +\infty, \quad y^*(-\infty, t) = -\infty$.

As a function of y,

$$G(x, y, t) = F(x, y, t) - x^2/2t \tag{35}$$

attains its absolute minimum at the same values of y as F. Since

$$G(x, y, t) = G(0, y, t) - xy/t \tag{36}$$

the lemma expresses properties of the points in the y, z-plane which the curve $z = G(0, y, t)$, with $t > 0$ fixed, has in common with the line of support $z = xy/t + c$ of a given slope x/t. As we saw, such a line that supports the curve from below exists for every value of the slope. $y_*(x, t)$ and $y^*(x, t)$ are the smallest and largest y-coordinates of the points which the line of support has in common with the curve.

Proof of the lemma. According to the definition of y^*

$$G(x, y, t) - G(x, y^*, t) \begin{cases} \geqq 0, & y < y^*, \\ > 0, & y > y^*, \end{cases} \tag{37}$$

where

$$y^* = y^*(x, t). \tag{38}$$

From (36) one obtains

$$G(x + a, y, t) - G(x + a, y^*, t) = G(x, y, t) - G(x, y^*, t) - \frac{a}{t}(y - y^*). \tag{39}$$

We identify y^* with (38) and let

$$a > 0.$$

Now the upper inequality (37) shows that the right hand side of (39) and, therefore, the left hand side, is positive for all $y < y^*$. As it vanishes at $y = y^*$ we infer that $G(x + a, y, t)$ as a function of y can attain its smallest value only for $y \geqq y^*$. In virtue of (35) this establishes property (a) if we let $x' = x + a$.

According to the lower inequality (37),

$$\frac{G(x, y, t) - G(x, y^*, t)}{y - y^*}$$

is a positive and continuous function of y if $y > y^*$. By (34) and (35) this expression tends to $+\infty$ as $y \to +\infty$. Hence, if $\epsilon > 0$ is chosen arbitrarily, this function has a positive minimum for $y \geqq y^* + \epsilon$. $a > 0$ can, therefore, be chosen such that the right hand side of (39) is positive whenever $y > y^* + \epsilon$. With this fixed value of a, the function of y on the left of (39) is thus positive outside the interval $y^* \leqq y \leqq y^* + \epsilon$ and vanishes at $y = y^*$. Hence $G(x + a, y, t)$ as a function of y can reach its minimum only in this interval, in other words $y^*(x + a, t)$ is not greater than $y^*(x, t) + \epsilon$ if $a > 0$ is sufficiently small. This proves the second of the properties (b). The first property (b) follows immediately by applying the second one to the function $F(-x, -y, t)$.

Only the first property (c) needs to be proved. Let m denote the smallest value of $G(0, y, t)$ for a fixed $t > 0$. The function of y

$$G(0, y, t) - m - \frac{x}{t}(y - A), \qquad \frac{x}{t} = G(0, A + 1, t) - m$$

is $\geqq 0$ for $y < A$ and $= 0$ for $y = A + 1$. Hence it attains its smallest value at some $y \geqq A$. According to (36), the function $G(x, y, t)$ where x has the indicated value must have the same property, in other words $y^*(x, t) \geqq A$ must hold for this value of x. Since A is arbitrary, it follows that $y^*(x, t)$ as a function of x takes on arbitrarily large values. The desired property then follows from the fact that $y^*(x, t)$ is a nowhere decreasing function of x and from property (a). This concludes the proof of lemma 1.

(a) and (b) signify that $y_*(x, t)$ and $y^*(x, t)$ are two nowhere decreasing functions of x with the same points x of discontinuity and with the same limit value on the left of x and the same limit value on the right of x. These limit values are $y_*(x, t)$ and $y^*(x, t)$ respectively. From the fact that a monotonic function has only a denumerable set of discontinuities, we infer: at any given moment $t > 0$, $y_*(x, t) = y^*(x, t)$ holds for all x with the possible exception of a denumerable set of values of x where $y_* < y^*$.

Lemma 2. *The minimum of* $F(x, y, t)$ *if* x, t *are fixed,*

$$m(x, t) = F(x, y_*(x, t), t) = F(x, y^*(x, t), t)$$

is a continuous function of x, t *in the semiplane* $t > 0$.

Proof. Let x', t' be an arbitrary fixed point in $t > 0$ and put

$$y_{,} = y_*(x', t'), \qquad y' = y^*(x', t'), \qquad m' = F(x', y_{,}, t') = F(x', y', t').$$

(34) and the continuity of F in y imply the existence of a positive number p such that

$$F(x', y, t') > m' + p \quad \text{whenever} \quad y < y_{,} - 1 \quad \text{or} \quad y > y' + 1.$$

F is continuous in all three variables and

$$F(x, y, t) \to +\infty \quad \text{as} \quad |y| \to \infty \tag{40}$$

holds uniformly with respect to x, t in the neighborhood of x', t'. Hence there exists a positive number q, $q < t'$, such that

$$F(x, y, t) > m' + (p/2)$$

holds whenever

$$y < y_{,} - 1 \text{ or } y > y' + 1, \qquad \text{and} \qquad |x - x'| + |t - t'| \leqq q.$$

On the other hand, there certainly exists an $r > 0$, $r < q$, such that

$$F(x, y', t) < m' + \frac{p}{2} \qquad \text{whenever } |x - x'| + |t - t'| \leqq r.$$

Hence $F(x, y, t) - m'$, as a function of y, attains its minimum in the interval

$$y_{,} - 1 \leqq y \leqq y' + 1 \qquad \text{whenever} \qquad |x - x'| + |t - t'| \leqq r.$$

We therefore may confine ourselves to the closed set

$$y_{,} - 1 \leqq y \leqq y' + 1, \qquad |x - x'| + |t - t'| \leqq r$$

in the space of x, y, t and to min F, as x, t are kept fixed, within this set. The fact that this function min F is continuous at $x = x'$, $t = t'$ is readily proved.

Lemma 3. *As functions of x and t, $y_*(x, t)$ and $y^*(x, t)$ are lower- and upper-semicontinuous, respectively, in the semiplane $t > 0$. At a point where $y_* = y^*$ both functions are continuous.*

Proof. Let x', t' be an arbitrary fixed point in $t > 0$. Let the point (x, t) run through a sequence of points that converge toward (x', t'). Since (40) holds uniformly with respect to x, t in a neighborhood of (x', t'), the values of y_* (and of y^* as well) along this sequence must be bounded. Denote by $y_{,}$ the limes inferior of the values y_* along the sequence. If in the inequality

$$F(x, y, t) \geqq F(x, y_*(x, t), t)$$

y is kept fixed, and if (x, t) runs toward (x', t') along a suitable subsequence, one infers that

$$F(x', y, t') \geqq F(x', y_{,}, t')$$

must hold for every y. From the definition of y_* and y^* it therefore follows that $y_*(x', t') \leqq y_{,}$. Hence $y_*(x, t)$ is lower-semicontinuous at x', t'. The statement concerning y^* is proved in the same way. This concludes the proof of lemma 3.

Theorem 3. *Let $u(x, t; \mu)$, $t > 0$, be the solution of* (1) *having arbitrarly given initial values $u_0(x)$ which satisfy* (6). *Let $y_*(x, t)$ and $y^*(x, t)$ be determined from these initial values as stated above. Then, for every x and $t > 0$,*

$$\frac{x - y^*(x, t)}{t} \leqq \liminf_{\substack{\mu=0\\ \xi=x\\ \tau=t}} u(\xi, \tau; \mu) \leqq \limsup_{\substack{\mu=0\\ \xi=x\\ \tau=t}} u(\xi, \tau; \mu) \leqq \frac{x - y_*(x, t)}{t} .$$

In particular,

$$\lim_{\substack{\mu=0\\ \xi=x\\ \tau=t}} u(\xi, \tau; \mu) = \frac{x - y_*(x, t)}{t} = \frac{x - y^*(x, t)}{t}$$

holds at every point, $t > 0$, in which $y_* = y^*$.

Proof. Obviously, we may write[5]

$$u(\xi, \tau; \mu) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{\xi - y}{\tau} \exp\left\{-\frac{P(\xi, y, \tau)}{\mu}\right\} dy}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{-\frac{P(\xi, y, \tau)}{\mu}\right\} dy} \tag{41}$$

where

$$P(\xi, y, \tau) = \tfrac{1}{2}[F(\xi, y, \tau) - m(\xi, \tau)]$$

and where

$$m(\xi, \tau) = F(\xi, y_*(\xi, \tau), \tau) = F(\xi, y^*(\xi, \tau), \tau)$$

is the smallest value of $F(\xi, y, \tau)$ with ξ, τ being fixed. P has the properties

$$\begin{aligned} &P(\xi, y, \tau) > 0 \quad \text{as} \quad y < y_*(\xi, \tau) \quad \text{and as} \quad y > y^*(\xi, \tau),\\ &P(\xi, y, \tau) = 0 \quad \text{as} \quad y = y_*(\xi, \tau) \quad \text{and as} \quad y = y^*(\xi, \tau). \end{aligned} \tag{42}$$

P is, in virtue of lemma 2, continuous in ξ, y, τ if $\tau > 0$ and

$$\lim_{|y|=\infty} y^{-2} P(\xi, y, \tau) = \frac{1}{4\tau} \tag{43}$$

holds uniformly with respect to ξ, τ in every closed set in $\tau > 0$.

We choose arbitrarily a point x, t in $t > 0$ and put in this fixed point

$$Y_* = y_*(x, t), \qquad Y^* = y^*(x, t).$$

We also choose arbitrarily a number $\epsilon > 0$. We take the positive numbers a, b, $a < t$, so small that

[5]We have avoided the application of the asymptotic Laplace-formula to numerator and denominator of this expression because that would have required an assumption which does not lie in the nature of things.

The form of the expression suggests a probability interpretation as Max A. Zorn remarked to the author. We presume that this interpretation will be brought out by the study of the stochastic process back of (1).

$$(44)\qquad l = \frac{x - Y^*}{t} - \epsilon < \frac{\xi - y}{\tau} < \frac{x - Y_*}{t} + \epsilon = L$$

holds whenever ξ, y, τ satisfy

$$(45)\qquad |\xi - x| + |\tau - t| < a \qquad \text{and} \qquad Y_* - 2b < y < Y^* + 2b.$$

By lemma 3, the number a can at the same time be chosen such that

$$(46)\qquad Y_* - b < y_*(\xi, \tau) \leqq y^*(\xi, \tau) < Y^* + b$$

holds whenever ξ, τ satisfy

$$(47)\qquad |\xi - x| + |\tau - t| < a.$$

We keep the numbers a, b fixed. The numerator on the right of (41) is, by (44) and (45), greater than

$$l \int_{-\infty}^{\infty} \exp\{\}\, dy + \int_{-\infty}^{Y_*-2b} \left(\frac{\xi - y}{\tau} - l\right) \exp\{\}\, dy + \int_{Y^*+2b}^{\infty} \left(\frac{\xi - y}{\tau} - l\right)$$

and less than

$$L \int_{-\infty}^{\infty} \exp\{\}\, dy + \int_{-\infty}^{Y_*-2b} \left(\frac{\xi - y}{\tau} - L\right) \exp\{\}\, dy$$

$$+ \int_{Y^*+2b}^{\infty} \left(\frac{\xi - y}{\tau} - L\right) \exp\{\}\, dy$$

whenever ξ, τ satisfy (47). The absolute values of the brackets (. . .) in the second and third terms can be majorised by

$$\omega(Y_* - y) \qquad \text{and} \qquad \omega(y - Y^*),$$

respectively, where ω is independent of ξ, τ and μ as long as (47) is satisfied. Clearly, the main part of theorem 3 will be proved if we can show that the two expressions

$$(48)\qquad \frac{\displaystyle\int_{\infty}^{Y_*-2b} (Y_* - y) \exp\{\}\, dy}{\displaystyle\int_{-\infty}^{y_*(\xi,\tau)} \exp\{\}\, dy}, \qquad \frac{\displaystyle\int_{Y^*+2b}^{\infty} (y - Y^*) \exp\{\}\, dy}{\displaystyle\int_{y^*(\xi,\tau)}^{\infty} \exp\{\}\, dy}$$

tend to zero as $\mu \to 0$ uniformly with respect to ξ, τ provided that (47) is fulfilled. Since both expressions can be treated in the same way, we may confine ourselves to the second one. With the exponential expression

$$\exp\left\{-\frac{P(\xi, y, \tau)}{\mu}\right\}$$

we deal in the following way. By virtue of (42), (43) and (46)

$$\frac{P(\xi, y, \tau)}{(y - Y^*)^2}$$

is positively bounded from below if ξ, y, τ satisfy $y > Y^* + 2b$ and (47) (remember that (46) is then fulfilled). If $A/2$ is a positive lower bound, we infer that the numerator of the second ratio (48) is less than

$$\int_{Y^*+2b}^{\infty} (y - Y^*) \exp\left\{-\frac{A}{2\mu}(y - Y^*)^2\right\} dy = \frac{\mu}{A} \exp\left\{-\frac{2Ab^2}{\mu}\right\}$$

if ξ, τ satisfy (47). On the other hand, the uniform continuity of P and (42) imply the existence of a fixed $\delta > 0$ such that $P < 2Ab^2$ holds whenever $y^*(\xi, \tau) < y < y^*(\xi, \tau) + \delta$ and $|\xi - x| + |\tau - t| < a$. Hence the denominator of the second ratio (48) is greater than the integral extended from y^* to $y^* + \delta$ which, in turn, is greater than

$$\delta \exp\left\{-\frac{2Ab^2}{\mu}\right\}$$

whenever (47) is satisfied. The second ratio is, therefore, less than $\mu/A\delta$ whenever ξ, τ satisfy (47). Hence we infer the truth of the statement concerning (48). Theorem 3 is thus proved.

We now define the function

$$u(x, t) = \lim_{\mu=0} u(x, t; \mu)$$

in every point x, t, $t > 0$, in which this limit exists. By virtue of theorem 3 this is the case in every point where $y_* = y^*$. These points constitute the normal case insofar as, for any given $t > 0$, the exceptional points form a denumerable set. Theorem 3 moreover implies that at any normal point, $t > 0$, $u(x, t)$ is not only defined but also continuous in both variables. From theorem 3 and from property (b) of lemma 1 one infers that generally

$$u(x - 0, t) = \frac{x - y_*(x, t)}{t}, \qquad u(x + 0, t) = \frac{x - y^*(x, t)}{t} \tag{49}$$

holds in each point of the semiplane $t > 0$. In particular,

$$u(x - 0, t) \geqq u(x + 0, t)$$

holds everywhere in $t > 0$. At a point in which these two limits coincide, $u(x, t)$ is defined and continuous in both variables.

Theorem 4. *At each point of $t > 0$ the inequalities*

$$\limsup_{y=y_*-0} \frac{\int_{y_*}^{y} u_0(\eta)\, d\eta}{y - y_*} \leqq u(x - 0, t) \leqq \liminf_{y=y_*+0} \frac{\int_{y_*}^{y} u_0(\eta)\, d\eta}{y - y_*}$$

hold if $y_* = y_*(x, t)$. These inequalities stay valid if y_* is replaced by y^* and $u(x - 0, t)$ by $u(x + 0, t)$.

Proof. This is an immediate consequence of theorem 3, of the fact that (8) as a function of y attains its minimum at $y = y_*$ and at $y = y^*$, and of (49).

If $u_0(y)$ has left and right limits at $y = y_*$ as well as at $y = y^*$ the inequalities are

$$u_0(y_* - 0) \leqq u(x - 0, t) \leqq u_0(y_* + 0)$$

and

$$u_0(y^* - 0) \leqq u(x + 0, t) \leqq u_0(y^* + 0).$$

In particular, we have $u_0(y_*) = u(x, t)$ if $y_* = y^*$ and if u_0 is continuous at $y = y_*$.

5. *Continued study of the limit function* $u(x,t)$

We now examine the relations between the limit function $u(x, t)$ and equation (32) which is the case $\mu = 0$ of Burgers' equation (1).

The characteristics of (32) are the solutions of (33),

$$x = a + bt, \qquad u = b.$$

They are the straight lines in the space of x, t, u which are parallel to the x, t-plane and whose x, t-projections have a slope dx/dt equal to the distance from that plane. We shall speak of a (generalized) solution $\tilde{u}(x, t)$ of (32) in an open set R at the x, t-plane, if $\tilde{u}(x, t)$ is single-valued in R and if, above R, each segment of a characteristic that has a point in common with the surface $u = \tilde{u}$ belongs entirely to it. If this is the case, the x, t-projections of the characteristic segments obviously form a continuous field in R. Conversely, a set of characteristic segments above R forms a single-valued function $\tilde{u}$ (and hence a generalized solution) in R if that field condition is satisfied by the x, t-projections in R. If $\tilde{u}$ is a generalized solution of class C' in R it actually solves (32) in R. However, the differentiability question which offers no difficulty in our case, will be left aside.

Returning to our particular function $u(x, t)$ we consider an arbitrary point x_1, t_1 in the semiplane $t > 0$ and let $y_1 = y_*(x_1, t_1)$, $y^1 = y^*(x_1, t_1)$. To this point we attach the two characteristics

$$x = x_1 + \frac{x_1 - y_1}{t_1}(t - t_1), \qquad u = \frac{x_1 - y_1}{t_1} \tag{50,}$$

and

$$x = x_1 + \frac{x_1 - y^1}{t_1}(t - t_1), \qquad u = \frac{x_1 - y^1}{t_1}. \tag{50'}$$

(50,) and (50′) are characterized by either one of the two following properties: They pass through the two limit-points x_1, t_1, u_1 and x_1, t_1, u^1 of the surface, respectively, where $u_1 = u(x_1 - 0, t_1)$ and $u^1 = u(x_1 + 0, t_1)$ (see (49)). Their x, t-projections pass through x_1, t_1 and meet the line $t = 0$ at $x = y_1$ and $x = y^1$, respectively. We call their open segments $0 < t < t_1$ the characteristic segments and characteristic x, t-segments, respectively, belonging to x_1, t_1 (and to the surface). In the normal case where $y_1 = y^1$ (50,) and (50′) coincide and simply represent the segment $0 < t < t_1$ of the characteristic through the point x_1, t_1, $u(x_1, t_1)$ of the surface.

Theorem 5. *At every point x, t of a characteristic x, t-segment $u(x, t)$ is defined and continuous. It has the constant value $u = dx/dt$ along the segment, that is, the surface $u = u(x, y)$ contains the full characteristic segment. Two characteristic x, t-segments (no matter whether they belong to the same point or to different points) have either no point in common or one is part of the other (remember that the characteristic segments are defined as open segments $0 < t < t_1$).*

Proof. Let the segment belong to the point x_1, t_1, $t_1 > 0$. Suppose it is the segment formed with y_1 (the other one with y^1 is treated in exactly the same way). Let

$$u_1 = \frac{x_1 - y_1}{t_1}.$$

According to the definition of y_* by means of F of (8),

$$F(x_1, y, t_1) - F(x_1, y_1, t_1) = \int_{y_1}^{y} \left[u_0(\eta) + \frac{\eta - x_1}{t_1} \right] d\eta \geqq 0$$

for all y. Now, for a point x, t on the segment,

$$x = x_1 - (t_1 - t)u_1 = y_1 + tu_1 = \frac{tx_1 + (t_1 - t)y_1}{t_1}, \qquad 0 < t < t_1, \tag{51}$$

and hence

$$\frac{\eta - x}{t} = \frac{\eta - x_1}{t_1} + \frac{t_1 - t}{t_1 t}(\eta - y_1).$$

Therefore,

$$F(x, y, t) - F(x, y_1, t) = \int_{y_1}^{y} \left[u_0(\eta) + \frac{\eta - x}{t} \right] d\eta$$

$$= F(x_1, y, t_1) - F(x_1, y_1, t_1) + \frac{t_1 - t}{2t_1 t}(y - y_1)^2.$$

This proves that $F(x, y, t)$ as a function of y reaches its minimum only at $y = y_1$ if x, t is an arbitrary point on the characteristic x, t-segment belonging to x_1, t_1, that is, that $y_*(x, t) = y^*(x, t) = y_1$ holds along this segment. By virtue of the remarks made at the end of the preceding paragraph and of theorem 3, $u(x, t)$ is continuous at all points of the segment and it has the value

$$u(x, t) = (x - y_1)/t$$

along it. In consequence of (51), this value is constant and equal to $u_1 = dx/dt$.

The last statement of the theorem remains to be proved. Consider two characteristic x, t-segments. They may belong to two different points x_1, t_1 and either segment may be of either kind (50_*) or (50^*). On each such straight line segment $u(x, t)$ is defined and equal to the slope dx/dt of the segment.

Hence, if the two segments have a point in common, they must have the same slope and consequently one must be contained in the other. This concludes the proof of theorem 5.

Lemma 4. *If $u(x, t_1)$, $t_1 > 0$ fixed, is continuous for $a \leqq x \leqq b$ the characteristic x, t-segments belonging to the points $a \leqq x \leqq b$, $t = t_1$ form a continuous field.*

Proof. In virtue of theorem 5 two such segments belonging to different points of $a \leqq x \leqq b$ can have no point in common. The slope of the segment which is $u(x, t_1)$ varies continuously with x. Hence the segments form a continuous field, q.e.d.

The characteristic x, t-segments of lemma 4 completely and simply cover the region contained between $t = 0$ and $t = t_1$ and between the two extreme segments belonging to the points (a, t_1) and (b, t_1). As mentioned before, the lower end points of the segments are the points $y_*(x, t_1) = y^*(x, t_1)$ on the line $t = 0$. As x increases this lower endpoint moves continuously and never to the left. $y_*(x, t_1)$ may well have an interval of constancy which means that the corresponding x, t-segments have a common lower endpoint, say y_0. Suppose that $u_0(y_0 \pm 0)$ both exist. (49) and theorem 4 then show that the initial values $u_0(y)$ must make an upward jump at $y = y_0$. It is seen that such a discontinuity of the initial values is, for $t > 0$, immediately dissolved into a linear change of $u(x, t)$, $u = (x - y_0)/t$.

Theorem 6. *The complementary set S in the semiplane $t > 0$ of the closure of the set of points x, t of discontinuity for $u(x, t)$ is either empty or open. In the latter case the part of the surface $u = u(x, t)$ above S is formed by segments $0 < t < t$ (in general t_1 varies with the segment) of characteristics. Their x, t-projections form a continuous field in S.*

This is an easy consequence of theorem 5 and lemma 4. If the initial values are merely continuous, it can presumably happen that S is empty. We leave the question of the structure of S aside and quite generally construct a set of continuous curves which fill the semiplane $t > 0$ and which contain the characteristic x, t-segments of the surface $u = u(x, t)$ as well as the "lines of discontinuity for $u(x, t)$".

To every point x, t in $t > 0$ there belongs a characteristic triangle formed by $t = 0$ and the two characteristic x, t-segments (50). The endpoints of its basis on $t = 0$ are $y_*(x, t)$ and $y^*(x, t)$. If x, t is a point of continuity for $u(x, t)$ the triangle is a single characteristic x, t-segment. In virtue of theorem 5, two characteristic triangles either lie outside of each other or one encloses the other. Let x_1, t_1 denote an arbitrarily given point in $t > 0$. We assert that on every level $t = t_2 > t_1$ there exists one and only one point $x = x_2$ whose characteristic triangle contains the given point x_1, t_1; it then contains the whole triangle belonging to x_1, t_1. To see this, consider an arbitrary point x, t_2. We infer from lemma 1,c that there exist values of x for which the corresponding characteristic triangle lies on the right of the point x_1, t_1 and that for

$x \to -\infty$ the triangle ultimately always lies on the left of that point. The values x of the first kind, therefore, have a greatest lower bound x_2. It follows from lemma 1,b that the right side of the triangle belonging to x_2, t_2 does not lie on the left of x_1, t_1. Its left side cannot lie on the right of x_1, t_1 because the same lemma would then lead to a contradiction of the definition of x_2. Hence x_2, t_2 satisfies the requirement. The uniqueness of this point on $t = t_2$ easily follows from the non-intersection property of the triangles. In this way a unique curve $x = x(t)$, $x_1 = x(t_1)$, is defined for all $t \geqq t_1$. It is readily seen that the triangles belonging to the points $x(t)$, t form, in the sense of inclusion, an increasing family of triangles. The continuity of $x(t)$ for $t \geqq t_1$ is geometrically evident.

As t increases, $y_*(x(t), t)$ never increases and $y^*(x(t), t)$ never decreases. If the curve $x = x(t)$ contains a point x_2, t_2 of continuity for $u(x, t)$, it obviously is an x, t-characteristic in the interval $t_1 \leqq t \leqq t_2$. These facts prove the first assertion of

Theorem 7. Each point x_1, t_1 in $t > 0$ uniquely determines a curve $x = x(t)$, $x_1 = x(t_1)$, for all $t \geqq t_1$ continuous and such that the characteristic triangles (triangular areas) belonging to the points of the curve form an increasing family of sets. At every $t \geqq t_1$

$$\lim_{\substack{t'=t+0\\ t''=t+0}} \frac{x(t'') - x(t')}{t'' - t'} = \frac{1}{2}\,[u(x - 0, t) + u(x + 0, t)]; \qquad x = x(t). \tag{52}$$

For any $t > t_1$

$$\lim_{\substack{t'=t-0\\ t''=t-0}} \frac{x(t'') - x(t')}{t'' - t'} \tag{53}$$

exists, but it may have a different value.

It suffices to prove (52) at $t = t_1$. Let $x' = x(t')$, $x'' = x(t'')$, $t'' > t' > t_1$, and

$$y_{,} = y_*(x', t'), \quad y' = y^*(x', t'); \qquad y_{,,} = y_*(x'', t''), \quad y'' = y^*(x'', t'');$$

$$y_1 = y_*(x_1, t_1), \quad y^1 = y^*(x_1, t_1).$$

These numbers satisfy the inequalities

$$y_{,,} \leqq y_{,} \leqq y_1 \leqq y^1 \leqq y' \leqq y''. \tag{54}$$

From lemma 3 and from what was said about the monotonicity of the two functions $y_*(x(t), t)$, $y^*(x(t), t)$ it follows that both functions of t are continuous on the right in t. Hence one infers that

$$y_{,,} \to y_1, \; y'' \to y^1 \qquad \text{as} \qquad t'' \to t_1. \tag{55}$$

If the curve has an initial segment which is characteristic, the assertion is trivial. We therefore suppose that $y_* < y^*$ for $t > t_1$. This implies that

$$y_{,,} \leqq y_{,} < y' \leqq y''. \tag{56}$$

According to the original definition of y_* and y^* by means of the function (8), the following inequalities hold for all values of y

$$\int_{y_{,}}^{y} I'\, d\eta \geqq 0, \qquad \int_{y_{,,}}^{y} I''\, d\eta \geqq 0 \tag{57}$$

where

$$I' = u_0(\eta) + \frac{\eta - x'}{t'}, \qquad I'' = u_0(\eta) + \frac{\eta - x''}{t''}.$$

The inequalities (57) stay valid if the lower limits of integration are replaced by y' and y'', respectively. We also have

$$\int_{y_{,}}^{y'} I'\, d\eta = 0, \qquad \int_{y_{,,}}^{y''} I''\, d\eta = 0. \tag{58}$$

If the first of these equations is subtracted from the second, the following equation results:

$$\int_{y_{,,}}^{y'} I''\, d\eta + \int_{y_{,}}^{y''} I'\, d\eta + \int_{y_{,}}^{y''} (I'' - I')\, d\eta = 0.$$

From (57), where $y_{,}$ is to be replaced by y', it is inferred that the third term of the last relation is $\leqq 0$. If this term is evaluated and divided by $y'' - y_{,}$ which, in virtue of (56), is certainly not zero, the inequality

$$t'(x'' - x') \geqq \left(x' - \frac{y_{,} + y''}{2}\right)(t'' - t')$$

is finally obtained. Interchanging the two equations (58), one arrives at another inequality

$$t''(x' - x'') \geqq \left(x'' - \frac{y_{,,} + y'}{2}\right)(t' - t'').$$

From the last two inequalities we infer the desired result using (55) and the continuity of $x(t)$. The value of (52) follows from (49).[6] That (53) exists follows from the monotonicity of y_* and y^* along $x = x(t)$. On the left, however, these functions need not be continuous in t. Theorem 7 is hereby completely proved.

A curve $x = x(t)$ which, for any of its points, has the property stated in theorem 7 is called a *line of discontinuity* for $u(x, t)$ if $y_* < y^*$ holds along it. While, a discontinuity line is uniquely determined by any of its points in the direction of increasing t this need not be so in the opposite direction. Two lines of discontinuity can, as t increases, merge at some moment. A discontinuity line, $t > t_0$, always has a definite initial point. This is readily seen from the

[6]This value of the slope of the discontinuity line was derived by Burgers—see footnote 1(a)—on much more general grounds. The differentiability of $x(t)$, proven here, was taken for granted by Burgers.

property of the characteristic triangles if $t_0 > 0$. If $t_0 = 0$ the additional (and easily proved) fact must be used that

$$\lim_{t=0} y_*(x, t) = \lim_{t=0} y^*(x, t) = x$$

holds uniformly in every finite x-interval. A discontinuity line may be called complete if it is not part of another such line. Again by considering the triangles one can prove that a line of discontinuity can be completed in at least one way. If no differentiability conditions are imposed on the initial values, it may well happen that a discontinuity line possesses infinitely many completions and that, moreover, the discontinuity lines are everywhere dense. Conditions under which this does not occur are mentioned below. It is, however, generally true that the complete lines of discontinuity are only denumerable in number. This follows from the fact stated above that on $t =$ const. there are only denumerably many points of discontinuity and from the fact that two such lines must differ at some rational value of t if they do not coincide.

The differentiability assumptions mentioned about $u_0(y)$ are these: Suppose that the line of y can be subdivided into intervals which may accumulate only at ∞ such that $u_0'(y)$ is monotonic in the interior of each interval and bounded if the interval is finite. $u_0(y \pm 0)$ exists in all points of division. u_0' may have intervals of constancy which, however, may accumulate at ∞ only. Under these assumptions about the initial values the following statements are true (proofs are omitted): An arbitrary rectangle $a \leqq x \leqq b$, $0 \leqq t \leqq T$ can contain points of but a finite number of different lines of discontinuity. Only a finite number of starting points of discontinuity lines can lie in such a rectangle. On a finite segment $t < T$ of a discontinuity line there can be only finitely many points in which other discontinuity lines merge with it. Only in these points, by the way, can the two derivatives (52) and (53) have different values.

Lines of discontinuity "in general" do originate in $t > 0$ even if $u_0'(y)$ is continuous throughout. The only exceptional case is the one in which $u_0'(y) \geqq 0$ everywhere. These statements are proved in the next section. The starting point of a discontinuity line that originates in $t > 0$ is "in general" a point of continuity of $u(x, t)$. We do not go into the details back of this statement and we confine ourselves to proving a typical converse statement. *A point x_1, t_1 of continuity of $u(x, t)$ in which either the left or right derivative $\partial u/\partial x$ exists and equals $-\infty$ is the starting point of a discontinuity line.* This is readily inferred from the following fact: Let x_1, t_1 be a point on the characteristic x, t-segment belonging to a point x_2, t_2 in $t > 0$ which is a point of continuity for $u(x, t)$. The inequality

$$\frac{u(x', t_1) - u(x_1, t_1)}{x' - x_1} \geqq -\frac{1}{t_2 - t_1} \tag{59}$$

is then fulfilled by values x' arbitrarily near x_1 on the left as well as on the right. To prove this fact we observe that the characteristic x, t-segments belonging to the points x'', t_2 converge toward the segment belonging to x_2, t_2 as $x'' \to x_2$. Denote by x', t_1 the point of intersection of such a segment with the line $t = t_1$. The value of $u(x, t)$ on such a segment equals its slope dx/dt which is equal to $(x'' - x')/(t_2 - t_1)$. The difference quotient in (59) therefore equals

$$\frac{(x'' - x') - (x_2 - x_1)}{(t_2 - t_1)(x' - x_1)} = \frac{1}{t_2 - t_1}\left[-1 + \frac{x'' - x_2}{x' - x_1}\right].$$

Since the x, t-segments cannot intersect, the second term in the square bracket is always positive. Hence (59) must hold as stipulated above.

6. *The function $u(x, t)$ and the initial values $u_0(y)$*

The connection between the characteristic segments of the preceding section which were defined by the surface $u = u(x, t)$, $t > 0$, and the characteristics that are directly defined by the initial curve is a matter that still needs some clarification. The explicit rule that connects $u(x, t)$ with the initial values is an accidental feature of Burgers' equation which has no analogue in other problems of similar nature. As in the preceding section we endeavor to give the results a form as independent as possible of that special feature.

We now suppose that the limits $u_0(y \pm 0)$ exist for every y, and we replace the condition at infinity (6) by the somewhat stronger one that

$$u_0(y) = 0(y), \qquad |y| \to \infty. \tag{60}$$

We assign a characteristic to each value of y for which $u_0(y - 0) \leqq u_0(y + 0)$ by the formula

$$x = y + vt, \qquad u = v$$

where v may have any value that satisfies

$$u_0(y - 0) \leqq v \leqq u_0(y + 0).$$

In particular, the single characteristic $x = y + u_0(y)t$, $u = u_0(y)$ is assigned to a value y of continuity for $u_0(y)$. No characteristic is assigned to a value for which $u_0(y - 0) > u_0(y + 0)$. These characteristics we call I-characteristics. The x, t, I-characteristics are their x, t-projections.

Theorem 4 now implies that every characteristic segment of the surface as defined by (50,) and (50′) is part of an I-characteristic. It remains to examine the converse question, if a given I-characteristic has an initial segment in common with the surface $u = u(x, t)$. This need not be true if the initial values satisfy no other local conditions besides the ones stated above. The form of such an additional differentiability condition that is most convenient for our purposes is a non-intersection condition or a field condition. *If an initial segment $0 < t < t_1$ of an x, t, I-characteristic is not met by any other x, t, I-characteristic then the surface contains the entire segment $0 < t < t_1$ of the I-characteristic.* This is easily seen on considering an arbitrary point x, t on that segment and a characteristic x, t-segment (in the sense of (50)) which belongs to this point. This latter segment is, as stated above, part of an x, t, I-characteristic. The hypothesis of the italicized statement clearly implies that this x, t, I-characteristic contains the original segment. Hence the latter must up to (x, t), and since this point was arbitrary on the segment, wholly lie on the surface.

The simplest and the only case in which no discontinuity appears is the one in which $u_0(y)$ is a nowhere decreasing function of y. In this case the x, t, I-characteristics never intersect and fill the entire semiplane $t > 0$. The surface $u(x, t)$ is formed by the I-characteristics.

In the following statement the hypothesis is a purely local one. *Suppose that the initial point $t = 0$ of an x, t, I-characteristic possesses a neighborhood, $t > 0$, in which it is never met by any other x, t, I-characteristic that starts in this neighborhood. Then the original x, t-characteristic has an initial segment in common with the surface.* To see this observe that the points of intersection with the x, t, I-characteristics that start outside of the neighborhood cannot come arbitrarily close to $t = 0$. This follows from the conditions of continuity for $u_0(y)$ and from the condition at infinity (60). The rest follows from the preceding italicized statement.

Suppose that, at $y = y_0$, $u_0(y)$ makes an upward jump. Consider an x, t, I-characteristic that belongs to this point and is not one of the two extreme x, t-characteristics of the bundle belonging to y_0. We assert that this x, t-characteristic has an initial segment in common with the surface. Indeed, it is not difficult to show that a neighborhood of $x = y_0$, $t = 0$ exists in which the preceding italicized statement becomes applicable.

An I-characteristic either has an initial segment in common with the surface or its initial point on $t = 0$ is the starting point of a discontinuity line. Suppose first that the initial point y_0 of that characteristic is a point of continuity for $u_0(y)$. Then no other x, t, I-characteristic starts at y_0. Hence it follows that $y_*(x, t)$ and $y^*(x, t)$ can never attain y_0 as a value because, otherwise, an x, t-characteristic (50) would begin at y_0 and because such a characteristic is always part of an I-characteristic. The curve $x(t)$ that starts at y_0 must, therefore, be a discontinuity line.[7] If u_0 is discontinuous at y_0 the argument is somewhat more involved; we omit the proof in this case.

We finally mention a criterion for the occurrence of discontinuities. *If two different x, t, I-characteristics intersect in $t > 0$, at least one of the two closed finite segments contains an interior point or a starting point of a line of discontinuity.* If no discontinuity line starts at the initial point of such a segment, we apply the preceding italicized statement. The segment therefore contains a subsegment which is a characteristic x, t-segment in the sense of the preceding section. If the longest subsegment of this kind is shorter than the original segment, then a discontinuity line starts at its end point. If this is the case for neither

[7]Our definition of the curve $x(t)$ through a given point x_0, t_0 defines it uniquely if $t_0 > 0$. This need not be the case if $t_0 = 0$ (consider, for instance, the case where $u_0(y)$ makes an upward jump at $y = x_0$). In the special case considered here we are, however, justified in speaking of "the" curve $x(t)$ issuing from $x = y_0$, $t = 0$.

original segment then both these segments must be x, t-projections of two surface-characteristics. Since their slopes are different the (constant) values of u on them differ. Hence u is discontinuous at the point of intersection. This completes the proof of the statement.

A point $y = y_0$ on $t = 0$ where $u_0(y - 0) > u_0(y + 0)$ is the starting point of a discontinuity line for $u(x, t)$. To see this we remember that no x, t, I-characteristic issues from such a point. As every characteristic x, t-segment is part of an x, t, I-characteristic neither $y_*(x, t)$ nor $y^*(x, t)$ can attain y_0 as a value. Hence a definite curve $x(t)$ starts from that point and this curve must be a discontinuity line.[7]

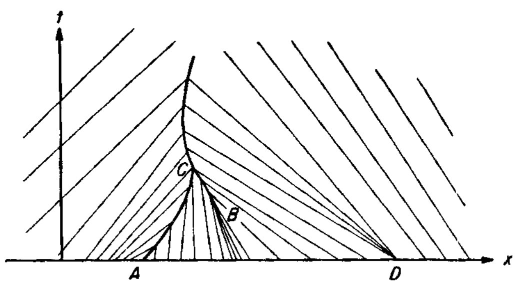

FIG. 1.

The figure shows an example where the initial values have a downward jump at A and an upward jump at D. A line of discontinuity starts at A. Another such line originates at B. The two lines merge at C.

7. *Behavior of $u(x,t)$ as $t \to \infty$.*

In section 3 we solved the problem how $u(x, t; \mu)$ behaves as $t \to \infty$ while $\mu > 0$ stays fixed. We refer to the formulation of the result given close to the end of that section. Right at the end it was stated how the result behaves if $\mu \to 0$. In hydrodynamics, the position of the problem in this order precisely corresponds to the problem of turbulent flow at high Reynolds numbers. This is the problem of the statistical long run behavior of a liquid of small viscosity. There is no reason to expect that the reversal of the order of the two limit processes leads to the same result. In the case of Burgers' equation one can easily show that the result is actually different.

We again assume the integral

$$M = \int_{-\infty}^{\infty} u_0(y)\, dy$$

to be finite. We show that the limit function

$$\lim_{t \to \infty} \sqrt{t}\, u(\bar{x}\sqrt{t}, t) \tag{61}$$

exists and that it may differ from the limit function mentioned at the end of section 3. Now, upon introducing new variables

$$x = \bar{x}\sqrt{t}, \qquad y = \bar{y}\sqrt{t}, \qquad z = \bar{z},$$

$$\sqrt{t}\, u(x, t) = \bar{u}(\bar{x}, t)$$

the geometrical rule that determines $u(x, t)$ from the curve $z = G(0, y, t)$ of (35) and (8) becomes this: If $\bar{y} = \bar{y}(\bar{x}, t)$ is the $\bar{y}$-coordinate of the point $\bar{y}, \bar{z}$ (we need only consider the cases with one such point) which the curve

$$\bar{z} = \int_{-\infty}^{\bar{y}\sqrt{t}} u_0(\eta)\, d\eta + \frac{\bar{y}^2}{2}$$

has in common with the lower support line of the slope $d\bar{z}/d\bar{y} = \bar{x}$ then $\bar{u} = \bar{u}(\bar{x}, t) = \bar{x} - \bar{y}$. The replacement of the lower limit 0 of integration by $-\infty$ is permitted because it only shifts the curve in the direction of $\bar{z}$. For large values of t the integral part differs little from 0 for $y < -\epsilon$ and from M for $y > \epsilon$. But the range of its values in the small interval $|\, y \,| < \epsilon$ is important for our purposes. Letting

$$N = \min_{\eta} \int_{-\infty}^{\eta} u_0(\eta)\, d\eta \tag{62}$$

we readily see that the limit curve which we have to consider is

$$\bar{z} = \begin{cases} \bar{y}^2/2 \quad , & \bar{y} < 0, \\ N \quad , & \bar{y} = 0, \\ M + \bar{y}^2/2, & \bar{y} > 0, \end{cases} \tag{63}$$

and that the limit (61) ($= \lim_{t=\infty} \bar{u}(\bar{x}, t)$) equals $\bar{x} - \bar{y}$ where $\bar{y}$ belongs to the point which the discontinuous curve (63) has in common with its lower support line of the slope $\bar{x}$. The curve is obtained by shifting the right half of the parabola $\bar{z} = \bar{y}^2/2$ by the amount M in the direction of $\bar{z}$ and by adding the isolated point $\bar{y} = 0$, $\bar{z} = N$. We have not added the analogous point corresponding to the maximum in (62). This point always lies in the convex hull of the curve and is therefore unimportant. This is not necessarily true for the other point, because

$$N \leqq \min\,(0, M). \tag{64}$$

A tangent to the continuous part of (63) has a slope equal to the $\bar{y}$-coordinate of the point of tangency. For two critical values

$$\bar{y}_1 = -\sqrt{-2N}, \qquad \bar{y}_2 = \sqrt{2(M - N)} \tag{65}$$

the tangent goes through the isolated point $\bar{y} = 0$, $\bar{z} = N$. Obviously, for any value of the slope $\bar{x}$ outside of the interval formed by (65), $\bar{u} = 0$. For any other value of $\bar{x}$ we have $\bar{u} = \bar{x}$ since the support line goes through that isolated point. This rule furnishes our limit function (61), $\bar{u} = \bar{u}(\bar{x})$,

$$\bar{u} = \begin{cases} 0, & \bar{x} < \bar{y}_1 , \\ \bar{x}, & \bar{y}_1 < \bar{x} < \bar{y}_2 , \\ 0, & \bar{x} > \bar{y}_2 . \end{cases}$$

Only in the exceptional case where the equality sign prevails in (64) does this function coincide with the function mentioned at the end of section 3.

Back of this discrepancy is the following fact (the proof of which is omitted here): We know that the moment

$$\int_{-\infty}^{\infty} u(x, t)\, dx$$

is an "integral" of Burgers' differential equation (1), $\mu > 0$. It is, essentially, the only such integral. After the passage to the limit $\mu \to 0$ it remains an integral. In this limit case, however, a new integral appears,

$$\min_x \int_{-\infty}^{x} u(x, t)\, dx.$$

8. *Integral form of the differential equations*

We first show that $u(x, t)$ satisfies the relation

$$\int_a^b u(\xi, t)\, d\xi = F(b, y_*(b, t), t) - F(a, y_*(a, t), t) \tag{66}$$

for all a and b. By virtue of lemma 2 this relation is valid if y_* is replaced by y^*. The relation holds if the initial values merely satisfy (6).

To prove it we use definition (8) of F and the definition of y^* and y_*. Letting

$$y_* = y_*(x, t), \qquad y'_* = y_*(x', t)$$

we have

$$\begin{aligned} F(x, y_*, t) - F(x', y'_*, t) &= [F(x, y_*, t) - F(x, y'_*, t)] \\ &\quad + [F(x, y'_*, t) - F(x', y'_*, t)] \\ &= [F(x, y_*, t) - F(x', y_*, t)] \\ &\quad + [F(x', y_*, t) - F(x', y'_*, t)]. \end{aligned}$$

In the first relation the first term on the right is $\leqq 0$; in the second relation the second term is $\geqq 0$. By (8) this means that the quantity

$$F(x, y_*(x, t), t) - F(x', y_*(x', t), t)$$

lies between the limits

$$(x - x') \frac{x + x' - 2y_*(x', t)}{2t}, \qquad (x - x') \frac{x + x' - 2y_*(x, t)}{2t}.$$

In virtue of (49) these limits equal

$$\frac{(x - x')^2}{2t} + (x - x')u(x' - 0, t), \qquad - \frac{(x - x')^2}{2t} + (x - x')u(x - 0, t)$$

respectively. Identifying x, x' with successive points of a subdivision of the the interval (a, b) and adding the resulting inequalities, (66) becomes obvious.

We use (66) to prove that

$$\lim_{t=0} \int_0^x u(\xi, t)\, d\xi = \int_0^x u_0(\eta)\, d\eta \tag{67}$$

holds uniformly in every finite x-interval, in other words, that the initial condition (9) is satisfied not only by the solution of (1) belonging to the initial values u_0 but also by the limit function obtained by letting $\mu \to 0$.

Obviously, $\min_y F \leqq F(x, x, t)$ and hence

$$\int_0^{y_*(x,t)} u_0(\eta)\, d\eta \leqq F(x, y_*(x, t), t) \leqq \int_0^x u_0(\eta)\, d\eta.$$

(67) follows from (66) and from the fact (stated in the second half of §5) that $y_*(x, t) \to x$ as $t \to 0$ holds uniformly in every x-interval.

We now turn to our main object and we assert: A solution u of (1), $\mu > 0$, which is regular in the semiplane $t > 0$ satisfies the relation (the double integrals are extended over the semiplane)

$$\iint u f_t\, dx\, dt + \iint \frac{u^2}{2} f_x\, dx\, dt + \mu \iint u f_{xx}\, dx\, dt = 0 \tag{68}$$

for each $f(x, t)$ such that f, f_x, f_t, f_{xx} are continuous in $t > 0$ and such that $f = 0$ holds outside some circle lying entirely within $t > 0$. This fact trivially results on multiplying (1) by f and on integrating over $t > 0$.

We are, however, more interested in the converse question. We consider (68), together with the requirement that it hold for arbitrary f as stipulated, as the primary form of the problem of defining u. Let us call a function u that solves (68) in this sense a generalized solution of (1) in $t > 0$. We assert: a generalized solution of (1) in $t > 0$ with the property that u, u_x, u_t, u_{xx} are continuous in $t > 0$ is a solution of (1) regular in $t > 0$. This fact also follows from (68) if we perform the integrations by parts backwards and observe that f is arbitrary within the specified function class.

The following theorem is deeper. *A generalized solution of* (1) *in* $t > 0$ *which is merely measurable and quadratically integrable in every closed rectangle* *in* $t > 0$ *coincides almost everywhere with a regular solution of* (1) *in* $t > 0$. This shows that nothing is gained by enlarging the function space to which u is confined. Later we will see that this is quite different in the limit case $\mu \to 0$. Only the idea of a proof of the theorem is briefly indicated here. It makes use of the additional supposition that for $|x|$ large, $u = e^{o(x^2)}$ holds uniformly in every t-interval. Appropriate approximations by functions f to the well known fundamental solution of $v_t + v_{xx} = 0$, the adjoint to the heat equation, lead to a quadratic integral equation for $u(x, t)$ which is satisfied almost everywhere. From this equation bounds can be derived for the difference quotients of the function $\bar{u}$ that solves the equation throughout and that coincides with u almost everywhere. In this way one can prove the differentiability properties required for u to render the transformation of (68) back to (1) trivial.

We now consider the limit case $\mu \to 0$. By a generalized solution u of (32), i.e. of (1), $\mu = 0$, in $t > 0$ we mean a function u that is measurable and quadratically integrable in every closed rectangle in the open semiplane $t > 0$ and that satisfies the relation

$$\iint \left[u g_t + \frac{u^2}{2} g_x \right] dx\, dt = 0, \tag{69}$$

where g is an arbitrary function of class C' in $t > 0$ that vanishes outside some circle lying entirely in $t > 0$. We assert: *Every limit function u obtained according to section* 4 *from the solution of* (1) *as* $\mu \to 0$ *is a generalized solution of* (1), $\mu = 0$. This can now be readily proved. From the results of section 4 it is inferred that the solutions of (1), for fixed initial values $u_0(x)$, converge almost everywhere in $t > 0$ as $\mu \to 0$. From the convergence theorem 3 it furthermore follows that every point x, t in $t > 0$ has a neighborhood in which the solutions of (1) stay uniformly bounded as $\mu \to 0$. In consequence of these facts the passage to the limit $\mu \to 0$ in (68) may be performed in the integrands (f being kept fixed). Thus it follows that (69) must hold for all functions f. That it must hold for all functions g, even if the class C' is replaced by the wider class of Lipschitz functions, follows from well known approximation theorems.

It is desirable to complete the picture and to show conversely that the limit function u constructed in section 4 is the only generalized solution of (1), $\mu = 0$, that satisfies the initial condition (67) uniformly in every finite x-interval. This uniqueness theorem is very likely to hold true, but we did not succeed in deriving the explicit formulae of section 4 directly from (69) and the initial condition.

The concept of the generalized solution of (1), $\mu = 0$, appears to furnish the simplest direct characterization of the limit case $\mu \to 0$ in the solutions of (1), whereas the case $\mu = 0$ of (1) in the literal sense cannot completely describe it.