Positive periodic solutions for abstract evolution equations with delay *

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Abstract

In this paper, we discuss the existence and asymptotic stability of the positive periodic mild solutions for the abstract evolution equation with delay in an ordered Banach space E,

$$u'(t) + Au(t) = F(t, u(t), u(t - \tau)), \quad t \in \mathbb{R},$$

where $A: D(A) \subset E \to E$ is a closed linear operator and -A generates a positive C_0 -semigroup $T(t)(t \geq 0)$, $F: \mathbb{R} \times E \times E \to E$ is a continuous mapping which is ω -periodic in t. Under the ordered conditions on the nonlinearity F concerning the growth exponent of the semigroup $T(t)(t \geq 0)$ or the first eigenvalue of the operator A, we obtain the existence and asymptotic stability of the positive ω -periodic mild solutions by applying operator semigroup theory. In the end, an example is given to illustrate the applicability of our abstract results.

Key Words: Evolution equations with delay; Positive periodic solutions; Existence and uniqueness; Asymptotic stability; Positive C_0 -semigroup

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1 Introduction and main results

The theory of partial differential equation with delay has extensive physical background and realistic mathematical model, and it has undergone a rapid development in the last fifty years. Such equations are often more realistic to describe natural phenomena than those without delay(see [6, 23]).

The problems concerning periodic solutions of partial differential equations with delay are an important area of investigation since they can take into account seasonal fluctuations occurring in the phenomena appearing in the models, and have been studied by some researchers in recent years. The existence and asymptotic stability of periodic solutions of evolution equation with delay have attracted much attention, see [4, 24, 16, 17, 18, 12, 7, 22, 14, 15].

Specilly, in [12], by using analytic semigroups theory and an integral inequality with delays, Li discussed the time periodic solution for the evolution equation with multiple delays in a Hilbert space H

$$u'(t) + Au(t) = F(t, u(t), u(t - \tau_1), \dots, u(t - \tau_n)), \quad t \in \mathbb{R},$$
 (1.1)

where $A:D(A)\subset H\to H$ is a positive definite selfadjoint operator, having compact resolvent and the first eigenvalue $\lambda_1>0,\ F:\mathbb{R}\times H^{n+1}\to H$ is a nonlinear mapping which is ω -periodic in t, and $\tau_1,\tau_2,\cdots,\tau_n$ are positive constants which denote the time delays. Under the following assumptions

(F1)
$$||F(t, v_0, v_1, \dots, v_n)|| \le \sum_{i=0}^n \beta_i ||v_i|| + K, t \in \mathbb{R}, (v_0, \dots, v_n) \in H^{n+1},$$

$$(F2) \sum_{i=0}^{n} \beta_i < \lambda_1,$$

(F3)
$$||F(t, v_0, v_1, \dots, v_n) - F(t, w_0, w_1, \dots, w_n)|| \le \sum_{i=0}^n \beta_i ||v_i - w_i||,$$

the author obtained the existence and uniqueness of time ω -periodic solutions to Eq. (1.1), where $\beta_0, \beta_1, \dots, \beta_n$ and K are positive constants. Moreover, strengthening the condition (F2) as follow

$$(F2^*) \beta_0 + \sum_{i=1}^n e^{\lambda_1 \tau_i \beta_i} < \lambda_1,$$

the unique time periodic solution was asymptotically stable. However, because of the limitation of the research space and the particularity of the operator, the results of the research are not universal, and sometimes the conditions (F1) and (F3) are not easy to verify in applications.

In [7], Kpoumiè et al discussed the existence of periodic solutions for the fol-

lowing nonautonomous partial functional differential equation with delay

$$u'(t) = A(t)u(t) + L(t, u_t) + F(t, u_t), \quad t \ge 0$$
(1.2)

in a Banach space X, where $(A(t))_{t\geq 0}$ is a family of linear operators on X, L and F are given continuous mappings and ω -periodic with respect to the first argument, the history u_t , for $t\geq 0$, is defined from $(-\infty,0]$ to X by

$$u_t(s) = u(t+s), \quad s \in (-\infty, 0].$$

By using Massera's approach and fixed point for multivalued maps, they proved the existence of an ω -periodic solution.

Recently, In [22], under suitable assumptions, such as the ultimate boundedness of the solutions of equations, Wang and Zhu established a theorem on periodic solutions to equations of this kind by using the Horn fixed-point theorem. In [14, 15], Liang et al also studied nonautonomous evolutionary equations with time delay and impulsive. Under the nonlinear term satisfying continuous and Lipschitzian, the proved the existence theorem for periodic mild solutions to the nonautonomous delay evolution equations by Horn's fixed point theorem or Sadovskii's Fixed Point Theorem. However, in all these works, the key assumption or process of prior boundedness of solutions was employed.

In many practice models, such as heat conduction equation, neutron transport equation, reaction diffusion equation, etc., only positive periodic solutions are significant. In [10], the existence and uniqueness of positive periodic mild solutions for the evolution equation without delay

$$u'(t) + Au(t) = F(t, u(t)), \quad t \in \mathbb{R}, \tag{1.3}$$

are obtained in an ordered Banach space E, where -A is the infinitesimal generator of a positive C_0 -semigroup, $F: \mathbb{R} \times E \to E$ is a continuous mapping which is ω -periodic in t. Recently, under the ordered conditions on the nonlinearity F, the existence and asymptotic stability of positive ω -periodic mild solutions for the evolution equation (1.3) have been obtained by applying operator semigroup theory, monotone iterative technique and some fixed point theorems in an ordered Banach space E, see [13]. However, to the best of our knowledge, there are few papers to study the existence and asymptotic stability of positive ω -periodic solutions for the evolution equation with delay. Furthermore, for the abstract evolution equation

without delay, the periodic solutions have been discussed by more authors, see [2, 3, 8, 9, 1, 11, 20, 25] and references therein.

Motivated by the papers mentioned above, by means of operator semigroup theory and some fixed point theorems, we will use a completely different method to improve and extend the results mentioned above, which will make up the research in this area blank.

Our discussion will be made in the framework of ordered Banach spaces. Let E be an ordered Banach space E, whose positive cone K is normal cone with normal constant N. Let $A:D(A)\subset E\to E$ is a closed linear operator and -A generates a positive C_0 -semigroup $T(t)(t\geq 0)$ in E, the nonlinear function $F:\mathbb{R}\times E\times E\to E$ is a continuous mapping and for every $x,y\in K$, F(t,x,y) is ω -periodic in t. In this paper, we consider the following abstract evolution equation with delay

$$u'(t) + Au(t) = f(t, u(t), u(t - \tau)), \qquad t \in \mathbb{R},$$
 (1.4)

We will study the existence and asymptotic stability of positive ω -periodic mild solutions for (1.4) under some new conditions by applying the Leray-Schauder fixed point theorem in an ordered Banach space E. More precisely, the nonlinear term satisfies order conditions concerning the growth exponent of the semigroup $T(t)(t \geq 0)$ or the first eigenvalue of the operator A.

For C_0 -semigroup $T(t)(t \ge 0)$, there exist M > 0 and $\gamma \in \mathbb{R}$ such that (see [19])

$$||T(t)|| \le Me^{\gamma t}, \quad t \ge 0. \tag{1.5}$$

Let

$$\nu_0 = \inf\{\gamma \in \mathbb{R} | \text{ There exists } M > 0 \text{ such that } ||T(t)|| \le Me^{\gamma t}, \ \forall t \ge 0\},$$

then ν_0 is called the growth exponent of the semigroup $T(t)(t \ge 0)$. Furthermore, ν_0 can be also obtained by the following formula

$$\nu_0 = \limsup_{t \to +\infty} \frac{\ln ||T(t)||}{t}.$$

If C_0 -semigroup T(t) is continuous in the uniform operator topology for every t > 0 in E, it is well known that ν_0 can also be determined by $\sigma(A)$ (see [21])

$$\nu_0 = -\inf\{\text{Re}\lambda | \lambda \in \sigma(A)\}, \tag{1.6}$$

where -A is the infinitesimal generator of C_0 -semigroup $T(t)(t \ge 0)$. We know that $T(t)(t \ge 0)$ is continuous in the uniform operator topology for t > 0 if $T(t)(t \ge 0)$ is compact semigroup (see [21]).

For the abstract delay evolution equation (1.4), we obtain the following results:

Theorem 1.1. Let -A generate an exponentially stable positive compact semigroup $T(t)(t \ge 0)$ in E, that is $\nu_0 < 0$. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H1) there are positive constants C_1, C_2 satisfying $C_1 + C_2 \in (0, |\nu_0|)$ and a function $h_0 \in C_{\omega}(\mathbb{R}, K)$ such that

$$F(t, x, y) \le C_1 x + C_2 y + h_0(t), \quad t \in \mathbb{R}, \ x, y \in K,$$

holds, then Eq.(1.4) has at least one positive ω -periodic mild solution u.

Theorem 1.2. Let -A generate an exponentially stable positive compact semigroup $T(t)(t \ge 0)$ in E. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H2) there are positive constants C_1, C_2 satisfying $C_1 + C_2 \in (0, |\nu_0|)$, such that for any $x_i, y_i \in K(i = 1, 2)$ with $x_1 \leq x_2, y_1 \leq y_2$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \le C_1(x_2 - x_1) + C_2(y_2 - y_1), \qquad t \in \mathbb{R},$$

holds, then Eq. (1.4) has a unique positive ω -periodic mild solution u.

Now, we strengthen the condition (H2) in Theorem 1.2, we can obtain the following asymptotic stability result of the periodic solution:

Theorem 1.3. Let -A generate an exponentially stable positive compact semigroup $T(t)(t \geq 0)$ in E. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H3) there are positive constants C_1, C_2 satisfying $C_1 + C_2 e^{-\nu_0 \tau} \in (0, |\nu_0|)$, such that for any $x_i, y_i \in K(i = 1, 2)$ with $x_1 \leq x_2, y_1 \leq y_2$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \le C_1(x_2 - x_1) + C_2(y_2 - y_1), \qquad t \in \mathbb{R},$$

holds, then the unique positive ω -periodic mild solution of Eq.(1.4) is globally asymptotically stable.

Furthermore, we assume that the positive cone K is regeneration cone. By the characteristic of positive semigroups (see [8]), for sufficiently large $\lambda_0 > -\inf\{Re\lambda|\lambda \in \sigma(A)\}$, we have that $\lambda_0 I + A$ has positive bounded inverse operator $(\lambda_0 I + A)^{-1}$. Since $\sigma(A) \neq \emptyset$, the spectral radius $r((\lambda_0 I + A)^{-1}) = \frac{1}{dist(-\lambda_0,\sigma(A))} > 0$. By the famous Krein-Rutmann theorem, A has the first eigenvalue λ_1 , which has a positive eigenfunction e_1 , and

$$\lambda_1 = \inf\{Re\lambda | \lambda \in \sigma(A)\},\tag{1.7}$$

that is $\nu_0 = -\lambda_1$. Hence, by Theorem 1.1, Theorem 1.2 and Theorem 1.3, we have the following results.

Corollary 1.4 Let -A generate an exponentially stable positive compact semigroup $T(t)(t \geq 0)$ in E. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H1') there are positive constants C_1, C_2 satisfying $C_1 + C_2 \in (0, \lambda_1)$ and a function $h_0 \in C_{\omega}(\mathbb{R}, K)$ such that

$$F(t, x, y) \le C_1 x + C_2 y + h_0(t), \quad t \in \mathbb{R}, \ x, y \in K,$$

holds, then Eq.(1.4) has at least one positive ω -periodic mild solution u.

Corollary 1.5 Let -A generate an exponentially stable positive compact semigroup $T(t)(t \geq 0)$ in E. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H2') there are positive constants C_1, C_2 satisfying $C_1 + C_2 \in (0, \lambda_1)$, such that for any $x_i, y_i \in K(i = 1, 2)$ with $x_1 \leq x_2, y_1 \leq y_2$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \le C_1(x_2 - x_1) + C_2(y_2 - y_1), \qquad t \in \mathbb{R},$$

holds, then Eq. (1.4) has a unique positive ω -periodic mild solution u.

Corollary 1.6 Let -A generate an exponentially stable positive compact semigroup $T(t)(t \geq 0)$ in E. Assume that $F : \mathbb{R} \times K \times K \to K$ is a continuous mapping which is ω -periodic in t. If the following condition

(H3') there are positive constants C_1, C_2 satisfying $C_1 + C_2 e^{\lambda_1 \tau} \in (0, \lambda_1)$, such that for any $x_i, y_i \in K(i = 1, 2)$ with $x_1 \leq x_2, y_1 \leq y_2$,

$$F(t, x_2, y_2) - F(t, x_1, y_1) \le C_1(x_2 - x_1) + C_2(y_2 - y_1), \qquad t \in \mathbb{R},$$

holds, then the unique positive ω -periodic mild solution of Eq. (1.4) is globally asymptotically stable.

Remark 1.7. In Corollary 1.4 and Corollary 1.5, since λ_1 is the first eigenvalue of A, the condition $C_1 + C_2 < \lambda_1$ in (H1') and (H2') cannot be extended to $C_1 + C_2 \le \lambda_1$. Otherwise, periodic problem (1.4) does not always have a mild solution. For example, $F(t, x, y) = \frac{\lambda_1}{2}x + \frac{\lambda_1}{2}y$.

Remark 1.8. It is clear that our results can also be extended to the evolution equation with multiple delays (1.1). In this case, the conditions (H1') and (H3') have improved the conditions (F1) and (F3), and our conditions are easy to verify in applications. Hence, our results of the positive periodic solutions, improve and generalize the results in [12]. On the other hand, we delete the Lipschitz conditions on nonlinearity. In this case, the prior estimate of solutions are not employed. Thus, compared with the existence results in [7, 14, 15], our conclusions are new in some respects in some respects.

The paper is organized as follows. Section 2 provides the definitions and preliminary results to be used in theorems stated and proved in the paper. The proofs of Theorems 1.1-1.3 are based on positive C_0 -semigroups theory, Leray-Schauder fixed point theorem and an integral inequality of Bellman, which will be given in Section 3. In the last section, we give an example to illustrate the applicability of the abstract results.

2 Preliminaries

In this section, we introduce some notions, definitions, and preliminary facts which are used through this paper.

Let J denote the infinite interval $[0, +\infty)$ and $h: J \to E$, consider the initial value problem of the linear evolution equation

$$\begin{cases} u'(t) + Au(t) = h(t), \ t \in J, \\ u(0) = x_0 \end{cases}$$
 (2.1)

It is well known [19, Chapter 4, Theorem 2,9], when $x_0 \in D(A)$ and $h \in C^1(J, E)$, the initial value problem (2.1) has a unique classical solution $u \in C^1(J, E) \cap C(J, E_1)$ expressed by

$$u(t) = T(t)x_0 + \int_0^t T(t-s)h(s)ds,$$
 (2.2)

where $E_1 = D(A)$ is Banach space with the graph norm $\|\cdot\|_1 = \|\cdot\| + \|A\cdot\|$. Generally, for $x_0 \in E$ and $h \in C(J, E)$, the function u given by (2.2) belongs to C(J, E) and it is called a mild solution of the linear evolution equation (2.1).

Let $C_{\omega}(\mathbb{R}, E)$ denote the Banach space $\{u \in C(\mathbb{R}, E) | u(t + \omega) = u(t), t \in \mathbb{R}\}$ endowed the maximum norm $||u||_C = \max_{t \in [0,\omega]} ||u(t)||$. Evidently, $C_{\omega}(\mathbb{R}, E)$ is also an order Banach space with the partial order " \leq " induced by the positive cone $K_C = \{u \in C_{\omega}(\mathbb{R}, E) | u(t) \geq \theta, t \in \mathbb{R}\}$ and K_C is also normal with the normal constant N.

Given $h \in C_{\omega}(\mathbb{R}, E)$, for the following linear evolution equation corresponding to Eq.(1.4)

$$u'(t) + Au(t) = h(t), \quad t \in \mathbb{R}, \tag{2.3}$$

we have the following result.

Lemma 2.1.([10]) If -A generates an exponentially stable positive C_0 -semigroup $T(t)(t \geq 0)$ in E, then for $h \in C_{\omega}(\mathbb{R}, E)$, the linear evolution equation (2.3) exists a unique positive ω -periodic mild solution u, which can be expressed by

$$u(t) = (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)h(s)ds := (Ph)(t), \tag{2.4}$$

and the solution operator $P: C_{\omega}(\mathbb{R}, E) \to C_{\omega}(\mathbb{R}, E)$ is a positive bounded linear operator with the spectral radius $r(P) \leq \frac{1}{|\nu_0|}$.

Proof. For any $\nu \in (0, |\nu_0|)$, there exists M > 0 such that

$$||T(t)|| \le Me^{-\nu t} \le M, \quad t \ge 0.$$
 (2.5)

In E, define the equivalent norm $|\cdot|$ by

$$|x| = \sup_{t>0} ||e^{\nu t}T(t)x||,$$

then $||x|| \le |x| \le M||x||$. By |T(t)| we denote the norm of T(t) in $(E, |\cdot|)$, then for $t \ge 0$, it is easy to obtain that $|T(t)| < e^{-\nu t}$. Hence, $(I - T(\omega))$ has bounded inverse operator

$$(I - T(\omega))^{-1} = \sum_{n=0}^{\infty} T(n\omega),$$
 (2.6)

and its norm satisfies

$$|(I - T(\omega))^{-1}| \le \frac{1}{1 - |T(\omega)|} \le \frac{1}{1 - e^{-\nu\omega}}.$$
 (2.7)

Set

$$x_0 = (I - T(\omega))^{-1} \int_0^{\omega} T(t - s)h(s)ds := Bh,$$
 (2.8)

then the mild solution u(t) of the linear initial value problem (2.1) given by (2.2) satisfies the periodic boundary condition $u(0) = u(\omega) = x_0$. For $t \in \mathbb{R}^+$, by (2.2) and the properties of the semigroup $T(t)(t \geq 0)$, we have

$$u(t+\omega) = T(t+\omega)u(0) + \int_0^{t+\omega} T(t+\omega-s)h(s)ds$$

$$= T(t)\Big(T(\omega)u(0) + \int_0^\omega T(\omega-s)h(s)ds\Big) + \int_0^t T(t-s)h(s-\omega)ds$$

$$= T(t)u(0) + \int_0^t T(t-s)h(s)ds = u(t).$$

Therefore, the ω -periodic extension of u on \mathbb{R} is a unique ω -periodic mild solution of Eq.(2.3). By (2.2) and (2.8), the ω -periodic mild solution can be expressed by

$$u(t) = T(t)B(h) + \int_0^t T(t-s)h(s)ds$$
$$= (I - T(\omega))^{-1} \int_{t-\omega}^t T(t-s)h(s)ds := (Ph)(t). \tag{2.9}$$

Evidently, by the positivity of semigroup $T(t)(t \geq 0)$, we can obtain that $P: C_{\omega}(\mathbb{R}, E) \to C_{\omega}(\mathbb{R}, E)$ is a positive bounded linear operator. By (2.7) and (2.9), we have

$$|(Ph)(t)| \le |(I - T(\omega))^{-1}| \int_{t-\omega}^{t} |T(t-s)h(s)| ds$$

$$\le \frac{1}{1 - e^{-\nu\omega}} \int_{t-\omega}^{t} e^{-\nu(t-s)} |h|_C ds$$

$$\le \frac{1}{\nu} |h|_C,$$

which implies that $|P| \leq \frac{1}{\nu}$. Therefore, $r(P) \leq |P| \leq \frac{1}{\nu}$. Hence, by the arbitrary of $\nu \in (0, |\nu_0|)$, we have $r(P) \leq \frac{1}{|\nu_0|}$. This completes the proof of Lemma 2.1. \square In the proof of our main results, we also need the following results.

Lemma 2.2.(Leray-Schauder fixed point theorem [5]) Let Ω be convex subset of Banach space E with $\theta \in \Omega$, and let $Q: \Omega \to \Omega$ be compact operator. If the set $\{u \in \Omega | u = \eta Q u, \ 0 < \eta < 1\}$ is bounded, then Q has a fixed point in Ω .

3 Proof of the main results

Proof of Theorem 1.1. Evidently, the normal cone K_C is a convex subset of Banach space $C_{\omega}(\mathbb{R}, E)$ and $\theta \in K_C$. Consider the operator Q defined by

$$Qu = (P \circ \mathcal{F})(u), \tag{3.1}$$

where

$$\mathcal{F}(u)(t) := F(t, u(t), u(t-\tau)), \quad u \in K_C. \tag{3.2}$$

From the positivity of semigroup of $T(t)(t \ge 0)$ and the conditions of Theorem 1.1, it is easy to see that $Q: K_C \to K_C$ is well defined. From (3.1) and (3.2), it follows that

$$(Qu)(t) = (I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)F(s, u(s), u(s-\tau))ds, \quad t \in \mathbb{R}.$$
 (3.3)

By the definition P, the positive ω -periodic mild solution of Eq.(1.4) is equivalent to the fixed point of the operator Q. In the following, we will prove Q has a fixed point by applying the famous Leray-Schauder fixed point theorem.

At first, we prove that Q is continuous on K_C . Let $\{u_m\} \subset K_C$ be a sequence such that $u_m \to u \in K_C$ as $m \to \infty$, so for every $t \in \mathbb{R}$, $\lim_{m \to \infty} u_m(t) = u(t)$. Since $F : \mathbb{R} \times K^{n+1} \to K$ is continuous, then for every $t \in \mathbb{R}$, we get

$$F(t, u_m(t), u_m(t-\tau)) \to F(t, u(t), u(t-\tau)), \quad m \to \infty.$$
(3.4)

By (3.3) and the Lebesgue dominated convergence theorem, for every $t \in \mathbb{R}$, we have

$$\| (Qu_{m})(t) - (Qu)(t) \|$$

$$= \| (I - T(\omega))^{-1} \Big(\int_{t-\omega}^{t} T(t-s) F(s, u_{m}(s), u_{m}(s-\tau)) ds$$

$$- \int_{t-\omega}^{t} T(t-s) F(s, u(s), u(s-\tau)) ds \Big) \|$$

$$\leq \| (I - T(\omega))^{-1} \| \cdot \int_{t-\omega}^{t} \| T(t-s) \| \cdot \| F(s, u_{m}(s), u_{m}(s-\tau))$$

$$- F(s, u(s), u(s-\tau)) \| ds$$

$$\leq CM \cdot \int_{t-\omega}^{t} \| F(s, u_{m}(s), u_{m}(s-\tau)) - F(s, u(s), u(s-\tau)) \| ds,$$

$$\leq CM \cdot \int_{t-\omega}^{t} \| F(s, u_{m}(s), u_{m}(s-\tau)) - F(s, u(s), u(s-\tau)) \| ds,$$

$$(3.5)$$

where $C = ||(I - T(\omega))^{-1}||$. Therefore, we can conclude that

$$||Qu_m - Qu|| \to 0, \quad m \to \infty.$$
 (3.6)

Thus, $Q: K_C \to K_C$ is continuous.

Subsequently, we show that Q maps every bounded set in K_C into a bounded set. For any R > 0, let

$$\overline{\Omega}_R := \{ u \in K_C | \|u\|_C \le R \}. \tag{3.7}$$

For each $u \in \overline{\Omega}_R$, from the continuity of F, we know that there exists $M_1 > 0$ such that

$$||F(t, u(t), u(t-\tau))|| \le M_1, \quad t \in \mathbb{R},$$
 (3.8)

hence, we get

$$\|(Qu)(t)\| = \|(I - T(\omega))^{-1} \int_{t-\omega}^{t} T(t-s)F(s, u(s), u(s-\tau))ds\|$$

$$\leq \|(I - T(\omega))^{-1}\| \int_{t-\omega}^{t} \|T(t-s)\| \cdot \|F(s, u(s), u(s-\tau))\|ds$$

$$\leq CM \int_{t-\omega}^{t} M_{1}ds$$

$$\leq CMM_{1}\omega := \overline{R}.$$

Therefore, $Q(\overline{\Omega}_R)$ is bounded.

Next, we demonstrate that $Q(\overline{\Omega}_R)$ is equicontinuous. For every $u \in \overline{\Omega}_R$, by the periodicity of u, we only consider it on $[0, \omega]$. Set $0 \le t_1 < t_2 \le \omega$, we get that

$$Qu(t_{2}) - Qu(t_{1})$$

$$= (I - T(\omega))^{-1} \int_{t_{2} - \omega}^{t_{2}} T(t_{2} - s) F(s, u(s), u(s - \tau)) ds$$

$$-(I - T(\omega))^{-1} \int_{t_{1} - \omega}^{t_{1}} T(t_{1} - s) F(s, u(s), u(s - \tau)) ds$$

$$= (I - T(\omega))^{-1} \int_{t_{2} - \omega}^{t_{1}} (T(t_{2} - s) - T(t_{1} - s)) F(s, u(s), u(s - \tau)) ds$$

$$-(I - T(\omega))^{-1} \int_{t_{1} - \omega}^{t_{2} - \omega} T(t_{1} - s) F(s, u(s), u(s - \tau)) ds$$

$$+(I - T(\omega))^{-1} \int_{t_1}^{t_2} T(t_2 - s) F(s, u(s), u(s - \tau)) ds$$
:= $I_1 + I_2 + I_3$,

It is clear that

$$||Qu(t_2) - Qu(t_1)|| \le ||I_1|| + ||I_2|| + ||I_3||. \tag{3.9}$$

Now, we only need to check $||I_i||$ tend to 0 independently of $u \in \overline{\Omega}_R$ when $t_2 - t_1 \to 0$, i = 1, 2, 3. From the definition of I_i , we can easily see

$$\begin{split} \|I_1\| & \leq C \cdot \int_{t_2 - \omega}^{t_1} \|(T(t_2 - s) - T(t_1 - s))\| \cdot \|F(s, u(s), u(s - \tau))\| ds \\ & \leq C M_1 \int_{t_2 - \omega}^{t_1} \|(T(t_2 - s) - T(t_1 - s))\| ds \\ & \rightarrow 0, \text{ as } t_2 - t_1 \rightarrow 0, \\ \|I_2\| & \leq C \cdot \int_{t_1 - \omega}^{t_2 - \omega} \|(T(t_1 - s))\| \cdot \|F(s, u(s), u(s - \tau))\| ds \\ & \leq C M M_1(t_2 - t_1) \\ & \rightarrow 0, \text{ as } t_2 - t_1 \rightarrow 0, \\ \|I_3\| & \leq C \cdot \int_{t_1}^{t_2} \|(T(t_2 - s))\| \cdot \|F(s, u(s), u(s - \tau))\| ds \\ & \leq C M M_1(t_2 - t_1) ds \\ & \rightarrow 0, \text{ as } t_2 - t_1 \rightarrow 0. \end{split}$$

As a result, $||Qu(t_2) - Qu(t_1)||$ tends to 0 independently of $u \in \overline{\Omega}_R$ as $t_2 - t_1 \to 0$, which means that $Q(\overline{\Omega}_R)$ is equicontinuous.

Now, we prove that $(Q\overline{\Omega}_R)(t)$ is relatively compact in K for all $t \in \mathbb{R}$. We define a set $(Q_{\varepsilon}\overline{\Omega}_R)(t)$ by

$$(Q_{\varepsilon}\overline{\Omega}_R)(t) := \{(Q_{\varepsilon}u)(t) | u \in \overline{\Omega}_R, \ 0 < \varepsilon < \omega, \ t \in \mathbb{R}\}, \tag{3.10}$$

where

$$(Q_{\varepsilon}u)(t) = (I - T(\omega))^{-1} \int_{t-\omega}^{t-\varepsilon} T(t-s)F(s, u(s), u(s-\tau))ds$$
$$= T(\varepsilon)(I - T(\omega))^{-1} \int_{t-\omega}^{t-\varepsilon} T(t-s-\varepsilon)F(s, u(s), u(s-\tau))ds.$$

Then the set $(Q_{\varepsilon}\overline{\Omega}_R)(t)$ is relatively compact in K since the operator $T(\varepsilon)$ is compact in K. For any $u \in \overline{\Omega}_R$ and $t \in \mathbb{R}$, from the following inequality

$$\|Qu(t) - Q_{\varepsilon}u(t)\|$$

$$= \|(I - T(\omega))^{-1} \left(\int_{t-\omega}^{t} T(t-s)F(s, u(s), u(s-\tau))ds - \int_{t-\omega}^{t-\varepsilon} T(t-s)F(s, u(s), u(s-\tau))ds \right) \|$$

$$\leq C \int_{t-\varepsilon}^{t} \|T(t-s)F(s, u(s), u(s-\tau))\|ds$$

$$\leq CMM_{1}\varepsilon, \tag{3.11}$$

one can obtain that the set $(Q\overline{\Omega}_R)(t)$ is relatively compact in K for all $t \in \mathbb{R}$.

Thus, the Arzela-Ascoli theorem guarantees that $Q: K_C \to K_C$ is a compact operator.

Finally, we prove the set $\Lambda(Q) := \{u \in K_C | u = \eta Q u, \forall 0 < \eta < 1\}$ is bounded. For every $u \in K_C$, by (3.2) and the condition (H1), we have

$$\theta \leq \mathcal{F}(u)(t) = F(t, u(t), u(t - \tau))$$

$$\leq C_1 u(t) + C_2 u(t - \tau) + h_0(t), \quad t \in \mathbb{R}. \tag{3.12}$$

Define an operator $\mathcal{B}: C_{\omega}(\mathbb{R}, K) \to C_{\omega}(\mathbb{R}, K)$ as following:

$$\mathcal{B}u(t) = C_1 u(t) + C_2 u(t - \tau), \quad t \in \mathbb{R}, \ u \in K_C.$$
(3.13)

It is easy to see that $\mathcal{B}: C_{\omega}(\mathbb{R}, K) \to C_{\omega}(\mathbb{R}, K)$ is a positive bounded linear operators satisfying $\|\mathcal{B}\| \leq C_1 + C_2$. Let $u \in \Lambda(Q)$, then there is a constant $\eta \in (0, 1)$ such that $u = \eta Q u$. Therefore, by the definition of Q, Lemma 2.1 and (3.12), we have

$$\theta \leq u(t) = \eta(Qu)(t) < (Qu)(t)$$

$$= P \circ \mathcal{F}(u)(t) \leq P(\mathcal{B}u(t) + h_0(t))$$

$$= \mathcal{B}Pu(t) + Ph_0(t) < \mathcal{B}P \circ Q(t) + Ph_0(t)$$

$$\leq cP(cPu(t) + Ph_0(t)) + Ph_0(t)$$

$$= \mathcal{B}^2 P^2 u(t) + \mathcal{B}P^2 h_0(t) + Ph_0(t).$$

inductively, we can see

$$u(t) \le \mathcal{B}^n P^n u(t) + \mathcal{P} h_0(t), \quad n = 1, 2, \cdots, \tag{3.14}$$

where, $\mathcal{P} = \mathcal{B}^{n-1}P^n + \mathcal{B}^{n-2}P^{n-1} + \cdots + \mathcal{B}P^2 + P$ is a bounded linear operator, and there exists a constant $M_2 > 0$ such that $\|\mathcal{P}\| \leq M_2$. Hence, by the normality of the cone K_C , we can see

$$||u||_{C} < N||\mathcal{B}^{n}|| \cdot ||P^{n}|| \cdot ||u||_{C} + M_{2}||h_{0}||_{C},$$

$$\leq N(C_{1} + C_{2})^{n} \cdot ||P^{n}|| \cdot ||u||_{C} + M_{2}||h_{0}||_{C}.$$

From the spectral radius of Gelfand formula $\lim_{n\to\infty} \sqrt[n]{\|P^n\|} = r(P) = \frac{1}{|\nu_0|}$, and the condition (H1), when n is large enough, we get that $(C_1 + C_2)^n \cdot \|P^n\| < \frac{1}{N}$, then

$$||u||_C < \frac{M_2||h_0||_C}{1 - N(C_1 + C_2)^n \cdot ||P^n||},$$
 (3.15)

which implies that $\Lambda(Q)$ is bounded. By the Leray-Schauder fixed point theorem of compact operator, the operator Q has at least one fixed point u in K_C , which is a positive ω -periodic mild solution of the delay evolution equation (1.4). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. From the condition (H3), it is easy to see that the condition (H1) holds. Hence by Theorem 1.1, Eq.(1.4) has positive ω -periodic mild solutions. Let $u_1, u_2 \in K_C$ be the positive ω -periodic solutions of Eq.(1.4), then they are the fixed points of the operator $Q = P \circ \mathcal{F}$. Let us assume $u_1 \leq u_2$, by the definition of \mathcal{F} and the condition (H3), for any $t \in \mathbb{R}$, we have

$$\mathcal{F}(u_2)(t) - \mathcal{F}(u_1)(t)$$

$$\leq C_1(u_2(t) - u_1(t)) + C_2(u_2(t - \tau) - u_1(t - \tau))$$

$$= \mathcal{B}(u_2(t) - u_1(t)),$$

where \mathcal{B} is defined by (3.13). Thus, we can obtain that

$$\theta \le u_2(t) - u_1(t) = (Qu_2)(t) - (Qu_1)(t)$$

$$= P((\mathcal{F}u_2)(t) - (\mathcal{F}u_1)(t)) \le \mathcal{B}P(u_2(t) - u_1(t))$$

$$\le \cdots \le \mathcal{B}^n P^n(u_2(t) - u_1(t)).$$

By the normality of the cone K_C , we can see

$$||u_2 - u_1||_C \le N||\mathcal{B}^n|| \cdot ||P^n|| \cdot ||u_2 - u_1||_C, \tag{3.16}$$

From the proof of Theorem 1.1, when n is large enough, $N \|\mathcal{B}^n\| \cdot \|P^n\| < 1$, so $\|u_2 - u_1\|_C = 0$, it follows that $u_2 \equiv u_1$. Thus, Eq.(1.4) has only one positive ω -periodic mild solution.

In order prove Theorem 1.3, we need discuss the existence and uniqueness of the initial value problem of the nonlinear delay evolution equation (1.4).

Let $C([-\tau, \infty), E)$ denote the Banach space endowed the maximum norm $||u||_C = \sup_{t \in [-\tau,\infty)} ||u(t)||$. For $u \in C([-\tau,\infty), E)$ and $t \in [0,\infty)$, we denote $u_t \in C([-\tau,0], E)$, $u_t(s) = u(t+s)$, $s \in [-\tau,0]$. Let $\varphi \in C([-\tau,0], E)$, we study the following initial value problem of the evolution equation with delay

$$\begin{cases}
 u'(t) + Au(t) = F(t, u(t), u(t - \tau)), & t \in J, \\
 u_0 = \varphi,
\end{cases}$$
(3.17)

where -A generates positive C_0 -semigroup $T(t)(t \ge 0)$ in E and $F: J \times K \times K \to K$ be continuous.

If there exists $u \in C([-\tau, \infty), E)$ satisfying $u(t) = \varphi(t)$ for $-\tau \le t \le 0$ and

$$u(t) = T(t)u(0) + \int_0^t T(t-s)F(s, u(s), u(s-\tau)), \ t \ge 0,$$
 (3.18)

then u is called a mild solution of the nonlinear initial value problem (3.17). Furthermore, when $\varphi \in C([-\tau, 0], K)$, it follows that $u(t) \geq \theta(t \in [-\tau, \infty))$ by the characteristic of positive semigroups.

For the nonlinear initial value problem (3.17), we have the following result.

Lemma 3.1. Let E be an ordered Banach space whose positive cone K is normal cone, -A generate a positive compact semigroup $T(t)(t \ge 0)$ in E. Assume that $F: \mathbb{R} \times K \times K \to K$ is continuous, $\varphi \in C([-\tau, 0], K)$. If F satisfies the condition (H3), then the initial value problem (3.17) has a unique positive mild solution $u \in C([-\tau, \infty), K)$.

Proof By the condition (H3), we have

$$F(t, x, y) \le C_1 x + C_2 y + F(t, \theta, \theta), \quad t \in [0, \infty), x, y \in K,$$
 (3.19)

namely, $||F(t, x, y)|| \le C_1 ||x|| + C_2 ||y|| + K$, where $K = \max_{t \in [0,\omega]} ||F(t, \theta, \theta)||$. Thus, by a standard argument as in [12, Theorem 3.1], we can prove that initial value problem (3.17) exists positive mild solution.

Next, we show the uniqueness. Let $u_1, u_2 \in C([-r, \infty), K)$ be the positive solutions of the initial value problem (3.17), hence they satisfy the initial value condition $u_1(t) = u_2(t) = \varphi(t)(-\tau \le t \le 0)$ and (3.18). Let us assume that $u_1 \le u_2$, by the condition (H3), for every $t \ge 0$, we have

$$u_{2}(t) - u_{1}(t)$$

$$= \int_{0}^{t} T(t-s) \Big(F(s, u_{2}(s), u_{2}(s-\tau)) - F(s, u_{1}(s), u_{1}(s-\tau)) \Big) ds$$

$$\leq \int_{0}^{t} T(t-s) \Big(C_{1}(u_{2}(s) - u_{1}(s)) + C_{2}(u_{2}(s-\tau) - u_{1}(s-\tau)) \Big) ds.$$

Define an operator $\mathcal{B}: C([-r,\infty),K) \to C([-r,\infty),K)$ as following:

$$\mathcal{B}u(t) = C_1 u(t) + C_2 u(t - \tau), \quad t \ge 0, \ u \in C([-\tau, \infty), K).$$

Clearly, \mathcal{B} is a linear bounded operator with $\|\mathcal{B}\| \leq C_1 + C_2$. Therefore,

$$u_2(t) - u_1(t) \le \int_0^t T(t-s)\mathcal{B}(u_2(s) - u_1(s))ds,$$

which implies that

$$||u_2(t) - u_1(t)|| \le \int_0^t ||T(t-s)|| \cdot (C_1 + C_2) \cdot ||u_2(s) - u_1(s)|| ds.$$

By the Gronwall-Bellman inequality, we have $||u_2(t) - u_1(t)|| \equiv 0 (t \geq 0)$. Hence, $u_1 \equiv u_2$. \square

The proof of Theorem 1.3 needs the following integral inequality of Bellman type with delay.

Lemma 3.2.([12])Let us assume that $\phi \in C([-r, \infty), J)$ and there exist positive constants c_1, c_2 , such that ϕ satisfy the integral inequality

$$\phi(t) \le \phi(0) + c_1 \int_0^2 \phi(s)ds + c_2 \int_0^t \phi(s - \tau)ds, \quad t \ge 0.$$
 (3.20)

Then $\phi(t) \leq \|\phi\|_{C[-\tau,0]} e^{(c_1+c_2)t}$ for every $t \geq 0$, where $\|\phi\|_{[-\tau,0]} = \max_{t \in [-\tau,0]} |\phi(t)|$.

Proof of Theorem 1.3. By Theorem 1.2, the delay evolution equation (1.4) has a unique positive ω -periodic mild solution $u^* \in C_{\omega}(\mathbb{R}, K)$. For any $\varphi \in C([-\tau, \infty), K)$, the initial value problem (3.17) has a unique global positive mild solution $u = u(t, \varphi) \in C([-r, \infty), K)$ by Lemma 3.1.

By the semigroup representation of the solutions, u^* and u satisfy the integral equation (3.18). Thus, by (3.18) and assumption (H3), for any $t \ge 0$, we have

$$u(t) - u^{*}(t) \leq T(t)(u(0) - u^{*}(0)) + \int_{0}^{t} T(t - s)(C_{1}(u(s) - u^{*}(s))) + C_{2}(u(s - \tau) - u^{*}(s - \tau)))ds.$$

$$(3.21)$$

Since $T(t)(t \ge 0)$ is an exponentially stable positive C_0 -semigroup, that is the growth exponent $\nu_0 < 0$, hence, by the property of semigroup, there is a number $M \ge 1$ such that

$$||T(t)|| \le Me^{\nu_0 t}, \quad t \ge 0.$$

We choose the equivalent norm $|\cdot|_0$ by

$$|x|_0 = \sup_{t>0} ||e^{\nu_0 t} T(t)x||,$$

then $||x|| \le |x|_0 \le M||x||$. Thus, we denote the norm of $T(t)(t \ge 0)$ in $(E, |\cdot|_0)$ by $|T(t)|_0$ and $|T(t)|_0 < e^{-\nu_0 t}$ for $t \ge 0$.

Now, by (3.21) and the normality of cone K in E, we have

$$|u(t) - u^{*}(t)|_{0}$$

$$\leq |T(t)|_{0} \cdot |u(0) - u^{*}(0)|_{0}$$

$$+ \int_{0}^{t} |T(t - s)|_{0} \Big(C_{1}|u(s) - u^{*}(s)|_{0} + C_{2}|u(s - \tau) - u^{*}(s - \tau)|_{0} \Big) ds$$

$$\leq e^{\nu_{0}t}|u(0) - u^{*}(0)|_{0}$$

$$+ \int_{0}^{t} e^{\nu_{0}(t - s)} \Big(C_{1}|u(s) - u^{*}(s)|_{0} + C_{2}|u(s - \tau) - u^{*}(s - \tau)|_{0} \Big) ds$$

$$\leq e^{\nu_{0}t}|u(0) - u^{*}(0)|_{0} + C_{1}e^{\nu_{0}t} \int_{0}^{t} e^{-\nu_{0}s} (|u(s) - u^{*}(s)|_{0}) ds$$

$$+ C_{2}e^{\nu_{0}(t - \tau)} \int_{0}^{t} e^{-\nu_{0}(s - \tau)} (|u(s - \tau) - u^{*}(s - \tau)|_{0}) ds.$$

For $t \in [-\tau, \infty)$, setting $\phi(t) = e^{-\nu_0 t} |u(t) - u^*(t)|_0$, from the inequality above, it follows that

$$\phi(t) \le \phi(0) + C_1 \int_0^t \phi(s)ds + C_2 e^{-\nu_0 \tau} \int_0^t \phi(s - \tau)ds. \tag{3.22}$$

Hence, by Lemma 3.2, we have

$$e^{-\nu_0 t} |u(t) - u^*(t)|_0 = \phi(t) \le C(\varphi) e^{(C_1 + Ce^{-\nu_0 \tau})t}, \quad t \ge 0,$$
 (3.23)

where $C(\varphi) = \max_{s \in [-\tau,0]} \{e^{-\nu_0 s} | \varphi(s) - u^*(s)|_0\}$. By the assumption (H3), $\sigma := -\nu_0 - (C_1 + Ce^{-\nu_0 \tau}) > 0$, and form (3.23) it follows that

$$|u(t) - u^*(t)|_0 \le C(\varphi)e^{-\sigma t} \to 0 \quad (t \to \infty).$$

Thus, the positive ω -periodic solution u^* is globally asymptotically stable and it exponentially attracts every positive solution of the initial value problem. This completes the proof of Theorem 1.3.

4 Application

In this section, we present one example, which indicates how our abstract results can be applied to concrete problems. Let $\overline{\Omega} \in \mathbb{R}^n$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Let

$$A(x,D)u = -\sum_{i,j=1}^{N} a_{ij}(x)D_iD_ju + \sum_{j=1}^{N} a_j(x)D_ju + a_0(x)u,$$
(4.1)

be a uniformly elliptic differential operator in $\overline{\Omega}$, whose coefficients $a_{ij}(x), a_j(x)$ $(i, j = 1, \dots, n)$ and $a_0(x)$ are Höder-continuous on $\overline{\Omega}$, and $a_0(x) \geq 0$. We let B = B(x, D) be a boundary operator on $\partial \Omega$ of the form:

$$Bu := b_0(x)u + \delta \frac{\partial u}{\partial \beta}, \tag{4.2}$$

where either $\delta = 0$ and $b_0(x) \equiv 1$ (Dirichlet boundary operator), or $\delta = 1$ and $b_0(x) \geq 0$ (regular oblique derivative boundary operator; at this point, we further assume that $a_0(x) \not\equiv 0$ or $b_0(x) \not\equiv 0$), β is an outward pointing, nowhere tangent vector field on $\partial\Omega$. Let λ_1 be the first eigenvalue of elliptic operator A(x, D) under the boundary condition Bu = 0. It is well known ([2, Theorem 1.16],) that $\lambda_1 > 0$.

Under the above assumptions, we discuss the existence, uniqueness and asymptotic stability of positive time ω -periodic solutions of the semilinear parabolic boundary value problem

$$\begin{cases}
\frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t),u(x,t-\tau)), & x \in \Omega, t \in \mathbb{R}, \\
Bu = 0, x \in \partial\Omega,
\end{cases} (4.3)$$

where $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ a local Hölder-continuous function which is ω -periodic in $t, \tau > 0$ denotes the time delay.

Let $E = L^p(\Omega)(p > 1)$, $K = \{u \in E | u(x) \ge 0 \text{ a.e. } x \in \Omega\}$, then E is an ordered Banach space, whose positive cone K is a normal regeneration cone. Define an operator $A: D(A) \subset E \to E$ by:

$$D(A) = \{ u \in W^{2,p}(\Omega) | B(x,D)u = 0, \ x \in \partial\Omega \}, \quad Au = A(x,D)u. \tag{4.4}$$

If $a_0(x) \geq 0$, then -A generates an exponentially stable analytic semigroup $T_p(t)(t \geq 0)$ in E (see [3]). By the maximum principle of elliptic operators, we know that $(\lambda I + A)$ has a positive bounded inverse operator $(\lambda I + A)^{-1}$ for $\lambda > 0$, hence $T_p(t)(t \geq 0)$ is a positive semigroup (see [8]). From the operator A(x, D) has compact resolvent in $L^p(\Omega)$, we obtain $T_p(t)(t \geq 0)$ is also a compact semigroup (see [19]). Therefore, by Corollary 1.4, we have the following result.

Theorem 4.1. Assume that $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a local Hölder-continuous function which is ω -periodic in t and satisfies $f(x,t,u,v) \geq 0$ for $(x,t,u,v) \in (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$. If the following condition holds:

(H4) there are constants $C_1, C_2, C_1 + C_2 \in (0, \lambda_1)$ and a function $h \in C_{\omega}(\overline{\Omega} \times \mathbb{R})$ satisfying $h(x, t) \geq 0$ such that

$$f(x, t, u, v) \le C_1 u + C_2 v + h(x, t), \quad (x, t) \in \overline{\Omega} \times \mathbb{R}, u, v \ge 0,$$

then the delay parabolic boundary value problem (4.3) has at least one positive ω -periodic solution $u \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$.

Proof Let $u(t) = u(\cdot,t)$, $F(t,u(t),u(t-\tau)) = f(\cdot,t,u(\cdot,t),u(\cdot,t-\tau))$, then the delay parabolic boundary value problem (4.3) can be reformulated as the abstract evolution equation (1.4) in E. From the assumption, it is easy to see that the conditions of Corollary 1.4 are satisfied. By Corollary 1.4, the delay parabolic boundary value problem (4.3) has a time positive ω -periodic mild solution $u \in C_{\omega}(\mathbb{R}, E)$. By the analyticity of the semigroup $T_p(t)(t \geq 0)$ and the regularization

method used in [3], we can see that $u \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$ is a classical time ω -periodic solution of the equation (4.3). This completes the proof of the theorem.

From Corollary 1.5 and Theorem 4.1, we obtain the uniqueness result.

Theorem 4.2. Assume that $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a local Hölder-continuous function which is ω -periodic in t and satisfies $f(x,t,u,v) \geq 0$ for $(x,t,u,v) \in (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$. If the following condition holds:

(H5) there are constants $C_1, C_2, C_1 + C_2 \in (0, \lambda_1)$ such that for $y_i, z_i \in K(i = 1, 2), y_1 \leq y_2, z_1 \leq z_2$,

$$f(x, t, y_2, z_2) - f(x, t, y_1, z_1) \le C_1(y_2 - y_1) + C_2(z_2 - z_1), \quad t \in \mathbb{R},$$

then the parabolic boundary value problem (4.3) has a unique positive ω -periodic solution $u^* \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$.

Let $\varphi \in C(\Omega \times [-\tau, \infty))$, define a mapping $t \mapsto \varphi(\cdot, t)$, then we can see $\varphi \in C([-\tau, 0], X)$. Consider the semilinear delay parabolic initial boundary value problem

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + A(x,D)u(x,t) = f(x,t,u(x,t),u(x,t-\tau)), & x \in \Omega, \ t \ge 0, \\ Bu = 0, \quad x \in \partial\Omega, \\ u(x,t) = \varphi(x,t), (x,t) \in \Omega \times [-\tau,0], \end{cases}$$

$$(4.5)$$

From Lemma 3.1, we can obtain the following existence and uniqueness results.

Lemma 4.1. Let $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a local Hölder-continuous function, and $f(x,t,u,v) \geq 0$ for every $(x,t,u,v) \in (\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$. If the following condition (H6) there are constants C_1, C_2 satisfying $C_1 + C_2 e^{\lambda_1 \tau} \in (0, \lambda_1)$, such that for any $y_i, z_i \in K(i = 1, 2)$ with $y_1 \leq y_2, z_1 \leq z_2$,

$$f(x, t, y_2, z_2) - f(x, t, y_1, z_1) \le C_1(y_2 - y_1) + C_2(z_2 - z_1), \quad (x, t) \in \Omega \times \mathbb{R},$$

holds, then the delayed parabolic initial boundary value problem (4.5) has a uniqueness positive solution $u \in C([-\tau, \infty), L^p(\overline{\Omega})) \cap C^{2,1}(\overline{\Omega} \times (0, \infty))$.

Hence, From Corollary 1.6, we can derive the asymptotic stability of the positive ω -periodic solution for the delay parabolic boundary value problem (4.3).

Theorem 4.3 Assume that $f: \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a local Hölder-continuous function which is ω -periodic in t and satisfies $f(x,t,u,v) \geq 0$ for $(x,t,u,v) \in$

 $(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+)$. If the condition (H6) holds, then the unique positive ω -periodic solution $u^* \in C^{2,1}(\overline{\Omega} \times \mathbb{R})$ of equation (4.3) is globally asymptotically stable and it exponentially attracts every positive solution of the initial value problem.

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