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SPDE in Hydrodynamic: Recent Progress and Prospects

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Preface

It is a commonly accepted fact in the mathematical scientific community that the rigorous understanding of turbulence and related questions in hydrodynamics is one of the most important problems in mathematics and one of the challenging tasks for the future development of the theory of partial differential equations in particular, but also of analysis in general. One of the central open problems, namely the well posedness of the 3D-Navier–Stokes system has been selected as one of the millennium problems and has resisted all attempts to solve it up to the present day.

Over more than half a century a lot of deep mathematics was developed to tackle these problems. One of the approaches was to use stochastic analysis based on modifying the equations (as e.g. Euler, Navier–Stokes and Burgers) adding a noise term. The idea here was to use the smoothing effect of the noise on the one hand, but also to discover new phenomena of stochastic nature on the other hand. In addition, this was also motivated by physical considerations, aiming at including perturbative effects, which cannot be modelled deterministically, due to too many degrees of freedom being involved, or aiming at taking into account different time scales of the components of the underlying dynamics. Today we look back on 30 years of mathematical work implementing probabilistic ideas into the area. During the last few years activities have become even more intense and several new groups in the world working on probability have turned their attention to these classes of highly interesting SPDEs of fundamental importance in Physics.

In a sentence, one of the purposes of the course was to understand the link between the Euler and Navier–Stokes equations or their stochastic versions and the phenomenological laws of turbulence. The idea can be better understood by analogy with Feynmans description of statistical mechanics: in that field on the one hand one has the Hamilton (or Schrödinger) equations for the dynamics of molecules and, on the other hand, the macroscopic laws of thermodynamics. In between there is the concept of Gibbs measures, so the theory looks like the ascent of a mountain, from Hamilton equations to Gibbs measures (here ergodicity is a central topic), and a descent from Gibbs measures to thermodynamics. Translating this viewpoint in the realm of turbulent fluid dynamics, on the one side we have the Navier–Stokes equations as dynamical equations (that we commonly accept as essentially correct). On the other side, we know a number of phenomenological laws, like the Kolmogorov's scaling (which he proposed in 1941, therefore called K41 scaling) or the multifractal scalings, which fit experimental and numerical data to some extent, but miss a rigorous foundation and presumably require some correction. If we aim at an analogy with statistical mechanics, the missing point is a concept of Gibbs measure linking these two extreme parts of the theory.

Three courses of eight hours each were delivered to develop these ideas, both for the deterministic and the stochastic case.

Sergio Albeverio presented an approach to (deterministic) Euler and stochastic Navier–Stokes equations in two dimensions based on invariant measures and renormalization methods. His last lecture was devoted to asymptotic methods for functional integrals.

Franco Flandoli started from some basic results on Navier–Stokes equations in three dimensions, discussing topics as existence of martingale solutions, construction of a transition semigroup, ergodicity and continuous dependence on initial conditions. One of the main results was a preliminary step to prove well posedness of the stochastic 3D-Navier–Stokes equations by showing the existence of a Markov selection. He also has presented a review of the Kolmogorov K41 scaling law and some rigorous results on it for the stochastic Navier–Stokes equations.

Finally, Yakov Sinai described some rigorous mathematical results for d-dimensional (determinisitic) Navier–Stokes systems. In this direction he explained the power series and diagrams method for the Fourier transform of Navier–Stokes equations and the Foias-Temam Theorem. He also presented some recent results on the one-dimensional Burgers equation with random forcing, that is, in the stochastic case.

Afternoon sessions were devoted to research seminars delivered by the participants.

We thank the lecturers and all participants for their contributions to the success of this Summer School.

Finally, we thank the CIME Scientific Committee for giving us the opportunity to organize this meeting and the CIME staff for their efficient and continuous help.

Bielefeld and Pisa 2008

Giuseppe Da Prato
Michael Röckner

Contents

Some Methods of Infinite Dimensional Analysis in Hydrodynamics: An Introduction

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1 Introduction

Mathematical modeling and numerical simulation for the study of fluids are topics of great interest, both for our understanding of the phenomena related to fluids and for applications. In fact, we are still far from a complete understanding of fluid phenomena. There is nowadays an increasing interplay of approaches based on deterministic modeling and on stochastic modeling. Already Leonardo da Vinci was fascinated, observed carefully and made drawings of the vortex formation in turbulent fluids. L. Euler formulated the equation of motion for the ideal case of inviscid fluids, H. Navier in 1822 and C.H. Stokes in 1845 introduced the most studied of fluid models, namely the one described by the "Navier–Stokes equations". These equations constitute a challenging prototype for non linear parabolic differential equations. At the same time they are the starting point for the building of discrete models used in numerical simulation of fluids. A deep mathematical analysis of the Navier–Stokes equations was initiated by J. Leray (1933). He, N. Kolmogorov and others also introduced and developed concepts used in what can be called a "theory of turbulence". N. Wiener, according to his autobiographical account, developed a theory of Brownian motion as a first step for constructing an infinite dimensional analysis, capable eventually to handle problems of turbulence. Developments in the study of fluids, in particular in their turbulent behaviour, are connected with areas like non linear functional analysis, the theory of dynamical systems, ergodic theory, the study of invariant measures and stochastic analysis. We shall not discuss here the derivation of Navier–Stokes equations, but just mention some recent work on it. There are indeed attempts of deriving the Navier–Stokes equations from microdynamics, but the theory is far from complete. The most ambitious starts from quantum dynamics, derives in a certain limit a suitable Hamiltonian reversible dynamics for particles, then from these by certain limit operations the (irreversible) Boltzmann equation and the Navier–Stokes equations; see, e.g., [ABGS00],

[NY03], [LM01]. The program of understanding turbulence phenomena starting from the Navier–Stokes equations is also still widely open, see, e.g. [Fri95].

In these lectures we shall concentrate on certain mathematical results concerning the case of deterministic Euler and stochastic Navier–Stokes equations for incompressible fluids. For general references we refer to [Fri95], [Tem83], [CF88], [VF88], [VKF79], [Lio96], [MP94], [NPe01], [Che96b], [Che98], [Che04], [CK04a], [Con95], [Con94], [Con01a], [Con01b], [FMRT01], [LR02], [MB02] [Tem84] and [Bir60] and for a discussion of challenging open problems see, e.g. [Fef06], [Can00], [Can04], [CF03], [Cho94], [Con95], [Con01a], [FMRT01], [Gal01], [ES00a], [Hey90], [Ros06] and [FMB03]. We shall concentrate particularly on the study of invariant measures associated with the above equations for fluids. On the one hand, this follows an analogy with the statistical mechanical approach to classical particle systems and ergodic theory, see, e.g. [Min00], [Rue69]. On the other hand, it follows Kolmogorov's suggestion, see e.g. [ER85], of adding small stochastic perturbations ("noise") in classical dynamical systems, so to construct invariant measures and then study what happens when removing the noise.

The content of our lecture is as follows: in Section 2 we shall study the deterministic Euler equation and construct certain natural invariant measures for it. We also relate this analysis with the study of a certain Hamiltonian system describing vortices ("vortex models"). In Section 3 we shall study the stochastic Navier–Stokes equation with Gaussian space-time white noise and its invariant measure. We also provide brief comments and bibliographical references concerning recent work in directions which are complementary to those described here.

2 The Euler Equation, its Invariants and Associated Invariant Measures

2.1 The Euler Equation

An Euler fluid is the particular case for vanishing viscosity of a fluid described by the Navier–Stokes equation (for an incompressible, i.e. divergenceless and homogeneous, i.e. constant density fluid). It is also called a "perfect fluid".

Its equations of motion express the conservation of mass ($\dot\rho + \operatorname{div}(\rho u) = 0$) which reduces for $\rho =$ constant in space and time to $\operatorname{div}(u) = 0$, and Newton's law. Here one takes into account that one has a "continuum of fluid particles" moving with coordinates $x(t)$ in d–dimensional space $\mathbb{R}^d$ associated with the (smooth) velocity field u: $\dot x(t) = u(t, x(t)) \in \mathbb{R}^d$ (t being the time and $x(t)$ taking values in a subset Λ of $\mathbb{R}^d$ which contains the fluid). The corresponding acceleration is given by:

$$\begin{aligned} \ddot{x} &= \frac{\partial}{\partial t}u(t,x(t)) + \sum_{i=1}^{d} \frac{\partial u}{\partial x_i}(t,x(t))u_i(t,x(t)) \\ &= \frac{\partial}{\partial t}u(t,x(t)) + (u(t,x(t))\cdot\nabla)u(t,x(t)) \equiv \frac{D}{Dt}u(t,x(t)). \end{aligned} \tag{1}$$

$\frac{D}{Dt}$ is the "material derivative". The total force acting on the particles of the fluid can be decomposed in a "stress force" $-\nabla p$ (p being the pressure and ∇ the gradient in $\mathbb{R}^d$) and an external force f. The Newton equation for fluids is then

$$\frac{D}{Dt}u = -\nabla p + f, \tag{2}$$

where $u = (u_1,\dots,u_d) = u(t,x)$, $x \in \Lambda \subset \mathbb{R}^d$, $t \geq 0$. This together with the $\operatorname{div} u = 0$ condition constitutes Euler's equation of motion for a fluid. One has to specify the boundary conditions, usually taken to be $u\cdot n = 0$ ($u \cdot n$ being the component of u normal to $\partial\Lambda$), which is natural when p is interpreted as the pressure in the fluid, or u periodic in the space variables, if Λ is identified with a torus. Moreover one has to specify initial conditions at $t = 0$.

Remarks 2.1.

(i). Euler's equation is an equation for an "ideal fluid", in particular it contains no viscosity term (which would be present if the stress tensor would be more realistic...). The condition $\operatorname{div} u = 0$ *leads by Liouville's theorem (cf. e.g. [AK98]) to volume (and thus also mass) conservation.*

(ii). The form (2) *of a Newton's equation leads naturally to a geometrical picture of Euler's equation as a Hamiltonian system. This has been exploited by P. and T. Ehrenfest and V. Arnold, see, e.g. [AK98], [MEF72], in particular through work by Ebin and Marsden to prove existence results for smooth solutions. The geometrical picture is further exploited e.g. [Ebi84] in [AK98] [Gli03] [Rap02a, Rap03, Rap05].*

(iii). Results on existence respectively uniqueness of solutions of Euler's equation in various spaces and various degrees of smoothness of initial conditions and of the force f have been established. The results are particularly strong for $d = 2$, see e.g. [Ebi84], [Kat67].

2.2 The Euler Equation in Terms of Vorticity

Let us first proceed heuristically assuming that there exist solutions of the Euler equation in the class of vector fields one is interested in. Set $\operatorname{rot} u \equiv \xi$ (ξ is called vorticity function) and from now on assume f is a gradient field (which is natural due to the Newton's equation point of view), i.e., there exists a scalar function $\psi : \mathbb{R}_+ \times \Lambda \to \mathbb{R}$ so that $f = \nabla\psi$. Then Euler's equation can be written as

$$\frac{D}{Dt}\xi = (\xi\cdot\nabla)u \tag{3}$$

with $\xi = \operatorname{rot} u$, and $u \cdot n = 0$ on $\partial\Lambda$ (resp. u periodic if Λ is a torus) and initial conditions. (3) is called "vorticity equation". In fact, to see that (2) implies (3) it suffices to realize that $\frac{1}{2}\nabla(u\cdot u) = u \times \operatorname{rot} u + (u\cdot\nabla)u$, hence

$$(u\cdot\nabla)u = \frac{1}{2}\nabla(u\cdot u) - u \times \operatorname{rot} u$$

Moreover

$$\begin{aligned}\operatorname{rot}(u\times\xi) &= (\xi\cdot\nabla)u - \xi(\nabla\cdot u) - (u\cdot\nabla)\xi + u(\nabla\cdot\xi)\\ &= (\xi\cdot\nabla)u - (u\cdot\nabla)\xi,\end{aligned}$$

where we used that $\nabla\cdot u = 0$ by the incompressibility condition and $\nabla\xi = 0$. Taking then rot of (2) we get, observing that $\operatorname{rot}\left(\frac{1}{2}\nabla(u\cdot u)\right) = 0$:

$$\frac{\partial}{\partial t}\xi = \operatorname{rot}(u\times\xi) = (\xi\cdot\nabla)u - (u\cdot\nabla)\xi,$$

hence $\frac{D}{Dt}\xi = (\xi\cdot\nabla)u$.

For $d = 2$ and for Λ a simply connected domain we can set, using $\operatorname{div} u = 0$,

$$u = \nabla^\perp\varphi,$$

with φ a scalar function, called "stream function", $\nabla^\perp = (-\partial_2, \partial_1)$. Moreover, for $d = 2$ the vorticity vector has only one non vanishing component. We write ξ for this scalar quantity (for $d = 2$): $\xi = \nabla^\perp\cdot u$. Since $\nabla^\perp\cdot u = \nabla^\perp\cdot\nabla^\perp\varphi = \Delta\varphi$, we get $\xi = \Delta\varphi$, so $(\xi\cdot\nabla)u = (\Delta\varphi\cdot\nabla)\nabla^\perp\varphi = 0$. In this case, (3) becomes

$$\frac{D}{Dt}\xi = 0 \tag{4}$$

This expresses the "conservation of vorticity" for $d = 2$. As an equation for φ, (4) reads:

$$\frac{\partial}{\partial t}\Delta\varphi = -\left((\nabla^\perp\varphi).\nabla\right)\Delta\varphi \tag{5}$$

(where we used $\xi = \Delta\varphi$, $(u\cdot\nabla)\xi = ((\nabla^\perp\varphi\cdot\nabla)\Delta\varphi)$).

Remark 2.1. *For a corresponding treatment in the case of non simply connected domains see [AHK89].*

2.3 The Conserved Quantities for the Euler Equation

Proposition 2.1. *Let Λ be either $\mathbb{R}^d$ or a bounded open subset of $\mathbb{R}^d$ with "smooth" boundary $\partial\Lambda$ (in which case one requires that the boundary condition on the Euler velocity u, $u\cdot n = 0$, is satisfied) or the d-dimensional torus $\mathbb{T}^d$. Let u be a classical solution of the Euler equation* (2) *(see, e.g. [Kat67, Ebi84, BA94]). Then the following functionals of u are time independent (i.e. are conserved):*

(i). the energy

$$E(u) = \frac{1}{2}\int_\Lambda u^2 dx$$

(for all $d \geq 1$)

(ii). for $d = 2$: the enstrophy

$$S(u) = \frac{1}{2}\int_\Lambda (\operatorname{rot} u)^2 dx$$

(iii). for $d = 2$: the g–functionals of the vorticity

$$S_g(u) = \int_\Lambda g(\operatorname{rot} u)\, dx,$$

for any $g \in C(\mathbb{R})$, such that the integral exists.

Proof. (a):

$$\frac{\partial}{\partial t}E(u) = \int_\Lambda u\frac{\partial u}{\partial t}dx = \int_\Lambda u(-(u.\nabla)u - \nabla p)dx,$$

where we used (2); integrating by parts and using the boundary conditions and $\operatorname{div} u = 0$, we obtain $\frac{\partial}{\partial t}E(u) = 0$.
(b), (c):

$$\begin{aligned}\frac{\partial}{\partial t}S_g(u) &= \int_\Lambda g'(\operatorname{rot} u)\frac{\partial}{\partial t}\operatorname{rot} u\ dx\\ &= -\int_\Lambda g'(\operatorname{rot} u)(u\cdot\nabla)\operatorname{rot} u\ dx = -\int_\Lambda (u\cdot\nabla)g(\operatorname{rot} u)\ dx,\end{aligned}$$

where we used Leibniz rule $\nabla g(h) = g'(h)\nabla h$ together with (4); integrating by parts and using the boundary conditions and $\operatorname{div} u = 0$, we obtain $\frac{\partial}{\partial t}S_g(u) = 0$.

Remarks 2.2.

(i). The integral S_g for $g(\lambda) = \lambda$ is called circulation.

(ii). The above are essentially all known conserved quantities, called also invariants, see e.g. [Ser84a, Ser84b, Cip99].

(iii). S_g can be expressed in terms of the vorticity ξ respectively stream function φ as follows:

$$S_g(u) = \int_\Lambda g(\xi)dx = \int_\Lambda g(\Delta\varphi)dx.$$

E can be expressed by the stream function φ as

$$E(u) = \frac{1}{2}\int_\Lambda (\nabla^\perp\varphi)^2 dx.$$

2.4 Heuristic Invariant Measures Associated with the Euler Equation

The first observation is that the heuristic "flat measure" du on the space of solutions at time t of the Euler equation (2) does not depend on t, because of the incompressibility conditions $\operatorname{div} u = 0$. This is a heuristic expression of the rigorous fact that the Euler flow $t \mapsto u(t,x)$ preserves volumes (see, e.g. [AK98]).

Let $I(u)$ be any of the conserved quantities of Proposition 2.1. Heuristically $\mu_I(du) =$ "$Z^{-1}\exp(-I(u))du$", with $Z \equiv$ "$\int_\Gamma e^{-I(u)}du$" and Γ being a space of solutions of (2), is an invariant (i.e. time independent) probability measure associated with (2).[1]

Remark 2.2. *From its heuristic expression in the case $I(u) = E(u)$ we see that $\mu_E(du(0,\cdot))$ can be realized rigorously as the Gaussian white noise measure (i.e. the cylinder measure) associated with $L^2(\Lambda,\mathbb{R}^d)$ (with mean zero and unit covariance) (see, e.g. [HKPS93]). Whether this measure is indeed invariant in some sense under the "Euler flow" $\tau_t: \ u(0,\cdot) \mapsto u(t,\cdot)$ is a subtle open question, due to the "bad support properties" of μ_E, τ_t being understood as a map in a space of generalized functions (in the support of μ_E). Henceforth we shall concentrate on the case $d = 2$, and for $I(u)$ of the form b) in Proposition 2.1 (which are "less singular").*

Let us consider for simplicity the case $\Lambda = \mathbb{T}^2$ (a 2–dimensional torus, identifiable with $[0,2\pi]\times[0,2\pi]$), with space periodic boundary conditions for the solution u of (2) (the general case of a bounded Λ is discussed in [AHK89]; a corresponding explicit discussion for $\Lambda = \mathbb{R}^2$ seems to be lacking). Introduce the complex Hilbert space $L^2(\Lambda)$. For $\varphi \in L^2(\Lambda)$ we have the Fourier expansion

$$\varphi(x) = \sum_{k\in\mathbb{Z}^2} \omega_k \frac{e^{ik\cdot x}}{2\pi}, \tag{6}$$

with $\omega \equiv (\omega_k) \in \ell^2(\mathbb{Z}^2)$ in the sense that $\omega_k \in \mathbb{C}$ for all $k \in \mathbb{Z}^2$ and $\|\omega\|^2_{\ell^2(\mathbb{Z}^2)} \equiv \sum_{k\in\mathbb{Z}^2} |\omega_k|^2 < \infty$.

Remarks 2.3.

(i). Correspondingly we have $u = \sum_k u_k e_k$, with $e_k(x) \equiv \frac{e^{ik\cdot x}}{2\pi}\frac{k^\perp}{|k|}$, $k \in \mathbb{Z}^2$, $k \neq 0$, $k^\perp = (-k_2, k_1)$. Then $u_k = i|k|\omega_k$. The reality of u is equivalent with $\overline{u_k} = -u_{-k}$, $k \in \mathbb{Z}^2$, since $\overline{\omega_k} = \omega_{-k}$ (with $^-$ meaning complex conjugation).

[1] This heuristics appears originally in work by L. Onsager and T.D. Lee and expanded in [Gal76], which provided the inspiration for the first rigorous work on this line in [ARdFHK79], [BF80], [BF81], to which we refer for further references, see also [Gla81], [Gla77], [KM80], [RS91] for further physical discussions along these lines.

(ii). The modification of φ by an additive constant does not change the relation $u = \nabla^{\perp}\varphi$. By this we see that we can assume, without losing generality, that $\int_{\Lambda} \varphi(x)dx = 0$. This corresponds to taking $\omega_0 = 0$.

Hence the above expansion (6) for φ can be written

$$\varphi(x) = \sum_{k \in \mathbb{Z}_0^2} \omega_k \frac{e^{ik\cdot x}}{2\pi}, \tag{7}$$

with $\overline{\omega_k} = \omega_{-k}$, $\mathbb{Z}_0^2 \equiv \mathbb{Z}^2 \backslash \{\mathbf{0}\}$. Let us express the invariants E, S of Proposition 2.1 in terms of the variables $(\omega_k)_{k \in \mathbb{Z}_0^2}$ (denoting them again by the symbols E, S):

$$E(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}_0^2} |k|^2 |\omega_k|^2 \ , \tag{8}$$

$$S(\omega) = \frac{1}{2} \sum_{k \in \mathbb{Z}_0^2} |k|^4 |\omega_k|^2. \tag{9}$$

Let us moreover remark that at least for the simple case where $I = E$ or $I = S$ a rigorous meaning can be immediately given to the measure $\mu_{\gamma I}$ as Gaussian product of measures, for any $\gamma > 0$. In fact in these cases

$$I(\omega) = \sum_{k \in \mathbb{Z}_+^2} |k|^{\alpha(I)} |\omega_k|^2$$

with $\alpha(I) = 2$ for $I = E$ and $\alpha(I) = 4$ for $I = S$. We have set $\mathbb{Z}_+^2 = \left\{k \in \mathbb{Z}_0^2 : k_1 > 0 \text{ or } \{k_1 = 0,\ k_2 > 0\}\right\}$, because it is enough to consider half of the indices k. Then

$$\mu_{\gamma I}(d\omega) = \bigotimes_{k \in \mathbb{Z}_+^2} \mu_\gamma^k(d\omega_k),$$

where μ_γ^k is the Gaussian measure on $\mathbb{C} \cong \mathbb{R} \times \mathbb{R}$ given by

$$\mu_\gamma^k(d\omega_k) = Z_k^{-1} e^{-\gamma |k|^{\alpha(I)} |\omega_k|^2} d\omega_k,$$

$$Z_k = \int_{\mathbb{C}} e^{-\gamma |k|^{\alpha(I)} |\omega_k|^2} d\omega_k$$

($|\omega_k|^2 = x_k^2 + y_k^2, d\omega_k = dx_k\, dy_k$ for $\omega_k = x_k + iy_k;\ x_k, y_k \in \mathbb{R}$). $\mu_{\gamma I}$ can be realized as a cylinder probability measure on $\mathbb{C}^{\mathbb{Z}_+^2}$. It is identifiable with the standard centered Gaussian measure $N(0, |\cdot|^2_{\mathcal{H}(\gamma I)})$, with $\mathcal{H}(\gamma I)$ the complex Hilbert space $\mathcal{H}(\gamma I) \equiv \left\{\omega = (\omega_k)_{k \in \mathbb{Z}_+^2} : I(\omega) < \infty\right\}$, with scalar product $(\omega, \omega')_{\gamma I} \equiv \gamma \sum_{k \in \mathbb{Z}^2_+} |k|^{\alpha(I)} \overline{\omega_k} \omega'_k$. It is well known that to $\mu_{\gamma I}$ one can give a meaning as a σ–additive probability measure on a larger space $\tilde{\mathcal{H}}(\gamma I) \supset \mathcal{H}(\gamma I)$, the embedding being an Hilbert–Schmidt operator (see, e.g. [Kuo75], [DPZ92]). The nontrivial problem is then to show that these rigorous measures are indeed invariant under the "Euler flow" in some rigorous sense, see Sections 2.8–2.10. In the next section we shall discuss the Euler equation in terms of the Fourier variables ω, in order to later on discuss the invariance of above $\mu_{\gamma I}$ under the "Euler flow".

2.5 The Euler Equation in Terms of the Fourier Components of the Stream Function

Proposition 2.2. *Let $\varphi(t)$ be a classical solution of the Euler equation* (5) *on the 2-torus $\mathbb{T}^2$, with initial condition $\varphi(0) = \varphi_0$. Let $\omega(t) = (\omega_k(t))_{k \in \mathbb{Z}^2_0}$ be the Fourier components of $\varphi(t)$:*

$$\varphi(t)(x) = \sum_{k \in \mathbb{Z}^2_0} \omega_k(t) \frac{e^{ik \cdot x}}{2\pi}, \ \ x \in \mathbb{T}^2,$$

with $\varphi_0(x) = \sum_{k \in \mathbb{Z}^2_0} \omega_k(0) \frac{e^{ik \cdot x}}{2\pi}$.
Then

$$\frac{d}{dt} \omega_k(t) = B_k(\omega(t)), \ \ k \in \mathbb{Z}^2_0 \tag{10}$$

with

$$B_k(\omega) \equiv \frac{1}{2\pi} \sum_{\substack{h \in \mathbb{Z}^2_0 \\ h \neq k}} c_{h,k} \omega_h \omega_{k-h}$$

$$c_{h,k} \equiv -\frac{(h^\perp \cdot k)(h \cdot k)}{|k|^2} + \frac{h^\perp \cdot k}{2} \qquad \text{for any } h \neq 0, h \neq k.$$

Viceversa, if the sequence $\omega \equiv (\omega_k)$ satisfies (10)*, then φ satisfies* (5).

Proof. This is easily seen by computation. For details see, e.g. [ARdFHK79].

Proposition 2.3. *Let B_k be as in Proposition 2.2. Then*

(i). $\frac{\partial}{\partial \omega_k} B_k(\omega) = 0$

(ii). $\overline{B_k(\omega)} = B_{-k}(\omega)$, *for all* $k \in \mathbb{Z}^2_0$

Proof. These properties are immediate consequences of the definition of B_k.

Remark 2.3. *These equations hold for $\omega = \omega(t)$, for all $t \geq 0$.*

Proposition 2.4. *Let $\omega(t)$ be as in Proposition 2.2. Then for all* $t \geq 0$:

(i). $\sum_{k\in\mathbb{Z}_0^2} |k|^2 B_k(\omega(t))\overline{\omega_k(t)} = 0,$ *if* $E(\omega(0)) < \infty$

(ii). $\sum_{k\in\mathbb{Z}_0^2} |k|^4 B_k(\omega(t))\overline{\omega_k(t)} = 0,$ *if* $S(\omega(0)) < \infty$.

Proof.

(i). Follows by computation from $\frac{d}{dt}E(\omega(t)) = 0$, bearing in mind (8).
(ii). Follows by computation from $\frac{d}{dt}S(\omega(t)) = 0$, bearing in mind (9).

Remarks 2.4.

(i). Independently of the time independence of $E(\omega)$ and $S(\omega)$, properties (i), (ii) can also be seen to hold by exploiting the particular form of B_k.

(ii). A corresponding result holds for the Galerkin approximation to the Euler equation, obtained by taking the equation system (4) only for $k \in I_N$ (I_N is the subset of $\mathbb{Z}_0^2$ so that $|k| \le N$ and $I_N = -I_N$), i.e.

$$\frac{d}{dt}\omega_k(t) = B_k^N(\omega(t)) \qquad \textit{for } k \in \mathbb{Z}_0^2, |k| \le N$$

with $B_k^N(\omega) = \frac{1}{2\pi} \sum_{\substack{h \\ 0<|h|\le N \\ 0<|k-h|\le N}} c_{h,k}\omega_h\omega_{k-h}$.

2.6 The Necessity of Looking for Singular Solutions. Divergence of the Energy with Respect to the Enstrophy Measure

For the discussion of the Euler equation it is necessary to relate the potentially invariant measures $\mu_{\gamma I}$ to the Euler equation itself. A first step is to give a meaning to $B_k(\omega)$ for all ω in the support of $\mu_{\gamma I}$. For the Gaussian cases $I = E$ and $I = S$, it turns out that the second one is better in this sense. We call $\mu_\gamma = \mu_{\gamma S}$ the "enstrophy measure" (with parameter $\gamma > 0$). We are going to show in the next section that $B_k \in L^2(\mu_\gamma)$ (for all $k \in \mathbb{Z}_0^2$, $\gamma > 0$), and thus $B_k(\omega)$ has a meaning for almost all ω in the support of μ_γ. Before doing so, let us however point out that the support of μ_γ is "bad" in the sense that the energy $E(\omega)$ is infinite for all ω in the support of μ_γ. In fact we have the following

Proposition 2.5. *Let $E^N(\omega) \equiv \frac{1}{2}\sum_{\substack{k\in\mathbb{Z}_0^2 \\ 0<|k|\le N}} |k|^2|\omega_k|^2$. Then $E^N \in L^2(\mu_\gamma)$ for all $N \in \mathbb{N}$ and $E^N \uparrow +\infty$ as $N \to \infty$. However $: E^N(\omega) :\equiv E^N(\omega) - \mathbb{E}_{\mu_\gamma}(E^N(\cdot))$ (with $\mathbb{E}_{\mu_\gamma}$ the expectation with respect to μ_γ) is in $L^2(\mu_\gamma)$ and converges in $L^2(\mu_\gamma)$ as $N \to \infty$.*

Remark 2.4. *The $L^2(\mu_\gamma)$–limit of $: E^N :$ as $N \to \infty$ is written $: E :$ and called the μ_γ-renormalized energy.*

Proof. $\mathbb{E}_{\mu_\gamma}(E^N(\cdot)) = \frac{1}{2}\sum_{\substack{k\in\mathbb{Z}_0^2 \\ 0<|k|\le N}} |k|^2 \mathbb{E}_{\mu_\gamma}(|\omega_k|^2)$. But $\mathbb{E}_{\mu_\gamma}(|\omega_k|^2) = \frac{2}{\gamma|k|^4}$ (from the definition of μ_γ). Hence $\mathbb{E}_{\mu_\gamma}(E^N)$ diverges logarithmically as $N \to \infty$. A similar computation shows, however, that $:E^N:$ is a Cauchy sequence in $L^2(\mu_\gamma)$, hence it converges to an $L^2(\mu_\gamma)$ limit $:E:$ (see [ARdFHK79] for details). By subsequences then $:E^N:(\omega) \to :E:(\omega)$ for μ_γ-a.e. ω, which implies, together with the above divergence of $\mathbb{E}_{\mu_\gamma}(E^N)$, that $E^N(\omega)$ diverges for μ_γ - a.e. ω, as $N \to \infty$.

Since the energy $E(\omega) = \lim_N E^N(\omega)$ is infinite for μ_γ-a.e. ω, the solutions of the Euler equation we are interested in are solutions of infinite energy. Some results for these singular (or generalized) solutions will be presented in Section 2.10. The technique exploits the invariant measure μ_γ; the situation is thus very different from the general analysis of solutions of Euler's equation with infinite energy (see, e.g. [Shn97]).

Remark 2.5. *One can show that* $e^{-\beta :E:} \in L^1(\mu_\gamma)$ *for all* $\beta > -\gamma$ *(see [ARdFHK79, AHK89, CC95]). It follows then that*

$$\mu_{\beta,\gamma}(d\omega) \equiv \frac{e^{-\beta :E:(\omega)}\mu_\gamma(d\omega)}{\int e^{-\beta :E:(\omega)}\mu_\gamma(d\omega)}$$

is a probability measure associated with both the energy and enstrophy. $\mu_{\beta,\gamma}$ *is heuristically invariant under the Euler flow* $\omega(0) \mapsto \omega(t)$ *(being constructed from invariant functions and the heuristic invariant flat measure). In the next section we shall discuss more closely the invariance of* $\mu_\gamma = \mu_{0,\gamma}$ *(and* $\mu_{\beta,\gamma}$*).*

2.7 Relation of the Enstrophy Measure μ_γ with the Euler Equation

Proposition 2.6. *For any* $k \in \mathbb{Z}_0^2$, B_k *is the* $L^2(\mu_\gamma)$*–limit for* $N \to \infty$ *of its Galerkin approximations* $B_k^N(\omega)$.

Proof. Set for simplicity $\mathbb{E}_{\mu_\gamma} \equiv \mathbb{E}$.
For $\omega = (\omega_k)_{k\in\mathbb{Z}_0^2} \in \operatorname{supp}\mu_\gamma$, the ω_k are independent and μ_γ^k–distributed random variables, i.e. Gaussian centered with covariance $\mathbb{E}(\omega_k\omega'_k) = \frac{2}{\gamma|k|^4}\delta_{kk'}$.
We have

$$\mathbb{E}\left(|B_k^N|^2\right) = \frac{1}{(2\pi)^2}\sum_{h,h'}^{N} c_{h,k}c_{h',k'}\mathbb{E}\left(\overline{\omega_h\omega_{k-h}}\omega_{h'}\omega_{k'-h'}\right)$$

where the sum is over the indices h, h' such that $0 < |h| \le N, 0 < |k-h| \le N, 0 < |h'| \le N, 0 < |k'-h'| \le N$. But it is well known that all the even moments of a Gaussian process can be computed in terms of its covariance.

Hence we get

$$\begin{aligned}\mathbb{E}\left(|B_k^N|^2\right) &= \tfrac{8}{(2\pi\gamma)^2} \sum\nolimits_{h\, \substack{h\neq k \\ |h|\leq N}} c_{h,k}^2 \tfrac{1}{|h|^4|k-h|^4} \\ &\leq \tfrac{8}{(2\pi\gamma)^2} \sum\nolimits_{h\in\mathbb{Z}_0^2\, h\neq k} c_{h,k}^2 \tfrac{1}{|h|^4|k-h|^4} < \infty\end{aligned}$$

(see [ARdFHK79] for details). Hence $B_k^N \in L^2(\mu_\gamma)$. Similarly one shows that $(B_k^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mu_\gamma)$, which proves the proposition.

Remark 2.6. *One can also show that $B_k \in L^p(\mu_\gamma)$, for any $1 \leq p < \infty$. In fact Cipriano [Cip99] has shown that $\sup_k \left(\mathbb{E}_{\mu_\gamma}|B_k|^p\right)^{1/(2p)} \equiv c_{p,\gamma} < \infty$. From this one sees that $\sum_k |k|^{2b}\mathbb{E}_{\mu_\gamma}|B_k|^{2p} < \infty$, for all $b < -1$.*

2.8 The Infinitesimal Invariance of μ_γ. Relation with the "Hopf Approach"

Hopf in [Hop52] introduced a general (heuristic) approach to the study of the equations of hydrodynamics, "lifting" the evolution equation from individual solutions to "statistical solutions". A rigorous implementation of this approach can be obtained as follows (see [ARdFHK79, AHK89]).

Let FC_b^∞ be the space of finitely based functions ("cylinder functions") of $\omega = (\omega_k)$ (smooth and bounded with all derivatives, on the base). Thus $F \in FC_b^\infty$ iff $\exists \tilde{F} \in C_b^\infty(\mathbb{C}^n)$, for some $n \in \mathbb{N}$, so that $F(\omega) = \tilde{F}(\omega_{k_1}, \dots, \omega_{k_n})$, for some $k_i \in \mathbb{Z}_+^2$, $i = 1, \dots, n$. The following proposition can easily be proved.

Proposition 2.7. *If $\omega(t)$ satisfies the Euler equation $\frac{d}{dt}\omega_k(t) = B_k(\omega(t))$, $k \in \mathbb{Z}_0^2$, and $F \in FC_b^\infty$, then*

$$\frac{d}{dt}F(\omega(t)) = BF(\omega(t)),$$

with $B \equiv \sum_k B_k \frac{\partial}{\partial\omega_k}$ (defined on FC_b^∞).

Remarks 2.5.

(i). One calls B the Liouville operator in $L^2(\mu_\gamma)$ (associated with the Euler equation). This describes the dynamics when looked through its action on cylindric smooth functions F. Proposition 2.6 assures that B is well defined on the set FC_b^∞. One calls the equation in Proposition 2.7 "Liouville equation" (associated with the Euler equation).

(ii). For $F \in FC_b^\infty$, $BF = \sum_k B_k \frac{\partial F}{\partial\omega_k}$ and the sum is finite.

Definition 2.1 (Infinitesimal invariance). *A probability measure ν on the space of sequences $\omega = (\omega_k)_{k\in\mathbb{Z}_+^2}$ is called infinitesimal invariant with respect to B (or with respect to the Euler flow) if $\int BF d\nu = 0$ for all $F \in FC_b^\infty$.*

Equivalent to this is $B^*1 = 0$, where 1 is the function identically one in $L^2(\nu)$ and $*$ is the adjoint in $L^2(\nu)$, as seen from

$$\int BF d\nu = \langle 1, BF\rangle_{L^2(\nu)} = \langle B^*1, F\rangle_{L^2(\nu)}.$$

Proposition 2.8.

(i). $B^* \supset -B$, i.e. (B, FC_b^∞) is skew-symmetric in $L^2(\mu_\gamma)$.
(ii). μ_γ is infinitesimal invariant with respect to B.

Proof.

(i). We have $B = \sum_k B_k \frac{\partial}{\partial\omega_k}$ on FC_b^∞. But $\left(B_k \frac{\partial}{\partial\omega_k}\right)^* \supset \left(\frac{\partial}{\partial\omega_k}\right)^* B_k^*$ on FC_b^∞ with $\left(\frac{\partial}{\partial\omega_k}\right)^* \supset -\frac{\partial}{\partial\omega_{-k}} + \gamma|k|^4\omega_k$. We know from Proposition 2.3, ii) that $B_k^* = B_{-k}$; then $B^* \supset \sum_k -\frac{\partial}{\partial\omega_{-k}} B_{-k} + \gamma\sum_k |k|^4\omega_k B_{-k}$. Bearing in mind Proposition 2.4, ii) and Proposition 2.3, i), we prove (i).

(ii). From the definition, we have infinitesimal invariance iff $B^*1 = 0$. By (i) $B^*1 = B1 = 0$ (the latter step is an easy consequence of the specific form of B).

Remarks 2.6.

(i). $\mu_\gamma^N \equiv \bigotimes_{0<|k|\le N} \mu_\gamma^k(d\omega_k)$ is infinitesimal invariant with respect to $B^N \equiv \sum_{0<|k|\le N} B_k^N \dfrac{\partial}{\partial\omega_k}$. In general, all the results on the Galerkin approximations are easily proved.

(ii). Instead of the definition domain FC_b^∞ for B, we could have taken other dense sets in $L^2(\mu_\gamma)$, e.g. of polynomial type (see [AHK89, AF02b]).

2.9 The Question of Uniqueness of Generators of the Euler Flow

Let us set $L \equiv iB$. L is densely defined (e.g. on FC_b^∞) in $L^2(\mu_\gamma)$, symmetric (i.e. $L^* \supset L$, which follows from $B^* \supset -B$). μ_γ is infinitesimal invariant under L in the sense that $\int LF d\mu_\gamma = 0 \ \forall F \in FC_b^\infty$. In addition L is real in $L^2(\mu_\gamma)$ in the sense that it is invariant under the following operation J of conjugation. J is defined in $L^2(\mu_\gamma)$ by

$$JF(\omega) \equiv \overline{F}(-\omega),\ \ F \in L^2(\mu_\gamma);$$

so $J^2F = F$. The J-symmetry of L is expressed by

$$JL = LJ \text{ on } FC_b^\infty.$$

By a theorem of von Neumann (see, e.g. [RS75]) L has then at least one self–adjoint extension in $L^2(\mu_\gamma)$. The question raised in [ARdFHK79, AHK89] was: how many different self–adjoint extensions of L do exist? Let us first remark that any such extension $\tilde{L}$, $\tilde{L}$ being self–adjoint, by Stone's theorem generates a strongly continuous unitary group $(e^{it\tilde{L}})_{t\in\mathbb{R}}$ in $L^2(\mu_\gamma)$. One calls $F \mapsto U_t F = e^{it\tilde{L}}F$, $F \in L^2(\mu_\gamma)$ a generalized Euler flow (associated with the Euler equation in the $L^2(\mu_\gamma)$-sense: this is a form of the Hopf–Koopman–von Neumann approach to the study of evolution equations and ergodic questions. See, e.g, [GGM80]).

Remark 2.7. *If L is essentially self–adjoint on FC_b^∞ in $L^2(\mu_\gamma)$ (i.e. its closure $\bar{L}$ is already self–adjoint), then there is only one self–adjoint extension of L, namely $\bar{L}$ itself. One can show, see [AF04b, AF03] that the generated Euler flow $F \mapsto e^{it\tilde{L}}F$ comes from a μ_γ–measurable point flow Φ_t, in the sense that there exists a μ_γ–measurable flow $\Phi_t :\ \omega(0) \mapsto \Phi_t(\omega(0)) \equiv \omega(t)$, which is μ_γ–preserving and so that $(e^{it\tilde{L}}F)(\omega) = F(\Phi_t^{-1}(\omega))$, for μ_γ–a.e. ω.*

The problem of essential self–adjointness of L seems to be still open. Partial results have however been obtained:

(a). In [AF02b] it is shown that L is dominated by a related self–adjoint operator H of the "Schrödinger type", in the sense that there exist constants $b > 1$, $C_b > 0$ so that

$$2|\langle F, LF\rangle_{L^2(\mu_\gamma)}| \le C_b\,\langle F, HF\rangle_{L^2(\mu_\gamma)},$$

with $H \equiv \sum_{k\in\mathbb{Z}^2_+} |k|^{2b}(\frac{\partial}{\partial\omega_k})^*\frac{\partial}{\partial\omega_k} + V(\omega)$, $V \equiv \sum_{k\in\mathbb{Z}^2_+} |k|^{-2b}|B_k|^2$.

(b). Essential self–adjointness of L is a subtle question, certain finite dimensional analogues of L are indeed not essential self–adjoint on corresponding domains. E.g. $i(x^2\frac{d}{dx} + \frac{d}{dx}x^2)$ on $C_0^\infty(\mathbb{R})$ is symmetric, real (with respect to $Jf(x) = -\bar{f}(x)$), but not essential self–adjoint in $L^2(\mathbb{R}, dx)$, having defect indices $(0,1)$ although it is dominated by the self–adjoint Schrödinger operator $-\frac{d^2}{dx^2} + x^4$, see [RS75, AF02b]. Similar results hold for the corresponding operators in $L^2(\mathbb{R}, \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}}dx)$. Related problems of essential self–adjointness arise for certain operators of quantum field theory, see [AFY04].

(c). The finite-dimensional operators $L^N = iB^N$ (i.e. the Galerkin approximations of Section 2.5) are essentially self-adjoint in $L^2(\mu_\gamma^N)$, when defined on C_b^∞, and have $\mu_\gamma^N(d\omega) = \bigotimes_{0<|k|\le N}\mu_\gamma^k(d\omega_k)$ as invariant measure. But the Galerkin dynamics is well defined globally in time. The problem of essential self-adjointness arises when we pass to the infinite dimensional setting, i.e. for $N \to \infty$. For similar problems, we mention the essential self-adjointness of a Liouville operator in spatial dimension $d = 1$ solved in [MPP78].

(d). A uniqueness result on extensions of L in a space different from $L^2(\mu_\gamma)$ has recently been obtained in [ABF].

(e). For further work on invariant measures for the Euler flow see [CDG85], [CC99], [Pul89]. For related work see also [CC06].

2.10 An Euler Flow in a Sobolev Space of Negative Index

Let us consider for any $p \in \mathbb{R}$ the Sobolev space

$$\mathcal{H}_2^p \equiv \left\{ \omega = (\omega_k)_{k \in \mathbb{Z}_+^2} : \sum_{k \in \mathbb{Z}_+^2} |k|^{2p} |\omega_k|^2 < \infty \right\};$$

for $p = 2$ this is a Hilbert space with scalar product $(\omega, \omega')_{\mathcal{H}_2^2} = S(\omega)$ (S being the enstrophy introduced in Sections 2.3, 2.4). For any $\varepsilon > 0$ the triple $(\mathcal{H}_2^2, \mathcal{H}_2^{1-\varepsilon}, \mu_\gamma)$ constitutes an abstract Wiener space (in L. Gross' sense); in fact the embedding $H_2^2 \subset H_2^{1-\varepsilon}$ is Hilbert–Schmidt, for the proof of this see [AC90], [HKPS93]. One has $\mu_\gamma(\mathcal{H}_2^1) = 0$ (another expression of the fact that the energy E is μ_γ–a.s. infinite) and $\mu_\gamma(\mathcal{H}_2^2) = 0$, but $\mu_\gamma(\mathcal{H}_2^{1-\varepsilon}) = 1$. One can take for support of the enstrophy measure μ_γ the space $\bigcap_{\varepsilon>0} \mathcal{H}_2^{1-\varepsilon}$.

By the way, the support of the energy measure μ_E, defined in Section 2.4, is $\bigcap_{\varepsilon>0} \mathcal{H}_2^{-\varepsilon}$ (for $d = 2$).

The following theorem was established in [AC90]:

Theorem 2.1. *There exists a probability space* $(\Omega, \mathcal{A}, P_\gamma)$ *and a pointwise flow* $\Phi(s,\omega) \equiv (\Phi_k(s,\omega))_{k \in \mathbb{Z}_+^2}$, $s \in \mathbb{R}$, $\omega \in \Omega$ *so that* $\Phi(\cdot, \omega) \in C(\mathbb{R}; \mathcal{H}_2^{1-\alpha})$ *for any* $\alpha > \frac{3}{2}$ *and* $\Phi_k(t,\omega) = \Phi_k(0,\omega) + \int_0^t B_k(\Phi(s,\omega))\, ds$ *for* P_γ*–a.e.* ω, $\forall k \in \mathbb{Z}_+^2$.
Moreover μ_γ *is invariant under* Φ*, in the sense that*

$$\int F(\Phi(t,\omega)) P_\gamma(d\omega) = \mathbb{E}_{\mu_\gamma} F \qquad \forall t, \forall F \in FC_b^\infty$$

Proof. It uses essentially the estimate $B_k \in L^2(\mu_\gamma)$, for details see [AC90].

Remarks 2.7.

(i). One shows that $: E :\in L^2(\mu_\gamma)$ *is an invariant function under* Φ.
(ii). According to the Remark 2.6, the $\mathcal{H}_2^b$*-norm of* $(B_k)_{k \in \mathbb{Z}_+^2}$ *is in* $L^2(\mu_\gamma)$ *for any* $b < -1$.
(iii). Theorem 2.1 gives the existence of solutions of the Euler equation μ_γ*–almost everywhere in a weak probabilistic sense (implying, in particular, a change of probabilistic space).*

2.11 Some Remarks on the Vortex Model and its Relations with the Euler Equation

Consider the vorticity $\xi = \operatorname{rot} u$ concentrated in a finite number n of distinct points:

$$\xi(t,x) = \sum_{j=1}^n \nu_j \delta_{x_j(t)}(x)$$

ν_j is the intensity of the j-th vortex, $x \in \mathbb{T}^2$ the 2D-torus. For an ideal fluid, the time evolution of these point vortices in the vortex model is given by

$$\nu_j \frac{d}{dt} x_j(t) = \nabla^{\perp}_{x_j} \sum_{\substack{h,\,l \\ h \neq l}}{}_1^n \nu_h \nu_l \; G(x_h(t) - x_l(t)), \quad j = 1, \dots, n \tag{11}$$

G is the Green's function of $-\Delta$ on the torus.

Remark 2.8. *Equation* (11) *has the structure of an Hamiltonian system, with Hamiltonian function*

$$H(x) = \frac{1}{2} \sum_{\substack{j,\,l \\ j \neq l}}{}_1^n \nu_j \nu_l G(x_j - x_l).$$

It has been studied by itself, see, e.g. [DP82], [Caf89]. Particular attention has been given to the case where the ν_j take only 2 values, say $\pm\alpha$, $\alpha > 0$. In this case there is a close relation (due to the special logarithmic singularity of G) with the Coulomb gas in 2 dimensions, which has been studied in connection with statistical mechanics [AHK73], [FS76], [CLMP92] (and plasma physics [AHKM85]) . Existence of Gibbs states $Z^{-1} e^{-\beta H(x)} dx_1 \dots dx_n$ with any number of vortices (resp. particles) for $0 < \beta < \frac{4\pi}{\alpha^2}$ has been shown, see e.g. [FR83], [Lio98].

We are particularly interested here in the relation of the vortex model with the Euler equation. This has been studied by Marchioro and Pulvirenti [MP94]. As far as we are concerned with invariant measures, this comes about through the consideration of invariance of the Lebesgue measure for the Euler equation, see the first Remark 2.1, as well as for (11), because of its Hamiltonian structure. Let us construct a "concrete" invariant measure considering any number of vortices. For this we define the compound Poisson measure Π on the space Γ of configurations of $\mathbb{T}^2$. Let for any $n \in \mathbb{N}$:

$$\Gamma^{(n)} \equiv \left\{ \xi = \sum_{l=1}^{n} \nu_l \delta_{x_l} \; : \nu_l \in \mathbb{R}_0, x_l \in \mathbb{T}^2, x_l \neq x_k \text{ for } l \neq k \right\}$$

(where $\mathbb{R}_0 \equiv \mathbb{R} \setminus \{0\}$). $\Gamma^{(n)}$ is looked upon as a space of n–point configurations in $\mathbb{T}^2$. Let $\tilde{\Lambda}^{(n)} = \big\{(\nu_i, x_i) \in (\mathbb{R}_0 \times \mathbb{T}^2)^n : i = 1, \cdots, n, x_i \neq x_k \text{ for } i \neq k\big\}$. There is a bijection $J^{(n)}$ of $\tilde{\Lambda}^{(n)} \mod S^{(n)}$ into $\Gamma^{(n)}$, where $S^{(n)}$ is the permutation group over $\{1, \cdots, n\}$. Let Θ be a finite positive measure on $\mathcal{B}(\mathbb{R}_0)$ such that $\int_{\mathbb{R}_0} (1 \wedge \nu^2) \Theta(d\nu) < \infty$. Set $\|\Theta\| = \int_{\mathbb{R}_0} \Theta(d\nu)$. Consider the measure $\sigma^{\otimes n} \equiv (\Theta \otimes \lambda)^{\otimes n}$ on $\mathcal{B}(\tilde{\Lambda}^{(n)} \mod S^{(n)})$, λ being the Lebesgue measure on $\mathbb{R}^2$. Let σ_n be the image of $\sigma^{\otimes n}$ on $\Gamma^{(n)}$. Set $\Gamma^{(0)} = \{\emptyset\}$, $\sigma_0 = \delta_{\{\emptyset\}}$. The space of configurations is by definition $\Gamma = \cup_{n=0}^{\infty} \Gamma^{(n)}$, defined as disjoint union of topological spaces with the corresponding Borel σ-algebra (see, e.g. [AKR98b], [AKR98a]). The compound Poisson measure Π on Borelians of Γ is defined by $\Pi = e^{-\|\Theta\|(2\pi)^2} \sum_{n=0}^{\infty} \frac{\sigma_n}{n!}$. In [AF03] it is shown that Π is invariant with respect to a unique vortex flow, well defined for Π–a.e. initial data. This is based on results by Dürr and Pulvirenti [DP82]. In fact in each component $\Gamma^{(n)}$ of Γ there is a σ_n–preserving flow, because the Lebesgue measure is invariant and the vortex intensities are constant during the motion. Hence there is a unique strongly continuous positivity preserving unitary group on $L^2(\Pi)$. Under the assumption $\int_{\mathbb{R}_0} \nu^2 \Theta(d\nu) < \infty$ one has $B^k \in L^p(\Pi)$, $1 \leq p < \infty$, $\frac{\partial B_k}{\partial \xi_k} = 0$ Π–a.e. ξ, $\bar{B}_k(\xi) = B_{-k}(\xi)$ Π–a.e. ξ, $k \in \mathbb{Z}_0^2$. Thus the Liouville operator $L = i \sum_k B_k \frac{\partial}{\partial \xi_k}$ in $L^2(\Pi)$ is Markov unique in the sense that there exists only one self–adjoint extension which generates a positivity preserving strongly continuous unitary group U_t in $L^2(\Pi)$, see [GGM80], [AF03]. Π is invariant unter U_t.

Remarks 2.8.

(i). *Π and μ_γ are singular, see [AF03], and $\mu_\gamma(\Gamma) = 0$.*

(ii). *See also [AF02a], [AF02b] for other measures of the Poisson type, which are heuristically invariant for the 2D Euler flow, also discussed in [BF80], [BF81], [AHKM85], [CDG85].*

(iii). *Stochastic perturbations of the vortex model are mentioned in [AF02a]. It would be interesting to study them in more details. Let us mention in passing that stochastic perturbations of 2–dimensional Euler equations have been studied in particular in [BF99], [Bes99], [MV00], [BP01], [Kim02], and in [CC99] (using non–standard analysis, see [AHKFL86]). Stochastic models for the study of formation of vortices have been developed [FG04], [FG05]. For related work see also, e.g., [BLS05].*

3 Stochastic Navier–Stokes Equation

The study of the deterministic Navier–Stokes equation is well known to present challenging problems, especially in the case of 3 dimensions. E.g. the global existence and uniqueness of classical solutions of this equations with smooth initial data is a famous open problem, see, e.g. [Sin05b, Sin05a], [Fef06], [Con01a], [CF03], [Soh01], [Zgl03]. In 2 space dimensions the situation is better understood, see, e.g. [BA94], [Can04], [Tem83], [Tem84], [KT01]. However the problem of the existence of invariant measures (different from those concentrated on stationary solutions [VF88]) is widely open. In 2 space dimensions there are results on the construction of invariant measures for Navier–Stokes equations stochastically perturbed by Gaussian white noise, which we are going to describe in detail. We are interested in the stochastic Navier–Stokes equation with a space-time white noise. We consider, as in Section 2, the spatial domain to be the torus $\mathbb{T}^2 = [0, 2\pi]^2$ (hence periodic boundary conditions are assumed); we point out that for other finite spatial domains, the boundary conditions for the Euler equation and for the Navier–Stokes equation are different (see e.g. [Bir60]). In Section 3.1 we present the stochastic Navier–Stokes equation. In Section 3.2 we introduce an infinitesimal invariant measure associated to this stochastic equation, as done in [AC90]; this is a Gaussian measure μ_ν, with covariance given in terms of the enstrophy (and of the viscosity parameter ν). Existence of a solution of this stochastic Navier–Stokes equation has been proved in two different ways: [AC90] considers a weak solution and [DPD02] a strong solution (weak and strong are to be understood in the probabilistic sense). We present the second approach in Section 3.3. Uniqueness of these strong solutions is given in Section 3.4, following [AF04b]. In the Appendix a technical lemma is presented.

3.1 The Navier–Stokes Equation with Space-Time White Noise

We consider an homogeneous incompressible viscous flow in $\mathbb{T}^2$ with periodic boundary conditions. Displaying the external force on the right hand side of the equation, we have

$$\begin{cases} \frac{\partial}{\partial t}u - \nu\Delta u + [u\cdot\nabla]u + \nabla p = f \\ \nabla\cdot u = 0 \\ u|_{t=0} = u_0 \end{cases} \tag{1}$$

The unknowns are $u = u(t,x), p = p(t,x)$. The definition domains of the variables are $t \geq 0, x \in \mathbb{T}^2$. $\nu > 0$ is the viscosity coefficient.

The mathematical setting is as in Section 2. We expand in Fourier series any periodic divergence-free vector u (see Section 2.4):

$$u(x) = \sum_{k\in\mathbb{Z}_0^2} u_k e_k(x), \qquad u_k \in \mathbb{C}, \quad \overline{u}_k = -u_{-k} \tag{2}$$

Note that $\{e_k\}_{k\in\mathbb{Z}_0^2}$ is a complete orthonormal system of the eigenfunctions (with corresponding eigenvalues $|k|^2$) of the operator $-\Delta$ in $[L_2^{\mathrm{div}}(\mathbb{T}^2)]^2 = \{u \in [L_2(\mathbb{T}^2)]^2 : \nabla\cdot u = 0, \int_{\mathbb{T}^2} u(x)\,dx = 0$, with the normal component of u being periodic on $\partial\mathbb{T}^2\}$. With respect to the Fourier components, the energy is given by $E = \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}|u_k|^2$ and the enstrophy by $S = \frac{1}{2}\sum_{k\in\mathbb{Z}_0^2}|k|^2|u_k|^2$.

Each e_k is a periodic divergence-free C^∞-vector function. The convergence of the series (2) depends on the regularity of the vector function u, and can be used to define Sobolev spaces as in the following definition.

Let $\mathcal{U}'$ be the space of zero mean value periodic divergence-free vector distributions. Any element $u \in \mathcal{U}'$ is uniquely defined by the sequence of the coefficients $\{u_k\}_{k\in\mathbb{Z}_+^2}$; indeed, by duality, $u_k = \langle u, e_{-k}\rangle$. In the following we often identify the space of vectors u and the space of sequences $\{u_k\}$, for $u = \sum_k u_k e_k$.

Following [BL76], we define the periodic divergence-free vector Sobolev spaces ($s \in \mathbb{R}, 1 \leq p \leq \infty$)

$$\mathcal{H}^s_p = \{ \ u = \sum_{k\in\mathbb{Z}^2_0} u_k e_k \in \mathcal{U}' : \sum_k u_k |k|^s e_k(\cdot) \in L_p(\mathbb{T}^2) \ \}$$

and the periodic divergence-free Besov spaces as real interpolation spaces

$$\begin{aligned} \mathcal{B}^s_{pq} = (\mathcal{H}^{s_0}_p, \mathcal{H}^{s_1}_p)_{\theta,q}, \qquad & s \in \mathbb{R}, 1 \le p, q \le \infty \\ & s = (1-\theta)s_0 + \theta s_1, \qquad 0 < \theta < 1 \end{aligned}$$

In particular, $\mathcal{B}^s_{2\,2} = \mathcal{H}^s_2$, the Hilbert spaces already defined in Section 2.10. (Notice, however, that we dealt there with the Euler equation in the unknown stream function φ, whereas here we deal with the stochastic Navier–Stokes equation in the unknown velocity u.) Moreover, $\mathcal{U}' = \cup_{s\in\mathbb{R},1\le p\le\infty}\mathcal{H}^s_p$ with the inductive topology.

Remark 3.1. *For $u = \sum_{k\in\mathbb{Z}^2_0} u_k e_k$, if we define*

$$\delta_j u = \sum_{2^{j-1}<|k|\le 2^j} u_k e_k \qquad \text{for } j \in \mathbb{N}$$

then $\delta_j u$ contains the Fourier components of u between 2^{j-1} and 2^j.
For $s \in \mathbb{R}, p, q \ge 1$ we then have (see [DPD02])

$$\mathcal{B}^s_{pq} = \left\{ u \in \mathcal{U}' : \sum_{j\in\mathbb{N}} 2^{qjs} \|\delta_j u\|^q_{L_p(\mathbb{T}^2)} < \infty \right\}$$

$\mathcal{B}^s_{pq}$ is a Banach space with the norm $\|u\|_{\mathcal{B}^s_{pq}} = \left(\sum_j 2^{qjs}\|\delta_j u\|^q_{L_p(\mathbb{T}^2)}\right)^{1/q}$.

We define the Stokes operator as

$$A = -\Delta$$

which is a linear operator in $\mathcal{H}^s_p$ with domain $\mathcal{H}^{s+2}_p$. It is an isomorphism from $\mathcal{H}^{s+2}_p$ to $\mathcal{H}^s_p$ ($s \in \mathbb{R}, 1 \le p < \infty$). For $u = \sum_k u_k e_k$ we have $Au = \sum_k u_k |k|^2 e_k$. Let P be the projector operator from the space of periodic vectors onto the space of periodic divergence-free vectors. Applying P to both sides of the first equation in the Navier–Stokes system (1), we get rid of the pressure term (see [Tem84]).

Let the bilinear operator $\tilde{B}$ be defined by

$$\begin{aligned} \tilde{B}(u,v) &= P\left[(u\cdot\nabla)v\right] \\ &= P\left[\nabla\cdot(u\otimes v)\right] \qquad \text{(by the divergence-free condition)} \\ &= P\left[\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix}\cdot\begin{pmatrix}u_1v_1 & u_1v_2\\ u_2v_1 & u_2v_2\end{pmatrix}\right], \end{aligned} \tag{3}$$

whenever it makes sense. For instance, a classical result is that $\tilde{B} : \mathcal{H}^1_2 \times \mathcal{H}^1_2 \to \mathcal{H}^{-1}_2$ (see, e.g., [Tem83]). For less regular vectors u and v, estimates on $\tilde{B}$ are given in Besov spaces (see, e.g., [Che96b, Che98]). The (optimal) regularity of $\tilde{B}$ is the key point to solve the Navier–Stokes equation, both in the deterministic and in the stochastic case.

We shall very often write shortly $\tilde{B}(u)$ for the quadratic term $\tilde{B}(u,u)$.

The stochastic Navier–Stokes equation we are interested in, has the following abstract Itô form

$$\begin{cases} d\,u(t) + [\nu A u(t) + \tilde{B}(u(t))]\,dt = dw(t), & t > 0 \\ u(0) = v_x. \end{cases} \tag{4}$$

$\{w(t)\}_{t\geq 0}$ is a Wiener process, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, which is cylindric in the space of finite energy $\mathcal{H}_2^0$, i.e.

$$w(t) = \sum_{k\in\mathbb{Z}_0^2} w_k(t) e_k$$

where $\{w_k\}_{k\in\mathbb{Z}_+^2}$ is a sequence of standard independent complex valued Wiener processes and $w_{-k} = -\overline{w}_k$ for $k \in \mathbb{Z}_+^2$ (for $k \in \mathbb{Z}_+^2$: $w_k(t) = a_k(t) + ib_k(t)$ and $\{a_k\}, \{b_k\}$ i.i.d. with $\mathbb{E}a_k a_j = \mathbb{E}b_k b_j = \delta_{kj}$). This is a process with continuous paths taking values in $\mathcal{H}_2^\sigma$ for any $\sigma < -1$ (see, e.g., [DPZ92]). In other terms, $dw(t)$ is a Gaussian space-time white noise.

We shall denote by $\mathbb{E}$ the expectation with respect to the measure $\mathbb{P}$.

Remark 3.2. *For noise which is more regular in space("coloured Gaussian noise") the techniques to analyse equation* (4) *are very different from ours. E.g. solutions with finite energy have been discussed with global existence in space dimensions 2 and 3, uniqueness being known only for d=2 (as for the deterministic case). Further typical results include existence and uniqueness of invariant measures and ergodicity (mostly for "many Fourier modes" but some also for "few Fourier modes") See, e.g., [Cru89a], [Cru89b], [BDPD04], [BG96], [BL04], [Car03], [Cha96], [CK04b], [CE06], [Cut03], [DPD03], [Fer99], [Fer01], [Fer03], [Fer06], [FG95], [FG98], [Fla], [Fla94], [Fla02], [Fla03], [FGGT05], [FR01], [LJS97], [Mel00], [MS02], [Pes85], [QY98], [Rob91], [Rob03], [EMS01], [ES00b], [BT73], [Cho78], [VF88], [BCF92], [FY92], [Kot95], [CG94], [FM95], [Fer97a], [ES00a], [KS01], [BKL01], [MR04], [BDS04], [MR05], [HM06].*

For $d = 3$ *there is an existence result on invariant measures and ergodicity in [DPD03] (for further results see the lectures by F. Flandoli).*

Equation (4) is equivalent to the following equations for the Fourier components

$$d\,u_k(t) + [\nu|k|^2 u_k(t) + \tilde{B}_k(u(t))]\,dt = dw_k(t), \qquad t > 0$$

where $\tilde{B}_k(u) = \frac{i}{2\pi}\sum_{h\in\mathbb{Z}_0^2, h\neq k} \tilde{c}_{h,k} u_h u_{k-h}$, with $\tilde{c}_{h,k} = \frac{(h^\perp\cdot k)|k|}{2|h||k|} - \frac{(h^\perp\cdot k)(h\cdot k)}{|h||k-h||k|}$.

3.2 The Gaussian Invariant Measure Given by the Enstrophy (and the Viscosity Parameter)

We shall consider the centered Gaussian measure μ_ν on the space of complex valued sequences $\{u_k\}_{k\in\mathbb{Z}^2_+}$, heuristically defined as the infinite product of centered Gaussian measures μ^k_ν on $\mathbb{C}\cong\mathbb{R}\times\mathbb{R}$

$$\mu_\nu(du)=\bigotimes_{k\in\mathbb{Z}^2_+}\mu^k_\nu(du_k) \tag{5}$$

where

$$\mu^k_\nu(du_k)=\frac{\nu|k|^2}{\pi}\,e^{-\nu|k|^2|u_k|^2}du_k$$

It is identifiable with the standard centered Gaussian measure $N(0,\nu|\cdot|_{\mathcal{H}^1_2})$. Let us denote by $\mathbb{E}_{\mu_\nu}$ the expectation with respect to this measure: $\mathbb{E}_{\mu_\nu}F\equiv\int_{\mathcal{U}'}F(u)\,\mu_\nu(du)$.

In particular

$$\mathbb{E}_{\mu_\nu}[u_k\overline{u_j}]=\begin{cases}\frac{1}{\nu|k|^2} & \text{if } k=j\\ 0 & \text{if } k\neq j\end{cases}$$

Heuristically we can write (5) as "$Z^{-1}\exp(-\nu S(u))du$", S being the enstrophy associated to the velocity field u: $S(u)=\sum_{k\in\mathbb{Z}^2_+}|k|^2|u_k|^2$. Thus, μ_ν corresponds to the Gaussian measure $\mu_{\nu S}$ of Section 2.4.

For later use we shall need more information on the support of the measure μ_ν. We have, for any integer n

$$\begin{aligned}\mathbb{E}_{\mu_\nu}\left(\|u\|^{2n}_{\mathcal{H}^{-\sigma}_{2n}}\right)&=\mathbb{E}_{\mu_\nu}\left(\int_{\mathbb{T}^2}\left|\sum_k u_k|k|^{-\sigma}e_k(x)\right|^{2n}\,dx\right)\\&=\int_{\mathbb{T}^2}\mathbb{E}_{\mu_\nu}\left(\left|\sum_k u_k|k|^{-\sigma}e_k(x)\right|^{2n}\right)\,dx\\&=c_n\left[\sum_k|k|^{-2\sigma}\mathbb{E}_{\mu_\nu}|u_k|^2\right]^n\end{aligned}\tag{6}$$

for some constant $c_n>0$. In these calculations we have used that, for any $\gamma_k\in\mathbb{C}$:

$$\mathbb{E}_{\mu_\nu}\left(\left|\sum_k u_k\gamma_k\right|^{2n}\right)=\frac{(2n)!}{2^n\,n!}\left[\sum_k|\gamma_k|^2\,\mathbb{E}_{\mu_\nu}(|u_k|^2)\right]^n \tag{7}$$

and the fact that $|e_k(x)|=\frac{1}{2\pi}$ for any $x\in\mathbb{T}^2$.

Since $\mathbb{E}_{\mu_\nu}(|u_k|^2)=\frac{1}{\nu|k|^2}$, the above calculation implies that there exists a positive constant c'_n such that

$$\mathbb{E}_{\mu_\nu}\left(\|u\|^2_{\mathcal{H}^{-\sigma}_{2n}}\right)\le\left(\mathbb{E}_{\mu_\nu}\|u\|^{2n}_{\mathcal{H}^{-\sigma}_{2n}}\right)^{\frac{1}{n}}\le\frac{1}{\nu}\,c'_n\sum_{k\in\mathbb{Z}^2_0}\frac{1}{|k|^{2+2\sigma}}$$

The latter series converges as soon as $\sigma > 0$. Hence $\mu_\nu(\mathcal{H}_{2n}^{-\sigma}) = 1$ for any $\sigma > 0$ and integer n. Since we are in a bounded domain, we have the embedding $\mathcal{H}_{2(n+1)}^{-\sigma} \subset \mathcal{H}_q^{-\sigma} \subset \mathcal{H}_{2n}^{-\sigma}$ for $2n < q < 2(n+1)$. Therefore

$$\mu_\nu(\mathcal{H}_q^{-\sigma}) = 1 \qquad \forall \sigma > 0, \quad 1 \leq q < \infty$$

and by interpolation (see the details in [AF04b], based on [BL76]) we get Besov spaces of full measure μ_ν:

Proposition 3.1. *For any $\nu > 0$*

$$\mu_\nu(\mathcal{B}_{pq}^{-\sigma}) = 1 \qquad \forall \sigma > 0, \quad 1 \leq p \leq q < \infty.$$

□

Remark 3.3. *It was already known from [ARdFHK79] that the space $\mathcal{H}_2^0$ of finite energy velocity vectors has not full measure with respect to μ_ν; in fact one has even $\mu_\nu(\mathcal{H}_2^0) = 0$.*

With calculations similar to (6) we can obtain that, $\mathbb{P}$-a.s., the paths w of the Wiener process $w(t)$ belong to $\mathcal{H}_p^{-1-\sigma}$ for $\sigma > 0$ and $1 \leq p < \infty$.

We collect the main properties of the Fourier components $\tilde{B}_k$. We look for uniform estimates for the sequence of finite approximations $\tilde{B}_k^N$. The extension to the infinite dimensional dynamics has to be checked carefully. In fact, we cannot deal directly with $\tilde{B}(u)$ for any u in the support of the measure μ_ν, since such u are too irregular for the quadratic term $\tilde{B}(u)$ to be defined. But all the components B_k are well defined.

Proposition 3.2. *For any $k \in \mathbb{Z}_0^2$*

$$\partial_k \tilde{B}_k = 0 \tag{8}$$

$$\overline{\tilde{B}_k} = -\tilde{B}_{-k} \tag{9}$$

$$\tilde{B}_k \in L^p(\mu_\nu) \qquad \text{for any } 1 \leq p < \infty \tag{10}$$

Indeed each component $\tilde{B}_k$ is the $L^p(\mu_\nu)$-limit (as $N \to \infty$) of the Galerkin approximations

$$\tilde{B}_k^N(u) = \sum_{\substack{h \\ 0<|h|,|k-h|\leq N}} \tilde{c}_{h,k} u_h u_{k-h}, \qquad k \in \mathbb{Z}_0^2, |k| \leq N, N \in \mathbb{N}$$

for which conservation of the enstrophy holds, that is

$$\sum_{\substack{k \\ 0<|k|\leq N}} \tilde{B}_k^N(u)|k|^2\overline{u_k} = 0, \qquad N \in \mathbb{N}.$$

□

Remarks 3.1.

(i). Let $\tilde{B}(u) = \sum_k \tilde{B}_k(u)e_k$. $\tilde{B}$, resp $\tilde{B}_k$ are the corresponding quantities $-B$, resp $-B_k$ discussed in Section 2.2 written for the variables u_k instead of the variables ω_k

(ii). The coefficients $\tilde{c}_{hk}$ are naturally related with the coefficients c_{hk} of Section 2.4.

Notice that the dynamics (4) is a "combination" of the Euler dynamics and of the stochastic Stokes dynamics. Therefore, if the stochastic linear dynamics $dz(t) + \nu Az(t)dt = dw(t)$ has as a unique invariant measure exactly μ_ν, then the whole dynamics (4) has μ_ν as infinitesimal invariant measure. We have that the stochastic Stokes equation corresponds to a system of uncoupled linear equations $dz_k(t) + \nu|k|^2 z_k(t)dt = dw_k(t)$. Each component has invariant measure corresponding to the law of the stationary process $z_k(t) = \int_{-\infty}^t e^{-(t-s)\nu|k|^2} dw_k(s)$; this is a stationary centered Gaussian process whose covariance is $1/(\nu|k|^2)$. Hence the infinite product ($k \in \mathbb{Z}^2_+$) of these Gaussian measures is an invariant measure for the stochastic Stokes equation. We point out that the proper choice of the noise and of the viscosity coefficient gives this expression for the invariant measure (for each $\nu > 0$ there exists a unique invariant measure μ_ν).

We can see this infinitesimal invariance of μ_ν also by introducing the Kolmogorov operator K associated to the stochastic equation (4) (see, e.g., [AF02b])

$$K = \sum_{k\in\mathbb{Z}^2_0} \Big[\frac{\partial}{\partial u_{-k}}\frac{\partial}{\partial u_k} + \big(-\nu k^2 u_k - \tilde{B}_k\big)\frac{\partial}{\partial u_k}\Big] \equiv -\sum_{k\in\mathbb{Z}^2_0} \frac{\partial^*}{\partial u_k}\frac{\partial}{\partial u_k} - \sum_{k\in\mathbb{Z}^2_0} \tilde{B}_k \frac{\partial}{\partial u_k}$$

where $(\frac{\partial}{\partial u_k})^*$ is the dual of the operator $\frac{\partial}{\partial u_k}$ in the space $L^2(\mu_\nu)$.

As done above for the Liouville operator L, we have that the Kolmogorov operator K is a linear operator in $L^2(\mu_\nu)$, well-defined on $D(K) = FC_b^\infty$. Indeed, $K = -Q + L$ with $Q = \sum_k (\frac{\partial}{\partial u_k})^* \frac{\partial}{\partial u_k}$, the positive symmetric Ornstein–Ulenbeck operator defined on FC_b^∞. The operator K, defined on $\mathcal{F}C_b^\infty$, is dissipative and closable. The measure μ_ν is infinitesimal invariant also for Q; hence for the sum $-Q + L$. (see [AF02b]).

Remark 3.4. *For other results on Kolmogorov's equation for stochastic Navier–Stokes, see e.g. [FG98, BDPD04, Sta07]. For particular results of uniqueness of closed extensions of K see [AF02b] and [ABF07]. For problem of uniqueness of other, somewhat related, generators in infinite dimensional spaces, see e.g. [AKR92, AKR95, AR95, BDPD04, DPD99, DPD07, DPT00, DV87, KR07, LR98, Ebe99, Sta99, Sta03, DP04, RS06].*

We now provide an estimate of the quadratic term $\tilde{B}$ with respect to the measure μ_ν. This will be useful later on.

Proposition 3.3. *For any viscosity $\nu > 0$ and any $1 \leq \gamma < \infty$, we have*

$$\int_{\mathcal{U}'} \|\tilde{B}(u)\|^{\gamma}_{\mathcal{H}_2^{-1-\varepsilon}} d\mu_\nu(u) < \infty \qquad \forall \varepsilon > 0. \tag{11}$$

Proof. See [AF04b]. We only point out that [AC90] showed $\|B\|_{\mathcal{H}_2^{-\alpha}} \in L^2(\mu_\gamma)$ for any $\alpha > \frac{3}{2}$. In [AF04b] that result was improved by showing that $\mathbb{E}_{\mu_\gamma}|B_k|^2 \leq \frac{c}{\pi^2\nu^2}\log|k|$ for any $|k| \geq 2$.

Remark 3.5. *According to the latter result, the nonlinear term $\tilde{B}(u)$ is defined for μ_ν-a.e. u. Since $\mu_\nu(\mathcal{H}_2^0) = 0$ but $\mu_\nu(\mathcal{H}_q^{-\varepsilon}) = 1$ ($\varepsilon > 0, 1 < q < \infty$), the elements u for which the nonlinear term $\tilde{B}(u)$ exists are (non regular) distributions. In [DPD02] it is explained that $\tilde{B}(u) \in L^\gamma(\mu_\nu; \mathcal{H}_2^{-1-\varepsilon})$ for $1 \leq \gamma < \infty$, $\varepsilon > 0$, as follows. Denote by $: u \otimes u :$ the renormalized square (Wick square), defined as $: u \otimes u := u \otimes u - \mathbb{E}_{\mu_\nu}(u \otimes u)$ (see, e.g., [Sim74]). Consider the finite dimensional approximations $u_N := \sum_{|k|\leq N} u_k e_k$; one has that $\sup_N \mathbb{E}_{\mu_\nu}\| : u_N \otimes u_N : \|^{\gamma}_{\mathcal{H}_2^{-\varepsilon}} < \infty$. Notice that $\nabla \cdot (: u_N \otimes u_N :) = \nabla \cdot (u_N \otimes u_N - \mathbb{E}_{\mu_\nu}(u_N \otimes u_N)) = \nabla \cdot (u_N \otimes u_N)$. Hence $\tilde{B}(u_N) P[\nabla \cdot (: u_N \otimes u_N :)]$ and in the limit $\tilde{B}(u) = P[\nabla \cdot (: u \otimes u :)]$ is well defined, i.e. $\tilde{B}(u) \in L^\gamma(\mu_\nu; \mathcal{H}_2^{-1-\varepsilon})$.*

Moreover, in [Deb02] there is a useful proposition providing this result in the Besov spaces, i.e. $\tilde{B}(u) \in L^\gamma(\mu_\nu; \mathcal{B}_{p\,q}^{-1-\varepsilon})$ for any $\varepsilon > 0$ and $\gamma, p, q \geq 1$. □

3.3 Existence of Strong Solutions

The results in this section are from [DPD02, Deb02].

Set

$$z(t) = \int_{-\infty}^{t} e^{-(t-s)\nu A} dw(s), \qquad t \in \mathbb{R}$$

which is a stationary solution of the stochastic Stokes equation

$$dz(t) + \nu A z(t) dt = dw(t) \tag{12}$$

This is a linear stochastic equation. We know from Section 3.2 that the invariant law $\mathcal{L}(z(t))$ is exactly the Gaussian measure given by the enstrophy and the viscosity parameter. We have $z \in C(\mathbb{R}; \mathcal{B}_{p\,q}^{-\sigma})$ $\mathbb{P}$-a.s. for any $\sigma > 0, p, q \geq 1$. This is shown by means of Kolmogorov's criterium, as in [DPZ92].

Then we define $v = u - z$. Taking the difference between (4) and (12) the additive noise disappears; hence v satisfies the random equation

$$\frac{dv(t)}{dt} + \nu A v(t) + \tilde{B}(v(t) + z(t)) = 0$$

Using the bilinearity of $\tilde{B}$, we can write this equation in the integral form

$$v(t) = e^{-t\nu A}(x - z(0)) - \int_0^t e^{-(t-s)\nu A}[\tilde{B}(v) + \tilde{B}(v,z) + \tilde{B}(z,v) + \tilde{B}(z)]\, ds \tag{13}$$

The term $\tilde{B}(z)$ is defined according to (11). Indeed, $\tilde{B}(z) \in L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})$ $\mathbb{P}$-a.s., because

$$\mathbb{E}\int_0^T \|\tilde{B}(z(t))\|^\gamma_{\mathcal{H}_2^{-1-\varepsilon}} dt = T\int_{\mathcal{U}'} \|\tilde{B}(u)\|^\gamma_{\mathcal{H}_2^{-1-\varepsilon}} d\mu_\nu(u)$$

Since the initial data are not smooth, it is not expected that the equation for v has a solution with paths in $C([0,T];\mathcal{H}_2^s)$ for $s \geq 0$. On the other hand, this is a parabolic equation for which it will be proved in Proposition 3.4 that the solution v exists in $C([0,T];\mathcal{B}_{pq}^{-\sigma}) \cap L^\beta(0,T;\mathcal{B}_{pq}^\alpha)$ for some $-\sigma < 0$ and $\alpha > 0$; this is enough to define all the terms in (13).

To have short notations, it is convenient to introduce the space

$$\mathcal{E} = C([0,T];\mathcal{B}_{pq}^{-\sigma}) \cap L^\beta(0,T;\mathcal{B}_{pq}^\alpha)$$

for $\sigma, \alpha > 0$ and $p, q, \beta \geq 1$.

This is a Banach space with norm $\|v\|_{\mathcal{E}} = \|v\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})} + \|v\|_{L^\beta(0,T;\mathcal{B}_{pq}^\alpha)}$

We have a local existence result.

Proposition 3.4. *Let the real parameters $\sigma, p, q, \alpha, \beta, \varepsilon, \gamma$ satisfy $2 \leq p, q < \infty$, $\varepsilon > 0$, $1 \leq \gamma, \beta < \infty$ and*

$$0 < \sigma < \alpha < \frac{2}{p} \tag{14}$$

$$\frac{1}{p} - \frac{1}{2} < \frac{\alpha}{2} - \frac{1}{\beta} < -\frac{\sigma}{2} \tag{15}$$

$$\frac{1}{\gamma} + \frac{\alpha}{2} + \frac{\varepsilon}{2} < \frac{1}{p} + \frac{1}{\beta} \tag{16}$$

$$\frac{\varepsilon}{2} < \frac{1}{\gamma} + \frac{\sigma}{2} + \frac{1}{p} \tag{17}$$

Then, given $T > 0$ for any $f \in L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})$, $z \in C([0,T];\mathcal{B}_{pq}^{-\sigma})$ and $v_0 \in \mathcal{B}_{pq}^{-\sigma}$, the equation

$$v(t) = e^{-t\nu A}v_0 - \int_0^t e^{-(t-s)\nu A}[\tilde{B}(v) + \tilde{B}(v,z) + \tilde{B}(z,v) + f]\, ds$$

has a unique solution in $C([0,T^*];\mathcal{B}_{pq}^{-\sigma}) \cap L^\beta(0,T^*;\mathcal{B}_{pq}^\alpha)$ provided $T^* \leq T$ is such that

$$T^* \leq C(\|v_0\|_{\mathcal{B}_{pq}^{-\sigma}} + \|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})} + \|f\|_{L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})})^{-1/\eta}$$

with

$$\eta = \frac{1}{2} - \frac{1}{p} + \frac{\alpha}{2} - \frac{1}{\beta}$$

and where C depends on $\nu, \sigma, p, q, \alpha, \beta, \varepsilon, \gamma$.

Proof. This result is based on the lemmas below. We only give a sketch, containing the main ideas, full justification is made by passing through Galerkin approximations. Define the mapping Φ by

$$(\Phi v)(t) = e^{-t\nu A}v_0 - \int_0^t e^{-(t-s)\nu A}[\tilde{B}(v) + \tilde{B}(v,z) + \tilde{B}(z,v) + f]\, ds.$$

The lemmas below show that $\Phi : \mathcal{E} \to \mathcal{E}$ and

$$\|\Phi v\|_{\mathcal{E}} \le c_1\|v_0\|_{\mathcal{B}_{pq}^{-\sigma}} + c_2\|f\|_{L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})} + c_3T^\eta \left(\|v\|_{\mathcal{E}}^2 + 2\|v\|_{\mathcal{E}}\|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}\right).$$

Moreover, if $\|v\|_{\mathcal{E}} \le R$, then $\|\Phi v\|_{\mathcal{E}} \le R$, provided

$$2Rc_3T^\eta \le 1 \quad \text{and} \quad R \ge 2(c_1\|v_0\|_{\mathcal{B}_{pq}^{\sigma}} + c_2\|f\|_{L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})} + \|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}).$$

Choosing a ball in $\mathcal{E}$ of radius R, the mapping Φ restricted to this ball is a contraction. In fact, given $v_1, v_2 \in \mathcal{E}$ with norms bounded by R we have

$$\begin{aligned}
&\|\Phi v_1 - \Phi v_2\|_{\mathcal{E}} \\
&\le c_3T^\eta(\|v_1\|_{\mathcal{E}}\|v_1 - v_2\|_{\mathcal{E}} + \|v_2\|_{\mathcal{E}}\|v_1 - v_2\|_{\mathcal{E}} + 2\|v_1 - v_2\|_{\mathcal{E}}\|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}) \\
&\le 2c_3T^\eta(R + \|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})})\|v_1 - v_2\|_{\mathcal{E}}.
\end{aligned}$$

We conclude that Φ has a unique fixed point v if

$$T < \left(2c_3(R + \|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})})\right)^{-1/\eta}$$

and this v is the solution of (13) in $[0, T]$.

Remark 3.6. *Since $\sigma > 0$, condition on (15) imposes that $p > 2$. This is the reason for working in Besov spaces, instead of the usual Hilbert spaces.* □

In the three following lemmas, the parameters fulfil the conditions (14)–(17) in Proposition 3.4.

Lemma 3.1. *If $v_0 \in \mathcal{B}_{pq}^{-\sigma}$, then $e^{-t\nu A}v_0 \in \mathcal{E}$ and*

$$\|e^{-t\nu A}v_0\|_{\mathcal{E}} \le c_1\|v_0\|_{\mathcal{B}_{pq}^{-\sigma}}$$

for some constant c_1.

Proof. These estimates follow from classical results on semigroup theory. Indeed

$$\|e^{-t\nu A}v_0\|_{\mathcal{B}_{pq}^{\alpha}} \le \frac{c}{t^{\frac{\alpha+\sigma}{2}}}\|v_0\|_{\mathcal{B}_{pq}^{-\sigma}} \qquad \text{for } t > 0 \text{ and } \alpha \ge -\sigma$$

So

$$\left(\int_0^T \|e^{-t\nu A}v_0\|_{\mathcal{B}_{pq}^{\alpha}}^\beta\, dt\right)^{1/\beta} \le c\|v_0\|_{\mathcal{B}_{pq}^{-\sigma}} T^{\frac{1}{\beta} - \frac{\alpha+\sigma}{2}}$$

if $\frac{1}{\beta} - \frac{\alpha+\sigma}{2} > 0$.

In the above proof and in the following, we denote by c different constants. When needed, we shall specify them by a subindex.

Lemma 3.2. *If* $f \in L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})$, *then* $\int_0^t e^{-(t-s)\nu A} f(s)ds \in \mathcal{E}$ *and*

$$\| \int_0^t e^{-(t-s)\nu A} f(s)ds\|_{\mathcal{E}} \le c_2 \|f\|_{L^\gamma(0,T;\mathcal{H}_2^{-1-\varepsilon})}$$

for some constant c_2.

Proof. We use

$$\|e^{-(t-s)\nu A} f(s)\|_{\mathcal{H}_2^{\alpha+1-\frac{2}{p}}} \le \frac{c}{(t-s)^{1+\frac{\alpha}{2}+\frac{\varepsilon}{2}-\frac{1}{p}}} \|f(s)\|_{\mathcal{H}_2^{-1-\varepsilon}}$$

and the embedding (see [BL76] Th. 6.5.1)

$$\mathcal{H}_2^{\alpha+1-\frac{2}{p}} \subset \mathcal{B}^\alpha_{pq} \qquad \text{for } p,q \ge 2$$

We deduce

$$\| \int_0^t e^{-(t-s)\nu A} f(s)ds\|_{\mathcal{B}^\alpha_{pq}} \le \int_0^t \frac{c}{(t-s)^{1+\frac{\alpha}{2}+\frac{\varepsilon}{2}-\frac{1}{p}}} \|f(s)\|_{\mathcal{H}_2^{-1-\varepsilon}} ds$$

By Young's inequality, the convolution integral is in $L^\beta(0,T)$ if

$$\frac{1}{\gamma} + \frac{\alpha}{2} + \frac{\varepsilon}{2} < \frac{1}{p} + \frac{1}{\beta}$$

For the second estimate, we have similarly

$$\| \int_0^t e^{-(t-s)\nu A} f(s)ds\|_{\mathcal{B}^{-\sigma}_{pq}} \le c \int_0^t \|e^{-(t-s)\nu A} f(s)\|_{\mathcal{H}_2^{-\sigma+1-\frac{2}{p}}} ds$$

$$\le c \int_0^t \frac{1}{(t-s)^{-\frac{\sigma}{2}+1-\frac{1}{p}+\frac{\varepsilon}{2}}} \|f(s)\|_{\mathcal{H}_2^{-1-\varepsilon}} ds$$

The convolution integral is bounded if

$$\frac{\varepsilon}{2} < \frac{1}{\gamma} + \frac{\sigma}{2} + \frac{1}{p}$$

Lemma 3.3. *Let* $v_1 \in L^\beta(0,T;\mathcal{B}^\alpha_{pq})$ *and* $v_2 \in C([0,T];\mathcal{B}^{-\sigma}_{pq})$. *Then for* $(i,j) = (1,2)$ *or* $(i,j) = (2,1)$ *we have* $\int_0^t e^{-(t-s)\nu A} \tilde{B}(v_i(s), v_j(s))ds \in \mathcal{E}$ *and*

$$\| \int_0^t e^{-(t-s)\nu A} \tilde{B}(v_i(s), v_j(s))ds\|_{\mathcal{E}} \le c_3 T^\eta \|v_1\|_{L^\beta(0,T;\mathcal{B}^\alpha_{pq})} \|v_2\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})}$$

for some constant c_3, *with* $\eta = \frac{1}{2} - \frac{1}{p} + \frac{\alpha}{2} - \frac{1}{\beta}$.

Proof. Choose $(i,j)=(1,2)$; but the same works, and is needed, for the other choice $(i,j)=(2,1)$.
We use product rules in Besov spaces (see [Che96b] Corollary 1.3.1):

$$\|v_i \otimes v_j\|_{\mathcal{B}_{pq}^{\alpha-\sigma-\frac{2}{p}}} \le c\|v_i\|_{\mathcal{B}_{pq}^{\alpha}}\|v_j\|_{\mathcal{B}_{pq}^{-\sigma}}$$

with $0<\sigma<\alpha$, $\alpha<\frac{2}{p}$, and $q\ge 1$ and some constant c which in the following can vary from line to line. Bearing in mind the expression (3) of the bilinear term $\tilde{B}$, we have

$$\|\tilde{B}(v_i,v_j)\|_{\mathcal{B}_{pq}^{\alpha-\sigma-\frac{2}{p}-1}} \le c\|v_i\|_{\mathcal{B}_{pq}^{\alpha}}\|v_j\|_{\mathcal{B}_{pq}^{-\sigma}}$$

Therefore

$$\|e^{-(t-s)\nu A}\tilde{B}(v_i,v_j)\|_{\mathcal{B}_{pq}^{\alpha}} \le \frac{c}{(t-s)^{\frac{\sigma}{2}+\frac{1}{p}+\frac{1}{2}}}\|v_i\|_{\mathcal{B}_{pq}^{\alpha}}\|v_j\|_{\mathcal{B}_{pq}^{-\sigma}}$$

From Young's inequality, it follows that $\int_0^t e^{-(t-s)\nu A}\tilde{B}(v_i(s),v_j(s))\,ds \in L^\beta(0,T;\mathcal{B}_{pq}^{\alpha})$ provided

$$\frac{1}{2} > \frac{\sigma}{2}+\frac{1}{p}$$

and

$$\begin{aligned}&\|\int_0^t e^{-(t-s)\nu A}\tilde{B}(v_i(s),v_j(s))\,ds\|_{L^\beta(0,T;\mathcal{B}_{pq}^{\alpha})}\\ &\le cT^{\frac{1}{2}-\frac{\sigma}{2}-\frac{1}{p}}\|v_i\|_{L^\beta(0,T;\mathcal{B}_{pq}^{\alpha})}\|v_j\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}\end{aligned}$$

To estimate the second norm, we proceed in the same way as above. We have

$$\begin{aligned}&\|\int_0^t e^{-(t-s)\nu A}\tilde{B}(v_i(s),v_j(s))\,ds\|_{\mathcal{B}_{pq}^{-\sigma}}\\ &\le c\int_0^t \frac{1}{(t-s)^{-\frac{\alpha}{2}+\frac{1}{p}+\frac{1}{2}}}\|\tilde{B}(v_i(s),v_j(s))\|_{\mathcal{B}_{pq}^{\alpha-\sigma-\frac{2}{p}-1}}ds\\ &\le \|v_j\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}\int_0^t \frac{1}{(t-s)^{-\frac{\alpha}{2}+\frac{1}{p}+\frac{1}{2}}}\|v_i(s)\|_{\mathcal{B}_{pq}^{\alpha}}ds\end{aligned}$$

Again by Young's inequality, we conclude that

$$\begin{aligned}&\|\int_0^t e^{-(t-s)\nu A}\tilde{B}(v_i(s),v_j(s))\,ds\|_{L^\infty(0,T;\mathcal{B}_{pq}^{-\sigma})}\\ &\le c\,T^{\frac{1}{2}-\frac{1}{\beta}+\frac{\alpha}{2}-\frac{1}{p}}\|v_j\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}\|v_i\|_{L^\beta(0,T;\mathcal{B}_{pq}^{\alpha})}\end{aligned}$$

if

$$\frac{1}{2}+\frac{\alpha}{2} > \frac{1}{\beta}+\frac{1}{p}$$

Theorem 3.1. *Let the real parameters $\sigma, p, q, \alpha, \beta$ satisfy $2 \le p, q < \infty$,$1 \le \beta < \infty$ and*

$$0 < \sigma < \alpha < \frac{2}{p}$$
$$\frac{1}{p} - \frac{1}{2} < \frac{\alpha}{2} - \frac{1}{\beta} < -\frac{\sigma}{2}$$

Then, given $T > 0$ for any μ_ν-a.e. $x \in \mathcal{B}^{-\sigma}_{pq}$ there exists a solution u^x to (4) such that

$$u^x \in C([0,T]; \mathcal{B}^{-\sigma}_{pq}) \qquad \mathbb{P} - a.s.$$

Moreover, for any $l \in \mathbb{N}$

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|u^x(t)\|^l_{\mathcal{B}^{-\sigma}_{pq}} \right) < \infty \tag{18}$$

Proof. We start from Proposition 3.4, giving the local in time solution v. Then $u = v + z$ is a pathwise solution of (4) on a time interval $[0, T^*]$, where the time T^* is a random time depending on the initial data. It is sufficient to have an a priori estimate in $C([0,T]; \mathcal{B}^{-\sigma}_{pq})$ in order to have global existence. We want to show that

$$\int_{\mathcal{U}'} \mathbb{E}(\sup_{t\in[0,T]} \|u^x(t)\|_{\mathcal{B}^{-\sigma}_{pq}}) d\mu_\nu(x) < \infty \tag{19}$$

This implies for μ_ν-a.e. x that $\sup_{t\in[0,T]} \|u^x(t)\|_{\mathcal{B}^{-\sigma}_{pq}} < \infty$, $\mathbb{P}$-a.s. Therefore the pathwise local in time construction can be iterated leading to a global solution.

We now prove (19). We have

$$u^x(t) = e^{-t\nu A}(x - z(0)) - \int_0^t e^{-(t-s)\nu A} \tilde{B}(u^x(s))\, ds + z(t)$$

We estimate the convolution integral as usual; then

(20)
$$\|u^x(t)\|_{\mathcal{B}^{-\sigma}_{pq}} \le c(\|x\|_{\mathcal{B}^{-\sigma}_{pq}} + \|z\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})}) + c\int_0^t (t-s)^{-1/2} \|\tilde{B}(u^x(s))\|_{\mathcal{B}^{-\sigma-1}_{pq}} ds$$

In the latter term, we use the Hölder inequality in time, take the expectation $\mathbb{E}$ and use again the Hölder inequality:

$$\begin{aligned}
&\mathbb{E}[\sup_{t\in[0,T]} \int_0^t (t-s)^{-1/2} \|\tilde{B}(u^x(s))\|_{\mathcal{B}^{-\sigma-1}_{pq}} ds] \\
&\qquad \le \mathbb{E}[(\int_0^t (t-s)^{-3/4} ds)^{2/3} (\int_0^T \|\tilde{B}(u^x(s))\|^3_{\mathcal{B}^{-\sigma-1}_{pq}} ds)^{1/3}] \\
&\qquad \le cT^{1/6} \left(\mathbb{E}[\int_0^T \|\tilde{B}(u^x(s))\|^3_{\mathcal{B}^{-\sigma-1}_{pq}} ds] \right)^{1/3}
\end{aligned}$$

Then, we integrate with respect to μ_ν, use Hölder inequality and the invariance of μ_ν; so

$$\begin{aligned}\int \mathbb{E}[\sup_{t\in[0,T]} \int_0^t (t-s)^{-1/2} \|\tilde{B}(u^x(s))\|_{\mathcal{B}_{pq}^{-\sigma-1}} ds]\, d\mu_\nu(x) \\ \le cT^{1/6} \left(\int \mathbb{E}[\int_0^T \|\tilde{B}(u^x(s))\|^3_{\mathcal{B}_{pq}^{-\sigma-1}} ds] d\mu_\nu(x) \right)^{1/3} \\ = cT^{1/2} \left(\int \|\tilde{B}(x)\|^3_{\mathcal{B}_{pq}^{-\sigma-1}}\, d\mu_\nu(x) \right)^{1/3}\end{aligned}$$

Coming back to (20), we have obtained that

$$\begin{aligned}\int \mathbb{E}\|u^x\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})}\, d\mu_\nu(x) \le c \int \|x\|_{\mathcal{B}_{pq}^{-\sigma}}\, d\mu_\nu(x) + c\mathbb{E}\|z\|_{C([0,T];\mathcal{B}_{pq}^{-\sigma})} \\ + cT^{1/2} \left(\int \|\tilde{B}(x)\|^3_{\mathcal{B}_{pq}^{-\sigma-1}}\, d\mu_\nu(x) \right)^{1/3}\end{aligned}$$

By Remark 3.5, the right hand side is finite.
Similar computations show the validity of (18). This concludes the proof.

3.4 Pathwise Uniqueness

The results in this section are from [AF04b].

Consider a solution u^x to (4) with random initial data x with probability distribution μ_ν. This is obtained pathwise as the limit of a subsequence of Galerkin approximations u_N (taking the limit as done in [DPD02]) and has invariant measure μ_ν, in the sense that

$$\int \mathbb{E} f(u^v(t))\, d\mu_\nu(v) = \int f(v)\, d\mu_\nu(v), \qquad \forall f \in L^1(\mu_\nu), t \ge 0 \tag{21}$$

The fact that μ_ν is invariant for the Galerkin approximations is an important tool in the proof of the existence (in the spaces considered in [DPD02] as well as in those considered in [AC90]). Moreover any solution u, obtained as the limit of a subsequence of Galerkin approximations, has μ_ν as invariant measure. It is natural to ask about uniqueness of this limit obtained from any subsequence of Galerkin approximations.

From now on, we consider as state space any Besov space $\mathcal{B}_{pq}^{-\sigma}$ of full measure μ_ν.

For μ_ν-a.e. x, a solution given by Theorem 3.1 enjoys ($\mathbb{P}$-a.s.) the property

$$\int_0^T \|\tilde{B}(u^x(t))\|^\gamma_{\mathcal{H}_2^{-1-\varepsilon}} dt < \infty \quad \forall T > 0, \varepsilon > 0, 1 \le \gamma < \infty \tag{22}$$

Let $\tilde{u}^x$ be any other process defined on the same probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, with the same properties given above for u^x and solving equation (4) with the same $\{\mathcal{F}_t\}$-Wiener process as for u^x. Define the difference $U^x = u^x - \tilde{u}^x$; then $U^x \in C([0,T]; \mathcal{B}_{pq}^{-\sigma})$. From now on we drop the dependence on x and work pathwise ($\mathbb{P}$-a.s.). U satisfies the equation

$$\begin{cases} \frac{d}{dt}U(t) + AU(t) = -\tilde{B}(u(t)) + \tilde{B}(\tilde{u}(t)), & t > 0 \\ U(0) = 0 \end{cases} \tag{23}$$

Bearing in mind the regularizing effect of the Stokes operator A, something more can be proven. More precisely, (22) grants that the right hand side of the first equation in (23) belongs to the space $L^\gamma(0,T; \mathcal{H}_2^{-1-\varepsilon})$ for any $1 \le \gamma < \infty, \varepsilon > 0$. By Proposition 3.5 in the Appendix one has that

$$U \in L^\gamma(0,T; \mathcal{H}_2^{1-\varepsilon}) \cap C([0,T]; \mathcal{B}_{2\,\gamma}^{1-\varepsilon-\frac{2}{\gamma}}) \tag{24}$$

This holds for any $\varepsilon > 0, 1 \le \gamma < \infty$. Hence we have proven that any solution U to equation (23) must belong to the functional space $\Sigma := \cap_{1 \le \gamma < \infty, \varepsilon > 0} \Sigma_{\gamma,\varepsilon}$, where $\Sigma_{\gamma,\varepsilon} := L^\gamma(0,T; \mathcal{H}_2^{1-\varepsilon}) \cap C([0,T]; \mathcal{B}_{2\,\gamma}^{1-\varepsilon-\frac{2}{\gamma}})$. Let us point out that for $2 \le p \le \gamma \le q$ we have $\mathcal{B}_{2\,\gamma}^{1-\varepsilon-\frac{2}{\gamma}} \subseteq \mathcal{B}_{p\,\gamma}^{-\varepsilon-\frac{2}{\gamma}+\frac{2}{p}} \subseteq \mathcal{B}_{p\,\gamma}^{-\varepsilon} \subseteq \mathcal{B}_{pq}^{-\varepsilon}$ and for $\varepsilon \le \sigma$ we have $\mathcal{B}_{pq}^{-\varepsilon} \subseteq \mathcal{B}_{pq}^{-\sigma}$; therefore $\mathcal{B}_{2\,\gamma}^{1-\varepsilon-\frac{2}{\gamma}} \subseteq \mathcal{B}_{pq}^{-\sigma}$. Thus the regularity specified in (24) is stronger than the regularity $U \in C([0,T]; \mathcal{B}_{pq}^{-\sigma})$ given by the definition of U itself.

Remark 3.7. *The regularizing effect of the Stokes operator is not enough to obtain more regularity in the stochastic equation* (4)*, because of the presence of the cylindric noise dw. As soon as the noise disappears in* (23)*, the solution is more regular. This is enough to get uniqueness.* □

Bearing in mind the bilinearity of the operator $\tilde{B}$, the equation for U can be written in the following form

$$\begin{cases} \frac{d}{dt}U(t) + AU(t) + \tilde{B}(u(t), U(t)) + \tilde{B}(U(t), \tilde{u}(t)) & = 0, \qquad t > 0 \\ U(0) = 0 \end{cases} \tag{25}$$

The function $U \equiv 0$ is a solution to (25). We are going to prove that this is the only solution of (25) in the class Σ.

To prove this, we first show that, given $u, \tilde{u} \in C([0,T]; \mathcal{B}_{pq}^{-\sigma})$, under the assumptions below there exists a unique solution U to the problem (25) belonging to a class less regular than Σ. This is proved in Theorem 3.2 below. From this, uniqueness in the smaller class Σ immediately follows. This concludes our proof that the unique solution for (23) is $U \equiv 0$. What remains to be proven is therefore the following

Theorem 3.2. *Let real numbers σ, a be given as well as $2 \leq p, q < \infty$, $1 \leq b < \infty$ satisfying the following conditions*

$$\begin{aligned} 0 < \sigma < a < \frac{2}{p} \\ \frac{\sigma}{2} + \frac{1}{p} < \frac{1}{2} \\ \frac{1}{b} + \frac{1}{p} < \frac{1}{2} + \frac{a}{2} \end{aligned}$$

Then, given $T > 0$, for any $u, \tilde{u} \in C([0,T]; \mathcal{B}^{-\sigma}_{pq})$ there exists a unique $U \in \mathcal{D} := C([0,T]; \mathcal{B}^{-\sigma}_{pq}) \cap L^b(0,T; \mathcal{B}^a_{pq})$ solution to the following problem

$$\begin{cases} \frac{d}{dt}U(t) + \nu AU(t) + \tilde{B}(u(t), U(t)) + \tilde{B}(U(t), \tilde{u}(t)) = 0, & t > 0 \\ U(0) = 0 \end{cases} \tag{26}$$

In particular, if U satisfies (26) then $U(t) = 0$ for all $t \in [0,T]$.

Proof. We consider the mild solution to (26) in the integral form (in the sense of, e.g., [DPZ92])

$$U(t) = -\int_0^t e^{-(t-\tau)\nu A}[\tilde{B}(u(\tau), U(\tau)) + \tilde{B}(U(\tau), \tilde{u}(\tau))]d\tau \tag{27}$$

We want to prove existence and uniqueness of a solution in $\mathcal{D}$ by a fixed point theorem. We proceed as in Proposition 3.4. Since the equation to deal with is linear in the unknown U, the estimate leads easily to the desired result. We have

$$\begin{aligned} \| \int_0^t e^{-(t-\tau)\nu A} \tilde{B}(u(\tau), U(\tau))\, d\tau \|_{L^b(0,T;\mathcal{B}^a_{pq})} \\ \leq c_4\, T^{\frac{1}{2} - \frac{\sigma}{2} - \frac{1}{p}} \|u\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})} \|U\|_{L^b(0,T;\mathcal{B}^a_{pq})} \end{aligned} \tag{28}$$

if $\frac{1}{2} > \frac{\sigma}{2} + \frac{1}{p}$.
Moreover

$$\begin{aligned} \| \int_0^t e^{-(t-\tau)\nu A} \tilde{B}(u(\tau), U(\tau))\, d\tau \|_{L^\infty([0,T];\mathcal{B}^{-\sigma}_{pq})} \\ \leq\ c_5 T^{\frac{1}{2} - \frac{1}{b} + \frac{a}{2} - \frac{1}{p}} \|u\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})} \|U\|_{L^b(0,T;\mathcal{B}^a_{pq})} \end{aligned} \tag{29}$$

if $\frac{1}{2} + \frac{a}{2} > \frac{1}{b} + \frac{1}{p}$.
Hence, if $U \in \mathcal{D}$ and the conditions on the parameters hold, then
$\int_0^t e^{-(t-\tau)\nu A} \tilde{B}(u(\tau), U(\tau))\, d\tau \in \mathcal{D}$.

We perform the same computations for $\tilde{B}(U, \tilde{u})$.

The mapping

$$U \mapsto -\int_0^t e^{-(t-\tau)\nu A}[\tilde{B}(u(\tau),U(\tau)) + \tilde{B}(U(\tau),\tilde{u}(\tau))]d\tau$$

is then a contraction in $\mathcal{D}_{T^*}$ with $T^* \leq T$ and such that

$$T^* < \min\left\{(c_4\ N_T)^{-1/(\frac{1}{2}-\frac{\sigma}{2}-\frac{1}{p})}, (c_5\ N_T)^{-1/(\frac{1}{2}-\frac{1}{b}+\frac{a}{2}-\frac{1}{p})}\right\} \tag{30}$$

where $N_T = \|u\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})} + \|\tilde{u}\|_{C([0,T];\mathcal{B}^{-\sigma}_{pq})}$. Hence on the interval $[0,T^*]$ there exists a unique solution U with the regularity specified in $\mathcal{D}$. One has $U(t) = 0$ for $0 \leq t < T^*$. Notice that the amplitude of the time interval for local existence depends only on the $C([0,T];\mathcal{B}^{-\sigma}_{pq})$-norms of u and $\tilde{u}$; therefore we can continue in such a way as to cover the time interval $[0,T]$ with a finite number of intervals of amplitude $\frac{3}{4}T^*$.
Since this holds for any finite T, the proof is achieved.

Choose now the parameters of Proposition 3.2 to be $p = q = b = 3$, $\sigma = \frac{1}{6}$, $a = \frac{1}{2}$. In this way, bearing in mind Proposition 3.1, we have fixed a set $\mathcal{B}^{-\sigma}_{pq}$ of initial data such that $\mu_\nu(\mathcal{B}^{-\sigma}_{pq}) = 1$ (but many other choices are possible); moreover the assumptions of Theorem 3.2 are satisfied. Choose also the parameters $\gamma = 3, \varepsilon = \frac{1}{6}$ for the regularity of (24). Finally, by an embedding theorem (see [BL76] Theorem 6.5.1) we have

$$\mathcal{B}^{1-\varepsilon-\frac{2}{\gamma}}_{2\,\gamma} \subset \mathcal{B}^{-\sigma}_{pq}$$
$$\mathcal{H}^{1-\varepsilon}_2 \subset \mathcal{B}^{a}_{pq}$$

Hence $\Sigma \subset \mathcal{D}$. And the uniqueness in $\mathcal{D}$ implies the uniqueness in Σ.
We have therefore proven the following

Theorem 3.3. *Pathwise uniqueness of the solutions to the stochastic Navier–Stokes equation with space-time Gaussian white noise* (4)*, for which μ_ν is an invariant measure, holds in the following precise sense: there exists a set $\mathcal{S} \subset \mathcal{U}'$ with $\mu_\nu(\mathcal{S}) = 1$ such that for μ_ν-a.e. $x \in \mathcal{S}$ the $C([0,T];\mathcal{S})$-valued paths of any two solutions of* (4)*, defined on the same probability space with the same Wiener process and having invariant measure μ_ν, coincide $\mathbb{P}$-a.s.*

A further result concerns the ergodicity of the stochastic Navier–Stokes flow given by the above equations. It is contained in [Deb02]:

Theorem 3.4. *The solution process u^x of Theorem 3.2 is exponentially $L^2(\mu_\nu)$ ergodic in the sense that*

$$\varphi(u^x(t)) \longrightarrow \bar{\varphi} \equiv \int \varphi d\mu_\nu$$

as $t \to \infty$, exponentially quickly, for all measurable $\varphi \in L^2(\mu_\nu)$.
I.e. $\exists \lambda > 0$ such that as $t \to \infty$:

$$\|\varphi(u^x(t))) - \bar{\varphi}\|_{L^2(\mu_\nu)} \leq e^{-\lambda t} \|\varphi - \bar{\varphi}\|_{L^2(\mu_\nu)} \tag{31}$$

The proof uses the fact that μ_ν is Gaussian and the classical Dirichlet operator L_{μ_ν} associated with μ_ν (cf. [Alb00]) has a spectral gap (of some length λ).

Remark 3.8. *Ergodicity for stochastic equations for fluids has been an intensively discussed topic in recent years. One of the main points has been to prove ergodicity also in the presence of noise "concentrated only in a few Fourier modes", see [HM06].*

Remark 3.9. *The problem of extending the above considerations to 3 dimensions having a flow for Euler deterministic resp. stochastic Navier–Stokes equations with an "explicit (physically relevant) invariant measure" is open. For the 3 dimensional space or the 3 dimensional torus the only explicit positive invariant functional for the deterministic Euler equation is the energy. But the corresponding Gaussian measure seems to have too singular support to be related in some way to the equation itself.*

3.5 Some Additional Remarks and Complements

In the whole presentation we have restricted our attention to incompressible fluids. We shall continue to do so, except for the remark 8 below.

1. Stochastic Euler equations have been studied e.g. in [Bes99], [BF99], [CFM07], [BP01], [CC99], [Kim02], [MV00], both with additive and multiplicative noise of the Gaussian type. Deterministic Euler equations are studied e.g. in [Lio98, Bre99]. Dissipative Euler equations are studied e.g. [BF00]. See also [CC06].
2. Deterministic Burgers equations with initial random conditions have been studied e.g. [AMS94], [LOR94a, LOR94b] with interesting connections with the study of large scale structures in astrophysics [ASZ82], [SZ89], see also e.g. [Sin91]. A useful tool in the study of the Burgers equation is the Cole-Hopf transformation to a heat equation see [ABHK85] for an early proposal for its use. The stochastic Burgers equation in one resp. higher dimensions has been studied with Gaussian white noise e.g. in [DPDT05], [DPDT94], with spatial periodic and white Gaussian in time [LNP00], [BCJL94], [LW01], [TRW03], [E01], see also [EKMS00]. An analytic approach (uniqueness, infinitesimal invariance) to stochastic Burgers equation has been developed in [RS06], [BR01], [DPDT94] Propagation of chaos results for Burgers equation has been obtained e.g., [Szn87], [Osa87]. Burgers equation with other types of noise (Lévy noise) has been discussed [TW06], [TW03], [TZ03].

3. Stochastic Navier–Stokes equations with other types of Gaussian noise have been studied, see the references in Remark 3.2. Recently also additive and multiplicative noises of Lévy type have been considered, see [ABW]. Ergodicity of finite dimensional approximations of the stochastic, Navier–Stokes equation is discussed by [Rom04]. For other studies on ergodicity of 2D Navier–Stokes equations with random forcing, see e.g., [Fer97a], [FM95], [HM06], [KS01], [Kuk06], [Mat99].
4. The study of the deterministic limit of stochastic Navier–Stokes (or Euler) equations when there is a small parameter $\varepsilon > 0$ in front of the noise and ε is sent to zero is largely open, see, however [ABHK85], [Cru89b], [Hab91] for some initial considerations.
5. The study of the behaviour of solution of stochastic (and deterministic) Navier–Stokes equations as the viscosity coefficient ν tends to zero (i.e. the passage Navier–Stokes $\rightarrow$ Euler) has been performed in several publications. It was suggested in [ABHK85] and developed e.g. in [TRW03], some remarks are also in [Cru89b] [Hab91]. In [Kuk04] on a 2D-torus and for an additive noise of the Gaussian white type in time and smooth in space it is shown that for a subsequence (ν_i) of viscosity parameters the double limit $\lim_{\nu_j \to 0} \lim_{T\to\infty} u_{\nu_j}(T+t)$ yields a stationary solution of the deterministic Euler equation. See also, e.g., [Che96a], [CMR98], [Fre97], [Swa71].
6. There exists a probabilistic approach to deterministic hydrodynamical equations. In particular the solution of deterministic Navier–Stokes equations can be expressed by the solution process associated with a backward Kolmogorov equation. This goes back to E. Nelson and has been developed by Belopolskaya and Daletski ('78-'90), Busnello [Bus99], [BFR05], [AB02], [AB06], [Oss05], [BRV01], see also [BDS04], [FR02], [Rap02b].
7. Other problems connected with Euler and Navier–Stokes equation relate to
   a) coupling with heat equation ("Bénard problem") see e.g. [Fer97b].
   b) approximations, see, e.g. [DG95]
   c) physical properties like turbulence see, e.g. [Cho94]
   d) optimal control [Bar98], [CFMT83], [GSS02]
8. The phenomena associated with compressible fluids are quite different from those associated with incompressible fluids. In particular blow up phenomena for good initial data have been intensively studied. There is indeed a large literature on compressible Burgers and Burgers-type equations, with resp. without viscosity term, as well as on deterministic and stochastic compressible Euler and Navier–Stokes equations see, e.g., [FY92], [Mas00], [Roz03], [Roz04].
9. There are interesting basic connections between singularity phenomena occuring in (stochastic) equation for fluids and those occuring in the study of wave propagation and in certain quantum fields. E.g. the invariant measure constructed in Section 2 have close similarities with those

constructed for wave propagation in [MV94], [CH97] and with the stochastic quantization equations studied in the theory of quantum fields, see, e.g. [AR95], [DPT05], [ABR], [ALZ06], [AR96], [BCM88], [JLM85], [MR], [Sim74]. Analogies are already apparent in the non linearities and the fact of renormalization needed, an instance of which we saw in Section 2, where we had to introduce the renormalized energy : E: (this is similar to renormalizing the non linear term in the stochastic quantization equation, see, e.g., [JLM85], [AR95], [AR96]). Also the form of invariant measures $\mu_{\gamma I}$ is similar to the one occuring in the theory of quantum fields. Other relations concern the analogy of the classical limit of quantum fields and the $\nu \searrow 0$ in hydrodynamics, see, e.g., [ABHK85]. Also quantum fields described by stochastic differential equations with Poisson noise have been considered (see, e.g., [AGY05]), they have analogies with the invariant measures for the Euler-Navier–Stokes equations discussed in [AF04a]. It is well known from quantum field theory that problems get worse with the dimension of space increasing and this is also so in stochastic hydrodynamics. A "renormalization group approach", see, e.g. [Sin05b], [Sin05a], [Fri95], (see also [Sin08]), has been considered to be helpful in handling these problems. In any case it is likely that progress in one of these areas, fluids resp. quantum fields, will be beneficial to the other area. These are most challenging areas for future mathematical research and, in particular, further development of the type of infinite dimensional stochastic analysis we tried to present in form of an introduction in these lectures.

3.6 Appendix

We give a result of regularity for parabolic equations.

Proposition 3.5. *Let $T \in (0,\infty]$, $1 < \gamma < \infty$ and $\sigma \in \mathbb{R}$. Let A be the Stokes operator described in Section 3.1.*
For any $f \in L^\gamma(0,T;\mathcal{H}_2^\sigma)$, the Cauchy problem

$$\begin{cases} \frac{d}{dt}X(t) + AX(t) = f(t), & t \in (0,T] \\ X(0) = 0 \end{cases}$$

has a unique solution $X \in \mathcal{W}^{1,\gamma}(0,T) \equiv \{X \in L^\gamma(0,T;\mathcal{H}_2^{\sigma+2}) : \frac{d}{dt}X \in L^\gamma(0,T;\mathcal{H}_2^\sigma)\}$. Moreover, the solution depends continuously on the data in the sense that there exists a constant $c_{\gamma,\sigma}$ such that

$$\left(\int_0^T [\|X(t)\|^\gamma_{\mathcal{H}_2^{\sigma+2}} + \|\tfrac{d}{dt}X(t)\|^\gamma_{\mathcal{H}_2^\sigma}]\,dt\right)^{1/\gamma} \le \left(c_{\gamma,\sigma}\int_0^T \|f(t)\|^\gamma_{\mathcal{H}_2^\sigma}dt\right)^{1/\gamma}$$

Finally, $X \in C_b([0,T];\mathcal{B}_{2\,\gamma}^{\sigma+2-\frac{2}{\gamma}})$.

Proof. The Stokes operator A is a positive self adjoint operator in $\mathcal{H}_2^\sigma$ with domain $\mathcal{H}_2^{\sigma+2}$ and it generates an analytic semigroup in $\mathcal{H}_2^\sigma$. Then the first part of the proposition is obtained applying Theorem 3.2 in [DV87]. Moreover, by interpolation we get that the space $\mathcal{W}^{1,\gamma}(0,T)$ is continuously embedded in the space $C_b([0,T];\mathcal{B}_{2\,\gamma}^{\sigma+2-\frac{2}{\gamma}})$, that is there exists a positive constant c such that

$$\|X\|_{C_b([0,T];\mathcal{B}_{2\,\gamma}^{\sigma+2-\frac{2}{\gamma}})} \leq c\,\|X\|_{\mathcal{W}^{1,\gamma}(0,T)} \tag{32}$$

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An Introduction to 3D Stochastic Fluid Dynamics

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1 Introduction

We consider a viscous, constant density, Newtonian fluid described by the stochastic Navier–Stokes equations on the torus $\mathcal{T} = [0, L]^3$, $L > 0$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = \nu \triangle u + \sum_{i=1}^{\infty} \sigma_i h_i(x) \dot{\beta}_i(t) \tag{1}$$

with $\mathsf{div} u = 0$ and periodic boundary conditions, with suitable fields $h_i(x)$ and independent Brownian motions $\beta_i(t)$. The notation $(u \cdot \nabla) u$ stands for the vector field with components $[(u \cdot \nabla) u]_j = \sum_{k=1}^{d} u_k \partial_k u_j$; the operation $\triangle u$ has to be understood componentwise. The fluid is described by the velocity field $u = u(t, x)$ (a random vector field) and the pressure field $p = p(t, x)$ (a random scalar field). The fluid in a torus is an artificial model, but the topics we are going to investigate are so poorly understood that it is meaningful to idealize the mathematics as much as possible, preserving only those aspects that we believe to be essential. The random white noise force $\sum_{i=1}^{\infty} \sigma_i h_i(x) \dot{\beta}_i(t)$ is also part of the idealization, not only for its specific form but more basically because body forces are usually either absent or gradient-like (as the gravitational force field) and not so roughly varying. The complex phenomena (related to turbulence) we want to investigate are usually caused by complicate boundary effects (think to the fluid below a grid), too difficult to be dealt with at present. Thus we hope that a white noise force may both simplify the investigation and produce some phenomena similar to those of more realistic fluid systems.

The parameter $\nu > 0$ is called the kinematic viscosity. We shall investigate sometimes the limit as $\nu \to 0$, without any rescaling with ν of the random force: this is a singular limit problem, hopefully similar to the more realistic boundary layer ones. The limit as $\nu \to 0$ with constant-amplitude force essentially corresponds to the limit of infinite Reynolds number.

The white noise force is assumed, for sake of simplicity, to be the superposition of independent perturbations of various modes: we shall assume that the $h_i(x)$'s are eigenfunctions of the Stokes operator and the $\beta_i(t)$'s are independent scalar Brownian motions. The most interesting physical situation is the case when only a few large modes are activated, those with smaller wave length. In such a case L has the meaning of length scale of the action of the external force. A force with a typical length scale is a model for a fluid which interacts with a boundary or an object. However, in some cases one is able to deal only with the case of several or infinitely many modes, for mathematical reasons.

Section 3 is devoted to a finite dimensional model that captures several features of equations (1). It covers the Galerkin approximations of (1), so its analysis represents a main step in view of the infinite dimensional system; even the results of Section 5 are only based on the finite dimensional facts of Section 3.

In Section 4 we take the limit as the dimension goes to infinity and treat the 3D stochastic Navier–Stokes system. We define solutions to the martingale problem, prove their existence, and (partially) describe how to extract Markov selections. Finally, as a short introduction to the theory of Da Prato and Debussche [23], we prove that every Markov process composed of martingale solutions has a Strong Feller like property, under the assumptions on the noise imposed in [23]. This is a property of continuous dependence on initial condition that represents a striking step forward with respect to the deterministic theory. Relevant references on the topics of this section are, among others, [8], [12], [15], [16], [18], [19], [24], [25], [26], [33], [35], [36], [40], [41], [42], [45], [57], [58], [59], [61], [63], [64], [66], [67].

Section 5 deals with turbulence, restricting the attention to the so called K41 theory. A definition of K41 scaling law is given and investigated, disproved in 2D, shown to be equivalent to hopefully more manageable properties in 3D, that could be better analyzed in the future to understand whether they are true or, more likely, how they should be modified. These notes are restricted to the 3D case, where we aim to describe a few first steps in the direction of relevant open problems. The theory in the 2D case is richer of well posedness results, even for cylindrical noise, ergodicity, control, non-viscous case and limit, see among others references [2], [3], [4], [6], [8], [9], [11], [13], [14], [17], [21], [22], [23], [28], [30], [31], [32], [47], [49], [50], [51], [55], [56], [62].

2 Abstract Framework and General Preliminaries

We describe here a minimal amount of preliminaries on spaces and operators appearing in fluid dynamics; see for instance [53] and [65] for extensive discussions. Let $\mathbb{L}^2(\mathcal{T})$ be the space of vector fields $u : \mathcal{T} \to \mathbb{R}^3$ with $L^2(\mathcal{T})$-components. For every $\alpha > 0$, let $\mathbb{H}^\alpha(\mathcal{T})$ be the space of fields $u \in \mathbb{L}^2(\mathcal{T})$ with components in the Sobolev space $H^\alpha(\mathcal{T}) = W^{\alpha,2}(\mathcal{T})$.

Let $\mathcal{D}^\infty$ be the space of infinitely differentiable divergence free periodic fields u on $\mathcal{T}$, with zero mean:

$$\int_{\mathcal{T}} u(x)\, dx = 0.$$

This zero mean condition plays somewhat the role of a boundary condition. Let H be the closure of $\mathcal{D}^\infty$ in the $\mathbb{L}^2(\mathcal{T})$-topology; it is the space of all fields $u \in \mathbb{L}^2(\mathcal{T})$ such that $\mathsf{div} u = 0$, $u \cdot n$ on the boundary is periodic (one can show that for divergence free fields the trace $u \cdot n$ on the boundary is a well defined $H^{-1/2}$-distribution), $\int_{\mathcal{T}} u(x)\, dx = 0$. We endow H with the inner product

$$\langle u, v\rangle_H = \frac{1}{L^3} \int_{\mathcal{T}} u(x) \cdot v(x)\, dx$$

and the associated norm $|.|_H$.

Let V (resp. $D(A)$) be the closure of $\mathcal{D}^\infty$ in the $\mathbb{H}^1(\mathcal{T})$-topology (resp. $\mathbb{H}^2(\mathcal{T})$-topology); it is the space of divergence free, zero mean, periodic elements of $\mathbb{H}^1(\mathcal{T})$ (resp. of $\mathbb{H}^2(\mathcal{T})$). The spaces V and $D(A)$ are dense and compactly embedded in H (Rellich theorem). Due to the zero mean condition we also have

$$\int_{\mathcal{T}} |Du(x)|^2\, dx \geq \lambda \int_{\mathcal{T}} |u(x)|^2\, dx$$

for every $u \in V$, for some positive constant λ (Poincarè inequality). So we may endow V with the norm

$$|u|_V^2 := \int_{\mathcal{T}} |Du(x)|^2\, dx$$

where $|Du(x)|^2 = \sum_{i,j=1}^3 \left(\frac{\partial u_i(x)}{\partial x_j}\right)^2$.

Let $A : D(A) \subset H \to H$ be the operator $Au = -\triangle u$ (componentwise). Notice that $\triangle u \in H$ because we are in the periodic case (otherwise we would need a projection on divergence free fields). Since A is a selfadjoint positive (unbounded) operator in H, there is a complete orthonormal system $\{h_i\}_{i\in\mathbb{N}} \subset H$ made of eigenfunctions of A, with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ ($Ah_i = \lambda_i h_i$). Notice that the positivity is due to the zero mean condition. We may take the Poincaré constant λ above equal to λ_1. Notice that we have

$$\langle Au, u\rangle_H = |u|_V^2$$

for every $u \in D(A)$, so in particular

$$\langle Au, u\rangle_H \geq \lambda |u|_H^2 .$$

On the torus we know explicitly eigenfunctions and eigenvalues (see example 3.1 below), and often it is useful to parametrize them by vector wave numbers instead of the index $i \in \mathbb{N}$.

Let V' be the dual of V; with proper identifications we have $V \subset H \subset V'$ with continuous dense injections, and the scalar product $\langle \cdot, \cdot\rangle_H$ extends to the dual pairing $\langle \cdot, \cdot\rangle_{V,V'}$ between V and V'. Denote the norms in H and V by $|\cdot|_H$ and $\|\cdot\|_V$ respectively.

Let $B(\cdot,\cdot) : V \times V \to V'$ be the bilinear operator defined as

$$\langle w, B(u,v)\rangle_{V,V'} = \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

for every $u, v, w \in V$. To clarify the definition, first take $u, v, w \in \mathcal{D}^\infty$ and notice that by Hölder inequality

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq \sum_{i,j=1}^{3} |u_i|_{L^{\alpha_1}(\mathcal{T})} |v_j|_{W^{1,\alpha_2}(\mathcal{T})} |w_j|_{L^{\alpha_3}(\mathcal{T})}$$

where $\alpha_i > 1$, $\sum_{i=1}^{3} \frac{1}{\alpha_i} = 1$. Sobolev embedding theorem says

$$|u|_{L^q(\mathcal{T})} \leq C(s,p,\mathcal{T}) |u|_{W^{s,p}(\mathcal{T})}, \quad \frac{1}{q} = \frac{1}{p} - \frac{s}{d}$$

Hence, for $p = 2$,

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq C \sum_{i,j=1}^{3} |u_i|_{W^{s_1,2}(\mathcal{T})} |v_j|_{W^{1+s_2,2}(\mathcal{T})} |w_j|_{W^{s_3,2}(\mathcal{T})}$$

($C = C(s_1, s_2, s_3, \mathcal{T})$) for $s_i \geq 0$ such that $\frac{3}{2} - \frac{1}{d}\sum_{i=1}^{3} s_i = 1$, namely $\sum_{i=1}^{3} s_i = \frac{d}{2}$. In particular

$$\left| \sum_{i,j=1}^{3} \int_{\mathcal{T}} u_i \frac{\partial v_j}{\partial x_i} w_j dx \right| \leq C \sum_{i,j=1}^{3} |u_i|_{W^{s_1,2}(\mathcal{T})} |v_j|_{W^{1,2}(\mathcal{T})} |w_j|_{W^{s_3,2}(\mathcal{T})}$$

with $s_1 + s_3 = \frac{d}{2}$. For $d = 3$ (but also $d = 2$ and 4) we may take $s_1 = s_3 = \frac{d}{4}$ and confirm that $B(\cdot,\cdot)$ can be extended to a bilinear mapping $B(\cdot,\cdot) : V \times V \to V'$.

We every $u, v \in \mathcal{D}^\infty$ we have

$$\langle B(u,v), v\rangle_H = \frac{1}{2} L^{-3} \int_{\mathcal{T}} (u(x) \cdot \nabla) |v(x)|^2 dx = 0$$

since $\mathsf{div} u = 0$, and this property extends from $\mathcal{D}^\infty$ to the various spaces of vector fields used in the sequel.

To get further estimates, it is useful to recall that the function $\alpha \mapsto \log |u|_{W^{\alpha,2}(\mathcal{T})}$ is convex:

$$|u|_{W^{\alpha s_1 + (1-\alpha)s_2,2}(\mathcal{T})} \leq |u|^{\alpha}_{W^{s_1,2}(\mathcal{T})} |u|^{(1-\alpha)}_{W^{s_2,2}(\mathcal{T})}$$

for $\alpha \in [0,1]$ (easy by Fourier analysis).

Among the infinitely many consequences of the previous computations we have

$$|\langle B(u,v),w\rangle| \leq C(\mathcal{T}) \sum_{i,j=1}^{d} |u_i|_{W^{\frac{d}{4},2}(\mathcal{T})} |v_j|_{W^{1,2}(\mathcal{T})} |w_j|_{W^{\frac{d}{4},2}(\mathcal{T})}$$

and

$$|u|_{W^{\frac{d}{4},2}(\mathcal{T})} \leq |u|_{L^2(\mathcal{T})}^{1-\frac{d}{4}} |u|_{W^{1,2}(\mathcal{T})}^{\frac{d}{4}}$$

thus

$$|\langle B(u,v),w\rangle| \leq C |u|_H^{1-\frac{d}{4}} \|u\|_V^{\frac{d}{4}} \|v\|_V |w|_H^{1-\frac{d}{4}} \|w\|_V^{\frac{d}{4}}.$$

We have proved:

Lemma 2.1. *In $d=3$,*

$$|\langle B(u,v),w\rangle| \leq C |u|_H^{1/4} \|u\|_V^{3/4} \|v\|_V |w|_H^{1/4} \|w\|_V^{3/4}.$$

We also need the following inequalities.

Lemma 2.2. *In $d=3$,*

$$|\langle Ax, B(x,x)\rangle_H| \leq C \|x\|_V^{3/2} |Ax|_H^{3/2}$$

for every $x \in D(A)$.

Proof. Due to the periodicity of vector fields that allows us to drop the boundary terms in the integrations by parts, for every $u,v \in \mathcal{D}^\infty$ we have

$$\begin{aligned}\langle Au, B(v,u)\rangle_H &= \sum_{i,j=1}^{3} \int_{\mathcal{T}} v_i \partial_i u_j \triangle u_j = -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k (v_i \partial_i u_j) \partial_k u_j \\ &= -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k v_i \partial_i u_j \partial_k u_j - \frac{1}{2} \sum_{i,j,k=1}^{3} \int_{\mathcal{T}} v_i \partial_i (\partial_k u_j)^2 \\ &= -\sum_{i,j,k=1}^{3} \int_{\mathcal{T}} \partial_k v_i \partial_i u_j \partial_k u_j\end{aligned}$$

since $\mathsf{div} v = 0$ and thus

$$\begin{aligned}\langle Au, B(u,u)\rangle_{H_L} &\leq C \sum_{i,k=1}^{3} \left(\int_{\mathcal{T}} |\partial_k u_i|^3 \right) \leq C |Du|_{L^3(\mathcal{T})}^3 \\ &\leq C |Du|_{W^{1/2,2}(\mathcal{T})}^3 \leq C |Du|_{L^2(\mathcal{T})}^{3/2} |Du|_{W^{1,2}(\mathcal{T})}^{3/2}.\end{aligned}$$

Lemma 2.3.

$$|AB(u,v)|_H \leq C \left(|Au| \|Av\|_V + |Av| \|Au\|_V \right)$$

$$\left| A^{1/2} B(u,v) \right|_H \leq C |Au| |Av|$$

$$|B(u,v)|_H \leq C \left(|Au| \left| A^{1/2} v \right| \wedge \left| A^{1/2} u \right| |Av| \right)$$

and for every $\gamma \in (0,1/2)$

$$\left|A^{\gamma}B\left(u,v\right)\right|_{H}\leq C\left(\left|Au\right|^{2}\left|A^{\gamma+\frac{1}{2}}v\right|^{2}\wedge\left|Av\right|^{2}\left|A^{\gamma+\frac{1}{2}}u\right|^{2}\right).$$

Proof. Up to multiplicative constants that we omit,

$$\begin{aligned}\left|AB\left(u,v\right)\right|_{H}&\leq\sqrt{\sum_{i,j=1}^{3}\int_{\mathcal{T}}\left[\triangle\left(u_{i}\partial_{i}v_{j}\right)\right]^{2}dx}\\&\leq\left|Dv\right|_{\infty}\left|Au\right|+\left|Du\right|_{\infty}\left|Av\right|+\left|u\right|_{\infty}\left\|Av\right\|_{V}\\&\leq\left|Au\right|\left\|Av\right\|_{V}+\left|Av\right|\left\|Au\right\|_{V}\end{aligned}$$

$$\begin{aligned}\left|A^{1/2}B\left(u,v\right)\right|_{H}&\leq\sqrt{\sum_{i,j=1}^{3}\int_{\mathcal{T}}\left[D\left(u_{i}\partial_{i}v_{j}\right)\right]^{2}dx}\\&\leq\left|Du\right|_{4}\left|Dv\right|_{4}+\left|u\right|_{\infty}\left|Av\right|\end{aligned}$$

$$\left|B\left(u,v\right)\right|_{H}^{2}\leq\sum_{i,j=1}^{3}\int_{\mathcal{T}}\left[u_{i}\partial_{i}v_{j}\right]^{2}dx\leq\left|u\right|_{4}^{2}\left|Dv\right|_{4}^{2}\leq\left|A^{1/2}u\right|\left|Av\right|$$

$$\left|B\left(u,v\right)\right|_{H}^{2}\leq\left|u\right|_{\infty}^{2}\left|Dv\right|_{2}^{2}\leq\left|Au\right|\left|A^{1/2}v\right|$$

and the last one follows by interpolation.

With the previous notations in mind, we (formally) rewrite equations (1) of Section 1 as an abstract stochastic evolution equation in H

$$du(t)+\left[\nu Au(t)+B\left(u(t),u(t)\right)\right]dt=\sum_{i=1}^{\infty}\sigma_{i}h_{i}d\beta_{i}\left(t\right). \tag{1}$$

The rigorous definition of solution will be given in Section 4 and is not entirely trivial; here we only anticipate that, as in the deterministic case, we have to interpret expressions in integral and weak form over test functions

$$\begin{aligned}&\left\langle u(t),\varphi\right\rangle_{H}+\int_{0}^{t}\nu\left\langle u(s),A\varphi\right\rangle_{H}ds-\int_{0}^{t}\left\langle B\left(u(s),\varphi\right),u(s)\right\rangle_{H}ds\\&=\left\langle u_{0},\varphi\right\rangle_{H}+\sum_{i=1}^{\infty}\sigma_{i}\left\langle h_{i},\varphi\right\rangle_{H}\beta_{i}\left(t\right)\end{aligned} \tag{2}$$

with $\varphi\in\mathcal{D}^{\infty}$. As a last general remark, we shall always assume at least

$$\sum_{i=1}^{\infty}\sigma_{i}^{2}<\infty$$

(H-valued Brownian motion), but some of the most interesting results will require

$$\sum_{i=1}^{\infty} \lambda_i \sigma_i^2 < \infty$$

to have certain regularities in $D(A)$.

3 Finite Dimensional Models

The reason for this Section is twofold: first we may illustrate a number of basic facts and open problems in a simple setting where the rigor is easy to control; second, most of these results are a preliminary technical step for the analysis of Sections 4 and 5.

3.1 Introduction and Examples

Consider a real *finite* dimensional Hilbert space H,

$$\dim H < \infty$$

endowed with norm $|.|_H$ and inner product $\langle ., . \rangle_H$. We stress that H is *not* the space introduced above but it is finite dimensional; we should write H_n to avoid misunderstandings, and similarly we should write A_n, B_n etc., but in the whole Section we never take the limit as $n \to \infty$ (except in very few well advertized places) so we drop the subscript n for sake of simplicity.

About the various constants involved in the following estimates, we say that a constant is *not* universal if it depends on $\dim H$, ν, the norm in H of A, or constants related to the continuity properties of B. When a constant is independent of these quantities, we call it *universal* and denote it generically by $C > 0$. The non-universal constants are not stable in the limit of the stochastic Navier–Stokes equations (Section 4) or in the limit as $\nu \to 0$ (Section 5).

Let A be a positive definite symmetric linear mapping in H,

$$\langle Ax, x \rangle_H \geq \lambda |x|_H^2$$

for every $x \in H$, where $\lambda > 0$ is a universal constant (Poincaré constant in our applications); $B(.,.) : H \times H \to H$ a bilinear mapping such that

$$\langle B(x, x), x \rangle_H = 0 \tag{1}$$

for every $x \in H$, Q a semi-definite symmetric matrix in H, $(W_t)_{t \geq 0}$ a Brownian motion in H defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P\right)$.

Remark 3.1. Sometimes we need a stronger version of (1):

$$\langle B(y,x), x\rangle_H = 0 \tag{2}$$

for every $x, y \in H$. This is equivalent to

$$\langle B(y,x), z\rangle_H = -\langle B(y,z), x\rangle_H \tag{3}$$

for every $x, y, z \in H$: if (2) holds, then

$$\begin{aligned} 0 &= \langle B(y,x+z), x+z\rangle_H \\ &= \langle B(y,z), x\rangle_H + \langle B(y,x), z\rangle_H \end{aligned}$$

so (3) is true. If (3) holds then, taking $z = x$, we get (2).

Since A is positive definite, $(u,v) \mapsto \langle Au, v\rangle_H$ is another inner product in H and

$$\|u\|_V := \sqrt{\langle Au, u\rangle_H}$$

is a norm in H and we have

$$\lambda |x|_H^2 \le \|u\|_V^2 \le C_A |x|_H^2$$

for a non universal constant C_A (the norm of A in H); the lower bound is universal.

Consider the stochastic differential equation (SDE) in H, with $\nu > 0$,

$$dX_t = [-\nu A X_t - B(X_t, X_t)]\, dt + \sqrt{Q} dW_t, \quad t \ge 0 \tag{4}$$

with initial condition given by an $\mathcal{F}_0$-measurable random variable $X_0 : \Omega \to H$. As usual, we interpret the equation in the integral sense

$$X_t = X_0 + \int_0^t [-\nu A X_s - B(X_s, X_s)]\, ds + \sqrt{Q} W_t. \tag{5}$$

Example 3.1. Our main example is the Galerkin approximation of equation (1) of Section 1. Given $L > 0$, on the torus $\mathcal{T} = [0, L]^3$, consider the complexification of the infinite dimensional spaces and operators introduced in the previous section, that we denote just in this example by $\mathcal{H}$, $\mathcal{V}$, $\mathcal{D}(\mathcal{A})$, $\mathcal{A}$, $\mathcal{B}$ to avoid superposition with the notations of the present chapter. Define the index sets

$$\Lambda^{(n)} = \left\{ (k,\alpha) \in \left(\frac{2\pi}{L}\mathbb{Z}^3\right) \times \{1,2\} : 0 < |k|^2 \le \left(\frac{2\pi}{L} n\right)^2 \right\}$$

and $\Lambda^{(\infty)} = \cup_n \Lambda^{(n)}$. The eigenvectors of $\mathcal{A}$ are given by

$$h_{k,\alpha}(x) := a_{k,\alpha} e^{ik\cdot x}, \quad x \in \mathcal{T}, \quad (k,\alpha) \in \Lambda^{(\infty)}$$

(with eigenvalues $\lambda_{k,\alpha} = |k|^2$), where, for every $k \in \mathbb{R}^3 \diagdown 0$, we have to choose an orthonormal basis $a_{k,\alpha}$, $\alpha = 1, 2$, of the orthogonal space to k (the space generated by the vectors $\left(e_\alpha - \frac{k_\alpha k}{|k|^2}\right)$, $\alpha = 1, 2, 3$). Let

$$\mathcal{H}^{(n)} = span\left\{h_{k,\alpha}; (k,\alpha) \in \Lambda^{(n)}\right\}$$

and let $\pi^{(n)}$ be the orthogonal projection of $\mathcal{H}$ on $\mathcal{H}^{(n)}$, which commutes with $\mathcal{A}$. With these notations, our main example of finite dimensional system is defined by the objects

$$\mathcal{H}^{(n)}, \mathcal{A}|_{\mathcal{H}^{(n)}}, \pi^{(n)}\mathcal{B}(.,.)|_{\mathcal{H}^{(n)}\times\mathcal{H}^{(n)}}$$

that we simply denote by H, A, $B(.,.)$.

Example 3.2. The famous Lorenz system in $\mathbb{R}^3$, with parameters $a, b, c > 0$,

$$\begin{cases} dx + (ax - ay)\,dt = \sigma_1 d\beta_1 \\ dy + (-bx + y + xz)\,dt = \sigma_2 d\beta_2 \\ dz + (cz - xy)\,dt = \sigma_3 d\beta_3 \end{cases}$$

fits into the framework of this section if $b = a \in (0, 1]$. The same is true for the Minea system:

$$\begin{cases} dx + \left(x + \delta\left(y^2 + z^2\right)\right) dt = \sigma_1 d\beta_1 \\ dy + (y - \delta xy)\,dt = \sigma_2 d\beta_2 \\ dz + (z - xz)\,dt = \sigma_3 d\beta_3. \end{cases}$$

Example 3.3. Another interesting example is the GOY model (from Gledzer, Ohkitani, Yamada), a particular case of the so called “shell model”. See [44] for an introduction and references. It is a simplified Fourier system where the interaction between different modes is preserved only between neighbor modes, and the complex valued variables $u_n(t) = u_{n,1}(t) + iu_{n,2}(t)$ are summaries of the Fourier coefficients. The finite dimensional model is defined, for $n = -1, 0, 1..., N, N+1, N+2$, by the constraints

$$u_{-1}(t) = u_0(t) = u_{N+1}(t) = u_{N+2}(t) = 0$$

and the equations for $n = 1, ..., N$

$$\begin{aligned} &du_n + \nu k_n^2 u_n dt + ik_n\left(\frac{1}{4}\overline{u}_{n-1}\overline{u}_{n+1} - \overline{u}_{n+1}\overline{u}_{n+2} + \frac{1}{8}\overline{u}_{n-1}\overline{u}_{n-2}\right) \\ &= \sigma_n d\beta_n \end{aligned}$$

where $k_n = 2^n k_0$, $k_0 > 0$ given. Some foundational results on the related infinite dimensional system can be found in [4].

3.2 A Priori Bounds

Throughout this section we assume that $(X_t)_{t\geq 0}$ is a continuous adapted solution of equation (4). In this section it is sufficient to work under condition (1) on B.

Lemma 3.1. *[L^2 bounds and energy equality]Assume $E\left|X_0\right|_H^2 < \infty$. Then, for every $T > 0$, we have*

$$E\left(\sup_{t\in[0,T]} |X_t|_H^2 + \nu \int_0^T \|X_s\|_V^2\, ds\right) \leq C_1\left(E\left|X_0\right|_H^2, TrQ, T\right) \tag{6}$$

where $C_1\left(E\left|X_0\right|_H^2, TrQ, T\right)$ is given by (12),

$$|X_t|_H^2 + 2\nu \int_0^t \|X_s\|_V^2\, ds = |X_0|_H^2 + TrQ\, t + M_t \tag{7}$$

where M_t is a square integrable martingale, and

$$\frac{1}{2}E\left|X_T\right|_H^2 + \nu E\int_0^T \|X_s\|_V^2\, ds = \frac{1}{2}E\left|X_0\right|_H^2 + \frac{1}{2}TrQ\, T. \tag{8}$$

Proof. **Step 1**. The function $f(x) = |x|_H^2$ has derivatives

$$Df(x) = 2x, \quad D^2 f(x) = 2 \cdot \mathrm{Id}$$

hence Itô formula and property (1) give us

$$|X_t|_H^2 = |X_0|_H^2 - \int_0^t 2\nu \left\langle AX_s, X_s\right\rangle_H ds + M_t + TrQ\, t \tag{9}$$

where

$$M_t = 2\int_0^t \left\langle X_s, \sqrt{Q}dW_s\right\rangle_H$$

is a local martingale.

Step 2. Let us prove that

$$E\int_0^T |X_t|_H^2\, dt < \infty \tag{10}$$

for every $T > 0$. The reader not interested in details but only in the main concepts may drop this technical point and go to step 3. To simplify, one may think that (10) has been imposed as an additional assumption in the lemma. However, it is conceptually interesting to notice that the assumptions of the lemma do not include any quantitative bound on the solution, but only on the data (the integrability of the initial condition and the gaussianity of the forcing term), and it is the equation itself, with its particular algebraic structure, that produce bounds on the solution.

We may localize M_t by an increasing sequence of stopping times (τ_n): $\tau_n \to \infty$ a.s. and $t \mapsto M_{t\wedge\tau_n}$ is a square integrable martingale for every n. To be more specific, we may take

$$\tau_n = \inf\left\{t \geq 0 : |X_t|_H^2 = n\right\}.$$

From (9) and the positivity of A we have

$$|X_{t\wedge\tau_n}|_H^2 \leq |X_0|_H^2 + M_{t\wedge\tau_n} + TrQ\ (t\wedge\tau_n) \tag{11}$$

and thus

$$E\left[|X_{t\wedge\tau_n}|_H^2\right] \leq E\,|X_0|_H^2 + TrQ\ t.$$

We also have

$$\int_0^{T\wedge\tau_n} |X_t|_H^2\, dt = \int_0^{T\wedge\tau_n} |X_{t\wedge\tau_n}|_H^2\, dt \leq \int_0^T |X_{t\wedge\tau_n}|_H^2\, dt$$

hence

$$E\int_0^{T\wedge\tau_n} |X_t|_H^2\, dt \leq \int_0^T E\,|X_{t\wedge\tau_n}|_H^2\, dt \leq T\left(E\,|X_0|_H^2 + TrQ\ T\right).$$

By the monotone convergence theorem we get (10).

Step 3. Having proved (10), M_t is now a square integrable martingale. Then we have (7) and then (8), which also implies the second part of the bound (6). To prove the first part of (6) we use the bound

$$|X_t|_H^2 \leq |X_0|_H^2 + |M_t| + TrQ\ t$$

coming from (7) due to the positivity of A. We have

$$\sup_{t\in[0,T]} |X_t|_H^2 \leq |X_0|_H^2 + 1 + \sup_{t\in[0,T]} |M_t|^2 + TrQ\ T$$

hence, by Doob's inequality $E\sup_{t\in[0,T]} |M_t|^2 \leq 4E\,|M_T|^2$, we have

$$\begin{aligned}&E\sup_{t\in[0,T]} |X_t|_H^2\\ &\leq E\,|X_0|_H^2 + 1 + 16E\int_0^T \langle QX_s, X_s\rangle\, ds + TrQ\ T.\end{aligned}$$

From (8) (with t in place of T) and the positivity of A we already know that

$$\sup_{t\in[0,T]} E\left[|X_t|_H^2\right] \leq E\,|X_0|_H^2 + TrQ\ T.$$

Hence we have (6) with

$$C_1\left(E\left|X_0\right|_H^2, TrQ, T\right) \tag{12}$$
$$:= E\left|X_0\right|_H^2 + 1 + 16TrQ\left(E\left|X_0\right|_H^2 + TrQ\,T\right) + TrQ\,T.$$

The proof is complete.

Remark 3.2. The bound (6) tells us the basic topologies where we have to look for solutions. With others below, it will give us the bounds, on the Galerkin approximations of the stochastic Navier–Stokes equations, needed to extract subsequences that converge in a proper way to pass to the limit in the equations.

Remark 3.3. The two identities (7) and (8) express energy balance laws. Let us comment the second one: we may think that the term $\frac{1}{2}TrQ$ is the mean rate of energy (mean energy per unit of time) injected into the system, $\nu E\int_0^T \|X_s\|_V^2\,ds$ is the mean energy dissipated on $[0,T]$, $\frac{1}{2}E\left|X_T\right|_H^2$ is the mean (kinetic) energy of the system.

We say that a stochastic process $(X_t)_{t\geq 0}$ is *stationary* if given any $0 \leq t_1 < ... < t_n$ and $s \geq 0$, the law of the r.v. $(X_{t_1+s}, ..., X_{t_n+s})$ (r.v. in H^n) is independent of s. In the following corollary in fact we just need that the covariance of X_t is independent of t.

Notice that in principle we should assume a quantitative bound like $E\left|X_0\right|_H^2 < \infty$ to start with, like in Lemma 3.1; but stationarity provides itself a mechanism for proper estimates (the property of stationarity is like an a priori bound itself). Somewhat related results can be found in [10] by a different approach.

A technical remark: in the proof of the following corollary there is a step where we use the fact that equation (4) has a unique solution for every square integrable $\mathcal{F}_0$-measurable initial condition X_0. This fact will be proved in a subsequent section. So, from a logical viewpoint, we should state this corollary only later on. We anticipate here to avoid repetitions.

Corollary 3.1. *If $(X_t)_{t\geq 0}$ is stationary then*

$$E\|X_t\|_V^2 = \frac{TrQ}{2\nu} \tag{13}$$

for every $t \geq 0$ (in particular $E\|X_t\|_V^2 < \infty$ for every stationary solution).

Proof. If $E\left|X_0\right|_H^2 < \infty$, from (8) and the stationarity we first have

$$2\nu\int_0^T E\|X_s\|_V^2\,ds = TrQ\,T$$

but also $E\|X_s\|_V^2$ is independent of t, whence the result. Therefore, in order to complete the proof, we have only to show that $E\left|X_0\right|_H^2 < \infty$ is true for every stationary solution.

Given $\varepsilon > 0$, let $R_\varepsilon > 0$ be such that $P\left(|X_0|_H^2 > R_\varepsilon\right) < \varepsilon$. Let $\Omega_\varepsilon \in \mathcal{F}$ be defined as $\Omega_\varepsilon = \left\{|X_0|_H^2 \leq R_\varepsilon\right\}$; we have $P(\Omega_\varepsilon) \geq 1 - \varepsilon$. Define $X_0^{(\varepsilon)}$ as X_0 on Ω_ε, 0 otherwise. Let $\left(X_t^{(\varepsilon)}\right)_{t\geq 0}$ be the unique solution of equation (4) with initial condition $X_0^{(\varepsilon)}$ (theorem 3.1 below). Just looking at the integral form of (4) (which has an elementary pathwise meaning) it is easy to realize that $X_\cdot^{(\varepsilon)}(\omega) = X_\cdot(\omega)$ for P-a.e. $\omega \in \Omega_\varepsilon$. For $\left(X_t^{(\varepsilon)}\right)_{t\geq 0}$ we have (8), hence

$$\frac{1}{T}\int_0^T E\left\|X_s^{(\varepsilon)}\right\|_V^2 ds \leq \frac{R_\varepsilon}{2\nu T} + \frac{TrQ}{2\nu}$$

(in a sense, we use here an idea of Chow and Hasminski [20]). Then, given $N > 0$,

$$\begin{aligned}
&E\left(\|X_0\|_V^2 \wedge N\right) \\
&= \frac{1}{T}\int_0^T E\left[\|X_s\|_V^2 \wedge N\right] ds \\
&= \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(\|X_s\|_V^2 \wedge N\right)\right] ds + \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon^c}\left(\|X_s\|_V^2 \wedge N\right)\right] ds \\
&\leq \frac{1}{T}\int_0^T E\left[1_{\Omega_\varepsilon}\left(\left\|X_s^{(\varepsilon)}\right\|_V^2 \wedge N\right)\right] ds + N\varepsilon \\
&\leq \frac{1}{T}\int_0^T E\left[\left\|X_s^{(\varepsilon)}\right\|_V^2\right] ds + N\varepsilon \\
&\leq \frac{R_\varepsilon}{2\nu T} + \frac{TrQ}{2\nu} + N\varepsilon.
\end{aligned}$$

It is now sufficient to take first the limit as $T \to \infty$, then as $\varepsilon \to 0$, finally as $N \to \infty$. The proof is complete.

Remark 3.4. Quantitative knowledge of the statistics of the stationary regime of a turbulent fluid is one of the most important and open problems of fluid dynamics (due in great part to the fact that a turbulent fluid is a non-equilibrium, although possibly stationary, system, so there are no general paradigm as the Gibbs one to describe its stationary regime; in mathematical terms, equation (4) is not gradient like, hence we do not know the density of its invariant measure explicitly in a simple Gibbs form). The identity (13) is a positive example in this direction. It has a very interesting interpretation in connection with the experimental fact that the rate of energy dissipation of a turbulent fluid has a finite limit when the viscosity goes to zero.

In the proof of existence of solutions we need a stopped version of part of the previous result. This is the only point in this section where we do not assume that X is a solution over the whole half-line $[0, \infty)$, but only on a random time interval.

Lemma 3.2. *Let $\tau \geq 0$ be a stopping time and $(X_t)_{t\geq 0}$ a continuous adapted process that P-a.s. satisfies (5) for $t \in [0, \tau(\omega)]$. Assume $E|X_0|_H^2 < \infty$. Then, for every $T > 0$, we have*

$$E\left(\sup_{t\in[0,T]} |X_{t\wedge\tau}|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right).$$

Proof. We may repeat step by step the previous proof, substituting everywhere the process $X_{t\wedge\tau}$ to X_t and stating every identity or inequality for $t \in [0, \tau(\omega)]$ only. But we profit of this repetition to give an alternative proof. Let

$$\tau_R' = \inf\left\{t \geq 0 : |X_t|_H^2 = R\right\}, \quad \tau_R = \tau_R' \wedge \tau$$

and notice that $\tau_R' \uparrow \tau$ as $R \to \infty$. We have (also for τ in place of τ_R)

$$\begin{aligned} X_{t\wedge\tau_R} &= X_0 + \int_0^{t\wedge\tau_R} [-\nu AX_s - B(X_s, X_s)]\, ds + \sqrt{Q}W_{t\wedge\tau_R} \\ &= X_0 + \int_0^t [-\nu AX_{s\wedge\tau_R} - B(X_{s\wedge\tau_R}, X_{s\wedge\tau_R})]\, 1_{s\leq\tau_R} ds \\ &+ \int_0^t 1_{s\leq\tau_R}\sqrt{Q}dW_s. \end{aligned}$$

Apply Itô formula to $|X_{t\wedge\tau_R}|_H^2$ and get

$$\begin{aligned} |X_{t\wedge\tau_R}|_H^2 &= |X_0|_H^2 - 2\int_0^t \langle \nu AX_{s\wedge\tau_R} - B(X_{s\wedge\tau_R}, X_{s\wedge\tau_R}), X_{s\wedge\tau_R}\rangle_H\, 1_{s\leq\tau_R} ds \\ &+ \widetilde{M}_t + TrQ\ (t\wedge\tau_R) \end{aligned}$$

and thus

$$|X_{t\wedge\tau_R}|_H^2 + 2\nu\int_0^t \|X_{s\wedge\tau_R}\|_V^2\, 1_{s\leq\tau_R} ds = |X_0|_H^2 + \widetilde{M}_t + TrQ\ (t\wedge\tau_R)$$

where

$$\widetilde{M}_t = 2\int_0^t \left\langle X_{s\wedge\tau_R}, 1_{s\leq\tau_R}\sqrt{Q}dB_s\right\rangle_H.$$

Then, by Doob's inequality along with the isometry formula of Itô integrals

$$\begin{aligned} E\sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2 &\leq E|X_0|_H^2 + 1 + E\sup_{t\in[0,u]} \left|\widetilde{M}_t\right|_H^2 + TrQ\ u \\ &\leq E|X_0|_H^2 + 1 + +TrQ\ u \\ &+ C \cdot TrQ \cdot E\int_0^u 1_{s\leq\tau_R} |X_{s\wedge\tau_R}|_H^2\, ds \end{aligned}$$

Therefore

$$E \sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2 \leq C + C\int_0^u E \sup_{t\in[0,s]} |X_{t\wedge\tau_R}|_H^2\, ds$$

which implies the result by Gronwall lemma applied to the function $f(t) = E\sup_{t\in[0,u]} |X_{t\wedge\tau_R}|_H^2$ and the independence of the constants of R. The proof is complete.

The statement of the following lemma is not the strongest possible one (because we claim (15) only for $p^* < p$), see the next remark. However we restrict ourselves to this result since its proof is now elementary, being very similar to the one just given above. Moreover, it will be sufficient for our purposes.

Lemma 3.3 (L^p estimates). *Assume $E|X_0|_H^p < \infty$ for some $p > 2$. Then, for every $T > 0$,*

$$E\int_0^T |X_s|_H^p\, ds < C_2\left(p, E|X_0|_H^p, TrQ, T\right) \tag{14}$$

and

$$E\sup_{t\in[0,T]} |X_t|_H^{p^*} < C_3\left(p, E|X_0|_H^p, TrQ, T\right) \tag{15}$$

where $p^* = p/2 + 1$ (which is also greater than 2) and the constants C_2 and C_3 are given in the proof.

Proof. **Step 1**. We consider now the function

$$f(x) = |x|_H^p = \left(|x|_H^2\right)^{p/2}$$

which has derivatives

$$\begin{aligned} Df(x) &= p\,|x|_H^{p-2}\, x, \\ D^2 f(x) &= p\,(p-2)\,|x|_H^{p-4}\, x\otimes x + p\,|x|_H^{p-2}\cdot \mathrm{Id} \end{aligned}$$

with the property

$$Tr\left[QD^2 f(x)\right] = p\,(p-1)\,|x|_H^{p-2}\, TrQ.$$

Itô formula and property (1) give us now

$$\begin{aligned} &|X_t|_H^p + p\nu\int_0^t |X_s|_H^{p-2}\langle AX_s, X_s\rangle_H\, ds \\ &= |X_0|_H^p + M_t^{(p)} + \frac{p\,(p-1)\,TrQ}{2}\int_0^t |X_s|_H^{p-2}\, ds \end{aligned} \tag{16}$$

where

$$M_t^{(p)} = p\int_0^t |X_s|_H^{p-2} \left\langle X_s, \sqrt{Q}dW_s \right\rangle_H$$

is a local martingale.

Step 2. The proof of (14) is now the same as the proof of (10), plus a simple iterative argument. Let p' be any number between 2 and the value of p declared in the assumptions of the lemma. By the same localization argument used in the previous proof, we get

$$E\left[|X_{t\wedge\tau_n}|_H^{p'}\right] \le E\left[|X_0|_H^{p'}\right] + \frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\,ds$$

and thus

$$\begin{aligned} E\int_0^{T\wedge\tau_n} |X_t|_H^{p'}\,dt \le E\int_0^T |X_{t\wedge\tau_n}|_H^{p'}\,dt \le TE\left[|X_0|_H^{p'}\right] \\ + T\frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\,ds \end{aligned}$$

which implies

$$E\int_0^T |X_t|_H^{p'}\,dt \le TE\left[|X_0|_H^{p'}\right] + T\frac{p'(p'-1)\,TrQ}{2} E\int_0^T |X_s|_H^{p'-2}\,ds \tag{17}$$

by the monotone convergence theorem. This inequality allows us to iterate a bound of the form (14) from a smaller value of p to a larger one, starting from $p = 2$ given in the previous lemma. More formally, let Π be the set of $p' \in [2, p]$ (p given in the claim of the lemma), such that, for some constant $C\left(p, E|X_0|_H^p, TrQ, T\right)$, we have

$$E\int_0^T |X_t|_H^{p'}\,dt \le C\left(p, E|X_0|_H^p, TrQ, T\right)$$

for all $T \ge 0$. The set Π is non empty ($2 \in \Pi$ by the previous lemma) and has the property that $a \in \Pi$, $a < p$ implies that $b = (a+2) \wedge p \in \Pi$ (from (17)). Then $p \in \Pi$, so (14) is proved.

Step 3. Unfortunately (14) does not imply that $M_t^{(p)}$ is a square integrable martingale, but this is true for $M_t^{(p^*)}$, since

$$\begin{aligned} (p^*)^2 \int_0^t |X_s|_H^{2(p^*-2)} \langle QX_s, X_s\rangle_H\,ds \\ \le (p^*)^2 |Q| \int_0^t |X_s|_H^{2(p^*-1)}\,ds = (p^*)^2 |Q| \int_0^t |X_s|_H^p\,ds. \end{aligned}$$

We can now repeat the proof of step 3 of the previous lemma:

$$\sup_{t\in[0,T]} |X_t|_H^{p^*} \le |X_0|_H^{p^*} + 1 + \sup_{t\in[0,T]} \left|M^{(p^*)}\right|^2 + \frac{p^*(p^*-1)\,TrQ}{2}\int_0^T |X_s|_H^{p^*-2}\,ds$$

hence, by Doob's inequality,

$$E\sup_{t\in[0,T]} |X_t|_H^{p^*} \le E\,|X_0|_H^{p^*} + 1 + 4\,(p^*)^2 \int_0^T |X_s|_H^{2(p^*-2)} \langle QX_s, X_s\rangle_H\,ds + \frac{p^*(p^*-1)\,TrQ}{2}\int_0^T |X_s|_H^{p^*-2}\,ds$$

which is easily bounded by a constant $C_3\left(p, E\,|X_0|_H^p, TrQ, T\right)$ due to (14). The proof is complete.

Corollary 3.2. *For a stationary solutions we have* $E\left[|X_t|_H^{p-2}\,\|X_t\|_V^2\right] < \infty$ *(hence in particular* $E\,|X_t|_H^p < \infty$*) for every* $p \ge 2$ *and*

$$E\left[|X_t|_H^{p-2}\,\|X_t\|_V^2\right] = \frac{(p-1)\,TrQ}{2\nu} E\left[|X_t|_H^{p-2}\right].$$

Proof. The proof is the same given above for $p = 2$ and it is based on identity (16), that one has to iterated in p.

Remark 3.5. Under the same assumptions we have the expected estimate

$$E\sup_{t\in[0,T]} |X_t|_H^p \le C\left(p, E\,|X_0|_H^p, TrQ, T\right)$$

for every $T > 0$. Its proof follows the lines of the proof of lemma 3.2 and makes use of Burkholder-Davis-Gundy inequality. See Flandoli and Gaterek [34], Appendix 1.

We have seen that the regularity of solutions can be improved in the direction of p-integrability on Ω. Less easy is to improve it in the direction of stronger topologies (in fact in finite dimensions they are all equivalent, but the equivalence is not stable in the passage to the limit). A natural question would be whether we have an estimate of the form

$$E\left(\sup_{t\in[0,T]} \|X_t\|_V^2 + \nu\int_0^T |AX_s|_H^2\,ds\right) \le C_1\left(E\,\|X_0\|_V^2, TrQ, T\right). \tag{18}$$

This is an open problem (the answer is positive in the 2D case). Nevertheless, under assumptions inspired to the 3D case we can state at least one result on the time integrability of $|AX_s|_H^2$, proved by [41] in the deterministic case (see a related result with a different proof in [52], Thm 3.6) and extended to the stochastic case by Da Prato and Debussche [23]. To avoid unnecessary complications due to the generality, let us assume that A and Q commute, so there exists a common orthonormal system $\{e_i\}$ of eigenvectors, with $Ae_i = \lambda_i e_i$, $Qe_i = \sigma_i^2 e_i$.

To understand assumption (19) below, notice that in a finite dimensional space H we always have $\langle Ax, B(x,x)\rangle_H \leq C_{A,B} |x|_H^2$ for a suitable nonuniversal constant $C_{A,B}$. On the contrary, the constant C in (19) is universal, see lemma 2.2. In 2D (always with periodic boundary conditions) the situation is entirely different since we have $\langle Ax, B(x,x)\rangle_H = 0$ (vorticity conservation for $\nu = 0$).

Lemma 3.4 (bounds on $|AX_t|_H$). *Assume that*

$$\langle Ax, B(x,x)\rangle_H \leq C \|x\|_V^{3/2} |Ax|_H^{3/2}, \quad x \in H \tag{19}$$

and

$$\sum_i \sigma_i^2 \lambda_i < \infty.$$

Then

$$E\int_0^T \frac{|AX_s|_H^2}{\left(1+\|X_s\|_V^2\right)^2} ds \leq C\frac{1}{\nu^4} E\int_0^T \|X_s\|_V^2 + \frac{1}{\nu}\left(1 + C\sum_i \sigma_i^2\lambda_i\right)$$

$$\sqrt[3]{\nu} E\int_0^T |AX_s|_H^{2/3} ds \leq C\left(T, \sum_i \sigma_i^2\lambda_i\right)\left(1 + \frac{1}{\nu}E\int_0^T \|X_s\|_V^2\right)$$

where, concerning the term $E\int_0^T \|X_s\|_V^2 ds$, we recall the bound (6).

Proof. Introduce the function $f : H \to \mathcal{R}$ defined as

$$f(x) = \frac{1}{1+\|x\|_V^2} = \left(1 + \langle Ax, x\rangle_H\right)^{-1}.$$

We have

$$Df(x) = -\left(1+\|x\|_V^2\right)^{-2} D\langle Ax, x\rangle_H = -2\frac{Ax}{\left(1+\|x\|_V^2\right)^2}$$

$$D^2 f(x) = 8\frac{Ax \otimes Ax}{\left(1+\|x\|_V^2\right)^3} - 2\frac{A}{\left(1+\|x\|_V^2\right)^2}.$$

Hence

$$\frac{1}{1+\|X_t\|_V^2} = \frac{1}{1+\|X_0\|_V^2} + 2\int_0^t \frac{\langle AX_s, \nu AX_s + B(X_s, X_s)\rangle_H}{\left(1+\|X_s\|_V^2\right)^2} ds$$
$$+\widetilde{M}_t + \frac{1}{2}\int_0^t g(X_s)\, ds$$

where

$$\widetilde{M}_t = -2\int_0^t \frac{\left\langle AX_s, \sqrt{Q}dW_s\right\rangle_H}{\left(1+\|X_s\|_V^2\right)^2}$$

is a local martingale and

$$g(x) = \sum_i \sigma_i^2 \left[8\frac{\langle Ax, e_i\rangle_H^2}{\left(1+\|x\|_V^2\right)^3} - 2\frac{\langle Ae_i, e_i\rangle_H}{\left(1+\|x\|_V^2\right)^2} \right].$$

In fact $\widetilde{M}_t$ is a square integrable martingale, because

$$\int_0^t \frac{\langle QAX_s, AX_s\rangle_H}{\left(1+\|X_s\|_V^2\right)^4} ds \le \int_0^t \frac{\left|A^{1/2}QA^{1/2}\right| \left|A^{1/2}X_s\right|_H^2}{\left(1+\|X_s\|_V^2\right)^4} ds$$

$$\le \sum_i \sigma_i^2 \lambda_i \int_0^t \frac{\|X_s\|_V^2}{\left(1+\|X_s\|_V^2\right)^4} ds \le \sum_i \sigma_i^2 \lambda_i < \infty.$$

We also have $|g(x)| \le C\sum_i \sigma_i^2 \lambda_i$, so, from

$$2\nu \int_0^t \frac{|AX_s|_H^2}{\left(1+\|X_s\|_V^2\right)^2} ds \le \frac{1}{1+\|X_t\|_V^2} - 2\int_0^t \frac{\langle AX_s, B(X_s,X_s)\rangle_H}{\left(1+\|X_s\|_V^2\right)^2} ds$$
$$-\widetilde{M}_t - \frac{1}{2}\int_0^t g(X_s)\, ds$$

the assumption on B and the martingale property of $\widetilde{M}_t$ we have

$$2\nu E\int_0^t \frac{|AX_s|_H^2}{\left(1+\|X_s\|_V^2\right)^2} ds$$
$$\le 1 + 2CE\int_0^t \frac{\|X_s\|_V^{3/2} |AX_s|_H^{3/2}}{\left(1+\|X_s\|_V^2\right)^2} ds + \frac{C}{2}\sum_i \sigma_i^2 \lambda_i.$$

Moreover,

$$E\int_0^T \frac{\|X_s\|_V^{3/2} |AX_s|^{3/2}}{\left(1+\|X_s\|_V^2\right)^2} dt \le \varepsilon\nu E\int_0^T \frac{|AX_s|^2}{\left(1+\|X_s\|_V^2\right)^2} + C_\varepsilon \frac{1}{\nu^3} E\int_0^T \|X_s\|_V^2$$

for every $\varepsilon > 0$ and for a suitable constant C_ε, due to the following Young inequality $(f, g \ge 0)$

$$fg \leq \varepsilon f^p + \frac{C}{\varepsilon^{1/(p-1)}} g^{p'}, p' = \frac{p}{p-1}.$$

With a universal choice of $\varepsilon > 0$ we have

$$\nu E \int_0^t \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} ds \leq 1 + C \frac{1}{\nu^3} E \int_0^t \|X_s\|_V^2 + C \sum_i \sigma_i^2 \lambda_i.$$

This implies the first inequality of the lemma. The second one simply follows from the following inequalities:

$$\sqrt[3]{\nu} E \int_0^T |AX_s|_H^{2/3} ds \leq \left(E \int_0^T \left(\nu \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} \right) dt \right)^{1/3} \cdot \left(E \int_0^T \left(1 + \|X_s\|_V^2\right) dt \right)^{2/3}$$

$$\leq \left(1 + C \frac{1}{\nu^3} E \int_0^t \|X_s\|_V^2 + C \sum_i \sigma_i^2 \lambda_i \right)^{1/3} \left(T + E \int_0^T \|X_s\|_V^2 dt \right)^{2/3}.$$

The proof is complete.

Remark 3.6. Under the assumption $E |X_0|_H^2 < \infty$ we know that $E \int_0^t \|X_s\|_V^2 ds$ is bounded by a universal constant, so the same is true for $E \int_0^T |AX_s|_H^{2/3} ds$. This implies $E \left[|AX_t|_H^{2/3} \right] < \infty$ for almost every t. If in addition the process is stationary, we have $E \left[|AX_t|_H^{2/3} \right] < \infty$ for every t. In terms of invariant measures μ of the limit infinite dimensional problem this will imply that $\mu(D(A)) = 1$.

Remark 3.7. For stationary X_s,

$$2E \frac{\langle AX_s, \nu AX_s + B(X_s, X_s) \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2} + \frac{1}{2} E g(X_s) = 0.$$

Notice that $|g(x)| \leq C \sum_i \sigma_i^2 \lambda_i$, so under the assumption that this quantity is finite and given, we may heuristically think that $Eg(X_s)$ converges to a nonzero value g_0 as $\nu \to 0$. Then we have

$$E \frac{|AX_s|_H^2}{\left(1 + \|X_s\|_V^2\right)^2} \sim \frac{1}{\nu} \left(\frac{g_0}{4} + E \frac{\langle AX_s, B(X_s, X_s) \rangle_H}{\left(1 + \|X_s\|_V^2\right)^2} \right).$$

Remark 3.8. Let us briefly understand that under assumption (19) it is not possible to obtain a bound of the form (18). Without all the details, from Itô formula for $d\|X_t\|_V^2$, we have

$$d\|X_t\|_V^2 + 2\nu |AX_t|_H^2 \leq |\langle AX_t, B(X_t, X_t)\rangle_H| \text{ plus other terms}$$

and

$$\begin{aligned}|\langle AX_t, B(X_t, X_t)\rangle_H| &\leq C\|X_t\|_V^{3/2} |AX_t|_H^{3/2} \\ &\leq \nu |AX_t|_H^2 + C\|X_t\|_V^6\end{aligned}$$

so we meet the differential inequality

$$d\|X_t\|_V^2 \leq C\|X_t\|_V^6 \text{ plus other terms}$$

that cannot be closed on a global time interval.

3.3 Comparison of Two Solutions and Pathwise Estimates

Having assumed an additive noise, it disappears when we write the equation for the difference of two solutions; this has some advantages. We can reach similar advantages for a single solution with the following trick: we consider the difference between the solution and an auxiliary process, usually the solution of the associated linear equation. Let us perform some of these computations in this section.

However, here we assume the stronger algebraic condition (2) on B.

Lemma 3.5. *Let* $\left(X_t^{(1)}\right)$ *and* $\left(X_t^{(2)}\right)$ *be two solutions on some interval* $[0, T]$ *and let us set* $V_t = X_t^{(1)} - X_t^{(2)}$. *Let* C_B *be a constant such that*

$$\langle B(x, y), x\rangle_H \leq C_B |x|_H^2 |y|_H$$

for every $x, y \in H$. *Then*

$$|V_t|_H \leq |V_0|_H e^{C_B \int_0^t \left|X_s^{(2)}\right|_H ds}.$$

Proof. We have

$$\frac{dV_t}{dt} + \nu AV_t + B\left(X_t^{(1)}, V_t\right) + B\left(V_t, X_t^{(2)}\right) = 0$$

whence

$$\frac{1}{2}\frac{d|V_t|_H^2}{dt} + \nu \langle AV_t, V_t\rangle_H = -\left\langle B\left(V_t, X_t^{(2)}\right), V_t\right\rangle_H \tag{20}$$

and thus

$$\frac{1}{2}\frac{d|V_t|_H^2}{dt} \leq C_B |V_t|_H^2 \left|X_t^{(2)}\right|_H.$$

The conclusion follows from Gronwall lemma.

The previous result is not stable in the limit of infinite dimensions. On the contrary, the assumption on B of the next lemma is stable in dimension 3.

Lemma 3.6. *Assume that*

$$\langle B(x,y), x\rangle_H \le C |x|_H^{1/2} \|x\|_V^{3/2} \|y\|_V$$

for every $x, y \in H$, where C is a universal constant. Then we have

$$|V_t|_H \le |V_0|_H \, e^{C \int_0^t \left\| X_s^{(2)} \right\|_V^4 ds}.$$

Proof. We restart from (20) and get now (we use Young inequality in the second step)

$$\begin{aligned}\frac{1}{2}\frac{d |V_t|_H^2}{dt} + \nu \|V_t\|_V^2 &\le C |V_t|_H^{1/2} \|V_t\|_V^{3/2} \left\| X_t^{(2)} \right\|_V \\ &\le \frac{1}{2} \|V_t\|_V^2 + C |V_t|_H^2 \left\| X_t^{(2)} \right\|_V^4 .\end{aligned}$$

Therefore

$$\frac{1}{2}\frac{d |V_t|_H^2}{dt} \le C |V_t|_H^2 \left\| X_t^{(2)} \right\|_V^4$$

which implies the claim of the lemma, again by Gronwall lemma.

From the viewpoint of the limit to infinite dimensions for 3D fluids, the problem in the first lemma is the constant C_B. On the contrary, the problem in the second lemma is the term $\int_0^t \left\| X_s^{(2)} \right\|_V^4 ds$, on which we do not have bounds which are stable with the dimension.

To summarize, we have shown two simple computations which imply uniqueness for the finite dimensional problem but are useless for 3D fluids. There exist very many variants of these computations in different topologies, but the result, until now, is always the same: either the constant or some norm of the solutions blow-up in the case of the 3D Navier–Stokes equation.

Exercise 3.1. In the application to the 2-dimensional Navier–Stokes equations the continuity properties of B are stronger, due to the improvement coming from Sobolev embedding theorem. One has

$$\langle B(x,y), x\rangle_H \le C \|y\|_V |x|_H \|x\|_V$$

for every $x, y \in H$, where C is a universal constant. Prove that

$$|V_t|_H \le |V_0|_H \, e^{C \int_0^t \left\| X_s^{(2)} \right\|_V^2 ds}.$$

Deduce a pathwise uniqueness result.

Exercise 3.2. (from Schmalfuss [60]). Continue the 2-dimensional case but consider a multiplicative noise of the form

$$G(X_t)\, dW_t$$

in place of the additive noise $\sqrt{Q}dW_t$. Assume that G is a Lipschitz continuous mapping from H to the space of linear bounded operators in H. Under this more general condition, one can prove an existence result along the same lines developed above. However, the uniqueness is more difficult, since the equation for the difference of two solutions $V_t = X_t^{(1)} - X_t^{(2)}$ is still an Itô equation, and an estimate on $|V_t|^2$ cannot simply be obtained by a pathwise application of Gronwall lemma. On the other side, the inequality that one gets from Itô formula for $|V_t|^2$ cannot be closed at the level of mean values since it contains cubic terms. Solve the problem using Itô formula for

$$e^{-C\int_0^t \left\|X_s^{(2)}\right\|_V^2 ds} \left|V_t\right|^2 .$$

In another form, this trick comes out again in the paper of DaPrato and Debussche [23].

3.4 Existence and Uniqueness, Markov Property

In the next theorem we shall show that equation (4) has a unique strong solution $(X_t^x)_{t\geq 0}$ for every initial condition $x \in H$, that depends measurably on x. We then may define the operators $P_t : B_b(H) \to B_b(H)$ as

$$(P_t\varphi)(x) = E\left[\varphi\left(X_t^x\right)\right].$$

Here $B_b(H)$ is the space of Borel bounded functions on H and $C_b(H)$ will be the space of continuous bounded ones. We prove also that (4) defines a Markov process in the sense that

$$E\left[\varphi\left(X_{t+s}^x\right)|\mathcal{F}_t\right] = (P_s\varphi)\left(X_t^x\right) \quad (P\text{-a.s.}).$$

for every $\varphi \in C_b(H)$, $t, s > 0$, $x \in H$. Taking the expectation in this identity one gets the semigroup property $P_{t+s} = P_t P_s$ on $C_b(H)$. We say that P_t is Feller if $P_t\varphi \in C_b(H)$ for every $\varphi \in C_b(H)$.

Theorem 3.1. *For every $\mathcal{F}_0$-measurable $X_0 : \Omega \to H$, there exists a unique continuous adapted solution $(X_t)_{t\geq 0}$ of equation (4) on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$. If the initial conditions x^n converge to x in H, the corresponding solutions converge P-a.s., uniformly in time on bounded intervals. Equation (4) defines a Markov process with the Feller property.*

Proof. Uniqueness and continuous dependence have been proved above in lemma 3.5 (of course such a result is not stable in the limit of infinite dimensions). This implies also the Feller property. Let us divide the proof of existence and Markov property in several steps. They are classical and are given for completeness. Preliminary, we remark that the proof of existence can be performed either by means of classical probabilistic arguments or by a pathwise analysis, due to the additivity of the noise. Let us give the probabilistic proof.

Step 1. (existence for bounded X_0). It is sufficient to prove the existence on $[0,T]$. Assume that $|X_0|_H \leq C$ for some constant $C>0$. For any $n>C$, let $B_n(.) : H \to H$ be a Lipschitz continuous function such that $B_n(x) = B(x,x)$ for every $|x|_H \leq n$. Consider then the equation

$$dX_t^{(n)} = \left[-\nu A X_t^{(n)} - B_n\left(X_t^{(n)}, X_t^{(n)}\right)\right] dt + \sqrt{Q} dW_t$$

with initial condition X_0. It has globally Lipschitz coefficients, so there exists a unique continuous adapted solution $\left(X_t^{(n)}\right)_{t\geq 0}$. The proof of this classical result can be done by contraction principle in $L^2(\Omega; C([0,T];H))$. Let τ_n be defined as

$$\tau_n = \inf\left\{t \geq 0 : \left|X_t^{(n)}\right|_H = n\right\} \wedge T.$$

Up to τ_n the solution $X_t^{(n)}$ is also a solution of the original equation: it is sufficient to observe the integral form of the equations. Therefore, by lemma 3.2, applicable since $E|X_0|_H^2 < \infty$, we have

$$E\left(\sup_{t\in[0,T]} \left|X_{t\wedge\tau_n}^{(n)}\right|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right).$$

In particular

$$E\left(1_{\{\tau_n<T\}} \left|X_{T\wedge\tau_n}^{(n)}\right|_H^2\right) \leq C_1\left(E|X_0|_H^2, TrQ, T\right)$$

which implies

$$P(\tau_n < T) \leq \frac{1}{n^2} C_1\left(E|X_0|_H^2, TrQ, T\right)$$

since $\left|X_{T\wedge\tau_n}^{(n)}\right|_H^2 = n^2$ on $\{\tau_n < T\}$. If $N>n$ then $\tau_N > \tau_n$ and

$$P\left(X_t^{(N)} = X_t^{(n)}, t\in[0,\tau_n]\right) = 1$$

Therefore, if $\tau_\infty := \sup_{n>C} \tau_n$, we may uniquely define a process $X_t^{(\infty)}$ for $t\in[0,\tau_\infty)$, equal to $X_t^{(n)}$ on $[0,\tau_n]$ for every n. Hence $X_t^{(\infty)}$ is a solution on $[0,\tau_\infty)$. But we have

$$P(\tau_\infty < T) \leq P(\tau_n < T) \leq \frac{C}{n^2}$$

for every n, hence $P(\tau_\infty < T) = 0$. Thus $X_t^{(\infty)}$ is a solution for $t\in[0,T-\varepsilon]$ for every small $\varepsilon>0$. Since T is arbitrary, we have proved global existence. Denote by $(X_t^x)_{t\geq 0}$ the unique solution with initial condition $x\in H$.

Step 2. (existence for general X_0). Let $\Omega_n \in \mathcal{F}$ be defined as $\Omega_n = \left\{ |X_0|_H^2 \leq n \right\}$. Define $X_0^{(n)}$ as X_0 on Ω_n, 0 otherwise. Let $\left(X_t^{(n)} \right)_{t \geq 0}$ be the unique solution of equation (4) with initial condition $X_0^{(n)}$. If $N > n$, then

$$P\left(\Omega_n \cap \left(X_t^{(N)} = X_t^{(n)} \text{ for every } t \geq 0 \right) \right) = P\left(\Omega_n \right).$$

We may then uniquely define a process $X_t^{(\infty)}$ on $\Omega' = \cup_n \Omega_n$ as $X_t^{(\infty)} = X_t^{(n)}$ on Ω_n. Looking at the equation in integral form, in particular at its pathwise meaning, it is clear that $X_t^{(\infty)}$ solves the equation on Ω'. But $P(\Omega') = 1$, hence we have proved the existence of a global solution.

Step 3. (Markov property). Given $x \in H$, $\varphi \in C_b(H)$, $t, s > 0$, we have to prove that

$$E\left[\varphi\left(X_{t+s}^x \right) Z \right] = E\left[\left(P_s \varphi \right) \left(X_t^x \right) Z \right]$$

for every bounded $\mathcal{F}_t$-measurable r.v. Z. By uniqueness

$$X_{t+s}^x = X_{t,t+s}^{X_t^x} \quad (P\text{-a.s.})$$

where $\left(X_{t_0,t}^\eta \right)_{t \geq t_0}$ denotes the unique solution on the time interval $[t_0, \infty)$, with the $\mathcal{F}_{t_0}$-measurable initial condition $X_{t_0,t_0}^\eta = \eta$. It is then sufficient to prove that

$$E\left[\varphi\left(X_{t,t+s}^\eta \right) Z \right] = E\left[\left(P_s \varphi \right) (\eta) Z \right]$$

for every H-valued $\mathcal{F}_t$-measurable r.v. η. By approximation (one has to use Lebesgue theorem and the fact that strong convergence of η_n in H implies that $\left(P_s \varphi \right) (\eta_n)$ converges P-a.s. to $\left(P_s \varphi \right) (\eta)$), it is sufficient to prove it for every r.v. η of the form $\eta = \sum_{i=1}^k \eta^{(i)} 1_{A^{(i)}}$ with $\eta^{(i)} \in H$ and $A^{(i)} \in \mathcal{F}_t$. By inspection (everything decomposes with respect to the partition $A^{(i)}$) one can see that it is sufficient to prove it for every deterministic element $\eta \in H$. Now the r.v. $X_{t,t+s}^\eta$ depends only on the increments of the Brownian motion between t and $t + s$, hence it is independent of $\mathcal{F}_t$. Therefore

$$E\left[\varphi\left(X_{t,t+s}^\eta \right) Z \right] = E\left[\varphi\left(X_{t,t+s}^\eta \right) \right] E[Z].$$

Since $X_{t,t+s}^\eta$ has the same law of X_s^η (by uniqueness), we have $E\left[\varphi\left(X_{t,t+s}^\eta \right) \right] = E\left[\varphi\left(X_s^\eta \right) \right]$ and thus

$$E\left[\varphi\left(X_{t,t+s}^\eta \right) Z \right] = \left(P_s \varphi \right) (\eta) E[Z] = E\left[\left(P_s \varphi \right) (\eta) Z \right].$$

The proof is complete.

3.5 Invariant Measures

Let P_t^* be the semigroup on the space of probability measures on H defined as

$$\left(P_t^* \mu \right) (f) = \mu\left(P_t f \right)$$

where we use the notation $\mu(f)$ for $\int_H f d\mu$. We have $P_{t+s}^* = P_t^* P_s^*$ for every $t, s \geq 0$ and P_0^* is the identity. If $(X_t)_{t\geq 0}$ is a solution and ν_t denotes the law of X_t, then $P_{t-s}^* \nu_s = \nu_t$ for every $t \geq s \geq 0$.

We say that a probability measure μ is invariant if $P_t^* \mu = \mu$ for every $t \geq 0$. Equivalently, if

$$\mu(\varphi) = \mu(P_t \varphi)$$

for every $t \geq 0$ and $\varphi \in C_b(H)$.

We recall that, given a metric space (X, d) with its Borel σ-field $\mathcal{B}$, a set of probability measures Λ on $(X, \mathcal{B})$ is *tight* if the following condition holds: for every $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset X$ such that $\mu(K_\varepsilon) > 1 - \varepsilon$ for every $\mu \in \Lambda$. Moreover, a set of probability measures Λ on $(X, \mathcal{B})$ is relatively compact if from every sequence $\{\mu_n\} \subset \Lambda$ one may extract a subsequence $\{\mu_{n_k}\}$ and find a probability measure μ on $(X, \mathcal{B})$ such that $\mu_{n_k} \to \mu$ weakly (by this we mean that $\mu_{n_k}(\varphi) \to \mu(\varphi)$ for every $\varphi \in C_b(H)$). If X is a Polish space, Prohorov theorem states that Λ is tight if and only if it is relatively compact. Notice that if X is compact, tightness is free and then also the relative compactness of Λ, but in our applications the metric space is H, so we need estimates to prove tightness.

Theorem 3.2. *There exists at least one invariant measure for (4), with the property*

$$\mu\left(\|.\|_V^2\right) \leq \frac{TrQ}{2\nu}. \tag{21}$$

Proof. **Step 1** (preparation). Following the general scheme attributed to Krylov and Bogoliubov, we consider a solution $(X_t)_{t\geq 0}$ with a suitable initial condition, say $X_0 = 0$, we denote the law of X_t by ν_t and introduce the time averages

$$\mu_T = \frac{1}{T}\int_0^T \nu_s ds = \frac{1}{T}\int_0^T P_s^* \nu_0 ds$$

or more explicitly $\mu_T(\varphi) = \frac{1}{T}\int_0^T \nu_s(\varphi) ds$ for every $\varphi \in C_b(H)$. The family of measures $\{\mu_T; T \geq 0\}$ is tight. Let us give two illuminating proofs of this fact.

Step 2 (first proof of tightness). We follow a clever argument of Chow and Khasminskii [20]. From the energy equality (8), taking into account the choice $X_0 = 0$, we know that

$$\frac{1}{T}\int_0^T E\|X_s\|_V^2 ds \leq \frac{TrQ}{2\nu}.$$

Notice that

$$\mu_T\left(\|.\|_V^2\right) = \frac{1}{T}\int_0^T \nu_s\left(\|.\|_V^2\right) ds = \frac{1}{T}\int_0^T E\|X_s\|_V^2 ds$$

(the first identity holds true for the test function $\|.\|_V^2 \wedge N$ by definition of μ_T, and then extends to the function $\|.\|_V^2$ by monotone convergence theorem). Hence

$$\mu_T\left(\|.\|_V^2\right) \leq \frac{TrQ}{2\nu} \tag{22}$$

and thus, by Chebyshev inequality,

$$\mu_T\left(\|x\|_V^2 \geq R^2\right) \leq R^{-2}\mu_T\left(\|.\|_V^2\right) \leq R^{-2}\frac{TrQ}{2\nu}.$$

This implies the tightness.

Step 3 (second proof of tightness). Equation (8), which reads

$$E\,|X_t|_H^2 + 2\nu\int_0^t E\,\|X_s\|_V^2\, ds = TrQ\; t$$

implies that $E\,|X_t|_H^2$ is differentiable and

$$\frac{dE\,|X_t|_H^2}{dt} + 2\nu E\,\|X_t\|_V^2 = TrQ.$$

Hence

$$\frac{dE\,|X_t|_H^2}{dt} \leq -2\nu\lambda E\,|X_t|_H^2 + TrQ.$$

This implies

$$E\,|X_t|_H^2 \leq e^{-2\nu\lambda t}E\,|X_0|_H^2 + \int_0^t e^{-2\nu\lambda(t-s)}TrQds \leq \frac{TrQ}{2\nu\lambda}.$$

The Gronwall-like inequality can be easily proved as Gronwall lemma, computing $\frac{d\left(e^{2\nu\lambda t}E|X_t|_H^2\right)}{dt}$ and integrating the result on $[0,t]$. From the previous inequality we have, as above,

$$\begin{aligned}\mu_T\left(|x|_H^2 \geq R^2\right) &= \frac{1}{T}\int_0^T \nu_s\left(|x|_H^2 \geq R^2\right) ds \\ &\leq \frac{1}{T}\int_0^T \nu_s\left(|.|_H^2\right) ds = \frac{1}{T}\int_0^T E\,|X_s|_H^2\, ds \leq \frac{TrQ}{2\nu\lambda}\end{aligned}$$

which yields the tightness. Notice that the result of this second method is weaker from the viewpoint of the topologies.

Step 4 (conclusion). From Prohorov theorem, there exists a sequence μ_{T_n} weakly convergent to a probability measure μ. Let us show that μ is invariant. We have, for every $f \in C_b(H)$, and using the fact that $P_t f \in C_b(H)$ by the Feller property,

$$\left(P_t^*\mu\right)(f) = \mu\left(P_t f\right) = \lim_{n\to\infty}\mu_{T_n}\left(P_t f\right) = \lim_{n\to\infty}\left(P_t^*\mu_{T_n}\right)(f)$$

and

$$\begin{aligned}P_t^* \mu_{T_n} = P_t^* \frac{1}{T_n} \int_0^{T_n} P_s^* \nu_0 ds = \frac{1}{T_n} \int_0^{T_n} P_{t+s}^* \nu_0 ds \\ = \frac{1}{T_n} \int_t^{t+T_n} P_\sigma^* \nu_0 d\sigma = \mu_{T_n} \\ -\frac{1}{T_n} \int_0^t P_\sigma^* \nu_0 d\sigma + \frac{1}{T_n} \int_{T_n}^{T_n+t} P_\sigma^* \nu_0 d\sigma.\end{aligned}$$

The last two terms converge weakly to zero. This proves $P_t^* \mu = \mu$, so the existence of an invariant measure is assured.

Finally, from (22), we get

$$\mu_{T_n} \left(\|.\|_V^2 \wedge N \right) \leq \frac{TrQ}{2\nu}$$

for every $N, n > 0$, hence

$$\mu \left(\|.\|_V^2 \wedge N \right) \leq \frac{TrQ}{2\nu}$$

for every $N > 0$, and thus we have (21) by monotone convergence theorem. The proof is complete.

Remark 3.9. If Q is invertible, the invariant measure is unique and ergodic. We refer to specialized text for definitions and results.

Remark 3.10. In specific examples one can say more about ergodicity: it holds also for certain degenerate noises. Consider our main example 3.1 on the Galerkin approximation of the d-dimensional Navier–Stokes equations. Weinan E and Mattingly [27] in $d = 2$ and Romito [59] in $d = 3$ proved that ergodicity is true if the noise is active at least on a very small number of modes (like 4), properly displaced to “generate” all other modes through the action of the drift. Let us mention also the work of Mattingly and Hairer [53] on the true 2D Navier–Stokes equations (ergodicity with very degenerate noise) and the references therein.

In the previous theorem we have constructed an invariant measure with the property (21). For this purpose we had to start Krylov-Bogoliubov scheme from a good initial condition. In fact, this is not necessary: the invariance itself provides the mechanism to prove (21).

Theorem 3.3. *All invariant measures have the property (21). In fact they also satisfy*

$$\mu \left(\|.\|_V^2 \right) = \frac{TrQ}{2\nu}.$$

Proof. Let μ be an invariant measure. If $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ is the filtered probability space where the Brownian motion is defined, consider the enlarged filtered probability space

$$\Omega' = \Omega \times H,\ \mathcal{F}' = \mathcal{F} \otimes \mathcal{B},\ \mathcal{F}'_t = \mathcal{F}_t \otimes \mathcal{B},\ P' = P \otimes \mu$$

with the new Brownian motion (W'_t) and the $\mathcal{F}'_0$-measurable r.v. X_0 defined as

$$W'_t(\omega, x) = W_t(\omega),\ X_0(\omega, x) = x.$$

The law of X_0 is μ. The unique solution (X_t) of (4) with initial condition X_0 is a stationary process, with the law of X_t equal to μ for every $t \geq 0$ (we do not give the details, the result is intuitively clear). Then we can apply corollary 3.1. Finally, from (8) we deduce that $\mu\left(\|.\|_V^2\right)$ is truly equal to $\frac{TrQ}{2\nu}$. The proof is complete.

Corollary 3.3. *Assume that A and Q commute and have eigenvalues λ_i and σ_i^2 respectively, with $\sum_i \sigma_i^2 \lambda_i < \infty$. Assume also that condition (19) holds true. Then all invariant measures μ have the property*

$$\mu\left[\frac{|Ax|_H^2}{\left(1+\|x\|_V^2\right)^2}\right] \leq C\frac{\sum_i \sigma_i^2}{2\nu^5} + \frac{1}{\nu}\left(1 + C\sum_i \sigma_i^2 \lambda_i\right)$$

$$\sqrt[3]{\nu}\mu\left[|Ax|_H^{2/3}\right] \leq C\left(1 + \frac{\sum_i \sigma_i^2}{2\nu^5}\right)\sum_i \sigma_i^2 \lambda_i$$

with universal constant $C > 0$.

3.6 Galerkin Stationary Measures for the 3D Equation

Our main concern are the stochastic Navier–Stokes equations (1) of Section 1. We work only with the Galerkin approximations and use the notations and definitions of Example 3.1 of Section 1, but restricted to real (not complex) spaces and operators. We assume for simplicity that the noise has the form

$$\sqrt{Q}W_t = \sum_{i=1}^{\infty} \sigma_i h_i(x)\,\beta_i(t)$$

where σ_i are real numbers, h_i are eigenfunctions of $\mathcal{A}$, β_i are independent Brownian motions. We do not define here the concept of solution to (1) (this will be done later on), but simply introduce a class of probability measures on $\mathcal{H}$ that we call *Galerkin stationary measures of equations* (1).

Let us say that a probability measure μ on $\mathcal{H}$ is a *cluster points* of Galerkin invariant measures if there exists a sequence $\{n_k\}$ diverging to infinity and for each n_k an invariant measure μ_{n_k} of the corresponding Galerkin approximation system, such that the sequence of measures $\{\mu_{n_k}\}$ weakly converges to μ on $\mathcal{H}$. Then we call Galerkin stationary measures of equations (1) every such cluster point. We denote by $\mathcal{P}_{NS}^{Galerkin}$ the set of all such probability measures on $\mathcal{H}$.

Theorem 3.4. *$P_{NS}^{Galerkin}$ is non empty. Every $\mu \in P_{NS}^{Galerkin}$ satisfies*

$$\mu\left(\|.\|_{\mathcal{V}}^2\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}. \tag{23}$$

Proof. For every n, let μ_n be an invariant measure of the corresponding Galerkin approximation system. We have

$$\mu_n\left(\|.\|_{\mathcal{V}}^2\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

so the family $\{\mu_n\}$ of measures is bounded in probability in $\mathcal{V}$, and thus it is tight in $\mathcal{H}$ since the space $\mathcal{V}$ is compactly embedded into $\mathcal{H}$. By Prohorov theorem, there is a subsequence $\{\mu_{n_k}\}$ weakly convergent to some probability measure μ on $\mathcal{H}$. Thus $\mathcal{P}_{NS}^{Galerkin}$ is non empty.

From the previous uniform bound we also have

$$\mu_n\left(\sum_{i=1}^m \lambda_i \left|\langle \cdot, h_i\rangle\right|^2 \wedge N\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

for every $N > 0$ and integer $m > 0$, where we observe that

$$\|.\|_{\mathcal{V}}^2 = \sum_{i=1}^\infty \lambda_i \left|\langle \cdot, h_i\rangle\right|^2 .$$

Now $\sum_{i=1}^m \lambda_i \left|\langle \cdot, h_i\rangle\right|^2 \wedge N \in C_b(\mathcal{H})$, hence we may take the limit as $n \to \infty$ and have

$$\mu\left(\sum_{i=1}^m \lambda_i \left|\langle \cdot, h_i\rangle\right|^2 \wedge N\right) \leq \frac{\sum_i \sigma_i^2}{2\nu}$$

which implies (23) by monotone converge theorem. In general, given $\mu \in P_{NS}^{Galerkin}$, by definition there is $\{\mu_{n_k}\}$ as above, so the previous argument applies, and (23) is proved for every $\mu \in P_{NS}^{Galerkin}$.

Remark 3.11. Unfortunately, even if for the finite dimensional approximations we have

$$\mu_n\left(\|.\|_{\mathcal{V}}^2\right) = \frac{\sum_i^{N_n} \sigma_i^2}{2\nu},$$

this equality does not pass to the limit because we cannot say that $\mu_n\left(\|.\|_{\mathcal{V}}^2\right)$ converges to $\mu\left(\|.\|_{\mathcal{V}}^2\right)$. We could have

$$\lim_{n\to\infty}\mu_n\left(\|.\|_{\mathcal{V}}^2\right)=\mu\left(\|.\|_{\mathcal{V}}^2\right)$$

(and then the equality above for μ) if

$$\begin{aligned}\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu\left(\|.\|_{\mathcal{V}}^2\right)\right| &\leq \left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right| \\ &+\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)-\mu\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right| \\ &+\left|\mu\left(\|.\|_{\mathcal{V}}^2\wedge N\right)-\mu\left(\|.\|_{\mathcal{V}}^2\right)\right|\end{aligned}$$

can be made small for large n. The last term is small for large N. The second term is unclear in general, but under the assumption of the next theorem it is small since we may have weak convergence of μ_n to μ in $\mathcal{V}$. But the problem is to have the first term uniformly small in n, for large N. We have

$$\begin{aligned}\left|\mu_n\left(\|.\|_{\mathcal{V}}^2\right)-\mu_n\left(\|.\|_{\mathcal{V}}^2\wedge N\right)\right| &= \int_{\|.\|_{\mathcal{V}}^2>N}\|x\|_{\mathcal{V}}^2\,d\mu_n(x) \\ &\leq \mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)^{1/p}\mu_n\left(\|.\|_{\mathcal{V}}^2>N\right)^{1/p'} \\ &\leq \mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)^{1/p}\left(\frac{\mu_n\left(\|.\|_{\mathcal{V}}^2\right)}{N}\right)^{1/p'}\end{aligned}$$

and this would be small uniformly small in n, for large N, if $\mu_n\left(\|.\|_{\mathcal{V}}^{2p}\right)\leq C$ for some $p>1$. But this is unknown. It can be proved in dimension 2.

We use now assumption (19), that holds true in the 3D case (lemma 2.2, plus the fact that the projection π_n is selfadjoint in $\mathcal{H}$ and commutes with $\mathcal{A}$).

Theorem 3.5. *If* $\sum_i\sigma_i^2\lambda_i<\infty$ *then* $\mu(D(\mathcal{A}))=1$ *for every* $\mu\in P_{NS}^{Galerkin}$ *and*

$$\sqrt[3]{\nu}\mu\left[|\mathcal{A}x|_{\mathcal{H}}^{2/3}\right]\leq C\left(1+\frac{\sum_i\sigma_i^2}{2\nu^5}\right)\sum_i\sigma_i^2\lambda_i.$$

Proof. We have both $\sum_i\sigma_i^2\lambda_i<\infty$ and condition (19), so, given $\mu\in P_{NS}^{Galerkin}$ and a sequence $\{\mu_{n_k}\}$ converging to μ, by corollary 3.3 we have

$$\sqrt[3]{\nu}\mu_{n_k}\left[|\mathcal{A}x|_{\mathcal{H}}^{2/3}\right]\leq C\left(1+\frac{\sum_i\sigma_i^2}{2\nu^5}\right)\sum_i\sigma_i^2\lambda_i$$

with a universal constant $C>0$. This easily implies the claim, with an argument already used in the previous proof. The proof is complete.

Let μ be a probability measure on $\mathcal{H}$. We say that it is *space homogeneous* if

$$\mu\left[f\left(u\left(\cdot-a\right)\right)\right]=\mu\left[f\left(u\right)\right] \tag{24}$$

for every $a\in\mathcal{T}_L$ and $f\in C_b\left(\mathcal{H}\right)$. We say it is *partial (or discrete) isotropic* if, for every rotation R that transforms the set of coordinate axes in itself, we have

$$\mu\left[f\left(u\left(R\cdot\right)\right)\right]=\mu\left[f\left(Ru(\cdot)\right)\right] \tag{25}$$

for all $f\in C_b\left(\mathcal{H}\right)$. This is the form of isotropy compatible with the symmetries of the torus. The same definitions apply to random fields, hence to $\sum_{i=1}^{\infty}\sigma_i h_i\beta_i\left(t\right)$ for given t. Notice that $\sum_{i=1}^{\infty}\sigma_i h_i\beta_i\left(t\right)$ is space homogeneous and partial isotropic for every $t\geq 0$ if and only if it is such for some t, being gaussian with covariance of the form Qt.

Theorem 3.6. *If $\sum_{i=1}^{\infty}\sigma_i h_i\beta_i\left(t\right)$ is space homogeneous and partial isotropic, then there exist $\mu\in P_{NS}^{Galerkin}$ that is space homogeneous and partial isotropic.*

Proof. There exist space homogeneous and partial isotropic invariant measures for the Galerkin approximations: it is sufficient to start the Krylov-Bogoliubov method from the initial condition equal to zero. Then their cluster points have the same property. The proof is complete.

The problem whether under the previous assumptions *all* elements of $P_{NS}^{Galerkin}$ are space homogeneous and partial isotropic, seems to be open (symmetry breaking).

4 Stochastic Navier–Stokes Equations in 3D

4.1 Concepts of Solution

Consider the abstract (formal) stochastic evolution equation (1) of Section 2 and its weak formulation over test functions (2) of that Section. From Lemma 2.1 we have

$$\begin{aligned}\int_0^t\left|\left\langle B\left(u_s,\varphi\right),u_s\right\rangle\right|ds &\leq\int_0^t C\left|u_s\right|_H^{1/2}\left\|u_s\right\|_V^{3/2}\left\|\varphi\right\|_V ds\\ &\leq C_\varphi\sup_{s\in[0,t]}\left|u_s\right|_H^{1/2}\int_0^t\left\|u_s\right\|_V^{3/2}ds\end{aligned} \tag{1}$$

hence the nonlinear term in (2) (Section 2) is well defined for functions u that live in $L^\infty\left(0,T;H\right)\cap L^2\left(0,T;V\right)$, $T>0$ (but many other spaces work as well, like $L^2\left(0,T;L^4\right)$).

As in the deterministic case, strong continuity of trajectories in H is an open problem. There will be strong continuity in weaker spaces, like $D(A)'$, and a uniform bound in H. Let H_σ be the space H with the weak topology. Since

$$C\left([0,T]\,;D(A)'\right)\cap L^{\infty}\left(0,T;H\right)\subset C\left([0,T]\,;H_{\sigma}\right)$$

(see lemma 4.6 below), the trajectories of the solutions will be at least weakly continuous in H. One could also prove strong continuity from the right at $t=0$ and for a.e. t, and in addition there is strong continuity in $\left[L^p\left(\mathcal{T}\right)\right]^3$ for $p<2$; we do not prove these results.

The following presentation is strongly inspired to [55].

Definition 4.1. *We call Brownian stochastic basis the object*

$$\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq0},Q,\left(\beta_i\left(t\right)\right)_{t\geq0,i\in\mathbb{N}}\right)$$

where $(W,\mathcal{F},Q)$ is a probability space, $\left(\mathcal{F}_t\right)_{t\geq0}$ a filtration, $\left(\beta_i\left(t\right)\right)_{t\geq0,i\in\mathbb{N}}$ a sequence of independent Brownian motions on $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq0},Q\right)$ (namely, the real valued processes β_i are independent, are adapted to $\left(\mathcal{F}_t\right)_{t\geq0}$, are continuous and null at $t=0$, and have increments $\beta_i\left(t\right)-\beta_i\left(s\right)$ that are $N\left(0,t-s\right)$-distributed and independent of $\mathcal{F}_s$).

Definition 4.2 (strong solutions). *Let $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq0},Q,\left(\beta_i\left(t\right)\right)_{t\geq0,i\in\mathbb{N}}\right)$ be a Brownian stochastic basis. Given $u_0:W\to H$, $\mathcal{F}_0$-measurable, we say that a $D(A)'$-valued process u on $(W,\mathcal{F},Q)$ is a strong solution of equation (1) with initial condition u_0 if:*

1. *u is a continuous adapted process in $D(A)'$ and*

$$u\left(.,\omega\right)\in L^{\infty}\left(0,T;H\right)\cap L^{2}\left(0,T;V\right)\quad Q\text{-a.s.}$$

for every $T>0$, and

2. *(2) is satisfied.*

Definition 4.3 (weak martingale solutions). *Given a probability measure μ_0 on H, a weak solution of equation (1) with initial law μ_0 consists of a Brownian stochastic basis $\left(W,\mathcal{F},\left(\mathcal{F}_t\right)_{t\geq0},Q,\left(\beta_i\left(t\right)\right)_{t\geq0,i\in\mathbb{N}}\right)$ and a $D(A)'$-valued process u on $(W,\mathcal{F},Q)$ such that*

[WM1] u is a continuous adapted process in $D(A)'$ and

$$u\left(.,\omega\right)\in L^{\infty}\left(0,T;H\right)\cap L^{2}\left(0,T;V\right)\quad Q\text{-a.s.}$$

for every $T>0$,
[WM2] (2) is satisfied
[WM3] $u_0:=u(0)$ has law μ_0.

Let us set

$$\Omega = C\left([0,\infty); D(A)'\right)$$

and denote by $(\xi_t)_{t\geq 0}$ the canonical process $(\xi_t(\omega) = \omega_t)$, by F the Borel σ-algebra in Ω and by F_t the σ-algebra generated by the events $(\xi_s \in A)$ with $s \in [0,t]$ and $A \in \mathcal{B}\left(D(A)'\right)$.

Definition 4.4 (solution to the martingale problem). *Given a probability measure μ_0 on H, we say that a probability measure P on (Ω, F) is a solution of the martingale problem associated to equation (1) with initial law μ_0 if*

[MP1]for every $T > 0$

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H + \int_0^T \|\xi_s\|_V^2 \, ds < \infty\right) = 1$$

[MP2] for every $\varphi \in \mathcal{D}^\infty$ the process M_t^φ defined P-a.s on (Ω, F) as

$$M_t^\varphi := \langle \xi_t, \varphi\rangle_H - \langle \xi_0, \varphi\rangle_H + \int_0^t \nu \langle \xi_s, A\varphi\rangle_H \, ds \\ - \int_0^t \langle B(\xi_s, \varphi), \xi_s\rangle_H \, ds$$

is square integrable and (M_t^φ, F_t, P) is a continuous martingale with quadratic variation

$$[M^\varphi]_t = \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i\rangle_H^2 \cdot t$$

[MP3] $\mu_0 = \pi_0 P$.

We have given the definition of strong solutions only for completeness, since unfortunately at present there is no result of existence of strong solutions for the 3D stochastic Navier–Stokes equation (except when identically $\sigma_i = 0$). In fact one can solve pathwise the equation with additive noise (see for instance [40]) and prove the existence of a measurable selection, but the existence of a *progressively* measurable selection remains an open problem. See also [57].

Therefore we concentrate on the other two notions. The term "weak martingale solution" has the following origin. In the theory of SDE's, weak solutions are those described by such a definition (let us say weak in the probabilistic sense). But in the theory of PDE's the term weak usually refers to some kind of distributional formulation (let us say weak in the deterministic sense). Here we have to mix-up both kind of weaknesses, and a way to remind that we mean weak also in the probabilistic sense is to add the qualification "weak martingale". No special martingale notion appear in the definition, but the next theorem of equivalence is a motivation for this choice of the name.

Theorem 4.1. *P is a solution of the martingale problem if and only if there exists a weak martingale solution with law P.*

Proof. **Step 1**. Let $\left(W, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, (\beta_i(t))_{t\geq 0, i\in\mathbb{N}}\right)$ and u be the objects in the definition of weak martingale solution. Let $\widetilde{P}$ be the law of u on (Ω, F). Let us prove that $\widetilde{P}$ solves the martingale problem. The main point is to prove [MP2]. With the notation $u(\gamma)$ for the function $(u(t,\gamma))_{t\geq 0}$, we have

$$\begin{aligned} M_t^{\varphi}(u(\gamma)) &= \langle u(t,\gamma), \varphi\rangle_H - \langle u(0,\gamma), \varphi\rangle_H \\ &\quad + \int_0^t \nu \langle u(s,\gamma), A\varphi\rangle_H ds - \int_0^t \langle B(u(s,\gamma), \varphi), u(s,\gamma)\rangle_H ds \end{aligned}$$

so for Q-a.e. $\gamma \in W$

$$M_t^{\varphi}(u(\gamma)) = \sum_{i=1}^{\infty} \sigma_i \langle \varphi, h_i\rangle_H \beta_i(t,\gamma).$$

First, $M^{\varphi}(t)$ is square integrable: since $\widetilde{P}$ is the law of u, we have

$$E^{\widetilde{P}}\left[M^{\varphi}(t)^2\right] = E^{P}\left[M^{\varphi}(t, u(.))^2\right] = \sum_{i=1}^{\infty} \sigma_i^2 \langle \varphi, h_i\rangle_H^2.$$

The other assertions of [MP2] are a consequence of lemma (4.1).

Step 2. Let now P be a solution of the martingale problem. Due to the special shape of the quadratic variation of M_t^{φ}, by Levy martingale characterization of the Brownian motion it follows that M_t^{φ} is a Brownian motion. Furthermore, $\beta_i(t,\omega) := M^{h_i}(t,\omega)$ is a sequence of independent Brownian motions on (Ω, F, F_t, P) and

$$M_t^{\varphi}(\omega) = \sum_{i=1}^{\infty} \sigma_i \langle \varphi, h_i\rangle_{Hi}(t,\omega).$$

This immediately implies [WM2], from [MP2].

Remark 4.1. The Brownian motions $\beta_i(t,\gamma)$ depend on P. Thus this proof does not provide a space with a simultaneous solution for every initial condition.

Remark 4.2. For equations with non-constant diffusion term, step 2 requires a representation theorem for martingales.

Lemma 4.1. *Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ be two filtered probability spaces and $X : \Omega \to \Omega'$ be a measurable mapping such that $P' = XP$; and such that $Z' \circ X$ is $\mathcal{F}_t$-measurable for every $\mathcal{F}'_t$-measurable*

Z'. Let $(M'_t)_{t\geq 0}$ be a continuous adapted process on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ such that $M_t := M'_t \circ X$ is a martingale on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$ and there is an increasing adapted process $(A_t)_{t\geq 0}$ on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$ such that $A_t \circ X = [M]_t$. Then $(M'_t)_{t\geq 0}$ is a martingale on $\left(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, P'\right)$, with quadratic variation $(A_t)_{t\geq 0}$.

The proof is left as an exercise.

4.2 Existence of Solutions to the Martingale Problem

The theorem of existence will be based on a classical Galerkin approximation scheme. For other purposes, like the existence of the so called suitable weak solutions (satisfying local energy inequalities) other approximations must be used, but we shall not take that direction (see Flandoli and Romito [38]).

Let H_n be the finite dimensional space spanned by the first N_n eigenvectors of A, with N_n increasing to infinity. We endow H_n with the inner product induced by $|.|_H$, and use the same notation. Let A_n be the restriction of A to H_n and $B_n(.,.) : H_n \times H_n \to H_n$ the continuous bilinear operator defined as

$$\langle B_n(u,v), w\rangle = \langle B(u,v), w\rangle$$

for every $u, v, w \in H_n$. We have also

$$B_n(u,v) = \pi_n B(u,v), \quad u, v \in H_n$$

where π_n is the orthogonal projection of H on H_n.

Consider the equation in H_n

$$dX_t^n + [\nu A_n X_t^n + B_n(X_t^n, X_t^n)]\, dt = \sum_{i=1}^{N_n} \sigma_i h_i d\beta_i. \tag{2}$$

The operators in this equation satisfy all the assumptions of the previous Section, so we may use all the results proved there. Of course we may take advantage only of those estimates having *universal* constants.

Theorem 4.2. *Assume $\sigma^2 := \sum_i \sigma_i^2 < \infty$. Let μ be a measure on H such that $m_2 := \int_H |x|_H^2 \mu(dx) < \infty$. Then there exists at least one solution to the martingale problem with initial condition μ.*

Proof. **Step 1** (a priori bounds on Galerkin approximations). Let

$$\left(W, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, (\beta_i(t))_{t\geq 0, i\in\mathbb{N}}\right)$$

be a Brownian stochastic basis supporting also an $\mathcal{F}_0$-measurable r.v. $u_0 : W \to H$ with law μ (to construct such a basis it is sufficient to use product spaces, as we did in theorem 3.3). Let $X_0^n := \pi_n u_0$.

For every n, there exist a unique continuous adapted solution $(X_t^n)_{t\geq 0}$ of equation (2) in H_n, with initial condition X_0^n. Under the embedding $H_n \subset H$ we have that $(X_t^n)_{t\geq 0}$ is a continuous adapted process in H, so it defines a measure P_n on $C([0,\infty);H)$, and thus on (Ω, F). In Section 3 we have proved

$$P_n\left[\sup_{t\in[0,T]}|\xi_t|_H^2 + \nu\int_0^T \|\xi_s\|_V^2\,ds\right] \leq C_1\left(m_2,\sigma^2,T\right)$$

We have used the fact that

$$Q\left[|X_0^n|_H^2\right] = \int_H |\pi_n x|_H^2\,\mu(dx) \leq m_2.$$

Moreover, in view of the time regularity, equation (2) has the form

$$X_t^n = X_0^n + J_t^n + \sum_{i=1}^{N_n}\sigma_i h_i d\beta_i$$

where

$$J_t^n = -\int_0^t [\nu A_n X_s^n + B_n(X_s^n, X_s^n)]\,ds$$

and we have, on one side,

$$Q\left\|\sum_{i=1}^{N_n}\sigma_n h_n \beta_\cdot\right\|_{W^{\alpha,p}(0,T;H)}^p < C$$

(C independent of n) for every $p>1$, $\alpha\in(0,1/2)$, $T>0$, from Corollary 4.2; on the other side, for J_t^n, chosen $\gamma\in(3/2,2)$, we have

$$\begin{aligned}
&\|J_\cdot^n\|_{W^{1,2}(0,T;D(A^{-\gamma}))}^2\\
&\leq C_\nu\int_0^T |A_n X_s^n|_{V'}^2\,ds + C\int_0^T |B_n(X_s^n,X_s^n)|_{D(A^{-\gamma})}^2\,ds\\
&\leq C_\nu\int_0^T \|X_s^n\|_V^2\,ds + C\sup_{s\in[0,T]}|X_s^n|_H^2\int_0^T\|X_s^n\|_V^2\,ds
\end{aligned}$$

since, for $x,\varphi\in\mathcal{D}^\infty$,

$$\begin{aligned}
|B_n(x,x)|_{D(A^{-\gamma})}^2 &= \sup_{|\varphi|_{D(A^\gamma)}\leq 1}\left|\langle B_n(x,x),\varphi\rangle_{D(A^{-\gamma}),D(A^\gamma)}\right|\\
&= \sup_{|\varphi|_{D(A^\gamma)}\leq 1}|\langle B(x,x),\varphi\rangle| \leq C\,|x|_H^2\,\|x\|_V^2
\end{aligned}$$

from the Sobolev embedding of $D(A^\gamma)$ in the continuous fields. Therefore

$$P_n\left[\|\xi\|_{W^{\alpha,2}(0,T;D(A^{-\gamma}))}\right] \leq C_4\left(\nu,m_2,\sigma^2,T\right)$$

for every $\alpha\in(0,1/2)$, $\gamma\in(3/2,2)$, $T>0$.

Step 2 (tightness). By Chebyshev inequality, given $\alpha \in (0, 1/2)$, $\gamma \in (3/2, 2)$, $T > 0$, for every $\varepsilon > 0$ there is a bounded set

$$B_\varepsilon \subset L^2(0,T;V) \cap W^{\alpha,2}\left(0,T;D\left(A^{-\gamma}\right)\right)$$

such that $P_n(B_\varepsilon) > 1 - \varepsilon$ for every n. From theorem 4.6, there is a compact set

$$K_\varepsilon \subset L^2(0,T;H)$$

such that $P_n(K_\varepsilon) > 1-\varepsilon$ for every n. From the boundedness of the law of J^n in $W^{1,2}(0,T;D(A^{-\gamma}))$ and of the law of the Brownian motion in $W^{\alpha,p}(0,T;H)$ for every $p > 1$ and $\alpha \in (0, 1/2)$, we may apply lemma 4.3 and have a compact set

$$K'_\varepsilon \subset C([0,T];D(A)')$$

such that $P_n(K'_\varepsilon) > 1-\varepsilon$ for every n. Therefore the family of measures $\{P_n\}$ is tight in $L^2(0,T;H)$ and in $C([0,T];D(A)')$, with their Borel σ-fields. Hence there exists a probability measure P on

$$C([0,T];D(A)') \cap L^2(0,T;H)$$

that is the weak limit in such spaces of a subsequence $\{P_{n_k}\}$.

Step 3 (P is a solution to the martingale problem). From the uniform estimates on $\{P_{n_k}\}$ in $L^2(0,T;V)$ and $L^\infty(0,T;H)$ we may deduce that P gives probability one to each one of these spaces and has bounds in the mean similar to those uniform of P_{n_k}. The details of this fact are rather tedious so we give only a sample in the next section, see lemma 4.8. This way we have checked property [MP1] in the definition of solution to the martingale problem.

Concerning [MP3], we have

$$P_{n_k}(\varphi) \to P(\varphi)$$

for every $\varphi \in C_b(C([0,T];D(A)'))$, hence in particular $\pi_0 P_{n_k} \to \pi_0 P$ as probability measures on $D(A)'$. But $\pi_0 P_{n_k}$ is the law of $\pi_n u_0$, which converges to μ since $\pi_n u_0$ converges Q-a.s. to u_0. Hence $\pi_0 P$ is μ.

Finally, let us check property [MP2]. Given $\varphi \in \mathcal{D}^\infty$, we have to prove that for every $t > s \geq 0$ and every F_s-measurable bounded r.v. Z, we have

$$\begin{aligned} P\left[(M_t^\varphi)^2\right] &< \infty \\ P\left[(M_t^\varphi - M_s^\varphi) Z\right] &= 0 \\ P\left[\left((M_t^\varphi)^2 - \varsigma_t - \left((M_s^\varphi)^2 - \varsigma_s\right)\right) Z\right] &= 0 \end{aligned}$$

where $\varsigma_t := \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$. For the measure P_{n_k} we know (by lemma 4.1) that $(M_t^{\varphi,n_k}, F_t, P_{n_k})$ is a square integrable martingale with quadratic variation

$$[M^{\varphi,n_k}]_t = \sum_{i=1}^{N_{n_k}} \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$$

where

$$M_t^{\varphi,n} := \langle \xi_t, \varphi \rangle_H - \langle \xi_0, \varphi \rangle_H + \int_0^t \nu \langle \xi_s, A\varphi \rangle_H ds - \int_0^t \langle B(\xi_s, \pi_n \varphi), \xi_s \rangle_H ds.$$

Thus $(M_t^{\varphi,n_k}, F_t, P_{n_k})$ is a Brownian motion and we have

$$\sup_k P_{n_k} \left[(M_t^{\varphi,n_k})^{2+\varepsilon} \right] < \infty$$
$$P_{n_k} \left[(M_t^{\varphi,n_k} - M_s^{\varphi,n_k}) Z \right] = 0$$
$$P_{n_k} \left[\left((M_t^{\varphi,n_k})^2 - \varsigma_t^{n_k} - \left((M_s^{\varphi,n_k})^2 - \varsigma_s^{n_k} \right) \right) Z \right] = 0$$

where $\varsigma_t^n := \sum_{i=1}^{N_n} \sigma_i^2 \langle \varphi, h_i \rangle_H^2 \cdot t$ and $\varepsilon > 0$. It is now sufficient to use lemma 4.2 below. The proof is complete.

Remark 4.3. In the case of noise depending on u, the passage to the limit (step 3, proof of [MP2]) is more involved and requires uniform estimates on p-moments of X_t^n, see [34].

Lemma 4.2. *On a Polish space X, if P_n converges weakly to P (in the sense of measures), $\varphi_n, \varphi : X \to \mathbb{R}$ are measurable and $\varphi_n(x_n) \to \varphi(x)$ for every $x \in X$ and any sequence $x_n \to x$, and*

$$P_n \left[|\varphi_n|^{1+\varepsilon} \right] \leq C$$

for some $\varepsilon, C > 0$, then $P[|\varphi|] < \infty$ and $P_n[\varphi_n] \to P[\varphi]$.

Proof. Let Y_n, Y be r.v. on a probability space (Ω, F, Q), with expectation E, with values in X, with laws P_n and P respectively, such that $Y_n \xrightarrow{X} Y$, Q-a.s. (Skorohod theorem). We have to prove that $E[|\varphi(Y)|] < \infty$ and $E[\varphi_n(Y_n)] \to E[\varphi(Y)]$. But we know that

$$E\left[|\varphi_n(Y_n)|^{1+\varepsilon} \right] \leq C$$

and $\varphi_n(Y_n) \to \varphi(Y)$, Q-a.s.; hence it is sufficient to apply Vitali convergence theorem.

For the definitions of space homogeneous and partial isotropic measure or random field, see section 3.6 above. The proof of the following facts is similar to the previous theorem and will be omitted. Under the assumptions of theorem 4.2 we have:

Theorem 4.3. *If μ and $\sum_{i=1}^{\infty} \sigma_i h_i \beta_i (1)$ are space homogeneous and partial isotropic, then there exists a solution P of the martingale problem with initial condition μ such that $\pi_t P$ is space homogeneous and partial isotropic for every $t \geq 0$.*

If $\sum_i \sigma_i^2 \lambda_i < \infty$ then there exists a solution P of the martingale problem with initial condition μ such that

$$P \left[\int_0^T \frac{|A\xi_s|_H^2}{\left(1 + \|\xi_s\|_V^2\right)^2} ds \right] < \infty.$$

In particular, $P(\xi_t \in D(A)) = 1$ for a.e. $t \geq 0$.

We complete the section by stating a result proved in [34], proof that is a variant of the previous one and we do not repeat. By *stationary* martingale solution we mean a solution P of the martingale problem that is shift invariant (in time) on Ω.

Theorem 4.4. *There exists at least one stationary martingale solution P_{stat}, with the following properties:*

$$P_{stat} \left[\|\xi_t\|_V^2 \right] \leq \frac{TrQ}{2\nu}, \quad P_{stat} \left[|\xi_t|_H^p \right] < \infty$$

for every $t \geq 0$ and $p \geq 2$.

Remark 4.4. In dimension $d = 2$ we have the identity

$$E^{P_{stat}} \|\xi_t\|_V^2 = \frac{TrQ}{2\nu}.$$

Theorem 4.5. *If $\sum_{i=1}^{\infty} \sigma_i h_i \beta_i (1)$ is space homogeneous and partial isotropic, there exists at least one stationary martingale solution P_{stat} such that $\pi_t P_{stat}$ is space homogeneous and partial isotropic for every $t \geq 0$.*

If $\sum_i \sigma_i^2 \lambda_i < \infty$ then there exists at least one stationary martingale solution P_{stat} such that for every $t \geq 0$ we have $P_{stat} (\xi_t \in D(A)) = 1$ and

$$P_{stat} \left[\frac{|A\xi_t|_H^2}{\left(1 + \|\xi_t\|_V^2\right)^2} \right] < \infty.$$

In the usual context of Markov dynamics, one introduces the notion of invariant measure (see above in the finite dimensional case). This can be done in dimension 2, but in 3D we have the obstacle of the lack of Markov property (at least a priori; see the next sections). Nevertheless, there are several ways to introduce measures that may play the role of invariant measures. In order to stress the difference w.r.t. the Markov set-up, we shall call them stationary measures. Recall the definition of Galerkin stationary measure given above; call $P_{NS}^{Galerkin}$ the set of such measures; we have proved that it is non empty. Here, having the existence of stationary solutions, it is meaningful to consider the measures of the form $\pi_t P_{stat}$, where P_{stat} is a stationary solution of the martingale problem; call $P_{NS}^{stationary}$ the set of such measures. One can prove the following relation:

$$P_{NS}^{Galerkin} \subset P_{NS}^{stationary}$$

However, the other relations are less clear (opposite inclusion, relation to invariant measures for Markov selections, etc.).

4.3 Technical Complements

We collect here some technical facts used in the previous section.

Sobolev spaces of fractional order

Let E be a (separable) Banach space, $p > 1$, $\alpha \in (0,1)$, $T > 0$, $W^{\alpha,p}(0,T;E)$ the (Sobolev) space of all $u \in L^p(0,T;E)$ such that

$$[u]^p_{W^{\alpha,p}(0,T;E)} := \int_0^T \int_0^T \frac{|u(t)-u(s)|_E^p}{|t-s|^{1+\alpha p}} dtds < \infty$$

endowed with the norm

$$\|u\|^p_{W^{\alpha,p}(0,T;E)} = \int_0^T |u(t)|_E^p \, dt + [u]^p_{W^{\alpha,p}(0,T;E)} .$$

One can show that $W^{\alpha,p}(0,T;E)$ is a (separable) Banach space. Moreover, if $\alpha p > 1$, $W^{\alpha,p}(0,T;E) \subset C^\gamma([0,T];E)$ for every $\gamma < \alpha p - 1$. We also have:

Lemma 4.3. *If $E \subset \widetilde{E}$ are two Banach spaces with compact embedding, $p > 1$ and $\alpha \in (0,1)$ satisfy $\alpha p > 1$, then $W^{\alpha,p}(0,T;E)$ is compactly embedded into $C\left([0,T];\widetilde{E}\right)$. Similarly, if $E_1, ..., E_n$ are compactly embedded into $\widetilde{E}$ and $p_1, ..., p_n > 1$, $\alpha_1, ..., \alpha_n \in (0,1)$ satisfy $\alpha_i p_i > 1$ for every $i = 1, ..., n$, then*

$$W^{\alpha_1,p_1}(0,T;E_1) + \cdots + W^{\alpha_n,p_n}(0,T;E_n)$$

is compactly embedded into $C\left([0,T];\widetilde{E}\right)$.

Theorem 4.6. *Let $E_0 \subset E \subset E_1$ be Banach spaces, E_0 and E_1 reflexive, E_0 compactly embedded in E, E continuously embedded into E_1. Given $p > 1$, $\alpha \in (0,1)$, $T > 0$, the space*

$$X := L^p(0,T;E_0) \cap W^{\alpha,p}(0,T;E_1)$$

is compactly embedded into $L^p(0,T;E)$.

Detailed proofs can be found in [34].

Gaussian measures in Hilbert spaces

Let X be the gaussian r.v. in H defined as

$$X = \sum_{i=1}^{\infty} \sigma_i h^{(i)} X_i$$

where $\left(h^{(i)}\right)$ is a c.o.s. in H, $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$ and (X_i) is a sequence of independent standard Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ with expectation E. Then:

Lemma 4.4. *For all* $t \in [0, \inf_i \frac{1}{2\sigma_i})$ *we have*

$$E\left[e^{t|X|_H^2}\right] = \exp\left(-\frac{1}{2}\sum_{i=1}^{\infty} \log\left(1 - 2\sigma_i^2 t\right)\right).$$

Proof. We leave the proof as an exercise, based on the formula

$$\frac{1}{\sqrt{2\pi\sigma_i^2}} \int_{-\infty}^{+\infty} \exp\left(tx^2 - \frac{x^2}{2\sigma_i^2}\right) dx = \frac{1}{\sqrt{1 - 2\sigma_i^2 t}}.$$

Corollary 4.1. *For every* $p > 1$ *we have*

$$E\left[|X|_H^p\right] \leq C_p \left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{p/2}.$$

Proof. If $p = 2m$ with a positive integer m, this follows from the lemma by differentiation. For $p \in (2m-1, 2m)$ we apply Hölder inequality:

$$E\left[|X|_H^p\right] \leq E\left[\left(|X|_H^p\right)^{\frac{2m}{p}}\right]^{\frac{p}{2m}} \leq C_{2m} \left(\sum_{i=1}^{\infty} \sigma_i^2\right)^{p/2}.$$

Remark 4.5. Applied to Gaussian martingales, this is the upper bound in Burkholder-Davies-Gundy (BDG) inequality. In fact, if we want to deal with non additive noise, we have to replace the present arguments with BDG inequality.

Corollary 4.2. *If* $B(t)$ *is a Brownian motion in* H *given by*

$$B(t) = \sum_{i=1}^{\infty} \sigma_i h^{(i)} \beta^{(i)}(t)$$

with $\sum_{i=1}^{\infty}\sigma_i^2<\infty$ *and* $\left(\beta^{(i)}(t)\right)$ *a sequence of independent standard Brownian motions, then for every* $p>1$, $\alpha\in(0,1/2)$, $T>0$,

$$E\left\|B\right\|^p_{W^{\alpha,p}(0,T;H)}<C(p,\alpha,T)\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}.$$

Proof. From the corollary above we have

$$\begin{aligned}&E\int_0^T\int_0^T\frac{|B(t)-B(s)|_H^p}{|t-s|^{1+\alpha p}}dtds\\&\le\int_0^T\int_0^T\frac{C_p\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}(t-s)^{p/2}}{|t-s|^{1+\alpha p}}dtds\\&=C_p\left(\sum_{i=1}^{\infty}\sigma_i^2\right)^{p/2}\int_0^T\int_0^T\frac{1}{|t-s|^{1+\left(\alpha-\frac{1}{2}\right)p}}dtds.\end{aligned}$$

The integral is finite since $1+\left(\alpha-\frac{1}{2}\right)p<1$. The proof is complete.

Remark 4.6. If we would not know yet that $B(t)$ has a.s. continuous trajectories, we could deduce it from the previous result.

Remarks on $\Omega=C([0,\infty);D(A)')$

The following and other similar results are used several times throughout this Section, often without mention. We discuss the following ones as a sample. The general idea of the following results is that apparently stronger topologies in $D(A)'$ and Ω define measurable sets and functions. First, H is a Borel set in $D(A)'$. Indeed, there is a c.o.s. $\{e_n\}$ in H made of elements of $D(A)$, such that, for an element $x\in D(A)'$, we have $x\in H$ if and only if $\sum\langle x,e_n\rangle^2<\infty$ (since $e_n\in D(A)$, $\langle x,e_n\rangle$ is a priori well defined for $x\in D(A)'$, so $\sum\langle x,e_n\rangle^2$ either converges or diverges to $+\infty$). Similarly, the (possibly infinite) function $x\mapsto|x|_H$ is measurable on $D(A)'$.

Lemma 4.5. $C([0,\infty);H)$ *is a Borel set in* Ω.

Proof. Fix a dense countable set D in $[0,\infty)$. For an $\omega\in\Omega$ we have $\omega\in C([0,\infty);H)$ if and only if it is uniformly continuous on every bounded subset of D: for every$N>0$, $n>0$ there is $m>0$ such that $t,s\in D\cap[0,N]$, $|t-s|<1/m$ implies $|\omega(t)-\omega(s)|_H<1/n$. This condition is expressed by means of countably many operations (recall also that $x\mapsto|x|_H$ is measurable on $D(A)'$).

Lemma 4.6. *Let $B([0,\infty);H)$ be the set of H-valued functions ω, bounded on every bounded set, i.e. such that for every $T>0$*

$$\sup_{t\in[0,T]} |\omega(t)|_H < \infty.$$

Then

$$B([0,\infty);H)\cap \Omega = C\left([0,\infty);H_\sigma\right)\cap \Omega.$$

Moreover

$$B([0,\infty);H)\cap \Omega \in F$$

and the (possibly infinite) function

$$f(\omega) = \sup_{t\in[0,T]} \sum_{n=1}^{\infty} \langle \omega(t), e_n\rangle^2$$

is measurable, for every $T>0$.

Proof. If $\omega \in B([0,\infty);H)\cap\Omega$ and $\varphi\in H$, then, given$T>0$, $t_0\in[0,T]$ and $\varepsilon>0$, let $\varphi'\in D(A)$ be such that $|\varphi-\varphi'|_H\le\varepsilon$, and take $\delta>0$ such that $|\omega(t)-\omega(t_0)|_{D(A)'}\le\varepsilon$ for $|t-t_0|\le\delta$, $t,t_0\in[0,T]$. We have

$$\begin{aligned}|\langle\omega(t)-\omega(t_0),\varphi\rangle| &\le |\langle\omega(t)-\omega(t_0),\varphi-\varphi'\rangle| \\ &\quad + |\langle\omega(t)-\omega(t_0),\varphi'\rangle| \\ &\le \varepsilon C+\varepsilon C.\end{aligned}$$

Viceversa, if $\omega\in C\left([0,\infty);H_\sigma\right)\cap\Omega$, then, for every $T>0$ and $\varphi\in H$,

$$\sup_{t\in[0,T]} |\langle\omega(t),\varphi\rangle| \le \infty.$$

So $(\omega(t))$ is a family of functionals on H that are pointwise equibounded. By Banach-Steinhaus theorem, they are equibounded in the operator norm, namely

$$\sup_{t\in[0,T]} |\omega(t)|_H^2 < \infty.$$

Finally, about the measurability, consider the sets

$$A_{N,R} = \left\{\omega\in\Omega : \sup_{t\in[0,T]} \sum_{n=1}^{N} \langle\omega(t),e_n\rangle^2 \le R\right\}.$$

They are measurable and

$$\bigcup_{R>0}\bigcap_{N>0} A_{N,R} = B([0,\infty);H)\cap\Omega.$$

The proof is complete.

Lemma 4.7. *Let $P \in \Pr(\Omega)$ be such that*

$$P\left(C\left([0,\infty); H_\sigma\right) \cap \Omega\right) = 1.$$

Then, given $t \geq 0$, the mapping $\omega \mapsto \omega(t)$, a priori F_t-measurable with values in $D(A)'$, has a P-modification on F_t that is F_t-measurable with values in $(H, \mathcal{B}(H))$.

Lemma 4.8. *Let $P_n \in \Pr(\Omega)$, weakly convergent to P. Assume that for every $T > 0$ there is a constant $C_T > 0$ such that*

$$P_n\left[\sup_{t\in[0,T]} |\omega(t)|_H^2\right] \leq C_T.$$

Then $P\left(B([0,\infty); H) \cap \Omega\right) = 1$ and

$$P\left[\sup_{t\in[0,T]} |\omega(t)|_H^2\right] \leq C_T.$$

Proof. Given $R, N > 0$, the functional

$$f_{N,R}(\omega) = \sup_{t\in[0,T]} \sum_{n=1}^{N} \langle \omega(t), e_n\rangle^2 \wedge R$$

belong to $C_b(H)$, so

$$\lim_{n\to\infty} P_n\left(f_{N,R}\right) = P\left(f_{N,R}\right).$$

Moreover,

$$f_{N,R}(\omega) \leq \sup_{t\in[0,T]} |\omega(t)|_H^2$$

so $P_n(f_{N,R}) \leq C$ and thus $P(f_{N,R}) \leq C$. By monotone convergence we get the result.

4.4 An Abstract Markov Selection Result

The topics of this and the following sections are quite technical and a complete treatment of them would exceed the reasonable size of this note. Therefore we limit the discussion to the main ideas. Details of this section can be found in [39].

Let $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ be a Gelfand triple of separable Hilbert spaces with continuous dense injections. In our application $\mathcal{V}$ will be $D(A)$. Denote by Ω the space $C([0,\infty); \mathcal{V}')$, with Borel σ-field $\mathcal{B}$, and for every $t \geq 0$ we set $\Omega^t := C([t,\infty); \mathcal{V}')$, with its Borel σ-field $\mathcal{B}^t$; clearly Ω^t is isomorphic to Ω by the natural map $\Phi_t : \Omega \to \Omega^t$, $(\Phi_t\omega)(t+s) = \omega(s)$ for every $s \geq 0$. Denote also by $(F_t)_{t\geq 0}$ the canonical filtration on Ω.

Given P on Ω, given $t>0$, there is an F_t-measurable (P-unique) function $\omega \mapsto P_\omega^{F_t}$, from Ω to $\Pr(\Omega^t)$, such that

$$P\left(A^{ext}\cap F\right)=\int_F P_\omega^{F_t}(A)\,dP(\omega)$$

for every $F\in F_t$ and $A\in\mathcal{B}^t$, where $A^{ext}\in\mathcal{B}$ is the set $\left\{\omega|_{[t,\infty)}\in A\right\}$. Moreover,

$$P_\omega^{F_t}(\xi_t=\omega(t))=1$$

for P-a.e. $\omega\in\Omega$. The existence of such function $P_\omega^{F_t}$ comes from the existence of a regular conditional probability distribution, which exists since Ω is Polish and F_t is countably generated. We also have:

Lemma 4.9. *Given P on Ω and an F_t-measurable function $\omega\mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$, such that*

$$Q_\omega(\xi_t=\omega(t))=1$$

for every $\omega\in\Omega$, there is a (unique) measure P^Q on Ω such that

$$P^Q(F)=P(F)\ \textit{for every}\ A\in F_t$$

$$\left(P^Q\right)_\omega^{F_t}=Q_\omega$$

for P^Q-a.e. $\omega\in\Omega$.

Details can be found in [62]. The idea is to define

$$P^Q(F\times A)=\int_F Q_\omega(A)\,dP(\omega)$$

and verify all the assertions.

Definition 4.5. *Let $\{P^x\}_{x\in H}\subset\Pr(\Omega)$ be a family of measures such that*

$$P^x(\Omega\cap C([0,\infty);\mathcal{H}_\sigma))=1.$$

We say it is Markov if for every $t\geq 0$

$$(P^x)_\omega^{F_t}=\Phi_t P^{\omega(t)}\ \textit{for}\ P^x\textit{-a.e.}\ \omega\in\Omega.$$

A priori $\omega\in\Omega$, so $P^{\omega(t)}$ could be not-well-defined, but we require the identity only for P^x-a.e. ω, and we know that P^x is supported by $\mathcal{H}$ valued functions. So the previous definition is meaningful.

Definition 4.6. *Let $\{C^x\subset\Pr(\Omega);x\in\mathcal{H}\}$ be a collection of families of measures such that $P(\Omega\cap C([0,\infty);\mathcal{H}_\sigma))=1$ for every $P\in C^x$. We say it is pre-Markov if for every $t\geq 0$ the following two assertions hold:*

i) for every $P\in C^x$,

$$(P)_\omega^{F_t} \in \Phi_t C^{\omega(t)} \text{ for } P\text{-a.e. } \omega \in \Omega$$

ii) for every $P \in C^x$, and every F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$*, such that $Q_\omega \in \Phi_t C^{\omega(t)}$ for every $\omega \in \Omega$,*

$$P^Q \in C^x.$$

Remark 4.7. If a family of singletons is pre-Markov, then it is Markov.

The set $\Pr(\Omega)$ with the weak converge is Polish. Denote by $Comp(\Pr(\Omega))$ the family of all compact sets in $\Pr(\Omega)$. It is a metric space and we can talk about measurability of functions from a measurable space to $Comp(\Pr(\Omega))$ (see [62] for details).

Remark 4.8. To understand the following proof it may be useful to recall the following well-known principle in the calculus of variations: if the functional to be maximized is "local", then every segment of a global maximizer is a maximizer of the corresponding segmented functional. To be more specific, assume $f^*(t)$ maximizes the functional $J_0(f) := \int_0^T \varphi(f(t))dt$ (under suitable assumptions on φ) with the constraint $f(0) = x^0$. Then, given $s \in (0,T)$, the segment $f^*|_{[s,T]}$ maximizes the functional $J_s(f) := \int_s^T \varphi(f(t))dt$ with the constraint $f(s) = f^*(s)$. Indeed, if this would not be true, if g is a function on $[s,T]$ with $g(s) = f^*(s)$ and $J_s(g) > J_s(f^*)$, then the function

$$\widetilde{f} = \begin{cases} f^* \text{ on } [0,s] \\ g \text{ on } [s,T] \end{cases}$$

has the property $J_0\left(\widetilde{f}\right) > J_0(f^*)$, contradicting the assumption that f^* was optimal for J_0.

Theorem 4.7. *Let $\{C^x \subset \Pr(\Omega); x \in \mathcal{H}\}$ be a pre-Markov family such that*

$$P(\Omega \cap C([0,\infty);\mathcal{H}_\sigma)) = 1$$

for every $P \in C^x$. If C^x is a convex compact set in $\Pr(\Omega)$ *for every $x \in \mathcal{H}$ and $x \mapsto C^x$ is measurable, then there exist a Markov selection.*

Proof. We essentially repeat the proof of [62], with minor topological remarks due to the infinite dimensions and different notations.

Step 1 (reduction of C^x by local functionals; preparation). Given a measurable pre-Markov family $\{C^x \subset \Pr(\Omega); x \in \mathcal{H}\}$, given $\lambda > 0$, let us define the operator $R_\lambda^+ : C_b(\mathcal{V}') \to C_b(\mathcal{V}')$ as

$$\left(R_\lambda^+ \varphi\right)(x) = \sup_{P \in C^x} J_{\varphi,\lambda}(P)$$

where, for any $\varphi \in C_b(\mathcal{V}')$ and $\lambda > 0$, the functional $J_{\varphi,\lambda}$ on $\Pr(\Omega)$ is defined as

$$J_{\varphi,\lambda}(P)=P\left[\int_0^\infty e^{-\lambda t}\varphi(\xi_t)\,dt\right].$$

The notation R_λ^+ is inspired by the particular case when C^x is a singleton and thus we have a Markov process $\{P^x\in\Pr(\Omega);x\in\mathcal{H}\}$; if it is sufficiently regular and L is its infinitesimal generator, then

$$R_\lambda^+=(\lambda-L)^{-1}$$

(a rigorous formulation of this sentence requires specification of function spaces and regularities that are not of interest here).

Since the function

$$\omega\mapsto\int_0^\infty e^{-\lambda t}\varphi(\xi_t(\omega))\,dt$$

is bounded and continuous on Ω, $J_{\varphi,\lambda}$ is continuous on $\Pr(\Omega)$. Therefore, given $x\in\mathcal{H}$, on the compact set C^x there is at least one maximizing element for $J_{\varphi,\lambda}$. Denote by $C^x_{\varphi,\lambda}$ the set of all such maximizing elements; thus

$$\left(R_\lambda^+\varphi\right)(x)=J_{\varphi,\lambda}\left(C^x_{\varphi,\lambda}\right).$$

Let us show that the family

$$\left\{C^x_{\varphi,\lambda}\subset\Pr(\Omega);x\in\mathcal{H}\right\}$$

is pre-Markov and has all the same properties of $\{C^x\subset\Pr(\Omega);x\in\mathcal{H}\}$. Clearly

$$P\left(\Omega\cap C\left([0,\infty);\mathcal{H}_\sigma\right)\right)=1$$

for every $P\in C^x_{\varphi,\lambda}$. The set $C^x_{\varphi,\lambda}$ is compact (it is the set of maximizing elements of a continuous mapping on a compact set). The mapping $x\mapsto C^x_{\varphi,\lambda}$ is measurable since the two mappings $x\mapsto C^x$ and $C^x\mapsto C^x_{\varphi,\lambda}$ are measurable (the last assertion comes from [62], lemma 12.1.7). Finally, $C^x_{\varphi,\lambda}$ is convex: given $P^i\in C^x_{\varphi,\lambda}$ and $\alpha_i\geq 0$, $i=1,2$, such that $\alpha_1+\alpha_2=1$, setting $P=\alpha_1P^1+\alpha_2P^2$, we have

$$J_{\varphi,\lambda}(P)=\alpha_1J_{\varphi,\lambda}\left(P^1\right)+\alpha_2J_{\varphi,\lambda}\left(P^2\right)$$

which implies that $P\in C^x_{\varphi,\lambda}$. Let us prove it is pre-Markov.

Step 2 (pre-Markov property, part 1). First, let us prove that for every $P\in C^x_{\varphi,\lambda}$,

$$P\left[\omega\in\Omega:(P)^{F_t}_\omega\in\Phi_tC^{\omega(t)}_{\varphi,\lambda}\right]=1. \tag{3}$$

As a preliminary remark, notice that $\omega\mapsto(P)^{F_t}_\omega$ is F_t-measurable with values in $\Pr(\Omega^t)$, and up to a P-modification $\omega\mapsto\Phi_tC^{\omega(t)}_{\varphi,\lambda}$ is F_t-measurable with values in compact sets in $\Pr(\Omega^t)$, because $\omega \mapsto \omega(t)$ is F_t-measurable with values in $\mathcal{H}$ and $x \mapsto \Phi_t C^x_{\varphi,\lambda}$ is measurable from $\mathcal{B}(\mathcal{H})$ to compact sets in $\Pr(\Omega^t)$. Therefore, by [62], lemma 12.1.9, the set

$$\left[\omega \in \Omega : (P)^{F_t}_\omega \in \Phi_t C^{\omega(t)}_{\varphi,\lambda}\right]$$

belongs to F_t. If (3) is not true, there is $A \in F_t$ such that $P(A) > 0$ and $(P)^{F_t}_\omega \notin \Phi_t C^{\omega(t)}_{\varphi,\lambda}$ for every $\omega \in A$, namely

$$J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) < \max_{C^{\omega(t)}} J_{\varphi,\lambda}$$

for every $\omega \in A$.

Choose an F_t-measurable selection from $\Omega \ni \omega \mapsto \Phi_t C^{\omega(t)}_{\varphi,\lambda}$, call it Q_ω, define the F_t-measurable mapping

$$R_\omega = \begin{cases} Q_\omega & \text{if } \omega \notin A \\ (P)^{F_t}_\omega & \text{if } \omega \in A \end{cases}$$

with values in $\Pr(\Omega^t)$, and define the probability measure P^R. We have

$$J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) < J_{\varphi,\lambda}\left(\Phi_t^{-1} Q_\omega\right) \tag{4}$$

for every $\omega \in A$.

We show now that $J_{\varphi,\lambda}(P^R) > J_{\varphi,\lambda}(P)$, which is a contradiction since P is a maximizer; the proof of (3) will be then complete, by contradiction. We have

$$\begin{aligned} J_{\varphi,\lambda}\left(P^R\right) &= P^R\left[P^R\left[\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds \,\middle|\, F_t\right]\right] \\ &= P^R\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t} P^R\left[P^R\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds \,\middle|\, F_t\right]\right] \\ &= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t} P\left[P^R\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds \,\middle|\, F_t\right]\right] \end{aligned}$$

and

$$\begin{aligned} &P^R\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds \,\middle|\, F_t\right](\omega) \\ &= \begin{cases} Q_\omega\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] & \text{if } \omega \notin A \\ (P)^{F_t}_\omega\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right] & \text{if } \omega \in A \end{cases} \\ &= \begin{cases} \Phi_t^{-1} Q_\omega\left[\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds\right] & \text{if } \omega \notin A \\ \Phi_t^{-1}(P)^{F_t}_\omega\left[\int_0^\infty e^{-\lambda s}\varphi(\xi_s)\,ds\right] & \text{if } \omega \in A \end{cases} \\ &= \begin{cases} J_{\varphi,\lambda}\left(\Phi_t^{-1} Q_\omega\right) & \text{if } \omega \notin A \\ J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)^{F_t}_\omega\right) & \text{if } \omega \in A \end{cases} \end{aligned}$$

so, by (4),

$$
\begin{aligned}
P\left[P^{R}\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\middle| F_t\right]\right] &> P\left[J_{\varphi,\lambda}\left(\Phi_t^{-1}(P)_\omega^{F_t}\right)\right]\\
&= P\left[(P)_\omega^{F_t}\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]\right]\\
&= P\left[P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\middle| F_t\right]\right]\\
&= P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right].
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_{\varphi,\lambda}\left(P^R\right) &> P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]\\
&= J_{\varphi,\lambda}(P).
\end{aligned}
$$

The proof of the first part of the pre-Markov property is complete.

Step 3 (pre-Markov property, part 2). Let us prove that for every $P \in C^x_{\varphi,\lambda}$, and every F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$, such that $Q_\omega \in \Phi_t C^{\omega(t)}_{\varphi,\lambda}$ for every $\omega \in \Omega$,

$$
P^Q \in C^x_{\varphi,\lambda}.
$$

We have, similarly to some of the above arguments,

$$
\begin{aligned}
J_{\varphi,\lambda}\left(P^Q\right) &= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[P^Q\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\middle| F_t\right]\right]\\
&= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[Q_\omega\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]\\
&\geq P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[(P)_\omega^{F_t}\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\right]
\end{aligned}
$$

since Q_ω is a maximizer,

$$
\begin{aligned}
&= P\left[\int_0^t e^{-\lambda s}\varphi(\xi_s)\,ds\right] + e^{-\lambda t}P\left[P\left[\int_t^\infty e^{-\lambda(s-t)}\varphi(\xi_s)\,ds\middle| F_t\right]\right]\\
&= J_{\varphi,\lambda}(P)
\end{aligned}
$$

hence P^Q is a maximizer.

Step 4 (iterative reduction to singletons) Let $\{\psi_j\}$ and $\{\theta_k\}$ be dense subsets of $(0,\infty)$ and $C_b(\mathcal{V}')$ respectively. Let $\{\varphi_n,\lambda_n\}$ be an enumeration of $\{\psi_j,\theta_k\}_{j,k}$. Given $x \in H$, let $C^x_{\varphi_1,\lambda_1}$ be the set of maximizers of J_{φ_1,λ_1} over C^x, $C^x_{\varphi_2,\lambda_2}$ be the set of maximizers of J_{φ_2,λ_2} over $C^x_{\varphi_1,\lambda_1}$, and so on. The sets

$C^x_{\varphi_n,\lambda_n}$ are a decreasing sequence of compact sets, hence they have non empty compact intersection, that we denote by $\widetilde{C}^x$.

The family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is pre-Markov: it is the intersection of a sequence of pre-Markov families (it is easy to check that the pre-Markov property is preserved by countable intersection).

Let us prove that $\widetilde{C}^x$ is a singleton; this will imply that the family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is Markov. If $P,Q\in\widetilde{C}^x$, then, for every n, $P,Q\in C^x_{\varphi_n,\lambda_n}$, hence

$$J_{\varphi_n,\lambda_n}(P)=J_{\varphi_n,\lambda_n}(Q)$$

This means

$$\int_0^\infty e^{-\theta_k t}P\left[\psi_j\left(\xi_t\right)\right]dt=\int_0^\infty e^{-\theta_k t}Q\left[\psi_j\left(\xi_t\right)\right]dt$$

for every j,k, and since $t\mapsto P\left[\psi_j\left(\xi_t\right)\right]$ and $t\mapsto Q\left[\psi_j\left(\xi_t\right)\right]$ are continuous, from the uniqueness of the Laplace transform we have

$$P\left[\psi_j\left(\xi_t\right)\right]=Q\left[\psi_j\left(\xi_t\right)\right]$$

for every t and j. Hence

$$P\left[\varphi\left(\xi_t\right)\right]=Q\left[\varphi\left(\xi_t\right)\right]$$

for every t and $\varphi\in C_b\left(V'\right)$.

Step 5 (conclusion). Let us summarize what we know: that for every $x\in\mathcal{H}$, for every $P,Q\in\widetilde{C}^x$, for every t and $\varphi\in C_b\left(\mathcal{V}'\right)$ we have $P\left[\varphi\left(\xi_t\right)\right]=Q\left[\varphi\left(\xi_t\right)\right]$. We have to prove the following statement: given $x\in\mathcal{H}$ and $P,Q\in\widetilde{C}^x$, for every n, every $0\leq t_1<...<t_n$ and every $\varphi_1,...,\varphi_n\in C_b\left(\mathcal{V}'\right)$,

$$P\left[\varphi_1\left(\xi_{t_1}\right)...\varphi_n\left(\xi_{t_n}\right)\right]=Q\left[\varphi_1\left(\xi_{t_1}\right)...\varphi_n\left(\xi_{t_n}\right)\right].$$

We prove it by induction. It is true for $n=1$. Assume it is true for n. Denote by $M_{t_1,....,t_n}$ the σ-field generated by $\xi_{t_1},...,\xi_{t_n}$. We have

$$\begin{aligned}&P\left[\varphi_1\left(\xi_{t_1}\right)...\varphi_n\left(\xi_{t_n}\right)\varphi_{n+1}\left(\xi_{t_{n+1}}\right)\right]\\&=P\left[\varphi_1\left(\xi_{t_1}\right)...\varphi_n\left(\xi_{t_n}\right)P\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1,....,t_n}\right]\right]\end{aligned}$$

so if we prove that

$$P\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1,....,t_n}\right]=Q\left[\varphi_{n+1}\left(\xi_{t_{n+1}}\right)|M_{t_1,....,t_n}\right],\quad P-a.s.$$

the proof will be complete, by the induction hypothesis (notice we cannot take simply F_{t_n} in place of $M_{t_1,....,t_n}$ because we have to apply here the induction hypothesis). With other notations, we have to prove that

$$P_\omega^{M_{t_1,....,t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right]=Q_\omega^{M_{t_1,....,t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right],\quad P-a.s.$$

for every $\varphi \in C_b(\mathcal{V}')$. If we had F_{t_n} here in place of $M_{t_1,\ldots,t_n}$, the proof would be complete by the main assumption, since a.s. we have $P_\omega^{F_{t_n}}, Q_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$. Let us use this fact.

We know that the family $\left\{\widetilde{C}^x\right\}_{x\in H}$ is pre-Markov, so there are sets N_P, $N_Q \in F_{t_n}$, with $P(N_P) = 0$ and $Q(N_Q) = 0$, such that $P_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ for every $\omega \notin N_P$ and $Q_\omega^{F_{t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ for every $\omega \notin N_Q$. Therefore

$$P_\omega^{F_{t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right] = Q_\omega^{F_{t_n}}\left[\varphi\left(\xi_{t_{n+1}}\right)\right]$$

for every $\omega \notin N_P \cup N_Q$.

We have

$$P_\omega^{M_{t_1,\ldots,t_n}}(.) = \int P_{\omega'}^{F_{t_n}}(.)\, P_\omega^{M_{t_1,\ldots,t_n}}(d\omega')$$

and $P_\omega^{M_{t_1,\ldots,t_n}}(\xi(t_n) = \omega(t_n)) = 1$, so the integral is a convex combination of elements of $\Phi_{t_n}\widetilde{C}^{\omega(t_n)}$. By the convexity property of $\Phi_{t_n}\widetilde{C}^{\omega(t_n)}$ we get $P_\omega^{M_{t_1,\ldots,t_n}} \in \Phi_{t_n}\widetilde{C}^{\omega(t_n)}$, and similarly for $Q_\omega^{M_{t_1,\ldots,t_n}}$. This implies $P = Q$ and the proof is complete.

Remark 4.9. As in [62], one can easily show that there exists a unique Markov selection if and only if for every $x \in \mathcal{H}$ the set C^x is a singleton.

Remark 4.10. With less easy notations one can prove the strong Markov property, under a pre-strong Markov property for C^x, see [62], [39].

Remark 4.11. A Markov selection allows us to define a Markov semigroup on $B_b(H)$:

$$(P_t\varphi)(x) = P^x[\varphi(\xi_t)].$$

The semigroup property is a consequence of the Markov property:

$$\begin{aligned}(P_{t+s}\varphi)(x) &= P^x[\varphi(\xi_{t+s})] = P^x[P^x[\varphi(\xi_{t+s})|F_s]] \\ &= P^x\left[P^{\xi_s}[\varphi(\xi_t)]\right] = (P_sP_t\varphi)(x).\end{aligned}$$

Remark 4.12. Let C^x be a pre-Markov family with the associated operator R_λ^+ defined in the previous proof. Let P_t be the semigroup generated by one of the Markov selections and let R_λ be the operator associated to it by

$$R_\lambda\varphi = \int_0^\infty e^{-\lambda t}(P_t\varphi)(x)\,dt.$$

If (λ_1, φ_1) is the first pair used in the selection procedure of P_t, then we have

$$R_{\lambda_1}\varphi_1 = R_{\lambda_1}^+\varphi_1.$$

In the particular case when C^x is the singleton P^x, we have $R_{\lambda_1} = R_{\lambda_1}^+$.

Remark 4.13. In our applications, from Itô formula we have

$$P^x\left[e^{-\lambda T}\theta\left(\xi_T\right)\right]=\theta\left(x\right)+\int_0^T e^{-\lambda t}P^x\left[\left(\left(L_0-\lambda\right)\theta\right)\left(\xi_t\right)\right]dt$$

for every θ of the form

$$\theta\left(x\right)=\psi\left(\left\langle v_1,x\right\rangle_{V,V'},\ldots\left\langle v_n,x\right\rangle_{V,V'}\right)$$

with $v_i\in V$ and $\psi\in C_b^2\left(\mathbb{R}^n\right)$; denote this class of functions by $\mathcal{F}C_b^2$. Here L_0 is defined as

$$\left(L_0\theta\right)\left(x\right)=\frac{1}{2}Tr\left[QD^2\theta\left(x\right)\right]-\left\langle D\theta\left(x\right),Ax+B\left(x,x\right)\right\rangle_{V,V'}$$

for $\theta\in\mathcal{F}C_b^2$ and $x\in V$. As $T\to\infty$ we get

$$\theta\left(x\right)=\int_0^\infty e^{-\lambda t}\left(P_t\left(\lambda-L_0\right)\theta\right)\left(x\right)dt.$$

Then $\lambda-L_0$ is injective,

$$\left|\theta\right|_\infty\leq\frac{1}{\lambda}\left|\left(\lambda-L_0\right)\theta\right|_\infty\quad\text{for every }\theta\in\mathcal{F}C_b^2$$

and for every

$$\varphi\in\mathcal{E}_\lambda:=Range\left(\left(\lambda-L_0\right)\mathcal{F}C_b^2\right)$$

we have

$$\left(\left(\lambda-L_0\right)^{-1}\varphi\right)\left(x\right)=\int_0^\infty e^{-\lambda t}\left(P_t\varphi\right)\left(x\right)dt=R_\lambda\varphi\left(x\right).$$

Moreover, while R_λ depends on the Markov process, L_0 does not. Therefore

$$\left(P_t\varphi\right)\left(x\right)$$

is independent of the Markov selection for every $\varphi\in\cap_{\lambda>0}\mathcal{E}_\lambda$. If $\cap_{\lambda>0}\mathcal{E}_\lambda$ would be a separating class, then we have uniqueness of the Markov selection and also of the martingale solutions. This is one of the several ways to see that density properties of the range of $\lambda-L_0$ over $\mathcal{F}C_b^2$ are related to uniqueness. See [58] for an example of rigorous use of this argument.

4.5 Markov Selection for the 3D Stochastic NSE's

Let us go back to equation (1). Verifying that martingale solutions of (1) satisfy the assumptions of the abstract theorem 4.7 is a very difficult task. On one side we need a definition of martingale solution that is stable by disintegration and recollection. On the other side, we have to prove compactness of

C^x, namely as a first step its tightness. For the tightness we need quantitative bounds on elements $P \in C^x$ and 3D Navier–Stokes equations have the unpleasant feature that it is not possible to perform computations on weak solutions, so we cannot prove bounds from the definition given in a previous section. The usual trick in similar problems is to include the bounds in the definition itself: on one side one can prove the existence of martingale solutions satisfying such bounds, on the other the bounds hold true by definition. But here comes the conflict with the first requirement, that solutions should be stable by disintegration and recollection. Indeed, quantitative properties on mean values are not stable by disintegration.

Therefore the idea is to include in the definition of martingale solution not the final mean energy inequality, but an instrument that implies it and is stable by disintegration and recollection. The main instrument that is stable is the concept of martingale. All properties that we express in terms of martingales, or sub or super - martingales, will be stable. This is the reason why we include in the definition of martingale solution a special *super-martingale* property that implies the mean energy inequality.

Unfortunately, even if the idea is clear, the details are still hard since we need energy inequalities over generic intervals $[s,t]$, not only $[0,t]$, since we have to disintegrate at time s and have the same property after time s. Here a detail emerges, namely that for 3D Navier–Stokes equations there is a problem to prove energy inequalities on $[s,t]$ for every s, while one can prove them for *almost every* s with respect to the Lebesgue measure.

Because of these details, the topic is very technical and we address to [39]. In this section we content ourselves with a *conditional* result. We introduce two concepts: enriched martingale problem and a.s. enriched martingale problem; in the definition of the first one we include a super-martingale property; in the definition of the second one we include an a.s. martingale property. Then we prove the existence of at least one solution to the a.s. enriched martingale problem. On the other side, *assuming the existence* of at least one solution to the enriched martingale problem, we prove the existence of a Markov selection.

In this section

$$\Omega = C\left([0,\infty); D(A)'\right).$$

Recall that $(\theta_t, F_t, P)_{t\geq 0}$ is a super-martingale if $P[\theta_t] < \infty$ for every $t \geq 0$ and

$$P[\theta_t 1_A] \leq P[\theta_s 1_A]$$

for every $t \geq s \geq 0$ and every $A \in F_s$.

We say that $(\theta_t, F_t, P)_{t\geq 0}$ is an *almost sure* (a.s.) *super-martingale* if $P[\theta_t] < \infty$ for every $t \geq 0$ and there exists a full Lebesgue measure set $S \subset [0,\infty)$ (namely a set $S \subset [0,\infty)$, with Lebesgue measure of $[0,\infty) \setminus S$ equal to zero), with $0 \in S$, such that

$$P[\theta_t 1_A] \leq P[\theta_s 1_A]$$

holds for every $s \in S$, every $t \geq s$, and every $A \in F_s$.

Definition 4.7 (solution of the enriched martingale problem). *Given a probability measure μ_0 on H, we say that a probability measure P on (Ω, F) is a solution of the enriched martingale problem associated to equation (1) with initial law μ_0 if*

[MP1]

$$P\left(\sup_{t\in[0,T]} |\xi_t|_H^2 + \int_0^T \|\xi_s\|_V^2\, ds < \infty\right) = 1$$

[MP2] for every $\varphi \in \mathcal{D}^\infty$, the process M_t^φ defined P-a.s. on (Ω, F) as

$$\begin{aligned} M_t^\varphi &:= \langle \xi_t, \varphi\rangle_H - \langle \xi_0, \varphi\rangle_H + \int_0^t \nu \langle \xi_s, A\varphi\rangle_H\, ds \\ &\quad - \int_0^t \langle B(\xi_s, \varphi), \xi_s\rangle_H\, ds; \end{aligned}$$

is P-square integrable and (M_t^φ, F_t, P) is a continuous martingale with quadratic variation

$$[M^\varphi]_t = \sum_{i=1}^\infty \sigma_i^2 \langle \varphi, h_i\rangle_H^2 \cdot t$$

[MP3] the process N_t defined P-a.s. on (Ω, F) as

$$N_t := |\xi_t|_H^2 + 2\nu \int_0^t \|\xi_s\|_V^2\, ds - |\xi_0|_H^2 - \sum_{i=1}^\infty \sigma_i^2\, t$$

is P-integrable and (N_t, F_t, P) is a super-martingale

[MP4] more generally, for every integer $n > 0$ the process $N_t^{(2n)}$ defined P-a.s. on (Ω, F) as

$$\begin{aligned} N_t^{(2n)} &:= |\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} \|\xi_s\|_V^2\, ds \\ &\quad - |\xi_0|_H^{2n} - n(2n-1) \sum_{i=1}^\infty \sigma_i^2 \int_0^t |\xi_s|_H^{2n-2}\, ds \end{aligned}$$

is P-integrable and (N_t, F_t, P) is a super-martingale

[MP5] $\mu_0 = \pi_0 P$.

Definition 4.8 (solution of the a.s. enriched martingale problem). *The definition of solution of the* ***a.s.*** *enriched martingale problem associated to equation (1) with initial law μ_0 is the same as the previous one, with the only difference that $N_t^{(2n)}$ are a.s. super-martingales, for $n \geq 1$.*

Remark 4.14. In the deterministic case, a basic concept is the validity of the energy inequality; the analog in the stochastic case is the super-martingale property [MP3]. It seems that the decreasing process of the Doob-Meyer decomposition of N_t is the extra dissipation process which could exist in 3D fluids (it is an open problem whether it is zero or not).

Similarly to Theorem 4.2 we have:

Theorem 4.8. *Given $x \in H$, there exists at least one solution to the a.s. enriched martingale problem with initial condition x.*

Proof. **Step 1**. The construction of P is the same done in Theorem 4.2.

Step 2. Let us check that P fulfills [MP3]. The property

$$P\left[|N_t|\right] < \infty$$

is a consequence of the estimate

$$P_{n_k}\left[\sup_{t\in[0,T]} |\xi_t|_H^2 + \nu \int_0^T \|\xi_s\|_V^2 \, ds\right] \leq C_1\left(m_2, \sigma^2, T\right)$$

proved in Section 3, that passes to the limit to P. We have to prove that there exists a full Lebesgue measure set $S \subset [0, \infty)$ with $0 \in S$, such that

$$P\left[\left(N_t - N_s\right) 1_A\right] \leq 0$$

holds for every $s \in S$, every $t \geq s$, and every $A \in F_s$. Namely, we need to have

$$P\left[\left(|\xi_t|_H^2 + 2\nu \int_s^t \|\xi_r\|_V^2 \, dr - |\xi_s|_H^2 - \sum_{i=1}^{\infty} \sigma_i^2 \ (t-s)\right) 1_A\right] \leq 0. \tag{5}$$

From the results of Section 3 we have

$$P_{n_k}\left[\left(|\xi_t|_H^2 + 2\nu \int_s^t \|\xi_r\|_V^2 \, dr - |\xi_s|_H^2 - \sum_{i=1}^{N_{n_k}} \sigma_i^2 \ (t-s)\right) 1_A\right] = 0.$$

Let us argue as in the proof of lemma 4.2. Let Y_{n_k}, Y be r.v. on a probability space $(\Sigma, \mathcal{G}, Q)$, with expectation E, with values in

$$X = C\left([0,T]; D(A)'\right) \cap L^2\left(0, T; H\right)$$

with laws P_{n_k} and P respectively, such that $Y_{n_k} \xrightarrow{X} Y$, Q-a.s. Let us prove that for every $t \geq s \geq 0$ there is a subsequence Y'_{n_k} of Y_{n_k} such that

$$E\left[1_A \int_s^t \|Y(r)\|_V^2 \, dr\right] \leq \liminf_{k\to\infty} E\left[1_A \int_s^t \left\|Y'_{n_k}(r)\right\|_V^2 \, dr\right]. \tag{6}$$

We know that

$$E\left[\int_s^t \|Y_{n_k}(r)\|_V^2\, dr\right] = P_{n_k}\left[\int_s^t \|\xi_r\|_V^2\, dr\right] \le C$$

uniformly in k, hence there is a subsequence Y'_{n_k} of Y_{n_k} that converges weakly to some $\widetilde{Y}$ in $L^2([s,t]\times\Sigma;V)$; relaxing to weaker common topologies we identify $Y=\widetilde{Y}$; then (6) holds true.

Let us prove that there is a subsequence Y''_{n_k} of Y'_{n_k} such that

$$E\left[\left|Y''_{n_k}(t)\right|_H^2\right] \to E\left[|Y(t)|_H^2\right] \quad \text{for a.e. } t\ge 0. \tag{7}$$

It follows from

$$E\left[\left|Y''_{n_k}(t)-Y(t)\right|_H^2\right] \quad \text{for a.e. } t\ge 0.$$

There exists such Y''_{n_k} since

$$E\int_0^T \left|Y'_{n_k}(t)-Y(t)\right|_H^2 dt \to 0 \tag{8}$$

and this convergence to zero is true since $\int_0^T |Y_{n_k}(t)-Y(t)|_H^2\, dt$ converges to zero Q-a.s. and we have uniform Q-integrability by the estimates ($p^*>2$)

$$E\sup_{t\in[0,T]} |Y_{n_k}(t)|_H^{p^*} < C_3\left(p, E|x|_H^p, TrQ, T\right)$$

proved in Section 3. Properties (6) and (7) imply (5) for a.e. t and s, $t\ge s\ge 0$. Given such an s, the extension to every $t>s$ comes from weak continuity in H of trajectories and Fatou lemma.

Step 3. Let us check that P fulfills [MP4]. Set $p=2n$. The proof that $P\left[\left|N_t^{(p)}\right|\right]<\infty$ is like in step 2 and to prove that $P\left[(N_t-N_s)1_A\right]\le 0$ the only novelty is to show that there is a subsequence Y'_{n_k} of, say, Y_{n_k} such that

$$E\left[\int_s^t |Y(r)|_H^{p-2}\, dr\right] = \lim_{k\to\infty} E\left[\int_s^t \left|Y'_{n_k}(r)\right|_H^{p-2} dr\right].$$

From (8) (for Y_{n_k}) there exists Y'_{n_k} which converges to Y a.s. in (t,σ) ($\sigma\in\Sigma$). Thus it is sufficient to apply Vitali convergence theorem; the uniform integrability is guaranteed always by the p^*-estimates of Section 3, where p^* is arbitrary since x is deterministic. The proof is complete.

Let us now prove a conditional result of Markov selection (see [39] fo a non conditional result). Let C^x be the collection of all solutions of the enriched martingale problem. Assume $C^x\neq\emptyset$ for every $x\in H$. Let us prove that $\{C^x\}_{x\in H}$ is pre-Markov and satisfies the assumptions of theorem 4.7, namely that:

Claim. The family $\{C^x \subset \Pr(\Omega)\,; x \in H\}$ has the following properties:

Lemma 4.10. *i) for every $P \in C^x$,*

$$P_\omega^{F_t} \in \Phi_t C^{\omega(t)} \text{ for } P\text{-a.e. } \omega \in \Omega$$

ii) for every $P \in C^x$, and every F_t-measurable function $\omega \mapsto Q_\omega$, from Ω to $\Pr(\Omega^t)$, such that $Q_\omega \in \Phi_t C^{\omega(t)}$ for every $\omega \in \Omega$,

$$P^Q \in C^x$$

iii) C^x is a convex compact set in $\Pr(\Omega)$ for every $x \in H$ and $x \mapsto C^x$ is measurable.

As a consequence, under the conditional assumption $C^x \neq \emptyset$, there exists a Markov selection from the family of all solutions of the martingale problem associated to equation (1).

The proof of the claim is rather lengthy and we only describe the idea (see [39] for the details). The difficult property in (iii) is the compactness, that can be proved as follows. Let $\{P_m^x\} \subset C^x$ be given. From the supermartingale properties we have

$$P_m^x\left[|\xi_t|_H^2 + 2\nu \int_0^t \|\xi_s\|_V^2\, ds - |x|_H^2 - \sum_{i=1}^\infty \sigma_i^2\, t\right] \leq 0$$

$$\begin{aligned} P_m^x\Bigg[&|\xi_t|_H^{2n} + 2n\nu \int_0^t |\xi_s|_H^{2n-2} \|\xi_s\|_V^2\, ds \\ &- |x|_H^{2n} - n(2n-1)\sum_{i=1}^\infty \sigma_i^2 \int_0^t |\xi_s|_H^{2n-2}\, ds\Bigg] \leq 0 \end{aligned}$$

and in addition we may use Doob's maximal inequality to estimate the supremum in time (recall that given a supermartingale X_t on a discrete set of times $t = 0, ..., T$ one has

$$P\left(\sup_{t\leq T} X_t \geq \lambda\right) \leq \frac{1}{\lambda}\left(E[X_0] + E\left[X_T^-\right]\right)$$

for every $\lambda > 0$). From these estimates one has bounds, uniform in m, similar to those of Galerkin approximations, and then the proof of the existence of a subsequence converging to some $P^x \in C^x$ is similar to the case treated above.

As to points (i) and (ii) of the lemma, we use the fact that the martingale and super-martingale properties are stable under disintegration and recombination; the same is true for properties having probability zero or one. Let us state the theoretical results in this direction that we have to use and omit some of the proofs, that can be found in [39] (some of the results are taken from [62], Thm. 1.2.10). This is the machinery to complete the proof.

The proof of the following lemma is easy.

Lemma 4.11. *Given $P \in \Pr(\Omega)$, a σ-field $G \subset \mathcal{B}$, a set $A \in \mathcal{B}$ and a measurable mapping $\omega \mapsto Q_\omega$ from (Ω, G) to $\Pr(\Omega)$, defined P^Q as*

$$P^Q = \int_\Omega Q_\omega dP(\omega)$$

the following statements are equivalent:
i) $P^Q(A) = 1$
ii) there is a P-null set $N \in G$ such that, for all $\omega \notin N$

$$Q_\omega(A) = 1.$$

The proof of the following two lemmas is less elementary; the first one is very similar to [62], Thm. 1.2.10; for the second one see [39].

Lemma 4.12. *Given $P \in \Pr(\Omega)$, two continuous adapted processes $\theta, \zeta : [0,\infty) \times \Omega \to \mathbb{R}$ and $t_0 \geq 0$, the following conditions are equivalent:*
i) $(\theta_t, F_t, P)_{t \geq t_0}$ is a P-square integrable martingale with quadratic variation $(\zeta_t)_{t \geq t_0}$
ii) there is a P-null set $N \in F_{t_0}$ such that, for all $\omega \notin N$, $\left(\theta_t, F_t, P_\omega^{F_{t_0}}\right)_{t \geq t_0}$ is a $P_\omega^{F_{t_0}}$-square integrable martingale with quadratic variation $(\zeta_t)_{t \geq t_0}$; and $P\left[P_\cdot^{F_{t_0}}[\zeta_t]\right] < \infty$ for every $t \geq t_0$.

Lemma 4.13. *Let $\alpha, \beta : [0,\infty) \times \Omega \to \mathbb{R}_+$ be two adapted processes, β being non decreasing, and let*

$$\theta = \alpha - \beta.$$

Assume θ is left lower semicontinuous. Given $P \in \Pr(\Omega)$ and $t_0 \geq 0$, the following conditions are equivalent:
i) $(\theta_t, F_t, P)_{t \geq t_0}$ is a super-martingale, $P[\alpha_t] < \infty$ and $P[\beta_t] < \infty$ for every $t \geq t_0$;
ii) there is a P-null set $N \in F_{t_0}$ such that, for all $\omega \notin N$, $\left(\theta_t, F_t, P_\omega^{F_{t_0}}\right)_{t \geq t_0}$ is a super-martingale, $P_\omega^{F_{t_0}}[\alpha_t] < \infty$ and $P_\omega^{F_{t_0}}[\beta_t] < \infty$ for every $t \geq t_0$; and $P\left[P_\cdot^{F_{t_0}}[\beta_t]\right] < \infty$ for every $t \geq t_0$.

4.6 Continuity in u_0 of Markov Solutions

Although the uniqueness of solutions to (1) is still an open problem (as in the deterministic case), striking results in the direction of the well posedness have been proved by Da Prato and Debussche [23]. Under proper assumptions on non degeneracy of the noise, they have proved the existence of a selection that depends continuously on the initial conditions (in the sense that the associated Markov semigroup is Strong Feller). They have also provided a direct solution of the Kolmogorov equation and certain gradient estimates that could be helpful in relation with the range problem of remark 4.13 and thus the uniqueness problem.

In this section we revisit their approach and prove that *every Markov* process associated to equation (1) has a Strong Feller like property of continuous dependence on initial conditions. We give here only a few details, and address the reader to [33].

Recall that a Markov operator P_t is Strong Feller if it maps bounded Borel functions in bounded continuous functions; here we always talk about a Strong Feller like property since it will turn out that P_t maps bounded Borel functions in continuous but possibly unbounded functions; and moreover the topology of continuity is that of $D(A)$.

It is difficult to figure out how non-uniqueness could be compatible with such a result if we compare this situation with that of measurable selections, as described by the following simple remark.

Remark 4.15. Let X, Y be two metric spaces, with Borel σ-fields $\mathcal{B}(X)$ and $\mathcal{B}(Y)$, and let Φ be a measurable multivalued mapping from X to Y. Assume that X has no isolated points. If every measurable selection from Φ is continuous, then Φ is univalued. Indeed, let φ be a measurable selection. Given $(x_0, y_0) \in X \times Y$ such that $y_0 \in \Phi(x_0)$, the function $\widetilde{\varphi} : X \to Y$ equal to φ on $X \setminus \{x_0\}$ and to y_0 at x_0, is a measurable selection too, hence it is continuous, hence

$$y_0 = \lim_{x \to x_0} \widetilde{\varphi}(x) = \lim_{x \to x_0} \varphi(x) = \varphi(x_0).$$

This means that Φ is univalued.

However, the situation with Markov selections is not so simple: the Markov structure is much more demanding than measurability only, to the extent that different Strong Feller Markov selections may exists, as in the example of exercise 6.7.7 of Stroock-Varadhan [62]. Thus the uniqueness problem remains open.

Example 4.1. We briefly recall the following example from [62]. Consider the equation on the real line

$$dX_t = \left(|X_t|^{1/4} \wedge 1 \right) dW_t, \quad X_0 = x. \tag{9}$$

Existence of solutions is not a problem. Until $X_t \neq 0$, there is also uniqueness. If $x = 0$, then $X_t \equiv 0$ is a solution; one can embed it into a Markov process: given $x \neq 0$, when the solution from x meets zero we glue it with the zero solution. Let us give another solution (of the martingale problem) from zero and another Markov process. Let $(B_t)_{t \geq 0}$ be a Brownian motion and let $(\tau_t^x(\omega))_{t \geq 0}$ be a solution of the equation

$$\frac{d\tau_t^x(\omega)}{dt} = \left(\left| x + B_{\tau_t^x(\omega)}(\omega) \right|^{1/4} \wedge 1 \right), \quad \tau_0^x = 0.$$

Then $X_t^x := x + B_{\tau_t^x}$ is a solution of (9) and is a Markov process. One can choose $\left(\tau_t^0(\omega)\right)_{t\geq 0}$ different from zero and have a process different from the first one described above. One can also show that they are Feller. Moreover, there are other Markov selections, described in [62].

Nevertheless, the fact that every Markov process is Strong Feller may have interesting consequences. In the following two remarks we describe informally two consequences.

Remark 4.16. If the equation is well posed for **one** initial condition, and the noise allows us to prove irreducibility, then the equation is well posed for **every** initial condition. The scheme of the proof is first to show, by irreducibility and the Markov property, that there is a dense set of initial conditions for which the equation is well posed. Such dense set then belongs to every Markov process; by the Strong Feller property, a priori different Markov processes will coincide on every initial conditions. Notice that in the deterministic case the well-posedness is known for sufficiently small and regular initial conditions. But the proofs of such results do not extend to the case of additive white noise, since preservation of smallness is impossible in such a case. This is an interesting dichotomy between the deterministic and the stochastic case.

Remark 4.17. Under the assumptions that produce the Strong Feller property and irreducibility, we have the following result: for every initial condition $x \in H$, for every solution P^x of the martingale problem from x (at least for those that are members of some Markov selection), at **every** time $t \geq 0$ we have

$$P^x\left(\xi_t \in D(A)\right) = 1. \tag{10}$$

Related easier and well-known results are: i) just under the assumption $\sum_{i=1}^{\infty} \lambda_i \sigma_i^2 < \infty$, without any other condition of nondegeneracy, (10) is true for a.e. t; ii) (10) is known at every time t for stationary solutions. To pass to the case of every solution P^x and every t we take any Markov selection $\left(\widetilde{P}^x\right)$, we prove it has an invariant measure μ with full support, from (ii) we deduce that given t, for μ-a.e. $x \in D(A)$ we have $\widetilde{P}^x\left(\xi_t \in D(A)\right) = 1$. Then we use Strong Feller to extend to every $x \in D(A)$. Finally we use (i) to extend to every $x \in H$. This proves the claim for every solution $\widetilde{P}^x$ being a member of a Markov selection. The result should be extendible to every solution P^x by the reconstruction theorem of [62], but the details should be investigated. Finally, notice that this property looks related to the fact, proved in [23], that the Kolmogorov equation is solvable for initial conditions defined only on $D(A)$.

To shorten a few details, we directly assume that Q has the form

$$Q = A^{-\alpha}, \quad 5/2 < \alpha < 3.$$

The assumption $\alpha > 5/2$ implies that $A^{\frac{1}{2}+\varsigma}\sqrt{Q}$ is Hilbert-Schmidt in H (the embedding of $H^{3/2+\varepsilon}$ into L^2 is Hilbert-Schmidt, in 3D) for some $\varsigma > 0$, so that the noise lives in $D\left(A^{\frac{1}{2}+\varsigma}\right)$. This will imply $z \in C([0,\infty); D(A))$ and other regularity properties. The assumption $\alpha < 3$, on the contrary, allows us to deal with $Q^{-1/2}D_x u_t^x$.

The first ingredient is the bunch of regular paths that every weak solution has for a positive local (random) time, when the initial condition is regular. Following [23] we work with $x \in D(A)$ but other choices seem possible (like $x \in V$). To introduce and analyze this bunch of regular paths, that we call the *regular plume*, we study pathwise equation (1). Notice that such a pathwise analysis is possible, for a given x, also for the solutions of the martingale problem that are probability measures in path space, since the theorem of equivalence describes them as the law of a pathwise solution on some Brownian filtered space (which may depend on x).

Consider the deterministic equation

$$u(t) + \int_0^t (Au(s) + B(u,u))\, ds = x + \omega(t)$$

(interpreted in weak form over test functions $\varphi \in \mathcal{D}^\infty$) and the corresponding Galerkin approximation (an equation in H_n)

$$u_n(t) + \int_0^t (Au_n(s) + \pi_n B(u_n,u_n))\, ds = \pi_n x + \pi_n \omega(t)$$

when $\omega \in \cap_{\alpha\in(0,1/2)} C^\alpha\left([0,\infty); D(A^{\frac{1}{2}+\varsigma})\right)$, $\varsigma > 0$ given in the assumptions on Q. Consider also the auxiliary Stokes equations

$$z(t) + \int_0^t Az(s)\, ds = \omega(t)$$

having the unique mild solution

$$z(t) = e^{-tA}\omega(t) - \int_0^t Ae^{-(t-s)A}(\omega(s) - \omega(t))\, ds.$$

From elementary arguments based on the analytic estimates $\left|A^\alpha e^{-tA}\right| \le \frac{C_{\alpha,T}}{t^\alpha}$ for $t \in (0,T)$, we have (see for instance [32] for details)

$$z \in C\left([0,\infty); D(A^{1+\varsigma-\varepsilon})\right)$$

for every $\varepsilon > 0$. In particular,

$$z \in C([0,\infty); D(A)).$$

Lemma 4.14. *Given $x \in D(A)$, there exists $t_0 > 0$ and a unique solution $u \in C([0, t_0); D(A))$; moreover, there is at least one weak solution*

$$u \in C([0, \infty; H_\sigma) \cap L^2_{loc}([0, \infty); V) \cap L^{2/3}_{loc}([0, \infty); D(A)).$$

Local uniqueness, on $[0, t_0)$, holds in the weak class too, thus any such weak solution coincides on $[0, t_0)$ with the unique $u \in C([0, t_0); D(A))$. Finally, given $T > 0$, if for a weak solution we have

$$\int_0^T |Au(t)|^2 \, dt < \infty$$

then there exists a unique solution $u \in C([0, T]; D(A))$.

Proof. The proof of this result is standard, so we omit the details; let us simply show, formally, that an a priori estimate in $C([0, t_0]; D(A))$ can be proved locally, and it holds true on an interval $[0, T]$ if $\int_0^T |Au(t)|^2 \, dt < \infty$. The new variable $v = u - z$ satisfies

$$v(t) + \int_0^t (Av(s) + B(u, u)) \, ds = x$$

hence

$$\frac{dv}{dt} + Av + B(u, u) = 0$$

$$\begin{aligned} \frac{d|Av(t)|^p}{dt} &= p|Av|^{p-2} \left\langle Av, A\frac{dv}{dt} \right\rangle \\ &= -p|Av|^{p-2} \langle Av, AAv + AB(u, u) \rangle \end{aligned}$$

and therefore

$$\frac{d|Av(t)|^p}{dt} + p|Av|^{p-2} \|Av\|_V^2 = -p|Av|^{p-2} \langle Av, AB(u, u) \rangle$$

$$\leq p|Av|^{p-2} \|Av\|_V \left| A^{1/2} B(u, u) \right| \leq Cp|Av|^{p-2} \|Av\|_V |Au|^2$$

from lemma 2.3; this implies

$$\frac{d|Av(t)|^p}{dt} \leq C|Av|^{p-2} |Au|^4 . \tag{11}$$

Hence in particular

$$\frac{d|Av(t)|^2}{dt} \leq C|Av|^2 |Av|^2 + C|Az|^4 .$$

It is not difficult to deduce the results from this estimate.

Given an initial condition $x \in D(A)$ and a corresponding weak solution u^x define

$$\tau^x = \infty \text{ if } \int_0^T |Au^x(t)|^2 \, dt < \infty \text{ for every } T \geq 0, \text{ otherwise}$$

$$\tau^x = \inf \left\{ T \geq 0 : \int_0^T |Au^x(t)|^2 \, dt = \infty \right\}$$

and notice that $\tau^x > 0$ (for $x \in D(A)$) because of the aforementioned results. A priori this definition depends on the weak solution, because of lack of global uniqueness.

Lemma 4.15. *The definition of τ^x is independent of the weak solution. It depends only on $u|_{[0,\tau^x)}$, that is unique and continuous in $D(A)$. Moreover, if $\tau^x < \infty$, then $\int_0^{\tau^x} |Au^x(t)|^2 \, dt = \infty$. Finally, τ^x coincides with $\widetilde{\tau}^x$ defined as $\widetilde{\tau}^x = \infty$ if u^x is locally bounded around t in $D(A)$ for every $t \geq 0$, otherwise*

$$\widetilde{\tau}^x = \inf \{ t \geq 0 : u^x \text{ locally bounded around } t \text{ in } D(A) \} .$$

Proof. Recall that $\int_0^T |Au^x(t)|^2 \, dt < \infty$ implies u^x regular and unique on $[0,T]$. Denote by τ_1^x, τ_2^x, the times associated to two weak solutions u_1^x and u_2^x. If $\tau_1^x = \infty$ then u_1^x is globally unique, hence $u_1^x \equiv u_2^x$ and $\tau_1^x = \tau_2^x$. Therefore (by symmetry) it is sufficient to consider the case $\tau_1^x, \tau_2^x < \infty$. In such a case $\int_0^T |Au_1^x(t)|^2 \, dt < \infty$ for every $T < \tau_1^x$, hence u_1^x is regular and unique on $[0,\tau^x)$, thus $u_2^x \equiv u_1^x$ on $[0,\tau^x)$. This implies $\int_0^T |Au_2^x(t)|^2 \, dt < \infty$ for every $T < \tau_1^x$, hence $\tau_2^x \geq \tau_1^x$. Reversing the role of τ_1^x and τ_2^x we prove the converse inequality, thus $\tau_1^x = \tau_2^x$.

If $\int_0^{\tau_1^x} |Au_1^x(t)|^2 \, dt$ would be finite, then u_1^x would be regular on $[0,\tau^x]$, in particular $u_1^x(\tau^x) \in D(A)$, hence it could be prolonged as a continuous function in $D(A)$ on some interval $[0,\tau^x+\varepsilon]$, $\varepsilon > 0$, contradicting the fact that $\int_0^{\tau_1^x+\varepsilon} |Au_1^x(t)|^2 \, dt = \infty$. Therefore $\int_0^{\tau_1^x} |Au_1^x(t)|^2 \, dt = \infty$.

Finally, $\tau^x \leq \widetilde{\tau}^x$ because $\int_0^T |Au^x(t)|^2 \, dt < \infty$ implies $u \in C([0,T]; D(A))$. Viceversa, if $T < \widetilde{\tau}^x$, then (by a covering argument) u is bounded on $[0,T]$ with values in $D(A)$, hence $\int_0^T |Au^x(t)|^2 \, dt < \infty$; therefore $T < \tau^x$, and thus $\widetilde{\tau}^x \leq \tau^x$. The proof is complete.

Lemma 4.16. *For $t < \tau^x$, we have*

$$u_n \to u \text{ in } C([0,t]; D(A))$$

(hence in particular $\int_0^t |Au_n(r)|^2 \, dr \to \int_0^t |Au(r)|^2 \, dr$) while for $t \geq \tau^x$ we have

$$\int_0^t |Au_n(r)|^2 \, dr \to \infty.$$

Proof. **Step 1**. On u_n we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|A\left(u_n-z_n\right)\right|^2+\left|A^{3/2}\left(u_n-z_n\right)\right|^2 \\
&=-\left\langle A^2\left(u_n-z_n\right),\pi_n B\left(u_n,u_n\right)\right\rangle \leq C\left|A^{3/2}\left(u_n-z_n\right)\right|\left|Au_n\right|^2 \\
&\leq \left|A^{3/2}\left(u_n-z_n\right)\right|^2+C\left|A\left(u_n-z_n\right)\right|^4+C\left|Az_n\right|^4
\end{aligned}$$

Hence, given $R,T>0$, there is $\triangle t>0$ such that for every $t_0\in[0,T]$

$$|Au_n(t_0)|\leq R\Rightarrow \sup_{t\in[t_0,t_0+\triangle t]}|Au_n(t)|\leq 2R.$$

Step 2. With the notations

$$\begin{aligned}
v_n\left(t\right)&=u_n(t)-u(t)\\
\widetilde{v}_n\left(t\right)&=\left(u_n(t)-u(t)\right)-\left(\pi_n-I\right)z\left(t\right)
\end{aligned}$$

we have

$$\widetilde{v}_n\left(t\right)+\int_0^t A\widetilde{v}_n\left(t\right)ds+\int_0^t\left[\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)\right]ds=x_n-x.$$

Formally (in some intermediate computations we use $\left\|Az\left(.\right)\right\|_V$ that we do not know to be finite; but in the final inequality (12) it disappears; then the rigorous proof can be done by an approximation, see [33] for details)

$$\frac{1}{2}\frac{d}{dt}\left|A\widetilde{v}_n\right|^2+\left|A^{3/2}\widetilde{v}_n\right|^2=-\left\langle A^2\widetilde{v}_n,\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)\right\rangle.$$

We have

$$\pi_n B\left(u_n,u_n\right)-B\left(u,u\right)=\left(\pi_n-I\right)B\left(u,u\right)+\pi_n\left[B\left(u_n-u,u_n\right)+B\left(u,u_n-u\right)\right]$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|A\widetilde{v}_n\right|^2+\left|A^{3/2}\widetilde{v}_n\right|^2\leq\left|\left\langle A\widetilde{v}_n,\left(\pi_n-I\right)AB\left(u,u\right)\right\rangle\right| \\
&+\left|\left\langle A\widetilde{v}_n,\pi_n AB\left(u_n-u,u_n\right)\right\rangle\right|+\left|\left\langle A\widetilde{v}_n,\pi_n AB\left(u,u_n-u\right)\right\rangle\right| \\
&\leq\left|A^{3/2}\widetilde{v}_n\right|^2+f_n+C\left|Av_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)
\end{aligned}$$

and thus

$$\frac{d}{dt}\left|A\widetilde{v}_n\right|^2\leq\widetilde{f}_n+C\left|A\widetilde{v}_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)$$

or in integral form

$$
\begin{aligned}
(12) \qquad \left|A\widetilde{v}_n(t)\right|^2 &\le \left|A\widetilde{v}_n(t_0)\right|^2 \\
&\quad + \int_{t_0}^{t} C\left|A\widetilde{v}_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)ds + \int_{t_0}^{t}\widetilde{f}_n ds
\end{aligned}
$$

where

$$
\begin{aligned}
f_n &= \left|\left(\pi_n - I\right)A^{1/2}B\left(u,u\right)\right|^2, \\
\widetilde{f}_n &= f_n + C\left|A\left(\pi_n - I\right)z\left(t\right)\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right).
\end{aligned}
$$

On any interval $[t_0, t_0+\triangle t]$ we have

$$
\begin{aligned}
\left|A\widetilde{v}_n(t)\right|^2 &\le \left(\left|A\widetilde{v}_n(t_0)\right|^2 + \int_{t_0}^{t_0+\triangle t}\widetilde{f}_n ds\right) \\
&\quad + \int_{t_0}^{t} C\left|A\widetilde{v}_n\right|^2\left(\left|Au_n\right|^2+\left|Au\right|^2\right)ds
\end{aligned}
$$

and we notice that $\int_{t_0}^{t_0+\triangle t}\widetilde{f}_n ds \to 0$ by Lebesgue theorem, if on that interval we can invoke the result of step 1.

Step 3. Given $u_0 \in D(A)$, $t_1 < \tau^{u_0}$, we have

$$
\int_0^{t_1}\left|Au\left(r\right)\right|^2 dr < \infty, \quad u \in C\left(\left[0,t_1\right]; D(A)\right).
$$

Let

$$
R := 1 + \sup_{t\in[0,t_1]}\left|Au\left(t\right)\right|
$$

and $\triangle t$ be given by step 1. We can apply the result of step 1 for every n, since $\left|A\pi_n u_0\right| \le R$.

By Gronwall lemma, for $t_0 = 0$, we have

$$
\left|A\widetilde{v}_n(t)\right|^2 \le e^{\int_0^t C\left(\left|Au_n\right|^2+\left|Au\right|^2\right)ds}\left(\left|A\widetilde{v}_n(0)\right|^2 + \int_0^{\triangle t}\widetilde{f}_n ds\right)
$$

and therefore

$$
\sup_{[0,\triangle t]}\left|Av_n\left(t\right)\right| \to 0.
$$

We have established the result of the first part of the lemma on $[0,\triangle t]$. Moreover, since $\left|Au\left(\triangle t\right)\right| \le R-1$, eventually in n we have $\left|Au_n\left(\triangle t\right)\right| \le R$. Hence we can apply the result of step 1 eventually in n on the interval $[\triangle t, 2\triangle t]$. By Gronwall lemma for $t_0 = \triangle t$, we have

$$
\left|A\widetilde{v}_n(t)\right|^2 \le e^{\int_{\triangle t}^t C\left(\left|Au_n\right|^2+\left|Au\right|^2\right)ds}\left(\left|A\widetilde{v}_n(\triangle t)\right|^2 + \int_{\triangle t}^{2\triangle t}\widetilde{f}_n ds\right)
$$

and therefore

$$\sup_{[\triangle t, 2\triangle t]} |Av_n(t)| \to 0.$$

In a finite number of steps we prove the first claim of the lemma.

Step 4. It is sufficient to consider the case $\tau^x < \infty$ and prove that

$$\int_0^{\tau^x} |Au_n(r)|^2 \, dr \to \infty.$$

By contradiction, assume there is a constant $C > 0$ and a subsequence (u_{n_k}) such that $\int_0^{\tau^x} |Au_{n_k}(r)|^2 \, dr \leq C$ for every k. In addition to the global usual estimates, this implies that there exists a further subsequence $\left(u'_{n_k}\right)$ and an element $u' \in L^2([0, \tau^x]; D(A))$ (beyond the usual regularities) such that $u'_{n_k} \to u'$ strongly in $L^2([0, \tau^x]; H)$, weakly in $L^2([0, \tau^x]; D(A))$, etc. Then it is possible to prove that u is a weak solution on $[0, \tau^x]$. On $[0, \tau^x)$ it must coincide with u. Hence $u \in L^2([0, \tau^x]; D(A))$, which contradicts the definition of τ^x. The proof is complete.

Let us now apply the previous results to the stochastic case. Given $\omega \in \Omega$, let $\tau(\omega) \in [0, \infty]$ be defined as

$$\tau(\omega) = \infty \text{ if } \int_0^T |A\omega(t)|^2 \, dt < \infty \text{ for every } T \geq 0, \text{ otherwise}$$

$$\tau(\omega) = \inf \left\{ T \geq 0 : \int_0^T |A\omega(t)|^2 \, dt = \infty \right\}$$

Definition 4.9 ((of regular plume)). *Given a Brownian stochastic basis*

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, Q, \left(\beta^{(i)}(t) \right)_{t \geq 0, i \in \mathbb{N}} \right),$$

given $x \in D(A)$, equation (1) can be uniquely solved pathwise on $[0, \tau)$, giving rise to a locally defined continuous process in $D(A)$. Its value, in H, at time τ is uniquely prescribed by weak continuity in H of any weak solution. The process $(u^x_{t \wedge \tau})_{t \geq 0}$ so defined will be called the regular plume from x associated to equation (1).

Definition 4.10 (of regularized semigroup). *Given a Brownian stochastic basis, with expectation E, and the regular plume $(u^x_{t \wedge \tau})_{t \geq 0}$ from any $x \in D(A)$, the regularized semigroup associated to equation (1) is defined as*

$$\begin{aligned} (S_t \varphi)(x) &= \int_{\{t < \tau^x\}} e^{-K \int_0^t |Au^x_r|^2 dr} \varphi(u^x_t) \, dQ \\ &= E\left[e^{-K \int_0^t |Au^x_r|^2 dr} \varphi(u^x_t) 1_{t < \tau^x} \right] \end{aligned}$$

(with the understanding that $e^{-\infty} = 0$) for every $t \geq 0$, $x \in D(A)$, $\varphi \in B_b(D(A))$. Here K is any positive constant, so we should write $S_t^{(K)}$, but we shall omit the superscript. Given any Markov selection $\{P^x\}_{x\in H}$, we also have

$$(S_t\varphi)(x) = P^x\left[e^{-K\int_0^t |A\xi_r|^2 dr}\varphi(\xi_t)\, 1_{t<\tau}\right].$$

Remark 4.18. Here and below we use the notation $E[X1_A]$ to denote $\int_A XdQ$; so X may be infinite or even not well defined on $\Omega \setminus A$.

Lemma 4.17. *Given $p > 0$, if K is sufficiently large, $(S_t\varphi)(x)$ is also well defined for every $t \geq 0$, $x \in D(A)$ and measurable $\varphi : D(A) \to \mathbb{R}$, such that*

$$|\varphi(x)| \leq C\left(1 + |Ax|^p\right)$$

for some $C > 0$, $p > 0$. In such a case we have

$$|(S_t\varphi)(x)| \leq C'\left(1 + |Ax|^p\right).$$

Proof. From (11) we have ($v_t = u_t^x - z_t$)

$$\begin{aligned}\frac{d|Av|^p}{dt} &\leq Cp|Av|^{p-2}|Au|^2\left(|Av|^2 + |Az|^2\right)\\ &\leq Cp|Av|^p|Au|^2 + Cp|Av|^{p-2}|Au|^2|Az|^2\\ &\leq C'p|Av|^p|Au|^2 + C'p|Au|^2|Az|^p.\end{aligned}$$

Hence, for $K = C'p$,

$$\begin{aligned}\left(e^{-K\int_0^t |Au|^2 dr}|Av|^p\right)' &\leq Ke^{-K\int_0^t |Au|^2 dr}|Au|^2|Az|^p\\ &\leq -C_z^{(p)}\left(e^{-K\int_0^t |Au|^2 dr}\right)'\end{aligned}$$

where $C_z^{(p)} = \sup_{t\in[0,T]}|Az(t)|^p$, which implies

$$e^{-K\int_0^t |Au|^2 dr}|Av(t)|^p \leq |Ax|^p + C_z^{(p)}$$

hence

$$e^{-K\int_0^t |Au|^2 dr}|Au(t)|^p \leq p\left(|Ax|^p + 2C_z^{(p)}\right) \tag{13}$$

which finally implies the result since $C_z^{(p)}$ has all finite moments.

Given a Brownian stochastic basis $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, Q, \left(\beta^{(i)}(t)\right)_{t\geq 0, i\in\mathbb{N}}\right)$ (with expectation E) on which we have constructed the regular plume, let $(u_t^{x,n})$ be the unique adapted continuous solution of the Galerkin approximation. Define the semigroup S_t^n on $B_b(D(A))$ as

$$(S_t^n\varphi)(x) = E\left[e^{-K\int_0^t |Au_r^{x,n}|^2 dr}\varphi(u_t^{x,n})\right].$$

Lemma 4.18. *If K is sufficiently large, for every continuous $\varphi : D(A) \to \mathbb{R}$, such that*

$$|\varphi(x)| \leq C_\varphi (1 + |Ax|)^k$$

for some $C > 0$, $k \geq 0$, we have

$$(S_t^n \varphi)(x) \to (S_t \varphi)(x)$$

for every $t \geq 0$, $x \in D(A)$.

Proof. If $k = 0$ (φ bounded), it is sufficient to use lemma 4.16 to check that, given x, t, Q-a.s., we have

$$e^{-K \int_0^t |Au_r^{x,n}|^2 dr} \varphi(u_t^{x,n}) \to e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) 1_{t<\tau^x} \tag{14}$$

as $n \to \infty$ (as explained above, the understanding of $e^{-K \int_0^t |Au_r^x|^2 dr} \varphi(u_t^x) 1_{t<\tau^x}$ is that it is zero for $t \geq \tau^x$, even if $\varphi(u_t^x)$ is not well defined) and then apply Lebesgue theorem. For a general k, (14) is still true for the same reason for $t < \tau^x$, while it requires more care for $t \geq \tau^x$. Indeed, for $t \geq \tau^x$, we know that $e^{-K \int_0^t |Au_r^{x,n}|^2 dr} \to 0$, but we need a control on the possible rate of explosion of $\varphi(u_t^{x,n})$. But, as in the previous proof, we have

$$e^{-K \int_0^t |Au_t^{x,n}|^2 dr} |Au_t^{x,n}|^k \leq C_k \left(|Ax|^k + C_z + C_z^{k/2} \right)$$

with the same constants. Choose $K' \geq 2K$, where K is the one of the latter estimate. Then

$$\begin{aligned} &e^{-K' \int_0^t |Au_r^{x,n}|^2 dr} |\varphi(u_t^{x,n})| \\ &\leq e^{-K \int_0^t |Au_r^{x,n}|^2 dr} C_\varphi \left(1 + C_k \left(|Ax|^k + C_z + C_z^{k/2} \right)\right) \end{aligned}$$

which goes to zero Q-a.s., as $n \to \infty$, Hence (14) is true also for $t \geq \tau^x$, with Q-probability one, with the constant K' at the exponent. From the same estimate we see that

$$e^{-K' \int_0^t |Au_r^{x,n}|^2 dr} |\varphi(u_t^{x,n})| \leq C_\varphi \left(1 + C_k \left(|Ax|^k + C_z + C_z^{k/2} \right)\right)$$

for every t, hence we can apply again Lebesgue theorem. The proof is complete.

Let us now prove a Strong Feller property for the regularized semigroup.

Lemma 4.19. *Given $k \geq 0$, if K is sufficiently large, for every measurable $\varphi : D(A) \to \mathbb{R}$, such that*

$$|\varphi(x)| \leq C_\varphi (1 + |Ax|)^k$$

for some $C > 0$, $k \geq 0$, we have

$$\begin{aligned} &|(S_t \varphi)(x) - (S_t \varphi)(y)| \\ &\leq \left[c \cdot C_\varphi \left(t^{\varepsilon - 1} + 1\right) (|Ax| + |Ay| + 1)^k \right] \cdot |A(x - y)| \end{aligned}$$

for every $t > 0$, $x, y \in D(A)$ and for some $\varepsilon > 0$.

Proof. **Step 1**. From [23] we know that (lemma 4.1)

$$|D_h S_t^n \varphi(x)| \le c \cdot C_\varphi \left(t^{\varepsilon-1}+1\right) |Ah| \left(|Ax|+1\right)^k \tag{15}$$

for a certain $\varepsilon > 0$. We sketch the proof below. Here by D_h we denote the derivative in the direction h. Hence, for $x, y \in D(A)$,

$$\begin{aligned}&|(S_t^n\varphi)(x) - (S_t^n\varphi)(y)| \\ &\le \left[c \cdot C_\varphi \left(t^{\varepsilon-1}+1\right) \left(|Ax|+|Ay|+1\right)^k\right] \cdot |A(x-y)| .\end{aligned}$$

Up to mollification of φ, we may apply the previous lemma and get the result.

Step 2. To have an idea of the role of the regularization given by the potential and of the assumption $\alpha < 3$, let us sketch the proof of (15). Following [23], from a variant of Bismut-Elworthy-Li formula we have

$$D_h (S_t^n\varphi)(x) = \frac{1}{t}(I_1 + I_2)$$

$$I_1 = E\left[e^{-K\int_0^t |Au_r^{x,n}|^2 dr}\varphi\left(u_t^{x,n}\right)\int_0^t \left\langle Q^{-1/2} D_h u_s^{x,n}, dW_s\right\rangle\right]$$

$$I_2 = E\int_0^t S_{t-s}^n \varphi\left(u_s^{x,n}\right) D_h\left[e^{-K\int_0^s |Au_r^{x,n}|^2 dr}\right] ds.$$

Let us treat only the (most difficult) term I_1. We have ($u_t = u_t^{x,n}$ for brevity)

$$I_1 \le E\left[e^{-K\int_0^t |Au_r|^2 dr} C_\varphi^2 \left(1+|Au_t|\right)^{2k}\right]^{1/2} E\left[\varsigma_t^2\right]^{1/2}$$

with

$$\varsigma_t := e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\int_0^t \left\langle Q^{-1/2} D_h u_s, dW_s\right\rangle .$$

The first factor can be treated by the analog of (13) for $u_t^{x,n}$. As to the second factor,

$$\begin{aligned}d\varsigma_t^2 &= -K\left|Au_t\right|^2 \varsigma_t^2 dt + 2\varsigma_t e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\left\langle Q^{-1/2} D_h u_t, dW_t\right\rangle \\ &\quad + e^{-\frac{K}{2}\int_0^t |Au_r|^2 dr}\left|Q^{-1/2} D_h u_t\right|^2 dt\end{aligned}$$

$$E\left[\varsigma_t^2\right] \le E\int_0^t e^{-\frac{K}{2}\int_0^s |Au_r|^2 dr}\left|A^{\alpha/2} D_h u_s\right|^2 ds.$$

We have thus to analyze the regularity of $\eta_t^{h,x,n} = D_h u_t^{x,n}$.

Step 3. We have ($\eta_t = \eta_t^{h,x,n}$ for brevity), for every $\beta \ge 0$

$$\frac{d\eta_t}{dt} + A\eta_t + \pi_n B(\eta_t, u_t) + \pi_n B(u_t, \eta_t) = 0, \quad \eta_0 = \pi_n h.$$

$$\frac{d\left|A^{\beta}\eta_t\right|^2}{dt}+2\left|A^{\beta+\frac{1}{2}}\eta_t\right|^2\leq 2\left|\left\langle A^{2\beta}\eta_t,B\left(\eta_t,u_t\right)+B\left(u_t,\eta_t\right)\right\rangle\right|$$

and from lemma 2.3, for $\beta\in(1/2,1)$,

$$\begin{aligned}\frac{d\left|A^{\beta}\eta_t\right|^2}{dt}+\left|A^{\beta+\frac{1}{2}}\eta_t\right|^2&\leq\left|A^{\beta-\frac{1}{2}}\left[B\left(\eta_t,u_t\right)+B\left(u_t,\eta_t\right)\right]\right|\\&\leq C\left|Au_t\right|^2\left|A^{\beta}\eta_t\right|^2\end{aligned}$$

which implies, for sufficiently large K,

$$\int_0^t e^{-\frac{K}{2}\int_0^s|Au_r|^2dr}\left|A^{\beta+\frac{1}{2}}\eta_s\right|^2 ds\leq\left|A^{\beta}h\right|^2.$$

For $\beta=\frac{\alpha-1}{2}$ we get the estimate $E\left[\varsigma_t^2\right]\leq\left|A^{\beta}h\right|^2$. This is the essence of the proof of (15).

The previous result has consequences on the Markov processes associated to equation (1) by means of the following variation of constant formula. Given a Markov selection $\{P^x\}_{x\in H}$ we associate to it the Markov semigroup P_t on $B_b(H)$ defined as

$$\left(P_t\varphi\right)(x)=P^x\left[\varphi\left(\xi_t\right)\right]$$

A priori, this semigroup depends on the selection, but it satisfies the same relation w.r.t. S_t.

Lemma 4.20. *Let $K>0$ be large enough. For every*

$$x\in D(A)\ \textit{and}\ \varphi\in B_b(H)$$

we have

$$P^x\left[\varphi\left(\xi_t\right)\right]=\left(S_t\varphi\right)(x)+\int_0^t\left(S_s\left(K\left|A\cdot\right|^2\left(P_{t-s}\varphi\right)(\cdot)\right)\right)(x)\,ds.$$

Proof. **Step 1**. We use the convention $e^{-\infty}=0$; $0\cdot\infty$ is not necessarily defined but $\int_a^b f(t)\,dt$ is well defined as soon as f is well defined a.s., and integrable.

Let $x\in D(A)$ and u_t^x be a weak solution of the deterministic equation (usual assumptions on ω). The main result of step 1 is to show that

$$e^{-K\int_0^t|Au_r^x|^2dr}-1=-\int_0^t e^{-K\int_0^s|Au_r^x|^2dr}K\left|Au_s^x\right|^2 ds$$

for every $t\geq 0$. For $t<\tau^x$ this is obvious. The proof for $t\geq\tau^x$ seems to be non trivial. We have $\int_0^t\left|Au_r^x\right|^2 dr=\infty$, then $e^{-K\int_0^t|Au_r^x|^2dr}-1=-1$; about the integral, the integrand for $s\in[0,\tau^x)$ is obviously defined and finite, while for $s\in[\tau^x,t]$ we have $e^{-K\int_0^s|Au_r^x|^2dr}=0$, $\left|Au_s^x\right|^2$ is finite a.s., hence

$e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2$ is well defined and equal to zero a.s.; in conclusion, for $t \geq \tau^x$ we have

$$\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2 \, ds = \int_0^{\tau^x} e^{-K\int_0^s |Au_r^x|^2 dr} |Au_s^x|^2 \, ds.$$

By (13) this integral is finite and

$$\begin{aligned}\int_0^{\tau^x} e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \, ds &= \lim_{\eta \uparrow \tau^x} \int_0^{\eta} e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \, ds \\ &= \lim_{\eta \uparrow \tau^x} \left(e^{-K\int_0^\eta |Au_r^x|^2 dr} - 1 \right).\end{aligned}$$

We know that $\int_0^{\tau^x} |Au_r^x|^2 \, dr = \infty$. By monotone convergence theorem,

$$\lim_{\eta \uparrow \tau^x} \int_0^{\eta} |Au_r^x|^2 \, dr = \lim_{\eta \uparrow \tau^x} \int_0^{\tau^x} |Au_r^x|^2 \, 1_{r \leq \eta} dr = \int_0^{\tau^x} |Au_r^x|^2 \, dr = \infty$$

hence $\lim_{\eta \uparrow \tau^x} \left(e^{-K\int_0^\eta |Au_r^x|^2 dr} - 1 \right) = -1$, and the identity is proved also for $t \geq \tau^x$.

Step 2. Take now a non negative $\varphi \in B_b(H)$ (by linearity, this is sufficient), $x \in D(A)$ and u_t^x be a weak martingale solution of the stochastic equation having law P^x (the element of the Markov selection under investigation). Let

$$\left(\Omega^x, \mathcal{F}^x, (\mathcal{F}_t^x)_{t \geq 0}, Q^x, \left(\beta_x^{(i)}(t) \right)_{t \geq 0, i \in \mathbb{N}} \right)$$

be the Brownian stochastic basis in the definition of the weak martingale solution u_t^x. From the identity of the previous step, Q^x-a.s. we have

$$\begin{aligned}\varphi(u_t^x) &= e^{-K\int_0^t |Au_r^x|^2 dr} 1_{t < \tau^x} \varphi(u_t^x) \\ &\quad + \int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \varphi(u_t^x) \, ds.\end{aligned}$$

The first two terms are clearly Q^x-integrable and equal to $(P_t\varphi)(x)$ and $(S_t\varphi)(x)$ resp., thus also the third one is Q^x-integrable and we have

$$\begin{aligned}(P_t\varphi)(x) = (S_t\varphi)(x) + \\ Q^x \left[\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K |Au_s^x|^2 \varphi(u_t^x) \, ds \right].\end{aligned}$$

The last term is, by monotone convergence, the limit as $N \to \infty$ of

$$Q^x \left[\int_0^t e^{-K\int_0^s |Au_r^x|^2 dr} K \left(|Au_s^x|^2 \wedge N \right) \varphi(u_t^x) \, ds \right]$$

which in turns is equal to

$$\int_0^t Q^x \left[e^{-K\int_0^s |Au_r^x|^2 dr} K \left(|Au_s^x|^2 \wedge N \right) \varphi(u_t^x) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) \varphi(\xi_t) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) P^x \left[\varphi(\xi_t) | F_s \right] \right] ds$$

The Markov property gives us (this is a crucial point)

$$P^x \left[\varphi(\xi_t) | F_s \right] = \left(P_{t-s}\varphi \right)(\xi_s).$$

Therefore the previous integral is equal to

$$\int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} K \left(|A\xi_s|^2 \wedge N \right) \left(P_{t-s}\varphi \right)(\xi_s) \right] ds$$
$$= \int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} 1_{s<\tau} K \left(|A\xi_s|^2 \wedge N \right) \left(P_{t-s}\varphi \right)(\xi_s) \right] ds$$

that converges, by monotone convergence, as $N \to \infty$, to

$$\int_0^t P^x \left[e^{-K\int_0^s |A\xi_r|^2 dr} 1_{s<\tau} K \left| A\xi_s \right|^2 \left(P_{t-s}\varphi \right)(\xi_s) \right] ds$$
$$= \int_0^t \left(S_s \left(K \left| A\cdot \right|^2 \left(P_{t-s}\varphi \right)(\cdot) \right) \right)(x)\, ds.$$

The proof is complete.

We can now prove the main result of this section.

Theorem 4.9. *Given a Markov selection $\{P^x\}_{x\in H}$, for every $\varphi \in B_b(H)$ and $x, y \in D(A)$ we have*

$$\left| P^x \left[\varphi(\xi_t) \right] - P^y \left[\varphi(\xi_t) \right] \right|$$
$$\leq c \left[t^{\varepsilon-1} + 1 + t^{\varepsilon} \left(|Ax| + |Ay| + 1 \right)^2 \right] \left| A(x-y) \right|.$$

Proof. From the variation of constant formula and lemma 4.19 we have

$$\left| P^x \left[\varphi(\xi_t) \right] - P^y \left[\varphi(\xi_t) \right] \right| \leq \left| \left(S_t\varphi \right)(x) - \left(S_t\varphi \right)(y) \right|$$
$$+ \int_0^t \left| \left(S_s \left(K \left| A\cdot \right|^2 P_{t-s} \right) \right)(x) - \left(S_s \left(K \left| A\cdot \right|^2 P_{t-s} \right) \right)(y) \right| ds$$
$$\leq \left[c \left(t^{\varepsilon-1} + 1 \right) \right] \left| A(x-y) \right|$$
$$+ \left[\int_0^t \left(s^{\varepsilon-1} + 1 \right) ds \cdot c \left(|Ax| + |Ay| + 1 \right)^2 \right] \left| A(x-y) \right|$$
$$\leq c \left[t^{\varepsilon-1} + 1 + t^{\varepsilon} \left(|Ax| + |Ay| + 1 \right)^2 \right] \left| A(x-y) \right|.$$

The proof is complete.

5 Some Topics on Turbulence

5.1 Introduction and a Few Keywords

We shall mainly refer to the ideas related to Kolmogorov and Obukhov theory developed around 1941 (shortly denoted by K41 theory, see [46]). It is an example of *phenomenology of turbulence*. Following Frisch [42], by this we mean that we create in ourselves a *mental image* of what a turbulent fluid could be, with the help of intuitive *geometric structures* that usually are called *eddies* (or *vortex filaments* when they have strongly elongated shapes, or *vortex pancakes* when they are more surface like, etc.). A typical intuition we have about them is that they *rotate* (in a complex way, not as rigid bodies), with a *typical velocity* U of rotation. We also idealize their shape and associate a *size* l to them (a *typical length scale* of the eddy, something like its diameter). When the structure is more filament (or pancake) like, there could be more than one typical length scales involved, but for the time being let us discuss the case of structures with only one length scale l.

A usual idealization is to think that l takes the values 2^{-n} for positive integers n (more physically we should say $l = l_0 2^{-n}$ where l_0 is a measure of the length of the whole space occupied by the fluid). We apply the intuitive correspondence between length l and *wave number* k saying that a structure of size l has wave number $k = l^{-1}$. Hence we have the different wave numbers $k = 2^n$ (or better $k = k_0 2^n$).

We may think that the various geometric structures *interact*. There are very many interactions that could take place. For the time being, let us forget the interaction between structures of the same size. Let us concentrate our attention on the *interaction between different scales*. Its description is not simple at all and to some extent not completely understood. Let us idealize it by thinking that structures of size $l = l_0 2^{-n}$ produce, by *instability*, smaller structures of size $l = l_0 2^{-n'}$ for some $n' > n$ (for instance, when an instability of the so called Kelvin-Helmoltz type occurs, a new vortex structure originates, and it has a smaller size than the typical size of the original part of fluid where the instability took place). This is the so called *direct cascade*. There could be an *inverse cascade*, in which several small structures merge to form a larger structure (Onsager provided a statistical mechanics explanation of this fact); this seems to be a relevant effect mostly for 2D fluids, so we do not discuss it.

Let us mention the fact that a relevant part of the cascade seems to be due to *vortex stretching*: an eddy undergoes geometric transformations that make it longer in a direction and thinner along the orthogonal plane, so from a blob like shape it gets a filament like shape. This would oblige us to introduce the multiple length scales of filaments now, so for simplicity we do not discuss this mechanism for a while.

The mental image proposed by Kolmogorov is roughly the following one. The turbulent fluid that we observe is entirely composed of eddies; for each eddy size $l = l_0 2^{-n}$, there are eddies of that size that *fill in the whole space* occupied by the fluid. A cascade of energy takes place from length l to length $l/2$ for every l: eddies of size l produce eddies of size $l/2$ by instability and in such a transformation the larger eddies transfer part of their energy to the smaller ones. *Energy is injected* by some mechanism at the largest scales (some solid object which moves in the fluid, some external force). Such *energy is transferred* from scale to scale, from the larger to the smaller scales. There is a length scale η, that we shall call *Kolmogorov dissipation scale*, such that eddies of scale l of the order of η or smaller dissipate very fast their energy (into heat), so the cascade mechanism described above stops. The rate of energy dissipated by such small eddies will be equal to the rate of energy injected at large scales, otherwise the system would be not in a stationary regime.

The viscosity of the fluid does not play a major role on structures with l essentially larger than η, while it is a basic ingredient at scales $l \sim \eta$ or smaller. If the viscosity is decreased, then the scale η becomes smaller, but always positive. If we keep unchanged the external forces acting at large scales, so the rate of energy injection remains constant when we send ν to zero, to preserve a stationary regime the rate of energy dissipation will remain also constant (in spite of the fact that, as we shall see, it is the product of ν which goes to zero, and the mean square gradient of the velocity).

If there is hope to discover universal statistical laws which hold for every turbulent fluid (independently of geometry of the region, particular features of the mechanism of injection of energy etc.), it is reasonable to expect that they will hold at quite small scales and when the viscosity is very small. For this reason the previous comment on the limit as $\nu \to 0$ is relevant: this is the "regime" where one looks for universal statistical laws. We need a mathematical model which incorporate the idea that when $\nu \to 0$ the energy injection and the energy dissipation remain constant. The model proposed above of the stochastic equation

$$dX_t = [-\nu A X_t - B(X_t, X_t)]\, dt + \sqrt{Q} dW_t$$

is in this direction; for the finite dimensional approximations we have the identity

$$\nu E \|X_t\|_V^2 = \frac{TrQ}{2}$$

for stationary solutions which states precisely that energy injection and the energy dissipation remain constant in the limit $\nu \to 0$. Unfortunately, for the 3D infinite dimensional model the equality above is an open problem, but perhaps this is just a technical issue related to our present poor understanding of the well posedness.

With these intuitive ideas in mind, Kolmogorov analyzed a specific statistical index of the turbulent fluid, the so called structure function of order 2, and proposed a formula for it. This formula is still now one of the few quantitative predictions that are close to experimental results (up to some approximation).

5.2 K41 Scaling Law: Heuristics and Unclear Issues

Let $u(t,x)$ be the velocity field of the fluid. Statistical quantities like the mean value or the covariance of this field are not expected to have universal behavior (their size, for instance, depends too much on the particular conditions of the fluid), except for some qualitative feature: we assume $x \mapsto u(t,x)$ to be *space homogeneous and isotropic.*

The velocity displacements $u(t,x+ry)-u(t,x)$ on the contrary could have universal statistical properties when r is small. So let us look at their second moment $E\left[|u(t,x+ry)-u(t,x)|^2\right]$. We assume to work on time-stationary fields, so this mean value is independent of t; we shall drop t and write $E\left[|u(x+ry)-u(x)|^2\right]$. Since we assume space homogeneity, this mean value will not depend on x; by isotropy, it will not depend on the particular direction y either. So take a unitary coordinate vector e and consider the quantity

$$S_2(r) = E\left[|u(re)-u(0)|^2\right]$$

(assume 0 is a point of the domain where the fluid lives). This is called *second-order structure function.* The structure function of order p is simply the same expression with the p power; we do not discuss it at the beginning.

Kolmogorov and Obukhov conjectured that $S_2(r)$ could have a universal behavior in r, for small r and small viscosity. Let us describe the argument by *dimensional analysis* that Kolmogorov proposed to obtain a formula for $S_2(r)$.

Let us denote by $[L]$ the dimension of a length, $[T]$ for time. Velocity has dimension $[L][T]^{-1}$, hence $S_2(r)$ has dimension $[L]^2[T]^{-2}$.

Let us assume that, when $\nu \to 0$, for small r the function $S_2(r)$ depends only on r and the rate of energy dissipation ε, as a power law. The latter is defined as

$$\varepsilon = \nu E\left[|Du(0)|^2\right]$$

(we advertise that this discussion is still heuristic; precise definitions will be given below for random fields on the torus). So the assumption is that

$$S_2(r) = C r^{\alpha} \varepsilon^{\beta}$$

for some adimensional constant C and some exponents α and β.

The dimension of ε is a little tricky to determine. A simple way is to think that the energy dissipated is dimensionally as the time derivative of the kinetic energy (this is clear from the Navier–Stokes equations). The kinetic energy $\frac{1}{2}E\left[|u(0)|^2\right]$ has dimension $[L]^2[T]^{-2}$, so its time derivative has dimension $[L]^2[T]^{-3}$. Thus ε has dimension $[L]^2[T]^{-3}$.

Using finally the fact that r has obviously dimension $[L]$, from the power assumption above we must have

$$\begin{aligned}[L]^2 [T]^{-2} &= [L]^{\alpha} \left([L]^2 [T]^{-3}\right)^{\beta} \\ &= [L]^{\alpha+2\beta} [T]^{-3\beta}.\end{aligned}$$

The only solution is $\beta = 2/3$ and $\alpha = 2 - 2\beta = 2/3$. Hence

$$S_2(r) = Cr^{2/3}\varepsilon^{2/3}. \tag{1}$$

In particular, the behavior in r is like $r^{2/3}$, or

$$\lim_{r\to 0} \frac{\log S_2(r)}{\log r} = \frac{2}{3}.$$

It is common to observe in (sophisticated enough) experiments that the log-log plot of $S_2(r)$ has a plateau of approximate slope $\frac{2}{3}$. As we said, this is still now one of the two best statistical laws compared to experiments (the other one is concerned with boundary layers).

We have just remarked that in experiments a slope close to $\frac{2}{3}$ is observed along a plateau of the curve, but not along the whole curve. So the previous argument has to be made a little more precise about the range of r where it is expected to be true. Going back to the mental image described in the previous section, we have seen that we may expect an energy cascade up to some dissipation scale η only; and the universal behavior that we try to describe with the structure function $S_2(r)$ refers to scales r in such a range where the cascade takes place, called *inertial range*. So the prescription of K41 theory is that the law (1) holds true in a range $r \in [\eta, r_0]$. We have now to determine η.

Let us find η again by a dimensional argument. Assume that η depends only on ε and the viscosity ν, as a power law. [Notice that these assumptions are a very strong idealization but they are reasonable: the rate of energy injection or dissipation ε has to play a role in the quantitative laws; for $S_2(r)$ there should be also an obvious dependence on r, while for η there should be a dependence on ν, as already remarked in the previous section.]

So we assume

$$\eta = C\nu^{\alpha}\varepsilon^{\beta}.$$

The dimension of η is $[L]$ and the dimension of ε is $[L]^2 [T]^{-3}$. The dimension θ of ν could be found from the relation $\varepsilon = \nu E\left[|Du(0)|^2\right]$: Du has dimension $[T]^{-1}$, so from $[L]^2 [T]^{-3} = \theta [T]^{-2}$ we deduce $\theta = [L]^2 [T]^{-1}$. Therefore the power relation imposes

$$\begin{aligned}[L] &= \left([L]^2 [T]^{-1}\right)^{\alpha} \left([L]^2 [T]^{-3}\right)^{\beta} \\ &= [L]^{2\alpha+2\beta} [T]^{-\alpha-3\beta}.\end{aligned}$$

The only solution is $\alpha = 3/4$ and $\beta = -1/4$. Thus the law for η is

$$\eta = C\nu^{3/4}\varepsilon^{-1/4}.$$

There are more refined arguments which support the power laws given here, but all of them are in any case based on unproved assumptions, never deduced from the Navier–Stokes equations.

In the sequel we give a rigorous definition of the K41 scaling law and prove some necessary and some sufficient conditions for it, with the hope to throw some light on this problem. The presentation is based on the work by Flandoli, Gubinelli, Hairer and Romito [37], but, to simplify, we avoid anomalous exponents and restrict ourselves to K41 theory. It is necessary to say that we do not believe K41 is exactly true for the Navier–Stokes equations. Neverthless, understanding necessary and/or sufficient conditions for K41 may help to start a rigorous investigation of such scaling laws.

The arguments just presented rely on some assumptions, namely the dependence of $S_2\,(r)$ and η only on certain variables in the form of a power law, that are unjustified. They may look natural:

- it is clear that $S_2\,(r)$ should depend on r and that η should depend on ν (we have $\lim_{\nu\to 0}\eta = 0$);
- it can be intuitively clear that both of them should depend on ε: by analogy with queuing theory, in the stationary regime, independently of the complexity of the queuing network, the rate of input is equal to the rate of output, and such rate is a basic parameter that affects several main quantities of the system;
- but it is not clear why no other quantity should be involved.

The agreement of K41 prediction with experimental results is good but not perfect. The same dimensional argument described above may be applied to 2D fluids (thin layers of fluids), where on the contrary the experiments do not confirm the K41 prediction. Essentially the explanation has something to do with the additional conservation law for 2D fluids, that is the conservation of enstrophy (for zero viscosity).

Another failure of the scaling argument above is when it is applied to the *structure function of order* p: here it is better to work with the *longitudinal* structure function to appreciate also the case of odd numbers (especially $p = 3$)

$$S_p^l\,(r) = E\left[\langle u(re) - u(0), e\rangle^p\right].$$

For even p the behavior in r is expected to be the same as that of

$$S_p\,(r) = E\left[\|u(re) - u(0)\|^p\right].$$

If we believe that the assumptions of K41 theory apply to $S_p^l\,(r)$ as well, by the same dimensional analysis we would find

$$S_p^l\,(r) \sim r^{p/3}.$$

On the contrary, if we "define" the numbers ζ_p as those for which

$$S_p^l(r) \sim r^{\zeta_p}$$

there is experimental evidence that the function

$$p \mapsto \zeta_p$$

is strictly concave for large p's, not equal to the line $p \mapsto p/3$. For $p = 3$ it seems both from the experimental viewpoint and from some heuristics that the correct value is really $\zeta_3 = 3/3 = 1$. For $p = 2$ the experiments give values ζ_2 very close to $2/3$, but possibly slightly larger. But for large p the experimental values of ζ_p, although not coinciding from one experiment to the other, is definitely smaller that $p/3$.

5.3 Definitions and Examples

Given the unitary torus $\mathcal{T} = [0,1]^d$, $d = 2,3$, recall the definitions of H and $D(A)$. We denote by $\mathcal{P}$ the class of all probability measures μ on H (with the Borel σ-algebra) such that $\mu(D(A)) = 1$, μ is space homogeneous and partial isotropic, and

$$\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right] < \infty.$$

Since $\mathbb{H}^2(\mathcal{T}) \subset \mathbb{C}(\mathcal{T})$ by Sobolev embedding theorem, the elements of $D(A)$ are continuous (have a continuous element in their equivalence class). Consequently, given $x_0 \in \mathcal{T}$, the mapping $u \mapsto u(x_0)$ is well defined on $D(A)$, with values in $\mathbb{R}^d$. In particular, any expression of the form

$$\mu\left[f\left(u(x_1), ..., u(x_n)\right)\right]$$

is well defined for given $x_1, ..., x_n \in \mathcal{T}$, given $\mu \in \mathcal{P}$, and suitable $f : \mathbb{R}^{nd} \to \mathbb{R}$ (for instance measurable non negative). It will follow that $S_2^\mu(r)$ is well defined (possibly infinite) for every $\mu \in \mathcal{P}$. On the contrary we cannot evaluate pointwise Du and D^2u, but we may use the quantities

$$\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right], \quad \mu\left[\int_{\mathcal{T}} \left\|D^2u(x)\right\|^2 dx\right]$$

the first of which is finite by assumption for $\mu \in \mathcal{P}$, while the second one is either finite or equal to $+\infty$.

For every $\mu \in \mathcal{P}$ we introduce the *second order structure function*

$$S_2^\mu(r) = \mu\left[\|u(r \cdot e) - u(0)\|^2\right] \tag{2}$$

for some coordinate unitary vector e, with $r > 0$ (the results proved below extend to the so called longitudinal structure function; we consider (2) to fix the ideas). The symmetries in $\mathcal{P}$ imply that $S_2^{\mu}(r)$ is independent of the coordinate unitary vector e, and the velocity difference could be taken at any other point x: $u(x+r\cdot e)-u(x)$ gives us the same result.

We are going to define K41 scaling law for a set $\mathcal{M}\subset\mathcal{P}\times\mathbb{R}_+$. The reason is that equation (1) may have (a priori) more than one stationary measure for any given ν and in certain claims it seems easier to consider a set of measures for a given ν. Given $\nu>0$ we use the notation $\mathcal{M}_\nu$ for the set section $\{\mu\in\mathcal{P}:(\mu,\nu)\in\mathcal{M}\}$.

Given $(\mu,\nu)\in\mathcal{P}\times\mathbb{R}_+$, we define the *mean energy dissipation rate* as

$$\varepsilon=\varepsilon(\mu,\nu):=\nu\cdot\mu\left[\int_{\mathcal{T}}\|Du(x)\|^2\,dx\right].$$

In the sequel, to simplify the exposition, we impose on families $\mathcal{M}\subset\mathcal{P}\times\mathbb{R}_+$ the following condition (*constant mean energy dissipation rate as the viscosity goes to zero*):

$$\varepsilon(\mu,\nu)=\varepsilon_0 \text{ for every } (\mu,\nu)\in\mathcal{M}. \tag{3}$$

This is true if we consider the finite dimensional models of Section 3 (with the identification of $\varepsilon(\mu,\nu)$ with $\nu\cdot\mu\left(\|\cdot\|_V^2\right)$). It remains true for the stochastic Navier–Stokes equation (1), if the dimension is $d=2$. Unfortunately, in 3D, it is an open problem, as illustrated in the section on Galerkin stationary measures. So, in a sense, we impose here an assumption that we do not know whether it is satisfied by our main example, the 3D case. We do this for simplicity of exposition: in [37] the assumption is partially removed.

Given $(\mu,\nu)\in\mathcal{P}\times\mathbb{R}_+$, we also define the quantity

$$\eta=\eta(\mu,\nu):=\nu^{3/4}\varepsilon(\mu,\nu)^{-1/4}.$$

Under the assumption (3) we simply have

$$\eta(\mu,\nu)=\nu^{3/4}\eta_0$$

where, for shortness, we have used the symbol $\eta_0={\varepsilon_0}^{-1/4}$.

Let us come to the definition of K41 scaling law. See [37] for a more general version, including also a correction to the 2/3 exponent.

Definition 5.1. *We say that a scaling law of K41 type holds true for a set* $\mathcal{M}\subset\mathcal{P}\times\mathbb{R}_+$ *if there exist* $\nu_0>0$, $C>c>0$, $r_0>0$ *such that the bound*

$$c\cdot r^{2/3}\le S_2^{\mu}(r)\le C\cdot r^{2/3}$$

holds for every pair $(\mu,\nu)\in\mathcal{M}$ *and every* r *such that* $\nu\in(0,\nu_0]$ *and* $\eta(\mu,\nu)<r<r_0$, *namely*

$$\nu^{3/4}\eta_0<r<r_0.$$

This is the mathematical formulation of K41 theory that we analyze. We shall prove some necessary conditions and some sufficient conditions for it. Let us insist on the fact that we do not claim that K41 is true. Presumably a version with an exponent larger than 2/3 exponent is true. The reason to state a definition is to attempt a rigorus investigation of this scaling property.

Before going into some rigorous results about this definition, let us ask ourselves a few preliminary apparently easy questions: can we give examples of functions $f(\nu, r)$ (we have the association in mind $f(\nu, r) = S_2^{\mu(\nu)}(r)$) such that

$$c \cdot r^{2/3} \leq f(\nu, r) \leq C \cdot r^{2/3} \text{ for } \nu^{3/4} < r < 1 \text{ and small } \nu?$$

It is important to realize that our usual way of thinking in mathematics is about limit properties. The previous property is not a limit one, but it is a property in an intermediate range, with some kind of uniformity as a parameter goes to a limit ($\nu \to 0$).

The easiest way to answer the previous question is by the example

$$f(\nu, r) = r^{2/3} \text{ for } \textit{all } (r, \nu).$$

But such an example cannot be related to our models. Indeed, we shall see below that, due to the property $\mu(D(A)) = 1$, we must have a regular behavior in $r \to 0$, for every given ν:

$$f(\nu, r) \sim r^2 \text{ as } r \to 0, \text{ for every } \nu.$$

More precisely, we shall see that essentially we have

$$f(\nu, r) = \frac{r^2}{\nu} \text{ for sufficiently small } r.$$

So let us refine our question and ask whether we may:

- find examples of functions $f(\nu, r)$ such that

  (4) $$C_1 \cdot r^{2/3} \leq f(\nu, r) \leq C_2 \cdot r^{2/3}$$

  for $C_3 \nu^{3/4} < r < 1$ and

  (5) $$C_4 \cdot r^2 \leq f(\nu, r) \leq C_5 \cdot r^2$$

  for $r < C_6 \nu^{3/4}$.

  This is not easy (unless we artificially define piecewise $f(\nu, r)$).

Example 5.1 (Negative example). Let us preliminary understand better the function

$$f_0(\nu, r) := \frac{r^2}{\nu}$$

which certainly satisfies the second part of the requirement. We have

$$f_0(\nu, r) = r^{2/3}\left(r\nu^{-3/4}\right)^{4/3}$$

so we have

$$C_1 \cdot r^{2/3} \leq f(\nu, r) \leq C_2 \cdot r^{2/3} \text{ for } C_3\nu^{3/4} < r < C_4\nu^{3/4}$$

for suitable constants. Apparently we get a property very close to the required one, but it holds true only on an interval of r whose boundary values are infinitesimal of the same order. This is a basic point: to have a meaningful definition of a scaling law we must ask its validity on an interval whose boundary points diverge one w.r.t. the other. What is meaningless in the previous 2/3 property on $C_3\nu^{3/4} < r < C_3'\nu^{3/4}$ is simply that a similar property holds true replacing 2/3 with any other exponent $\alpha \in (0, 2)$. Indeed we have

$$f_0(\nu, r) = r^{\alpha}\left(r\nu^{-\frac{1}{2-\alpha}}\right)^{2-\alpha}$$

hence

$$C_1' \cdot r^{\alpha} \leq f_0(\nu, r) \leq C_2' \cdot r^{\alpha} \text{ for } C_3'\nu^{\frac{1}{2-\alpha}} < r < C_4'\nu^{\frac{1}{2-\alpha}}.$$

Summarizing, $f_0(\nu, r) = \frac{r^2}{\nu}$ clearly does not have any interesting non-integer scaling law, as we see from its definition, and it shows in addition that in the definition of a scaling law it is necessary to impose that the range of r has boundary points that diverge one from each other.

Example 5.2 (Positive example). This example arose in the computations made of a random vortex model of Flandoli and Gubinelli, see [36]. Consider the function

$$f(\nu, r) = \int_\eta^1 l^{2/3}\left(\frac{l \wedge r}{l}\right)^2 \frac{dl}{l}$$

with $\eta = \nu^{3/4}$. We have

$$r \leq \eta \Rightarrow f(\nu, r) = \int_\eta^1 l^{2/3}\left(\frac{r}{l}\right)^2 \frac{dl}{l} = \frac{3}{4}r^2\left[\nu^{-1} - 1\right]$$

which gives us (5). On the other hand,

$$\begin{aligned} r \in [\eta, 1] \Rightarrow f(\nu, r) &= \int_\eta^r l^{2/3}\frac{dl}{l} + \int_r^1 l^{2/3}\left(\frac{r}{l}\right)^2 \frac{dl}{l} \\ &= \frac{9}{4}r^{2/3} - \frac{3}{2}\nu^{1/2} - \frac{3}{4}r^2 \end{aligned}$$

which is bounded above and below by the order $r^{2/3}$ since $r \in \left[\nu^{3/4}, 1\right]$ $(\nu^{1/2} \leq r^{2/3})$. This implies (4).

5.4 Brownian Eddies and Random Vortex Filaments

Before entering into some rigorous results around K41 scaling law for the stochastic Navier–Stokes equations, let us get more intuition from a phenomenological model. This model is a priori given, in the sense that it does not come from the Navier–Stokes equations, but it is nevertheless defined in rigorous mathematical terms and it is an attempt to describe some of the numerical observations of vortex filaments obtained in the last 15 years. The main source of motivation has been the book of Chorin [19], where discrete vortex filaments based on paths of self-avoiding walk are investigated. Other related works are [10], [37].

Let $(W_t)_{t\geq 0}$ be a 3D Brownian motion. Let $\rho : \mathbb{R}^3 \to \mathbb{R}$ be the function $\rho(x) = \exp\left(-\|x\|^2\right)$ (there is a lot of freedom in the choice of ρ, this is just a convenient example). Given $\ell > 0$, rescale ρ as

$$\rho_\ell(x) = \rho\left(\frac{x}{\ell}\right) = \exp\left(-\frac{\|x\|^2}{\ell^2}\right).$$

Let $K_\ell(x) : \mathbb{R}^3 \to \mathbb{R}^3$ be the field

$$K_\ell(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \rho_\ell(y)\frac{x-y}{|x-y|^3}dy.$$

Remark 5.1. If $\gamma : \mathbb{R} \to \mathbb{R}^3$ is the curve $\gamma(t) = (0,0,t)$, then the vector field (interpret it as a velocity field)

$$u_{Burgers}(x) := \int_{-\infty}^{+\infty} K_\ell(x-\gamma(t)) \wedge \dot{\gamma}(t)\,dt$$

is called a Burgers vortex, and is easily seen to be a rotating field around the z-axis, with some decay at infinity. It is also given by the Biot-Savart law with respect to the "vorticity" field $\xi(x)$:

$$u_{Burgers}(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \wedge \xi_{Burgers}(y)\,dy$$

$$\xi_{Burgers}(y) := \int_{-\infty}^{+\infty} \rho_\ell(y-\gamma(t)) \wedge \dot{\gamma}(t)\,dt.$$

The number ℓ is a measure of the "cross section" of the vortex "tube".

We repeat the mathematics of the previous example but starting from the Brownian motion $(W_t)_{t\geq 0}$ in place of the line γ.

Definition 5.2. *Let us call Brownian eddy at scale $l > 0$ the following random field $(W)_\ell^\perp(x)$:*

$$(W)_\ell^\perp(x) := \frac{1}{\ell^2}\int_0^{\ell^2} K_\ell(x-W_t) \wedge dW_t \qquad x \in \mathbb{R}^3.$$

Remark 5.2. We use the notation $(.)_{\cdot}^{\perp}$ because the field $(W)_{\ell}^{\perp}(x)$ is somewhat orthogonal to the trajectory of the Brownian motion.

Remark 5.3. Let us explain the rigorous use of the notation. Given a stochastic process $(X) = (X_t)_{t \geq 0}$, if the stochastic integral is well define we may introduce the associated random field

$$(X)_{\ell}^{\perp}(x) := \frac{1}{\ell^2} \int_0^{\ell^2} K_{\ell}(x - X_t) \wedge dX_t \qquad x \in \mathbb{R}^3.$$

Such an integral is well defined for instance for every semimartingale (X). The notation $(.)_{\ell}^{\perp}$ denotes a mapping from processes to random fields; for instance, if we write

$$(X_0 + W)_{\ell}^{\perp}(x)$$

where X_0 is a 3D random variable, we understand the vector field associated to the process $X_0 + W$, that is a Brownian motion starting from position X_0.

Remark 5.4. It may help the intuition to figure out the shape of a Brownian eddy. The Brownian motion has sometimes long excursions: along them a Brownian eddy looks like an irregular Burgers vortex. On the contrary, most often the trajectory of a Brownian motion is very much folded around itself: in such a case the Brownian eddy is more blob-like. The number ℓ is a measure of the size, and also of the smoothness. Notice finally that the typical displacement of a Brownian motion in a time ℓ^2 is of the order ℓ, that is the size of the kernel K_ℓ: therefore for most of the trajectories of the Brownian motion the associated eddy is as long as large, eddy-like more than filament-like.

Remark 5.5. One can verify by stochastic analysis that the field $(W)_{\ell}^{\perp}(x)$ is very regular (it has C^∞-realizations) and

$$\mathsf{div}\,(W)_{\ell}^{\perp}(x) = 0.$$

The reason why this field turns out to be interesting is the following scaling property.

Lemma 5.1. *For every $\lambda, \ell > 0$,*

$$(W)_{\ell}^{\perp}(\lambda x) \stackrel{\mathcal{L}}{=} (W)_{\ell/\lambda}^{\perp}(x)$$

the equality in law being at the level of random fields.

Proof.

$$\begin{aligned}
(W)_{\ell}^{\perp}(\lambda x) &= \frac{1}{\ell^2} \int_0^{\ell^2} K_{\ell}\left(\lambda\left(x - \frac{W_t}{\lambda}\right)\right) \wedge \lambda d\left(\frac{W_t}{\lambda}\right) \\
&= \frac{1}{(\ell/\lambda)^2} \int_0^{\ell^2} K_{\ell/\lambda}\left(x - \frac{W_t}{\lambda}\right) \wedge d\left(\frac{W_t}{\lambda}\right)
\end{aligned}$$

since

$$K_\ell(\lambda x) = \lambda K_{\ell/\lambda}(x).$$

The processes $\left(\frac{W_t}{\lambda}\right)$ and $\left(W_{t/\lambda^2}\right)$ have the same law, hence

$$(W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \frac{1}{(\ell/\lambda)^2}\int_0^{\ell^2} K_{\ell/\lambda}\left(x - W_{t/\lambda^2}\right) \wedge d\left(W_{t/\lambda^2}\right)$$

where it is not difficult to see that the equality in law is at the level of random fields (namely jointly in different locations x). Finally, simple arguments on time change in stochastic integrals show that

$$(W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \frac{1}{(\ell/\lambda)^2}\int_0^{(\ell/\lambda)^2} K_{\ell/\lambda}(x - W_t) \wedge d(W_t).$$

The proof is complete.

Remark 5.6. In particular,

$$(W)_\ell^\perp(\ell x) \stackrel{\mathcal{L}}{=} (W)_1^\perp(x).$$

This says that the velocities we observe in $(W)_\ell^\perp$ are the same as those of $(W)_1^\perp$. The energy will be much smaller: the "support" of an ℓ-eddy is roughly of order ℓ^3, hence its kinetic energy is roughly of order ℓ^3 times the kinetic energy of $(W)_1^\perp$.

Remark 5.7. The analogous definition of fractional Brownian eddy at scale $\ell > 0$ and Hurst parameter $H \in (0,1)$ would be

$$\left(W^H\right)_\ell^\perp(x) = \frac{1}{\ell^2}\int_0^{\ell^{1/H}} K_\ell\left(x - W_t^H\right) \wedge dW_t^H \qquad x \in \mathbb{R}^3$$

whenever the integral is well defined, where $\left(W_t^H\right)$ is a fractional Brownian motion in $\mathbb{R}^3$ with Hurst parameter $H \in (0,1)$. With the same proof one can show that

$$\left(W^H\right)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \left(W^H\right)_{\ell/\lambda}^\perp(x).$$

Indeed, the only difference in the proof is that now the processes $\left(\frac{W_t^H}{\lambda}\right)$ and $\left(W_{t/\lambda^{1/H}}\right)$ have the same law.

With the same proof of the previous lemma we have:

Lemma 5.2. *Given a non anticipating 3D r.v. X_0, for every $\lambda, \ell > 0$,*

$$(X_0 + W)_\ell^\perp(\lambda x) \stackrel{\mathcal{L}}{=} \left(\frac{X_0}{\lambda} + W\right)_{\ell/\lambda}^\perp(x)$$

the equality in law being at the level of random fields.

With the help of the previous objects we may define more complex random fields. Let $\left\{X_0^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of 3D i.i.d. random variables, $\left\{W^{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of 3D independent Brownian motions, $\left\{\ell_{(n)}\right\}_{n\in\mathbb{N}}$ be a sequence of positive i.i.d. random variables, with all these objects independent one of each others. Then define *formally* the series

$$u(x) = \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(X_0^{(n)} + W^{(n)}\right)_{\ell_{(n)}}^{\perp} (x) \,.$$

Assume *formally* that

- $X_0^{(n)}$ are uniformly distributed in $\mathbb{R}^3$
- $\ell_{(n)}$ are distributed according to $\frac{d\ell}{\ell^4}$ on $(0,\infty)$.

Exercise 5.1. Understand intuitively that the natural distribution to have space filling of eddies of every size is $\frac{d\ell}{\ell^4}$ and not $\frac{d\ell}{\ell^3}$. The "invariance" below of the law of $\left(X_0^{(n)}, \ell_{(n)}\right)$ by homotheties is a technical explanation.

These sentences are not rigorous as they stand since they refer to measures which are only σ-finite, but they may be made rigorous by using Poisson point processes.

The intuitive geometric idea about $u(x)$ is that at every (small) interval of scales $[\ell, \ell + \triangle\ell]$ we see the space filled in of vortex eddies of size in $[\ell, \ell + \triangle\ell]$. And $u(x)$ is the velocity field associated to such a fluid composed of many eddies of every size.

Let us show *formally* that

$$u(\lambda x) \overset{\mathcal{L}}{=} \lambda^{1/3} u(x). \tag{6}$$

By the lemma above we have

$$\begin{aligned} u(\lambda x) &\overset{\mathcal{L}}{=} \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(\frac{X_0^{(n)}}{\lambda} + W^{(n)}\right)_{\ell_{(n)}/\lambda}^{\perp} (x) \\ &= \lambda^{1/3} \sum_{n=1}^{\infty} \left(\frac{\ell_{(n)}}{\lambda}\right)^{1/3} \left(\frac{X_0^{(n)}}{\lambda} + W^{(n)}\right)_{\ell_{(n)}/\lambda}^{\perp} (x) \\ &\overset{\mathcal{L}}{=} \lambda^{1/3} \sum_{n=1}^{\infty} \ell_{(n)}^{1/3} \left(X_0^{(n)} + W^{(n)}\right)_{\ell_{(n)}}^{\perp} (x) = \lambda^{1/3} u(x) \end{aligned}$$

where we have used the formal fact that the joint law of $\left(X_0^{(n)}, \ell_{(n)}\right)$ is invariant by homotheties:

$$E\left[\varphi\left(\frac{X_0^{(n)}}{\lambda}, \frac{\ell_{(n)}}{\lambda}\right)\right] = \int_0^\infty \int_{\mathbb{R}^3} \varphi\left(\frac{x}{\lambda}, \frac{\ell}{\lambda}\right) dx \frac{d\ell}{\ell^4}$$

$$\overset{x'=\frac{x}{\lambda}}{=} \int_0^\infty \lambda^3 \int_{\mathbb{R}^3} \varphi\left(x', \frac{\ell}{\lambda}\right) dx' \frac{d\ell}{\ell^4} = \int_{\mathbb{R}^3} \int_0^\infty \varphi\left(x', \frac{\ell}{\lambda}\right) \frac{d\left(\ell/\lambda\right)}{\left(\ell/\lambda\right)^4} dx'$$

$$\overset{\ell'=\frac{\ell}{\lambda}}{=} \int_{\mathbb{R}^3} \int_0^\infty \varphi\left(x', \ell'\right) \frac{d\ell'}{\left(\ell'\right)^4} dx' = E\left[\varphi\left(X_0^{(n)}, \ell_{(n)}\right)\right].$$

Unfortunately all these computations are not rigorous since the series defining $u(x)$ does not converge! That something is wrong can be immediately guessed from the fact that (6) implies that either $u(x)$ is identically zero, or that in some sense it is identically infinite. Indeed as $\lambda \to 0$, if we accept continuity, we get $u(0) \overset{\mathcal{L}}{=} 0 \cdot u(0)$; and on the other side $u(x)$ should be a space homogeneous random field, so $u(x) \overset{\mathcal{L}}{=} 0 \cdot u(x)$ at every x. Even without continuity, any meaning of stationarity implies that $u(x)$ and $u(\lambda x)$ should have certain equal quantities, and this is compatible only with the multiplier $\lambda = 1$. In addition, certainly it is not reasonable to believe that the field u written above is identically zero, so we have to conclude that in a sense it is infinite everywhere. We do not make this argument rigorous, since the final result is a negative one and there is no intuitive hope that the behavior is better than the one just described.

However, having a random field with the self-similarity property (6) would give us an example of random field with a K41-type property over an infinite range of r:

$$\begin{aligned} S_2(r) &= E\left[\left\|u\left(r \cdot e\right) - u\left(r \cdot 0\right)\right\|^2\right] \\ &= E\left[\left\|r^{1/3} u(e) - r^{1/3} u\left(0\right)\right\|^2\right] = r^{1/3} S_2\left(1\right). \end{aligned}$$

If $S_2(1)$ where different from zero and finite (but it is infinite) we would get K41, even without limitations on r.

All of this is formal but very instructive. One should come back to this example after having learned more about the scaling transformations that we shall perform on the stochastic Navier–Stokes equations.

In fact the only problem with the previous objects is that $\ell_{(n)}$ have distributions that extend to infinite too much, so that there are arbitrarily large and intense eddies. It is sufficient to cut-off ℓ and we get a rigorous example of random field that has the K41 scaling law. But it is not exactly self-similar: only "at small distances" (see Kupiainen [50]).

Theorem 5.1. *Given a positive real number $\ell_{\max}$, consider the σ-finite measure $\frac{d\ell}{\ell^4} 1_{l \in (0, \ell_{\max}]}$. Assume that the r.v. $\ell_{(n)}$ are distributed according to this measure. Then the random field $u(x)$ above is well defined, it has all finite* *moments, it is space homogeneous and isotropic, and its second order structure function $S_2(r)$ is bounded above and below (uniformly in $r \in (0,1)$, say) by the function*

$$f(r) = \int_0^1 \ell^{2/3} \left(\frac{\ell \wedge r}{\ell} \right)^2 \frac{d\ell}{\ell}$$

and therefore by $r^{2/3}$.

The proof is very technical (based on Burkholder-Davis-Gundy inequality, strong Markov property, arguments of potential theory similar to those of the theory of the Brownian sausage) and may be found in [36]. The meaning of the r.v.'s distributed according to only σ-finite measures is rigorously given in [36] by means of Poisson point processes, as we have already said; this is a quite technical issue so we do not give the details here.

The following modification of the previous theorem, again proved in [36], which includes a cut-off at viscous scales, may also be of interest.

Theorem 5.2. *For every $\nu \in (0, \ell_{\max})$ consider the σ-finite measure $\frac{d\ell}{\ell^4} 1_{l \in (\eta, \ell_{\max}]}$ where*

$$\eta = \eta(\nu) = \nu^{3/4}.$$

Assume that the r.v. $\ell_{(n)}$ are distributed according to this measure. Then the same conclusions of the previous theorem hold but with the function

$$f(\nu, r) = \int_\eta^1 \ell^{2/3} \left(\frac{\ell \wedge r}{\ell} \right)^2 \frac{d\ell}{\ell}$$

and therefore satisfies K41 scaling law (see section 5.2).

Is it the idealization proposed by this Brownian eddies model close to reality? We do not know the answer, simply it looks like the mental image described by Kolmogorov [46]. Let us only mention that other models give the same result. We may introduce more filament-like random vortices of the form

$$u_{\text{single}}^{(X_0, \ell, T, U)}(x) = \frac{U}{\ell^2} \int_0^T K_\ell (x - X_0 - W_t) \wedge dW_t.$$

If T is larger than ℓ^2 the displacement of the Brownian trajectory is typically longer than the cross-section ℓ. With these fields as building objects one may still construct random field with K41 law but not eddy-like. Notice that numerical simulations in the last 15 years have often shown that a turbulent fluid is rich of elongated vortex filaments (but their relevance for the statistics is not proved).

In favor of the previous model of Brownian eddies we may quote the local self-similarity, which seems to be one of the observed features of turbulent fluids and is also related to the scaling properties of the stochastic Navier–Stokes equations, as we shall see. In this respect, if we believe that more filament like objects are also present in the fluid, they could constitute a secondary object, but maybe important to explain certain intermittent features and corrections to K41.

5.5 Necessary Conditions for K41

Let us leave random vortex filaments and go back to the rigorous analysis of K41 property. We give some general results and then their application to the stochastic Navier–Stokes equations.

The first results of this subsection apply to suitable families of probability measures, without any use of the Navier–Stokes equations. They will be applied to stochastic Navier–Stokes equations at the end of the section.

Given a measure $\mu \in \mathcal{P}$, $\mu \neq \delta_0$, we introduce the number $\theta = \theta(\mu)$ defined by the identity

$$\theta^2 = \frac{\mu\left[\int_{[0,1]^d} \|Du(x)\|^2 dx\right]}{\mu\left[\int_{[0,1]^d} \|D^2u(x)\|^2 dx\right]} \tag{7}$$

with the understanding that $\theta = 0$ when $\mu\left[\int_{[0,1]^d} \left\|D^2u(x)\right\|^2 dx\right] = \infty$ and $\theta = 1$ when $\mu = \delta_0$. θ has the dimension of a length and we interpret it as *an estimate of the length scale where dissipation is more relevant.* Indeed, very roughly, from

$$\frac{\int_{\mathcal{T}} \left\|D^2u(x)\right\|^2 dx}{\int_{\mathcal{T}} \|Du(x)\|^2 dx} \sim \frac{\sum |k|^2 \left(|k|^2 |\widehat{u}(k)|^2\right)}{\sum |k|^2 |\widehat{u}(k)|^2}$$

we see that $\theta(\mu)^{-2}$ has the meaning of typical square wave length of dissipation (looking at $|k|^2 |\widehat{u}(k)|^2$ as a sort of distribution in wave space of the dissipation).

Lemma 5.3. *For every $\mu \in \mathcal{P}$ such that $\theta(\mu) > 0$ we have*

$$\frac{1}{4d} \cdot r^2 \leq \frac{S_2^\mu(r)}{\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 dx\right]} \leq r^2 \text{ for every } r \in (0, \frac{\theta(\mu)}{4d}]. \tag{8}$$

Proof. We have to use Taylor formula, but the measures μ are concentrated a priori only on $W^{2,2}$ vector fields. For sake of brevity, we give the proof under the additional assumption that

$$\mu\left(D(A) \cap C^2(\mathcal{T})\right) = 1$$

for all the measures μ involved. In [37] one may found the proof in the general case, performed by mollification.

By space homogeneity of μ

$$\mu\left[\|u(re)-u(0)\|^2\right] \le r^2 \int_0^1 \mu\left[\|Du(\sigma e)\|^2\right] d\sigma$$
$$= r^2 \mu\left[\|Du\|^2\right]$$

and thus the right-hand inequality of (8) is proved for every $r > 0$.

On the other side, for smooth vector fields we have

$$u(re) - u(0) = Du(0)\, re + r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma$$

and thus

$$\mu\left[\|Du \cdot re\|^2\right] \le 2\mu\left[\|u(re)-u(0)\|^2\right]$$
$$+ 2\mu\left[\left\|r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma\right\|^2\right].$$

Again from space homogeneity of μ,

$$\mu\left[\left\|r^2 \int_0^1 D^2 u(\sigma e)(e,e)\, d\sigma\right\|^2\right] \le r^4 \mu\left[\left\|D^2 u\right\|^2\right]$$

and from discrete isotropy we have (see the appendix of [37])

$$\mu\left[\|Du \cdot e\|^2\right] = \frac{1}{d}\mu\left[\|Du\|^2\right].$$

Therefore

$$\mu\left[\|u(re)-u(0)\|^2\right] \ge \frac{r^2}{2d}\mu\left[\|Du\|^2\right] - r^4 \mu\left[\left\|D^2 u\right\|^2\right].$$

Therefore, by definition of $\theta(\mu)$,

$$S_2(r) \ge \left(\frac{1}{2d} - \frac{r^2}{\theta(\mu)}\right)\mu\left[\|Du\|^2\right] \cdot r^2.$$

This implies the left-hand inequality of (8) for $r \in (0, \frac{\theta(\mu)}{4d}]$. The proof is complete.

Theorem 5.3. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ be a set with the following scaling property: there is a function $\widetilde{\eta} : \mathcal{M} \to \mathbb{R}_+$ (the length scale of the scaling property), with*

$$\lim_{\nu \to 0} \sup_{\mu \in \mathcal{M}_\nu} \widetilde{\eta}(\mu, \nu) = 0,$$

a scaling exponent $\alpha \in (0,2)$ *and constants* $C_2 \geq C_1 > 0$, $\nu_0 > 0$, $r_0 > 0$ *such that*

$$C_1 \cdot r^{\alpha} \leq S_2^{\mu}(r) \leq C_2 \cdot r^{\alpha} \quad \text{for } r \in [\widetilde{\eta}(\mu,\nu), r_0] \tag{9}$$

for every $\nu \in (0,\nu_0)$ *and every* $\mu \in \mathcal{M}_\nu$. *Let* $\theta(\mu)$ *be the dissipation length scale defined above.*

Then the two length scales $\theta(\mu)$ *and* $\widetilde{\eta}(\mu,\nu)$ *are related by the following property: there exist* $C > 0$, $\nu_1 > 0$ *such that*

$$\theta(\mu) \leq C \cdot \widetilde{\eta}(\mu,\nu) \tag{10}$$

for every $\nu \in (0,\nu_1)$ *and every* $\mu \in \mathcal{M}_\nu$.

For the proof we address to [37]; we do not repeat it here since it is intuitively rather clear that (8) is in contradiction with (9) if the ranges of r where the two properties hold overlap, so we need the bound (10).

Remark 5.8. The divergence of the range of r's in the definition (9) of a scaling law is essential to have a non trivial definition. If, on the contrary, we simply ask that the scaling law holds on a bounded interval $r \in [C_3\eta_\nu, C_4\eta_\nu]$, we have a definition without real interest, as it is explained by the example of section 5.1.

Let us finally state two general consequences of the previous theorem, that we shall apply to stochastic Navier–Stokes equations.

Corollary 5.1. *Given a family* $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, *if*

$$\inf_{(\mu,\nu)\in\mathcal{M}} \theta(\mu) > 0$$

then no scaling law in the sense of the previous theorem may hold true.

Corollary 5.2. *Let* $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ *be a family with the K41 scaling law, in the sense of definition 5.1. Then there exists* $\nu_0 > 0$ *and* $C > 0$ *such that*

$$\mu\left[\int_{\mathcal{T}} \left\|D^2 u(x)\right\|^2 dx\right] \geq C\varepsilon_0^{3/2} \cdot \nu^{-5/2}$$

for every $\nu \in (0,\nu_0)$ *and every* $\mu \in \mathcal{M}_\nu$.

Proof. From (10), the definition of $\eta(\mu,\nu)$ and the definition of $\theta^2(\mu)$ we have

$$\frac{\mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 dx\right]}{\mu\left[\int_{\mathcal{T}} \|D^2 u(x)\|^2 dx\right]} \leq C\nu^{3/2}\eta_0^2$$

Thus, from the definition of ε_0,

$$\frac{\varepsilon_0}{\mu\left[\int_{\mathcal{T}} \|D^2 u(x)\|^2 dx\right]} \leq C\nu^{5/2}\eta_0^2.$$

This implies the claim of the Corollary. The proof is complete.

Remark 5.9. Dimensional analysis says that ν has dimension $[L]^2 [T]^{-1}$, ε has dimension $[L]^2 [T]^{-3}$, so $\varepsilon_0^{3/2} \cdot \nu^{-5/2}$ has dimension $[L]^{-2} [T]^{-2}$, the correct dimension of $\mu \left[\int_{\mathcal{T}} \left\| D^2 u(x) \right\|^2 dx \right]$.

Remark 5.10. The previous and next corollaries are based only on the scaling exponents of $\eta(\mu, \nu)$, not on the exponent $2/3$ in definition 5.1. Therefore, any other $\alpha \in (0,2)$ in place of $2/3$ would give us the same result.

Let us first apply the general results to disprove K41 in the 2D case. The following result is well known.

Lemma 5.4. *Let μ be an invariant measure of (1) $(d = 2)$ such that*

$$\mu \left[\int_{\mathcal{T}} \| Du(x) \|^2 dx \right] < \infty.$$

Then $\mu \in \mathcal{P}_0$ and

$$\nu \cdot \mu \left[\int_{\mathcal{T}} \| Du(x) \|^2 dx \right] = \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i^2$$

$$\nu \cdot \mu \left[\int_{\mathcal{T}} \| D\mathsf{curl} u(x) \|^2 dx \right] = \frac{1}{2} \sum_{i=1}^{\infty} \sigma_i^2 \lambda_i.$$

Since $\int_{\mathcal{T}} \left\| D^2 u \right\|^2 dx = \int_{\mathcal{T}} \| D\mathsf{curl} u \|^2 dx$, we readily have:

Corollary 5.3. *In 2D, there exists a positive constant θ_0, independent of ν, such that*

$$\theta(\mu) = \theta_0$$

for every invariant measure $\mu \in \mathcal{P}$ of (1). Hence a family of invariant measures $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ of (1) cannot have any scaling law (in the sense of (9).

Remark 5.11. Under our assumptions on the noise, invariant measures of (1) that belong to $\mathcal{P}$ certainly exist. In principle there could be others not in $\mathcal{P}$, but this cannot happen in all those cases when uniqueness of the invariant measure is known (see [53] and the references therein).

Remark 5.12. Consider equation (1) without the nonlinear term (called Stokes equations):

$$du(t) + \nu A u(t) dt = \sum_{i=1}^{\infty} \sigma_i h_i d\beta_i(t).$$

in dimension $d = 2, 3$. Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$ be a family of invariant measures for it. Then the same results of the previous theorem hold true. The proof is the same. Alternatively, one may work componentwise in h_i and prove easily the claims.

Let us treat now the 3D case. Recall the concept of *Galerkin stationary measures* introduced at the end of Section 3 and the notations $\mathcal{P}_{NS}^{Galerkin}(\nu)$ for the set of all such measures and $\mathcal{S}^n(\nu)$ for the invariant measures of the approximating Galerkin system (we underline here the dependence on ν).

Given $u \in V$, let S_u be the tensor with $L^2(\mathcal{T})$ components

$$S_u = \frac{1}{2}\left(Du + Du^T\right)$$

(called stress tensor). The scalar field

$$S_u(x)\,\mathsf{curl}u(x) \cdot \mathsf{curl}u(x)$$

describes the *stretching of the vorticity field*. If we set $\xi = \mathsf{curl}u$, then formally we have

$$\frac{\partial \xi}{\partial t} + (u \cdot \nabla)\xi = \nu \triangle \xi + S_u \xi + \sum_{i=1}^{\infty} \sigma_i (\mathsf{curl}h_i)\, \dot{\beta}_i(t).$$

A *formal* application of Itô formula yields the inequality

$$\nu \cdot \mu \int_{\mathcal{T}} \|D\mathsf{curl}u\|^2 dx \le \mu \int_{\mathcal{T}} S_u \mathsf{curl}u \cdot \mathsf{curl}u\, dx + \frac{1}{2}\sum_{i=1}^{\infty} \sigma_i^2 \lambda_i. \tag{11}$$

for $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ (in fact formally the identity). Along with the general results proved above we would get

$$\mu\left[\int_{\mathcal{T}} S_u(x)\,\mathsf{curl}u(x) \cdot \mathsf{curl}u(x)\, dx\right] \ge C\varepsilon_0^{3/2}\nu^{-3/2}. \tag{12}$$

This would be the final result of this section, having an interesting physical interpretation. However, we cannot prove this result in this form, without further assumptions. We give, without proof (see [37]), two results around (12): theorem 5.4 reformulates it for the coarse graining scheme given by Galerkin approximations; theorem 5.4 expresses the most natural statement directly for $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ but it requires an additional unproved regularity assumption.

Theorem 5.4. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law. Then there exists $\nu_0 > 0$ and $C > 0$ such that*

$$\liminf_{k\to\infty} \mu_{n_k}\left[\int_{\mathcal{T}} S_u \mathsf{curl}u \cdot \mathsf{curl}u dx\right] \ge C\varepsilon_0^{3/2}\nu^{-3/2}$$

for every $\nu \in (0,\nu_0)$, every $\mu \in \mathcal{M}_\nu$ and every sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that μ_{k_n} converges to μ on H.

Lemma 5.5. *If $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$ is the weak limit of a sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that $\mu_{n_k}\left[\|\cdot\|_V^{2+\varepsilon}\right] \leq C$ for some $\varepsilon, C > 0$, then*

$$\nu \cdot \mu\left[\int_{\mathcal{T}} \|Du(x)\|^2 \, dx\right] = \frac{1}{2}\sum_{i=1}^{\infty} \sigma_i^2.$$

If in addition $\mu_{n_k}\left[\|\cdot\|_V^{3+\varepsilon}\right] \leq C$ then (11) holds true.

Corollary 5.4. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law. Assume that every μ in $\mathcal{M}$ is the weak limit of a sequence $\mu_{n_k} \in \mathcal{S}^{k_n}(\nu)$ such that*

$$\mu_{n_k}\left[\|\cdot\|_V^{3+\varepsilon}\right] \leq C$$

for some $\varepsilon, C > 0$. Then there exists $\nu_0 > 0$ and $C > 0$ such that (12) holds for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.

Remark 5.13. If K41 scaling law holds then vortex stretching must be intense. Heuristically, no geometrical depletion of such stretching may occur (in contrast to the 2D case where the stretching term is zero because $\mathsf{curl}u(x)$ is aligned with the eigenvector of eigenvalue zero of $S_u(x)$): indeed, if we extrapolate the behavior $E\left[|Du|^2\right] \sim \frac{1}{\nu}$ as $Du \sim \frac{1}{\sqrt{\nu}}$, $\mathsf{curl}u \sim \frac{1}{\sqrt{\nu}}$, then we get $E\left[S_u \mathsf{curl}u \cdot \mathsf{curl}u\right] \sim \frac{1}{\nu\sqrt{\nu}}$ if there is no help from the geometry. Another way to explain this idea is the following sort of generalized Hölder inequality (for the proof, see [37]).

Corollary 5.5. *Let $\mathcal{M} \subset \mathcal{P} \times \mathbb{R}_+$, with $\mathcal{M}_\nu \subset \mathcal{P}_{NS}^{Galerkin}(\nu)$, be a family with the K41 scaling law, fulfilling the assumptions of theorem 5.4. Then there exists $\nu_0 > 0$ and $C > 0$ such that*

$$\left(\mu \int_{\mathcal{T}} \|Du\|^2 \, dx\right)^{1/2} \leq C\left(\mu\left[\int_{\mathcal{T}} \|S_u \mathsf{curl}u \cdot \mathsf{curl}u\|^2 \, dx\right]\right)^{1/3}$$

for every $\nu \in (0, \nu_0)$ and every $\mu \in \mathcal{M}_\nu$.

5.6 A Condition Equivalent to K41

We continue with the notations and concepts just introduced in the last section on the 3D case. Let $u(t, x)$ be a solution of equation (1) *on the unitary torus* ($L = 1$). We analyze the K41 property for it. Given $L > 0$, consider the new fields (see Kupiainen [50])

$$u_L(t, x) = L^{1/3} u(L^{-2/3}t, L^{-1}x)$$

and $p_L(t, x) = L^{2/3} p\left(L^{-2/3}t, L^{-1}x\right)$. To help the intuition, think that L is large so we blow-up the solution u. Formally, these fields satisfy the equations on the torus of size L

$$\frac{\partial u_L}{\partial t} + (u_L \cdot \nabla) u_L + \nabla p_L = \nu_L \triangle u_L + \sum_{i=1}^{\infty} \sigma_i h_i \left(\frac{x}{L}\right) \dot{\beta}_i^L (t) \tag{13}$$

where h_i were the eigenfunctions of the Stokes operator on the unitary torus, $\beta_i^L(t)$ are the independent Brownian motions

$$\beta_i^L(t) = L^{1/3} \beta_i \left(L^{-2/3} t\right)$$

and

$$\nu_L = \nu L^{4/3}.$$

The heuristic proof of this fact is a simple exercise: all terms $\frac{\partial u_L}{\partial t}$, $(u_L \cdot \nabla) u_L$, etc. are equal to $L^{-1/3}$ times the analogous terms $\frac{\partial u}{\partial t}$, $(u \cdot \nabla) u$, etc., and formally

$$\dot{\beta}_i^L(t) = L^{1/3} \dot{\beta}_i \left(L^{-2/3} t\right) L^{-2/3} = L^{-1/3} \dot{\beta}_i \left(L^{-2/3} t\right).$$

The same computation can be performed for the more general transformation

$$u_{(\lambda,\alpha)}(t,x) = \lambda^{\alpha} u\left(\lambda^{\alpha+1} t, \lambda x\right), \quad p_{(\lambda,\alpha)}(t,x) = \lambda^{2\alpha} p\left(\lambda^{\alpha+1} t, \lambda x\right)$$

but the previous choice of exponents is the only one such that the energy input per unit time and space is the same for every L, or λ (no coefficient depending on the scale parameter appears in front of the noise). Heuristically, if we believe in a uniform (not spiky, not intermittent) cascade picture of the energy (without essential inverse cascade), this invariance of the energy input should imply that the small scale properties of (1) (on the unitary torus) and (13) are the same, namely that they are invariant under this transformation; so we should expect that in the stationary regime $u_L(x)$ and $u(x)$ have approximatively the same law. But this would imply that $L^{1/3} u(L^{-1}x)$ and $u(x)$ have approximatively the same law, namely $u(Lx) \sim L^{1/3} u(x)$. Such scaling property would imply K41.

Let us stress again that not only we cannot prove claims like $u(Lx) \sim L^{1/3} u(x)$ but we do not believe they are exactly true. Presumably the correct result is closer to $u(Lx) \sim L^{1/3+k} u(x)$ for some $k > 0$.

Let us denote by $\mathcal{P}_{NS}^{Galerkin}(\nu)$ the family of Galerkin stationary measures for (1) on the unitary torus. Similarly, given L and a number $\tilde{\nu}$ (not necessarily equal to $\nu L^{4/3}$), let us denote by $\mathcal{P}_{NS}^{Galerkin}(\tilde{\nu}, L)$ the family of Galerkin stationary measures for equation (13) on the torus of size L, where we replace the symbol ν_L by $\tilde{\nu}$.

Let us denote by $\mathcal{P}_{NS}^{Galerkin} \times \mathbb{R}_+$ the set of *all pairs* (μ, ν) such that $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$. Similarly, let us denote by $\widetilde{\mathcal{P}}_{NS}^{Galerkin} \times \mathbb{R}_+^2$ the set of *all triples* $(\widetilde{\mu}, \tilde{\nu}, L)$ such that $\widetilde{\mu} \in \widetilde{\mathcal{P}}_{NS}^{Galerkin}(\tilde{\nu}, L)$.

Let $\mathcal{P}$ be the set of measures of the previous sections relative to the unitary torus. Let $\widetilde{\mathcal{P}}_L$ for the set of probability measures analogous to $\mathcal{P}$, but on the torus $[0, L]^3$. Denote by $\widetilde{\mathcal{P}} \times \mathbb{R}_+^2$ the set of all triples $(\widetilde{\mu}, \tilde{\nu}, L)$ such that $(\tilde{\nu}, L) \in \mathbb{R}_+^2$ and $\widetilde{\mu} \in \widetilde{\mathcal{P}}_L$. In the next definition and later on we use the notation $\widetilde{\mu}\left[\|u(e) - u(0)\|^2\right]$ when $\widetilde{\mu} \in \widetilde{\mathcal{P}}_L$ (and other similar mean values): this means

$$\widetilde{\mu}\left[\|u(e) - u(0)\|^2\right] = \int_{H_L} \|u(e) - u(0)\|^2 \, d\widetilde{\mu}(u)$$

where H_L is the usual space H but on the torus $[0, L]^3$.

The following condition seems interesting since it looks rather qualitative, in contrast to the definition of the K41 law, and shows that the exponent 2/3 arises from the scaling properties of the stochastic Navier–Stokes equations. Also the exponent in the range of r's arises spontaneously from this transformation.

Condition. *A subset $\tilde{\mathcal{M}} \subset \widetilde{\mathcal{P}} \times \mathbb{R}_+^2$ is said to satisfy Condition A if there exist $\tilde{\nu}_0 > 0$, $L_0 > 0$, $C > c > 0$ such that*

$$c \le \widetilde{\mu}\left[\|u(e) - u(0)\|^2\right] \le C \tag{14}$$

for every $(\widetilde{\mu}, \tilde{\nu}, L) \in \tilde{\mathcal{M}}$ with $\tilde{\nu} \le \tilde{\nu}_0$, $L \ge L_0$.

Theorem 5.5. *The set $\widetilde{\mathcal{P}}_{NS}^{Galerkin} \times \mathbb{R}_+^2$ satisfies Condition A if and only if the set $\mathcal{P}_{NS}^{Galerkin} \times \mathbb{R}_+$ has a scaling law of K41 type, in the sense of Definition 5.1.*

Proof. **Step 1** (preparation). The proof is simple but notationally non trivial. The statement of K41 property involves two parameters, (ν, r), subject to the following constraints:

$$\nu \le \nu_0 \,, \qquad r \in [\eta_0 \nu^{3/4}, r_0].$$

Hence we deal with the region

$$K_{\nu_0, r_0} = \left\{(\nu, r) \in \mathbb{R}_+^2 : \nu < \nu_0 \,, r \in [\left(\eta_0 \nu^{3/4}, r_0\right)\right\}$$

It is not restrictive to assume $r_0 = \nu_0^{3/4} \eta_0$, so the region K_{ν_0, r_0} looks like a right-angled triangle with a round hypotenuse (we suggest the reader to draw a picture of this set in the plane $\nu - r$).

Condition A involves two other parameters, $(\tilde{\nu}, L)$, subject to the constraint

$$\tilde{\nu} \le \tilde{\nu}_0, \qquad L \ge L_0.$$

Hence, in condition A, we deal with the region

$$D_{\tilde{\nu}_0, L_0} = \left\{(\tilde{\nu}, L) \in \mathbb{R}_+^2 : \tilde{\nu} < \tilde{\nu}_0, L > L_0\right\}.$$

Such a region is a vertical semi-strip open upwards. Let us introduce the transformation $f : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined as

$$f(\nu, r) = \left(\nu r^{-4/3}, r^{-1}\right)$$

which is invertible, with inverse given by

$$f^{-1}(\tilde{\nu}, L) = \left(\tilde{\nu} L^{-4/3}, L^{-1}\right).$$

We have

$$f(K_{\nu_0,r_0}) = D_{\tilde{\nu}_0,L_0}$$

if $\tilde{\nu}_0 = \eta_0^{-4/3}$, $L_0 = r_0^{-1}$. The piece of the curve $r = \eta_0 \nu^{3/4}$ pertaining to the boundary of K_{ν_0,r_0} is mapped into the vertical half-line $\tilde{\nu} = \tilde{\nu}_0, L > L_0$ of the boundary of $D_{\tilde{\nu}_0,L_0}$. The horizontal boundary segment of K_{ν_0,r_0} is mapped into the horizontal boundary segment of $D_{\tilde{\nu}_0,L_0}$. The vertical boundary segment of K_{ν_0,r_0} is mapped into the vertical half-line $\tilde{\nu} = \eta_0^{-4/3}, L > L_0$ of $D_{\tilde{\nu}_0,L_0}$.

Given any $L > 0$, let us also consider the mapping $S_L : \mathcal{H}_L \to \mathcal{H}$ defined by (see the scaling transformation above)

$$u(x) = L^{-1/3} \widetilde{u}(Lx).$$

It is possible to prove rigorously (see [37]) that

$$S_L\left(\widetilde{\mathcal{P}}_{NS}^{Galerkin}(\tilde{\nu}, L)\right) = \mathcal{P}_{NS}^{Galerkin}(\nu)$$
$$\text{for every } (\nu, \tilde{\nu}, L) \in \mathbb{R}^3_+ \text{ such that } \nu = \tilde{\nu} L^{-4/3}.$$

The heuristic has been given above before the theorem.

Step 2 (Condition A implies K41). Given $\tilde{\nu}_0, L_0$ in the definition of Condition A, choose ν_0, r_0 such that $f(K_{\nu_0,r_0}) \subset D_{\tilde{\nu}_0,L_0}$. Given $(\nu, r) \in K_{\nu_0,r_0}$ and $\mu \in \mathcal{P}_{NS}^{Galerkin}(\nu)$, let $(\tilde{\nu}, L) = f(\nu, r)$, so that $\nu = \tilde{\nu} L^{-4/3}$, and denote by $\widetilde{\mu}$ the measure in $\widetilde{\mathcal{P}}_{NS}^{Galerkin}(\tilde{\nu}, L)$ such that $\mu = S_{r^{-1}} \widetilde{\mu}$. We have

$$\begin{aligned} S_2^{\mu}(r) &= \int_H \|u(re) - u(0)\|^2 \, d\mu(u) \\ &= \int_H \|u(re) - u(r0)\|^2 \, d(S_{r^{-1}} \widetilde{\mu})(u) \\ &= \int_{H_{r^{-1}}} \left\| r^{1/3} \widetilde{u}(e) - r^{1/3} \widetilde{u}(0) \right\|^2 d\widetilde{\mu}(\tilde{u}) \\ &= r^{2/3} \int_{H_{r^{-1}}} \|\widetilde{u}(e) - \widetilde{u}(0)\|^2 \, d\widetilde{\mu}(\tilde{u}). \end{aligned}$$

By Condition A, $\int_{H_{r^{-1}}} \|\widetilde{u}(e) - \widetilde{u}(0)\|^2 \, d\widetilde{\mu}(\tilde{u})$ is bounded between two constants, hence we get K41.

Step 3 (K41 implies Condition A). The proof proceeds like step 2 but in the opposite direction and is left to the reader.

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Mathematical Results Related to the Navier–Stokes System

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1 Introduction

The d–dimensional Navier–Stokes System, $d \geqslant 2$, is written usually for $d+1$ unknown functions $u(x,t) = (u_1(x,t), u_2(x,t), \dots, u_d(x,t))$, $p(x,t)$ where $x = (x_1, \dots, x_d)$, $t \geqslant 0$ and it has the form

$$\begin{aligned} &\frac{\partial u_i}{\partial t} + \sum_{k=1}^{d} \frac{\partial u_i}{\partial x_k} u_k(x,t) = \nu \Delta u_i - \frac{\partial p}{\partial x_i} + f_i(x,t), \qquad i = 1, \dots, d \\ &\operatorname{div} u = \sum_{i=1}^{d} \frac{\partial u_i}{\partial x_i} = 0. \end{aligned} \tag{1}$$

The last equation is called the incompressibility condition and assumes the density $\rho \equiv 1$. The coefficient ν is called the viscosity. In this paper we take $\nu = 1$ unless something else is mentioned.

The vector $u(x,t)$ describes the velocity of the moving gas or liquid, $p = p(x,t)$ is the pressure, $f = (f_1(x,t), \dots, f_d(x,t))$ is the vector of external forces which is a given function of x, t. Usually people consider three cases: I) $x \in \mathbb{R}^d$; II) $x \in \mathbb{T}^d$; III) $x \in Q \subset \mathbb{R}^d$ where Q is a compact domain with a smooth boundary and $u(x,t) = 0$, $x \in \partial Q$ is a non-slip boundary condition.

It is believed that (1) describes the dynamics of an uni-phase gas. So the phenomena like clouds, rain, snow, etc. are not described by the system (1) and require more complicate equations. In the first part of these lectures we deal mostly with the case I and $f \equiv 0$. Thus we study the dynamics of a viscous fluid on the whole space when external forcing is absent. Presumably, in this case we have a reasonable approximation to the dynamics of a dry air in a big desert which has a purely kinematic character. However, in the deserts such phenomena like tornados are possible and it is conceivable that solutions of (1) can describe them to some extent.

The basic property of (1) is the energy inequality. We shall derive it for complex solutions of (1). Write down the system (1) in a sightly modified form

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{d}\frac{\partial \bar u_i}{\partial x_k}u_k(x,t)=\nu\Delta u_i-\frac{\partial p}{\partial x_i}+f_i(x,t),\qquad i=1,\dots,d \tag{$1'$}$$

in ($1'$) it is assumed that $u(x,t)$ is a complex-valued function of (x,t). Put

$$E(u)=\frac{1}{2}\int |u(x,t)|^2dx \tag{2}$$

where $|u(x,t)|=\sum\limits_{i=1}^{d}u_i(x,t)\bar u_i(x,t)$. Then from ($1'$) under assumption $f\equiv 0$

$$\frac{dE}{dt}=\frac{1}{2}\int\limits_{\mathbb{R}^d}\left[\sum_{i=1}^{d}\frac{\partial u_i(x,t)}{\partial t}\bar u_i(x,t)+\frac{\partial \bar u_i(x,t)}{\partial t}u_i(x,t)\right]dx$$

$$=\frac{1}{2}\int\limits_{\mathbb{R}^d}\left[\nu\sum_{i=1}^{d}\Delta u_i(x,t)\bar u_i(x,t)+\nu\sum_{i=1}^{d}\Delta\bar u_i(x,t)u_i(x,t)\right.$$

$$-\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{\partial \bar u_i(x,t)}{\partial x_k}u_k(x,t)\bar u_i(x,t)-\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{\partial u_i(x,t)}{\partial x_k}\bar u_k(x,t)u_i(x,t)$$

$$\left.-\sum_{i=1}^{d}\frac{\partial p(x,t)}{\partial x_i}\bar u_i(x,t)-\sum_{i=1}^{d}\frac{\partial \bar p(x,t)}{\partial x_i}u_i(x,t)\right]$$

The integration by parts of the first two terms gives $-2\nu\int\sum\limits_{i=1}^{d}|\nabla u_i|^2dx$ which is always negative. The third term $\sum\limits_{i=1}^{d}\sum\limits_{k=1}^{d}\frac{\partial \bar u_i(x,t)}{\partial x_k}u_k(x,t)\bar u_i(x,t)=\frac{1}{2}\sum\limits_{i=1}^{d}\frac{\partial}{\partial x_i}\sum\limits_{k=1}^{d}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_k}\bar u_i^2(x,t)u_k(x,t)$ and after the integration by parts gives zero in view of incompressibility condition. The same is true for $\sum\limits_{i=1}^{d}\sum\limits_{k=1}^{d}\frac{\partial u_i(x,t)}{\partial x_k}u_i(x,t)\bar u_k(x,t)$.

The last term also is zero in view of the incompressibility condition. Thus

$$\frac{dE}{dt}=-2\nu\int\sum_{i=1}^{d}|\nabla u_i(x,t)|^2dx\leqslant 0$$

which is the main energy inequality.

As was mentioned before, in the first part of these lectures we consider $d=3$ and $f\equiv 0$. It will be convenient to make the Fourier transform

$$v(k,t) = \int_{\mathbb{R}^3} \exp\{-i\langle k,x\rangle\} u(x,t)dx$$

and to write down the equation for $v(k,t)$ equivalent to (1):

(3)

$$v(k,t) = e^{-|k|^2 t} v(k,0) + i \int_0^t \exp\{-(t-s)|k|^2\} ds \int_{\mathbb{R}^3} \langle k, v(k-k',s)\rangle P_k v(k',s) dk'$$

The incompressibility condition takes the form $v(k,t) \perp k$ for any $k \neq 0$. For this reason the pressure does not appear in (3) but instead we consider the space of functions $v(k)$, $v(k) \perp k$, $k \in \mathbb{R}^3$ as the phase space of the dynamical system corresponding to (3).

The properties of solutions of (3) depend on the functional space in which (3) is studied. We shall call strong solutions of (3) on the interval $[0,T]$ functions $v(k,t)$, $0 \leqslant t \leqslant T$, such that the integrals

$$\int_{\mathbb{R}^3} |v(k-k',s)||v(k',s)|dk'$$

are uniformly bounded in s and the rhs equals the lhs. This definition is slightly different from the one proposed by T. Kato (see [K1]).

We need the subspaces $\Phi(\alpha,\omega)$ introduced in [S1]. Namely

Definition 1. $\{v(k),\ k \in \mathbb{R}^3\} \in \Phi(\alpha,\omega)$ if for some constants C, D

1. $|v(k)| \leqslant \dfrac{C}{|k|^\alpha}$, $|k| \leqslant 1$;
2. $|v(k)| \leqslant \dfrac{D}{|k|^\omega}$, $|k| \geqslant 1$.

In this definition instead of 1 we could take any positive number and α, ω satisfy the inequalities $\alpha \geqslant 2$, $\omega < 3$. Infimum of all possible $C + D$ can be considered as some norm in the space $\Phi(\alpha,\omega)$.

In the spaces $\Phi(\alpha,\omega)$, $\alpha > 2$, $\omega < 3$ a local existence theorem is valid (see [S1]). Namely, for any $v(k,0) \in \Phi(\alpha,\omega)$ one can find $t_0 = t_0(v(k,0))$ such that (3) has a unique solution on $[0,t_0]$ belonging to the space $\Phi(\alpha,\omega)$.

In the space $\Phi(2,2)$ even a stronger statement is valid. Let $v(k,0) = \dfrac{c(k,0)}{|k|^2}$ and $\sup|c(k,0)| = \|c(k,0)\| \leqslant c_0$ where c_0 is sufficiently small. Then there exists the unique solution $v(k,t)$ of (3) defined for all $t > 0$. This theorem was proven by Le Jan and Sznitman (see [LJS]) and by M. Cannone and F. Planchon (see [CP]). In [S1] a short proof of this theorem was given.

Hypothesis 1. For "typical" $v(k,0) = \dfrac{c(k,0)}{|k|^2}$ and large $\|c(k,0)\|$ even the local existence theorem in not valid.

For any $d > 3$ the space of functions $v(k,0) = \dfrac{c(k,0)}{|k|^{d-1}}$ has the same properties as $\Phi(2,2)$ for $d = 3$ and is called critical space.

HYPOTHESIS 2. A limiting orbit of any solution of (3) from $\Phi(2,2)$ with small initial condition is periodic (or fixed point).

After Fourier transform the functions $v(k,t)$ take values in the space $\mathbb{C}^3$. We shall allow $v(k,t)$ to be arbitrary $\mathbb{C}^3$–valued functions on $\mathbb{R}^3$, which is equivalent to the assumption that we allow to consider complex-valued solutions of (1).

Take $v(k,0) = \dfrac{c(k,0)}{|k|^{\alpha}} \in \Phi(\alpha,\alpha)$, $\alpha = 2 + \mathbb{E}$ and $\mathbb{E} > 0$, $\|c(k,0)\| = 1$ and introduce a one-parameter family of initial conditions $v_A(k,0) = \dfrac{Ac(k,0)}{|k|^{\alpha}}$ where A is a complex number. According to the local existence theorem an interval of time where the solution exists is $t_0(A)$. In [S2] the following theorem was proven.

Theorem 1.1. *The local existence theorem is valid on the time interval* $[0,t]$ *such that* $|\lambda| \leqslant \lambda_0(\alpha)$ *where* $\lambda = At^{\frac{\mathbb{E}}{2}}$ *and* $\lambda_0(\alpha)$ *is an absolute constant depending only on* α*.*

The proof of this theorem is based on the method of iterations. In the case of (3) the iterations are defined by the formula

$$c^{(n)}(k,t) = \exp\{-t|k|^2\}c(k,0) + i\int_0^t \exp\{-(t-s)|k|^2\}ds \int_{\mathbb{R}^3} \frac{\langle k, c^{(n-1)}(k-k',s)\rangle P_k c^{(n-1)}(k',s)dk'}{|k-k'|^{\alpha}|k'|^{\alpha}} \tag{4}$$

First we show that if $|\lambda| \leqslant \lambda_0(\alpha)$ then $\|c^{(n)}\| \leqslant 2A$ for all $n \geqslant 1$. Then we show that the $c^{(n-1)} \to c^{(n)}$ given by (4) is a contraction. This gives the result.

A similar statement in the 2–dimensional case was proven by M. Arnold for $\dfrac{1}{2} < \alpha < 1$ (see [A1]).

2 Power Series and Diagrams for the Navier–Stokes-System

As was mentioned before, in the first part of these lectures the previous results show $\lambda = At^{\frac{\mathbb{E}}{2}}$ is the ruling parameter in $\Phi(\alpha,\alpha)$. Therefore it is natural to construct power series of λ which give solutions of (3). We write such a series in the form:

(1)

$$v_A(k,t)=A\left(\exp\{-t|k|^2\}v(k,0)+\sum_{p\geqslant 1}A^p\int_0^t\exp\{-(t-s)|k|^2\}s^{\frac{p\mathbb{E}}{2}}g_p(k\sqrt{s},s)ds\right)$$

Put $\tilde{k}=k\sqrt{s}$, $\tilde{k}'=k'\sqrt{s}$, $s_1=s\tilde{s}_1$, $s_2=s\tilde{s}_2$. Then

$$g_1(\tilde{k},s)=i\int_{\mathbb{R}^3}\frac{\langle\tilde{k},c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_k c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)e^{-|\tilde{k}'|^2-|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha|\tilde{k}'|^\alpha}$$

$$g_2(\tilde{k},s)=$$
$$i\left[\int_0^t\tilde{s}_1^{\mathbb{E}}d\tilde{s}_1\int_{\mathbb{R}^3}\frac{\langle\tilde{k},g_1((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\tilde{s}_1)\rangle P_k c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)e^{-|\tilde{k}'|^2-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}'|^\alpha}\right.$$
$$\left.+\int_0^t\tilde{s}_2^{\mathbb{E}}d\tilde{s}_2\int_{\mathbb{R}^3}\frac{\langle\tilde{k},c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_k g_1(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)e^{-(1-\tilde{s}_2)|\tilde{k}'|^2-|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}\right]$$

and in the general case

(2)

$$g_p(\tilde{k},s)=$$
$$i\left[\int_0^t\tilde{s}_1^{\frac{p\mathbb{E}}{2}}d\tilde{s}_1\int_{\mathbb{R}^3}\frac{\langle\tilde{k},c\left(\frac{\tilde{k}-\tilde{k}'}{\sqrt{s}},0\right)\rangle P_k g_{p-1}(\tilde{k}'\sqrt{\tilde{s}_1},s\tilde{s}_1)e^{-(1-\tilde{s}_1)|\tilde{k}'|^2-|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}-\tilde{k}'|^\alpha}\right.$$
$$+\sum_{\substack{p_1,p_2\geqslant 1\\ p_1+p_2=p-1}}\int_0^t\tilde{s}_1^{\frac{(p_1+1)\mathbb{E}}{2}}d\tilde{s}_1\int_0^t\tilde{s}_2^{\frac{(p_2+1)\mathbb{E}}{2}}d\tilde{s}_2\int_{\mathbb{R}^3}\langle\tilde{k},g_{p_1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_1},s\tilde{s}_1)\rangle$$
$$P_k g_{p_2}(\tilde{k}'\sqrt{\tilde{s}_2},s\tilde{s}_2)e^{-(1-\tilde{s}_2)|\tilde{k}'|^2-(1-\tilde{s}_1)|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'$$
$$\left.+\int_0^t\tilde{s}_2^{\frac{p\mathbb{E}}{2}}d\tilde{s}_2\int_{\mathbb{R}^3}\frac{\langle\tilde{k},g_{p-1}((\tilde{k}-\tilde{k}')\sqrt{\tilde{s}_2},s\tilde{s}_2)\rangle P_k c\left(\frac{\tilde{k}'}{\sqrt{s}},0\right)e^{-|\tilde{k}'|^2-(1-\tilde{s}_2)|\tilde{k}-\tilde{k}'|^2}d\tilde{k}'}{|\tilde{k}'|^\alpha}\right]$$

We shall discuss these relations under the

Main Assumption:

The function $c(k,0)\equiv 0$ if $|k|\geqslant R$. If $|k|\leqslant R$ then $|c(k,0)|\leqslant C|k|^\alpha$. Here C and R are arbitrary positive constants.

This assumption means that the initial velocity $v(k,0)\equiv 0$ if $|k|\geqslant R$ and $|v(k,0)|\leqslant C$ if $|k|\leqslant R$.

Important Remark. Take any bounded function with compact support $v(k,0)$. We can embed it into any space $\Phi(\alpha,\alpha)$ and the corresponding function

$c(k,0)$ will depend on α: $v(k,0) = \dfrac{c^{(\alpha)}(k,0)}{|k|^\alpha}$. The series also depends on α. However, it is easy to check that the result $v(k,t)$ does not depend on α.

In [S2] the estimates of $g_p(\tilde{k},s)$ were derived which imply that the series (1) converges if $|\lambda| \leqslant \lambda_1(\alpha)$ where λ_1 is a constant depending only on α.

The series (1) can be expressed as a sum of multi-dimensional integrals which we call diagrams.

Consider any term in (2). Using (2) we can express g_{p_1}, g_{p_2} through g_{q_1}, g_{q_2}, $q_1 < p_1$; $q_2 < p_2$ and so on. As a result g_p can be written as a sum of multi-dimensional integrals involving the products of p initial conditions $c(\cdot,0)$ with different values of the arguments.

Each integral corresponds to some choice of terms in (2). It is easy to show that the number of diagrams grows with p not faster than exponentially.

We shall explain in more detail the structure of diagrams (see [S3]). Each diagram is determined by a scheme. Any scheme is a sequence of partitions. We start with the set $[1,2,\dots,p{+}1] = \Delta^{(0)}$ and decompose it onto two subsets $\Delta^{(j_1)}$, $j_1 = 1$ or 2, where $\Delta^{(1)} = [1,\dots,p_1]$, $\Delta^{(2)} = [p_1+1,\dots,p+1]$.

In the same way each subset $\Delta^{(j_1)}$ can be decomposed onto two subsets $\Delta^{(j_1,j_2)}$, $j_2 = 1$ or 2 and so on.

Elements of partitions which appear in this way are denoted by $\Delta^{(j_1,j_2,\dots,j_m)}$, $p(j_1,j_2\dots,j_m)$ is the number of integer points belonging to $\Delta^{(j_1,j_2,\dots,j_m)}$.

An element $\Delta^{(j_1,j_2,\dots,j_m)}$ is called final if it has only one point and thus it cannot be decomposed onto smaller parts. The whole sequence of partitions is called a scheme.

Take a scheme $S(p{+}1)$. For each non-final $\Delta^{(j_1,j_2,\dots,j_m)}$ introduce a variable $\tilde{s}(j_1,\dots,j_m)$, $0 \leqslant \tilde{s}(j_1,\dots,j_m) \leqslant 1$. The integral

$$\Lambda(S(p+1)) = \prod\nolimits^{(nf)} \int_0^1 (\tilde{s}(j_1,\dots,j_m))^{\frac{p(j_1,\dots,j_m)\mathbb{E}}{2}} d\tilde{s}(j_1,\dots,j_m)$$

is called the partition function of the diagram, $\prod^{(nf)}$ is the product over all non-final $\Delta^{(j_1,\dots,j_m)}$. There are diagrams for which $\Lambda(S(p+1))$ decay faster that exponentially and there are the other ones for which $\Lambda(S(p{+}1))$ decay exponentially with p. Introduce the rescaling $\tilde{s}(j_1,\dots,j_m) = 1 - \dfrac{2\theta(j_1,\dots,j_m)}{p(j_1,\dots,j_m)\mathbb{E}}$ and write

$$\begin{aligned}&\prod\nolimits^{(nf)} (\tilde{s}(j_1,\dots,j_m))^{\frac{p(j_1,\dots,j_m)\mathbb{E}}{2}}\\ &= \Lambda(S(p+1))\prod\nolimits^{(nf)}\left(1+\frac{2}{p(j_1,\dots,j_m)\mathbb{E}}\right)\left(1-\frac{2\theta(j_1,\dots,j_m)}{p(j_1,\dots,j_m)\mathbb{E}}\right)^{\frac{p(j_1,\dots,j_m)\mathbb{E}}{2}} \cdot \\ &\hfill d\theta(j_1,\dots,j_m)\end{aligned}$$

This formula shows that for large p the distribution of $\theta(j_1, \ldots, j_m)$ is close to exponential.

The whole integral corresponding to a diagram is a double integral and the outer integration is the integration over all variables $\theta(j_1, \ldots, j_m)$ for non-final $\Delta^{(j_1,\ldots,j_m)}$. For each $\Delta^{(j_1,\ldots,j_m)}$ introduce the variable $\tilde{k}(j_1, \ldots, j_m)$ such that $\tilde{k}(j_1, \ldots, j_m) = \tilde{k}(j_1, \ldots, j_m, 1) + \tilde{k}(j_1, \ldots, j_m, 2)$. The inner integrations are the integrations over all variables $\tilde{k}(j_1, \ldots, j_m)$ satisfying the last relation. There is a Gaussian factor under the sign of integration which we shall describe. The integration goes from $\Delta^{(j_1,\ldots,j_m,j)}$, $j = 1, 2$ to $\Delta^{(j_1,\ldots,j_m)}$.

Assume that for $\Delta^{(j_1,\ldots,j_m,j)}$ we have the Gaussian factors $\exp\{-r(j_1, \ldots, j_m, j)|\tilde{k}(j_1, \ldots, j_m, j)|^2\}$. First we add the Gaussian factor $\exp\left\{-\frac{2\theta(j_1,\ldots,j_m,j)}{p(j_1,\ldots,j_m,j)\mathbb{E}}|\tilde{k}(j_1, \ldots, j_m, j)|^2\right\}$. This suggests that $r(j_1, \ldots, j_m, j)$ can be written in the form $r(j_1, \ldots, j_m, j) = \frac{\rho(j_1,\ldots,j_m,j)}{p(j_1,\ldots,j_m,j)}$ so that $\rho(j_1, \ldots, j_m, j)$ takes values $O(1)$. The first step can be written as

$$r'(j_1, \ldots, j_m, j) = r(j_1, \ldots, j_m, j) + \frac{2\theta(j_1, \ldots, j_m, j)}{p(j_1, \ldots, j_m, j)\mathbb{E}}$$

or

$$\rho'(j_1, \ldots, j_m, j) = \rho(j_1, \ldots, j_m, j) + \frac{2\theta(j_1, \ldots, j_m, j)}{\mathbb{E}} \tag{3}$$

The next step follows from the formula (see [S2] or [S3])

$$\begin{aligned} a_1|\tilde{k}(j_1, \ldots, j_m, 1)|^2 + a_2|\tilde{k}(j_1, \ldots, j_m, j, 2)|^2 &= \frac{a_1 a_2}{a_1 + a_2}|\tilde{k}(j_1, \ldots, j_m)|^2 \\ &+ (a_1 + a_2)\left|\tilde{k}(j_1, \ldots, j_m, 2) - \frac{a_1}{a_1 + a_2}\tilde{k}(j_1, \ldots, j_m)\right|^2 \end{aligned} \tag{4}$$

This shows that

$$r(j_1, \ldots, j_m) = \frac{1}{\frac{1}{r'(j_1,\ldots,j_m,1)} + \frac{1}{r'(j_1,\ldots,j_m,2)}}$$

or

$$\begin{aligned} \frac{1}{\rho(j_1, \ldots, j_m)} &= \frac{1}{\rho'(j_1, \ldots, j_m, 1)} \frac{p(j_1, \ldots, j_m, 1)}{p(j_1, \ldots, j_m)} \\ &+ \frac{1}{\rho'(j_1, \ldots, j_m, 2)} \frac{p(j_1, \ldots, j_m, 2)}{p(j_1, \ldots, j_m)} \end{aligned} \tag{5}$$

The last expression can be considered as a two-dimensional version of the famous Gauss map in the theory of dynamical systems. The second term in (4) is used as the weight with respect to which the integration goes.

The last part in the inner integration is the product $\prod^{(f)}$ over all final elements and each factor is $c\left(\frac{\tilde{k}(j_1,\ldots,j_m,j)}{\sqrt{s(j_1,\ldots,j_m,j)}}, 0\right)$ which stays either under the inner product or under the sign $P_{\tilde{k}(j_1,\ldots,j_m)}$ depending on the structure of the diagram. Also $s(j_1,\ldots,j_m) = s\prod_{r=1}^{m}\tilde{s}(j_1,\ldots,j_r)$.

It turns out that the estimates of the diagrams depend on the behavior of the partition function. A diagram is called simple if in each partition one of the elements is final. It was shown in [S3] that simple diagrams decay faster than exponentially. Presumably this result can be extended to the class of diagrams where each partition has one element with less than d_1 points where d_1 is a fixed number.

Quite different type of behavior is displayed by another class of diagrams. Choose constants d_2, $0 < d_2 < \frac{1}{2}$, d_3 and consider the diagrams where $d_2 \leqslant \frac{p(j_1,\ldots,j_m,1)}{p(j_1,\ldots,j_m)} \leqslant 1-d_2$ for all elements of partitions which have more than d_3 points. Such diagrams are called short because the number of floors in the related tree is less than $(const)\log_2 p$. Partition functions of short diagrams decay exponentially. In [S3] we described some approach which allows to study and to estimate short diagrams for large p. This approach resembles the renormalization group method in statistical mechanics.

3 Foias-Temam Theorem for 2D–Navier–Stokes System with Periodic Boundary Condition

Probably this case is the simplest in the whole mathematical theory of Navier–Stokes system. Foias and Temam proved in [FT] a remarkable theorem which says that for any sufficiently smooth initial condition the solution of (1) is real-analytic for all $t > 0$. We shall reproduce here this theorem following our joint paper with J.Mattingly (see [MS] and [ES]).

We shall use the Fourier series:

$$u(x,t) = \sum_{k\in\mathbb{Z}^d} v_k(t)\exp\{2\pi i\langle k,x\rangle\}$$

Then the system of equation for the Fourier modes takes the form

$$\frac{dv_k(t)}{dt} = -|k|^2 v_k(t) - 2\pi i\sum_{k_1\in\mathbb{Z}^d}\langle k, v_{k_1}(t)\rangle P_k v_{k-k_1}(t) + f_k \tag{1}$$

Here f_k are Fourier coefficients of the external force. For simplicity we assume that f_k do not depend on t and are different from zero only for finitely many values of k. The sum $\Omega(\{v_k\}) = \sum_k |k|^2|v_k|^2$ is called the enstrophy. Below we consider the case $d = 2$ or 3.

Lemma 3.1. *Assume that* $|v_k(0)| \leqslant \frac{D_1}{|k|^\gamma}$ *for some* $D_1 < \infty$ *and* $\gamma \geqslant \frac{d}{2}+1$ *and for a solution* $\{v_k(t), 0 \leqslant t \leqslant T\}$ *the enstrophy* $\Omega(\{v_k(t)\}) \leqslant \Omega_0$. *Then one can find another constant* D_2 *such that*

$$|v_k(t)| \leqslant \frac{D_2}{|k|^\gamma}, \qquad 0 \leqslant t \leqslant T$$

for all $k \neq 0$.

Proof: For simplicity we assume that all $v_k(t)$ and f_k are pure imaginary and therefore we can consider the real-valued version of (1). Take a sufficiently large K. Then we can find a constant $D_2(K) = D_2$ such that

$$|v_k(t) \leqslant \frac{\sqrt{\Omega_0}}{|k|}| \leqslant \frac{D_2(K)}{|k|^\gamma}, \qquad |k| \leqslant K.$$

We shall prove that $|v_k(t)| \leqslant D_2(K)|k|^{-\gamma}$ for all k, $|k| > K$, provided that K is large enough.

Suppose that this is wrong and

$$|v_{\bar{k}}(\bar{t})| = \frac{D_2(K)}{|\bar{k}|^\gamma}.$$

We may assume that $\bar{t}$ is the least value of $\bar{t}$ for which this is true (however, see the Remark at the end of the proof) and consider the case

$$v_{\bar{k}}(\bar{t}) = \frac{D_2(K)}{|\bar{k}|^\gamma}.$$

The other case when there is a minus in this relation can be discussed in a similar way. We must have

$$\frac{dv_{\bar{k}}(\bar{t})}{dt} \geqslant 0.$$

We shall come to a contradiction if we show that the viscous term in (1) dominates and the rhs of (1) is negative.

I. $|k_1| \leqslant \frac{1}{2}|\bar{k}|$. In this case $|\bar{k} - k_1| \geqslant \frac{1}{2}|\bar{k}|$ and therefore $|v_{\bar{k}-k_1}||\bar{k} - k_1| \leqslant D_2(K)\frac{1}{|\bar{k} - k_1|^{\gamma-1}}$.

We can write

$$\left| \sum_{|k_1| \leqslant \frac{1}{2}|\bar{k}|} \langle \bar{k}, v_{k_1} \rangle P_{\bar{k}} v_{\bar{k}-k_1} \right| \leqslant \sum_{|k_1| \leqslant \frac{1}{2}|\bar{k}|} |v_{k_1}||\bar{k} - k_1||v_{\bar{k}-k_1}|$$

$$\leqslant D_2 2^{\gamma-1} |\bar{k}|^{-\gamma+1} \sum_{|k_1| \leqslant \frac{1}{2}|\bar{k}|} |v_{k_1}|$$

and by Cauchy-Schwartz inequality

$$\sum_{|k_1|\leqslant\frac{1}{2}|\bar{k}|} |v_{k_1}| \leqslant \sum_{\substack{k_1\neq 0\\ |k_1|\leqslant\frac{1}{2}|\bar{k}|}} |k_1||v_{k_1}|\frac{1}{|k_1|}$$

$$\leqslant \sqrt{\sum_{k_1}|k_1|^2|v_{k_1}|^2}\sqrt{\sum_{\substack{k_1\neq 0\\ |k_1|\leqslant\frac{1}{2}|\bar{k}|}}\frac{1}{|k_1|^2}} \leqslant \sqrt{\Omega_0}Const|\bar{k}|^{d-2}$$

In the two-dimensional case we shall have $\ln|\bar{k}|$. It is clear that last expression is much smaller than the viscous term $|\bar{k}|^{2-\gamma}D_2$ if K is large enough.

II. $\frac{1}{2}|\bar{k}| \leqslant |k_1| \leqslant 2|\bar{k}|$. In this case

$$\sum_{\frac{1}{2}|\bar{k}|\leqslant|k_1|\leqslant 2|\bar{k}|} |v_{k_1}||\bar{k}-k_1||v_{\bar{k}-k_1}| \leqslant \frac{2^\gamma D_2(K)}{|\bar{k}|^\gamma}\sum_{\frac{1}{2}|\bar{k}|\leqslant|k_1|\leqslant 2|\bar{k}|} |v_{\bar{k}-k_1}||\bar{k}-k_1|.$$

Using the same arguments as before we get

$$\sum_{\frac{1}{2}|\bar{k}|\leqslant|k_1|\leqslant 2|\bar{k}|} |v_{\bar{k}-k_1}||\bar{k}-k_1| \leqslant \sqrt{\Omega_0}|\bar{k}|^{\frac{3}{2}}.$$

Again we see that the viscous term dominates.

III. $|k_1| > 2|\bar{k}|$. Here $|\bar{k}-k_1| > |\bar{k}|$ and

$$\sum_{|k_1|>2|\bar{k}|} |v_{k_1}||\bar{k}-k_1||v_{\bar{k}-k_1}| \leqslant \sum_{|k_1|>2|\bar{k}|}\frac{D_2^2 Const}{|k_1|^{d+3}} \leqslant \frac{D_2^2 Const}{|\bar{k}|^3}$$

and the viscous term dominates. Lemma is proven. □

Remark. One has to prove that in our situation $\bar{t} > 0$. This follows from the method of iteration (see above) and from the construction of solutions for small t.

Now we can formulate and prove the main theorem.

Theorem 3.1. *Let us assume that $\{v_k(0)\}$ is such that $|v_k(0)| \leqslant \frac{D_1}{|k|^\gamma}$, $k \neq 0$ and for a solution $\{v_k(t)\}$ the enstrophy $\Omega \leqslant \Omega_0$, $0 \leqslant t \leqslant T_0$. Then one can find positive numbers α, D_3 such that $|v_k(t)| \leqslant \frac{D_3\exp\{-\alpha t|k|\}}{|k|^\gamma}$.*

The proof goes essentially in the same way as the proof of Lemma 3.1. First we take a large K and find $\alpha(K) = \alpha$ such that $|v_k(t)| \leqslant \frac{2D_1}{|k|^\gamma}e^{-\alpha t|k|}$, $0 \leqslant t \leqslant T_0$ for all k, $|k| \leqslant K$. Introduce $v_k^{(1)}(t) = e^{\alpha t|k|}v_k(t)$. Then we can write down the system of equations for $v_k^{(1)}(t)$ and prove that the viscous term again dominates for $|k| > K$. The arguments are the same as in the proof of Lemma 3.1. The details are left for the reader.

4 Burgers System and $1-D$ Inviscid Burgers Equation with Random Forcing

Burgers system differs from the Navier–Stokes system by the absence of the pressure term. In the d-dimensional case it has the form:

$$\frac{\partial u_i}{\partial t}+\sum_{k=1}^{d}\frac{\partial u_i}{\partial x_k}u_k=\nu\Delta u_i+\frac{\partial F(x,t)}{\partial x_i} \tag{1}$$

Here $u=(u_1,\ldots,u_d)$ is the velocity vector, F is the potential of external forces. The viscosity ν again equals to 1. There is no incompressibility condition and, for this reason, no pressure term. We shall consider (1) with periodic boundary conditions.

For the system (1) one can prove the existence and uniqueness of strong solutions (G.A. Seregin, private communication). The system (1) has the following remarkable property: if $u(x,0)$ is a gradient of some function then $u(x,t)$ for all $t>0$ is a gradient of some function. Let us write $u=-2\frac{\nabla\varphi}{\varphi}$. Then $\varphi(x,t)$ satisfies the heat equation

$$\frac{\partial\varphi(x,t)}{\partial t}=\Delta\varphi(x,t)+F(x,t)\varphi(x,t) \tag{2}$$

whose solution can be written with the help of Feynman-Kac formula. The transition from u to φ is called the Hopf-Cole substitution. However, presumably it was known much before the works of Hopf and Cole. In [S4] the case of random potential F was considered. With the help of methods of statistical mechanics the existence and uniqueness of stationary measure for the related Markov process was proved. It would be interesting to extend the results of [S2] to the case of hyperbolic systems, i.e. to the case of systems of conservation laws perturbed by viscous terms (see a very interesting but concise survey paper [B1] by Bressan about systems of conservation laws).

Now we shall consider $d=1$ and $\nu=0$ which is called one-dimensional inviscid Burgers equation. The function $F(x,t)=\sum\limits_{|k|\leqslant K}\sigma_k'\sin 2\pi kxB_k'(t)+\sum\limits_{|k|\leqslant K}\sigma_k''\cos 2\pi kxB_k''(t)$ where B_k', B_k'' are independent white noises, σ_k' and σ_k'' are constants. The equation (1) determines a Markov process in the functional space of periodic functions $u(x)$ and we shall discuss the problem of existence and uniqueness of stationary measure for this process. It is well known that in a typical situation solutions of (1) have discontinuities of the first kind. Therefore the phase space of this process should be the Skorokhod space.

The equation (1) was studied in the paper [EKMS1] by Weinan E, K. Khanin, A. Mazel and myself. The construction of the stationary measure was done with the help of the so-called "One force – One solution principle" which we explain below.

Consider a piece-wise continuously differentiable function $x=\{x(t),\,t\geqslant 0\}$ with values in S^1 and introduce the formal expression

$$A(x) = \int_{-\infty}^{0} \left[\frac{1}{2} \left(\frac{dx}{dt} \right)^2 + F(x,t) \right] dt$$

which we shall call action. The last integral is a stochastic integral which should be understood in the sense of Ito calculus.

DEFINITION 4.1. A function $\bar{x}$ is called a one-sided minimizer if for any x such that $\bar{x}(t) = x(t)$ for all $t \leqslant t_0$ for some t_0 the difference $A(x) - A(\bar{x}) \geqslant 0$.

Clearly, the difference is well-defined. It is easy to show that if $\bar{x}$ is a one-dimensional minimizer then it satisfies the Euler-Lagrange equation

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -\frac{\partial F(x,t)}{\partial x}$$

which should be treated as a system of stochastic differential equations. The first result is the following theorem.

Theorem 4.1. *With probability* 1 *for every $x_0 \in S^1$ there exists a one-sided minimizer $\bar{x}(t)$, $t \leqslant 0$, such that $\bar{x}(0) = x_0$.*

These minimizers have many remarkable properties.

Theorem 4.2. *With probability* 1 *the one-sided minimizers are pair-wise disjoint. More precisely, if $\bar{x}'$, $\bar{x}''$ are two minimizers then $\bar{x}'(t) \neq \bar{x}(t)''$ for all $t < 0$.*

The usual methods of variational calculus allow us to prove the absence of two values t_1, t_2 such that $\bar{x}'(t_j) = \bar{x}''(t_j)$, $j = 1, 2$. The absence of one value t_1 requires more subtle arguments and uses the random character of the forcing.

Put $u(x,0) = \left. \frac{d\bar{x}(t)}{dt} \right|_{\substack{t=0 \\ \bar{x}(0)=x}}$ and $u(x,s)$ is the same function $u(\cdot,0)$ constructed for the shifted potential $B(x, t+s)$. The so-called Lax-Oleinik variational principle (see [L], [O]) says that $u(x,s)$ is a weak solution of (1). One can easily see that the induced probability distribution of $u(\cdot,0)$ generates a stationary measure for our Markov process. It is possible to show that this measure is unique. This statement is an illustration of the principle "one force — one solution".

Using the described construction one can study properties of typical realizations wrt this measure.

For example for any s with probability 1 the set where $u(x,s)$ is discontinuous is finite for every s. Having a realization of the random potential $F(x,t)$ consider the set of $(x,s) \in S^1 \times \mathbb{R}^1$ where $u(x,s)$ is discontinuous. It has the from of skeleton containing a special curve $y = y(s)$, $-\infty < s < \infty$, which is called the main shock. Other components of this set are compact ribs which sooner or later merge with the main shock. The main shock is unique with probability 1.

A curve $\bar{y}(t)$, $-\infty < t < \infty$ is called the two-sided minimizer if it gives minimum to all compact perturbation of the integral

$$\int_{-\infty}^{\infty} \left[\frac{1}{2} \left(\frac{d\bar{y}}{dt} \right)^2 + F(\bar{y}(t), t) \right] dt$$

It is possible to show that with probability 1 the two-sided minimizer exists and is unique. It satisfies the Euler-Lagrange equation

$$\frac{d\bar{y}}{dt} = \bar{v}, \qquad \frac{d\bar{v}}{dt} = F(\bar{y}, t) \tag{3}$$

and with probability 1 is a hyperbolic solution of the system (3) in the sense of theory of dynamical systems. In particular, it has stable and unstable manifolds and the set $\{x, u(x, 0)\}$ is a subset of the unstable manifold of the two-sided minimizer.

Using the described construction one can study the probability distributions of various random variables important from the point of view of physical applications (see [EKMS2], [ES] and other references given there).

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