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L^p stability for entropy solutions of scalar conservation laws with strict convex flux

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Abstract

Here we consider the scalar convex conservation laws in one space dimension with strictly convex flux which is in C^1. Existence, uniqueness and L^1 contractivity were proved by Kružkov [14]. Using the relative entropy method, Leger showed that for a uniformly convex flux and for the shock wave solutions, the L^2 norm of a perturbed solution relative to the shock wave is bounded by the L^2 norm of the initial perturbation. Here we generalize the result to L^p norm for all $1 \leqslant p < \infty$. Also we show that for the non-shock wave solution, L^p norm of the perturbed solution relative to the modified N-wave is bounded by the L^p norm of the initial perturbation for all $1 \leqslant p < \infty$.

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1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Associated with f, consider the following conservation laws

$$\begin{aligned} u_t + f(u)_x &= 0 \quad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x) \quad x \in \mathbb{R} \end{aligned} \tag{1.1}$$

where $u_0 \in L^\infty(\mathbb{R})$. $u \in L^\infty_{loc}(\mathbb{R} \times (0,\infty))$ is called a solution to (1.1) if u is a weak solution satisfying the entropy condition of Kružkov [14]. Existence and uniqueness of solution were proved by Lax–Oleinik in the case of uniformly convex flux f and Kružkov [14] in the case of general locally Lipschitz continuous flux f (for details see [12,14,18]). Furthermore, Kružkov [14], Keyfitz [13] proved the map $S_t u_0(x) = u(x,t)$ forms an L^1-contractive semigroup for all $u_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$. That is, for given $u_0, v_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, for $t > 0$

$$\|S_t u_0 - S_t v_0\|_{L^1(\mathbb{R})} \leqslant \|u_0 - v_0\|_{L^1(\mathbb{R})}. \tag{1.2}$$

In general, this result is false for the systems (see Temple [20]) and L^1-stability result holds for almost contractive semi-group structure (see Bressan et al. [3]). We refer [17], for L^2 stability type results for the systems of conservation laws.

Using the relative entropy method developed by Dafermos, DiPerna [4–10] and Filippov theory [11], Leger [16] proved the following

Theorem 1.1. *Let $C_L > C_R$ and $f'' > 0$. Define*

$$\phi(x) = \begin{cases} C_L & \text{if } x < 0, \\ C_R & \text{if } x > 0, \end{cases} \tag{1.3}$$

$$\sigma = \frac{f(C_L) - f(C_R)}{C_L - C_R}. \tag{1.4}$$

Let $u_0 \in L^\infty(\mathbb{R})$ be such that $u_0 - \phi \in L^2(\mathbb{R})$. Then there exists a Lipschitz continuous curve $x(t)$ with $x(0) = 0$, $x(t) = O(t^{1/2})$ as $t \to \infty$ such that

$$\big\|S_t u_0 - \phi\big(\cdot - x(t) - \sigma t\big)\big\|_{L^2(\mathbb{R})} \leqslant \|u_0 - \phi\|_{L^2(\mathbb{R})}. \tag{1.5}$$

It is to be remarked that the proof of the theorem needs the following hypotheses:

(1) $f'' > 0$.
(2) $C_L > C_R$.
(3) Lager remarks that his proof works only for L^2 stability case.

Therefore the natural questions one would like to ask are the following:

Q1: Is $f'' > 0$ is necessary? For example is the result is true for $f(u) = |u|^q$, $1 < q < \infty$ or for any convex function (non-uniformly convex)?
Q2: What happens if $C_L \leqslant C_R$?
Q3: Can one get L^p stability for all $1 < p < \infty$?

In this present paper, using the Structure Theorem (see [1]) for the entropy solutions, we analyze all the above questions under suitable conditions on u_0.

In order to state the main result, throughout this paper, we assume that f satisfies the following assumptions.

(1) $f \in C^1(\mathbb{R})$ and is strictly convex (in order to write the Lax–Oleinik formula, we do not need to assume the uniform convexity of the flux f, we only need f' to be strictly monotone, for example $f(u) = |u|^q$, $1 < q < \infty$)

(2) Due to maximum principle property of the scalar conservation laws, one can always assume that the flux f is of superlinear growth, i.e.

$$\lim_{|u|\to\infty} \frac{f(u)}{|u|} = \infty. \tag{1.6}$$

Then we have the following:

Theorem 1.2. *Let $1 \leqslant p < \infty$ and $u_0 \in L^\infty(\mathbb{R})$. Let ϕ be as in (1.3) and σ as in (1.4).*

(I) *Assume that for all $x \in \mathbb{R}$, $C_R \leqslant u_0(x) \leqslant C_L$ and*

$$\|u_0 - \phi\|_{L^p(\mathbb{R})} < \infty. \tag{1.7}$$

1. *Then there exists a uniformly Lipschitz curve $\xi(t)$ with $\xi(0) = 0$ such that for $t > 0$*

$$\big\| S_t u_0 - \phi\big(\cdot - \xi(t)\big)\big\|_{L^p(\mathbb{R})} \leqslant \|u_0 - \phi\|_{L^p(\mathbb{R})}. \tag{1.8}$$

2. *Let $C_L > C_R$ and u_0 be continuous outside a compact interval and satisfies*

$$\lim_{x\to\infty} \big|u_0(x) - \phi(x)\big| = 0,$$

then

$$\lim_{t\to\infty} \left|\frac{\xi(t)}{t} - \sigma\right| = 0. \tag{1.9}$$

Furthermore there exists an $M > 0$ depending only on $\|u_0\|_\infty$ and f and a uniformly Lipschitz continuous function η such that for all $t > 0$,

$$\left|\frac{\eta(t)}{t} - \sigma\right| \leqslant M \tag{1.10}$$

and

$$\int_{-\infty}^{\infty} \big|S_t u_0(x) - \phi(x - \sigma t)\big|^p \, dx \leqslant \int_{-\infty}^{\infty} \big|u_0(x) - \phi\big(x - \eta(t)\big)\big|^p \, dx. \tag{1.11}$$

(II) *Let* $C_L < C_R$, $A > 0$, $\bar{u}_0 \in L^\infty(-A, A)$ *and*

$$u_0(x) = \begin{cases} C_L & \text{if } x \leqslant -A, \\ C_R & \text{if } x > A, \\ \bar{u}_0(x) & \text{if } x \in (-A, A). \end{cases} \tag{1.12}$$

Then there exist $N_1 : \mathbb{R} \times (0, \infty) \to [C_L, C_R]$ *and* $N_0 : \mathbb{R} \to [C_L, C_R]$ *such that*

(i) N_0 *is a non-decreasing function with* $N_0(-A) = C_L$, $N_0(A) = C_R$.

(ii) N_1 *is the solution of* (1.1) *with* N_0 *as its initial condition satisfying*

$$\left\| S_t u_0 - N_1(\cdot, t) \right\|_{L^p(\mathbb{R})} \leqslant \|u_0 - N_0\|_{L^p(\mathbb{R})}. \tag{1.13}$$

Remark 1.3. N_1 is basically a rarefaction.

2. Preliminaries

Let $f \in C^1(\mathbb{R})$ be a strictly convex function with superlinear growth and denote f^* the convex dual of f given by

$$f^*(u) = \max_{q \in \mathbb{R}} \left\{ qu - f(q) \right\}.$$

Let $\alpha \in \mathbb{R}$ and $v_0(x) = \int_0^x u_0(\theta)\, d\theta$ be the primitive of u_0 and denote

$$v(x,t) = \min_{y \in \mathbb{R}} \left\{ v_0(y) + t f^*\left(\frac{x-y}{t} \right) \right\}, \tag{2.1}$$

$$\begin{aligned} ch(x,t) &= \left\{ y \in \mathbb{R}\colon\ y \text{ is a minimizer in (2.1)} \right\} \\ &= \text{characteristic set}, \end{aligned} \tag{2.2}$$

$$y_+(x,t) = \max\left\{ y\colon\ y \in ch(x,t) \right\}, \tag{2.3}$$

$$y_-(x,t) = \min\left\{ y\colon\ y \in ch(x,t) \right\}, \tag{2.4}$$

$$\xi_-(t,\alpha) = \inf\left\{ x \in \mathbb{R}\colon\ y_-(x,t) \geqslant \alpha \right\}, \tag{2.5}$$

$$\xi_+(t,\alpha) = \sup\left\{ x \in \mathbb{R}\colon\ y_+(x,t) \leqslant \alpha \right\}, \tag{2.6}$$

$$\xi_-(t,0) = \xi(t). \tag{2.7}$$

First recall the Hopf, Lax–Oleinik [12] results in the following

Theorem 2.1. (1) v *is a Lipschitz continuous function with Lipschitz constant depending only on* $\|u_0\|_\infty$ *and* f. *It is a unique viscosity solution of the following Hamilton–Jacobi equation*

$$\begin{aligned} v_t + f(v_x) &= 0 \quad x \in \mathbb{R},\ t > 0, \\ v(x,0) &= v_0(x) \quad x \in \mathbb{R}. \end{aligned} \tag{2.8}$$

(2) *$ch(x,t)$, y_+, y_- exist and $x \mapsto y_\pm(x,t)$ are non-decreasing functions with $y_+(x,t) = y_-(x,t)$ for a.e. x.*

(3) *Let $u = \frac{\partial v}{\partial x}$, then u is the solution of (1.1) and for a.e. x, u satisfies*

$$f'\big(u(x,t)\big) = \frac{x - y_+(x,t)}{t} = \frac{x - y_-(x,t)}{t}. \tag{2.9}$$

(4) *Non-intersecting property (NIP): Let for $i = 1, 2$, $y_i \in ch(x_i, t_i)$ and $r_i(\theta) = x_i + \frac{x_i - y_i}{t_i}(\theta - t_i)$. Then r_1 and r_2 cannot intersect for $0 < \theta < \min\{t_1, t_2\}$ unless $r_1 \equiv r_2$ in the common domain of definition.*

(5) *From NIP and (2.9) it can be deduced easily the following*

$$y_-(x,t) = \lim_{\xi \uparrow x} y_+(\xi,t), \qquad y_+(x,t) = \lim_{\xi \downarrow x} y_-(\xi,t), \tag{2.10}$$

$$f'\big(u(x-,t)\big) = \frac{x - y_-(x,t)}{t}, \qquad f'\big(u(x+,t)\big) = \frac{x - y_+(x,t)}{t}. \tag{2.11}$$

Furthermore, if x is a point of differentiability of $y_\pm(x,t)$ and $y_\pm(x,t)$ is a point of continuity of u_0, then

$$f'\big(u_0\big(y_\pm(x,t)\big)\big) = \frac{x - y_\pm(x,t)}{t}.$$

Let $s_\pm(\theta)$ be the extreme characteristic at (x,t) given by

$$s_\pm(\theta) = x + f'\big(u(x\pm,t)\big)(\theta - t). \tag{2.12}$$

Then for $0 < \theta < t$,

$$u\big(s_\pm(\theta)\pm, t\big) = u(x\pm, t). \tag{2.13}$$

Definition 2.2 *(Regular characteristic line).* Let $u_0 \in L^\infty(\mathbb{R})$ and u be the solution given in Theorem 2.1. Let $r(t) = \alpha + tf'(p)$. Then r is called a regular characteristic line if for all $t > 0$

$$u\big(r(t)+,t\big) = u\big(r(t)-,t\big) = p. \tag{2.14}$$

Definition 2.3 *(Asymptotically single shock packet (ASSP)).* Let u_0, u be as in Definition 2.2. Let $C_1 < C_2$, $r_i(t) = C_i + tf'(p)$ for $i = 1, 2$ and

$$D(C_1, C_2, p) = \big\{(x,t)\colon\ r_1(t) < x < r_2(t)\big\}. \tag{2.15}$$

Then $D(C_1, C_2, p)$ is called an ASSP if:

(i) For $i = 1, 2$, r_i are regular characteristic lines.
(ii) For all $(x,t) \in D(C_1, C_2, p)$

$$y_\pm(x,t) \in [C_1, C_2]. \tag{2.16}$$

(iii) $D(C_1, C_2, p)$ does not contain regular characteristic line.

Then recall the following Structure Theorem proved in [1] (see p. 11, Theorem 2.7).

Theorem 2.4 *(Structure Theorem). Let C_L, C_R and u_0 be as in (1.12) and let u be the corresponding solution of (1.1). Let $R_-(t) = \xi_-(t, -A)$, $R_+(t) = \xi_+(t, A)$, ξ are as in (2.5), (2.6) and (2.7). Then*:

1. *$R_\pm(t), \xi(t)$ are Lipschitz continuous functions with Lipschitz constant depending only on $\|u_0\|_\infty$ and f satisfying $R_-(t) \leqslant \xi(t) \leqslant R_+(t)$, $\xi(0) = 0$, $R_+(0) = A$ and $R_-(0) = -A$. Furthermore if $x \leqslant R_-(t)$, then $y_-(x,t) \leqslant -A$, if $x \geqslant R_+(t)$ then $y_+(x,t) \geqslant A$, $y_-(\xi(t),t) \leqslant 0 \leqslant y_+(\xi(t),t)$ and*

$$f'\big(u(x,t)\big) = \begin{cases} f'(C_L) = \frac{x-y_-(x,t)}{t} & \text{if } x < R_-(t), \\ f'(C_R) = \frac{x-y_+(x,t)}{t} & \text{if } x > R_+(t). \end{cases} \tag{2.17}$$

2. *Shock solution*: *Let $C_L > C_R$, then there exists an (x_0, T_0) such that*

$$x_0 = R_-(T_0) = R_+(T_0) = \xi(T_0) \tag{2.18}$$

and for $t > T_0$

$$R_-(t) = R_+(t) = x_0 + \sigma(t - T_0). \tag{2.19}$$

3. *Let $C_L \leqslant C_R$, then there exist $-A \leqslant B_1 \leqslant B_2 \leqslant A$ and $\{D_i = D(C_{1,i}, C_{2,i}, p_i)\}_{i\in I}$, a countable collection of disjoint ASSP such that*:
   (i) *$R_-(t) < R_+(t)$ for all $t > 0$.*
   (ii) *Let $\Gamma_1(t) = B_1 + tf'(C_L)$, $\Gamma_2(t) = B_2 + tf'(C_R)$, then for $i = 1, 2$, Γ_i is a regular characteristic line with*

$$R_-(t) \leqslant \Gamma_1(t) \leqslant \Gamma_2(t) \leqslant R_+(t) \quad \text{for all } t > 0. \tag{2.20}$$

   (iii) *Let*

$$E = \big\{(x,t)\colon\ \Gamma_1(t) < x < \Gamma_2(t)\big\}. \tag{2.21}$$

Then for all $i \in I$, $D_i \subset E$. Let

$$R = E \setminus \bigcup_{i\in I} D_i, \tag{2.22}$$

then R consists of regular characteristic lines and $x \mapsto u(x,t)$ is continuous on $R_t = \{x\colon (x,t) \in R\}$. Furthermore, if (x,t) lies on a regular characteristic line $r(t) = \alpha + tf'(p)$, then

$$u(x,t) = p. \tag{2.23}$$

4. *Let*

$$
\begin{aligned}
F_- &= \{(x,t):\ x \leqslant R_-(t)\},\\
F_+ &= \{(x,t):\ x \geqslant R_+(t)\},\\
D_- &= \{(x,t):\ R_-(t) < x < \Gamma_1(t)\},\\
D_+ &= \{(x,t):\ \Gamma_2(t) < x < R_+(t)\}.
\end{aligned}
$$

Then clearly, $F_\pm$ are closed sets, $D_\pm$ are open sets with

$$\mathbb{R} \times (0,\infty) = F_- \cup F_+ \cup D_- \cup D_+ \bigcup_{i\in I} D_i \cup R.$$

Define

$$
N(x,t) = \begin{cases}
C_L & \text{if } (x,t) \in F_-,\\
\frac{x-B_1}{t} & \text{if } (x,t) \in D_-,\\
p_i & \text{if } (x,t) \in D_i,\\
u(x,t) & \text{if } (x,t) \in R,\\
\frac{x-B_2}{t} & \text{if } (x,t) \in D_+,\\
C_R & \text{if } (x,t) \in F_+.
\end{cases}
$$

Then $x \mapsto N(x,t)$ is a continuous non-decreasing function on $\{x:\ (x,t) \notin F_\pm\}$ taking values in $[C_L, C_R]$ and having jumps at $R_\pm(t)$. This represents the Lax N-wave (see [15]) and satisfies

$$\lim_{t\to\infty} \int_{-\infty}^{\infty} \left|u(x,t) - N(x,t)\right| dx = 0.$$

For the shock solution it has been proved before in [19,4]. See Section 4 for the illustration of the Structure Theorem.

3. Proof of Theorem 1.2

Before going to the proof of the theorem, let us recall some properties of the characteristic curves $\xi_\pm(t,\alpha)$ from [1,2].

Properties of $\xi_\pm(t,\alpha)$:

(1) $t \mapsto \xi_\pm(t,\alpha)$ are Lipschitz continuous functions.
(2) Let $\alpha, \beta \in \mathbb{R}$ and $T > 0$. Suppose one of $\xi_\pm(t,\alpha)$ denoted by $\xi(t,\alpha)$ and one of $\xi_\pm(t,\beta)$ denoted by $\xi(t,\beta)$ satisfy at $t = T$, $\xi(T,\alpha) = \xi(T,\beta)$, then for all $t > T$, $\xi(t,\alpha) = \xi(t,\beta)$.
(3) Let $y_\pm(t,\alpha) = y_\pm(\xi_-(t,\alpha),t)$, $Y_\pm(t,\alpha) = y_\pm(\xi_+(t,\alpha),t)$. Then $t \mapsto y_-(t,\alpha), Y_-(t,\alpha)$ are non-increasing functions and $t \mapsto y_+(t,\alpha), Y_+(t,\alpha)$ are non-decreasing functions.
Furthermore if $y(t,\alpha)$ denote one of the four functions $y_\pm(t,\alpha), Y_\pm(t,\alpha)$ and let $\xi(t,\alpha)$ be the corresponding characteristic curve such that $y(t,\alpha)$ is bounded as $t \to \infty$. Then by monotonicity,

$$\lim_{t\to\infty} y(t,\alpha) = G, \tag{3.1}$$

$$\lim_{t\to\infty} \frac{\xi(t,\alpha) - y(t,\alpha)}{t} = f'(p), \tag{3.2}$$

exist and the line

$$r(\theta) = G + \theta f'(p) \tag{3.3}$$

is a regular characteristic line.

Proof. For (1) and (2) see (4) and (6) of Lemma 3.1 of [1]. Proof for (3), see (1), (6) and (10) of Lemma 3.4 of [1]. □

Lemma 3.1. *Let $C_L > C_R$ and u_0 be a continuous function on $(M_0, \infty) \cup (-\infty, -M_0)$ for some $M_0 > 0$ and satisfy*

$$\lim_{|x|\to\infty} \left|u_0(x) - \phi(x)\right| = 0, \tag{3.4}$$

$$\int_{-\infty}^{\infty} \left|u_0(x) - \phi(x)\right| dx < \infty. \tag{3.5}$$

Let $\xi(t) = \xi_-(t, 0)$ and σ be as in (1.4). Then

$$\lim_{t\to\infty} \left|\frac{\xi(t)}{t} - \sigma\right| = 0. \tag{3.6}$$

Proof. Proof consists of several steps.

Step 1. Let $\epsilon > 0$ be such that $C_L > C_R + 2\epsilon$. Choose $M_1 > M_0$ such that

$$u_0(x) > C_L - \epsilon \quad \text{if } x \leqslant -M_1, \tag{3.7}$$

$$u_0(x) \leqslant C_R + \epsilon \quad \text{if } x \geqslant M_1. \tag{3.8}$$

Let $\xi_1(t) = \xi_-(t, -M_1)$, $\xi_2(t) = \xi_+(t, M_1)$. Then there exists $T > 0$ such that

$$\xi_1(T) = \xi_2(T). \tag{3.9}$$

Suppose not, then $\xi_1(t) < \xi_2(t)$ for all $t > 0$. From the non-intersecting property of the characteristics, it follows that

$$-M_1 \leqslant y_+\big(\xi_1(t), t\big) \leqslant y_-\big(\xi_2(t), t\big) \leqslant M_1. \tag{3.10}$$

Then from (3.1), (3.2) and (3.3)

$$\lim_{t\to\infty}\big(y_+\big(\xi_1(t),t\big),y_-\big(\xi_2(t),t\big)\big)=(A_+,A_-), \tag{3.11}$$

$$\lim_{t\to\infty}\left(\frac{\xi_1(t)-y_+(\xi_1(t),t)}{t},\frac{\xi_2(t)-y_-(\xi_2(t),t)}{t}\right)=\big(f'(p_+),f'(p_-)\big) \tag{3.12}$$

exist and the lines

$$r_+(\theta)=A_++\theta f'(p_+),\qquad r_-(\theta)=A_-+\theta f'(p_-) \tag{3.13}$$

are regular characteristic lines and $r_+(\theta)\leqslant r_-(\theta)$.

Case (i). Suppose $\{y_-(\xi_1(t),t)\}$ is bounded as $t\to\infty$. Then again from (3.1) and (3.2), we have

$$\lim_{t\to\infty} y_-\big(\xi_1(t),t\big)=B_+, \tag{3.14}$$

$$\lim_{t\to\infty}\frac{\xi_1(t)-y_-(\xi_1(t),t)}{t}=f'(P_+) \tag{3.15}$$

exist. Since $\{y_\pm(\xi_1(t),t)\}$ is bounded as $t\to\infty$, it follows from (3.12) and (3.14)

$$f'(P_+)=\lim_{t\to\infty}\frac{\xi_1(t)}{t}=f'(p_+)$$

and hence $P_+=p_+$. Since u_0 is continuous in $(-\infty,M_1)$ and $t\mapsto\xi_1(t)$ is Lipschitz continuous function, it follows from (5) of Theorem 2.1, for a.e. $t>0$

$$f'\big(u_0\big(y_+\big(\xi_1(t),t\big)\big)\big)=\frac{\xi_1(t)-y_+(\xi_1(t),t)}{t}.$$

From (3.7), $u_0(y_+(\xi_1(t),t))\geqslant C_L-\epsilon$ and hence

$$\begin{aligned}f'(C_L-\epsilon)\leqslant\overline{\lim_{t\to\infty}}\,f'\big(u_0\big(y_+\big(\xi(t),t\big)\big)\big)&=\overline{\lim_{t\to\infty}}\,\frac{\xi_1(t)-y_+(\xi_1(t),t)}{t}\\&=f'(P_+).\end{aligned} \tag{3.16}$$

Hence $f'(p_+)=f'(P_+)\geqslant f'(C_L-\epsilon)$.

Case (ii). Suppose $\lim_{t\to\infty}y_-(\xi_1(t),t)=-\infty$.

Then from (5) of Theorem 2.1, for a.e. $t>0$

$$f'\big(u_0\big(y_-\big(\xi_1(t),t\big)\big)\big)=\frac{\xi_1(t)-y_-(\xi_1(t),t)}{t}$$

and hence

$$f'(C_L)=\lim_{t\to\infty}\frac{\xi_1(t)-y_-(\xi_1(t),t)}{t}. \tag{3.17}$$

Since $y_\pm(\xi_1(t), t) \in ch(\xi_1(t), t)$ and hence from (2.1), we have

$$\int_0^{y_-(\xi_1(t),t)} u_0(x)\,dx + tf^*\left(\frac{\xi_1(t) - y_-(\xi_1(t),t)}{t}\right) = \int_0^{y_+(\xi_1(t),t)} u_0(x)\,dx \tag{3.18}$$

$$+ tf^*\left(\frac{\xi_1(t) - y_+(\xi_1(t),t)}{t}\right). \tag{3.19}$$

From (3.5) and (3.19) we have

$$\begin{aligned}
&\frac{C_L y_-(\xi_1(t),t)}{t} + f^*\left(\frac{\xi_1(t) - y_-(\xi_1(t),t)}{t}\right) - f^*\left(\frac{\xi_1(t) - y_+(\xi_1(t),t)}{t}\right) \\
&\quad = \frac{1}{t}\int_0^{y_+(\xi_1(t),t)} u_0(x)\,dx - \frac{1}{t}\int_0^{y_-(\xi_1(t),t)} \big(u_0(x) - C_L\big)\,dx \\
&\quad = O\left(\frac{1}{t}\right), \\
&C_L\left(\frac{y_-(\xi_1(t),t) - \xi_1(t)}{t}\right) + C_L\frac{\xi_1(t)}{t} + f^*\left(\frac{\xi_1(t) - y_-(\xi_1(t),t)}{t}\right) \\
&\quad - f^*\left(\frac{\xi_1(t) - y_+(\xi_1(t),t)}{t}\right) = O\left(\frac{1}{t}\right).
\end{aligned} \tag{3.20}$$

From (3.12), (3.17) and letting $t \to \infty$ in (3.20) to obtain

$$-C_L f'(C_L) + C_L f'(p_+) + f^*\big(f'(C_L) - f^*\big(f'(p_+)\big)\big) = 0.$$

Since $f^*(f'(p)) = pf'(p) - p$ and hence the above equation gives

$$f(C_L) = f(p_+) + (C_L - p_+)f'(p_+)$$

and hence $p_+ = C_L$, by strict convexity of f. Hence in conclusion, we have

$$f'(p_+) \geqslant f'(C_L - \epsilon). \tag{3.21}$$

Similarly we have

$$f'(p_-) \leqslant f'(C_R + \epsilon). \tag{3.22}$$

Since $C_L - \epsilon > C_R + \epsilon$ and hence $f'(p_+) > f'(p_-)$. Therefore the characteristic lines γ_+ and γ_- intersect, which is a contradiction. This proves step 1.

Step 2. Following limit holds

$$\lim_{t\to\infty}\big(y_-\big(\xi(t),t\big), y_+\big(\xi(t),t\big)\big) = (-\infty, \infty). \tag{3.23}$$

Suppose $\{y_-(\xi(t),t)\}$ is bounded as $t\to\infty$. Then from (3.1), (3.2) and (3.3)

$$\lim_{t\to\infty} y_-\big(\xi(t),t\big)=G,$$
$$\lim_{t\to\infty}\frac{\xi(t)-y_-(\xi(t),t)}{t}=f'(p)$$

exist and $\gamma(\theta)=G+\theta f'(p)$ is a characteristic line with $\gamma(t)\leqslant\xi(t)$ for all t. Let $-M_1<\min(-M_0,G)$, then by step 1, there exists $T>0$ such that $\xi_1(T)=\xi(T)$ and hence there exists $T_1\leqslant T$ such that $\gamma(T_1)=\xi_1(T_1)$. Since γ is a regular characteristicline hence $G=y_-(\gamma(T_1),T_1)=y_-(\xi_1(T_1),T_1)\leqslant -M_1<G$, which is a contradiction. This proves step 2.

Step 3. Let us denote $p_\pm(t)$ by

$$f'\big(p_\pm(t)\big)=\frac{\xi(t)-y_\pm(\xi(t),t)}{t}.$$

From step 2, choose $T_1>0$ such that for $t\geqslant T_1$,

$$y_-\big(\xi(t),t\big)<-M_0<M_0<y_+\big(\xi(t),t\big).$$

Then from (5) of Theorem 2.1, for a.e. $t>T_1$,

$$f'\big(u_0\big(y_-\big(\xi(t),t\big)\big)\big)=\frac{\xi(t)-y_-(\xi(t),t)}{t}=f'\big(p_-(t)\big), \tag{3.24}$$
$$f'\big(u_0\big(y_+\big(\xi(t),t\big)\big)\big)=\frac{\xi(t)-y_+(\xi(t),t)}{t}=f'\big(p_+(t)\big). \tag{3.25}$$

Therefore from (3.4), we have

$$\lim_{t\to\infty}\big(f'\big(p_-(t)\big),f'\big(p_+(t)\big)\big)=\big(f'(C_L),f'(C_R)\big). \tag{3.26}$$

Since $y_\pm(\xi(t),t)\in ch(\xi(t),t)$, hence from (2.1), we have

$$\begin{aligned}
&\int_0^{y_-(\xi(t),t)} u_0(x)\,dx+tf^*\big(f'\big(p_-(t)\big)\big)=\int_0^{y_+(\xi(t),t)} u_0(x)\,dx+tf^*\big(f'\big(p_+(t)\big)\big),\\
&\frac{1}{t}\big(C_Ly_-\big(\xi(t),t\big)-C_Ry_+\big(\xi(t),t\big)\big)=f^*\big(f'\big(p_+(t)\big)\big)-f^*\big(f'\big(p_-(t)\big)\big)\\
&\qquad+\frac{1}{t}\int_0^{y_+(\xi(t),t)}(u_0-C_R)\,dx-\frac{1}{t}\int_0^{y_-(\xi(t),t)}(u_0-C_L)\,dx.
\end{aligned} \tag{3.27}$$

From (3.24) and (3.25) we have

$$\frac{1}{t}\big(C_L y_-\big(\xi(t),t\big) - C_R y_+\big(\xi(t),t\big)\big) = (C_L - C_R)\frac{\xi(t)}{t} - C_L f'\big(p_-(t)\big) + C_R f'\big(p_+(t)\big). \tag{3.28}$$

From (3.5), (3.27) and (3.28), we have

$$\begin{aligned}(C_L - C_R)\left(\frac{\xi(t)}{t} - \sigma\right) &= C_L f'\big(p_-(t)\big) - C_R f'\big(p_+(t)\big) + f^*\big(f'\big(p_+(t)\big)\big)\\ &\quad - f^*\big(f'\big(p_-(t)\big)\big) - f(C_L) + f(C_R) + O\left(\frac{1}{t}\right)\\ &= f\big(p_-(t)\big) - f(C_L) + \big(C_L - p_-(t)\big)f'\big(p_-(t)\big)\\ &\quad - f\big(p_+(t)\big) + f(C_R) + \big(p_+(t) - C_R\big)f'\big(p_+(t)\big) + O\left(\frac{1}{t}\right).\end{aligned}$$

Since $C_L > C_R$ and from (3.26) we obtain

$$\lim_{t\to\infty}\left(\frac{\xi(t)}{t} - \sigma\right) = 0.$$

Hence the lemma.

Let u be the solution of (1.1). Let η_1 and η_2 be convex functions such that

$$0 = \eta_1(C_L) = \min_{\theta\in\mathbb{R}} \eta_1(\theta), \tag{3.29}$$

$$0 = \eta_2(C_R) = \min_{\theta\in\mathbb{R}} \eta_2(\theta). \tag{3.30}$$

Let

$$F_1(u) = \int_{C_L}^{u} \eta_1'(\theta) f'(\theta)\, d\theta, \qquad F_2(u) = \int_{C_R}^{u} \eta_2'(\theta) f'(\theta)\, d\theta. \tag{3.31}$$

Since u is an entropy solution, hence u satisfies in the sense of distribution for $i = 1, 2$

$$\frac{\partial}{\partial t}\eta_i(u) + \frac{\partial}{\partial x}F_i(u) \leqslant 0. \qquad \square \tag{3.32}$$

Proof of Theorem 1.2. I. Let $C_R < C_L$, $C_R \leqslant u_0(x) \leqslant C_L$ for all $x \in \mathbb{R}$ (for illustration see Fig. 2).

1. **Step 1.** First assume that u_0 be as in (1.12). Let (x_0, T_0) be as in (2) of Theorem 2.4, then for $T \geqslant T_0$, we have

$$R_-(T) = R_+(T) = x_0 + \sigma(T - T_0). \tag{3.33}$$

Let $\Delta(T)$ denote the triangle with vertices $(R_-(T),T)$, $(y_-(R_-(T),T),0)$, $(y_+(R_-(T),T),0)$. Let r_1 and r_2 be the lines joining between $(R_-(T),T)$, $(y_-(R_-(T),T),0)$ and $(R_-(T),T)$, $(y_+(R_-(T),T),0)$ respectively and from (2.17),

$$r_1(\theta)=R_-(T)+f'(C_L)(\theta-T),\qquad r_2(\theta)=R_-(T)+f'(C_R)(\theta-T). \tag{3.34}$$

From the non-intersecting property of the characteristic it follows that for $T>T_0$, $r_1(\theta)<R_-(\theta)$, $r_2(\theta)>R_+(\theta)$ for all $\theta\in(0,T)$. Hence from (2.17) for all $\theta\in(0,T)$

$$u\big(r_1(\theta),\theta\big)=C_L,\qquad u\big(r_2(\theta),\theta\big)=C_R. \tag{3.35}$$

Claim. Let $(x,t)\in\Delta(T)$ and denote $u_\pm=u(x\pm,t)$ and $s_\pm(\theta)=x+f'(u\pm)(\theta-t)$ be the extreme characteristics at (x,t). Then

$$\int_{-\infty}^{x}\big|u(y,t)-C_L\big|^p\,dy\leqslant\int_{-\infty}^{y_-(x,t)}\big|u_0(y)-C_L\big|^p\,dy, \tag{3.36}$$

$$\int_{x}^{\infty}\big|u(y,t)-C_R\big|^p\,dy\leqslant\int_{y_+(x,t)}^{\infty}\big|u_0(y)-C_R\big|^p\,dy. \tag{3.37}$$

Let P_+ be the parallelogram with vertices $(y_-(R_-(T),T),0)$, $(y_-(x,t),0)$, (x,t), $(r_1(t),t)$. Then integrating (3.32) for $i=1$ over P_+ to obtain

$$\begin{aligned}0\geqslant{}&\int_{r_1(t)}^{x}\eta_1\big(u(y,t)\big)\,dy-\int_{y_-(R_-(T),T)}^{y_-(x,t)}\eta_1\big(u_0(y)\big)\,dy\\&+\int_0^t\bigg(\frac{dr_1}{d\theta}\eta_1\big(u\big(r_1(\theta)+,\theta\big)\big)-F_1\big(u\big(r_1(\theta)+,\theta\big)\big)\bigg)\,d\theta\\&+\int_0^t\bigg(F_1\big(u\big(s_-(\theta)-,\theta\big)\big)-\frac{ds_-(\theta)}{d\theta}\eta_1\big(u\big(s_-(\theta)-,\theta\big)\big)\bigg)\,d\theta.\end{aligned} \tag{3.38}$$

From (3.29), (3.31) and (3.35), it follows that

$$\eta_1\big(u_1\big(r_1(\theta)+,\theta\big)\big)=\eta_1(C_L)=0=F_1(C_L).$$

From (2.17), $u_1(s_-(\theta)-,\theta)=u_-$ for all $\theta\in(0,t)$. From the hypothesis and using maximum principle we have $C_R\leqslant u_+\leqslant u_-\leqslant C_L$ (see Fig. 1) and therefore

$$F_1\big(u\big(s_-(\theta)-,\theta\big)\big)-\frac{ds_-(\theta)}{d\theta}\eta_1\big(u\big(s_-(\theta)-,\theta\big)\big)=\int_{C_L}^{u_-}\eta_1'(\theta)f'(\theta)\,d\theta-f'(u_-)\eta_1(u_-)$$

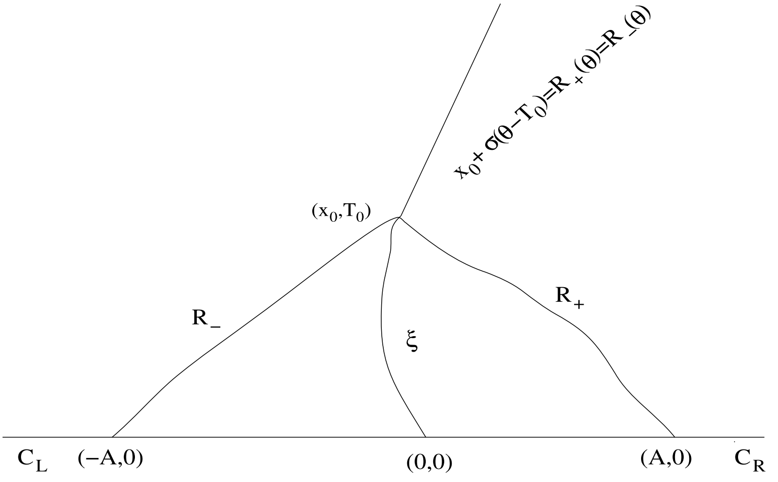

Fig. 1. $C_L > C_R$.

$$= -\int_{u_-}^{C_L} \eta_1'(\theta) f'(\theta)\, d\theta - f'(u_-)\eta_1(u_-)$$
$$\geqslant f'(u_-)\eta_1(u_-) - f'(u_-)\eta_1(u_-) = 0. \quad (3.39)$$

Hence last two integrals are non-negative in (3.38) and therefore we obtain

$$\int_{r_1(t)}^{x} \eta_1\big(u(y,t)\big)\, dy \leqslant \int_{y_-(R_-(T),T)}^{y_-(x,t)} \eta_1\big(u_0(y)\big)\, dy. \quad (3.40)$$

Since $u(y,t) = C_L$ for $y < r_1(t)$ and $u_0(y) = C_L$ for $y < y_-(R_-(T),T)$ and $\eta_1(C_L) = 0$, hence the above inequality implies that

$$\int_{-\infty}^{x} \eta_1\big(u(y,t)\big)\, dy \leqslant \int_{-\infty}^{y_-(x,t)} \eta_1\big(u_0(y)\big)\, dy. \quad (3.41)$$

Now take $\eta_1(u) = |u - C_L|^p$ in (3.41) to yield (3.36). Similarly (3.37) follows and hence the claim.

Now take $x = \xi(t)$ in (3.36) and (3.37) and observe that $y_-(\xi(t),t) \leqslant 0 \leqslant y_+(\xi(t),t)$. Therefore (3.36) and (3.37) yield

$$\int_{-\infty}^{\infty} \big|u(y,t) - \phi\big(y - \xi(t)\big)\big|^p\, dy \leqslant \int_{-\infty}^{\infty} \big|u_0(y) - \phi(y)\big|^p\, dy. \quad (3.42)$$

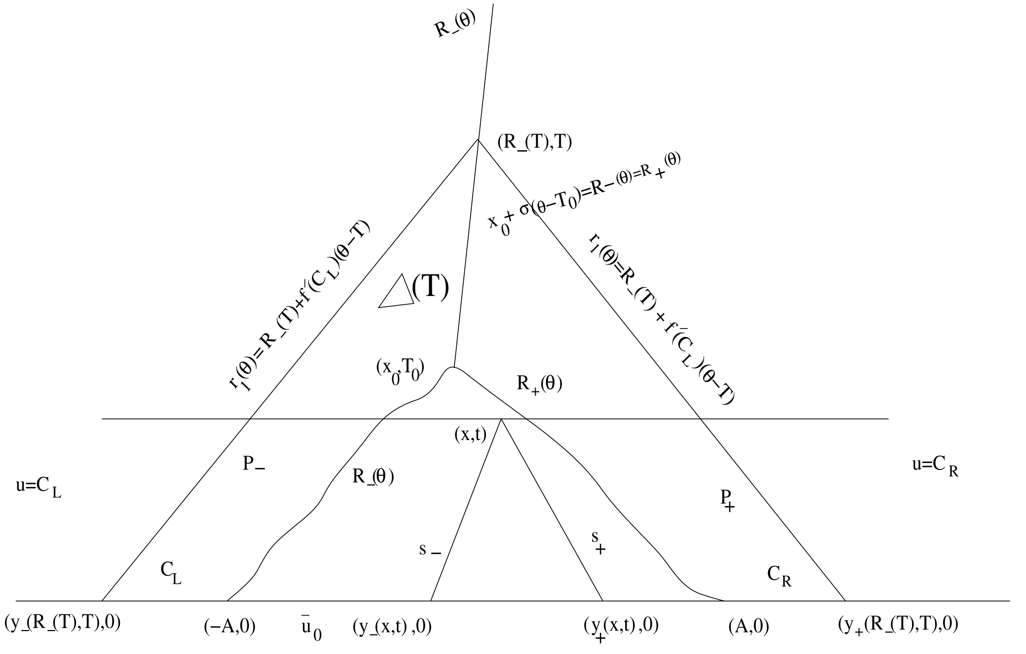

Fig. 2. $C_L > C_R$.

Step 2. Now let

$$\int_{-\infty}^{\infty} \left|u_0(x) - \phi(x)\right|^p dx < \infty. \tag{3.43}$$

For $M > 0$, define

$$u_{0,M}(x) = \begin{cases} u_0(x) & \text{if } |x| \leqslant M, \\ C_L & \text{if } x < -M, \\ C_R & \text{if } x > M. \end{cases}$$

Let u_M be the corresponding solution to (1.1) and ξ_M be as in (2.7). Since $u_{0,M} \to u_0$ in L^1_{loc} and hence for a.e. $t > 0$, a.e. x, $u_M(x,t) \to u(x,t)$. From (1) of Theorem 2.4 and Ascoli–Arzela's Theorem, for a sequence $M_k \to \infty$, $\xi_{M_k} \to \xi$ uniformly on compact subsets. Then from (3.42) and using Fatou's Lemma, we have for a.e. $t > 0$

$$\begin{aligned}
\int_{-\infty}^{\infty} \left|u(x,t) - \phi\big(x - \xi(t)\big)\right|^p dx &\leqslant \varliminf_{M\to\infty} \int_{-\infty}^{\infty} \left|u_M(x,t) - \phi\big(x - \xi_M(t)\big)\right|^p dx \\
&\leqslant \lim_{M\to\infty} \int_{-\infty}^{\infty} \left|u_{0,M}(x) - \phi(x)\right|^p dx
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \left|u_0(x) - \phi(x)\right|^p dx. \tag{3.44}$$

Let t be arbitrary. Then from the Lax–Oleinik explicit formula and from non-intersecting property, for any sequence $t_k \downarrow t$ such that for a.e. x, $\lim_{t_k \downarrow t} y_\pm(x, t_k) = y_\pm(x, t)$. Hence $u(x, t_k) \to u(x, t)$ for a.e. x. Now choose $t_k \downarrow t$ such that (3.44) holds at t_k and again Fatou's Lemma implies (3.44) holds for all $t > 0$.

Step 3. Let $C_L = C_R$.

First assume that u_0 satisfies (1.12) and define

$$u_{0,M}(x) = \begin{cases} C_L + \frac{1}{M} & \text{if } x < -A, \\ u_0(x) & \text{if } -A < x < A, \\ C_R & \text{if } x > A, \end{cases}$$

$$\phi_M(x) = \begin{cases} C_L + \frac{1}{M} & \text{if } x < 0, \\ C_R & \text{if } x > 0. \end{cases}$$

Then $u_{0,M} \to u_0$ in L^1_{loc} and hence the corresponding solution u_M of (1.1) converges to u a.e. (x, t). By letting $\to \infty$, (3.42) follows from the similar arguments as in the earlier case. From (3.42) and one more approximation gives (3.42) for all u_0. This proves (1.8). (1.12) follows from Lemma 3.1 and hence (1).

2. Let $y_\pm(t) = y_\pm(\sigma t, t)$, then from (1) of Theorem 2.1, there exists $M > 0$ depending only on $\|u_0\|_\infty$ and f such that for any $x \in \mathbb{R}$,

$$\left|\frac{x - y_\pm(t)}{t}\right| \leqslant M.$$

Let $\eta(t) = \frac{y_-(\sigma t, t) + y_+(\sigma t, t)}{2}$ and $x = \sigma t$. Then the above inequality implies that

$$\left|\frac{\eta(t)}{t} - \sigma\right| = \left|\frac{y_+(t) + y_-(t)}{2t} - \sigma\right| \leqslant M.$$

Let u_0 be as in (1.12). Let $x = \sigma t$ in (3.36) and (3.37) to obtain

$$\begin{aligned} \int_{-\infty}^{\sigma t} \left|u(x, t) - \phi(x - \sigma t)\right|^p dx &\leqslant \int_{-\infty}^{y_-(t)} \left|u_0(x) - C_L\right|^p dx \\ &\leqslant \int_{-\infty}^{\eta(t)} \left|u_0(x) - C_L\right|^p dx \end{aligned}$$

and

$$\int_{\sigma t}^{\infty} \left|u(x,t)-\phi(x-\sigma t)\right|^p dx \leqslant \int_{y_+(t)}^{\infty} \left|u_0(x)-C_R\right|^p dx$$
$$\leqslant \int_{\eta(t)}^{\infty} \left|u_0(x)-C_R\right|^p dx.$$

Adding these two inequalities we have

$$\int_{-\infty}^{\infty} \left|u(x,t)-\phi(x-\sigma t)\right|^p dx \leqslant \int_{-\infty}^{\infty} \left|u_0(x)-\phi\big(x-\eta(t)\big)\right|^p dx. \tag{3.45}$$

Let $u_{0,M}$ be as defined earlier and u_M be the corresponding solution. Let $y_{M,\pm}$ be the extreme characteristic points at (x,t) with data $u_{0,M}$. For $M_1 < M_2$, $u_{0,M_1}(x) = u_{0,M_2}(x)$ for $x \in [-M_1, M_1]$ and hence for any compact set $K \subset \mathbb{R} \times (0,\infty)$, there exists an $M(K) > 0$ such that for all $M > M(K)$, $u_M(x,t) = u(x,t)$, $y_{M,\pm}(x,t)$ for all $(x,t) \in K$ where u is the solution of (1.1) with initial u_0. Hence for all (x,t), we have

$$\lim_{M\to\infty} y_{M,\pm}(x,t) = y_\pm(x,t),$$
$$\lim_{M\to\infty} \eta_M(t) = \eta(t).$$

Furthermore for M large

$$\begin{aligned} \left|u_{0,M}(x)-\phi\big(x-\eta_M(t)\big)\right| &\leqslant \left|u_{0,M}(x)-\phi(x)\right| + \left|\phi(x)-\phi\big(x-\eta(t)\big)\right| \\ &\leqslant \left|u_0(x)-\phi(x)\right| + |C_L - C_R|\chi_{[-|\eta(t)|,|\eta(t)|]}(x). \end{aligned} \tag{3.46}$$

Let $M_0 > 0$, then from (3.45) and dominated convergence theorem

$$\begin{aligned} \int_{-M_0}^{M_0} \left|u(x,t)-\phi(x-\sigma t)\right|^p dx &\leqslant \lim_{M\to\infty} \int_{-\infty}^{\infty} \left|u_M(x,t)-\phi(x)\right|^p dx \\ &\leqslant \lim_{M\to\infty} \int_{-\infty}^{\infty} \left|u_{0,M}(x)-\phi\big(x-\eta_M(t)\big)\right|^p dx \\ &= \lim_{M\to\infty} \int_{-\infty}^{\infty} \left|u_{0,M}(x)-\phi\big(x-\eta(t)\big)\right|^p dx \\ &= \int_{-\infty}^{\infty} \left|u_0(x)-\phi\big(x-\eta(t)\big)\right|^p dx. \end{aligned}$$

This proves (2).

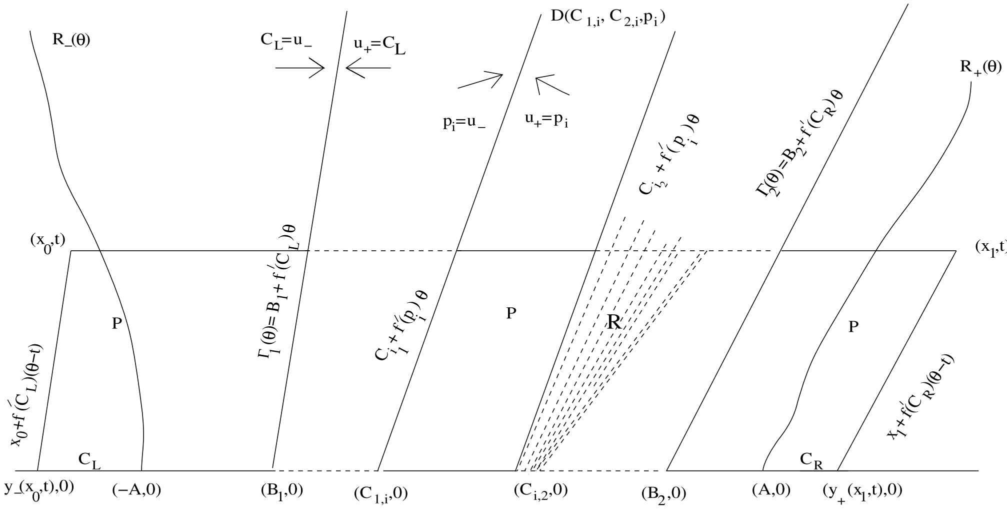

Fig. 3. $C_L \leqslant C_R$.

II. Let $C_L < C_R$ (see Fig. 3 for clear illustration).
From the Structure Theorem, decompose $\mathbb{R} \times (0, \infty)$ by

$$\mathbb{R} \times (0, \infty) = F_- \cup F_+ \cup D_- \cup D_+ \bigcup_{i \in I} D_i \cup R,$$
$$D_i = D(C_{1,i}, C_{2,i}, p_i).$$

Define

$$N_1(x,t) = \begin{cases} N(x,t) & \text{if } (x,t) \notin D_- \cup D_+, \\ C_L & \text{if } (x,t) \in D_-, \\ C_R & \text{if } (x,t) \in D_+, \end{cases} \tag{3.47}$$

$$N_0(x) = \begin{cases} C_L & \text{if } x < B_1, \\ p_i & \text{if } x \in [C_{1,i}, C_{2,i}], \\ C_R & \text{if } x > B_2, \\ p_x & \text{if } x \in [B_2, B_1] \setminus \bigcup_{i \in I}(C_{1,i}, C_{2,i}) \end{cases} \tag{3.48}$$

where p_x is defined as follows: for $x \notin \cup(C_{1,i}, C_{2,i})$, there exists a regular characteristic line r such that $r(0) = x$.

Define

$$f'(p_x) = \min\{r'(0)\colon\ r(0) = x,\ r \text{ is a regular characteristic line}\}.$$

Because of non-intersecting property of characteristic, $x \mapsto N_0(x)$ is a non-decreasing function.

Step 1. Inequality in ASSP: Let η be a convex function such that $0 = \eta(0) = \min_{\theta \in \mathbb{R}} \eta(\theta)$. Let

$$\begin{aligned} \eta_i(u) &= \eta(u - p_i), \\ F_i(u) &= \int_{p_i}^{u} \eta_i'(\theta) f'(\theta)\, d\theta. \end{aligned} \tag{3.49}$$

Then $\eta_i(p_i) = F_i(p_i) = 0$. Let $r_{j,i}(\theta) = C_{j,i} + \theta f'(p_i)$ and P be the parallelogram with vertices $(C_{1,i}, 0)$, $(C_{2,i}, 0)$, $(r_2(t), t)$, $(r_1(t), t)$. Integrating (3.32) in P to obtain

$$0 \geqslant \int_{r_{1,i}(t)}^{r_{2,i}(t)} \eta_i\big(u(x,t)\big)\, dx - \int_{C_{1,i}}^{C_{2,i}} \eta_i\big(u_0(x)\big)\, dx.$$

Since $u(r_{1,i}(\theta)+, \theta) = u(r_{2,i}(\theta)-, \theta) = p_i$ and $\eta(p_i) = F_i(p_i) = 0$. Hence

$$\int_{r_{1,i}(t)}^{r_{2,i}(t)} \eta\big(u(x,t) - p_i\big)\, dx \leqslant \int_{C_{1,i}}^{C_{2,i}} \eta\big(u_0(x) - p_i\big)\, dx. \tag{3.50}$$

Also on R, $u(x,t) = N(x,t)$ and hence $\eta(u(x,t) - N(x,t)) = 0$. Therefore

$$\begin{aligned} \int_{\Gamma_1(t)}^{\Gamma_2(t)} \eta\big(u(x,t) - N(x,t)\big)\, dx &= \sum_i \int_{r_{1,i}(t)}^{r_{2,i}(t)} \eta\big(u(x,t) - p_i\big)\, dx \\ &\leqslant \sum_i \int_{C_{1,i}}^{C_{2,i}} \eta\big(u_0(x) - p_i\big)\, dx \\ &\leqslant \int_{B_1}^{B_2} \eta\big(u_0(x) - N_0(x)\big)\, dx. \end{aligned} \tag{3.51}$$

Step 2. Inequality in $D_\pm \cup F_\pm$: Let $x_0 < R_-(t)$. Then the line between (x_0, t) and $(y_-(x_0,t), 0)$ lies in F_- and hence $u(x,t) = C_L$ on this line. Let P be the parallelogram with vertices at $(y_-(x_0,t), 0)$, $(B_1, 0)$, $(\Gamma_1(t), t)$, (x_0, t) and $\eta_1(u) = \eta(u - C_L)$, $F_1(u) = \int_{C_L}^{u} \eta_1'(\theta) f'(\theta)\, d\theta$. Since $u(\Gamma_1(t)-, t) = C_L$ and hence η_1 and F_1 vanish on Γ_1 and the line joining between (x_0, t) and $(y_-(x_0,t), 0)$. Hence integrating the entropy inequality (3.32) in P to obtain

$$0 \geqslant \int_{x_0}^{\Gamma_1(t)} \eta\big(u(x,t) - C_L\big)\, dx - \int_{y_-(x_0,t)}^{B_1} \eta\big(u_0(x) - C_L\big)\, dx$$

and this gives

$$\int_{-\infty}^{\Gamma_1(t)} \eta\big(u(x,t) - N(x,t)\big)\,dx \leqslant \int_{-\infty}^{B_1} \eta\big(u_0(x) - N_0(x)\big)\,dx. \tag{3.52}$$

Similarly

$$\int_{\Gamma_2(t)}^{\infty} \eta\big(u(x,t) - N(x,t)\big)\,dx \leqslant \int_{B_2}^{\infty} \eta\big(u_0(x) - N_0(x)\big)\,dx. \tag{3.53}$$

Combining (3.51) to (3.53) to obtain

$$\int_{-\infty}^{\infty} \eta\big(u(x,t) - N(x,t)\big)\,dx \leqslant \int_{-\infty}^{\infty} \eta\big(u_0(x) - N_0(x)\big)\,dx. \tag{3.54}$$

In particular if $\eta(u) = |u|^p$, then

$$\int_{-\infty}^{\infty} \big|u(x,t) - N(x,t)\big|^p\,dx \leqslant \int_{-\infty}^{\infty} \big|u_0(x) - N_0(x)\big|^p\,dx. \tag{3.55}$$

This proves the theorem. □

4. Examples

Here we give 2 examples to illustrate the Structure Theorem.

I. Let

$$u_0(x) = \begin{cases} C_L & \text{if } x < -A, \\ \alpha & \text{if } -A < x < A, \\ C_R & \text{if } x > A, \end{cases}$$

where $C_L > \alpha > C_R$. Then the solution u of (1.1) is given by

$$u(x,t) = \begin{cases} C_L & \text{if } x < \sigma_1 t - A,\ t < T_0, \\ \alpha & \text{if } \sigma_1 t - A < x < \sigma_2 t + A,\ t < T_0, \\ C_R & \text{if } x > \sigma_2 t + A,\ t < T_0 \end{cases}$$

and

$$u(x,t) = \begin{cases} C_L & \text{if } x < x_0 + \sigma(t - T_0),\ t > T_0, \\ C_R & \text{if } x > x_0 + \sigma(t - T_0),\ t > T_0 \end{cases}$$

where

$$\sigma_1 = \frac{f(C_L) - f(\alpha)}{C_L - \alpha}, \qquad \sigma_2 = \frac{f(C_R) - f(\alpha)}{C_R - \alpha}, \qquad \sigma = \frac{f(C_L) - f(C_R)}{C_L - C_R},$$
$$T_0 = \frac{2A}{\sigma_1 - \sigma_2}, \qquad x_0 = \sigma_1 T_0 - A.$$

II. $C_L > C_R$.

$$\gamma = \frac{1}{\sqrt{1 + (f'(C_L))^2}}\big(-1, f'(C_L)\big), \qquad \tilde{\gamma} = \frac{1}{\sqrt{1 + (f'(C_R))^2}}\big(1, -f'(C_R)\big),$$
$$\sigma = \frac{f(C_L) - f(C_R)}{C_L - C_R}.$$

Let $f(u) = \frac{u^2}{2}$ and let $A_1 < A_2$ and $I = (A_1, A_2)$. Let

$$u_0(x) = \begin{cases} 0 & \text{if } x < A_1 \text{ or } x > A_2, \\ -1 & \text{if } \frac{A_1+A_2}{2} < x < A_2, \\ 1 & \text{if } A_1 < x < \frac{A_1+A_2}{2} \end{cases}$$

and denote $u(x, t, I)$ the solution of (1.1) and given by

$$u(x,t,I) = \begin{cases} 0 & \text{if } x < A_1 \text{ or } x > A_2, \\ 1 & \text{if } A_1 - t < x < \frac{A_1+A_2}{2},\ t < T_0 = \frac{A_2-A_1}{2}, \\ -1 & \text{if } \frac{A_1+A_2}{2} < x < A_2,\ t < T_0, \\ \frac{x-A_1}{t} & \text{if } t > T_0,\ A_1 < x < \frac{A_1+A_2}{2}, \\ \frac{x-A_2}{t} & \text{if } t > T_0,\ \frac{A_1+A_2}{2} < x < A_2. \end{cases}$$

Let $A_1 < A_2 < A_3 < A_4 < A_5$ and $I_1 = (A_2, A_3)$, $I_2 = (A_4, A_5)$. Let

$$u_0(x) = \begin{cases} -10 & \text{if } x < A_1, \\ 0 & \text{if } A_1 < x < A_2 c, \\ 1 & \text{if } A_2 < x < \frac{A_1+A_2}{2},\ A_4 < x < \frac{A_4+A_5}{2}, \\ -1 & \text{if } \frac{A_1+A_2}{2} < x < A_2, \text{ or } \frac{A_4+A_5}{2} < x < A_5, \\ 0 & \text{if } A_3 < x < A_4, \\ 7 & \text{if } x > A_5. \end{cases}$$

In this case we have $B_1 = A_1$, $B_2 = A_5$, $R_-(t) = \Gamma_1(t) = A_1 - 10t$, $R_+(t) = \Gamma_2(t) = A_5 + 7t$.

$$D_1 = D(A_2, A_3, 0) = \big\{(x,t)\colon\ A_2 < x < A_3\big\},$$
$$D_2 = D(A_4, A_5, 0) = \big\{(x,t)\colon\ A_4 < x < A_5\big\},$$
$$R = [A_1, A_2] \times (0, \infty) \cup [A_3, A_4] \times (0, \infty) \cup \big\{(x,t)\colon\ A_1 - 10t \leqslant x \leqslant A_1\big\}$$
$$\cup \big\{(x,t)\colon\ A_5 \leqslant x \leqslant A_5 + 10t\big\},$$

$$N_1(x,t) = \begin{cases} 10 & \text{if } x < A_1 - 10t, \\ \frac{x-A_1}{t} & \text{if } A_1 - 10t < x < A_1, \\ \frac{x-A_5}{t} & \text{if } A_5 < x < A_5 + 7t, \\ 7 & \text{if } A_5 + 7t < x, \\ 0 & \text{otherwise}, \end{cases}$$

$$N_0(x) = \begin{cases} -10 & \text{if } x \leqslant A_1, \\ 0 & \text{if } A_1 < x \leqslant A_5, \\ 7 & \text{if } x > A_5. \end{cases}$$

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