ANALYSIS OF SPECTRAL METHODS FOR BURGERS' EQUATION

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Abstract.

This is the second in a series of three papers in which spectral and pseudospectral methods are analyzed for a class of time dependent nonlinear partial differential equations. In this paper, we first present a general framework for analyzing numerical methods for the evolutionary equations which admit semigroup formulations. This framework is then applied to spectral and pseudospectral methods for Burgers' equation, using trigonometric series, Chebyshev and Legendre polynomials. Optimal order of convergence is obtained which implies the spectral accuracy of these methods.

§ 0 Introduction

In this paper, we first present a general framework for analyzing numerical methods for the evolutionary equations which admit semigroup formulations. This framework is then applied to spectral and pseudospectral methods for Burgers' equation, using trigonometric series, Chebyshev and Legendre polynomials. Optimal order of convergence under suitable norms is proved for these methods in the sense that the error in the numerical solution is of the same order as the error in the initial approximation. As a consequence, we know that these methods achieve spectral accuracy if the exact solution to the differential equation is smooth enough.

There are many papers concerning the analysis of spectral methods. For linear evolutionary problems, we mention the important work of Kreiss and Oliger [8], Gottlieb and Orszag [6], Majda, McDonough and Osher [14]. For the nonlinear problems in incompressible flow, we mention the work of Canuto, Maday, Quarteroni and others which lays the foundation of this subject. The fundamental ingredients of their work seem to be the basic approximation theory for the spectral expansion and interpolation in the setting of Sobolev spaces ([1], [3], [16]) and the variational techniques for problems of elliptic type, including the steady-state Navier-Stokes equation [1], [11], 12], [13]. The abstract framework of Brezzi, Rappaz and Raviart plays an essential role in the analysis of these nonlinear problems [2].

Relatively little has been done for the full time-dependent Navier-stokes equation. A review of the available results is contained in [5] and is not going to be repeated here. The purpose of this paper is to show that the ideas in § 3 of [5] can be developed into a general framework which could be useful for other problems. We will show this by applying this framework to various types of spectral and pseudospectral methods for Burgers' equation.

Our fundamental tools are the semigroup formulation and the variation of constants formula. In § 1, we set up the problem and prove our main theorem (Theorem 1.1) in this framework. In § 2, § 3, § 4, we consider the application of this framework to the numerical solutions of Burgers' equation by Fourier, Legendre and Chebyshev methods respectively. These methods admit semigroup formulations if we define the underlying spaces and infinitesimal generators suitably. Our approach provides an alternative to the conventional one of using energy estimates.

This work was motivated when I was visiting IMA last fall. It is my pleasure here to thank the IMA staff for their hospitality. I am grateful to Professor Engquist who arranged the visit and who has shown constantly his interest on this work and encouragement. I am also grateful to Dr. Okamoto in IMA for pointing out to me reference [7].

§ 1. A General Framework for the Approximation of Evolutionary Equations

1.1 Linear Second-order Parabolic Equations.

We consider a general second order parabolic problem represented by the following abstract evolutionary equation: (higher order equations can be treated similarly).

$$\begin{cases} \dfrac{du}{dt} + Au = f(t) \\ \\ u(0) = a \end{cases} \tag{1.1}$$

in the space $X = L^2_\omega(\Omega)$. We have in mind the application to spectral methods, so we take general weighted Sobolev spaces with weight $\omega(x) > 0$. As usual, we define

$$(\varphi,\psi)_\omega = \int_\Omega \varphi(x)\psi(x)\omega(x)\,dx \quad , \quad \text{for} \quad \varphi,\psi \in L^2_\omega(\Omega)$$

$$\|\varphi\|^2_{0,\omega} = (\varphi,\varphi)_\omega$$

and for $k \in \mathbb{Z}$,

$$H^k_\omega(\Omega) = \{u \in L^2_\omega(\Omega), \qquad D^\alpha u \in L^2_\omega(\Omega) \quad , \quad \text{for} \quad |\alpha| \le k\}$$

with the norm

$$\|u\|_{k,\omega} = \Big(\sum_{|\alpha|\le k} \|D^\alpha u\|^2_{0,\omega}\Big)^{1/2}$$

$$H^k_{0,\omega}(\Omega) = \qquad \text{the closure of} \qquad C^\infty_0(\Omega) \qquad \text{in} \qquad H^k_\omega(\Omega).$$

For a non integer $s \in \mathbb{R}^+$, H^s_ω and $H^s_{0,\omega}$ are defined by interpolation [18]. $H^{-s}_\omega(\Omega) =$ dual of $H^s_{0,\omega}(\Omega)$ in the L^2_ω - inner product. For $v(t) \in C\ ([0,T], H^\sigma_\omega(\Omega))$, we denote

$$|||v(t)|||_{\sigma,\omega} = \sup_{0\le t\le T} \quad ||v(t)||_{\sigma,\omega}$$

The operator A in (1.1) is defined via a bilinear form $a(.,.) : H^1_{0,\omega}(\Omega) \times H^1_{0,\omega}(\Omega) \to R$ satisfying,

$$(A1) \qquad a(u,u) \ge c_0\,||u||^2_{1,\omega} \quad , \qquad \text{for} \qquad u \in H^1_{0,\omega}(\Omega)$$

$$(A2) \qquad |a(u,v)| \le c_1\,||u||_{1,\omega}\,||v||_{1,\omega} \quad , \qquad \text{for} \qquad u,v \in H^1_{0,\omega}(\Omega)$$

For $u \in \mathcal{D}(A) = H^2_\omega \cap H^1_{0,\omega}(\Omega)$, we define Au by, $Au \in X$, and $(Au,v)_\omega = a(u,v)$ for any $v \in H^1_{0,\omega}(\Omega)$.

These conditions are satisfied in general by a parabolic equation. From these conditions we immediately get,

Lemma 1.1. Suppose (A1) and (A2) are satisfied. Then $-A$ is the infinitesimal generator of an analytic semigroup in X, denoted by $\{e^{-tA}\}_{t\ge 0}$.

For a proof of this lemma, see Kato [7].

Now the usual estimates for analytic semigroups hold for $\{e^{-tA}\}_{t\ge0}$, namely, for $\alpha \ge 0 \quad , \quad t > 0$

$$\|A^{\alpha} e^{-tA}\| \leq C_{\alpha} t^{-\alpha} \tag{1.2}$$

Here and in the following, the operator norms are taken in the space of linear operators from X to X. We also have the following estimate on the resolvent of $A, R(\lambda, A) = (\lambda I - A)^{-1}$. Namely, there is a constant $\delta \in (\frac{\pi}{2}, \pi)$, such that $\sigma(A) \supset \sum_{\delta} = \{\xi \in \mathbb{C}, \quad |\arg \xi| < \delta\}$, where $\sigma(A)$ is the resolvent set of A, and for $0 \leq \alpha \leq 1, \quad \lambda \in \sum_{\delta}$,

$$\|A^{\alpha} R(\lambda, A)\| \leq \frac{C_{\alpha}}{|\lambda|^{1-\alpha}} \tag{1.3}$$

Let $\{V_N\}$ be a sequence of finite dimensional subspaces of $H^1_{0,\omega}(\Omega)$. Let $a_N(\cdot,\cdot), V_N \times V_N \to \mathbb{R}$ be a discretization of the bilinear form $a(\cdot,\cdot)$ satisfying,

(A1′) $\quad a_N(\varphi,\varphi) \geq C'_0 \; \|\varphi\|^2_{1,\omega} \quad , \qquad$ for $\quad \varphi \in V_N$

(A2′) $\quad |a_N(\varphi,\psi)| \leq C'_1 \; \|\varphi\|_{1,\omega} \; \|\psi\|_{1,\omega}, \qquad$ for $\quad \varphi, \psi \in V_N$

We consider the following approximation scheme for (1.1)

$$\begin{cases} u_N(t): \quad [0,T] \to V_N \\ \\ \dfrac{d\,u_N}{dt} + A_N\, u_N = f_N(t) \\ \\ u_N(0) = a_N \end{cases} \tag{1.4}$$

where A_N is defined by: $\quad A_N : V_N \to V_N$

$$(A_N\, v, \varphi)_{\omega} = a_N(v,\varphi) \qquad \text{for any} \qquad v, \varphi \in V_N.$$

$f_N(t) \in C\ ([0,T], V_N)$ is an approximation of $f(t)$.

Conditions (A1′) and (A2′) guarantee that the numerical scheme (1.4) has some smoothing properties as shown in the following lemma.

LEMMA 1.2. There exists a constant $\delta \in (\frac{\pi}{2}, \pi)$, such that $\sigma(A_N) \supset \sum_{\delta} = \{\xi \in \mathbb{C}, |\arg \xi| < \delta\}$. Moreover, we have the following estimates for $0 \leq \alpha \leq 1, \quad \lambda \in \sum_{\delta}$,

(1.5a)
$$\|A_N^\alpha \, e^{-A_N t} \, \varphi\|_{0,\omega} \leq \frac{C_\alpha}{t^\alpha} \|\varphi\|_{0,\omega}, \qquad \varphi \in V_N$$

(1.5b)
$$\|A_N^\alpha \, R(\lambda, A_N)\| \leq \frac{C_\alpha}{|\lambda|^{1-\alpha}}$$

where $C_\alpha{}'$ s are constants independent of N and φ.

This lemma is proved in [10].

In the following we will use C to denote generic constants which may have different values in different locations. The subscript ω will be omitted if $\omega(x) \equiv 1$.

1.2 Nonlinear Parabolic Equations.

In this subsection, we will consider the nonlinear evolutionary equation in X.

(1.6)
$$\begin{cases} \dfrac{du}{dt} + Au + F(t,u) = 0 \\ \\ u(0) = a \end{cases}$$

where A is the operator considered in the previous subsection. We will assume that this system has a unique solution which satisfies some regularity requirements specified below.

We consider an approximation scheme in the following form,

(1.7)
$$\begin{cases} u_N(t) : [0,T] \to V_N \\ \\ \dfrac{du_N}{dt} + A_N \, u_N + F_N(t, \; u_N) = 0 \\ \\ u_N(0) = P_N a \in V_N \end{cases}$$

where $F_N(t,v) : [0,T] \times V_N \to V_N$ is the approximation of the nonlinear term, P_N is an operator (a projection operator in some sense) chosen in each application,

$$P_N : X = L^2_\omega(\Omega) \to V_N$$

By using the variation of constants formula, we can write (1.6) and (1.7) in integral forms

$$u(t) = e^{-At}a - \int_0^t \quad e^{-A(t-s)} \; F\,(s, u(s))\, ds \tag{1.8a}$$

$$u_N(t) = e^{-A_N t}P_N a - \int_0^t \quad e^{-A_N(t-s)} \; F_N\,(s, u_N(s))\, ds \tag{1.8b}$$

Here (1.8a) makes sense under some smoothness conditions for u and F.

Next lemma is the version of Gronwall's inequality we are going to use. Many inequalities of this type can be found in [9]. The version to be used here is proved in [15].

LEMMA 1.3. Let T, α, β, ν be positive constants. $0 < \nu < 1$. Then for any continuous functions $f : [0, T] \to [0, \infty)$ satisfying

$$f(t) \leq \alpha + \beta \int_0^t \quad (t-s)^{-\nu} \; f\,(s) ds, \qquad 0 \leq t \leq T$$

we have

$$f(t) \leq C\alpha \exp\{C\beta^{\frac{1}{1-\nu}} \; t\}, \qquad \text{for} \qquad 0 \leq t \leq T.$$

with a positive constant C which depends only on ν.

We can now state the main results in this section.

THEOREM 1.1. Let $0 \leq \alpha, \beta \leq 1, 0 \leq t \leq T$. Assume that the semi-discrete approximation (1.7) satisfies:

$$\epsilon_N = \sup_{0 \leq t, \tau \leq T} \{\|A_N^\alpha(P_N e^{-A(t-\tau)}u(\tau) - e^{-A_N(t-\tau)}P_N u(\tau))\|_{0,\omega} + \tag{X}$$

$$+ \| \int_\tau^t A_N^\alpha\{P_N e^{-A(t-s)}F(s, u(s)) - e^{-A_N(t-s)} \; F_N\,(s, P_N u(s))\}\, ds\|_{0,\omega}\}$$

$$\to 0, \qquad \text{as} \qquad N \to +\infty$$

(Y)
$$\|A_N^{\alpha-\beta}\ \{F_N(s,v)-F_N(s,\varphi)\}\ \|_{0,\omega} \le L(\|A_N^\alpha v\|_{0,\omega},\ \|A_N^\alpha \varphi\|_{0,\omega})\ \|A_N^\alpha (v-\varphi)\|_{0,\omega}$$

$$\text{for} \qquad 0 \le s \le T, \qquad v,\varphi \in V_N$$

where $L(\cdot,\cdot)$ is an increasing function with respect to each argument and $u(t)$ is the true solution to (1.1).

(Z)
$$\|\|A_N^\alpha P_N u(t)\|\|_{0,\omega} \le K$$

Then there exist constants $N_o(T), K_o(T)$, such that for $N > N_o(t)$, (1.7) has a unique solution $u_N(t) : [0,T] \to V_N$. Moreover, we have for $0 \le t \le T$

(1.9a)
$$\|A_N^\alpha u_N(t)\|_{0,\omega} \le K+1$$

(1.9b)
$$\|A_N^\alpha (u_N(t) - P_N u(t))\|_{0,\omega} \le K_o(T)\,\epsilon_N$$

Before proving the theorem, we make some remarks.

REMARK 1. Assumption (X) says that our numerical method should converge for the linear problems (discussed in previous subsection) in some sense and the approximation for the nonlinear term should be consistent. Assumption (Y) plays the role of a stability condition as we will see in the proof. Notice that it is a very weak form of the stability assumption which corresponds to the local Lipschitz condition on F needed for the existence of a local solution of (1.6) [9].

REMARK 2. Theorem 1.1 actually gives a way of extending approximation results for linear problems to those of the nonlinear problems. It is in the spirit of Brezzi, Rappaz and Raviart's framework for steady equations.

We will first prove a local version of Theorem 1.1.

LEMMA 1.4. Under the assumptions of Theorem 1.1, there exist constants N_0, r and t_0, such that for $0 \le t \le t_0$, (1.8b) has a unique solution $u_N(t)$. Moreover we have

(1.10a)
$$\|A_N^\alpha u_N(t)\|_{0,\omega} \le K+1$$

(1.10b) $$\|A_N^\alpha(u_N(t) - P_N u(t))\|_{0,\omega} \leq \epsilon_N\, r$$

for $0 \leq t \leq t_0$

Proof. Set $S^0 = \{v(t) : [0, t_0] \to v_N, \qquad v(0) = P_N\, a,$

$$\sup_{0 \leq t \leq t_0} \|A_N^\alpha(v(t) - P_N\, u(t))\|_{0,\omega} \leq 1\}$$

For $\varphi \in C\,([0, t_0], V_N)$, define

$$|\varphi|_\alpha = \sup_{0 \leq t \leq t_0} \|A_N^\alpha \varphi\|_{0,\omega}$$

For $v \in S^0$, define for $0 \leq t \leq t_0$

$$Gv(t) = e^{-A_N t} P_N a - \int_0^t e^{-A_N(t-s)}\, F_N\,(s, v(s))\, ds$$

It is clear that solving (1.8b) is equivalent to finding a fixed point of G in S^0. We will do this by using the contraction principle.

First we check that $P_N u(t)$ satisfies approximately the fixed point equation, $v = Gv$. In fact, from (1.8a), we have

$$\begin{aligned} P_N(u(t) &= P_N e^{-At} a - \int_0^t P_N e^{-A(t-s)}\, F(s, u(s))\, ds \\ &= e^{-A_N t} P_N a - \int_0^t e^{-A_N(t-s)}\, F_N(s, P_N u(s))\, ds \\ &+ (P_N e^{-At} a - e^{-A_N t} P_N a) \\ &- \int_0^t \{P_N e^{-A(t-s)}\, F(s, u(s)) - e^{-A_N(t-s)}\, F_N\,(s, P_N u(s))\}\, ds \\ &= G P_N u(t) + R \end{aligned}$$

¿From assumption (X), $\|A_N^\alpha R\|_{0,\omega} \to 0$, as $N \to +\infty$

Next, let $v, \varphi \in S^0$, then

$$Gv - P_N u = Gv - GP_N u - R$$

$$\|A_N^\alpha(Gv - P_N u)\|_{0,\omega} \leq \epsilon_N + \|A_N^\alpha(Gv - GP_N u)\|_{0,\omega}$$

$$\leq \epsilon_N + \| \int_0^t A_N^\beta e^{-A_N(t-s)} A_N^{\alpha-\beta}\{F_N(s, v(s)) - F_N(s, P_N u)\}\, ds \,\|_{0,\omega}$$

$$\leq \epsilon_N + C_\beta \int_0^t (t-s)^{-\beta} \, \|A_N^{\alpha-\beta}\, \{F_N(s, v(s)) - F_N(s, P_N u)\}\|_{0,\omega}\, ds$$

Notice that $\|A_N^\alpha v(t)\|_{0,\omega} \leq \|A_N^\alpha P_N u(t)\|_{0,\omega} + 1$. From assumption (Y), (Z), we get

$$\|A_N^{\alpha-\beta}\{F_N(s, v(s)) - F_N\,(s, P_N u(s))\}\|_{0,\omega}$$

$$\leq L\,(\|A_N^\alpha P_N u(t)\|_{0,\omega} + 1\;,\;\; \|A_N^\alpha P_N u(t)\|_{0,\omega} + 1)\|A_N^\alpha(v - P_N u)\|_{0,\omega}$$

$$\leq L_0\; \|A_N^\alpha(v - P_N u)\|_{0,\omega} \leq L_0$$

where $L_0 = L\,(K+1, K+1)$

Therefore $\|A_N^\alpha\,(Gv - P_N u)\|_{0,\omega} \leq \epsilon_N + C_\beta \int_0^t (t-s)^{-\beta}\, ds$

$$\leq \epsilon_N + \frac{C_\beta L_0}{1-\beta} t_0^{1-\beta}$$

Similarly, we have

$$\|A_N^{\alpha-\beta}\{F_N(s, v(s)) - F_N(s, \varphi(s))\}\|_{0,\omega} \leq L_0 \|A_N^\alpha\,(v - \varphi)\|_{0,\omega}$$

and

$$\|A_N^\alpha(Gv(t) - G\varphi(t))\|_{0,\omega} \leq \| \int_0^t A_N^\beta e^{-A_N(t-s)} A_N^{\alpha-\beta}\; \{F_N(s, v(s)) - F_N(s, \varphi(s))\}\, ds\|_{0,\omega}$$

$$\leq \frac{C_\beta L_0}{1-\beta}\; t_0^{1-\beta} \sup_{0 \leq t \leq t_0} \|A_N^\alpha(v(s) - \varphi(s))\|_{0,\omega}$$

i.e. $|Gv - G\varphi|_\alpha \leq \frac{C_\beta L_0}{1-\beta}\; t_0^{1-\beta}\; |v - \varphi|_\alpha$

Now if we choose t_0, such that

$$\frac{C_\beta L_0}{1-\beta}\ t_0^{1-\beta} < \frac{1}{2}$$

and N_0, such that for $N > N_0$, $\quad \epsilon_N < \frac{1}{2}$. Then

$$|Gv - P_N u|_\alpha < \frac{1}{2}$$

$$|Gv - G\varphi|_\alpha < \frac{1}{2}\ |v-\varphi|_\alpha$$

Notice that Gv solves the linear equation

$$\begin{cases} \dfrac{d}{dt}\ Gv + A_N Gv + F_N(t, v(t)) = 0 \\ \\ Gv(0) = P_N a \end{cases}$$

Therefore $Gv \in C\left([0, t_0], V_N\right)$. Hence G defines a contraction on S^0. We conclude that G has a unique fixed point $u_N(t)$ in S^0 which solves (1.8b).

(1.10a) is obviously satisfied by $u_N(t)$. Let's prove (1.10b).

Let $e_N(t) = P_N u(t) - u_N(t)$, then

$$e_N(t) = R + P_N u(t) - G u_N(t)$$

By using the same estimates as we had before, we get

$$\|A_N^\alpha e_N(t)\|_{0,\omega} \leq \epsilon_N + C_\beta L_0 \int_0^t (t-s)^{-\beta} \|A_N^\alpha e_N(s)\|_{0,\omega}\ ds$$

Using Lemma 1.3, we get, for $0 \leq t \leq t_0$

$$\|A_N^\alpha e_N(t)\|_{0,\omega} \leq \epsilon_N r$$

where r is a constant depending on β only. this proves the Lemma. $\square$

Proof of Theorem 1.1.

We will use an induction argument. Specifically, we will prove that the constructive procedure used in the previous lemma works for $[0,\ t_0]$, $[t_0,\ 2t_0], \ldots$ with the following estimates

$$\|A_N^\alpha(u_N(t)-P_Nu(t))\|_{0,\omega} \le \epsilon_N r(c_0 r+1)^n \qquad \text{for} \qquad n\,t_0 \le t \le (n+1)\,t_0$$

where $u_N(t)$ is the constructed solution in $[n\,t_0,\,(n+1)\,t_0]$. Of course for $nt_0 > T$, the procedure stops at $[(n-1)t_0, T]$.

Let $t^n = nt_0$. Lemma 1.4 proves the above statement for $n = 0$. Assume this can be done for $[0,t^0], [t^0,2t^0], \ldots [t^{n-1},t^n]$, we proceed to show that the same thing is true on $[t^n, t^{n+1}]$.

Let $S^n = \{v \in C\left([t^n,t^{n+1}], V_N\right), \qquad v(t^n) = u_N(t^n)$

$$\sup_{t^n \le t \le t^{n+1}} \|A_N^\alpha(v-P_Nu)\|_{0,\omega} \le 1\}$$

where $u_N(t^n)$ is given by the construction on $[t^{n-1},t^n]$.

Define $|v|_\alpha = \sup\limits_{t^n \le t \le t^{n+1}} \|A_N^\alpha v\|_{0,\omega}$

For $v \in S^n$, define

$$Gv = e^{-A_N(t-t^n)}u_N(t^n) - \int_{t^n}^t e^{-A_N(t-s)}\, F_N\,(s, v(s))\,ds$$

Our purpose is to solve the fixed point equation $v = Gv$ on S^n. First we check that $P_N(u(t))$ satisfies this equation approximately.

Indeed from (1.8a), we have

$$P_Nu(t) = P_Ne^{-A(t-t^n)}u(t^n) - \int_{t^n}^t P_Ne^{-A(t-s)}\,F\,(s,u(s))\,ds$$

$$= GP_Nu(t) + R'$$

where $R' = P_Ne^{-A(t-t^n)}u(t^n) - e^{-A_N(t-t^n)}u_N(t^n)$

$$-\int_{t^n}^t P_Ne^{-A(t-s)}\,F\,(s,u(s))\,ds + \int_{t^n}^t e^{-A_N(t-s)}\,F_N\,(s,P_Nu(s))\,ds$$

Therefore

$$\|A_N^\alpha R'\|_{0,\omega} \le \|A_N^\alpha e^{-A_N(t-t^n)}(P_N u(t^n) - u_N(t^n))\|_{0,\omega} + \epsilon_N$$

$$\le C_0 \,\|A_N^\alpha(P_N u(t^n) - u_N(t^n))\|_{0,\omega} + \epsilon_N$$

¿From the induction assumption, we get,

$$\|A_N^\alpha R'\|_{0,\omega} \le C_0 r(C_0 r + 1)^{n-1}\,\epsilon_N + \epsilon_N \le \epsilon_N\,(C_0 r + 1)^n$$

Next, let $v, \varphi \in S^n$. Similar to what we did in the lemma, we can get

$$|Gv - P_N\, u|_\alpha < (C_0 r + 1)^n \epsilon_N + \frac{1}{2}$$

$$|Gv - G\varphi|_\alpha < \frac{1}{2}\,|v - \varphi|_\alpha$$

Choose $N_0(T)$, such that for $N > N_0(T)$,

$$(C_0\, r + 1)^n \in_N < \frac{1}{2} \tag{1.11}$$

Then G defines a contraction on S^n. Therefore we get a fixed point, again denoted by $u_N(T)$, which is the solution of (1.8b) on $[t^n, t^{n+1}]$. The estimates of $P_N u(t) - u_N(t)$ on $[t^n, t^{n+1}]$ follows from the same argument as we used in the lemma.

Now the induction argument is complete.

Notice that we are considering a finite time interval $[0, T], (C_0 r + 1)^n \le (C_0 r + 1)^{T/t_0}$, which is a fixed number, say $K_0'(T)$. Assumption (1.11) is fulfilled if we let $\epsilon_N < \frac{1}{2K_0'(T)}$. Take $K_0(T) = r\ \ K_0'(T)$ leads to (1.9b). (1.9a) is obvious from our construction.

By patching together these local solutions we get a function defined on $[0, T]$, again denoted by $u_N(t)$. It follows from standard ODE argument that $u_N(t)$ is indeed the unique solution of (1.7), and it is smooth in time. □

§ 2. Fourier-Collocation Method for Burgers' Equation

The equation we are going to study in this section is a standard model in fluid mechanics, namely the Burgers' equation.

$$
\begin{cases}
u_t - u_{xx} + \frac{1}{2}(u^2)_x = f \quad , \quad t > 0, \quad x \in I = (-\pi, \pi) \\
u(x,0) = a(x) \\
\text{periodic boundary condition}
\end{cases}
\tag{2.1}
$$

For $m \geq 0$, define

$$
H_p^m = \{u = \sum_{k \in \mathbb{Z}} C_k e^{ikx} \quad , \quad \bar{C}_k = C_{-k}, \tag{2.2}
$$

$$
\|u\|_m^2 = \sum_{k \in \mathbb{Z}} (1 + |k|)^{2m} |C_k|^2 < +\infty\}
$$

For $m < 0$, define

$$
H_p^m = \quad \text{the dual of} \quad H_p^{-m}
$$

For $u \in H_p^2 = D(A)$ define $Au = -u_{xx} + u \in H_p^0$. It is easy to see that A can be extended as a positive definite linear self-adjoint operator on H_p^0. Therefore we can define the powers of A, A^α, for $\alpha \in \mathbb{R}$. Furthermore, we have $D(A^\alpha) = H_p^{2\alpha}$ and there exist constants C_1 and C_2, such that

$$
C_1 \|u\|_{2\alpha} \leq \|A^\alpha u\|_0 \leq C_2 \|u\|_{2\alpha} \quad , \quad \text{for any} \quad u \in D(A^\alpha) \tag{2.3}
$$

Now (2.1) can be written in the form,

$$
(2.4) \qquad \begin{cases} u(t): \ [0,T] \to H_p^1 \\ \dfrac{du}{dt} + Au + F(t,u) = 0 \\ u(0) = a \end{cases}
$$

where $F(t,u) = \frac{1}{2}(u^2)_x - u - f(t)$

Note that $-A$ generates an analytic semigroup (denoted by $\{e^{-tA}\}_{t\geq 0}$) in H_p^0, and the following estimates hold for $\alpha \geq 0$,

$$
(2.5) \qquad \|A^\alpha e^{-tA}\| \leq M_\alpha t^{-\alpha} e^{-\delta t}
$$

with some constants δ and M_α.

S_N is the space of trigonometric polynomials of degree N

$$
S_N = \{u = \sum_{|k|\leq N} C_k e^{ikx}\}
$$

For a continuous periodic function $\varphi(x)$ on I, we define its Fourier interpolatant $P_c\varphi(x)$ by,

$$
P_c\varphi(x) \in S_N, \quad P_c\varphi(x_j) = \varphi(x_j)
$$

where $x_j = \frac{2\pi}{2N+1} j$ for $|j| \leq N$. More precisely,

$$
(2.6) \qquad P_c\varphi(x) = \sum_{|k|\leq N} C_k e^{ikx}
$$

where $C_k = h \sum_{|j|\leq N} \varphi(x_j) e^{-ix_j k}, \qquad h = \frac{2\pi}{2N+1}$.

The approximation property of P_c is given in the following lemma.

LEMMA 2.1 [3]. There exists a constant C, such that for $u \in H_p^m$, $m > \frac{1}{2}$, $0 \leq \mu \leq m$, we have

$$
(2.7) \qquad \|u - P_c u\|_\mu \leq C N^{\mu - m} \, \|u\|_m
$$

We are going to consider the Fourier-Collocation approximation of (2.1) formulated as the following problem

$$
(2.8) \qquad \begin{cases} \text{Find} \quad u_N(x,t) \in C([0,T], S_N), \quad s.t. \quad \text{for} \quad |j| \leq N \\ \\ \dfrac{\partial u_N}{\partial t}(x_j) - \dfrac{\partial^2 u_N}{\partial x^2}(x_j) + u_N(x_j)\,\dfrac{\partial u_N}{\partial x}(x_j) = f(x_j) \\ \\ u_N(x_j, 0) = a(x_j) \end{cases}
$$

Equation (2.8) can be equivalently written as

$$
\frac{\partial u_N}{\partial t} - \frac{\partial^2 u_N}{\partial x^2} + \frac{1}{2}\, P_c\, (u_N^2)_x = P_c\, f(t)
$$

This can be put into the dynamical form (1.7) by taking $A_N = A$, $F_N(t, v) = \frac{1}{2}\, P_c\,(v_x^2) - v - P_c f(t)$ (denoted also by $F_c(t, v)$) for $v \in S_N$.

Before proving the main results of this section, we state a lemma which will be used in the theorem below.

LEMMA 2.2. $\|P_c \varphi\|_0 \leq 3\|\varphi\|_0$ for $\varphi \in S_{2N}$ where P_c is the Fourier interpolation operator in S_N.

This lemma is a simple consequence of the aliasing expansion [17].

Now we can prove our main estimates for the Fourier-Collocation method.

THEOREM 2.1. Assume for $\sigma \geq 2$, $u(t) \in C([0,T], H_p^\sigma), f(t) \in C([0,T], H_p^{\sigma-1})$. Then there exist $K_0(T), N_0(T)$, such that for $N > N_0(T)$, (2.8) has a unique solution $u_N(t) \in C([0,T], S_N)$.

Furthermore, we have

(2.9a) $$|||u_N(t)|||_1 \leq |||u(t)|||_1 + 1$$

(2.9b) $$\|u_N(t) - u(t)\|_1 \leq \frac{K_0(T)}{N^{\sigma-1}} \left(|||u(t)|||_\sigma^2 + \|a\|_\sigma + |||f(t)|||_{\sigma-1}\right)$$

Proof. First of all, we note that with very little change, the framework developed in the previous section carries over to the case of periodic boundary conditions. It is easy to see that the basic assumptions (A1) and (A2) are satisfied by the operator A. Next we observe that, the restriction of A to S_N gives A_N. Therefore, for a proper version of the conditions (X), (Y) and (Z), we can replace A_N by A, and $P_N u(t)$ by $u(t)$. This allows us to obtain more accurate estimates. Now let's estimate ϵ_N for $\alpha = \frac{1}{2}$.

¿From (2.7), we have

$$\|A^{\frac{1}{2}} e^{-A(t-\tau)}(u(\tau) - P_c\, u(\tau))\|_0 \leq C\, \|u(\tau) - P_c u(\tau)\|_1 \leq \frac{C}{N^{\sigma-1}}\, \|u(\tau)\|_\sigma$$

Apply (2.7) to $(u^2)_x$ and use the fact that $\|u^2\|_\sigma \leq c\, \|u\|_\sigma^2$, we get

$$\begin{aligned} &\| \int_\tau^t A^{\frac{1}{2}} e^{-A(t-s)} \{F(s, u(s)) - F_c(s, u(s))\}\, ds \|_0 \\ &\leq C \int_\tau^t (t-s)^{-\frac{1}{2}} \{\|(u^2)_x - P_c\, (u^2)_x\, \|_0 + \|f - P_c f\, \|_0\}\, ds \\ &\leq \frac{C}{N^{\sigma-1}} \left(|||u(s)|||_\sigma^2 + |||f(s)|||_{\sigma-1}\right) . \end{aligned}$$

Hence, we have

(2.10) $$\epsilon_N \leq \frac{C}{N^{\sigma-1}} \left(|||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)||||_{\sigma-1}\right)$$

Next we check condition (Y) for $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$. For $v, \varphi \in S_N$, from Lemma 2.2, we have

$$\|F_c(s,\varphi) - F_c(s,v)\|_0 \leq \frac{1}{2}\, \|P_c\,(\varphi_x^2 - v_x^2)\|_0 + \|v-\varphi\|_0$$

$$\leq C\,(\|\varphi_x^2 - v_x^2\|_0 + \|v-\varphi\|_0)$$

$$\leq C\,(\|\varphi + v\|_1\; \|\varphi - v\|_1 + \|\varphi - v\|_0)$$

$$\leq C\,(1 + \|A^{\frac{1}{2}}\varphi\|_0 + \; \|A^{\frac{1}{2}}v\|_0)\; \|A^{\frac{1}{2}}(v-\varphi)\|_0$$

Hence (Y) is satisfied. It is easy to see that (Z) is satisfied. Now (2.9a) and (2.9b) follows from (2.10) and (1.9a) and (1.9b). □

REMARK 1. Another version of the Fourier-Collocation method appeared in the literature [12], uses the following formulation

$$(2.11)\qquad \begin{cases} \text{Find} \qquad u_N(t) : [0,T] \to S_N \\[2ex] \dfrac{\partial u_N}{\partial t} - \dfrac{\partial^2 u_N}{\partial x^2} + \dfrac{1}{2}\,(P_c u_N^2)_x = P_c\, f(t) \\[2ex] u_N(0) = P_c a \end{cases}$$

With this formulation, the above argument carries through. Therefore an optimal error estimate follows directly from Theorem 1.1. Furthermore, for this formulation, it is straightforward to prove an additional optimal L^2-estimates under the assumptions in Theorem 2.1

$$(2.12)\qquad \|u(t) - u_N(t)\|_0 \leq \frac{K_0(T)}{N^\sigma}\;(|||u(t)|||_\sigma^2 + \|a\|_\sigma + |||f(t)|||_{\sigma-1})$$

for $N > N_0(T)$, where $N_0(T)$ and $K_0(T)$ are constants independent of N.

§ 3. Legendre Methods for Bergers' Equation

Spectral and pseudospectral methods using Legendre polynomials are potentially another type of methods having spectral accuracy. They are good candidates for problems with general boundary conditions. Perhaps the disadvantage of these methods, compared with Fourier methods or Chebyshev methods, is that we don't have fast transform devices for them. This, however, might be minimized with the advance of parallel computers.

3.1 Legendre-Galerkin Method for Burgers' Equation.

The equation we will deal with is the following

$$
(3.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{2}\,(u^2)_x = f(t,x) \quad , \qquad t > 0, \quad x \in [-1,1] = I \\ u(-1,t) = u(1,t) = 0 \\ u(x,0) = a(x) \end{cases}
$$

Since the weight function $\omega(x) = 1$, we will use the conventional Sobolev spaces. For $u, v \in H_0^1(I)$, define

$$
(3.2) \qquad a(u,v) = \int_I u_x v_x \, dx
$$

and for $u \in D(A) = H^2 \cap H_0^1(I)$, define

$$
(3.3) \qquad Au = -u_{xx}
$$

It's clear that assumptions (A1) and (A2) are satisfied. A can be extended as a self-adjoint, positive definite operator on $L^2(I)$. Therefore its fractional powers are well-defined. Furthermore, we have, for $u \in D(A^\alpha)$, $\quad \alpha \in \mathbb{R}$

$$
(3.4) \qquad \frac{1}{C}\,\|u\|_{2\alpha} \le \|A^\alpha u\|_0 \le C\,\|u\|_{2\alpha}
$$

where C is a constant depending only on α. As a consequence of (1.2) and (3.4), we have, for $\sigma \geq 0, v \in D(A^{\frac{\sigma}{2}})$,

$$\|e^{-At}v\|_\sigma \leq C\,\|v\|_\sigma \tag{3.5}$$

Now (3.1) can be written in a dynamical form,

$$\begin{cases} u(t) : [0,T] \to H_0^1(I) \\ \\ \dfrac{du}{dt} + Au + F(t,u) = 0 \\ \\ u(0) = a \end{cases} \tag{3.6}$$

where $F(t,u) = \frac{1}{2}\,(u^2)_x - f(t)$

Next, we turn into the Legendre-Galerkin approximation of (3.1). Let

$$S_N = \{\varphi, \quad \varphi \quad \text{is a polynomial on} \quad [-1,1] \quad \text{of degree} \quad \leq N\}.$$

$$V_N = S_N \cap H_0^1(I)$$

$$P_N : L^2(I) \to V_N \quad \text{is the} \quad L^2-\text{projection operator:} \qquad \text{for} \quad v \in L^2(I),$$

$$(v - P_N\,v, \varphi) = 0, \quad \text{for} \quad \varphi \in V_N$$

The Legendre-Galerkin method can be formulated as the following problem

$$\begin{cases} \text{Find} \quad u_N(t) : [0,T] \to V_N, \quad \text{such that for every} \quad \varphi \in V_N, \\ \\ \left(\dfrac{\partial u_N}{\partial t} - \dfrac{\partial^2 u_N}{\partial x^2}, \varphi\right) + \dfrac{1}{2}\,((u_N^2)_x, \varphi) = (f, \varphi) \\ \\ u_N(0) = P_N a \end{cases} \tag{3.7}$$

For $v, \varphi \in V_N$, let $a_N(v,\varphi) = -\int_I \frac{\partial^2 v}{\partial x^2}\,\varphi\,dx = a(v,\varphi)$. Then assumptions $(A1')$ and $(A2')$ are trivially satisfied. Furthermore, our numerical scheme can be put into the form of (1.7),

$$
\left[\begin{array}{l}
u_N(t) : [0,T] \to V_N \\
\\
\dfrac{du_N}{dt} + A_N u_N + F_N\,(t, u_N) = 0 \\
\\
u_N(0) = P_N a
\end{array}\right.
\tag{3.8}
$$

where for $v \in V_N$, $\quad F_N(t,v) = P_N\left(\frac{1}{2}\,(v_x^2) - f(t)\right), A_N$ is defined in § 1 via $a_N(\cdot,\cdot)$.

Before going into the framework set up by Theorem 1.1, let's first establish some properties of the operator A and A_N.

LEMMA 3.1. a) For $0 \le \alpha \le \frac{1}{2}$, there is a constant C, such that for any $\varphi \in V_N$,

$$
\|A^\alpha \varphi\|_0 \le C\ \|A_N^\alpha \varphi\|
\tag{3.9a}
$$

b) , For $0 \le \alpha \le 1$, there is a constant C, such that for any $\varphi \in V_N$,

$$
\|A_N^\alpha \varphi\|_0 \le C\ \|A^\alpha \varphi\|_0
\tag{3.9b}
$$

Proof. a) Let $\varphi \in V_N$, then

$$
(A^{\frac{1}{2}}\,\varphi, A^{\frac{1}{2}}\,\varphi) = (A\varphi,\varphi) = a(\varphi,\varphi) = (A_N\varphi,\varphi) = (A_N^{\frac{1}{2}}\,\varphi, A_N^{\frac{1}{2}}\,\varphi)
$$

Hence

$$
\|A^{\frac{1}{2}}\,\varphi\|_0 \le \|A_N^{\frac{1}{2}}\,\varphi\|_0
$$

Obviously (3.9a) holds for $\alpha = 0$. For $0 < \alpha < \frac{1}{2}$, (3.9a) follows from Heinz-Kato inequality.

The proof of (3.9b) is similar. □

LEMMA 3.2. Following estimates hold,

a) For $v \in H^\sigma \cap H_0^1(I)$, $\quad 0 \le \mu \le 1, \quad \mu \le \sigma$

$$\|v - P_N v\|_\mu \le C N^{\frac{3}{2}\mu - \sigma} \|v\|_\sigma \tag{3.10a}$$

b) For $g \in H^{\sigma-2}$, $\quad 0 \le \mu \le 1 \le \sigma$

$$\|(P_N A^{-1} - A_N^{-1} P_N) g\|_\mu \le C N^{\frac{3}{2}\mu - \sigma} \|g\|_{\sigma-2} \tag{3.10b}$$

c) For $g \in H^{-1}(I)$,

$$\|A_N^{-\frac{1}{2}} P_N g\|_0 \le C \|A^{-\frac{1}{2}} g\|_0 \le C \|g\|_{-1} \tag{3.10c}$$

d) For $\varphi \in V_N$, $\quad 0 \le \alpha \le \frac{1}{2}$.

$$\|A_N^{-\alpha} \varphi\|_0 \le C \|A^{-\alpha} \varphi\|_0 \tag{3.10d}$$

Proof. (3.10a) is proved in [3]. (3.10b) follows from (3.10a) and Theorem 1.6 in [11]. If we let $v_N = A_N^{-1} P_N g, v = A^{-1} g$, then it is easy to see that $a(v_N, v_N) \le a(v, v)$. This proves the first half of (3.10c) whereas the second half is even more obvious. (3.10d) follows from (3.10c) and the Heinz-Kato inequality.

LEMMA 3.3. Let $\alpha = \frac{1}{2}$. Assume that for $\sigma \ge 2, u(t) \in C\left([0,T], H^\sigma \cap H_0^1\right), f(t) \in C\left([0,T], H^{\sigma-1}\right)$. Then there is a constant C, such that

$$\epsilon_N \le C N^{\frac{3}{2} - \sigma} \left(|||u(t)|||_\sigma^2 + |||u(t)|||_\sigma + |||f(s)|||_{\sigma-1} \right) \tag{3.11}$$

Proof. We first recall a resolvent identity (see [10])

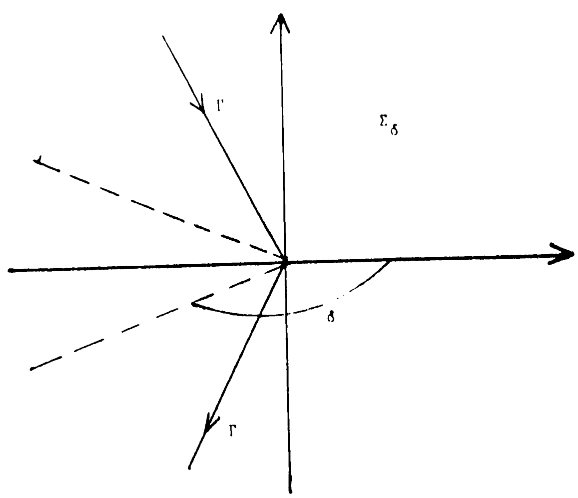
$$(3.12)\qquad P_N R(\lambda, A) - R(\lambda, A_N)\, P_N = A_N R(\lambda, A_N)(P_N A^{-1} - A_N^{-1} P_N) A R(\lambda, A)$$

From the Dunford integral formula, we get

$$(3.13)\qquad P_N e^{-tA} a - e^{-tA_N} P_N a = \int_\Gamma e^{\lambda t}[P_N R(\lambda, A) - R(\lambda, A_N)\, P_N]\, a d\lambda$$

$$= \int_\Gamma e^{\lambda t} A_N R(\lambda, A_N)[P_N A^{-1} - A_N^{-1}\, P_N] A R(\lambda, A)\, a d\lambda$$

where the path of integration in the complex plane is shown in the figure.



Therefore, from (3.10a) and (3.10b) we obtain

$$(3.14)\quad \|A_N^{\frac{1}{2}}(P_N e^{-At} a - e^{-A_N t} P_N a)\|_0 \le C \int_\Gamma |e^{\lambda t}|\ \|A_N R_N\|\ \|(P_N A^{-1} - A_N^{-1} P_N) A R a\|_1\, d\lambda$$

$$\le C N^{\frac{3}{2}-\sigma} \int_\Gamma |e^{\lambda t}|\ \|R(\lambda, a) a\|_\sigma\, d\lambda$$

$$\le C N^{\frac{3}{2}-\sigma} \int_\Gamma |e^{\lambda t}|\ \|R(\lambda, A)\|\ \|a\|_\sigma\, d\lambda$$

$$\le C N^{\frac{3}{2}-\sigma} \int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|}\, d\lambda \cdot \|a\|_\sigma \le C N^{\frac{3}{2}-\sigma}\ \|a\|_\sigma.$$

where R_N and R represent $R(\lambda, A_N)$ and $R(\lambda, A)$ respectively, and we have used (1.3) and (1.5b). We also get

$$\|A_N^{\frac{1}{2}}(P_N e^{-At}a - e^{-A_N t}P_N a)\|_0 \leq CN^{\frac{3}{2}-\sigma} \int_\Gamma |e^{\lambda t}| \, \|A^{\frac{1}{2}} R(\lambda, A)\| \, \|a\|_{\sigma-1} \, d\lambda$$

(3.15)

$$\leq CN^{\frac{3}{2}-\sigma} \|a\|_{\sigma-1} \int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|^{\frac{1}{2}}} \, d\lambda \leq Ct^{-\frac{1}{2}} N^{\frac{3}{2}-\sigma} \, \|a\|_{\sigma-1}.$$

¿From these we get

$$\|A_N^{\frac{1}{2}}(P_N e^{-A(t-\tau)}u(\tau) - e^{-A_N(t-\tau)}P_N u(\tau))\|_0 \leq CN^{\frac{3}{2}-\sigma} \, \|u(\tau)\|_\sigma \tag{3.16}$$

$$\|A_N{}^{\frac{1}{2}}(P_N e^{-A(t-s)}(u^2(s))_x - e^{-A_N(t-s)}P_N(u^2(s))_x)\|_0 \leq \tag{3.17}$$

$$\leq C\,(t-s)^{-\frac{1}{2}} N^{\frac{3}{2}-\sigma} \, \|u^2(s)_x\| \leq C\,(t-s)^{-\frac{1}{2}} N^{\frac{3}{2}-\sigma} \, \|u(s)\|_\sigma^2$$

On the other hand, we have

(3.18)

$$\|A_N^{\frac{1}{2}} e^{-A_N(t-s)} P_N\{u^2(s)_x - (P_N u(s))_x^2\}\|_0 \leq C\,(t-s)^{-\frac{1}{2}} \|u^2(s)_x - (P_N u(s))_x^2\|_0$$

$$\leq C\,(t-s)^{-\frac{1}{2}} \|u^2(s) - (P_N u(s))^2\|_1$$

$$\leq C\,(t-s)^{-\frac{1}{2}} \|u(s) + P_N u(s)\|_1 \, \|u(s) - P_N u(s)\|_1$$

$$\leq C\,(t-s)^{-\frac{1}{2}} N^{\frac{3}{2}-\sigma} \, \|u(s)\|_2 \, \|u(s)\|_\sigma.$$

Here we have used the fact that $\|P_N u(s)\|_1 \leq C \, \|u(s)\|_2$. Similarly, we get

$$\|A_N^{\frac{1}{2}}(P_N e^{-A(t-s)} f(s) - e^{-A_N(t-s)} P_N f(s))\|_0 \leq CN^{\frac{3}{2}-\sigma}(t-s)^{-\frac{1}{2}} \, \|f(s)\|_{\sigma-1} \tag{3.19}$$

Combining (3.16) - (3.19), we get (3.11). $\square$

Our next lemma deals with condition (Y) in Theorem 1.1

LEMMA 3.4. Condition (Y) in Theorem 1.1 is satisfied for $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$.

Proof. Let $v, \varphi \in V_N$. From (3.4) and (3.9a) we get,

$$\|A_N^{\alpha-\beta}(F_N(s,v) - F_N(s,\varphi))\|_0 \le C\ \|P_N(v_x^2 - \varphi_x^2)\|_0$$

$$\le C\ \|v_x^2 - \varphi_x^2\|_0 \le C\ \|v^2 - \varphi^2\|_1 \le C\ \|v+\varphi\|_1\ \|v-\varphi\|_1$$

$$\le C\ (\|A_N^{\frac{1}{2}}v\|_0 + \|A_N^{\frac{1}{2}}\varphi\|_0)\|A_N^{\frac{1}{2}}(v-\varphi)\|_0 \qquad \square$$

Now we can prove our main result for Legendre-Galerkin method.

THEOREM 3.1. Assume that for $\sigma \ge 2, u(t) \in C\ ([0,T], H^\sigma \cap H_0^1), f(t) \in C\ ([0,T], H^{\sigma-1})$. Then there exist constants $N_0(T)$ and $K_0(T)$, such that for $N > N_0(T)$, the Legendre-Galerkin approximation $u_N(t)$ exists for $0 \le t \le T$. Furthermore, we have, for $0 \le t \le T$

$$\|u(t) - u_N(t)\|_1 \le K_0(T) N^{\frac{3}{2}-\sigma}\ (|||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}) \tag{3.20}$$

Proof. First we note that condition (Z) is also satisfied because

$$\|A_N^{\frac{1}{2}} P_N u(t)\|_0 \le C\ \|P_N u(t)\|_1 \le C\ \|u(t)\|_2$$

Now we can use Theorem 1.1 to obtain the first half of Theorem 3.1 together with an estimate

$$\|P_N u(t) - u_N(t)\|_1 \le C N^{\frac{3}{2}-\sigma}(\ |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}). \tag{3.21}$$

(3.20) is a direct consequence of (3.21) and (3.10a). $\square$

THEOREM 3.2. Under the same assumptions as in Theorem 4.1, we have the following estimate for $N > N_0(T)$, $\quad 0 \le t \le T$

$$\text{(3.22)} \qquad \|u(t) - u_N(t)\|_0 \le CN^{-\sigma}(|||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}).$$

Proof. Let $e_N(t) = u(t) - u_N(t)$. Using variation of constants formula, we get

$$e_N(t) = (e^{-At}a - e^{-A_N t}P_N a) - \frac{1}{2}\int_0^t \{e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_N(u_x^2)\}\, ds$$

$$-\frac{1}{2}\int_0^t e^{-A_N(t-s)}P_N\{u_x^2 - (u_N^2)_x\}\, ds \quad +$$

$$+\int_0^t (e^{-A(t-s)}f(s) - e^{-A_N(t-s)}P_N f(s))\, ds =$$

$$= I_1 + I_2 + I_3 + I_4.$$

¿From (3.10a) and (3.13), we obtain,

$$\|P_N e^{-At}a - e^{-A_N t}P_N a\|_0 \le CN^{-\sigma}\int_\Gamma |e^{\lambda t}|\, \|R(\lambda, A)a\|_\sigma\, d\lambda$$

$$\le CN^{-\sigma}\int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|}\, \|a\|_\sigma\, d\lambda \le CN^{-\sigma}\|a\|_\sigma$$

$$\|P_N e^{-At}a - e^{-A_N t}P_N a\|_0 \le C\,t^{-\frac{1}{2}}N^{-\sigma}\ \|a\|_{\sigma-1}.$$

Therefore,

$$\|I_1\|_0 = \|e^{-At}a - e^{-A_N t}P_N a\|_0 \le CN^{-\sigma}\|a\|_\sigma$$

$$\|e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_N(u_x^2)\|_0 \le C\ (t-s)^{-\frac{1}{2}}N^{-\sigma}\ \|u_x^2\|_{\sigma-1}$$

$$\le C\ (t-s)^{-\frac{1}{2}}N^{-\sigma}\|u\|_\sigma^2$$

$$\|I_2\|_0 \le CN^{-\sigma}\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_\sigma^2\, ds \le CN^{-\sigma}\ |||u(s)|||_\sigma^2.$$

Similarly, $\|I_4\|_0 \leq CN^{-\sigma} \; |||f(s)|||_{\sigma-1}$.

Notice that from (3.10c), we have

$$\|A_N^{-\frac{1}{2}} P_N(u^2 - u_N^2)_x\|_0 \leq C \; \|(u^2 - u_N^2)_x\|_{-1} \leq C \; \|u^2 - u_N^2\|_0$$

Hence,

$$\|I_3\|_0 \leq C \int_0^t (t-s)^{-\frac{1}{2}} \; \|u^2 - u_N^2\|_0 \, ds$$

$$\leq C \int_0^t (t-s)^{-\frac{1}{2}} \; \|u + u_N\|_{L^\infty} \; \|u - u_N\|_0 \, ds$$

$$\leq L \int_0^t (t-s)^{-\frac{1}{2}} \, \|e_N(s)\|_0 \, ds$$

where L depends on $|||u(s)|||_2$. Combining these results, we get

$$\|e_N(t)\|_0 \leq CN^{-\sigma}(\; |||u(s)\,|||_\sigma^2 \; + \; |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}) \; +$$

$$+ \; L \int_0^t (t-s)^{-\frac{1}{2}} \; \|e_N(s)\|_0 \, ds.$$

Using Lemma 1.3, we obtain (3.22). □

3.2 Legendre-Collocation Method.

Now we can consider the pseudospectral method using Legendre polynomials which is computationally more efficient than the previous one. In the pseudospectral version, collocation is used instead of projection. For continuous functions $\varphi(x), \psi(x)$, we define the interpolation operator $P_c : \; C^0(I) \to S_N$ by,

(3.23) $$P_c\varphi(x_j) = \varphi(x_j), \qquad 0 \leq j \leq N$$

and a discrete inner product $(\cdot,\cdot)_N$

$$(3.24) \qquad (\varphi, \psi)_N = \sum_{j=0}^{N} \omega_j \varphi(x_j)\psi(x_j)$$

where $\{(x_j, \omega_j)\}_{j=0}^{N}$ is the set of nodes and weights of the Gauss-Lobatto integration formula relative to the weight function $w(x) = 1$. It's well-known that for $\varphi \in S_{2N-1}$

$$(3.25) \qquad \int_I \varphi(x)\, dx = \sum_{j=0}^{N} \varphi(x_j)\omega_j$$

¿From (3.25), we have for $\varphi \cdot \psi \in S_{2N-1}$,

$$(3.26) \qquad (\varphi, \psi)_N = (\varphi, \psi)$$

Furthermore, P_c is uniquely determined by

$$(\varphi, \psi)_N = (P_c\varphi, \psi)_N \qquad \text{for any} \qquad \varphi, \psi \in C(I)$$

The Legendre-Collocation approximation for (3.1) is given by the following problem

$$(3.27) \qquad \begin{cases} \text{Find} \quad u_N(t) : [0,T] \to V_N \quad , \qquad \text{such that for} \qquad 1 \leq j \leq N-1 \\ \dfrac{\partial}{\partial t} u_N(x_j, t) - \dfrac{\partial^2 u_N}{\partial x^2}(x_j, t) + \dfrac{1}{2}(P_c u_N^2)_x(x_j, t) = f(x_j, t) \\ u_N(x_j, 0) = a(x_j), \qquad 0 \leq j \leq N \end{cases}$$

Or equivalently

$$(3.28) \qquad \begin{cases} (\dfrac{\partial u_N}{\partial t}, \varphi)_N - (\dfrac{\partial^2 u_N}{\partial x^2}, \varphi)_N + \dfrac{1}{2}((P_c u_N^2)_x, \varphi)_N = (f, \varphi)_N \quad , \quad \text{for any} \quad \varphi \in V_N \\ u_N(0) = P_c a \end{cases}$$

For $v \in V_N$, define $A_N v \in V_N$ by

$$(A_N v, \varphi)_N = a_N(v, \varphi) = -(v_{xx}, \varphi)_N \quad , \quad \text{for any} \quad \varphi \in V_N$$

¿From (3.25) we know that

$$a_N(v, \varphi) = a(v, \varphi) \qquad \text{for} \qquad v, \varphi \in V_N$$

So conditions $(A1')$ and $(A2')$ are satisfied and we can use the framework in § 1.

Now (3.28) can be written in a dynamical form.

$$\text{(3.29)} \qquad \begin{cases} u_N(t) : [0, T] \to V_N \\ \dfrac{du_N}{dt} + A_N u_N + F_c(t, u_N) = 0 \\ u_N(0) = P_c \; a \end{cases}$$

where $F_c(t, \varphi) = \frac{1}{2} (P_c \varphi^2)_x - P_c f(t) \quad , \quad \text{for} \qquad \varphi \in V_N$

The analysis of the Legendre-Collocation method is parallel to the Legendre- Galerkin method as we will see. Lemma 3.1 carries over whereas Lemma 3.2 has to be changed to

LEMMA 3.5. Following estimates hold

a) For $\quad 0 \le \mu \le \sigma, \quad \sigma > \frac{1}{2}, \quad v \in H^\sigma$

$$\text{(3.30a)} \qquad \|v - P_c v\|_\mu \le C N^{2\mu + \frac{1}{2} - \sigma} \; \|v\|_\sigma$$

b) For $\quad \sigma \ge 2, \quad g \in H^{\sigma - 2}$

$$\text{(3.30b)} \qquad \|\,(P_c A^{-1} - A_N^{-1} P_c)\, g\|_1 \le C N^{\frac{5}{2} - \sigma} \; \|g\|_{\sigma - 2}$$

Proof. (3.30a) was proved in [2]. To prove (3.30b), let $A^{-1} g = v, \quad A_N^{-1} P_c g = v_N$, i.e. $Av = g, \quad A_N v_N = P_c g$. Then for any $\varphi \in V_N$, we have

$$a(v,\varphi) = (g,\varphi)$$

$$a(v_N,\varphi) = (g,\varphi)_N$$

Take $\varphi = \chi - V_N$, where $\chi \in V_N$ satisfies $\|v-\chi\|_1 = \inf_{\varphi \in V_N} \|v-\varphi\|_1$, then

$$a(v-v_N, v-v_N) = a(v-v_N, v-\chi) + (g, \chi - v_N) - (g, \chi - v_N)_N \tag{3.31}$$

Use the following result from [10]: for $\varphi \in V_N$

$$|(g,\varphi) - (g,\varphi)_N| \le C\ \|\varphi\|_0 \left(\|g - P_c g\|_0 + \|g - P_{N-1} g\|_0\right)$$

we obtain,

$$|(g,\chi - v_N) - (g,\chi - v_N)_N| \le C\ \|\chi - v_N\|_0 (\|g - P_c g\|_0 + \|g - P_{N-1} g\|_0)$$

$$\le C\ \|v - v_N\|_1\ (\|g - P_c g\|_0 + \|g - P_{N-1} g\|_0)$$

Go back to (3.31), we get (use Poincare inequality).

$$\beta\|v - v_N\|_1 \le a(v-v_N, v-v_N) \le C\,\|v - v_N\|_1\ (\ \|v-\chi\|_1 + \|g - P_c g\|_0 +$$

$$+ \|g - P_{N-1} g\|_0)$$

or

$$\|v - v_N\|_1 \le C\ (\|v-\chi\|_1 + \|g - P_c g\|_0 + \|g - P_{N-1} g\|_0)$$

$$\le CN^{1-\sigma}\ \|v\|_\sigma + CN^{\frac{5}{2}-\sigma}\ \|g\|_{\sigma-2} \le CN^{\frac{5}{2}-\sigma}\ \|g\|_{\sigma-2}. \qquad \square$$

LEMMA 3.6. Assume that for $\sigma > \frac{1}{2}, a \in H^\sigma$, then we have

$$\|P_c e^{-At} a - e^{-A_N t} P_c a\|_1 \le C N^{\frac{5}{2}-\sigma} \|a\|_\sigma \tag{3.32a}$$

$$\|P_c e^{-At} a - e^{-A_N t} P_c a\|_1 \le C N^{\frac{5}{2}-\sigma} t^{-\frac{1}{2}} \|a\|_{\sigma-1}. \tag{3.32b}$$

Proof. From (3.9a) and (3.13), (3.30b), we get

$$\|P_c e^{-At} a - e^{-A_N t} P_c a\|_1 \le C \|A_N^{\frac{1}{2}} (P_c e^{-At} a - e^{-A_N t} P_c a)\|_0$$

$$\le C \int_\Gamma |e^{\lambda t}| \, \|A_N R_N\| \, \|(P_c A^{-1} - A_N^{-1} P_c) A R a\|_1 \, d\lambda$$

$$\le C N^{\frac{5}{2}-\sigma} \int_\Gamma |e^{\lambda t}| \, \|R(\lambda, A) a\|_\sigma \, d\lambda \le C N^{\frac{5}{2}-\sigma} \|a\|_\sigma$$

The proof of (3.32b) is similar. □

LEMMA 3.7. Assume that for $\sigma > \frac{5}{2}$, $u(t) \in C([0,T], H^\sigma \cap H_0^1)$, $f(t) \in C([0,T], H^{\sigma-1})$. Then we have

$$\epsilon_N \le C N^{\frac{5}{4}-\frac{\sigma}{2}} |||u(s)|||_\sigma^2 + C N^{\frac{5}{2}-\sigma} \{ |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1} \}. \tag{3.33}$$

Proof. From (3.32a), we get

$$\|A_N^{\frac{1}{2}} (P_c e^{-A(t-\tau)} u(t) - e^{-A_N (t-\tau)} P_c u(\tau))\|_0 \le C N^{\frac{5}{2}-\sigma} \|u(\tau)\|_\sigma.$$

To treat the second part in ϵ_N is more complicated. First, let's note that from (3.9a), we get easily, for $0 \le \alpha \le \frac{1}{2}$

$$\|A_N^{-\alpha} \varphi\|_0 \le C \|A^{-\alpha} \varphi\|_0, \qquad \text{for} \quad \varphi \in V_N \tag{3.9a$'$}$$

As a consequence, we have, for $\varphi \in V_N$

$$\|A_N^{-\frac{3}{8}} \varphi_x\|_0 \leq C \; \|A^{-\frac{3}{8}} \varphi_x\|_0 \leq C \; \|\varphi_x\|_{-\frac{3}{4}} \leq C \; \|\varphi\|_{\frac{1}{4}}.$$

Let $\sigma' = \frac{\sigma}{2} - \frac{1}{4}$, then

$$\|A_N^{\frac{1}{2}} e^{-A_N}(t-s)\{P_c\,(P_c u)^2 - (P_c u)^2\}_x\|_0 \leq$$

$$\leq \|A_N^{\frac{7}{8}} e^{-A_N(t-s)}\| \; \|A_N^{-\frac{3}{8}}\{P_c\,(P_c u)^2 - (P_c u)^2\}_x\|_0$$

$$\leq C \; (t-s)^{-\frac{7}{8}} \; \|P_c\,(P_c u)^2 - (P_c u)^2\|_{\frac{1}{4}} \leq$$

$$\leq C\,(t-s)^{-\frac{7}{8}} \, N^{1-\sigma'} \; \|P_c\, u\|^2_{\sigma'} \leq C\,(t-s)^{-\frac{7}{8}} N^{\frac{5}{4}-\frac{\sigma}{2}} \; \|u\|^2_\sigma$$

We also have

$$\|A_N^{\frac{1}{2}} e^{-A_N(t-s)}\{(P_c u)^2_x - P_c(u^2_x)\}\|_0 \leq C\,(t-s)^{-\frac{1}{2}} \|(P_c u)^2_x - P_c\,(u^2_x)\|_0$$

$$\leq C\,(t-s)^{-\frac{1}{2}}\,(\,\|u^2_x - (\,P_c u)^2_x\|_0 + \|\,(u^2)_x - P_c\,(u^2_x)\|_0)$$

$$\leq C\,(t-s)^{-\frac{1}{2}}(\|u^2 - (\,P_c u)^2\|_1 + \|(u^2)_x - P_c\,(u^2_x)\|_0)$$

$$\leq C\,(t-s)^{-\frac{1}{2}} N^{\frac{5}{2}-\sigma} \|u\|^2_\sigma$$

Finally, from (3.32b), we get

$$\|A_N^{\frac{1}{2}}\{P_c e^{-A(t-s)}(u^2_x) - e^{-A_N(t-s)} P_c\,(u^2_x)\}\|_0 \leq c\,(t-s)^{-\frac{1}{2}} N^{\frac{5}{2}-\sigma} \; \|u(s)\|^2_\sigma$$

$$\|A_N^{\frac{1}{2}}\{P_c e^{-A(t-s)} f(s) - e^{-A_N(t-s)} P_c f(s)\}\|_0 \leq C N^{\frac{5}{2}-\sigma}(t-s)^{-\frac{1}{2}} \; \|f(s)\|_{\sigma-1}$$

Now (3.33) follows readily from these estimates.

LEMMA 3.8. Condition (Y) in Theorem 1.1 is satisfied for $\alpha = \frac{1}{2}, \beta = \frac{7}{8}$.

Proof. Let $v, \varphi \in V_N$

$$\|A_N^{\alpha-\beta}\{F_c(s,v) - F_c(s,\varphi)\}\|_0 = \frac{1}{2}\ \|A_N^{-\frac{3}{8}}\{P_c v^2 - P_c\varphi^2\}_x\|_0$$

$$\leq C\ \|P_c(v^2-\varphi^2)\|_{\frac{1}{4}} \leq C\ \|v^2-\varphi^2\|_1 \leq C\ \|v+\varphi\|_1\ \ \|v-\varphi\|_1$$

$$\leq C\ (\ \|A_N^{\frac{1}{2}}v\|_0 + \|A_N^{\frac{1}{2}}\varphi\|_0)\ \|A_N^{\frac{1}{2}}\ (v-\varphi)\|_0 \qquad \square$$

THEOREM 3.3. Assume for $\sigma > \frac{5}{2}$, $u(t) \in C\ ([0,T],\ H^\sigma \cap H_0^1), f(t) \in C\ ([0,T], H^{\sigma-1})$. Then there exist constants $N_0(T), K_0(T)$ such that for $N > N_0(T)$, the Legendre-Collocation approximation $u_N(t)$ exists on $[0,T]$, and following estimate holds.

$$\|u(t) - u_N(t)\|_1 \leq K_0(T)N^{\frac{5}{4}-\frac{\sigma}{2}}\ |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}). \tag{3.34}$$

Proof. (3.34) is a direct consequence of Theorem 1.1, (3.33) and (3.30a). The only assumption in Theorem 1.1 to be checked is (Z). This is easy to accomplish because,

$$\|A_N^{\frac{1}{2}}P_c u(t)\|_0 \leq C\ \|P_c u(t)\|_1 \leq C\ \|u(t)\|_\sigma. \qquad \square$$

However, (3.34) is not a satisfactory result because it is much worse than (3.30a). this is circumvented in the next theorem by comparing $u_N(t)$ directly with $u(t)$.

THEOREM 3.4. Under the assumptions of Theorem 3.3, we have for $N > N_0(T)$.

$$\|u(t) - u_N(t)\|_1 \leq CN^{\frac{5}{2}-\sigma}\,(\ |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}). \tag{3.35}$$

Proof. Let $e_N(t) = u(t) - u_N(t)$, then

$$e_N(t) = e^{-At}a - e^{-A_N t}P_c a - \frac{1}{2}\int_0^t \{e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_c\,(u_x^2)\}\,ds$$

$$-\frac{1}{2}\int_0^t e^{-A_N(t-s)}\{P_c\,(u_x^2) - (P_c u_N^2)_x\}\,ds +$$

$$+\int_0^t \{e^{-A(t-s)}f(s) - e^{-A_N(t-s)}P_c f(s)\}\,ds$$

$$= I_1 + I_2 + I_3 + I_4$$

¿From (3.30a) and (3.32a), we have

$$\|I_1\|_1 \le CN^{\frac{5}{2}-\sigma}\ \|a\|_\sigma$$

$$\|I_2\|_1 \le C\int_0^t (t-s)^{-\frac{1}{2}} N^{\frac{5}{2}-\sigma}\ \|u_x^2\|_{\sigma-1}\,ds \le CN^{\frac{5}{2}-\sigma}\ |||u(s)|||_\sigma^2$$

$$\|I_4\|_1 \le C\int_0^t (t-s)^{-\frac{1}{2}} N^{\frac{5}{2}-\sigma}\|f(s)\|_{\sigma-1}\,ds \le CN^{\frac{5}{2}-\sigma}|||f(s)|||_{\sigma-1}$$

In Lemma 3.7, we proved

$$\|A_N^{\frac{1}{2}}e^{-A_N(t-s)}\{(P_c u^2)_x - P_c\,(u_x^2)\}\,\|_0 \le C\,(t-s)^{-\frac{1}{2}}N^{\frac{5}{2}-\sigma}\ \|u\|_\sigma^2$$

On the other hand, we also have

$$\| \int_0^t A_N^{\frac{1}{2}} e^{-A_N(t-s)} \{P_c u^2 - P_c u_N^2\}_x \, ds \,\|_0 \le C \int_0^t (t-s)^{-\frac{7}{8}} \, \|A_N^{-\frac{3}{8}} (P_c u^2 - P_c u_N^2)_x \,\|_0 \, ds$$

$$\le C \int_0^t (t-s)^{-\frac{7}{8}} \, \|P_c (u^2 - u_N^2)\|_{\frac{1}{4}} \, ds$$

$$\le C \int_0^t (t-s)^{-\frac{7}{8}} \, \|u^2 - u_N^2\|_1 \, ds$$

$$\le C \int_0^t (t-s)^{-\frac{7}{8}} \, \|u + u_N\|_1 \, \|e_N(s)\|_1 \, ds$$

$$\le C L(u,f) \int_0^t (t-s)^{-\frac{7}{8}} \, \|e_N(s)\|_1 \, ds$$

where $L(u,f) = |||u(s)|||_\sigma^2 + |||u(s)|||_\sigma + |||f(s)|||_{\sigma-1}$ from Theorem 3.3.

Combining these results, we get

$$\|e_N(t)\|_1 \le C N^{\frac{5}{2}-\sigma} L(u,f) + C L(u,f) \int_0^t (t-s)^{-\frac{7}{8}} \, \|e_N(s)\|_1 \, ds. \tag{3.36}$$

Now (3.35) follows from (3.36) and Lemma 1.3. $\square$

§ 4. Chebyshev Methods for Burgers' Equation

The ideas in Chebyshev methods are very similar to those of the Legendre methods, except that instead of Legendre polynomials, Chebyshev polynomials are used as the base function. With Chebyshev polynomials, FFT can be invoked to reduce the operation count. The analysis of Chebyschev methods gives rise to new difficulties because a singular weight is used in the integration formula. Nevertheless, the arguments to be used is parallel to the ones for Legendre methods. Therefore we will skip some of the details and emphasize their differences.

4.1 Chebyshev-Galerkin Method.

Define the Chebyshev weight function as

$$\omega(x) = (1 - x^2)^{-\frac{1}{2}} \qquad \text{on} \qquad I = (-1, 1) \tag{4.1}$$

For $u, v \in H^1_{0,\omega}(I)$, define

$$a(u, v) = \int_I u_x (v\omega)_x \, dx \tag{4.2}$$

Some basic properties of this bilinear form were proved by Canuto and Quarteroni [4].

Lemma 4.1. There exist positive constants $\beta_1, \beta_2, \beta_3$, such that for any $v \in H^1_{0,\omega}(I)$, we have

$$\|v\|_{0,\omega} \leq \beta_1 \, \|v_x\|_{0,\omega} \tag{4.3a}$$

$$a(v, v) \geq \beta_2 \, \|v\|^2_{1,\omega} \tag{4.3b}$$

$$a(u, v) \leq \beta_3 \, \|u\|_{1,\omega} \, \|v\|_{1,\omega} \tag{4.3c}$$

For $u \in D(A) = H^2_\omega \cap H^1_{0,\omega}(I)$, define

$$Au = -u_{xx} \tag{4.4}$$

We see that $(Au, v)_\omega = a(u, v)$, for $u \in D(A)$, $\quad v \in H^1_{0,\omega}$. So the basic assumptions (A1) and (A2) are satisfied because of Lemma 4.1, and we conclude that A generates an analytic semigroup, the fractional powers of A are well-defined.

LEMMA 4.2. a) For $\alpha > \frac{1}{2}$, $H^\alpha_\omega(I)$ is continuously imbedded in $L^\infty(I)$

b) For $\alpha \geq 1$, $H^\alpha_\omega(I)$ is a Banach algebra, i.e.

$$\|uv\|_{\alpha,\omega} \leq C\ \|u\|_{\alpha,\omega}\ \|v\|_{\alpha,\omega}, \qquad \text{for} \qquad u, v \in H^\alpha_\omega(I) \tag{4.5}$$

c) For $\alpha \geq 0$, there is a constant C, such that for for $u \in D(A^\alpha)$,

$$\frac{1}{C}\ \|u\|_{2\alpha,\omega} \leq \|A^\alpha u\|_{0,\omega} \leq C\ \|u\|_{2\alpha,\omega} \tag{4.6}$$

For a proof of these results, see [11].

Next we turn to the Chebyshev-Galerkin method. The projection operator P_N is now defined by:

$$P_N : \quad L^2_\omega(I) \to V_N$$

$$(V - P_N v, \varphi)_\omega = 0 \quad , \quad \text{for} \quad v \in L^2_\omega(I), \quad \varphi \in V_N$$

For $v, \varphi \in V_N$, let $a_N(v,\varphi) = -\int v_{xx}\varphi\omega dx = a(v,\varphi)$. With this setup, the formulation of the Chebyshev-Galerkin method is the same as that of the Legendre-Galerkin method. Assumptions $(A1')$ and $(A2')$ are satisfied.

Now we proceed to the approximation properties of this method. Similar to Lemma 3.2 and (3.14), (3.15), we have (see also [3], [4]).

LEMMA 4.1. Following estimates hold

a) For $0 \leq \mu \leq 1 \leq \sigma$, $v \in H^\sigma_\omega \cap H^1_{0,\omega}(I)$

$$\|v - P_N v\|_{\mu,\omega} \leq CN^{\frac{3}{2}\mu-\sigma}\ \|v\|_{\sigma,\omega} \tag{4.7a}$$

b) For $0 \leq \mu \leq 1 \leq \sigma$, $g \in H^{\sigma-2}_\omega(I)$

$$\|(P_N A^{-1} - A_N^{-1} P_N) g\,\|_{\mu,\omega} \leq CN^{\frac{3}{2}\mu-\sigma}\|g\|_{\sigma-2} \tag{4.7b}$$

c) For $\sigma \geq 2$, $a \in H^\sigma_\omega \cap H^1_{0,\omega}(I)$

$$\|A_N^{\frac{1}{3}}(P_N e^{-At} - e^{-A_N t} P_N)\, a\,\|_{0,\omega} \leq CN^{1-\sigma}\|A\|_{\sigma,\omega} \tag{4.7c}$$

(4.7d) $$\|A_N^{\frac{1}{3}}(P_N e^{-At} - e^{-A_N t}P_N)\, a\,\|_{0,\omega} \le CN^{1-\sigma}t^{-\frac{1}{2}}\|a\|_{\sigma-1,\omega}$$

For a proof of (4.7a), see [3]. The proof of (4.7b), (4.7c) and (4.7d) is similar to that of (3.10b), (3.14) and (3.15) respectively, and is therefore omitted.

Because the bilinear form $a(\cdot,\cdot)$ is nonsymmetric, estimates of the type (3.9a) and (3.9a') become more delicate. Nevertheless, we can prove

LEMMA 4.2. For $0 \le \alpha < \frac{1}{2}$, there exists a constant C, such that for $\varphi \in V_N$,

(4.8a) $$\|A^{\alpha}\varphi\|_{0,\omega} \le C\ \|A_N^{\alpha}\varphi\|_{0,\omega}$$

(4.8b) $$\|A_N^{-\alpha}\varphi\|_{0,\omega} \le C\ \|A^{-\alpha}\varphi\|_{0,\omega}$$

Proof. We will use the setup in § 3 of [7]. We can view A_N as a regularly accretive operator on the space V_N. Its real part H_N is a self-adjoint operator on V_N defined via a symmetric bilinear form $b(\cdot,\cdot)$ by, $H_N: \quad V_N \to V_N$

(4.9) $$(H_N v,\varphi)_{0,\omega} = b(v,\varphi) = \frac{1}{2}(a(v,\varphi)+a(\varphi,v)) \quad , \quad \text{for} \quad v,\varphi \in V_N$$

Then from Theorem 3.1 of [7], we get for $0 \le \alpha < \frac{1}{2}$, $\varphi \in V_N$

(4.10) $$\|H_N^{\alpha}\varphi\|_{0,\omega} \le (1-\tan\frac{\pi\alpha}{2})^{-1}\ \|A_N^{\alpha}\varphi\|_{0,\omega}$$

On the other hand, from (4.3b) and (4.6), we get

$$\|H_N^{\frac{1}{2}}\varphi\|_{0,\omega}^2 = (H_N\varphi,\varphi)_\omega = a(\varphi,\varphi) \ge \beta_2\ \|\varphi\|_{1,\omega}^2 \ge \tilde{\beta}\ \|A^{\frac{1}{2}}\varphi\|_{0,\omega}^2$$

Therefore we have

$$\|A^{\frac{1}{2}}\varphi\|_{0,\omega} \le C\ \|H_N^{\frac{1}{2}}\varphi\|_{0,\omega}$$

Now (4.8a) follows from Heinz-Kato inequality, together with (4.10)

To prove (4.8b), let's first remark that, parallel to (4.8a), we also have, for $0 \le \alpha < \frac{1}{2}$,

$$\|(A^*)^\alpha\varphi\|_{0,\omega} \leq C\ \|(A_N^*)^\alpha\varphi\|_{0,\omega} \quad , \quad \text{for} \quad \varphi \in V_N$$

where A^* and A_N^* are the adjoint operator of A and A_N respectively.

For $g \in V_N$, let $v = (A_N^*)^{-\alpha} g \in V_N, \quad g_1 = (A^*)^\alpha v,$

$$\|A_N^{-\alpha}\varphi\|_{0,\omega} = \sup_{g \in V_N} \frac{(A_N^{-\alpha}\varphi, g)_\omega}{\|g\|_{0,\omega}} = \sup_{g \in V_N} \frac{(\varphi, (A_N^*)^{-\alpha} g)_\omega}{\|g\|_{0,\omega}} = \sup \frac{(\varphi, v)}{\|(A_N^*)^\alpha v\|}$$

$$\leq C \sup \frac{(\varphi, v)_\omega}{\|(A^*)^\alpha v\|_{0,\omega}} = C \sup_{g_1 \in L^2_\omega} \frac{(A^{-\alpha}\varphi, g_1)_\omega}{\|g_1\|_{0,\omega}} \leq C\ \|A^{-\alpha}\varphi\|_{0,\omega} \qquad \square$$

LEMMA 4.3. Let $\alpha = \frac{1}{3}$. Assume for $\sigma \geq 2, u(t) \in C\ ([0,T], H^\sigma_\omega \cap H^1_{0,\omega}(I))$, $f(t) \in C\ ([0,T], H^{\sigma-1}_\omega(I))$, then there exists a constant C, such that,

$$\text{(4.11)} \qquad \epsilon_N \leq C N^{1-\sigma}(\ |||u(s)|||^2_{\sigma,\omega} + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega} \quad , \quad \text{for} \quad 0 \leq t \leq T.$$

The proof of this lemma is very similar to the one given for Lemma 3.3.

LEMMA 4.4. Condition (Y) is satisfied for $\alpha = \frac{1}{3}, \beta = \frac{5}{6}$

Proof. First we remark that similar to (3.10c), we have for $g \in H^{-1}_\omega(I)$,

$$\|A_N^{-\frac{1}{2}} P_N g\|_{0,\omega} \leq C\ \|g\|_{-1,\omega}$$

Let $v, \varphi \in V_N$,

$$\|A_N^{-\frac{1}{2}}\{F_N(s,v) - F_N(s,\varphi)\}\|_{0,\omega} \leq \|A_N^{-\frac{1}{2}} P_N(v_x^2 - \varphi_x^2)\|_{0,\omega}$$

$$\leq C\ \|v^2 - \varphi^2\|_{0,\omega} \leq C\ \|v + \varphi\|_{L^\infty}\ \|v - \varphi\|_{0,\omega}$$

$$\leq \|C\ A^{\frac{1}{3}}(v + \varphi)\|_{0,\omega}\ \|v - \varphi\|_{0,\omega}$$

$$\leq C\ (\|A_N^{\frac{1}{3}} v\|_{0,\omega} + \|A_N^{\frac{1}{3}}\varphi\|_{0,\omega})\ \|A_N^{\frac{1}{3}}(v - \varphi)\|_{0,\omega} \qquad \square$$

Now we are ready to state our main results on Chebyshev-Galerkin method. The proofs are the same as those of Theorem 3.1, 3.2.

THEOREM 4.1. Assume that for $\sigma \geq 2, u(t) \in C\ ([0,T],\ H_w^\sigma \cap H_{0,\omega}^1(I)), f(t) \in C\ ([0,T], H_w^\sigma \cap H_{0,\omega}^1(I))$. Then there exist constants $N_0(T), K_0(T)$, such that for $N > N_0(T)$, the Chebyshev-Galerkin approximation $u_N(t)$ of $u(t)$ exists on $[0,T]$. Furthermore, we have for $0 \leq t \leq T$

$$\|u(t) - u_N(t)\|_{2/3,\omega} \leq K_0(T) N^{1-\sigma}(|||u(s)|||_{\sigma,\omega}^2 + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}) \tag{4.12}$$

THEOREM 4.2. Under the same assumptions as in Theorem 4.1, we also have for $N > N_0(T), \quad 0 \leq t \leq T$,

$$\|u(t) - u_N(t)\|_{0,\omega} \leq C N^{-\sigma}(\ |||u(s)|||_{\sigma,\omega}^2 + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}) \tag{4.13}$$

4.2 Chebyshev-Collocation Method.

For a continuous function $\varphi(x)$, we define its Chebyshev interpolatant $P_c\varphi(x) \in S_N$ by,

$$P_c : C\ (\bar{I}) \rightarrow S_N$$

$$P_c\varphi(x_j) = \varphi(x_j) \qquad 0 \leq j \leq N$$

where $x_j = \cos \frac{\pi j}{N} \quad (0 \leq j \leq N)$ are the collocation points for Chebyshev method. The weights in the corresponding Gauss-Lobatto integration formula are denoted by ω_j. We define a discrete inner product on $C\ (\bar{I})$ by, for $\varphi(x),\ \psi(x) \in C\ (\bar{I})$

$$(\varphi, \psi)_{N,\omega} = \sum_{j=0}^{N} \varphi(x_j)\psi(x_j)\omega_j$$

The interpolation operator P_c and the discrete inner product enjoy the same properties as those for the Legendre method, as we enumerated in § 3.2. Also the Chebyshev-Collocation approximation of (3.1) and the operator A_N can be defined in a similar fashion, e.g. we have for $v, \varphi \in V_N$,

$$(A_N v, \varphi)_{N,\omega} = a_N(v, \varphi) = -(v_{xx}, \varphi)_{N,\omega} = -(v_{xx}, \varphi)_\omega = a(v, \varphi).$$

Conditions $(A1')$ and $(A2')$ are satisfied.

Now we proceed to the analysis of this method. Similar to Lemma 3.5, we have

LEMMA 4.5. Following estimates hold,

a) For $0 \leq \mu \leq \sigma, \quad \sigma > \frac{1}{2}, \quad v \in H^\sigma_\omega(I)$.

$$\|v - P_c v\|_{\mu,\omega} \leq C N^{2\mu-\sigma} \|v\|_{\sigma,\omega} \tag{4.14a}$$

b) For $\sigma \geq 2, \quad g \in H^{\sigma-2}_\omega(I)$.

$$\|(P_c A^{-1} - A_N^{-1} P_c) g\|_1 \leq C N^{2-\sigma} \|g\|_{\sigma-2} \tag{4.14b}$$

$$\|(P_c A^{-1} - A_N^{-1} P_c) g\|_0 \leq C N^{1-\sigma} \|g\|_{\sigma-2} \tag{4.14c}$$

(4.14a) is proved in [3]. The argument for (3.30b) also proves (4.14 b). (4.14c) follows directly from (4.14 b) and the Nitsche's trick.

Similar to Lemma 3.4, we have the following estimates for the linear problem.

LEMMA 4.6. Assume that for $\sigma > \frac{1}{2}, \quad a \in H^\sigma_\omega \cap H^1_{0,\omega}(I)$, then we have

$$\|P_c e^{-At} a - e^{-A_N t} P_c a\|_{1,\omega} \leq C N^{2-\sigma} \|a\|_{\sigma,\omega} \tag{4.15a}$$

$$\|P_c e^{-At} a - e^{-A_N t} P_c a\|_{1,\omega} \leq C N^{2-\sigma} t^{-\frac{1}{2}} \|a\|_{\sigma-1,\omega} \tag{4.15b}$$

LEMMA 4.7. Assume that for $\sigma > 2, \quad u(t) \in C\,([0,T], H^\sigma_\omega \cap H^1_{0,\omega}(I)), f(t) \in C\,([0,T], H^{\sigma-1}_\omega(I))$. Then we have, for $\alpha = \frac{1}{3}$

$$\epsilon_N \leq N^{\frac{2}{3}-\frac{\sigma}{2}} \left(|||u(s)|||^2_{\sigma,\omega} + |||u(s)|||_{\sigma.\omega} + |||f(s)|||_{\sigma-1,\omega} \right) \tag{4.16}$$

Proof. From (4.15a), we get

$$\|A_N^{\frac{1}{3}}(P_c e^{-A(t-\tau)}u(\tau) - e^{-A_N(t-\tau)}P_c u(\tau))\|_{0,\omega} \le CN^{2-\sigma}\ \|u(\tau)\|_{\sigma,\omega}$$

The second part of ϵ_N is dealt with according to the following identity,

$$(P_c\ (P_c u)^2)_x = P_c\ (u_x^2) + \{(P_c u)_x^2 - P_c\ (u_x^2)\} + \{P_c\ (P_c u)^2 - (P_c u)^2\}_x \tag{4.17}$$

¿From (4.15)a), we know

$$\|A_N^{\frac{1}{3}}\,(P_c e^{-A(t-s)}u_x^2 - e^{-A_N(t-s)}P_c u_x^2)\|_{0,\omega} \le CN^{2-\sigma}(t-s)^{-\frac{1}{2}}\ \|u(s)\|_{\sigma,\omega}^2$$

We also have, similar to the case in Lemma 3.7

$$\|A_N^{\frac{1}{3}}e^{-A_N(t-s)}\{(P_c u)_x^2 - P_c\ (u_x^2)\}\ \|_{0,\omega} \le C\ (t-s)^{-\frac{1}{3}}\ \|\,(P_c u)_x^2 - P_c\ (u_x^2)\|_{0,\omega}$$

$$\le C\ (t-s)^{-\frac{1}{2}}(\|u_x^2 - (P_c u)_x^2\|_{0,\omega} + \|u_x^2 - P_c\ (u_x^2)\ \|_{0,\omega})$$

$$\le C\ (t-s)^{-\frac{1}{2}}N^{2-\sigma}\ \|u\|_{\sigma,\omega}^2$$

To estimate the last term, we first note that Lemma 4.2 still holds (the same proof works through) for our new A_N defined via the discrete inner product. Therefore, for $\varphi \in V_N$, we have

$$\|A_N^{-\frac{1}{3}}\varphi_x\|_{0,\omega} \le C\ \|A^{-\frac{1}{3}}\varphi_x\|_{0,\omega} \le C\ \|\varphi_x\|_{-\frac{2}{3},\omega} \le C\ \|\varphi\|_{\frac{1}{3},\omega}$$

Let $\sigma' = \frac{\sigma}{2}$, then

$$\|A_N^{\frac{1}{3}}e^{-A_N(t-s)}\{P_c(P_c u)^2 - (P_c u)^2\}_x\ \|_{0,\omega} \le \|A_N^{\frac{2}{3}}e^{-A_N(t-s)}\|\ \|A_N^{-\frac{1}{3}}\{P_c\ (P_c u)^2 - (P_c u)^2\}_x\|_{0,\omega}$$

$$\le C\ (t-s)^{-\frac{2}{3}}\|P_c\ (P_c u)^2 - (P_c u)^2\|_{\frac{1}{3},\omega} \le C\ (t-s)^{-\frac{2}{3}}N^{\frac{2}{3}-\sigma'}\ \|P_c u\|_{\sigma',\omega}^2$$

$$\le C\ (t-s)^{-\frac{2}{3}}N^{\frac{2}{3}-\sigma'}\ \|u\|_{\sigma,\omega}^2$$

Combining these estimates and the estimate

$$\|A_N^{\frac{1}{3}}(P_c e^{-A(t-s)} f(s) - e^{-A_N(t-s)} P_c f(s))\|_{0,\omega} \leq C N^{2-\sigma}(t-s)^{-\frac{1}{2}} \|f(s)\|_{\sigma,\omega}$$

we get (4.16).

Next, we check condition (Y) in Theorem 1.1

LEMMA 4.8. Condition (Y) in Theorem 1.1 is satisfied for $\alpha = \frac{1}{3}, \beta = \frac{5}{6}$.

Proof. We will make use of the following inequality which can be shown directly by using the intrinsic norms for $H_\omega^s(I), \frac{1}{2} < s < 1$ (see [18])

$$\text{If} \quad f, g \in H_\omega^s(I), \quad s > \frac{1}{2}, \qquad \text{then} \quad f \cdot g \in H_\omega^s(I) \quad \text{and}$$

$$\|fg\|_{s,\omega} \leq C\ (\ \|f\|_{L^\infty}\ \|g\|_{s,\omega} + \|g\|_{L^\infty}\ \|f\|_{s,\omega}) \tag{4.18}$$

Now let $v, \varphi \in V_N$

$$\|A_N^{\alpha-\beta}(F_c\ (s,v) - F_c\ (s,\varphi))\|_{0,\omega} \leq C \|\ A_N^{-\frac{1}{3}}(P_c v^2 - P_c \varphi^2)_x\|_{0,\omega}$$

$$\leq C\ \|P_c\ (v^2 - \varphi^2)\|_{\frac{1}{3},\omega} \quad \leq C\ \|v^2 - \varphi^2\|_{\frac{2}{3},\omega}$$

$$\leq C\ (\|v+\varphi\|_{L^\infty}\ \|v-\varphi\|_{\frac{2}{3},\omega} + \|v-\varphi\|_{L^\infty}\ \|v+\varphi\|_{\frac{2}{3},\omega})$$

$$\leq C\ \|v+\varphi\|_{\frac{2}{3},\omega}\ \|v-\varphi\|_{\frac{2}{3},\omega}$$

$$\leq C\ (\ \|A_N^{\frac{1}{3}} v\|_{0,\omega} + \|A_N^{\frac{1}{3}}\varphi\|_{0,\omega})\|A_N^{\frac{1}{3}}(v-\varphi)\|_{0,\omega} \qquad \square$$

Now we can state our main results on Chebyshev-Collocation method. We omit the proofs because they are the same as those of Theorem 3.3, and Theorem 3.4.

THEOREM 4.3. Assume for $\sigma > 2, u(t) \in C\left([0,T], H^\sigma_\omega \cap H^1_{0,\omega}(I)\right), f(t) \in C\left([0,T], H^{\sigma-1}_\omega(I)\right)$. Then there exist constants $N_0(T), K_0(T)$, such that for $N > N_0(T)$, the Chebyshev-Collocation approximation $u_N(t)$ of $u(t)$ exists on $[0,T]$, and the following estimate holds for $0 \le t \le T$.

$$\|u(t) - u_N(t)\|_{\frac{2}{3},\omega} \le K_0(T) N^{\frac{2}{3}-\frac{\sigma}{2}} \left(|||u(s)|||^2_{\sigma,\omega} + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}\right) \tag{4.19}$$

THEOREM 4.4. Under the same assumptions as in Theorem 4.3, we have for $N > N_0(T)$,

$$\|u(t) - u_N(t)\|_{1,\omega} \le C N^{2-\sigma} \left(|||u(s)|||^2_{\sigma,\omega} + |||u(s)|||_{\sigma,\omega} + |||f(s)|||_{\sigma-1,\omega}\right) \tag{4.20}$$

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