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# A Partial Nonlinear Extension of Lax-Richtmyer Approximation Theory

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A PARTIAL NONLINEAR EXTENSION OF LAX-RICHTMYER APPROXIMATION THEORY

by

ARADHANA KUMARI

A dissertation submitted to the Graduate Faculty in Mathematics in partial fulfillment of the requirements for the degree of Doctor of Philosophy, The City University of New York

2017

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This manuscript has been read and accepted by the Graduate Faculty in Mathematics in satisfaction of the dissertation requirement for the degree of Doctor of Philosophy.

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## Abstract

## A PARTIAL NONLINEAR EXTENSION OF LAX-RICHTMYER APPROXIMATION THEORY

by

ARADHANA KUMARI

Adviser: Professor Dennis Sullivan

Lax and Richtmyer developed a theory of algorithms for linear initial value problems that guarantees, under certain circumstances, the convergence to numerical solution of initial value problem. The assumptions are first that the difference equations (algorithms) approximate the differential equations under study (this is called consistency) and, secondly, that the initial value problem be well-posed (which means that the solutions exist, are unique and depend continuously on initial data). Under these assumptions the stability condition (which requires that errors in the algorithm do not accumulate nor increase as one iterates the algorithm) is necessary and sufficient for convergence in a certain uniform sense for arbitrary initial data. In this work we will extend certain aspects of their work to the nonlinear context. We drop the PDE and the well-posedness assumptions at first and add the " $\beta$ -axioms" that will guarantee convergence [ Theorems 2 and 3 ] of algorithm orbits in a projective limit of finite dimensional spaces. A conjecture for a partial converse that some stability is a consequence of convergence for a natural class of nonlinear algorithms where the deviation of these non-linear algorithms from being linear is itself a bilinear map. When the algorithms satisfy consistency with a PDE initial value problem we obtain the definition of a new kind of numerical solution and their existence [Theorem 6] given said algorithms.

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## Indroduction

In the paper of “Survey of the Stability of Linear Finite Difference Equations” [2] Lax and Richtmyer develop the theory to obtain the numerical solution of the initial value problem

$$\frac{du(t)}{dt} = Au(t) \quad 0 \leq t \leq T \quad (1)$$

$$u(0) = u_0 \quad (2)$$

where  $u_0$  represents a preassigned initial state of the system and  $A$  denotes a linear operator that transforms the element  $u$  into the element  $Au$  by spatial differentiations, matrix-vector multiplications and the like.

A basic question is whether these numerical solutions converge to the true solution of the initial value problem as the scale is refined. In order to formally phrase the answer to this basic question, one introduces some definitions:

- Well-posedness : An initial value problem is said to be *well-posed* if there are unique solutions to it for a dense set of initial data and this solution depends continuously on the initial data.
- Consistency : A sequence of finite difference approximation is said to be *consistent* if the difference approximation approximates the differential equations under study.
- Stability : A sequence of finite difference approximation is said to be *stable* if these algorithms together with all its composition of certain order is uniformly bounded.
- Convergence : We say that the family of finite difference approximation provides a *convergent approximation* for the initial value problem if the magnitude of the difference

between the numerical solution (obtained from the algorithms) and the true solution goes to zero (in some uniform sense) as the scale goes to zero.

**Theorem 1. (*Lax and Richtmyer*)** *Given a properly posed initial value problem (1), (2) and a finite difference approximation to it that satisfies the consistency condition, stability is a necessary and sufficient condition that difference approximation be a convergent approximation.*

Note, that in the above initial value problem the operator  $A$  is a linear operator and the sequence of finite difference approximations are also linear. We will study the non-linear difference approximation which are stable under iteration whose first iterate satisfies the properties relative to adjacent approximation levels ( $\beta$ -axioms). When the parameter  $\beta > 1$ , the algorithms converge to continuous time dependent path in the projective limit space of the finite dimensional vector space. When  $\beta > 2$ , the algorithms converges to an almost everywhere differentiable path in the projective limit space.

## Section I

### Construction of a path in the projective limit space

Consider a tower of normed finite dimensional vector spaces ,

$\dots \rightarrow C_n \dots \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0$ , with projective limit space  $C$ ,

where the  $P_n : C_{n+1} \rightarrow C_n$  are linear and norm non-increasing.

For each  $n$ , let  $A_n$  be a self mapping of  $C_n$  (properties of  $A_n$  are described below).

We consider orbits of iterates of  $A_n$  of length  $2^n$ ,  $v_n, A_n(v_n), A_n^2(v_n), A_n^3(v_n), \dots, A_n^{2^n}(v_n)$ .

For each  $n$ , we consider the graph of the equally spaced piecewise linear mapping:  $[0, 1] \rightarrow C_n$  connecting successive points of the orbit of  $A_n$  at level  $n$ . We refer to the graph as the path at level  $n$ .

**Definition 1.** *A sequence of points  $a_n$  have a limit point in  $C$  if for each  $k$  the projections of the points from the higher levels have a unique limit point. Note that the unique limit points are related by the projections and define the limit point in  $C$ .*

For each level  $k$ , project the path from each higher level to this level  $k$ . We will write axioms below to insure that all of these paths at level  $k$  will accumulate to a unique limit path at level  $k$ . As  $k$  varies these limit paths by construction map to one another by the projections  $P_n$ . Thus they will define a mapping  $[0, 1] \rightarrow C$ , namely a path in the projective limit space of the tower.

**Definition 2.**  *$\beta$ -almost semigroup property: The mappings  $A_n$  satisfy the  $\beta$ -almost semigroup property if*

$$\|A_n(v) - P_n(A_{n+1}^2(u))\| = k_1\left(\frac{1}{2}\right)^{n\beta}, \text{ for all } n, u \text{ and } v, \text{ where } u \in C_{n+1}, v \in C_n \text{ and } P_n(u) = v.$$

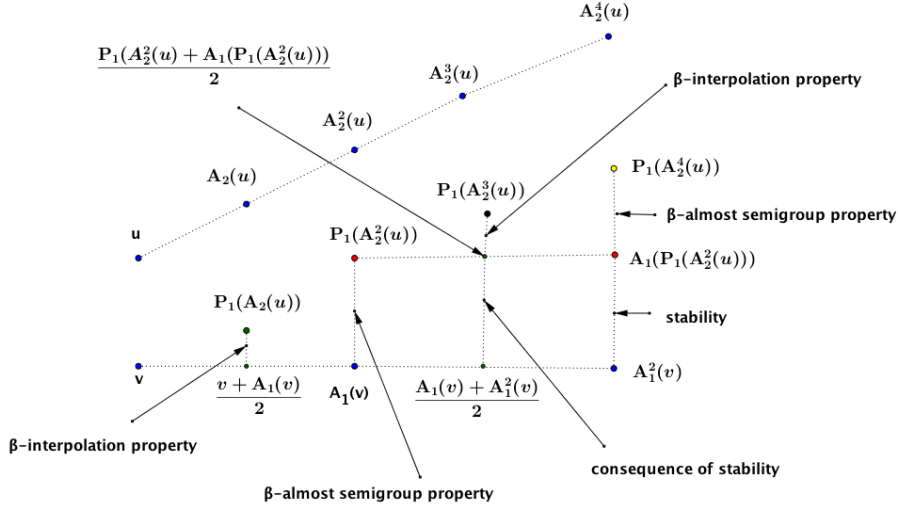


Figure 1: Triangle diagram-I

**Definition 3.**  *$\beta$ -Interpolation property:* The mappings  $A_n$  satisfy the  $\beta$ -interpolation property if  $\|\frac{1}{2}[v + A_n(v)] - P_n(A_{n+1}(u))\| = k_2(\frac{1}{2})^{n\beta}$ , for all  $n$ ,  $u$  and  $v$ , where  $u \in C_{n+1}$ ,  $v \in C_n$  and  $P_n(u) = v$ .

**Definition 4.** *Stability:* We say that  $\{A_n\}$  is a stable sequence if

$$\|A_n^k(v_1) - A_n^k(v_2)\| \leq k_3\|v_1 - v_2\|, \text{ for all } n, \text{ for all } v_1 \in C_n \text{ and } v_2 \in C_n, \text{ where } 0 \leq k \leq 2^n.$$

**Triangle Diagram:** This diagram indicates how to estimate the distances associated to dotted vertical lines using the  $\beta$ -almost semigroup property,  $\beta$ -Interpolation property and stability.

Now in the Figure 1, in the right most, there are total two vertical lines, top one is controlled by  $\beta$ -almost semigroup property and the bottom one is controlled by stability. In the above picture  $P_1$  is the projection map from  $C_2$  to  $C_1$ . The norm difference between the path at level 2 after projecting it to level 1 and the path at level 1 is either  $k_1(\frac{1}{2})^\beta + k_3k_1(\frac{1}{2})^\beta = 2 \max(k_1, k_3) (\frac{1}{2})^\beta$  (when we consider the right most vertical lines) or  $k_2(\frac{1}{2})^\beta + k_1(\frac{1}{2})^\beta = 2 \max(k_1, k_2) (\frac{1}{2})^\beta$  (when we consider the second right most vertical

lines.)

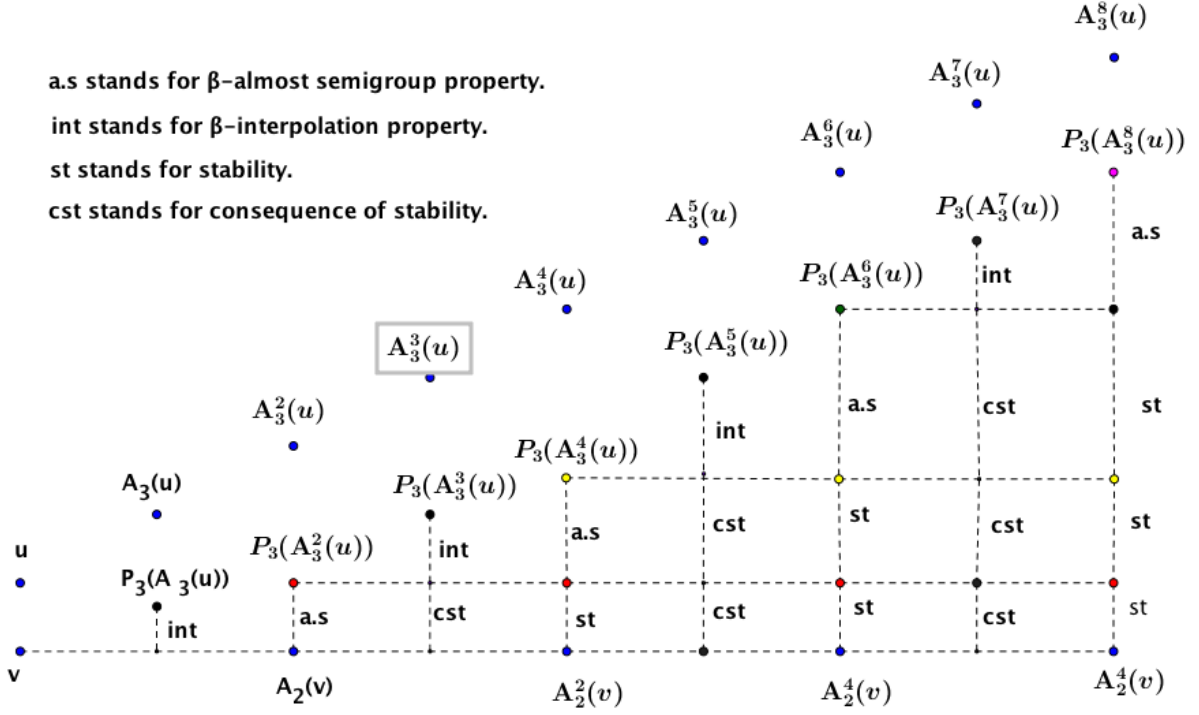


Figure 2: Triangle diagram-II

In the Figure 2, in the right most, there are four vertical lines, from the top, the first one is controlled by  $\beta$ - almost semigroup property(a.s) and other three are controlled by stability (st). And in the second right most, there are total four vertical lines, from the top, the first one is controlled by  $\beta$ -interpolation property and other three are controlled by consequence of stability (cst). The norm difference after projecting this path from level 3 to level 2 and the path at level 2 is  $k_1(\frac{1}{2})^{2\beta} + k_3k_1(\frac{1}{2})^{2\beta} + k_3k_1(\frac{1}{2})^{2\beta} + k_3k_1(\frac{1}{2})^{2\beta} = 4 \max(k_1k_3) (\frac{1}{2})^{2\beta} = 2^2 \max(k_1, k_3) (\frac{1}{2})^{2\beta}$  (when we consider the right most vertical lines) or  $k_2(\frac{1}{2})^{2\beta} + k_1(\frac{1}{2})^{2\beta} + k_1(\frac{1}{2})^{2\beta} + k_1(\frac{1}{2})^{2\beta} = 4 \max(k_1, k_2) (\frac{1}{2})^{2\beta} = 2^2 \max(k_1, k_2) (\frac{1}{2})^{2\beta}$  (when we consider the second right most vertical lines).

In general the difference between the path at level  $n + 1$  after projecting this path to level  $n$  and the path at level  $n$  is given by  $k_1(\frac{1}{2})^{n\beta} + k_3k_1(\frac{1}{2})^{n\beta} + k_3k_1(\frac{1}{2})^{n\beta} + k_3k_1(\frac{1}{2})^{n\beta} + \dots + k_3k_1(\frac{1}{2})^{n\beta} = 2^n \max(k_1, k_3) (\frac{1}{2})^{n\beta} = 2^n \max(k_1, k_3) (\frac{1}{2})^{n\beta}$  (when we consider the right most

vertical lines)

or  $k_2(\frac{1}{2})^{n\beta} + k_1(\frac{1}{2})^{n\beta} + k_1(\frac{1}{2})^{n\beta} + k_1(\frac{1}{2})^{n\beta} + \dots + k_1(\frac{1}{2})^{n\beta} = 2^n \max(k_1, k_2) (\frac{1}{2})^{n\beta} = 2^n \text{Max}(k_1, k_2) (\frac{1}{2})^{n\beta}$   
 (when we consider the second right most vertical lines).

**Definition 5.** *Two directed lines are nearly pointing in the same direction if there are ordered pairs of points distance  $d$  apart on each line so that after some translation of the lines the distance between the initial points and final points are small compare to  $d$ .*

Consider the following **four point diagram**

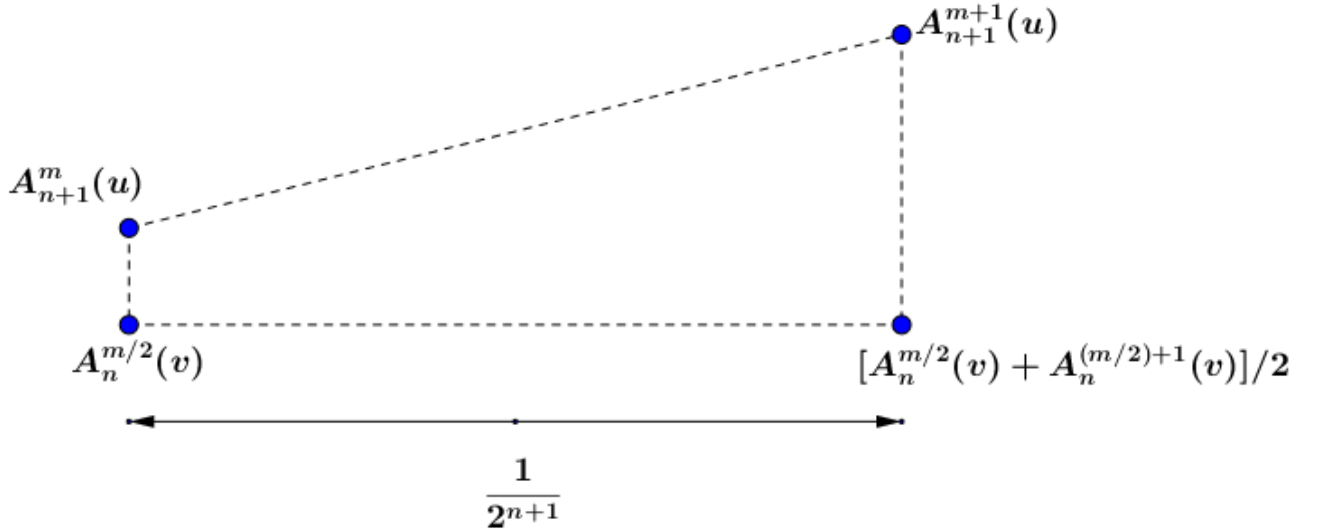


Figure 3: Four point diagram

In the Figure 3, the vertical distance between the points  $A_n^{m/2}(v)$  and the point which is obtained by projecting the point  $A_{n+1}^m(u)$  from the level  $n + 1$  to level  $n$  is at most  $\text{Max}(k_1, k_3) 2^n (\frac{1}{2})^{n\beta}$  where  $m$  is even. Again from above the triangle diagram calculation the distance between the point  $\frac{A_n^{m/2} + A_n^{m/2+1}(v)}{2}$  and the point obtained after projecting the



point  $A_{n+1}^{m+1}(u)$  from level  $n + 1$  to level  $n$  is at most  $\text{Max}(k_2, k_3) 2^n \left(\frac{1}{2}\right)^{n\beta}$ . The vertical distances  $\text{Max}(k_1, k_3) 2^n \left(\frac{1}{2}\right)^{n\beta}$  and  $\text{Max}(k_2, k_3) 2^n \left(\frac{1}{2}\right)^{n\beta}$  divided by  $\frac{1}{2^{n+1}}$  is small compare to  $\frac{1}{2^{n+1}}$ , when  $\beta > 2$ . Hence the two directed lines, one passing from the points  $A_{n+1}^m(u)$  and  $A_{n+1}^{m+1}(u)$  and the another passing from the points  $A_n^{\frac{m}{2}}(v)$  and  $\frac{A_n^{\frac{m}{2}} + A_n^{\frac{m}{2}+1}(v)}{2}$  nearly points in the same direction.

**Theorem 2.** *If  $\beta > 1$  then tower of paths constructed from the piecewise linear paths obtained from the orbits of  $A_n$ , where the  $A_n$  satisfies the  $\beta$ -almost semigroup property,  $\beta$ -interpolation property and stability defines a continuous path in the projective limit space.*

*Proof.* : We fix a level  $n$  and project the tower of paths above the level  $n$  to this fixed level. The norm difference between the path at level  $n$  and the path at level  $n + 1$  after projecting onto level  $n$  is at most  $\max(k_1, k_2, k_3) 2^n \left(\frac{1}{2}\right)^{\beta n}$ . And the norm difference between the path at level  $n + 1$  and the path at level  $n + 2$  after projecting the path from level  $n + 2$  to level  $n + 1$  is at most  $\max(k_1, k_2, k_3) 2^{n+1} \left(\frac{1}{2}\right)^{(n+1)\beta}$ , this difference between the paths which are level  $n + 1$  and  $n + 2$  is preserved when we project both the paths level  $n$  as our projection map is distance decreasing. Therefore the norm difference between the paths at level  $n$  and  $n + k$  after bringing them on a fixed level  $n$  is at most  $c 2^n \left(\frac{1}{2}\right)^{n\beta} + c 2^{n+1} \left(\frac{1}{2}\right)^{(n+1)\beta} + \dots + c 2^{n+k-1} \left(\frac{1}{2}\right)^{(n+k-1)\beta}$ , where  $c = \max(k_1, k_2, k_3)$ , which is a part of a convergent series  $\sum \left(\frac{1}{2}\right)^{n(\beta-1)}$ , when  $\beta$  is bigger than one, hence the tower of paths converges at fixed level  $n$  and the convergence is uniform, as  $n$  arbitrary hence the tower of path converges at each fixed level. Since each path in this tower is continuous so the limiting path is continuous at each level. And hence we have a continuous limiting path in the projective limit space.  $\square$

We proved in theorem 2, for  $\beta > 1$ , that the tower of paths constructed from the piecewise linear paths which are obtained from the orbits of the mappings  $A_n$ , where  $A_n$  satisfies the

$\beta$ -almost semigroup property and  $\beta$ -interpolation property and stability defines a continuous path in the projective limit space. Differentiability of the path in the projective limit space means that after projecting this tower of paths at each fixed level, the limiting path at each fixed level is differentiable.

**Theorem 3.** *If  $\beta > 2$ , then the tower of paths constructed from the piecewise linear paths which are obtained from the orbits of the mappings  $A_n$ , where  $A_n$  satisfies the  $\beta$ -almost semigroup property,  $\beta$ -interpolation property and stability defines an almost everywhere differentiable path in the projective limit space.*

*Proof.* : We will show that if  $\beta > 2$  then the path in the projective limit space is lipschitz and hence it is almost everywhere differentiable. We first calculate the difference between the slope coming from two adjacent paths at level  $n$  and level  $n + 1$ . And then we project these adjacent paths to level 1 by a linear and distance decreasing map and hence difference in slope is not increased. Let the first path in this tower of paths has derivative  $m$  (that is slope  $m$ ), slope of second path which has two linear piece, by the four point diagram has the derivative which lies between  $m - k(\frac{1}{2})^{\beta-2}$  and  $m + k(\frac{1}{2})^{\beta-2}$ . Each of these two linear piece gives birth of two new linear piece and derivative of each piece lies between  $m - k(\frac{1}{2})^{\beta-2} - k(\frac{1}{2^2})^{\beta-2}$  and  $m + k(\frac{1}{2})^{\beta-2} + k(\frac{1}{2^2})^{\beta-2}$ . The path at  $n$ -th level has  $2^n$  linear piece and the derivative of each piece lies between  $m - (\frac{1}{2})^{\beta-2} - (\frac{1}{2^2})^{\beta-2} - \dots - (\frac{1}{2^n})^{\beta-2}$  and  $m + (\frac{1}{2})^{\beta-2} + (\frac{1}{2^2})^{\beta-2} + \dots + (\frac{1}{2^n})^{\beta-2}$ . The sequence of slope coming from the sequence of paths is given by  $m, m \pm (\frac{1}{2})^{\beta-2}, m \pm (\frac{1}{2})^{\beta-2} \pm (\frac{1}{2^2})^{\beta-2}, \dots, m \pm (\frac{1}{2})^{\beta-2} \pm (\frac{1}{2^2})^{\beta-2} \pm \dots \pm (\frac{1}{2^n})^{\beta-2} \pm \dots$ . When  $\beta > 2$ , this sequence of slope function converges except possibly on the dyadic rational. And these slope function is uniformly bounded by  $m + (\frac{1}{2})^{\beta-2} + (\frac{1}{2^2})^{\beta-2} + \dots + (\frac{1}{2^n})^{\beta-2} \pm \dots$  and hence the limiting path is lipschitz. Similarly we can show that after bringing the tower of path at any other level and using the four point diagram the sequence of direction converges and it is uniformly bounded and so lipschitz. Therefore the path in the projective

limit space is lipschitz and hence it is almost everywhere differentiable.

□

# Section II

## Equicontinuous Maps

**Definition 6.** Let  $X$  and  $Y$  be topological vector spaces and  $\Gamma$  a collection of linear mappings from  $X$  to  $Y$ . We say that  $\Gamma$  is equicontinuous at  $0$  if for every neighborhood  $U$  of  $0$  in  $Y$  there corresponds a neighborhood  $E$  of  $0$  in  $X$  such that  $T(E) \subset U$  for all  $T \in \Gamma$ . We say that  $\Gamma$  is equicontinuous at  $x_0$  in  $X$  if for every neighborhood  $U$  of  $0$  in  $Y$  there corresponds a neighborhood  $E$  of  $0$  in  $X$  such that  $T(x_0 + E) \subset T(x_0) + U$  for all  $T \in \Gamma$ .

**Theorem 4. (Banach-Steinhaus)** Let  $X$  and  $Y$  be topological vector spaces. The topology on  $X$  is coming from a metric space and  $X$  is complete with respect to this metric.  $\Gamma$  a collection of continuous linear mappings from  $X$  to  $Y$ . If for each fixed  $x \in X$ , the set  $\Gamma(x) = \{T(x) : T \in \Gamma\}$  is bounded in  $Y$ , then  $\Gamma$  is equicontinuous.

**Definition 7.** Let  $X$ ,  $Y$  and  $Z$  be topological vector spaces a map  $B$  of  $X \times Y$  into  $Z$  is called bilinear, if for each  $x \in X$  and each  $y \in Y$ , the partial mappings  $B_x : y \rightarrow B(x, y)$  and  $B_y : x \rightarrow B(x, y)$  are linear.

**Definition 8.** A family  $\{B_i\}_{i \in I}$  of bilinear maps from  $X \times Y$  into  $Z$  is equicontinuous at the origin  $(0, 0)$  if given any neighborhood  $W$  of the origin  $0$  in  $Z$  there exist a neighborhood  $U \times V$  of  $(0, 0)$  in  $X \times Y$  such that  $B_i(U \times V) \subset W$  for all  $i \in I$ .

**Definition 9.** A bilinear map  $B$  from  $X \times Y$  into  $Z$  is separately continuous if all the partial maps  $B_x$  and  $B_y$  are continuous.

**Definition 10.** A family  $\{B_i\}_{i \in I}$  of bilinear maps from  $X \times Y$  into  $Z$  is separately equicontinuous if for each  $x \in X$  and each  $y \in Y$  the families  $\{(B_i)_x : B_i \in \{B_i\}_{i \in I}\}$  and  $\{(B_i)_y : B_i \in \{B_i\}_{i \in I}\}$  are equicontinuous.

**Theorem 5. Equicontinuity for family of bilinear maps:** [1] Let  $X, Y$  and be metrisable and complete topological vector spaces,  $Z$  any topological vector space. A set  $M$  of bilinear mappings of  $X \times Y$  into  $Z$  is equicontinuous if  $B \in M$  are separately continuous, and for every  $(x, y)$  in  $X \times Y$ , the set  $M(x, y)$  of images  $B(x, y)$  for  $B \in M$ , is a bounded subset of  $Z$ .

**Remark:** In linear Lax-Richtmyer theory the stability is defined in terms of equicontinuity of a set of difference algorithms. In order to prove the converse in the Lax-Richtmyer theorem: that is, given a well posed problem and a family of consistent difference algorithms associated with it, convergence of the difference algorithms implies the algorithms are stable, the idea was to observe that the difference algorithms (which are linear maps and continuous) are pointwise bounded and hence by Banach-Steinhaus theorem are equicontinuous and hence stable.

We have explained above a similar phenomenon for bilinear maps to be equicontinuous. We hope to use this idea of bilinear equicontinuity to show this converse aspect of Lax-Richtmyer theory holds for non-linear problems for which the difference algorithms satisfy “the deviation of the algorithm from being linear is itself bilinear.”

**Conjecture:** A family of continuous difference algorithm which converges and the deviation of it from being linear is a bilinear map, is a stable algorithm.

# Section III

## Consistency and Numerical solution

In this section we discuss about consistency and a new definition of numerical solution.

Usually a PDE is defined on some Banach space of initial conditions. An example is the space of  $C^{1+\alpha}$  differential forms. Assume that we have a PDE defined on  $C^{1+\alpha}$  differential forms defined on manifold. We want to make a discrete model [8], for this we take a finer and finer triangulation/cubulation of our space and we integrate the differential form on these finer and finer triangles/cubes to obtain cochains. So now we have a tower of vector spaces which are the cochains defined on these sequence of finer and finer subdivisions of the manifold and we have a map of the Banach space of differential forms into the tower by integration. It is onto at each level so gives a dense embedding of the Banach space of forms into the projective limit space. Thus the PDE will be defined on a dense subspace of the limit space.

**Definition 11.** *A PDE initial value problem is given below, where  $A$  is a non-linear.*

$$\frac{du}{dt} = A(u), \quad t \in [0, T], \quad (3)$$

$$u(0) = u_0, \quad (4)$$

**Definition 12.** *Consider  $A_n$  as defined in section I.*

$$\frac{A_n(q_n) - q_n}{\frac{1}{2^n}}$$

is called the difference quotient obtained from  $A_n$  at  $q_n$  at level  $n$ .

**Definition 13.** *Consistent difference algorithm:* A collection of algorithm  $\{A_n\}$  (as defined in section I) is said to be consistent with the PDE given by (3) and (4) if following holds: given any point  $v$  in the projective limit space which lies in the domain of  $A$  (given in (3)), we construct  $\left\{ \frac{A_n(v_n) - v_n}{\frac{1}{2^n}} \right\}$  the difference quotient sequence, this difference quotient sequence might not be coherent, we fix a level  $k$  and project the difference quotient sequence at this level and take the limit, we do this construction at each fixed level, we consider this limit point sequence, which by construction is coherent. If this limit point sequence (call difference quotient sequence) is equal to the sequence  $\{P_n(A(v))\}$  then  $\{A_n\}$  is said to be consistent with the PDE given by (3) and (4).

**Definition 14.** *Numerical Solution:* A parametrized path in the projective limit space is said to be a numerical solution for the PDE given by (3) and (4) if whenever a point  $q = \{q_n\}$  in the path (this is a point in the projective limit space) lies in the domain of  $A$  then the difference quotient sequence is equal to the sequence  $\{P_n(A(q))\}$ .

**Theorem 6.** A numerical solution obtained from a consistent algorithm which lies in the domain of  $A$  ( $A$  given in (3)) is a true solution of the PDE given by (3) and (4).

Proof: It follows from definition of numerical solution and the fact the path entirely lies in the domain of  $A$  and hence by definition of consistency it satisfies the right hand side of the PDE and hence it is a true solution.

# Appendix

$\{\mathbb{R}^\infty, d\}$  is a complete metric space. [9]

Define  $d$  on  $\mathbb{R}^\infty$  as

$$d((a_1, a_2, a_3, a_4, \dots), (b_1, b_2, b_3, \dots)) = \sup_i \left\{ \min\{|a_1 - b_1|, 1\}, \frac{\min\{|a_2 - b_2|, 1\}}{2}, \frac{\min\{|a_3 - b_3|, 1\}}{3}, \dots, \right\} \\ \dots(1).$$

Claim:  $d$  is a metric on  $\mathbb{R}^\infty$ .

Proof: The distance between two distinct sequences is positive as we are taking supremum over positive numbers. If two sequences are same then the minimum between 1 and absolute value difference of their coordinates is zero. Hence distance between two same sequence is zero. And the distance between two sequences is zero only when they are same.

$d$  is symmetric.

$d$  satisfies the triangle inequality:

we want to show that  $d(x, y) \leq d(x, z) + d(z, y)$ , where  $x, y, z \in \mathbb{R}^\infty$ .

Notice  $|a - b| = |a - c + c - b| < |a - c| + |c - b|$

therefore  $\min\{|a - b|, 1\} = \min\{|a - c + c - b|, 1\} < \min\{|a - c|, 1\} + \min\{|c - b|, 1\}$

$$\frac{\min\{|x_n - y_n|, 1\}}{n} \leq \frac{\min\{|x_n - z_n|, 1\}}{n} + \frac{\min\{|z_n - y_n|, 1\}}{n}$$



$$\sup_n \frac{\min\{|x_n - y_n|, 1\}}{n} \leq \sup_n \frac{\min\{|x_n - z_n|, 1\}}{n} + \sup_n \frac{\min\{|z_n - y_n|, 1\}}{n}$$

therefore  $d(x, y) \leq d(x, z) + d(z, y)$ .

Hence  $d$  is a metric on the countable product of  $\mathbb{R}$ .

Claim 2: The metric topology induce the product topology on the countable product of  $\mathbb{R}$ .

Idea is take an open set  $U$  in the metric topology and we claim there exist an open set  $V$  of the product topology such that  $V \subset U$ . For this, pick a point  $x$  in  $U$  take an open ball  $B_d(x, \epsilon)$  in the metric  $d$ , choose  $N$  large enough such that  $\frac{1}{N} \leq \epsilon$ , then the set  $(x_1 - \epsilon, x_1 + \epsilon) \times (x_2 - \epsilon, x_2 + \epsilon) \times (x_3 - \epsilon, x_3 + \epsilon) \times \dots \times \mathbb{R} \times \mathbb{R} \dots$  is an open set in the product topology and it is contained in  $B_d(x, \epsilon)$ . Hence  $V \subset U$ .

Next we want to show that given any open set  $D$  in the product topology there exist an open set  $E$  in the metric topology such that  $E \subset D$ . Let  $D = \prod V_i$ , where after finitely many index  $i = j_1, j_2, j_3, \dots, j_k$  all the  $V_i$ 's are  $\mathbb{R}$ . Let point  $x = \{x_i\} \in \prod V_i$  in the product topology and consider the basic open set  $(x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2) \times (x_3 - \epsilon_3, x_3 + \epsilon_3) \times \dots (x_k - \epsilon_k, x_k + \epsilon_k) \times \mathbb{R} \times \mathbb{R} \times \dots$  in the product topology. For  $i = j_1, j_2, j_3, \dots, j_k$  we choose  $\epsilon_{j_1} \leq 1, \epsilon_{j_2} \leq 1, \dots, \epsilon_{j_k} \leq 1$ .

Define  $\epsilon = \min \left\{ \frac{\epsilon_i}{i} \mid i = j_1, j_2, j_3, \dots, j_k \right\}$ . Then the ball  $B_d(x, \epsilon)$  in the metric topology is contained in the basic open set  $(x_1 - \epsilon_1, x_1 + \epsilon_1) \times (x_2 - \epsilon_2, x_2 + \epsilon_2) \times (x_3 - \epsilon_3, x_3 + \epsilon_3) \times \dots (x_k - \epsilon_k, x_k + \epsilon_k) \times \mathbb{R} \times \mathbb{R} \times \dots$  of metric topology, which is contained in the open set  $\prod V_i$  of the product topology.

Claim 2:  $\mathbb{R}^\infty$  is a complete metric space under the metric  $d$  given in (1).

Proof: Take a cauchy sequence in  $\mathbb{R}^\infty$  then project it on each factor, notice each projection is uniformly continuous (more than that it is lipschitz) hence it takes cauchy sequences to

cauchy sequences. And  $\mathbb{R}$  is complete metric space with respect to the metric defined as minimum between 1 and absolute value of the two points. Hence cauchy sequence converge in each factor. Hence the starting cauchy sequence in the countable product space converge.

Note:  $\{\mathbb{R}^\infty, d\}$  is a **complete metric space** and  $\mathbb{R}^\infty$  is a vector space. Hence we can define the notion equicontinuous family of bilinear maps from  $\mathbb{R}^\infty \times \mathbb{R}^\infty$  to  $\mathbb{R}$ .

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