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SOME NON-LINEAR EVOLUTION EQUATIONS ;

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Introduction.

In this paper we consider evolution equations of the form

$$(1) \qquad A(t)\,u(t) + u''(t) + \beta(t;\, u(t),\, u'(t)) = f(t), \qquad 0 \leq t \leq T$$

$(u' = du/dt,\ u'' = d^2u/dt^2)$, where each $A(t)$ is an unbounded formally self-adjoint (1) linear operator, which is in practice an elliptic partial differential operator subject to appropriate boundary conditions. The operator $\beta(t;\, u,\, u')$ depends non-linearly on u and u' and is, in some sense, close to a " monotonic " (or " dissipative ") operator.

Various examples of equations of type (1) where β is a non-linear operator arise in physics. For instance :

(*a*) If $A(t) = -\Delta$, $\beta(t;\, u,\, u') = u^3$ — or, more generally, any positive odd power (so that β depends only on u) — the equation arises in quantum field theory : *cf.* JÖRGENS [7], SEGAL [18], [19]; systems of equations of this type also occur in this connection;

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(1) We could slightly generalize by assuming that only the principal part of $A(t)$ is self-adjoint.

(*b*) If $A(t) = -\Delta$, $\beta(t; u, u') = |u'| u'$ (so that β depends only on u'), the equation represents a classical vibrating membrane with the resistance proportional to the velocity (2).

We shall give some sufficient conditions on $A(t)$ and on $\beta(t; u, u')$ so that (1), together with the initial conditions

$$(2) \qquad u(0) = u_0, \qquad u'(0) = u_1 \qquad (u_0, u_1 \text{ given}),$$

is a well-posed problem, i. e., so that there exists a unique solution in the *whole* interval $[0, T]$; of course this implies corresponding results on a half-line $[0, \infty)$. For some quite general *local* results, we refer to LERAY [8], DIONNE [5] and SOBOLEVSKIJ [20].

We shall also briefly consider first-order evolution equations of the type

$$(3) \qquad A(t)\, u(t) + u'(t) + \beta(t; u(t)) = f(t),$$

where $A(t)$ and $\beta(t; u)$ are as above, but $A(t)$ need not be formally self-adjoint.

In order to solve (1), (2), we begin by considering a sequence of equations which approximate (1) and in which the non-linear terms are bounded in an appropriate space. Generally, the most convenient approximate equations are obtained by the standard method introduced, for non-linear equations, by HOPF [6]. The difficulty is to pass to the limit; this can be overcome in essentially two different ways.

In Part I, we find sufficiently many *a priori* estimates on the solutions of the approximate equations to obtain — via *compactness* arguments — *strong* convergence in appropriate spaces. The passage to the limit in the non-linear terms is now possible, while simply weak convergence would not have been enough.

This process is applied in sections 1.1 to 1.4 to the general equation (1) in which $A(t)$ is (roughly) " elliptic " and the non-linear term is independent of u and depends in a " monotonic " way on u'. In section 1.5, we apply this result to the equation

$$(4) \qquad -\Delta u + u'' + h_1(x, t)\, |u'|^{\rho-1} u' = f \qquad (\rho > 1),$$

where $h_1 \geqq 0$, by introducing L^p-spaces with respect to the measure $h_1(x, t)\, dx$. In this connection, we could also consider much more general non-linearities by using Orlicz spaces, but we simply refer to somewhat

(2) This example, which was at the origin of the present work, was mentioned to us by L. AMERIO in Varenna, May 1963. A related equation with a somewhat " weaker " non-linearity was studied by PRODI [17]. Some similar equations were considered by YAMAGUTI and MIZOHATA ([24], [25], [26]).

related problems in Višik [22]. Section 1.6 treats more general equations, and in section 1.7, the boundary conditions may themselves be non-linear.

In section 1.8 we prove a partial regularity theorem for equation (4) (with $h_1 = \text{constant}$). In sections 1.9 and 1.10, we give two examples in which $\beta(u, u')$ depends on *both* u and u'. In order to avoid making the treatment too ponderous, we have not put the results of these last sections into a general framework.

In Part II, we exploit more directly the *monotonicity* property of the non-linearities, without proving strong convergence and using a minimum of *a priori* estimates. This kind of argument was used recently for equations involving bounded operators, particularly non-linear integral equations, by Minty ([14], [15]). It was first applied to partial differential operators of elliptic and parabolic type by Browder ([2], [3]). For equations of *parabolic* type, we give a result in section 2.7 which extends that of Browder [4] in several respects (3). Our method [for equation (3)], while technically different from Browder's, follows roughly the same pattern. It can be applied directly to parabolic partial differential equations. Compactness methods have also been used by Višik [23] to solve this kind of equation.

In the case of equations of type (1) with $\beta(t; u, u') = \beta(t; u')$ a " monotonic " function of u', there are certain non-trivial technical difficulties in this kind of monotonicity argument which do not appear in the parabolic case. Specifically, we have assumed, in Part II, the annoying condition $dA(t)/dt \leq 0$ (taken in the appropriate sense), a condition which was not necessary in Part I. On the other hand, this method allows us to *weaken* the regularity hypotheses on the initial data and on the coefficients appearing in the equation as well as on the right-hand side of the equation. The general theory is given in sections 2.1 to 2.5, new examples being given in the following section. A typical one is initial boundary-value problems for the equation

$$-\Delta_x u + u'' + \sum_\sigma (D_\sigma)^* \{ |D_\sigma u'|^{\rho-1} D_\sigma u' \} = f,$$

($\rho > 1$) in the cylinder $(0, T) \times \Omega$, where the operators D_σ are arbitrary linear differential operators in the x-variables with smooth coefficients and where $(D_\sigma)^*$ denotes the formal adjoint of D_σ. The fact that the Laplacian is *strongly* elliptic is not used in the argument.

Some of the results given here were announced in [13] or were presented at the Stanford Colloquium [Ocotober 1963] or the Collège de France [December 1963].

(3) T. Kato [27] has another kind of extension by a different method.

PART I.

Compactness Method.

1.1. Hypotheses.

We are given two Hilbert spaces V and H such that $V \subset H$ and the inclusion mapping of V into H is continuous. In addition, we have a Banach space W and a one-parameter family of Banach spaces $W(t)$, $0 \leq t \leq T$, contained in H, such that, for each t, W is a dense subset of $W(t)$ and the inclusion mapping is continuous. We assume that $V \cap W$ is dense in H.

The scalar products in V and H shall be denoted by $((v_1, v_2))$ and (h_1, h_2), and the norms by $\| v \|$ and $| h |$, respectively. The spaces may be either all real or all complex.

If X is a Banach space with norm $\| \ \|_X$, we denote by $L^p(0, T; X)$ the space of (classes of) functions (real, or complex) f which are L^p over $(0, T)$ with values in X, provided with the usual norm $(1 \leq p < \infty)$

$$\left(\int_0^T \| f(t) \|_X^p \, dt \right)^{\frac{1}{p}},$$

and the usual modification in case $p = \infty$.

HYPOTHESIS I. — For every $t \in [0, T]$, we are given a bilinear (sesquilinear in the complex case) form on $V \times V$

$$u, v \rightarrow a(t; u, v) \qquad (u, v \in V)$$

which is hermitian $\left(a(t; u, v) = \overline{a(t; v, u)}\right)$ and which satisfies :

(1) $$a(t; v, v) \geq c_1 [v]^2 \qquad (v \in V),$$

where $[v]$ is a continuous pseudo-norm on V such that the norm on V

$$([v]^2 + | v |^2)^{\frac{1}{2}}$$

is equivalent to the norm $\| v \|$.

(2) For every $u, v \in V$, the function $t \rightarrow a(t; u, v)$ is twice continuously differentiable; we set

$$\frac{d}{dt} a(t; u, v) = a'(t; u, v), \qquad \frac{d^2}{dt^2} a(t; u, v) = a''(t; u, v).$$

HYPOTHESIS II. — For fixed $t \in [0, T]$, we are given a continuous linear (conjugate-linear in the complex case) form on $W(t)$

$$v \rightarrow b(t; w, v) \qquad (v \in W(t)),$$

where w is a fixed element of $W(t)$ (the dependence on w being in general non-linear) which satisfies

(1) $\operatorname{Re} b(t; v, v) \geqq 0 \quad (v \in W(t))$;
$\operatorname{Re} b(t; w, w-v) - \operatorname{Re} b(t; v, w-v) \geqq 0 \quad (v, w \in W(t))$. (3 bis)

(2) For $u, v, w \in V \cap W$, we assume the existence of the limit

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}[b(t; u+\varepsilon v, w) - b(t; u, w)] = b_*(t; u; v, w);$$

where $b_*(t; u; v, w)$ is a bilinear (sesquilinear) form in v and w and is a continuous function of u when u is restricted to finite-dimensional subspaces of $V \cap W$ (4).

(3) For every $v, w \in V \cap W$, $\frac{d}{dt} b(t; v, w) = b'(t; v, w)$ exists and is a jointly continuous function of t and v when $0 \leqq t \leqq T$ and v is restricted to finite-dimensional subspaces of $V \cap W$. Moreover,

$$2\,|\,b'(t; v, w)| \leqq \operatorname{Re} b_*(t; v; w, w) + c_2\{\operatorname{Re} b(t; v, v) + |\,v\,|^2 + |\,w\,|^2\},$$

where c_2 is a constant independent of t, v and w.

(4) If $g \in L^\infty(0, T; V)$, $g(t) \in W(t)$ a. e., $b(t; g(t), g(t))$ is a measurable function of t and

$$\operatorname{Re}\int_0^T b(t; g(t), g(t))\,dt < \infty;$$

then for every $v \in V \cap W$, $t \to b(t; g(t), v)$ is an integrable function on $(0, T)$.

Hypothesis III. — Suppose we are given a sequence of functions g_m, continuous with values in $V \cap W$, such that

$$g_m \to g \quad \text{in the weak-star topology of } L^\infty(0, T; V) \quad (5)$$

and

$$\frac{dg_m}{dt} = g'_m \to g' \quad \text{in the weak-star topology of } L^\infty(0, T; H)$$

and

$$\operatorname{Re}\int_0^T b(t; g_m(t), g_m(t))\,dt$$

(3 bis) More generally, we may assume that
$$\operatorname{Re} b(t; w, w-v) - \operatorname{Re} b(t; v, w-v) \geqq -k\,|\,w-v\,|^2, \qquad v, w \in W(t),$$
where k is some constant.

(4) In case $b(t; u, v)$ is independent of t, we can omit Hypothesis II (2) as well as II (3), by a variant of the proof to be given below.

(5) That is, for all $h \in L^1(0, T; V)$,
$$\int_0^T ((g_m(t), h(t)))\,dt \to \int_0^T ((g(t), h(t)))\,dt.$$

remains bounded in m. Then $g(t) \in W(t)$ a. e., $b(t;\, g(t),\, g(t))$ is a measurable function of t,

$$\operatorname{Re} \int_0^T b(t;\, g(t)\, g(t))\, dt < \infty$$

and a subsequence $\{ g_\nu \}$ of $\{ g_m \}$ can be extracted such that

$$b(t;\, g_\nu(t),\, v) \to b(t;\, g(t),\, v)$$

in the sense of distributions over $(0, T)$ for every $v \in V \cap W$.

Before stating the last hypothesis, we define V' to be the space of continuous (conjugate-) linear forms on V; i. e., the (anti-) dual of V. Since $v \to a(t;\, u,\, v)$ is a continuous (conjugate-) linear form on V, we may write :

$$a(t;\, u,\, v) = (A(t)\, u,\, v) \quad \text{for all } v \in V,$$

where $A(t)\, u \in V'$. Then we define $D(A(t))$ as the set of all u in V such that $A(t)\, u \in H$. Alternatively, we could define $D(A(t))$ as the set of those elements u in V such that the form is continuous on V when V is provided with the topology of H. We provide $D(A(t))$ with the norm

$$\| u \|_{D(A(t))} = \{ | u |^2 + | A(t)\, u |^2 \}^{\frac{1}{2}}.$$

HYPOTHESIS IV.

(1) $V \cap W$ is separable.

(2) $D(A(0)) \cap W$ is dense in $D(A(0))$.

We also define $W(t)'$ as the (anti-) dual of $W(t)$. Since $v \to b(t;\, u,\, v)$ is a continuous (conjugate-) linear form on $W(t)$, we may write

$$b(t;\, u,\, v) = (\beta(t)\, u,\, v) \quad \text{for all } v \in W(t),$$

where $\beta(t)\, u \in W(t)'$. We define $\mathcal{B}_t$ as the set of all u in $W(t)$ such that $\beta(t)\, u \in H$.

1.2. **An existence-uniqueness theorem.**

THEOREM 1.1. — *Let V, H, W, $W(t)$, $a(t;\, u,\, v)$, $b(t;\, u,\, v)$ be given, satisfying the above hypotheses. Let f, u_0, u_1 be given, satisfying*

$$f \in L^1(0, T;\, H), \qquad f' = \frac{df}{dt} \in L^1(0, T;\, H), \tag{1.1}$$

$$u_0 \in D(A(0)) \quad (^6), \tag{1.2}$$

$$u_1 \in V \cap \mathcal{B}_0. \tag{1.3}$$

---

[6] Hypothesis IV (2), is not necessary in case $u_0 \in D(A(0)) \cap W$.

Then there exists one and only one function u which satisfies

$$(1.4)\quad \left\{\begin{array}{c} u \in L^\infty(0, T; V), \\ u' = \dfrac{du}{dt} \in L^\infty(0, T; V), \\ u'' = \dfrac{d^2 u}{dt^2} \in L^\infty(0, T; H), \\ u'(t) \in W(t) \text{ a. e.,} \\ b(t; u'(t), u'(t)) \textit{ is a measurable function of } t \textit{ and} \\ \operatorname{Re} \displaystyle\int_0^T b(t; u'(t), u'(t))\, dt < \infty; \end{array}\right.$$

$$(1.5)\quad \left\{\begin{array}{l} (u''(t), v) + a(t; u(t), v) + b(t; u'(t), v) = (f(t), v) \\ \quad \textit{for all} \quad v \in V \cap W(t), \qquad \textit{for } \text{a. e. } t \in (0, T); \end{array}\right.$$

$$(1.6)\qquad u(0) = u_0, \qquad u'(0) = u_1.$$

[Conditions (1.6) make sense because $u'' \in \mathrm{L}^\infty(0, T; H)$.]

In case $V \cap W$ is dense in V and in W, (1.5) may be replaced by the more suggestive equation

$$(1.5)'\qquad u''(t) + A(t)\, u(t) + \beta(t)\, u'(t) = f(t) \quad \text{a. e.,}$$

where $A(t) : V \to V'$ and $\beta(t) : W(t) \to W(t)'$ are defined as above.

1.3. Proof of the uniqueness.

Note that Hypothesis I (2) implies the existence of a constant c_1 such that

$$(1.7)\qquad |a(t; u, v)| + |a'(t; u, v)| + |a''(t; u, v)| \leqq c_1 \|u\| . \|v\|,$$

because of the uniform boundedness principle.

Now let u_1 and u_2 be two solutions of (1.4), (1.5), (1.6) and set $w(t) = u_1(t) - u_2(t)$. Then, for fixed t, for every $v \in V \cap W(t)$,

$$a(t; w(t), v) + (w''(t), v) + b(t; u_1'(t), v) - b(t; u_2'(t), v) = 0.$$

Since $w(t) \in V \cap W(t)$ a. e., we may take $v = w'(t)$ in this equation; Hypothesis II then implies that

$$\operatorname{Re}\{ a(t; w(t), w'(t)) + (w''(t), w'(t)) \} \leqq 0 \quad \text{a. e.}$$

This may be written as

$$\frac{d}{dt}\{ |w'(t)|^2 + a(t; w(t), w(t)) \} - a'(t; w(t), w(t)) \leqq 0 \quad \text{a. e.}$$

Integrating from 0 to t and using Hypothesis I, we obtain

$$|w'(t)|^2+[w(t)]^2 \leq c_2 \int_0^t \{[w(\sigma)]^2+|w(\sigma)|^2\}\, d\sigma.$$

But

$$w(t)=\int_0^t w'(\sigma)\, d\sigma$$

implies that

$$\int_0^t |w(\sigma)|^2\, d\sigma \leq c_3 \int_0^t |w'(\sigma)|^2\, d\sigma.$$

Therefore

$$|w'(t)|^2+[w(t)]^2 \leq c_4 \int_0^t \{|w'(\sigma)|^2+[w(\sigma)]^2\}\, d\sigma \quad \text{a. e.}$$

This implies that $|w'(t)|=0$ a. e., so that $w'(t)=0$ a. e. and $w(t)=0$. This completes the proof of the uniqueness assertion of Theorem 1.1.

Several hypotheses, including III and IV in their entirety, were not used in the above proof. However, we refrain from enumerating them since a much stronger uniqueness theorem (under somewhat different hypotheses) will be given later.

1.4. Proof of the existence.

The first step is to construct finite-dimensional approximations to the differential equation in a well-known manner. We construct a basis (i. e., a linearly independent set whose finite linear combinations are dense) of $V \cap W$ as follows. The initial data u_0 and u_1 are given satisfying (1.2) and (1.3); $u_1 \in V \cap W$. Define $y_1=u_1$. Define $y_2=u_0$ in case $u_0 \in W$; if $u_0 \notin W$, Hypothesis IV implies the existence of a sequence $\{y_2, y_3, \ldots\}$ contained in $D(A(0)) \cap W$, hence in $V \cap W$, which converges to u_0 in the topology of $D(A(0))$. Throw away from the sequence $\{y_1, y_2, \ldots\}$ all vectors which depend linearly on the preceding ones. Complete the remaining sequence of vectors to a basis $\{w_1, w_2, \ldots\}$ of $V \cap W$, utilizing Hypothesis IV (1). Then u_1 is an element of the basis (unless $u_1=0$), and u_0 is a limit in $D(A(0))$ of basis elements $\{u_{0m}\}$.

We also define

$$u_{2m}=P_m[f(0)-\beta(0)\, u_1-A(0)\, u_{0m}] \tag{1.8}$$

where P_m is the orthogonal projection in H onto the subspace generated by $w_1, \ldots, w_m$. Since

$$|f(0)| \leq c_3 \int_0^T (|f(t)|+|f'(t)|)\, dt,$$

$$|A(0)\, u_{0m}| \leq c_3 \|u_0\|_{D(A(0))},$$

we have

(1.9) $$|u_{2m}|^2 \leqq c_4 K(f, u_0, u_1),$$

where

$$K(f, u_0, u_1) = \left| \int_0^T (|f(t)| + |f'(t)|)\,dt \right|^2 + \|u_0\|^2_{D(A(0))} + \|u_1\|^2 + |\beta_0(u_1)|^2.$$

Now denote by $u_m(t)$ the solution of the non-linear differential system

(1.10) $$\begin{cases} (u''_m(t), w_j) + a(t; u_m(t), w_j) + b(t; u'_m(t), w_j) = (f(t), w_j) \\ \qquad (j = 1, 2, \ldots, m), \end{cases}$$

(1.11) $$u_m(t) \in \text{range of } P_m \text{ for all } t \in [0, T],$$

(1.12) $$u_m(0) = P_m u_0, \qquad u'_m(0) = P_m u_1.$$

This solution exists in some interval $0 \leqq t \leqq \delta_m$ [because of Hypothesis II (2)]; the *a priori* estimates to be given below will show that the interval of existence is in fact $0 \leqq t \leqq T$.

The standard *a priori* (" energy ") estimates are now obtained by replacing w_j in equation (1.10) by $u'_m(t)$ [i. e., multiplying (1.10) by the appropriate scalar function and summing on j]. We obtain

$$\frac{1}{2}\frac{d}{dt}\{|u'_m(t)|^2 + a(t; u_m(t), u_m(t))\} - \frac{1}{2} a'(t; u_m(t), u_m(t)) + \operatorname{Re} b(t; u'_m(t), u'_m(t)) = \operatorname{Re}(f(t), u'_m(t)).$$

Integrating this equation from 0 to t and using (1.7) gives :

(1.13) $$\begin{cases} |u'_m(t)|^2 + [u_m(t)]^2 + 2\operatorname{Re}\displaystyle\int_0^t b(\sigma; u'_m(\sigma), u'_m(\sigma))\,d\sigma \\ \leqq c_5 \Big\{ |u'_m(0)|^2 + [u_m(0)]^2 \\ \qquad + \displaystyle\int_0^t [|f(\sigma)|.|u'_m(\sigma)| + \|u_m(\sigma)\|^2]\,d\sigma \Big\}. \end{cases}$$

But, just as in the uniqueness proof, we have

$$\int_0^t |u_m(\sigma)|^2\,d\sigma \leqq c_6 |u_m(0)|^2 + c_6 \int_0^t |u'_m(\sigma)|^2\,d\sigma.$$

Therefore, the right-hand side of (1.13) is bounded by

$$c_7 K(f, u_0, u_1) + c_7 \int_0^t \{|u'_m(\sigma)|^2 + [u_m(\sigma)]^2\}\,d\sigma;$$

so that

$$(1.14)\qquad |u'_m(t)|^2 + [u_m(t)]^2 + 2\,\mathrm{Re}\int_0^t b(\sigma;\, u'_m(\sigma),\, u'_m(\sigma))\,d\sigma \leq c_8\,K(f,\, u_0,\, u_1).$$

Each term on the left-hand side of (1.14) must therefore be bounded. The identity

$$u_m(t) = u_m(0) + \int_0^t u'_m(\sigma)\,d\sigma$$

then implies that

$$(1.15)\qquad |u_m(t)| \leq c_9\,K(f,\, u_0,\, u_1).$$

Further *a priori* estimates are obtained as follows. We differentiate (1.10) with respect to t, and then replace w_j by $u''_m(t)$. This gives (suppressing the t)

$$\begin{aligned}(u'''_m,\, u''_m) &+ a(u'_m,\, u''_m) + a'(u_m,\, u''_m)\\ &+ b'(u'_m,\, u''_m) + b_*(u'_m,\, u''_m,\, u''_m) = (f',\, u''_m).\end{aligned}$$

Taking real parts and integrating from 0 to t gives

$$(1.16)\qquad \left\{\begin{aligned} &|u''_m(t)|^2 + a(t;\, u'_m(t),\, u'_m(t))\\ &\qquad + 2\,\mathrm{Re}\int_0^t b_*(\sigma;\, u'_m(\sigma),\, u''_m(\sigma),\, u''_m(\sigma))\,d\sigma\\ &\quad = |u''_m(0)|^2 + a(t;\, u'_m(0),\, u'_m(0))\\ &\qquad + \int_0^t \mathrm{Re}\,\{\, 2(f',\, u''_m) - 2\,b'(u'_m,\, u''_m)\\ &\qquad\qquad - a'(u'_m,\, u'_m) - 2\,a'(u_m,\, u''_m)\,\}\,d\sigma.\end{aligned}\right.$$

The identity

$$\begin{aligned} a'(t;\, u_m(t),\, u''_m(t)) &= \frac{d}{dt}\,a'(t;\, u_m(t),\, u'_m(t))\\ &\quad - a'(t;\, u'_m(t),\, u'_m(t)) - a''(t;\, u_m(t),\, u'_m(t))\end{aligned}$$

then implies, by virtue of the hypotheses on a and b, that the right-hand side of (1.16) is bounded by

$$\begin{aligned} &c_{10}\,K(f,\, u_0,\, u_1) + \int_0^t \mathrm{Re}\,b_*(\sigma;\, u'_m,\, u''_m,\, u''_m)\,d\sigma\\ &\qquad + c_{10}\int_0^t \mathrm{Re}\,\{\, b(\sigma,\, u'_m,\, u'_m) + \|u'_m\|^2 + |u''_m|^2 + \|u_m\|^2\,\}\,d\sigma\\ &\qquad + c_{10}\,\|u_m(t)\|\,.\,\|u'_m(t)\|.\end{aligned}$$

The last term is estimated by $\delta \| u'_m(t) \|^2 + c_{11} \| u_m(t) \|^2$, where δ may be chosen as small as we please. By using (1.14) and (1.15), (1.16) becomes:

$$| u''_m(t) |^2 + [u'_m(t)]^2 + \operatorname{Re} \int_0^t b_*(\sigma; u'_m(\sigma); u''_m(\sigma), u''_m(\sigma))\, d\sigma$$
$$\leq c_{12} K(f, u_0, u_1) + c_{12} \int_0^t \{ [u'_m(\sigma)]^2 + | u''_m(\sigma) |^2 \}\, d\sigma.$$

But $\operatorname{Re} b_*(\sigma; u'_m(\sigma); u''_m(\sigma), u''_m(\sigma)) \geq 0$ by Hypothesis II (1) because it is a limit of non-negative terms. It follows that

$$(1.17) \qquad | u''_m(t) |^2 + [u'_m(t)]^2 \leq c_{13} K(f, u_0, u_1).$$

Now we pass to the limit. The estimates (1.14), (1.15) and (1.17) imply that a subsequence $\{ u_\nu \}$ can be extracted from $\{ u_m \}$ such that:

$u_\nu \to u$ in the weak-star topology of $L^\infty(0, T; V)$;
$u'_\nu \to u'$ in the weak-star topology of $L^\infty(0, T; V)$;
$u''_\nu \to u''$ in the weak-star topology of $L^\infty(0, T; H)$.

Hypothesis III therefore implies that $u'(t) \in W(t)$ a. e. and

$$\operatorname{Re} \int_0^T b(t; u'(t), u'(t))\, dt < \infty$$

and $b(t; u'_\nu(t), v) \to b(t; u'(t), v)$ as $\nu \to \infty$ in the sense of distributions over $(0, T)$, for all $v \in V \cap W$.

It remains to show (1.5), since (1.6) is immediate. In equation (1.10), we put $m = \nu$ and let $\nu \to \infty$, thereby obtaining:

$$(u''(t), w_j) + a(t; u(t), w_j) + b(t; u'(t), w_j) = (f(t), w_j) \quad \text{a. e.}$$

The latter equation holds for all j, therefore for any finite linear combination of the w_j's. Now $v \to b(t; u'(t), v)$ is a continuous functional on $W(t)$ by Hypothesis II (for fixed t); since $V \cap W$ is dense in $V \cap W(t)$, a final passage to the limit yields (1.5). This completes the proof of the theorem.

1.5. The prime example.

Let Ω be any open set in E^n and let $H = L^2(\Omega)$. By $H^k(\Omega)$ we mean $\{ u \mid D^\alpha u \in L^2(\Omega)$ for $|\alpha| \leq k \}$, and by $H_0^k(\Omega)$ we mean the closure in $H^k(\Omega)$ of the smooth functions with compact support. We let $V = H_0^1(\Omega)$ and define

$$a(t; u, v) = a(u, v) = \int_\Omega \sum_{i=1}^n \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i}\, dx \qquad (u, v \in V).$$

Let h_0 and h_1 be two given functions on $[0, T]\times\Omega$ which are non-negative, bounded and such that $\partial h_0/\partial t = h'_0$ and $\partial h_1/\partial t = h'_1$ are continuous, h'_0 is bounded and h'_1 is bounded by a constant multiple of h_1. For any $t\in[0, T]$, let $W(t)$ be the space of all functions u defined on Ω, measurable with respect to the measure $h_1(x, t)\,dx$, such that the norm (with ρ fixed > 1)

$$\|u\|_{W(t)} = \left(\int_\Omega |u(x)|^{\rho+1} h_1(x, t)\; dx\right)^{\frac{1}{\rho+1}} + \left(\int_\Omega |u(x)|^2\, dx\right)^{\frac{1}{2}}$$

is finite; furnish $W(t)$ with the above norm. Let $W = L^{\rho+1}(\Omega)\cap L^2(\Omega)$. Let $\varphi(z) = |z|^{\rho-1} z$ ($z =$ complex number) and define

$$b(t;\, u, v) = \int_\Omega h_1(x, t)\, \varphi(u(x))\, \overline{v(x)}\, dx + \int_\Omega h_0(x, t)\, u(x)\, \overline{v(x)}\, dx$$

for $u,\ v\in W(t)$.

Let us verify the hypotheses of paragraph 1.1. Hypothesis I is satisfied if we define

$$[v]^2 = \int_\Omega \sum_{i=1}^{n} \left|\frac{\partial v}{\partial x_i}\right|^2 dx.$$

For Hypothesis II, we require the

LEMMA 1.1. — *The function* $\varphi(z) = |z|^{\rho-1} z$ *satisfies :*

(a) $\operatorname{Re}[\varphi(z)\,\bar z] \geqq c_{10}\,|z|^{\rho+1}$;

(b) $|\varphi(z)| \leqq c_{11}\,|z|^{\rho}$;

(c) $\operatorname{Re}[\varphi'_z(\zeta)\,\bar\zeta] \geqq c_{12}\,|z|^{\rho-1}\,|\zeta|^2$;

(d) $|\varphi'_z(\zeta)| \leqq c_{13}\,|z|^{\rho-1}\,|\zeta|$,

where the c_i *are positive constants and where* $\varphi'_z(\zeta)$ *denotes the derivative of* φ *at the point* z *in the direction* ζ.

Parts (a) and (b) are obvious (in fact $c_{10} = c_{11} = 1$); (c) and (d) follow easily from the identity

$$\varphi'_z(\zeta) = |z|^{\rho-1}\zeta + (\rho - 1)\,|z|^{\rho-3}\operatorname{Re}(z\bar\zeta)\, z;$$

in fact, we may take $c_{12} = 1$, and $c_{13} = \rho$.

On the other hand, by the mean-value theorem,

$$\varphi(z) - \varphi(\zeta) = \varphi'_\xi\,(z - \zeta)$$

where ξ lies on the line segment joining z and ζ; hence (c) and (d) imply

(c)′ $\operatorname{Re}[\varphi(z) - \varphi(\zeta)]\,[\overline{z - \zeta}] \geqq c_{12}\,|\xi|^{\rho-1}\,|z-\zeta|^2 \geqq 0$,

and

$(d)'$ $\quad |\varphi(z)-\varphi(\zeta)| \leq c'_{13}[|z|^{\rho-1}+|\zeta|^{\rho-1}]|z-\zeta|$.

The only properties we shall use of the function φ are that it is once continuously differentiable and that it satisfies (a), (b), (c), (d).

Hypothesis II (1) follows from (a) and $(c)'$. Verify now that

$$b_*(t; u, v, w) = \int_\Omega \{h_1(x, t)\, \varphi'_{u(x)}(v(x))\, \overline{w(x)} + h_0(x, t)\, v(x)\, \overline{w(x)}\}\, dx$$

and

$$b'(t; u, v) = \int_\Omega \left\{\left(\frac{\partial h_1}{\partial t}\right)\varphi(u)\,\bar{v} + \left(\frac{\partial h_0}{\partial t}\right) u\bar{v}\right\} dx.$$

Therefore

$$2\,|b'(t; u, v)| \leq c_{14} \int_\Omega \{|\varphi(u(x))|\, h_1(x, t) + |u(x)|\}\, |v(x)|\, dx.$$

Lemma 1.1 (b), together with Schwarz's inequality, then yields

$$2\,|b'(t, u, v)| \leq \delta \int_\Omega |u(x)|^{\rho-1}\, |v(x)|^2 h_1(x, t)\, dx$$
$$+ c_{15} \int_\Omega |u(x)|^{\rho+1} h_1(x, t)\, dx + c_{15} \int_\Omega (|u(x)|^2 + |v(x)|^2)\, dx,$$

where δ may be chosen as small as we wish (c_{15} depending on δ). An application of Lemma 1.1 therefore yields Hypothesis II (3).

Hypothesis II (4) is satisfied because of the inequality

$$|b(t; g(t), v)| \leq c_{16} \left(\int_\Omega |g|^{\rho+1} h_1\, dx\right)^{\frac{1}{\rho+1}} \left(\int_\Omega |v|^{\rho+1} h_1\, dx\right)^{\frac{1}{\rho+1}}$$
$$+ c_{16} \left(\int_\Omega |g|^2\, dx\right)^{\frac{1}{2}} \left(\int_\Omega |v|^2\, dx\right)^{\frac{1}{2}},$$

($g(t) \in W(t)$ a. e.) and Lemma 1.1 (a).

In order to verify the crucial Hypothesis III, we take a sequence of function $\{g_m\}$ satisfying :

$$g_m \to g \quad \text{weakly-star in } L^\infty(0, T; H_0^1(\Omega)),$$
$$g'_m \to g \quad \text{weakly-star in } L^\infty(0, T; L^2(\Omega)),$$
$$\sup_m \int_0^T \int_\Omega h_1(x, t)\, |g_m(x, t)|^{\rho+1}\, dx\, dt < \infty.$$

Let $L^q((0, T) \times \Omega; h_1)$ denote the space of functions whose q^{th} powers are integrable with weight function $h_1(x, t)$ $(1 \leq q < \infty)$. Then $\{g_m\}$ is bounded in $L^{\rho+1}((0, T) \times \Omega; h_1)$, and [by Lemma 1.1 (*b*)] $\{\varphi(g_m)\}$ is bounded in $L^{(\rho+1)/\rho}((0, T) \times \Omega; h_1)$. Therefore we may extract a subsequence $\{g_\nu\}$ such that

$$g_\nu \to g \quad \text{weakly in } L^{\rho+1}((0, T) \times \Omega; h_1),$$

$$\varphi(g_\nu) \to \psi \quad \text{weakly in } L^{\frac{\rho+1}{\rho}}((0, T) \times \Omega; h_1),$$

and moreover such that

$$g_\nu \to g \quad \textit{strongly} \text{ in } L^2(K)$$

for a given compact subset K of $(0, T) \times \Omega$. The latter conclusion is possible because $H_0^1(\Omega)$ is compact in $L^2(\Omega)$. Now, extracting a further subsequence if necessary, $g_\nu \to g$ a. e., in K, hence $\varphi(g_\nu) \to \varphi(g)$ a. e. in K. This is enough to imply that $\psi = \varphi(g)$. Hence

$$\varphi(g_\nu) \to \varphi(g) \quad \text{weakly in } L^{\frac{\rho+1}{\rho}}((0, T) \times \Omega; h_1).$$

This proves Hypothesis III.

As for Hypothesis IV, it is clear that $H_0^1(\Omega) \cap L^{\rho+1}(\Omega)$ is separable, being homeomorphic to a subspace of $L^2(\Omega) \times L^{\rho+1}(\Omega)$. Finally, if we assume the boundary Γ of Ω to be sufficiently smooth, we have $D(A(0)) = H^2(\Omega) \cap H_0^1(\Omega)$, so that $D(A(0)) \cap L^{\rho+1}(\Omega)$ is dense in $D(A(0))$.

Theorem 1.1 may therefore be applied to yield the following result. *Given :*

$u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *if* Γ *is sufficiently regular;*

$u_0 \in D(A(0)) \cap L^{\rho+1}(\Omega)$ *otherwise* [*cf.* footnote (2)];

$$u_1 \in H_0^1(\Omega) \quad \textit{and} \quad \int_\Omega h_1(x, 0)^2 \,|\, u_1(x) \,|^{2\rho} \, dx < \infty;$$

$$f, f' \in L^1(0, T; L^2(\Omega)).$$

Then there exists one and only one function u which satisfies

$$\begin{cases} u \in L^\infty(0, T; H_0^1(\Omega)), \\ u' \in L^\infty(0, T; H_0^1(\Omega)) \cap L^{\rho+1}((0, T) \times \Omega; h_1), \\ u'' \in L^\infty(0, T; L^2(\Omega)); \end{cases}$$

$$\begin{cases} u'' - \Delta u + h_0 u' + h_1 \,|\, u' \,|^{\rho-1} u' = f, \\ u\,|_{t=0} = u_0, \qquad u'\,|_{t=0} = u_1. \end{cases}$$

1.6. A second example.

We take $H = L^2(\Omega)$, Ω open in E^n; $V =$ any closed subspace of $H^m(\Omega)$ which contains the smooth functions with compact support; $W =$ the closure in $W^{\mu, \rho+1}(\Omega)$ of $V \cap W^{\mu, \rho+1}(\Omega)$, where μ is a given integer *less* *than* m; $W(t) = W$ for all t; m is any given positive integer. In general, we define

$$W^{k,q}(\Omega) = \{ u \mid D^\alpha u \in L^q(\Omega), \mid \alpha \mid \leq k \}$$

and

$$H^k(\Omega) = W^{k,2}(\Omega)$$

with the usual norms. We take

$$a(t; u, v) = \sum_{|\alpha|, |\beta| \leq m} \int_\Omega a_{\alpha\beta}(x, t) D^\alpha u(x) \overline{D^\beta (v(x))}\, dx,$$

where $a_{\alpha\beta}(x, t) = \overline{a_{\beta\alpha}(x, t)}$, each $a_{\alpha\beta}$ has two continuous derivatives in t, both of which are bounded in $(0, T) \times \Omega$ together with $a_{\alpha\beta}$ itself. Assume the " coercivity " condition :

$$a(t; v, v) \geq c_{17} [v]^2, \qquad v \in V; \qquad c_{17} > 0;$$

where

$$[v]^2 = \sum_{|\alpha| = m} \int_\Omega | D^\alpha v(x) |^2\, dx.$$

Finally, we take $(\mu \leq m - 1)$:

$$b(t; u, v) = \sum_{|\alpha| \leq \mu} \int_\Omega \varphi(D^\alpha u(x)) \overline{D^\alpha v(x)}\, dx,$$

where φ satisfies the hypotheses of Lemma 1.1.

Hypotheses I and II are easily verified. Let us consider Hypothesis III. We are given a sequence of functions $\{ g_m \}$ satisfying :

$$\begin{aligned} g_m &\to g \quad \text{weakly in} \quad L^\infty(0, T; V), \\ g'_m &\to g' \quad \text{weakly in} \quad L^\infty(0, T; L^2(\Omega)), \\ \{ g_m \} &\quad \text{is bounded in } L^{\rho+1}(0, T; W). \end{aligned}$$

Hence there exists a subsequence $\{ g_\sigma \}$ such that

$$\begin{aligned} D_x^\alpha g_\sigma &\to D_x^\alpha g \quad \text{weakly in } L^{\rho+1}((0, T) \times \Omega), \\ \varphi(D_x^\alpha g_\sigma) &\to \psi_\alpha \quad \text{weakly in } L^{\frac{\rho+1}{\rho}}((0, T) \times \Omega) \quad \text{for } |\alpha| \leq \mu, \\ D_x^\alpha g_\sigma &\to D_x^\alpha g \quad \textit{strongly} \text{ in } L^2(K),\ K \text{ any compact set in } (0, T) \times \Omega. \end{aligned}$$

Hence $\psi_\alpha = \varphi(D_x^\alpha g)$ and Hypothesis III is satisfied as in the preceding example. Now $V \cap W$ is separable, and Hypothesis IV (2) will be discussed below.

Theorem 1.1 therefore yields the following result. *Given :*

$$f \in L^1(0, T; L^2(\Omega)), \qquad f' \in L^1(0, T; L^2(\Omega));$$
$$u_0 \in D(A(0)) \cap W^{\mu, \rho+1}(\Omega) \quad [\textit{cf.} \text{ footnote } (^2)],$$
$$u_1 \in V \cap \mathcal{B}_0.$$

There exists a unique function u satisfying :

$$\begin{cases} u \in L^\infty(0, T; V), \qquad u' \in L^\infty(0, T; V), \\ u'' \in L^\infty(0, T; L^2(\Omega)), \\ u' \in L^{\rho+1}(0, T; W^{\mu, \rho+1}(\Omega)); \end{cases}$$

$$\begin{cases} u'' + A(t)u + \displaystyle\sum_{|\alpha| \leq \mu} (-1)^{|\alpha|} D^\alpha(|D^\alpha u'|^{\rho-1} D^\alpha u') = f(t), \\ \qquad u|_{t=0} = u_0, \qquad u'|_{t=0} = u_1; \end{cases}$$

and satisfying the (in general, non-linear) boundary conditions which, in a purely formal way, follow from the fact that

$$(A(t)\, u(t), v) + \sum_{|\alpha| \leq \mu} (-1)^{|\alpha|} (D^\alpha(|D^\alpha u'(t)|^{\rho-1} D^\alpha u'(t)), v)$$
$$= a(t; u(t), v) + \sum_{|\alpha| \leq \mu} (|D^\alpha u'(t)|^{\rho-1} D^\alpha u'(t), D^\alpha v),$$

for every $v \in V$.

We mention that

$$|D^\alpha u_1|^{\rho-1} D^\alpha u_1 \in H_0^\mu(\Omega) \quad \text{for all } |\alpha| \leq \mu$$

is a *sufficient* condition that $u_1 \in \mathcal{B}_0$.

If we assume that [Hypothesis IV (2)]

$$(1.18) \qquad D(A(0)) \cap W^{\mu, \rho+1}(\Omega) \quad \text{is dense in } D(A(0)),$$

then we may take $u_0 \in D(A(0))$. Now,

$$A(t)\, v(x) = \sum_{|\alpha|, |\beta| \leq m} (-1)^{|\beta|} D^\beta[a_{\alpha\beta}(x, t)\, D^\alpha v(x)].$$

If the coefficients $a_{\alpha\beta}$ and the boundary Γ of Ω are sufficiently smooth, if Ω is bounded, and if V is defined by sufficiently " smooth " boundary conditions, then $D(A(0)) \subset H^{2m}(\Omega)$ and the smooth functions in $\overline{\Omega}$ are dense in $D(A(0))$; hence (1.18) is satisfied. For further discussion of boundary conditions in certain cases, *see* [9].

1.7. A non-linear boundary condition.

As a final example, we take $H = L^2(\Omega)$, $V = H^1(\Omega)$, and

$$W = W(t) = \{ w \mid w \in H^1(\Omega) \cap L^{\rho_1+1}(\Omega),\ w|_\Gamma \in L^{\rho_2+1}(\Gamma) \},$$

ρ_1 and $\rho_2 > 1$. We assume the boundary Γ of Ω to be sufficiently smooth to define the " trace " $w|_\Gamma$ of an element w of $H^1(\Omega)$; Γ is furnished with the usual surface measure $d\sigma$.

We set (as in the prime example)

$$a(t;\, u,\, v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} dx, \qquad [v]^2 = a(t;\, v,\, v);$$

we define

$$b(t;\, u,\, v) = k_1 \int_\Omega |u|^{\rho_1-1} u\bar{v}\, dx + k_2 \int_\Gamma |u|^{\rho_2-1} u\bar{v}\, d\sigma,$$

k_1 and k_2 positive.

We note that

$$b_*(u;\, w,\, v) = k_1 \int_\Omega (\varphi_1)'_{u(x)}\, w(x)\, \overline{v(x)}\, dx + k_2 \int_\Gamma (\varphi_2)'_{u(x)}\, w(x)\, \overline{v(x)}\, d\sigma,$$

where $\varphi_1(z) = |z|^{\rho_1-1}z$ and $\varphi_2(z) = |z|^{\rho_2-1}z$ and $(\varphi_j)'_z(\zeta)$ is the derivative of φ_j at the point z in the direction ζ $(j = 1, 2)$. Hypothesis II (1) follows from Lemma 1.1. To verify Hypothesis II (4), let $g \in L^\infty(0, T; H^1(\Omega))$ and

$$\operatorname{Re} \int_0^T b(t;\, g(t),\, g(t))\, dt < \infty.$$

Then $g \in L^{\rho_1+1}((0, T) \times \Omega)$ and $g|_\Gamma \in L^{\rho_2+1}((0, T) \times \Gamma)$.

In order to verify Hypothesis III, we consider a sequence of functions $\{ g_m \}$ which satisfies :

$$\begin{array}{ll} g_m \to g & \text{weakly-star in } L^\infty(0, T; H^1(\Omega)), \\ g'_m \to g' & \text{weakly-star in } L^\infty(0, T; L^2(\Omega)), \\ g_m & \text{remains in a bounded set of } L^{\rho_1+1}((0, T) \times \Omega), \\ g_m|_\Gamma & \text{remains in a bounded set of } L^{\rho_2+1}((0, T) \times \Gamma). \end{array}$$

We may then extract a subsequence $\{ g_\nu \}$ such that :

$$\begin{array}{rl} g_\nu \to g & \textit{strongly} \text{ in } L^2(0, T; H^s(K)), \\ g_\nu \to g & \text{weakly in } L^{\rho_1+1}((0, T) \times \Omega), \\ \varphi_1(g_\nu) \to \psi_1 & \text{weakly in } L^{\frac{\rho_1+1}{\rho_1}}((0, T) \times \Omega), \\ g_\nu|_\Gamma \to g|_\Gamma & \text{weakly in } L^{\rho_2+1}((0, T) \times \Omega), \\ \varphi_2(g_\nu)|_\Gamma \to \psi_2 & \text{weakly in } L^{\frac{\rho_2+1}{\rho_2}}((0, T) \times \Omega), \end{array}$$

where K is any compact subset of $[0, T] \times \overline{\Omega}$ with sufficiently smooth boundary and s is a fixed number,

$$\frac{1}{2} < s < 1.$$

(Here we have used the fact that $H^1(K)$ is compact in $H^s(K)$ if K is compact). As in paragraph 1.5, it follows that $\psi_1 = \varphi_1(g)$.

Now let M be a smooth compact piece of Γ, Γ assumed to be smooth. There is then a compact set $K \subset \overline{\Omega}$ whose smooth boundary ∂K contains M. But the trace operator $f \to f|_{\partial K}$ is continuous from $H^s(K)$ into $L^2(\partial K)$ $(1/2 < s < 1)$. Therefore,

$$g_\nu|_{\partial K} \to g|_{\partial K} \quad \textit{strongly} \text{ in } L^2(0, T; L^2(\partial K)).$$

It follows that $\psi_2 = \varphi_2(g)$ a. e. on M, hence on Γ. So Hypothesis III is satisfied.

Finally note that $D(A(0)) \subset H^2(\Omega)$ and $D(A(0)) \subset W$ is dense in $D(A(0))$ provided again that Γ is sufficiently smooth. Also $\mathcal{B}_0 \supset H_0^1(\Omega) \cap L^{2\rho_1}(\Omega)$. We therefore obtain the following result.

There exists a unique function u which satisfies :

$$(1.19)\quad \begin{cases} \begin{cases} u \in L^\infty(0, T; H^1(\Omega)), \\ u' \in L^\infty(0, T; H^1(\Omega)) \cap L^{\rho_1+1}((0, T) \times \Omega), \\ u'|_\Gamma \in L^{\rho_2+1}((0, T) \times \Gamma), \\ u'' \in L^\infty(0, T; L^2(\Omega)); \end{cases} \\ u'' - \Delta u + k_1 |u'|^{\rho_1 - 1} u' = f; \\ \begin{cases} u(0) = u_0 \in H^2(\Omega), \quad \dfrac{\partial u_0}{\partial n} = 0 \quad \text{on } \Gamma; \\ u'(0) = u_1 \in H_0^1(\Omega) \cap L^{2\rho_1}(\Omega); \\ \qquad f, f' \in L^1(0, T; L^2(\Omega)), \end{cases} \end{cases}$$

with the boundary condition

$$(1.20)\quad \frac{\partial u}{\partial n} + k_2 \left|\frac{\partial u}{\partial t}\right|^{\rho_2 - 1} \frac{\partial u}{\partial t} = 0 \quad \text{on } (0, T) \times \Gamma.$$

This boundary condition can be interpreted in the following sense. We assume that $f \in L^p((0, T) \times \Omega)$ for some $p > 1$. It follows from the equation (1.19) that Δu belongs to $L^q((0, T) \times \Omega)$, where $q > 1$, if Ω is bounded. In that case $\dfrac{\partial u}{\partial n}$ can be defined as an element of $L^q\left(0, T; W^{-1-\frac{1}{q}, q}(\Gamma)\right)$, as in III [11]; hence (1.20) makes sense.

1.8. A regularity theorem.

In this section, we shall prove some regularity for problem (1.21) below.

THEOREM 1.2. — *Let Ω be an open set in E^n with smooth boundary. Let f, u_0, u_1 be given, satisfying*

$$f \in L^1(0, T; L^2(\Omega)),$$

either $f \in L^1(0, T; H_0^1(\Omega))$ *or else* $f' \in L^1(0, T; L^2(\Omega))$,

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega),$$
$$u_1 \in H_0^1(\Omega).$$

Then the solution u of the problem [7] :

$$(1.21) \qquad \begin{cases} u \in L^\infty(0, T; H_0^1(\Omega)), \\ u' \in L^\infty(0, T; L^2(\Omega)) \cap L^{\rho+1}((0, T) \times \Omega), \\ u'' - \Delta u + k |u'|^{\rho-1} u' = f, \\ u(0) = u_0, \qquad u'(0) = u_1 \end{cases}$$

(where $k > 0$, $\rho > 1$) satisfies the conditions

$$(1.22) \qquad u \in L^\infty(0, T; H^2(\Omega)),$$
$$(1.23) \qquad u' \in L^\infty(0, T; H_0^1(\Omega)),$$
$$(1.24) \qquad u'' \in L^\infty(0, T; L^2(\Omega)) + L^{\frac{\rho+1}{\rho}}((0, T) + \Omega).$$

Moreover, if we assume that $u_1 \in L^{\rho+1}(\Omega)$ and $f \in L^2((0, T) \times \Omega)$ [8], *then the solution satisfies*

$$(1.25) \qquad \begin{cases} u' \in L^{2\rho}((0, T) \times \Omega) \cap L^\infty(0, T; L^{\rho+1}(\Omega)), \\ u'' \in L^2((0, T) \times \Omega). \end{cases}$$

If $u_1 \in L^{2\rho}(\Omega)$ and $f' \in L^1(0, T; L^2(\Omega))$, then we may conclude that

$$(1.26) \qquad u' \in L^\infty(0, T; L^{2\rho}(\Omega)) \quad \textit{and} \quad u'' \in L^\infty(0, T; L^2(\Omega)).$$

PROOF. — We shall show, *without* using Part II, that there exists a unique solution of problem (1.21) with the stated properties. This solution must then be the same as the one given in Part II, section 2.6.

Let A denote the self-adjoint operator on $L^2(\Omega)$ with domain $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ which is given by $Av = -\Delta v$ for $v \in D(A)$.

---

(7) The existence and uniqueness of the solution to this problem is proved in Part II, section 2.6. (Existence could also be derived from the result in section 1.5.)

(8) This last hypothesis is redundant in case $f' \in L^1(0, T; L^2(\Omega))$.

Let $\{E(S) \mid S = \text{Borel set on the line}\}$ be its spectral measure. Let P_m be the projection $E((-m, m))$, $m = 1, 2, \ldots$. We then define the approximate solution $u_m(t)$ as the solution of the equation

$$(1.27) \qquad u''_m(t) + P_m A P_m u_m(t) + P_m \beta P_m u'_m(t) = P_m f(t)$$

with the initial conditions

$$u_m(0) = P_m u_0, \qquad u'_m(0) = P_m u_1,$$

where $\beta v = k |v|^{\rho-1} v$. The operator $P_m \beta P_m$ is a bounded operator on the range of P_m with bounded Fréchet-derivative and $P_m A P_m$ is a bounded linear operator. By the method of successive approximations, it is easy to see that this problem has a unique solution such that $u_m(t) \in P_m$ and $u'_m(t) \in P_m$, for all t in some interval $0 \leq t \leq \varepsilon$, $\varepsilon > 0$ (*cf.* SEGAL [18]). It will follow from the estimates below that the solution in fact exists globally, i. e., in $0 \leq t \leq T$.

In case Ω is bounded, P_m is a finite-dimensional projection, and $u_m(t)$ is an approximate solution as defined in section 1.3 where the " basis " $\{w_1, w_2, \ldots\}$ consists of the eigenfunctions of A. In case $\Omega = E^n$, the approximate equation (1.27) is a variant of that used by SEGAL [19].

Since $u_m(t)$, $u'_m(t)$ lie in the range of P_m, we may rewrite (1.27) as :

$$(1.28) \qquad u''_m(t) + A\, u_m(t) + P_m \beta\, u'_m(t) = P_m f(t).$$

Taking inner products of this equation with $u'_m(t)$, we conclude, just as in section 1.3, that

$\{u_m\}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$,
$\{u'_m\}$ is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^{\rho+1}((0, T) \times \Omega)$.

Now, in case $f \in L^1(0, T; H_0^1(\Omega))$, we take inner products of (1.28) with $A u'_m(t) = -\Delta u'_m(t)$. Setting

$$a(u, v) = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} dx,$$

we obtain

$$(1.29) \qquad a(u''_m, u'_m) + (\Delta u_m, \Delta u'_m) + a(\beta u'_m, u'_m) = a(f, u'_m),$$

because $u''_m(t) \in H_0^1(\Omega)$ and $\beta(u'_m)$ is also zero on the boundary of Ω. But

$$\operatorname{Re} a(\beta u'_m, u''_m) = \sum_{i=1}^{n} \operatorname{Re} \int_\Omega \frac{\partial}{\partial x_i} (\beta u'_m) \frac{\partial u'_m}{\partial x_i} dx \geq 0.$$

Therefore, taking real parts of (1.29) and integrating from o to t, we obtain

$$a(u'_m(t), u'_m(t)) + |\Delta u_m(t)|^2$$
$$\leq 2\,\mathrm{Re}\int_0^t a(f(s), u(s))\,ds + a(P_m u_1, P_m u_1) + |\Delta P_m u_0|^2.$$

Using the fact that the projections P_m are uniformly bounded in $H_0^1(\Omega)$, as well as in $L^2(\Omega)$, we conclude that

$$(1.30)\qquad \begin{cases} \{u'_m\} & \text{is bounded in } L^\infty(0, T; H_0^1(\Omega)), \\ \{\Delta u_m\} & \text{is bounded in } L^\infty(0, T; L^2(\Omega)). \end{cases}$$

We must pass to the limit. We define

$$D(A^N) = \{ v \mid v \in D(A),\ A^j v \in D(A) \text{ for } j = 0, \ldots, N-1 \};$$

it is a Hilbert space when provided with the graph norm. By Sobolev's inequality, there exists a positive integer N such that

$$D(A^N) \subset H^{2N}(\Omega) \subset L^p(\Omega) \qquad (p = \rho + 1).$$

The operators P_m, which are uniformly bounded from $D(A^N)$ to itself, must therefore also be uniformly bounded from $D(A^N)$ to $L^p(\Omega)$. Hence, by duality, their restrictions to $L^2(\Omega) \cap L^{p'}(\Omega)$ must be uniformly bounded from $L^{p'}(\Omega)$ to $D(A^N)'$ [= dual or anti-dual of $D(A^N)$]. It follows from this remark that $u''_m = P_m f - \Delta u_m - P_m \beta u'_m$ remains in a bounded set of $L^{p'}(0, T; D(A^N)') + L^\infty(0, T; L^2(\Omega))$. Hence

$$(1.31)\qquad \{u''_m\} \text{ remains bounded in } L^{p'}(0, T; D(A^N)').$$

It follows from (1.10), (1.31) and a compactness theorem of AUBIN [1] that, for any given compact subset of K of Ω, there exists a subsequence of $\{u'_m\}$ which converges *strongly* in $L^2(0, T; L^2(K))$, for instance. Therefore, a further subsequence converges almost everywhere in $(0, T) \times K$.

Now we may take weakly convergent subsequences. There exists a subsequence $\{u_\nu\}$ such that :

$$\begin{aligned} u_\nu &\to u && \text{in the weak-star topology of } L^\infty(0, T; H^2(\Omega)), \\ u'_\nu &\to u' && \text{in the weak-star topology of } L^\infty(0, T; H_0^1(\Omega)), \\ u'_\nu &\to u' && \text{in the weak topology of } L^{\rho+1}((0, T) \times \Omega), \\ |u'_\nu|^{\rho-1} u'_\nu &\to \psi && \text{in the weak topology of } L^{\frac{\rho+1}{\rho}}((0, T) \times \Omega). \end{aligned}$$

The almost-everywhere convergence implies just as in section 1.5 that $\psi = | u' |^{\rho-1} u'$ a. e. Therefore, as usual, $u(t)$ is a solution of problem (1.21) and satisfies the additional conditions (1.22) and (1.23). Therefore,

$$u'' = f + \Delta u - k | u' |^{\rho-1} u' \in L^\infty (0, T; L^2(\Omega)) + L^{\rho'}((0, T) \times \Omega).$$

By the same method as in section 1.3, it is easy to prove the uniqueness of the solution under these conditions.

The second case is to prove the same result under the assumption

$$f' \in L^1(0, T; L^2(\Omega)).$$

In this case, we take the derivative with respect to t of equation (1.28) and then take inner products with

$$v_m(t) = u''_m(t) + P_m \beta\, u'_m(t) = P_m f(t) - A\, u_m(t).$$

We obtain :

$$(v'_m, v_m) + (A\, u'_m, u''_m) + (A\, u'_m, P_m \beta\, u'_m) = (P_m f', v_m).$$

Taking real parts, integrating from 0 to t, and noting that P_m commutes with A, we find that

$$| v_m(t) |^2 + a(u'_m(t), u'_m(t)) + 2 \operatorname{Re} \int_0^t a(u'_m, \beta\, u'_m)\, ds$$
$$= 2 \operatorname{Re} \int_0^t (f', v_m)\, ds + | v_m(0) |^2 + a(P_m u_1, P_m u_1).$$

But once again $\operatorname{Re} a(u'_m, \beta\, u'_m) \geq 0$, and $v_m(0) = P_m f(0) - \Delta P_m u_0$ is bounded in $L^2(\Omega)$; so we may infer that :

$\{ u'_m \}$ is bounded in $L^\infty(0, T; H^1_0(\Omega))$;
$\{ v_m \}$ hence $\{ \Delta u'_m \}$ also, is bounded in $L^\infty(0, T; L^2(\Omega))$.

This is exactly the same result as in the previous case; hence we may conclude the argument just as before.

Now we shall prove (1.25) under the additional assumption that $u_1 \in L^{\rho+1}(\Omega)$ and $f \in \mathrm{L}^2(0, T; L^2(\Omega))$. Since the union of the ranges of the projections P_m is a dense subset of $D(A^N)$ and $D(A^N)$ is itself a dense subset of $L^{\rho+1}(\Omega)$, we may find a sequence $\{ u_{1k} \}$ such that u_{1k} converges to u_1 strongly in $L^{\rho+1}(\Omega) \cap H^1_0(\Omega)$ and each u_{1k} is in the range of the projection P_m for m sufficiently large. We may assume for the sake of convenience that the sequence is so chosen that u_{1m} is in the range of P_m.

Now redefine the approximate solution $u_m(t)$ as the solution of equation (1.28) with the initial conditions $u_m(0) = P_m u_0$ and $u'_m(0) = P_m u_{1m} = u_{1m}$. The previous *a priori* estimates are then obtained without change. Next, we take inner products of equation (1.28) with $u''_{mk}(t)$ to obtain

$$| u''_m |^2 + a(u_m, u''_m) + (\beta u'_m, u''_m) = (f, u''_m).$$

But $a(u_m, u''_m) = \frac{d}{dt} a(u_m, u'_m) - a(u'_m, u'_m)$. Therefore,

$$\int_0^t | u''_m |^2 \, ds + \frac{1}{2} \operatorname{Re} a(u_m(t), u_m(t)) - \int_0^t a(u'_m, u'_m) \, ds + \frac{1}{\rho} \| u_m(t) \|^{\rho+1}_{L^{\rho+1}(\Omega)}$$
$$= \operatorname{Re} \int_0^t (f, u''_m) \, ds + \frac{1}{2} \operatorname{Re} a(u_0, u_{1m}) + \frac{1}{\rho} \| u_{1m} \|^{\rho+1}_{L^{\rho+1}(\Omega)}.$$

But $\{ u_{1m} \}$ is bounded in $L^{\rho+1}(\Omega) \cap H^1_0(\Omega)$, and u_m and u'_m are both bounded in $L^\infty(0, T; H^1_0(\Omega))$ by the previous estimates. We may therefore infer that :

$$\{ u''_m \} \quad \text{is bounded in } L^2((0, T) \times \Omega),$$
$$\{ u'_m \} \quad \text{is bounded in } L^\infty(0, T; L^{\rho+1}(\Omega)).$$

Taking a weakly convergent subsequence, we conclude that

$$u'' \in L^2((0, T) \times \Omega) \qquad \text{and} \qquad u' \in L^\infty(0, T; L^{\rho+1}(\Omega)).$$

Therefore,

$$k | u' |^{\rho-1} u' = f + \Delta u - u'' \in L^2((0, T) \times \Omega),$$

so that $u' \in L^{2\rho}((0, T) \times \Omega)$.

Finally, in case $u_1 \in L^{2\rho}(\Omega)$ and $f' \in L^1(0, T; L^2(\Omega))$, the existence results of section 1.5, together with uniqueness, show that $u'' \in L^\infty(0, T; L^2(\Omega))$. Therefore,

$$k | u' |^{\rho-1} u' = f + \Delta u - u'' \in L^\infty(0, T; L^2(\Omega)),$$

so that $u' \in L^\infty(0, T; L^{2\rho}(\Omega))$.

1.9. **Another type of equation** (I).

In this section we shall prove :

THEOREM 1.3. — *Let f be given in $L^1(0, T; L^2(\Omega))$,*

$$u_0 \in H^1_0(\Omega) \cap L^{2\rho}(\Omega) \qquad \textit{and} \qquad u_1 \in L^2(\Omega),$$

all of which are supposed to be real-valued. We assume that Ω is an open set in E^n with smooth boundary. Let $\rho > 1$, $k > 0$. Then there exists a unique function which satisfies :

$$(1.32)\qquad \begin{cases} u \in L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^{2\rho}(\Omega)), \\ u' \in L^\infty(0, T; L^2(\Omega)), \\ \qquad u'' - \Delta u + k\,|u|^{\rho-1} u' = f, \\ \qquad u(0) = u_0, \qquad u'(0) = u_1. \end{cases}$$

REMARK 1. — If $\rho \geqq 2$, we have $L^{2\rho}(\Omega) \cap L^2(\Omega) \subset L^{2\rho-2}(\Omega)$, so that $u \in L^\infty(0, T; L^{2\rho-2}(\Omega))$. Hence

$$|u|^{\rho-1} \in L^\infty(0, T; L^2(\Omega)) \qquad \text{and} \qquad |u|^{\rho-1} u' \in L^\infty(0, T; L^1(\Omega)).$$

If, on the other hand, $\rho \leqq 2$, similar reasoning shows that $|u|^{\rho-1} u' \in L^\infty(0, T; L^{2/\rho}(\Omega))$. Therefore, $u'' = f + \Delta u - k\,|u|^{\rho-1} u'$ is an element of $L^1(0, T; L^1_{\text{loc}}(\Omega) + H^{-1}(\Omega))$, so that $u'(0)$ makes sense.

REMARK 2. — If the conditions on u_0 and u_1 are weakened to

$$u_0 \in L^{\frac{2n\rho}{n+2}} \cap L^2(\Omega), \qquad u_1 \in H^{-1}(\Omega),$$

then we can prove the existence and uniqueness of an extremely weak solution u of the above initial boundary-value problem with

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^{\rho+1}((0, T) \times \Omega)$$

and $\int_0^t u(s)\,ds$ an element of $L^\infty(0, T; H_0^1(\Omega))$. We omit the proof which is a variant of the one given below for Theorem 1.3 and uses Theorem 2.1.

REMARK 3. — Formally, the term $k\,|u|^{\rho-1} u'$ can be obtained by taking the t-derivative of $k\rho^{-1}\,|w'|^{\rho-1} w'$ and then replacing w' by u in the result. The study of (1.32) can therefore be formally reduced to the study of the equation

$$w'' - \Delta w + k\rho^{-1}\,|w'|^{\rho-1} w' = F,$$

which has already been solved. This approach was suggested to us by J. LERAY.

PROOF. — Firstly, we note that the relation

$$(1.33)\qquad \frac{\partial}{\partial t}\left(\frac{1}{\rho}|\psi|^{\rho-1}\psi\right) = |\psi|^{\rho-1}\psi'$$

[taken in the sense of distributions in $(0, T)\times\Omega$] is valid for all $\psi\in\Psi$, where Ψ is defined as follows : If $\rho\geq 2$, Ψ is the class of all real-valued functions ψ satisfying

$$\psi\in L^\infty(0, T; L^{2\rho-2}(\Omega)), \qquad \psi'\in L^\infty(0, T; L^2(\Omega));$$

if $1<\rho<2$, Ψ is defined as the class of all ψ satisfying

$$\psi\in L^\infty(0, T; L^2(\Omega)), \qquad \psi'\in L^\infty(0, T; L^2(\Omega)).$$

To prove (1.33), we write ψ as the limit of a sequence $\{\psi_j\}$ of smooth functions in Ψ, this limit being taken in the obvious topology for Ψ. Then, ψ_j being smooth, we have

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}|\psi_j|^{\rho-1}\psi_j\right)=|\psi_j|^{\rho-1}\psi'_j.$$

It suffices to show the convergence as $j\to\infty$ of each side of the latter equation in the sense of distributions. But, if $\rho\geq 2$,

$$\begin{aligned}
|\psi_j|^{\rho-1}\psi_j &\to |\psi|^{\rho-1}\psi && \text{in } L^\infty\left(0, T; L^{2-\frac{2}{\rho}}(\Omega)\right),\\
|\psi_j|^{\rho-1} &\to |\psi|^{\rho-1} && \text{in } L^\infty(0, T; L^2(\Omega)),
\end{aligned}$$

and

$$|\psi_j|^{\rho-1}\psi'_j\to|\psi|^{\rho-1}\psi' \qquad \text{in } L^\infty(0, T; L^1(\Omega)).$$

Similar statements hold if $1<\rho<2$. This proves (1.33).

Secondly, we *define* w_0 as the unique solution in $H^1_0(\Omega)$ of the equation

$$(1.34)\qquad (1-\Delta)w_0=-\frac{k}{\rho}|u_0|^{\rho-1}u_0-u_1.$$

The right-hand side is an element if $L^2(\Omega)$; since the boundary of Ω is smooth (" uniformly smooth " at infinity, in case Ω is unbounded), we know that

$$w_0\in H^1_0(\Omega)\cap H^2(\Omega).$$

Now we refer to Theorem 1.2 to obtain a solution w of the equation

$$(1.35)\qquad w''-\Delta w+\frac{k}{\rho}|w'|^{\rho-1}w'=-w_0+\int_0^t f(\sigma)\,d\sigma,$$

subject to the conditions

$$(1.36)\qquad \begin{cases} w(0)=w_0, \qquad w'(0)=u_0;\\ w\in L^\infty(0, T; H^2(\Omega));\end{cases}$$

$$(1.37)\qquad \begin{cases} w\ \in L^\infty(0, T; H^1_0(\Omega)),\\ w'\in L^\infty(0, T; H^1_0(\Omega))\cap L^\infty(0, T; L^{2\rho}(\Omega)),\\ w''\in L^\infty(0, T; L^2(\Omega)).\end{cases}$$

Therefore, we may take the t-derivative of equation (1.35) and make use of (1.33) to obtain

$$w''' - \Delta w' + k |w'|^{\rho-1} w'' = f.$$

Furthermore, $w''(0) = u_1$ by (1.35) and (1.34). Hence $u = w'$ is the required solution.

Conversely, let u be any function satisfying the conditions of the theorem. We *define* w by

$$w(t) = w_0 + \int_0^t u(\sigma)\, d\sigma$$

and we integrate both sides of equation (1.32) in t. As we saw in Remark 1, u'' is an element of $L^1(0, T; H^{-1}(\Omega) + L^1(\Omega))$ if $\rho \geqq 2$; so that, for each t in $(0, T)$, $\int_0^t u''(\sigma)\, d\sigma$ makes sense as an element of $H^{-1}(\Omega) + L^1(\Omega)$ and equals $u'(t) - u'(0) = w''(t) - u_1$. If $1 < \rho < 2$, we simply replace $L^1(\Omega)$ by $L^{2/\rho}(\Omega)$ in the preceding argument. Similarly, $\int_0^t \Delta u(\sigma)\, d\sigma$ makes sense in $H^{-1}(\Omega)$ and equals $\Delta \int_0^t u(\sigma)\, d\sigma = \Delta w(t) - \Delta w_0$. Finally (*cf.* Remark 1), $u \in \Psi$; so that (1.33) implies

$$\int_0^t |u|^{\rho-1} u'\, d\sigma = \frac{1}{\rho} |w'(t)|^{\rho-1} w'(t) - \frac{1}{\rho} |u_0|^{\rho-1} u_0.$$

Therefore we finally obtain

$$w'' - \Delta w + \frac{k}{\rho} |w'|^{\rho-1} w' - u_1 + \Delta w_0 - \frac{k}{\rho} |u_1|^{\rho-1} u_1 = \int_0^t f(\sigma)\, d\sigma.$$

Making use of (1.34), we conclude that w satisfies equation (1.35). Therefore, referring to Theorem 1.1, w is the unique solution to the problem (1.35), (1.36), (1.37). This shows that u is unique.

1.10. **Another type of equation** (II).

THEOREM 1.4 A. — *Let f be given in $L^1(0, T; L^2(\Omega))$ such that $f' \in L^1(0, T; L^2(\Omega))$. Let u_0, u_1 be given, satisfying :*

$$u_0 \in H_0^1(\Omega) \cap L^{2\tau}(\Omega), \qquad \Delta u_0 \in L^2(\Omega),$$
$$u_1 \in H_0^1(\Omega) \cap L^{2\rho}(\Omega),$$

where

$$\rho \geqq \tau \geqq 2.$$

We assume that Ω is an open set in E^n and that either the boundary of Ω is smooth or else $u_0 \in L^{\rho+1}(\Omega)$. Then there exists a function u which satisfies :

$$\begin{cases} u \in L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^{\tau+1}(\Omega)), \\ u' \in L^\infty(0, T; H_0^1(\Omega)) \cap L^{\rho+1}(0, T; L^{\rho+1}(\Omega)), \\ u'' \in L^\infty(0, T; L^2(\Omega)); \end{cases}$$

$$-\Delta u + u'' + \beta(u') + \gamma(u) = f,$$

where $\beta(u') = |u'|^{\rho-1}u'$ *and* $\gamma(u) = |u|^{\tau-1}u$;

$$u(0) = u_0, \qquad u'(0) = u_1.$$

PROOF. — We choose a " basis " $w_1, w_2, \ldots$ of $H_0^1(\Omega) \cap L^{\rho+1}(\Omega)$ as follows (*cf.* section 1.4). We let $w_1 = u_1$ (unless $u_1 \equiv 0$); in case $u_0 \in L^{\rho+1}(\Omega)$, we let $w_2 = u_0$ (unless u_0 depends linearly on u_1) and define the remainder of the basis elements arbitrarily. In case $u_0 \notin L^{\rho+1}(\Omega)$, we choose $w_2, w_3, \ldots$ such that

$$w_j \in H_0^1(\Omega) \cap L^{\rho+1}(\Omega) \cap L^{2\tau}(\Omega),$$
$$\Delta w_j \in L^2(\Omega) \qquad (j \geq 2)$$

and such that there exist finite linear combinations u_{0m} of the w_j's such that

$$u_{0m} \to u_0 \quad \text{in} \quad H_0^1(\Omega) \cap L^{2\tau}(\Omega),$$
$$\Delta u_{0m} \to \Delta u_0 \quad \text{in} \quad L^2(\Omega).$$

The latter choice is possible if, for instance, we assume the boundary Γ of Ω to be smooth enough that the space of twice continuously differentiable functions in $\overline{\Omega}$, with compact support in $\overline{\Omega}$ and zero on Γ, is dense in $D(-\Delta) \cap L^{2\tau}(\Omega)$. In case $u_0 \in L^{\rho+1}(\Omega)$ we define $u_{0m} = u_0$ for all m.

As in section 1.4, we denote by P_m the orthogonal projection in $L^2(\Omega)$ onto the subspace generated by $w_1, \ldots, w_m$. We define

$$u_{2m} = P_m[f(0) + \Delta u_{0m} - \beta(u_1) - \gamma(u_{0m})].$$

We denote by $u_m(t)$ the solution of

$$\begin{aligned}(1.38) \quad & (u''_m(t), w_k) + a(u_m(t), w_k) \\ & + (\beta(u'_m(t)), w_k) + (\gamma(u_m(t)), w_k) = (f(t), w_k) \qquad (k = 1, \ldots, m),\end{aligned}$$

where

$$a(u, v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial \bar{v}}{\partial x_i} dx,$$

subject to the initial conditions

$$u_m(0) = u_{0m}, \qquad u'_m(0) = u_1.$$

From (1.38) we easily deduce

$$\frac{1}{2}\frac{d}{dt}(|u'_m(t)|^2 + a(u_m(t), u_m(t))) + \frac{1}{\tau+1}\frac{d}{dt}\int_\Omega |u_m(x, t)|^{\tau+1}\,dx$$
$$+ \operatorname{Re}(\beta(u'_m(t)), u'_m(t)) = \operatorname{Re}(f(t), u'_m(t));$$

whence :

$$(1.39)\quad \begin{cases} u_m & \text{remains in a bounded set of} \\ & \qquad L^\infty(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^{\tau+1}(\Omega)), \\ u'_m & \text{remains in a bounded set of} \\ & \qquad L^\infty(0, T; L^2(\Omega)) \cap L^{\rho+1}((0, T) \times \Omega). \end{cases}$$

Now denote $\varphi(z) = |z|^{\rho-1}z$ and $\psi(z) = |z|^{\tau-1}z$ (z complex number). Denote by $\psi'(z; \zeta_1)$ the derivative of $\psi(z)$ in the direction ζ_1 and by $\psi''(z; \zeta_2, \zeta_1)$ the derivative of $\psi'(z; \zeta_1)$ as a function of z in the direction ζ_2; similarly for $\varphi(z)$. We now take the t-derivative of equation (1.38), obtaining

$$(u'''_m(t), w_k) + a(u'_m(t), w_k) + (\varphi'(u'_m(t); u''_m(t)), w_k)$$
$$+ (\psi'(u_m(t); u'_m(t)), w_k) = (f'(t), w_k).$$

Therefore,

$$(1.40)\quad |u''_m(t)|^2 + a(u'_m(t), u'_m(t)) + 2\operatorname{Re}\int_0^t (\varphi'(u'_m(\sigma); u''_m(\sigma)), u''_m(\sigma))\,d\sigma$$
$$+ 2\operatorname{Re}\int_0^t (\psi'(u_m(\sigma); u'_m(\sigma)), u''_m(\sigma))\,d\sigma$$
$$= 2\operatorname{Re}\int_0^t (f'(\sigma), u''_m(\sigma))\,d\sigma + |u''_m(0)|^2 + a(u_1, u_1).$$

But we deduce from (1.38) that $u''_m(0) = u_{2m}$ remains in a bounded set of $L^2(\Omega)$. Furthermore,

$$2\operatorname{Re}\int_0^t (\psi'(u_m(\sigma); u'_m(\sigma)), u''_m(\sigma))\,d\sigma$$
$$= \operatorname{Re}(\psi'(u_m(t); u'_m(t)), u'_m(t)) - \operatorname{Re}(\psi'(u_{0m}; u_1), u_1)$$
$$- \operatorname{Re}\int_0^t (\psi''(u_m(\sigma), u'_m(\sigma), u'_m(\sigma)), u'_m(\sigma))\,d\sigma.$$

But explicit calculation shows that

$$0 \leq \operatorname{Re}[\psi(z; \zeta)\bar{\zeta}] \leq \tau |z|^{\tau-1}|\zeta|^2$$

and

$$(1.41) \qquad |\psi''(z;\zeta_1,\zeta_2)\bar{\zeta}_3| \leqq \tau(\tau-1)\,|z|^{\tau-2}\,|\zeta_1|.|\zeta_2|.|\zeta_3.$$

Hence

$$|\operatorname{Re}(\psi'(u_{0m};u_1),u_1)| \leqq \tau \int_\Omega |u_{0m}|^{\tau-1}|u_1|^2\,dx.$$

Using Hölder's inequality with exponents $\frac{\tau+1}{\tau-1}$ and $\frac{\tau+1}{2}$, we find this to be bounded because $u_1 \in L^{\rho+1}(\Omega)\cap L^2(\Omega)\subset L^{\tau+1}(\Omega)$ and $\{u_{0m}\}$ is bounded in $L^{2\tau}(\Omega)\cap L^2(\Omega)\subset L^{\tau+1}(\Omega)$. By (1.41), we have

$$\begin{aligned}
&\int_0^t |(\psi''(u_m;u'_m,u'_m),u'_m)|\,d\sigma \\
&\leqq \tau(\tau-1)\int_0^t\int_\Omega |u_m|^{\tau-2}|u'_m|^3\,dx\,d\sigma \\
&\leqq c_1\int_0^t\int_\Omega \{|u_m|^{\tau+1}+|u'_m|^{\tau+1}\}\,dx\,d\sigma \\
&\leqq c_2\int_0^t\int_\Omega \{|u'_m|^2+|u'_m|^{\rho+1}\}\,dx\,d\sigma + c_2\sup_t\int_\Omega |u_m|^{\tau+1}\,dx,
\end{aligned}$$

where we have used $\tau \leqq \rho$ and $\frac{1}{p}+\frac{1}{q}=1$, $p=\frac{\tau+1}{\tau-2}$, $q=\frac{\tau+1}{3}$. The latter expression is bounded. Consequently, we conclude from (1.40) that

$$|u''_m(t)|^2 + a(u'_m(t),u'_m(t)) \leqq c_3 + c_3\int_0^t |(f'(\sigma),u''_m(\sigma))|\,d\sigma.$$

Therefore,

$$(1.42) \qquad \begin{cases} u''_m & \text{remains in a bounded set of } L^\infty(0,T;L^2(\Omega)), \\ u'_m & \text{remains in a bounded set of } L^\infty(0,T;H_0^1(\Omega)). \end{cases}$$

The theorem now follows from (1.39) and (1.42) as in section 1.5.

We have the following partial uniqueness result for this equation.

THEOREM 1.4B. — *The solution in Theorem 1.4A is unique if: all the functions are real-valued, τ is an odd integer; and*

$$(1.43) \qquad \frac{1}{2}-\frac{1}{n} \leqq \frac{3}{2(\tau+1)}-\frac{1}{2(\rho+1)} \qquad \text{if } n \geqq 2,$$

except that we exclude equality here if $n=2$.

PROOF. — Let u_1 and u_2 be two solutions and let $w = u_1 - u_2$. Substracting the corresponding differential equations, multiplying by w' and integrating the result, yields :

$$(1.44) \quad a(w(t), w(t)) + |w'(t)|^2 + 2\int_0^t (\beta u'_1 - \beta u'_2, w')\, d\sigma \\ + 2\int_0^t \int_\Omega [\psi(u_1) - \psi(u_2)]\, w'\, dx\, d\sigma = 0.$$

Now, we can write

$$\psi(u_1) - \psi(u_2) = u_1^\tau - u_2^\tau = vw,$$

where

$$v = \sum_{j=0}^{\tau=1} u_1^j u_2^{\tau-j} \geqq 0.$$

Therefore the last term in (1.44) can be integrated by parts as follows :

$$2\int_0^t \int_\Omega vww'\, dx\, d\sigma = \int_\Omega v(x, t)\, w(x, t)^2\, dx - \int_0^t \int_\Omega v' w^2\, dx\, d\sigma \\ \geqq -c_4 \int_0^t \int_\Omega [|u_1|^{\tau-2} + |u_2|^{\tau-2}][|u'_1| + |u'_2|]\, w^2\, dx\, d\sigma,$$

where c_4 is a positive constant. Letting q be defined by

$$\frac{2}{q} + \frac{\tau-2}{\tau+1} + \frac{1}{\rho+1} = 1,$$

Hölder's inequality implies that

$$\int_\Omega |u_1|^{\tau-2}\, |u'_1|\, w^2\, dx \leqq \|u_1\|_{\tau+1}^{\tau-2}\, \|u'_1\|_{\rho+1}\, \|w\|_q^2,$$

where $\|\ \|_r$ denotes the norm in $L^r(\Omega)$. Hence an application of Hölder's inequality to the t-integral with $p = \rho + 1$, $p' = (\rho + 1)/\rho$, gives

$$\int_0^t \int_\Omega vww'\, dx\, d\sigma \geqq -c_5 \left\{ \sup_t \left[\|u_1\|_{\tau+1}^{\tau-2} + \|u_2\|_{\tau+1}^{\tau-2} \right] \right\} \\ \times \left\{ \int_0^t [\|u'_1\|_p^p + \|u'_2\|_p^p]\, d\sigma \right\}^{\frac{1}{p}} \left\{ \int_0^t \|w(\sigma)\|_q^{2p'}\, d\sigma \right\}^{\frac{1}{p'}} \\ \geqq -c_6 \left\{ \int_0^t \|w(\sigma)\|_q^{2p'}\, d\sigma \right\}^{\frac{1}{p'}},$$

since $u_1, u_2 \in L^\infty(0, T; L^{\tau+1}(\Omega))$ and $u_1', u_2' \in L^p((0, T)\times\Omega)$. But, by hypothesis (1.43), $\frac{1}{q} \geq \frac{1}{2} - \frac{1}{n}$, so that Sobolev's inequality gives :

$$\| w(\sigma) \|_q^2 \leq c_7 \{ a(w(\sigma), w(\sigma)) + | w(\sigma) |^2 \}.$$

Putting these estimates into (1.44) yields

$$a(w(t), w(t)) + | w'(t) |^2 \leq c_8 \left[\int_0^t \{ a(w(\sigma), w(\sigma)) + | w(\sigma) |^2 \}^{p'} \right]^{\frac{1}{p'}},$$

which implies

$$\{ a(w)(t), w(t)) + | w'(t) |^2 \}^{p'} \leq c_9 \int_0^t \{ a(w(\sigma), w(\sigma)) + | w'(\sigma) |^2 \}^{p'} d\sigma.$$

Hence $w' = w = 0$.

Finally, we present a case not included in the preceding results for which we have both existence and uniqueness.

THEOREM 1.4 C. — *In case*

$$n = 3, \qquad \tau = 3, \qquad \rho > 1,$$

there exists a unique solution which has the same properties as in Theorem 1.4 A.

PROOF. — The existence proceeds along the same lines as before, the crucial point being the estimation of

$$\mathrm{Re} \int_0^t (\psi'(u_m(\sigma); u_m'(\sigma)), u_m''(\sigma))\, d\sigma.$$

This term is bounded by

$$3 \int_0^t \int_\Omega | u_m |^2 | u_m' | . | u_m'' | \, dx\, d\sigma \leq 3 \int_0^t \| u_m(\sigma) \|_6^2 \| u_m'(\sigma) \|_6 \| u_m''(\sigma) \|_2 \, d\sigma.$$

We now use Sobolev's inequality : $\| u \|_6 \leq c_{10} \| u \|$, where $\| \quad \|$ denotes the norm in $H_0^1(\Omega)$. Since $\{ u_m \}$ is bounded in $L^\infty(0, T; H_0^1(\Omega))$, the expression under consideration is bounded by :

$$c_{11} \int_0^t \{ \| u_m'(\sigma) \|^2 + | u_m''(\sigma) |^2 \} \, d\sigma.$$

Once again this is enough to imply (1.42) and therefore existence.

To prove uniqueness, we note that

$$\left| \int_0^t \int_\Omega [\psi(u_1) - \psi(u_2)]\, \overline{w}' \, dx\, d\sigma \right| \leq 3 \int_0^t \int_\Omega (|u_1|^2 + |u_2|^2) | w | . | w' | \, dx\, d\sigma$$

$$\leq c_{12} \int_0^t \{ \| w(\sigma) \|^2 + | w'(\sigma) |^2 \} \, d\sigma,$$

using Sobolev's inequality again. Putting this into (1.44), we obtain

$$a(w(t), w(t)) + |w'(t)|^2 \leq c \int_0^t \{ a(w(\sigma), w(\sigma)) + w'(\sigma) \} \, d\sigma,$$

which proves uniqueness.

PART II.

Monotonicity Method.

2.1. **Hypotheses.**

We are given Hilbert spaces V and H and a Banach space W which satisfy

$$V \subset H, \qquad W \subset H,$$

where each space to the left of an inclusion sign is dense in the space to its right and the corresponding inclusion map is continuous. Furthermore, we assume that $V \cap W$ is separable and that $V \cap W$ is a dense subset of V and of W and that W is reflexive.

If X is any Banach space dense in H, with the inclusion map of X into H continuous, X' denotes the space of conjugate-linear (linear, in the real case) functionals defined and continuous on the space X; H is identified with its own dual H'. Notations for inner products and norms will be the same as in Part I. In particular, (v_1, v_2) denotes the inner product between an element of such a space X and an element of its dual X'; if $v_1, v_2 \in H$, this is the ordinary inner product in H. The following inclusion relations hold :

$$V \cap W \subset V \subset H \subset V'(V \cap W)',$$
$$V \cap W \subset W \subset H \subset W' \subset (V \cap W)'.$$

Besides these spaces we are given a family of sesquilinear (bilinear, in the real case) forms

$$u, v \to a(t; u, v) \qquad (u, v \in V)$$

defined on $V \times V$, $t \in [0, T]$, which are assumed to satisfy :

(i) $a(t; u, v) = \overline{a(t; v, u)}$;

(ii) $a(t; v, v) \geq c_1 [v]^2$, where $([v]^2 + |v|^2)^{1/2}$ is a norm on V which is equivalent to $\| v \|$;

(iii) For each $v \in V$, the function

$$t \to a(t; v, v)$$

is once continuously differentiable;

(iv) $a'(t; v, v) = \dfrac{t}{dt} a(t; v, v) \leq 0 \qquad (v \in V, t \in [0, T])$.

We note that Hypothesis (iii) implies that the function $t \to a(t; u, v)$ is once continuously differentiable for every $u, v \in V$; hence there exists a constant c_2 such that

$$| a(t; u, v) | + | a'(t; u, v) | \leq c_2 \| u \| . \| v \| \qquad (u, v \in V).$$

We note also that since the map

$$v \to a(t; u, v)$$

is continuous and conjugate-linear, there corresponds to each $u \in V$ a unique element $A(t) u \in V'$ such that

$$(A(t)\, u, v) = a(t; u, v) \qquad (v \in V).$$

Finally, we are given a family of (non-linear) transformations $\beta(t)$ which map W into W' [almost every $t \in (0, T)$] and satisfy the following conditions.

(j) For almost every t, $\beta(t)$ is weakly continuous from finite-dimensional subsets of W into W';

(jj) If $f(.) \in L^p(0, T; W)$, then $\beta(.) f(.)$ is an element of $L^{p'}(0, T; W')$; the mapping $f(.) \to \beta(.) f(.)$ sends bounded sets of $L^p(0, T; W)$ into bounded sets of $L^{p'}(0, T; W')$ and its restriction to lines in $L^p(0, T; W)$ is weakly continuous. Throughout this part p is a fixed number greater than one;

(jjj) $\operatorname{Re}(\beta(t)\, v, v) + c_3 | v |^p \geq c_4 \| v \|_W^p$, for $v \in W$, almost every $t \in (0, T)$; where c_3 and c_4 are positive constants;

(jv) $\operatorname{Re}(\beta(t)\, u - \beta(t)\, v, u - v) \geq 0$ a. e. $\qquad (u, v \in W)$.

2.2. An existence-uniqueness theorem.

THEOREM 2.1. — *Assume the existence of spaces, forms and transformations satisfying the above hypotheses. If we are given*

$$u_0 \in V, \qquad u_1 \in H,$$
$$f = f_1 + f_2, \qquad f_1 \in L^1(0, T; H), \qquad f_2 \in L^{p'}(0, T; W');$$

there exists one and only one function u which satisfies :

$$(2.1) \qquad \begin{cases} u \in L^\infty(0, T; V), \\ u' = \dfrac{\partial u}{\partial t} \in L^\infty(0, T; H) \cap L^p(0, T; W); \\ A(t)\, u(t) + u''(t) + \beta(t)\, u'(t) = f(t) \quad \text{a. e.}; \\ u(0) = u_0, \qquad u'(0) = u_1. \end{cases}$$

Moreover, the mapping $(u_0, u_1, f) \to u$ *sends bounded sets into bounded sets; and the mapping* $(u_0, u_1, f_1) \to u$, *for fixed* f_2, *is continuous in the appropriate topologies* [*cf.* equation (2.11) below] [9].

We note that equation (2.1) implies

$$u'' \in L^1(0, T; H) + L^{p'}(0, T; W') + L^\infty(0, T; V'),$$

so that $u'(.)$ is continuous with values in $(V \cap W')$ and $u'(0)$ makes sense.

2.3. Proof of existence.

We take any basis (i. e., any linearly independent set whose finite linear combinations are dense) $\{w_j\}$ of $V \cap W$. We denote by $u_m(t)$ the unique solution of the ordinary differential system

$$(2.2) \quad \begin{cases} (u''_m(t), w_j) + a(t; u_m(t), w_j) + (\beta(t)\, u'_m(t), w_j) = (f(t), w_j) \\ \qquad (j = 1, \ldots, m), \\ u_m(0) = P_m u_0, \qquad u'_m(0) = P_m u_1, \end{cases}$$

where P_m is the projection onto the subspace spanned by $w_1, \ldots, w_m$. [Here we have used Hypothesis (j).] The solution is defined in some interval $0 \leq t \leq \delta_m$, $\delta_m \leq T$.

In (2.2) we may replace w_j by $u'_m(t)$ to obtain

$$(2.3) \quad \begin{aligned} &|u'_m(t)|^2 + a(t; u_m(t), u_m(t)) + 2\,\mathrm{Re}\int_0^t (\beta u'_m, u'_m)\, d\sigma \\ &= |u'_m(0)|^2 + a(0; u_m(0), u_m(0)) \\ &\quad + \int_0^t a'(\sigma; u_m(\sigma), u_m(\sigma))\, d\sigma + 2\,\mathrm{Re}\int_0^t (f, u'_m)\, d\sigma. \end{aligned}$$

Hypotheses (ii), (iii) and (jjj) imply that $(0 \leq t \leq T)$:

$$(2.3)' \quad \begin{aligned} &|u'_m(t)|^2 + [u_m(t)]^2 + 2\,\mathrm{Re}\int_0^t (\beta u'_m, u'_m)\, d\sigma \\ &\leq |u'_m(0)|^2 + c_5 \|u_m(0)\|^2 \\ &\quad + c_5 \int_0^t \|u_m(\sigma)\|^2\, d\sigma + 2\int_0^t |(f(\sigma), u'_m(\sigma))|\, d\sigma. \end{aligned}$$

[9] One can also treat by the same method non-linear terms where $\beta = \beta_1 + \beta_2$, β_j mapping $L^{p_j}(0, T; W_j)$ into $L^{p'_j}(0, T, W_j)$ $(j = 1, 2)$, where $p_1 = p_2$ and W_1 and W_2 are two Banach spaces of the above type.

Now $f = f_1 + f_2$, where $f_1 \in L^1(0, T; H)$ and $f_2 \in L^{p'}(0, T; W')$. We have, for any $\delta > 0$,

$$2\int_0^t |(f_1, u'_m)| \, d\sigma \leq \delta^{-1} \left\{ \int_0^t |f_1(\sigma)| \, d\sigma \right\}^2 + \delta \sup_{0 \leq \sigma \leq t} |u'_m(\sigma)|^2.$$

On the other hand, we define

$$\| f_2 \|^{p'} = \int_0^t \| f_2(\sigma) \|_{W'}^{p'} \, d\sigma.$$

Then, by Hölder's inequality and Hypothesis (jjj),

$$2\int_0^t |(f_2, u'_m)| \, d\sigma \leq 2 \| f_2 \| \left\{ \int_0^t \| u'_m(\sigma) \|_W^p \, d\sigma \right\}^{\frac{1}{p}}$$

$$\leq c_6 \| f_2 \| \left\{ \int_0^t \mathrm{Re}(\beta u'_m, u'_m) \, d\sigma \right\}^{\frac{1}{p}} + c_6 \| f_2 \| \left\{ \int_0^t | u'_m(\sigma) |^p \, d\sigma \right\}^{\frac{1}{p}}.$$

The first term on the right-hand side of the last inequality is estimated using Young's inequality

$$ab \leq c_\delta a^{p'} + \delta b^p \qquad (a, b \geq 0, \delta > 0);$$

in the second term we use Young's inequality for $p = p' = 2$ together with

$$\left\{ \int_0^t | u'_m(\sigma) |^p \, d\sigma \right\}^{\frac{2}{p}} \leq T^{\frac{2}{p}} \sup_{0 \leq \sigma \leq t} | u'_m(\sigma) |^2.$$

We obtain, for any $\delta > 0$,

$$2\int_0^t |(f_2, u'_m)| \, d\sigma$$

$$\leq c_7 \{ \| f_2 \|^{p'} + \| f_2 \|^2 \} + c_8 \delta \int_0^t \mathrm{Re}(\beta u'_m, u'_m) \, d\sigma + c_8 \delta \sup_{0 \leq \sigma \leq t} | u'_m(\sigma) |^2,$$

where c_7 (but not c_8) depends on δ. Finally, we use the identity

$$u_m(t) = u_m(0) + \int_0^t u'_m(\sigma) \, d\sigma,$$

which implies [by (ii)] that

$$\| u_m(t) \|^2 \leq c_9 \sup_{0 \leq \sigma \leq t} \{ | u'_m(\sigma) |^2 + [u_m(\sigma)]^2 \}.$$

Putting all these estimates into (2.3)′, and defining

$$K(f_1, f_2, u_0, u_1) = \left(\int_0^T |f_1(\sigma)|\,d\sigma\right)^2 + \|f_2\|^2 + \|f_2\|^{p'} + \|u_0\|^2 + |u_1|^2,$$

we conclude that

$$\begin{aligned} &|u'_m(t)|^2 + \|u_m(t)\|^2 + 2\int_0^t \operatorname{Re}(\beta u'_m, u'_m)\,d\sigma \\ &\quad \leqq c_{10} K(f_1, f_2, u_0, u_1) + c_{10}\int_0^t \{|u'_m(\sigma)|^2 + \|u_m(\sigma)\|^2\}\,d\sigma \\ &\quad + \delta c_{11}\int_0^t \operatorname{Re}(\beta u'_m, u'_m)\,d\sigma + \delta c_{11} \sup_{0 \leqq \sigma \leqq t} |u'_m(\sigma)|^2, \end{aligned}$$

where c_{10} (but not c_{11}) depends on δ. Now choose $\delta = (2c_{11})^{-1}$. It follows that

$$|u'_m(t)|^2 + \|u_m(t)\|^2 + \operatorname{Re}\int_0^t (\beta u'_m, u'_m)\,d\sigma \leqq c_{12} K(f_1, f_2, u_0, u_1). \tag{2.4}$$

This implies that $\delta_m = T$, that $\{u_m\}$ is a bounded set in $L^\infty(0, T; V)$ and that $\{u'_m\}$ is bounded in $L^\infty(0, T; H)$. By (jjj), we have

$$\int_0^T \|u'_m(\sigma)\|_W^p\,d\sigma \leqq c_{14}\left\{\operatorname{Re}\int_0^T (\beta u'_m, u'_m)\,d\sigma + T \sup_{0 \leqq \sigma \leqq T} |u'_m(\sigma)|^p\right\},$$

which is bounded. So u'_m is also bounded in $L^p(0, T; W)$. Therefore $\{\beta(u'_m)\}$ is bounded in $L^{p'}(0, T; W')$. [Here we have used Hypotheses (jj) and (jjj).] We may therefore extract a subsequence $\{u_\nu\}$ from $\{u_m\}$ such that :

$$\begin{aligned} u_\nu &\to u && \text{in the weak-star topology of } L^\infty(0, T; V), \\ u'_\nu &\to u' && \text{in the weak-star topology of } L^\infty(0, T; H), \\ u'_\nu &\to u' && \text{in the weak topology of } L^p(0, T; W), \\ \beta(u'_\nu) &\to \psi && \text{in the weak topology of } L^{p'}(0, T; W'). \end{aligned}$$

(It is easy to show that the first three limits have the relationship stated.)

Letting $m = \nu$ in equation (2.2) and $\nu \to \infty$ yields the following equation for $u(t)$:

$$\frac{d}{dt}(u'(t), w_j) + a(t; u(t), w_j) + (\psi(t), w_j) = (f(t), w_j) \qquad (j = 1, 2, \ldots),$$

taken in the sense of distributions over $(0, T)$. However, since $(f(t), w_j) - (\psi(t), w_j) - a(t; u(t), w_j)$ is a measurable function of t for each j, so is $(d/dt)\,(u'(t), w_j)$, so that this equation is valid almost everywhere. Hence

$$u'' + A(.)\,u + \psi = f \tag{2.5}$$

and $u'' \in L^1(0, T; (V \cap W)')$. It also follows that

$$u(0) = u_0 \quad \text{and} \quad u'(0) = u_1. \tag{2.6}$$

The former relation is true since $u_\nu(0) \to u(0)$ weakly in H while $u_\nu(0) \to u_0$ strongly in H. To prove the latter relation in (2.6), let

$$\varphi = \sum_{j=1}^{m} \varphi_j \otimes w_j$$

where $\varphi_j \in C^{(1)}(0, T)$ and $\varphi_j(T) = 0$. We note that

$$\int_0^T (u''_\nu, \varphi)\,dt = -\int_0^T (u'_\nu, \varphi')\,dt - (u'_\nu(0), \varphi(0)).$$

On the other hand,

$$\int_0^T (u''_\nu, \varphi)\,dt = \int_0^T [(f, \varphi) - a(t; u_\nu, \varphi) - (\beta(t)\,u'_\nu, \varphi)]\,dt,$$

which converges to

$$\int_0^T [(f, \varphi) - a(t; u, \varphi) - (\psi, \varphi)]\,dt = \int_0^T (u'', \varphi)\,dt$$

as $\nu \to \infty$. Therefore

$$\int_0^T (u'', \varphi)\,dt = -\int_0^T (u', \varphi')\,dt - (u_1, \varphi(0)).$$

It follows that $(u_1, \varphi(0)) = (u'(0), \varphi(0))$, where $\varphi(0)$ is in fact an arbitrary linear combination of $w_1, \ldots, w_m$. Hence $u_1 = u'(0)$.

It remains to show that $\psi = \beta u'$. For the time being, let us assume the

LEMMA 2.1. — *Let*

$$w \in L^2(0, T; V),$$

$$\left\{\begin{array}{l} w' \in L^\infty(0, T; H) \cap L^p(0, T; W); \\ A(t)\,w(t) + w''(t) = F(t) \quad \text{a. e.}; \\ w(0) = w_0 \in V, \qquad w'(0) = w_1 \in H, \\ F \in L^{p'}(0, T; W') + L^1(0, T; H). \end{array}\right. \tag{2.7}$$

Then for a. e. t,

$$a(t;\, w(t),\, w(t)) + |\, w'(t)\,|^2 \geqq a(0;\, w_0,\, w_0) + |\, w_1\,|^2 + \int_0^t a'(\sigma;\, w(\sigma),\, w(\sigma))\, d\sigma$$
$$+ 2\,\mathrm{Re}\int_0^t (F(\sigma),\, w'(\sigma))\, d\sigma.$$

Equality holds in case $w_0 = w_1 = 0$ (10).

We apply this lemma to the present situation with $w = u$, $F = f - \psi$ to yield the inequality

$$(2.8)\qquad a(t;\, u(t),\, u'(t)) + |\, u'(t)\,|^2$$
$$\geqq a(0;\, u_0,\, u_0) + |\, u_1\,|^2 + \int_0^t a'(\sigma;\, u(\sigma),\, u(\sigma))\, d\sigma$$
$$+ 2\,\mathrm{Re}\int_0^t (f(\sigma) - \psi(\sigma),\, u'(\sigma))\, d\sigma$$

almost everywhere. We choose a sequence of numbers $\{ t_k \}$ such that (2.8) holds for $t = t_k$ and such that $t_k \to T$. By the familiar " diagonal " procedure, (2.4) implies that a subsequence $\{ u_\mu \}$ can be extracted from $\{ u_\nu \}$ such that

$$u_\mu(t_k) \to u\,(t_k) \quad \text{weakly in } V,$$
$$u'_\mu(t_k) \to u'(t_k) \quad \text{weakly in } H$$

as $\mu \to \infty$, for $k = 1, 2, \ldots.$ To show that these limits are as asserted, we repeat the method used to prove (2.6).

Now in equation (2.3), we put $m = \mu$ and take inferior limits as $\mu \to \infty$; this yields the inequality

$$|\, u'(t)\,|^2 + a(t;\, u(t),\, u(t)) + \liminf_{\mu \to \infty} 2\,\mathrm{Re}\int_0^t (\beta\, u'_\mu,\, u'_\mu)\, d\sigma$$
$$\leqq |\, u_1\,|^2 + a(0;\, u_0,\, u_0) + \int_0^t a'(\sigma;\, u(\sigma),\, u(\sigma))\, d\sigma + 2\,\mathrm{Re}\int_0^t (f(\sigma),\, u'(\sigma))\, d\sigma,$$

for $t = t_k$. Here we have used, in particular, Hypothesis (iv). Comparing this inequality with (2.8), we conclude that

$$\mathrm{Re}\int_0^t (\psi(\sigma),\, u'(\sigma))\, d\sigma \geqq \liminf_{\mu \to \infty} \mathrm{Re}\int_0^t (\beta(\sigma)\, u'_\mu(\sigma),\, u'_\mu(\sigma))\, d\sigma$$

for $t = t_k$.

---

(10) Equality also holds more generally, but it is not necessary for present purposes.

If φ is an arbitrary element of $L^p(o, T; W)$, we have

$$-\operatorname{Re}\int_0^t (\beta\varphi, u')\,d\sigma = -\lim_{\mu\to\infty} \operatorname{Re}\int_0^t (\beta\varphi, u'_\mu)\,d\sigma$$

and

$$-\operatorname{Re}\int_0^t (\psi, \varphi)\,d\sigma = -\lim_{\mu\to\infty} \operatorname{Re}\int_0^t (\beta u'_\mu, \varphi)\,d\sigma.$$

We add the last three relationships and add the quantity

$$\operatorname{Re}\int_0^t (\beta\varphi, \varphi)\,d\sigma$$

to both sides of the result, obtaining :

$$\operatorname{Re}\int_0^t (\psi - \beta\varphi, u' - \varphi)\,d\sigma \geqq \liminf_{\mu\to\infty} \operatorname{Re}\int_0^t (\beta u'_\mu - \beta\varphi, u'_\mu - \varphi)\,d\sigma$$

for $t = t_k$. But the right-hand side of this inequality is non-negative; therefore, letting $t_k \to T$,

$$\operatorname{Re}\int_0^T (\psi - \beta\varphi, u' - \varphi)\,d\sigma \geqq o.$$

Proceeding as in [14] (proof of Theorem 4, p. 344), we now let $\varphi = u' - \lambda\varphi_1$, where $\lambda > o$ and $\varphi_1 \in L^p(o, T; W)$:

$$\operatorname{Re}\int_0^T (\psi - \beta(u' - \lambda\varphi_1), \varphi_1)\,d\sigma \geqq o.$$

Letting $\lambda \to o$ [and using Hypothesis (jj)] gives :

$$\operatorname{Re}\int_0^T (\psi - \beta u', \varphi_1) \geqq o$$

for all $\varphi_1 \in L^p(o, T; W)$. This is absurd unless

$$\psi(t) = \beta(t)\,u'(t) \quad \text{a. e.}$$

2.4. **Proof of Lemma** 2.1.

Let $o < s < t < T$. Let θ_n be identically one in $[s, t]$, zero in $\left(-\infty, s - \frac{1}{n}\right)$ and in $\left(t + \frac{1}{n}, \infty\right)$, and linear in the remaining two intervals of the real line; n is a positive integer. Let η be an infinitely differentiable function on the line with compact support, $\eta(t) = \eta(-t)$, and let

$$\eta_k(t) = k\,\eta(kt), \qquad k \text{ positive integer;}$$

then η_k converges to the delta function as $k \to \infty$. We assume that k is large enough that $(\theta_n w') \star \eta_k \star \eta_k$ vanishes near T and near 0.

Using a variant of a method of [12] ([11]), we take inner products of both sides of equation (2.7) with

$$\theta_n[(\theta_n w') \star \eta_k \star \eta_k] = \theta_n[(\theta_n w) \star \eta'_k \star \eta_k - (\theta'_n w) \star \eta_k \star \eta_k],$$

which is infinitely differentiable with values in V and W. We obtain :

$$(2.9)\quad \operatorname{Re}\int_0^T \{ a(\sigma;\, \theta_n w,\, (\theta_n w) \star \eta'_k \star \eta_k) - a(\sigma;\, \theta_n w,\, (\theta'_n w) \star \eta_k \star \eta_k) + (\theta_n w'',\, (\theta_n w') \star \eta_k \star \eta_k) \}\, d\sigma = \operatorname{Re}\int_0^T \theta_n (F,\, (\theta_n w') \star \eta_k \star \eta_k)\, d\sigma.$$

[Note that $w'' \in L^1(0, T; H) + L^{p'}(0, T; W') + L^2(0, T; V')$, so that the inner products make sense.]

We denote $v = \theta_n w$, and write

$$a(t;\, u, v) = ((\mathcal{A}(t), u, v)) \qquad (u, v \in V),$$

where $\mathcal{A}(t)$ is a bounded operator in V. We can then write the first term in (2.9) as $X_1 + X_2$, where

$$\begin{aligned} X_1 &= \operatorname{Re}\int_0^T a(\sigma;\, (v \star \eta_k)(\sigma),\, (v \star \eta'_k)(\sigma))\, d\sigma \\ &= \frac{1}{2} a(v \star \eta_k,\, v \star \eta_k)\Big|_0^T - \frac{1}{2}\int_0^T a'(\sigma;\, v \star \eta_k,\, v \star \eta_k)\, d\sigma \\ &= -\frac{1}{2}\int_0^T a'(\sigma;\, (v \star \eta_k)(\sigma),\, (v \star \eta_k)(\sigma))\, d\sigma, \end{aligned}$$

and

$$\begin{aligned} X_2 &= \operatorname{Re}\int_0^T \{ ((\mathcal{A} v,\, v \star \eta'_k \star \eta_k)) - ((\mathcal{A}(v \star \eta_k),\, v \star \eta'_k)) \}\, d\sigma \\ &= -\operatorname{Re}\int_0^T \left(\left(\frac{d}{dt}[(\mathcal{A} v) \star \eta_k - \mathcal{A}(v \star \eta_k)],\, v \star \eta_k \right)\right) d\sigma. \end{aligned}$$

A vector-valued Friedrichs' Lemma (*cf.*, for example, [9], p. 72) implies that $X_2 \to 0$ as $k \to \infty$.

([11]) Another variant of the same method is used for linear problems by G. Torelli [21].

The third term on the left-hand side of (2.9) equals $X_3 + X_4$, where

$$X_3 = \int_0^T ((\theta_n w')' \star \eta_k, (\theta_n w') \star \eta_k)\, d\sigma = 0$$

and

$$X_4 = -\int_0^T ((\theta'_n w') \star \eta_k, (\theta_n w') \star \eta_k)\, d\sigma.$$

Therefore, letting $k \to \infty$, we obtain from (2.9)

$$-\int_0^T \theta_n \theta'_n \{ a(w, w) + |w'|^2 \}\, d\sigma = \frac{1}{2} \int_0^T \theta_n^2 \{ a'(w, w) + (F, w') \}\, d\sigma.$$

But if $h \in L^1(0, T)$, then the integrals

$$\begin{aligned} -\int_0^T \theta_n \theta'_n h \, d\sigma = \; & n \int_t^{t+\frac{1}{n}} [1 - n(\sigma - t)]\, h(\sigma)\, d\sigma \\ & - n \int_{s-\frac{1}{n}}^{s} [1 + n(\sigma - s)]\, h(\sigma)\, d\sigma \end{aligned}$$

converge as $n \to \infty$ to $\frac{1}{2}(h(t) - h(s))$ for almost every s, t. Therefore

$$(2.10) \quad \begin{aligned} & a(t; w(t), w(t)) + |w'(t)|^2 \\ & = a(s; w(s), w(s)) + |w'(s)|^2 + \int_s^t a'(\sigma; w(\sigma), w(\sigma))\, d\sigma \\ & \qquad + 2 \operatorname{Re} \int_s^t (F(\sigma), w'(\sigma))\, d\sigma \end{aligned}$$

almost everywhere.

Now choose any $s_q \to 0$ and t such that (2.10) holds for t and $s = s_q$. Then

$$a(s_q; w(s_q), w(s_q)) + |w'(s_q)|^2$$

is a bounded function of q. Also, the inequality

$$|w(s)| \leq |w_0| + T . \operatorname*{ess\,sup}_{[0,\, T]} |w'|$$

shows that $|w(s_q)|$ is bounded. Hence $\{ w(s_q) \}$ is bounded in V and $\{ w'(s_q) \}$ is bounded in H, so that we can extract a subsequence $\{ s_r \}$ of $\{ s_q \}$ such that

$$\begin{aligned} w(s_r) &\to w_0 \quad \text{weakly in } V, \\ w'(s_r) &\to w_1 \quad \text{weakly in } H \end{aligned}$$

as $r \to \infty$. [To show the limits are as stated, note that w is continuous with values in H and w' is continuous with values in $(V \cap W)'$.] The desired inequality therefore follows by letting $s = s_r \to 0$ in (2.10).

If $w_0 = w_1 = 0$, define $w(\sigma)$, $A(\sigma)$ and $F(\sigma)$ to be identically zero for $\sigma < 0$. Then (2.10) holds also if $s < 0$. This reduces to the desired equality.

2.5. Proof of uniqueness and of continuity with respect to the data.

Let u and v be two solutions of the respective equations

$$u'' + Au + \beta u' = f, \qquad v'' + Av + \beta v' = g,$$

where $f, g \in L^1(0, T; H) + L^{p'}(0, T; W')$.

For uniqueness, let $f = g$ and $u(0) = v(0)$, $u'(0) = v'(0)$. We apply Lemma 2.1 to $w = u - v$, $w_0 = 0$, $F = \beta(v') - \beta(u')$ to obtain the equality :

$$a(t; w(t), w(t)) + |w'(t)|^2$$
$$= \int_0^t a'(\sigma; w(\sigma), w(\sigma))\, d\sigma - 2 \operatorname{Re} \int_0^t (\beta u' - \beta v', w')\, d\sigma \leqq 0.$$

So $|w'(t)|^2 \leqq 0$, $w'(t) = 0$, $w = 0$.

Now assume only that

$$f - g \in L^1(0, T; H).$$

We shall show that, if w denotes $u - v$, we have

$$|w'(t)|^2 + \|w(t)\|^2 \leqq c_{13} K(w(0), w'(0), f - g), \tag{2.11}$$

where

$$K(w_0, w_1, h) = \|w_0\|^2 + |w_1|^2 + \left\{ \int_0^T |h(\sigma)|\, d\sigma \right\}^2.$$

Let u_m be defined by the differential system (2.2), where $u_0 = u(0)$, $u_1 = u'(0)$, and let v_m be defined by the corresponding system with u_0, u_1 and f replaced by $v(0)$, $v'(0)$ and g, respectively. Define $w_m = u_m - v_m$. We substract the defining equations for u_m and v_m and then replace w_j by $w'_m(t)$, thus obtaining

$$|w'_m(t)|^2 + a(t; w_m(t), w_m(t)) + 2 \operatorname{Re} \int_0^t (\beta u'_m - \beta v'_m, w'_m)\, d\sigma$$
$$= |w'_m(0)|^2 + a(0; w_m(0), w_m(0)) + \int_0^t a'(\sigma; w_m(\sigma), w_m(\sigma))\, d\sigma$$
$$+ 2 \operatorname{Re} \int_0^t (f - g, w'_m)\, d\sigma \quad \text{a. e.}$$

Using Hypotheses (iv) and (jv), it follows that

$$\begin{aligned} &| w'_m(t) |^2 + a(t; w_m(t), w_m(t)) \\ &\quad \leq | w'_m(0) |^2 + a(0; w_m(0), w_m(0)) + 2 \operatorname{Re} \int_0^t (f - g, w'_m)\, d\sigma \end{aligned}$$

Taking, for fixed t, weakly convergent subsequences as before (and using the uniqueness result), we conclude that

$$\begin{aligned} &| w'(t) |^2 + a(t; w(t), w(t)) \\ &\quad \leq | w'(0) |^2 + a(0; w(0), w(0)) + 2 \operatorname{Re} \int_0^t (f - g, w')\, d\sigma. \end{aligned}$$

for almost every t. This implies (2.11) by the usual procedure. This completes the proof of Theorem 2.1.

2.6. **Application to non-linear partial differential equations.**

Let Ω be any open set in E^n. Let $H = L^2(\Omega)$. Let

$$a(t; u, v) = a(u, v) = \sum_{\substack{|\alpha| \leq m \\ |\gamma| \leq m}} \int a_{\alpha\gamma}(x) D^\alpha u(x) \overline{D^\gamma v(x)}\, dx,$$

where $a_{\alpha\gamma} \in L^\infty(\Omega)$, $\overline{a_{\alpha\gamma}(x)} = \overline{a_{\gamma\alpha}(x)}$ a. e. Let $\tilde{V}$ be *any* linear submanifold of the space $\mathcal{D}(\overline{\Omega})$, which consists of all C^∞-functions defined in $\overline{\Omega}$ with compact support in $\overline{\Omega}$. We *assume* that

$$a(v, v) \geq 0 \qquad \text{for } v \in \tilde{V}.$$

If we now define

$$\| v \| = \{ a(v, v) + | v |^2 \}^{\frac{1}{2}}, \qquad [v] = \{ a(v, v) \}^{\frac{1}{2}} \qquad (v \in \tilde{V})$$

and define V as the completion of $\tilde{V}$ with respect to the norm $\| v \|$, then V is a Hilbert space under this norm and (i)-(iv) of section 2.1 are obviously satisfied.

The unbounded operator $A(t) = A$ does not depend on t in the present case. It is the partial differential operator in Ω defined by

$$A_d f = A f = \sum_{\substack{|\alpha| \leq m \\ |\gamma| \leq m}} (-1)^{|\gamma|} D^\gamma (a_{\alpha\gamma}(x) D^\alpha f);$$

its domain (as an operator on H) is defined by *boundary conditions* which themselves depend on the choice of $\tilde{V}$ (and, of course, of the coefficients $a_{\alpha\gamma}$).

We now introduce the space W. For each σ in a finite set J, let D_σ be *any* linear differential operator with smooth coefficients and let $(D_\sigma)^*$ denote its formal adjoint. Let $\rho > 1$ and $p = \rho + 1$. For each $\sigma \in J$, let a non-negative function k_σ, $k_\sigma \in L^\infty(\Omega)$, be given. We *define* the space $\mathcal{W}$ as the collection of all functions v in $L^2(\Omega)$ such that

$$\int_\Omega |D_\sigma v(x)|^p k_\sigma(x)\, dx < \infty$$

for all $\sigma \in J$; this space is provided with the obvious norm. Let $\mathcal{W}_0$ be the closure in $\mathcal{W}$ of the subspace $\mathcal{D}(\Omega)$ of C^∞-functions with compact support in Ω. Finally, W is any closed subspace of $\mathcal{W}$ such that

$$\mathcal{W}_0 \subset W \subset \mathcal{W}.$$

The intersection $V \cap W$ is separable. Indeed, it is contained in $H^m(\Omega) \cap W$ with continuous inclusion mapping. Since the latter space is isometric to a direct sum of L^p-spaces with various separable measures, it is separable and therefore so is $V \cap W$.

We assume that $V \cap W$ *is dense in* V *and in* W. Here are three cases when this assumption is valid.

EXAMPLE 1. — $\tilde{V} = \mathcal{D}(\Omega)$ (i. e., the C^∞-functions with compact support in the open set Ω) and $W = \mathcal{W}_0$. Then $\mathcal{D}(\Omega)$ is contained in both V and W and is dense in each of them (by definition).

EXAMPLE 2. — $V = H^m(\Omega)$ and $W = \mathcal{W} = W^{k,p}(\Omega)$, where $W^{k,p}(\Omega)$ denotes the Sobolev space and the boundary Γ of Ω is sufficiently smooth. Under the last condition, the smooth functions in $\overline{\Omega}$ with compact support in $\overline{\Omega}$ are contained in $V \cap W$ and are dense in both V and W.

EXAMPLE 3. — $V = H_0^m(\Omega)$ and W is the closure in $\mathcal{W} = W^{k,p}(\Omega)$ of $H_0^m(\Omega) \cap W^{k,p}(\Omega)$, where $k > m$. In this case, $V \cap W$ contains $\mathcal{D}(\Omega)$, so that it is dense in V. On the other hand, if Ω has a smooth boundary, $V \cap W$ contains all the smooth functions in W (*cf.* NIKOLSKI: [16]), and it is therefore dense in W.

Note that the space V can be the same in Examples 1 and 3, although the space W is different. It should not, however, be concluded that V and W can be chosen independently of one another. For instance, we *cannot* take $V = H_0^m(\Omega)$ and $W = \mathcal{W}$ if $\Omega \neq E^n$, $m > 0$, and, for some $\sigma \in J$, D_σ has positive order and k_σ is not identically zero a. e. For, in that case, $V \cap W$ is not dense in W. The choices of V (i. e., $\tilde{V}$) and of W determine the boundary conditions of the differential equation.

Now, for each $\sigma \in J$, let h_σ be a positive, bounded, measurable function defined on the cylinder $(0, T) \times \Omega$, and bounded away from zero

$$0 < c_1 \leq h_\sigma(x, t) \leq c_2 \quad \text{a. e.} \quad (^{12}).$$

Finally, $\beta(t)$ is defined as follows. Let $\varphi(z) = |z|^{p-1} z$ and let

$$b(t; u, v) = \sum_{\sigma \in J} \int_\Omega \varphi(D_\sigma u(x)) \overline{D_\sigma v(x)}\, h_\sigma(x, t)\, h_\sigma(x)\, dx,$$

for $u, v \in W$. Since $v \to b(t; u, v)$ is a conjugate-linear map and

$$(2.12) \qquad | b(t; u, v) | \leq c_{10} \| u \|_W^p \| v \|_W,$$

it follows that $\beta(t) u$ is uniquely determined as an element of W' by the equation

$$b(t; u, v) = (\beta(t) u, v) \qquad (v \in W).$$

Now Hypotheses (j) and (jj) follows easily from the definition of $\beta(t)$ and from estimate (2.12). Indeed, if $u, v \in L^p(0, T; W)$,

$$\| \beta u \|_{L^{p'}(0, T; W')} \leq c_{10} \| u \|^p_{L^p(0, T; W)}.$$

Hypotheses (jjj) and (jv) follow from the corresponding properties (*cf.* Lemma 1.1) of the numerical function $\varphi(z) = |z|^{p-1} z$.

The general theory can therefore be applied to the present situation. *Let* $u_0 \in V$, $u_1 \in H = L^2(\Omega)$, $f = f_1 + f_2$,

$$f_1 \in L^1(0, T; L^2(\Omega)), \qquad f_2 \in L^{p'}(0, T; W').$$

Then there exists one and only one function u such that

$$(2.13) \qquad \begin{cases} u \in L^\infty(0, T; V), \\ u' = \dfrac{\partial u}{\partial t} \in L^\infty(0, T; L^2(\Omega)), \\ u' \in L^p(0, T; W); \\ u'' + A u + \beta(t) u' = f \quad \textit{in} \ (0, T) \times \Omega; \\ u(x, 0) = u_0(x), \qquad u'(x, 0) = u_1(x) \quad \textit{in} \ \Omega \quad (^{13}). \end{cases}$$

---

(12) If we weaken this assumption to : $0 < c_1(\varepsilon) \leq h_\sigma(x, t) \leq c_2$ in $(\varepsilon, T-\varepsilon) \times \Omega$ a. e. for all $\varepsilon > 0$, then we can still prove the corresponding results by exactly the same method if we introduce spaces $W(t)$ with weight functions $k_\sigma(x)\, h_\sigma(x, t)$ (*cf.* Part I).

(13) Assume $f_{1j} \to f_1$ in $L^1(0, T; L^2(\Omega))$, $f_{2j} \to f_2$ in $L^{p'}(0, T; W')$ (strongly); if u_j is the solution of (2.13) with $f = f_{1j} + f_{2j}$ and $u_0 = u_1 = 0$, then one can prove that there is a subsequence such that

$$D_\sigma u'_j \to D_\sigma u' \quad \text{a. e. for the measure } k_\sigma\, dx.$$

Equation (2.13) *is taken in the sense of distributions in* $t \in (0, T)$ *with values in* $(V \cap W)'$ [because $(V \cap W)'$ is not necessarily a space of distributions on Ω].

Some of the boundary conditions are contained in equation (2.13). More precisely, let us write (2.13) in its " weak " form : if v is any element of $V \cap W$, then

$$(2.14) \qquad \frac{d^2}{dt^2}(u(t), v) + a(u(t), v) + b(t; u'(t), v) = (f(t), v).$$

Now set

$$\beta_d(t) f = \sum_{\sigma \in J} (D_\sigma)^* (\varphi(D_\sigma f) h_\sigma(t) k_\sigma),$$

all the derivatives being taken in the sense of distributions in Ω (which we indicate by the subscript " d "). Since, *in particular*, (2.14) is valid for every $v \in \mathcal{D}(\Omega)$, we have

$$(2.15) \qquad u'' + A_d u + \beta_d(t) u' = f$$

in the sense of distributions in $(0, T) \times \Omega$ assuming that the cœfficients $a_{\alpha\beta}$ are $C^\infty(\Omega)$ or that $a(u, v)$ is coercive.

Therefore, $A_d u + \beta_d(t) u' = f - u''$ is the sum of elements of $L^1(0, T; H)$, $L^{p'}(0, T; W')$ and of $(-u'')$, which is a distribution on $(0, T)$ with values in H (even in V); consequently, for every $v \in V \cap W$, $(A_d u(t) + \beta_d(t) u'(t), v)$ makes sense as a distribution over $(0, T)$ and equals $(f(t), v) - \frac{d^2}{dt^2}(u(t), v)$. Comparing this to (2.14), we obtain

$$(2.16) \qquad \begin{cases} (A_d u(t) + \beta_d(t) u'(t), v) = a(u(t), v) + b(t; u'(t), v) \\ \qquad\qquad \text{for every} \quad v \in V \cap W. \end{cases}$$

Conclusion. — *u satisfies* (2.15), *subject to the initial conditions* $u(x, 0) = u_0(x)$, $u'(x, 0) = u_1(x)$ *and subject to the boundary conditions :*

(I) $u \in L^\infty(0, T; V)$;

(II) $u' \in L^p(0, T; W)$;

(III) u satisfies (2.16).

Example 1 : $V = H_0^m(\Omega)$. — Then the boundary condition implied by (I) is

$$D^\alpha u(x, t) = 0 \quad \text{for } |\alpha| \leq m - 1 \qquad [x \in \Gamma, t \in (0, T)]$$

(taken in a generalized sense, of course). In this case, (2.16) reduces to :

$$\int_\Omega (\beta_d(t) u') \bar{v} \, dx = b(t; u'(t), v) \qquad (v \in V \cap W).$$

Moreover, if we take $W = \mathcal{W}_0$, then (2.16) is automatically satisfied and (II) means that certain space-derivatives (depending on the D_σ's) of u' are zero on $(0, T) \times \Gamma$, in a generalized sense.

EXAMPLE 2 : $V = H^m(\Omega)$, and the space $\mathcal{W}$ involves no derivatives; i. e., the operators D_σ are just multiplications by functions of x. Then (I) and (II) involve no condition at the boundary and the remaining boundary condition (III) reduces to :

$$\int_\Omega (A_d u)\bar{v}\,dx = a(u(t), v) \qquad (v \in V \cap W).$$

This means that u satisfies Neumann-type boundary conditions.

EXAMPLE 3. — Let us now take $V = H^1(\Omega)\ A_d = -\Delta$,

$$a(u, v) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_i}\,dx;$$

let J consist of a single element σ and let $D_\sigma = \partial/\partial x_1$, $h_\sigma = h$, $k_\sigma = k$ and finally let $W = \mathcal{W}$. Once again, the only boundary condition is (III), which can be written in this case as :

$$\int_\Omega (-\Delta u)\bar{v}\,dx - \int_\Omega \frac{\partial}{\partial x_1}\left(hk\varphi\left(\frac{\partial^2 u}{\partial x_1\,\partial t}\right)\right)\bar{v}\,dx$$

$$= \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \bar{v}}{\partial x_i}\,dx + \int_\Omega hk\varphi\left(\frac{\partial^2 u}{\partial x_1\,\partial t}\right)\cdot\frac{\partial \bar{v}}{\partial x_1}\,dx.$$

If ν denotes the unit outer normal to Γ and ν_1 denotes its first component (i. e., with respect to the variable x_1), this boundary condition can be written as

$$\frac{\partial u}{\partial \nu} + \nu_1\varphi\left(\frac{\partial^2 u}{\partial x_1\,\partial t}\right)hk = 0 \quad \text{on } (0, T)\times\Gamma.$$

Thus the boundary conditions are, in general, non-linear.

REMARK. — We could also consider cases when p (or $\rho = p - 1$) depends on $\sigma \in J$.

2.7. **A remark on first-order equations.**

We are given a pair of Hilbert spaces H and V and a reflexive Banach space W with the following properties : both V and W are contained in H with each inclusion mapping continuous, $V \cap W$ is separable and is dense in V and in W.

Secondly, we have a family of sesquilinear forms

$$u, v \to a(t; u, v)$$

defined on $V \times V$ for almost every $t \in (0, T)$. We assume that :

(i) $t \to a(t; u, v)$ is measurable, for each $u, v \in V$;

(ii) there is a constant c_1 such that

$$|a(t; u, v)| \leqq c_1 \|u\| . \|v\| \qquad [u, v \in V, t \in (0, T)];$$

(iii) there are constants λ_1, c_2 ($c_2 > 0$) such that

$$\operatorname{Re} a(t; v, v) + \lambda_1 |v|^2 \geqq c_2 \|v\|^2 \qquad [v \in V, t \in (0, T)].$$

If we define $A(t)$ as usual by $(A(t)u, v) = a(t; u, v)$ for all $u, v \in V$, then $A(t)$ is a linear mapping of V into V', and (i) and (ii) imply that the mapping $u(.) \to A(.)u(.)$ is continuous from $L^2(0, T; V)$ into $L^2(0, T; V')$.

Thirdly, we are given a family of maps $\beta(t) : W \to W'$, defined for almost every $t \in (0, T)$, which satisfy :

(j) $\beta(t)$ is continuous from finite-dimensional subspaces of W to the weak topology of W';

(jj) there exists a constant λ_2 such that

$$\operatorname{Re}(\beta(t)u - \beta(t)v, u - v) + \lambda_2 |u - v|^2 \geqq 0$$

for all $u, v \in W$;

(jjj) there exist constants λ_3, c_3 and c_4 ($c_4 > 0$) such that ($p \geqq 1$) :

$$\operatorname{Re}(\beta(t)v - \beta(t)0, v) + \lambda_3 |v|^2 + c_3 |v|^p \geqq c_4 \|v\|_W^p$$

for all $v \in V \cap W$;

(jv) the map $u(.) \to \beta(.)$ sends bounded sequences of $L^p(0, T; W)$ into bounded sets of $L^{p'}(0, T; W')$, and it is weakly continuous when restricted to lines of $L^p(0, T; W)$.

THEOREM 2.2. — *Given $u_0 \in H$ and*

$$f \in L^1(0, T; H) + L^2(0, T; V') + L^{p'}(0, T; W').$$

Under the above hypotheses, there exists a unique function u such that

$$u \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^p(0, T; W), \tag{2.17}$$

$$u' \in L^1(0, T; H) + L^2(0, T; V') + L^{p'}(0, T; W'), \tag{2.18}$$

$$u(0) = u_0; \tag{2.19}$$

$$u'(t) + A(t)u(t) + \beta(t)u(t) = f(t) \quad \text{a. e.} \tag{2.20}$$

REMARKS :

(i) We could just as well consider a finite sum of operators like β, with different p's and different W's. More generally, we could consider an abstract space F, $F \subset L^{\infty}(0, T; H)$ such that β maps F into its dual F' and is " coercive " relative to the norm of F.

(ii) This theorem is formulated in such a way that it can be easily applied to partial differential equations in a fashion analogous to that of section 2.6.

(iii) We may assume, in the proof of the theorem, that $\beta(t)\,0 = 0$. For, in the contrary case, we may simply replace $\beta(t)\,v$ by $\beta(t)\,v - \beta(t)\,0$ and $f(t)$ by $f(t) - \beta(t)\,0$.

(iv) We may also assume, in he proof, that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Otherwise, we could just as well consider the analogous problem with $u(t)$ replaced by $\exp(kt)\,u(t) = w(t)$, $A(t)$ replaced by $A(t) + \frac{k}{2}$, $\beta(t)\,v$ replaced by $e^{-kt}\beta(t)\,(e^{kt}v) + \frac{k}{2}v$, and $f(t)$ replaced by $\exp(-kt)\,f(t)$. Then we would choose k sufficiently large, and the solution of the latter problem would immediately imply the solution of the original one. Therefore, we assume from now on that

$$\text{(2.21)} \qquad \operatorname{Re} a(t; v, v) \geq c_2 \| v \|^2 \qquad (v \in V);$$

$$\text{(2.22)} \qquad \operatorname{Re}(\beta(t)\,u - \beta(t)\,v,\ u - v) \geq 0 \qquad (u, v \in W);$$

$$\text{(2.23)} \qquad \operatorname{Re}(\beta(t)\,v, v) + c_3 | v |^p \geq c_4 \| v \|_W^p \qquad (v \in V \cap W).$$

(v) We show as in LIONS [10] that if u is any function satisfying (2.17) and (2.18), then it is equal a. e. to a continuous function from $[0, T]$ to H [so that, in particular, condition (2.19) makes sense] and that the integration by parts formula $(0 \leq a \leq b \leq T)$:

$$\int_a^b \{ (u', v) + (u, v') \}\, dt = (u(b), v(b)) - (u(a), v(a))$$

is valid for all functions v satisfying the same conditions as u.

PROOF OF UNIQUENESS. — Let u and v be solutions of the respective equations

$$u' + A(t)\,u + \beta(t)\,u = f, \qquad v' + A(t)\,v + \beta(t)\,v = g,$$

where u and f satisfy the conditions of Theorem 2.2 and v and g satisfy the corresponding conditions. Let $w = u - v$. Subtracting the equations for u and for v, and taking scalar products with $w(t)$, we obtain

$$(w'(t), w(t)) + (A(t)\,w(t), w(t)) + (\beta(t)\,u(t) - \beta(t)\,v(t), w(t)) = (f(t) - g(t), w(t)) \quad \text{a. e.}$$

By (2.21) and (2.22), this implies that

$$(2.24)\qquad \frac{d}{dt}|w(t)|^2 + c_5\,\|w(t)\|^2 \leq 2\,\mathrm{Re}(f(t)-g(t),\, w(t)) \quad \text{a. e.}$$

If $f=g$ and $u(0)=v(0)$, $|w(t)|^2$ is then a non-increasing function of t; since $w(0)=0$, $w=0$. This proves uniqueness.

If $f-g \in L^1(0, T; H) + L^2(0, T; V)$, then (2.24) implies the following continuity of the solution with respect to the data (*cf.* section 2.5) :

$$|w(t)|^2 + \int_0^t \|w(\sigma)\|_V^2\, d\sigma$$
$$\leq c_6 \left\{ |w(0)|^2 + \left[\int_0^T |f-g|\, d\sigma\right]^2 + \int_0^T \|f-g\|_{V'}^2\, d\sigma \right\}.$$

PROOF OF EXISTENCE. — Let $\{w_1, w_2, \ldots\}$ be a " basis " of $V \cap W$. We define u_{0m} as a finite linear combination of $w_1, \ldots, w_m$ such that $u_{0m} \to u_0$ in H (strongly). We define the " approximate solution " $u_m(t) = \sum g_{im}(t)\, w_0$ as the solution of

$$(2.25)\qquad \begin{cases} (u'_m(t), w_j) + (A(t)\,u_m(t), w_j) + (\beta(t)\,u_m(t), w_j) = (f(t), w_j) \\ \qquad (j = 1, \ldots, m), \\ \qquad u_m(0) = u_{0m}. \end{cases}$$

As usual, we obtain :

$$(2.26)\qquad |u_m(t)|^2 + 2\,\mathrm{Re}\int_0^t (A(\sigma)\,u_m(\sigma), u_m(\sigma))\, d\sigma$$
$$+ 2\,\mathrm{Re}\int_0^t (\beta(\sigma)\,u_m(\sigma), u_m(\sigma))\, d\sigma$$
$$= |u_{0m}|^2 + 2\,\mathrm{Re}\int_0^t (f(\sigma), u_m(\sigma))\, d\sigma.$$

Let us write $f = f_1 + f^2 + f_3$, where

$$f_1 \in L^1(0, T; H), \qquad f_2 \in L^2(0, T; V'), \qquad f_3 \in L^{p'}(0, T; W');$$

and write $\|f_k\|$ for the norm of f_k in the space to which it belongs ($k = 1, 2, 3$). Then, in analogy wuth the procedure in section 2.3, we see that there exists, for any $\delta > 0$, a constant c_7 (depending on δ) such that

$$\left|\int_0^t (f_1, u_m)\, d\sigma\right| \leq \delta \sup_{0 \leq \sigma \leq t} |u_m(\sigma)|^2 + c_7\,\|f_1\|^2,$$

$$\left|\int_0^t (f_2, u_m)\, d\sigma\right| \leq \delta \int_0^t \|u_m(\sigma)\|_V^2\, d\sigma + c_7\,\|f_2\|^2,$$

and

$$\left| \int_0^t (f_3, u_m)\, d\sigma \right| \leqq \delta \operatorname{Re} \int_0^t (\beta(\sigma)\, u_m(\sigma),\ u_m(\sigma))\, d\sigma$$
$$+ \delta \sup_{0 \leqq \sigma \leqq t} |u_m(\sigma)|^2 + c_7 \{ \| f_3 \|^{p'} + \| f_3 \|^2 \}.$$

[In the last estimate, we have used (2.23).] These estimates with δ chosen sufficiently small, together with (2.21), yield the inequality :

$$| u_m(t) |^2 + \int_0^t \| u_m(\sigma) \|_V^2\, d\sigma + \operatorname{Re} \int_0^t (\beta(\sigma)\, u_m(\sigma),\ u_m(\sigma))\, d\sigma$$
$$\leqq c_8 \{ | u_{0m} |^2 + \| f_1 \|^2 + \| f_2 \|^2 + \| f_3 \|^2 + \| f_3 \|^{p'} \}.$$

Using (2.23), we conclude that :

$$\text{(2.27)} \quad \begin{cases} u_m \text{ remains in a bounded set of} \\ \qquad L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^p(0, T; W). \end{cases}$$

Therefore, we can extract from $\{ u_m \}$ a subsequence $\{ u_\nu \}$ such that :

$$\text{(2.28)} \quad \begin{cases} u_\nu \to u & \text{in the weak topology of } L^2(0, T;\ V), \text{ and of} \\ L^p(0, T;\ W) & \text{and in the weak-star topology of } L^\infty(0, T;\ H); \\ \beta u_\nu \to g & \text{in the weak topology of } L^{p'}(0, T;\ W'); \\ u_\nu(T) \to \varkappa & \text{in the weak topology of } H. \end{cases}$$

By the same technique as in section 2.3, we show that

$$\text{(2.29)} \qquad u' + A(t)\, u + g = f,$$

that $u' \in L^1(0, T;\ H) + L^2(0, T;\ V') + L^{p'}(0, T;\ W')$, and that $u(0) = u_0$, $u(T) = \varkappa$. By remark (iv), we may assume that u is continuous on $[0, T]$ with values in H. It remains to prove that $g = \beta(u)$.

The technique is as before, but with some simplifications. Taking $m = \nu$ and $t = T$ in (2.26) and taking the limit inferior as $\nu \to \infty$ of both sides, we get

$$\text{(2.30)} \qquad | u(T) |^2 + 2 \operatorname{Re} \int_0^T (A(\sigma)\, u(\sigma),\ u\ (d\sigma))\sigma$$
$$+ \liminf_{\nu \to \infty} 2 \operatorname{Re} \int_0^T (\beta(\sigma)\, u_\nu(\sigma),\ u_\nu(\sigma))\, d\sigma$$
$$\leqq | u_0 |^2 + 2 \operatorname{Re} \int_0^T (f(\sigma),\ u(\sigma))\, d\sigma.$$

On the other hand, by remark (v),

$$2\,\mathrm{Re}\int_0^T (u'(\sigma),\, u(\sigma))\, d\sigma = |\, u(T)\,|^2 - |\, u_0\,|^2.$$

Therefore, if we take inner products of equation (2.29) with u and integrate, we obtain

$$|\, u(T)\,|^2 - |\, u_0\,|^2 + 2\,\mathrm{Re}\int_0^T (A(\sigma)\, u(\sigma),\, u(\sigma))\, d\sigma + 2\,\mathrm{Re}\int_0^T (g(\sigma),\, u(\sigma))\, d\sigma$$
$$= 2\,\mathrm{Re}\int_0^T (f(\sigma),\, u(\sigma))\, d\sigma.$$

Comparing this with (2.30), we get

$$\mathrm{Re}\int_0^T (g(\sigma),\, u(\sigma))\, d\sigma \geqq \liminf_{\nu\to\infty} \mathrm{Re}\int_0^T (\beta(\sigma)\, u_\nu(\sigma),\, u_\nu(\sigma))\, d\sigma.$$

Letting φ be an arbitrary element of $L^p(0, T; W)$, we deduce using (2.28) as in section 2.3, that

$$\mathrm{Re}\int_0^T (g(\sigma) - \beta(\sigma)\,\varphi(\sigma),\, u(\sigma) - \varphi(\sigma))\, d\sigma \geqq 0 \qquad (t = t_k).$$

By the same device as before, this implies that $g = \beta u$. This completes the proof of Theorem 2.2.

Added in Proof.

1° The end of the existence Proofs of Theorems 2.1 and 2.2 consists in showing that if $g_\nu \to g$ weakly in $L^p(0, t; W)$ and $\beta g_\nu \to \psi$ weakly in $L^{p'}(0, t; W')$, and

$$(1) \qquad \mathrm{Re}\langle \psi,\, g\rangle \geqq \liminf \mathrm{Re}\langle \beta g_\nu,\, g_\nu\rangle.$$

[where $\langle \psi, g\rangle = \int_0^t \langle \psi(\sigma), g(\sigma)\rangle\, d\sigma$, $0 \leqq t \leqq T$], then $\psi = \beta g$ a. e. in $(0, t)$.

As shown to us by E. de Giorgi, in case for instance

$$(2) \qquad \beta g = |\, g\,|^{\rho-1} g$$

we have

$$(3) \qquad \langle \beta g, g\rangle = \|\, \beta g\,\|_{L^{p'}(0,\, t;\, W')}\, \|\, g\,\|_{L^p(0,\, t;\, W)}$$

and then

$$\mathrm{Re}\langle \psi,\, g\rangle \geqq \|\, \psi\,\|_{L^{p'}(0,\, t;\, W')}\, \|\, g\,\|_{L^p(0,\, t;\, W)}.$$

Therefore these are all equalities and either $\liminf \| \beta g_\nu \| = \| \psi \| = 0$ [norms in $L^{p'}(0, t; W')$, in which case $g = 0$, either $\liminf \| g_\nu \| = \| g \|$ [norms in $L^p(0, t; W)$] and then, thanks to the uniform convexity of $L^p(0, t; W)$, $g_\nu \to g$ strongly and the result follows.

2° For the equations

$$(4) \qquad -\Delta u + u'' + \beta(u') = f.$$

a generalization of our hypotheses on β is made in G. Andreassi, G. Torelli (to appear).

3° For (4), with β given by (2), the existence of *periodic* solutions is proved, by G. Prodi (to appear).

4° For (4), with β given by (2), $\rho = 2$, the existence of *almost periodic*, solutions is proved, if $n \leqq 5$, by G. Prouse (to appear).

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