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Convergence of the spectral Galerkin method for the stochastic reaction–diffusion–advection equation

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ABSTRACT

We study the convergence of the spectral Galerkin method in solving the stochastic reaction–diffusion–advection equation under different Lipschitz conditions of the reaction function f. When f is globally (locally) Lipschitz continuous, we prove that the spectral Galerkin approximation strongly (weakly) converges to the mild solution of the stochastic reaction–diffusion–advection equation, and the rate of convergence in H_r-norm is $(\frac{1}{2}-r)^-$, for any $r \in [0, \frac{1}{2})$ $(r \in (\frac{1}{2}-\frac{1}{2d}, \frac{1}{2}))$. The convergence analysis in the local Lipschitz case is challenging, especially in the presence of an advection term. We propose a new approach based on the truncation techniques, which can be easily applied to study other stochastic partial differential equations. Numerical simulations are also provided to study the convergence of Galerkin approximations.

1. Introduction

Consider the one-dimensional stochastic reaction–diffusion–advection equation with additive noise:

$$du - \left(Au - f(u) + \frac{\partial g(u)}{\partial x}\right)dt = dw, \qquad x \in (0,1), \quad t > 0, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in [0,1], \tag{1.3}$$

where the operator $A : D(A) \mapsto L_2(0,1)$ is defined as

$$Au := \partial_x^2 u, \tag{1.4}$$

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with its domain $D(A) := \{v \in H_2(0,1) \mid v(0) = v(1) = 0\}$. w represents a cylindrical Wiener process on $L_2(0,1)$, f and g describe the reaction and advection, respectively. The stochastic equation (1.1) covers many stochastic evolution equations and has various applications. When $g(u) = 0$, it reduces to the stochastic reaction diffusion equation; additionally if $f(u) = u^3 - u$, (1.1) is the stochastic Allen–Cahn equation which is often used in modeling phase field problems [7,14]. If $f(u) = 0$, (1.1) becomes the stochastic diffusion advection equation; furthermore, if $g(u) = u^2/2$, it is the stochastic Burgers' equation [8,3,4]. When $f(u) = g(u) = 0$, (1.1) reduces to the stochastic heat equation [9,13,5]. The existence and uniqueness of the solutions to the stochastic reaction–diffusion–advection equation (1.1)–(1.3) can be found in the literature [16] and references therein.

In this paper, we study the convergence of the spectral Galerkin method in solving the stochastic reaction–diffusion–advection equation (1.1)–(1.3). Recently, there has been growing interest in numerical approximations to the Cauchy problems associated to (1.1)–(1.3), and special attention has been given to the spatial discretization [17,20,1,2,6,10,12,14]. For example, Liu proposes a Galerkin approximation to the stochastic Allen–Cahn equation and proves that for any $r \in [0, \frac{1}{2})$, the rate of convergence in H_r-norm is $(\frac{1}{2} - r)^-$ [14]. In [1], Alabert and Gyöngy introduce a finite difference scheme for the stochastic Burgers' equation, and they show that in L_2-norm the convergence rate of this scheme is $\frac{1}{2}^-$. The same convergence rate is also obtained in [12] when the spectral collocation method is used. Later, Blömker and Jentzen present a Galerkin approximation to the stochastic Burgers' equation and prove that the rate of convergence in L_∞-norm is $\frac{1}{2}^-$ [2]. To the best of our knowledge, no numerical study has yet been reported on the stochastic reaction–diffusion–advection equation of the general form in (1.1)–(1.3). Here, we propose a spectral Galerkin method and study its rate of convergence in solving (1.1)–(1.3). We remark that due to the presence of both f and g terms, the approaches in [14,2] for studying the convergence of Galerkin methods fail to be directly applied in our case, and thus we will introduce a new approach for convergence analysis.

The main results of this paper can be summarized as follows. With the assumption that g' is bounded and globally Lipschitz continuous, in the first part we prove that when f is globally Lipschitz continuous, the Galerkin approximation strongly converges to the mild solution of (1.1)–(1.3); the rate of convergence in H_r-norm is $(\frac{1}{2} - r)^-$, for any $r \in [0, \frac{1}{2})$. In the second part, we consider the case where f is locally Lipschitz continuous and prove that the Galerkin approximation converges in probability to the mild solution of (1.1)–(1.3). The rate of convergence is $(\frac{1}{2} - r)^-$ in H_r-norm, for any $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$, where d is the degree of the polynomial function f (see Assumption 2'). Numerical experiments are also presented to study the convergence of the Galerkin approximation.

One main novelty of this paper is to introduce a new approach to study the convergence of the finite dimensional Galerkin approximation, when both reaction and advection terms are involved and additionally the reaction term f is locally Lipschitz continuous. In the literature, different approaches are used to study the convergence of Galerkin approximations to the stochastic Allen–Cahn equation (i.e., with only local Lipschitz function f) [14] and stochastic Burgers' equation (i.e., with only advection term) [1,12,2]. On the one hand, due to the appearance of advection term g, the approach used in convergence analysis for the stochastic Allen–Cahn equation [14] fails in our case. On the other hand, the convergence of the Galerkin approximation to the stochastic Burgers' equation is studied under the L_2 or L_∞-topology [1,12,2]. But, with the local Lipschitz function f in (1.1), these topologies do not work for (1.1)–(1.3) and consequently, the methods for convergence analysis in [1,12,2] can not be directly applied in our study. Motivated by the studies in [15,18], we propose a new approach to analyze the convergence of the spectral Galerkin method. This approach is based on a truncated version of (1.1)–(1.3) where both the reaction and advection terms have better regularity than their original form.

The paper is organized as follows. In Section 2, we first introduce our mathematical setting and assumptions and present some preliminary results. Then, we introduce the finite dimensional spectral Galerkin approximation to the solution of (1.1)–(1.3). The convergence of Galerkin approximation is studied in Section 3 when the function f is globally Lipschitz continuous and in Section 4 when f is locally Lipschitz continuous. Numerical examples are also presented to study the convergence of Galerkin approximation. In Section 5, we make some concluding remarks.

2. Assumptions, preliminaries, and Galerkin approximation

In Section 2.1, we introduce the main mathematical setting and assumptions. Some preliminary results on the semigroup e^{tA} and on the solution of (1.1)–(1.3) are presented in Section 2.2. In Section 2.3, we introduce the spectral Galerkin approximation to the solution of the stochastic reaction–diffusion–advection equation (1.1)–(1.3).

2.1. Mathematical setting and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Denote $L_p(0,1)$ as the Lebesgue space of real functions on $(0,1)$ with norm $\|\cdot\|_p$. Denote $H_r(0,1)$ as the Sobolev space, i.e., a subspace of $L_2(0,1)$, with norm $\|\cdot\|_{H_r}$, and let $H_r^0(0,1) = \{u \in H_r(0,1) \,|\, u(0) = u(1) = 0\}$.

Choose $\{\phi_k(x) = \sqrt{2}\sin(k\pi x),\ k \in \mathbb{N}\}$ to be an orthonormal basis of the Hilbert space $L_2(0,1)$. For the linear operator A in (1.4), there is

$$A\phi_k(x) = \lambda_k \phi_k(x), \qquad \text{with} \quad \lambda_k = -\left(k\pi\right)^2,$$

i.e., $\{(\lambda_k, \phi_k),\ k \in \mathbb{N}\}$ are eigenvalues and eigenfunctions of A. Then, we have

$$Av = \sum_{k=1}^{\infty} \lambda_k \langle v, \phi_k\rangle \phi_k(x), \qquad v \in D(A), \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product in the Hilbert space. Define $(-A)^s$ as the interpolation spaces of the operator $(-A)$ for $s \in \mathbb{R}$, with its domain

$$D((-A)^s) = \left\{ v = \sum_{k=1}^{\infty} \langle v, \phi_k\rangle \phi_k \,\middle|\, \|v\|^2_{D((-A)^s)} := \|(-A)^s v\|_2^2 = \sum_{k=1}^{\infty} (-\lambda_k)^{2s} \langle v, \phi_k\rangle^2 < +\infty \right\}.$$

Similarly, for any $v \in D((-A)^s)$, we have

$$(-A)^s v = \sum_{k=1}^{\infty} (-\lambda_k)^s \langle v, \phi_k\rangle \phi_k(x).$$

For $v \in H_r(0,1)$, the norm

$$\|v\|^2_{H_r} := \|v\|^2_{D((-A)^{r/2})} + \|v\|_2^2.$$

Denote $\{\beta_k(t),\ k \in \mathbb{N}\}$ as a sequence of independent Brownian motions in the space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, the cylindrical Wiener process w can be written as

$$w = \sum_{k=1}^{\infty} \beta_k(t) \phi_k(x). \tag{2.2}$$

In the following, we make some assumptions on the initial condition u_0 and nonlinear terms f and g:

Assumption 1 (Initial condition). The initial condition $u_0(x) \in H_1^0(0,1)$.
Assumption 2 (Nonlinearity f). $f : \mathbb{R} \mapsto \mathbb{R}$ is globally Lipschitz continuous, i.e., for any $x, y \in \mathbb{R}$, $\exists\, K_1 > 0$, s.t. $|f(x) - f(y)| \le K_1|x - y|$.
Assumption 2' (Nonlinearity f). $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous. Furthermore, it is a polynomial of degree $d = 2n + 1$ (for $n \in \mathbb{N}^+$) with positive leading order coefficient.
Assumption 3 (Nonlinearity g). $g' : \mathbb{R} \mapsto \mathbb{R}$, denoting as the derivative of function g, is bounded and globally Lipschitz continuous, i.e., for any $x, y \in \mathbb{R}$, $\exists\, K_2, K_3 > 0$, s.t. $|g'(x)| \le K_2$ and $|g'(x)-g'(y)| \le K_3|x-y|$.

Assumptions 2 and 2' will be used in Sections 3 and 4, respectively, while Assumptions 1 and 3 will be fulfilled throughout the paper.

2.2. Preliminaries

Denote e^{tA} (for $t \ge 0$) as a semigroup generated by the operator A on $L_2(0,1)$. It is well known that e^{tA} is an analytical semigroup. The smoothing properties of the semigroup e^{tA} can be summarized in the following lemma [19]:

Lemma 2.1. *Let $1 \le m \le n$. For any $t > 0$ and $v \in L_m(0,1)$, there exists a constant $0 < C < \infty$, such that*

$$\|e^{tA}v\|_n \le C\, t^{(1/n-1/m)}\|v\|_m, \tag{2.3}$$

$$\|(-A)^\gamma e^{tA}v\|_m \le C\, t^{-\gamma}\|v\|_m, \qquad \textit{for} \ \ \gamma \ge 0. \tag{2.4}$$

The detailed proof of Lemma 2.1 can be found in [19, p. 25]. The mild solution of (1.1)–(1.3) can be written as:

$$u(t) = e^{tA}u_0 - \int_0^t e^{(t-s)A}f(u(s))\,ds + \int_0^t e^{(t-s)A}(-A)^{1/2}g(u(s))\,ds + \int_0^t e^{(t-s)A}\,dw(s), \qquad t \ge 0. \tag{2.5}$$

The well-posedness of the solution (2.5) can be found in the literature [16, Theorem 1.1 and Remark 1.2]. In addition, it shows in [16] that when Assumptions 1, 2' and 3 are satisfied, there is

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_{H_r}^p < \infty, \qquad 0 \le r < 1/2, \qquad p \ge 2. \tag{2.6}$$

This provides a regularity estimate for the mild solution (2.5). A simple application of (2.6) and the Chebyshev inequality yields the following proposition:

Proposition 2.1. *Suppose that Assumptions 1, 2' and 3 are satisfied. Then for any $T > 0$, the solution $u(x,t)$ of (1.1)–(1.3) satisfies that*

$$\lim_{K\to\infty} \mathbb{P}\Big(\sup_{t\in[0,T]} \|u(t)\|_{H_r} \ge K \Big) = 0, \qquad 0 \le r < 1/2. \tag{2.7}$$

Here we omit the proof of Proposition 2.1 for brevity. In contrast to (2.6), (2.7) provides a weaker regularity estimate on the solution $u(x,t)$ in (2.5). Proposition 2.1 will be used in Section 4 for the convergence analysis when the function f is locally Lipschitz continuous.

2.3. Galerkin approximation

For $N \in \mathbb{N}^+$, define a finite-dimensional space

$$Q_N = \operatorname{span}\{\phi_1, \phi_2, \ldots, \phi_N\}.$$

Let $P_N : L_2 \mapsto Q_N$ denote a projection operator, such that

$$P_N v = \sum_{k=1}^{N} \langle v, \phi_k \rangle \phi_k, \qquad \text{for} \quad v \in L_2(0,1).$$

It is easy to prove that $\|P_N\|_{L_2 \mapsto L_2}$ is uniformly bounded [14], i.e., there exists a constant $C > 0$, such that $\|P_N\|_{L_2 \mapsto L_2} \le C$, for any $N \in \mathbb{N}^+$.

Let $U_N \in Q_N$ denote a Galerkin approximation to the solution $u(x,t)$ of (1.1)–(1.3), which satisfies the following stochastic partial differential equation (PDE):

$$dU_N - \left[AU_N - P_N f(U_N) + P_N\big((-A)^{1/2} g(U_N)\big)\right] dt = P_N dw, \tag{2.8}$$

$$U_N(x,0) = P_N u_0(x), \qquad x \in [0,1]. \tag{2.9}$$

By the solvability of stochastic differential equations in finite-dimensional spaces [9,13], we can obtain the existence and uniqueness of the strong solution to (2.8)–(2.9), which has the following integral form:

$$U_N(t) = e^{tA} P_N u_0 - \int_0^t e^{(t-s)A} P_N f(U_N(s))\, ds + \int_0^t e^{(t-s)A} P_N \big((-A)^{1/2} g(U_N(s)\big)\, ds + \int_0^t e^{(t-s)A} P_N dw(s). \tag{2.10}$$

Remark 2.1. Usually, the Galerkin approximation $U_N(x,t)$ satisfying (2.8)–(2.9) is not equivalent to the projection of $u(x,t)$ on Q_N, i.e., $U_N(x,t) \neq P_N u(x,t)$. However, when both functions f and g are linear, $U_N(x,t) = P_N u(x,t)$.

In Sections 3 and 4, we will study the convergence of the Galerkin approximation $U_N(x,t)$ in (2.10) to the solution $u(x,t)$ in (2.5) under different Lipschitz conditions of function f.

3. The global Lipschitz case

In this section, we study the convergence of the N-dimensional Galerkin approximation (2.10) to the solution of the stochastic reaction–diffusion–advection equation (1.1)–(1.3), when f is globally Lipschitz continuous as stated in Assumption 2. We will start with the properties of the Wiener process w. Denote

$$W_A(t) := \int_0^t e^{(t-\tau)A}\, dw(\tau), \qquad t \ge 0.$$

It is easy to show that W_A is a Gaussian process, and it is mean-square continuous with values in $L_2(0,1)$. Furthermore, W_A has a version which is, a.s. for $\omega \in \Omega$, γ-Hölder continuous with respect to (x,t) for any $\gamma \in [0, 1/4)$ [5].

In the following lemma, we will prove that W_A is continuous in H_r-norm:

Lemma 3.1. *For $r \in [0, \frac{1}{2})$, $\|W_A(t)\|_{H_r}$ is continuous, a.s. for $\omega \in \Omega$.*

Proof. We start with proving that $(-A)^{r/2}W_A(x,t)$ is Hölder continuous in (x,t). From the definition of W_A and (2.2), we have

$$(-A)^{r/2}W_A(x,t) = \sum_{k=1}^{\infty} (-\lambda_k)^{r/2}\phi_k(x) \int_0^t e^{\lambda_k(t-\tau)}\beta_k(\tau),$$

and thus

$$(-A)^{r/2}\big[W_A(x,t) - W_A(y,t)\big] = \sum_{k=1}^{\infty} (-\lambda_k)^{r/2}\big(\phi_k(x) - \phi_k(y)\big) \int_0^t e^{\lambda_k(t-\tau)}\beta_k(\tau).$$

Using the definition of stochastic integral, we get

$$\begin{aligned}\mathbb{E}\big|(-A)^{r/2}\big[W_A(x,t) - W_A(y,t)\big]\big|^2 &= \sum_{k=1}^{\infty} (-\lambda_k)^{r}\big|\phi_k(x) - \phi_k(y)\big|^2 \int_0^t e^{2\lambda_k(t-\tau)}d\tau \\ &= \sum_{k=1}^{\infty} \frac{1}{2}(-\lambda_k)^{r-1}\big|\phi_k(x) - \phi_k(y)\big|^2. \qquad (3.1)\end{aligned}$$

Similarly, we have

$$\begin{aligned}&\mathbb{E}\big|(-A)^{r/2}\big[W_A(x,t) - W_A(x,s)\big]\big|^2 \\ &= \sum_{k=1}^{\infty}(-\lambda_k)^{r}\big|\phi_k(x)\big|^2\Big[\int_s^t e^{2\lambda_k(t-\tau)}\,d\tau + \int_0^s \Big(e^{\lambda_k(t-\tau)} - e^{\lambda_k(s-\tau)}\Big)^2\,d\tau\Big] \\ &= \sum_{k=1}^{\infty}\frac{1}{2}(-\lambda_k)^{r-1}|\phi_k(x)|^2\Big[2\big(1 - e^{2\lambda_k(t-s)}\big) + \big(e^{\lambda_k t} - e^{\lambda_k s}\big)^2\Big]. \qquad (3.2)\end{aligned}$$

For any $\gamma \in [0,1]$, there exist constants $0 < c_\gamma, d_\gamma < \infty$, such that

$$|\phi_k(x) - \phi_k(y)| \le c_\gamma(-\lambda_k)^{\gamma/2}|x-y|^\gamma, \qquad |e^{-x} - e^{-y}| \le d_\gamma|x-y|^{2\gamma}, \qquad (3.3)$$

for $x, y \in \mathbb{R}$. From (3.1)–(3.3), we obtain

$$\mathbb{E}\big|(-A)^{r/2}[W_A(x,t) - W_A(y,s)]\big|^2 \le C\Big[|x-y|^{2\gamma}\sum_{k=1}^{\infty}(-\lambda_k)^{r-1+\gamma} + |t-s|^{2\gamma}\sum_{k=1}^{\infty}(-\lambda_k)^{r-1+2\gamma}\Big], \qquad (3.4)$$

for any $r \in [0, \frac{1}{2}]$ and $\gamma \in [0,1]$.

Let's choose $\gamma \in (0, \frac{1}{4} - \frac{r}{2})$. Since $\lambda_k = -k^2\pi^2$ and then $\sum_{k=1}^{\infty}(-\lambda_k)^p < \infty$, for any $p < -\frac{1}{2}$, we obtain from (3.4) that

$$\mathbb{E}\big|(-A)^{r/2}[W_A(x,t) - W_A(y,s)]\big|^2 \le C\,(|x-y|^2 + |t-s|^2)^\gamma.$$

Noticing that $(-A)^{r/2}[W_A(x,t) - W_A(y,s)]$ is a Gaussian random variable, we have

$$\mathbb{E}\left|(-A)^{r/2}[W_A(x,t) - W_A(y,s)]\right|^{2m} \le C\left(|x-y|^2 + |t-s|^2\right)^{m\gamma},$$

for any $m \in \mathbb{N}^+$. Using the Kolmogorov test, we deduce that $(-A)^{r/2}W_A(x,t)$ is Hölder continuous with respect to (x,t) of exponent $\gamma - \frac{2}{m}$. As m can be arbitrary large, $(-A)^{r/2}W_A(x,t)$ is Hölder continuous of exponent γ, for any $\gamma \in (0, \frac{1}{4} - \frac{r}{2})$. Hence, $(-A)^{r/2}W_A(x,t)$ is bounded on the domain of (x,t). Using the bounded convergence theorem, it is easy to prove that $\|(-A)^{r/2}W_A(t)\|_2$ is continuous with respect to t. □

The convergence of the spectral Galerkin approximation to $W_A(t)$ can be summarized in the following proposition:

Proposition 3.1. *Let $0 \le r < \frac{1}{2}$ and $1 \le p < \infty$. For any $\alpha \in (0, 1-2r)$, there is*

$$\mathbb{E} \sup_{t\in[0,T]} \|(I - P_N)W_A(t)\|^p_{H_r} \le \frac{C}{N^{\alpha p/2}}, \tag{3.5}$$

where C is a constant independent of N.

Proposition 3.1 shows that as $N \to \infty$, the spectral Galerkin approximation to $W_A(t)$ is strongly convergent. Moreover, its convergence rate in H_r-norm is $(\frac{1}{2} - r)^-$, for any $r \in [0, \frac{1}{2})$. The proof of Proposition 3.1 can be found in [14, p. 367], which is based on the regularity estimates on Gaussian process.

Next, we present our main theorem of this section:

Theorem 3.1. *Suppose that Assumptions 1, 2 and 3 are satisfied, and $U_N(x,t)$ is the spectral Galerkin approximation to the solution $u(x,t)$ of the stochastic problem (1.1)–(1.3). Let $0 \le r < \frac{1}{2}$ and $2 \le p < \infty$. Then for any $\alpha \in (0, 1-2r)$, there is*

$$\mathbb{E} \sup_{t\in[0,T]} \|u(t) - U_N(t)\|^p_{H_r} \le \frac{C}{N^{\alpha p/2}}, \tag{3.6}$$

where C is a constant independent of N.

Proof. Let $e_N(t) = u(t) - U_N(t)$. From (2.5) and (2.10), we obtain

$$\begin{aligned}
e_N(t) &= e^{tA}\left[u_0 - P_N u_0\right] + \int_0^t e^{(t-s)A}\left[(-A)^{1/2}g(u(s)) - P_N(-A)^{1/2}g(U_N(s))\right]ds \\
&\quad + \int_0^t e^{(t-s)A}\left[f(u(s)) - P_N f(U_N(s))\right]ds + \int_0^t e^{(t-s)A}(I - P_N)\,dw(s) \\
&= \underbrace{e^{tA}\left[u_0 - P_N u_0\right]}_{I_0} + \underbrace{\int_0^t e^{(t-s)A}P_N\left[(-A)^{1/2}g(u(s)) - (-A)^{1/2}g(U_N(s))\right]ds}_{I_{g_1}} \\
&\quad + \underbrace{\int_0^t e^{(t-s)A}(I - P_N)(-A)^{1/2}g(u(s))\,ds}_{I_{g_2}} + \underbrace{\int_0^t e^{(t-s)A}(I - P_N)f(u(s))\,ds}_{I_{f_2}}
\end{aligned}$$

$$+\underbrace{\int_0^t e^{(t-s)A} P_N\big[f(u(s)) - f(U_N(s))\big]ds}_{I_{f_1}} + \underbrace{\int_0^t e^{(t-s)A}\,(I - P_N)\;dw(s)}_{I_w}.$$

The estimate of I_w can be found in Proposition 3.1. In the following, we will focus on the estimates of the other terms. We will first prove (3.6) for the case of $p \geq 4$ and then extend it to $p \geq 2$. For the term I_0, since

$$e^{tA}u(x,t) = \sum_{k=1}^{\infty} e^{\lambda_k t}\,\langle u, \phi_k\rangle \phi_k(x),$$

and $\lambda_k = -(k\pi)^2$ for any $k \in \mathbb{N}$, we have

$$\begin{aligned}\big\|(-A)^{r/2} I_0(t)\big\|_2^p &= \Big\|\sum_{k=N+1}^{\infty} (-\lambda_k)^{r/2}\, e^{\lambda_k t}\,\langle u_0, \phi_k\rangle \phi_k\Big\|_2^p \\ &\leq \frac{e^{p\lambda_{N+1}t}}{(-\lambda_{N+1})^{p(1-r)/2}} \left(\sum_{k=N+1}^{\infty} (-\lambda_k)\,\langle u_0, \phi_k\rangle^2\right)^{p/2} \\ &= \frac{e^{p\lambda_{N+1}t}}{(-\lambda_{N+1})^{p(1-r)/2}} \big\|(-A)^{1/2}(u_0 - P_N u_0)\big\|_2^p,\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]} \big\|(-A)^{r/2} I_0(t)\big\|_2^p &\leq \frac{1}{(-\lambda_{N+1})^{p(1-r)/2}}\,\big\|(-A)^{1/2}(u_0 - P_N u_0)\big\|_2^p \\ &\leq \frac{C}{(N+1)^{p(1-r)}}\,\big\|u_0 - P_N u_0\big\|_{H_1}^p. \end{aligned} \tag{3.7}$$

Using the Minkowski inequality and the property of the semigroup e^{tA} in (2.4) with $m = 2$ and $\gamma = r/2$, we obtain

$$\begin{aligned}\big\|(-A)^{r/2} I_{f_1}(t)\big\|_2 &\leq \int_0^t \big\|e^{(t-s)A}(-A)^{r/2} P_N\big[f(u(s)) - f(U_N(s))\big]\big\|_2 ds \\ &\leq C\int_0^t \frac{1}{(t-s)^{r/2}} \big\|P_N\big[f(u(s)) - f(U_N(s))\big]\big\|_2 ds \\ &\leq C\int_0^t \frac{1}{(t-s)^{r/2}} \big\|f(u(s)) - f(U_N(s))\big\|_2 ds.\end{aligned}$$

Noticing that f satisfies the global Lipschitz condition in Assumption 2, we further obtain

$$\begin{aligned}\big\|(-A)^{r/2} I_{f_1}(t)\big\|_2 &\leq C\int_0^t \frac{1}{(t-s)^{r/2}} \big\|u(s) - U_N(s)\big\|_2 ds \\ &\leq C\Big(\int_0^t \big\|u(s) - U_N(s)\big\|_2^p ds\Big)^{1/p}\end{aligned}$$

$$\leq C \left(\int_0^t \left\| (-A)^{r/2} \left[u(s) - U_N(s) \right] \right\|_2^p ds \right)^{1/p},$$

where the Hölder inequality is used for $r/2 + 1/p < 1$ and the last inequality results from the Poincaré inequality. It immediately leads to

$$\begin{aligned} \mathbb{E} \sup_{t \in [0,T]} \left\| (-A)^{r/2} I_{f_1}(t) \right\|_2^p &\leq C\, \mathbb{E} \int_0^T \left\| (-A)^{r/2} \left[u(s) - U_N(s) \right] \right\|_2^p ds \\ &\leq C \int_0^T \mathbb{E} \sup_{\tau \in [0,s]} \left\| (-A)^{r/2} \left[u(\tau) - U_N(\tau) \right] \right\|_2^p ds. \end{aligned} \tag{3.8}$$

The estimates on the term I_{g_1} can be done following the similar lines as above. Using the Minkowski inequality and the property of the semigroup e^{tA} in (2.4) with $m = 2$ and $\gamma = (r+1)/2$, we get

$$\begin{aligned} \left\| (-A)^{r/2} I_{g_1}(t) \right\|_2 &\leq \int_0^t \left\| e^{(t-s)A} (-A)^{r/2} P_N \left[(-A)^{1/2} g(u(s)) - (-A)^{1/2} g(U_N(s)) \right] \right\|_2 ds \\ &\leq C \int_0^t \frac{1}{(t-s)^{(r+1)/2}} \left\| P_N \left[g(u(s)) - g(U_N(s)) \right] \right\|_2 ds \\ &\leq C \int_0^t \frac{1}{(t-s)^{(r+1)/2}} \left\| g(u(s)) - g(U_N(s)) \right\|_2 ds \\ &\leq C \int_0^t \frac{1}{(t-s)^{(r+1)/2}} \left\| u(s) - U_N(s) \right\|_2 ds, \end{aligned}$$

where the last inequality follows from the fact that g satisfies the globally Lipschitz continuous condition, which is implied by Assumption 3. Using the Hölder inequality for $r/2 + 1/p < 1/2$ and Poincaré inequality gives

$$\begin{aligned} \left\| (-A)^{r/2} I_{g_1}(t) \right\|_2^p &\leq C \int_0^t \left\| u(s) - U_N(s) \right\|_2^p ds \\ &\leq C \int_0^t \left\| (-A)^{r/2} \left[u(s) - U_N(s) \right] \right\|_2^p ds, \end{aligned}$$

which leads to

$$\mathbb{E} \sup_{t \in [0,T]} \left\| (-A)^{r/2} I_{g_1}(t) \right\|_2^p \leq C \int_0^T \mathbb{E} \sup_{\tau \in [0,s]} \left\| (-A)^{r/2} \left[u(\tau) - U_N(\tau) \right] \right\|_2^p ds. \tag{3.9}$$

To study I_{f_2} and I_{g_2}, we first consider the following estimates. For any $\theta \in (0,1)$, we use the property of the semigroup $e^{\theta t A}$ in (2.4) with $m = 2$ and $\gamma = r/2$ to obtain

$$\big\|(-A)^{r/2}e^{tA}(I-P_N)f(u(t))\big\|_2 = \big\|(-A)^{r/2}\, e^{\theta tA}\, e^{(1-\theta)tA}(I-P_N)f(u(t))\big\|_2$$
$$\leq C(\theta t)^{-r/2}\big\|e^{(1-\theta)tA}(I-P_N)f(u(t))\big\|_2.$$

Following the similar lines as those used in estimating I_0, it is easy to prove that

$$\big\|e^{(1-\theta)tA}(I-P_N)f(u(t))\big\|_2 \leq e^{\lambda_{N+1}(1-\theta)t}\|f(u(t))\|_2.$$

Hence, for any $\theta \in (0,1)$, there is

$$\big\|(-A)^{r/2}e^{tA}(I-P_N)f(u(t))\big\|_2 \leq C(\theta t)^{-r/2}e^{\lambda_{N+1}(1-\theta)t}\|f(u(t))\|_2. \tag{3.10}$$

Using the Minkowski inequality, the Hölder inequality, and (3.10), we get

$$\begin{aligned}\big\|(-A)^{r/2}I_{f_2}(t)\big\|_2 &\leq \int_0^t \big\|e^{(t-s)A}(-A)^{r/2}(I-P_N)f(u(s))\big\|_2\, ds \\ &\leq C\int_0^t \big[\theta(t-s)\big]^{-r/2}e^{\lambda_{N+1}(1-\theta)(t-s)}\|f(u(s))\|_2\, ds \\ &\leq C\left(\int_0^t (\theta s)^{-rq/2}e^{\lambda_{N+1}(1-\theta)sq}\, ds\right)^{1/q}\left(\int_0^t \|f(u(s))\|_2^p\, ds\right)^{1/p},\end{aligned}$$

where $\theta \in (0,1)$ and $1/p + 1/q = 1$. By the definition of the Gamma function, we further obtain

$$\begin{aligned}\|(-A)^{r/2}I_{f_2}(t)\|_2 &\leq C\,(-\lambda_{N+1})^{r/2-1/q}\left(\int_0^t \|f(u(s))\|_2^p\, ds\right)^{1/p} \\ &= \frac{C}{(N+1)^{2-2/p-r}}\left(\int_0^t \|f(u(s))\|_2^p\, ds\right)^{1/p},\end{aligned}$$

as the fact $r/2 + 1/p < 1$ with $p \geq 4$. From Assumption 2 that f is globally Lipschitz continuous, we have that there exists a constant $C > 0$, such that $|f(u)| \leq C(1 + |u|)$. Hence, we get

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|(-A)^{r/2}I_{f_2}(t)\|_2^p &\leq \frac{C}{(N+1)^{2p-2-pr}}\,\mathbb{E}\int_0^T \big(\|u(s)\|_2^p + 1\big)ds \\ &\leq \frac{C}{(N+1)^{2p-2-pr}}\left(\mathbb{E}\sup_{t\in[0,T]}\|(-A)^{r/2}u(t)\|_2^p + 1\right),\end{aligned}$$

where the last inequality results from the Poincaré inequality. By (2.6), we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|(-A)^{r/2}I_{f_2}(t)\|_2^p \leq \frac{C}{(N+1)^{2p-2-pr}}. \tag{3.11}$$

Similarly, using the Minkowski and Hölder inequalities and (3.10), we get

$$\|(-A)^{r/2}I_{g_2}(t)\|_2 \leq \int_0^t \big\|e^{(t-s)A}(-A)^{r/2}(I-P_N)\,(-A)^{1/2}g(u(s))\big\|_2\, ds$$

$$\leq C \int_0^t \big[\theta(t-s)\big]^{-(r+1)/2} e^{\lambda_{N+1}(1-\theta)(t-s)} \|g(u(s))\|_2 \, ds$$

$$\leq C \left(\int_0^t (\theta s)^{-(r+1)q/2} e^{\lambda_{N+1}(1-\theta)sq} \, ds \right)^{1/q} \left(\int_0^t \|g(u(s))\|_2^p \, ds \right)^{1/p}$$

$$\leq \frac{C}{(N+1)^{1-2/p-r}} \left(\int_0^t \|g(u(s))\|_2^p \, ds \right)^{1/p},$$

as $r/2 + 1/p < 1/2$ when $p \geq 4$. Noticing that the function g is globally Lipschitz continuous and following the same procedure in obtaining (3.11), we obtain

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \|(-A)^{r/2} I_{g_2}(t)\|_2^p &\leq \frac{C}{(N+1)^{p-2-pr}} \Big(\mathbb{E} \sup_{t\in[0,T]} \|(-A)^{r/2} u(t)\|_2^p + 1 \Big) \\ &\leq \frac{C}{(N+1)^{p-2-pr}}. \end{aligned} \tag{3.12}$$

Combining (3.5), (3.7), (3.8), (3.9), (3.11) and (3.12), we obtain

$$\begin{aligned} \mathbb{E} \sup_{t\in[0,T]} \big\|(-A)^{r/2}\big[u(t) - U_N(t)\big]\big\|_2^p \leq C\Big(&\frac{1}{(N+1)^{\min\{p(1-r),\, (p-2-pr),\, \alpha p/2\}}} \\ &+ \int_0^T \mathbb{E} \sup_{\tau\in[0,s]} \big\|(-A)^{r/2}\big[u(\tau) - U_N(\tau)\big]\big\|_2^p \, ds \Big), \end{aligned}$$

for $p \geq 4$, where the constant C depends on the initial condition u_0, but it is independent of N. It is easy to find that $\min\{p(1-r), (p-2-rp), \alpha p/2\} = \alpha p/2$, as $p \geq 4$, $0 \leq r < \frac{1}{2}$, and $0 < \alpha < 1 - 2r$. Therefore, using the Gronwall's inequality immediately yields (3.6) for $p \geq 4$.

It implies that for $p \geq 4$, the error e_N in norm $L_p(\Omega; L_\infty([0,T]; H_r(0,1)))$ is of order $O(N^{-\frac{\alpha}{2}})$. Using the Hölder inequality, we can obtain that for $p = 2$ or 3,

$$\Big(\mathbb{E} \sup_{t\in[0,T]} \|u(t) - U_N(t)\|_{H_r}^p \Big)^{1/p} \leq \Big(\mathbb{E} \sup_{t\in[0,T]} \|u(t) - U_N(t)\|_{H_r}^4 \Big)^{1/4} \leq \frac{C}{N^{\frac{\alpha}{2}}}, \tag{3.13}$$

that is, (3.6) is valid for any $p \geq 2$. □

Theorem 3.1 shows the strong convergence of the spectral Galerkin approximation $U_N(x,t)$ to the solution $u(x,t)$ of (1.1)–(1.3), as $N \to \infty$. Moreover, it gives the rate of convergence $(\frac{1}{2} - r)^-$ in H_r-norm, for any $r \in [0, \frac{1}{2})$. Even though the convergence rates in Proposition 3.1 and Theorem 3.1 are the same, the domain of p in Theorem 3.1 ($2 \leq p < \infty$) is more restrictive than that in Proposition 3.1 ($1 \leq p < \infty$), which is mainly caused by the terms f and g.

3.1. Numerical example

In this section, we numerically study the convergence of spectral Galerkin approximation by simulating the stochastic reaction–diffusion–advection equation (1.1)–(1.3). We choose $f(u) = \sin(u)$ and $g(u) = 1/(1+u^2)$, which satisfy Assumptions 2 and 3, respectively. Then, we consider the following stochastic evolution equation:

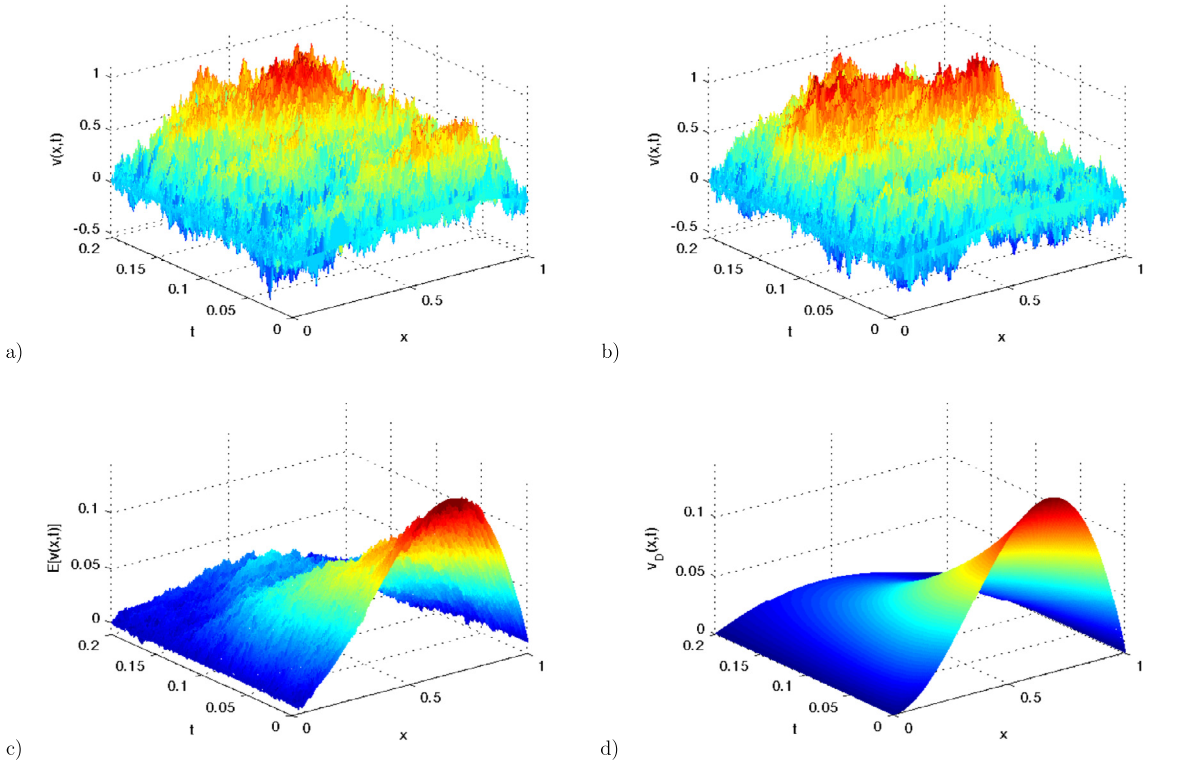

Fig. 1. a)–b). Numerical approximation of $u(x,t)$ from two independent realizations. c). The expected value $\mathbb{E}[u(x,t)]$ by the mean of 1000 independent realizations. d). The solution of the deterministic problem corresponding to (3.14)–(3.16).

$$du = \left(\partial_x^2 u - \sin(u) + \frac{\partial}{\partial x}\left(\frac{1}{1+u^2}\right)\right)dt + dw, \tag{3.14}$$

$$u(0,t) = u(1,t) = 0, \qquad t \in (0,T], \tag{3.15}$$

$$u(x,0) = x^2(1-x) \qquad x \in [0,1], \tag{3.16}$$

where w is a cylindrical Wiener process on $L_2([0,1];\mathbb{R})$. We discretize (3.14)–(3.16) by the spectral Galerkin approximation in space and by the exponential Euler scheme in time [6,10,11,2]. Here we focus on the spatial errors from the Galerkin approximations, and thus a very small time step $\tau = 10^{-4}$ is used in our simulations, so that the errors from temporal discretization can be neglected.

Choose an integer $M > 0$, and define the time sequence $t_n = m\tau$ for $0 \le m \le M$. For a given $N > 1$, we define the mesh size $h = 1/N$ and denote the grid points $x_k = kh$ for $0 \le k \le N$. Let $U_N(x,t)$ denote the Galerkin approximation to $u(x,t)$, and $U_{N,k}^m := U_N(x_k, t_m)$ represents the numerical approximation of U_N at the point (x_k, t_m). In Fig. 1, we present the Galerkin approximations of the solution $u(x,t)$ from two independent realizations (see Fig. 1 a–b), where $t \in [0, 0.2]$ and $N = 1024$. In addition, we compute the expected value $\mathbb{E}[u(x,t)]$ by the mean of 1000 independent realizations (see Fig. 1 c) and the numerical solutions of the deterministic problem corresponding to (3.14)–(3.16) (see Fig. 1 d). We see that due to the noise w, the solution of (3.14)–(3.16) varies from different realizations. However, their expected value $\mathbb{E}[u(x,t)]$ converges to the solution of the deterministic equation.

Next, we study the convergence of the Galerkin approximation $U_N(x,t)$. Let u_k^m denote the "exact" solution at the point (x_k, t_m), and it is computed by using a very fine mesh size $h = 1/2048$ and a small time step $\tau = 10^{-4}$. Then, the error is computed by

$$\mathrm{Error}_1 := \left[\mathbb{E} \sup_{0 \le m \le M} \left(\sum_{k=1}^{N-1} \left(1 + (-\lambda_k)^r\right) \left|\widehat{u}_k^m - \widehat{U}_{N,k}^m\right|^2 \right)^{p/2}\right]^{1/p}, \tag{3.17}$$

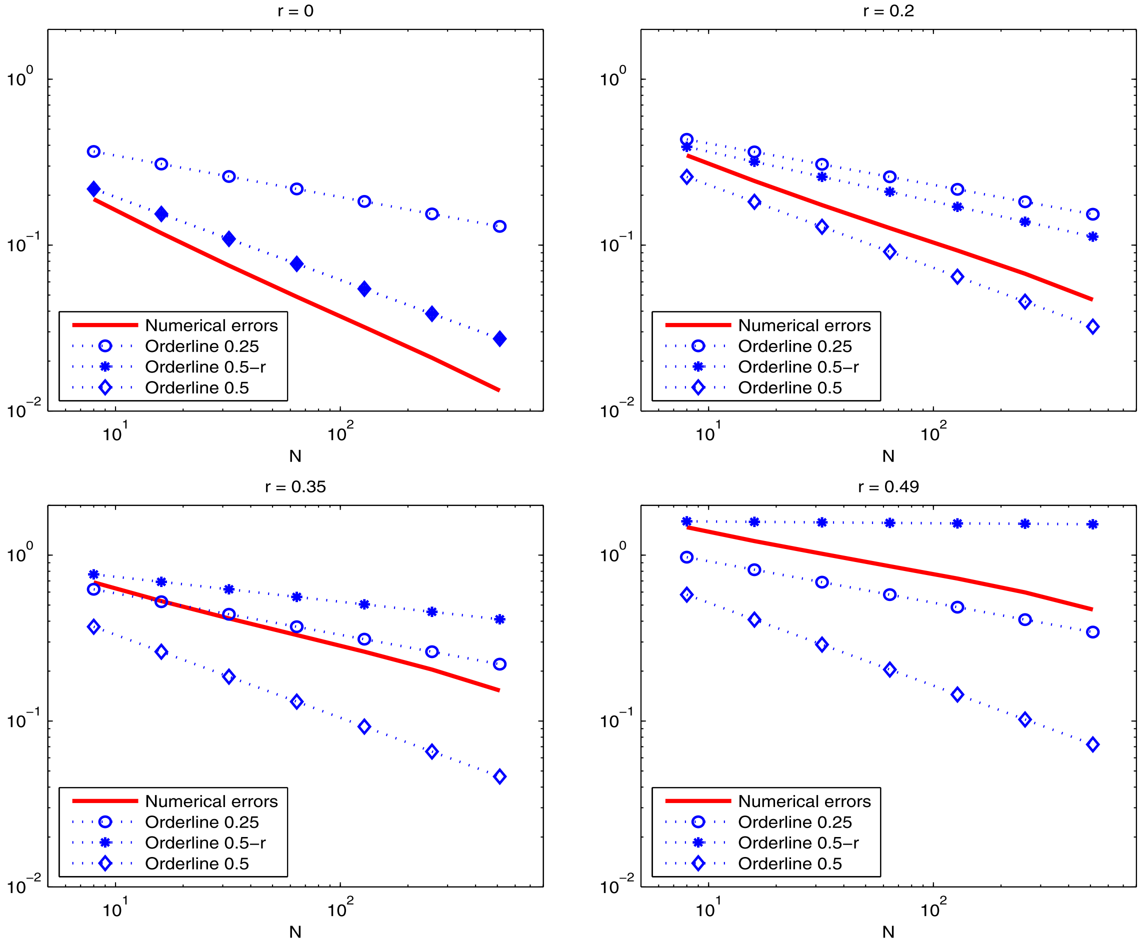

Fig. 2. Errors of the Galerkin approximation to the solution of (3.14)–(3.16), where $N = \{8, 16, 32, 64, 128, 256, 512\}$ and the errors are computed from (3.17).

where $\widehat{U}^m_{N,k}$ represents the k-th coefficient of the discrete sine transform of the vector $(U^m_{N,1}, \dots, U^m_{N,N-1})^T$. In Fig. 2, we present log–log plots of the errors for $r = 0, 0.2, 0.35,$ and 0.49, where $N = \{8, 16, 32, 64, 128, 256, 512\}$. We take $p = 5$ in (3.17), but our extensive simulations show that the convergence rate is the same for $p \geq 2$. For the sake of easy comparison, we also illustrate the order line with rates of 0.25, $0.5 - r$, and 0.5 in our plots. From Fig. 2, we see that for each r, the errors decreases as N increases, which implies that the Galerkin approximation converges to the solution of (3.14)–(3.16) as $N \to \infty$. Furthermore, the convergence rate in H_r-norm becomes larger as r becomes smaller, which corresponds to a larger slope in the plot. However, for each r, the convergence rate from numerical simulations is larger than the theoretical results in Theorem 3.1. For instance, when $r = 0$, the convergence rate is about 0.6. More numerical studies will be carried out in our future work to further explore this.

4. The local Lipschitz case

In this section, we study the convergence of the spectral Galerkin approximation $U_N(x,t)$ to the solution of (1.1)–(1.3), when the nonlinearity f is locally Lipschitz continuous, satisfying the conditions in Assumption 2'. Due to the lack of regularity, the local Lipschitz condition of f makes the convergence analysis much more challenging than that in Theorem 3.1. Here, we generalize the method used in [15,18] for studying the temporal convergence of the numerical approximations to stochastic PDEs and propose a new approach for spatial convergence analysis. The key idea here is to truncate the locally Lipschitz continuous function so as to improve its regularity properties. In the following, we will begin with introducing the truncated problem.

Let $r \in (0, \frac{1}{2})$. For any constant $R > 0$, we can define a nonnegative function $K_R \in C_0^\infty(\mathbb{R})$ satisfying that $|K_R| \le 1$, e.g.,

$$K_R(x) = \begin{cases} 1 & \text{if } \ x < R, \\ 0 & \text{if } \ x \ge R + 1. \end{cases}$$

For $u \in H_r(0,1)$ and f satisfying Assumption 2', we define

$$f_R(u) = K_R(\|u\|_{H_r}) f(u),$$

which can be viewed as a truncation of function f. It is easy to show that $|f_R(u)| \le |f(u)|$. Similarly, we define

$$g_R(u) = K_R(\|u\|_{H_r}) g(u),$$

for g satisfying Assumption 3. It is easy to prove that there exist constants $C_1 := C_1(R) > 0$ and $C_2 := C_2(R) > 0$, such that for any $u, v \in H_r(0,1)$,

$$\begin{aligned} \|f_R(u) - f_R(v)\|_2 &\le C_1 \|u - v\|_{H_r}, \\ \|g_R(u) - g_R(v)\|_2 &\le C_2 \|u - v\|_{H_r}. \end{aligned}$$

The above inequalities provide the Lipschitz conditions for the truncated functions f_R and g_R in L_2-norm. Now, we consider the following stochastic PDE:

$$du_R - \left(Au_R - f_R(u_R) + \frac{\partial g_R(u_R)}{\partial x} \right) dt = dw, \qquad x \in (0,1), \quad t > 0, \tag{4.1}$$

$$u_R(0,t) = u_R(1,t) = 0, \qquad t \ge 0, \tag{4.2}$$

$$u_R(x,0) = u_0(x), \qquad x \in [0,1], \tag{4.3}$$

where the operator A and the Wiener process w are the same as those defined in the problem (1.1)–(1.3). In fact, (4.1)–(4.3) can be viewed as a truncated version of (1.1)–(1.3). To distinguish it from u satisfying (1.1)–(1.3), we use the notation u_R in (4.1)–(4.3). From [16], we get that (4.1)–(4.3) has a unique solution of the following form:

$$u_R(t) = e^{tA} u_0 - \int_0^t e^{(t-s)A} f_R(u_R(s))\, ds + \int_0^t e^{(t-s)A} (-A)^{1/2} g_R(u_R(s))\, ds + \int_0^t e^{(t-s)A}\, dw(s), \tag{4.4}$$

for $t \ge 0$. Let $U_{R,N} \in Q_N$ denote the Galerkin approximation to the solution (4.4), which satisfies

$$dU_{R,N} - \left[AU_{R,N} - P_N f_R(U_{R,N}) + P_N\big((-A)^{1/2} g_R(U_{R,N})\big) \right] dt = P_N dw, \tag{4.5}$$

$$U_{R,N}(x,0) = P_N u_0(x). \tag{4.6}$$

Following a similar discussion as in Section 2.3, we obtain that (4.5)–(4.6) has the unique solution as

$$U_{R,N}(t) = e^{tA}P_N u_0 - \int_0^t e^{(t-s)A}P_N\left[f_R(U_{R,N}(s))\right]ds + \int_0^t e^{(t-s)A}P_N\left[(-A)^{1/2}g_R(U_{R,N}(s))\right]ds + \int_0^t e^{(t-s)A}P_N\,dw(s), \qquad t \geq 0. \tag{4.7}$$

In the following proposition, we will show that the Galerkin approximation $U_{R,N}(x,t)$ in (4.7) strongly converges to $u_R(x,t)$ in (4.4).

Proposition 4.1. *Suppose that Assumptions 1, 2' and 3 are satisfied, and $U_{R,N}(x,t)$ is the spectral Galerkin approximation to the solution $u_R(x,t)$ of the truncated problem (4.1)–(4.3). Let $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$ with d being the degree of polynomial function f in Assumption 2'. Then for any $R > 0$, $p \geq 4$, and $\alpha \in (0, 1-2r)$, we have*

$$\mathbb{E}\sup_{t\in[0,T]}\|u_R(t) - U_{R,N}(t)\|^p_{H_r} \leq \frac{C_R}{N^{\alpha p/2}}, \tag{4.8}$$

where C_R is defined as

$$C_R = CR^{dp}\exp(R^{dp}) \tag{4.9}$$

with C a constant independent of N and R.

Proof. Let $e_{R,N}(t) = u_R(t) - U_{R,N}(t)$. From (4.4) and (4.7), we obtain

$$\begin{aligned} e_{R,N}(t) &= e^{tA}\left[u_0 - P_N u_0\right] + \int_0^t e^{(t-s)A}\left[f_R(u_R(s)) - P_N f_R(U_{R,N}(s))\right]ds \\ &\quad + \int_0^t e^{(t-s)A}\left[(-A)^{1/2}g_R(u_R(s)) - P_N\left((-A)^{1/2}g_R(U_{R,N}(s))\right)\right]ds + \int_0^t e^{(t-s)A}(I-P_N)\,dw(s) \\ &= \underbrace{e^{tA}\left[u_0 - P_N u_0\right]}_{I_0} + \underbrace{\int_0^t e^{(t-s)A}P_N\left[f_R(u_R(s)) - f_R(U_{R,N}(s))\right]ds}_{I_{f_{R1}}} \\ &\quad + \underbrace{\int_0^t e^{(t-s)A}(I-P_N)\,f_R(u_R(s))\,ds}_{I_{f_{R2}}} + \underbrace{\int_0^t e^{(t-s)A}P_N(-A)^{1/2}\left[g_R(u_R(s)) - g_R(U_{R,N}(s))\right]ds}_{I_{g_{R1}}} \\ &\quad + \underbrace{\int_0^t e^{(t-s)A}(I-P_N)\,(-A)^{1/2}\left[g_R(u_R(s))\right]ds}_{I_{g_{R2}}} + \underbrace{\int_0^t e^{(t-s)A}(I-P_N)\,dw(s)}_{I_w}. \end{aligned}$$

Since $u_0(x)$ satisfies Assumption 1 and w is a Wiener process, the estimates of I_0 and I_w are the same as those in Theorem 3.1, which can be found in (3.7) and (3.5), respectively.

Now we consider the estimate of $I_{f_{R1}}$. Without loss of generality, we assume that $\|U_{R,N}(s)\|_{H_r} \leq \|u_R(s)\|_{H_r}$ for $s \in [0,T]$. Using the Minkowski inequality and the property of the semigroup e^{tA} in (2.4) with $m = 2$ and $\gamma = r/2$, we get

$$\begin{aligned}\big\|(-A)^{r/2} I_{f_{R1}}(t)\big\|_2 &\leq C \int_0^t \frac{1}{(t-s)^{r/2}} \big\|K_R(\|u_R(s)\|_{H_r}) f(u_R(s)) - K_R(\|U_{R,N}(s)\|_{H_r}) f(U_{R,N}(s))\big\|_2 \, ds \\ &\leq C \int_0^t \frac{1}{(t-s)^{r/2}} \Big[K_R(\|u_R(s)\|_{H_r}) \, \big\|f(u_R(s)) - f(U_{R,N}(s))\big\|_2 \\ &\quad + \big\|[K_R(\|u_R(s)\|_{H_r}) - K_R(\|U_{R,N}(s)\|_{H_r})] f(U_{R,N}(s))\big\|_2\Big] ds. \end{aligned} \tag{4.10}$$

Next, we will estimate the two terms in (4.10) separately. Since f satisfies Assumption 2', we have

$$\begin{aligned}&K_R(\|u_R(s)\|_{H_r}) \|f(u_R(s)) - f(U_{R,N}(s))\|_2 \\ &\leq C K_R(\|u_R(s)\|_{H_r}) \big\| \big[u_R(s) - U_{R,N}(s)\big] \big[\big(u_R(s)\big)^{d-1} + \big(U_{R,N}(s)\big)^{d-1} + 1\big] \big\|_2 \\ &\leq C K_R(\|u_R(s)\|_{H_r}) \|u_R(s) - U_{R,N}(s)\|_m \Big(\|u_R(s)\|^{d-1}_{n(d-1)} + \big\|U_{R,N}(s)\big\|^{d-1}_{n(d-1)} + 1 \Big) \\ &\leq C(R^{d-1} + 1) \|u_R(s) - U_{R,N}(s)\|_{H_r}, \end{aligned} \tag{4.11}$$

where the second inequality results from the Hölder inequality with $\frac{1}{m} + \frac{1}{n} = \frac{1}{2}$, and the last inequality is obtained by using the definition of K_R and the Sobolev embedding theorem. From the Sobolev embedding theorem, $H_r \subset L_q$ if $\frac{1}{q} > \frac{1}{2} - r$. Hence, to obtain (4.11), we require that $\frac{1}{m} > \frac{1}{2} - r$ and $\frac{1}{n(d-1)} > \frac{1}{2} - r$, i.e., requiring $r > \frac{1}{2} - \frac{1}{2d}$, which is fulfilled by the assumption of this proposition.

Notice that the definition of K_R implies that if $\|u_R(s)\|_{H_r} \geq \|U_{R,N}(s)\|_{H_r} \geq (R+1)$, then

$$\big\|[K_R(\|u_R(s)\|_{H_r}) - K_R(\|U_{R,N}(s)\|_{H_r})] f(U_{R,N}(s))\big\|_2 = 0.$$

This leads to

$$\begin{aligned}&\big\| \big[K_R(\|u_R(s)\|_{H_r}) - K_R(\|U_{R,N}(s)\|_{H_r})\big] f(U_{R,N}(s))\big\|_2 \\ &\leq C \big| \|u_R(s)\|_{H_r} - \|U_{R,N}(s)\|_{H_r} \big| \, \big\|f(U_{R,N}(s))\big\|_2 \\ &\leq C(R^d + 1) \|u_R(s) - U_{R,N}(s)\|_{H_r}, \end{aligned} \tag{4.12}$$

where the last inequality is obtained from Assumption 2' and the Sobolev embedding theorem. Combining (4.11) and (4.12) and using the Poincaré inequality, we get

$$\big\|(-A)^{r/2} I_{f_{R1}}(t)\big\|_2 \leq C(R^d + 1) \int_0^t \frac{1}{(t-s)^{r/2}} \big\|(-A)^{r/2} \big[u_R(s) - U_{R,N}(s)\big]\big\|_2 \, ds.$$

It follows from the Hölder inequality

$$\big\|(-A)^{r/2} I_{f_{R1}}(t)\big\|_2^p \leq C(R^{dp} + 1) \int_0^t \big\|(-A)^{r/2} \big[u_R(s) - U_{R,N}(s)\big]\big\|_2^p \, ds,$$

provided that $r/2 + 1/p < 1$. From it, we get

$$\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}I_{f_{R1}}(t)\right\|_2^p \le C(R^{dp}+1)\int_0^T \mathbb{E}\sup_{\tau\in[0,s]}\left\|(-A)^{r/2}\left[u_R(\tau)-U_{R,N}(\tau)\right]\right\|_2^p ds. \tag{4.13}$$

The estimate of $I_{g_{R1}}$ is similar to that for $I_{f_{R1}}$, and here we will only outline the main steps. Using the Minkowski inequality and the property of e^{tA} in (2.4) with $m=2$ and $\gamma=(r+1)/2$, we obtain

$$\begin{aligned}
\|(-A)^{r/2}I_{g_{R1}}(t)\|_2 &\le C\int_0^t \frac{1}{(t-s)^{(r+1)/2}}\big\|K_R(\|u_R(s)\|_{H_r})g(u_R(s)) \\
&\quad - K_R(\|U_{R,N}(s)\|_{H_r})g(U_{R,N}(s))\big\|_2\, ds \\
&\le C\int_0^t \frac{1}{(t-s)^{(r+1)/2}}\Big[K_R(\|u_R(s)\|_{H_r})\ \big\|g(u_R(s))-g(U_{R,N}(s))\big\|_2 \\
&\quad + \big\|[K_R(\|u_R(s)\|_{H_r})-K_R(\|U_{R,N}(s)\|_{H_r})]g(U_{R,N}(s))\big\|_2\Big] ds \\
&\le C(R+1)\int_0^t \frac{1}{(t-s)^{(r+1)/2}}\big\|(-A)^{r/2}\left[u_R(s)-U_{R,N}(s)\right]\big\|_2 ds,
\end{aligned}$$

which leads to

$$\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}I_{g_{R1}}(t)\right\|_2^p \le C(R^{p}+1)\int_0^T \mathbb{E}\sup_{\tau\in[0,s]}\left\|(-A)^{r/2}\left[u_R(\tau)-U_{R,N}(\tau)\right]\right\|_2^p ds, \tag{4.14}$$

provided that $r/2+1/p<1/2$.

Now we move to the estimates on $I_{f_{R2}}$ and $I_{g_{R2}}$. Following the same lines as those for estimating I_{f_2} in Theorem 3.1, we get

$$\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}I_{f_{R2}}(t)\right\|_2^p \le \frac{C}{(N+1)^{2p-2-pr}}\left(\mathbb{E}\sup_{t\in[0,T]}\left\|K_R(\|u_R(t)\|_{H_r})f(u_R(t))\right\|_2^p\right),$$

provided that $r/2+1/p<1$. From Assumption 2', the definition of K_R, and the Sobolev embedding theorem, we have $\|K_R(\|u\|_{H_r})f(u)\|_2 \le C(R^d+1)$. Hence,

$$\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}I_{f_{R2}}(t)\right\|_2^p \le \frac{C\left(R^{dp}+1\right)}{(N+1)^{2p-2-pr}}. \tag{4.15}$$

Similarly, we can obtain the estimate of $I_{g_{R2}}$ as

$$\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}I_{g_{R2}}(t)\right\|_2^p \le \frac{C\left(R^{p}+1\right)}{(N+1)^{p-2-pr}}, \tag{4.16}$$

provided that $r/2+1/p<1/2$.

Hence, combining (3.5), (3.7), (4.13)–(4.16), we can get

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\left\|(-A)^{r/2}\left[u_R(t)-U_{N,R}(t)\right]\right\|_2^p &\le \frac{C(R^{dp}+1)}{(N+1)^{\min\{p(1-r),\,(p-2-pr),\,\alpha p/2\}}} \\
&\quad + C(R^{dp}+1)\left(\int_0^T \mathbb{E}\sup_{\tau\in[0,s]}\left\|(-A)^{r/2}\left[u_R(\tau)-U_{N,R}(\tau)\right]\right\|_2^p ds\right),
\end{aligned}$$

where the constant C depends on the initial condition, but independent of R and N. Since $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$, $p \geq 4$ and $\alpha \in (0, 1-2r)$, we have $\min\{p(1-r), (p-2-rp), \alpha p/2\} = \alpha p/2$. By the Gronwall inequality, we get

$$\mathbb{E} \sup_{t \in [0,T]} \left\|(-A)^{r/2}\left[u_R(t) - U_{R,N}(t)\right]\right\|_2^p \leq \frac{CR^{dp}\exp(R^{dp})}{N^{\alpha p/2}}.$$

Using the Poincaré inequality immediately yields (4.8), with C_R defined in (4.9). $\square$

Proposition 4.1 shows that as $N \to \infty$, the spectral Galerkin approximation $U_{R,N}(x,t)$ strongly converges to the solution $u_R(x,t)$ of the truncated problem (4.1)–(4.3). Furthermore, its convergence rate is $(\frac{1}{2} - r)^-$ in H_r-norm, for $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$.

Remark 4.1. The function f considered in Theorem 3.1 is globally Lipschitz continuous, while f_R in Proposition 4.1 is Lipschitz continuous in L_2-norm, which is much weaker than the globally Lipschitz condition. As a result, the error bound obtained in (4.8) depends on the truncation parameter R, which is the main difference from (3.6). As we will see later, the dependence of R in the error bound introduces challenges to prove the convergence of $U_N(x,t)$ in (2.10) to $u(x,t)$ in (2.5). Moreover, the local Lipschitz condition also requires $r > \frac{1}{2} - \frac{1}{2d}$, while in Theorem 3.1 we have $r \geq 0$.

In the following lemma, we show the left-continuity of $\|U_{N,R}(t)\|_{H_r}$ in time if $\|U_{N,R}(t)\|_{H_r} < R$.

Lemma 4.1. *Suppose that Assumptions 1, 2' and 3 are satisfied, and $U_{R,N}(x,t)$ is the spectral Galerkin approximation to the solution $u_R(x,t)$ of the truncated problem (4.1)–(4.3). Let $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$ with d the degree of polynomial function f in Assumption 2', and the constants $R > 0$ and $\tau > 0$. If $\|U_{R,N}(t)\|_{H_r} < R$ for $t < \tau$, then $\|U_{N,R}(\tau)\|_{H_r} < R$.*

Proof. From (4.7), we have for any $\delta \in (0, \tau)$,

$$\begin{aligned} U_{R,N}(\tau) = e^{\delta A} U_{R,N}(\tau - \delta) - \underbrace{\int_{\tau-\delta}^{\tau} e^{(\tau-s)A} P_N[f_R(U_{R,N}(s))]\, ds}_{I_{f_\delta}} \\ + \underbrace{\int_{\tau-\delta}^{\tau} e^{(\tau-s)A} P_N[(-A)^{1/2} g_R(U_{R,N}(s))]\, ds}_{I_{g_\delta}} + \underbrace{\int_{\tau-\delta}^{\tau} e^{(\tau-s)A} P_N dw(s)}_{I_{w_\delta}}. \end{aligned} \tag{4.17}$$

Using the Minkowski inequality, the Hölder inequality, and (3.10), we get

$$\begin{aligned} \left\|(-A)^{r/2} I_{f_\delta}(\tau)\right\|_2 &\leq \int_{\tau-\delta}^{\tau} \left\|e^{(\tau-s)A}(-A)^{r/2} P_N f_R(U_{R,N}(s))\right\|_2 ds \\ &\leq C \int_{\tau-\delta}^{\tau} (\tau - s)^{-r/2} \left\|f_R(U_{R,N}(s))\right\|_2 ds \\ &\leq C \left(\int_{\tau-\delta}^{\tau} (\tau-s)^{-rq/2}\, ds\right)^{1/q} \left(\int_{\tau-\delta}^{\tau} \left\|f_R(U_{R,N}(s))\right\|_2^p ds\right)^{1/p} \end{aligned}$$

$$= C\,\delta^{1/q-r/2}\left(\int_{\tau-\delta}^{\tau}\left\|f_R(U_{R,N}(s))\right\|_2^p\,ds\right)^{1/p},$$

where $1/p+1/q=1$ and $1/q-r/2>0$. From the definition of f_R, we have that if $\|U_{R,N}\|_{H_r}>(R+1)$, then $f_R(U_{R,N})=0$; otherwise, if $\|U_{R,N}\|_{H_r}\le(R+1)$, we have

$$\left\|f_R(U_{R,N}(s))\right\|_2^p\le\left\|f(U_{R,N}(s))\right\|_2^p\le C\left(\left\|U_{R,N}(s)\right\|_{2d}^{dp}+1\right)\le C\,\left(R^{dp}+1\right),$$

where the last inequality follows from the Sobolev embedding theorem $H_r\subset L_{2d}$, if $\frac{1}{2}-\frac{1}{2d}<r$. Therefore,

$$\left\|(-A)^{r/2}I_{f_\delta}(\tau)\right\|_2\le C\,\delta^{1-r/2}(R^{dp}+1)^{1/p}. \tag{4.18}$$

Similarly, we can get

$$\left\|(-A)^{r/2}I_{g_\delta}(\tau)\right\|_2\le C\delta^{(1-r)/2}(R^p+1)^{1/p}, \tag{4.19}$$

where $1/p+1/q=1$ and $(1-r)/2>0$. Using the Poincaré inequality, we obtain from (4.17)–(4.19) that

$$\left\|U_{R,N}(\tau)\right\|_{H_r}\le\left\|e^{\delta A}U_{R,N}(\tau-\delta)\right\|_{H_r}+\left\|I_{w_\delta}(\tau)\right\|_{H_r}+C\left[\delta^{1-r/2}(R^{dp}+1)^{1/p}+\delta^{(1-r)/2}(R^p+1)^{1/p}\right],$$

with $1/p+1/q=1$ and $(1-r)/2>0$. Letting $\delta\to0^+$ and by the continuity of $\|W_A(t)\|_{H_r}$ in Lemma 3.1, we immediately obtain

$$\left\|U_{R,N}(\tau)\right\|_{H_r}\le\left\|U_{R,N}(\tau-\delta)\right\|_{H_r}<R.\quad\square$$

In the following proposition, we discuss the relation between the solution of the truncated problem (4.1)–(4.3) and that of the original stochastic problem (1.1)–(1.3).

Proposition 4.2. *Let $r\in(0,\frac{1}{2})$. For $R>0$ and $T>0$, we define*

$$\tau_R:=\inf\{t\in[0,T]\mid\|u_R(t)\|_{H_r}\ge R\}\wedge T,$$
$$\tau_{R,N}:=\inf\{t\in[0,T]\mid\|U_{R,N}(t)\|_{H_r}\ge R\}\wedge T.$$

Suppose that Assumptions 1, 2' and 3 are satisfied. Denote $U_N(x,t)$ as the spectral Galerkin approximation to the solution $u(x,t)$ of the stochastic problem (1.1)–(1.3) and $U_{R,N}(x,t)$ as the spectral Galerkin approximation to the solution $u_R(x,t)$ of the truncated problem (4.1)–(4.3). Then, we have

- *The sequence $\{\tau_R\}_{R>0}$ is non-decreasing, and $\lim_{R\to\infty}\tau_R=T$, a.s.*
- *The sequence $\{\tau_{R,N}\}_{R>0}$ is non-decreasing, and $\lim_{R\to\infty}\tau_{R,N}=T$, a.s.*
- *For $\forall\,t\le\tau_R$, $u_R(x,t)=u(x,t)$, a.s.*
- *For $\forall\,t\le\tau_{R,N}$, $U_{R,N}(x,t)=U_N(x,t)$, a.s.*

By Lemma 3.1 and Propositions 4.1 and 4.2, we are able to obtain the following lemma on the boundedness of the Galerkin approximate sequence U_N:

Lemma 4.2. *Suppose that Assumptions 1, 2' and 3 are satisfied, and $U_N(x,t)$ is the spectral Galerkin approximation to the solution $u(x,t)$ of the stochastic problem (1.1)–(1.3). Let $r\in(\frac{1}{2}-\frac{1}{2d},\frac{1}{2})$ with d the degree of polynomial function f in Assumption 2'. Then we have for $T>0$,*

$$\lim_{N\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq \bar{R}(N) \right\} = 0, \tag{4.20}$$

where $\bar{R}(N)$ *satisfies that*

$$\lim_{N\to\infty} \bar{R}(N) = \infty, \tag{4.21}$$

$$\lim_{N\to\infty} \frac{\bar{R}^{dp}\exp\left(\bar{R}^{dp}\right)}{N^{\alpha p/2}} = 0, \quad \textit{for} \quad \alpha \in (0, 1-2r), \;\; p \geq 4. \tag{4.22}$$

Proof. Let $R > 0$ and denote

$$e_N(x,t) := u(x,t) - U_N(x,t), \qquad e_{R,N}(x,t) := u_R(x,t) - U_{R,N}(x,t).$$

For $\varepsilon \in (0,1)$ and $p \geq 4$, we get

$$\begin{aligned}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \subset & \left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\} \\ & \bigcup \left[\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} < \varepsilon^p \right\} \bigcap \left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \right].\end{aligned}$$

Note that the second event in the above inclusion satisfies

$$\left[\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} < \varepsilon^p \right\} \bigcap \left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \right] \subset \left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} > (R-\varepsilon) \right\}.$$

Hence, we have

$$\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \subset \left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\} \bigcup \left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} > (R-\varepsilon) \right\}. \tag{4.23}$$

Now, we focus on the first term of the right hand side of (4.23). First, we can obtain

$$\begin{aligned}\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\} \subset & \left[\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1) \right\} \right. \\ & \bigcup \left\{ \|W_A(t)\|_{H_r} \text{ is not continuous} \right\} \bigcup \left(\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} < (R-1) \right\} \right. \\ & \left. \left. \bigcap \left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\} \bigcap \left\{ \|W_A(t)\|_{H_r} \text{ is continuous} \right\} \right) \right].\end{aligned} \tag{4.24}$$

In fact, the last event in the above inclusion satisfies

$$\begin{aligned}& \left(\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} < (R-1) \right\} \bigcap \left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\} \right. \\ & \left. \quad \bigcap \left\{ \|W_A(t)\|_{H_r} \text{ is continuous} \right\} \right) \subset \left\{ \sup_{t\in[0,T]} \|e_{R,N}(t)\|^p_{H_r} \geq \varepsilon^p \right\}.\end{aligned} \tag{4.25}$$

To prove (4.25), we denote $\tau^* = \inf\left\{ t \in [0,T] \,\middle|\, \|e_N(t)\|^p_{H_r} \geq \varepsilon^p \right\}$. When the event of the left hand side of (4.25) happens, we get

$$\|U_N(t)\|_{H_r} < R - 1 + \varepsilon < R, \qquad \text{for } t \in [0, \tau^*).$$

Comparing (2.8)–(2.9) with (4.5)–(4.6) and noticing the definitions of f_R and g_R, we get

$$U_N = U_{R,N} \qquad t \in [0, \tau^*),$$

which immediately leads to

$$U_N = U_{R,N} \qquad t \in [0, \tau^*], \tag{4.26}$$

by Lemma 4.1. Moreover, comparing (1.1)–(1.3) with (4.1)–(4.3) and noticing the definitions of f_R and g_R, we get when the event of the left hand side of (4.25) happens, there is

$$u = u_R, \qquad t \in [0, T]. \tag{4.27}$$

Combining (4.26) and (4.27), we obtain

$$\sup_{t\in[0,T]} \|e_{R,N}(t)\|_{H_r}^p \geq \|e_{R,N}(\tau^*)\|_{H_r}^p \geq \varepsilon^p.$$

Hence, from (4.24) and (4.25), we get

$$\begin{aligned}\Big\{\sup_{t\in[0,T]} \|e_N(t)\|_{H_r}^p \geq \varepsilon^p\Big\} &\subset \Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1)\Big\} \\ &\bigcup \Big\{\sup_{t\in[0,T]} \|e_{R,N}(t)\|_{H_r}^p \geq \varepsilon^p\Big\} \bigcup \Big\{\|W_A(t)\|_{H_r} \text{ is not continuous}\Big\}.\end{aligned} \tag{4.28}$$

Combining (4.28) with (4.23), we obtain

$$\begin{aligned}\Big\{\sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R\Big\} \subset \Big(&\Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} > (R-\varepsilon)\Big\} \bigcup \Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1)\Big\} \\ &\bigcup \Big\{\sup_{t\in[0,T]} \|e_{R,N}(t)\|_{H_r}^p \geq \varepsilon^p\Big\} \bigcup \Big\{\|W_A(t)\|_{H_r} \text{ is not continuous}\Big\}\Big).\end{aligned}$$

It implies that

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R\Big\} &\leq \mathbb{P}\Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} > (R-\varepsilon)\Big\} \\ &+ \mathbb{P}\Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1)\Big\} + \mathbb{P}\Big\{\sup_{t\in[0,T]} \|e_{R,N}(t)\|_{H_r}^p \geq \varepsilon^p\Big\},\end{aligned}$$

where we have used the conclusion in Lemma 3.1 that $\|W_A(t)\|_{H_r}$ is continuous for $t \in [0, T]$. Since $\varepsilon \in (0, 1)$, by the Chebyshev inequality and Proposition 4.1, we can further obtain

$$\begin{aligned}\mathbb{P}\Big\{\sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R\Big\} &\leq 2\mathbb{P}\Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1)\Big\} + \frac{1}{\varepsilon^p}\mathbb{E}\sup_{t\in[0,T]} \|e_{R,N}(t)\|_{H_r}^p \\ &\leq 2\,\mathbb{P}\Big\{\sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (R-1)\Big\} + \frac{CR^{dp}\exp(R^{dp})}{\varepsilon^p N^{\alpha p/2}},\end{aligned}$$

for $\alpha \in (0, 1-2r)$ and $p \geq 4$, where the constant C independent of N. The above inequality is true for any $R > 0$. If we choose $R = \bar{R}(N)$ satisfying (4.21) and (4.22), we immediately get

$$\lim_{N\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq \bar{R}(N) \right\} \leq 2 \lim_{N\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq (\bar{R}(N)-1) \right\} = 0,$$

by Proposition 2.1. □

With Proposition 4.1 and Lemma 4.2, we are equipped to prove our second main theorem on the convergence of the Galerkin approximation $U_N(x,t)$ to $u(x,t)$, when the function f is locally Lipschitz continuous.

Theorem 4.1. *Suppose that Assumptions 1, 2' and 3 are satisfied, and $U_N(x,t)$ is the spectral Galerkin approximation to the solution $u(x,t)$ of the stochastic problem (1.1)–(1.3). Let $4 \leq p < \infty$, and $r \in (\frac{1}{2}-\frac{1}{2d}, \frac{1}{2})$ with d being the degree of polynomial function f in Assumption 2'. Then for any $\alpha \in (0, 1-2r)$, there is*

$$\lim_{N\to\infty} \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u(t)-U_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} = 0. \tag{4.29}$$

Proof. Let

$$e_N(x,t) = u(x,t) - U_N(x,t), \qquad e_{R,N}(x,t) = u_R(x,t) - U_{R,N}(x,t).$$

For any $R > 0$, we have

$$\begin{aligned}\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} \subset \Bigg[&\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq R \right\} \bigcup \left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \\ &\bigcup \left(\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} < R \right\} \bigcap \left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} < R \right\} \bigcap \left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} \right) \Bigg].\end{aligned}$$

Following similar lines in Lemma 4.2 to obtain (4.28), we can get

$$\begin{aligned}\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} \subset \Bigg[&\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq R \right\} \\ &\bigcup \left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} \bigcup \left\{ \sup_{t\in[0,T]} \|e_{R,N}(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} \Bigg],\end{aligned}$$

which implies that

$$\begin{aligned}\mathbb{P}\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} &\leq \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq R \right\} \\ &+ \mathbb{P}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} + \mathbb{P}\left\{ \sup_{t\in[0,T]} \|e_{R,N}(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\}.\end{aligned}$$

Since $\alpha \in (0, 1-2r)$, we choose a sufficiently small δ, such that $\alpha + \delta \in (0, 1-2r)$. Then by the Chebyshev inequality and Proposition 4.1, we obtain

$$\begin{aligned}&\mathbb{P}\left\{ \sup_{t\in[0,T]} \|e_N(t)\|^p_{H_r} \geq \frac{1}{N^{\alpha p/2}} \right\} \\ &\leq \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq R \right\} + \mathbb{P}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} + \frac{N^{(\alpha+\delta)p/2}}{N^{\delta p/2}} \mathbb{E} \sup_{t\in[0,T]} \|e_{R,N}(t)\|^p_{H_r} \\ &\leq \mathbb{P}\left\{ \sup_{t\in[0,T]} \|u(t)\|_{H_r} \geq R \right\} + \mathbb{P}\left\{ \sup_{t\in[0,T]} \|U_N(t)\|_{H_r} \geq R \right\} + \frac{C_R}{N^{\delta p/2}},\end{aligned} \tag{4.30}$$

where we have used $\alpha + \delta$ instead of α in Proposition 4.1, and the constant C_R is defined as in (4.9). Note that the above inequality holds for any $R > 0$. Now we choose $R := R(N)$ depending on N, and

$$\lim_{N\to\infty} R(N) = \infty, \qquad \lim_{N\to\infty} \frac{C_R}{N^{(\delta\wedge\alpha)p/2}} = 0.$$

Taking the limit of $N \to \infty$ and using Proposition 2.1 and Lemma 4.2 immediately leads to (4.29). □

Remark 4.2. Theorem 4.1 does not hold for the Burgers' type equations (also indicated by Assumption 3), since they do not satisfy the regularity estimates in (2.7). It is shown in [3, Theorem 3.1] and [4, Theorem 3.2] that the solutions of the Burgers' equation belong to the space consisting of all continuous adapted process on $[0, T]$ with value on $L_2(0, 1)$, such that $\mathbb{E} \sup_{t\in[0,T]} \|u(t)\|_2^p < +\infty$, which is weaker than the space required by the regularity estimate (2.7).

4.1. Numerical example

In this section, we study the convergence of the spectral Galerkin approximation by directly solving the stochastic reaction–diffusion–advection equation (1.1)–(1.3). We choose $f(u) = u^3 - 2u$ satisfying the local Lipschitz condition in Assumption 2' and $g(u) = 1/(1 + u^2)$ as used in Section 3.1. Then, we consider the following stochastic problem:

$$du = \left(\Delta u - u^3 + 2u + \frac{\partial}{\partial x}\left(\frac{1}{1+u^2}\right)\right) dt + dw, \tag{4.31}$$

$$u(0, t) = u(1, t) = 0, \quad t \in (0, T], \tag{4.32}$$

$$u(x, 0) = x^2(1 - x), \quad x \in [0, 1], \tag{4.33}$$

where w is a cylindrical Wiener process on $L_2([0, 1]; \mathbb{R})$. Similarly, (4.31)–(4.33) are discretized by the Galerkin approximation in space and by the exponential Euler scheme in time [6,10,11,2]. The notations and numerical parameters are the same as those used in Section 3.1.

Fig. 3 demonstrates the Galerkin approximations of the solution $u(x, t)$ from two independent realizations (see Fig. 3 a–b), the expected values $\mathbb{E}[u(x, t)]$ by the mean of 1000 independent realizations (see Fig. 3 c), and the numerical solution of the deterministic equation corresponding to (4.31)–(4.33) (see Fig. 3 d), where we choose $T = 0.2$ and $N = 1024$. Similar to the observations in Section 3.1, we find that the solutions of stochastic problem (4.31)–(4.33) are different from different realizations, and the mean of the solution converges to the solution of its deterministic counterpart.

To study the convergence of the Galerkin approximation, we first prepare reference solutions by choosing the mesh size $h = 1/2048$ and time step $\tau = 10^{-4}$. In Theorem 4.1, the convergence of Galerkin approximation and its rate are discussed in probability. Here, we compute the pathwise approximate error in the following form:

$$\text{Error}_2 := \left[\sup_{0\le m\le M} \left(\sum_{k=1}^{N-1} \left(1 + (-\lambda_k)^r\right) \left|\widehat{u}_k^m - \widehat{U}_{N,k}^m\right| \right)^{p/2} \right]^{1/p}. \tag{4.34}$$

We remark that numerically it is very challenging to directly verify the probability as stated in (4.29). In our simulations, we choose $p = 5$. Fig. 4 shows the log–log plots of the pathwise approximation error (4.34) for $r = 0.35$ and $r = 0.49$, where $N = \{8, 16, 32, 64, 128, 256, 512\}$. From it, we find that as $N \to \infty$, the pathwise approximate error decrease, i.e., the Galerkin approximation converges to the solution of (4.31)–(4.33) as N increases. Similar to the case in Section 3.1, the convergence rate in H_r-norm is larger when r is smaller.

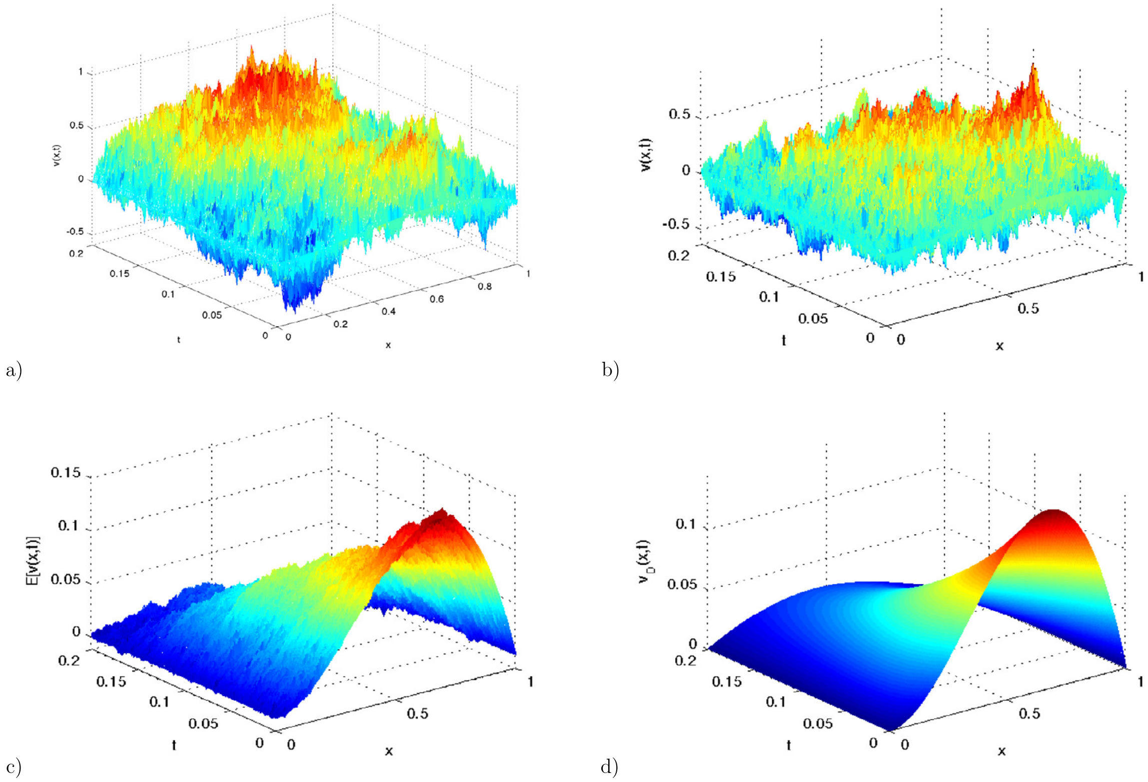

Fig. 3. a)–b). Numerical approximation of $u(x,t)$ from two independent realizations. c). The expected value $\mathbb{E}[u(x,t)]$ by the mean of 1000 independent realizations. d). The solution of the corresponding deterministic equation of (4.31)–(4.33).

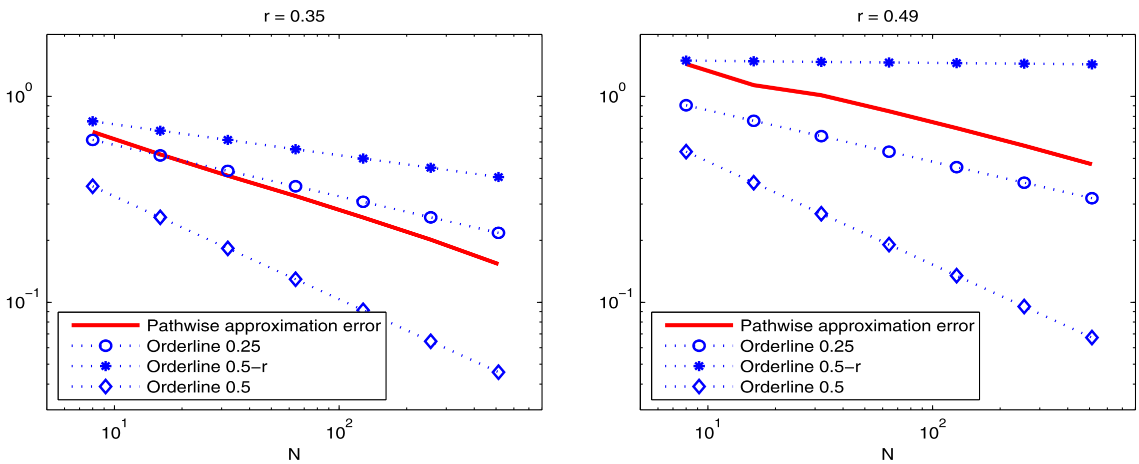

Fig. 4. Pathwise approximation errors of the Galerkin approximation to the solution of (4.31)–(4.33), where $N = \{8, 16, 32, 64, 128, 256, 512\}$ and the errors are computed from (4.34).

5. Conclusion

We studied the convergence of spectral Galerkin approximation to the solution of the stochastic reaction–diffusion–advection equation, under different Lipschitz conditions of the reaction function f. When f is globally Lipschitz continuous, we proved that the Galerkin approximation strongly converges to the mild solution of the stochastic reaction–diffusion–advection equation, with the rate of convergence $(\frac{1}{2} - r)^-$ in H_r-norm, for any $r \in [0, \frac{1}{2})$. While f is locally Lipschitz continuous, we found that the Galerkin approximation converges to the mild solution in probability, and the convergence rate in H_r-norm is $(\frac{1}{2} - r)^-$, for $r \in (\frac{1}{2} - \frac{1}{2d}, \frac{1}{2})$, where d is the leading degree of the polynomial f. Compared to the global Lipschitz case, the convergence analysis for the local Lipschitz case is more challenging, especially with the presence of the advection term. The existing approaches of spatial convergence analysis are for the problem with only f or g term, so they can not be directly applied to study the stochastic reaction–diffusion–advection equation. Hence, we proposed a new approach which is mainly based on the convergence analysis of the Galerkin approximation for a truncated problem of (1.1)–(1.3). The proposed approach and techniques can be also applied to the convergence analysis of other stochastic partial differential equations.

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