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Research Article

Abstract Semilinear Evolution Equations with Convex-Power Condensing Operators

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By using the techniques of convex-power condensing operators and fixed point theorems, we investigate the existence of mild solutions to nonlocal impulsive semilinear differential equations. Two examples are also given to illustrate our main results.

1. Introduction

This paper is concerned with the existence of mild solutions for the following impulsive semilinear differential equations with nonlocal conditions

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \ t \neq t_i, \\ u(0) = g(u), \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, 2, \ldots, p, \\ 0 < t_1 < t_2 < \cdots < t_p < T, \end{gathered} \tag{1}$$

where $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of strongly continuous semigroup $S(t)$ for $t > 0$ in a real Banach space X and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition. f and g are X-valued functions to be given later.

As far as we know, the first paper dealing with abstract nonlocal initial value problems for semilinear differential equations is due to [1]. Because nonlocal conditions have better effect in the applications than the classical initial ones, many authors have studied the following type of semilinear differential equations under various conditions on $S(t)$, f, and g:

$$\begin{gathered} u'(t) = Au(t) + f(t, u(t)), \quad t \in [0, T], \\ u(0) = g(u). \end{gathered} \tag{2}$$

For instance, Byszewski and Lakshmikantham [2] proved the existence and uniqueness of mild solutions for nonlocal semilinear differential equations when f and g satisfy Lipschitz type conditions. In [3], Ntouyas and Tsamatos studied the case with compactness conditions. Byszewski and Akca [4] established the existence of solution to functionaldifferential equation when the semigroup is compact and g is convex and compact on a given ball. Subsequently, Benchohra and Ntouyas [5] discussed the second-order differential equation under compact conditions. The fully nonlinear case was considered by Aizicovici and McKibben [6], Aizicovici and Lee [7], Aizicovici and Staicu [8], García-Falset [9], Paicu and Vrabie [10], Obukhovski and Zecca [11], and Xue [12, 13].

Recently, the theory of impulsive differential inclusions has become an important object of investigation because of its wide applicability in biology, medicine, mechanics, and control theory and in more and more fields. Cardinali and Rubbioni [14] studied the multivalued impulsive semilinear differential equation by means of the Hausdorff measure of noncompactness. Liang et al. [15] investigated the nonlocal impulsive problems under the assumptions that g is compact, Lipschitz, and g is not compact and not Lipschitz, respectively. All these studies are motivated by the practical interests of nonlocal impulsive Cauchy problems. For a more detailed bibliography and exposition on this subject, we refer to [14–18].

The present paper is motivated by the following facts. Firstly, the approach used in [9, 12, 13, 19, 20] relies on the assumption that the coefficient l of the function f about the measure of noncompactness satisfies a strong inequality, which is difficult to be verified in applications. Secondly, in

[21], it seems that authors have considered the inequality restriction on coefficient function $l(t)$ of f may be relaxed for impulsive nonlocal differential equations. However, in fact, they only solve the classical initial value problems $u(0) = u_0$ rather than the nonlocal initial problems $u(0) = u_0 + g(u)$. For more details, one can refer to the proof of Theorem 3.1 in [21] (see the inequalities (3.3) and (3.4) in page 5 and the estimations of the measure of noncompactness in page 6 and page 7 of [21]).

Therefore, we will continue to discuss the impulsive nonlocal differential equations under more general assumptions. Throughout this work, we mainly use the property of convex-power condensing operators and fixed point theorems to obtain the main result (Theorem 10). Indeed, the fixed point theorem about the convex-power condensing operators is an extension for Darbo-Sadovskii's fixed point theorem. But the former seems more effective than the latter at times for some problems. For example, in [22] we ever applied the former to study the nonlocal Cauchy problem and obtained more general and interesting existence results. Based on the results obtained, we discuss the impulsive nonlocal differential equations. Fortunately, applying the techniques of convex-power condensing operators and fixed point theorems solves the difficulty involved by coefficient restriction that is, the constraint condition for the coefficient function $l(t)$ of f is unnecessary (see Theorem 10). Therefore, our results generalize and improve many previous ones in this field, such as [9, 12, 13, 19, 20].

The outline of this paper is as follows. In Section 2, we recall some concepts and facts about the measure of noncompactness, fixed point theorems, and impulsive semilinear differential equations. In Section 3, we obtain the existence results of (1) when g is compact in $PC([0,T];X)$. In Section 4, we discuss the existence result of (1) when g is Lipschitz continuous, while Section 5 contains two illustrating examples.

2. Preliminaries

Let E be a real Banach space, we introduce the Hausdorff measure of noncompactness α defined on each bounded subset Ω of E by

$$\alpha(\Omega) = \inf\left\{r > 0;\ \text{there are finite points}\ x_1, x_2, \ldots, x_n \in E \text{ with } \Omega \subset \bigcup_{i=1}^{n} B(x_i, r)\right\}. \tag{3}$$

Now we recall some basic properties of the Hausdorff measure of noncompactness.

Lemma 1. *For all bounded subsets Ω, Ω_1, and Ω_2 of E, the following properties are satisfied:*

(1) *Ω is precompact if and only if $\alpha(\Omega) = 0$;*

(2) *$\alpha(\Omega) = \alpha(\overline{\Omega}) = \alpha(\operatorname{co}\Omega)$, where $\overline{\Omega}$ and $\operatorname{co}\Omega$ mean the closure and convex hull of Ω, respectively;*

(3) *$\alpha(\Omega_1) \le \alpha(\Omega_2)$ when $\Omega_1 \subset \Omega_2$;*

(4) *$\alpha(\Omega_1 \bigcup \Omega_2) \le \max\{\alpha(\Omega_1), \alpha(\Omega_2)\}$;*

(5) *$\alpha(\lambda\Omega) = |\lambda|\alpha(\Omega)$, for any $\lambda \in R$;*

(6) *$\alpha(\Omega_1 + \Omega_2) \le \alpha(\Omega_1) + \alpha(\Omega_2)$, where $\Omega_1 + \Omega_2 = \{x + y; x \in \Omega_1, y \in \Omega_2\}$;*

(7) *if $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of nonempty bounded closed subsets of E and $\lim_{n\to\infty}\alpha(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in E.*

The map $Q : D \subset E \to E$ is said to be α-condensing if for every bounded and not relatively compact $B \subset D$, we have $\alpha(QB) < \alpha(B)$ (see [23]).

Lemma 2 (see [9]: Darbo-Sadovskii). *If $D \subset E$ is bounded closed and convex, the continuous map $Q : D \to D$ is α-condensing, then the map Q has at least one fixed point in D.*

In the sequel, we will continue to generalize the definition of condensing operator. First of all, we give some notations.

Let $D \subset E$ be bounded closed and convex, the map $Q : D \to D$, and $x_0 \in D$ for every $B \subset D$, set

$$\begin{gathered} Q^{(1,x_0)}(B) = Q(B), \\ Q^{(n,x_0)}B = Q\left(\overline{\operatorname{co}}\left\{Q^{(n-1,x_0)}B, x_0\right\}\right), \quad n = 2, 3, \ldots, \end{gathered} \tag{4}$$

where $\overline{\operatorname{co}}$ means the closure of the convex hull.

Now we give the definition of a kind of new operator.

Definition 3. Let $D \subset E$ be bounded closed and convex, the map $Q : D \to D$ is said to be α-convex-power condensing if there exist $x_0 \in D$, $n_0 \in N$ and for every bounded and not relatively compact $B \subset D$, we have

$$\alpha\left(Q^{(n_0,x_0)}(B)\right) < \alpha(B). \tag{5}$$

From this definition, if $\alpha(Q^{(n_0,x_0)}(B)) = \alpha(B)$, one obtains $B \subset E$ as relatively compact.

Subsequently, we give the fixed point theorem about the convex-power condensing operator.

Lemma 4 (see [23]). *If $D \subset E$ is bounded closed and convex, the continuous map $Q : D \to D$ is α-convex-power condensing, then the map Q has at least one fixed point in D.*

Throughout this paper, let $(X, \|\cdot\|)$ be a real Banach space. We denote by $C([0,T];X)$ the Banach space of all continuous functions from $[0,T]$ to X with the norm $\|u\| = \sup\{\|u(t)\|, t \in [0,T]\}$ and by $L^1([0,T];X)$ the Banach space of all X-valued Bochner integrable functions defined on $[0,T]$ with the norm $\|u\|_1 = \int_0^T \|u(t)\|dt$. Let $PC([0,T];X) = \{u : u$ is a function from $[0,T]$ into X such that $u(t)$ is continuous at $t \ne t_i$ and the left continuous at $t = t_i$ and the right limit $u(t_i^+)$ exists for $i = 1, 2, \ldots, p\}$. It is easy to check that $PC([0,T];X)$ is a Banach space with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t \in [0,T]\}$ and $C([0,T];X) \subseteq PC([0,T];X) \subseteq L^1([0,T];X)$. Moreover, we denote β by the

Hausdorff measure of noncompactness of X, denote β_c by the Hausdorff measure of noncompactness of $C([0,T];X)$ and denote β_{pc} by the Hausdorff measure of noncompactness of $PC([0,T];X)$.

Throughout this work, we suppose the following

(H_A) The linear operator $A : D(A) \subseteq X \to X$ generates an equicontinuous C_0-semigroup $\{S(t) : t \geq 0\}$. Hence, there exists a positive number M such that $\|S(t)\| \leq M$.

For further information about the theory of semigroup of operators, we may refer to some classic books, such as [24–26].

To discuss the problem (1), we also need the following lemma.

Lemma 5. *If $W \subseteq PC([0,T];X)$ is bounded, then one has*

$$\sup_{t\in[0,T]} \beta(W(t)) \leq \beta_{pc}(W), \tag{6}$$

where $W(t) = \{u(t); u \in W\} \subset X$.

Lemma 6 (see [27]). *If $W \subseteq C([0,T];X)$ is bounded, then for all $t \in [0,T]$,*

$$\beta(W(t)) \leq \beta_c(W). \tag{7}$$

Furthermore, if W is equicontinuous on $[0,T]$, then $\beta(W(t))$ is continuous on $[0,T]$ and

$$\beta_c(W) = \sup\{\beta(W(t)) : t \in [0,T]\}. \tag{8}$$

Since C_0-semigroup $S(t)$ is said to be equicontinuous, the following lemma is easily checked.

Lemma 7. *If the semigroup $S(t)$ is equicontinuous and $w \in L^1([0,T];R^+)$, then the set $\{\int_0^t S(t-s)u(s)ds, \|u(s)\| \leq w(s)$ for a.e. $s \in [0,T]\}$ is equicontinuous for $t \in [0,T]$.*

Definition 8. A function $u \in PC([0,T];X)$ is said to be a mild solution of the nonlocal problem (1), if it satisfies

$$\begin{aligned} u(t) = {} & S(t)g(u) + \int_0^t S(t-s) f(s,u(s))\,ds \\ & + \sum_{0<t_i<t} S(t-t_i) I_i(u(t_i)), \quad 0 \leq t \leq T. \end{aligned} \tag{9}$$

In addition, let r be a finite positive constant, and set $B_r := \{x \in X : \|x\| \leq r\}$ and $W_r := \{u \in C([0,T];X) : u(t) \in B_r,$ for all $t \in [0,T]\}$.

3. g Is Compact

In this section, we state and prove the existence theorems for the nonlocal impulsive problem (1). First, we give the following hypotheses:

(H_f)

(1) $f : [0,T] \times X \to X$ is a Carathéodory function; that is, for all $x \in X$, $f(\cdot,x) : [0,T] \to X$ is measurable and for a.e. $t \in [0,T]$, $f(t,\cdot) : X \to X$ is continuous;

(2) for finite positive constant $r > 0$, there exists a function $\alpha_r \in L^1(0,T;R)$ such that

$$\|f(t,x)\| \leq \alpha_r(t) \tag{10}$$

for a.e. $t \in [0,T]$ and $x \in B_r$;

(3) there exists a function $l(t) \in L^1(0,T;R)$ such that

$$\beta(f(t,D)) \leq l(t)\beta(D) \tag{11}$$

for a.e. $t \in [0,T]$ and every bounded subset $D \subset B_r$;

(H_g) $g : PC([0,T];X) \to X$ is a continuous and compact mapping; furthermore, there exists a positive number N_1 such that $\|g(u)\| \leq N_1$, for any $u \in W_r$;

(H_I) $I_i : X \to X$ is a continuous and compact mapping for every $i = 1,2,\ldots,p$;

(H_r) $M(N_1 + \|\alpha_r\|_{L^1} + \sup_{u\in W_r}\sum_{i=1}^p \|I_i(u(t_i))\|) \leq r$.

Remark 9. The mapping f is said to be L^1-Carathéodory if the assumption $(H_f)(1)(2)$ is satisfied.

Theorem 10. *If the hypotheses (H_A), $(H_f)(1)(2)(3)$, (H_g), (H_I), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on $[0,T]$.*

To prove the above theorem, we need the following lemma.

Lemma 11. *If the condition (H_r) holds, then for arbitrary bounded set $B \subset W_r$, we have*

$$\begin{aligned} & \beta\left(\int_0^t S(t-s) f(s,B(s))\,ds\right) \\ & \quad \leq 4M \int_0^t \beta(f(s,B(s)))\,ds, \quad t \in [0,T]. \end{aligned} \tag{12}$$

This proof is quite similar to that of Lemma 3.1 in [20]; we omit it.

Proof of Theorem 10. We consider the operator $Q : PC([0,T];X) \to PC([0,T];X)$ defined by

$$\begin{aligned} (Qu)(t) = {} & S(t)g(u) + \int_0^t S(t-s) f(s,u(s))\,ds \\ & + \sum_{0<t_i<t} S(t-t_i) I_i(u(t_i)), \quad 0 \leq t \leq T. \end{aligned} \tag{13}$$

It is easy to see that the fixed points of Q are the mild solutions of nonlocal impulsive semilinear differential equation (1). Subsequently, we shall prove that Q has a fixed point by using Lemma 4.

We shall first prove that Q is continuous on $PC([0,T];X)$. In fact, let $\{u_n\}_{n=1}^{+\infty} \subset PC([0,T];X)$ be an arbitrary sequence satisfying $\lim_{n\to+\infty} u_n = u$ in $PC([0,T];X)$. It follows from Definition 8 that

$$\|Qu_n - Qu\| \le M\left(\|g(u_n) - g(u)\| + \int_0^T \|f(s,u_n(s)) - f(s,u(s))\|\, ds + \sum_{i=1}^{p} \|I_i(u_n(t_i)) - I_i(u(t_i))\| \right). \tag{14}$$

According to the continuity of f in its second argument, for each $s \in [0,T]$, we have the following:

$$\lim_{n\to+\infty} f(s,u_n(s)) = f(s,u(s)). \tag{15}$$

In addition, g and I_i are all continuous for each $i = 1,2,\ldots,p$, and hence, the Lebesgue dominated convergence theorem implies

$$\lim_{n\to+\infty} \|Qu_n - Qu\| = 0. \tag{16}$$

Namely, Q is continuous on $PC([0,T];X)$.

Subsequently, we claim that $QW_r \subseteq W_r$. Actually, by (H_r), we obtain

$$\begin{aligned}\|(Qu)(t)\| &\le \|S(t)g(u)\| + \int_0^t \|S(t-s)f(s,u(s))\|\, ds + \sum_{0<t_i<t} \|S(t-t_i)I_i(u(t_i))\| \\ &\le M\left(N_1 + \|\alpha_r\|_{L^1} + \sup_{u\in W_r}\sum_{i=1}^{p} \|I_i(u(t_i))\| \right) \\ &\le r,\end{aligned} \tag{17}$$

for any $u \in W_r \subseteq PC([0,T];X)$. Thus, $QW_r \subseteq W_r$.

Now we demonstrate that QW_r is equicontinuous for any $t > 0$. Let $t \in (0,T]$, $\varepsilon > 0$. Since g is compact, $g(W_r)$ is relatively compact; that is, there is a finite family $\{x_j\}_{j=1}^{m} \subset g(W_r)$ such that for any $u \in W_r$, there exists some $j \in \{1,2,\ldots,m\}$ such that

$$2M\|g(u) - x_j\| \le \frac{\varepsilon}{5}. \tag{18}$$

On the other hand, as $S(t)$ is equicontinuous at $t > 0$, we can choose $\delta_1 \in (0,t)$ such that

$$\begin{gathered}\|(S(t+\delta') - S(t))x_j\| \le \frac{\varepsilon}{5}, \\ \sum_{i=1}^{p} \|(S(t+\delta'-t_i) - S(t-t_i))I_i(u(t_i))\| \le \frac{\varepsilon}{5},\end{gathered} \tag{19}$$

for each $\delta' \in R$, $|\delta'| < \delta_1$, uniformly for $u \in W_r$, and $j \in \{1,2,\ldots,m\}$. By $(H_f(2))$, it can be obtained that there exists $\delta_2 \in (0,t)$ such that

$$M\int_t^{t+\delta'} \|f(s,u(s))\|\, ds \le \frac{\varepsilon}{5}, \tag{20}$$

for each $\delta' \in R$, $|\delta'| < \delta_2$, uniformly for $u \in W_r$. Furthermore, by Lemma 7, we get that there exists $\delta_3 \in (0,t)$ such that

$$\int_0^t \|(S(t+\delta'-s) - S(t-s))f(s,u(s))\|\, ds \le \frac{\varepsilon}{5}, \tag{21}$$

for each $\delta' \in R$, $|\delta'| < \delta_3$, uniformly for $u \in W_r$. Thus, there exists $h = \min\{\delta_1,\delta_2,\delta_3\}$ such that

$$\begin{aligned}&\|(Qu)(t+h') - (Qu)(t)\| \\ &\le \|(S(t+h') - S(t))g(u)\| + \sum_{i=1}^{p} \|(S(t+h'-t_i) - S(t-t_i))I_i(u(t_i))\| \\ &\quad + \int_0^t \|(S(t+h'-s) - S(t-s))f(s,u(s))\|\, ds + \int_t^{t+h'} \|S(t+h'-s)f(s,u(s))\|\, ds \\ &\le \|(S(t+h') - S(t))(g(u) - x_j)\| + \|(S(t+h') - S(t))x_j\| \\ &\quad + \sum_{i=1}^{p} \|(S(t+h'-t_i) - S(t-t_i))I_i(u(t_i))\| \\ &\quad + \int_0^t \|(S(t+h'-s) - S(t-s))f(s,u(s))\|\, ds + \int_t^{t+h'} \|S(t+h'-s)f(s,u(s))\|\, ds \\ &\le 2M\|g(u) - x_j\| + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + M\int_t^{t+h'} \|f(s,u(s))\|\, ds \\ &\le \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} \\ &= \varepsilon,\end{aligned} \tag{22}$$

for each $h' \in R$, $|h'| < h$, uniformly for $u \in W_r$. Therefore, QW_r is equicontinuous at $t > 0$.

Similarly, we can conclude that QW_r is also equicontinuous at $t = 0$. Thus, QW_r is equicontinuous on $[0,T]$.

Set $W = \overline{\mathrm{co}}(QW_r)$. It is obvious that W is equicontinuous on $[0,T]$ and Q maps W into itself.

Next, we shall prove that $Q : W \to W$ is a convex-power condensing operator. Take $x_0 \in W$; by the definition of convex-power condensing operator, we shall show that there exists a positive integral n_0 such that

$$\beta_{pc}\left(Q^{(n_0,x_0)}B\right) < \beta_{pc}(B) \tag{23}$$

if $B \subset W$ is not relatively compact. In fact, by using the conditions (H_g) and (H_I), we get from Lemma 11 that

$$\begin{aligned}\beta\left(\left(Q^{(1,x_0)}B\right)(t)\right) &= \beta\left(\left(QB\right)(t)\right)\\ &\leq \beta\left(S(t)\,g(B)\right)\\ &\quad+\beta\left(\int_0^t S(t-s)\,f(s,B(s))\,ds\right)\\ &\quad+\beta\left(\sum_{i=1}^{p}S(t-t_i)\,I_i(B(t_i))\right)\\ &= 4M\int_0^t \beta\left(f(s,B(s))\right)ds\\ &\leq 4M\int_0^t l(s)\,\beta(B(s))\,ds\\ &\leq 4M\int_0^t l(s)\,ds\beta_{pc}(B).\end{aligned} \tag{24}$$

Since $l(t) \in L^1(0,T;R^+)$, there exists a continuous function $\omega : [0,T] \to R^1$ such that for any $0<\varepsilon<1$,

$$\int_0^T |l(s)-\omega(s)|\,ds < \varepsilon. \tag{25}$$

Then

$$\begin{aligned}&\beta\left(\left(Q^{(1,x_0)}B\right)(t)\right)\\ &\leq 4M\left(\int_0^t |l(s)-\omega(s)|\,ds+\int_0^t|\omega(s)|\,ds\right)\beta_{pc}(B)\\ &\leq 4M\left(\varepsilon+N_2t\right)\beta_{pc}(B),\end{aligned} \tag{26}$$

where $N_2=\max\{|\omega(t)| : t\in[0,T]\}$. Hence,

$$\begin{aligned}&\beta\left(\left(Q^{(2,x_0)}B\right)(t)\right)\\ &\leq \beta\left(S(t)\,g\left(\overline{\text{co}}\left\{\left(Q^{(1,x_0)}B\right),x_0\right\}\right)\right)\\ &\quad+\beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\text{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right)\\ &\quad+\beta\left(\sum_{i=1}^{p}S(t-t_i)\,I_i\left(\overline{\text{co}}\left\{\left(Q^{(1,x_0)}B\right)(t_i),x_0(t_i)\right\}\right)\right)\\ &=\beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\text{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right)\\ &\leq 4M\int_0^t\beta\left(f\left(s,\overline{\text{co}}\left\{\left(Q^{(1,x_0)}B\right)(s),x_0(s)\right\}\right)\right)ds\\ &\leq 4M\int_0^t l(s)\,\beta\left(\left(Q^{(1,x_0)}B\right)(s)\right)ds\\ &\leq 4M\int_0^t\left(|l(s)-\omega(s)|+|\omega(s)|\right)4M\left(\varepsilon+N_2s\right)\beta_{pc}(B)\,ds\\ &\leq 4^2M^2\left[\varepsilon\left(\varepsilon+N_2t\right)+N_2\left(t\varepsilon+N_2\frac{t^2}{2}\right)\right]\beta_{pc}(B)\\ &=4^2M^2\left[\varepsilon^2+C_2^1\varepsilon\left(N_2t\right)+\frac{\left(N_2t\right)^2}{2!}\right]\beta_{pc}(B).\end{aligned} \tag{27}$$

(28)

Thus,

$$\begin{aligned}&\beta\left(\left(Q^{(3,x_0)}B\right)(t)\right)\\ &\leq \beta\left(S(t)\,g\left(\overline{\text{co}}\left\{\left(Q^{(2,x_0)}B\right),x_0\right\}\right)\right)\\ &\quad+\beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\text{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right)\\ &\quad+\beta\left(\sum_{i=1}^{p}S(t-t_i)\,I_i\left(\overline{\text{co}}\left\{\left(Q^{(2,x_0)}B\right)(t_i),x_0(t_i)\right\}\right)\right)\\ &=\beta\left(\int_0^t S(t-s)\,f\left(s,\overline{\text{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)ds\right)\\ &\leq 4M\int_0^t\beta\left(f\left(s,\overline{\text{co}}\left\{\left(Q^{(2,x_0)}B\right)(s),x_0(s)\right\}\right)\right)ds\\ &\leq 4M\int_0^t l(s)\,\beta\left(\left(Q^{(2,x_0)}B\right)(s)\right)ds\\ &\leq 4M\int_0^t\left(|l(s)-\omega(s)|+|\omega(s)|\right)\\ &\quad\times 4^2M^2\left[\varepsilon^2+C_2^1\varepsilon\left(N_2s\right)+\frac{\left(N_2s\right)^2}{2!}\right]ds\beta_{pc}(B)\\ &\leq 4^3M^3\left[\varepsilon\left(\varepsilon^2+C_2^1\varepsilon\left(N_2t\right)+\frac{\left(N_2t\right)^2}{2!}\right)\right.\\ &\qquad\left.+N_2\int_0^t\left(\varepsilon^2+C_2^1\varepsilon\left(N_2s\right)+\frac{\left(N_2s\right)^2}{2!}\right)ds\right]\beta_{pc}(B)\\ &=4^3M^3\left[\varepsilon^3+C_3^1\varepsilon^2\left(N_2t\right)+C_3^2\varepsilon\frac{\left(N_2t\right)^2}{2!}\right.\\ &\qquad\left.+\frac{\left(N_2t\right)^3}{3!}\right]\beta_{pc}(B),\end{aligned} \tag{29}$$

and hence, by the method of mathematical induction, for any positive integer n and $t\in[0,T]$, we obtain

$$\begin{aligned}&\beta\left(\left(Q^{(n,x_0)}B\right)(t)\right)\\ &\leq 4^nM^n\left[\varepsilon^n+C_n^1\varepsilon^{n-1}\left(N_2t\right)\right.\\ &\qquad\left.+C_n^2\varepsilon^{n-2}\frac{\left(N_2t\right)^2}{2!}+\cdots+\frac{\left(N_2t\right)^n}{n!}\right]\beta_{pc}(B).\end{aligned} \tag{30}$$

Therefore, for any positive integer n, we have

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \le 4^n M^n \left[\varepsilon^n + C_n^1 \varepsilon^{n-1}(N_2T) + C_n^2\varepsilon^{n-2}\frac{(N_2T)^2}{2!} + \cdots + \frac{(N_2T)^n}{n!}\right]\beta_{pc}(B). \quad (31)$$

Since $\lim_{n\to+\infty}[\varepsilon^{n-1}n(n/(n-1))^{n-1}]^{1/n} = \varepsilon$, it follows from the Stirling Formula (see [28]) that

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \le o\left(\frac{1}{n^s}\right)4^n M^n \beta_{pc}(B), \quad n \longrightarrow \infty,\ \forall s > 1, \quad (32)$$

and hence, there exists sufficiently large positive integer n_0 such that

$$\beta_{pc}\left(Q^{(n_0,x_0)}B\right) < \beta_{pc}(B), \quad (33)$$

which shows that $Q : W \to W$ is a convex-power condensing operator. From Lemma 4, we get that Q has at least one fixed point in W; that is, (1) has at least one mild solution $u \in W$. This completes the proof. □

Remark 12. The technique of constructing convex-power condensing operator plays a key role in the proof of Theorem 10, which enables us to get rid of the strict inequality restriction on the coefficient function $l(t)$ of f. However, in many previous articles, such as [9, 12, 13, 19, 20], the authors had to impose a strong inequality condition on the integrable function $l(t)$, as they used Darbo-Sadovskii's fixed point theorem only. Thus, our result extends and complements those obtained in [9, 12, 13, 19, 20] and has more broad applications.

Remark 13. If we use the following assumption instead of $(H_f)(3)$:

$(H_f)(3')$ there exists a constant $l > 0$ such that

$$\beta(f(t,D)) \le l\beta(D) \quad (34)$$

for a.e. $t \in [0,T]$ and every bounded subset $D \subset B_r$,

we may use the same method to obtain

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \le 4^n M^n l^n \frac{T^n}{n!}\beta_{pc}(B), \quad (35)$$

for any $B \subset W$. Thus, there exists a large enough positive integral n_0 such that

$$4^{n_0} M^{n_0} l^{n_0}\frac{T^{n_0}}{n_0!} < 1; \quad (36)$$

namely,

$$\beta_{pc}\left(Q^{(n,x_0)}B\right) \le \beta_{pc}(B), \quad B \subset W. \quad (37)$$

Therefore, we can get the following consequence.

Theorem 14. *If the hypotheses (H_A), $(H_f)(1)(2)(3')$, (H_g), (H_I), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on* $[0,T]$.

4. g Is Lipschitz Continuous

In this section, by applying the proof of Theorem 10 and Darbo-Sadovskii's fixed point theorem, we give the existence of mild solutions of the problem (1) when the nonlocal condition g is Lipscitz continuous in $PC([0,T];X)$.

We give the following hypotheses:

(H_g') there exists a constant $k > 0$ such that

$$\|g(u) - g(v)\| \le k\|u - v\|, \quad \text{for } u, v \in PC([0,T];X); \quad (38)$$

(H_I') $I_i : X \to X$ is Lipschitz continuous with Lipschitz constant k_i, for $i = 1, 2, \ldots, p$.

Theorem 15. *If the hypotheses (H_A), $(H_f)(1)(2)(3)$, (H_g'), (H_I'), and (H_r) are satisfied, then the nonlocal problem* (1) *has at least one mild solution on $[0,T]$ provided that $M(k+\sum_{i=1}^{p}k_i+4\|l\|_{L^1}) < 1$.*

Proof of Theorem 15. Given $x \in W$, let's first consider the following Cauchy initial problem:

$$\begin{gathered} u'(t) = Au(t) + f(t,u(t)), \quad t \in [0,T],\ t \ne t_i, \\ \Delta u(t_i) = I_i(x(t_i)), \quad i = 1, 2, \ldots, p, \\ 0 < t_1 < t_2 < \cdots < t_p < T, \\ u(0) = g(x). \end{gathered} \quad (39)$$

From the proof of Theorem 10, we can easily see that there exists at least one mild solution to (39). Define $G : W \to W$ by that Gx is the mild solution to (39). Then

$$(Gx)(t) = S(t)g(x) + \sum_{0<t_i<t} S(t-t_i)I_i(x(t_i)) + \int_0^t S(t-s)f(s,(Gx)(s))\,ds. \quad (40)$$

Now, we will show that G is β_{pc}-condensing on W. According to Lemma 11, for any bounded subset $B \subset W$, we deduce

$$\begin{aligned} &\beta((GB)(t)) \\ &\le \beta(S(t)g(B)) + \sum_{0<t_i<t} S(t-t_i)I_i(B(t_i)) \\ &\quad + \beta\left(\int_0^t S(t-s)f(s,(GB)(s))\,ds\right) \\ &\le Mk\beta_{pc}(B) + M\sum_{i=1}^{p}k_i\beta_{pc}(B) \\ &\quad + 4M\int_0^t \beta(f(s,(GB)(s)))\,ds \end{aligned}$$

$$\begin{aligned}&\leq M\left(k+\sum_{i=1}^{p}k_i\right)\beta_{pc}(B)\\&\quad+4M\int_0^t l(s)\,\beta((GB)(s))\,ds\\&\leq M\left(k+\sum_{i=1}^{p}k_i\right)\beta_{pc}(B)\\&\quad+4M\|l\|_{L^1}\beta_{pc}(GB),\end{aligned}\tag{41}$$

which implies that

$$\beta_{pc}(GB)\leq\frac{M\left(k+\sum_{i=1}^{p}k_i\right)}{1-4M\|l\|_{L^1}}\beta_{pc}(B).\tag{42}$$

In addition, since $M(k+\sum_{i=1}^{p}k_i+4\|l\|_{L^1})<1$, it follows that the mapping K is a β_{pc}-condensing operator on W. In view of Lemma 2, the mapping G has at least one fixed point in W, which produces a mild solution for the nonlocal impulsive problem (1). □

Remark 16. Similarly, one can show that the conclusion of Theorem 15 remains valid provided that hypothesis $(H_f)(3)$ is replaced by condition $(H_f)(3')$.

Remark 17. In Theorem 15, we do not assume the compactness of nonlocal item g. Under the Lipschitz assumption, we make full use of the conclusion of Theorem 10, the properties of noncompact measure and the technique of fixed point to deal with the solution operator G.

Remark 18. Recently, the existence results for fractional differential equations have been widely studied in many papers. For more details on this theory one can refer to [29, 30] and references therein. It should be pointed out that the techniques and ideas in this paper can also be used to study fractional equations. In the future, we will also try to investigate to nonlocal controllability of impulsive differential equations by applying the similar techniques, methods, and compactness conditions. Further discussions on this topic will be in our consequent papers.

5. Examples

In this section, we shall give two examples to illustrate Theorems 10 and 15.

Example 1. Consider the following semilinear parabolic system:

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,x)=-A(x,D)\,u(t,x)+F(t,u(t,x)),\\&\qquad\qquad t\in[0,T],\ x\in\Omega,\ t\neq t_i,\\&D^{\alpha}u(t,x)=0,\quad t\in[0,T],\ x\in\partial\Omega\quad\text{for }|\alpha|\leq m,\\&u(t_i^{+},x)-u(t_i^{-},x)=I_i(u(t_i,x)),\quad i=1,2,\ldots,p,\\&u(0,x)=\int_{\Omega}\int_0^T\Gamma\left(t,x,x',u\left(t,x'\right)\right)dt\,dx',\quad x\in\Omega,\end{aligned}\tag{43}$$

where Ω is a bounded domain in R^n $(n\geq 1)$ with smooth boundary $\partial\Omega$, $A(x,D)u=\Sigma_{|\alpha|\leq 2m}a_{\alpha}(x)D^{\alpha}u$ is strongly elliptic, $F:[0,T]\times R\to R$, and $\Gamma:[0,T]\times\Omega\times\Omega\times R\to R$.

Let $X=L^2(\Omega)$ and define the operator $A:D(A)\subseteq X\to X$ by

$$\begin{aligned}D(A)&=H^{2m}(\Omega)\cap H_0^m(\Omega),\\Au&=-A(x,D)\,u.\end{aligned}\tag{44}$$

Then the operator A is an infinitesimal generator of an equicontinuous C_0-semigroup $S(t)$ on X (see [26]).

Suppose that the function $\Gamma:[0,T]\times\Omega\times\Omega\times R\to R$ satisfies the following conditions:

(i) the Carathéodory condition, that is, $\Gamma(t,x,x',r)$, is a continuous function about r for a.e. $(t,x,x')\in[0,T]\times\Omega\times\Omega$; $\Gamma(t,x,x',r)$ is measurable about (t,x,x') for each fixed $r\in R$;

(ii) $|\Gamma(t,x,x',r)-\Gamma(t,\overline{x},x',r)|\leq\mu_k(t,x,\overline{x},x')$ for all $(t,x,x',r),(t,\overline{x},x',r)\in[0,T]\times\Omega\times\Omega\times R$ with $|r|\leq k$, where $\mu_k\in L^1([0,T]\times\Omega\times\Omega\times R;R^+)$ satisfies $\lim_{x\to\overline{x}}\int_{\Omega}\int_0^T\mu_k(t,x,\overline{x},x')dt\,dx'=0$, uniformly in $\overline{x}\in\Omega$;

(iii) $|\Gamma(t,x,x',r)|\leq(\delta/Tm(\Omega))|r|+\Phi(t,x,x')$ for all $r\in R$, where $\Phi\in L^2([0,T]\times\Omega\times\Omega;R^+)$ and $\delta>0$.

We assume the following.

(1) $f:[0,T]\times X\to X$ is defined by

$$f(t,z)(x)=F(t,z(x)),\quad x\in\Omega.\tag{45}$$

Moreover, for given $r>0$, there exist two integrable functions ϕ_r, $\psi:[0,T]\to R$ such that $\|f(t,z)\|\leq\phi_r(t)$ and $\beta(f(t,D))\leq\psi(t)\beta(D)$ for a.e. $t\in[0,T]$, $z\in B_r$ and every bounded subset $D\subset B_r$;

(2) $g:C([0,T];X)\to X$ is defined by

$$g(u)(x)=\int_{\Omega}\int_0^T\Gamma\left(t,x,x',u\left(t,x'\right)\right)dt\,dx',\quad x\in\Omega.\tag{46}$$

From Theorem 4.2 in [31], we get directly that g is well defined and is a completely continuous operator by the above conditions about the function Γ.

(3) $I_i:X\to X$ is a continuous and compact function for each $i=1,2,\ldots,p$, defined by

$$I_i(u)(x)=I_i(u(x)).\tag{47}$$

Let us observe that the problem (43) may be reformulated as the abstract problem (1) under the above conditions. By using Theorem 10, the problem (43) has at least one mild solution $u\in C([0,T];L^2(\Omega))$ provided that the hypothesis (H_r) holds.

Example 2. Consider the following partial differential system:

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x)=\frac{\partial}{\partial x}u(t,x)+F(t,u(t,x)),\\
&\qquad t\in[0,T],\ x\in\Omega,\ t\neq t_i,\\
&u(t,x)=0,\quad t\in[0,T],\ x\in\partial\Omega,\\
&u(t_i^+,x)-u(t_i^-,x)=I_i(u(t_i,x)),\quad i=1,2,\dots,p,\\
&u(0,x)=\sum_{j=1}^{q}c_j u(s_j,x),
\end{aligned}\tag{48}$$

where Ω is a bounded domain in $R^n\,(n\geq 1)$ with smooth boundary $\partial\Omega, F:[0,T]\times R\to R$, and c_j and s_j both are given real numbers for $j=1,2,\dots,q$.

Let $X=C(\overline{\Omega})$, and define the operator $A:D(A)\subseteq X\to X$ by

$$\begin{aligned}
D(A)&=\{v\in X:v'\in X, v(x)=0, x\in\partial\Omega\},\\
Av&=v'.
\end{aligned}\tag{49}$$

As is known to all, the operator A is an infinitesimal generator of the semigroup $S(t)$ defined by $S(t)x(s)=x(t+s)$ for each $x\in X$. Here, $S(t)$ is equicontinuous but is not compact.

We now suppose the following.

(1) $f:[0,T]\times X\to X$ is defined by

$$f(t,y)(x)=F(t,y(x)),\quad x\in\Omega.\tag{50}$$

Moreover, for given $r>0$, there exist two integrable functions $\varphi_r,\omega:[0,T]\to R$ such that $\|f(t,y)\|\leq\varphi_r(t)$ and $\beta(f(t,D))\leq\varpi(t)\beta(D)$ for a.e. $t\in[0,T]$, $y\in B_r$, and every bounded subset $D\subset B_r$;

(2) $g:PC([0,T];X)\to X$ is defined by

$$\begin{aligned}
g(u)(x)&=\sum_{j=1}^{q}c_j u(s_j,x),\\
0<s_1&<s_2<\cdots<s_q<T,\ x\in\Omega.
\end{aligned}\tag{51}$$

Then g is Lipschitz continuous with constant $k=\sum_{j=1}^{q}|c_j|$; that is, the assumption H_g' is satisfied.

(3) $I_i:X\to X$ is a continuous function for each $i=1,2,\dots,p$, defined by

$$I_i(u)(x)=I_i(u(x)).\tag{52}$$

Here we take $I_i(u(x))=(\alpha_i|u(x)|+t_i)^{-1}$, $\alpha_i>0$, $i=1,2,\dots,p$, $0<t_1<t_2<\cdots<t_q<T$, $x\in\Omega$. Then I_i is Lipschitz continuous with constant $k_i=\alpha_i/t_i^2$, $i=1,2,\dots,p$; that is, the assumption H_I' is satisfied.

Let us observe that (48) may be rewritten as the abstract problem (1) under the above conditions. If the following inequality

$$M\left(\sum_{j=1}^{q}|c_j|+\sum_{i=1}^{p}\frac{\alpha_i}{t_i^2}+4\|\varpi\|_{L^1}\right)<1\tag{53}$$

holds, then according to Theorem 15, the impulsive problem (48) has at least one mild solution in $PC([0,T];X)$.

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