Burgers turbulence

Jérémie Bec

CNRS, Observatoire de la Côte d'Azur

Nice, France

Multidimensional Burgers equation

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} + \nabla F$$

$$\boldsymbol{u}(\boldsymbol{x}, t) \in \mathbb{R}^d \qquad \boldsymbol{x} \in \mathbb{Z}^d, \mathbb{R}^d$$

$\boldsymbol{u}(\boldsymbol{x}, t_0) = \boldsymbol{u}_0(\boldsymbol{x})$ initial condition

or

$F(\boldsymbol{x}, t) =$ external forcing potential

random

Burgers **turbulence**

If $\boldsymbol{u}_0(\boldsymbol{x}) = \nabla \Psi_0(\boldsymbol{x})$, then $\boldsymbol{u}(\boldsymbol{x}, t) = \nabla \Psi(\boldsymbol{x}, t)$ at any later time

Ψ solves **Hamilton-Jacobi** equation $\partial_t \Psi + \frac{1}{2} |\nabla \Psi|^2 = \nu \nabla^2 \Psi + F$

Cole-Hopf transformation $\mathcal{Z} = \exp\left[\Psi/(2\nu)\right]$

$\partial_t \mathcal{Z} = \nu \nabla^2 \mathcal{Z} + \frac{1}{2\nu} F \, \mathcal{Z}$ **Schrödinger** equation (at imaginary times)

References

- Gurbatov, Malakhov & Saichev, *Nonlinear random waves and turbulence in nondispersive media*, Manchester UP 1991
- Woyczynski, *Burgers-KPZ turbulence: Gottingen lecture*, Springer 1998
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- E & Vanden Eijnden, Statistical theory for the Burgers equation in the inviscid limit, Comm. Pure Appl. Math. 2000
- E, Khanin, Mazel & Sinai, *Invariant measures for Burgers equation with stochastic forcing*, Ann. Math. 2000
- Fathi, *Weak KAM theorem in Lagrangian dynamics*, CUP 2003
- Iturriaga & Khanin, *Burgers turbulence and random Lagrangian systems,* Commun. Math. Phys. 2003
- Bec & Khanin, *Burgers turbulence*, Phys. Rep. 2007, nlin.CD/

Short history

- **1939:** Burgers introduces a 1D model for turbulence
  same type of “hydrodynamic nonlinearity”, same invariances

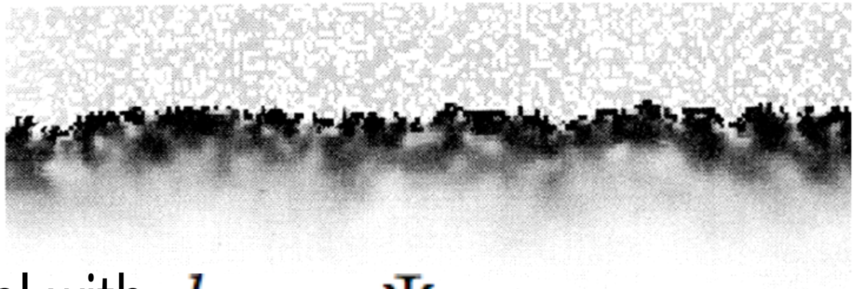
- **1950’s:** Cole and Hopf show that it is integrable
  does not reproduce a fundamental aspect of turbulence: chaoticity
  but still used as nonlinear hyperbolic conservation law (e.g. P.D. Lax)

- **1980’s:** reappears in statistical physics and astrophysics
  under a multi-dimensional or random form

- **End 1990’s:** becomes a benchmark for turbulence
  to test numerical methods, closures, statistical tools, mathematical construction of an invariant measure

Burgers equation in statistical physics

- **Deposition / Interface growth**
  Hamilton-Jacobi equation with
  δ-correlated forcing potential



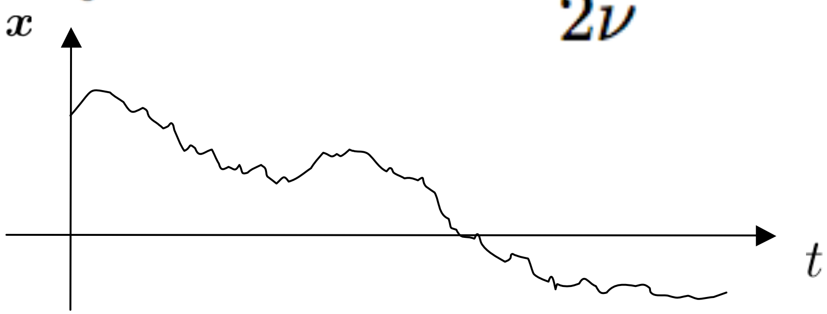
  ⇔ **Kardar-Parisi-Zhang** model with $h = -\Psi$

  (see book by Barabási & Stanley, CUP 1995)

- **Directed polymers in a random medium**

$$\partial_t \mathcal{Z} = \nu \nabla^2 \mathcal{Z} + \frac{1}{2\nu} F\, \mathcal{Z}$$

equation for the partition function of an elastic string in the random potential $F/2\nu$



(Bouchaud, Mezard & Parisi, PRE 1995)

Time t ⇒ "preferential" direction
Space x ⇒ transverse directions
Elastic modulus $c = 1/(2\nu)$

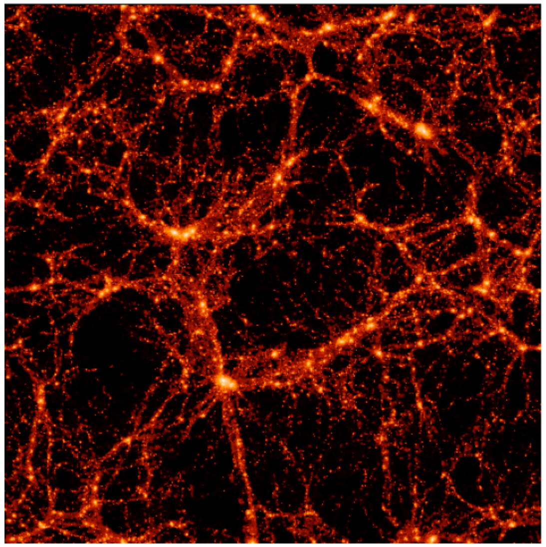
- **Models for vehicular traffic flows** (Chowdhury, Santen & Schadschneider, Phys. Rep. 2000)

Large-scale structures of the Universe

- After decoupling (baryons / photons): **Vlasov-Poisson** kinetics

$$\partial_t f + \frac{1}{ma^2}\nabla_x \cdot (f\boldsymbol{p}) - m\nabla_p \cdot (f\nabla_x \Phi) = 0$$

$$\nabla_x^2 \Phi = 4\pi\mathcal{G}a^2(\rho - \bar{\rho})$$

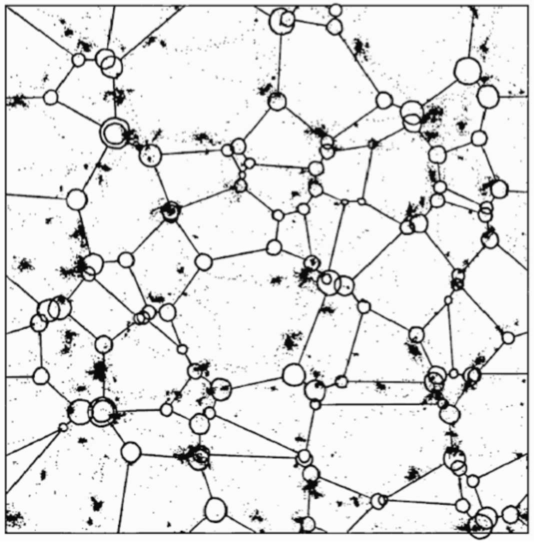


- Zel'dovich 1970 : Initial distribution = mono-kinetic + potential at leading order

$\Longrightarrow$ **Adhesion model** (Gurbatov & Saichev 1984)

- Good approximation to understand the distribution of matter at large scales

books by Peebles (Princeton University Press 1993) or by Coles & Lucchin (Wiley & sons, 1995)



from Kofman *et al.* 1992

Benchmark for hydrodynamic turbulence

- **Numerics:** test methods (most literature)
- **Physics:** - Understand the singularities and their statistical signatures
  - Test universality of small scales w.r.t. large-scale forcin
  - Test methods borrowed from other fields (e.g. treatment of dissipative anomaly by a field theoretical operator product expansion)
- **Maths:** - Construct a statistically stationary state (invariant measure)
  - Weak Kolmogorov-Arnold-Moser (KAM) theory
  - Random Lagrangian systems
- Burgers equation appears asymptotically in many problems:
  - Compressible turbulence
  - Inelastic granular gases (Ben Naïm)
  - Random nonlinear waves in nondispersive media (Gurbatov & Saichev)

Unforced Burgers equation

$$F = 0$$

Ψ_0 random and smooth (twice differentiable)

Inviscid limit of unforced Burgers

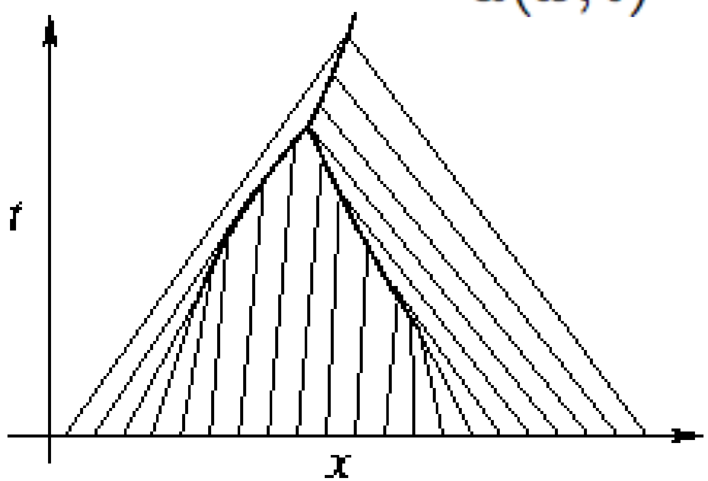
$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nu\nabla^2\boldsymbol{u} \qquad \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) = -\nabla\Psi_0(\boldsymbol{x})$$

- Limit of vanishing viscosity: $\nu \to 0$

$$\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} = 0$$

means that velocity is conserved along fluid particle trajectories $\boldsymbol{x}(t)$ (characteristics) solutions of $\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x},t)$

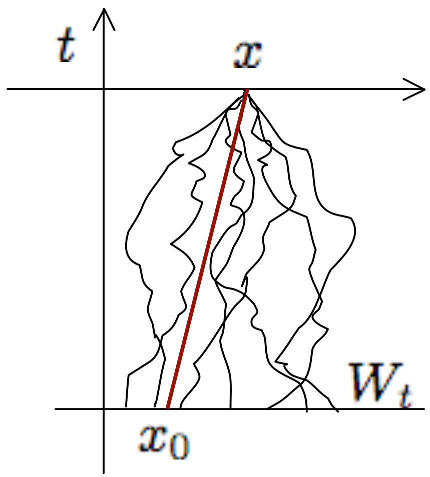
$$\boldsymbol{x} = \boldsymbol{x}_0 + t\boldsymbol{u}_0(\boldsymbol{x}_0)$$
$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x}_0)$$



What's happening after the first crossing of trajectories?

Viscosity/entropic solutions

- Limit of vanishing viscosity defines a unique solution (**viscosity** or **entropic** solution) of the inviscid Burgers equation
- **Cole-Hopf** $\boldsymbol{u} = -2\nu\nabla\ln\Theta \;\Rightarrow$ heat equation $\partial_t\Theta = \nu\nabla^2\Theta$
- Feynman-Kac: $\Theta(\boldsymbol{x},t) = \langle\exp[\Psi_0(\boldsymbol{W}_t)/(2\nu)]\rangle$



$\boldsymbol{W}_t \;=\; d$ -dimensional Brownian motion

with $\begin{cases} \boldsymbol{W}_0 = \boldsymbol{x} \\ \langle W_t^i W_s^j\rangle = 2\nu\min(s,t)\delta^{ij} \end{cases}$

- $\nu\to 0$: saddle point $\Rightarrow$ **Maximum principle**

$$\Psi(\boldsymbol{x},t) = \max_{\boldsymbol{x}_0}\left[\Psi_0(\boldsymbol{x}_0) - \frac{|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2t}\right]$$

- "Euler-Lagrange" equations = characteristics but maximum principle allows choosing the **viscosity** solution

Singularities

$$\Psi(\boldsymbol{x},t) = \max_{\boldsymbol{x}_0}\left[\Psi_0(\boldsymbol{x}_0) - \frac{|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2t}\right]$$

- If Ψ_0 is smooth and generic, the maximum is attained almost everywhere for a unique value of $\boldsymbol{x}_0$
- Generically, the set of points where the maximum is attained for two or more distinct values of $\boldsymbol{x}_0$ form a manifold of co-dimension 1

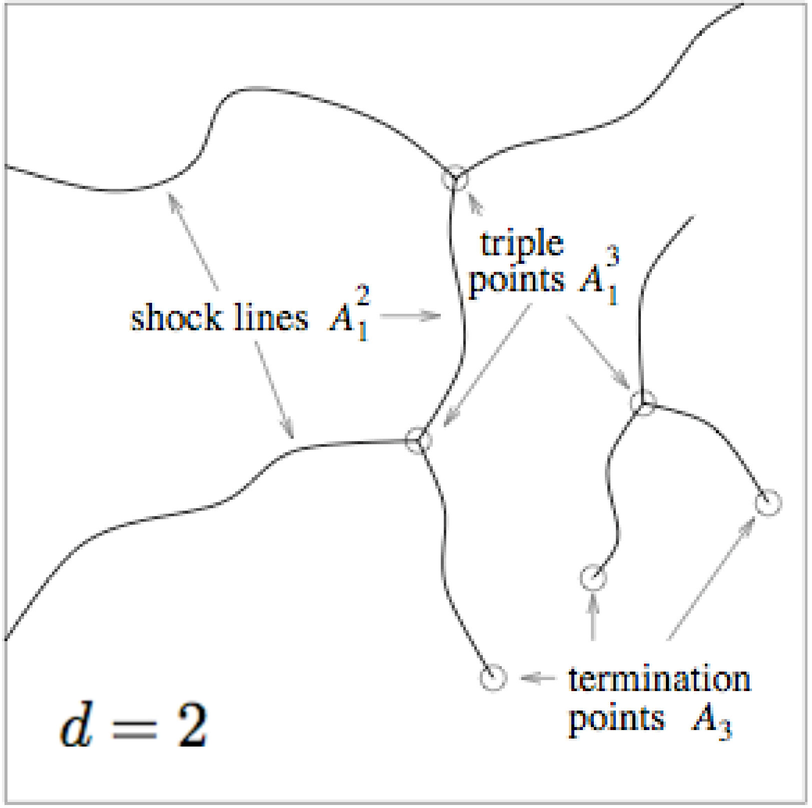
  $=$ **shocks** $=$ discontinuities of the velocity field
- Isolated points for $d=1$, curves for $d=2$, surfaces for $d=3$
- Locations where the minimum is attained for n distinct values of $\boldsymbol{x}_0$ form a sub-manifold of co-dimension $n-1$
- $+$ other singularities when the maximum is degenerate

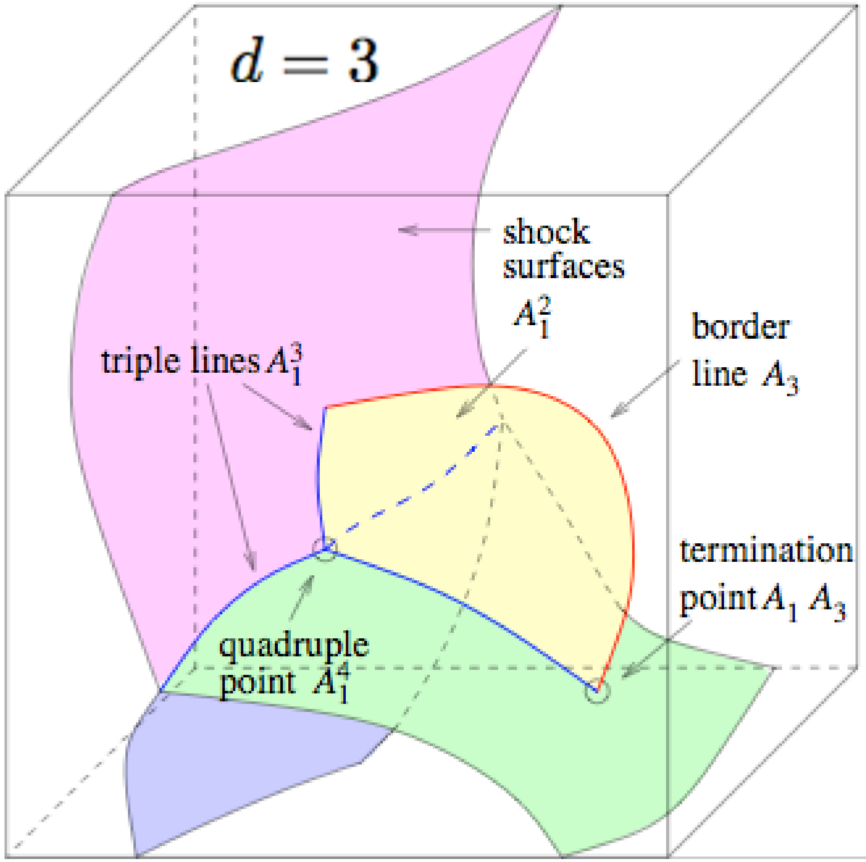
Classification

- Arnold, Baryshnikov, and Bogaevski (1991) proposed a classification of all singularities and their metamorphoses in 1, 2 and 3D

A_n^m — m: number of points where the maximum is attained; n: multiplicity of the maximum

A_1^2 = shocks, A_3 = termination points of a shock line

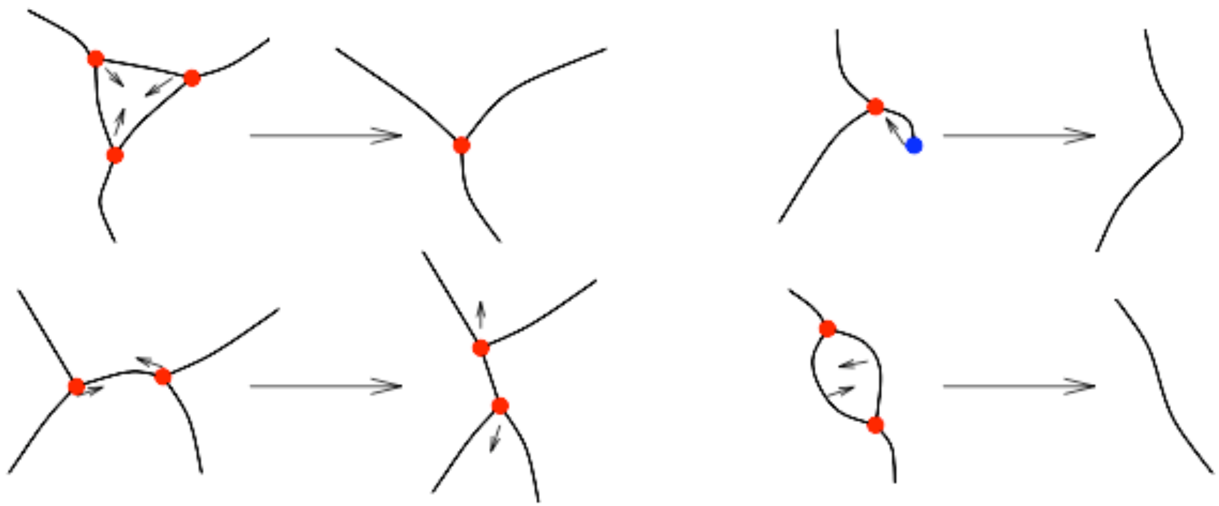




Metamorphoses

- The singularities of co-dimension $d+1$ appear at discrete times
  Irreversibility of Burgers equation restricts admissible metamorphoses

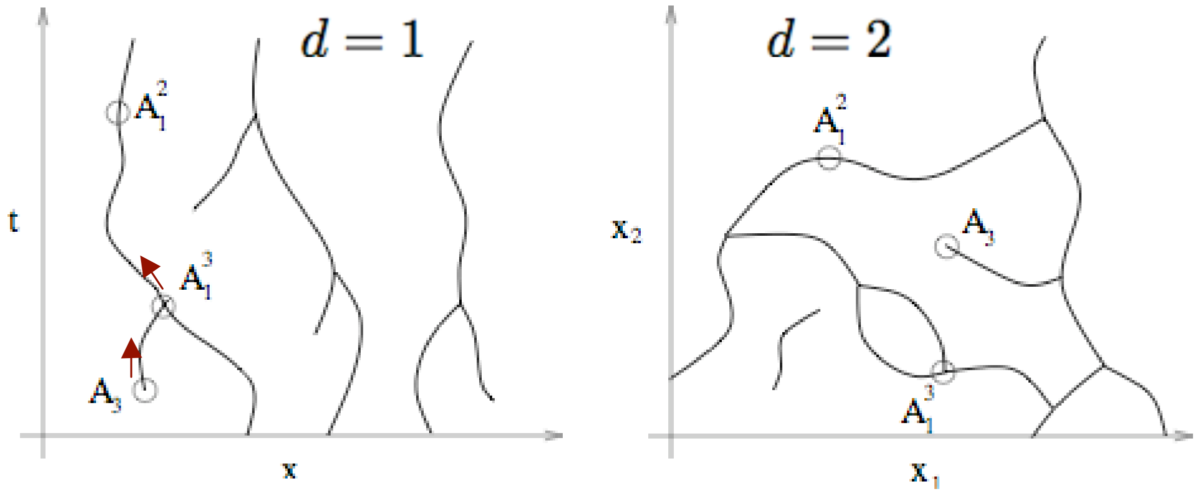
  Bogaevski (2002): right after the bifurcation the singular manifold has to be **locally contractible** (homotopic to a point)

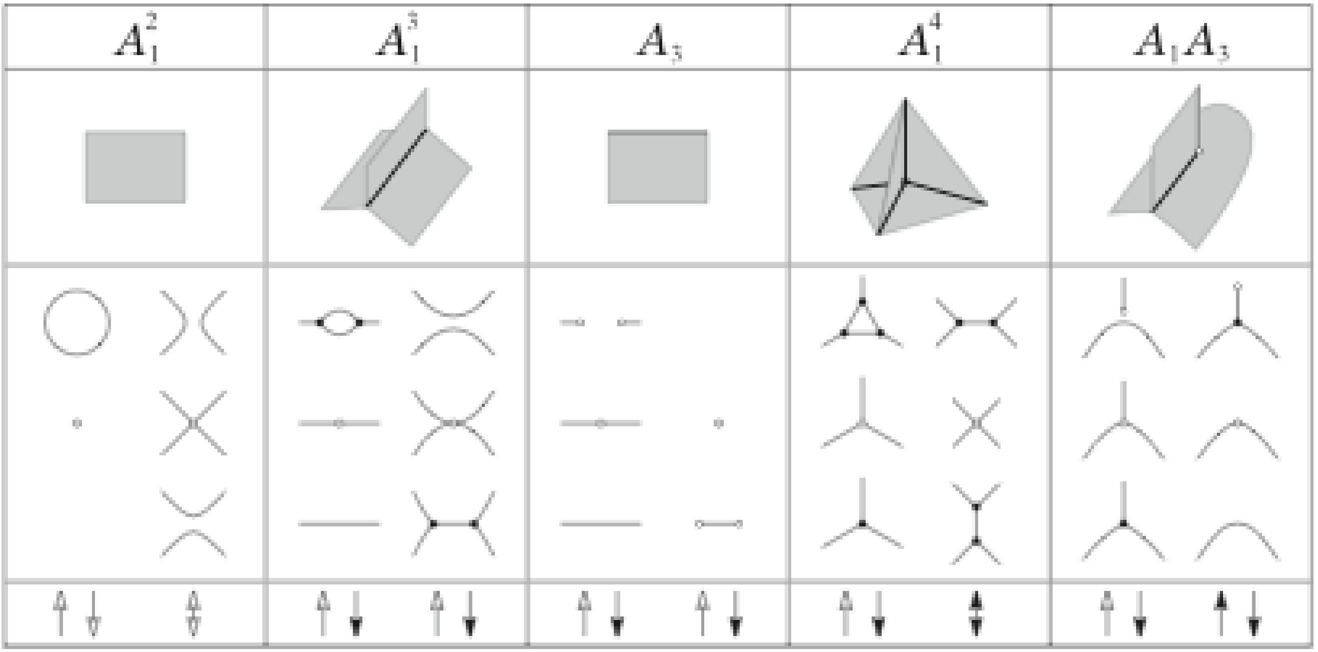


Applies to all **entropic** solutions to Hamilton-Jacobi with a **convex** Hamiltonian

Similarities and restrictions

Metamorphoses in dimension d are generically present in dimension $d+1$





Shocks and energy dissipation

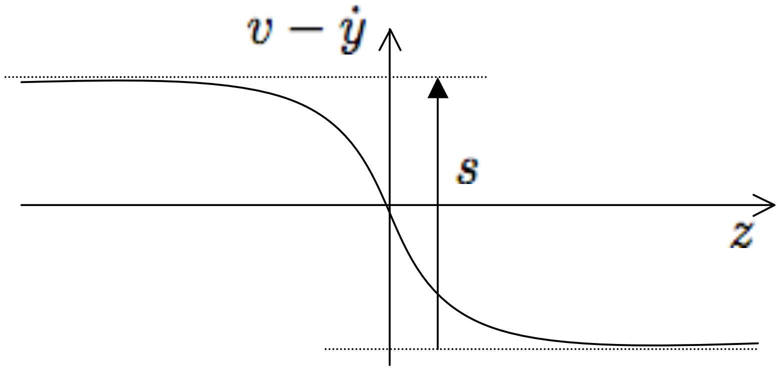
The discontinuities in the solution dissipate kinetic energy:

Matched asymptotics

Assume $y(t)$ is the position of a shock at time

Perturbative expansion in the limit of small $\nu > 0$

$$z = \frac{x - y(t)}{\nu} \qquad u(x,t) = v\,(z,t) = v_0(z,t) + \nu v_1(z,t) + \nu^2 v_2(z,t) + \cdots$$

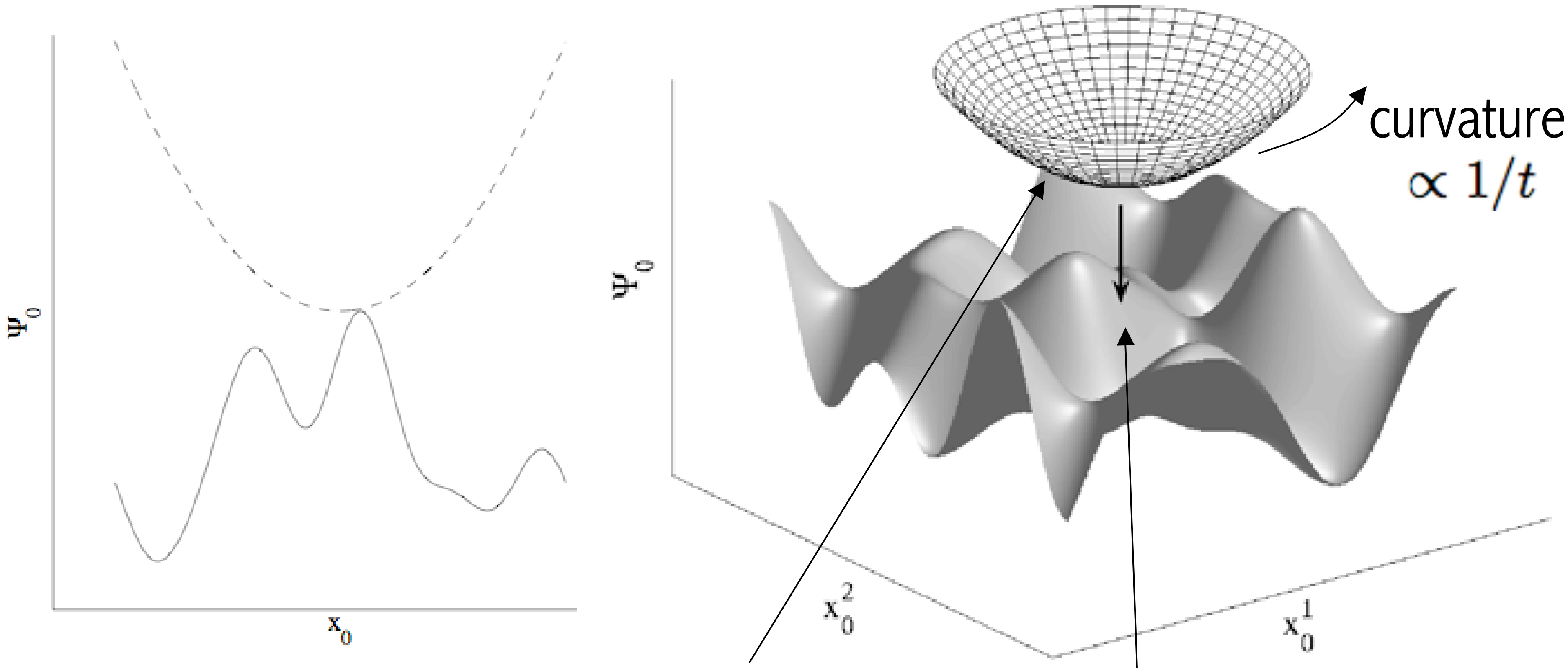


$$\dot{y} = (u_+ + u_-)/2$$
$$s \equiv u_- - u_+$$
$$v_0(z,t) = \dot{y} - \frac{s}{2}\tanh\left(\frac{sz}{4}\right)$$

$$\frac{d}{dt}\int u^2(x)dx = -\nu\int(\partial_x u)^2 dx \simeq -\frac{\nu s^2}{4}\int\left[\partial_x \tanh\left(\frac{s(x-y)}{4\nu}\right)\right]^2 dx$$
$$\simeq -\frac{s^3}{16}\int\frac{dz}{\cosh^4 z}$$

Geometrical constructions

$$\Psi(\boldsymbol{x},t) = \max_{\boldsymbol{x}_0}\left[\Psi_0(\boldsymbol{x}_0) - \frac{|\boldsymbol{x}-\boldsymbol{x}_0|^2}{2t}\right]$$



Place where it touches
$=$ Lagrangian antecedent of x

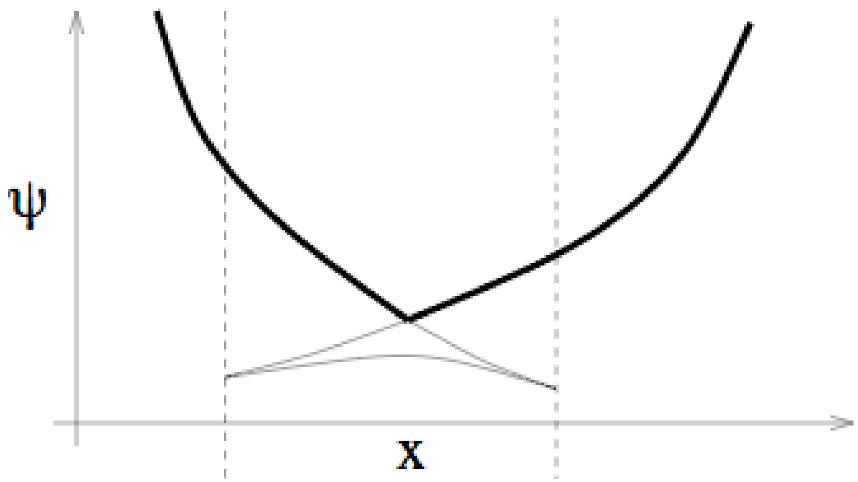
Paraboloid centred at x

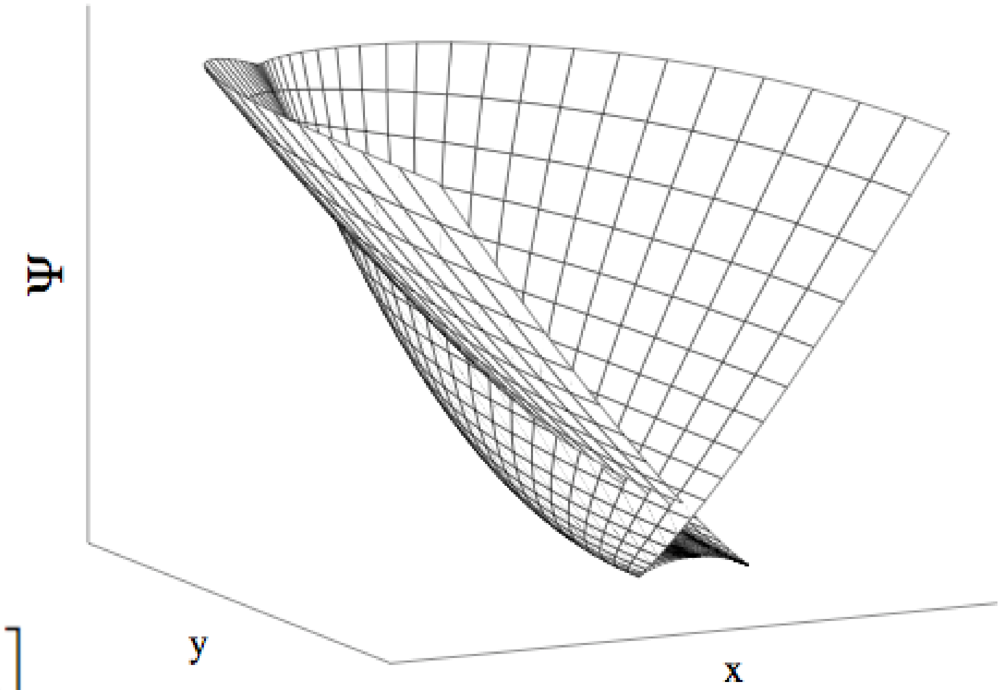
Geometrical constructions

- **Potential Lagrangian manifold**

  (d-1)-dimensional manifold of $(\boldsymbol{x}, \Psi)$ parameterized by $\boldsymbol{x}_0$

$$\begin{cases} \boldsymbol{x} = \boldsymbol{x}_0 - t\nabla\Psi_0(\boldsymbol{x}_0) \\ \Psi = \Psi_0(\boldsymbol{x}_0) - \dfrac{t}{2}|\nabla\Psi_0(\boldsymbol{x}_0)|^2 \end{cases}$$





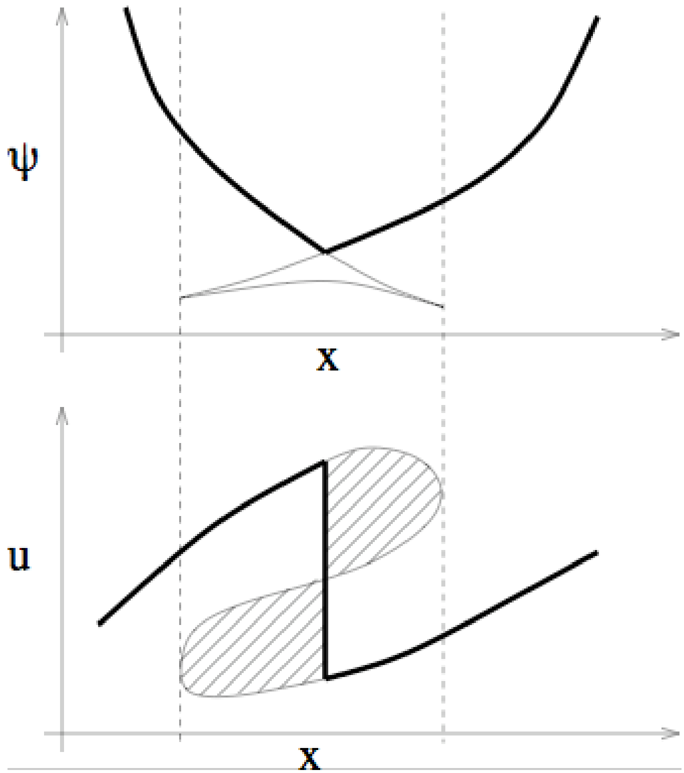
$$\Psi(\boldsymbol{x}, t) = \max_{\boldsymbol{x}_0} \left[\Psi_0(\boldsymbol{x}_0) - \frac{|\boldsymbol{x} - \boldsymbol{x}_0|^2}{2t} \right]$$

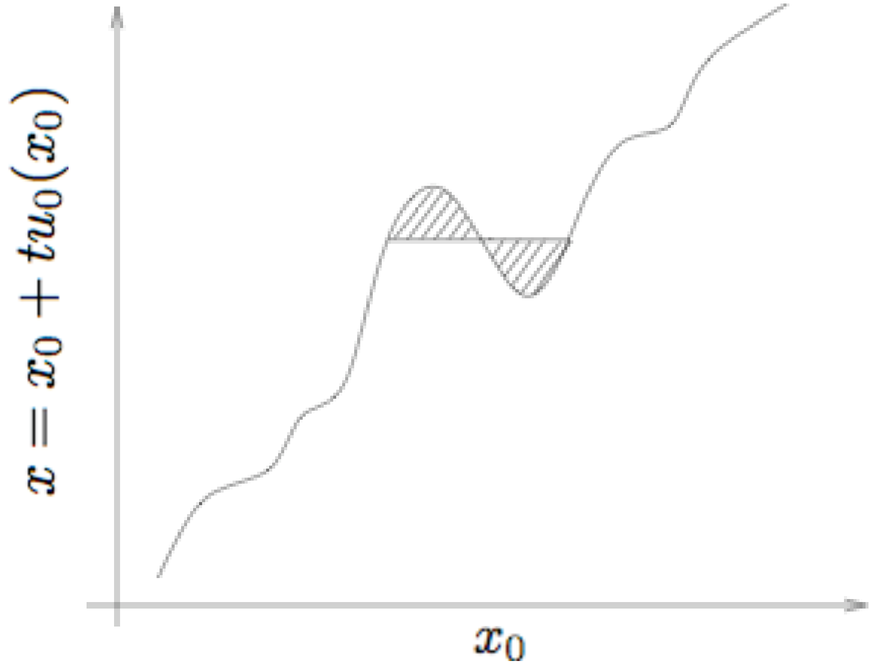
Geometrical constructions

- **Maxwell rule in 1D:** Ψ is continuous at singularities

$$\begin{cases} x = x_0 + tu_0(x_0) \\ u = u_0(x_0) \end{cases}$$

Lagrangian map $x_0 \mapsto x$



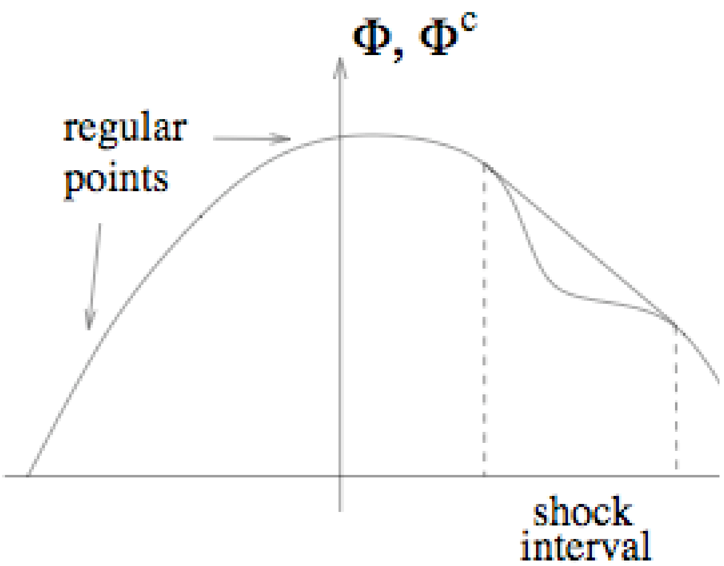


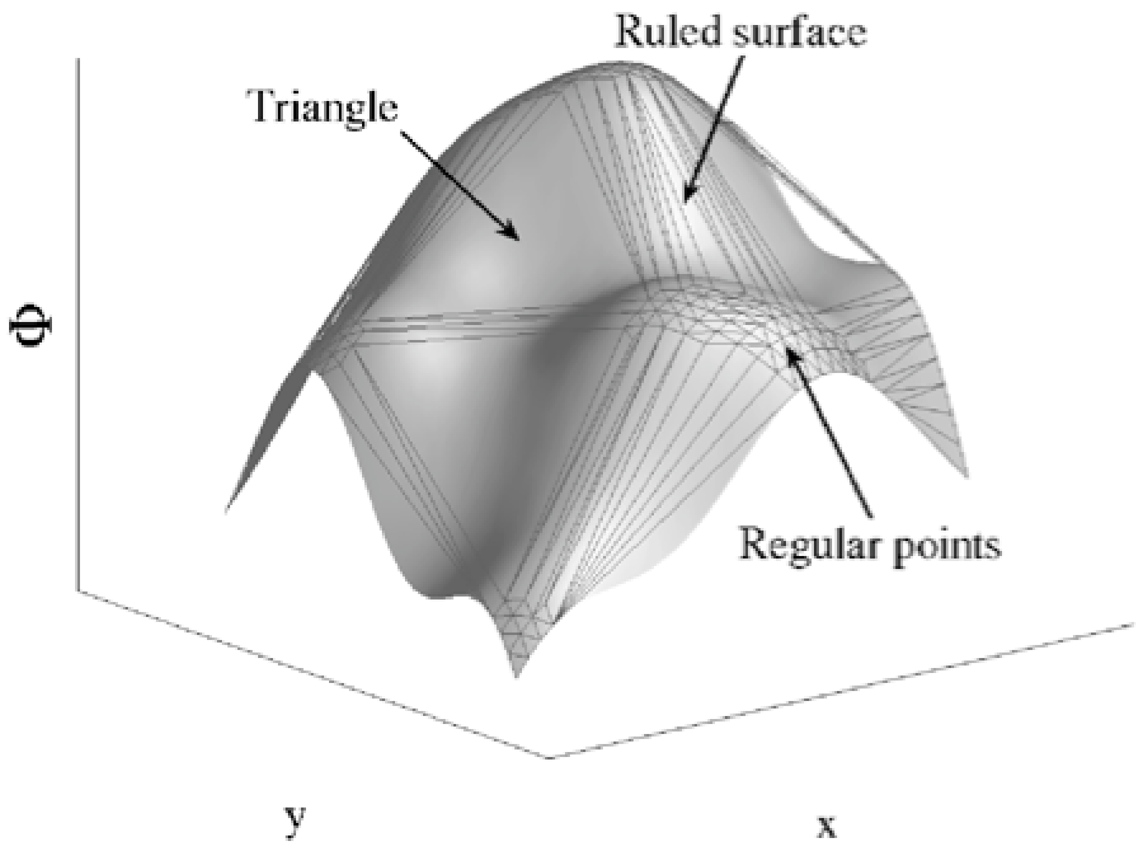
Geometrical constructions

- **Lagrangian potential:** $\Phi(\boldsymbol{x}_0, t) = t\Psi_0(\boldsymbol{x}_0) - \frac{1}{2}|\boldsymbol{x}_0|^2$

$$t\Psi(\boldsymbol{x}, t) + \frac{1}{2}|\boldsymbol{x}|^2 = \max_{\boldsymbol{x}_0}[\Phi(\boldsymbol{x}_0, t) - \boldsymbol{x}\cdot\boldsymbol{x}_0] \quad = \text{Legendre transform}$$
$$= \Phi(\boldsymbol{x}_0^\star, t) - \boldsymbol{x}\cdot\boldsymbol{x}_0^\star$$

$\boldsymbol{x}_0^\star$ solution of $\nabla_{\boldsymbol{x}_0}\Phi^c(\boldsymbol{x}_0^\star, t) - \boldsymbol{x} = 0$ where Φ^c is the convex hull of Φ

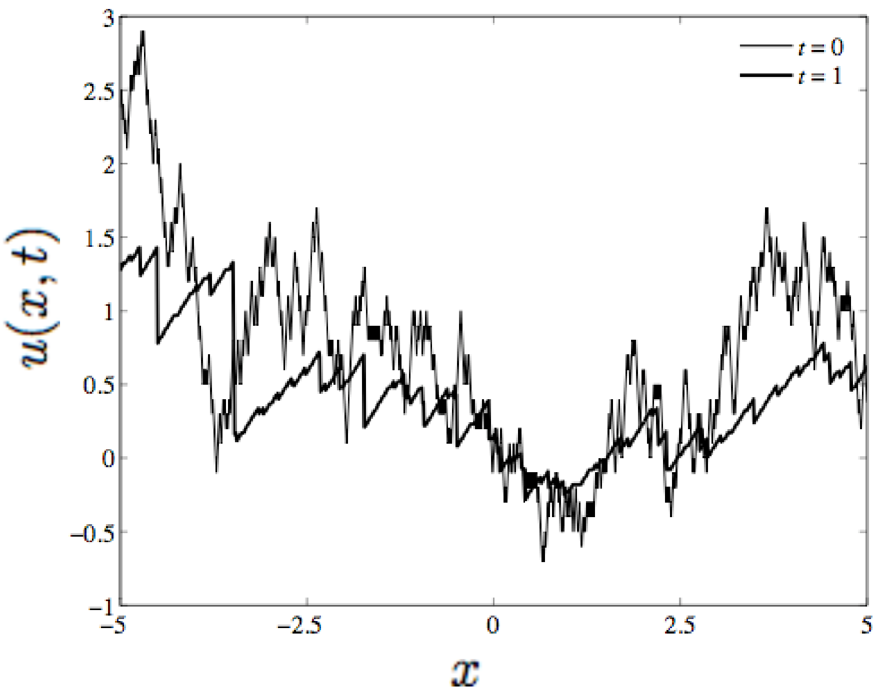


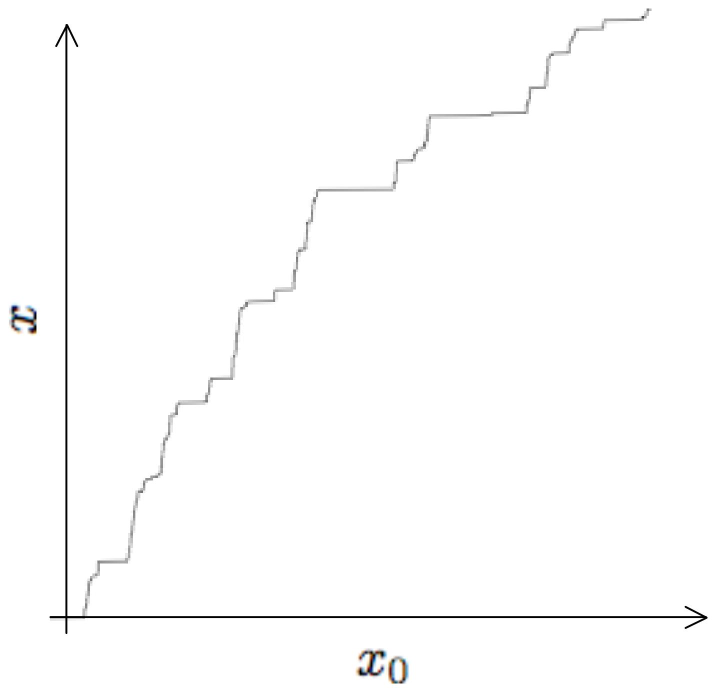


1D Burgers with Brownian velocities

She, Aurell, Frisch (CMP 1992); Sinai (CMP 1992);
Vergassola, Dubrulle, Frisch & Noullez (A&A 1994)

shocks are dense





Lagrangian regular points (particles which have not been captured by a shock) form a fractal set of Hausdorff dimension 1/2

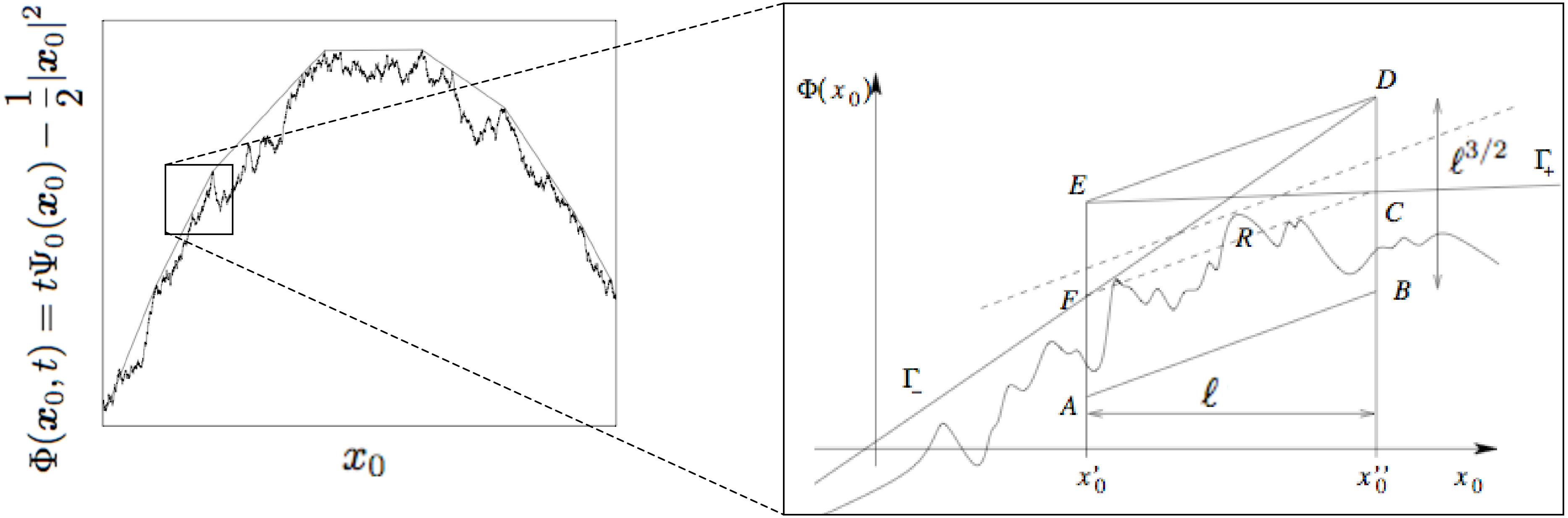
$\Rightarrow$

The graph of the Lagrangian map is a Devil's staircase

1D Burgers with Brownian velocities

In terms of the **Lagrangian potential**:

A point is regular if the graph of $\Phi(x_0, t)$ is below its tangent at this point



Probability that an interval of size ℓ contains at least one regular point

R regular $\Leftrightarrow$
- *Box:* cross (FC) and stay below (ED); enter (AF) [resp. exit (CB)] with slope larger [resp. smaller] than that of Γ_- [resp. Γ_+]
- *Left:* graph below the half line Γ_-
- *Right:* graph below the half line Γ_+

$u_0(x_0)$ Markov process $\Rightarrow$ *Box*, *Left* and *Right* are independent

$$P^{\rm reg}(\ell) = {\rm Prob}(Box) \times {\rm Prob}(Left) \times {\rm Prob}(Right)$$

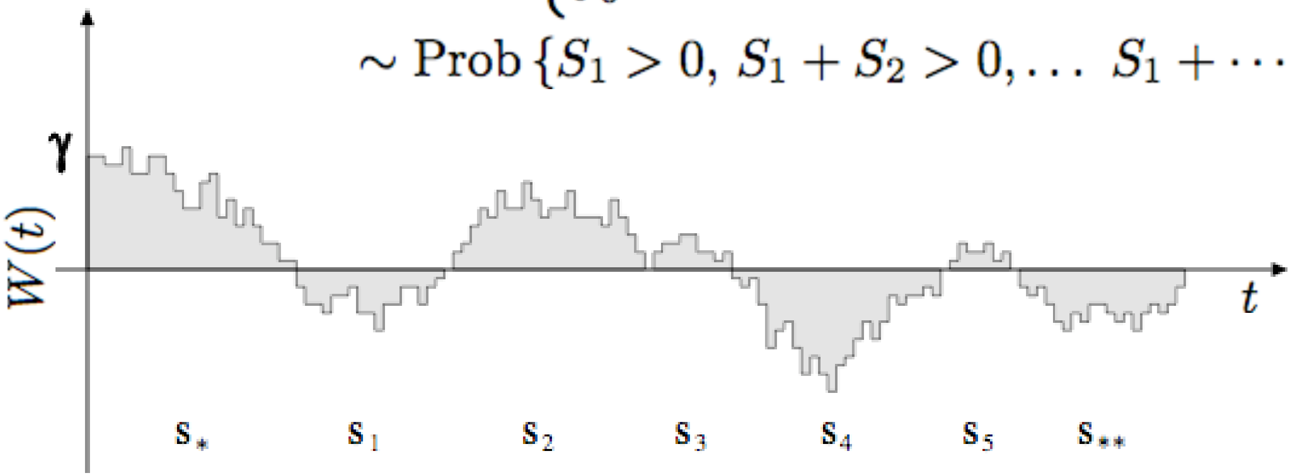
Choice of boxe's sizes $\Rightarrow$ ${\rm Prob}(Box) = const. \propto \ell^0$

Symetry $x_0 \mapsto -x_0 \quad \Rightarrow {\rm Prob}(Left) = {\rm Prob}(Right)$

What remains is ${\rm Prob}(Right)$, i.e. that the graph remains below

$$\Gamma_+ : x_0 \mapsto \Phi(x_0'') + \beta\ell^{3/2} + [(d^2\Phi/d^2x_0)(x_0'') + \gamma\ell^{1/2}](x_0 - x_0'')$$

$$\begin{aligned}
{\rm Prob}(Right) &= {\rm Prob}\left\{\int_0^x [u_0(x_0) + \gamma\ell^{1/2}]dx_0 + \beta\ell^{3/2} + x^2/2 > 0 \quad \forall 0 < x < 1\right\} \\
&= {\rm Prob}\left\{\int_0^T W(t)dt > -\beta \quad \forall 0 < T < \ell^{-1}\right\} \\
&\sim {\rm Prob}\left\{S_1 > 0,\ S_1 + S_2 > 0, \ldots\ S_1 + \cdots + S_n > 0\right\} \sim n^{1/2} \sim \ell^{1/4}
\end{aligned}$$



$$\Rightarrow \boxed{P^{\rm reg}(\ell) \sim \ell^{1/2}}$$

Transport of mass in Burgers

- Burgers equation with smooth initial data coupled to the continuity equation
$$\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

- Lagrangian formulation: $\partial_t \boldsymbol{X}(\boldsymbol{x}_0, t) = \boldsymbol{u}(\boldsymbol{X}(\boldsymbol{x}_0, t), t)$

(i.e. $\boldsymbol{X}(\boldsymbol{x}_0, t) = \boldsymbol{x}_0 + t\boldsymbol{u}_0(\boldsymbol{x}_0)$ in the inviscid limit)

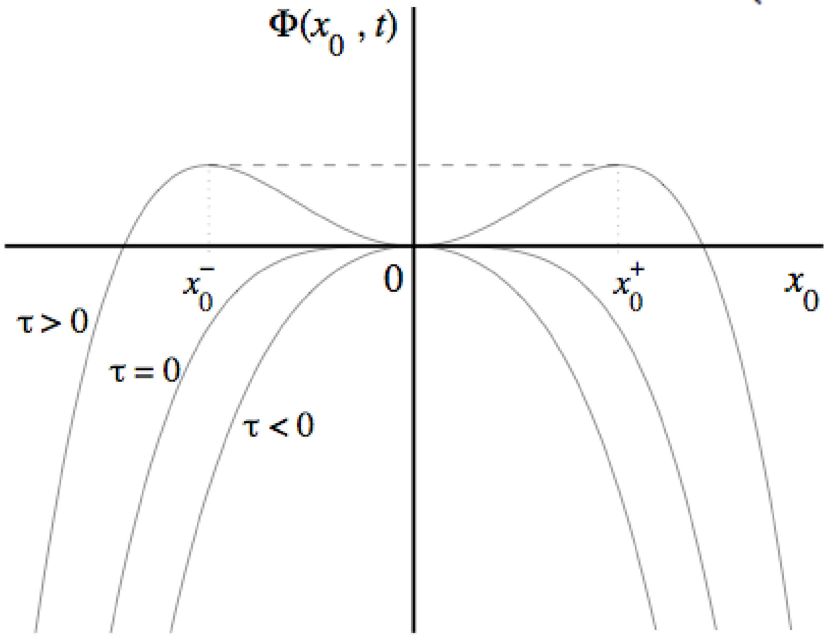
$$\rho(\boldsymbol{x}, t) = \frac{\rho_0(\boldsymbol{x}_0)}{J(\boldsymbol{x}_0, t)} \quad \text{where} \quad \begin{cases} \boldsymbol{x} = \boldsymbol{X}(\boldsymbol{x}_0, t) \\ J(\boldsymbol{x}_0, t) = \det[\partial X^i / \partial x_0^j] \end{cases}$$

When the Jacobian J vanishes (inside the shocks), the density is infinite and mass accumulates

Power-law tails in density PDF

Frisch, Bec & Villone (Phys. D 2001)

- Large but finite densities are attained near A_3 singularities (shock formation in 1D, shock edges in higher dimensions) where the maximum is degenerate
- **1D**: $\rho(X(x_0,t),t) = \dfrac{\rho_0(x_0)}{1 - t(d^2\Psi_0)/(dx_0^2)}$



Shock formation at:

$$\frac{d^3\Psi_0}{dx_0^3}(x_0^\star) = 0 \qquad t^\star = \left[\frac{d^2\Psi_0}{dx_0^2}(x_0^\star)\right]^{-1}$$

Normal form:

$$\Phi_0(x_0,t) = \frac{1}{2}\tau x_0^2 - \zeta x_0^4 \qquad \zeta > 0$$

$$\rho(X(x_0,t),t) = \frac{\rho_0}{12\zeta x_0^2 - \tau}$$

$\rho > \mu \Rightarrow |\tau| < \dfrac{\rho_0}{\mu}$ and $|x| < \dfrac{1}{2\sqrt{3\zeta}}\left[\dfrac{\rho_0}{\mu}\right]^{3/2} \Rightarrow$ $\boxed{\mathrm{Prob}(\rho > \mu) \propto \mu^{-5/2}}$

Velocity gradient PDF

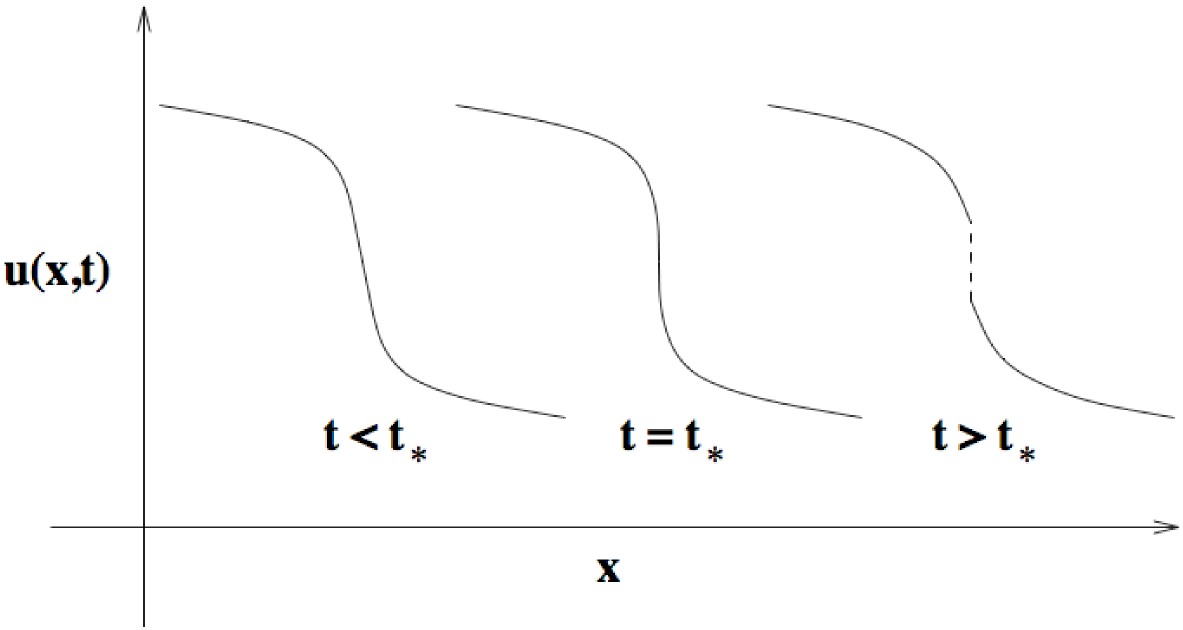
Bec & Frisch (PRE 2001)

- Densities and velocity gradients both involve inverse of Jacobian

$$\rho(X(x_0,t),t) = \frac{\rho_0(x_0)}{1 - t(d^2\Psi_0)/(dx_0^2)}$$

$$\partial_x u(X(x_0 t),t) = \frac{-(d^2\Psi_0)/(dx_0^2)}{1 - t(d^2\Psi_0)/(dx_0^2)}$$

$\Rightarrow \quad \mathrm{Prob}(\partial_x u < \xi) \propto |\xi|^{-5/2}$ for ξ large negative



Signature of **preshocks**

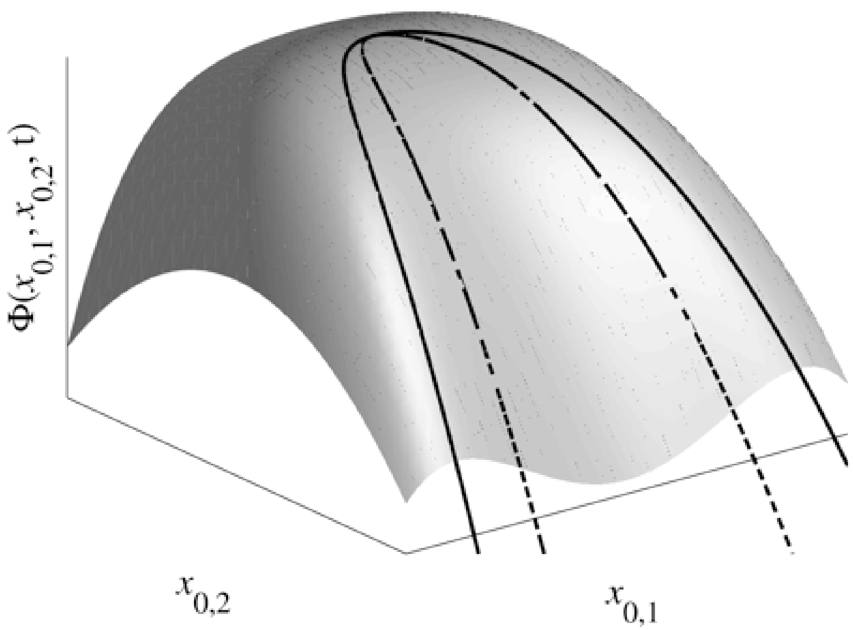
Led to a controversy in the forced case (see tomorrow lecture)

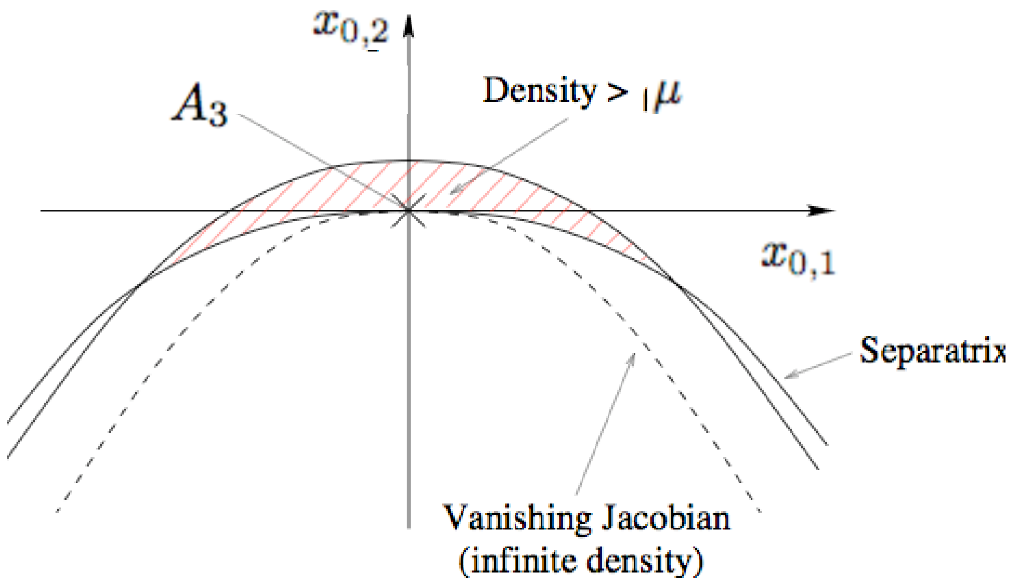
Multidimensional case

- Same law for density applies to higher dimension where A_3 singularities are generically present and correspond to shock edges
- Normal form: $\Phi_0(\boldsymbol{x}_0, t) \simeq -\zeta x_{0,1}^4 + \sum_{2\leq j\leq d} \left[-\frac{\mu_j}{2} x_{0,j}^2 + \beta_j x_{0,j} x_{0,1}^2\right]$

$\beta \cdot \boldsymbol{y}_0$, where $\boldsymbol{y}_0 = (x_{0,2}, \dots\ x_{0,d})$, plays the same role as time in 1D

$$\mathrm{Prob}(\rho > \mu) \propto \underbrace{\mu^{-3/2}}_{x_{0,1}} \times \underbrace{\mu^{-1}}_{\boldsymbol{\beta}\cdot\boldsymbol{y}_0} \times \underbrace{1 \times \cdots \times 1}_{\text{rest of } \boldsymbol{y}_0} \times \underbrace{1}_{\text{time}}$$



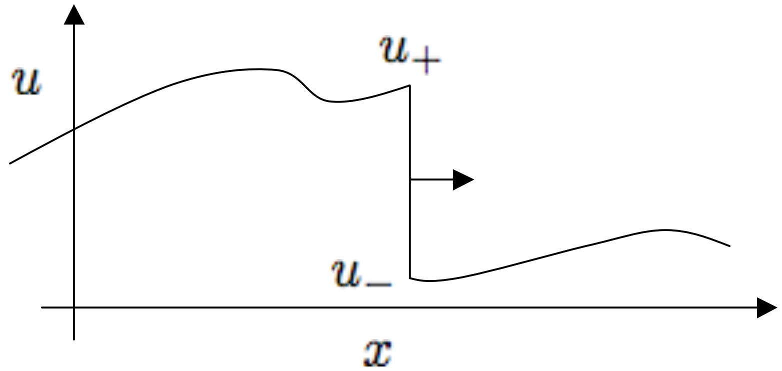


Evolution of matter inside shocks

- How the singularities and the mass they contain evolve with time?
- **1D:** $X(t)$ = shock location. Mass cannot escape from shocks.

Rankine-Hugoniot:

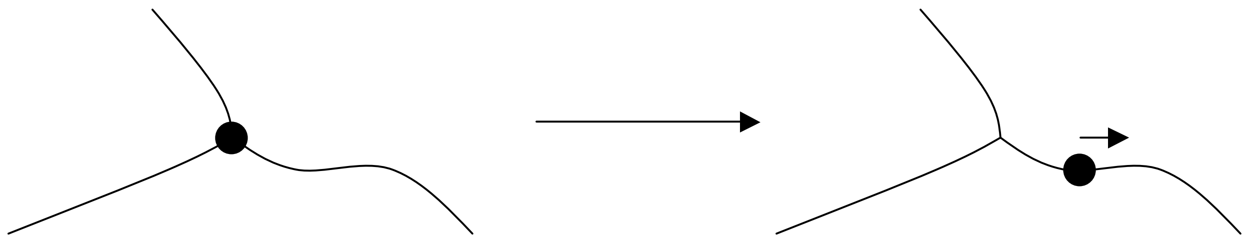
$$\frac{dX}{dt} = \frac{1}{2}(u_+ + u_-)$$



- Higher dimensions:

- Is there any equivalent of Rankine-Hugoniot?

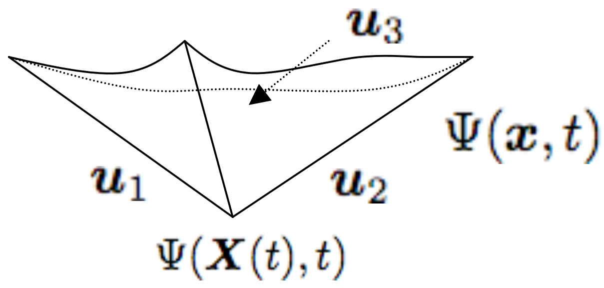
- Are there mechanisms by which mass concentration can escape the singularity that formed it?

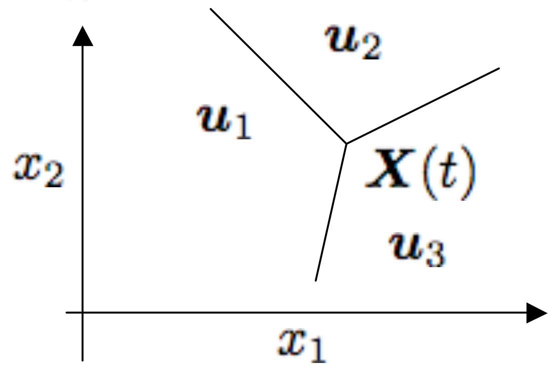


Dynamics of shocks

- $\boldsymbol{X}(t)$ = position of an A_1^n singularity

Locally: $\Psi(\boldsymbol{x},t) = \Psi(\boldsymbol{X}(t),t) + \max_{j=1..n}[\boldsymbol{u}_j \cdot (\boldsymbol{X}(t)-\boldsymbol{x})] + o(\|\boldsymbol{x}-\boldsymbol{X}(t)\|)$





For $\boldsymbol{x} \in \Omega_j$, $\boldsymbol{u}_j \cdot (\boldsymbol{X}(t)-\boldsymbol{x})$ is maximal and $\boldsymbol{u} = \boldsymbol{u}_j$

$\forall m \leq n$ the set of points s.t. $\boldsymbol{u}_{i_1} \cdot (\boldsymbol{X}-\boldsymbol{y}) = \cdots = \boldsymbol{u}_{i_m} \cdot (\boldsymbol{X}-\boldsymbol{y})$
with $1 \leq i_1 < \cdots < i_m \leq n$ forms a singular submanifold of co-dim m

Evolution: $\Psi(\boldsymbol{x},t+\delta t) \simeq \Psi(\boldsymbol{X}(t),t) + \max_{\boldsymbol{y}} \max_{j=1..n}[\boldsymbol{u}_j \cdot (\boldsymbol{X}(t)-\boldsymbol{y}) - \frac{1}{2\delta t}\|\boldsymbol{x}-\boldsymbol{y}\|^2]$

Local structure preserved: the eq. for the submanifold becomes

$$\boldsymbol{u}_{i_1} \cdot (\boldsymbol{X}-\boldsymbol{y}) + \frac{\delta t}{2}\|\boldsymbol{u}_{i_1}\|^2 = \cdots = \boldsymbol{u}_{i_m} \cdot (\boldsymbol{X}-\boldsymbol{y}) + \frac{\delta t}{2}\|\boldsymbol{u}_{i_m}\|^2$$

$\Longrightarrow$ Equation for $\boldsymbol{X}(t)$: $\|d\boldsymbol{X}/dt - \boldsymbol{v}_1\| = \cdots = \|d\boldsymbol{X}/dt - \boldsymbol{v}_n\|$

Dynamics of matter

Bogaevsky (2004)

- $\dfrac{d\boldsymbol{X}^\nu}{dt} = \boldsymbol{u}^\nu(\boldsymbol{X}^\nu, t)$ where $\boldsymbol{u}^\nu$ solves the **viscous** Burgers equation

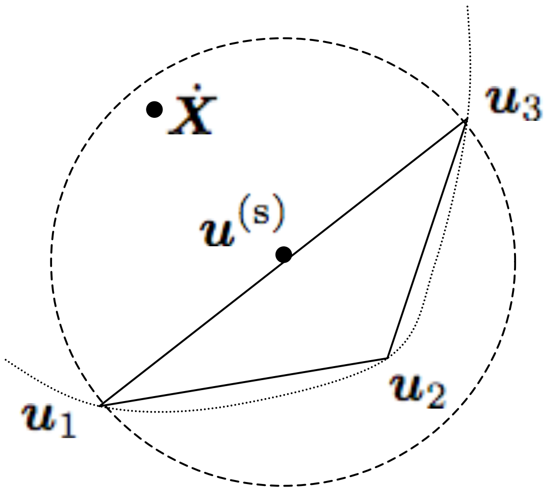
The dynamics 'inside shocks' is understood as $\boldsymbol{X}(t) = \lim\limits_{\nu \to 0} \boldsymbol{X}^\nu(t)$

- $\boldsymbol{X}(t)$ has a one-sided time derivative: $\dfrac{d^+}{dt}\boldsymbol{X}(t) = \lim\limits_{\Delta t \to 0+} \dfrac{\boldsymbol{X}(t+\Delta t) - \boldsymbol{X}(t)}{\Delta t}$

and $\dfrac{d^+}{dt}\boldsymbol{X}(t) = \boldsymbol{u}^{(\mathrm{s})}(\boldsymbol{X}(t), t)$ where $\boldsymbol{u}^{(\mathrm{s})}$ is defined also at singularities

- **Variational definition**: for A_1^n singularities, $\boldsymbol{u}^{(\mathrm{s})}$ is where

$\min\limits_{\boldsymbol{v}} \max\limits_{1 \le i \le n} \|\boldsymbol{v} - \boldsymbol{u}_i\|^2$ is attained $\Rightarrow$ center of the minimum ball covering all the $\boldsymbol{u}_i$'s



Triple point: $\|\dot{\boldsymbol{X}} - \boldsymbol{v}_1\| = \|\dot{\boldsymbol{X}} - \boldsymbol{v}_2\| = \|\dot{\boldsymbol{X}} - \boldsymbol{v}_3\|$

$\Rightarrow$ $\dot{\boldsymbol{X}}$ circumcenter of the triangle

$\dot{\boldsymbol{X}} \neq \boldsymbol{u}^{(\mathrm{s})}$ when the triangle is obtuse