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A Mathematical Approach to one Dimensional Burgers' Equation

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Abstract

The one dimensional Burgers' equation $\frac{\partial u}{\partial t} = u\,\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}$ is solved numerically using a finite element method, where a combination of cubic B-splines is used as an approximating function. Different comparisons for the test problem in hand, are made to validate the proposed numerical technique. Results obtained from the proposed numerical scheme are found to be in good agreement with the exact solution.

1. Introduction

Bateman [1] in 1915 first introduced Burger's equation and later describing a mathematical model of turbulence, it was proposed by Burgers' [2]. Due to the extensive work of Burger it is known as Burgers' equation. We consider the Burgers' equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \varepsilon\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le l, \;\; t > 0 \tag{1}$$

With boundary conditions

$$u(0,t) = f(x) \tag{2}$$

$$u(l,t) = h(x) \qquad for \;\; t > 0 \tag{3}$$

and initial condition

$$u(x,0) = g(x) \qquad for\; 0 \le x \le l, \tag{4}$$

Where $\vartheta > 0$ is coefficient of kinematic viscosity and *x,t* are spatial and temporal variables respectively. Equation (1) is considered to be as an approach to study turbulence, shock wave and gas dynamics. Analytical solution of Burgers' equation involves series solution that converges very slowly for small values of the viscosity constant ϑ [3]. So, Burgers' equation is taken as a model not only to test the numerical methods but also to obtain the numerical solution of equation for small values of viscosity. The nonlinear term $u\frac{\partial u}{\partial x}$ makes it more interesting to study. It has attracted many researchers to develop the solutions of Burgers' equation. Two different analytical solutions of Burgers' equation have been found for a restricted set of arbitrary initial and boundary conditions [4,5]. Many others have tried numerical schemes such as Rubin and Grave [6] used cubic spline functions and quasilinearization for the numerical solution of Burgers' equation.Gardner et al. [7] used Petrov-Galerkin method by a quadratic B-spline spatial finite elements and they also used a least-square technique using linear space-time finite elements. A numerical solution was discussed using Adomain method by Abbasbandy and Darvishi [8]. They also solved the problem by time discretization of Adomain's decomposition method [9]. Implicit-finite difference schemes together with splitting up technique was set up using interpolation cubic splines to obtain the numerical solution [10]. Darvishi and Javidi [11] studied a numerical solution of Burgers' equation by pseudospectral method and Darvishi's preconditioning.

Varoglu and Finn developed a finite element method based on a weighted residual formulation [12]. Finite difference and cubic spline finite element methods were used by Caldwell and Smith [13]. A generalized boundary element approach by Kakuda and Tosaka [14] and a linear space-time finite element method based on least square approach is carried out by Nguyen and Rynen [15]. Kapoor and Dhawan [16] presented the mumerical technique based on B-spline functions. A variable mesh cubic B-spline technique is developed for the shock-like solution of the Burgers' equation [17]. Since B-splines have many numerical and geometrical properties [18], we use B-spline basis function for finding the solution of Burgers' equation in the present work.

2. Solution Procedure

B-splines were first introduced by Schoenberg in [19]-[20]. A B-spline curve is a piecewise polynomial connected continuously by the small curve segments. These piecewise defined functions allow a large number of control points and also maintain continuity. Ageneral B-spline curve is

$$u(x) = \sum_{i=1}^{N} a_i \, N_{iK}(x)$$

Where $N_{iK}(x)$ is a special function of order k called B-spline. It has a particular property of having compact support. B-spline of order one are step functions defined by

$$N_{iK} = \begin{cases} 1\,;\, x\epsilon[x_i, x_{i+1}] \\ 0;\, otherwise \end{cases}$$

and an efficient construction of B-splines of order $K > 1$ is given by the recurrence relation of Curry and Schoenberg [22].

$$N_{i,K} = \frac{x-x_i}{x_{i+k-1}-x_i} N_{iK}(x) + \frac{x_{i+k}-x}{x_{i+k}-x_{i+1}} N_{i+1,K-1}(x) \tag{5}$$

It introduces the knots $x_{i\,;}\ i = 1, \dots N + k$. The effciency of the B-spline method for solving differential problems depends crucially on the choice of the B-spline basis function and the approximation technique. Since the Galerkin method satiesfies the differential equation on average, it is the preferred technique for the spline approximation of a large variety of problems. In the next section we discuss the solution procedure adopted to solve the Burgers' equation numerically using cubic B-spline basis functions in detail.

3. Solution Procedure

We use B-spline basis functions for the solution of Burgers' equation (1)-(4). The interval $[0, l]$ is divided into N finite elements with equal length $\Delta x = x_{m+1} - x_m$ such that $0 = x_0 < x_1 < \cdots < x_N = l$. The set of splines $\{N_0, N_{-1}, \dots, N_N\}$ is is taken to form a basis for the functions defined on the given domain. $N_m(x)$ are cubic B-spline basis functions. Cubic B-splines $N_m, (m = -1, \dots N + 1)$ at knots x_m to form a basis over the problem domain are defined by [23]

$$N_m(x) = \frac{1}{h^3}\begin{cases} (x - x_{m-2})^3 & [x_{m-2}\ , x_{m-1}], \\ h^3 + +3h^2(x - x_{m-1}) + 3h(x - x_{m-1})^2 - 3(x - x_{m-1})^3, & [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1} - x) + 3h(x_{m+1} - x)^2 - 3(x_{m+1} - x)^3, & [x_m, x_{m+1}], \\ \left(x_{m+2} - x\right)^3 & [\,x_{m+1}, x_{m+2}], \\ 0 & otherwise, \end{cases} \tag{6}$$

Where $(m = -1, \dots N + 1)$ and $h = x_{m+1} - x_m$ for all m. The splines $N_m(x)$ and its two derivatives vanish outside the interval $[x_{m-2}, x_{m+2}]$. The spline values $N_m{}', N'_m, N''_m$ at the knots are given by the following Table.

x	x_{m-2}	x_{m-1}	x_m	x_{m+1}	x_{m+2}
N_m	0	1	4	1	0
N_m	0	$3/_h$	0	$-3/_h$	0
N_m	0	$6/_{h^2}$	$-12/_{h^2}$	$6/_{h^2}$	0

The finite elements for the problem are identified with the interval $[x_m, x_{m+1}]$ and the element nodes with knots x_m, x_{m+1} . Using the table values, we have the nodal parameters u_m, u_m as

$$u_m = \sigma_{m-1} + 4\sigma_m + \sigma_{m+1}$$
$$u_m = \frac{3}{h}(\sigma_{m+1} - \sigma_{m-1}) \tag{7}$$

We transform the cubic B-splines into element shape functions over the finite intervals $[0, \mathrm{h}]$ using a local coordinate system $\pi = \mathrm{x} - \mathrm{x_m}, 0 \leq \pi \leq \mathrm{h}$. Over $[0, \mathrm{h}]$ the cubic B-splines in terms of π are given by

$$N_{m-1} = \left(1 - \frac{\pi}{h}\right)^3$$
$$N_m = 1 - 3\left(\frac{\pi}{h}\right) + 3\left(1 - \frac{\pi}{h}\right)^2 - 3\left(1 - \frac{\pi}{h}\right)^3$$
$$N_{m+1} = 1 + 3\left(\frac{\pi}{h}\right)^2$$
$$N_{m+2} = \left(\frac{\pi}{h}\right)^2 \tag{8}$$

Taking B-spline basis functions (6) we have the approximate solution of the form

$$u = U_0 + \sum_{i=1}^{N} \sigma_i \ N_i\,, \qquad 0 \leq x \leq l \tag{9}$$

Where N_i are cubic B-spline basis functions, σ_i are unknown coefficients and U_0 is associated with the boundary conditions, using (9) in (1) gives

$$\sum_i \frac{d\sigma_i}{dt} N_i + \left(U_0 + \sum_{i=1}^{N} \sigma_i \ N_i\right) \left(\frac{dU_0}{dx} + \sum_i \sigma_i \frac{dN_i}{dx}\right) - \varepsilon \left(\frac{d^2U_0}{dx^2} + \sum_i \sigma_i \frac{d^2N_i}{dx^2}\right) = 0 \tag{10}$$

using the homogeneous boundary conditions $u(0,t) = u(l,t) = 0$, (10) gives us finite element equation in the matrix form as

$$X^e\dot{\sigma} + (Y^e + \varepsilon Z^e)\sigma = 0 \tag{11}$$

Assembling contributions from all the elements we have

$$X\dot{\sigma} + (Y + \varepsilon Z)\sigma = 0 \tag{12}$$

where $\sigma = (\sigma_{m-1}, \sigma_m, \sigma_{m+1}, \sigma_{m+2})$ and the element matrices obtained after some manipulations are given by

$$X_{ij} = \int_0^l N_i N_j \, d\pi = \frac{1}{140}\begin{pmatrix} 20 & 129 & 60 & 1 \\ 129 & 1188 & 933 & 60 \\ 60 & 933 & 1188 & 129 \\ 1 & 60 & 129 & 20 \end{pmatrix}$$

$$Z_{ij} = \int_0^l N'_i N_j \, d\pi = \frac{3}{10}\begin{pmatrix} 6 & 7 & -12 & -1 \\ 7 & 34 & -29 & -12 \\ -12 & -29 & 34 & 7 \\ -1 & -12 & 7 & 6 \end{pmatrix} \quad (13)$$

and $Y = \sum_i \sigma_i \int_0^l N_i N'_j N_k dx$. To start with the iteration process, we get the initial values by using Galerkin procedure to the initial data as

$$\int_0^1 (u - u_0) N_i \, dx = 0 \quad (14)$$

which gives

$$\int_0^l \left(U_0 + \sum_{i=1}^{N} \sigma_i \; N_i \right) N_j dx = \int_0^l u_0 N_i \, dx$$

$$\sum_{i=1}^{N} \sigma_i \int_0^l N_i \, N_j dx = \int_0^l u_0 N_i \, dx - \int_0^l U_0 N_i \, dx$$

which gives us system of equations in the matrix form as

$$X\sigma = A \quad (15)$$

Where $A = \int_0^l u_0 N_i \, dx - \int_0^l U_0 N_i \, dx$. Thus the system of equation given by (13) can be solved by taking initial values from (15) and the general row of each matrix has the form

$$X = \frac{1}{140}(1, 120, 1191, 2416, 1191, 120, 1),$$

$$Y = \frac{1}{420}\{(5,108,129,10,0,0,0)\sigma, -(21,1944,8130,3888,129,0,0)\sigma, -(-21,0,17841\; 35682,8130,108,0)\sigma, (5,1944,17841,0,-17841,1944,-5)\sigma, (0,108,8130,35682,17841,0,-21)\sigma, (0,0,129,3888,8130,1944,21)\sigma\; (0,0,0,10,129,108,5)\sigma\}$$

$$Z = -\frac{1}{10}(3, 72, 45, -240, 45, 72, 3)$$

Where $\sigma = (\sigma_{m-1}, \sigma_m, \sigma_{m+1}, \sigma_{m+2})$. In the next section we discuss the numerical results obtained using the given technique.

4. Numerical experiments and Results

We have the Burgers equation (1)-(3), taking with the given initial condition $U_0 = 0, l = 1, u_0 = 1$, with the given initial condition

$$u(x,0) = u_0 \sin\left(\frac{\pi x}{l}\right) \quad (16)$$

so the element matrix A is expressed as

$$\mathrm{A} = \int_0^1 \left(\mathrm{u}_0 \sin\left(\frac{\pi x}{l}\right) N_i\right) dx$$

Thus the exact solution of Burgers' equation (1) with the initial and boundary conditions
(15) and (3) is given by [23] as

$$u(x,t) = \frac{4\pi\varepsilon}{l} \frac{\sum_{n=1}^{\infty} n I_n\left(\frac{\mathrm{u_0 l}}{2\pi\varepsilon}\right) e^{-n^2\pi^2\varepsilon t/l^2} \sin\left(\frac{n\pi x}{l}\right)}{I_0\left(\frac{\mathrm{u_0 l}}{2\pi\varepsilon}\right) + 2\sum_{n=1} I_n\left(\frac{\mathrm{u_0 l}}{2\pi\varepsilon}\right) e^{-n^2\pi^2\varepsilon t/l^2} \cos\left(\frac{n\pi x}{l}\right)} \quad (17)$$

where I_0 and I_n are the modified Bessel functions of the first kind. The harmony between the exact solutions and numerical solutions of the problem for different values of ε at different times Table 1. The effect of choosing different mesh sizes on the numerical solution with different values of ε is given in Table 2-3. The plots of the numerical solutions obtained for values of viscosity ranging from large to very small were shown in Figs. 1-6. As it is expected, the method of solution presented provides high accuracy.

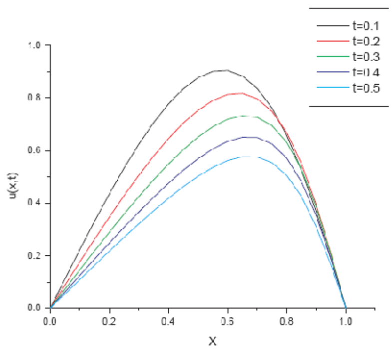

Figure 1: Solution compared at different times with $\varepsilon = 0.07$.

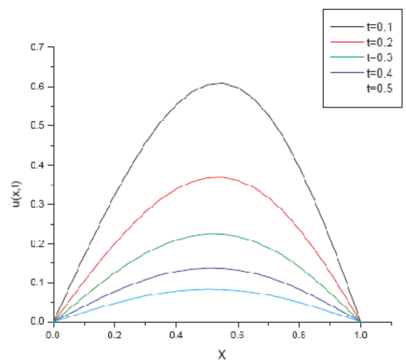

Figure 2: Solution compared at different times with $\varepsilon = 0.1$.

Table 1: Comparison of numerical results with exact solution for different values of ε at different times

x	t	Exact	Numerical	Exact	Numerical	Exact	Numerical
0.1	0.10	0.1095381	0.1095491	0.195411	0.195433	0.235941	0.235965
	0.15	0.0678845	0.0678911	0.164326	0.164343	0.210946	0.210967

	0.20	0.0419291	0.0419333	0.140611	0.140625	0.190735	0.190754
	0.25	0.0257959	0.0257984	0.121576	0.121588	0.174059	0.174076
	0.30	0.0158251	0.0158266	0.105747	0.105757	0.160068	0.160084
	0.35	0.0096902	0.0096912	0.092265	0.092274	0.148162	0.148176
	0.4	0.0059268	0.0059274	0.080607	0.080615	0.137907	0.137923
	0.45	0.0036224	0.0036227	0.070434	0.070441	0.128982	0.128994
	0.50	0.0022130	0.0022132	0.061515	0.061521	0.121144	0.121156
0.3	0.10	0.2918963	0.2919032	0.538782	0.538794	0.664325	0.664344
	0.15	0.1798872	0.1798914	0.455944	0.455956	0.604602	0.604622
	0.20	0.1106223	0.1106252	0.390724	0.390711	0.553155	0.553172
	0.25	0.0678596	0.0678612	0.337307	0.337316	0.508888	0.508904
	0.30	0.0415533	0.0415543	0.292446	0.292454	0.470661	0.470675
	0.35	0.0254155	0.0254161	0.254153	0.254106	0.437459	0.437473
	0.4	0.0155339	0.0155342	0.220975	0.220981	0.408434	0.408447
	0.45	0.0094900	0.0094902	0.192187	0.192192	0.382892	0.382904
	0.50	0.0057961	0.0057962	0.167093	0.167097	0.360271	0.360283
0.5	0.10	0.3715773	0.3715784	0.731441	0.731444	0.947414	0.947423
	0.15	0.2268242	0.2268242	0.626336	0.626343	0.900098	0.900109
	0.20	0.1384734	0.1384743	0.537755	0.537758	0.848365	0.848377
	0.25	0.0845376	0.0845376	0.462545	0.462548	0.796762	0.796775
	0.30	0.0516131	0.0516153	0.398346	0.398302	0.747713	0.747725
	0.35	0.0315078	0.0315078	0.343202	0.343204	0.702267	0.702282
	0.4	0.0192355	0.0192355	0.295836	0.295837	0.660711	0.660723
	0.45	0.0117432	0.0117432	0.255062	0.255061	0.622944	0.622956
	0.50	0.0071692	0.0071692	0.219934	0.219931	0.588696	0.588707
0.7	0.10	0.309905	0.309898	0.660422	0.660411	0.934133	0.934121
	0.15	0.187274	0.187273	0.573702	0.573693	0.969585	0.969578
	0.20	0.113469	0.113467	0.493072	0.493064	0.980079	0.980079
	0.25	0.068933	0.068932	0.421518	0.421514	0.968982	0.968985
	0.30	0.041955	0.041954	0.359502	0.359495	0.943045	0.943051
	0.35	0.025565	0.025565	0.306404	0.306399	0.908697	0.908705
	0.4	0.015589	0.015589	0.261215	0.261215	0.870589	0.870597
	0.45	0.009510	0.009510	0.222855	0.222852	0.831633	0.831642
	0.50	0.005803	0.005803	0.1909309	0.190305	0.793493	0.793503

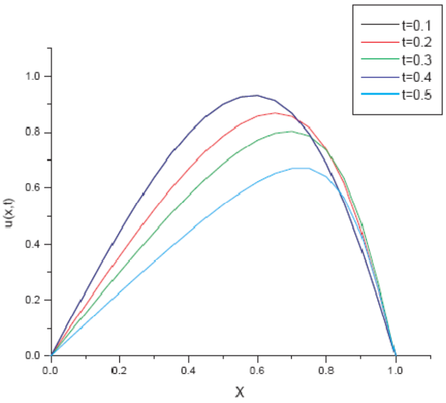

Figure 3: Solution compared at different times with $\varepsilon = 0.5$.

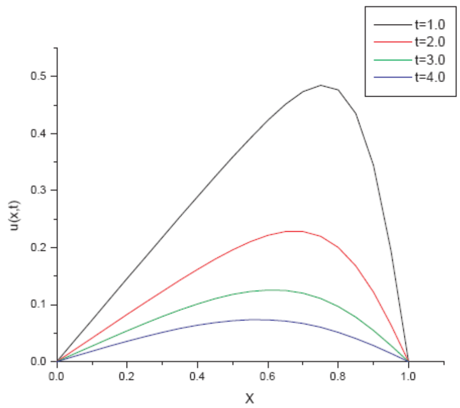

Figure 4: Solution compared at different times with $\varepsilon = 1.0$

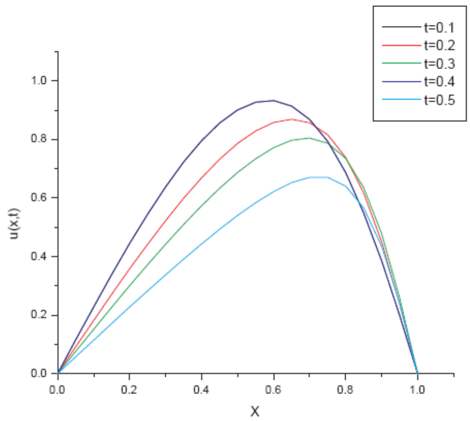

Figure 5: Solution compared at different times with $\varepsilon = 0.01$.

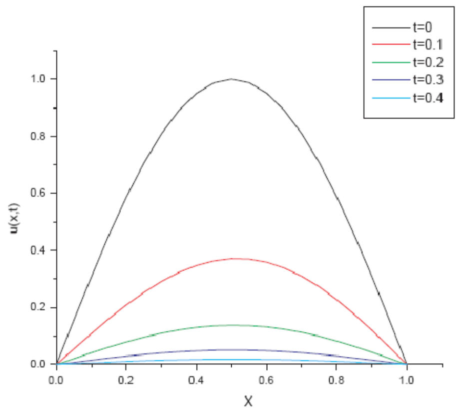

Figure 6: Solution compared at different times with $\varepsilon = 0.05$.

Table 2: Comparison of numerical results with exact solution for $\varepsilon = 0.01$ at t=0.1 taking different mesh sizes.

x	Exact	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.005$	$\Delta t = 0.025$	$\Delta t = 0.01$
0.10	0.235941	0.235941	0.235936	0.235935	0.235931	0.235935
0.15	0.350605	0.350604	0.350597	0.350595	0.350588	0.350588
0.20	0.461225	0.461224	0.461215	0.461213	0.461203	0.461204
0.25	0.566328	0.566326	0.566316	0.566313	0.566303	0.566304
0.30	0.664325	0.664324	0.664312	0.664309	0.664298	0.664299
0.35	0.753479	0.753478	0.753466	0.753463	0.753451	0.753453
0.40	0.831864	0.831862	0.831851	0.831848	0.831837	0.831838
0.45	0.897323	0.897319	0.897309	0.897307	0.897297	0.897298

0.50	0.947414	0.947413	0.947406	0.947404	0.947396	0.947397
0.55	0.979392	0.979391	0.979386	0.979385	0.979381	0.979381
0.60	0.990156	0.990155	0.990155	0.990155	0.990154	0.990154
0.65	0.976283	0.976283	0.976287	0.976288	0.976291	0.976291
0.70	0.934133	0.934133	0.934143	0.934145	0.934153	0.934152
0.75	0.860131	0.860134	0.860146	0.860145	0.860161	0.860157
0.80	0.751086	0.751107	0.751118	0.751115	0.751124	0.751117
0.85	0.605072	0.604858	0.604949	0.604953	0.605004	0.605108
0.90	0.429297	0.429432	0.429131	0.429744	0.429642	0.429357
0.95	0.229397	0.229073	0.229359	0.229472	0.229172	0.229927
1.00	0.000158	0.0001596	0.0001588	0.0001591	0.0001588	0.0001592

Table 3: Comparison of numerical results with exact solution for $\varepsilon = 0.2, \mathrm{t} = 0.1$ taking different mesh sizes.

x	Exact	$\Delta t = 0.002$	$\Delta t = 0.005$	$\Delta t = 0.03$	$\Delta t = 0.02$	$\Delta t = 0.01$
0.10	0.209429732	0.209421822	0.209422481	0.209421163	0.209416556	0.209363847
0.15	0.310577265	0.310565702	0.310566666	0.310564739	0.310557994	0.310480933
0.20	0.407378036	0.407363173	0.407364412	0.407361935	0.407353266	0.407254213
0.25	0.498273521	0.498255824	0.498257299	0.498254352	0.498244027	0.498126088
0.30	0.581612641	0.581592679	0.581594342	0.581591015	0.581579372	0.581446337
0.35	0.655632058	0.655610492	0.655612289	0.655608695	0.655596115	0.655452376
0.40	0.718441832	0.718419391	0.718421265	0.718417525	0.718404432	0.718254863
0.45	0.768021224	0.767998683	0.768000561	0.767996806	0.767983665	0.767833447
0.50	0.802232373	0.802210526	0.802212347	0.802208706	0.802195965	0.802050364
0.55	0.818863185	0.818842793	0.818844489	0.818841091	0.818829193	0.818693232
0.60	0.815714768	0.815696485	0.815698009	0.815694962	0.815684297	0.815562411
0.65	0.790751476	0.790735822	0.790737125	0.790734516	0.790725383	0.790621003
0.70	0.742329983	0.742317262	0.742318322	0.742316202	0.742308781	0.742223955
0.75	0.669512751	0.669503023	0.669503834	0.669502213	0.669496538	0.669431659
0.80	0.572445886	0.572438958	0.572439535	0.572438384	0.572434338	0.572388125
0.85	0.452740701	0.452736174	0.452736551	0.452735796	0.452733155	0.452702949
0.90	0.313752567	0.313749943	0.313750162	0.313749725	0.313748194	0.313730681
0.95	0.160625604	0.160624427	0.160624525	0.160624329	0.160623643	0.160615792

5. Refrences

[1] H.Bateman, Some recent researches on the motion of fluids, Monthly Weather Rev. 43(1915) 163-170.

[2] J.M.Burgers, A mathematical model illustrating the theory of turbulence, Adv. Appl. Mech. 1(1948) 171-199.

[3] E.L.Miller, Predictor-corrector studies of Burgers' model of turbulent flow, M.S. thesis, University of Delaware, Newark, DE, (1966).

[4] J.D.Cole, A quasi-linear parabolic equation in aerodynamics, Quart. Applied Mathe-matics, 9(1951), 225-236.
[5] E.Hopf, The partial differential equation $u_t + uu_x = \varepsilon u_{xx}$, Communications on Pure and Applied Mathematics, 3(1950) 201 – 230.
[6] S.G.Rubin and R.A.Graves, Viscous flow solutions with a cubic spline approximation, Computer. & Fluids 3(1975) 1-36.
[7] G.A.Gardner, A.Dogan, A Petrov-Galerkin finite element scheme for Burgers' equation , Arab. J. Sci. Eng. 22(1997) 99-109.
[8] S.Abbasbandy and M.T.Darvishi, A numerical solution of Burgers' equation by modified Adomain method, Appl. Math. Compt. 163 (2005) 1265-1272.
[9] S.Abbasbandy and M.T.Darvishi, A numerical solution of Burgers' equation by time discretization of Adomain method, Applied Mathematics and Computation 170 (1), (2005) 95-102.
[10] P.C.Jain and D.N.Holla, Numerical solution of coupled Burgers' equation, Int.J.Non-linear Mech, 13 (1976), 213-222.
[11] M.T. Darvishi and M. Javidi, A numerical solution of Burgers equation by pseudo spectral method and Darvishis preconditioning, Applied Mathematics and Computation 173(1) (2006)421-429.
[12] E.Varoglu and W.D.L.Finn, Space-time finite elements incorporating characteristics for the Burgers' equation, International Journal of Numerical Methods in Engineering. 16(1983) 171-184.
[13] J.Caldwell, P.Smith, Solution of Burgers' equation with a large Reynolds number, Appl. Math. Modelling 6(1982) 381-385.
[14] K.kakuda, N.Tosaka, The generalized boundary element approach to Burgers' equation, Int. J.Numer. Methods Eng. 29(1990) 245-261.
[15] H.Nguyen, J.Reynen, A space-time finite element approach to Burgers' equation, in: C.Taylor, E.Hinton, D.R.J.Owen, E.Onate (Eds.), Numerical Methods for Non- Linear problems, Vol. 2, Pineridge Publishers, Swansa, (1982), 718-728.
[16] S.Kapoor and S.Dhawan, A Computational Technique For The Solution Of Burgers' Equation, Int. J. of Appl. Math. and Mech. 6(3), (2010) 84-95.
[17] P.C.Jain and B.L.Lohar, Cubic spline technique for coupled non-linear parabolic equations, Comp. Math with applications, 5 (1979) 179-185.
[18] G.Farin, Curves and surfaces for computer aided geometric Design, inc. second ed. Academic Press (1990).
[19] I.J.Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Quart. Appl. Math. 4(1946), 45-99, 112-141.
[20] I.J.Schoenberg, Cradinal interpolation and spline functions, J. Approximation Theory, 2(1969), 167-206.
[21] C.de Boor, A Practicle guide to splines, Springer, (1978).
[22] H.B.Curry and I.J.Schoenberg, On polya frequency functions IV: The fundamental spline functions and their limits, J.Anal. Math. 17(1966), 71-107.
[23] P.M.Prenter, Splines and Variational Methods, Wiley, New York, (1975).
[24] Debnath, Nonlinear Partial Differential Equations for Scientists and Engineers (Basle:Birkhauser), (1989).