Volume 245

A Series of Lecture Notes in Pure and Applied Mathematics

Stochastic Partial Differential Equations and Applications – VII

Edited by
Giuseppe Da Prato
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Stochastic Partial Differential Equations and Applications – VII

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Chapman & Hall/CRC
Taylor & Francis Group
Boca Raton London New York

Published in 2006 by
Chapman & Hall/CRC
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

Printed in the United States of America on acid-free paper
10 9 8 7 6 5 4 3 2

International Standard Book Number-10: 0-8247-0027-9 (Hardcover)
International Standard Book Number-13: 978-0-8247-0027-0 (Hardcover)
Library of Congress Card Number 2005049749

Library of Congress Cataloging-in-Publication Data

Stochastic partial differential equations and applications--VII / edited by Giuseppe Da Prato and Luciano Tubaro.
p. cm. -- (Lecture notes in pure and applied mathematics ; 245)
Includes bibliographical references.
ISBN 0-8247-0027-9
1. Stochastic partial differential equations--Congresses. I. Da Prato, Giuseppe. II. Tubaro, L. (Luciano), 1947- III. Lecture notes in pure and applied mathematics ; v. 245.

QA 274.25.S754 2002
519.2--dc22 2005049749

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Preface

The seventh meeting on "Stochastic Partial Differential Equations and Applications" was held in Levico Terme (Trento) in January 2004. This conference brought together a particularly distinguished and representative group of researchers in the field.

The topics discussed included:

1. Stochastic partial differential equations: general theory and applications
2. Finite- and infinite-dimensional diffusion processes
3. Stochastic calculus
4. Theory of interacting particles
5. Quantum probability
6. Stochastic control

The aim of this book is to present several new results, often in a review form, in order to provide an overview of the state-of-the-art research in the field.

This conference was financed by

- Centro Internazionale per la Ricerca Matematica (CIRM),
- Istituto Nazionale di Alta Matematica, Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (INDAM-GNAMPA),
- Mathematics Department of the University of Trento, and
- Programmi di Ricerca Scientifica di Rilevante Interesse Nazionale (PRIN 2002) "Equazioni di Kolmogorov, Equazioni Differenziali alle Derivate Parziali ed Applicazioni alla Dinamica delle Popolazioni e alla Teoria dei Campi Quantistici."

The Organizing Committee (D. Nualart, E. Pardoux, M. Röckner, and the editors) would like to express their warmest thanks to the Secretary of the CIRM, Augusto Micheletti, for his continuous assistance.

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1 Weak, Strong, and Four Semigroup Solutions of Classical Stochastic Differential Equations: An Example

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1.1 Introduction

The structure of the present note is the following. In section (1.2) we recall some known facts about strong and weak solutions of stochastic differential equations and we refer to [KarShre88] for more information. In section (1.3) we motivate the definition of the four semigroup solution of an SDE. Finally, in section (1.4) we describe the four semigroups canonically associated to the Ornstein–Uhlenbeck process.

The main point of the four semigroup solution, which is intermediate between weak and strong solutions, is that it reduces the theory of stochastic flows to the study of a particular class of semigroups. This fact was exploited in [AcKo99b], [AcKo00b] to prove the existence of the infinite volume flow of a class of interacting particle systems by means of Hille–Yoshida type estimates. The usual existence criteria for classical or quantum flows were not applicable to this cases.

Here we deal only with scalar-valued processes but the fact that the theory can be applied to arbitrary vector-valued (including infinite-dimensional) processes supports our hope that, combining this approach with some analytical estimates due to Röckner, one could prove existence results for stochastic flows which cannot be handled with the present techniques.

1.2 Weak and strong solutions of SDE

We will only consider real-valued processes and all L^p-spaces considered ($1 \le p \le \infty$) will be complex valued). For any Hilbert space $\mathcal{H}$, we denote $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$.

Definition 1.1 $(\Omega, \mathcal{F}, P)$ fi $(\mathcal{F}_{t]})$ $(\mathcal{F})$ $(\mathcal{F}_{t]})$ $W_t : \Omega \to \mathbb{R}$

(i) $(W_t - W_s)$ $(\mathcal{F}_{t]})$ $|t-s|$

(ii) $\forall\, t \in \mathbb{R}_+$ $P - \forall\, \omega \in \Omega$ $t \mapsto W_t(\omega)$

Definition 1.2 $(\Omega, \mathcal{F}, P)$ fi $(\mathcal{F}_{t]})$ $(\mathcal{F}_{t]})$ (W_t) $b, \sigma : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$dX_t = bdt + \sigma dW. \tag{1.1}$$

$X \equiv (X_t)$ fi $(\Omega, \mathcal{F}, P)$ $(\mathcal{F}_{t]})$

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \tag{1.2}$$

Definition 1.3 ( )

$$X \equiv (X_t) \ ; \quad \hat{W} \equiv (\hat{W}_t) \ ; \ (\hat{\Omega}, \hat{\mathcal{F}}, \hat{P}) \ ; \quad (\hat{\mathcal{F}}_{t]})$$

(i) $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$

(ii) $(\hat{\mathcal{F}}_{t]})$ fi $(\hat{\mathcal{F}})$

(iii) X $(\hat{\mathcal{F}}_{t]})$

(iv) $\hat{W}$ $(\hat{\mathcal{F}}_{t]})$

(v) X fi

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)d\hat{W}_s. \tag{1.3}$$

Thus the main difference between a strong and a weak solution of an SDE is that, in the strong case the Brownian motion is given a priori and the solution is adapted to the filtration generated by it and by its initial data, while in the weak case the BM is built from the solution and is adapted to the filtration generated by it. A typical example of an equation admitting a weak solution which is not strong is

$$dX_s = \text{ sgn } (X_s)dB_s.$$

Definition 1.4 b, σ L $C^\infty(\mathbb{R}_+, \mathbb{R})$, $\Omega = C(\mathbb{R}_+; \mathbb{R})$, $\mathcal{F} =$ $(\Omega), (\mathcal{F}_{t]})$ fi L $(\Omega, \mathcal{F})$

$$f(X_t) - \int_0^t (Lf)(X_s)ds =: M_t \quad ; \quad \forall\, f \in C^\infty(\mathbb{R}_+, \mathbb{R})$$

$(\mathcal{F}_{t]}) - P$

It is known that, under general conditions, given a solution X of the martingale problem for L there exist two measurable functions b, σ, with b and $|\sigma|$ uniquely determined by L, such X is a weak solution of (1.1).

1.3 The four semigroup solution

Let $W = (W_t)$ be a given Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with associated filtration $(\mathcal{F}_{t]})$ and associated L^2-space

$$\Gamma = L^2(\Omega, \mathcal{F}, P).$$

Let $X_0 = X(0)$ be the initial data of equation (1.1). We assume that X_0 is a random variable with distribution equivalent to the Lebesgue measure. Therefore X_0 can be identified to the self-adjoint multiplication operator on $L^2(\mathbb{R})$ given by

$$\mathrm{id}(x) := x \qquad ; \qquad x \in \mathbb{R}. \tag{1.4}$$

In the same way $L^\infty(\mathbb{R})$ is identified to the algebra of multiplication by bounded measurable functions acting on $L^2(\mathbb{R})$. When ambiguities are possible, we write M_f to distinguish between $f \in L^\infty(\mathbb{R})$ and the corresponding multiplication operator.

If (X_t) is a strong solution of equation (1.1), then the map

$$j_t : f \in L^\infty(\mathbb{R}) \to j_t(f) := M_{f(X_t)} \in L^\infty(\mathbb{R} \times \Omega) \subseteq \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \tag{1.5}$$

is a w^*-continuous Markov flow of random multiplication operators. According to the convenience one can replace the algebra $L^\infty(\mathbb{R})$ by other algebras such as $C_b(\mathbb{R})$, cylindrical functions, ...

Conversely, given j_t, the process (X_t) is uniquely determined by the relation

$$j_t(\mathrm{id}) = X_t \tag{1.6}$$

and by the fact that the function id, defined by (1.4), is a limit, in the strong operator operator topology on $\mathcal{B}\left(L^2(\mathbb{R})\right)$, of bounded measurable functions.

The flow equation for $f(X_t)$

$$df(X_t) = \sigma f'(X_t)dW_t + \left(\frac{\sigma^2}{2} f''(X_t) - bX_t f'(X_t)\right) dt \tag{1.7}$$

when translated in terms of the corresponding flow of multiplication operators ($j_t(f) = M_{f(X_t)}$) becomes

$$dj_t(f) = i[p(\sigma), j_t(f)]dw_t - \frac{1}{2}[p(\sigma), [p(\sigma), j_t(f)]]dt + i[p(b_\sigma), j_t(f)]dt \tag{1.8}$$

where f runs in a suitable domain in $L^\infty(\mathbb{R})$ and, for any differentiable function g we use the notations

$$[a, b] := ab - ba \tag{1.9}$$

$$p(g) := \frac{1}{2}(gp + pg) \tag{1.10}$$

$$p := \frac{1}{i}\partial_x. \tag{1.11}$$

Equation (1.8) continues to have a meaning if we replace the multiplication operator f by an arbitrary bounded operator A on $L^2(\mathbb{R})$ for which all the commutators make sense. This gives (formally) a quantum extension of the equation of a classical diffusion flow. Formally any diffusion has a quantum extension; analytically this is false even in one dimension (cf. [FagMon96]). Furthermore, even at a formal level, there are several quantum extensions of a classical diffusion flow (e.g., in (1.8) one can replace $p(\sigma)$ and $p(b_\sigma)$ by $p(\sigma) + u$ and $p(b_\sigma) + v$, where u, v are arbitrary multiplication operators, without changing the classical flow). See [Fag99] Section 4.2 for a detailed discussion including the d-dimensional case. In the present note we only discuss the classical case for which it can be proved that the results below do not depend on the choice of this extension.

Let us introduce the notations

$$\psi_0 := 1 \qquad ; \qquad \psi_{\chi_{[0,t]}} = e^{-\int_0^t dW_s - \frac{t}{2}} \in L^2(\Omega, \mathcal{F}, P) \tag{1.12}$$

where 1 denotes the constant function equal to 1 in $L^2(\Omega, \mathcal{F}, P)$. Moreover, if $\varphi, \psi \in L^2(\Omega, \mathcal{F}, P)$ are arbitrary vectors, the map

$$b \otimes B \in \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \mapsto b\langle \varphi, B\psi \rangle \in \mathcal{B}\left(L^2(\mathbb{R})\right)$$

has a unique extension to a bounded linear map denoted

$$A \in \mathcal{B}\left(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)\right) \mapsto \langle \varphi, A\psi \rangle \in \mathcal{B}\left(L^2(\mathbb{R})\right).$$

One can show that this map is an extension of the time zero conditional expectation $E_{0]}$ onto the σ-algebra of the initial condition X_0 of equation (1.1), i.e., if $A = f(X_t)$ then

$$\langle \varphi, f(X_t)\psi \rangle = E_{0]}(\varphi \cdot f(X_t) \cdot \psi)$$

where, for $\omega \in \Omega$, $(\varphi \cdot f(X_t) \cdot \psi)(\omega)$ is the multiplication by: $\varphi(\omega) f(X_t((\omega))\psi(\omega)$ (replacing $X_t(\omega)$ by $X_t(x, \omega)$ — the solution of (1.1) starting at $x \in \mathbb{R}$ — one would obtain a multiplication operator on $L^2(\mathbb{R})$). With these notations one can define four 1-parameter linear maps

$$P^t_{00}, P^t_{01}, P^t_{01}, P^t_{11} : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$$

through the prescription

$$\begin{pmatrix} P^t_{00}(f)(X_0) & P^f_{01}(f)(X_0) \\ P^t_{01}(f)(X_0) & P^t_{11}(f)(X_0) \end{pmatrix} := \begin{pmatrix} \langle \psi_0, j_t(f)\psi_0 \rangle & \langle \psi_0, j_t(f)\psi_{\chi_{[0,t]}} \rangle \\ \langle \psi_{\chi_{[0,t]}}, j_t(f)\psi_0 \rangle & \langle \psi_{\chi_{[0,t]}}, j_t(f)\psi_{\chi_{[0,t]}} \rangle \end{pmatrix}$$

$$= \begin{pmatrix} E_{0]}(f_1(X_t)) & E_{0]}(f_2(X_t)e^{-\int_0^t dw_s - \frac{t}{2}}) \\ E_{0]}(f_2(X_t)e^{-\int_0^t dw_s - \frac{t}{2}}) & E_{0]}(f_s(X_t)e^{-2\int_0^t dw_s - t}) \end{pmatrix} \tag{1.13}$$

The second identity in (1.13) shows that $P^t_{01} = P^t_{01}$ and that all the $P^t_{\alpha\beta}$ are positivity preserving. Both properties are not obvious from the first identity and in the quantum case they are not true in general.

It can be proved [AcKo99b], [AcKo00b] that $P^t_{00}, P^t_{01}, P^t_{01}, P^t_{11}$ are w^*-continuous semigroups and that they uniquely determine the classical flow $j_t(f) = f(X_t)$ in the sense that they uniquely determine all the partial scalar products (in $L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)$)

$$\langle e^{\int_\mathbb{R} g_s dW_s}, f(X_t)e^{\int_\mathbb{R} h_s dW_s} \rangle = E_{0]}\left(e^{\int_\mathbb{R} g_s dW_s} \cdot f(X_t) \cdot e^{\int_\mathbb{R} h_s dW_s}\right)$$

for any choice of $g, h \in L^2(\mathbb{R})$. By the totality of the exponential martingales these products uniquely determine the flow $f(X_t)$.

By the Hille–Yoshida theorem also the generators of these four semigroups

$$\begin{pmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{pmatrix}$$

uniquely determine the classical flow $j_t(f) = f(X_t)$ in the same sense.

It is clear from (1.13) that P^t_{00} is the usual Markov semigroup associated to the process (X_t) and the other ones are perturbations of it. The formal generators of these semigroups are easily determined using (1.13) and the Ito formula

$$L_{00} = \frac{1}{2}\sigma^2\partial_x^2 - b\partial_x \tag{1.14}$$

$$L_{01} = L_{10} = L_{00} + \sigma\partial_x \tag{1.15}$$

$$L_{11} = L_{00} + \sigma\partial_x + 1. \tag{1.16}$$

However, it should be emphasized that the sums in (1.15) and (1.16) are only formal in the sense that it may happen that the integral expressions (1.13) are well defined while the domains of the corresponding differential operators have zero intersections. The situation is exactly analogous to what happens in the usual Feynman–Kac or Girsanov formula. In fact the semigroups (1.13) are Girsanov perturbations of the basic Markov semigroup.

We sum up our conclusions in the following theorem.

Theorem 1.1 (X_t) *( ) fi ( ), ( ), ( )* w^* *(ff)*

Remark 1.1 One can prove the converse of the above statement if the following linear combinations of the above four generators:

$$\theta_0 := L_{00} = b\partial_x + \frac{1}{2}\sigma^2\partial_x^2 \qquad ; \qquad \theta_2 := \sigma\partial_x \tag{1.17}$$

have a common core containing a sequence of functions (necessarily twice continuously differentiable) converging to the multiplication operator by the function id, defined by (1.4), strongly on a core of this operator.

In view of the above result the following definition is quite natural.

Definition 1.5 $b, \sigma,$ $W,$ $(\Omega, P, \mathcal{F}, (\mathcal{F}_{t]})$ *( ) fi ( ), ( ), ( )* w^* *(ff)*

1.4 The four semigroups canonically associated to the Ornstein–Uhlenbeck process

The classical (one-dimensional) Ornstein–Uhlenbeck process is the real-valued stochastic process satisfying

$$dX_t = \sigma dW_t - bX_t dt \qquad ; \qquad X(0) = X_0$$

where σ, b are positive constants and $(W_t)_{t\geq 0}$ is the standard Wiener process on $(\Omega, \mathcal{F}, P)$. Equivalently

$$X_t = X_0 \mathrm{e}^{-bt} + \sigma \int_0^t \mathrm{e}^{-b(t-s)} dW_s.$$

The associated flow $j_t : L^\infty(\mathbb{R}) \to L^\infty(\Omega, \mathcal{F}, P)$; $t \geq 0$ is defined by

$$j_t(f) := f(X_t)$$

and, for $f \in C^2(\mathbb{R})$ it satisfies the equation

$$df(X_t) = \sigma f'(X_t)dW_t + \left(\frac{\sigma^2}{2} f''(X_t) - bX_t f'(X_t)\right) dt. \tag{1.18}$$

In this case (i.e., with σ and b constants) the four operators defined by (1.14), (1.15), and (1.16) are effectively generators of strongly (and not only w^*-) continuous semigroups which can be written explicitly by applying Mehler's formula to the two perturbations of L_{00}

$$P_{00}^t(f)(x) = \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt}x + \sqrt{(1-\mathrm{e}^{-bt})/b}\, y\right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy \tag{1.19}$$

$$P_{01}^t(f)(x) = \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt}x + \sqrt{(1-\mathrm{e}^{-bt})/b}\,y + \sigma t\right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy \tag{1.20}$$

$$P_{10}^t(f) = P_{01}^t(f) \tag{1.21}$$

$$P_{11}^t(f)(x) = \mathrm{e}^t \int_{-\infty}^{+\infty} f\left(\mathrm{e}^{-bt}x + \sqrt{(1-\mathrm{e}^{-bt})/b}\,y + \sigma t\right) \frac{\mathrm{e}^{-y^2/2\sigma^2}}{\sqrt{2\pi}\sigma} dy. \tag{1.22}$$

Indeed, for a smooth f (which is bounded by assumption), the right-hand sides of all these identities are differentiable in t and lead to the correct partial differential equation which has a unique solution.

Remark 1.2 Although the semigroups (1.20), (1.21) are simple perturbations of the semigroup (1.19), it is worth noticing that the differentiation operator on L^∞ $f \mapsto \sigma f'$ is not relatively bounded with respect to the generator (1.19). Indeed, for $\sigma^2 = 2, b = 2$, there exists a sequence (f_n) of smooth functions vanishing at infinity and satisfying

$$\|f_n\|_\infty = \pi/(2\sqrt{n}), \quad \|f_n'\|_\infty \geq \mathrm{e}^{-1}\log(1+n), \quad \|L_0^0(f_n)\|_\infty = 1/\sqrt{n}.$$

Therefore the generators (1.20), (1.21) are simple but not "regular" perturbations of the generator (1.19).

References

[AcKo99b] L. Accardi, S.V. Kozyrev, On the structure of Markov flows, Chaos, Solitons and Fractals 12 (2001), 2639–2655, Volterra Preprint N. 393 November (1999).

[AcKo00b] L. Accardi, S.V. Kozyrev, Quantum interacting particle systems, Lectures given at the Volterra–CIRM International School: Quantum interacting particle systems, Levico Terme, 23–29 September 2000, in: QP–PQ XIV Accardi L., F. Fagnola (eds.) "Quantum interacting particle systems," World Scientific (2002), Preprint Volterra, N. 431 September (2000).

[FagMon96] F. Fagnola, R. Monte, A quantum extension of the semigroup of Bessel processes. Mat. Zametki 60 (5) (1996), 519–537.

[Fag99] F. Fagnola, Quantum Markov semigroups and quantum flows, Proyecciones 18 (1999), 1–144.

[FagWil03] F. Fagnola, S.J. Wills, Solving quantum stochastic differential equations with unbounded coefficients, J. Funct. Anal. 198 (2) (2003), 279–310.

[KarShr88] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, I, Springer, New York, (1988).

2 Feynman Path Integrals for Time-Dependent Potentials

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2.1 Introduction

Let us consider the Schrödinger equation describing the time evolution of the state of a quantum particle moving in a potential V

$$\begin{cases} i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\Delta\psi + V\psi \\ \psi(0,x) = \psi_0(x) \end{cases} \tag{2.1}$$

where $m > 0$ is the mass of the particle, $\hbar$ is the reduced Planck constant, $t \geq 0$, $x \in \mathbb{R}^d$. The aim of the present work is to give a rigorous mathematical meaning to the Feynman path integral representation of the solution of the Cauchy problem (2.1) in the case where the potential depends explicitly on the time variable t

$$\psi(t,x) = \text{``} \int_{\{\gamma|\gamma(t)=x\}} e^{\frac{i}{\hbar}S_t(\gamma)}\psi_0(\gamma(0))D\gamma \text{''} \tag{2.2}$$

(where $S_t(\gamma) = \frac{m}{2}\int_0^t |\dot\gamma(s)|^2 ds - \int_0^t V(s,\gamma(s))ds$ is the classical action of the system evaluated along the path γ and $D\gamma$ an heuristic "flat" measure on the space of paths, see e.g., [23, 17] for a discussion of Feynman's approach and its applications). In the physical and in the mathematical literature several rigorous mathematical realizations of the heuristic "Feynman complex measure"

$$\left(\int e^{\frac{i}{\hbar}S_t(\gamma)}D\gamma\right)^{-1} e^{\frac{i}{\hbar}S_t(\gamma)}D\gamma$$

can be found, for instance, by means of analytic continuation of probabilistic Wiener integrals [16, 34, 15, 28, 18], via white-noise calculus [25, 20], via non standard analysis [4], or as "infinite-dimensional oscillatory integrals" (see [8, 9, 19, 2]). In the following we shall focus on the latter approach, which is particularly interesting as it allows the rigorous implementation of an infinite-dimensional version of the stationary phase method and the corresponding study of the asymptotic behavior of the solution of the Schrödinger equation in the limit where $\hbar$ is considered as a small parameter approaching zero. We remark that some interesting results concerning the rigorous Feynman path integral representation of the solution of (2.1) in the case V depending explicitly on time have already been obtained by means of the white-noise approach [20, 24, 32, 33] and by means of the analytic continuation approach (see [27] and references therein). In [37, 38, 39] the Schrödinger equation with a harmonic oscillator Hamiltonian with a time-dependent frequency has been considered. In [6, 7] the infinite-dimensional oscillatory integral approach has been applied to the rigorous Feynman path integral representation of the solution of a stochastic Schrödinger equation with a time-dependent Hamiltonian.

In section 2.2 we recall some classical results, that is, the definitions and the main theorems on finite- and infinite-dimensional oscillatory integrals. In sections 2.3 and 2.4 the Schrödinger equation for a linearly forced harmonic oscillator and for a harmonic oscillator with a time-dependent frequency is considered. In the latter case the problem is solved by adopting a suitable transformation of the time and space variables which allows mapping, both in the classical and in the quantum case, the solution of the time-independent harmonic oscillator to the solution of the time-dependent one. In section 2.5 the potentials considered in sections 2.3 and 2.4 are perturbed with a bounded (time-dependent) potential, satisfying suitable assumptions.

2.2 Infinite-dimensional oscillatory integrals

In this section we recall for later use the definitions and the main results on infinite-dimensional oscillatory integrals; for more details we refer to [2, 8, 19]. In the following we will denote by $\mathcal{H}$ a (finite or infinite-dimensional) real separable Hilbert space, whose elements are denoted by $x, y \in \mathcal{H}$ and the scalar product with $\langle x, y \rangle$. $f : \mathcal{H} \to \mathbb{C}$ will be a function on $\mathcal{H}$ and $L : D(L) \subseteq \mathcal{H} \to \mathcal{H}$ an invertible, densely defined and self-adjoint operator. Let us denote by $\mathcal{M}(\mathcal{H})$ the Banach space of the complex bounded variation measures on $\mathcal{H}$, endowed with the total variation norm, that is

$$\mu \in \mathcal{M}(\mathcal{H}), \qquad \|\mu\| = \sup \sum_i |\mu(E_i)|,$$

where the supremum is taken over all sequences $\{E_i\}$ of pairwise disjoint Borel subsets of $\mathcal{H}$, such that $\cup_i E_i = \mathcal{H}$. $\mathcal{M}(\mathcal{H})$ is a Banach algebra, where the product of two measures $\mu * \nu$ is by definition their convolution

$$\mu * \nu(E) = \int_{\mathcal{H}} \mu(E - x)\nu(dx), \qquad \mu, \nu \in \mathcal{M}(\mathcal{H})$$

and the unit element is the vector δ_0.

Let $\mathcal{F}(\mathcal{H})$ be the space of complex functions on $\mathcal{H}$ which are Fourier transforms of measures belonging to $\mathcal{M}(\mathcal{H})$, that is

$$f : \mathcal{H} \to \mathbb{C} \qquad f(x) = \int_{\mathcal{H}} e^{i\langle x, \beta \rangle} \mu_f(d\beta) \equiv \hat{\mu}_f(x).$$

$\mathcal{F}(\mathcal{H})$ is a Banach algebra of functions, where the product is the pointwise one; the unit element is the function 1, i.e., $1(x) = 1 \; \forall x \in \mathcal{H}$ and the norm is given by $\|f\| = \|\mu_f\|$.

Let us suppose first of all that $\mathcal{H}$ is finite dimensional, i.e., $\mathcal{H} = \mathbb{R}^n$. In this case, following Hörmander [26], the "oscillatory integral"

$$\int e^{\frac{i}{2\hbar}\langle x, x \rangle} f(x) dx, \qquad \hbar > 0$$

is defined as the limit of a sequence of regularized, hence absolutely convergent, Lebesgue integrals. More precisely, we have for such integrals, called after normalization, "Fresnel integrals."

Definition 2.1 [illegible] $f : \mathbb{R}^n \to \mathbb{C}$ [illegible] $\phi \in \mathcal{S}(\mathbb{R}^n)$ [illegible] $\phi(0) = 1$ [illegible]

$$\lim_{\epsilon \to 0} (2\pi i \hbar)^{-n/2} \int e^{\frac{i}{2\hbar}\langle x, x \rangle} f(x) \phi(\epsilon x) dx \tag{2.3}$$

[illegible] ϕ [illegible] f [illegible]

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx. \tag{2.4}$$

In the case where the Hilbert space $\mathcal{H}$ is infinite dimensional, the oscillatory integral is defined as the limit of a sequence of finite-dimensional approximations.

Definition 2.2 [illegible] $f : \mathcal{H} \to \mathbb{C}$ [illegible] P_n [illegible] $\mathcal{H}$, [illegible] $P_n \le P_{n+1}$ [illegible] $P_n \to 1$ [illegible] $n \to \infty$ (1 [illegible] $\mathcal{H}$) [illegible] fi [illegible]

$$(2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x) d(P_n x),$$

[illegible] fi [illegible]

$$\lim_{n\to\infty} (2\pi i\hbar)^{-n/2} \int_{P_n\mathcal{H}} e^{\frac{i}{2\hbar}\langle P_n x, P_n x\rangle} f(P_n x) d(P_n x) \tag{2.5}$$

[illegible] $\{P_n\}$ [illegible] f [illegible]

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} f(x)dx.$$

An “operational characterization” of the largest class of “Fresnel integrable functions” is still an open problem, even in finite dimension, but one can find some interesting subsets of it, such as $\mathcal{F}(\mathcal{H})$.

Theorem 2.1 [illegible] $L : \mathcal{H} \to \mathcal{H}$ [illegible] $(I-L)$ [illegible] $f : \mathcal{H} \to \mathbb{C}$ [illegible] μ_f [illegible] $\mathcal{H}$ [illegible] $e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)$ [illegible] fi [illegible] σ [illegible]

$$\widetilde{\int} e^{\frac{i}{2\hbar}\langle x,x\rangle} e^{-\frac{i}{2\hbar}\langle x,Lx\rangle} f(x)dx = (\det(I-L))^{-1/2} \int_{\mathcal{H}} e^{-\frac{i\hbar}{2}\langle \alpha,(I-L)^{-1}\alpha\rangle} \mu_f(d\alpha) \tag{2.6}$$

[illegible] $\det(I-L) = |\det(I-L)| e^{-\pi i \,\mathrm{Ind}\,(I-L)}$ [illegible] $(I-L)$, $|\det(I-L)|$ [illegible] $((I-L))$ [illegible] $(I-L)$, [illegible]

2.3 The linearly forced harmonic oscillator

Let us consider the Schrödinger equation (2.1) for a linearly forced harmonic oscillator, i.e., let us assume that the potential V is of the type “quadratic plus linear” and that the linear part depends explicitly on time

$$H = -\frac{\hbar^2}{2m}\Delta + V(t,x), \quad V(t,x) = \frac{1}{2}x\Omega^2 x + f(t)\cdot x, \quad x \in \mathbb{R}^d \tag{2.7}$$

where Ω is a positive symmetric constant $d \times d$ matrix with eigenvalues Ω_j, $j = 1, \ldots, d$, and $f : I \subset \mathbb{R} \to \mathbb{R}^d$ is a continuous function. This potential is particularly interesting from a physical point of view, as it is used in simple models for a large class of processes as the vibration–relaxation of a diatomic molecule in gas kinetics and the interaction of a particle with the field oscillators in quantum electrodynamics. Feynman calculated heuristically the Green function for (2.7) in his famous paper on the path integral formulation of quantum mechanics [22]. Our aim is to give meaning to the Feynman path integral representation of the solution of equation (2.1) (with $m = 1$, for simplicity, and V given by (2.7))

$$\psi(t,x) = \int_{\gamma(t)=x} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \frac{i}{2\hbar}\int_0^t \gamma(s)\Omega^2\gamma(s)ds - \frac{i}{\hbar}\int_0^t f(s)\cdot\gamma(s)ds}\psi_0(\gamma(0))d\gamma \tag{2.8}$$

in terms of a well-defined infinite-dimensional oscillatory integral on the Cameron–Martin space $\mathcal{H}_t$, i.e., the Hilbert space of absolutely continuous paths $\gamma : [0,t] \to \mathbb{R}^d$, such that $\gamma(t) = 0$, and square integrable weak derivative $\int_0^t |\dot\gamma(s)|^2 ds < \infty$, endowed with the inner product $\langle\gamma_1,\gamma_2\rangle = \int_0^t \dot\gamma_1(s)\cdot\dot\gamma_2(s)ds$. We recall that a similar result has been obtained in the case $d = 1$ by means of the white-noise approach [20]. Let $L : \mathcal{H}_t \to \mathcal{H}_t$ be the trace-class symmetric operator on $\mathcal{H}_t$ given by

$$(L\gamma)(s) = \int_s^t ds' \int_0^{s'} (\Omega^2\gamma)(s'')ds'', \qquad \gamma \in \mathcal{H}_t.$$

One can easily verify that $\langle\gamma_1, L\gamma_2\rangle = \int_0^t \gamma_1(s)\Omega^2\gamma_2(s)ds$. Moreover, if $t \neq (n+1/2)\pi/\Omega_j$, $n \in \mathbb{Z}$ and Ω_j any eigenvalue of Ω, $(I - L)$ is invertible with

$$\begin{aligned}(I-L)^{-1}\gamma(s) = \gamma(s) - \Omega\int_s^t \sin[\Omega(s'-s)]\gamma(s')ds' \\ + \sin[\Omega(t-s)]\int_0^t [\cos\Omega t]^{-1}\Omega\cos(\Omega s')\gamma(s')ds',\end{aligned} \tag{2.9}$$

and

$$\det(I-L) = \det(\cos(\Omega t))$$

(for more details see [19]). With these notations, formula (2.8) can be written in the following way:

$$\begin{aligned}\psi(t,x) = \int_{\gamma(t)=0} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \frac{i}{2\hbar}\int_0^t (\gamma(s)+x)\Omega^2(\gamma(s)+x)ds} \\ e^{-\frac{i}{\hbar}\int_0^t f(s)\cdot(\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma = e^{-i\frac{t}{2\hbar}x\Omega^2 x}e^{-i\frac{x}{2\hbar}\cdot\int_0^t f(s)ds} \\ \widetilde{\int}_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle}e^{i\langle v,\gamma\rangle}e^{i\langle w,\gamma\rangle}\psi_0(\gamma(0)+x)d\gamma\end{aligned} \tag{2.10}$$

where $w, v \in \mathcal{H}_t$ are defined by

$$w(s) \equiv \frac{\Omega^2 x}{2\hbar}(s^2 - t^2), \qquad v(s) \equiv \frac{1}{\hbar}\int_t^s \int_0^{s'} f(s'')ds''ds', \tag{2.11}$$

$s \in [0,t]$.

Under the assumption that $\psi_0 \in \mathcal{F}(\mathbb{R}^d)$ it is possible to prove that the functional on $\mathcal{H}_t$ given by $\gamma \mapsto \psi_0(\gamma(0)+x)$ belongs to $\mathcal{F}(\mathcal{H}_t)$. In fact if $\psi_0(x) = \int_{\mathbb{R}^d} e^{ik\cdot x}\mu_0(dk)$, then

$$\psi_0(\gamma(0)+x) = \int_{\mathcal{H}_t} e^{i\langle\eta,\gamma\rangle}\mu_{\psi_0}(d\eta),$$

where

$$\int_{\mathcal{H}_t} f(\gamma)\mu_{\psi_0}(d\gamma) = \int_{\mathbb{R}^d} e^{ik\cdot x} f(kG_0)\mu_0(dk) \tag{2.12}$$

and, for any $k \in \mathbb{R}^d$, kG_0 is the element in $\mathcal{H}_t$ such that $\langle kG_0, \gamma\rangle = k \cdot \gamma(0)$; that is, $kG_0(s) = k(t-s)$. In this case the functional $\gamma \mapsto e^{i\langle v,\gamma\rangle} e^{i\langle w,\gamma\rangle}\psi_0(\gamma(0)+x)$ belongs to $\mathcal{F}(\mathcal{H}_t)$ and the infinite-dimensional oscillatory integral (2.10) on $\mathcal{H}_t$ can be explicitly computed by means of the Parseval-type equality (2.6)

$$\begin{aligned}&\widetilde{\int}_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{i\langle v,\gamma\rangle} e^{i\langle w,\gamma\rangle}\psi_0(\gamma(0)+x)d\gamma \\ &\qquad = (\det(I-L))^{-1/2}\int_{\mathcal{H}_t} e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}\delta_v * \delta_w * \mu_{\psi_0}(d\gamma)\end{aligned} \tag{2.13}$$

and we have

$$\begin{aligned}\psi(t,x) = \frac{e^{-i\frac{t}{2\hbar}x\Omega^2 x} e^{-i\frac{x}{2\hbar}\cdot\int_0^t f(s)ds}}{\sqrt{\det(\cos(\Omega t))}} &\int_{\mathbb{R}^d} e^{ik\cdot x} \\ &e^{-\frac{i\hbar}{2}\langle(v+w+kG_0),(I-L)^{-1}(v+w+kG_0)\rangle}\mu_0(dk).\end{aligned} \tag{2.14}$$

If $\psi_0 \in \mathcal{S}(\mathbb{R}^d)$, we can proceed further and compute explicitly the Green function $G(0,t,x,y)$

$$\psi(t,x) = \int_{\mathbb{R}^d} G(0,t,x,y)\psi_0(y)dy,$$

where

$$\begin{aligned}G(0,t,x,y) &= (2\pi i\hbar)^{-d/2}\sqrt{\det\left(\frac{\Omega}{\sin(\Omega t)}\right)} e^{\frac{i\Omega\sin(\Omega t)^{-1}}{2\hbar}(x\cos(\Omega t)x + y\cos(\Omega t)y - 2xy)} \\ &e^{-\frac{i}{\hbar}x\sin(\Omega t)^{-1}\int_0^t \sin(\Omega s)f(s)ds - \frac{i}{\hbar}y(\int_0^t\cos(\Omega s)f(s)ds - \cos(\Omega t)\sin(\Omega t)^{-1}\int_0^t\sin(\Omega s)f(s)ds)} \\ &e^{\frac{i}{\hbar}\Omega^{-1}(\frac{1}{2}\cos(\Omega t)\sin(\Omega t)^{-1}(\int_0^t\sin(\Omega s)f(s)ds)^2 - \int_0^t\sin(\Omega s)f(s)ds\int_0^t\cos(\Omega s)f(s)ds)} \\ &e^{\frac{i}{\hbar}\Omega^{-1}\int_0^t\cos(\Omega s)f(s)\int_s^t\sin(\Omega s')f(s')ds'ds}.\end{aligned} \tag{2.15}$$

One can easily verify by a direct computation that (2.15) is the Green function for the Schrödinger equation with the time-dependent Hamiltonian (2.7).

Remark 2.1 Our result can be obtained even if the initial assumption on the continuity of the function $f : [0,t] \to \mathbb{R}^d$ is weakened. In fact it is sufficient to assume that the function v in (2.11) belongs to $\mathcal{H}_t$; that is, $\int_0^t |\int_0^s f(s')ds'|^2 ds < \infty$.

2.4 The Schrödinger equation with time-dependent harmonic part

Let us consider the Schrödinger equation with a harmonic oscillator Hamiltonian with a time-dependent frequency

$$H = -\frac{\hbar^2}{2m}\Delta + \frac{1}{2}x\Omega^2(t)x, \qquad x \in \mathbb{R}^d \tag{2.16}$$

where $\Omega : [0,t] \to L(\mathbb{R}^d, \mathbb{R}^d)$ is a continuous map from the time interval $[0,t]$ to the space of symmetric positive $d \times d$ matrices. This problem has been analyzed by several authors (see, for instance, [31, 35] and references therein) as an approximate description for the vibration of complex physical systems, as well as an exact model for some physical phenomena; the motion of an ion in a Paul trap; the quantum mechanical description of highly cooled ions, the emergence of nonclassical optical states of light owing to a time-dependent dielectric constant; or even in cosmology for the study of a three-dimensional isotropic harmonic oscillator in a spatially flat universe such that $g_{ij} = R(t)\delta_{ij}$, with $R(t)$ being the scale factor at time t.

If $d = 1$, it is possible to solve the Schrödinger equation with Hamiltonian (2.16) (and also the corresponding classical equation of motion) by adopting a suitable transformation of the time and space variables which allows mapping of the solution of the time-independent harmonic oscillator to the solution of the time-dependent one (see [37, 38, 39] and references therein). Let us consider the classical equation of motion for the time-dependent harmonic oscillator (2.16)

$$\ddot{u}(s) + \Omega^2(s)u(s) = 0. \tag{2.17}$$

Let u_1 and u_2 be two independent solutions of (2.17) such that $u_1(0) = \dot{u}_2(0) = 0$ and $u_2(0) = \dot{u}_1(0) = 1$; then it easy to prove that the Wronskian $w(u_1, u_2) = u_1\dot{u}_2 - \dot{u}_1 u_2$ is a constant function $w = 1$. Let us define the function $\xi := u_1^2 + u_2^2$; then one proves that $\xi(s) > 0$ $\forall s$ and it satisfies the following differential equation:

$$2\xi\ddot{\xi} - \dot{\xi}^2 + 4\xi^2 - 4 = 0.$$

Moreover, the function $\eta : [0, \infty] \to \mathbb{R}$

$$\eta(s) = \int_0^s \xi(\tau)^{-1} d\tau$$

is well defined and strictly increasing. One verifies that

$$u(s) = \xi(s)^{1/2}(A\cos(\eta(s)) + B\sin(\eta(s))) \tag{2.18}$$

is the general solution of the classical equation of motion (2.17). In other words, by rescaling the time variable $s \mapsto \eta(s)$ and the space variable $x \mapsto \xi^{-1/2}x$ it is possible to map the solution of the equation of motion for the time-independent harmonic oscillator $\ddot{u}(s)+u(s) = 0$ into the solution of (2.17). In other words, it is possible to find (see, for instance, [29] for more details) a general canonical transformation $(x, p, t) \mapsto (X, P, \tau)$, given by

$$\begin{cases} X = \xi(t)^{-1/2}x \\ \frac{d\tau(t)}{dt} = \xi(t)^{-1} \\ P = \frac{dX}{d\tau} = (\xi^{1/2}\dot{x} - \frac{1}{2}\xi^{-1/2}\dot{\xi}x) \end{cases} \tag{2.19}$$

and the Hamiltonian is given by $H(X, P, \tau) = \frac{1}{2}(P^2 + X^2)$, while the generating function of the transformation $(x, p, t) \mapsto (X, P, \tau)$ is given by $F(x, P, t) = \xi(t)^{-1/2}xP + \frac{\xi(t)^{-1}\dot{\xi}}{4}x^2$ and the transformation is given more explicitly as

$$\begin{cases} p = \frac{\partial}{\partial x}F(x, P, t) \\ X = \frac{\partial}{\partial P}F(x, P, t) \\ H(X, P; \tau)\dot{\tau} = H(x, p; t) + \frac{\partial}{\partial t}F(x, P, t). \end{cases} \tag{2.20}$$

A similar result holds also in the quantum case. In fact by considering the Schrödinger equations for the time-independent and time-dependent harmonic oscillator, respectively,

$$(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\Delta - \frac{1}{2}x^2)\phi(t, x) = 0, \tag{2.21}$$

$$(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\Delta - \frac{1}{2}\Omega^2(t)x^2)\psi(t,x) = 0, \tag{2.22}$$

where $\phi(t,x)$ and $\psi(t,x)$ are continuously differentiable with respect to t and twice continuously differentiable with respect to x, it is possible to prove the following [37].

Theorem 2.2 $\phi(t,x)$ (2.21)

$$\psi(t,x) = \xi(t)^{-1/4}\exp[i\dot{\xi}(t)x^2/4\hbar\xi(t)]\phi(\eta(t),\xi(t)^{-1/2}x)$$

(2.22)

In an analogous way it is possible to prove that, denoting by $K_{TI}(t,0;x,y)$ and $K_{TD}(t,0;x,y)$, the Green functions for the Schrödinger equations (2.21) and (2.22), respectively, the following holds:

$$K_{TD}(t,0;x,y) = K_{TI}(\eta(t),0;\xi(t)^{-1/2}x,y). \tag{2.23}$$

It is interesting to note that the "correction term" $\xi(t)^{-1/4}\exp[i\dot{\xi}(t)x^2/4\hbar\xi(t)]$ in Theorem 2.2 can be interpreted in terms of the classical canonical transformation (2.20) (see [29] for more details).

The aim of the present section is to give a rigorous mathematical meaning to the Feynman path integral representation of the solution of equation (2.22) by means of a well-defined infinite-dimensional oscillatory integral associated with the Cameron–Martin space $\mathcal{H}_t$ and to prove by means of it formula (2.23). A similar result has been obtained in the framework of the white-noise approach [24].

Let us consider the following linear operator $L : \mathcal{H}_t \to \mathcal{H}_t$:

$$(L\gamma)(s) = -\int_t^s \int_0^r \Omega^2(u)\gamma(u)du dr, \qquad \gamma \in \mathcal{H}_t.$$

One can easily verify that L is self-adjoint and positive; moreover, for any $\gamma_1, \gamma_2 \in \mathcal{H}_t$ one has

$$\langle \gamma_1, L\gamma_2 \rangle = \int_0^t \gamma_1(s)\Omega^2(s)\gamma_2(s)ds.$$

Moreover, by using formula (2.18), it is possible to prove that, if $t \neq \eta^{-1}(\pi/2 + n\pi)$, $n \in \mathbb{N}$, the operator $I - L$ is invertible and its inverse is given by

$$\begin{aligned}(I-L)^{-1}\gamma(s) = \Big[-&\frac{\sin(\eta(t))}{\cos(\eta(t))}\Big(\int_0^t \xi(s')^{1/2}\cos(\eta(s'))\ddot{\gamma}(s')ds' + \\ &+ \dot{\gamma}(0)\Big) - \int_t^s \xi(s')^{1/2}\sin(\eta(s'))\ddot{\gamma}(s')ds'\Big]\xi(s)^{1/2}\cos(\eta(s)) + \\ &+ \Big[\int_0^t \xi(s')^{1/2}\cos(\eta(s'))\ddot{\gamma}(s')ds' + \dot{\gamma}(0) + \\ &+ \int_t^s \xi(s')^{1/2}\cos(\eta(s'))\ddot{\gamma}(s')ds'\Big]\xi(s)^{1/2}\sin(\eta(s)).\end{aligned} \tag{2.24}$$

The Fredholm determinant of the operator $I - L$ can be computed by exploiting the general relation between infinite-dimensional determinants of the form $\det(I + \epsilon L)$, $\epsilon \in \mathbb{C}$ and where L is of trace class, and finite-dimensional determinants associated with the solution of a certain Sturm–Liouville problem [1, 39]. According to [1], by using the fact that $v(s) = L\gamma(s)$ is the unique solution of the problem

$$\begin{cases} \ddot{v}(s) = -\Omega^2(s)\gamma(s), & s \in (0,t), \\ \dot{v}(0) = 0, \quad v(t) = 0 \end{cases} \tag{2.25}$$

and by using the ellipticity of the problem (2.25), one proves that the range of L is contained in $H^3((0,t);\mathbb{R})$, the Sobolev space of functions belonging to $L^2((0,t);\mathbb{R})$ which derivatives up to order 3 belong also to $L^2((0,t);\mathbb{R})$; hence L is a trace-class operator. Moreover, by considering the solution K_ϵ of the initial value problem

$$\begin{cases} \ddot{K}_\epsilon(s) + \epsilon\Omega^2(s)K_\epsilon(s) = 0, \\ \dot{K}_\epsilon(0) = 0, \quad K_\epsilon(0) = 1 \end{cases} \tag{2.26}$$

one has

$$K_\epsilon(t) = \det(I - \epsilon L).$$

By substituting $\epsilon = 1$ in (2.26) and by using formula (2.18) for the general solution of the differential equation (2.17) one has

$$\det(I - L) = \xi(t)^{1/2}\cos(\eta(t)). \tag{2.27}$$

Let us consider now the vectors $G_0, u \in \mathcal{H}_t$ given by

$$G_0(s) = t - s, \qquad u(s) = \frac{x}{\hbar}\int_t^s \int_0^u \Omega^2(r)drdu,\ x \in \mathbb{R}.$$

With the notations introduced so far and by assuming that the initial vector ψ_0 belongs to $\mathcal{F}(\mathbb{R})$, so that $\psi_0 = \hat{\mu}_0$, the heuristic Feynman path integral representation for the solution of the Schrödinger equation (2.1) with the time-dependent Hamiltonian (2.16)

$$\psi(t,x) = \text{“}\int_{\{\gamma|\gamma(t)=0\}} e^{\frac{i}{2\hbar}\int_0^t \dot{\gamma}^2(s)ds - \frac{i}{2\hbar}\int_0^t \Omega^2(s)(\gamma(s)+x)^2 ds}\psi_0(\gamma(0)+x)D\gamma\text{”}$$

can be rigorously realized as the infinite-dimensional oscillatory integral associated with the Cameron–Martin space $\mathcal{H}_t$

$$\psi(t,x) = e^{\frac{ix^2}{2\hbar}\int_0^t \Omega^2(s)ds} I_t, \qquad I_t = \widetilde{\int_{\mathcal{H}_t}} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)\gamma\rangle} e^{i\langle u,\gamma\rangle}\hat{\mu}_{\psi_0}(\gamma)d\gamma$$

where μ_{ψ_0} is given by formula (2.12). By the Parseval-type equality, I_t can be explicitly computed and one has

$$I_t = \det(I - L)^{-1/2}\int_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\langle\gamma,(I-L)^{-1}\gamma\rangle}\delta_u * \mu_{\psi_0}(d\gamma). \tag{2.28}$$

By assuming $\psi_0 \in \mathcal{S}(\mathbb{R})$, we can proceed further and compute explicitly the Green function of the problem; that is

$$\psi(t,x) = \int_{\mathbb{R}} K_{TD}(t,0;x,y)\psi_0(y)dy.$$

By substituting in (2.28) formulae (2.24) and (2.27), and performing a simple calculation we obtain

$$K_{TD}(t,0;x,y) = \xi(t)^{-1/4} e^{\frac{ix^2}{4\hbar}\xi(t)^{-1}\dot{\xi}(t)} \frac{e^{\frac{i}{2\hbar}\left(\frac{\cos(\eta(t))}{\sin(\eta(t))}\right)(\xi(t)^{-1}x^2+y^2) - \frac{2\xi(t)^{-1/2}xy}{\sin(\eta(t))}}}{(2\pi i\hbar\sin(\eta(t)))^{1/2}}$$

and by recalling the well-known formula for the Green function $K_{TI}(t,0;x,y)$ of the Schrödinger equation with a time-independent harmonic oscillator Hamiltonian (see, e.g., [40])

$$K_{TI}(t,0;x,y) = \frac{e^{\frac{i}{2\hbar}\left(\frac{\cos(t)}{\sin(t)}(x^2+y^2) - \frac{2xy}{\sin(t)}\right)}}{(2\pi i\hbar\sin(t))^{1/2}}$$

one can verify directly formula (2.23).

Remark 2.2 The case where $d > 1$ is more complicated. In fact neither a transformation formula analogous to (2.19) exists in general, nor a formula analogous to (2.23) relating the Green function of the Schrödinger equation with a time-dependent resp. time-independent harmonic oscillator potential (see, for instance, [38, 39] for some partial results in this direction).

2.5 Bounded perturbations in the Schrödinger equation

Let us consider now a quantum mechanical Hamiltonian of the following form:

$$H = H_0 + V(t, x), \tag{2.29}$$

where H_0 is of the type (2.7) or (2.16) and $V : [0, t] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following assumptions:

1. For each $s \in [0, t]$, the application $V(s, \cdot\,) : \mathbb{R}^d \to \mathbb{R}$ belongs to $\mathcal{F}(\mathbb{R}^d)$, i.e. $V(s, x) = \int_{\mathbb{R}^d} e^{ikx}\sigma_s(dk)$, $\sigma_s \in \mathcal{M}(\mathbb{R}^d)$.
2. The application $s \in [0, t] \mapsto \sigma_s \in \mathcal{M}(\mathbb{R}^d)$ is continuous in the norm $\|\cdot\|$ of the Banach space $\mathcal{M}(\mathbb{R}^d)$.

Note that condition (1) implies that for each $s \in [0, t]$, the function $V(s, \cdot\,) : \mathbb{R}^d \to \mathbb{R}$ is bounded. Moreover, by condition (2) one can easily verify that the application $s \in [0, t] \mapsto V(s, \cdot) \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ in continuous in the sup-norm.

Under the assumptions above it is possible to prove that the application

$$\gamma \in \mathcal{H}_t \mapsto V(s, \gamma(s) + x)$$

belongs to $\mathcal{F}(\mathcal{H}_t)$, more precisely it is the Fourier transform of the complex bounded variation measure μ_s on the Cameron–Martin space $\mathcal{H}_t$ given by

$$\mu_s(I) = \int_{\mathbb{R}^d} e^{ikx}\chi_I(kGs)\sigma_s(dk) \qquad I \in \mathcal{B}(\mathcal{H}_t),$$

where χ_I is the characteristic function of the Borel set I and, for any $k \in \mathbb{R}^d$, kG_s is the element in $\mathcal{H}_t$ given by $kG_s(s') = k(t - s)$ for $s' \le s$ and $kG_s(s') = k(t - s')$ for $s' > s$. As a consequence also the application

$$\gamma \in \mathcal{H}_t \mapsto e^{-\frac{i}{\hbar}\int_r^s V(u,\gamma(u)+x)du}, r, s \in [0, t]$$

belongs to $\mathcal{F}(\mathcal{H}_t)$. Let us denote by ν_r^s the bounded variation measure on $\mathcal{H}_t$ associated to it.

Under the assumptions the initial datum ψ_0 belongs to $L^2(\mathbb{R}^d) \cap \mathcal{F}(\mathbb{R}^d)$, the infinite-dimensional oscillatory integral associated with the Cameron–Martin space $\mathcal{H}_t$

$$\int_{\gamma(t)=0} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \frac{i}{2\hbar}\int_0^t (\gamma(s)+x)\Omega^2(s)(\gamma(s)+x)ds} \\ e^{-\frac{i}{\hbar}\int_0^t f(s)\cdot(\gamma(s)+x)ds} e^{-\frac{i}{\hbar}\int_0^t V(s,\gamma(s)+x)ds}\psi_0(\gamma(0) + x)d\gamma$$

(with $\Omega^2(s) = \Omega^2$ independent of s if $f \neq 0$) giving the rigorous mathematical realization of the Feynman path integral representation of the solution of the Schrödinger equation with time-dependent Hamiltonian (2.29), with H_0 is is given by (2.7) (if $\Omega^2(s) = \Omega^2$ and $f \neq 0$) and by (2.16), respectively, are well defined. In the following the detailed proof is given in the particular case where the “free Hamiltonian” H_0 is given by (2.7), but the same reasonings can be repeated in the case where H_0 is given by (2.16).

If $\{\mu_u : a \leq u \leq b\}$ is a family in $\mathcal{M}(\mathcal{H}_t)$, we shall let $\int_a^b \mu_u du$ denote the measure on $\mathcal{H}_t$ given by

$$f \to \int_a^b \int_{\mathcal{H}_t} f(\gamma)\mu_u(d\gamma)du$$

whenever it exists (it exists, e.g., if f is continuous and we have that $u \mapsto \mu_u$ is continuous from $[0,t]$ to $\mathcal{M}(\mathcal{H}_t)$, the space of measures on $\mathcal{H}_t$ with the norm given by the total variation $\|\mu_u\|$). Since for any continuous path γ

$$\exp\Big(-\frac{i}{\hbar}\int_0^t V(s,\gamma(s))ds\Big) = \\ 1 - \frac{i}{\hbar}\int_0^t V(u,\gamma(u))\exp\Big(-\frac{i}{\hbar}\int_0^u V(s,\gamma(s))ds\Big)du,$$

we get

$$\nu_0^t = \delta_0 - \frac{i}{\hbar}\int_0^t (\mu_u * \nu_0^u)du \tag{2.30}$$

where δ_0 is the Dirac measure at $0 \in H_t$.

For $r, s \in [0,t]$, let λ_r^s and η_r^s be the measures on $\mathcal{H}_t$ whose Fourier transforms when evaluated at $\gamma \in \mathcal{H}_t$ are, respectively, $e^{-\frac{i}{\hbar}\int_r^s f(u)\gamma(u)du}$ and $e^{-\frac{i}{\hbar}\int_r^s x\Omega^2\gamma(u)du}$. We set for $t > 0$ and $x \in \mathbb{R}^d$

$$U(t,0)\psi_0(x) = \int_{\gamma(t)=0} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \frac{i}{2\hbar}\int_0^t (\gamma(s)+x)\Omega^2(\gamma(s)+x)ds} \\ e^{-\frac{i}{\hbar}\int_0^t f(s)\cdot(\gamma(s)+x)ds} e^{-\frac{i}{\hbar}\int_0^t V(s,\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma \tag{2.31}$$

and

$$U_0(t,0)\psi_0(x) = \int_{\gamma(t)=0} e^{\frac{i}{2\hbar}\int_0^t |\dot\gamma(s)|^2 ds - \frac{i}{2\hbar}\int_0^t (\gamma(s)+x)\Omega^2(\gamma(s)+x)ds} \\ e^{-\frac{i}{\hbar}\int_0^t f(s)\cdot(\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma. \tag{2.32}$$

By the Parseval-type equality we have

$$U(t,0)\psi_0(x) = e^{-i\frac{t}{2\hbar}x\Omega^2 x} e^{-i\frac{x}{\hbar}\cdot\int_0^t f(s)ds} \\ (\det(I-L))^{-1/2}\int_{\mathcal{H}_t} e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}\eta_0^t * \lambda_0^t * \nu_0^t * \mu_{\psi_0}(d\gamma),$$

where μ_{ψ_0} is given by (2.12). By applying equation (2.30) we obtain

$$U(t,0)\psi_0(x) = C(t)\int_{\mathcal{H}_t} e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}\eta_0^t * \lambda_0^t * \mu_{\psi_0}(d\gamma) \\ -\frac{i}{\hbar}C(t)\int_0^t\int_{\mathcal{H}_t} e^{-\frac{i\hbar}{2}\langle\gamma,(I-L)^{-1}\gamma\rangle}\eta_0^t * \lambda_0^t * \mu_u * \nu_0^u * \mu_{\psi_0}(d\gamma)du,$$

where $C(t) = e^{-i\frac{t}{2\hbar}x\Omega^2 x}e^{-i\frac{x}{\hbar}\cdot\int_0^t f(s)ds}(\det(I-L))^{-1/2}$. By applying the Parseval-type equality in the other direction we get

$$U(t,0)\psi_0(x) = U_0(t,0)\psi_0(x) - \frac{i}{\hbar}e^{-i\frac{t}{2\hbar}x\Omega^2 x}e^{-i\frac{x}{\hbar}\cdot\int_0^t f(s)ds}$$
$$\int_0^t\int_{\mathcal{H}_t} e^{\frac{i}{2\hbar}\int_0^t|\dot\gamma(s)|^2ds - \frac{i}{2\hbar}\int_0^t\gamma(s)\Omega^2\gamma(s)ds}e^{-\frac{i}{\hbar}\int_0^t f(s)\gamma(s)ds}e^{-\frac{i}{\hbar}\int_0^t x\Omega^2\gamma(s)ds}$$
$$V(u,\gamma(u)+x)e^{-\frac{i}{\hbar}\int_0^u V(s,\gamma(s)+x)ds}\psi_0(\gamma(0)+x)d\gamma du. \quad (2.33)$$

Denoting by $\mathcal{H}_{r,s}$ the Cameron–Martin space of paths $\gamma : [r,s] \to \mathbb{R}^d$, we have $\mathcal{H}_t \equiv \mathcal{H}_{0,t} = \mathcal{H}_{0,u} \oplus \mathcal{H}_{u,t}$; indeed each $\gamma \in \mathcal{H}_t$ can uniquely be associated to a couple (γ_1,γ_2), with $\gamma_1 \in \mathcal{H}_{0,u}$ and $\gamma_2 \in \mathcal{H}_{u,t}$, $\gamma(s) = \gamma_2(s)$ for $s \in [u,t]$ and $\gamma(s) = \gamma_1(s) + \gamma_2(u)$ for $s \in [0,u)$. By means of these notations and by Fubini's theorem for oscillatory integrals (see [8, 19]) equation (2.33) can be written in the following form:

$$U(t,0)\psi_0(x) = U_0(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t\int_{\mathcal{H}_{u,t}} e^{\frac{i}{2\hbar}\int_u^t|\dot\gamma_2(s)|^2ds}$$
$$e^{-\frac{i}{2\hbar}\int_u^t(\gamma_2(s)+x)\Omega^2(\gamma_2(s)+x)ds}e^{-\frac{i}{\hbar}\int_u^t f(s)(\gamma_2(s)+x)ds}V(u,\gamma(u)_2+x)$$
$$\int_{\mathcal{H}_{0,u}} e^{\frac{i}{2\hbar}\int_0^u|\dot\gamma_1(s)|^2ds - \frac{i}{2\hbar}\int_0^u(\gamma_1(s)+\gamma_2(u)+x)\Omega^2(\gamma_1(s)+\gamma_2(u)+x)ds}$$
$$e^{-\frac{i}{\hbar}\int_0^u f(s)(\gamma_1(s)+\gamma_2(u)+x)ds}e^{-\frac{i}{\hbar}\int_0^u V(s,\gamma_1(s)+\gamma_2(u)+x)ds}$$
$$\psi_0(\gamma_1(0)+\gamma_2(u)+x)d\gamma_1 d\gamma_2 du$$

and by equations (2.31) and (2.32)

$$U(t,0)\psi_0(x) = U_0(t,0)\psi_0(x) - \frac{i}{\hbar}\int_0^t U_0(t,u)(V(u)U(u,0)\psi_0)(x)du.$$

Now the iterative solution to the latter integral equation is the Dyson series for $U(t,0)$, which coincides with the convergent power series expansion for the solution of the Schrödinger equation (2.1) with the time-dependent Hamiltonian (2.29).

Remark 2.3 A similar result can be obtained under the assumption that the potential is the Laplace transform of a complex bounded measure (with some restrictions on its growth at infinity), see [3] and [32] for rigorous Feynman path integrals defined for this class. In particular, in [32] by means of the white-noise approach a class of potential $V(s,x) = \int_{\mathbb{R}^d} e^{\alpha\cdot x}d\mu_s(\alpha)$ is considered, where μ denotes a family of complex measures on $\mathbb{R}^d$ labeled by the parameter $s \in [0,t]$ such that $\int_0^t\int_{\mathbb{R}^d} e^{C|\alpha|}d\mu_(\alpha,s)ds < \infty\ \forall C > 0$ and μ is of one of the following forms:

$$\mu_s(\alpha) = \sum_{j=1}^k \mu_j(\alpha)\rho_j(s)$$

with $k \in \mathbb{N}$, μ_j complex Borel measures on $\mathbb{R}^d$ and $\rho_j \in C^0(\mathbb{R},\mathbb{C})$ for all $j = 1,\dots,k$, or

$$\mu_s(\alpha) = \rho(\alpha,s)d\alpha$$

where $\rho : \mathbb{R}^d\times[0,t] \to \mathbb{C}$, with $\rho(\alpha,\cdot)$ continuous on $[0,t]$ for all $\alpha \in \mathbb{R}^d$ and $\sup_{s\in[0,t]}|\rho(\alpha,s)||e^{C\alpha}|$ in $L^1(\mathbb{R}^d,d\alpha)$ for all $C > 0$.

Remark 2.4 In [5] by means of a well-defined infinite-dimensional oscillatory integral on the Hilbert space $\mathcal{H}_t \times L_t$, i.e., the product space of the Cameron–Martin space $\mathcal{H}_t$ and $L_t = L^2([0,t])$, a rigorous mathematical meaning to the "phase space Feynman path integrals," i.e., to the Hamiltonian version of the Feynman heuristic formula, has been given. These results can been generalized using the results in the present chapter to the case where the potential depends explicitly from position x, momentum p, and time t in the following way:

$$V(x,p,t) = V_1(p,t) + V_2(x,t),$$

where $V_2(x,t)$ satisfies the assumptions 1. and 2. of page 15 and V_1 is such that the functional on the Hilbert space L_t

$$p(s)_{s\in[0,t]} \in \mathcal{H}_t \mapsto \int_0^t V_1(p(s),s)ds$$

is the Fourier transform of a complex bounded variation measure on L_t.

2.6 Future developments

In [10, 11, 12] a class of finite- and infinite-dimensional oscillatory integrals with phase functions of polynomial growth is studied. The main motivation is the rigorous realization of the Feynman path integral representation of the solution of the Schrödinger equation with a quartic potential. In [10, 11, 12] the quartic potential is time independent; the extension of these results to the case where the quartic potential is time dependent is in preparation [13].

Acknowledgments

Very stimulating discussions with Luciano Tubaro are gratefully acknowledged, as well as the hospitality of the Mathematics Institutes in Bonn and Trento. This research is part of the I.D.A. project of I.N.D.A.M, supported by P.A.T. (Trento) and M.I.U.R. (Italy).

References

[1] S. Albeverio, A.M. Boutet de Monvel-Berthier, Z. Brzeźniak. Stationary phase method in infinite dimensions by finite dimensional approximations: applications to the Schrödinger equation. **4**, 469-502, 1995.

[2] S. Albeverio and Z. Brzeźniak. Finite-dimensional approximation approach to oscillatory integrals and stationary phase in infinite dimensions. , **113**(1), 177-244, 1993.

[3] S. Albeverio, Z. Brzeźniak, Z. Haba. On the Schrödinger equation with potentials which are Laplace transform of measures. , **9**(1), 65-82, 1998.

[4] S. Albeverio, J.E. Fenstad, R. Høegh-Krohn, T. Lindstrøm. Nonstandard methods in stochastic analysis and mathematical physics. **122**, Academic Press, Inc., Orlando, FL, 1986.

[5] S. Albeverio, G. Guatteri, S. Mazzucchi. Phase space Feynman path integrals , **43**, 2847-2857, 2002.

[6] S. Albeverio, G. Guatteri, S. Mazzucchi. Representation of the Belavkin equation via Feynman path integrals. , **125**, 365-380, 2003.

[7] S. Albeverio, G. Guatteri, S. Mazzucchi. Representation of the Belavkin equation via phase space Feynman path integrals. , **7**(4), 507-526, 2004.

[8] S. Albeverio and R. Høegh-Krohn. Springer-Verlag, Berlin, 1976. Lecture Notes in Mathematics, Vol. 523.

[9] S. Albeverio and R. Høegh-Krohn. Oscillatory integrals and the method of stationary phase in infinitely many dimensions, with applications to the classical limit of quantum mechanics. , **40**(1), 59-106, 1977.

[10] S. Albeverio and S. Mazzucchi. , **129**(1), 1-23, 2005.

[11] S. Albeverio and S. Mazzucchi. Feynman path integrals for polynomially growing potentials. , **221**(1), 83-121, 2005.

[12] S. Albeverio and S. Mazzucchi. Generalized infinite-dimensional Fresnel integrals. , **338**(3), 255-259, 2004.

[13] S. Albeverio and S. Mazzucchi. Time-dependent quartic potentials — an approach by Feynman path integrals.

[14] S. Albeverio and S. Mazzucchi. Some new developments in the theory of path integrals, with applications to quantum theory. , **115**(112), 191-215, 2004.

[15] R. Azencott, H. Doss, L'équation de Schrödinger quand h tend vers zéro: une approche probabiliste. (French) [The Schrödinger equation as h tends to zero: a probabilistic approach], (Marseille, 1983), 1–17, Lecture Notes in Math., 1109, Springer, Berlin, 1985.

[16] R.H. Cameron. A family of integrals serving to connect the Wiener and Feynman integrals. , **39**, 126-140, 1960.

[17] P. Cartier and C. DeWitt-Morette. Functional integration. , **41**(6), 4154-4187, 2000.

[18] D.M. Chung. Conditional analytic feynman integrals on Wiener spaces. **112**, 479-488, 1991.

[19] D. Elworthy and A. Truman. Feynman maps, Cameron-Martin formulae and anharmonic oscillators. , **41**(2), 115–142, 1984.

[20] M. De Faria, J. Potthoff, L. Streit. The Feynman integrand as a Hida distribution. , **32**, 2123-2127, 1991.

[21] D. Fujiwara, N. Kumano-Go. Smooth functional derivation of Feynman path integrals by time slicing approximations. , **129**(1), 57-79, 2005.

[22] R. Feynman. Space-time approach to non-relativistic quantum mechanics. , **20**, 367-387, 1948.

[23] R.P. Feynman, A.R. Hibbs. Mcgraw-Hill, New York, 1965.

[24] M. Grothaus, D. Khandekar, J.L. da Silva, L. Streit. The Feynman integral for time-dependent anharmonic oscillators. , **38**(6), 3278-3299, 1997.

[25] T. Hida, H.H. Kuo, J. Potthoff, L. Streit. Kluwer, Dordrecht, 1995.

[26] L. Hörmander. Fourier integral operators I. , **127**(1), 79-183, 1971.

[27] G.W. Johnson, M.L. Lapidus, Oxford University Press, New York, 2000.

[28] G. Kallianpur, D. Kannan, R.L. Karandikar. Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron Martin formula. , **21**, 323-361, 1985.

[29] H. Kanasugi, H. Okada. Systematic treatment of general time-dependent harmonic oscillator in classical and quantum mechanics. , **16**(2), 384-388, 1975.

[30] T. Kato, H. Tanabe. On the analyticity of solution of evolution equations. , **4**, 1-4, 1967.

[31] D.C. Khandekar, S. V. Lawande. Exact propagator for a time-dependent harmonic oscillator with and without a singular perturbation. , **9**(5), 949-960, 1995.

[32] T. Kuna, L. Streit, W. Westerkamp. Feynman integrals for a class of exponentially growing potentials. , **39**(9), 4476-4491, 1998.

[33] A. Lascheck, P. Leukert, L. Streit, W. Westerkamp. Quantum mechanical propagators in terms of Hida distribution. , **33**, 221-232, 1993.

[34] E. Nelson. Feynman integrals and the Schrödinger equation. , **5**, 332-343, 1964.

[35] P. Pechukas, J.C. Light. On the exponential form of time-displacement operators in quantum mechanics. , **44**(10), 3897-3912, 1966.

[36] M. Reed, B. Simon. Academic Press, New York, 1975.

[37] J. Rezende. Quantum systems with time dependent harmonic part and the Morse index. , **25**(11), 3264-3269, 1984.

[38] J. Rezende. Time-dependent linear hamiltonian systems and quantum mechanics. , **38**, 117-127, 1996.

[39] J. Rezende. Feynman integrals and Fredholm determinants , **35**(8), 4357-4371, 1994.

[40] B. Simon. Academic Press, New York-London, 1979.

3 The Irreducibility of Transition Semigroups and Approximate Controllability

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3.1 The main result

Consider the stochastic differential equation in a Hilbert space H

$$\begin{aligned} &dX + AXdt + \gamma X dt = \sqrt{Q}\, dW \\ &X(0) = x_0. \end{aligned} \tag{3.1}$$

Here $\sqrt{Q}\, dW$ is a colored noise with covariance Q, $Q \in L(H,H)$, $Q = Q^*$, $\operatorname{Tr} Q < \infty$, defined on some probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ and with values in H. We shall assume further that

$$\sqrt{Q} \in L(U, X), \tag{3.2}$$

where U is a Hilbert space, $H \subset U$ and the injection of H into U is Hilbert–Schmidt.

As regards the nonlinear operator $A : D(A) \subset H \to H$ we shall assume that

(i) A is m-accretive in $H \times H$ and $(Ax - Ay, x - y) \geq \omega |Ax - Ay|^2_{X'}$, $\forall x, y \in D(A)$ where X is a Hilbert space such that $X \subset H \subset X'$ (the dual space of X) algebraically and topologically. Finally, $\gamma \in R$ is a given constant.

We note that A could be multivalued as well.

The transition semigroup P_t associated to (3.1) (if exists) is given by

$$P_t(\varphi) = E\varphi(X(t,\cdot)),\ t \geq 0,\ \varphi \in C_b(H). \tag{3.3}$$

P_t is said to be irreducible if

$$P(|X(T,x_0) - x_1| \geq r) < 1,\ \forall T > 0,\ x_0, x_1 \in \overline{D(A)},\ r > 0. \tag{3.4}$$

If ν is an invariant measure associated to P_t, then by (3.4) it follows that ν is full, i.e.,

$$\nu(B(x,r)) > 0,\ \forall x,\ r > 0,$$

where $B(x,r)$ is the ball of center x and radius r.

Theorem 3.1 (i) P_t
(3.1) P_t

Here $|\cdot|$ is the norm of H.

3.1.1 Examples

1. The two-phase Stefan problem (see V. Barbu & G. Da Prato [1])

$$\begin{cases} dx - \Delta\beta(x)dt = \sqrt{Q}\,dw & \text{in } \mathcal{O}\times[0,\infty) \\ \beta(x) = 0 & \text{on } \partial\mathcal{O}\times[0,\infty). \end{cases}$$

In this case $H = H^{-1}(\mathcal{O})$, $\mathcal{O}$ an open bounded subset of R^n, $A = -\Delta\beta$, $D(A) = \{u \in H^{-1}(\mathcal{O}) \cap L^1_{\text{loc}}(\mathcal{O}),\ \beta(u) \in H^1_0(\mathcal{O})\}$ where β is a monotonically increasing continuous function satisfying the condition

$$(\beta(u) - \beta(v))(u - v) \geq \omega|\beta(u) - \beta(v)|^2 \tag{3.5}$$

or, equivalently

$$|\beta(u) - \beta(v)| \leq \omega^{-1}|u - v|,\ \forall u, v \in R. \tag{3.6}$$

In the special case, $\beta(u) = \alpha_1 u$ for $u < 0$, $\beta(u) = 0$ on $[0,\rho]$, $\beta(u) = \alpha_2(u - \rho)$ for $u > \rho$; this reduces to classical two-phase Stefan problem (see V. Barbu and G. Da Prato [3]). We note that in this case assumption (i) holds with $X = H^1(\mathcal{O}) \cap H^1_0(\mathcal{O})$.

2. Diffusion with flux on the boundary
This is an equation of the form

$$\begin{aligned} & dX - \Delta X\,dt = \sqrt{Q}\,dW && \text{in } \mathcal{O}\times(0,\infty) \\ & \frac{\partial X}{\partial\nu} + \beta(X) = 0 && \text{on } \partial\mathcal{O}\times(0,\infty) \\ & X(0) = x(\xi) && \text{in } \mathcal{O} \end{aligned} \tag{3.7}$$

where β is a monotonically continuous function satisfying condition (3.6). Equation (3.7) can be written in the abstract form (3.1) where

$$\begin{aligned} & H = L^2(\mathcal{O}),\ A = -\Delta, \\ & D(A) = \left\{x \in H^2(\mathcal{O});\ \frac{\partial X}{\partial\nu} + \beta(X) = 0 \text{ a.e. on } \partial\mathcal{O}\right\}, \\ & X = H^1(\mathcal{O}),\ X' = (H^1(\mathcal{O}))'. \end{aligned}$$

Then assumption (i) holds.

3. Porous media equation

$$\begin{aligned} & dX - \Delta\psi(X)dt = \sqrt{Q}\,dw && \text{in } \mathcal{O}\times[0,\infty), \\ & \psi(X) = 0 && \text{on } \partial\mathcal{O}\times[0,\infty), \\ & X(0) = x_0 && \text{in } \mathcal{O}. \end{aligned} \tag{3.8}$$

Here ψ is a monotonically increasing continuous function with polynomial growth and in particular of the form

$$\psi(r) = r|r|^{2m-1}.$$

(For a treatment of equation (3.8) we refer to G. Da Prato and M. Röckner [6], V. Barbu, V. Bogachev, G. Da Prato, and M. Röckner [4]). In particular in the latter paper, it is established via controllability arguments a irreducibility result for the associated invariant measure for $d < 2(r+1)(r-1)^{-1}$.

Equation (3.8) is of the form (3.1) where $H = H^{-1}(\mathcal{O})$, $A - \Delta\psi$, $D(A) = \{u \in H^{-1}(\mathcal{O}), \psi(u) \in H_0^1(\mathcal{O})\}$. If ψ is Lipschitzian on $\{r;\ |r| \geq R\}$, then condition (i) is satisfied.

For other irreducibility results we refer to S. Cerrai [5] (the case of semilinear parabolic equations) and to the book [7] by G. Da Prato and J. Zabczyk. In the work [2] the irreducibility of the transition semigroup associated with the reaction diffusion equation with reflexion is proved.

3.2 Approximate controllability

It is well known (see [7]) that the irreducibility is closely related to the approximate controllability of the deterministic equation associated with (3.1). Consider the equation

$$\begin{aligned} &y' + Ay + \gamma y = Bu,\ r > 0 \\ &y(0) = x_0 \end{aligned} \tag{3.9}$$

where $B \in L(U, H)$, A is m-accretive and

$$\ker\{B^*\} = \{0\}. \tag{3.10}$$

Proposition 3.1 (3.9) $x_0 \in \overline{D(A)}$, $x_1 \in \overline{D(A)}$, $\varepsilon > 0$, $T > 0$, $\exists\, u \in C([0, T]; U)$

$$|y(T) - x_1| \leq \varepsilon. \tag{3.11}$$

Proof No. 1 First, one shows that for each $x_0 \in \overline{D(A)}$ and all $x_1 \in D(A)$, $\exists\, v \in L^2(0, T; H)$ such that

$$y(T) = x_1.$$

Consider the feedback system

$$\begin{aligned} &z' + Az + \gamma z = v,\ t > 0 \\ &z(0) = x_0 \\ &v(t) = -\rho \operatorname{sgn}(z(t) - x_1). \end{aligned} \tag{3.12}$$

$$\operatorname{sgn} z = \frac{z}{|z|},\ \operatorname{sgn} 0 = B(0, 1) = \{x;\ |x| \leq 1\}. \tag{3.13}$$

We get

$$\frac{1}{2}\ \frac{d}{dt}\ |z(t) - x_1|^2 + \rho|z(t) - x_1| \leq \gamma|z(t) - x_1|^2.$$

This yields

$$|z(t) - x_1| = 0 \quad \text{for } t \geq T$$

if

$$\rho \geq \gamma|x_0 - x_1|e^{-\gamma T} - |Ax_1|(e^{-\gamma T} - 1).$$

Next, by assumption (3.10) it follows that

$$\{Bu;\ u \in \mathbb{C}(0, T; U)\} \text{ is dense in } L^2(0, T; H)$$

which implies via standard device (3.11).

Proof of irreducibility We have

$$\begin{aligned} &X(t,x_0)-y(t)+\int_0^t (AX(s,x_0)-Ay(s))ds \\ &\qquad +\gamma\int_0^t (X(s,x_0)-y(s))ds=\sqrt{Q}(W(t)-V(t)) \\ &V(t)=\int_0^t u(s)ds. \end{aligned} \tag{3.14}$$

This yields ($B=\sqrt{Q}$)

$$\begin{aligned} &X(t,x_0)-y(t)+\int_0^t e^{-\gamma(t-s)}(AX(s,x_0)-Ay(s))ds \\ &=BW(t)-BV(t)-\gamma\int_0^t e^{-\gamma(t-s)}(BW(s)-BV(s))ds. \end{aligned} \tag{3.15}$$

Multiplying (3.15) by $AX-Ay$ and integrating on $(0,t)$, yields

$$\begin{aligned} &\int_0^t (X(s)-y(s),AX(s)-Ay(s))ds+\frac{1}{2}\ e^{-\gamma t}|Z(t)|^2 \\ &\qquad \le \int_0^t (BW-BV,AX-Ay)ds \\ &+\gamma\int_0^t \left(AX(s)-Ay(s),\int_0^s e^{-\gamma(t-s)}(BW(\tau)-BV(\tau))d\tau\right) \end{aligned} \tag{3.16}$$

where

$$Z(t)=\int_0^t e^{\gamma s}(AX(s)-Ay(s))ds,\ \ t\ge 0.$$

This yields

$$\begin{aligned} &\int_0^T (X(s)-y(s),AX(s)-Ay(s))ds+|z(T)|^2 \\ &\le \left(|BW-BV|_{L^2(0,T;X)}\left(\int_0^T |AX-Ay|^2_{X'}dt\right)^{1/2}\right). \end{aligned}$$

Finally, by assumption (i)

$$\int_0^T (X(s)-y(s),AX(s)-Ay(s))ds+|z(T)|^2\le c|BW-BV|^2_{L^2(0,T;X)}.$$

Recalling that in virtue of (2.7)

$$\begin{aligned} X(T,x_0)-y(T)&=B(W(T)-V(T))-e^{-\gamma T}z(T) \\ &-\gamma\int_0^T e^{-\gamma(T-s)}(BW(s)-B(V(s)) \end{aligned}$$

we get

$$\begin{aligned} &|X(T,x_0)-y(T)| \\ &\le c(|BW(T)-V(T)|+|BW-BV|_{L^2(0,T;X)}) \\ &\le c(|BW(T)-V(T)|+|W-V|_{L^2(0,T;U)}). \end{aligned} \tag{3.17}$$

By approximate controllability $\exists\, v \in C([0,T];U)$ such that $|y(T)-x_1| \le \varepsilon$. This yields

$$|X(T,x_0)-x_1| \le \varepsilon + c(|B(W(T)-V(T)| - |W-V|_{L^2(0,T;U)}). \tag{3.18}$$

Hence

$$\begin{gathered} P[|X(T,x_0)-x_1| \ge r] \\ \le P\left[|BW(T)-BV(T)| + |W-V|_{L^2(0,T;X)} \ge \frac{r-\varepsilon}{c}\right] < 1 \end{gathered}$$

because $\{BW-BV, W-V\}$ is a nondegenerate Gaussian random variable in $H \times L^2(0,T;U)$. This completes the proof.

3.3 Fullness for invariant measures

Consider the case of an invariant measure ν for the Kolmogorov operator N_0 associated with equation (3.1) given by

$$\nu = w^* - \lim_{n\to\infty} \nu_n$$

where ν_n is the invariant measure associated with the finite-dimensional approximation

$$\begin{aligned} &dX_n + A_n X_n dt + \gamma X_n dt = P_n\sqrt{Q}dw \\ &X_n(0) = P_n x_0 = x_0^n. \\ &P_n x = \sum_{i=1}^n (x,e_i)e_i,\ \{e_i\} \text{ is an orthonormal basis} \\ &\qquad\qquad \text{of eigenfunctions of } A_0 : H \to H \\ &A_n = P_n A. \end{aligned} \tag{3.19}$$

In this case we need the following asymptotic controllability lemma.

Lemma 3.1 $\varepsilon > 0$ $x_0, x_1 \in \overline{D(A)}$, $\exists\, u_\varepsilon^n \in L^2(0,T;U)$

$$|y_n(T) - P_n x_1| \le \varepsilon \tag{3.20}$$

$$u_\varepsilon^n \overset{n\to\infty}{\longrightarrow} u_\varepsilon \quad L^2(0,T;U). \tag{3.21}$$

Proof For each n consider the equation

$$\begin{aligned} &z_n' + A_n z_n + \gamma z_n + \rho\,\mathrm{sgn}(z_0 - P_n x_1) \ni 0 \\ &z_n(0) = P_n x_0. \end{aligned}$$

We set

$$v_n(t) = -\rho\,\mathrm{sgn}(z_n(t) - P_n x_1)$$

and note that

$$v_n(t) \overset{n\to\infty}{\longrightarrow} v(t) \ \text{ strongly in } L^2(0,T;H).$$

Next, define $B_n = P_n\sqrt{Q}$

$$u_\varepsilon^n = \arg\min\{|B_n u - v_n|^2_{L^2(0,T;H)} + \varepsilon |u|^2_{L^2(0,T;U)}\}.$$

It turns out that $\{u_\varepsilon^n\}$ satisfies the conditions of Lemma 3.1. (Complete proof can be found in [2], [3].)

We obtain as above

$$P[|X_n(T,x_0^n) - P_n x_1) \geq r] \leq P[|W(T) - V_n(T)| + |W - V_n|_{L^2(0,T;U)} \\ \geq \frac{1}{c}(r-\varepsilon)] \leq \delta_n < 1,$$

independent of n. This implies that

$$P_T^n(\chi_{B(P_n x_1, r)}(x_0^n)) \geq 1 - \rho.$$

Hence if ν_n is the invariant measure associated with P_t^n, we have

$$\nu_n(B(P_n x_1, r)) = \int_{H_n} P_T^n \chi_{B(B_n, x_1, r)}(P_n x_0)\nu_n(dx_0) > \eta > 0$$

and since $\nu_n \to \nu$ weak star, we infer that $\nu(B(x_1,r)) > 0$ as claimed.

References

[1] V. Barbu, G. Da Prato, The two phase stochastic Stefan problem, , 124 (2002), 544-560.

[2] V. Barbu, G. Da Prato, Irreducibility of the transition semigroup associated with the stochastic obstacle problem (to appear).

[3] V. Barbu, G. Da Prato, Irreducibility of the transition semigroup associated with the two phase Stefan problem (to appear).

[4] V. Barbu, V. Bogachev, G. Da Prato, M. Röckner, L^2-solutions to the Kolmogorov equations associated to stochastic porous media equations (to appear).

[5] S. Cerrai, Second order PDE's in finite and infinite dimension. A probabilistic approach, Lecture Notes in Math, 1762, Springer-Verlag, Berlin, 2001.

[6] G. Da Prato, M. Röckner, Weak solutions to porous media equation (to appear).

[7] G. Da Prato, J. Zabczyk, , London Math. Soc. Lecture Notes, 229, Cambridge University Press, Cambridge, 1996.

4 Gradient Bounds for Solutions of Elliptic and Parabolic Equations

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Suppose that for every $t \in [0,1]$, we are given a strictly positive definite symmetric matrix $A(t) = (a^{ij}(t))$ and a measurable vector field $x \mapsto b(t,x) = (b^1(t,x), \ldots, b^n(t,x))$.

Let L_t be the elliptic operator on $\mathbb{R}^d$ given by

$$L_t u(x) = \sum_{i,j \le d} a^{ij}(t,x)\partial_{x_i}\partial_{x_j} u(x) + \sum_{i \le d} b^i(t,x)\partial_{x_i} u(x). \tag{4.1}$$

Suppose that A and b satisfy the following hypotheses:

(Ha) $\sup_{t\in[0,1]}\big(\|A(t)\| + \|A(t)^{-1}\|\big) < \infty$, $\sup_{t\in[0,1]} \|b(t,\,\cdot\,)\|_{L^p(U)} < \infty$ for every ball U in $\mathbb{R}^d$ with some $p > d$, $p \ge 2$.

(Hb) b is [illegible] in the following sense: for every $t \in [0,1]$ and every $h \in \mathbb{R}^d$, there exists a measure zero set $N_{t,h} \subset \mathbb{R}^d$ such that

$$\big(b(t,x+h) - b(t,x), h\big) \le 0 \quad \text{for all } x \in \mathbb{R}^d \setminus N_{t,h}.$$

(Hc) For every $t \in [0,1]$, there exists a [illegible] V_t for L_t, i.e., a nonnegative C^2-function V_t such that $V_t(x) \to +\infty$ and $L_t V_t(x) \to -\infty$ as $|x| \to \infty$.

We consider the parabolic equation

$$\frac{\partial u}{\partial t} = L_t u, \quad u(0,x) = f(x), \tag{4.2}$$

where f is a bounded Lipschitz function. A locally integrable function u on $[0,1] \times \mathbb{R}^d$ is called a solution if, for every $t \in (0,1]$, one has $u(t,\,\cdot\,) \in W^{1,2}_{loc}(\mathbb{R}^d)$, the functions $\partial_{x_i}\partial_{x_j} u$ and $b^i \partial_{x_i} u$ are integrable on the sets $[0,1] \times K$ for every cube K in $\mathbb{R}^d$, and for every $\varphi \in C_0^\infty(\mathbb{R}^d)$ and all $t \in [0,1]$ one has

$$\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} f(x)\varphi(x)\,dx + \int_0^t \int_{\mathbb{R}^d} L_s\varphi(x)\,u(s,x)\,dxds.$$

In the case where A and b are independent of t, so that we have a single operator L, Hypotheses (Ha) and (Hc) imply (see [6] and [8]) that there exists a unique probability measure μ on $\mathbb{R}^d$ such that μ has a strictly positive continuous weakly differentiable density ϱ, $|\nabla\varrho| \in L^p_{loc}(\mathbb{R}^d)$, and $L^*\mu = 0$ in the following weak sense:

$$\int Lu\,d\mu = 0 \quad \text{for all } u \in C_0^\infty(\mathbb{R}^d).$$

The closure $\overline{L}$ of L with domain $C_0^\infty(\mathbb{R}^d)$ in $L^1(\mu)$ generates a Markov semigroup $\{T_t\}_{t\geq 0}$ for which μ is invariant. Let $D(\overline{L})$ denote the domain of $\overline{L}$ in $L^1(\mu)$ and let $\{G_\lambda\}_{\lambda>0}$ denote the corresponding resolvent, i.e., $G_\lambda = (\lambda - \overline{L})^{-1}$. The restrictions of T_t and G_λ to $L^2(\mu)$ are contractions on $L^2(\mu)$. In particular, if $v \in D(\overline{L})$ is such that $\lambda v - \overline{L}v = g \in L^2(\mu)$, then $v \in L^2(\mu)$. Moreover, it follows by [8, Theorem 2.8] that one has $v \in H^{2,2}_{loc}(\mathbb{R}^d)$ and $\overline{L}v = Lv$ almost everywhere, so that one has a.e.

$$\lambda v - Lv = g. \tag{4.3}$$

In fact, due to our assumptions on the coefficients of L one has even $v \in W^{p,2}_{loc}(\mathbb{R}^d)$ (see [10]). It has been shown in [3] that for every function $f \in L^1(\mu)$ that is Lipschitzian with constant C and all $t, \lambda > 0$, the continuous version of the function $T_t f$ is Lipschitzian with constant C, and the continuous version of $G_\lambda f$ is Lipschitzian with constant $\lambda^{-1}C$. Here we establish pointwise estimates in both cases and prove their parabolic analogue. The main results of this work are the following two theorems.

Theorem 4.1. A b t (Ha), (Hb), (Hc) $f \in L^1(\mu)$ $t, \lambda > 0$, $T_t f$ $G_\lambda f$

$$|\nabla T_t f(x)| \leq T_t|\nabla f|(x) \qquad |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda} G_\lambda |\nabla f|(x) \tag{4.4}$$

$$\sup_{x,t} |\nabla T_t f(x)| \leq \sup_x |\nabla f(x)|, \quad \sup_x |\nabla G_\lambda f(x)| \leq \frac{1}{\lambda}\sup_x |\nabla f(x)|. \tag{4.5}$$

Theorem 4.2. A b (Ha), (Hb), (Hc) f u (4.2) t

$$\sup_x |\nabla u(t,x)| \leq \sup_x |\nabla f(x)|. \tag{4.6}$$

In the case where $A = I$ and $b = 0$, estimate (4.6) has been established in [12], [13] for solutions of boundary problems in bounded domains. It should be noted that gradient estimates of the type

$$\sup_x |\nabla u(x,t)| \leq C(t) \sup_x |f(x)|$$

for solutions of parabolic equations have been obtained by many authors, see, e.g., [1], [2], [11], [15], and the references therein. Such estimates do not require (Hb) and one has $C(t) \to +\infty$ as $t \to 0$ or $t \to +\infty$. In contrast to this type of estimates, our theorems mean a contraction property on Lipschitz functions rather than a smoothing property. It is likely that some results of the cited works, established for sufficiently regular b, can be extended to more general drifts satisfying just (Ha), but not (Hb).

A short proof of the following result can be found in [3].

Lemma 4.1. b ff $\alpha > 0$,

$$\big(b(x+h) - b(x), h\big) \leq -\alpha(h,h) \qquad x, h \in \mathbb{R}^d$$

$\lambda > 0$ f,

$$|\nabla G_\lambda f| \leq G_\lambda |\nabla f|.$$

$\sup_x |\nabla G_\lambda f(x)| \leq \lambda^{-1} \sup_x |\nabla f(x)|$

Proof of Theorem 4.1. The estimate with the suprema has been proved in [3], and the stronger pointwise estimate can be derived from that proof. For the reader's convenience, instead of recursions to the steps of the proof in [3] we reproduce the whole proof and explain why it yields a stronger conclusion. We recall that if a sequence of functions on $\mathbb{R}^d$ is uniformly Lipschitzian with constant L and bounded at a point, then it contains a subsequence that converges uniformly on every ball to a function that is Lipschitzian with the same constant. Therefore, approximating f in $L^1(\mu)$ by a sequence of bounded smooth functions f_j with

$$\sup_x |\nabla f_j(x)| \le \sup_x |\nabla f(x)|,$$

it suffices to prove (4.5) for smooth bounded f. Moreover, due to Euler's formula $T_t f = \lim_n \left(\frac{t}{n} G_{\frac{t}{n}}\right)^n f$, it suffices to establish the resolvent estimate. First, we construct a suitable sequence of smooth strongly dissipative Lipschitzian vector fields b_k such that $b_k \to b$ in $L^p(U, \mathbb{R}^d)$ for every ball U as $k \to \infty$. Let $\sigma_j(x) = j^{-d}\sigma(x/j)$, where σ is a smooth compactly supported probability density. Let $\beta_j := b * \sigma_j$. Then β_j is smooth and dissipative and $\beta_j \to b$, $j \to \infty$, in $L^p(U, \mathbb{R}^d)$ for every ball U. For every $\alpha > 0$, the mapping $I - \alpha\beta_j$ is a homeomorphism of $\mathbb{R}^d$ and the inverse mapping $(I - \alpha\beta_j)^{-1}$ is Lipschitzian with constant α^{-1} (see [9]). Let us consider the Yosida approximations

$$F_\alpha(\beta_j) := \alpha^{-1}\bigl((I - \alpha\beta_j)^{-1} - I\bigr) = \beta_j \circ (I - \alpha\beta_j)^{-1}.$$

It is known (see [9, Ch. II]) that $|F_\alpha(\beta_j)(x)| \le |\beta_j(x)|$, the mappings $F_\alpha(\beta_j)$ converge locally uniformly to β_j as $\alpha \to 0$, and one has

$$\bigl(F_\alpha(\beta_j)(x) - F_\alpha(\beta_j)(y), x - y\bigr) \le 0.$$

Thus, the sequence $b_k := F_{\frac{1}{k}}(b * \sigma_k) - \frac{1}{k} I$, $k \in \mathbb{N}$, is the desired one. For every $k \in \mathbb{N}$, let L_k be the elliptic operator defined by (4.1) with the same constant matrix A and drift b_k in place of b. Let $\mu_k = \varrho_k\, dx$ be the corresponding invariant probability measure and let $G_\lambda^{(k)}$ denote the associated resolvent family on $L^1(\mu_k)$. Since b_k is smooth, Lipschitzian and strongly dissipative, $v_k := G_\lambda^{(k)} f$ is smooth, bounded, Lipschitzian, and

$$\sup_x |v_k(x)| \le \frac{1}{\lambda} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v_k(x)| \le \frac{1}{\lambda} \sup_x |\nabla f(x)|$$

by the lemma. Moreover, for every ball $U \subset \mathbb{R}^d$, the functions v_k are uniformly bounded in the Sobolev space $W^{2,2}(U)$, since the mappings $|b_k|$ are bounded in $L^p(U)$ uniformly in k and f is bounded. This follows from the fact that for any solution $w \in W^{2,2}(U)$ of the equation $\sum\limits_{i,j \le d} a^{ij}\partial_{x_i}\partial_{x_i} w + \sum\limits_{i \le d} b^i \partial_{x_i}\partial_{x_i} w - \lambda w = g$ one has $\|w\|_{W^{2,2}(U)} \le C\|w\|_{L^2(U)}$, where C is a constant that depends on U, A, and the quantity $\kappa := \|g\|_{L^2(U)} + \||b|\|_{L^p(U)}$ in such a way that as a function of κ it is locally bounded. Thus, the sequence $\{v_k\}$ contains a subsequence, again denoted by $\{v_k\}$, that converges locally uniformly to a bounded Lipschitzian function $v \in W^{2,2}_{loc}(\mathbb{R}^d)$ such that

$$\sup_x |v(x)| \le \lambda^{-1} \sup_x |f(x)| \quad \text{and} \quad \sup_x |\nabla v(x)| \le \lambda^{-1} \sup_x |\nabla f(x)|,$$

and, in addition, the restrictions of v_k to any ball U converge to $v|_U$ weakly in $W^{2,2}(U)$.

Let $\widehat{L}$ be the elliptic operator with the same second order part as L, but with drift is $\widehat{b} = 2A\nabla\varrho/\varrho - b$. Then by the integration by parts formula

$$\int \psi L\varphi \, d\mu = \int \varphi \widehat{L}\psi \, d\mu \quad \text{for all } \psi, \varphi \in C_0^\infty(\mathbb{R}^d).$$

In addition, for any $\lambda > 0$, the ranges of $\lambda - L$ and $\lambda - \widehat{L}$ on $C_0^\infty(\mathbb{R}^d)$ are dense in $L^1(\mu)$. The operator $\widehat{L}$ also generates a Markov semigroup on $L^1(\mu)$ with respect to which μ is invariant. The corresponding resolvent is denoted by $\widehat{G}_\lambda$. For the proofs we refer to [7, Proposition 2.9] or [14, Proposition 1.10(b)] (see also [8, Theorem 3.1]).

Now we show that $v = G_\lambda f$. Note that $\varrho_k \to \varrho$ uniformly on balls according to [5],[6]. Hence, given $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in a ball U, we have

$$\int [\lambda v - Lv - f]\varphi\varrho\, dx = \lim_{k\to\infty} \int [\lambda v_k - L_k v_k - f]\varphi\varrho_k\, dx = 0$$

by weak convergence of v_k to v in $W^{2,2}(U)$ combined with convergence of b_k to b in $L^p(U, \mathbb{R}^d)$. Therefore, by the integration by parts formula

$$\int v(\lambda\varphi - \widehat{L}\varphi)\, d\mu = \int f\varphi\, d\mu$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$. The function $G_\lambda f$ is bounded and satisfies the same relation, so it remains to recall that if a bounded function u satisfies $\int u(\lambda\varphi - \widehat{L}\varphi)\, d\mu = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, then $u = 0$ a.e., since $(\lambda - \widehat{L})\big(C_0^\infty(\mathbb{R}^d)\big)$ is dense in $L^1(\mu)$.

Now we turn to the pointwise estimate $|\nabla G_\lambda f(x)| \le \lambda^{-1} G_\lambda |\nabla f|(x)$. Suppose first that $f \in C_0^\infty(\mathbb{R}^d)$. The desired estimate holds for every $G_\lambda^{(k)}$ in place of G_λ. It has been shown above that $v = G_\lambda f$ is a weak limit of $v_k = G_\lambda^{(k)} f$ in $W^{2,2}(U)$ for every ball U. In addition, the functions $G_\lambda^{(k)}|\nabla f|$ converge weakly in $W^{2,2}(U)$ to the function $G_\lambda|\nabla f|$, which is also clear by the above reasoning. Since the embedding of $W^{2,2}(U)$ into $W^{2,1}(U)$ is compact, we may assume, passing to a subsequence, that $\nabla G_\lambda^{(k)} f(x) \to \nabla G_\lambda f(x)$ and $G_\lambda^{(k)}|\nabla f|(x) \to G_\lambda|\nabla f|(x)$ almost everywhere on U. Hence we arrive at the desired estimate. If f is Lipschitzian and has bounded support, we can find uniformly Lipschitzian functions $f_n \in C_0^\infty(\mathbb{R}^d)$ vanishing outside some ball such that $f_n \to f$ uniformly and $\nabla f_n \to \nabla f$ a.e. Then, by the same reasons as above, one has $G_\lambda|\nabla f_n| \to G_\lambda|\nabla f|$ and $\nabla G_\lambda f_n \to \nabla G_\lambda f$ in $L^2(U)$. Passing to an almost everywhere convergent subsequence we obtain a pointwise inequality. Finally, in the case of a general Lipschitzian function $f \in L^1(\mu)$, we can find uniformly Lipschitzian functions ζ_n such that $0 \le \zeta_n \le 1$ and $\zeta_n(x) = 1$ if $|x| \le n$. Let $f_n = f\zeta_n$. By the previous step we have

$$|\nabla G_\lambda f_n(x)| \le \lambda^{-1} G_\lambda |\nabla f_n|(x).$$

The functions f_n are uniformly Lipschitzian. Hence, for every ball U, the sequence of functions $G_\lambda f_n|_U$ is bounded in the norm of $W^{2,2}(U)$. In addition, the functions $G_\lambda|\nabla f_n|$ on U converge to $G_\lambda|\nabla f|$ in $L^2(U)$, since $|\nabla f_n| \to |\nabla f|$ in $L^2(\mu)$ by the Lebesgue dominated convergence theorem. Therefore, the same reasoning as above completes the proof. □

Proof of Theorem 4.2. Suppose first that A is piecewise constant, i.e., there exist finitely many intervals $[0, t_1)$, $[t_1, t_2)$,..., $[t_n, 1]$ such that $A(t) = A_k$ whenever $t_{k-1} \le t < t_k$, where each A_k is a strictly positive symmetric matrix. In addition, let us assume that there exist vector fields b_k such that $b(t, x) = b_k(x)$ whenever $t_{k-1} \le t < t_k$. Then we obtain a solution u by successively applying the semigroups $T_t^{(k)}$ generated by the elliptic operators with the diffusion matrices A_k and drifts b_k, i.e.,

$$u(t, x) = T_{t-t_{k-1}} T_{t_{k-1}} \cdots T_{t_1} f(x) \quad \text{whenever } t \in [t_{k-1}, t_k).$$

The conclusion of Theorem 4.2 in this case follows by Theorem 4.1. Our next step is to approximate A and b by mappings of the above form in such a way that the corresponding sequence of solutions would converge to a solution of our equation. Let us observe that, for an arbitrary sequence of such solutions u_k corresponding to piecewise constant in time coefficients, for every compactly supported function φ on $\mathbb{R}^d$, the functions

$$t \mapsto \int_{\mathbb{R}^d} \varphi(x) u_k(t,x)\, dx \tag{4.7}$$

are uniformly Lipschitzian provided that the operator norms of the matrix functions A_k are uniformly bounded and that the $L^p(K)$-norms of the vector fields $b_k(t,\,\cdot\,)$ are uniformly bounded for every fixed cube K in $\mathbb{R}^d$. This is clear, because (4.2) can be written as

$$\int_{\mathbb{R}^d} \varphi(x) u(t,x)\, dx = \int_0^t \int_{\mathbb{R}^d} [L_s\varphi(x)\, u(s,x) + \varphi(x) b^i(s,x)\partial_{x_i} u(s,x)]\, dx\, ds,$$

where in the case $u = u_k$ we have

$$|u(s,x)| \le \sup |f(x)| \quad \text{and} \quad |\nabla_x u(s,x)| \le \sup |\nabla f(x)|.$$

One can choose a subsequence in $\{u_k\}$ that converges to some function u on $[0,1]\times\mathbb{R}^d$ in the following sense: for every cube K in $\mathbb{R}^d$, the restrictions of the functions u_k to $[0,1]\times K$ converge weakly to u in the space $L^2([0,1], W^{2,2}(K))$, where each u_k is regarded as a mapping $t \mapsto u_k(t,\,\cdot\,)$ from $[0,1]$ to $W^{2,2}(K)$. Passing to another subsequence we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x) u_k(t,x)\, dx = \int_{\mathbb{R}^d} \varphi(x) u(t,x)\, dx$$

for all $t \in [0,1]$ and all smooth compactly supported φ. Indeed, for a given function φ this is possible due to the uniform Lipschitzness of the functions (4.7). Then our claim is true for a countable family of functions φ, which, on account of the uniform boundedness of u_k, yields the claim for all φ. Therefore, it remains to find approximations A_k and b_k such that, for every function $\psi \in C[0,1]$, the integrals

$$\int_0^1 \psi(s) \int_{\mathbb{R}^d} [L_s^{(k)}\varphi(x)\, u_k(s,x) + \varphi(x) b_k^i(s,x)\partial_{x_i} u_k(s,x)]\, dx\, ds$$

would converge to the corresponding integral with A, b, and u. Clearly, it suffices to obtain the desired convergence for suitable countable families of functions φ_i and ψ_j. Let us fix two sequences $\{\psi_j\} \subset C[0,1]$ and $\{\varphi_i\} \subset C_0^\infty(\mathbb{R}^d)$ with the following property: every compactly supported square-integrable function v on $[0,1]\times\mathbb{R}^d$ can be approximated in L^2 by a sequence of finite linear combinations of products $\psi_j\varphi_i$. Let us consider the functions

$$\alpha_{i,j,k}(t) := a^{ij}(t)\psi_k(t), \quad \beta_{i,j,k}(t) := \psi_k(t) \int_{\mathbb{R}^d} b^i(s,x)\varphi_j(x)\, dx,$$

$$\theta_{k,i}(t) = \int_{[-k,k]^d} b_i(t,x)^2\, dx.$$

Let $\mathcal{F}$ denote the obtained countable family of functions extended periodically from $[0,1)$ to $\mathbb{R}$ with period 1. It is well known that, for almost every $s \in [0,1)$, the Riemannian sums $R_n(\theta)(s) = 2^{-n} \sum\limits_{k=1}^{2^n} \theta(s + k2^{-n})$ converge to the integral of θ over $[0,1]$ for each $\theta \in \mathcal{F}$. It follows that one can find points $t_{n,l}$, $l = 1, \ldots, N_n$, $n \in \mathbb{N}$, such that

$$0 = t_{n,0} < t_{n,1} < t_{n,2} < \cdots < t_{n,N_n} = 1$$

and, for every $\theta \in \mathcal{F}$, letting $\theta_n(t) := \theta(t_{n,l})$ whenever $t_{n,l-1} \le t < t_{n,l}$, one has

$$\int_0^1 \theta_n(t)\, dt \to \int_0^1 \theta(t)\, dt.$$

To this end, we pick a common point s_0 of convergence of the Riemann sums $R_n(\theta)(s_0)$ to the respective integrals and let $t_{n,l} = s_0 + l2^{-n}$ (mod1). By using the points $t_{n,l}$, one obtains the desired piecewise constant approximations of A and b. Namely, let $A_n(t) = A(t_{n,l})$ and $b_n(t,x) = b(t_{n,l}, x)$ whenever $t_{n,l-1} \le t < t_{n,l}$. As explained above, passing to a subsequence, we may assume that the corresponding solutions u_n converge to a function u such that, for every cube $K = [-m,m]^d$ in $\mathbb{R}^d$ and every $t \in (0,1]$, one has

$$u(t,\,\cdot\,)|_K \in W^{2,2}(K), \qquad \int_0^1 \|u(t,\,\cdot\,)\|^2_{W^{2,2}(K)}\,dt < \infty,$$

and for any function $\zeta \in L^2([0,1]\times K)$ there holds the equalities

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 \int_K \zeta(t,x) u_n(t,x)\,dx\,dt &= \int_0^1 \int_K \zeta(t,x) u(t,x)\,dx\,dt,\\
\lim_{n\to\infty} \int_0^1 \int_K \zeta(t,x) \partial_{x_i}\partial_{x_j} u_n(t,x)\,dx\,dt &= \int_0^1 \int_K \zeta(t,x) \partial_{x_i}\partial_{x_j} u(t,x)\,dx\,dt,\\
\lim_{n\to\infty} \int_0^1 \int_K \zeta(t,x) \partial_{x_i} u_n(t,x)\,dx\,dt &= \int_0^1 \int_K \zeta(t,x) \partial_{x_i} u(t,x)\,dx\,dt,\\
\lim_{n\to\infty} \int_0^1 \int_K b_i^n(t,x)^2\,dx\,dt &= \int_0^1 \int_K b_i(t,x)^2\,dx\,dt.
\end{aligned}$$

Note that for every cube K in $\mathbb{R}^d$, the restrictions of the functions b_n^i to $[0,1]\times K$ converge to the restriction of b^i in the norm of $L^2([0,1]\times K)$. This is clear from the last displayed equality, which gives convergence of L^2-norms, along with convergence of the Riemann sums $R_n(\beta_{i,j,k})(s_0)$ to the integral of $\beta_{i,j,k}$ over $[0,1]$, which yields weak convergence (we recall that if a sequence of vectors h_n in a Hilbert space H converges weakly to a vector h and the norms of h_n converge to the norm of h, then there is norm convergence). It follows that for any $\psi \in C[0,1]$ and any $\varphi \in C_0^\infty(\mathbb{R}^d)$ with support in $[-m,m]^d$, we have

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 \psi(t) a_n^{ij}(t) \int_{\mathbb{R}^d} \partial_{x_i}\partial_{x_j}\varphi(x) u_n(t,x)\,dx\,dt&\\
= \int_0^1 \psi(t) a^{ij}(t) \int_{\mathbb{R}^d} \partial_{x_i}\partial_{x_j}\varphi(x) u(t,x)\,dx\,dt.&
\end{aligned}$$

In addition,

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 \psi(t) \int_{\mathbb{R}^d} \varphi(x) \partial_{x_i} u_n(t,x) b_n^i(t,x)\,dx\,dt&\\
= \int_0^1 \psi(t) \int_{\mathbb{R}^d} \varphi(x) \partial_{x_i} u(t,x) b^i(t,x)\,dx\,dt.&
\end{aligned}$$

This follows by norm convergence of b_n^i to b^i and weak convergence of $\varphi\partial_{x_i} u_n$ to $\varphi\partial_{x_i} u$ in $L^2([0,1]\times[-m,m]^d)$. Therefore, for every $\varphi \in C_0^\infty(\mathbb{R}^d)$, one has

$$\int_{\mathbb{R}^d} \varphi(x) u(t,x)\,dx\,dt = \int_{\mathbb{R}^d} \varphi(x) f(x)\,dx + \int_0^t \int_{\mathbb{R}^d} \varphi(x) L_t u(t,x)\,dx\,dt$$

for almost all $t \in [0,1]$, since the integrals of both sides multiplied by any function $\psi \in C_0^\infty(0,1)$ coincide. Taking into account the continuity of both sides (the left-hand side is Lipschitzian as explained above), we conclude that the equality holds for all $t \in [0,1]$. $\square$

Acknowledgments
This work has been supported in part by the RFBR project 04-01-00748, the DFG Grant 436 RUS 113/343/0(R), the INTAS project 03-51-5018, the Scientific Schools Grant 1758.2003.1, the DFG–Forschergruppe "Spectral Analysis, Asymptotic Distributions, and Stochastic Dynamics," the BiBoS–reseach centre, and the research programme "Analisi e controllo di equazioni di evoluzione deterministiche e stocastiche" from the Italian "Ministero della Ricerca Scientifica e Tecnologica."

References

[1] M. Bertoldi, S. Fornaro, Gradient estimates in parabolic problems with unbounded coefficients. Preprint del Dipartimento di Matematica, Università di Parma, no. 316, 2003.

[2] M. Bertoldi, S. Fornaro, L. Lorenzi, Gradient estimates for parabolic problems with unbounded coefficients in nonconvex domains. Tübinger Berichte zur Funkt. Analysis. H. 13. 2003–2004. S. 14–45.

[3] V.I. Bogachev, G. Da Prato, M. Röckner, Z. Sobol, Global gradient bounds for dissipative diffusion operators. C.R. Acad. Sci. Paris, Sér. I, 339 (2004), 277–282.

[4] V.I. Bogachev, N.V. Krylov, M. Röckner, Regularity of invariant measures: the case of non-constant diffusion part. J. Funct. Anal. 138 (1996), no. 1, 223–242.

[5] V.I. Bogachev, N.V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. Comm. Partial Diff. Equations 26 (2001), no. 11–12, 2037–2080.

[6] V.I. Bogachev, M. Röckner, A generalization of Hasminskii's theorem on existence of invariant measures for locally integrable drifts. Theory Probab. Appl. 45 (2000), no. 3, 417–436.

[7] V.I. Bogachev, M. Röckner, W. Stannat, Uniqueness of invariant measures and maximal dissipativity of diffusion operators on L^1. In: Infinite dimensional stochastic analysis, pp. 39–54, P. Clement et al. eds. Royal Netherlands Academy of Arts and Sciences, Amsterdam, 2000.

[8] V.I. Bogachev, M. Röckner, W. Stannat, Uniqueness of solutions of elliptic equations and uniqueness of invariant measures of diffusions. Sbornik Math. 193 (2002), no. 7, 945–976.

[9] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland, Amsterdam–London; American Elsevier, New York, 1973.

[10] M. Chicco, Solvability of the Dirichlet problem in $H^{2,p}(\Omega)$ for a class of linear second order elliptic partial differential equations. Boll. Un. Mat. Ital. (4) 4 (1971), 374–387.

[11] S. Fornaro, G. Metafune, E. Priola, Gradient estimates for Dirichlet parabolic problems in unbounded domains. J. Diff. Eq. 205 (2004), 329–353.

[12] C.S. Kahane, A gradient estimate for solutions of the heat equation. Czechoslovak Math. J. 48 (123) (1998), no. 4, 711–725.

[13] C.S. Kahane, A gradient estimate for solutions of the heat equation. II. Czechoslovak Math. J. 51 (126) (2001), no. 1, 39–44.

[14] W. Stannat, (Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 28 (1999), no. 1, 99–140.

[15] F.-Y. Wang, Gradient estimates of Dirichlet heat semigroups and application to isoperimetric inequalities. Ann. Probab. 32 (2004), no. 1A, 424–440.

5 Asymptotic Compactness and Absorbing Sets for Stochastic Burgers' Equations Driven by Space–Time White Noise and for Some Two-Dimensional Stochastic Navier–Stokes Equations on Certain Unbounded Domains

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5.1 Introduction

In the last decade there has been a growing interest in the ergodic properties of infinite dimensional systems governed by stochastic partial differential equations (SPDEs). In particular, existence of attractors for two-dimensional (2-D) Navier–Stokes equations (NSEs)in bounded domains both driven by real and additive noise has been established, see e.g., [BrzCapFl93], [CrFl94], and [Schmalfuss92]. Recently in a joint work with Y. Li we have generalized the results from [CrFl94] and [Schmalfuss92] to the case of unbounded domains. We observed there that the method of asymptotical compactness used by us should work also for equations in bounded domains with much rougher noise than the original methods could handle. The main motivation of this chapter is to show that this is indeed the case for the one dimensional (1D) stochastic Burgers' equations with additive space–time white noise. Even for readers mainly interested in the Navier–Stokes equations it could be useful to study the Burgers' equations case. Contrary to some recent works on stochastic Burgers' equations, see [DaPrZab6] and references therein, our approach is very similar to the approach we use for the NSEs in [BrzLi02]. The second motivation is to show that it also works for certain special form of two-dimensional (2D) stochastic NSEs with multiplicative noise. In fact, using a generalization of a recent result [CapCutl99] to bounded and unbounded domains, we show the existence of a compact invariant set for such problems. Full proofs of the results presented in this section will be published elsewhere. We should make it clear that we only study functional version of stochastic NSEs. For questions related to the pressure we refer the reader to [LRS04].

5.2 Random dynamical systems-short introduction

This section is a revised and compact version of Sections 2 and 3 from [BrzLi02]. See [Arnold98] for a comprehensive presentation of the theory.

A triple $\mathfrak{T} = (\Omega, \mathcal{F}, \vartheta)$ is called measurable dynamical system (DS) iff $(\Omega, \mathcal{F})$ is a measure space and $\vartheta : \mathbb{R} \times \Omega \ni (t, \omega) \mapsto \vartheta_t\omega \in \Omega$ is a measurable map such that for all $t, s \in \mathbb{R}$, $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$. A quadruple $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is called a metric DS iff $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathfrak{T}' := (\Omega, \mathcal{F}, \vartheta)$ is measurable DS such that for each $t \in \mathbb{R}$, the map $\vartheta_t : \Omega \ni \omega \mapsto \vartheta(t, \omega) \in \Omega$ preserves $\mathbb{P}$.

Suppose also that (X, d) is a complete and separable metric space and $\mathcal{B}$ is its Borel σ-field. Let $\mathbb{R}^+ = [0, \infty)$. Given a metric DS $\mathfrak{T}$ and a Polish space X as above, a measurable map $\varphi : \mathbb{R}^+ \times \Omega \times \mathrm{X} \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in \mathrm{X}$ is called a measurable **random dynamical system** (RDS) (on X over $\mathfrak{T}$), iff $\varphi(0, \omega) = id$, for all $\omega \in \Omega$ and the following **cocycle** property is satisfied: $\varphi(t + s, \omega) = \varphi(t, \vartheta_s\omega) \circ \varphi(s, \omega)$ for all $s, t \in \mathbb{R}^+$.

An RDS φ is said to be continuous, differentiable, or analytic iff for all $(t, \omega) \in \mathbb{R}^+ \times \Omega$, $\varphi(t, \cdot, \omega) : \mathrm{X} \to \mathrm{X}$ is continuous, differentiable or analytic, respectively. Similarly, an RDS φ is said to be time continuous iff for all $\omega \in \Omega$ and for all $x \in \mathrm{X}$ $\varphi(\cdot, x, \omega) : \mathbb{R}^+ \to \mathrm{X}$ is continuous. Because our interest lies in nonlocally compact metric spaces (in particular, in infinite-dimensional Banach spaces), our definition of continuous RDS differs from Definition 1.1.2 in [Arnold98]. Let us recall, see [CastVal77], that for two sets $A, B \subset X$ the Hausdorff metric is defined by $\rho(A, B) := \max\{d(A, B), d(B, A)\}$, where $d(A, B) = \sup_{x\in A} d(x, B)$. In fact, when ρ is restricted to the family $\mathcal{C}$ of all closed subsets of X, it is a metric. From now on, $\mathcal{X}$ will denote the σ-field on $\mathcal{C}$ generated by open sets with respect to the Hausdorff metric ρ, see e.g., [BrzCapFl93], [CastVal77], or [Crauel95]. A set valued map $C : \Omega \to \mathcal{C}$, where $(\Omega, \mathcal{F})$ is a measurable space and (X, d) is a complete separable metric space, is called measurable or a **closed random set** iff C is $(\mathcal{F}, \mathcal{X})$-measurable. Let $\varphi : \mathbb{R}^+ \times \Omega \times \mathrm{X} \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in \mathrm{X}$ be a measurable RDS on a Polish space (X, d) over a metric DS $\mathfrak{T}$. A random set B is called φ**-forward**, resp. **strictly forward** invariant, iff for all $\omega \in \Omega$, $\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subseteq B(\omega)$ for all $t \geq 0$, or, respectively, $\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) = B(\omega)$, $t \geq 0$. The Ω**-limit set** of a random set B, is the random set

$$\Omega(B, \omega) = \Omega_B(\omega) = \bigcap_{T\geq 0} \overline{\bigcup_{t\geq T} \varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega)}. \tag{5.1}$$

Obviously $\Omega_B(\omega)$ is a closed random set. There are examples of RDSs and random sets B for which $\Omega(B, \omega)$ is an empty set. One can easily show that $y \in \Omega_B(\omega)$ iff there exists sequences: $t_n \to \infty$, $\{x_n\} \subset B(\vartheta_{-t_n}\omega)$ such that $\varphi(t_n, \vartheta_{-t_n}\omega)x_n \to y$.

A random set $K(\omega)$ is said to (a) **attract**, (b) **absorb**, or (c) ρ**-attract** a random set $B(\omega)$ iff for all $\omega \in \Omega$, respectively, (a) $\lim_{t\to\infty} d(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0$; (b) there exists a time $t_B(\omega)$ such that $\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset K(\omega)$, for $t \geq t_B(\omega)$; or, (c) $\lim_{t\to\infty} \rho(\varphi(t, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega), K(\omega)) = 0$.

Let us observe that if a random set K absorbs a random set B, then K ρ-attracts B and if K ρ-attracts B, then K attracts B. By replacing ω by $\vartheta_{-s}\omega$ and t by $t - s$ one can prove that a random set K absorbs a random set B iff for all $\omega \in \Omega$, there exists a random time τ_B such that $\varphi(t - s, \vartheta_{-t}\omega)B(\vartheta_{-t}\omega) \subset K(\vartheta_{-s}\omega)$ for $t \geq s + \tau_B$.

The following definition is new in the framework of RDS. It is motivated, in particular, by the following works: [Ladyzhenskaya91], [Ghidaglia94], and [Rosa98].

Definition 5.1 φ X ***asymptotically compact*** *ff* $\omega \in \Omega$, $B \subset \mathrm{X}$, (t_n) $t_n \to \infty$, $\{x_n\}_n \subset B$, $\{\varphi(t_n, \vartheta_{-t_n}\omega)x_n : n \in \mathbb{N}\}$ X

Let us point out that the definition of asymptotical compactness given in [CrDebFl97] is not equivalent to the above. Moreover, the definition from [CrDebFl97] appears not to be applicable to SPDEs in unbounded domains.

From now on we will assume that $\mathfrak{T} = (\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is a metric DS, X is a separable Banach space, and φ is a continuous, asymptotically compact RDS on X (over $\mathfrak{T}$). We list now some basic properties of ω-limit sets, see [BrzLi02] for proofs.

Theorem 5.1 $B \subset \mathrm{X}$ $\omega \in \Omega$, $\Omega_B(\omega)$ B, φ

A very important consequence of the previous result is the existence of invariant measures for RDS φ. Let me first recall, see [Arnold98] Remark 1.1.8, that the **skew product** of a measurable DS $\mathfrak{T}$ with an RDS φ on a Polish space X over $\mathfrak{T}$ is the map

$$\Theta : \mathbb{R}^+ \times \Omega \times \mathrm{X} \ni (t, \omega, x) \mapsto (\vartheta(t, \omega), \varphi(t, \omega)x) =: \Theta_t(\omega, x) \in \Omega \times \mathrm{X}. \tag{5.2}$$

It is known that if Θ is the skew product of $\mathfrak{T}$ with φ, then the triple $\widehat{\mathfrak{T}} := (\Omega \times \mathrm{X}, \mathcal{F} \otimes \mathcal{B}, \Theta)$ is a measurable DS. Conversely, if $\mathfrak{T}$ is a measurable DS, $\vartheta : \mathbb{R}^+ \times \Omega \times \mathrm{X} \to \mathrm{X}$ is measurable, the function Θ defined by (5.2), and the triple $\widehat{\mathfrak{T}}$ is a measurable DS, then φ is an RDS on X over $\mathfrak{T}$. If φ is an RDS over a metric DS $\mathfrak{T}$, a probability measure μ on $(\Omega \times \mathrm{X}, \mathcal{F} \otimes \mathcal{B})$ is called an invariant measure for φ iff (a) Θ_t preserves μ, i.e., $\Theta_t(\mu) = \mu$ for each $t \in \mathbb{R}^+$; (b) the first marginal of μ is equal to $\mathbb{P}$, i.e., $\pi_\Omega(\mu) = \mathbb{P}$, where $\pi_\Omega : \Omega \times \mathrm{X} \ni (\omega, x) \mapsto \omega \in \Omega$. Since by Corollary 4.4 in [CrFl94], if an RDS φ on a Polish space X has an invariant compact random set $K(\omega)$, $\omega \in \Omega$, it also has an invariant probability measure, we have the following.

Corollary 5.1 φ μ $(\Omega \times \mathrm{X}, \mathcal{F} \otimes \mathcal{B})$

If $f : \mathrm{X} \to \mathbb{R}$ is a bounded and Borel measurable function, then we put

$$(P_t f)(x) = \mathbb{E}\, f(\varphi(t, x)), \quad t \geq 0,\ x \in \mathrm{X}.$$

One can show, see again [BrzLi02], that the family $(P_t)_{t \geq 0}$ is Feller, i.e., $P_t f \in C_b(\mathrm{X})$ if $f \in C_b(\mathrm{X})$. Moreover, if the RDS φ is time continuous, then for any $f \in C_b(\mathrm{X})$, $(P_t f)(x) \to f(x)$ as $t \searrow 0$. Finally, a Borel probability measure μ on X is called an invariant measure for the semigroup $(P_t)_{t \geq 0}$ iff $P_t^* \mu = \mu$, $t \geq 0$, where $(P_t^* \mu)(\Gamma) = \int_{\mathrm{H}} P_t(x, \Gamma)\, \mu(dx)$ for $\Gamma \in \mathcal{B}(\mathrm{H})$ and the $P_t(x, \cdot)$ is the transition probability, $P_t(x, \Gamma) = P_t(1_\Gamma)(x)$, $x \in B$. A Feller invariant probability measure for an RDS φ on H is, by definition, an invariant probability measure for P_t defined above. It is proved in [CrFl94], see Corollary 4.4, that if an RDS φ on a Polish space X has an invariant compact random set $K(\omega)$, $\omega \in \Omega$, then there exists a Feller invariant probability measure μ. Thus we have the following.

Corollary 5.2 φ

The uniqueness of an invariant Borel probability measure (and thus its independence of the set B) and the existence of a global attractor remain open questions.

5.3 Wiener and Ornstein–Uhlenbeck processes

Suppose that $\mathrm{H} = L^2(0, 1)$ and $\mathrm{A} = -\Delta$ with $D(\mathrm{A}) = H_0^{1,2}(0, 1) \cap H^{2,2}(0, 1)$, where $H^{k,p}(0, 1)$, for $k \in \mathbb{N}$ and $p \in [1, \infty)$, denotes the Sobolev space of all $f \in L^p(0, 1)$ whose weak derivatives $D^j u$, $j \in \{1, \ldots, k\}$ belong to $L^p(0, 1)$ as well. Equivalently, $H^{k,p}(0, 1)$ is the space of all $u \in C^{k-1}([0, 1])$ such that $u^{(k-1)}$ is absolutely continuous and whose derivative belongs to $L^p(0, 1)$. $H_0^{1,2}(0, 1)$ is the space of those $u \in H^{1,2}(0, 1)$ that satisfy $u(0) = u(1) = 0$. The norm in H will be denoted by $|\cdot|$ (or by $|\cdot|_{L^2(0,1)}$ in danger of ambiguity). It is well known that A is a positive self-adjoint operator on H, that $-\mathrm{A}$ is an infinitesimal generator of an analytic semigroup $(e^{-t\mathrm{A}})_{t \geq 0}$, and that the semigroup $(e^{-t\mathrm{A}})_{t \geq 0}$ has a unique restriction or extension to an analytic semigroup on $L^p(0, 1)$ for all $p \in [0, 1)$. These various semigroups and their corresponding infinitesimal generators will be be simply denoted by $(e^{-t\mathrm{A}})_{t \geq 0}$ and $-\mathrm{A}$.

We set $\mathrm{X} := L^4(0,1)$ and choose $\delta \in (\frac{1}{4}, \frac{1}{2})$. Then we observe that the map $\mathrm{A}^{-\delta} : \mathrm{H} \to \mathrm{X}$ is γ-radonifying. Indeed, A is a self-adjoint operator in H with compact inverse, eigenvalues $\pi^2 j^2$, $j = 1, 2, 3, \dots$ (and eigenvectors $e_j = \sin(j\pi\cdot)$, $j = 1, 2, 3, \dots$), X is of type 2 and $\sum_j |\mathrm{A}^{-\delta} e_j|^2_{L^4(0,1)} < \infty$. Denote next by E the completion of X with respect to the image norm $|x|_{\mathrm{E}} = |A^{\delta} x|_{\mathrm{X}}$, for $x \in \mathrm{X}$. In fact it is well known, see e.g., [Brzezniak95] for a simple explanation, that $\mathrm{E} = H^{-2\delta,4}(0,1)$ which a dual of the Sobolev space $H_0^{2\delta,4/3}(0,1)$ and that E is M-type 2 (hence separable) Banach space. For $\xi \in (0, \frac{1}{2})$ define

$$C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E}) := \{\omega \in C(\mathbb{R}, \mathrm{E}) : \sup_{t,s \in \mathbb{R}} \frac{|\omega(t) - \omega(s)|_{\mathrm{E}}}{|t - s|^{\xi}(1 + |t| + |s|)^{\frac{1}{2}}} < \infty,\ \omega(0) = 0\}.$$

It is standard to prove that $C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$ is a Banach space with norm

$$\|\omega\|_{C^{\xi}_{1/2}(\mathbb{R},\mathrm{E})} = \sup_{t,s \in \mathbb{R}} \frac{|\omega(t) - \omega(s)|_{\mathrm{E}}}{|t - s|^{\xi}(1 + |t| + |s|)^{\frac{1}{2}}}.$$

Despite the fact that the space $C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$ is non separable, the closure of $\{\omega \in C_0^{\infty}(\mathbb{R}, \mathrm{E}) : \omega(0) = 0\}$ in $C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$, denoted by $\Omega(\xi, \mathrm{E})$, is separable. We denote by $\mathcal{F}$ the Borel σ-algebra on $\Omega(\xi, \mathrm{E})$.

We also need a separable Banach space $C_{1/2}(\mathbb{R}, \mathrm{X})$ of all continuous functions $\omega : \mathbb{R} \to \mathrm{X}$ such that

$$\|\omega\|_{C_{1/2}(\mathbb{R},\mathrm{X})} := \sup_{t \in \mathbb{R}} \frac{|\omega(t)|_{\mathrm{X}}}{1 + |t|^{\frac{1}{2}}} < \infty.$$

One can show, see [Hairer05] for a similar problem in a 1D case, that for $\xi \in (0, \frac{1}{2})$, there exists a Borel probability measure $\mathbb{P}$ on $\Omega(\xi, \mathrm{E})$ such that the canonical process $(w_t)_{t \in \mathbb{R}}$, defined by

$$w_t(\omega) := \omega(t), \quad \omega \in \Omega(\xi, \mathrm{E}) \tag{5.3}$$

is a two-sided Wiener process such that the Cameron–Martin (or reproducing kernel Hilbert) space the law $\mathcal{L}(w_1)$ is equal to H. Let for $t \in \mathbb{R}$, $\mathcal{F}_t := \sigma\{w_s : s \le t\}$. Since for each $t \in \mathbb{R}$ the map $z \circ i_t : \mathrm{E}^* = H_0^{2\delta,4/3} \to L^2(\Omega(\xi, \mathrm{E}), \mathcal{F}_t, \mathbb{P})$, where $i_t : \Omega(\xi, \mathrm{E}) \ni \gamma \mapsto \gamma(t) \in \mathrm{E}$, satisfies $\mathbb{E}|z \circ i_t|^2 = t|z|^2_{L^2(0,1)}$, there exists a unique extension of $z \circ i_t$ to a bounded linear map $W(t) : \mathrm{H} \to L^2(\Omega(\xi, \mathrm{E}), \mathcal{F}_t, \mathbb{P})$. Moreover, the family $(W_t)_{t \in \mathbb{R}}$ is an $-$H-cylindrical Wiener process in the sense of, e.g., [BrzPesz01].

On both spaces $C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$ and $\Omega(\xi, \mathrm{E})$ we consider a flow $\vartheta = (\vartheta_t)_{t \in \mathbb{R}}$ defined by

$$\vartheta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

It is obvious that for each $t \in \mathbb{R}$, ϑ_t preserves $\mathbb{P}$.

Finally, we define a family, indexed by $\alpha \ge 0$, of Ornstein–Uhlenbeck processes (with fixed $\xi \in (\delta, \frac{1}{2})$) by, with $A_\alpha = A + \alpha I$

$$z_\alpha(\omega)(t) := \int_{-\infty}^{t} A_\alpha^{1+\delta} e^{-(t-r)A_\alpha} A_\alpha^{-\delta} (\theta_r \omega)(t - r)\, dr,\ t \in \mathbb{R} \tag{5.4}$$

for $\omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$. Notice that the choice of ξ implies that $(A + \alpha I)^{-\delta} \omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{X})$ when $\omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$. Hence, if $\omega \in C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E})$, then $z_\alpha(\omega) \in C_{1/2}(\mathbb{R}, \mathrm{X})$ and the map $z_\alpha : C^{\xi}_{1/2}(\mathbb{R}, \mathrm{E}) \ni \omega \mapsto z_\alpha(\omega) \in C_{1/2}(\mathbb{R}, \mathrm{X})$ is linear and bounded. For $\zeta \in C_{1/2}(\mathbb{R}, \mathrm{X})$ we define C_0-group $(\tau_s)_{s \in \mathbb{R}}$ of linear contractions on $C_{1/2}(\mathbb{R}, \mathrm{X})$ by

$$(\tau_s \zeta) = \zeta(t + s), \quad t, s \in \mathbb{R}.$$

One can prove, see [BrzLi02], that for $s \in \mathbb{R}$,

$$\tau_s \circ z_\alpha = z_\alpha \circ \vartheta_s. \tag{5.5}$$

Hence, if μ_α denotes the law of the process $(z_\alpha(t))_{t\in\mathbb{R}}$ defined on the probability space $\Omega(\xi, \mathrm{E}), \mathbb{P}$, then μ is invariant with respect to the transformations τ_s, $s \in \mathbb{R}$. In particular, $\mu_\alpha(t) = \mu_\alpha(0)$, for all $t \in \mathbb{R}$, where $\mu_\alpha(t)$ is the image of the measure μ_α by the evaluation operator $i_t : C_{1/2}(\mathbb{R}, \mathrm{X}) \ni \zeta \mapsto \zeta(t) \in \mathrm{X}$. In fact, one can show, see [BrzLi02], the following.

Proposition 5.1 $z_\alpha(t)$, $t \in \mathbb{R}$

$$dz_\alpha(t) + (A + \alpha I)z_\alpha \, dt = dW(t), \quad t \in \mathbb{R},$$

$t \in \mathbb{R}$, $z_\alpha(t) = \int_{-\infty}^t e^{-(t-s)(A+\alpha I)} \, dW(s)$, X, $C > 0$

$$\begin{aligned}
\mathbb{E}|z_\alpha(t)|^2_{\mathrm{X}} &= \mathbb{E}\left|\int_{-\infty}^t e^{(\mathrm{A}+\alpha I)(s-t)} \, dW(s)\right|^2_{\mathrm{X}} \le C \int_{-\infty}^t \|e^{(\mathrm{A}+\alpha)(s-t)}\|^2_{R(\mathrm{H},\mathrm{X})} \, ds \\
&= C \int_0^\infty e^{-2\alpha s} \|e^{-s\mathrm{A}}\|^2_{R(\mathrm{H},\mathrm{X})} \, ds,
\end{aligned} \tag{5.6}$$

$R(\mathrm{K}, \mathrm{X})$ γ K X

$\mathbb{E}|z_\alpha(t)|^2_{\mathrm{X}}$ 0 $\alpha \to \infty$

5.4 Stochastic 1D Burgers' equations with additive noise

In this section we keep the notation introduced in Section 5.3. Our aim is to study the following stochastic Burgers' equations:

$$du + [\mathrm{A}u + B(u)] \, dt = f dt + dW(t), \quad t \ge 0 \tag{5.7}$$

with the initial condition

$$u(0) = u_0, \tag{5.8}$$

where $B(u) = uu_x = \frac{1}{2}\frac{d}{dx}(u^2)$, $u_0 \in \mathrm{H} = L^2(0,1)$, $\mathrm{V} = H_0^{1,2}(0,1)$, and $f \in \mathrm{V}' = H^{-1,2}(0,1)$. Let us notice that V is a Hilbert space with norm $\|u\|^2 := \int_0^1 |\nabla u(x)|^2, dx$ which is equivalent with the norm inherited from the Sobolev space $H^{1,2}(0,1)$, i.e., $|\!|\!|u|\!|\!|^2 = \int_0^1 |\nabla u(x)|^2 dx + \int_0^1 |u(x)|^2, dx$. In fact, $\|u\|^2 \ge \lambda_1 |u|^2$, where λ_1 is the smallest eigenvalue of A. Finally, $(W(t))_{t\in\mathbb{R}}$, is the H-cylindrical Wiener process introduced in the previous section. Let us observe that we could also consider K-cylindrical Wiener process with the Hilbert space K even bigger than H. The above problem (5.7) is of a similar form as the stochastic NSEs, however, with one essential difference. The nonlinearity B defined above still satisfies the condition

$$(B(u), u) = 0, \ u \in \mathrm{V}$$

but its symmetric bilinear counterpart defined by $B(u,v) = \frac{1}{2}\frac{d}{dx}(uv)$ no longer satisfies the stronger condition $(B(u,v), v) = 0$ (even for very smooth functions u, v).

We have the following definition of a solution to problems (5.7) and (5.8).

Definition 5.2 H $(\mathcal{F}_t)_{t\geq 0}$ $u(t)$, $t \geq 0$ ( ) ( ) ff

$$\sup_{0\leq t\leq T} |u(t)|^2 + \int_0^T |u(t)|^4_{L^4(0,1)}\, dt < \infty \qquad T > 0 \tag{5.9}$$

$\psi \in \mathrm{V} \cap H^{2,2}(0,1)$ $t > 0$,

$$\begin{aligned}
(u(t),\psi) - (u_0,\psi) \quad &- \quad \int_0^t (u(s),\Delta\psi)\, ds - \frac{1}{2}\int_0^t (u^2(s),\nabla\psi)\, ds \\
&= \quad \int_0^t (f,\psi)\, ds + \langle \psi, W(t)\rangle.
\end{aligned}$$

In order to prove local existence of solutions we need the following result in which by $\mathcal{H}^{1,2}(0,T)$ we denote the Banach space of all u belonging to $L^2(0,T;\mathrm{V})$ such that its weak time derivative u' belongs to $L^2(0,T;\mathrm{V}')$.

Lemma 5.1 $u \in \mathcal{H}^{1,2}(0,T)$ $z \in L^4(0,T;L^4(0,1))$,

$$\int_0^T |B(u(t))|^2_{\mathrm{V}'}\, dt \quad\leq\quad \frac{1}{2}T^{\frac{1}{2}}\|u\|^4, \tag{5.10}$$

$$\int_0^T |B(z(t))|^2_{\mathrm{V}'}\, dt \quad\leq\quad \frac{1}{4}\int_0^T |z(t)|^4_{L^4}\, dt. \tag{5.11}$$

Proof

The integration by parts formula together with the Hölder inequality implies that if $p,q \in [2,\infty]$ and $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, then

$$|\langle B(u,v),\phi\rangle| = \frac{1}{2}|\int u(x)v(x)\phi_x(x)\, dx| \leq \frac{1}{2}|u|_{L^p}|v|_{L^q}|\nabla\phi|, \quad u,v,\phi \in V. \tag{5.12}$$

In particular, (5.12) implies that B can be uniquely extended to a bounded bilinear form from $L^4(0,1)$ to V' (of norm $\leq \frac{1}{2}$). Because also, see Theorem 9.3 in [Friedman69]

$$|u|^4_{L^4} \leq 2|\nabla u|_{L^2}|u|^3_{L^2}, u \in H^{1,2}_0(0,1), \tag{5.13}$$

we infer that

$$\begin{aligned}
\int_0^T |Bu(t)|^2_{\mathrm{V}'}\, dt \quad &\leq \quad \frac{1}{2^2}\int_0^T |u(t)|^4_{L^4}\, dt \leq \frac{1}{2}\int_0^T |\nabla u(t)|_{L^2}|u(t)|^3_{L^2}\, dt \\
&\leq \quad \frac{1}{2}\sup_{0\leq t\leq T}|u(t)|^3_{L^2}T^{\frac{1}{2}}\Big(\int_0^T |\nabla u(t)|^2_{L^2}\, dt\Big)^{\frac{1}{2}} \\
&\leq \quad \frac{1}{2}T^{\frac{1}{2}}\|u\|^4_{\mathcal{H}^{1,2}(0,T)}, \quad u \in \mathcal{H}^{1,2}(0,T).
\end{aligned} \tag{5.14}$$

The proof of (5.11) is even simpler.

A tool for studying the existence and uniqueness of solutions for the problems (5.7) and (5.8) is the following.

Proposition 5.2 $z \in L^4(0,T;L^2(0,1))$, $g \in L^2(0,T;\mathrm{V}')$, $v_0 \in \mathrm{H}$, $v \in \mathcal{H}^{1,2}(0,T)$

$$\frac{dv}{dt} \quad+\quad Av + B(v,z) + B(z,v) + B(v,v) = g, \quad t \geq 0, \tag{5.15}$$

$$v(0) \quad=\quad v_0. \tag{5.16}$$

$K^2 := e^{2\int_0^T |z(s)|^4\,ds}$, $L^2 := |v_0|^2 + 2\int_0^T |g(s)|^2_{\mathrm{V}'}\,ds$, $M^2 = |v_0|^2 + 9KL\int_0^T |z(t)|^4\,dt + \int_0^T |g(t)|^2_{\mathrm{V}'}\,dt$, $N := |g|_{L^2(0,T;\mathrm{V}')} + 2KLM|z|^2_{L^4(0,T;\mathrm{H})} + \frac{T^{1/4}}{\sqrt{2}}K^{3/2}L^{1/2}$,

$$\sup_{t\in[0,T]} |v(t)|^2 \leq K^2L^2, \tag{5.17}$$

$$\int_0^T |\nabla v(t)|^2\,dt \leq M^2, \tag{5.18}$$

$$\int_0^T |v'(t)|^2_{\mathrm{V}'}\,dt \leq N^2, \tag{5.19}$$

$$\int_0^T |v(t)|^4_{L^4(0,1)}\,dt \leq 2T^{1/2}K^3L^3M. \tag{5.20}$$

$L^2(0,T;\mathrm{V}') \times \mathrm{H} \ni (g, v_0) \mapsto v \in \mathcal{H}^{1,2}(0,T)$, v

Proof

Define a family $K(t)$, $t \in [0,T]$, of linear operators from V to V′ by

$$K(t)u := B(z(t), u), \quad u \in \mathrm{V}.$$

In view of (5.12) with $p = 2$ and $q = \infty$ and of the classical inequality $|u|^2_{L^\infty} \leq 2|u|_{L^2}|\nabla u|_{L^2}$, $v \in \mathrm{V}$, we have, for $\mathcal{H}^{1,2}(0,T)$

$$\begin{aligned}\int_0^T |B(z(t),u(t))|^2_{\mathrm{V}'}\,dt &\leq \frac{1}{2}\int_0^T |z(t)|^2_{L^2}|u(t)|_{L^2}|\nabla u(t)|_{L^2}\,dt \\ &\leq \frac{1}{2}\sup_{0\leq t\leq T}|u(t)|\Big(\int_0^T |z(t)|^4\,dt\Big)^{\frac{1}{2}}\Big(\int_0^T |\nabla u(t)|\,dt\Big)^{\frac{1}{2}} \quad (5.21)\\ &\leq \frac{1}{2}\Big(\int_0^T |z(t)|^4\,dt\Big)^{\frac{1}{2}}\|u\|_{\mathcal{H}^{1,2}(0,T)}.\end{aligned}$$

The above, in view of a slight modification of Lemma 2 from [Brzezniak91], implies that the linear part of problems (5.15) and (5.16) is an isomorphism between $\mathcal{H}^{1,2} \times \mathrm{H}$ and $L^2(0,T;\mathrm{V}')$. Because of (5.10), the method borrowed from [Brzezniak91], see the proof of Theorem 7, yields existence of $T_1 \in (0,T]$ and a unique $v \in \mathcal{H}^{1,2}(0,T_1)$ which solves (5.15) and (5.16) on the interval $[0,T_1]$. Moreover, for some constant C independent of v_0 and g, $T_1 \geq C(|v_0|^2 + \int_0^{T_1}|g(s)|^2\,ds)^{-\frac{1}{2}} \geq C(|v_0|^2 + \int_0^T |g(s)|^2\,ds)^{-\frac{1}{2}}$. Hence, it suffices to show that the H-norm of $v(t)$ remains bounded on the interval $[0,T_1]$. As it is standard, we use Lemma III.1.2 from [Temam84] to infer that

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2 + |\nabla v(t)|^2 + \langle B(z,v) + B(v,z), v\rangle = \langle g(t), v(t)\rangle, \text{ on } (0,T_1). \tag{5.22}$$

Since $|\langle B(z,v)+B(v,z),v\rangle| \leq |z||v|_{L^\infty}|\nabla v| \leq \sqrt{2}|z||v|^{\frac{1}{2}}|\nabla v|^{3/2} \leq \frac{3}{4}|\nabla v|^2 + |z|^4|v|^2$, the norm on V is the $|\nabla\cdot|$ norm and $|\langle g,v\rangle| \leq \frac{1}{4}|\nabla v|^2 + |g|^2_{\mathrm{V}'}$, we infer that

$$\frac{d}{dt}|v(t)|^2 \leq 2|z(t)|^4|v(t)|^2 + 2|g(t)|^2_{\mathrm{V}'}, \text{ on } (0,T_1). \tag{5.23}$$

Hence, by invoking the Gronwall lemma we infer that

$$|v(t)|^2 \leq e^{2\int_0^t |z(s)|^4\,ds}|v_0|^2 + 2\int_0^t e^{2\int_s^t |z(\sigma)|^4\,d\sigma}|g(s)|^2_{\mathrm{V}'}\,ds, \quad t \in [0,T_1].$$

From the last inequality we easily infer that $|v(t)|^2 \leq K^2(|v_0|^2 + 2\int_0^T |g(s)|^2_{\mathrm{V}'}\, ds)$ for all $t \in [0, T_1]$, which proves the requested boundedness of $|v(\cdot)|^2$ on $[0, T_1]$. Therefore, there exists a solution $v \in \mathcal{H}^{1,2}(0, T)$ to the problems (5.15) and (5.16). The uniqueness of solutions follows from the very first part of the proof.

A by-product of the above argument is also a proof of (5.17).

By modifying the argument leading inequality (5.23), i.e., using inequality $|z||v|_{L^\infty}|\nabla v| \leq \sqrt{2}|z||v|^{\frac{1}{2}}|\nabla v|^{3/2} \leq \frac{1}{4}|\nabla v|^2 + 27|z|^4|v|^2$ we can easily show that

$$\frac{d}{dt}|v(t)|^2 + |\nabla v(t)|^2 \leq 27|z(t)|^4|v(t)|^2 + 2|g(t)|^2_{\mathrm{V}'}, \text{ on } (0, T). \tag{5.24}$$

From it we can easily deduce inequality (5.18). Inequality (5.19) follows from the triangle inequality applied to (5.7) and inequalities (5.17), (5.18), (5.21), and (5.14). Inequality (5.20) follows from inequalities (5.13), (5.17) and (5.18). The last statement can be proved easily following the method from [Brzezniak91].

Remark 5.1 Our existence proof is different than the original proof from [DaPrDebTem94]. It seems to be simpler. We have clarified the definition of a solution.

Since, with $\lambda_1 = \pi^2$ being the smallest eigenvalue of A, $|\nabla v(t)|^2 \geq \lambda_1 |v|^2$ for $v \in \mathrm{V}$, from (5.24) we get that $\frac{d}{dt}|v(t)|^2 \leq (-\lambda_1 + 27|z(t)|^4)|v(t)|^2 + 2|g(t)|^2_{\mathrm{V}'}$, on $(0, T)$, from which by a simple use of the Gronwall lemma we infer the following corollary.

Corollary 5.3 $t \in [0, T]$

$$|v(t)|^2 \leq e^{\int_0^t (-\lambda_1 + 27|z(\sigma)|^4)\, d\sigma}|v_0|^2 + 2\int_0^t e^{\int_s^t (-\lambda_1 + 27|z(\sigma)|^4)\, d\sigma}\, |g(s)|^2_{\mathrm{V}'}\, ds. \tag{5.25}$$

Another consequence of the previous result is that the map $L^2(0, T; \mathrm{V}') \times \mathrm{H} \ni (g, v_0) \mapsto v \in \mathcal{H}^{1,2}(0, T)$ is not only real analytic but also continuous in the weak topologies. To be precise we have the following result in which by $v(\cdot, x)$ we denote the unique solution to the problem (5.7) with the initial condition $v(0, x) = x \in \mathrm{H}$ and $z \in L^4_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{L}^4(0, 1))$.

Corollary 5.4 $T > 0$ $x_n \to x$ H $v(\cdot, x_n) \to v(\cdot, x)$ $L^2(0, T; \mathrm{V})$ $\phi \in \mathrm{H}$ $(v(\cdot, x_n), \phi) \to (v(\cdot, x), \phi)$ $[0, T]$ $n \to \infty$

Proof

Because the sequence $\{x_n\}$ is bounded in H, from Proposition 5.2 we infer that the sequence $(v_n)_n$, where $v_n := v(\cdot, x_n)$, is bounded in $\mathcal{H}^{1,2}(0, T)$. Hence, there exists a subsequence $v_{n'}$ and $\tilde{v} \in L^\infty(0, T; \mathrm{H}) \cap L^2(0, T; \mathrm{V})$ such that $v_{n'} \to \tilde{v}$ weakly star in $L^\infty(0, T; \mathrm{H})$ and weakly in $L^2(0, T; \mathrm{V})$ and, by the compactness Theorem III.2.1 from [Temam84], strongly in $L^\infty(0, T; \mathrm{H})$. One can then prove that $\tilde{v}$ is a solution of (5.7) with initial condition $\tilde{v}(0) = x$. By the uniqueness of solution to (5.7) and (5.8) we infer that $\tilde{v} = v$. Hence, in particular, $v_{n'} \to v$ weakly in $L^2(0, T; \mathrm{V})$. Let us notice that we have in fact proved that from each subsequence of the sequence $(v_n)_n$ we can choose a sub-subsequence that is weakly convergent to v in $L^2(0, T; \mathrm{V})$. Since the weak topology on any ball in a Hilbert space, hence in $L^2(0, T; \mathrm{V})$, is metrizable, we infer that the sequence $(v_n)_n$ itself is weakly convergent to v in $L^2(0, T; \mathrm{V})$.

The proof of the second part is preceded by two observations. First, because of (5.17), the sequence $\langle v_n(\cdot), \phi \rangle$ is bounded in $C([0, T])$ for any $\phi \in \mathrm{H}$. Second, if $\phi \in \mathrm{H}$, then for $t_1 < t_2 \in [0, T]$, $|\langle v_n(t_2), \phi \rangle - \langle v_n(t_1), \phi \rangle| \leq \|\phi\|(t_2 - t_1)^{1/2}\big(\int_{t_1}^{t_2} |v_n'(s)|^2_{\mathrm{V}'}\, ds\big)^{1/2}$, so because of (5.19), the sequence $\langle v_n(\cdot), \phi \rangle$ is uniformly continuous on $[0, T]$. Hence, by the Arzela–Ascoli theorem, it is relatively compact in $C([0, T])$. Moreover, by what we have proved earlier, the sequence (v_n) is relatively compact in $L^2(0,T;\mathrm{H})$. Hence, if $\phi \in \mathrm{V}$, from any subsequence $\langle v_{n'}(\cdot),\phi\rangle$ we can choose a sub-subsequence $\langle v_{n''}(\cdot),\phi\rangle$, a function $\eta \in C([0,T])$ and a subset $\Gamma \subset [0,T]$ of full Lebesgue measure such that $\langle v_{n''}(t),\phi\rangle \to \langle v(t),\phi\rangle$ for all $t \in \Gamma$ and $\langle v_{n''}(\cdot),\phi\rangle \to \eta$ uniformly on $[0,T]$. Since $\langle v(\cdot),\phi\rangle \in C([0,T])$, we infer that $\langle v_{n''}(\cdot),\phi\rangle \to \langle v(\cdot),\phi\rangle$ uniformly on $[0,T]$. Hence, $\langle v_n(\cdot),\phi\rangle \to \langle v(\cdot),\phi\rangle$ uniformly on $[0,T]$. Since V is dense in H, the last statement in conjunction with (5.17) concludes the proof of the second part.

It can be shown that the Definition 5.2) of a solution to problems (5.7) and (5.8) is equivalent to the following one, which is modeled on another paper [Flandoli94].

Definition 5.3 [illegible] H [illegible] $(u(t))_{t\geq 0}$ [illegible] $\alpha \geq 0$, $u(t) = v_\alpha(t) + z_\alpha(t)$, [illegible] $t \geq 0$, [illegible] $v_\alpha(t) = v_\alpha(t,\omega)$ [illegible] $v(0) = u_0 - z_\alpha(0)$ [illegible] $g = g_\alpha(t) = f + \alpha z_\alpha(t) - B(z_\alpha(t))$, $t \geq 0$

Let us observe that for any fixed $\alpha \geq 0$ there exists a unique solution to (5.7) and (5.8). Indeed, this follows from Proposition 5.2 because a.s. $z_\alpha(\cdot) \in C_{1/2}(\mathbb{R},\mathrm{X}) \subset L^4_{\mathrm{loc}}(\mathbb{R}^+, L^4(0,1))$ so that by Lemma 5.1, $B(z_\alpha(\cdot),z_\alpha(\cdot)) \in L^2_{\mathrm{loc}}(\mathbb{R}^+,\mathrm{V}')$.

In order to justify Definition 5.3 we will show its independence from the parameter α. Let $u_0 \in \mathbb{H}$ be fixed. We need to show that $v_\alpha(t) + z_\alpha(t) = v_0(t) + z_0(t)$, $t \geq 0$ for each $\omega \in \Omega(\xi,\mathrm{E})$, where z_α is defined by (5.4) and $v_\alpha(t) = v_\alpha(t,\omega)$ is the solution of (5.15) such that $v(0) = u_0 - z_\alpha(0)$ and the force $g = g_\alpha(t) = f + \alpha z_\alpha(t) - B(z_\alpha(t), z_\alpha(t))$, $t \geq 0$. Since for each $T > 0$, the maps $\Omega(\xi,\mathrm{E}) \ni \omega \mapsto v_\alpha \in \mathcal{H}^{1,2}(0,T)$ and $\Omega(\xi,\mathrm{E}) \ni \omega \mapsto z_\alpha \in L^4(0,T;L^4(0,1))$ are continuous and $C_0^\infty(\mathbb{R},\mathrm{E})$ is dense in $\Omega(\xi,\mathrm{E})$, it enough to prove that equality for $\omega \in C_0^\infty(\mathbb{R},\mathrm{E})$. But for such an ω, $z_\alpha(\omega)(\cdot)$ is the solution of

$$\frac{dz_\alpha(t)}{dt} + (A+\alpha I)z_\alpha = \dot{\omega}(t), \quad t \in \mathbb{R}, \tag{5.26}$$

and hence $\bar{u} := u_\alpha(\omega,\cdot) - u_\beta(\omega,\cdot)$ is a solution of

$$\frac{d(\bar{u}(t))}{dt} + A\bar{u}(t) + [B(u^\alpha(t)) - B(u^\beta(t))] = 0,$$

with initial condition $\bar{u}(0) = 0$. Applying then an appropriate version of (5.22), the inequalities (5.12) (with $p = 8$) and (5.13) as well as the Hölder inequality we can find $C > 0$ such that

$$\frac{d}{dt}|\bar{u}(t)|^2 + |\nabla\bar{u}(t)|^2 \leq C\big(|u_\alpha(t)|_{L^4}^{8/7} + |u_\beta(t)|_{L^4}^{8/7}\big)|\bar{u}(t)|^2 \text{ on } (0,T). \tag{5.27}$$

Because $u_\alpha, u_\beta \in C([0,T];L^4(0,1)) \subset L^{8/7}(0,T;L^4(0,1))$ we infer, by applying the Gronwall lemma, that $\bar{u}(t) = 0$, for all $t \in [0,T]$.

Since by the second part of Lemma 5.1, $B(z_\alpha) \in L^2_{\mathrm{loc}}(0,\infty;\mathrm{V}')$ for each $\omega \in \Omega(\xi,\mathrm{E})$, from Proposition 5.2 applied to force $g(t) = g_\alpha(t) := f + \alpha z_\alpha(t) - B(z_\alpha(t), z_\alpha(t))$, $t \geq 0$ we infer the following.

Theorem 5.2 [illegible] $u_0 \in \mathrm{H}$ [illegible] $u(t)$, $t \geq 0$ [illegible]

Since the construction of the solution given above is pathwise, as a by-product of it we can show that the map $\varphi : \mathbb{R}^+ \times \Omega(\xi,\mathrm{E}) \times \mathrm{H} \to \mathrm{H}$ defined by

$$(t,\omega,x) \mapsto v_\alpha(t, z_\alpha(\omega))(x - z_\alpha(\omega)(0)) + z_\alpha(\omega)(t), \tag{5.28}$$

is well defined (so, in particular, independent of the choice of α). One can show that the quadruples $\mathfrak{T} = (\Omega(\xi, \mathrm{E}), \mathcal{F}, \mathbb{P}, \vartheta)$ and $\hat{\mathfrak{T}} = \left(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\right)$, where $\hat{\Omega}(\xi, \mathrm{E}) = \bigcap_{n=0}^{\infty} \Omega_n(\xi, \mathrm{E})$ and $\Omega_n(\xi, \mathrm{E})$ the set of those $\omega \in \Omega(\xi, \mathrm{E})$ for which*

$$\lim_{k\to\infty} \frac{1}{k} \int_{-k}^{0} |z_n(s)|^4 \, ds = \mathbb{E}|z_n(0)|^4, \tag{5.29}$$

are metric DS's. Moreover, $\lim_{n\to\infty} \int_{\hat{\Omega}(\xi,\mathrm{E})} |z_n(\omega)(0)|^2 \, d\hat{\mathbb{P}}(\omega) = 0$ for $\omega \in \hat{\Omega}(\xi, \mathrm{E})$. Finally, we have the following.

Theorem 5.3 φ H $\mathfrak{T}$ $\hat{\mathfrak{T}}$

The following are the main results of this section.

Theorem 5.4 φ $\mathfrak{T}$ ( ) ( )

We will show below that it is enough to prove the following.

Proposition 5.3 φ H $\mathfrak{T} = \left(\hat{\Omega}(\xi, \mathrm{E}), \hat{\mathcal{F}}, \hat{\mathbb{P}}, \hat{\vartheta}\right)$ $B \subset \mathrm{H}$ $K(\omega)$ φ

In what follows we denote by $v(\cdot, x)$ the unique solution to the problem (5.7) with the initial condition $v(0, x) = x \in \mathrm{H}$, with $z \in L^4_{\mathrm{loc}}(\mathbb{R}^+, \mathbb{L}^4(0, 1))$.

Proof of Theorem 5.4

Let $B \subset \mathrm{H}$ be a bounded set. In view of Proposition 5.3, it is sufficient to prove that there exist a bounded closed random set $K(\omega) \subset \mathrm{H}$ which absorbs B. Following [Flandoli94] we choose $\alpha \geq 0$ such that $\mathbb{E}|z_\alpha(0)|^4 < \frac{\lambda_1}{54}$ and fix the stationary O-U process $z_\alpha(t)$, $t \in \mathbb{R}$ defined everywhere on $\hat{\Omega}(\xi, \mathrm{E})$. One should point out that the reason for choosing such a value of the parameter α is to make the linearization at the origin of the problem (5.15) asymptotically stable.

Let $\omega \in \Omega$ is fixed, $s \leq 0$ and $x \in \mathrm{H}$ be given, v be the solution of (5.15) on $[s, 0]$ with the initial data $v(s) = x - z_\alpha(s)$, and external force $g_\alpha = f + \alpha z_\alpha - B(z_\alpha)$. From (5.25) we get for $t \in [s, T]$

$$\begin{aligned} |v(0)|^2 \quad &\leq \quad e^{\lambda_1 s + 27\int_s^0 |z_\alpha(t)|^4 \, dt} |v(s)|^2 \\ &+ \quad 2\int_s^0 |g_\alpha(t)|^2_{\mathrm{V}'} e^{\lambda_1 t + 27\int_t^0 |z_\alpha(r)|^4 \, dr} \, dt \\ &\leq \quad 2|x|^2 e^{\lambda_1 s + 27\int_s^0 |z_\alpha(t)|^4 \, dt} + 2|z_\alpha(s)|^2 e^{\lambda_1 s + 27\int_s^0 |z_\alpha(t)|^4 \, dt} \\ &+ \quad 2\int_s^0 |g_\alpha(t)|^2_{\mathrm{V}'} \, e^{\lambda_1 t + 27\int_t^0 |z_\alpha(r)|^4 \, dr} \, dt. \end{aligned} \tag{5.30}$$

First, we shall prove the lemma that follows.

Lemma 5.2 $\alpha \geq 0$ $\mathbb{E}|z_\alpha(0)|^4 < \frac{\lambda_1}{54}$

$$\overline{\lim}_{s\to-\infty} |z_\alpha(s)|^2 e^{\lambda_1 s + 27\int_s^0 |z_\alpha(r)|^4 \, dr} \quad = \quad 0, \tag{5.31}$$

$$\int_{-\infty}^{0} (1 + |z_\alpha(s)|^4_{L^4(0,1)}) e^{\lambda_1 s + 27\int_s^0 |z_\alpha(r)|^4 \, dr} \, ds < \infty. \tag{5.32}$$

*Since the X-valued process $z_\alpha(t)$, $t \in \mathbb{R}$ is a stationary and ergodic, by the Strong Law of Large Numbers, the set $\Omega(n, \mathrm{E})$ is of full measure.

Proof of Lemma 5.2

By the construction of the space $\hat{\Omega}(\xi, \mathrm{E})$ we get

$$\lim_{t\to\infty} \frac{1}{-(-t)} \int_{-t}^{0} |z_\alpha(s)|^4\, ds = \mathbb{E}|z_\alpha(0)|^4 < \frac{\lambda_1}{54}.$$

Therefore, for t big enough, we have

$$\int_{-t}^{0} (-\lambda_1 + 27|z_\alpha(s)|^4)\, ds \le -\frac{\lambda_1 t}{2}. \tag{5.33}$$

On the other hand, because $z_\alpha \in C_{1/2}(\mathbb{R}, L^4(0,1))$, we can find $\rho_2 \ge 0$ and $s_0 < 0$ such that $\frac{|z_\alpha(s)|}{|s|} \le \rho_2$ for $s \le s_0$. Hence by (5.33)

$$\begin{aligned}
\overline{\lim}_{t\to\infty} |z(-t)|^2_{L^4(0,1)} e^{\int_{-t}^{-k}(-\lambda_1+27|z(s)|^4)\,ds} &\le \overline{\lim}_{t\to\infty} |t|^{-2} |z(-t)|^2_{L^4(0,1)} \\
\overline{\lim}_{t\to\infty} |t|^2 e^{\int_{-t}^{-k}(-\lambda_1+27|z(s)|^4)\,ds} &\le \rho_2^2 \overline{\lim}_{t\to\infty} |t|^2\, e^{-\frac{\lambda_1(t-k)}{2}} = 0. \qquad (5.34)
\end{aligned}$$

This finishes the proof of Lemma 5.2.

We claim that it is enough to prove that

$$r_1^2(\omega) < \infty, \quad \text{for all } \omega \in \hat{\Omega}(\xi, \mathrm{E}), \tag{5.35}$$

where

$$\begin{aligned}
r_1(\omega)^2 &= 2 + 2\sup_{s\le 0} |z_\alpha(s)|^2 e^{\lambda_1 s + 27\int_s^0 |z_\alpha(t)|^4\, dt} \qquad (5.36) \\
&+ \int_{-\infty}^{0} |g_\alpha(t)|^2_{\mathrm{V}'} e^{\lambda_1 t + 27\int_t^0 |z_\alpha(r)|^4\, dr}\, dt.
\end{aligned}$$

Indeed, since the set B is bounded in H, there exists $\rho > 0$ such that $|x| \le \rho$ for $x \in B$. Since by (5.33) we can find $t_\rho(\omega) \le 0$ such that $\rho^2 e^{\lambda_1 s + 27\int_s^0 |z_\alpha(r)|^4\, dr} \le 1$ provided that $s \le t_\rho(\omega)$, we infer by (5.30), that $|v(0,\omega; s, x - z_\alpha(s))|^2 \le r_1^2(\omega)$ provided $|x| \le \rho$ and $s \le t_\rho(\omega)$. Therefore, for all $\omega \in \Omega(\xi, \mathrm{E})$

$$|u(0, s; \omega, x)| \le |v(0, s; \omega, x - z_\alpha(s))| + |z_\alpha(0)| \le r_2(\omega),$$

where $r_2(\omega) = r_1(\omega) + |z_\alpha(0,\omega)|$. From (5.35) and the assumptions we infer that for all $\omega \in \Omega$, $r_2(\omega) < \infty$. Hence we can define $K(\omega) = \{u \in \mathrm{H} : |u| \le r_2(\omega)\}$ and thus conclude the proof of the lemma.

In order to prove (5.35) we observe that equality (5.31) and the fact that $z_\alpha \in C_{1/2}(\mathbb{R}, \mathrm{H})$, we may infer that the first term on the right-hand side (RHS) of (5.35) is finite. We also immediately see that because of (5.33), $\int_{-\infty}^0 |f|^2 e^{\lambda_1 t + 27\int_t^0 |z_\alpha(r)|^4\, dr}\, dt$ is finite. Thus it remains to show that $\int_{-\infty}^0 \left[|z_\alpha(t)|^2_{\mathrm{V}'} + |B(z_\alpha(t))|^2_{\mathrm{V}'}\right] e^{\lambda_1 t + 27\int_t^0 |z_\alpha(r)|^4\, dr}\, dt$ is finite as well. Since $|z_\alpha(t)|^2_{\mathrm{V}'} + |B(z_\alpha(t))|^2_{\mathrm{V}'} \le C|z_\alpha(t)|^2 + C|z_\alpha(t)|^4_{L^4(0,1)}$ this follows from (5.32).

Main ideas of the proof of Proposition 5.3

We keep the choice of $\alpha \ge 0$ from the beginning of the proof of Theorem 5.4. Let $B \subset \mathrm{H}$ be a closed bounded set. Let us choose and fix $\omega \in \hat{\Omega}(\xi, \mathrm{E})$. Let $t_n \nearrow \infty$ and $(x_n)_n$ be any sequence taking values in B. Our aim is to construct a subsequence of the sequence $\varphi(t_n, \vartheta_{-t_n}\omega)x_n$ which is strongly convergent in H (obviously to some element of B).

By the assumptions we can find a closed and bounded random set $K(\omega)$ which absorbs B. Therefore we can find $N_1(\omega) \in \mathbb{N}$ such that $\varphi(t_n, \vartheta_{-t_n}\omega)B \subset K(\omega)$ for $n \ge N_1(\omega)$. Since bounded and closed sets in H are sequentially weakly compact, we can find a subsequence of the original sequence which is weakly convergent to some $y_0 \in$ H. We denote this subsequence as the original sequence. Since $z_\alpha(0) \in$ H, we infer that

$$\varphi(t_n, \vartheta_{-t_n}\omega)x_n - z_\alpha(0) \to y_0 - z_\alpha(0) \text{ weakly in H.}$$

This implies that $|y_0 - z_\alpha(0)| \le \liminf_{n\to\infty} |\varphi(t_n, \vartheta_{-t_n}\omega)x_n - z_\alpha(0)|$ and because H is a Hilbert space, if we prove that

$$|y_0 - z_\alpha(0)| \ge \overline{\lim}_{n\to\infty}|\varphi(t_n, \vartheta_{-t_n}\omega)x_n - z_\alpha(0)|, \tag{5.37}$$

then we would infer that $\varphi(t_n, \vartheta_{-t_n}\omega)x_n - z_\alpha(0) \to y_0 - z_\alpha(0)$ strongly in H what in turn would conclude the proof. In fact it is enough to prove (5.37) for some sub-subsequence.

First, we construct a negative trajectory, i.e., a sequence $(y_n)_{n=-\infty}^0$ such that $y_n \in K(\theta_n\omega)$, $n \in \mathbb{Z}^-$, and $y_k = \varphi(k-n, \theta_n\omega)y_n$, $n < k \le 0$ (and y_0 is the element of H constructed earlier). We also construct a decreasing sequence of subsequences $\{n^{(k)}\} \subset \{n^{(k-1)}\}$, $k = 1, 2, \dots$ such that

$$\varphi(-k + t_{n^{(k)}}, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} \to y_{-k} \quad \text{weakly in H,} \quad \text{as } n^{(k)} \to \infty. \tag{5.38}$$

We only show how to construct $y_{-1} \in K(\theta_{-1}\omega)$; the rest is just induction. Since $K(\vartheta_{-1}\omega)$ absorbs B, there exists a constant $N_2(\omega) \in \mathbb{N}$, such that

$$\{\varphi(-1 + t_n, \vartheta_{1-t_n}\vartheta_{-1}\omega)x_n : n \ge N_2(\omega)\} \subset K(\vartheta_{-1}\omega).$$

Then there exists a subsequence $\{n^{(1)}\}$ and $y_{-1} \in K(\vartheta_{-1}\omega)$ satisfying

$$\varphi(-1 + t_{n^{(1)}}, \vartheta_{-t_{n^{(1)}}}\omega)x_{n^{(1)}} \to y_{-1} \text{ weakly in H.} \tag{5.39}$$

The cocycle property, with $t = 1, s = t_n - 1$, and ω being replaced by $\vartheta_{-t_n}\omega$, reads as follows: $\varphi(t_n, \vartheta_{-t_n}\omega) = \varphi(1, \vartheta_{-1}\omega)\varphi(-1 + t_n, \vartheta_{-t_n}\omega)$. Hence, by Corollary 5.4 and (5.39), we infer that $\varphi(1, \vartheta_{-1}\omega)y_1 = y_0$.

Next, as in [Ghidaglia94] and [Rosa98], we put $[u, v] = ((u, v)) - \frac{\lambda_1}{2}(u, v)$, for any $u, v \in$ V. Clearly, $[\cdot, \cdot]$ is an inner product on V and the norm generated by it is equivalent to the norm $\|\cdot\|$. By adding and subtracting $\frac{\lambda_1}{2}|v(t)|^2$ to the equation (5.22) we get, with $b(z, v, u) = -\langle B(z, v) + B(v, z), u\rangle$ and $v = v(t)$,

$$\frac{d}{dt}|v|^2 + \lambda_1|v|^2 = 2\Big\{b(z_\alpha(t), v, v) - [v]^2 + \langle g_\alpha(t), v\rangle + \langle f, v\rangle\Big\}, \tag{5.40}$$

from which we infer that for $t \ge \tau$,

$$\begin{aligned}|v(t)|^2 = |v(\tau)|^2 e^{-\lambda_1(t-\tau)} \quad &+ \quad 2\int_\tau^t e^{-\lambda_1(t-s)}\Big\{b(z_\alpha(t), v(t), v(t)) \\ &+ \quad \langle g_\alpha(s), v(s)\rangle + \langle f, v(s)\rangle - [v(s)]^2\Big\}\, ds.\end{aligned} \tag{5.41}$$

Using the last equality we can show exactly as in [BrzLi02] that for some constant $C > 0$ and some nonnegative function $h \in L^1(0, \infty)$ and all $k \in \mathbb{N}$

$$\overline{\lim}_{n^{(k)}\to\infty}|\varphi(t_{n^{(k)}} - k, \vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z_\alpha(-k)|^2 e^{-\lambda_1 k} \le \int_{-\infty}^{-k} h(s)\, ds. \tag{5.42}$$

Moreover, also from (5.41) we get the following important equality:

$$
\begin{aligned}
|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z_\alpha(0)|^2 = e^{-\lambda_1 k} \qquad\qquad (5.43)\\
|\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z_\alpha(-k)|^2 - 2\int_{-k}^0 e^{\lambda_1 s}[v^{n^{(k)}}(s)]^2\,ds\\
+2\int_{-k}^0 e^{\lambda_1 s}\Big\{b(z_\alpha(s),v^{n^{(k)}}(s),v^{n^{(k)}}(s)) + \langle g_\alpha(s),v^{n^{(k)}}(s)\rangle + \langle f,v^{n^{(k)}}(s)\rangle\Big\}\,ds,
\end{aligned}
$$

where $v^{n^{(k)}}$ is the solution to (5.7) on the time interval $[-k,\infty)$ with initial condition $v(-k)=\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}}-z_\alpha(-k))$, i.e., $v^{n^{(k)}}(s)=v(s,-k;\omega,\varphi(t_{n^{(k)}}-k,\vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}}-z_\alpha(-k))$, $s\in[-k,0]$. Let us also denote $\tilde{y}_k=y_k-z_\alpha(-k)$ and $v_k(s)=v(s,-k;\omega,y_{-k}-z_\alpha(-k))$, $s\in(-k,0)$. Notice that then from (5.41) we infer that

$$
\begin{aligned}
|y_0-z_\alpha(0)|^2 &= |\varphi(k,\vartheta_{-k}\omega)y_k - z_\alpha(0)|^2 = |v(0,-k;\omega,y_k-z_\alpha(-k))|^2\\
&= |y_k-z_\alpha(-k)|^2 e^{-\lambda_1 k} + 2\int_{-k}^0 e^{\lambda_1 s}\Big\{\langle g_\alpha(s),v_k(s)\rangle \qquad (5.44)\\
&+ \langle B(v_k(s),z_\alpha(s)),v_k(s)\rangle + \langle f,v_k(s)\rangle - [v_k(s)]^2\Big\}\,ds.
\end{aligned}
$$

From equation (5.38) and Corollary 5.4 we infer that $v^{n^{(k)}}(\cdot)\to v_k$ weakly in $L^2(-k,0;\mathrm{V})$ and therefore we get $\int_{-k}^0 e^{\lambda_1 s}\langle g_\alpha(s),v^{n^{(k)}}(s)\rangle\,ds\to\int_{-k}^0 e^{\lambda_1 s}\langle g_\alpha(s),v_k(s)\rangle\,ds$ and $\int_{-k}^0 e^{\lambda_1 s}\langle f,v^{n^{(k)}}(s)\rangle\,ds\to\int_{-k}^0 e^{\lambda_1 s}\langle f,v_k(s)\rangle\,ds$, as $n^{(k)}\to\infty$.

On the other hand, we can find a subsequence of $\{v^{n^{(k)}}\}$, for which we do not introduce of notation, such that $v^{n^{(k)}}\to v_k$ strongly in $L^2(-k,0;\mathrm{H})$ and so $\int_{-k}^0 e^{\lambda_1 s}b(v^{n^{(k)}}(s),z_\alpha(s)),v^{n^{(k)}}(s))\,ds\to\int_{-k}^0 e^{\lambda_1 s}b(v_k(s),z_\alpha(s),v_k(s))\,ds$. Finally, since the norms $[\cdot]$ and $\|\cdot\|$ are equivalent on V so that $(\int_{-k}^0 e^{\lambda_1 s}[\cdot]^2\,ds)^{1/2}$ is a norm in $L^2(-k,0;\mathrm{V})$ equivalent to the standard one, we infer that

$$
\overline{\lim}_{n^{(k)}\to\infty}\Big\{-\int_{-k}^0[v^{n^{(k)}}(s)]^2\,ds\Big\}\le-\int_{-k}^0 e^{\lambda_1 s}[v_k(s)]^2\,ds.
$$

Taking the $\overline{\lim}_{n^{(k)}\to\infty}$ of (5.43) and combining it with (5.44) we arrive at

$$
\begin{aligned}
&\overline{\lim}_{n^{(k)}\to\infty}|\varphi(t_{n^{(k)}},\vartheta_{-t_{n^{(k)}}}\omega)x_{n^{(k)}} - z_\alpha(0)|^2 \le |y_0-z_\alpha(0)|^2 \qquad (5.45)\\
&+ \int_{-\infty}^{-k}h(s)\,ds - |y_k-z_\alpha(-k)|^2e^{-\lambda_1 k}\le|y_0-z_\alpha(0)|^2+\int_{-\infty}^{-k}h(s)\,ds.
\end{aligned}
$$

To conclude we use the diagonal process $(m_j)_{j=1}^\infty$ defined by $m_j=j^{(j)}$, $j\in\mathbb{N}$. Then for each $k\in\mathbb{N}$, the sequence $(m_j)_{j=k}^\infty$ is a subsequence of the sequence $(n^{(k)})$ and hence by (5.45), $\overline{\lim}_j|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega)x_{m_j}-z_\alpha(0)|^2\le\int_{-\infty}^{-k}h(s)\,ds+|y_0-z_\alpha(0)|^2$. By taking the $k\to\infty$ limit in the last inequality we infer that $\overline{\lim}_j|\varphi(t_{m_j},\vartheta_{-t_{m_j}}\omega)x_{m_j}-z_\alpha(0)|^2\le|y_0-z_\alpha(0)|^2$ which proves the claim in (5.37).

5.5 Stochastic 2D Navier–Stokes equations with multiplicative noise

We consider a domain $D \subset \mathbb{R}^2$ with smooth boundary ∂D. By $u(t,x) \in \mathbb{R}^2$ and $p(t,x) \in \mathbb{R}$ we denote, respectively, the velocity and the pressure of the an incompressible viscous fluid† at the point $x \in D$ at time $t \geq 0$. The flow of fluid subject to internal noise and determined by the following stochastic initial-boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla)u + \nabla p = f + g(u(t))\, \dot{w}(t) & \text{in } D, \\ \mathrm{div} u = 0 & \text{in } D, \\ u = 0 & \text{on } \partial D, \\ u(\cdot, 0) = u_0 & \text{in } D. \end{cases} \tag{5.46}$$

In the above $f : \mathbb{R}^+ \times D \to \mathbb{R}^2$ is an external deterministic body force and g, depending on both the position and the velocity, is the stochastic force, see, e.g., [BrzCapFl91], [BrzCapFl92], and [MikRoz04].

We assume that the Poincaré inequality holds on D, i.e., there exists $\lambda_1 > 0$ such

$$\int_D \phi^2\, dx \leq \frac{1}{\lambda_1} \int_D |\nabla \phi|^2\ dx, \quad \forall \phi \in C_0^\infty(D, \mathbb{R}^2). \tag{5.47}$$

We will use the standard mathematical framework of the NSEs, see, e.g., [Temam84]. The basic functional space is the Lebesgue space $\mathbb{L}^2(D) := L^2(D, \mathbb{R}^2)$ with scalar product $(u, v) = \sum_j \int_D (u_j(x) v_j(x))\, dx$ and norm $|\cdot| = (\cdot,\cdot)^{1/2}$. We will also need the Sobolev space $\mathbb{H}^{kp}(D) = H^{k,p}(D, \mathbb{R}^2)$, $k \in \mathbb{N}$, and $p \in [1, \infty)$ of all $L^p(D, \mathbb{R}^2)$ whose weak derivatives up to order k belong to $L^p(D, \mathbb{R}^2)$ as well. Let $\mathcal{V}$ be the space of all $\phi \in C_0^\infty(D, \mathbb{R}^2)$ such that ϕ is solenoidal (i.e., $\mathrm{div}\phi = 0$). The closure of $\mathcal{V}$ in $\mathbb{L}^2(D)$, respectively, in $\mathbb{H}^{1,2}(D)$, will be denoted by H, resp. V. The scalar product norms in those two spaces are those inherited from $\mathbb{L}^2(D)$, resp. $\mathbb{H}^{1,2}(D)$. Because of the assumption (5.47), the original norm on V is equivalent to the norm $\|\cdot\|$ induced by the scalar product

$$((u, v)) = \int_D \sum_{j=1}^{2} \nabla u_j \cdot \nabla v_j\, dx = (\nabla u, \nabla v), \quad u, v \in \mathrm{V}. \tag{5.48}$$

A bilinear form $a : \mathrm{V} \times \mathrm{V} \to \mathbb{R}$ is defined by

$$a(u, v) := (\nabla u, \nabla v), \quad u, v \in \mathrm{V}, \tag{5.49}$$

i.e., a is simply a (new) scalar product in V. Since a is V-continuous, by the Riesz lemma, there exists a unique linear operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$, where V' is the dual of V, such that $a(u, v) = \langle \mathcal{A}u, v \rangle$, for $u, v \in \mathrm{V}$. Since moreover, the form a is obviously V-coercive, by the Lax–Milgram theorem, the operator $\mathcal{A} : \mathrm{V} \to \mathrm{V}'$ is an isomorphism. Since V is densely and continuously embedded into H, and we can identify H with its dual H', we have

$$\mathrm{V} \subset \mathrm{H} = \mathrm{H}' \subset \mathrm{V}'.$$

We then define an unbounded linear operator A in H by $D(\mathrm{A}) = \{u \in \mathrm{V} : \mathcal{A}u \in \mathrm{H}\}$, $\mathrm{A}u := \mathcal{A}u$, $u \in D(\mathrm{A})$. A is a self-adjoint operator in H and $(Au, u) = \|u\|^2, \quad u \in \mathrm{V}$. $D(\mathrm{A})$ endowed with the $|\mathrm{A}\cdot|$ norm is a Hilbert space and A is an isomorphism of $D(\mathrm{A})$ onto H. It is well known that $D(A) = \mathrm{V} \cap \mathbb{H}^{1,2}(D)$. Let $P : \mathbb{L}^2(D) \to \mathrm{H}$ be the orthogonal projection;

†of constant density (assumed to be equal to 1) and of constant viscosity $\nu > 0$.

then $\mathrm{A}u = -P\Delta u, \quad u \in D(\mathrm{A})$, $\mathrm{V} = D(\mathrm{A}^{1/2})$, see [Temam97] and the graph norm on $D(\mathrm{A})$ is equivalent to the norm $|\mathrm{A}\cdot|$ and, see [Temam97]

$$\langle \mathrm{A}u, u\rangle = ((u,u)) = \|u\|^2 = |\nabla u|^2, \quad u \in D(\mathrm{A}). \tag{5.50}$$

Next, we define the following fundamental trilinear form:

$$b(u,v,w) = \int_D u\nabla v w\, dx = \sum_{i,j=1}^{2} \int_D u^i(x) D_i v^j(x) w^j(x)\, dx.$$

If u, v are such that the linear map $b(u,v,\cdot)$ is continuous on V, the corresponding operator will be denoted by $B(u,v)$. We will also denote, somehow with a bit of notational abuse, $B(u) = B(u,u)$. Note that if $u, v \in \mathrm{H}$ are such that $(u\nabla)v = \sum_j u_j D_j v \in \mathbb{L}^2(D)$, then $B(u,v) = P(u\nabla v)$. It is well known that $b(u,v,v) = 0$, for $u \in \mathrm{V}$, $v \in \mathbb{H}_0^{1,2}(D)$ and, among many similar ones, there is $C > 0$ such that

$$|b(u,v,w)| \le C|u|^{1/2}|\nabla u|^{1/2}|\nabla v||w|^{1/2}|\nabla w|^{1/2}, \quad u,v,w \in \mathrm{V}. \tag{5.51}$$

Since $|v|_{\mathbb{L}^4(D)} \le 2^{1/4}|v|^{1/2}_{\mathbb{L}^2(D)}|\nabla v|^{1/2}_{\mathbb{L}^2(D)}$, $v \in H_0^{1,2}(D)$, from the Hölder inequality we have the following:

$$|b(u,v,w)| \le |u|_{\mathbb{L}^4(D)}|\nabla v|_{\mathbb{L}^2(D)}|w|_{\mathbb{L}^4(D)}, \quad u,v,w \in H_0^{1,2}(D). \tag{5.52}$$

It follows that b is a bounded trilinear map from $\mathbb{L}^4(D) \times \mathrm{V} \times \mathbb{L}^4(D)$ to $\mathbb{R}$. Moreover, b has a unique extension to a bounded trilinear map from $\mathbb{L}^4(D) \times (\mathbb{L}^4(D) \cap \mathrm{H}) \times \mathrm{V}$ to $\mathbb{R}$. Hence, B maps $\mathbb{L}^4(D) \cap \mathrm{H}$ (and so V) into V' and

$$|B(u)|_{\mathrm{V}'} \le C_1|u|^2_{\mathbb{L}^4(D)} \le 2^{1/2}C_1|u||\nabla u| \le C_2|u|^2_V, \quad u \in \mathrm{V}. \tag{5.53}$$

Our aim is to study the following functional form of the Navier–Stokes equations (5.46):

$$\begin{cases} du + \{\nu \mathrm{A}u + B(u)\}\, dt = f\, dt + g(u)dW(t), \quad t \ge 0 \\ u(0) = u_0, \end{cases} \tag{5.54}$$

where we assume that $u_0 \in \mathrm{H}$, $f \in \mathrm{V}'$, and $W(t), t \in \mathbb{R}$ is a two-sided real-valued Wiener process defined on some filtered and complete probability space $\mathfrak{A} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}}, \mathbb{P})$. Let us now list those assumtions from [CapCutl99] that we will need.

(F1) $f \in \mathrm{V}'$;

(G1) $|g(u)|^2 \le \alpha(1 + |u|^2) + \beta\|u\|^2$, $u \in \mathrm{V}$ with $\beta \in (0, 2\nu)$;

(G2) $|g(u) - g(v)|^2 \le c|u - v|^2) + \beta\|u - v\|^2$, $u \in \mathrm{V}$ with $\beta \in (0, 2\nu)$;

(G3) $(g(u), v) = -(u, g(v))$, for all $u, v \in \mathrm{V}$.

Let us notice that (G3) is equivalent to two conditions: $(g(u), u) = 0$ and $(g(u) - g(v), u - v) = 0$ for all $u, v \in \mathrm{V}$. The conditions (G1)–(G3) are satisfied, in particular, by $g = B(h, \cdot)$ for $h \in D(\mathrm{A})$ with $|\mathrm{A}(h)| < 2\nu$. The following is a generalization of some results from [CapCutl99], e.g., Theorem 5.5, where only the periodic case (i.e., when D is a 2D torus) is studied.

Theorem 5.5 $(\Omega, \mathcal{F}, \mathcal{F}_{t\in\mathbb{R}}, \mathbb{P})$ $w(t)$, $t \in \mathbb{R}$ $w(0) = 0$, $\mathcal{F}_{t\ge s}$ $s \in \mathbb{R}$, $\varphi : \mathbb{R}^+ \times \Omega \times \mathrm{H} \to \mathrm{H}$ $u(t,\omega) = \varphi(t-s, u_0, \omega)$, $t \ge s$, $\omega \in \Omega$ *( )*

( ) $t \geq s$ $\omega \in \Omega$,

$$|u(t)|^2 + c_1 \int_s^t \|u(r)\|^2 \, dr \;\; \leq \;\; |u(s)|^2 e^{\alpha(t-s)} + c_4 e^{\alpha(t-s)-1}. \tag{5.55}$$

( ) $\alpha < \lambda_1(2\nu - \beta)$, $t \geq s$ $\omega \in \Omega$,

$$|u(t)|^2 \;\; \leq \;\; |u(s)|^2 e^{-c_3(t-s)} + c_5 \left(1 - e^{-c_3(t-s)}\right). \tag{5.56}$$

( ) $t \geq s$, $v_1, v_2 \in \mathrm{H}$ $\omega \in \Omega$,

$$|u_2(t) - u_1(t)|^2 + c_1 \int_s^t \|u_2(r) - u_1(r)\|^2 \, dr \leq |u_2(s) - u_1(s)|^2 c(t - s, |u_1(s)|). \tag{5.57}$$

The next theorem is the main result in this section. Let us point out that in the periodic case and under much stronger assumptions on f and g it was proved in [CapCutl99] the existence of an attractor.

Theorem 5.6 g φ $B \subset \mathrm{H}$, $\Omega_B(\omega)$ φ B ϕ $(\Omega \times \mathrm{H}, \mathcal{F} \otimes \mathcal{B})$

The main idea of the proof is to notice that there exists $\delta > 0$ such that the norm $[u,u] := \nu\|u\|^2 - \frac{\delta}{2}|u|^2 - \frac{1}{2}|gu|^2$ is equivalent to the $\|\cdot\|$ norm. Hence the energy equality, associated with (5.54), see [BrzCapFl92]

$$|u(t)|^2 + 2\nu \int_s^t \|u(r)\|^2 \, dr = |u(s)|^2 + 2 \int_s^t (f, u(r)) \, dr + \int_s^t |gu(r)|^2 \, dr$$

can be rewritten in the following form:

$$|u(t)|^2 + \delta \int_s^t |u(r)|^2 \, dr = |u(s)|^2 + 2 \int_s^t ((f, u(r)) - [u(r), u(r)]) \, dr,$$

from which we infer that

$$|u(t)|^2 = |u(0)|^2 e^{-\delta t} + 2 \int_0^t e^{-\delta(t-s)} \left((f, u(r)) - [u(r), u(r)]\right) dr. \tag{5.58}$$

Acknowledgments

I would like to thank the organizers of the meeting in Trento (2004), where a preliminary version of this chapter was presented, for their kind invitation.

References

[Arnold98] L. Arnold, RANDOM DYNAMICAL SYSTEMS, Springer-Verlag, Berlin, Heidelberg, New York, 1998.

[Brzezniak91] Z. Brzeźniak, , Universitatis Iagellonicae Acta Mathematica, Fasciculus *XXVIII*, 111-124 (1991).

[Brzezniak95] Z. Brzeźniak, , Potential Analysis **4**, no. 1, 1–45 (1995).

[Brzezniak97] Z. Brzeźniak, Stochastics and Stochastics Reports **61**, 245–295 (1997).

[BrzCapFl91] Z. Brzeźniak, M. Capiński and F. Flandoli, , Mathematical Models in Applied Sciences **1**, 41–59 (1991).

[BrzCapFl92] Z. Brzeźniak, M. Capiński and F. Flandoli, , Stochastic Analysis and Applications **105**, 523–532 (1992).

[BrzCapFl93] Z. Brzeźniak, M. Capiński and F. Flandoli, , Probability Theory and Related Fields **95**, 87–102 (1993).

[BrzLi02] Z. Brzeźniak and Y. Li, , Department of Mathematics, the University of Hull, Vol (*XV*), No.3, 2002, Trans. AMS (to appear).

[BrzLi04] Z. Brzeźniak and Y. Li, , pp. 78-11 in Recent Developments in Stochastic Analysis and Related Topics, Proceedings of the First Sino-German Conference on Stochastic Analysis (A Satellite Conference of ICM 2002) Beijing, China, 29 August–3 September 2002, edts. S. Albeverio, Zhi-Ming Ma & M. Röckner, World Scientific 2004.

[BrzPesz01] Z. Brzeźniak and S. Peszat, , Annals of Probability **29**, no. 4, 1796–1832 (2001).

[CapCutl99] M. Capiński and N.J. Cutland, , Probability Theory and Related Fields **115**, no. 1, 121–151 (1999).

[CastVal77] C. Castaing and M. Valadier, Convex Analysis and Measurable Multifunctions, Lecture Notes in Mathematics **580**, Springer, Berlin, 1977.

[Crauel95] H. Crauel, Random Probability Measures on Polish Spaces, Habilitationsschrift, Bremen, 1995.

[CrDebFl97] H. Crauel, A. Debussche and F. Flandoli, , Journal of Dynamics and Differential Equations, **9**(2), 307–341 (1997).

[CrFl94] H. Crauel and F. Flandoli, Probability Theory and Related Fields, **100**, 365–393 (1994).

[DaPrDeb02] G. Da Prato and A. Debussche, Journal of Functional Analysis **196**, no. 1, 180–210 (2002).

[DaPrDebTem94] G. Da Prato, A. Debussche and R. Temam, , NoDEA (Nonlinear Differential Equations Applications) **1**, no. 4, 389–402 (1994).

[DaPrZab6] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society Lecture Note Series 229, Cambridge University Press, Cambridge, 1996.

[EckHai01] J.-P. Eckmann and M. Hairer, Nonlinearity **14**, No.1, 133–151 (2001).

[Flandoli94] F. Flandoli, NoDEA **1**, 403–423 (1994).

[FlTor95] F. Flandoli and V.M. Tortorelli, Stochastics and Stochastics Reports **55**, no. 1-2, 141–165 (1995).

[Friedman69] A. Friedman, Partial differential equations, Holt, Rinehart and Winston, Inc., New York, Montreal, Quebec, London, 1969.

[Ghidaglia94] J.M. Ghidaglia, , Journal of Differential Equations **110**, no. 2, 356–359 (1994).

[Hairer05] M. Hairer, , Annals of Probability **33**, no. 2, 703–758 (2005).

[Heywood80] J. G. Heywood, Indiana University Mathematics Jorunal **29**, no. 5, 639–681 (1980).

[KelSchm99] H. Keller and B. Schmalfuss, , preprint, the University of Bremen (1999).

[Ladyzhenskaya91] O. Ladyzhenskaya, Attractors for semigroups and evolution equations, Lezioni Lincee, Cambridge University Press, Cambridge, 1991.

[LRS04] J. Langa, J. Real and J. Simon, , Applied Mathematics and Optimization **48** no. 3, 195–210 (2003).

[LMag72] J. L. Lions and E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. 1, Springer Verlag, Berlin, Heidelberg, New York, 1972.

[MikRoz04] R. Mikulevicius and B.L. Rozovskii, , SIAM Journal Mathematics Analysis **35**, no. 5, 1250–1310 (2004).

[Rosa98] R. Rosa, , Nonlinear Analysis **32**, 71–85 (1998).

[Schmalfuss92] B. Schmalfuss, Backward cocycles and attractors of Stochastic Differential Equations," pp. 185–192 in International Seminar on Applied Mathematics–Nonlinear Dynamics: Attractor Approximation and Global Behaviour, edts. Reitmann, T. Riedrich and N. Koksch, 1992.

[Temam84] R. Temam, Navier-Stokes Equations, North-Holland Publish Company, Amsterdam, 1979.

[Temam97] R. Temam, Infinite-Dimensional Dynamical Systems in Mechanics and Physics, Second Edition, Springer, New York, 1997.

6 A Characterization of Approximately Controllable Linear Stochastic Differential Equations*

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6.1 Statement of the problem and of the main result

The objective of the chapter is to study controllability for the following linear stochastic differential equation:

$$\begin{cases} dy(t) = \Big(Ay(t)dt + Bu(t)\Big)dt + \sum_{i=1}^{m} C_i y(t) d\beta_i(t) \\ y(0) = x, \end{cases} \tag{6.1}$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^d, \mathbb{R}^n)$, $C_i \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $i = 1, ..., m$, and the processes $\{\beta_1, ..., \beta_m$, i=1,...,m$\}$ are independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{E}, \mathbb{P})$. We denote by $\{\mathcal{F}_t : t \geq 0\}$ the filtration they generate, augmented with all $\mathbb{P}$-null sets of $\mathcal{E}$.

A process $u : \Omega \times [0, +\infty[\to \mathbb{R}^d$ is said to be an *admissible control* if it is $(\mathcal{F}_t)$-predictable and such that $\mathbb{E} \int_0^T |u(s)|^2 ds < +\infty$, for all $T > 0$.

As it is well known, under the above assumptions, for all initial datum $x \in \mathbb{R}^n$ and all admissible control u, equation (6.1) (intended in Ito sense) admits a unique predictable solution y with continuous trajectories. Moreover, this solution is square integrable over all compact time intervals, i.e., for all $T > 0$, $\mathbb{E} \sup_{s \in [0,T]} |y(s)|^2 < +\infty$. Such a solution (representing the state in the system) will be denoted by $y(\cdot, x, u)$.

Definition 6.1 *We say that equation (6.1) is* approximately controllable *if for all $x \in \mathbb{R}^n$, all $T > 0$, all $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$, and all $\varepsilon > 0$ there exists an admissible control u such that $\mathbb{E}|y(T, x, u) - \eta|^2 \leq \epsilon$.*

Moreover, we say that equation (6.1) is approximately null controllable *if the above condition holds in the particular case $\eta = 0$.*

We also give the following definition.

Definition 6.2 *Given $m+1$ linear operators $L, M_1, ..., M_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, a linear subspace $V \subset \mathbb{R}^n$ is said to be $(L; M_1, ..., M_m)$-strictly invariant if $LV \subset \mathrm{Span}\langle V, M_1 V, ..., M_m V\rangle$.*

Remark 6.1 We notice that $V \subset \mathbb{R}^n$ is $(L; M_1, ..., M_m)$-strictly invariant if and only if there exists operators $K_1, ..., K_m \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ such that $K_i V \subset V$ and $(L + M_1 K_1 + \cdots + M_m K_m)V \subset V$ or, equivalently, if and only if for all $v \in V$ there exist $w_1, ..., w_m \in V$ such that $Lv + M_1 w_1 + \cdots + M_m w_m \in V$.

*Research is supported by European Community's Human Potential Program under contract HPRN-CT-2002-00281, Evolution Equations.

Remark 6.2 Given an arbitrary linear subspace $V \subset \mathbb{R}^n$ it is easy to compute the largest linear subspace of V which is $(L; M_1, ..., M_m)$-strictly invariant (see also [12] and [17] for similar computations in wider generality). Indeed, if we put

$$V_0 = V;\ V_{i+1} = \{v \in V_i : Lv \in \mathrm{Span}\langle V_i, M_1 V_i, ..., M_m V_i\rangle\} = L^{-1}(\mathrm{Span}\langle V_i, M_1 V_i, ..., M_m V_i\rangle) \cap V_i,$$

then V_n is the required maximal $(L; M_1, ..., M_m)$-strictly invariant subspace of V.

Let us now present the main result of the present chapter.

Theorem 6.1 *The following assertions are equivalent:*

1. *Equation (6.1) is approximately controllable.*
2. *Equation (6.1) is approximately null controllable.*
3. *The largest* $(A^*; C_1^*, ..., C_m^*)$*-strictly invariant subspace of* $\operatorname{Ker} B^*$ *is the origin.*

Remark 6.3 Remark 6.2 implies that condition *3* is easily computable.

6.2 Remarks on related literature

In [7] (see also [9]) S. Peng has studied the "exact controllability" and "exact terminal controllability" of the following stochastic linear equation with control acting on the noise term as well:

$$dy(t) = \Big(Ay(t)dt + Bu(t)\Big)dt + \Big(Cy(t) + Du(t)\Big)d\beta(t). \tag{6.2}$$

In particular in [7], it is shown that equation (6.2) is exactly terminal controllable (that is, for each final condition η in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ there exists an initial datum $y(0)$ in $\mathbb{R}^n$ and an admissible control u such that $y(T) = \eta$, $\mathbb{P}$-almost surely (a.s.)) if and only if D has full rank. Then algebraic conditions of Kalman type under which equation (6.2) with full rank D is exactly terminal controllable (that is, each final condition can be exactly replicated starting from any initial datum. Our result can be seen as a counterpart of the ones described above. Namely, here we do not allow the control to act on the noise ($D = 0$ in (6.2). Consequently, we cannot expect to have exact terminal controllability (or, a fortiori, exact controllability). Nevertheless, we prove that if we weaken our request from exact to approximate controllability, then the condition can be satisfied and, in particular, if satisfied if and only if computable algebraic conditions on A, B, and C hold.

We also notice that Definition 6.2 is a modification of the definition of (A, B) invariant subspaces that has been introduced in [17] and then generalized in [12], [13], [14]. In particular in [16], this last concept was used to obtain, by algebraic Riccati equation methods, a characterization of stochastic linear equations admitting a feedback that stabilizes the system for all noise intensities.

Finally, we notice that the property of approximate controllability treated here is tightly connected to the one of exact null controllability. See [10] for the relation between this last property and backward stochastic differential equations or Riccati equations, both in finite and infinite-dimensional spaces; moreover, see [3] for a review on partial result in the direction of proving null controllability of specific infinite dimensional stochastic evolution equations.

6.3 The dual equation

Let us consider the following backward stochastic differential equation:

$$\begin{cases} dp(t) = -\Big(A^*p(t) + \sum_{i=1}^{m} C_i^* q^i(t)\Big)dt + \sum_{i=1}^{m} q^i(t)d\beta_i(t) \\ p(T) = \eta. \end{cases} \tag{6.3}$$

It is shown in [8] (se also [1] and [2] for earlier results in the control theory framework) that for all $T > 0$ and all $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ there exists a unique $m+1$-tuple of $\mathbb{R}^n$-valued predictable processes $(p, q^1, ..., q^m)$ such that equation (6.3) is satisfied, p has continuous trajectories and it holds

$$\mathbb{E} \sup_{s\in[0,T]} |p(s)|^2 < +\infty, \quad \mathbb{E}\int_0^T |q^i(s)|^2 ds < +\infty, \; i = 1, ..., m.$$

The following proposition specifies the connection between the above equation and controllability of equation (6.1).

Proposition 6.1 *Equation (6.1) is approximately controllable if and only if for all $T > 0$, every solution to equation (6.3) verifying $B^*p(s) = 0$, $\mathbb{P}$-a.s., $\forall s \in [0,T]$ is trivial, (i.e., is such that $p(s) = 0$, $\mathbb{P}$-a.s. for all $s \in [0,T]$).*

Moreover, equation (6.1) is approximately null controllable if and only if for all $T > 0$, every solution to equation (6.3) satisfying $B^*p(s) = 0$, $\mathbb{P}$-a.s., $\forall s \in [0,T]$, also verifies $p(0) = 0$.

Proof For fixed $T > 0$ we deduce from Itô's formula that

$$d\langle p(s), y(s,x,u)\rangle = \langle p(s), Bu(s)\rangle ds + \sum_{i=1}^{m} \big(\langle q^i(s), y(s,x,u)\rangle + \langle p(s), C_i y(s,x,u)\rangle\big)\, d\beta^i(s),$$

where $\langle \cdot\,, \cdot\rangle$ denotes the inner product in $\mathbb{R}^n$. Since

$$\mathbb{E}\left(\int_0^T \big(\langle q^i(s), y(s,x,u)\rangle + \langle p(s), C_i y(s,x,u)\rangle\big)^2 ds\right)^{1/2} < +\infty, \quad i = 1, ..., m,$$

we can compute the mean value and obtain

$$\mathbb{E}\langle p(T), y(T,x,u)\rangle - \mathbb{E}\langle p(0), x\rangle = \mathbb{E}\int_0^T \langle p(s), Bu(s)\rangle ds. \tag{6.4}$$

Let $L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$ be the space of all predictable processes $u : \Omega \times [0, +\infty] \to \mathbb{R}^d$ satisfying $\mathbb{E}\int_0^T |u(s)|^2 ds < +\infty$, endowed with the natural norm, and define the linear operator

$$M_T : L^2_{\mathcal{P}}([0,T], \mathbb{R}^d) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n), \qquad M_T u := y(T, u, 0). \tag{6.5}$$

It is evident that equation (6.1) is approximately controllable if and only if, for all $T > 0$, the image of M_T is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$. Moreover, by relation (6.4) (with $x = 0$ and $p(T)$ being equal to an arbitrary $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$ we deduce that $M_T^*\eta = B^*p$. The first part of the claim follows by observing that the image of M_T is dense if and only if the kernel of M_T^* is trivial and by noticing that, thanks to the uniqueness and the continuity of the solution to equation (6.3), $\eta = 0$ if and only if $p(s) = 0$ $\mathbb{P}$-a.s. for all $s \in [0,T]$.

As far as the second part of the proposition is concerned we introduce the linear operator

$$L_T : \mathbb{R}^d \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n), \qquad L_T x := y(T, 0, x) \tag{6.6}$$

and notice that equation (6.1) is approximately null controllable if and only if, for all $T > 0$, $L_T[\mathbb{R}^n] \subset \overline{M_T[L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)]}$.

Then again by relation (6.4) (now with $u = 0$) we get that $L_T^* \eta = p(0)$, and the claim follows recalling that $L_T[\mathbb{R}^n] \subset \overline{M_T[L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)]}$ if and only if $\mathrm{Ker}[M_T^*] \subset \mathrm{Ker}[L_T^*]$. □

Now we interprete equation (6.3) as a forward equation. More precisely, we consider the equation

$$\begin{cases} dp(t) = -\Big(A^* p(t) + \sum_{i=1}^m C_i^* q^i(t)\Big) dt + \sum_{i=1}^m q^i(t) d\beta_i(t) \\ p(0) = \theta. \end{cases} \tag{6.7}$$

For all $\theta \in \mathbb{R}^n$ and $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, there exists a unique predictable solution p with continuous trajectories verifying, for all $T > 0$, $\mathbb{E} \sup_{s \in [0,T]} |p(s)|^2 \, ds < +\infty$. We denote this solution by $p(\cdot, q^1, ..., q^m, \theta)$.

By existence and uniqueness of solutions to equation (6.3) we get that for all $T > 0$ and $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}^n)$, there exists unique $\theta \in \mathbb{R}^n$ and unique $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$ such that $p(\cdot, q^1, ..., q^m, \theta) = \eta$. Thus Proposition 6.1 can be reformulated as follows.

Proposition 6.2 *Equation (6.1) is approximately controllable if and only if for all $T > 0$, all $\theta \in \mathbb{R}^n$, and all $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, for which $B^* p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$ it holds $p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$.*

Moreover, equation (6.1) is approximately null controllable if and only if for all $T > 0$, all $\theta \in \mathbb{R}^n$, and all $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, such that $B^* p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. $\forall s \in [0,T]$, it holds $\theta = 0$.

6.4 Local in time viability

Proposition 6.2 justifies our interest in the following concept.

Definition 6.3 *A linear subspace $V \subset \mathbb{R}^n$ is said to be* locally in time viable *(l.i.t.v.) with respect to equation (3.5) if for all $\theta \in V$ there exists a $T > 0$ and $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in V$ $\mathbb{P}$-a.s. for all $s \in [0,T]$.*

Moreover, the set of all $\theta \in V$ for which there exists a $T > 0$ and $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in V$ $\mathbb{P}$-a.s. for all $s \in [0,T]$, is called the local viability kernel *of V.*

Note that the above notion of local (in time) viability slightly differs from the local (in space) viability defined and studied in [6].

We recall here some basic facts on Riccati equations and linear quadratic optimal control related to equation (6.7). The reader can find proofs (in a much wider generality), for instance, in [17] or [18].

For an arbitrarily fixed subspace $V \subset \mathbb{R}^n$ let Π_V denote the orthogonal projection on $V^\perp$ (the orthogonal of V). For all $N \geq 1$, we consider the following Riccati equation with values in $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$:

$$\begin{cases} P_N'(s) = -AP_N(s) - P_N(s)A^* - \sum_{i=1}^m P_N(s)C_i^*[I + P_N(s)]^{-1}C_iP_N(s) + N\Pi_V, \ t \geq 0, \\ P_N(0) = 0. \end{cases} \tag{6.8}$$

The above equation admits a unique continuous solution with values in the cone of linear symmetric nonnegative operators in $\mathbb{R}^n$. Moreover, for all $t > 0$ the sequence $\{P_N(t) : N \in \mathbb{N}\}$ increases in N.

The following equation, satisfied for all $0 \leq t \leq T$, is known as *fundamental relation*:

$$\begin{aligned} \mathbb{E}\langle P_N(T-t)p_t, p_t\rangle &= E\int_t^T \left[N|\Pi_V p_s|^2 + \sum_{i=1}^m |q^i(s)|^2\right] ds \\ &- \sum_{i=1}^m \mathbb{E}\int_t^T \left|[I + P_N(T-s)]^{1/2}\left[[I + P_N(T-s)]^{-1}C_iP(T-s)p(s) - q^i(s)\right]\right|^2 ds \end{aligned} \tag{6.9}$$

where we use the short-writing $p_s = p(s, q^1, ..., q^m, \theta)$. This relation is obtained by applying Itô's formula to the product of $P_N(T-t)p_t$ with p_t.

Using (6.9) one can immediately deduce that

$$\langle P_N(T)\theta, \theta\rangle \leq \mathbb{E}\int_0^T \left[N|\Pi_V p(s, q^1, ..., q^m, \theta)|^2 + \sum_{i=1}^m |q^i(s)|^2\right] ds \tag{6.10}$$

for arbitrary $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$.

Moreover, for all $\theta \in \mathbb{R}^n$ there exist suitable $\overline{q}^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, for which the second term in (6.9) vanishes

$$\langle P_N(T)\theta, \theta\rangle = \mathbb{E}\int_0^T \left[N|\Pi_V p(s, \overline{q}^1, ..., \overline{q}^m, \theta)|^2 + \sum_{i=1}^m |\overline{q}^i(s)|^2\right] ds. \tag{6.11}$$

We will call these controls $\overline{q}^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$ optimal.

Proposition 6.3 *The viability kernel of V with respect to equation (3.5) has the following representation:*

$$\left\{\theta \in V | \exists T > 0 : \lim_{N\to\infty} \langle P_N(T)\theta, \theta\rangle < +\infty\right\}. \tag{6.12}$$

Proof If θ is in the viability kernel and $q^i \in L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$ are such that $p(s, q^1, ..., q^m, \theta) \in V$, $\mathbb{P}$-a.s., for all $s \in [0,T]$, then by (6.10) $\langle P_N(T)\theta, \theta\rangle \leq \sum_{i=1}^m \mathbb{E}\int_0^T |q^i(s)|^2 ds$ $\forall N \in \mathbb{N}$.

Vice versa, choosing for every $N \in \mathbb{N}$ the optimal set of controls $(\overline{q}^1_N, ..., \overline{q}^m_N)$ we get

$$\langle P_N(T)\theta, \theta\rangle = \mathbb{E}\int_0^T \left[N|\Pi_V p(s, \overline{q}^1, ..., \overline{q}^m_N, \theta)|^2 + \sum_{i=1}^m |\overline{q}^i_N(s)|^2\right] ds.$$

Thus the sequences $\{\overline{q}^i_N : N \in \mathbb{N}\}$, $i = 1, ..., m$, are bounded in $L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$ and, consequently, for a suitable subsequence of $\{(\overline{q}^1_N, \ldots, \overline{q}^m_N) : N \in \mathbb{N}\}$ (that, abusing notations, will still be denoted by $\{(\overline{q}^1_N, \ldots, \overline{q}^m_N) : N \in \mathbb{N}\}$) we can assume that, for some $\overline{q}^i$ in $L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$, $i = 1, ..., m$, $\overline{q}^i_N \rightharpoonup \overline{q}^i$ weakly in $L^2_{\mathcal{P}}([0,T], \mathbb{R}^d)$. Moreover, since equation (6.7) is affine in q^1,...,q^m, we have $p(s, \overline{q}^1_N, ..., \overline{q}^m_N, \theta) \rightharpoonup p(s, \overline{q}^1, ..., \overline{q}^m, \theta)$.

Consequently, by (6.11)

$$\langle P_N(T)\theta, \theta\rangle \geq N\mathbb{E}\int_0^T |\Pi_V p(s, \overline{q}_N^1, ..., \overline{q}_N^m, \theta)|^2 ds$$

and we can conclude $\Pi_V p(s, \overline{q}^1_, ..., \overline{q}^m_, \theta) = 0$, $\mathbb{P}$-a.s. for every $s \in [0, T]$. □

The next result that will be essential in the following, is now very easy to prove. See [11] for a different proof in a much more general nonlinear situation but with bounded control space. While in [11], the viability kernel is always closed. This is not necessarily the case in our concept (when the set is not a finite-dimensional linear space).

Theorem 6.2 *The viability kernel of an arbitrary subspace $V \subset \mathbb{R}^n$ is locally in time viable.*

Proof Fix θ in the viability kernel and let $T > 0$ and $q^i \in L^2_{\mathcal{P}}([0, T], \mathbb{R}^d)$, $i = 1, ..., m$ such that $p(s, q^1, ..., q^m, \theta) \in V$, $\mathbb{P}$-a.s. for every $s \in [0, T]$. Then, for every $t < T$, by (6.9)

$$\mathbb{E}\langle P_N(T-t)p(t, q^1, ..., q^m, \theta), p(t, q^1, ..., q^m, \theta)\rangle \leq \sum_{i=1}^m \mathbb{E}\int_t^T |q^i(s)|^2 ds.$$

Thus, by monotone convergence, $\mathbb{E}\, lim_{N+\infty}\langle P_N(T-t)p(t, q^1, ..., q^m, \theta), p(t, q^1, ..., q^m, \theta)\rangle < +\infty$ and we can conclude with the help of Proposition 6.3 that $p(t, q^1, ..., q^m, \theta)$ belongs, $\mathbb{P}$-a.s. to the viability kernel of V. □

Remark 6.4 In the above argument we prove something more precise than the claim of the theorem. Namely, we show that for all θ in the viability kernel of V, all $T > 0$, and all $q^i \in L^2_{\mathcal{P}}([0, T], \mathbb{R}^d)$, $i = 1, ..., m$, for which $\{p(s, q^1, ..., q^m, \theta), s \in [0, T]\} \subset V$, $\mathbb{P}$-a.s., we even have that $\{p(s, q^1, ..., q^m, \theta), s \in [0, T]\}$ is $\mathbb{P}$-almost surely a subset of the viability kernel of V.

6.5 Proof of the main result

We start by showing that our problem can now be reduced to the computation of a viability kernel.

Theorem 6.3 *The following assertions are equivalent:*

1. *Equation (6.1) is approximately controllable.*
2. *Equation (6.1) is approximately null controllable.*
3. *The viability kernel of* $\mathrm{Ker}B^*$ *is trivial (i.e., it contains only the origin).*

Proof *2* ⇔ *3* is just a reformulation of Proposition 6.2. Moreover, *1* ⇒ *2* is evident by definition. Thus it remains to prove that *3* ⇒ *1*. This will be done by exploiting Proposition 6.2.

Assume that the viability kernel of KerB* is trivial and let $q^i \in L^2_{\mathcal{P}}([0, T], \mathbb{R}^d)$, $i = 1, ..., m$, such that $p(s, q^1, ..., q^m, \theta) \in \mathrm{Ker}\, B^*$, $\mathbb{P}$-a.s. for every $s \in [0, T]$. Then we know that $\theta = 0$ but also, see Remark 6.4, that $p(s, q^1, ..., q^m, \theta)$ belongs to the viability kernel of $\mathrm{Ker}B^*$. But since this last set coincides with the origin, we conclude that $p(s, q^1, ..., q^m, \theta) = 0$, $\mathbb{P}$-a.s. for all s in $[0, T]$. □

Then we only need to characterize the local in time viability in an explicit way. We refer the reader to [4] and to the appendix of [5] for other characterizations of stochastic viability for nonlinear systems with bounded control space.

Theorem 6.4 *A subspace $V \subset \mathbb{R}^n$ is locally in time viable if and only if it is $(A^*; C_1^*, ..., C_m^*)$-strictly invariant.*

Proof We start proving that any $(A^*; C_1^*, ..., C_m)$-strictly invariant subspace is locally in time viable.

For this we just notice that there exist linear operators K_i, $i = 1, ..., m$ such that $K_i V \subset V$ and $(A^* + C_1^* K_1 + \cdots + C_m^* K_m)V \subset V$. Thus, for any $\theta \in V$, we consider the following forward linear equation:

$$\begin{cases} d\widetilde{p}(t) = -\Big(A^* + \sum_{i=1}^{m} C_i^* K^i\Big)\widetilde{p}(t)dt + \sum_{i=1}^{m} K^i \widetilde{p}(t) d\beta_i(t), \\ \widetilde{p}(0) = \theta, \end{cases}$$

which solution $\widetilde{p}$ is clearly in V. If we set $q^i = K^i \widetilde{p}$, then $p(t, q^1, ..., q^m, \theta) = \widetilde{p}(t) \in V$ for all $t > 0$.

Vice versa, assume now that $\{p(s, q^1, ..., q^m, \theta), s \in [0, T]\} \subset V$ $\mathbb{P}$-a.s., for suitable θ in V, $T > 0$ and q^i in $L^2_{\mathcal{P}}([0, T], \mathbb{R}^d)$, $i = 1, ..., m$.

If we multiply equation (6.7) by (Π_V) and write $p_t = p(t, q^1, ..., q^m, \theta)$, we get

$$\begin{cases} d(I - \Pi_V)p_t = -(I - \Pi_V)\Big(A^* p_t + \sum_{i=1}^{m} C_i^* q^i(t)\Big)dt + \sum_{i=1}^{m} (I - \Pi_V) q^i(t) d\beta_i(t), \\ p(0) = \theta. \end{cases} \tag{6.13}$$

Since $(I - \Pi_V)p_t = 0$, $\mathbb{P}$-a.s. for every $t > 0$, a successive computation of the quadratic variation in $[0, t]$ of the components of $(I - \Pi_V)p(t)$ yields $\int_0^t |(I - \Pi_V)q^i(s)|^2 ds = 0$, $\mathbb{P}$-a.s. for every $t \in [0, T]$, $i = 1,..., m$. Thus $q^i_s \in V$, $\mathbb{P}$-a.s. for almost every $s \in [0, T]$.

Coming back to equation (6.13) we get $(I - \Pi_V)\Big(A^* p(t) + \sum_{i=1}^m C_i^* q^i(t)\Big) = 0$, $\mathbb{P}$-a.s. for almost every $t \in [0, T]$. If now W is the linear space

$$\left\{\theta \in V | \exists \xi_1, ..., \xi_m \in V : A^*\theta + \sum_{i=1}^{m} C_i^* \xi_i \in V\right\}$$

we have $p_s \in W$ $\mathbb{P}$-a.s. for almost every $t \in [0, T]$. But since p has continuous trajectories and W is closed, we get $\theta \in W$ and this completes the proof. □

Let us conclude the proof of Theorem 6.1.

By Theorem 6.2, the viability kernel of KerB^* is locally in time viable thus, by Theorem 6.4, it is $(A^*; C_1^*, ..., C_m^*)$-strictly invariant. Vice versa, again by Theorem 6.4, any $(A^*; C_1^*, ..., C_m^*)$-strictly invariant subspace of KerB^* is locally in time viable and consequently included in the viability kernel of KerB^*. So we can conclude that the viability kernel of KerB^* is the largest $(A^*; C_1^*, ..., C_m^*)$-strictly invariant subspace of KerB^*. The claim follows immediately by Theorem 6.3.

References

[1] J.-M. Bismut, Linear quadratic optimal stochastic control with random coefficients. *SIAM J. Contr. Optim.*, **14** (1976), 419–444.

[2] J.-M. Bismut, Contrôle des systèmes linéaires quadratiques: applications de l'intégrale stochastique. In *Séminaire de Probabilités, XII.* Lecture Notes in Math., 649, Springer, Berlin, 1978.

[3] V. Barbu, G. Tessitore, Considerations on the controllability of stochastic linear heat equations, in *Stochastic partial di erential equations and applications (Trento, 2002)*, Lecture Notes in Pure and Appl. Math., 227, pp. 39–51, Dekker, New York, 2002.

[4] R. Buckdahn, S. Peng, M. Quincampoix, C. Rainer, Existence of stochastic control under state constraints. *C. R. Acad. Sci.*, Sér. I. Paris, t. 327 (1998), 17–22.

[5] R. Buckdahn, P. Cardaliaguet, M. Quincampoix, A representation formula for the mean curvature motion. *SIAM J. Math. Anal.*, **33** (2001), 827–846.

[6] R. Buckdahn, M.Quincampoix, C. Rainer, A. Rascanu, Stochastic control with exit time and constraints, application to small time attainability of sets. *Appl. Math. Optim.,* **49** (2004), 99–112.

[7] Y. Liu, S. Peng, Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems. *Automatica,* **38** (2002), 1417–1423.

[8] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation. *Syst. Control Lett.*, **14** (1990), 55–61.

[9] S. Peng, Backward stochastic differential equation and exact controllability of stochastic control systems. *Progr. Natur. Sci. (English Ed.)*, **4** (1994), 274–284.

[10] M. Sirbu, G. Tessitore, Null controllability of an infinite dimensional SDE with state and control-dependent noise. *Syst. Control Lett.*, **44** (2001), 385–394.

[11] M. Quincampoix, C. Rainer, Stochastic control and compatible subsets of contraints. *Bull. Sci. Math.*, **129**(1) (2005), 39-55.

[12] J.C. Willems, Almost A(mod B)-invariant subspaces. *Astérisque,* **75–76** (1980), 239–248.

[13] J.C. Willems, Almost invariant subspaces: an approach to high gain feedback design – Part I: almost controlled invariant subspaces. *IEEE Trans. Automat. Control.,* **AC-26** (1981), 235–252.

[14] J.C. Willems, Almost invariant subspaces: an approach to high gain feedback design – Part II: almost controlled invariant subspaces. *IEEE Trans. Automat. Control.,* **AC-27** (1982), 1071–1085.

[15] J.L. Willems, J.C. Willems, Feedback stabilizability for stochastic systems with state and control depending noise, *Automatica*, **12** (1976), 277–283.

[16] J.L. Willems, J.C. Willems, Robust stabilization of uncertain systems. *SIAM J. Contr. Optim.*, **21** (1983), 342–372.

[17] W.M. Wonham, *Linear Multivariable Control: A Geometric Approach*, 2nd ed., Springer Verlag, Berlin, New York, 1979.

[18] J. Yong, X. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations.* Springer Verlag, Berlin, New York, 1999.

7 Asymptotic Behavior of Systems of Stochastic Partial Differential Equations with Multiplicative Noise

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7.1 Introduction

We are here interested with the problem of existence and uniqueness of the invariant measure for the following class of reaction–diffusion systems perturbed by a multiplicative noise:

$$\left\{\begin{array}{rl} \dfrac{\partial u_i}{\partial t}(t,\xi) = & \mathcal{A}_i\, u_i(t,\xi) + f_i(\xi, u_1(t,\xi),\ldots,u_r(t,\xi)) \\[10pt] & \displaystyle + \sum_{j=1}^r g_{ij}(\xi, u_1(t,\xi),\ldots,u_r(t,\xi)) Q_j \frac{\partial w_j}{\partial t}(t,\xi), \qquad t \geq 0, \qquad \xi \in \overline{\mathcal{O}}, \\[10pt] u_i(0,\xi) = & x_i(\xi), \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}_i\, u_i(t,\xi) = 0, \qquad t \geq 0, \qquad \xi \in \partial\mathcal{O}. \end{array}\right. \tag{7.1}$$

Here $\mathcal{O}$ is a bounded open set of $\mathbb{R}^d$, with $d \geq 1$, having regular boundary, and $\partial w_i(t)/\partial t$ are independent space–time white noises defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

For each $i = 1, \ldots, r$, the operator $\mathcal{A}_i$ is defined by

$$\mathcal{A}_i(\xi, D) := \sum_{h,k=1}^d a^i_{hk}(\xi) \frac{\partial^2}{\partial \xi_h \partial \xi_k} + \sum_{h=1}^d b^i_h(\xi) \frac{\partial}{\partial \xi_h}, \qquad \xi \in \overline{\mathcal{O}}.$$

The coefficients a^i_{hk} and b^i_h are taken of class $C^1(\overline{\mathcal{O}})$ and for any $\xi \in \overline{\mathcal{O}}$ the matrix $[a^i_{hk}(\xi)]$ is nonnegative, symmetric, and uniformly positive definite. Moreover, $\mathcal{B}_i$ is an operator acting on the boundary either of Dirichlet or of conormal type.

The Q_i's are bounded linear operators from $L^2(\mathcal{O})$ into itself, which are not assumed to be Hilbert–Schmidt and in the case of space dimension $d = 1$ can be taken equal to identity. This means that in dimension $d = 1$ we can consider systems perturbed by white noise and in dimension $d > 1$ we have clearly to color the noise but we do not assume any trace-class property for its covariance.

Finally, concerning the nonlinear terms, we assume that

$$f := (f_1, \ldots, f_r) : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r, \qquad g := [g_{ij}] : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$$

are continuous in both variables and are Lipschitz-continuous in the second variable, uniformly with respect to the first. Notice that here we are not assuming any restriction on the linear growth of G or on any its degeneracy (for example, we can take $g_{ij}(\xi, u) = \lambda_{ij} u_j$, for some $\lambda_{ij} \in \mathbb{R}$).

In this chapter we show that the transition semigroup P_t associated with system (7.1) admits a unique invariant measure μ which is ergodic and strongly mixing. Here we are not assuming the diffusion term g in front of the noise to be nondegenerate, and hence it is not possible to prove any smoothing effect of P_t in order to apply the Doob theorem which in turn implies the uniqueness of the invariant measure (see for all details [6]).

In [3] we have studied a class of stochastic reaction–diffusion equations with multiplicative noise, in which the diffusion term in front of the noise may vanish and the deterministic part of the equation is not necessary asymptotically stable. Developing some arguments introduced by Mueller in [10] we have shown that the L^1-norm of the difference of two solutions starting from any two different initial data converges $\mathbb{P}$-a.s. to zero, as time goes to infinity. But the method followed in [3] seems to apply only to one single equation (in space dimension $d = 1$). Actually such a method is based on comparison and at present it is not clear how to extend it to systems.

In [12] Sowers has proved the existence and uniqueness of the invariant measure for a class of reaction–diffusion equations in space dimension $d = 1$ perturbed by a noise of multiplicative type. In his paper he has considered one single equation and has assumed the reaction term f to be Lipschitz continuous with Lipschitz constant L_f, the second order operator $\mathcal{A}$ to be of negative type $-\lambda$, with $\lambda > L_f$, and the multiplication term g in front of the noise bounded from above and below (that is, $0 < g_0 \leq g(x) \leq g_1 < \infty$, for all $x \in \mathbb{R}$) and sufficiently small (that is, $g_1 \leq \epsilon$, for some $\epsilon > 0$). In [10], the result of Sowers is extended to the case of a multiplication term g which is still bounded both from above and below, but not necessarily small.

In his paper Sowers proves that the family of probability measures $\{\mathcal{L}(u^x)\}$ is tight in $C([0,\infty); C(\overline{\mathcal{O}}))$ for any $x \in C(\overline{\mathcal{O}})$ and shows the mean square convergence to zero of the sup-norm of the difference of two solutions $u^x(t) - u^y(t)$, starting form any two initial data $x, y \in C(\overline{\mathcal{O}})$. In the present chapter we extend these results to the case of systems in any space dimension, with a diffusion coefficient which is not bounded, neither from above nor from below.

But in order to have this extension we have to assume some stronger dissipativity condition for the second order operator $\mathcal{A}$. Namely, we assume that there exist suitable nonnegative constants α_1, α_2, α_3, and δ such that

$$\lambda > \max\{\, \alpha_1 L_f,\ \alpha_2 M_g^\delta,\ \alpha_3 L_g^\delta \,\}, \tag{7.2}$$

where L_f and L_g are the Lipschitz constants of f and g, respectively, and M_g is defined by

$$M_g := \sup_{\xi \in \overline{\mathcal{O}},\ \sigma \in \mathbb{R}^d} \frac{|g(\xi,\sigma)|}{1 + |\sigma|^\beta},$$

for some $\beta \in [0,1]$. Note that if $\beta < 1$, we can take $\alpha_2 = 0$.

It is important to notice that, due to condition (7.2) we can assume λ to be arbitrarily close to zero, if we take L_f, L_g and M_g sufficiently small; on the other hand, if L_f, L_g and M_g are given, we can prove ergodicity by taking λ sufficiently large. Moreover, we would like to stress that, since g is not bounded, not even the uniform estimate of solutions in the Hölder norm (which implies the existence of an invariant measure) is straightforward.

7.2 Assumptions and preliminaries

In what follows we shall denote by H the separable Hilbert space $L^2(\mathcal{O}; \mathbb{R}^r)$, with $r \geq 1$, endowed with the scalar product

$$\langle x, y \rangle_H := \int_{\mathcal{O}} \langle x(\xi), y(\xi) \rangle_{\mathbb{R}^r}\, d\xi = \sum_{i=1}^r \int_{\mathcal{O}} x_i(\xi) y_i(\xi)\, d\xi = \sum_{i=1}^r \langle x_i, y_i \rangle_{L^2(\mathcal{O})}$$

and the corresponding norm $|\cdot|_H$. For any other $p \geq 1$ the usual norm in $L^p(\mathcal{O};\mathbb{R}^r)$ is denoted by $|\cdot|_p$. If $\epsilon > 0$ and $p \geq 1$, we denote by $|\cdot|_{\epsilon,p}$ the norm in $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$

$$|x|_{\epsilon,p} := |x|_p + \sum_{i=1}^r \int_{\mathcal{O}\times\mathcal{O}} \frac{|x_i(\xi) - x_i(\eta)|^p}{|\xi - \eta|^{\epsilon p + d}}\, d\xi\, d\eta.$$

We denote by E the Banach space $C(\overline{\mathcal{O}};\mathbb{R}^r)$, endowed with the sup-norm

$$|x|_E := \left(\sum_{i=1}^r \sup_{\xi\in\overline{\mathcal{O}}} |x_i(\xi)|^2\right)^{\frac{1}{2}},$$

and the duality $\langle\cdot,\cdot\rangle_E$.

Next, for any $x, y \in E$ we define

$$\langle \delta_x, y\rangle_E := \begin{cases} \dfrac{1}{|x|_E} \displaystyle\sum_{i=1}^r x_i(\xi_i) y_i(\xi_i) & \text{if } x \neq 0 \\ \\ \langle \delta, y\rangle_E & \text{if } x = 0, \end{cases} \tag{7.3}$$

where δ is any element of $E^\star$ having norm equal 1 and $\xi_1, \dots, \xi_r \in \overline{\mathcal{O}}$ are such that $|x_i(\xi_i)| = |x_i|_{C(\overline{\mathcal{O}})}$, for all $i = 1, \dots, r$. Note that

$$\delta_x \in \partial\,|x|_E := \{\, h^\star \in E^\star;\ |h^\star|_{E^\star} = 1,\ \langle h, h^\star\rangle_E = |h|_E \,\}$$

(see, e.g., [1, Appendix A] for all definitions and details).

In what follows we shall denote by A the realization in H of the differential operator $\mathcal{A} = (\mathcal{A}_1, \dots, \mathcal{A}_r)$ endowed with the boundary conditions $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_r)$. Also, we can assume without any loss of generality that A is a nonpositive and self-adjoint operator which generates an analytic semigroup e^{tA} with dense domain. If this is not the case, A can be written as the sum of an operator C which fulfills these properties and of a first order operator L which can be treated as a lower order perturbation. Actually, we can take C and L as the realizations in H of the operators $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_r)$ and $\mathcal{L} = (\mathcal{L}_1, \dots, \mathcal{L}_r)$, respectively, endowed with the boundary conditions $\mathcal{B} = (\mathcal{B}_1, \dots, \mathcal{B}_r)$, where for any $i = 1, \dots, r$

$$\mathcal{L}_i(\xi, D) := \sum_{h=1}^d \left(b_h^i(\xi) - \sum_{k=1}^d \frac{\partial}{\partial \xi_k} a_{hk}^i(\xi)\right) \frac{\partial}{\partial \xi_h}, \qquad \xi \in \mathcal{O},$$

and by difference

$$\mathcal{C}_i := \mathcal{A}_i - \mathcal{L}_i$$

(for all details we refer to [2, Section 2 and Section 3]).

Due to this decomposition, we can also assume that e^{tA} may be extended to a nonnegative one-parameter contraction semigroup on $L^p(\mathcal{O};\mathbb{R}^r)$, for all $1 \leq p \leq \infty$. These semigroups are strongly continuous for $1 \leq p < \infty$ and are consistent, so that we shall denote all of them by e^{tA}. Moreover, if we consider the part of A in the space of continuous functions E, it generates an analytic semigroup e^{tA} (which has no dense domain, in general, due to the boundary conditions).

Next, we notice that for any $t, \epsilon > 0$ and $p \geq 1$, the semigroup e^{tA} maps $L^p(\mathcal{O};\mathbb{R}^r)$ into $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$ and

$$|e^{tA}x|_{\epsilon,p} \leq c\,(t\wedge 1)^{-\frac{\epsilon}{2}} |x|_p, \qquad x \in L^p(\mathcal{O};\mathbb{R}^r), \tag{7.4}$$

for some constant c independent of p. Then, as $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$ embeds into $L^\infty(\mathcal{O};\mathbb{R}^r)$, for any $\epsilon > d/p$, we have that e^{tA} maps H into $L^\infty(\mathcal{O};\mathbb{R}^r)$, for any $t > 0$, and

$$|e^{tA}x|_\infty \leq c\,(t \wedge 1)^{-\frac{d}{4}}|x|_H, \qquad x \in H.$$

This means that e^{tA} is *ultracontractive*.

As a consequence of ultracontractivity and of the boundedness of $\mathcal{O}$, as proved in [7, Theorems 2.1.4 and 2.1.5] we have that e^{tA} is compact on $L^p(\mathcal{O};\mathbb{R}^r)$ for all $1 \leq p \leq \infty$ and $t > 0$. The spectrum $\{-\alpha_k\}_{k\in\mathbb{N}}$ of A is independent of p and e^{tA} is analytic on $L^p(\mathcal{O};\mathbb{R}^r)$, for all $1 \leq p \leq \infty$. Concerning the complete orthonormal system of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$, we have that $e_k \in E$, for any $k \in \mathbb{N}$.

In what follows we shall assume that the operator $Q := (Q_1, \ldots, Q_r)$ fulfills the following condition.

Hypothesis 7.1 *The bounded linear operator $Q : H \to H$ is nonnegative and diagonal with respect to the orthonormal basis $\{e_k\}$ which diagonalizes A, with eigenvalues $\{\lambda_k\}$. Moreover, if $d \geq 2$*

$$\kappa_Q := \sum_{k=1}^\infty \lambda_k^\rho\, |e_k|_\infty^2 < \infty, \tag{7.5}$$

with

$$\rho < \frac{2d}{d-2}, \qquad d \geq 2.$$

Remark 7.1 1. In the case of the Laplace operator on the square endowed with Dirichlet boundary conditions, the L^∞-norms of the eigenfunctions e_k are equi-bounded, so that condition (7.5) becomes

$$\sum_{k=1}^\infty \lambda_k^\rho < \infty. \tag{7.6}$$

But in general we only have $|e_k|_\infty \leq c\,k^\alpha$, for some $\alpha \geq 0$ depending on A and on the domain $\mathcal{O}$.

2. In [2, Hypothesis 1 and Hypothesis 3.1] we have assumed that the sup-norm of the eigenfunctions e_k are equi-bounded and (7.6) holds. Here in (7.5) we combine together these two conditions in a more general way.

Concerning the multiplication term in front of the noise we assume the following hypothesis.

Hypothesis 7.2 *The mapping $g : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is continuous and for any $\sigma, \rho \in \mathbb{R}^r$*

$$\sup_{\xi\in\overline{\mathcal{O}}} |g(\xi,\sigma) - g(\xi,\rho)|_{\mathcal{L}(\mathbb{R}^r)} \leq L_g\,|\sigma - \rho|, \tag{7.7}$$

for some constant $L_g \geq 0$.

If for any $x, y : \overline{\mathcal{O}} \to \mathbb{R}^r$, we set

$$(G(x)y)(\xi) := g(\xi, x(\xi))y(\xi), \qquad \xi \in \mathcal{O},$$

due to the condition above we have that the mapping $G : E \to \mathcal{L}(E)$ is Lipschitz continuous. Moreover, it is immediate to check that G is also Lipschitz continuous both as a mapping from H into $L^1(\mathcal{O};\mathbb{R}^r)$ and as a mapping from $L^\infty(\mathcal{O};\mathbb{R}^r)$ into H (see [2, (3.4) and (3.5)]).

As we said in the introduction, $\partial w_j(t)/\partial t$ are independent space–time white noises defined on the same stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, for each $j = 1, \ldots, r$. Thus, if we set $w(t) = (w_1(t), \ldots, w_r(t))$, we have that $w(t)$ can be written as

$$w(t) := \sum_{k=1}^{\infty} e_k \beta_k(t), \qquad t \geq 0,$$

where $\{e_k\}$ is the complete orthonormal system in H which diagonalizes A and $\{\beta_k(t)\}$ is a sequence of mutually independent standard real Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

Now, if X is any Banach space, for $T > 0$ and $p \geq 1$, we define $L^p(\Omega; C_b((0,T]; X))$ as the set of all $\mathcal{F}_t$-adapted X-valued processes u having $\mathbb{P}$-a.s. continuous and bounded trajectories in $(0,T]$ and such that

$$|u|^p_{L_{T,p}(X)} := \mathbb{E} \sup_{t \in [0,T]} |u(t)|^p_X < \infty.$$

$L^p(\Omega; C_b((0,T]; X))$ is a Banach space, endowed with the norm $|\cdot|_{L_{T,p}(X)}$. Moreover, we denote by $L^p(\Omega; C([0,T]; X))$ the subspace of processes u which take values in $C([0,T]; X)$, $\mathbb{P}$-a.s.

With these notations, for any fixed $u \in L^p(\Omega; C_b((0,T]; X))$ the solution $\gamma(u)$ of the system

$$\left\{ \begin{array}{l} \displaystyle \frac{\partial \gamma_i}{\partial t}(t,\xi) = \mathcal{A}_i\, \gamma_i(t,\xi) + \sum_{j=1}^{r} g_{ij}(\xi, u_1(t,\xi), \ldots, u_r(t,\xi))\, Q_j \frac{\partial w_j}{\partial t}(t,\xi), \qquad \xi \in \overline{\mathcal{O}}, \\[10pt] \gamma_i(0,\xi) = 0, \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}_i\, \gamma_i(t,\xi) = 0, \quad \xi \in \partial \mathcal{O}, \qquad t \geq 0, \end{array} \right. \tag{7.8}$$

for any $i = 1, \ldots, r$, is given by

$$\gamma(u)(t) = \int_0^t e^{(t-r)C} G(r, u(r)) Q\, dw(r), \qquad t \geq 0.$$

Notice that it is possible to adapt the proof of [2, Lemma 4.1 and Theorem 4.2] to the more general case we are considering in the present chapter (see Remark 7.1 for a comparison between condition (7.5) and [2, Hypothesis 1 and Hypothesis 3.1]) and prove that γ is a contraction in $L^p(\Omega; C([0,T]; E))$.

Namely, under Hypotheses 7.1 and 7.2 we have the following result.

Theorem 7.1 *There exists $\bar{p} \geq 1$ such that γ maps the space $L^p(\Omega; C_b((0,T]; E))$ into the space $L^p(\Omega; C([0,T]; E))$, for any $p \geq \bar{p}$, and for any $u, v \in L^p(\Omega; C([0,T]; E))$*

$$|\gamma(u) - \gamma(v)|_{L_{T,p}(E)} \leq c_p^\gamma(T) |u - v|_{L_{T,p}(E)}, \tag{7.9}$$

for some continuous increasing function c_p^γ such that $c_p^\gamma(0) = 0$.

In particular, by a fixed point argument, if both the reaction term $f = 0$ and the initial datum x vanish, system (7.1) admits a unique mild solution in $L^p(\Omega; C([0,T]; E))$, for any $p \geq 1$ and $T > 0$.

Concerning the reaction term f the following condition is assumed.

Hypothesis 7.3 *The mapping $f : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r$ is continuous and for any $\sigma, \rho \in \mathbb{R}^r$*

$$\sup_{\xi \in \overline{\mathcal{O}}} |f(\xi, \sigma) - f(\xi, \rho)| \leq L_f\, |\sigma - \rho|, \tag{7.10}$$

for some constant $L_f \geq 0$.

In particular, if we define the operator F by setting for any $x : \overline{\mathcal{O}} \to \mathbb{R}^r$

$$F(x)(\xi) := f(\xi, x(\xi)), \qquad \xi \in \overline{\mathcal{O}},$$

we have that F is Lipschitz continuous in $L^p(\mathcal{O};\mathbb{R}^r)$, for any $1 \le p \le \infty$, and also in E.

We recall that a process $u \in L^p(\Omega; C_b((0,T];E))$ is a *mild* solution of system (7.1) if

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A}F(u(s))\,ds + \int_0^t e^{(t-s)A}G(u(s))\,dw(s).$$

By assuming the conditions above it is known that the following result holds (for a proof see [6, Theorem 5.3.1] and [11]; see also [2, Theorem 5.3, Proposition 5.6]).

Theorem 7.2 *Under Hypotheses 7.1, 7.2 and 7.3, for any $p \ge 1$ and $T > 0$ and for any initial datum $x \in E$ system (7.1) admits a unique mild solution $u^x \in L^p(\Omega; C_b((0,T];E))$ such that*

$$\mathbb{E} \sup_{t \in [0,T]} |u^x(t)|_E^p \le c_p(T)\,(1 + |x|_E^p). \tag{7.11}$$

Moreover, the mapping

$$x \in E \mapsto u^x \in L^p(\Omega; C_b((0,T];E))$$

is uniformly continuous.

Now, since for any $x \in E$ we have a unique mild solution for system (7.1), we can associate to it its transition semigroup by setting for any $x \in E$ and for any φ belonging to $B_b(H)$, the space of bounded Borel functions defined on H with values in $\mathbb{R}$,

$$P_t\varphi(x) := \mathbb{E}\,\varphi(u^x(t)), \qquad t \ge 0.$$

Notice that, as the solution u^x depends continuously on its initial datum $x \in E$, P_t is a Feller semigroups; that is, it maps the subspace of continuous functions into itself.

7.3 Main result

In this section we show that the semigroup P_t associated with system (7.1) admits an invariant measure which is unique, ergodic, and strongly mixing. To this purpose we have to assume a suitable dissipativity conditions on the operator A.

Hypothesis 7.4 *There exists $\lambda > 0$ such that*

$$\langle Ah, h^\star \rangle_E \le -\lambda\,|h|_E,$$

for any $h \in D(A)$ and $h^\star \in \partial\,|h|_E$.

In particular, we have $e^{tA} = e^{-t\lambda}\,e^{t(A+\lambda)}$ and the semigroup $e^{t(A+\lambda)}$ satisfies all conditions described for e^{tA} in Section 7.2.

As proved in [2, Theorem 4.2, Proposition 4.5], if $u \in L^p(\Omega; C_b((0,T];E))$ and $\gamma(u)$ is the solution of system (7.8), due to Hypothesis 7.4 we have

$$\mathbb{E} \sup_{t \in [0,T]} |\gamma(u)(t) - \gamma(v)(t)|_E^p \le c_{p,\lambda}^\gamma(T)\,\mathbb{E} \sup_{t \in [0,T]} |u(t) - v(t)|_E^p,$$

for some continuous increasing function $c_{p,\lambda}^\gamma \in L^\infty(0,\infty)$. This implies, in particular, that for any $u, v \in L^p(\Omega; C_b((0,\infty);E))$

$$\mathbb{E} \sup_{t \ge 0} |\gamma(u)(t) - \gamma(v)(t)|_E^p \le |c_{p,\lambda}^\gamma|_\infty\,\mathbb{E} \sup_{t \ge 0} |u(t) - v(t)|_E^p. \tag{7.12}$$

Moreover, in [2, Theorem 4.2, Proposition 4.5] it is also shown that $c^{\gamma}_{p,\lambda}(0) = 0$ and

$$\lim_{\lambda\to\infty} |c^{\gamma}_{p,\lambda}|_\infty = 0.$$

The following result generalizes an analogous result proved in [12] for one single equation in space dimension $d = 1$ with a diffusion bounded both from above and from below.

Lemma 7.1 *Assume Hypotheses 7.1 to 7.4. Then there exists $\bar{p} > 1$ such that for any $t > 0$, $p \geq \bar{p}$, and $0 < \delta < \lambda$ and for any $u, v \in L^p(\Omega; C_b((0,t]; E))$*

$$\sup_{s\leq t} e^{\delta p\, s}\mathbb{E}\, |\gamma(u)(s) - \gamma(v)(s)|^p_E \leq c_{1,p}\, \frac{L^p_g}{(\lambda-\delta)^{c_{2,p}}} \sup_{s\leq t} e^{\delta p\, s}\mathbb{E}\, |u(s) - v(s)|^p_E\,, \tag{7.13}$$

where L_g is the Lipschitz constant of the di usion coe cient g introduced in (7.7) and $c_{1,p}$ and $c_{2,p}$ are some positive constants.

Proof By using a factorization argument (see, e.g., [5, Theorem 8.3]), for any $t \geq 0$ we have

$$\gamma(u)(t) - \gamma(v)(t) = \frac{\sin \pi\alpha}{\pi} \int_0^t (t-s)^{\alpha-1} e^{(t-s)A} v_\alpha(s)\, ds,$$

where

$$v_\alpha(s) := \int_0^s (s-\sigma)^{-\alpha} e^{(s-\sigma)A} \left[G(u(\sigma)) - G(v(\sigma))\right] Q\, dw(\sigma),$$

and $\alpha \in (0, 1/2)$. As shown in [2, Theorem 4.2], according to (7.4) and to the Hölder inequality for any $\alpha > 1/p$ and $\eta < 2(\alpha - 1/p)$ and for any $\delta > 0$ we have

$$|\gamma(u)(t) - \gamma(v)(t)|^p_{\eta,p} \leq c^p_\alpha \left(\int_0^t (t-s)^{\alpha-\frac{\eta}{2}-1} e^{-\lambda(t-s)} |v_\alpha(s)|_p\, ds\right)^p$$

$$\leq c^p_\alpha \left(\int_0^t s^{(\alpha-\frac{\eta}{2}-1)\frac{p}{p-1}} e^{-(\lambda-\delta)s}\, ds\right)^{p-1} \int_0^t e^{-(\lambda-\delta)(t-s)} e^{-\delta p(t-s)} |v_\alpha(s)|^p_p\, ds$$

$$\leq c^p_\alpha\, \Gamma^{p-1}\left(1 + \frac{(\alpha-\eta/2-1)p}{p-1}\right) (\lambda-\delta)^{1-\frac{\alpha-\eta/2}{p}} e^{-\delta pt} \int_0^t e^{-(\lambda-\delta)(t-s)} e^{\delta ps} |v_\alpha(s)|^p_p\, ds$$

(here $\Gamma(x)$ denotes the Gamma function at $x > 0$). Thus, if $\eta > d/p$, that is, $\alpha > (d+2)/2p$, due to the Sobolev embedding theorem

$$\begin{aligned} &e^{\delta pt}\, \mathbb{E}\, |\gamma(u)(t) - \gamma(v)(t)|^p_E \\ &\leq c^p_\alpha\, \Gamma^{p-1}\left(1 + \frac{(\alpha-\eta/2-1)p}{p-1}\right) (\lambda-\delta)^{-\frac{\alpha-\eta/2}{p}} \sup_{s\leq t} e^{\delta ps}\mathbb{E}\, |v_\alpha(s)|^p_p. \end{aligned} \tag{7.14}$$

Now, if ϱ is the constant introduced in Hypothesis 7.1, we can find some $\bar{p}$ large enough such that for any $p \geq \bar{p}$

$$\frac{d+2}{p} + \frac{d(\varrho-2)}{2\varrho} < 1.$$

This implies that there exists some $\bar{\alpha} \in (0, 1/2)$ such that for any $p \geq \bar{p}$

$$\bar{\alpha} > \frac{d+2}{2p} \quad \text{and} \quad 2\bar{\alpha} + \frac{d(\varrho-2)}{2\varrho} < 1.$$

Then, proceeding as in the proof of [2, Theorem 4.2], for any $\xi \in \overline{\mathcal{O}}$ we have

$$\mathbb{E}\,|v_{\bar\alpha}(s,\xi)|^p \leq c\,\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\kappa_Q^{\frac{p}{\rho}}\,\mathbb{E}\left(\int_0^s (s-\sigma)^{-2\bar\alpha}e^{-2\lambda(s-\sigma)}\right.$$

$$\left.\left(\sum_{k=1}^\infty |e^{(s-\sigma)(A+\lambda)}\left([G(u(\sigma))-G(v(\sigma))]\,e_k\right)(\xi)|^{2\varsigma}|e_k|_\infty^{-2(\varsigma-1)}\right)^{\frac{1}{\varsigma}}\,d\sigma\right)^{\frac{p}{2}},$$

and then

$$e^{\delta ps}\,\mathbb{E}\,|v_{\bar\alpha}(s)|_p^p \leq c\,L_g^p\,\left(\frac{p(p-1)\kappa_Q^{\frac{2}{\rho}}}{2}\right)^{\frac{p}{2}}$$

$$\mathbb{E}\left(\int_0^s (s-\sigma)^{-(2\bar\alpha-\frac{d}{2\varsigma})}e^{-2(\lambda-\delta)(s-\sigma)}e^{2\delta\sigma}|u(\sigma)-v(\sigma)|_E^2\,d\sigma\right)^{\frac{p}{2}}$$

$$\leq c\,L_g^p\,\left(\frac{p(p-1)\kappa_Q^{\frac{2}{\rho}}}{2}\right)^{\frac{p}{2}}\left(\int_0^s \sigma^{-(2\bar\alpha-\frac{d}{2\varsigma})\frac{p}{p-2}}\,e^{-2(\lambda-\delta)\sigma}\,d\sigma\right)^{\frac{p-2}{2}}$$

$$\int_0^s e^{-2(\lambda-\delta)(s-\sigma)}e^{\delta p\sigma}\,\mathbb{E}\,|u(\sigma)-v(\sigma)|_E^p\,d\sigma.$$

This implies

$$e^{\delta ps}\mathbb{E}\,|v_{\bar\alpha}(s)|_p^p \leq c\,L_g^p\,\left(\frac{p(p-1)\kappa_Q^{\frac{2}{\rho}}}{2}\right)^{\frac{p}{2}}\,\Gamma^{\frac{p-2}{p}}\left(1-(2\bar\alpha-\frac{d}{2\varsigma})\frac{p}{p-2}\right)$$

$$[2(\lambda-\delta)]^{(2\bar\alpha-\frac{d}{2\varsigma})-2\left(1-\frac{1}{p}\right)}\sup_{\sigma\leq s}e^{\delta p\sigma}\,\mathbb{E}\,|u(\sigma)-v(\sigma)|_E^p.$$

Therefore, due to (7.14) we can conclude

$$e^{\delta pt}\,\mathbb{E}\,|\gamma(u)(t)-\gamma(v)(t)|_E^p \leq c_p\,L_g^p\,(\lambda-\delta)^{-c_{2,p}}\,\sup_{s\leq t}e^{\delta ps}\,\mathbb{E}\,|u(s)-v(s)|_E^p,$$

where

$$c_p := c\,c_{\bar\alpha}^p\,\Gamma^{p-1}\left(1+\frac{(\bar\alpha-\eta/2-1)p}{p-1}\right)\Gamma^{\frac{p-2}{p}}\left(1-(2\bar\alpha-\frac{d}{2\varsigma})\frac{p}{p-2}\right)$$

$$\left(\frac{p(p-1)\kappa_Q^{\frac{2}{\rho}}}{2}\right)^{\frac{p}{2}}2^{(2\bar\alpha-\frac{d}{2\varsigma})-2\left(1-\frac{1}{p}\right)} \tag{7.15}$$

and

$$c_{2,p} := \frac{\bar\alpha-\eta/2}{p}-(2\bar\alpha-\frac{d}{2\varsigma})-2\left(1-\frac{1}{p}\right)>0. \tag{7.16}$$

Remark 7.2 With the same arguments used in the proof of the Lemma above we can prove that if

$$\sup_{\xi\in\overline{\mathcal{O}}} |g(\xi,\sigma)|_{\mathcal{L}(\mathbb{R}^r)} = M_g\,(1+|\sigma|^{\beta}),\qquad \sigma\in\mathbb{R}^r,$$

for some $\beta\in[0,1]$ and $c>0$, then for any $p\geq\bar{p}$, $u\in L^p(\Omega;C_b((0,t];E))$ and $\delta>0$

$$\sup_{s\leq t} e^{\delta p\, s}\mathbb{E}\,|\gamma(u)(s)|_E^p \leq c_{1,p}\,\frac{M_g^p}{(\lambda-\delta)^{c_{2,p}}}\,\sup_{s\leq t} e^{\delta p\, s}\left(1+\mathbb{E}\,|u(s)|_E^{\beta p}\right), \tag{7.17}$$

for some positive constants $c_{1,p}$ and $c_{2,p}$, with $c_{2,p}$ given by (7.16).

Theorem 7.3 *Assume Hypotheses 7.1 to 7.4. Moreover, assume that*

$$\sup_{\substack{\xi\in\overline{\mathcal{O}}\\ \sigma\in\mathbb{R}^r}} \frac{|g(\xi,\sigma)|_{\mathcal{L}(\mathbb{R}^r)}}{1+|\sigma|^{\beta}} =: M_g<\infty, \tag{7.18}$$

for some $\beta\in[0,1]$. *Then*

1. *If* $\beta<1$ *and* $\lambda>L_f$, *for any* $p\geq 1$
$$\sup_{t\geq 0}\mathbb{E}\,|u^x(t)|_E^p\leq c_p\,(1+|x|_E^p) \tag{7.19}$$
and there exists $\bar{\theta}\in(0,1)$ *such that for any* $a>0$
$$\sup_{t\geq a}\mathbb{E}\,|u^x(t)|_{C^{\bar{\theta}}(\overline{\mathcal{O}};\mathbb{R}^r)}<+\infty. \tag{7.20}$$

2. *If* $\beta=1$, (7.19) *and* (7.20) *are still valid for any* $p\geq\bar{p}$, *under the further condition*
$$k_{1,p}\,\frac{L_f^p}{\lambda^p}+k_{2,p}\,\frac{M_g^p}{\lambda^{c_{2,p}}}<1,$$
for suitable positive constants $k_{1,p}$ *and* $k_{2,p}$.

Proof By setting $v:=u^x-\gamma(u^x)$, we have that v solves the problem

$$\frac{dv}{dt}(t)=Av(t)+F(u^x(t)),\qquad v(0)=x.$$

Therefore, due to Hypotheses 7.4 and to the Lipschitz continuity of F, we have

$$\frac{d^-}{dt}|v(t)|_E\leq\left\langle Av(t),\delta_{v(t)}\right\rangle_E+\left\langle F(u^x(t)),\delta_{v(t)}\right\rangle_E\leq-\lambda\,|v(t)|_E+L_f\,|u^x(t)|_E+|F(0)|_E.$$

By comparison, this yields

$$|v(t)|_E\leq e^{-\lambda t}|x|_E+L_f\int_0^t e^{-\lambda(t-s)}|u^x(s)|_E\,ds+\frac{|F(0)|_E}{\lambda},$$

and hence, as $u^x(t)=v(t)+\gamma(u^x)(t)$, for any $p\geq 1$ and $0<\delta<\lambda-L_f/2$ and for any $\epsilon>0$ we have

$$\begin{aligned}|u^x(t)|_E^p&\leq c_{p,\epsilon}\,|\gamma(u^x)(t)|_E^p+c_{p,\epsilon}\left(e^{-\lambda pt}|x|_E^p+\frac{|F(0)|_E^p}{\lambda^p}\right)\\
&+(1+\epsilon)L_f^p\left(\int_0^t e^{-\lambda(t-s)}|u^x(s)|_E\,ds\right)^p\leq c_{p,\epsilon}\,|\gamma(u^x)(t)|_E^p+c_{p,\epsilon}\left(e^{-\lambda pt}|x|_E^p+\frac{|F(0)|_E^p}{\lambda^p}\right)\\
&+(1+\epsilon)L_f^p\left(\frac{p-1}{(\lambda-\delta)p}\right)^{p-1}\int_0^t e^{-\delta p(t-s)}|u^x(s)|_E^p\,ds\end{aligned}$$

for some constant $c_{p,\epsilon} > 0$. According to (7.17), this implies that for any $p \geq \bar{p} \vee 2$

$$\begin{aligned} e^{\delta pt}\,\mathbb{E}\,|u^x(t)|_E^p &\leq c_{p,\epsilon}\,e^{\delta pt}\,\mathbb{E}\,|\gamma(u^x)(t)|_E^p + c_{p,\epsilon}\,\left(|x|_E^p + \frac{e^{\delta pt}|F(0)|_E^p}{\lambda^p}\right)\\ &+ (1+\epsilon)L_f^p\,\left(\frac{p-1}{(\lambda-\delta)p}\right)^{p-1}\int_0^t e^{\delta ps}\,\mathbb{E}\,|u^x(s)|_E^p\,ds\\ &\leq c_{p,\epsilon}\,c_{1,p}\frac{M_g^p}{(\lambda-\delta)^{c_{2,p}}}\sup_{s\leq t} e^{\delta ps}\,\left(1+\mathbb{E}\,|u^x(s)|_E^{\beta p}\right) + c_{p,\epsilon}\,\left(|x|_E^p + \frac{e^{\delta pt}|F(0)|_E^p}{\lambda^p}\right)\\ &+ (1+\epsilon)L_f^p\,\left(\frac{p-1}{(\lambda-\delta)p}\right)^{p-1}\int_0^t \sup_{\sigma\leq s} e^{\delta p\sigma}\,\mathbb{E}\,|u^x(\sigma)|_E^p\,ds. \end{aligned} \tag{7.21}$$

Now, if $\beta < 1$, due to the Young inequality for any $\eta > 0$ we can find some $c_\eta > 0$ such that

$$\begin{aligned} e^{\delta pt}\,\mathbb{E}\,|u^x(t)|_E^p &\leq \eta\,\alpha_\epsilon \sup_{s\leq t} e^{\delta ps}\;\mathbb{E}\,|u^x(s)|_E^p + e^{\delta pt}\,\left[\frac{c_{p,\epsilon}\,|F(0)|_E^p}{\lambda^p} + c_\eta\,\alpha_\epsilon\right] + c_{p,\epsilon}\,|x|_E^p\\ &+ (1+\epsilon)L_f^p\,\left(\frac{p-1}{(\lambda-\delta)p}\right)^{p-1}\int_0^t \sup_{\sigma\leq s} e^{\delta p\sigma}\mathbb{E}\,|u^x(\sigma)|_E^p\,ds, \end{aligned}$$

where

$$\alpha_\epsilon := c_{p,\epsilon}\,c_{1,p}\frac{M_g^p}{(\lambda-\delta)^{c_{2,p}}} \leq c_{p,\epsilon}\,c_{1,p}M_g^p\left(\frac{2}{L_f}\right)^{c_{2,p}} =: K_\epsilon.$$

Therefore, if we take $\eta_\epsilon > 0$ such that $\eta < 1/K_\epsilon$ for any $\eta \leq \eta_\epsilon$ and define

$$h(t) := \sup_{s\leq t} e^{\delta ps}\,\mathbb{E}\,|u^x(s)|_E^p, \tag{7.22}$$

$$k_{\eta,\epsilon}(t) := \frac{c_{p,\epsilon}}{1-\eta\,K_\epsilon}\,\left[\frac{|F(0)|_E^p}{\lambda^p} + \frac{c_\eta\,\alpha_\epsilon}{c_{p,\epsilon}} + e^{-\delta pt}|x|_E^p\right],$$

and

$$N_{\delta,\epsilon,\eta} := \frac{1+\epsilon}{1-\eta\,K_\epsilon}L_f^p\,\left(\frac{p-1}{p}\right)^{p-1}\,(\lambda-\delta)^{1-p},$$

then we have

$$h(t) \leq N_{\delta,\epsilon,\eta}\int_0^t h(s)\,ds + e^{\delta pt}\,k_{\eta,\epsilon}(t).$$

Thanks to the Gronwall lemma this yields

$$h(t) \leq e^{\delta pt}\,k_{\eta,\epsilon}(t) + N_{\delta,\epsilon,\eta}\int_0^t e^{N_{\delta,\epsilon,\eta}(t-s)}\,e^{\delta ps}k_{\eta,\epsilon}(s)\,ds,$$

so that

$$\mathbb{E}\,|u^x(t)|_E^p \leq k_{\eta,\epsilon}(t) + N_{\delta,\epsilon,\eta}\int_0^t e^{-(\delta p - N_{\delta,\epsilon,\eta})(t-s)}k_{\eta,\epsilon}(s)\,ds, \qquad t\geq 0. \tag{7.23}$$

Now, the function

$$f_{\epsilon,\eta}(\delta) := \delta p - \frac{1+\epsilon}{1-\eta\,K_\epsilon}L_f^p\,\left(\frac{p-1}{p}\right)^{p-1}\,(\lambda-\delta)^{1-p}, \qquad \delta \in\,(0,\lambda - L_f/2),$$

attains its maximum at

$$\bar{\delta} := \lambda - L_f \frac{p-1}{p} \left(\frac{1+\epsilon}{1-\eta\, K_\epsilon} \right)^{1/p} \in \, (0, \lambda - L_f/2),$$

since

$$\frac{p-1}{p} \left(\frac{1+\epsilon}{1-\eta\, K_\epsilon} \right)^{1/p} > \frac{1}{2},$$

for $p \geq 2$. As we are assuming $\lambda > L_f$, we can find $\bar{\epsilon} > 0$ and $\bar{\eta} < \eta_{\bar{\epsilon}}$ such that

$$f_{\epsilon,\eta}(\bar{\delta}) = p \left(\lambda - L_f \left(\frac{1+\bar{\epsilon}}{1-\bar{\eta}\, K_{\bar{\epsilon}}} \right)^{1/p} \right) > 0,$$

and, since

$$\|k_{\bar{\eta},\bar{\epsilon}}\|_\infty < +\infty,$$

we can conclude that (7.19) holds for $p \geq \bar{p} \vee 2$. Estimate (7.19) for any $p \geq 1$ follows from the Hölder inequality.

Next, assume that $\beta = 1$. If we fix $\epsilon_p > 0$ such that

$$(1+\epsilon_p) \left(\frac{p-1}{p} \right)^{p-1} < 1,$$

and take $\delta = \lambda/2$, due to (7.21) for any $p \geq \bar{p}$ we have

$$\begin{aligned}
&e^{\delta p t}\, \mathbb{E}\, |u^x(t)|_E^p \leq \alpha_{\lambda,p} \sup_{s \leq t} e^{\delta p s}\; \mathbb{E}\, |u^x(s)|_E^p \\
&+ e^{\delta p t} \left[\frac{c_{p,\epsilon_p}\, |F(0)|_E^p}{\lambda^p} + \alpha_{\lambda,p} + e^{-\delta p t} c_{p,\epsilon_p}\, |x|_E^p \right] + \frac{L_f^p 2^{p-1}}{\lambda^{p-1}} \int_0^t \sup_{\sigma \leq s} e^{\delta p \sigma} \mathbb{E}\, |u^x(\sigma)|_E^p\, ds,
\end{aligned}$$

where

$$\alpha_{\lambda,p} := c_{p,\epsilon_p}\, c_{1,p} \frac{M_g^p 2^{c_{2,p}}}{\lambda^{c_{2,p}}}.$$

Therefore, if we take λ such that $\alpha_{\lambda,p} < 1$, we have

$$h(t) \leq e^{\delta p t}\, k_{\lambda,\delta}(t) + N_{\lambda,\delta} \int_0^t e^{\delta p s}\, h(s)\, ds, \qquad t \geq 0,$$

with $h(t)$ defined as in (7.22) (for $\delta = \lambda/2$) and

$$k_{\lambda,p}(t) := \frac{1}{1-\alpha_{\lambda,p}} \left[\frac{c_{p,\epsilon_p}\, |F(0)|_E^p}{\lambda^p} + \alpha_{\lambda,p} + e^{-\frac{\lambda}{2} p t} c_{p,\epsilon_p}\, |x|_E^p \right],$$

and

$$N_{\lambda,p} := \frac{1}{1-\alpha_{\lambda,p}} \frac{L_f^p 2^{p-1}}{\lambda^{p-1}} = \frac{1}{1 - c_{p,\epsilon_p}\, c_{1,p} M_g^p (\lambda/2)^{-c_{2,p}}} \frac{L_f^p 2^{p-1}}{\lambda^{p-1}}$$

As in (7.23) we can conclude

$$\mathbb{E}\, |u^x(t)|_E^p \leq k_{\lambda,p}(t) + N_{\lambda,p} \int_0^t e^{-(\frac{\lambda}{2} p - N_{\lambda,p})(t-s)} k_{\lambda,p}(s)\, ds, \qquad t \geq 0.$$

This means that we have (7.19) if

$$\frac{\lambda p}{2} - N_{\lambda,p} = \frac{\lambda p}{2}\left[1 - \frac{1}{1 - c_{p,\epsilon_p}\, c_{1,p} M_g^p (\lambda/2)^{-c_{2,p}}}\left(\frac{L_f 2}{\lambda}\right)^p \frac{1}{p}\right] > 0.$$

Therefore, if we set

$$k_{1,p} := \frac{2^p}{p}, \qquad k_{2,p} := c_{p,\epsilon_p} c_{1,p} 2^{c_{2,p}},$$

we have that

$$\frac{\lambda p}{2} - N_{\lambda,p} > 0,$$

if

$$k_{1,p}\,\frac{L_f^p}{\lambda^p} + k_{2,p}\,\frac{M_g^p}{\lambda^{c_{2,p}}} < 1.$$

Finally, once we have (7.19), estimate (7.20) follows as in [2, Theorem 6.2].

Remark 7.3 If g has growth strictly less than linear (that is, $\beta < 1$), then the *size* of g does not play any role in order to have estimates (7.19) and (7.20). Actually, they hold for any $\lambda > L_f$, exactly as in the deterministic case (when $g = 0$). If g has linear growth (that is, $\beta = 1$), then we have to assume that $\lambda > f_p(L_f, M_g)$, for some function f_p which is increasing with respect to both variables and such that

$$f_p(x,0) = k_{1,p}^{1/p}\, x, \qquad f_p(0,y) = k_{2,p}^{1/c_{2,p}}\, y^{p/c_{2,p}},$$

for some constants $k_{1,p}$ and $k_{2,p}$ defined in the proof of Theorem 7.3. This means, in particular, that we can assume λ to be arbitrarily close to zero, if we take L_f and M_g sufficiently small. On the other hand, if L_f and/or M_g are given, we can prove (7.19) and (7.20) by taking λ sufficiently large.

From the theorem above, by standard arguments we have that there exists an invariant for the semigroup P_t associated with system (7.1). In the next theorem we show that two solutions starting from any two different initial data converge the one to the other as time t goes to infinity. This will imply that the invariant measure is unique.

Theorem 7.4 *Under Hypotheses 7.1 to 7.4, for any $p \geq \bar{p}$ there exist two positive constants $h_{1,p}$ and $h_{2,p}$ such that if*

$$h_{1,p}\,\frac{L_f^p}{\lambda^p} + h_{2,p}\,\frac{L_g^p}{\lambda^{c_{2,p}}} < 1,$$

then for any initial data $x, y \in E$

$$\lim_{t\to\infty} E\,|u^x(t) - u^y(t)|_E^{\bar{p}} = 0. \tag{7.24}$$

Proof If we define $\rho(t) := u^x(t) - u^y(t)$ and $v(t) := \rho(t) - [\gamma(u^x) - \gamma(u^y)](t)$, we have

$$\frac{dv}{dt}(t) = Av(t) + F(u^x(t)) - F(u^y(t)), \qquad v(0) = x - y.$$

Hence, according to Hypothesis 7.4 and to the Lipschitz continuity of F we get

$$\frac{d^-}{dt}|v(t)|_E \leq \left\langle Av(t), \delta_{v(t)}\right\rangle_E + \left\langle F(u^x(t)) - F(u^y(t)), \delta_{v(t)}\right\rangle_E$$

$$\leq -\lambda\,|v(t)|_E + L_f\,|u^x(t) - u^y(t)|_E.$$

By comparison, this yields

$$|v(t)|_E \le e^{-\lambda t}|x-y|_E + L_f \int_0^t e^{-\lambda(t-s)}|\rho(s)|_E\,ds,$$

and hence, for any $p\ge 1$ and $\eta>0$ we get

$$|\rho(t)|_E^p \le 2^{p-1}(1+\eta)\,|\gamma(u^x)(t)-\gamma(u^y)(t)|_E^p + c_{p,\eta}\,e^{-\lambda p t}|x-y|_E^p$$
$$+\,2^{p-1}\,(1+\eta)\,L_f^p\left(\int_0^t e^{-\lambda(t-s)}|\rho(s)|_E\,ds\right)^p,$$

for some constant $c_{p,\eta}>0$. As in the proof of Theorem 7.3, for any $\delta\in\,(0,\lambda)$ this yields

$$e^{\delta p t}\,\mathbb{E}\,|\rho(t)|_E^p \le 2^{p-1}(1+\eta)\,e^{\delta p t}\,\mathbb{E}\,|\gamma(u^x)(t)-\gamma(u^y)(t)|_E^p + c_{p,\eta}\,|x-y|_E^p$$
$$+\,2^{p-1}(1+\eta)\,\frac{L_f^p}{(\lambda-\delta)^{p-1}}\,\left(\frac{p-1}{p}\right)^{p-1}\int_0^t e^{\delta p s}\mathbb{E}\,|\rho(s)|_E^p\,ds.$$

Then, due to (7.13), for any $p\ge\bar p$ we have

$$e^{\delta p t}\,\mathbb{E}\,|\rho(t)|_E^p \le 2^{p-1}(1+\eta)\,c_{1,p}\frac{L_g^p}{(\lambda-\delta)^{c_{2,p}}}\sup_{s\le t} e^{\delta p s}\,\mathbb{E}\,|\rho(s)|_E^p + c_{p,\eta}\,|x-y|_E^p$$
$$+\,2^{p-1}(1+\eta)\,\frac{L_f^p}{(\lambda-\delta)^{p-1}}\,\left(\frac{p-1}{p}\right)^{p-1}\int_0^t e^{\delta p s}\,\mathbb{E}\,|\rho(s)|_E^p\,ds.$$

Thus, if we set $\delta=\lambda/2$ and

$$\alpha_{\lambda,p} := 2^{p-1}\,c_{1,p}\frac{L_g^p 2^{c_{2,p}}}{\lambda^{c_{2,p}}},$$

and

$$\beta_{\lambda,p} := 4^{p-1}\,\frac{L_f^p}{\lambda^{p-1}}\,\left(\frac{p-1}{p}\right)^{p-1},$$

and define as in the proof of Theorem 7.3

$$h(t) := \sup_{s\le t} e^{\frac{\lambda}{2}ps}\,\mathbb{E}\,|\rho(s)|_E^p,$$

we have

$$h(t)\le (1+\eta)\alpha_{\lambda,p}\,h(t) + c_{p,\eta}\,|x-y|_E^p + (1+\eta)\,\beta_{\lambda,p}\int_0^t h(s)\,ds.$$

Now, if we assume that

$$\alpha_{\lambda,p} = 2^{c_{2,p}+p-1}\,c_{1,p}\frac{L_g^p}{\lambda^{c_{2,p}}} < 1,$$

there exists some $\bar\eta$ such that $\alpha_{\lambda,p}(1+\eta)<1$, for any $\eta\ge\bar\eta$, so that

$$h(t)\le \frac{c_{p,\eta}}{1-(1+\eta)\alpha_{\lambda,p}}\,|x-y|_E^p + \frac{(1+\eta)\,\beta_{\lambda,p}}{1-(1+\eta)\alpha_{\lambda,p}}\int_0^t h(s)\,ds.$$

By the Gronwall lemma this implies

$$h(t)\le \frac{c_{p,\eta}}{1-(1+\eta)\alpha_{\lambda,p}}\,|x-y|_E^p\,\exp\left(\frac{(1+\eta)\,\beta_{\lambda,p}}{1-(1+\eta)\alpha_{\lambda,p}}\,t\right),\qquad t\ge 0,$$

and hence, recalling how $h(t)$ was defined,

$$\mathbb{E}\,|\rho(t)|_E^p \leq \frac{c_{p,\eta}}{1-(1+\eta)\alpha_{\lambda,p}}\,|x-y|_E^p\,\exp\left(\left(\frac{\lambda p}{2}-\frac{(1+\eta)\,\beta_{\lambda,p}}{1-(1+\eta)\alpha_{\lambda,p}}\right)\,t\right).$$

Therefore, if we set

$$h_{1,p} := 2^{2p+1}\left(\frac{p-1}{p}\right)^p\frac{1}{p-1},\qquad h_{2,p} := 2^{p+c_{2,p}+1}c_{1,p},$$

and assume

$$h_{1,p}\,\frac{L_f^p}{\lambda^p}+h_{2,p}\,\frac{L_g^p}{\lambda^{c_{2,p}}}<1,$$

for some $\eta \leq \bar{\eta}$ sufficiently small we have

$$\frac{\lambda p}{2}-\frac{(1+\eta)\,\beta_{\lambda,p}}{1-(1+\eta)\alpha_{\lambda,p}}=\frac{\lambda p}{2}\left[1-\frac{(1+\eta)2^{2p+1}\left(\frac{p-1}{p}\right)^p\frac{1}{p-1}\frac{L_f^p}{\lambda^p}}{1-(1+\eta)2^{p+c_{2,p}+1}c_{1,p}\frac{L_g^p}{\lambda^{c_{2,p}}}}\right]>0$$

and the thesis follows.

By standard arguments (for all details see [6]), as a consequence of Theorem 7.3 and Theorem 7.4 we have the following result.

Corollary 7.1 *Assume that Hypotheses 7.1 to 7.4 hold. Then there exist suitable nonnegative constants α_1, α_2, α_3, and δ such that if L_f, L_g are the Lipschitz constants of f and g, respectively, and M_g is defined by* (7.18) *and if*

$$\lambda > \max\{\,\alpha_1\,L_f,\ \alpha_2\,M_g^\delta,\ \alpha_3\,L_g^\delta\,\}, \tag{7.25}$$

then the transition semigroup P_t associated with system (7.1) *admits a unique invariant measure which is ergodic and strongly mixing.*

Remark 7.4

1. From Theorem 7.3 one sees that if

   $$M_g = \sup_{\xi\in\overline{\mathcal{O}}//\sigma\in\mathbb{R}^r}\frac{|g(\xi,\sigma)|}{1+|\sigma|^\beta},$$

   with $\beta<1$, then we can take $\alpha_2=0$.

2. It can useful to notice that due to condition (7.25) in order to have existence and uniqueness of the invariant measure for P_t we can assume λ to be arbitrarily close to zero, if we take L_f, M_g, and L_g sufficiently small. On the other hand, if L_f, M_g and L_g are given, we can prove the uniform estimates (7.19) and (7.20) and the convergence result (7.24) by taking λ sufficiently large.

References

[1] S. Cerrai, Second Order PDE's in Finite and Infinite Dimension. A Probabilistic Approach, Lecture Notes in Mathematics Series 1762, Springer Verlag (2001).

[2] S. Cerrai, *Stochastic reaction-di usion systems with multiplicative noise and non-Lipschitz reaction term*, Probability Theory and Related Fields 125 (2003), 271-304.

[3] S. Cerrai, *Stabilization by noise for a class of stochastic reaction-di usion equations*, to appear in Probability Theory and Related Fields.

[4] S. Cerrai, M. Röckner, *Large deviations for invariant measures of stochastic reaction-di usion systems with multiplicative noise and non-Lipschitz reaction term*, Annales de l'IHP (Probabilités et Statistiques) 41(1) (2005), 69-105.

[5] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge (1992).

[6] G. Da Prato, J. Zabczyk, Ergodicity for Infinite Dimensional Systems, London Mathematical Society, Lecture Notes Series 229, Cambridge University Press, Cambridge (1996).

[7] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge University Press, Cambridge (1989).

[8] M. Hairer, *Exponential mixing properties of stochastic PDEs through asymptotic coupling*, Probability Theory and Related Fields 124 (2002), 345-380.

[9] J. Mattingly, *Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics*, Communications in Mathematical Physics 230 (2002), 421-462.

[10] C. Mueller, *Coupling and invariant measures for the heat equation with noise*, Annals of Probability 21 (1993), 2189-2199.

[11] S. Peszat, *Existence and uniqueness of the solution for stochastic equations on Banach spaces*, Stochastics and Stochastics Reports 55 (1995), 167-193.

[12] R. Sowers, *Large deviations for the invariant measure of a reaction-di usion equation with non-Gaussian perturbations*, Probability Theory and Related Fields 92 (1992), 393-421.

8 On $L^1(H,\mu)$-Properties of Ornstein–Uhlenbeck Semigroups

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8.1 Introduction

Let H and K be real separable Hilbert spaces and consider a linear stochastic differential equation in H

$$\begin{cases} dX = AXdt + BdW \\ X_0 = x \in H \end{cases} . \tag{8.1}$$

We assume that A generates on H a strongly continuous semigroup (S_t); W is a standard cylindrical Wiener process on K and $B \in L(K,H)$ (bounded), $B \neq 0$. Let $Q = BB^*$ and

$$Q_t x = \int_0^t S_s Q S_s^* x ds, \quad x \in H, \quad t \in (0,+\infty].$$

If for $t \in (0,+\infty)$, the operators Q_t are of trace class, then the solution to (8.1) is given by the formula

$$X_t(x) = S_t x + \int_0^t S_{t-s} B dW_s, \quad t \geq 0. \tag{8.2}$$

The process X is Gaussian and Markov and it is called an Ornstein–Uhlenbeck (O-U) process in H.

In this note our basic assumption is the following:

$$\int_0^\infty \operatorname{tr} S_s Q S_s^* ds < +\infty \tag{8.3}$$

which is equivalent to the existence of an invariant measure for (8.1). If (8.3) holds, then the Gaussian measure $\mu = \mathcal{N}(0, Q_\infty)$ on H with mean 0 and the covariance operator Q_∞ is an invariant measure for the O-U process X given by (8.2) (see [D-Z; S]).

The O-U semigroup (R_t), i.e., the transition semigroup for the O-U process X in (8.2) is given by

$$R_t \phi(x) = E\phi(X_t(x)) = \int_H \phi(S_t x + y)\, \mu_t(dy),$$
$$\phi \in B_b(H) \quad \text{(Borel bounded)},$$

where $\mu_t = \mathcal{N}(0, Q_t)$. Then (R_t) is a positivity preserving C_0-semigroup of contractions on $L^p(H,\mu)$ for $1 \leq p < \infty$ (see, e.g., [D-Z; P]).

We define the following class of cylindrical functions:

$$\mathbb{F}C_b^\infty := \{\phi : H \to \mathbb{R} : \phi(x) = f(\langle x, h_1\rangle, \ldots, \langle x, h_m\rangle), \quad \text{for some} \quad m \in \mathbb{N}$$
$$\text{and} \quad h_1, \ldots, h_m \in \operatorname{dom}(A^*), \quad f \in C_b^\infty(\mathbb{R}^m)\}.$$

and the differential operator for $\phi \in \mathbb{F}C_b^\infty$

$$\mathcal{L}^0\phi(x) := \frac{1}{2}\operatorname{tr}\left(QD^2\phi(x)\right) + \langle Ax, D\phi(x)\rangle, \quad x \in \operatorname{dom}(A),$$

where D denotes the Fréchet derivative.

It was proved in [Ch-G; E, Lem. 1] (see also [D-Z; P]) that under (8.3) the generator $\mathcal{L}$ of (R_t) in $L^p(H,\mu)$, $1 \le p < \infty$, is the closure of $\mathcal{L}^0$, and, moreover, $\mathbb{F}C_b^\infty$ is invariant for (R_t).

Two classes of O-U semigroups have been intensely studied for many years. The first one is the class of symmetric O-U semigroups, which is important because of applications in physics. Recall that symmetric transition semigroups correspond to reversible processes. The second one is the class of strongly Feller O-U semigroups, which is important in the theory of Kolmogorov equations because of smoothing properties of such semigroups.

Regularity properties in $L^p(H,\mu)$, for $1 < p < \infty$ were studied for various classes of O-U semigroups in many papers (e.g., [G-Ch,...], [F], [D-F-Z], [D-Z, P] Sec. 10 and references therein). Properties of O-U semigroups in $L^1(H,\mu)$ may completely differ from those in $L^p(H,\mu)$ for $1 < p < \infty$. For $H = \mathbb{R}^n$ a result on the L^1-spectrum of the O-U generator in [M-P-Pr] implies that, e.g., the compactness and analycity of (R_t) fail in L^1. It is known that a strongly Feller R_t, $t > 0$, maps $L^p(H,\mu)$ into $C^\infty(H) \cap W^{1,p}(H)$ for $1 < p < \infty$ ([Ch-G; E], [D-F-Z]) but by [D-F-Z] $R_t f$ may not be even continuous if $f \in L^1(H,\mu)$ and $\dim H = \infty$. It follows easily from Prop. 2.2. b) in this note that R_t does not map $L^1(H,\mu)$ into the Sobolev space $W^{1,1}(H)$.

The aim of this chapter is to investigate certain properties of O-U semigroups in $L^1(H,\mu)$. In Sec. 8.2 we briefly review known results on the compactness, hypercontractivity, and $W^{n,p}_{Q_\infty}$-regularity of O-U semigroups in $L^p(H,\mu)$ for $1 < p < \infty$ and we give direct simple proofs of lack of these properties in $L^1(H,\mu)$. New results are proved in Sec. 8.3 and Sec. 8.4. In Sec. 8.3 we describe the $L^1(H,\mu)$-spectrum of $\mathcal{L}$ for a certain class of O-U semigroups that contains (nonpathological) strongly Feller semigroups and we thus extend the finite-dimensional result of [M-P-Pr]. A corollary on convergence to equilibrium is proved in Sec. 8.4 (Thm. 8.3). We also recall the corresponding result in $L^p(H,\mu)$ for $1 < p < \infty$ (Thm. 8.2). Example 8.1 shows that the situation in infinite dimension is completely different from that in finite dimension.

In the last part of this section we briefly summarize certain properties of general O-U semigroups proved in [Ch-G; R and Q] (and slightly extended in [Ch] to the case of possibly noninjective Q_∞).

Let $H_0 = Q_\infty^{1/2}(H)$, $H_1 = \overline{H}_0$ (the closure of H_0 in H), and $Q_\infty^{-1/2}$ denote the pseudoinverse of $Q_\infty^{1/2}$. Then $Q_\infty^{-1/2} : H_0 \to H_1$ and $H_0 = Q_\infty^{1/2}(H_1)$.

Lemma 8.1 *We have*

(i) $S_t(H_0) \subset H_0$ *for each* $t \ge 0$.

(ii) The family of operators $S_0(t) = Q_\infty^{-1/2} S_t Q_\infty^{1/2}|_{H_1}$, $t \ge 0$, *is a* C_0*-semigroup of contractions on* H_1.

(iii) For each t, *the adjoint* $S_0^*(t) = \overline{Q_\infty^{1/2} S_t^* Q_\infty^{-1/2}}$.

(iv) Let A_0 *be the generator of* $(S_0(t))$. *Then* $H_2 = Q_\infty^{1/2}(\operatorname{dom} A^*)$ *is a core for* A_0^* *in* H_1.

(v) $\ker Q_\infty \subset \ker Q$.

For $h \in H_1$ we denote by φ_h the μ measurable linear functional on H_1 s.t. $\varphi_h(x) = \left\langle h, Q_\infty^{-1/2} x \right\rangle$ for $x \in H_0$ and let $E_h(x) = \exp\left(\varphi_h(x) - \frac{1}{2}\|h\|^2\right)$, $x \in H_1$. Since $\mu(H_1) = 1$, for each $1 \le p \le \infty$ the spaces $L^p(H,\mu)$ and $L^p(H_1,\mu)$ are isometrically isomorphic. Then

$$\int_{H_1} \varphi_h^2 d\mu = \|h\|^2 = \int_H \varphi_h^2 d\mu \quad \text{and}$$
$$\int_{H_1} E_h d\mu = 1 = \int_H E_h d\mu.$$

Let $L^2(H_1,\mu) = \oplus_{n=0}^{\infty} \mathcal{H}_n$ be the Ito–Wiener decomposition (e.g., [W], [D-Z; P]). By I_n we denote the orthogonal projection in $L^2(H_1,\mu)$ onto $\mathcal{H}_n$.

If $T \in L(H_1)$, then the operator $Q_\infty^{1/2} T Q_\infty^{-1/2}$ is bounded on H_0 and hence it can be uniquely extended to a μ-measurable linear transformation T_{Q_∞} on H_1 such that

$$\int_{H_1} \|T_{Q_\infty} x\|^2 \mu(dx) = \operatorname{tr} Q_\infty^{1/2} T T^* Q_\infty^{1/2}.$$

Lemma 8.2 *([Ch-G; Q]). We have*

(i) $R_t \phi(x) = \int_{H_1} \phi\left((S_0(t))_{Q_\infty} x + (I - S_0(t) S_0^*(t))_{Q_\infty}^{1/2} y\right) \mu(dy) :=$
$\Gamma(S_0^*(t))(x)$ *for μ a.a.* $x \in H_1$, $\phi \in L^p(H_1,\mu)$, $1 \le p < \infty$.

(ii) $R_t I_n(\varphi_h^n) = I_n\left(\varphi_{S_0^*(t)h}^n\right)$, $h \in H_1$.

(iii) $R_t E_h = E_{S_0^*(t)h}$ $h \in H_1$.

8.2 Compactness and hypercontractivity

Let (R_t) be an O-U semigroup and fix $t > 0$. If R_t is a compact operator on $L^2(H,\mu)$, then by interpolation R_t is compact on $L_H^p = L^p(H,\mu)$ for all $1 < p < \infty$ ([Ch-G; E]). Prop. 8.1 below shows that the compactness fails for $p = 1$. (This follows from Thm. 3 in [Ch-G; Q] but we give a direct argument because of its simplicity.) By [Ch-G; Q], R_t is compact on L_H^p for $1 < p < \infty$ iff $S_0(t)$ is a compact operator and

$$\|S_0(t)\| < 1. \tag{8.4}$$

Moreover, by [Ch-G; R or Q], (8.4) is equivalent to the following condition:

$$Q_t^{1/2}(H) = Q_\infty^{1/2}(H). \tag{8.5}$$

Recall also that if R_t is strongly Feller (sF for short) and (8.3) holds, then $S_t(H) \subset Q_t^{1/2}(H)$ and (8.5) is satisfied (see [D-Z; S or P]). Hence $S_0(t) = \left(Q_\infty^{-1/2} S_t\right) Q_\infty^{1/2}$ is compact, as the superposition of the compact operator $Q_\infty^{1/2}$ and the bounded operator $Q_\infty^{-1/2} S_t$, and therefore R_t is compact on L_H^p for $1 < p < \infty$ (comp. [D-Z; E or P] for a different proof of the compactness).

By [Ch-G; Q] and [F] condition (8.5) is equivalent to the hypercontractivity of R_t on L_H^p, $1 < p < \infty$. (R_t is hypercontractive on L^p if for a certain $q > p$, $R_t : L_H^p \to L_H^q$ is a contraction.) The hypercontractivity also fails in L_H^1, which follows from Nelson's conditions ([Ch-G; Q] Thm. 2, [S]). A direct simple proof is given in Prop. 8.2a below, where we follow [N].

(8.5) also implies that for all $1 < p < \infty$ and $n = 1, 2, \dots$, $R_t : L^p_H \to W^{n,p}_{Q_\infty}$ is bounded. Prop. 8.2b) below shows that this is not true for $p = 1$.

Recall that the Sobolev space $W^{1,p}_{Q_\infty}(H)$, $1 \le p < \infty$, is defined as the completion of space $\mathcal{P}(H_0)$ of polynomials in $L^p(H, \mu)$ with respect to the norm

$$\|\phi\|^p_{W^{1,p}_{Q_\infty}} = \|\phi\|^p_{L^p_H} + \int_H \left\| Q^{1/2}_\infty D\phi \right\|^p_{H_1} d\mu,$$

where

$$\mathcal{P}(H_0) = \operatorname{lin}\{\varphi^n_h : n = 0, 1, 2, \dots, \quad h \in H_0\}.$$

We say that

i) (R_t) is a compact (etc.) semigroup, if for every $t > 0$, R_t is a compact operator (etc.),

ii) (R_t) is an eventually compact (etc.) semigroup, if for a certain $a > 0$ and every $t \ge a$, R_t is compact (etc.) (see [E-Na]).

Proposition 8.1 (comp. [Ch-G; Q] Thm. 3)**.** *Assume (8.3) and let $(R_t)_{t\ge 0}$ be an arbitrary O-U semigroup acting in the scale of $L^p(H, \mu)$, $1 \le p < \infty$. Then $(R_t)_{t\ge 0}$ is not a compact semigroup on $L^1(H, \mu)$.*

Proof Let $R_t = \Gamma(S^*_0(t))$, $t \ge 0$, and fix $t > 0$ and $h \in H_1$ such that $S^*_0(t) \ne 0$ and $\|S^*_0(t)h\| = \alpha > 0$. Let for $n = 1, 2, \dots$

$$\Psi_n(x) = E_{nh}(x) = \exp\left(n\varphi_h(x) - \frac{1}{2}n^2\|h\|^2\right).$$

Then by [Ch-G; Q] (see also Introduction)

$$R_t\Psi_n(x) = E_{nS^*_0(t)h}(x)$$

and

$$\|\Psi_n\|_{L^1_H} = \|R_t\Psi_n\|_{L^1_H} = 1.$$

Suppose that $R_t : L^1(H, \mu) \to L^1(H, \mu)$ is a compact operator. Then we can choose a subsequence $(R_t\Psi_{m_n})$ convergent in $L^1(H, \mu)$ to a certain ϕ of norm one. Since $\varphi_{S^*_0(t)h} \in L^2(H, \mu)$, for μ almost all (a.a.) $x \in H$, $\varphi_{S^*_0(t)h}(x)$ is finite and for such x

$$\log R_t\Psi_n(x) = n\varphi_{S^*_0(t)h}(x) - n^2\frac{\alpha^2}{2} \to -\infty,$$

as $n \to \infty$. Hence $R_t\Psi_n(x) \to 0$ for μ almost all (a.a.) $x \in H$ and we obtain a contradiction. ■

Remark 8.1 We have actually proved that if $S^*_0(t) \ne 0$, then $\Gamma(S^*_0(t))$ is not a compact operator on $L^1(H, \mu)$. Note that

$$\Gamma(0)\varphi(x) = \int_H \varphi(y)\,\mu(dy), \quad \text{for} \quad \varphi \in L^1_H,$$

$$\text{i.e.,} \quad \Gamma(0) : L^1_H \to \mathbb{R}\cdot\mathbf{1} \subset L^p_H, \quad 1 \le p \le \infty,$$

and, in particular, $\Gamma(0)$ is a compact operator on L^1_H. This observation is used in Sect. 8.4, where we give a simple example of an O-U semigroup (R_t) that is eventually compact on L^p_H for all $1 \le p < \infty$.

Proposition 8.2 *Assume (8.3) and let $R_t = \Gamma(S^*_0(t))$, $t \ge 0$, be an O-U semigroup acting on the scale of $L^p(H, \mu)$-spaces, $1 \le p < \infty$. Fix $t_0 \in (0, \infty)$. If $S^*_0(t_0) \ne 0$ then* *a) For every $q>1$, $R_{t_0}: L_H^1 \to L_H^q$ is unbounded.*

b) $R_{t_0}: L_H^1 \to W_{Q_\infty}^{1,1}(H)$ is unbounded.

Proof As in the proof of Prop. 8.1 consider the functions $\Psi_n = E_{nh}$, $n=1,2,\ldots$, for a fixed $h \in H_1$ with $\|S_0^*(t_0)h\| = \alpha > 0$. Then $R_{t_0}\Psi_n(x) = E_{nS_0^*(t_0)h}(x)$, $\|\Psi_n\|_{L_H^1} = 1$ and, $\|R_{t_0}\Psi_n\|_{L_H^q} = \exp\left\{\frac{1}{2}(q-1)n^2\alpha^2\right\}$, $q \ge 1$. Hence for $q>1$, $\sup_n \|R_{t_0}\Psi_n\|_{L_H^q} = +\infty$ and (a) follows. To prove (b) first observe that for $g \in H_0$, the function $e^{\varphi_g(x)} = \exp\left\langle Q_\infty^{-1/2}g, x\right\rangle$ is Fréchet differentiable (in x) on H and

$$De^{\varphi_g}(x) = Q_\infty^{-1/2} g e^{\varphi_g(x)}, x \in H,$$

which implies that

$$Q_\infty^{1/2} De^{\varphi_g} = g e^{\varphi_g}$$

for $g \in H_0$ and consequently for $g \in H_1$. Therefore

$$Q_\infty^{1/2} D(R_t\Psi_n) = nS_0^*(t)\, hR_t\Psi_n$$

and

$$\left\| Q_\infty^{1/2} DR_{t_0}\Psi_n \right\|_{L^1(H,\mu;H_1)} = n\alpha.$$

Hence $\sup_n \|R_{t_0}\Psi_n\|_{W_{Q_\infty}^{1,1}} = +\infty$, which proves b). ■

8.3 L^1 spectrum of $\mathcal{L}$

The spectrum in $L^2(H,\mu)$ of the O-U generator $\mathcal{L}$ of symmetric and compact semigroup (R_t) was described in [Ch-G;Q] Sec. 2 by second quantization method. Recently, this result has been generalized in [Ne], where the L^p-spectrum, $1<p<\infty$, of the O-U generator is characterized for hypercontractive and eventually norm continuous semigroup (R_t). For $H=\mathbb{R}^n$ the full description of the L^p-spectrum of $\mathcal{L}$, $1 \le p < \infty$, has been given in [M-P-Pr]. By this result the L^1-spectrum of $\mathcal{L}$ completely differs from the L^p-spectrum, $p \ne 1$, which is the same for all $1<p<\infty$.

It has been proved in [M-P-Pr] that if $H=\mathbb{R}^n$, (S_t) is exponentially stable and the corresponding O-U semigroup (R_t) is strongly Feller, then the spectrum of its generator $\mathcal{L}$ acting in $L^1(\mathbb{R}^n,\mu)$ is of the form

$$\operatorname{Sp}_{L^1}\mathcal{L} = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\} = \overline{\mathbb{C}}_-.$$

Below we prove a certain generalization of this result. As usual, $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$.
Theorem 8.1 Let (8.3) hold and assume that A_0^* has an eigenvalue $\lambda \in \mathbb{C}_-$. Then

$$\operatorname{Sp}_{L_H^1}\mathcal{L} = \overline{\mathbb{C}}_-$$

and every $\alpha \in \mathbb{C}_-$ is an eigenvalue of $\mathcal{L}$ in L_H^1.
Proof We consider two cases: $\lambda \in \mathbb{R}$ or $\lambda \in \mathbb{C}\backslash\mathbb{R}$ and reduce the problem to $H=\mathbb{R}$ or $H=\mathbb{R}^2$, respectively.
Part 1. Let $\lambda \in \mathbb{R}$, $\lambda<0$ and $h \in H_1$ be a corresponding eigenvector of norm one. In the notation of Introduction (Lem. 8.2) we have

$$R_t I_n(\varphi_h^n) = I_n\left(\varphi_{S_0^*(t)h}^n\right)$$

and

$$I_n\left(\varphi_h^n\right)(x)=\sqrt{n!}\chi_n\left(\varphi_h(x)\right), \quad x\in H,$$

where χ_n, $n=0,1,2,\ldots$, are the real Hermite polynomials (see, e.g., [Ch-G; Q] or [D-Z; P] p. 193). Hence

$$R_t\left(\chi_n\circ\varphi_h\right)=e^{n\lambda t}\chi_n\left(\varphi_h\right). \tag{8.6}$$

Since $\{\chi_n\}_{n=0}^{\infty}$ is an orthonormal basis (ONB for short) in $\mathcal{H}=L^2\left(\mathbb{R},\mathcal{N}(0,1)\right)$, the formula

$$U_tf=\sum\nolimits_{n=0}^{\infty}e^{n\lambda t}\left\langle f,\chi_n\right\rangle\chi_n, \quad f\in\mathcal{H} \tag{8.7}$$

defines a self adjoint H-S semigroup on $\mathcal{H}$ and its generator $\mathcal{A}$ satisfies

$$\mathcal{A}\chi_n(u)=n\lambda\chi_n(u)=|\lambda|\left(\chi_n''(u)-u\chi_n'(u)\right), \quad u\in\mathbb{R}.$$

(The last equality follows by the properties of the Hermite polynomials — see, e.g., [D-Z; S], p. 189.) Hence for an arbitrary polynomial $f\in\mathcal{P}(\mathbb{R})$

$$\mathcal{A}f(u)=|\lambda|\left(f''(u)-uf'(u)\right), \quad u\in\mathbb{R}.$$

Then the generator $\mathcal{A}$ coincides on $\mathcal{P}(\mathbb{R})$ with the O-U generator $\mathcal{L}^{\lambda}$ on $\mathcal{H}$ corresponding to the stochastic equation in $\mathbb{R}$

$$dz_t=\lambda z_t+\sqrt{-2\lambda}dw_t.$$

Since $\mathcal{P}(\mathbb{R})$ is a core for $\mathcal{L}^{\lambda}$, $\mathcal{A}=\mathcal{L}^{\lambda}$.
From (8.6) and (8.7) we obtain

$$R_t\left(f\circ\varphi_h\right)(x)=\left(U_tf\right)\left(\varphi_h(x)\right), \quad \text{for} \quad f\in\mathcal{H}, \quad x\in H. \tag{8.8}$$

Observe that for an H-valued random variable ξ with the distribution μ, $\varphi_h(\xi)$ has $\mathcal{N}(0,1)$ distribution and hence for $1\le p<\infty$

$$\int_H\left|f\left(\varphi_h(x)\right)\right|^p\mu(dx)=\int_{\mathbb{R}}|f(u)|^p\mathcal{N}(0,1)(du).$$

Therefore equality (8.8) extends to all $f\in L^1_{\mathbb{R}}=L^1\left(\mathbb{R},\mathcal{N}(0,1)\right)$. From this we deduce that

$$\begin{gathered}\text{if} \quad f\in\operatorname{dom}_{L^1_{\mathbb{R}}}(\mathcal{A}), \quad \text{then} \quad f\circ\varphi_h\in\operatorname{dom}_{L^1_H}(\mathcal{L}) \quad \text{and}\\ \mathcal{L}\left(f\circ\varphi_h\right)=(\mathcal{A}f)\circ\varphi_h.\end{gathered} \tag{8.9}$$

By the one-dimensional result ([Da-S] Thm. 3, see also [M-P-Pr] Thm. 5.1) every $\alpha\in\mathbb{C}_-$ is an eigenvalue of the O-U generator $\mathcal{A}$ considered in $L^1_{\mathbb{R}}$. Let $f_\alpha\in L^1_{\mathbb{R}}$ be a corresponding eigenvector. Then by (8.9), $\mathcal{L}\left(f_\alpha\circ\varphi_h\right)=\alpha\left(f_\alpha\circ\varphi_h\right)$ and hence every $\alpha\in\mathbb{C}_-$ is an eigenvalue of $\mathcal{L}$ in L^1_H. Since the spectrum of a closed operator is closed, we obtain $\overline{\mathbb{C}}_-\subset\operatorname{Sp}_{L^1_H}(\mathcal{L})$. On the other hand, $\operatorname{Sp}_{L^1_H}(\mathcal{L})\subset\overline{\mathbb{C}}_-$, since $\mathcal{L}$ generates a semigroup of contractions in L^1_H. This finishes the proof of Part 1.

Part 2. Let $\lambda=\beta+i\gamma$, where $\beta<0$, $\gamma\ne0$, and let $\widetilde{h}+i\widetilde{g}$ be a corresponding eigenvector, $\widetilde{h},\widetilde{g}\in\operatorname{dom}\left(A_0^*\right)$ in H_1. It is easy to check that $\widetilde{h}$ and $\widetilde{g}$ are linearly independent,

$$\begin{aligned}A_0^*\widetilde{h}&=\beta\widetilde{h}-\gamma\widetilde{g},\\ A_0^*\widetilde{g}&=\gamma\widetilde{h}+\beta\widetilde{g},\end{aligned} \tag{8.10}$$

and by the spectral properties of semigroups,

$$S_0^*(t)\,\widetilde{h} = e^{\beta t}\left(\cos(\gamma t)\,\widetilde{h} - \sin(\gamma t)\,\widetilde{g}\right), \qquad (8.11)$$
$$S_0^*(t)\,\widetilde{g} = e^{\beta t}\left(\sin(\gamma t)\,\widetilde{h} + \cos(\gamma t)\,\widetilde{g}\right).$$

Hence $G = \operatorname{lin}\left\{\widetilde{h}, \widetilde{g}\right\}$ is a two-dimensional real subspace invariant for $S_0^*(t), t \geq 0$. Let $\{h, g\}$ be an ONB in G and let $V = A_0^*|_G$. Then $T_t = e^{Vt} = S_0^*(t)|_G$. G with the ONB $\{h, g\}$ is identified as $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$ and V and T_t are treated as operators on $\mathbb{R}^2$. Let

$$T_t = \begin{bmatrix} \tau_t^{11} & \tau_t^{12} \\ \tau_t^{21} & \tau_t^{22} \end{bmatrix}.$$

Since T_t is a contraction, the formula

$$\begin{aligned} U_t f(x) &= \Gamma_{\mathbb{R}^2}(T_t) f(x) \\ &= \int_{\mathbb{R}^2} f\left(T_t^* x + \sqrt{I - T_t^* T_t}\, y\right) \mathcal{N}(0, I)(dy), \end{aligned}$$

$$\text{for any} \quad f \in L^p_{\mathbb{R}^2} = L^p\left(\mathbb{R}^2, \mathcal{N}(0, I)\right), \quad 1 \leq p < \infty,$$

defines a contraction U_t on $L^p_{\mathbb{R}^2}$. By (8.11), (T_t) is exponentially stable. Then

$$\widetilde{Q} = -(V + V^*),$$

obtained from the Liapunov equation satisfies

$$\widetilde{Q} = \widetilde{Q}^*, \qquad \int_0^\infty T_s^* \widetilde{Q} T_s\, ds = I,$$

$$\widetilde{Q}_t = \int_0^t T_s^* \widetilde{Q} T_s\, ds = I - T_t^* T_t.$$

Moreover $\widetilde{Q} > 0$, because V and V^* generate contraction semigroups and they are invertible by (8.10). This yields the equality for $f \in L^p_{\mathbb{R}^2}$

$$U_t f(x) = \int_{\mathbb{R}^2} f(T_t^* x + z)\, \mathcal{N}\left(0, \widetilde{Q}_t\right) dz,$$

which implies that (U_t) is the transition semigroup of the solution to the equation in $\mathbb{R}^2$

$$dX_t = V^* X_t dt + \sqrt{-(V + V^*)}\, dW_t.$$

Let

$$\chi_{kl}(u) = \chi_k(u_1)\,\chi_l(u_2), \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

(the Hermite polynomials in $\mathbb{R}^2$), and let $f_{e_j}(u) = \langle u, e_j \rangle = u_j$, $j = 1, 2$. Then by Lem. 8.2 ii) with $R_t = U_t$ we have

$$\begin{aligned} U_t(\chi_{kl})(u) &= \chi_k\left(f_{T_t e_1}(u)\right) \cdot \chi_l\left(f_{T_t e_2}(u)\right) \\ &= \chi_k\left(\tau_t^{11} u_1 + \tau_t^{21} u_2\right) \cdot \chi_l\left(\tau_t^{12} u_1 + \tau_t^{22} u_2\right), \quad u \in \mathbb{R}^2, \end{aligned}$$

and for $x \in H$

$$\begin{aligned} R_t\left(\chi_{kl} \circ (\varphi_h, \varphi_g)\right)(x) &= \chi_k\left(\varphi_{S_0^*(t)h}(x)\right) \cdot \chi_l\left(\varphi_{S_0^*(t)g}(x)\right) \\ &= \chi_k\left(\varphi_{T_t h}(x)\right) \chi_l\left(\varphi_{T_t g}(x)\right) \\ &= \chi_k\left(\tau_t^{11}\varphi_h(x) + \tau_t^{21}\varphi_g(x)\right) \cdot \chi_l\left(\tau_t^{12}\varphi_h(x) + \tau_t^{22}\varphi_g(x)\right) \\ &= U_t\left(\chi_{kl}\right)\left(\varphi_h(x), \varphi_g(x)\right). \end{aligned}$$

Since $\{\chi_{kl}\}_{k,l=1}^{\infty}$ is an ONB in $L^2_{\mathbb{R}^2}$, we obtain the identity for every $f \in L^2_{\mathbb{R}^2}$

$$R_t\left(f\left(\varphi_h, \varphi_g\right)\right)(x) = \left(U_t f\right)\left(\varphi_h(x), \varphi_g(x)\right), \quad x \in H. \tag{8.12}$$

By the same argument as in Part 1 the identity (8.12) extends to all $f \in L^1_{\mathbb{R}^2}$, which implies (8.9). The generator $\mathcal{A}$ of (U_t) for the O-U process in $\mathbb{R}^2$ satisfies all the assumptions of Thm. 5.1 in [M-P-Pr] and hence by this result every $\alpha \in \mathbb{C}_-$ is an $L^1_{\mathbb{R}^2}$ eigenvalue of $\mathcal{A}$. Then we complete the proof as in Part 1. ■

Remark 8.2 If $\mathrm{Sp}_{L^1_H}\,\mathcal{L} = \overline{\mathbb{C}}_-$, then by Thm. II., 4.18 in [E-Na] the semigroup (R_t) is not eventually norm continuous in $L_1(H, \mu)$.

As a consequence of Thm. 8.1 we obtain a slight generalization of Thm. 5.1 in [M-P-Pr], since in Cor. 8.1 we also allow of $\det Q_\infty = 0$.

Corollary 8.1 If $H = \mathbb{R}^n$ and (8.3) holds, then

$$\mathrm{Sp}_{L^1_{\mathbb{R}^n}}\,\mathcal{L} = \overline{\mathbb{C}}_-.$$

Proof Since $\dim\, H < \infty$, we have $Q_\infty^{1/2}(H) = H_0 = \overline{H}_0 = Q_\infty^{1/2}(H_0)$ and $Q_\infty^{1/2}$ is a bijection on H_0. In particular, $S_0^*(t) = Q_\infty^{1/2} S_t^* Q_\infty^{-1/2}$. Let $x \in H_0$ and $y = Q_\infty^{-1/2} x$. As a consequence of (8.3) we have for all $t > 0$

$$\begin{gathered} Q_\infty = Q_t + S_t Q_\infty S_t^*, \quad \text{and hence} \\ \langle x, x\rangle = \langle Q_t y, y\rangle + \langle S_0^*(t)x, S_0^*(t)x\rangle. \end{gathered}$$

If $t \to \infty$, $\langle Q_t y, y\rangle \to \langle Q_\infty y, y\rangle = \langle x, x\rangle$ and hence $\|S_0^*(t)x\| \to 0$, $x \in H_0$. Since $\dim H_0 < \infty$, the last yields

$$\mathrm{Sp}\left(A_0^*\right) \subset \mathbb{C}_-$$

and the conclusion follows from Thm. 8.1. ■

Corollary 8.2 Assume (8.3). Let (R_t) be an eventually sF (strongly Feller) O-U semigroup and let

$$\omega_0 = \lim_{t \to \infty} \frac{1}{t} \log \|S_t\| > -\infty. \tag{8.13}$$

Then

$$\mathrm{Sp}_{L^1_H}\,\mathcal{L} = \overline{\mathbb{C}}_-.$$

Proof Since (R_t) is an event, sF semigroup, for a certain $a > 0$ and all $t \geq a$ the inclusions hold (e.g., [D-Z; S])

$$S_t(H) \subset Q_t^{1/2}(H) \subset Q_\infty^{1/2}(H) = H_0. \tag{8.14}$$

By [D-Z; S or P] condition (8.14) implies that S_t and $S_0(t) = \left(Q_\infty^{-1/2} S_t\right) Q_\infty^{1/2}$ are H-S operators for $t \geq a$ and (8.5) holds. It follows, e.g., from Prop. IV., 2.2. in [E-Na] that for every $t > 0$, the spectral radius of S_t, $\mathrm{Rad}(S_t) = e^{\omega_0 t} > 0$ and hence a nonzero λ_t is in $\mathrm{Sp}(S_t)$. Then by spectral properties of eventually compact semigroup there exists an eigenvalue β and a corresponding eigenvector h_β of A such that $S_t h_\beta = e^{t\beta} h_\beta$, $t \geq 0$ (e.g.,

[Da] Thm. 2.20, [E-Na] Cor. IV, 3.8 (i)). By (8.14) $h_\beta \in H_0$ and hence we can define $g_\beta = Q_\infty^{-1/2} h_\beta \in \overline{H}_0$. Then $S_0(t)\, g_\beta = e^{t\beta} g_\beta$, $t \geq 0$, and hence $\overline{e^{t\beta}}$ is an eigenvalue of $S_0^*(t)$, $t \geq 0$ which implies that $\overline{\beta}$ is an eigenvalue of A_0^*. By (8.5), $\|S_0^*(t)\| < 1$, hence $\operatorname{Re}\overline{\beta} < 0$. Therefore the corollary follows from Thm. 8.1. ■

Corollary 8.3 Assume (8.3). If the O-U semigroup (R_t) is compact in $L^2(H,\mu)$ (in particular sF) and symmetric ($R_t = R_t^*$), then

(a) (R_t) is compact and analytic semigroup in $L^p(H,\mu)$, for $1 < p < \infty$.

(b) In $L^1(H,\mu)$ the semigroup (R_t) is neither compact nor differentiable and

$$\operatorname{Sp}_{L^1_H} \mathcal{L} = \overline{\mathbb{C}}_-.$$

Proof By [Ch-G; Q and S], $R_t = R_t^*$ iff $S_0(t) = S_0^*(t)$. Moreover, for $t > 0$, R_t is compact in L^2 iff $S_0(t)$ is compact and $\|S_0(t)\| < 1$. Then (a) follows by [Ch-G; Q] and (b) is an immediate consequence of Thm. 8.1 and Rem. 8.2. ■

8.4 Exponential convergence to equilibrium

By a result in [Ch-G; N], the condition

$$Q_\infty^{1/2}(H) \subset Q^{1/2}(H) \tag{8.15}$$

is equivalent to the existence of a gap in the L^2_H-spectrum of $\mathcal{L}$, that is for a certain $\delta > 0$

$$\operatorname{Sp}_{L^2_H} \mathcal{L} \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq -\delta\}, \tag{8.16}$$

and moreover (8.15) implies the uniform exponential convergence to equilibrium in $L^p(H,\mu)$ for every $1 < p < \infty$.

We recall below the suitable result from [Ch-G; N] and next show that this convergence fails in $L^1(H,\mu)$.

$$\text{Let} \quad \overline{\phi} = \int_H \phi \, d\mu \quad \text{for} \quad \phi \in L^1_H \quad \text{and}$$
$$L_0^p(H,\mu) = \left\{\phi \in L^p(H,\mu) : \overline{\phi} = 0\right\}, \quad 1 \leq p < \infty.$$

Clearly, L_0^p is a closed subspace of L_H^p and $R_t(L_0^p) \subset L_0^p$ (since $\overline{R_t\phi} = \overline{\phi}$). By, e.g., [Z 1] the inclusion (8.15) is equivalent to the following condition.

For a certain $a > 0$

$$\left\|Q^{1/2}x\right\| \geq a \left\|Q_\infty^{1/2}x\right\| \quad \text{for all} \quad x \in H. \tag{8.17}$$

Theorem 8.2 ([Ch-G; N] Thm. 3.3)**.** Assume (8.3). Then for any fixed $a > 0$, the following conditions are equivalent:

(i) (8.17) holds for a.

(ii) $\|S_0(t)\| \leq \exp\left(-\frac{a^2}{2}t\right)$, $t \geq 0$.

(iii) (8.16) holds for $\delta = \frac{a^2}{2}$.

(iv) $\|R_t\phi\|_{L^2_H} \leq \exp\left(-\frac{a^2}{2}t\right) \|\phi\|_{L^2_H}$, $\phi \in L_0^2$, $t \geq 0$.

Moreover, if (8.17) holds then for every $p \in (1, \infty)$.

(v) $\int_H \left|R_t\phi - \overline{\phi}\right|^p d\mu \le \exp\left(-\frac{pa^2}{\max(p,p')}t\right) \cdot \int_H \left|\phi - \overline{\phi}\right|^p d\mu$, where $p' = \frac{p}{p-1}$, $t \ge 0$, $\phi \in L^p_H$.

Remark 8.3 Condition (v) can be written equivalently as

$$\|R_t\|_{L_0^p \to L_0^p} \le \exp\left(-\frac{a^2}{\max(p,p')}t\right), \quad 1 < p < \infty, \quad t \ge 0, \tag{8.18}$$

and it follows from (iv) and the Riesz–Thorin interpolation theorem. Moreover, Thm. 8.2 holds true without the assumption that Q_∞ is injective ([Ch] Cor. 2.6). Condition (b) in Thm. 8.3 below says that in $L^1(H,\mu)$ there is no uniform exponential convergence to equilibrium but the limit estimate for $p = 1$ obtained from (8.18) is the best one in L_0^1.
Theorem 8.3 Let (8.3) and (8.15) hold. Then

$$\int_H \left|R_t\phi - \overline{\phi}\right| d\mu \to 0 \quad \text{as} \quad t \to \infty, \quad \text{for} \quad \phi \in L^1_H. \tag{8.19}$$

If moreover, the point spectrum of A_0^* is nonempty, then

$$\|R_t\|_{L_0^1 \to L_0^1} = 1 \quad \text{for} \quad t \ge 0. \tag{8.20}$$

Proof By Thm. 8.2, (8.19) is true for $\phi \in L^2_H$ and hence (8.19) holds for $\phi \in L^1_H$ by density. To prove (8.20) note that by Thm. 8.2 for a certain $a > 0$, $\operatorname{Sp} A_0^* \subset \left\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \le -\frac{a^2}{2}\right\}$. Then by Thm. 8.1 every real $\alpha < 0$ is an L^1_H-eigenvalue of $\mathcal{L}$ and let $f_\alpha \in L^1_H$ with $\|f_\alpha\|_{L^1_H} = 1$ be an eigenvector corresponding to α. Then for a fixed $\alpha < 0$

$$R_t f_\alpha - \overline{f}_\alpha = e^{\alpha t} f_\alpha - \overline{f}_\alpha \to \overline{f}_\alpha \quad \text{in} \quad L^1_H, \quad \text{as} \quad t \to \infty,$$

and hence by (8.19), $\overline{f}_\alpha = 0$, i.e, $f_\alpha \in L_0^1$. Taking $\alpha_n \to 0^-$, we obtain for a fixed $t \ge 0$

$$\|R_t f_{\alpha_n}\|_{L_0^1} = e^{\alpha_n t} \to 1 \quad \text{as} \quad n \to \infty.$$

This proves (8.20). ■
Corollary 8.4 Under the assumptions of Cor. 8.3 the conclusions of Thm. 8.3 hold true.

A simple example below illustrates the difference between the case of $\dim H < \infty$ and the case of $\dim H = \infty$. This example is of some importance in mathematical finance (see [Z 2]) and in [Ch-G; N] and [Ch] we investigated the hypercontractivity of the corresponding O-U semigroup (R_t).
Example 8.1 In the space $H = L^2(0,1)$ consider the equation

$$\begin{cases} dX_t = AX_t dt + b dw_t, & t \ge 0 \\ X_0 = x \in H, \end{cases}$$

where the operator

$$A = \frac{d}{du} \quad \text{with} \quad \operatorname{dom}(A) = \left\{x \in H^1(0,1) : x(1) = 0\right\}$$

generates the semigroup

$$S(t)\,x(u) = \begin{cases} x(t+u), & \text{if} \quad t+u \le 1 \\ 0, & \text{if} \quad t+u > 1, \end{cases}$$

(w_t) is a Wiener process in $\mathbb{R}$ and $b \in H$, $b \neq 0$ (for simplicity we can take $b = \mathbf{1}$). For $t \geq 1$, $S_t = 0$, hence $Q_\infty = Q_1$ and $S_0^*(t) = 0$ for $t \geq 1$. Then the corresponding O-U semigroup (R_t) and its generator $\mathcal{L}$ injoy the following properties.

For $t \geq 1$, $R_t\phi = \int_H \phi d\mu$, for $\phi \in L^p(H,\mu)$, $1 \leq p < \infty$. Therefore for each $t \geq 1$: R_t is compact and hypercontractive in $L^p(H,\mu)$ for all $1 \leq p < \infty$ (actually R_t is a contraction from L^1_H to L^∞_H); R_t is sF and maps L^1_H into $C^\infty(H)$ (and into $\bigcap_{n\geq 1}\bigcap_{p\geq 1} W^{n,p}(H)$). Note that $\mathrm{Sp}(A_0^*) = \varnothing$,

$$\mathrm{Sp}_{L^p_H}(R_t) = \{0,1\} \quad \text{and} \quad \mathrm{Sp}_{L^p_H}(\mathcal{L}) = \{0\}$$
$$\text{for all} \quad 1 \leq p < \infty.$$

It has been proved in [Ch] that for $0 < t < 1$, (8.5) is not satisfied. Hence for $0 < t < 1$ and any $1 \leq p < \infty$, R_t is not hypercontractive or compact in L^p_H, or sF.

References

[Ch] A. Chojnowska-Michalik, Transition semigroups for stochastic semilinear equations on Hilbert spaces, Diss. Math. **396** (2001).

[Ch-G; E] A. Chojnowska-Michalik and B. Goldys, Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces, Probab. Theory Relat. Fields **102** (1995), 331-356.

[Ch-G; N] A. Chojnowska-Michalik and B. Goldys, Nonsymmetric Ornstein-Uhlenbeck generators, Proceedings of Colloquium on Infinite Dimensional Stochastic Analysis, Symposium of the Royal Netherlands Academy, Amsterdam, 1999.

[Ch-G; Q] A. Chojnowska-Michalik and B. Goldys, Nonsymmetric Ornstein-Uhlenbeck semigroup as second quantized operator, J. Math. Kyoto Univ. **36** (1996), 481-498.

[Ch-G; R] A. Chojnowska-Michalik and B. Goldys, On regularity properties of nonsymmetric Ornstein-Uhlenbeck semigroup in L^p spaces, Stoch. Stoch. Rep. **59** (1996), 183-209.

[Ch-G; S] A. Chojnowska-Michalik and B. Goldys, Symmetric Ornstein-Uhlenbeck semigroups and their generators, Probab. Theory Relat. Fields **124** (2002), 459-486.

[D-F-Z] G. Da Prato, M. Fuhrman and J. Zabczyk, A note on regularizing properties of Ornstein-Uhlenbeck semigroups in infinite dimensions, in Stochastic Partial Differential Equations and Applications, Da Prato, G. and Tubaro, L. (editors), Dekker, New York, 2002, 167-182.

[D-Z; S] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge Univ. Press, Cambridge, 1992.

[D-Z; E] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, Cambridge Univ. Press, Cambridge, 1996.

[D-Z; P] G. Da Prato and J. Zabczyk, Second Order Partial Differential Equations in Hilbert Spaces, Cambridge Univ. Press, Cambridge, 2002.

[Da] E.B. Davies, One-Parameter Semigroups, Academic Press, London, 1980.

[Da-S] E.B. Davies and B. Simon, L^1-properties of intrinsic Schrődinger semigroups, J. Funct. Anal. **65** (1986), 126-146.

[E-Na] K. Engel and R. Nagel, One-parameter semigroups for linear evolution equations, Graduate Texts Math. **194**, Springer, 2000.

[F] M. Fuhrman, Hyprercontractivité des semigroupes d'Ornstein-Uhlenbeck non symétriques, C.R. Acad. Sci. Paris, Sér. I, Math. **321** (1995), 929-932.

[M-P-Pr] G. Metafune, D. Pallara and E. Priola, Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures, J. Funct. Anal. **196** (2002), 40-60.

[N] E. Nelson, Probability theory and euclidean field theory, in Constructive Quantum Field Theory, Velo, G. (editor) Lect. Notes in Physics **25**, Springer, New York, 1973, 94-124.

[Ne] J.M.A.M. van Neerven, Second quantization and the L^pspectrum of nonsymmetric Ornstein-Uhlenbeck operators, preprint, 2004.

[S] B. Simon, The $P\,(\phi)_2$ Euclidean (Quantum) Field Theory, Princeton University Press, Princeton, NJ, 1974.

[W] S. Watanabe, Stochastic Differential Equations and Malliavin Calculus, Springer-Verlag, New York, 1984.

[Z 1] J. Zabczyk, Mathematical Control Theory: An Introduction, Systems and Control: Foundations and Applications, Birkhäuser, Boston, 1992.

[Z 2] J. Zabczyk, Stochastic invariance and consistency of financial models, Rend. Mat. Acc. Lincei **11** (2000), 67-80.

9 Intertwining and the Markov Uniqueness Problem on Path Spaces

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9.1 Malliavin calculus on $\mathcal{C}_0\mathbb{R}^m$ and $\mathcal{C}_{x_0}M$

9.1.1 Notation

Let M be a compact Riemannian manifold of dimension n. Fix $T > 0$ and x_0 in M. Let $\mathcal{C}_{x_0}M$ denote the smooth Banach manifold of continuous paths

$$\sigma : [0, T] \to M \text{ such that } \sigma_0 = x_0$$

furnished with its Brownian motion measure μ_{x_0}. However, most of what follows works for a class of more general, possible degenerate, diffusion measures.

Let $\mathcal{C}_0\mathbb{R}^m$ be the corresponding space of continuous $\mathbb{R}^m$-valued paths starting at the origin, with Wiener measure $\mathbb{P}$, and let H denote its Cameron–Martin space: $H = L_0^{2,1}\mathbb{R}^m$ with inner product $\langle \alpha, \beta\rangle_H = \int_0^T \langle \dot\alpha(s), \dot\beta(s)\rangle_{\mathbb{R}^m}\, ds$.

As a Banach manifold $\mathcal{C}_{x_0}M$ has tangent spaces $T_\sigma M$ at each point σ, given by

$$T_\sigma M = \{v : [0, T] \to TM \,\big|\, v(0) = 0, v \text{ is continuous }, v(s) \in T_{\sigma(s)}M, s \in [0, T]\}.$$

Each tangent space has the uniform norm induced on it by the Riemannian metric of M. As an analogue of H there are the "Bismut tangent spaces" $\mathcal{H}_\sigma$ defined by

$$\mathcal{H}_\sigma = \{v \in T_\sigma \mathcal{C}_{x_0}M \,\Big|\, /\!/_s^{-1} v(s) \in L_0^{2,1} T_{x_0}M, 0 \leqslant s \leqslant T\}$$

where $/\!/_s$ denotes parallel translation of $T_{x_0}M$ to $T_{\sigma(s)}M$ using the Levi–Civita connection.

9.1.2 Malliavin calculus on $\mathcal{C}_0\mathbb{R}^m$

To have a calculus on $\mathcal{C}_0\mathbb{R}^m$ the standard method is to choose a dense subspace, $\mathrm{Dom}(d^H)$, of Fréchet differentiable functions (or elements of the first chaos) in $L^2(\mathcal{C}_0\mathbb{R}^m; \mathbb{R})$. By differentiating in the H-directions we obtain the H-derivative operator $d^H : \mathrm{Dom}(d^H) \to L^2\,(\mathcal{C}_0\mathbb{R}^m;\ H^*)$. By the Cameron –Martin integration by parts formula this operator is closable. Let $d : \mathrm{Dom}(d) \to L^2(\mathcal{C}_0\mathbb{R}^m; H^*)$ be its closure and write $\mathbb{D}^{2,1}$ for its domain with its graph norm and inner product.

From work of Shigekawa and Sugita, [Sugita], $\mathbb{D}^{2,1}$ does not depend on the (sensible) choice of initial domain $\mathrm{Dom}(d^H)$ and, moreover, if a function is weakly differentiable with weak derivative in L^2, in a sense described below, then it is in $\mathbb{D}^{2.1}$. In particular, if $\mathrm{Dom}(d^H)$ consists of the polygonomial cylindrical functions, then $\mathbb{D}^{2,1}$ contains the space BC^1 of bounded functions with bounded continuous Fréchet derivatives.

9.1.3 Malliavin calculus on $\mathcal{C}_{x_0}M$

If $f : \mathcal{C}_{x_0}M \to \mathbb{R}$ is Fréchet differentiable with differential $(df)_\sigma : T_\sigma\mathcal{C}_{x_0}M \to R$ at the point σ, define $(d^H f)_\sigma : \mathcal{H}_\sigma \to \mathbb{R}$ by restriction. Choosing a suitable domain $\mathrm{Dom}(d^H)$ in L^2 the integration by parts results of [Driver] imply closability and we obtain a closed operator $d : \mathrm{Dom}(d) \subset L^2(\mathcal{C}_{x_0}M;\mathbb{R}) \to L^2\mathcal{H}^*$, for $L^2\mathcal{H}^*$ the space of L^2-sections of the dual "bundle" $\mathcal{H}^*$ of $\mathcal{H}$. Let $\mathbb{D}^{2.1}$ or $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})$ denote the domain of this d furnished with its graph norm and inner product. Possible choices for the initial domain $\mathrm{Dom}(d^H)$ include the following:

(i) C^∞ Cyl, the space of C^∞ cylindrical functions;

(ii) BC^1, the space of BC^1 bounded functions with first Fréchet derivatives bounded;

(iii) BC^∞, the space of infinitely Fréchet differentiable functions all of whose derivatives are bounded.

One fundamental question is whether such different choices of the initial domain lead to the same space $\mathbb{D}^{2,1}$. At the time of writing this question appears to still be open. There is a gap in the proof suggested in [Elworthy-Li3] as will be described in Section 9.3 below. However, the techniques given there do show that choices (i) and (iii) above lead to the same $\mathbb{D}^{2,1}$.

From now on we shall assume that choice (i) has been taken. We use $\nabla : \mathrm{Dom}(d) \to L^2\mathcal{H}$ defined from d using the canonical isometry of $\mathcal{H}_\sigma$ with its dual space $\mathcal{H}^*_\sigma$. This requires the choice of a Riemannian structure on $\mathcal{H}$; for this see below. Let $\mathrm{div} : \mathrm{Dom}(\mathrm{div}) \subset L^2\mathcal{H} \to L^2(\mathcal{C}_{x_0}M;\mathbb{R})$ denote the adjoint of $-\nabla$. Then if $f \in \mathrm{Dom}(d)$ and $v \in \mathrm{Dom}(\mathrm{div})$, we have

$$\int df(v)d\mu_{x_0} = -\int f\,\mathrm{div}(v)d\mu_{x_0} = \int \langle \nabla f, v\rangle.d\mu_{x_0}.$$

Using these we get the self-adjoint operator Δ defined to be $\mathrm{div}\,\nabla$. Another basic open question is whether this is essentially self-adjoint on C^∞ Cyl. From the point of view of stochastic analysis it would be almost as good for it to have Markov uniqueness. Essentially this means that there is a unique diffusion process on $\mathcal{C}_{x_0}M$ whose generator $\mathcal{A}$ agrees with Δ on C^∞ cylindrical functions, see [Eberle]. Another characterization of this is given below.

Finally, there is the question of the existence of "local charts" for $\mathcal{C}_{x_0}M$ which preserve, at least locally, this sort of differentiability. The stochastic development maps $\mathfrak{D} : \mathcal{C}_0\mathbb{R}^m \to \mathcal{C}_{x_0}M$ appear not to have this property, [XD-Li]. The Itô maps we use seem to be the best substitute for such charts.

9.2 The approach via Itô maps and main results

9.2.1 Itô maps as a charts

As in [Aida-Elworthy] and [Elworthy-LeJan-Li] take a stochastic differential equation (SDE) on M

$$dx_t = X(x_t) \circ dB_t, \quad 0 \le t \le T \tag{9.1}$$

with our given initial value x_0. Here $(B_t, 0 \le t \le T)$ is the canonical Brownian motion on $\mathbb{R}^m$ and $X(x)$ is a linear map from $\mathbb{R}^m$ to the tangent space T_xM for each x in M, smooth in x. Choose the SDE with the following properties:

SDE1 The solutions to (1) are Brownian motions on M.

SDE2 For each $e \in \mathbb{R}^m$ the vector field $X(-)e$ has covariant derivative which vanishes at any point x where e is orthogonal to the kernel of $X(x)$.

This can be achieved, for example, by using Nash's theorem to obtain an isometric immersion of M into some $\mathbb{R}^m$ and taking $X(x)$ to be the orthogonal projection onto the tangent space; see [Elworthy-LeJan-Li].

Let $\mathcal{I} : \mathcal{C}_0\mathbb{R}^m \to \mathcal{C}_{x_0}M$ denote the Itô map $\omega \mapsto x_{\cdot}(\omega)$ with $\mathcal{I}_t(\omega) = x_t(\omega)$. Then $\mathcal{I}$ maps $\mathbb{P}$ to μ_{x_0}. Set

$$\mathcal{F}^{x_0} = \sigma\{x_s : 0 \le s \le T\}$$

$$\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}} = \{f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R} \ s.t. \ f \in \mathbb{D}^{2,1} \text{ and } f \text{ is } \mathcal{F}^{x_0}\text{-measurable}\}.$$

Also consider the isometric injection $\mathcal{I}^* : L^2(\mathcal{C}_{x_0}M;\mathbb{R}) \to L^2(\mathcal{C}_0\mathbb{R}^m;\mathbb{R})$ given by $f \mapsto f \circ \mathcal{I}$.

9.2.2 Basic results

Theorem 9.1 $\mathcal{I}^*$ $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})$ $\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$

Theorem 9.2 $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})] = \mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$

Theorem 9.3 $f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ $\mathrm{Dom}(\Delta)$ $\mathcal{F}^{x_0}$ f $\mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$.

From Theorem 9.3 we see that $\mathrm{BC}^2 \subset \mathbb{D}^{2,1}$ on $\mathcal{C}_{x_0}M$. Theorem 9.2 is a consequence of Theorem 9.4 below.

Problem 9.1 $\{f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ s.t. f $\mathrm{Dom}(\Delta)$ $\mathcal{F}^{x_0}$ $\}$ $\mathbb{D}^{2,1}_{\mathcal{F}^{x_0}}$.

Problem 9.1 is open. An affirmative answer would imply Markov uniqueness by the theorems above.

9.2.3 A stronger possibility

Problem 9.2 $f \in \mathbb{D}^{2,1}$ $\mathbb{E}\{f|\mathcal{F}^{x_0}\} \in \mathbb{D}^{2,1}$.

Problem 9.2 is open: there is a gap in the "proof" in [Elworthy-Li3]. It is true for f an exponential martingale or in a finite chaos space. An affirmative answer would imply an affirmative answer to Problem 9.1 and Markov uniqueness.

9.2.4 Markov uniqueness and weak differentiability

Let $\mathbb{D}^{2,1}\mathcal{H}$ and $\mathbb{D}^{2,1}\mathcal{H}^*$ be the spaces of $\mathbb{D}^{2,1}$-H-vector fields and H-1-forms on $\mathcal{C}_{x_0}M$, respectively, with their graph norms (see details below). Write

$$\begin{aligned}
\mathrm{Cyl}^0\mathcal{H}^* &= \text{linear span } \{gdk | g, k : \mathcal{C}_{x_0}M \to \mathbb{R} \text{ are in } \mathcal{C}^\infty \text{ Cyl}\} \\
W^{2,1} &= \mathrm{Dom}(d^* \mid \mathbb{D}^{2,1}\mathcal{H}^*)^* \\
{}^0W^{2,1} &= \mathrm{Dom}(d^* \mid \mathrm{Cyl}^0\,\mathcal{H}^*)^*.
\end{aligned}$$

Then $\mathbb{D}^{2,1} \subseteq W^{2,1} \subseteq {}^0W^{2,1}$. From [Eberle] we have

$$\text{Markov uniqueness} \Longleftrightarrow \mathbb{D}^{2,1} = {}^0W^{2,1}. \tag{9.2}$$

We claim

Theorem 9.4 $f \in W^{2,1}$ $\mathcal{C}_{x_0}M \iff \mathcal{I}^*(f) \in W^{2,1}$ $\mathcal{C}_0\mathbb{R}^m$

$$W^{2,1} = {}^0W^{2,1}.$$

If $f \in W^{2,1}$ it has a "weak derivative" $df \in L^2\Gamma\mathcal{H}$ defined by $\int df(V)d\mu_{x_0} = -\int f \operatorname{div} V d\mu_{x_0}$ for all $V \in \mathbb{D}^{2,1}\mathcal{H}$. See Section 9.3.4 below where the proof of Proposition 9.2 also demonstrates one of the implications of Theorem 9.4A.

An important step in the proof of Part B is the analogue of a fundamental result of [Kree-Kree] for $\mathcal{C}_0\mathbb{R}^m$.

Theorem 9.5 $\mathcal{C}_{x_0}M$ $\operatorname{div} : \mathbb{D}^{2,1}\mathcal{H} \to L^2(\mathcal{C}_{x_0}M;\mathbb{R})$.

9.3 Some details and comments on the proofs

We will sketch some parts of the proofs. The full details will appear, in greater generality, in [Elworthy-Li2].

9.3.1 To prove Theorem 9.3

For $f : \mathcal{C}_0\mathbb{R}^m \to \mathbb{R}$ in $\mathbb{D}^{2,1}$ take its chaos expansion

$$f = \sum_{k=1}^{\infty} I^k(\alpha^k) = \sum_{k=1}^{N} I^k(\alpha^k) + R_{N+1} \tag{9.3}$$

say. This converges in $\mathbb{D}^{2,1}$ as is well known, e.g., see [Nualart].

Set $\mathbb{E}\{I^k(\alpha^k)|\mathcal{F}^{x_0}\} = J^k(\alpha^k)$. Then

$$\mathbb{E}\{f|\mathcal{F}^{x_0}\} = \sum_{k=1}^{\infty} J^k(\alpha^k). \tag{9.4}$$

The right-hand side converges in L^2. An equivalent probem to Problem 9.2 is

Problem 9.3 ( ) $\mathbb{D}^{2,1}$.

If f is $\mathcal{F}^{x_0}$-measurable and in the domain of Δ, it is not difficult to show that there is convergence in $\mathbb{D}^{2,1}$, using the Lemma 9.1 below. Moreover, $\sum_{k=1}^{N} J^k(\alpha^k) \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$. Therefore by Theorem 9.1 we see $f \in \mathcal{I}^*[\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M;\mathbb{R})]$. Again this uses the basic result (c.f. [Elworthy-Yor], [Aida-Elworthy], [Elworthy-LeJan-Li]).

Lemma 9.1 $K^\perp(x) : \mathbb{R}^m \to \mathbb{R}^m$ $X(x)$ x M $(\alpha_s, 0 \le s \le T)$ $\mathbb{L}(\mathbb{R}^m;\mathbb{R}^p)$

$$\mathbb{E}\left\{\int_0^T \alpha_s(dB_s)\middle| \mathcal{F}^{x_0}\right\} = \int_0^T \mathbb{E}\{\alpha_s|\mathcal{F}^{x_0}\}K^\perp(x_s)\,dB_s.$$

9.3.2 The Riemannian structure for $\mathcal{H}$

Let $\mathrm{Ric}^\sharp : TM \to TM$ correspond to the Ricci curvature tensor of M, and $W_s : T_{x_0}M \to T_{x_s}M$ the damped, or "Dohrn-Guerra", parallel translation, defined for v_0 in $T_{x_0}M$ by

$$\begin{aligned} \frac{\mathbb{D}W_s(v_0)}{ds} &= 0 \\ W_0(v_0) &= v_0. \end{aligned}$$

Here $\frac{\mathbb{D}}{ds} = \frac{D}{ds} + \frac{1}{2}\mathrm{Ric}^\sharp$. Define $\langle v^1, v^2\rangle_\sigma = \int_0^T \langle \frac{\mathbb{D}}{ds}v^1, \frac{\mathbb{D}}{ds}v^2\rangle_{\sigma_s} ds$ and let $\boldsymbol{\nabla}$ denote the damped Markovian connection of [Cruzeiro-Fang]; see [Elworthy-Li2] for details.

For each $0 \le t \le T$ the Itô map $\mathcal{I}_t : H \to T_{x_t}M$ is infinitely differentiable in the sense of Malliavin calculus, with derivative $T_\omega\mathcal{I}_t : H \to T_{x_t(\omega)}M$ giving rise to a continuous linear map $T_\omega\mathcal{I} : H \to T_{x.(\omega)}M$ defined almost surely for $\omega \in \mathcal{C}_0\mathbb{R}^m$. For $\sigma \in \mathcal{C}_{x_0}M$ define $\overline{T\mathcal{I}}_\sigma : H \to \mathcal{H}_\sigma$ by

$$\overline{T\mathcal{I}}_\sigma(h)_s = \mathbb{E}\{T\mathcal{I}_s(h)|x. = \sigma\}.$$

From [Elworthy-LeJan-Li] this does map into the Bismut tangent space and gives an orthogonal projection onto it. It is given by

$$\frac{\mathbb{D}}{ds}\overline{T\mathcal{I}}_\sigma(h)_s = X(\sigma(s))(\dot{h}_s)$$

and has right inverse $\boldsymbol{Y}_\sigma : \mathcal{H}_\sigma \to H$ given by

$$\boldsymbol{Y}_\sigma(v)_t = \int_0^t Y_{\sigma(s)}(\frac{\mathbb{D}}{ds}v_s)ds,$$

for $Y_x : T_xM \to \mathbb{R}^m$ the right inverse of $X(x)$ defined by $Y_x = X(x)^*$.

It turns out, [Elworthy-Li2], that for suitable H-vector fields V on $\mathcal{C}_{x_0}M$, the covariant derivative is given by $\boldsymbol{\nabla}_u V = \overline{T\mathcal{I}}_\sigma(d(\boldsymbol{Y}_-(V(-)))_\sigma(u))$, for $u \in T_\sigma\mathcal{C}_{x_0}M$, and we define V to be in $\mathbb{D}^{2,1}\mathcal{H}$ iff $\sigma \mapsto \boldsymbol{Y}_\sigma(V(\sigma))$ is in $\mathbb{D}^{2,1}(\mathcal{C}_{x_0}M; H)$.

9.3.3 Continuity of the divergence

There is also a continuous linear map $\overline{T\mathcal{I}(-)} : L^2(\mathcal{C}_0\mathbb{R}^m; H) \to L^2\mathcal{H}$ defined by $\overline{T\mathcal{I}(U)}(\sigma)_s = \mathbb{E}\{T_-\mathcal{I}_s(U(-))|x.(-) = \sigma\}$, [Elworthy-Li1]. Another fundamental and easily proved result follows

Proposition 9.1 H fi U $\mathcal{C}_0\mathbb{R}^m$ Dom(div) $\overline{T\mathcal{I}(U)}$ Dom(div) $\mathcal{C}_{x_0}M$

$$\mathbb{E}\{\mathrm{div}\, U|\mathcal{F}^{x_0}\} = (\mathrm{div}\, \overline{T\mathcal{I}(U)}) \circ \mathcal{I} \tag{9.5}$$

Theorem 9.5 follows easily from Proposition 9.1 by observing that if $V \in \mathbb{D}^{2,1}\mathcal{H}$ then, from Theorem 9.1, $\mathcal{I}^*(\boldsymbol{Y}_-V(-)) \in \mathbb{D}^{2,1}$. By [Kree-Kree] this implies that $\mathcal{I}^*(\mathbf{Y}_-(V(-))$ is in Dom(div). Since

$$\overline{T\mathcal{I}(\mathcal{I}^*(\mathbf{Y}_-(V(-)))} = V$$

Proposition 9.1 assures us that $V \in \mathrm{Dom}(\mathrm{div})$. Moreover

$$\mathrm{div}\, V(x.) = \mathbb{E}\{\mathrm{div}\, \mathcal{I}^*(\boldsymbol{Y}_-(V(-)))|\mathcal{F}^{x_0}\}. \tag{9.6}$$

Theorem 9.4A can be deduced from Proposition 9.1 together with the following lemma.

Lemma 9.2 fi V $\mathcal{C}_0\mathbb{R}^m$ $\overline{T\mathcal{I}(V)} \in \mathbb{D}^{2,1}\mathcal{H}$ $\mathbb{D}^{2,1}$

9.3.4 Intertwining and weak differentiability

To see how weak differentiability relates to intertwining by our Itô maps we have:

Proposition 9.2 [illegible] $f \in W^{2,1}$ [illegible] df [illegible]

$$(\mathrm{df})_\sigma = \mathbb{E}\{d(\mathcal{I}^*(f))_\omega | x.(\omega) = \sigma\} \boldsymbol{Y}_\sigma. \tag{9.7}$$

Proof Let $V \in \mathbb{D}^{2,1}\mathcal{H}$. Then for $f \in W^{2,1}$, by equation (9.6) and then by Theorem 9.4A,

$$\begin{aligned}
\int_{\mathcal{C}_{x_0}M} f \operatorname{div}(V) d\mu &= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \operatorname{div}(V) \circ \mathcal{I}\, d\mathbb{P} \\
&= \int_{\mathcal{C}_0\mathbb{R}^m} \mathcal{I}^*(f) \operatorname{div} \mathcal{I}^*(\boldsymbol{Y}_-(V(-)))\, d\mathbb{P} \\
&= -\int_{\mathcal{C}_0\mathbb{R}^m} d(\mathcal{I}^*(f))_\omega (\boldsymbol{Y}_{x.(\omega)}(V(x.(\omega)))\, d\mathbb{P}(\omega) \\
&= -\int_{\mathcal{C}_{x_0}M} \mathbb{E}\{d(\mathcal{I}^*(f))_\omega | x.(\omega) = \sigma\} \boldsymbol{Y}_\sigma(V(\sigma))\, d\mu_{x_0}(d\sigma)
\end{aligned}$$

as required. □

Acknowledgments

This research was partially supported by EPSRC research grant GR/H67263, NSF research grant DMS 0072387, and a Royal Society Leverhulme Trust Senior Research Fellowship. It benefited from our contacts with many colleagues, especially S. Aida, S. Fang, Y. LeJan, Z.-M. Ma, and M. Röckner. K.D.E wishes to thank L. Tubaro and the Mathematics Department at Trento for their hospitality with excellent facilities during February and March 2004.

References

[Aida-Elworthy] S. Aida and K.D. Elworthy, Differential calculus on path and loop spaces. 1. Logarithmic Sobolev inequalities on path spaces. [illegible], 321:97–102, 1995.

[Cruzeiro-Fang] A.B. Cruzeiro and S. Fang, Une inégalité l^2 pour des intégrales stochastiques anticipatives sur une variété riemannienne. [illegible], 321:1245–1250, 1995.

[Driver] B.K. Driver, A Cameron-Martin type quasi-invariance theorem for Brownian motion on a compact Riemannian manifold. [illegible], 100:272–377, 1992.

[Eberle] A. Eberle, Uniqueness and Non-Uniqueness of Semigroups Generated by Singular Diffusion Operators. Lecture Notes in Mathematics, 1718. Springer-Verlag, Berlin, 1999.

[Elworthy-LeJan-Li] K.D. Elworthy, Y. LeJan, and Xue-Mei Li, [illegible]. Lecture Notes in Mathematics, 1720. Springer, New York, 1999.

[Elworthy-Li1] K.D. Elworthy and Xue-Mei Li, Special Itô maps and an L^2 Hodge theory for one forms on path spaces. In [illegible] ([illegible]), pages 145–162. Amer. Math. Soc., 2000.

[Elworthy-Li2] K.D. Elworthy and Xue-Mei Li, Itô maps and analysis on path spaces. Preprint. 2005.

[Elworthy-Li3] K.D. Elworthy and Xue-Mei Li, Gross-Sobolev spaces on path manifolds: uniqueness and intertwining by Itô maps. [illegible], 337:741–744, 2003.

[Elworthy-Yor] K.D. Elworthy and Yor, Conditional expectations for derivatives of certain stochastic flows. In Sem. de Prob. XXVII. Lecture Notes in Maths. 1557, Eds: Azéma, J. and Meyer, P.A. and Yor, M., 159–172, 1993.

[Kree-Kree] M. Krée and P. Krée, Continuité de la divergence dans les espaces de Sobolev relatifs l'espace de Wiener. (French) [Continuity of the divergence operator in Sobolev spaces on the Wiener space] [illegible], 296 no. 20, 833–836, 1983.

[XD-Li] Li, Xiang Dong, Sobolev spaces and capacities theory on path spaces over a compact Riemannian manifold. (English. English summary) [illegible], 125 no. 1, 96–134, 2003.

[Nualart] D. Nualart, [illegible]. Springer-Verlag, New York, 1995.

[Sugita] H. Sugita, On a characterization of the Sobolev spaces over an abstract Wiener space. [illegible], 25 no. 4, 717-725, 1985.

10 On Some Problems of Regularity in Two-Dimensional Stochastic Hydrodynamics

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10.1 Introduction

We are interested in the Navier–Stokes equation perturbed by a time white noise. In the literature there are many papers on the subject, dealing with problems of existence, uniqueness, and regularity of solutions, of invariant measures, of asymptotic behavior, and so on.

In a bounded spatial domain, the solutions interesting from the physical viewpoint are that of finite energy velocity vectors. The minimal assumptions on the noise to have a well-defined two-dimensional dynamics in the space of finite energy velocity vectors have been investigated in [Fl], improving results by [BT] and [VF]. In [Fl], the stochastic term is an additive noise and the spatial domain is a smooth bounded subset of $\mathbb{R}^2$ with vanishing velocity on the boundary. On the other hand, when the spatial domain is the torus, i.e., a square with periodic boundary conditions, solutions with infinite energy velocity vectors exist (see [ARHK, AC, DPD, AF]; they are interesting because, in the setting considered by all these papers, they are stationary solutions with respect to an invariant measure of Gaussian type.

In order to introduce our result, some technical detail is required. [Fl] deals with a noise $A^{-\varepsilon}dw(t)$, where $\varepsilon > \frac{1}{4}$. [ARHK, AC, DPD, AF] deal with a noise $dw(t)$ (same w as [Fl] and $\varepsilon = 0$, so there is no regularizing operator in front of the "basic" noise); moreover, they work in different spatial domains, as we said. Consequently, the techniques used in these two groups of papers are very different.

In [Fe97] it has been pointed out that, in the periodic case, solutions of finite energy can be obtained even for a noise slightly more regular than the cylindrical one, i.e., $A^{-\varepsilon}dw(t)$ with $\varepsilon > 0$. This fills the gap between $\varepsilon = 0$ and $\varepsilon > \frac{1}{4}$. Here we make precise this result and, moreover, the regularity of the solution is expressed in term of ε. This is the new result (see Theorem 10.1); it might be useful in the study of the limit as $\varepsilon \downarrow 0$. We postpone this analysis to a future work. Finally, we remember that the opposite analysis for big (positive) ε has been investigated in the periodic case by [Fe99].

10.2 Navier–Stokes equation: the periodic case

We consider the equations governing the motion of an homogeneous incompressible viscous fluid in the two-dimensional torus $\mathbb{T}^2 = [0, 2\pi]^2$

$$\begin{cases} \frac{\partial}{\partial t}u(t,\xi) - \nu\Delta u(t,\xi) + [u(t,\xi)\cdot\nabla]u(t,\xi) + \nabla p(t,\xi) = \varphi(t,\xi) \\ \nabla\cdot u(t,\xi) = 0 \\ u(0,\xi) = u_0(\xi) \end{cases} \tag{10.1}$$

with periodic boundary condition. The definition domains of the variables are $t \geq 0, \xi = (\xi_1, \xi_2) \in \mathbb{T}^2$. The unknowns are the velocity vector field $u = u(t, \xi)$ and the pressure scalar field $p = p(t, \xi)$. Here $\Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2}$, $\nabla = (\frac{\partial}{\partial \xi_1}, \frac{\partial}{\partial \xi_2})$, and $\cdot$ is the scalar product in $\mathbb{R}^2$. The viscosity ν is a strictly positive constant; u_0 and φ are the data.

We define the mathematical setting as follows. Consider any periodic divergence-free vector distribution u. Since $\nabla \cdot u = 0$, there exists a periodic scalar distribution ψ, called the stream function, such that

$$u = \nabla^\perp \psi \equiv (-\tfrac{\partial \psi}{\partial \xi_2}, \tfrac{\partial \psi}{\partial \xi_1}). \tag{10.2}$$

We decompose ψ in Fourier series with respect to the complete orthonormal system in $L_2(\mathbb{T}^2)$ given by $\{\frac{1}{2\pi} e^{ik\cdot\xi}\}_{k \in \mathbb{Z}^2}$

$$\psi(\xi) = \sum_{k \in \mathbb{Z}^2} \psi_k \frac{e^{ik\cdot\xi}}{2\pi}, \qquad \psi_k \in \mathbb{C},\ \overline{\psi_k} = \psi_{-k}. \tag{10.3}$$

By (10.2) we get that u has the following Fourier series representation:

$$u(\xi) = \sum_{k \in \mathbb{Z}_0^2} u_k e_k(\xi), \qquad u_k \in \mathbb{C},\ \overline{u_k} = -u_{-k} \tag{10.4}$$

where $e_k(\xi) = \frac{k^\perp}{2\pi |k|} e^{ik\cdot\xi}$. Here $k^\perp = (-k_2, k_1)$, $|k| = \sqrt{k_1^2 + k_2^2}$, and $\mathbb{Z}_0^2 = \{k \in \mathbb{Z}^2 : |k| \neq 0\}$. We define also $\mathbb{Z}_+^2 = \{k \in \mathbb{Z}_0^2 : k_1 > 0 \text{ or } \{k_1 = 0, k_2 > 0\}\ \}$ to consider half of the sequence (containing the same information as the whole sequence).

Each e_k is a periodic divergence-free C^∞-vector function. The convergence of the series (10.4) depends on the regularity of the vector function u, and can be used to define Sobolev spaces as in the following definition.

Let $\mathcal{U}'$ be the space of zero mean value periodic divergence-free vector distributions. Any element $u \in \mathcal{U}'$ is uniquely defined by the sequence of the coefficients $\{u_k\}_{k \in \mathbb{Z}_+^2}$; indeed, by duality, $u_k = \langle u, e_{-k} \rangle$, since each e_k is a periodic divergence-free and infinitely differentiable function. Following [Tr], we define the periodic divergence-free vector Sobolev spaces $(s \in \mathbb{R}, 1 < p < \infty)$

$$\mathcal{H}_p^s = \{\ u \underset{\mathcal{U}'}{=} \sum_{k \in \mathbb{Z}_0^2} u_k e_k : \sum_{k \in \mathbb{Z}_0^2} u_k (1 + |k|^2)^{s/2} e_k(\cdot) \in [L_p(\mathbb{T}^2)]^2\ \}$$

which are Banach spaces with norms $\|u\|_{\mathcal{H}_p^s} = \| \sum_k u_k (1 + |k|^2)^{s/2} e_k \|_{[L_p(\mathbb{T}^2)]^2}$.

The Hilbert space $\mathcal{H}_2^s$ is isomorphic to the space of complex valued sequences $\{u_k\}_{k \in \mathbb{Z}_+^2}$ such that $\sum_k |u_k|^2 |k|^{2s} < \infty$. Set $\mathcal{L}_p = \mathcal{H}_p^0$.

Let Π be the projector operator from the space of periodic vectors onto the space of periodic divergence-free vectors. Applying Π to both sides of the first equation in the Navier–Stokes system (10.1), we get rid of the pressure term (see, e.g., [Te]). The other terms lead to introduce the two following operators.

The Stokes operator is defined as

$$A = -\Pi \Delta, \quad D(A) = \mathcal{H}_2^2 \tag{10.5}$$

which is a linear unbounded self-adjoint operator in $\mathcal{L}_2$. For $u = \sum_{k \in \mathbb{Z}_0^2} u_k e_k$ we have $Au = \sum_k u_k |k|^2 e_k$ ($\{e_k\}$ and $\{|k|^2\}$ are the sequences of the eigenfunctions and eigenvalues of the Stokes operator, respectively. Note that $\{e_k\}$ is a complete orthonormal system in

$\mathcal{L}_2$.) Since A is a strictly positive operator, all the power operators A^s are well defined ($s \in \mathbb{R}$): $A^s u = \sum_k u_k |k|^{2s} e_k$, $D(A^s) = \mathcal{H}_2^{2s}$. Moreover, A^s is an isomorphism from $\mathcal{H}_p^{s+2}$ to $\mathcal{H}_p^s$.

The operator B is defined as the bilinear operator

$$B(u,v) = \Pi\left[(u \cdot \nabla)v\right] \tag{10.6}$$

whenever it makes sense. For instance, a classical result is that $B : \mathcal{H}_2^1 \times \mathcal{H}_2^1 \to \mathcal{H}_2^{-1}$ (see, e.g., [Te]).

By the incompressibility condition, we have

$$\langle B(u,v), z\rangle = -\langle B(u,z), v\rangle \tag{10.7}$$

where $\langle B(u,v), z\rangle = \int_{\mathbb{T}^2} [(u(\xi)\cdot\nabla)v(\xi)]\cdot z(\xi)\, d\xi$ for $u, v, z \in \mathcal{H}_2^1$.

Let us now introduce the stochastic Navier–Stokes equation we are interested in. It has the following abstract Itô form:

$$\begin{cases} d\,u(t) + [\nu A u(t) + B(u(t),u(t))]\,dt = f(t)\,dt + A^{-\varepsilon} dw(t), \quad t > 0 \\ u(0) = u_0. \end{cases} \tag{10.8}$$

The right-hand side has two components: the deterministic f and the stochastic $A^{-\varepsilon} dw(t)$. More precisely, we assume $\varepsilon > 0$, $\{w(t)\}_{t\ge 0}$ is a Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}_{t\ge 0}$, which is cylindrical in the space of finite energy $\mathcal{L}_2$, i.e.

$$w(t) = \sum_{k\in\mathbb{Z}_0^2} \beta_k(t) e_k \tag{10.9}$$

where $\{\beta_k\}_{k\in\mathbb{Z}_0^2}$ is a sequence of standard independent complex-valued Wiener processes (for $k \in \mathbb{Z}_+^2$ this means $\beta_k(t) = \beta_k^{(1)}(t) + i\beta_k^{(2)}(t)$, where the $\beta_k^{(j)}$'s are i.i.d. (independent identically distributed) real-valued standard Wiener processes; for $-k \in \mathbb{Z}_+^2$ the β_k's are defined by the condition $\beta_k = -\overline{\beta_{-k}}$ providing $w(t)$ to be real).

We denote by E the expectation with respect to the measure P.

10.3 Definition of generalized solution

We define a [illegible] as a process whose paths are regular enough for the dynamics to exist in the space of finite energy (initial velocity $u_0 \in \mathcal{L}_2$ is considered); implicitely this requires also that the nonlinear term is well defined.

Definition 10.1 [illegible] u [illegible] $[0,T]$ [illegible] (10.8) [illegible]

$$u \in C([0,T];\mathcal{L}_2) \cap L^2(0,T;\mathcal{L}_4) \quad P-a.s.$$

[illegible] u [illegible] P [illegible]

$$\langle u(t),\phi\rangle + \nu\int_0^t \langle u(s), A\phi\rangle ds - \int_0^t \langle B(u(s),\phi), u(s)\rangle ds$$

$$= \langle u_0,\phi\rangle + \int_0^t \langle f(s),\phi\rangle ds + \langle A^{-\varepsilon} w(t),\phi\rangle$$

$$\forall t\in[0,T] \text{ [illegible] } \forall \phi \in \mathcal{H}_2^2.$$

[illegible] u [illegible]

Remark 10.1 All the terms in the previous equality make sense and this equation corresponds to that of formulation (10.8). In fact

$$|\langle B(u,u),\phi\rangle| \underset{\text{by (10.7)}}{=} |-\langle B(u,\phi),u\rangle| \leq c\,\|\phi\|_{\mathcal{H}_2^1}\,\|u\|_{\mathcal{L}_4}^2 \tag{10.10}$$

by the Hölder's inequality and because the norms $\|\nabla\phi\|_{\mathcal{L}_2}$ and $\|\phi\|_{\mathcal{H}_2^1}$ are equivalent. This shows that $B:\mathcal{L}_4\times\mathcal{L}_4\to\mathcal{H}_2^{-1}$.

As to the noise term $\langle A^{-\varepsilon}w(t),\phi\rangle$, it makes sense because the process $A^{-\varepsilon}w$ has $C([0,T];\mathcal{H}_2^{-1})$-valued paths. (To see this, use the equality $E\|A^{-\varepsilon}w(t)\|^2_{\mathcal{H}_2^{-1}} = 4\sum_{k\in\mathbb{Z}_+^2}|k|^{-4\varepsilon-2}$. Moreover, notice that $A^{-\varepsilon}w$ is a cylindrical noise in the space $\mathcal{H}_2^{2\varepsilon}$, the embedding $\mathcal{H}_2^{2\varepsilon}\subset\mathcal{H}_2^{-1}$ is Hilbert–Schmidt, and conclude as in Section 4.3.1 by [DPZ].) □

This definition is based on that by [Fl]; the difference is that [Fl] deals with the Dirichlet boundary conditions for the velocity field, whereas here the periodic boundary conditions are assumed. As we shall see in the next section, the mathematical properties of the Stokes eigenfunctions in the periodic case allow us to get good estimates in the Lebesgue space $\mathcal{L}_4$; otherwise, in [Fl] the spatial regularity $\mathcal{H}_2^{1/2+\tilde\varepsilon}$ is required. Keeping in mind the embedding $H_2^{1/2}(\mathbb{T}^2)\subset L_4(\mathbb{T}^2)$, we conclude that our definition is more general (but meaningful only in the periodic case).

Other regularity results in different domains are given also in [BrLi].

10.4 Existence and uniqueness result

The results on solutions to equation (10.8) are obtained as suggested first by [Fl], analyzing two auxiliary equations: the linear stochastic Stokes equation (see (10.11) below) and the nonlinear equation obtained by making the difference between equation (10.8) and equation (10.11). With respect to previous works with this technique, here we specify a different regularity for the linear process.

The linear equation

Let us consider the stochastic Stokes equation

$$d\,z(t)+\nu Az(t)\,dt = A^{-\varepsilon}dw(t), \qquad t>0 \tag{10.11}$$

(the nonlinearity in (10.8) is neglected). It has a unique stationary solution, given by the stochastic convolution

$$z(t)=\int_{-\infty}^t e^{-(t-s)\nu A}A^{-\varepsilon}dw(s). \tag{10.12}$$

It can be developed in series as

$$z(t)=\sum_{k\in\mathbb{Z}_0^2}\int_{-\infty}^t e^{-(t-s)\nu|k|^2}|k|^{-2\varepsilon}d\beta_k(s)\,e_k \equiv \sum_{k\in\mathbb{Z}_0^2} z_k(t)\,e_k. \tag{10.13}$$

We estimate the spatial regularity of the process z. First, given any integer n and positive $\tilde\varepsilon$ consider

$$E\,\|z(t)\|^{2n}_{\mathcal{H}_{2n}^{2\tilde\varepsilon}} = E\,\|z_1(t)\|^{2n}_{H_{2n}^{2\tilde\varepsilon}(\mathbb{T}^2)} + E\,\|z_2(t)\|^{2n}_{H_{2n}^{2\tilde\varepsilon}(\mathbb{T}^2)}\,.$$

We estimate each component of the vector $z(t)$, showing the computations only for the first one $z_1(t)$. We have

$$E\,\|z_1(t)\|^{2n}_{\mathcal{H}_{2n}^{2\tilde\varepsilon}} = \int_{\mathbb{T}^2} E\left|\sum_{k\in\mathbb{Z}_0^2}|k|^{2\tilde\varepsilon}z_k(t)(-k_2)\frac{e^{ik\cdot\xi}}{2\pi|k|}\right|^{2n}d\xi.$$

We rewrite the series

$$\begin{aligned}\sum_{k\in\mathbb{Z}_0^2}|k|^{2\tilde\varepsilon}\frac{k_2}{2\pi|k|}z_k(t)e^{ik\cdot\xi} &= \sum_{k\in\mathbb{Z}_+^2}|k|^{2\tilde\varepsilon}\frac{k_2}{2\pi|k|}[z_k(t)e^{ik\cdot\xi}-z_{-k}(t)e^{-ik\cdot\xi}]\\ &= \sum_{k\in\mathbb{Z}_+^2}\frac{|k|^{2\tilde\varepsilon}k_2}{\pi|k|}\left[\int_{-\infty}^t e^{-(t-s)\nu|k|^2}|k|^{-2\varepsilon}d\beta_k^{(1)}(s)\cos(k\cdot\xi)\right.\\ &\qquad\left.-\int_{-\infty}^t e^{-(t-s)\nu|k|^2}|k|^{-2\varepsilon}d\beta_k^{(2)}(s)\sin(k\cdot\xi)\right].\end{aligned}$$

For fixed t and ξ, it is the sum of independent centered Gaussian real random variables. Their variance is given by

$$E\left[\frac{|k|^{2\tilde\varepsilon}k_2}{\pi|k|}\int_{-\infty}^t e^{-(t-s)\nu|k|^2}|k|^{-2\varepsilon}d\beta_k^{(1)}(s)\cos(k\cdot\xi)\right]^2=\frac{(k_2)^2}{2\pi^2\nu|k|^{4\varepsilon-4\tilde\varepsilon+4}}\cos^2(k\cdot\xi)$$

and

$$E\left[\frac{|k|^{2\tilde\varepsilon}k_2}{\pi|k|}\int_{-\infty}^t e^{-(t-s)\nu|k|^2}|k|^{-2\varepsilon}d\beta_k^{(2)}(s)\sin(k\cdot\xi)\right]^2=\frac{(k_2)^2}{2\pi^2\nu|k|^{4\varepsilon-4\tilde\varepsilon+4}}\sin^2(k\cdot\xi).$$

Therefore*

$$E\left|\sum_{k\in\mathbb{Z}_0^2}|k|^{2\tilde\varepsilon}\frac{k_2}{2\pi|k|}z_k(t)e^{ik\cdot\xi}\right|^{2n}=(2n-1)!!\left(\sum_{k\in\mathbb{Z}_+^2}\frac{(k_2)^2}{2\pi^2\nu|k|^{4\varepsilon-4\tilde\varepsilon+4}}\right)^n.$$

The latter sum is convergent as soon as $4\varepsilon-4\tilde\varepsilon>0$.

Therefore we have that given $\tilde\varepsilon<\varepsilon$, for any finite n, $z(t)\in\mathcal{H}_{2n}^{2\tilde\varepsilon}$ P-a.s. The continuity of the trajectories holds (see [DPZ], Th. 5.9). We sum up these properties in the following.

Proposition 10.1 [illegible] $\varepsilon>0$ [illegible] $\tilde\varepsilon<\varepsilon$, [illegible] z [illegible] (10.12) [illegible] $\mathcal{H}_p^{2\tilde\varepsilon}$ [illegible] $2\le p<\infty$, P [illegible]

Remark 10.2 By interpolation, the spatial regularity is extended into the Besov spaces. Indeed they can be defined as real interpolation spaces

$$\begin{aligned}\mathcal{B}^s_{p\,q}=(\mathcal{H}_p^{s_0},\mathcal{H}_p^{s_1})_{\theta,q}, &\qquad s\in\mathbb{R},1<p,q<\infty\\ &\qquad s=(1-\theta)s_0+\theta s_1,\qquad 0<\theta<1.\end{aligned}$$

(For the theory of interpolation spaces see, e.g., [BeL]).

We keep in mind that, for $\varepsilon=0$, [DPD] deals with $z\in C([0,T];\mathcal{B}^{-s}_{pq})$, for $s>0, 2\le p\le q<\infty$; this is obtained in the above case when $\varepsilon\downarrow 0$. □

*Given independent real random variables X_j, with $X_j\sim\mathcal{N}(0,v_j)$, it is elementary to show that

$$E\big[\sum_j X_j\big]^{2n}=(2n-1)!!\big(\sum_j v_j\big)^n.$$

The Navier–Stokes equation

Instead of analyzing equation (10.8), define the process $v = u - z$; it satisfies the following system:

$$\begin{cases} \frac{d}{dt}v(t) + \nu Av(t) + B(v(t), v(t)) & +B(v(t), z(t)) + B(z(t), v(t)) \\ & = f(t) - B(z(t), z(t)), \qquad t > 0 \\ v(0) = u_0 - z(0). \end{cases} \tag{10.14}$$

The equation is nonlinear and random (the process z appears), but there is no more the noise $A^{-\varepsilon}dw(t)$.

Working pathwise, we have by Proposition 10.1 that $z \in C([0,T];\mathcal{L}_4)$, so $B(z,z) \in C([0,T];\mathcal{H}_2^{-1})$. Usual techniques of a priori estimates (see [Fl], modifying the proof using only the $\mathcal{L}_4$-space regularity; this is successful as already remarked in [Fe97]) given that for any $u_0 \in \mathcal{L}_2$ there exists a unique solution v to system (10.14): $v \in C([0,T];\mathcal{L}_2) \cap L^2(0,T;\mathcal{H}_2^1)$. By interpolation $v \in L^4(0,T;\mathcal{H}_2^{1/2})$; by the embedding $H_2^{1/2}(\mathbb{T}^2) \subset L_4(\mathbb{T}^2)$ we also have that $v \in L^4(0,T;\mathcal{L}_4)$. Therefore there `exists` a process $u = v+z \in C([0,T];\mathcal{L}_2) \cap L^4(0,T;\mathcal{L}_4)$. This is a generalized solution to problem (10.8), as defined in Section 10.3. This regularity grants `uniqueness`. (For the proof, see [Fe03]. The key points are first that the estimates there depending on $u \in L^4(0,T;D(A^{1/4}))$ hold also if $u \in L^4(0,T;\mathcal{L}_4)$; moreover, this "regularity in time (i.e., $u \in L^4(0,T;D(A^{1/4}))$) is useful in order to prove uniqueness".)

Proposition 10.2 [illegible] $\varepsilon > 0$ [illegible] $u_0 \in \mathcal{L}_2$ [illegible] $f \in L^2(0,T;\mathcal{H}_2^{-1})$ [illegible]

$$u \in C([0,T];\mathcal{L}_2) \cap L^4(0,T;\mathcal{L}_4) \qquad P-a.s.$$

[illegible] (10.8)

Now we want to express the regularity of this generalized solution as depending on the parameter $\varepsilon > 0$. The case $\varepsilon = 0$ has been considered in previous papers: [ARHK, AC] constructed the Gaussian invariant measure of the enstrophy and proved existence of a weak solution (weak in the probabilistic sense), [DPD] proved the existence of a strong solution, and [AF] proved its uniqueness. The solution is defined for almost all initial velocity with respect to the invariant measure of the enstrophy. The solution by [DPD] has paths in the space $C([0,T];\mathcal{B}_{pq}^{-s})$ for any $s > 0$ and $2 \le p \le q < \infty$. The spatial regularity is of negative order ($-s < 0$), i.e., infinite energy velocity vectors are considered.

As soon as we take $\varepsilon > 0$, the solution is more regular as stated above; i.e., we work in the space of finite energy velocity vectors. In order to compare the result for $\varepsilon = 0$ with that for $\varepsilon > 0$ we make precise the regularity of the solution u when $\varepsilon > 0$. According to Proposition 10.1, $z \in C([0,T];\mathcal{H}_p^{2\tilde{\varepsilon}})$ for any $\tilde{\varepsilon} < \varepsilon$ and $2 \le p < \infty$. By interpolation, as already done before, $v \in L^{2/(2\tilde{\varepsilon}+1-\frac{2}{p})}(0,T;\mathcal{H}_2^{2\tilde{\varepsilon}+1-\frac{2}{p}})$. By the embedding $\mathcal{H}_2^{2\tilde{\varepsilon}+1-\frac{2}{p}} \subset \mathcal{H}_p^{2\tilde{\varepsilon}}$ ($p \ge 2$) we conclude that the unique solution to (10.14) is such that

$$v \in C([0,T];\mathcal{L}_2) \cap L^2(0,T;\mathcal{H}_2^1) \cap L^{2/(2\tilde{\varepsilon}+1-\frac{2}{p})}(0,T;\mathcal{H}_p^{2\tilde{\varepsilon}}) \tag{10.15}$$

for any $\tilde{\varepsilon} < \varepsilon$, $2 \le p < \infty$.

Combining the regularity of v and of z, we obtain the following result.

Theorem 10.1 [illegible] $\varepsilon > 0$ [illegible] $u_0 \in \mathcal{L}_2$ [illegible] $f \in L^2(0,T;\mathcal{H}_2^{-1})$ [illegible]

$$u \in C([0,T];\mathcal{L}_2) \cap L^4(0,T;\mathcal{L}_4) \cap L^{2/(2\tilde{\varepsilon}+1-\frac{2}{p})}(0,T;\mathcal{H}_p^{2\tilde{\varepsilon}}) \qquad P-a.s.$$

([illegible] $\tilde{\varepsilon} < \varepsilon$ [illegible] $2 \le p < \infty$) [illegible] (10.8)

We remark that formally for $\varepsilon = 0$ (so $\tilde{\varepsilon} < 0$) the parameter p has to be greater than 2, if we want the exponent $2/(2\tilde{\varepsilon}+1-\frac{2}{p})$ to be positive; therefore in the limit $\varepsilon = 0$ we cannot expect to work any longer in Hilbert spaces, but in the spaces $\mathcal{H}_p^{2\tilde{\varepsilon}}$ with $p > 2, \tilde{\varepsilon} < 0$.

References

[AC] Albeverio S., Cruzeiro A.B. (1990). Global flows with invariant (Gibbs) measures for Euler and Navier-Stokes two dimensional fluids, [illegible] 129, 431-444.

[AF] Albeverio S., Ferrario B. (2004). Uniqueness of solutions of the stochastic Navier–Stokes equation with invariant measure given by the enstrophy, [illegible] 23, no.2, 1632-1649.

[ARHK] Albeverio S., Ribeiro de Faria M., Høegh–Krohn R. (1979). Stationary measures for the periodic Euler flow in two dimensions, [illegible] 20, no. 6, 585-595.

[BT] Bensoussan A., Temam R. (1973). Équations stochastiques du type Navier–Stokes, [illegible] 13, 195-222.

[BeL] Bergh L., Löfström J. (1976). [illegible], Springer, Berlin-Heidelberg-New York.

[BrLi] Brzeźniak Z., Li Y. (2002). Asymptotic compactness of 2D stochastic Navier-Stokes equations on some unbounded domains, [illegible], Department of Mathematics, the University of Hull, Vol. XV, no. 3.

[DPD] Da Prato G., Debussche A. (2002). 2D-Navier–Stokes equations driven by a space-time white noise, [illegible] 196, no. 1, 180-210.

[DPZ] Da Prato G., Zabczyk J. (1992). [illegible], Cambridge University Press.

[Fe97] Ferrario B. (1997). The Bénard problem with random perturbations: dissipativity and invariant measures, [illegible] 4, no. 1, 101-121.

[Fe99] Ferrario B. (1999). Stochastic Navier–Stokes equations: analysis of the noise to have a unique invariant measure, [illegible], 177, 331-347.

[Fe03] Ferrario B. (2003). Uniqueness result for the 2D Navier–Stokes equation with additive noise, [illegible] 75, no.6, 435-442.

[Fl] Flandoli F. (1994). Dissipativity and invariant measures for stochastic Navier–Stokes equations, [illegible] 1, no. 4, 403-423.

[Te] Temam R. (1983). [illegible], SIAM, Philadelphia.

[Tr] Triebel H. (1973). Über die Existenz von Schauderbasen in Sobolev-Besov-Räumen. Isomorphiebeziehungen, [illegible] 44, 83-100.

[VF] Vishik M.J., Fursikov A.V. (1988). [illegible], Kluwer Academic Publishers, Dordrecht.

11 Two Models of K41

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11.1 Introduction

Let $u(x)$ be the velocity of a fluid at point $x \in \mathbb{R}^3$. Assume that $u(x)$ is a random field (namely, a family of random variables, indexed by $x \in \mathbb{R}^3$, all defined on a probability space $(\Omega, \mathcal{A}, P)$ with expectation E). Assume that $u(x)$ is homogeneous and isotropic, so that the law of $u(x+re) - u(x)$ is independent of $x \in \mathbb{R}^3$ and $e \in \mathbb{R}^3$, with $|e| = 1$. Consider the structure function of order $p > 0$

$$S_p(r) = E\left[|u(x+re) - u(x)|^p\right]$$

with $r > 0$. [The literature often considers the longitudinal structure function, where $u(x+re) - u(x)$ is projected along e, but this distinction is immaterial here.] The function $S_p(r)$ is one of the main objects of investigation in statistical fluid dynamics and one of the aims is to discover its scaling properties for turbulent fluids, with the hope to observe universal behaviors. The scaling exponent

$$\zeta_p = \lim_{r \to 0} \frac{\log S_p(r)}{\log r}$$

is of major interest.

In his 1941 paper [10], Kolmogorov predicted

$$\zeta_p = \frac{p}{3}. \tag{11.1}$$

In fact [10] treats only the case $p = 2$, and the result $\zeta_2 = \frac{2}{3}$ is in excellent agreement with experimental observations. But the (intuitive) arguments in [10] leading to $\zeta_2 = \frac{2}{3}$ apply also for $p > 2$ and yield $\zeta_p = \frac{p}{3}$, which, on the contrary, is not confirmed by experiments. The experimental data, although do not allow a precise fitting, clearly indicate that the function

$$p \mapsto \zeta_p$$

is a sort of concave function below the line $p \mapsto \frac{p}{3}$. A good explanation of this correction to K41 is still missing, although several arguments and phenomenological models have been devised, in particular, the multifractal model (see [7] for a careful discussion).

This short note has the only aim to indicate two mathematical models having the K41 scaling (11.1). The first model is based on stochastic partial differential equations (SPDEs) of Navier–Stokes and Euler type and is inspired by the presentation of Kupiainen [8]. We advertise that, at present, this model is not rigorous.

The second model is an ad-hoc-constructed ensemble of random vortex filaments, having the K41 scaling. It is rigorous, but the relation with fundamental physical laws like the Navier–Stokes equations is unknown. At least, it is based on geometrical objects (vortex filaments) that have, in numerical visualizations, shapes quite similar to those observed in direct numerical simulations of Navier–Stokes equations at high Reynold numbers, see for instance, [1], [11], [13]. This ensemble of vortex filaments is a particular case of those studied in [6], along lines inspired initially by the work of Chorin [2]; we emphasize here this particular case since it has simpler scaling properties that, at least formally, allow us to argue by self-similarity. Let us mention that the scope of [6] is wider: indeed it covers also the multifractal model under suitable choices of certain parameters.

The reason to list these two examples of models with K41 properties is an attempt to understand which kind of idealizations are behind it, when rigorous or semirigorous models are devised (the arguments in [10] are purely phenomenological). The hope for the future is to identify the source of the necessary correction.

11.1.1 ζ_p by self-similarity

In this section we introduce a concept of self-similarity which easily implies a scaling of K41 type. Unfortunately, this concept cannot be rigorously applied in our context, because it is incompatible with space homogeneity, another property fulfilled by our random fields: self-similar space homogeneous fields are either trivial (identically zero) or non-well-defined (identically infinite). However, the power of suggestion of this concept of self-similarity is strong and we shall see below that, inspired by this concept, we may rigorously construct a random field with K41 scaling.

In the following definition of blow-up, think that we take a small $\lambda > 0$ and observe the field with a zoom at scale λ, rescaling also the amplitude of the field by a factor λ^α, α a given real number (negative in our applications). Given a random field $u(x)$, we call α-[illegible] u [illegible] $\lambda > 0$ the new random field

$$u_{(\lambda,\alpha)}(x) = \lambda^\alpha u(\lambda x).$$

Denote by $\overset{\mathcal{L}}{=}$ the equality in law.

Definition 11.1 [illegible] α [illegible] $\lambda > 0$ [illegible] α [illegible] u [illegible] $\lambda > 0$ $u_{(\lambda,\alpha)}(x) \overset{\mathcal{L}}{=} u(x)$, [illegible]

$$u(\lambda x) \overset{\mathcal{L}}{=} \lambda^{-\alpha} u(x)$$

([illegible] x, [illegible])

Under the assumption of α-self-similarity, if the moments are finite (in fact we only need moments of the [illegible], not of the solution itself), we have

$$E\left[|u(re) - u(0)|^p\right] = r^{-\alpha p} E\left[|u(e) - u(0)|^p\right].$$

If we adopt the very particular definition $S_p(r) := E\left[|u(re) - u(0)|^p\right]$ (centered at $x = 0$, which is particular since we do not assume space homogeneity), we obtain that the limit ζ_p exists; [illegible] $E\left[|u(e) - u(0)|^p\right] \neq 0$, then ζ_p is given by

$$\zeta_p = -\alpha p.$$

These computations are correct but useless in our context: indeed, first notice that $u(0) = 0$ for an α-self-similar field with $\alpha \neq 0$; then, since in all our examples we shall deal with homogeneous (translation invariant) fields u, we conclude that u is identically zero. The other possibility, unfortunately nonrigorous at present, is that u is in a sense identically "infinite": such case would be very suggestive in connection with the example below of the Euler equation forced by white noise, whose stationary solution seems to be infinite because of lack of dissipation (formally such stationary solution seems to be self-similar and homogeneous).

Remark 11.1 Suitable generalizations of the previous notion of self-similarity seem possible, in the direction of requiring a self-similarity only at small scales, or at the level of the increments. In such cases the relevant assumption becomes that $E\left[\left|u\left(e\right)-u\left(0\right)\right|^{p}\right]$ is finite and not zero. This is work in progress.

11.2 Looking for K41 in stochastic Navier–Stokes equations

11.2.1 The equations

Let us stress the fact that almost all arguments described in this section are only [illegible]. For this reason we state the results as "[illegible]" instead of "theorems."

Let us consider the stochastic Navier–Stokes equations in $\mathbb{R}^3$, for $t \geq 0$

$$\frac{\partial u\left(t,x\right)}{\partial t}+\left(u\left(t,x\right)\cdot\nabla\right)u\left(t,x\right)+\nabla p\left(t,x\right)=\nu\triangle u\left(t,x\right)+f\left(t,x\right)$$

with the condition $divu = 0$, where $f\left(t,x\right)$ is white noise in time, possibly correlated in space. Formally

$$f\left(t,x\right)=\frac{\partial W}{\partial t}\left(t,x\right)$$

where $t \mapsto W\left(t,.\right)$ is a Brownian motion in a suitable function space with covariance tensor

$$q\left(x,y\right):=E\left[W\left(1,x\right)\otimes W\left(1,y\right)\right].$$

The literature contains a number of foundational results on such equations, see, for instance, [3], [12] and references therein, but the results of well-posedness and ergodicity we need to continue our discussion are still open. Therefore we avoid to put precise assumptions and proceed at an intuitive level.

Assume stationarity and isotropy in space

$$q\left(x,y\right)=q(|x-y|)$$

for some tensor $q(r)$, $r>0$.

11.2.2 Scaling transformations

To simplify the understanding, let us recall the result in the case of regular force f. We again consider the blow-up of $u(t,x)$, where now time has to be rescaled in a particular way for coherence between the terms $\frac{\partial u}{\partial t}$ and $\left(u\cdot\nabla\right)u$.

Claim 11.1 [illegible] $\alpha\in\mathbb{R}$, $\lambda>0$ [illegible]

$$u_{(\lambda,\alpha)}\left(t,x\right)=\lambda^{\alpha}u\left(\lambda^{\alpha+1}t,\lambda x\right),\quad p_{(\lambda,\alpha)}\left(t,x\right)=\lambda^{2\alpha}p\left(\lambda^{\alpha+1}t,\lambda x\right),$$
$$f_{(\lambda,\alpha)}\left(t,x\right)=\lambda^{2\alpha+1}f\left(\lambda^{\alpha+1}t,\lambda x\right)$$

[illegible] $\mathbb{R}^3$, [illegible] $t\geq 0$,

$$\frac{\partial u_{(\lambda,\alpha)}}{\partial t}+\left(u_{(\lambda,\alpha)}\cdot\nabla\right)u_{(\lambda,\alpha)}+\nabla p_{(\lambda,\alpha)}=\nu_{(\lambda,\alpha)}\triangle u_{\lambda}+f_{(\lambda,\alpha)}$$

[illegible]

$$\nu_{(\lambda,\alpha)}=\nu\lambda^{\alpha-1}.$$

The reader may easily verify this claim, by a formal computation. In the case of white-noise force, we have

$$f_{(\lambda,\alpha)}(t,x) = \lambda^{2\alpha+1}\frac{\partial W}{\partial t}\left(\lambda^{\alpha+1}t,\lambda x\right).$$

Introduce the process

$$W_{(\lambda,\alpha)}(t,y) := \lambda^{-\frac{\alpha+1}{2}}W\left(\lambda^{\alpha+1}t,y\right).$$

It has the same law as $W(t,y)$

$$W_{(\lambda,\alpha)}(t,y) \stackrel{\mathcal{L}}{=} W(t,y).$$

We have

$$\frac{\partial W_{(\lambda,\alpha)}}{\partial t}(t,y) = \lambda^{\frac{\alpha+1}{2}}\frac{\partial W}{\partial t}\left(\lambda^{\alpha+1}t,y\right).$$

Hence

$$f_{(\lambda,\alpha)}(t,x) = \lambda^{\frac{3\alpha+1}{2}}\frac{\partial W_{(\lambda,\alpha)}}{\partial t}(t,\lambda x).$$

Therefore, up to equality in law, we have the following.

Claim 11.2

$$\frac{\partial u_{(\lambda,\alpha)}}{\partial t} + \left(u_{(\lambda,\alpha)}\cdot\nabla\right)u_{(\lambda,\alpha)} + \nabla p_{(\lambda,\alpha)} = \nu_{(\lambda,\alpha)}\triangle u_{(\lambda,\alpha)} + \lambda^{\frac{3\alpha+1}{2}}\frac{\partial W}{\partial t}(t,\lambda x).$$

[illegible] $\lambda^{\frac{3\alpha+1}{2}}\frac{\partial W}{\partial t}(t,\lambda x)$ [illegible] $q_{(\lambda,\alpha)}(x,y) = \lambda^{3\alpha+1}q(\lambda(x-y))$ [illegible] $q_{(\lambda,\alpha)}(x,y)$ [illegible] $q_{(\lambda,\alpha)}(|x-y|)$, [illegible]

$$q_{(\lambda,\alpha)}(r) = \lambda^{3\alpha+1}q(\lambda r). \tag{11.2}$$

We describe now two arguments leading to K41 result (11.1).

11.2.3 Argument 1

Consider the $\nu \to 0$ limit of the previous stochastic Navier–Stokes equations, namely, the stochastic Euler equation

$$\frac{\partial u}{\partial t} + (u\cdot\nabla) + \nabla p = \frac{\partial W}{\partial t} \tag{11.3}$$

($divu = 0$) with the special noise (see the acknowledgments)

$$q(r) = q$$

independent of r. We advise the reader that what we are going to describe is extremely formal.

[illegible]. Take

$$\alpha = -\frac{1}{3}$$

and the corresponding scaled (stationary) processes $u_{(\lambda,\alpha)}$, $p_{(\lambda,\alpha)}$. They satisfy the equation

$$\frac{\partial u_{(\lambda,\alpha)}}{\partial t} + \left(u_{(\lambda,\alpha)}\cdot\nabla\right)u_{(\lambda,\alpha)} + \nabla p_{(\lambda,\alpha)} = \frac{\partial W}{\partial t}(t,\lambda x).$$

The white noise $\frac{\partial W}{\partial t}(t,\lambda x)$ has the covariance

$$q_{(\lambda,\alpha)}(x,y) = q(\lambda(x-y)) = q$$

i.e., it is equal to W in law. By the uniqueness-in-law assumption, we have, for the stationary solution

$$u_{(\lambda,\alpha)} \stackrel{\mathcal{L}}{=} u$$

and therefore (11.1) "holds" as explained in Section 11.1.1. Notice that, due to time stationarity, we may compute the structure function by means of the field $u\,(0,x)$. Besides many other obscure points, one the problems is that this field is expected to be space homogeneous and self-similar, hence trivial or "infinite" by the comments of Section 11.1.1. Our conjecture is that the crucial property to be proved is that the following expression, suitably defined, is finite and not zero:

$$E\left[|u\,(0,e) - u\,(0,0)|^2\right]$$

($u(t,x)$ being the stationary solution). This would be related to ζ_2 at least.

Also, the assumption above of existence and uniqueness of a stationary solution is of course full of troubles. First, equation (11.3) is not dissipative, so we cannot expect the existence of a stationary solution by easy methods. It could be that Euler equation in the turbulent regime has a form of dissipation, but this is unknown, although sometimes conjectured. It could be that individual solutions $u\,(t,x)$ (from a given initial condition) do not approach an equilibrium and have energy increasing to infinity, but the moments of the increments $E\left[|u\,(t,e) - u\,(t,0)|^2\right]$ remain bounded and different from zero (this is a form of the previous conjecture); in such a case there could be the possibility to prove a suitable form of self- similarity and deduce K41 scaling (see the remark at the end of Section 11.1.1). It could be that statements can be rigorously formulated as a limit from $\nu > 0$. We do not know an answer.

A second problem with the previous assumption is the uniqueness. A third one, less apparent, is that the constant covariance corresponds to the constant in space noise, so the noise can be absorbed in the term ∇p; so, for instance, in principle the solution $u \equiv 0$ is a solution if the initial condition is zero, an apparent paradox under such a strong noise. Thus one has to restrict properly the space where we look for solutions. All these items are open at present.

11.2.4 Argument 2

Kolmogorov [10] based part of his argument on the assumption that, in the limit $\nu \to 0$, the mean dissipation energy (density)

$$\varepsilon := \nu E\left[|\nabla u\,(0,0)|^2\right]$$

converges to a limit different from zero, and that the structure function $S_p\,(r)$, in the limit $\nu \to 0$ and for small r, depends only on ε and r.

By Itô formula (in fact one has to apply it to approximations on bounded sets $[-n,n]^3$, where the covariance of the noise has finite trace, and then take the limit $n \to \infty$)

$$\varepsilon = \frac{1}{2}\, Trace\, q(0).$$

Therefore, a reformulation of Kolmogorov assumption, for the stochastic Navier–Stokes equations, is that

$$S_p\,(r) \text{ [illegible] } q(0)$$

in the limit $\nu \to 0$ and for small r. Let us adopt this assumption.

From (11.2), for the rescaled solution we have

$$q_{(\lambda,\alpha)}(0) = \lambda^{3\alpha+1} q(0).$$

Hence for

$$\alpha = -\frac{1}{3}$$

we discover that $q_{(\lambda,\alpha)}(0)$ is independent of λ, and therefore $S_p^{(\lambda,\alpha)}(r)$ is independent of λ, hence equal to $S_p(r)$. We get again (11.1).

Notice that this second argument is based on an exogenous assumption, external to the stochastic Navier–Stokes equations; on the other side it is not based on stationary solutions of Euler equation with noise. The way to express that the result is true in the limit $\nu \to 0$ and for small r is mathematically less clear.

11.3 A model of K41 by vortex filaments

11.3.1 A single random vortex filament

Following [4], [5], [6] (as a translation in the continuum of the lattice models of [2]; or, from another viewpoint, as a randomized version of the so-called Burgers' vortices), let us consider a three-dimensional (3D) random field $\xi_{\text{single}}(x)$ on $\mathbb{R}^3$ defined as

$$\xi_{\text{single}}(x) = \xi_{\text{single}}^{(X_0,l,T,U)}(x) = \frac{U}{l^2}\int_0^T \rho_l(x - X_t) \circ dX_t$$

where:

- $X_t = X_0 + W_t$, $(W_t)_{t\geq 0}$ is a 3D Brownian motion, $X_0 \in \mathbb{R}^3$ is a given point.
- $\rho_l(x) = \rho\left(\frac{x}{l}\right)$, $\rho(x) = \exp\left(-|x|^2\right)$ (for instance).
- $(l, T, U) \in (0,\infty)^3$.

The intuitive picture is that of a tubular structure around the irregular core X_t, with cross section of radius $\sim l$, the tube being the support of a vorticity field $\xi_{\text{single}}(x)$, with the direction of the vorticity $\xi_{\text{single}}(x)$ being a local mean of "dX_t." The shape (in numerical simulations) of the isosurfaces of $\xi_{\text{single}}(x)$ looks very similar to the extremely complex and filament-like structures reported in [1], obtained there by direct numerical simulations of the 3D Navier–Stokes equations.

The factor $\frac{U}{l^2}$ in front of the integral has been chosen of this particular form so that, from later computations (see [6]), U has the interpretation of typical velocity intensity of the fluid near the vortex filament.

The random field $\xi_{\text{single}}(x)$ has been introduced mainly for interpretation, but the rigorous analysis is based on the associated velocity field $u_{\text{single}}(x)$ defined as

$$u_{\text{single}}(x) = u_{\text{single}}^{(X_0,l,T,U)}(x) = \frac{U}{l^2}\int_0^T k_l(x - X_t) \wedge dX_t$$

with $k_l = \nabla\varphi_l$ where φ_l is the solution of $\triangle\varphi_l = \rho_l$. Itô and Stratonovich integral here give us the same result, as explained in [6].

11.3.2 Ensemble of vortex filaments and main result

We have in mind a random velocity (r.v.) field of the form

$$u\left(x\right)=\sum_{i=1}^{\infty}\frac{U_i}{l_i^2}\int_0^{T_i}k_{l_i}\left(x-X_0^{(i)}-W_t^{(i)}\right)\wedge dW_t^{(i)}.$$

It should be the velocity field associated to a vorticity field composed of infinitely many vortex filaments, of different length T_i, thickness l_i, and intensity U_i.

We take independent Brownian motions $\left(W_t^{(i)}\right)$, starting from independent uniformly distributed points $X_0^{(i)}$, with independent random parameters (l_i, T_i, U_i). Sometimes, for a minor notational advantage, we shall work with $\left(l_i, \sqrt{T_i}, U_i\right)$. We shall specialize to the case

$$U_i = l_i^{1/3}, \quad T_i = l_i^2,$$

with l_i being independent r.v. distributed according to the measure $l^{-4}dl$ or a truncation of it.

Two of the previous elements are not well defined: uniformly distributed $X_0^{(i)} \in \mathbb{R}^3$ and $l^{-4}dl$-distributed r.v. l_i (their "laws" are only σ-finite). Since we deal with infinitely many of them, independent, we may give a rigorous definition by means of Poisson random measures. The rigorous presentation can be found in [6]. Since it is classical, we hope the reader may accept here, for the benefit of the intuition, that we continue to speak of uniformly distributed $X_0^{(i)} \in \mathbb{R}^3$ and $l^{-4}dl$-distributed l_i.

With this agreement on our formally language, we can prove the following result.

Theorem 11.1 [illegible] *fi* [illegible]

$$u\left(x\right)=\sum_{i=1}^{\infty}l_i^{-\frac{5}{3}}\int_0^{l_i^2}k_{l_i}\left(x-X_0^{(i)}-W_t^{(i)}\right)\wedge dW_t^{((i)}$$

[illegible] $X_0^{(i)}$ [illegible] $l_i > 0$ [illegible] $1_{l\in(0,l_{\max})}l^{-4}dl$ $(l_{\max} < \infty)$ [illegible] $\left(W_t^{(i)}\right)$ [illegible] *fi* [illegible] $\zeta_p = \frac{p}{3}$

The rigorous proof is given in [6]. The aim of the next section is to give a formal argument in favor of the following "fact": the random field described in the theorem, but in the case when $l_i > 0$ are distributed as $l^{-4}dl$ (no cut-off at large scales), is self-similar with exponent $\alpha = -1/3$; hence formally has the K41 scaling property, as explained in Section 11.1.1. The bad point is that such random field is not well defined: in a sense it is identically "infinite," due to the infinite contribution of the arbitrarily large structures. On the contrary, with the cut-off $1_{l\in(0,l_{\max})}l^{-4}dl$, u is finite, and its small scale self-similarity is preserved (in intuitive terms), yielding the desired scaling.

11.3.3 Scaling property of a single vortex

Later on we shall take $\beta = \frac{5}{3}$, but we express some of the steps in terms of β for sake of clarity. In the following lemma $\left(X_0, l, \sqrt{T}\right)$ is a given (deterministic) point of $\Lambda = R^3 \times (0,\infty) \times (0,\infty)$.

Lemma 11.1 [illegible]

$$u_{single}^{\left(X_0,l,\sqrt{T}\right)}\left(x\right)=l^{-\beta}\int_0^T k_l\left(x-X_0-W_t\right)\wedge dW_t.$$

[illegible] $\lambda > 0$

$$\lambda^{\alpha} u_{single}^{\left(X_0, l, \sqrt{T}\right)}(\lambda x) \stackrel{\mathcal{L}}{=} \lambda^{\alpha-\beta+2} u_{single}^{\lambda^{-1}\left(X_0, l, \sqrt{T}\right)}(x).$$

Proof By obvious facts and the equality in law between the processes $\left(\lambda^{-1} W_t\right)$ and $\left(W_{\lambda^{-2}t}\right)$, we have

$$\begin{aligned}
&\lambda^{\alpha} l^{-\beta} \int_0^T k_l\left(\lambda x - X_0 - W_t\right) \wedge dW_t \\
&= \lambda^{\alpha+1} l^{-\beta} \int_0^T k_l\left(\lambda\left(x - \lambda^{-1} X_0 - \lambda^{-1} W_t\right)\right) \wedge d\left(\lambda^{-1} W_t\right) \\
&\stackrel{\mathcal{L}}{=} \lambda^{\alpha+1} l^{-\beta} \int_0^T k_l\left(\lambda\left(x - \lambda^{-1} X_0 - W_{\lambda^{-2}t}\right)\right) \wedge d\left(W_{\lambda^{-2}t}\right).
\end{aligned}$$

Using the definition of Itô integral it is not difficult to see that the latter expression is equal to

$$\lambda^{\alpha+1} l^{-\beta} \int_0^{\lambda^{-2}T} k_l\left(\lambda\left(x - \lambda^{-1} X_0 - W_t\right)\right) \wedge dW_t.$$

Since $k_l(x) = l \cdot k\left(\frac{x}{l}\right)$

$$k_l(\lambda x) = \lambda k_{\frac{l}{\lambda}}(x).$$

Hence we get

$$\lambda^{\alpha-\beta+2}\left(\lambda^{-1} l\right)^{-\beta} \int_0^{\left(\lambda^{-1}\sqrt{T}\right)^2} k_{\lambda^{-1} l}\left(x - \lambda^{-1} X_0 - W_t\right) \wedge dW_t.$$

This is equal to $\lambda^{\alpha-\beta+2} u_{\text{single}}^{\lambda^{-1}\left(X_0, l, \sqrt{T}\right)}(x)$. The proof is complete. ■

Corollary 11.1 [illegible] $\beta = \frac{5}{3}$, $\alpha = -\frac{1}{3}$, [illegible]

$$\lambda^{\alpha} u_{single}^{\left(X_0, l, \sqrt{T}\right)}(\lambda x) \stackrel{\mathcal{L}}{=} u_{single}^{\lambda^{-1}\left(X_0, l, \sqrt{T}\right)}(x).$$

11.3.4 Scaling properties of the full velocity field

We describe the scaling argument at a certain level of generality and then specialize to the case of theorem 11.1.

Let $(N_A)_{A \in \mathcal{B}(\Lambda)}$ be a Poisson random field on $\Lambda = R^3 \times (0, \infty) \times (0, \infty)$, with intensity σ-finite measure ν. This means that to every ν-finite Borel set $A \subset \Lambda$ a Poisson r.v. N_A is associated, with parameter $\nu(A)$, and the variables N_A are independent on disjoint sets; moreover, the mapping $A \mapsto N_A$ has a version that is almost surely (a.s.) a measure. Formally, we may think to have a sequence of independent random variables $\left(X_0^{(i)}, l_i, \sqrt{T_i}\right)$ with values in Λ, distributed according to ν (but ν is only σ-finite). We define the full velocity random field as

$$u(x) = \sum_{i=1}^{\infty} l_i^{-\beta} \int_0^{T_i} k_{l_i}\left(x - X_0^{(i)} - W_t^{(i)}\right) \wedge dW_t^{(i)}$$

where $\left(W_t^{(i)}\right)$ is a sequence of independent 3D Brownian motions, also independent from the sequence $\left(X_0^{(i)}, l_i, \sqrt{T_i}\right)$. In [6] it has been proved that $u(x)$ is a well-defined random variable for every x, with finite moments, under assumptions that include those of theorem 11.1. In addition, it has been proved that the random field $u(x)$ is stationary and isotropic.

Claim 11.3 $\beta = \frac{5}{3}$, $\alpha = -\frac{1}{3}$ ν fi

$$\nu\left(\lambda^{-1}A\right) = \nu(A)$$

$\lambda > 0$ A

$$\lambda^{\alpha} u(\lambda x) \overset{\mathcal{L}}{=} u(x).$$

$$\zeta_p = \frac{p}{3}.$$

We have stated this result as a claim, since it is not rigorous: under the prescribed assumption on ν, the random field u is not well defined. If we take it as a formal expression and perform formal computations, we verify the claim. This is not a proof of anything, but a rather convincing argument behind Theorem 11.1.

The "proof" of the claim goes as follows. Use the notation $u_{\text{single}}^{\left(X_0^{(i)},l_i,\sqrt{T_i}\right),W^{(i)}}$ similarly to $u_{\text{single}}^{\left(X_0,l,\sqrt{T}\right)}$. Since

$$u(x) = \sum_{i=1}^{\infty} u_{\text{single}}^{\left(X_0^{(i)},l_i,\sqrt{T_i}\right),W^{(i)}}(x),$$

by Corollary 11.1 we have

$$\lambda^{\alpha} u(\lambda x) \overset{\mathcal{L}}{=} \sum_{i=1}^{\infty} u_{\text{single}}^{\lambda^{-1}\left(X_0^{(i)},l_i,\sqrt{T_i}\right),W^{(i)}}(x).$$

Since the law of $\left(X_0^{(i)}, l_i, \sqrt{T_i}\right)$ is invariant by homothety, we have

$$\lambda^{-1}\left(X_0^{(i)}, l_i, \sqrt{T_i}\right) \overset{\mathcal{L}}{=} \left(X_0^{(i)}, l_i, \sqrt{T_i}\right)$$

and these random vectors are independent, so the equality in law holds true for the full sequences. We get

$$\sum_{i=1}^{\infty} u_{\text{single}}^{\lambda^{-1}\left(X_0^{(i)},l_i,\sqrt{T_i}\right),W^{(i)}}(x) \overset{\mathcal{L}}{=} \sum_{i=1}^{\infty} u_{\text{single}}^{\left(X_0^{(i)},l_i,\sqrt{T_i}\right),W^{(i)}}(x)$$

and the "proof" is complete.

Remark 11.2 The result of the previous Claim holds true, in particular, for the measure ν defined on smooth functions φ with compact support in Λ as

$$\begin{aligned} &\int_{R^3}\int_0^{\infty}\int_0^{\infty} \varphi\left(X_0, l, \sqrt{T}\right) d\nu\left(X_0, l, \sqrt{T}\right) \\ &= \int_{R^3}\int_0^{\infty} \varphi(X_0, l, l)\, l^{-4} dl dX_0. \end{aligned}$$

Proof The property $\nu\left(\lambda^{-1}A\right) = \nu(A)$ is equivalent to

$$\begin{aligned} &\int_{R^3}\int_0^{\infty}\int_0^{\infty} \varphi\left(\lambda X_0, \lambda l, \lambda\sqrt{T}\right) d\nu\left(X_0, l, \sqrt{T}\right) \\ &= \int_{R^3}\int_0^{\infty}\int_0^{\infty} \varphi\left(X_0, l, \sqrt{T}\right) d\nu\left(X_0, l, \sqrt{T}\right). \end{aligned}$$

Using the definition of ν, what we have to prove is

$$\int_{R^3}\int_0^\infty \varphi\left(\lambda X_0, \lambda l, \lambda l\right) l^{-4} dl dX_0 = \int_{R^3}\int_0^\infty \varphi\left(X_0, l, l\right) l^{-4} dl dX_0.$$

This is true by change of variables, and the proof is complete. ■

The formal proof of the self-similarity idea behind theorem 11.1 is complete.

Acknowledgment

Some conceptual mistakes were present in a first draft of this work. Massimiliano Gubinelli helped to clarify the assumption on the covariance of the noise in Argument 1, Section 11.2.3, that was wrong before, and proposed part of the content of Remark 11.1.

References

[1] J.B. Bell and D.L. Marcus, Vorticity intensification and transition to turbulence in the three-dimensional Euler equations. [illegible], 147, no. 2, 371–394, 1992.

[2] A.J. Chorin, [illegible], volume 103 of [illegible]. Springer-Verlag, New York, 1994.

[3] G. Da Prato, A. Debussche, Ergodicity for the 3D stochastic Navier-Stokes equations, [illegible], (9) 82, no. 8, 877–947, 2003.

[4] F. Flandoli, On a probabilistic description of small scale structures in 3D fluids. [illegible], 38, no. 2, 207–228, 2002.

[5] F. Flandoli and M. Gubinelli, The Gibbs ensemble of a vortex filament. [illegible], 122, no. 3, 317–340, 2002.

[6] F. Flandoli and M. Gubinelli, Statistics of a vortex filament model, preprint, 2004.

[7] U. Frisch, [illegible]. Cambridge University Press, Cambridge, 1995. The legacy of A.N. Kolmogorov.

[8] A.J. Kupiainen, Lessons for turbulence. [illegible], Part I, 316–333, 2002.

[9] M. Ghil, R. Benzi, and G. Parisi, ed., [illegible], Amsterdam, 1985. Proc. Int. School on Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics, North-Holland.

[10] A.N. Kolmogorov, The local structure of turbulence in incompressible viscous fluid for very large Reynolds numbers. [illegible], 434(1890):9–13, 1991. Translated from the Russian by V. Levin, Turbulence and stochastic processes: Kolmogorov's ideas 50 years on.

[11] M. Lesieur, P. Bégou, E. Briand, A. Danet, F. Delcayre, and J. L. Aider, Coherent-vortex dynamics in large-eddy simulations of turbulence. [illegible], 4, 016, 2003.

[12] R. Mikulevicius, B.L. Rozovskii, Stochastic Navier-Stokes equations for turbulent flows. [illegible], 35, no. 5, 1250–1310, 2004.

[13] A. Vincent and M. Meneguzzi, The spatial structure and statistical properties of homogeneous turbulence. [illegible], 225, 1–25, 1991.

12 Exponential Ergodicity for Stochastic Reaction–Diffusion Equations*

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12.1 Introduction

Let (X_t) be a Markov process evolving in the state–space E and possessing a unique invariant measure μ^*. The question when and in what sense the transition measures $P(t, x, \cdot)$ converge to μ^* and the problem of finding the exact rate of convergence are well known, see for example [27]. The aim of this chapter is to review some recent results of this type for Markov processes defined by semilinear stochastic equations in separable Banach spaces. We will concentrate on cases when the forementioned properties hold in a very strong sense, namely, the transition measures converge to the invariant measure exponentially fast in the norm of total variation. We prove the uniform exponential ergodicity and the V-uniform ergodicity (see below for the precise formulation) for a general class of strongly Feller and irreducible Markov processes and then specify these results in the case of stochastic reaction–diffusion equations. Some consequences for the existence and lower estimate of the spectral gap in the space $L^p(\mu^*)$ are also discussed.

Though convergence to the invariant measure in the metric of total variation for infinite-dimensional systems has been investigated earlier (see, e.g., the monograph [9] or the survey paper [23]), little is known about the speed of convergence. Jacquot and Royer [19] proved exponential ergodicity for semilinear parabolic equations with bounded drift, Shardlow [34] applied the theory of Meyn and Tweedie to obtain V-uniform ergodicity for some semilinear equations in Hilbert spaces. Hairer [15], [16] and Goldys and Maslowski [11] proved, under different sets of conditions, uniform exponential ergodicity for equations with drifts growing faster than linearly. Recently, Goldys and Maslowski [12] have found some explicit bounds on convergence constants (including the convergence rate) for semilinear equations with additive noise (some lower estimates on spectral gap have been proved as well) and [13] have shown V-uniform ergodicity for nondegenerate stochastic two-dimensional (2D) Navier–Stokes and Burgers' equations.

This chapter consists of five sections including Introduction. In Section 12.2, basic definitions are given and the general results on V-uniform ergodicity and uniform exponential ergodicity are stated. Some of these results appear for the first time in the present form and the proofs are given in such cases. However, Section 12.2 is to a large extent based on earlier authors' papers [11], [12], [13] and a nice idea borrowed from the Shardlow's paper [34]. The hypotheses are rather general; the Markov process defined by the equation is supposed to be strongly Feller and irreducible and a certain compactness condition (12.2) is assumed to hold. Then the V-uniform ergodicity and uniform exponential ergodicity follow from ultimate boundedness and uniform ultimate boundedness of solutions, respectively

*This work was partially supported by the ARC Discovery Grant DP0346406, the UNSW Faculty Research Grant PS05345 and the GAČR Grant 201/04/0750.

(Theorem 12.1). Exponential convergence in L^p sense and existence of a spectral gap are obtained as corollaries (Theorems 12.2 and 12.3).

Section 12.3 is technical and provides a fairly general method to verify the compactness Hypothesis 12.2, which is especially useful for stochastic reaction–diffusion equations. The idea may be interesting for itself, because the assertion that is proved in Proposition 12.1 is sometimes needed in proofs of mere existence of invariant measure (see, e.g., [9], Theorem 6.1.2). The proof itself is related to an earlier result on weak Feller property by Maslowski and Seidler [24].

Section 12.4 contains a summary of results from the recent paper [12] on explicit bounds for constants in the convergences for both V-uniform ergodicity and uniform exponential ergodicity.

In Section 12.5, the results from previous sections are applied to stochastic reaction-diffusion equations. Two situations are considered: a one-dimensional (1D) second order semilinear parabolic equation with a multiplicative noise term equipped with Dirichlet boundary conditions and an analogous multidimensional equation with an additive noise. The first case is elaborated in detail (Subsection 12.5.1), the basic setting and some estimates being taken from Peszat [30]. In the multidimensional case the example is basically taken from [11] (where, however, only uniform exponential convergence is considered).

It should be noted that there exist several other important concepts of exponential or geometric ergodicity which are not discussed in this chapter. As mentioned above, we obtained some corollaries to exponential convergence and the spectral gap in L^p, which has been studied by numerous authors, cf. [1], [2], [3], [6], [14], [17]. The case of exponentially fast convergence in the metric defined by weak convergence of measures has been also extensively studied (some related results may be found, e.g., in the monograph [9]). Let us mention the paper by Mattingly [26] where exponential convergence to equilibrium in a norm intermediate between the total variation metric and Wasserstein metric has been obtained for stochastic Navier–Stokes equation.

12.2 Ergodicity–general results

In this section simple proofs of V-uniform ergodicity and uniform exponential ergodicity are given, the assumptions being formulated in the form which is convenient for applications to stochastic partial differential equations (SPDE's).

Let $E = (E, |\cdot|_E)$ be a real separable Banach space and denote by B, P, and $b\mathrm{B}$ the Borel σ-algebra of E, the space of Borel probability measures on E, and the space of bounded Borel functions, respectively. The space of bounded continuous functions on E will be denoted by $C_b(E)$. In the sequel we use the standard notation

$$\langle \phi, \nu \rangle = \int_E \phi d\nu, \quad \phi \in b\mathrm{B}, \nu \in \mathrm{P}.$$

Let $(X_t)_{t\geqslant 0}$ be a Markov process taking values in E

$$P_t\phi(x) = \mathbb{E}_x\phi\left(X_t\right), \quad \phi \in b\mathrm{B}, \quad x \in E, \quad t \geqslant 0,$$

and

$$P(t, x, \Gamma) = P_t I_\Gamma(x).$$

Let P_t^* denote the adjoint Markov semigroup, i.e.

$$P_t^*\nu(\Gamma) = \int_\Gamma P(t, x, \Gamma)\nu(dx), \quad \nu \in \mathrm{P}, \quad \Gamma \in \mathrm{B}, \quad t \geqslant 0.$$

An invariant measure $\mu^* \in \mathrm{P}$ is defined as a stationary point of the semigroup (P_t^*), i.e.

$$P_t^* \mu^* = \mu^*, \quad t \geqslant 0.$$

Obviously, $P_t^* \nu$ may be interpreted as the probability distribution of X_t in the case when the initial distribution is ν, and the invariant measure, if it exists, is a stationary distribution.

Hypothesis 12.1 [illegible] (P_t) [illegible]
([illegible] *)* $P_t\,(b\mathrm{B}) \subset C_b(E)$ [illegible] $t > 0$ [illegible]
([illegible] *)* $P_t I_U(x) > 0$ [illegible] $t > 0$, $x \in E$ [illegible] $U \subset E$ [illegible]

Hypothesis 12.2 [illegible] $r > 0$ [illegible] $T_0 > 0$ [illegible] $K \subset E$ [illegible]

$$\inf_{x \in B_r} P\,(T_0, x, K) > 0,$$

[illegible] $B_r = \{y \in E : |y|_E \leqslant r\}$

Hypothesis 12.2 is often helpful in adapting finite-dimensional results (for example, the classical Krylov–Bogolyubov proof of the existence of invariant measure) to infinite-dimensional processes. In Proposition 12.1 a method to verify Hypothesis 12.2 is proposed for a rather large class of stochastic evolution equations, cf. also examples in Section 12.5.

Let $V : E \to [1, \infty)$ be a measurable function. We will denote by $b_V\mathrm{B}$ the space of Borel-measurable functions $\phi : E \to \mathbb{R}$ such that

$$\|\phi\|_V := \sup_{x \in E} \frac{|\phi(x)|}{V(x)} < \infty.$$

The following concepts are well known (see, e.g., [27]).

Definition 12.1 [illegible] (X_t) [illegible] (P_t) [illegible] V [illegible] $C, \alpha > 0$ [illegible] $t \geqslant 0$ [illegible] $x \in E$

$$\sup_{\|\phi\|_V \leqslant 1} |P_t \phi(x) - \langle \phi, \mu^* \rangle| \leqslant C V(x) e^{-\alpha t}. \tag{12.1}$$

[illegible] $V \equiv 1$ [illegible] (X_t) [illegible]

Remark 12.1 In terms of the norm $\|\cdot\|_{var}$ of the total variation of signed measures on B, V-uniform ergodicity obviously implies

$$\|P_t^* \nu - \mu^*\|_{var} \leqslant C L_\nu e^{-\alpha t}, \quad t \geqslant 0, \quad \nu \in \mathrm{P}, \tag{12.2}$$

where $L_\nu = \langle V, \nu \rangle \leqslant \infty$, while the uniform exponential ergodicity is equivalent to

$$\|P_t^* \nu - \mu^*\|_{var} \leqslant C e^{-\alpha t}, \quad t \geqslant 0, \quad \nu \in \mathrm{P}. \tag{12.3}$$

Theorem 12.1 [illegible]
([illegible] *)* [illegible] p [illegible] (X_t) [illegible]

$$\mathbb{E}_x\, |X_t|_E^p \leqslant k |x|_E^p e^{-\omega t} + c, \quad t \geqslant 0, x \in E, \tag{12.4}$$

[illegible] $p > 0$, $\omega > 0$ [illegible] $k \in \mathbb{R}$ [illegible] μ^* [illegible] (P_t) [illegible] V [illegible] $V(x) = 1 + |x|_E^p$
([illegible] *)* [illegible] p [illegible] (X_t) [illegible]

$$\mathbb{E}_x\, |X_T|_E^p \leqslant M, \quad x \in E, \; T \geqslant \hat{T}, \tag{12.5}$$

[illegible] $p \geqslant 1$, $M < \infty$ [illegible] $\hat{T} > 0$ [illegible] (P_t) [illegible]

Proof of (i) First, we prove that for each $\tau > 0$ the skeleton $(X_{n\tau})$ has geometric drift toward B_r; that is, for some $\hat{\lambda} < 1$ and $b \in \mathbb{R}$

$$\mathbb{E}_x V\left(X_\tau\right) \leqslant \hat{\lambda} V(x) + b I_{B_r}(x), \quad x \in E, \tag{12.6}$$

where τ, r are such that

$$\tau > -\frac{1}{\omega} \log \frac{1}{4k}, \quad r > \left(4\left(c + \frac{1}{2}\right)\right)^{1/p}. \tag{12.7}$$

Indeed, we have

$$\begin{aligned} \mathbb{E}_x \left|X_\tau\right|_E^p + 1 &\leqslant k|x|_E^p e^{-\omega\tau} + c + 1 \leqslant \frac{1}{4}|x|_E^p + c + 1 \\ &\leqslant \frac{1}{2}\left(1 + |x|_E^p\right) - \frac{1}{4}|x|_E^p + c + \frac{1}{2} \leqslant \frac{1}{2} V(x) + \left(c + \frac{1}{2}\right) I_{B_r}(x), \end{aligned} \tag{12.8}$$

for $x \in E$, hence (12.6) holds with $\hat{\lambda} = \frac{1}{2}$ and $b = c + \frac{1}{2}$.

We will show that B_r is a nontrivial small set for a suitable skeleton of the process (X_t). By Hypothesis 12.1 the transition measures $P(t, x, \cdot)$ are equivalent for all $t > 0$ and $x \in E$. Thus, setting $\psi(\cdot) = P(1, x_0, \cdot)$ for a fixed $x_0 \in E$ it is easy to see that any skeleton (X_{nt_0}), $t_0 > 0$, is ψ-irreducible. Therefore (cf. Lemma 2 in [18]) or Theorem 5.2.2 in [27]) there exists a small set $S \in \mathrm{B}$ such that $\psi(S) > 0$, that is

$$P\left(1, x_0, S\right) > 0, \tag{12.9}$$

and

$$\inf_{x \in S} P(T, x, \Gamma) \geqslant \lambda(\Gamma), \quad \Gamma \in \mathrm{B}, \tag{12.10}$$

for some $T > 0$ and a nonnegative measure λ satisfying $\lambda(S) > 0$. Invoking Hypothesis 12.2 we obtain from the Chapman–Kolmogorov equality

$$\begin{aligned} \inf_{x \in B_r} P\left(2T + T_0, x, \Gamma\right) &\geqslant \inf_{x \in B_r} \int_S P(T, y, \Gamma) P\left(T + T_0, x, dy\right) \\ \geqslant \lambda(\Gamma) \inf_{x \in B_r} P\left(T + T_0, x, S\right) &\geqslant \lambda(\Gamma) \inf_{x \in B_r} \int_K P(T, y, S) P\left(T_0, x, dy\right). \end{aligned} \tag{12.11}$$

By (12.9) and the equivalence of transition measures we have $P(T, y, S) > 0$ for all $y \in K$. By Hypothesis 12.1 the function $y \to P(T, y, S)$ is continuous, hence bounded away from zero on K. By Hypothesis 12.2 it follows that for $\Gamma \in \mathrm{B}$

$$\inf_{x \in B_r} P\left(2T + T_0, x, \Gamma\right) \geqslant \delta_1 \lambda(\Gamma) \inf_{x \in B_r} P\left(T_0, x, K\right) \geqslant \delta_2 \lambda(\Gamma), \tag{12.12}$$

for some $\delta_1, \delta_2 > 0$. From (12.4) and (12.12) it easily follows that there exists an invariant measure $\mu^* \in \mathrm{P}$ (see, e.g., [25]), which is unique due to Hypothesis 12.1. Also, (12.12) means that B_r is a small set for any chain $\left(X_{nm(2T+T_0)}\right)$ with arbitrary $m \geqslant 1$. Let us fix $m \geqslant 1$ such that $T_1 = m\left(2T + T_0\right) > \tau$. By Step 1 of the proof we find that the chain $Y_n = X_{nT_1}$, $n \geqslant 1$, is V-uniformly ergodic, that is

$$\sup_{\|\phi\|_V \leqslant 1} \left|P_{nT_1}\phi(x) - \langle \phi, \mu^* \rangle\right| \leqslant C_0 e^{-n\omega T_1} V(x), \quad x \in E, n \geqslant 1, \tag{12.13}$$

for some $C_0, \omega > 0$, where μ^* is the invariant measure for the chain (Y_n). It follows that

$$\sup_{\|\phi\|_V \leqslant 1} \left|P_{nT_1+s}\phi(x) - \langle \phi, \mu^* \rangle\right| \leqslant \sup_{\|\phi\|_V \leqslant 1} \left|P_s\left(P_{nT_1}\phi - \langle \phi, \mu^* \rangle\right)(x)\right|$$

$$\leqslant C_0 e^{-nT_1\omega}\mathbb{E}_x\left(1+|X_s|_E^p\right) \leqslant C\left(1+|x|_E^p\right)e^{-(nT_1+s)\omega}, \tag{12.14}$$

for some $C>0$ and each $x\in E$, $n\geqslant 1$ and $s\in[0,T_1)$.

Proof of (ii) Using the uniform ultimate boundedness condition (12.5) it may be shown as in Step 2 above that the whole space E is a small set for the chain $\left(X_{n(2T+\hat{T})}\right)$. Hence, the chain is uniformly geometrically ergodic (cf. Theorem 16.0.2 in [27]) and the uniform exponential ergodicity of the semigroup (P_t) easily follows (cf. Theorem 2.4 of [11] for details).

Note that Hypothesis 12.2 may be weakened: we may assume that it is satisfied only for some r large enough (and not for each $r>0$), which, however, depends on V and the constant c from the ultimate boundedness condition (12.4) (cf. (12.7)).

Our next aim is to examine ergodicity of the Markov semigroup (P_t) in the space $L^p = L^p(E,\mu^*)$, $q\in[1,\infty)$, with the norm denoted by $\|\cdot\|_p$. Note that Hypothesis 12.1 implies that the transition measures $P(t,x,\cdot)$ and μ^* are equivalent for all $t>0$ and $x\in E$. Since the σ-algebra B is countably generated, there exists, for each $t>0$, a version of the transition density $p(t,x,y)=\frac{dP(t,x,\cdot)}{d\mu^*}(y)$ that is $\mathrm{B}\otimes\mathrm{B}$ measurable (cf. [27], Theorem 5.2.1) and thereby (P_t^*) is a C_0-semigroup on L^p for all $p\in[1,\infty)$ and $\|P_t^*\|_{L^p\to L^p}=1$ (cf. [12], Lemma 7.1). It is known that (P_t) extends to a contraction semigroup on L^p for $p\in[1,\infty]$ and is a C_0-semigroup if $p<\infty$.

Let L_p be the generator of the semigroup (P_t) in L^p. We say that L_p has the spectral gap in L^p if there exists $\delta>0$ such that

$$\sigma(L_p)\cap\{\lambda:\mathrm{Re}\lambda>-\delta\}=\{0\}.$$

The largest δ with this property is denoted by $\mathrm{gap}\,(L_p)$.

Theorem 12.2 [illegible] $P_t = P_t^*$ [illegible] $t\geqslant 0$ [illegible] $t\geqslant 0$

$$\|P_t\phi-\langle\phi,\mu^*\rangle\|_2\leqslant e^{-\alpha t}\|\phi\|_2, \quad \phi\in L^2, \tag{12.15}$$

[illegible] α [illegible]

The above theorem is a simple corollary to our Theorem 12.1 and Theorem 2.1 in [33] (cf. Corollary 7.4 in [12]).

Theorem 12.3 [illegible] $p>1$

$$\mathrm{gap}\,(L_p)\geqslant\frac{\alpha}{p},$$

[illegible] $\phi\in L^p$

$$\|P_t\phi-\langle\phi,\mu^*\rangle\|_p\leqslant C_p e^{-\alpha t/p}\|\phi\|_p, \quad t\geqslant 0, \tag{12.16}$$

[illegible] $\alpha>0$ [illegible] (P_t) [illegible] L^2 [illegible] $p=1$

The proof of this theorem is based on Theorem 12.1 and an interpolation argument (cf. Theorem 7.2 of [12]).

Remark 12.2 (i) In Theorem 12.3 it is essential that the ultimate boundedness is assumed to be uniform. The usual ultimate boundedness condition (12.4) is not sufficient (cf. [10] for an example of one-dimensional (1D) Ornstein–Uhlenbeck semigroup with $\sigma(L_1)=\{\lambda:\mathrm{Re}\lambda\leqslant 0\}$).

(ii) In Theorem 12.2 the condition of symmetry of (P_t) may not be removed (cf. Example 9.1 of [12]).

12.3 A remark on the compactness hypothesis

In this section we consider a stochastic evolution equation

$$\left\{\begin{array}{l} dX_t = (AX_t + F(X_t))\,dt + G(X_t)\,Q^{1/2}dW_t, \\ X_0 = x, \end{array}\right. \tag{12.17}$$

where A is linear operator in H with the domain $\mathrm{dom}(A) \subset H$, $(H, |\cdot|)$ being a real separable Hilbert space with continuous embedding $E \hookrightarrow H$. We assume that (W_t) is a standard cylindrical Wiener process on H defined on a filtered probability space $(\Omega, \mathrm{F}, (\mathrm{F}_t), \mathbb{P})$, the operator $Q = Q^* \in \mathrm{L}(H)$ is nonnegative and $Q \geqslant 0$. Moreover, we assume that $F : E \to H$ and $GQ^{1/2} : E \to \mathrm{L}(H)$.

Equation (12.17) is a fairly general model for the study of stochastic evolution equations. Our aim is to apply the abstract theory developed in Section 12.2 to the solution (X_t) of (12.17). To this end we need to know first that the process (X_t) has the Markov property, which usually follows from the existence and uniqueness of the solutions to (12.17) in an appropriate sense, see, e.g., [8] and for recent results in this direction [29]. We need also to show that the process (X_t) satisfies Hypotheses 12.1 and 12.2. There seems to be no general procedure to accomplish this task but certain methods have been developed for more specific equation of the form (12.17), as we shall see in Section 12.5. On the present level of generality we only prove Proposition 12.1 below, which may be useful in case when equation (12.17) may be approximated by a sequence of "nice" equations (for SPDEs it typically means a truncation and/or smoothing of F). Let us consider a sequence of equations

$$\left\{\begin{array}{l} dX_n^x(t) = (AX_n^x(t) + F_n(X_n^x(t)))\,dt + G_n(X_n^x(t))\,Q^{1/2}dW_t, \\ X_n^x(0) = x \in E, \end{array}\right. \tag{12.18}$$

where A is a generator of the C_0-semigroup (S_t) on E, $F_n : E \to E$ is Lipschitz for each n and G_n is such that for each $T > 0$ and all $t \leqslant T$

$$\begin{aligned} &\mathbb{E}\left|\int_0^t S_{t-r}\left(G_n(\zeta_r) - G_n(\chi_r)\right)Q^{1/2}dW_r\right|_E^p \\ &\qquad \leqslant \mathbb{E}\int_0^t k_n(t-r)\,|\zeta_r - \chi_r|_E^p\,dr, \end{aligned} \tag{12.19}$$

holds for certain $p \geqslant 2$, $k_n \in L^1_{loc}(0,\infty)$, and all progressively measurable processes $\zeta, \chi \in L^p(\Omega, C(0,T;E))$. For the results on maximal inequalities which yield (12.19) see, e.g., [4], [5], [30], and [31].

Finally, we assume that for each n equation (12.18) has a unique mild solution, i.e., an E-valued process satisfying

$$X_n^x(t) = S_t x + \int_0^t S_{t-r}F_n(X_n^x(r))\,dr + \int S_{t-r}G_n(X_n^x(r))\,Q^{1/2}dW_r, \quad t \geqslant 0, \tag{12.20}$$

and $X_n^x \in L^p(\Omega, C(0,T;E))$. The transition measures defined by equation (12.18) will be denoted by $P^n(t,x,\cdot)$.

Proposition 12.1 [illegible] E [illegible] $S_t : E \to E$ [illegible] $t > 0$ [illegible] $r > 0$ [illegible] $T_0 > 0$ [illegible]

$$\lim_{n\to\infty} \sup_{x \in B_r} \|P(T_0,x,\cdot) - P^n(T_0,x,\cdot)\|_{var} = 0 \tag{12.21}$$

[illegible] $P = (P(t,x,\cdot))$ [illegible] P [illegible]

Proof In view of (12.21) it suffices to show that for each fixed $n \geqslant 1$, $r > 0$, $T > 0$, and $\epsilon > 0$ there exists a compact set $K \subset E$ such that

$$\inf_{x \in B_r} P^n(T, x, K) \geqslant 1 - \epsilon. \tag{12.22}$$

By Theorem 2.2 of [24] the operator P_T^n (which is defined by the kernel $P^n(T, \cdot, \cdot)$) maps the space $C_b(E)$ into the space of weakly sequentially continuous functions on E. By the definition of the narrow topology, this implies that $x \longmapsto P^n(T, x, \cdot)$ is a continuous mapping from B_r equipped with the weak topology to the space of probability measures on $(E, |\cdot|_E)$ endowed with the narrow topology. Since B_r is weakly compact, the set $\{P^n(T, x, \cdot),\, x \in B_r\}$ is narrowly compact, and (12.22) follows.

In [24], the space E is Hilbertian and the condition (12.19) takes a slightly different form. The proof is not affected by these changes, but for completeness we repeat here the simple arguments that yield weak sequential continuity of $P_T^n f$ for $f \in C_b(E)$.

Take arbitrary $(x_m)_{m \geqslant 0} \subset B_r$, such that

$$x_m \to x_0 \quad \text{weakly in } E.$$

By (12.19) and (12.20)

$$\begin{aligned} &\mathbb{E}\, |X_n^{x_m}(t) - X_n^{x_0}(t)|_E^p \\ &\leqslant C_n \left(|S_t (x_m - x_0)|_E^p + \mathbb{E} \int_0^t (1 + k_n(t - r)) \, |X_n^{x_m}(r) - X_n^{x_0}(r)|_E^p \, dr \right), \end{aligned} \tag{12.23}$$

for all $t \leqslant T$ and a certain $C_n < \infty$. By the generalized Gronwall lemma (see, e.g., [7], Corollary 8.11) we obtain for $t \leqslant T$

$$\mathbb{E}\, |X_n^{x_m}(t) - X_n^{x_0}(t)|_E^p \leqslant C_n \left(|S_t (x_m - x_0)|_E^p + \sum_{j=1}^{\infty} V_n^j \left(|S_\cdot (x_m - x_0)|_E^p \right) \right), \tag{12.24}$$

where V_n is the integral Volterra operator

$$V_n y(s) = \int_0^s C_n \left(1 + k_n(s - r)\right) y(r) dr, \quad s > 0, \quad y \in L^1_{loc}(0, \infty).$$

As the semigroup $S_t : E \to E$ is compact, we have $|S_t (x_m - x_0)|_E \to 0$ for $m \to \infty$ and it is easy to conclude that

$$\lim_{m \to \infty} \mathbb{E}\, |X_n^{x_m}(t) - X_n^{x_0}(t)|_E^p = 0, \quad t \leqslant T.$$

By a standard argument it follows that $P^n(T, x_m, \cdot) \to P^n(T, x_0, \cdot)$ in the narrow topology.

Remark 12.3 It is possible to modify Proposition 12.1 so that nonreflexive state spaces could be considered if we assume more about approximating equations (12.18). Namely, suppose that $S_t : H \to E$ is a compact operator for each $t > 0$, $\|S_t\|_{H \to E} \in L^1_{loc}(0, \infty)$, and let equation (12.18) have a solution in $L^p(\Omega \times (0, T); E)$ for each $x \in H$. Assume also, that (12.19) holds with k_n constant for progressively measurable processes $\zeta, \chi \in L^p(\Omega \times (0, T); E)$. Then Proposition 12.1 remains true. To show this, we can use the same proof with $x_m \in B_r$, $m \geqslant 1$, $x_0 \in H$, and τ denoting now the weak topology of H. We find that the mapping $\left(\overline{B_r}, \tau\right) \ni x \to P^n(T, x, \cdot) \in (\mathrm{P}, \rightharpoonup)$ is continuous, where $\overline{B_r}$ is the closure of B_r in H.

12.4 Explicit bounds on convergence

In what follows we restate the results on exact bounds for the V-uniform ergodicity with $V(x) = |x|_E + 1$, and exponential ergodicity as proved in [12], i.e., we find explicit estimates on the constants C and α in (12.1) and (12.3), in case when the process (X_t) is a solution of a stochastic evolution equation with additive noise. These results are based on finding a specific lower measure λ for a small set $S = B_r$ (or $S = E$ in case of uniform exponential ergodicity) of the form $\lambda(dy) = \gamma(y)\mu_1(dy)$, where the measure μ_1 is Gaussian and γ is a positive lower estimate of the density $\frac{dP(1,x,\cdot)}{\delta\mu_1}$ on B_r. However, more specific hypotheses will be needed. We consider an equation

$$\begin{cases} dX_t = (AX_t + F(X_t))\,dt + Q^{1/2}dW_t, \\ X_0 = x \in E, \end{cases} \tag{12.25}$$

which is a version of equation (12.17) with $G(x) = I$ for all $x \in E$. We assume that A generates a C_0-semigroup (S_t) on H.

Hypothesis 12.3 [illegible]

$$Q_t = \int_0^t S_s Q S_s^* \, ds \tag{12.26}$$

[illegible] H [illegible]

$$\operatorname{im}(S_t) \subset \operatorname{im}\left(Q_t^{1/2}\right), \quad t > 0. \tag{12.27}$$

Obviously, (12.26) and (12.27) imply that the Ornstein–Uhlenbeck process

$$Z_t^x = S_t x + \int_0^t S_{t-s} Q^{1/2} dW_s, \quad t \geqslant 0,$$

is well defined, strongly Feller, and irreducible on H.

Hypothesis 12.4 [illegible] $\tilde{A}$ [illegible] A [illegible] E

$$\tilde{A} = A|\operatorname{dom}\left(\tilde{A}\right), \quad \operatorname{dom}\left(\tilde{A}\right) = \{y \in \operatorname{dom}(A) \cap E : Ay \in E\},$$

[illegible] C_0 [illegible] E [illegible] (S_t) [illegible] (Z_t^0) [illegible] E [illegible] E [illegible]

Hypothesis 12.5 [illegible]

$$\int_0^1 \left\|Q_t^{-1/2} S_t Q^{1/2}\right\|_{HS} dt < \infty,$$

[illegible] $\|\cdot\|_{HS}$ [illegible] H [illegible] $\dim(H) < \infty$ [illegible] $\alpha \in (0,1)$ [illegible] $\beta < \frac{1+\alpha}{2}$ [illegible]

$$\int_0^1 t^{-\alpha} \left\|S_t Q^{1/2}\right\|_{HS}^2 dt < \infty, \tag{12.28}$$

[illegible]

$$\left\|Q_t^{-1/2} S_t\right\| \leqslant \frac{c}{t^\beta}, \quad t \in (0,1], \tag{12.29}$$

[illegible]

Hypothesis 12.6 *(i)* [illegible] $F : E \to E$ [illegible] E [illegible] $x \in E$ [illegible] E [illegible]

(ii) [illegible] $\mathrm{im}(F) \subset \mathrm{im}\left(Q^{1/2}\right)$ [illegible] $\tilde{F} : E \to H$ [illegible] $Q^{1/2}\tilde{F} = F$ [illegible]

$$\left|\tilde{F}(x)\right| \leqslant K\left(1 + |x|_E^m\right), \quad x \in E, \tag{12.30}$$

[illegible] $K, m > 0$

Set $\mu_1 = N(0, Q_1)$. The following lower estimate of the transition density of the process (X_t) has been proved in [12]. The constants c_1, c_2 and the mapping Λ may be further specified in various particular cases.

Proposition 12.2 [illegible] $x \in E$

$$\frac{dP(1,x,\cdot)}{d\mu_1}(y) \geqslant c_1 \exp\left(-c_2|x|_E^\kappa - \Lambda(y)\right) \quad \mu_1 - a.e.,$$

[illegible] $\Lambda : \mathrm{M}_1 \to \mathbb{R}$ [illegible] $\mathrm{M}_1 \in \mathrm{B}$, $\mu_1(\mathrm{M}_1) = 1$, $\kappa = \max(2, 2m)$ [illegible] $c_1, c_2 > 0$ [illegible] A, Q [illegible] K, m [illegible]

The following result (cf. [12]) enables us to verify the ultimate boundedness conditions (12.4) and (12.5) in terms of the coefficients of Equation (12.25). By $< \cdot, \cdot >_{E,E^*}$ we denote the duality between E and E^* and by $\partial|\cdot|_E$ the subdifferential of the norm $|\cdot|_E$.

Theorem 12.4 [illegible]

$$k(\delta) := \sup_{t \geqslant 0} \mathbb{E}\left|Z_t^0\right|_E^\delta < \infty, \quad \delta > 0 \tag{12.31}$$

[illegible] $x \in \mathrm{dom}\left(\tilde{A}\right)$ [illegible] $x^* \in \partial|x|_E$ [illegible] $k_1, k_2, k_3, s > 0$ [illegible] $\epsilon \geqslant 0$ [illegible]

$$\left\langle \tilde{A}x + F(x+y), x^* \right\rangle_{E,E^*} \leqslant -k_1|x|_E^{1+\epsilon} + k_2|y|_E^s + k_3, \quad y \in E. \tag{12.32}$$

[illegible]

(i) [illegible] $\epsilon = 0$ [illegible] $p = k = 1$, $\omega = k_1$ [illegible] $c = k(1) + k_1^{-1}\left(k_2 k(s) + k_2\right)$

(ii) [illegible] $\epsilon > 0$ [illegible] $p = 1$, $\hat{T} \geqslant 1$ [illegible]

$$M = k(1) + \max\left(\left(\frac{2\left(k_2 k(s) + k_2\right)}{k_1}\right)^{1+\epsilon}, \left(2 + \frac{1}{k_1\epsilon}\right)^{1/\epsilon}\right).$$

Now we define some constants that are used to establish bounds on the convergence rate. Recalling (12.4) and (12.5) we take arbitrary $R > 4c$, $r > 4\left(c + \frac{1}{2}\right)$, and define

$$t_0 = -\frac{1}{k_1}\log\left(\frac{R}{2rk_0} - \frac{c}{rk}\right), \quad T = \max\left(t_0 + 1, -\frac{1}{k_1}\log\frac{1}{4k}\right), \quad b = c + \frac{1}{2}, \tag{12.33}$$

and

$$\delta = \frac{1}{2}c_1 e^{-c_2 R^\kappa} \int_{B_r} e^{-\Lambda(y)} \mu_1(dy), \tag{12.34}$$

where c_1, c_2, κ, and Λ are as in Proposition 12.2. The following is our key result.

Lemma 12.1 [illegible] ( ) [illegible] $p = 1$ [illegible]

( ) [illegible] $\Gamma \in B$

$$\inf_{x \in B_r} P(T, x, \Gamma) \geqslant \delta \bar{\mu}(\Gamma),$$

[illegible]

$$\bar{\mu}(\Gamma) = \left(\int_{B_r} e^{-\Lambda(y)} \mu_1(dy) \right)^{-1} \int_{B_r \cap \Gamma} e^{-\Lambda(y)} \mu_1(dy),$$

[illegible] B_r

( ) [illegible] $x \in E$

$$\mathbb{E}_x \left(|X_T|_E + 1 \right) \leqslant \frac{1}{2} \left(|x|_E + 1 \right) + b I_{B_r}(x),$$

[illegible] (X_{nT}) [illegible] B_r [illegible] $\frac{1}{2}$ [illegible] b [illegible] $V(x) = |x|_E + 1$

Now we are in a position to use the results on explicit convergence bounds from [28]. Following [28] we set

$$v = r + 1, \quad \gamma_c = \delta^{-2} \left(4b + \delta v \right),$$

$$\hat{\lambda} = \left(\frac{1}{2} + \gamma_c \right) (1 + \gamma_c) < 1, \quad \hat{b} = v + \gamma_c, \quad \bar{\xi} = 4b^2 \frac{4 - \delta^2}{\delta^5},$$

and

$$M_c = \frac{1}{\left(1 - \hat{\lambda}\right)^2} \left(1 - \hat{\lambda} + \hat{b} + \hat{b}^2 + \bar{\xi} \left(\hat{b} \left(1 - \hat{\lambda} \right) + \hat{b}^2 \right) \right).$$

We may choose a constant $\rho \in (1 - M_c, 1)$ which provides the geometric convergence rate for the chain (X_{nT}).

Theorem 12.5 [illegible] (X_t) [illegible] ( ) [illegible] ( ) [illegible] $p = 1$ [illegible] (X_t) [illegible] V [illegible] $V(x) = |x|_E + 1$ [illegible] ( ) [illegible] ( ) [illegible]

$$\alpha = -\frac{1}{T} \log \rho, \quad \text{and} \quad C = (1 + \gamma_c) \frac{\rho}{\rho + M_c - 1} (c + k + 1) e^{-\log \rho}.$$

[illegible] ( ) [illegible] (X_t) [illegible] ( ) [illegible] $\alpha = -\frac{1}{2} \log(1 - \delta) > 0$ [illegible] δ [illegible] ( ) [illegible] $R = 2M$ [illegible] $r = \infty$ [illegible] $C = 2(1 - \delta)^{-1}$

Corollary 12.1 [illegible] ( ) [illegible] ( ) [illegible] ( ) [illegible]

( ) V [illegible] $V(x) = |x|_E + 1$ [illegible] $\epsilon = 0$ [illegible] ( )))

( ) [illegible] $\epsilon > 0$ [illegible] ( ))

[illegible] α [illegible] C [illegible] F [illegible] ( ) [illegible] ( ) [illegible] K, m, k_1, k_2, k_3, s [illegible] ϵ

Proof See Theorems 6.3 and 6.4 in [12].

Remark 12.4 Under the conditions of Theorem 12.5, the conclusions of Theorems 12.2 and 12.3 on L^p-ergodicity and spectral gap hold true, the convergence rate α being the same as in Theorem 12.2 and 12.3, respectively.

12.5 Reaction–diffusion equations

We will begin with a study of 1D case.

12.5.1 One-dimensional equation

In this section we consider the equation

$$\begin{cases} \frac{\partial u}{\partial t}(t,\zeta) = Lu(t,\zeta) + f(u(t,\zeta)) + g(u(t,\zeta))\frac{\partial^2 W}{\partial t \partial \zeta}, & (t,\zeta) \in (0,\infty)\times(0,1), \\ u(0,\zeta) = x(\zeta), & u(t,0)=u(t,1)=0, \end{cases} \tag{12.35}$$

where

$$L\phi(\zeta) = \frac{\partial}{\partial \zeta}\left(a(\cdot)\frac{\partial}{\partial \zeta}\phi(\cdot)\right)(\zeta) + b(\zeta)\frac{\partial}{\partial \zeta}\phi(\zeta) + c(\zeta)\phi(\zeta), \quad \zeta \in (0,1),$$

with $a,b,c \in C([0,1])$, $a(\zeta) \geqslant a_0 > 0$. We assume that there exist $k>0$ and $\gamma \geqslant 1$ such that the function $f:\mathbb{R}\to\mathbb{R}$ satisfies the following conditions:

$$|f(\zeta)| \leqslant k\left(1+|\zeta|^{\gamma}\right), \tag{12.36}$$

$$|f(\zeta)-f(\eta)| \leqslant k|\zeta-\eta|\left(1+|\zeta|^{\gamma-1}+|\eta|^{\gamma-1}\right), \tag{12.37}$$

$$f(\zeta+\eta)\operatorname{sign}(\zeta) \leqslant k\left(1+|\zeta|+|\eta|^{\gamma-1}\right), \quad \zeta,\eta\in\mathbb{R}. \tag{12.38}$$

The diffusion coefficient $g:\mathbb{R}\to\mathbb{R}$ is a Lipschitz function satisfying the condition

$$0 < \delta_1 \leqslant g(\zeta) \leqslant \delta_2, \quad \zeta \in \mathbb{R}, \tag{12.39}$$

for some δ_1, δ_2. The formal system (12.35) may be rewritten in the form (12.17), where $H = L^2(0,1)$, $A = L$ with $\operatorname{dom}(A) = H^2(0,1)\cap H_0^1(0,1)$, and F and G are the Nemytskii operator and the multiplication operator associated with f and g, respectively. Finally, the noise term $\frac{\partial^2 W}{\partial t\partial\zeta}$ is formally replaced with $\frac{d}{dt}Q^{1/2}dW_t$, where (W_t) is a cylindrical Wiener process on H and $Q = Q^* \geqslant 0$ is a bounded operator on H with bounded inverse (cf. [30] for details). Let $\mathrm{K}_T^{\theta}(L^q)$ denote the Banach space of $L^q(0,1)$-valued continuous and adapted processes (Y_t) endowed with the norm

$$|||Y|||_{\theta,T} = \left(\mathbb{E}\sup_{t\leqslant T}|Y_t|_q^{\theta}\right)^{1/\theta},$$

where $|\cdot|_q$ denotes the norm in $L^q = L^q(0,1)$. Under the above conditions we have the following:

Theorem 12.6

$$q \in (\max(\gamma,2),\infty), \quad \text{and} \quad \theta > 2q\gamma. \tag{12.40}$$

$x \in L^q(0,1)$ $T>0$ $\mathrm{K}_T^{\theta}(L^q)$

Our aim is to apply the results obtained in the previous sections to the Markov process (X_t) defined by equation (12.35) in the space $E = L^q(0,1)$. In order to do this we have to verify Hypotheses 12.1 and 12.2 and the uniform boundedness conditions (12.4) or (12.5). We start with three estimates which are based (as well as our basic setting) on the results contained in [30]. We denote by (X_t^x) the solution starting from $X_0^x = x \in E$ and put

$$Z_t^x = \int_0^t S_{t-r}G\left(X_r^x\right)Q^{1/2}dW_r, \quad V_t^x = X_t^x - Z_t^x, \quad t\geqslant 0.$$

Lemma 12.2 (a) $T > 0$ $r > 0$

$$\sup_{x \in B_r} \mathbb{E} \sup_{t \leqslant T} |Z_t^x|_{\gamma q}^\theta < \infty,$$

q θ

(b) $T > 0$ $\hat{c}_T > 0$ $\zeta, \chi \in \mathrm{K}_T^\theta (L^q)$ θ

$$\mathbb{E} \sup_{t \leqslant T} \left| \int_0^T S_{t-s} \left(G(\zeta_s) - G(\chi_s) \right) Q^{1/2} dW_s \right|_q^\theta \leqslant \hat{c}_T \mathbb{E} \int_0^T |\zeta_s - \chi_s|_q^\theta \, ds. \tag{12.41}$$

(c) $k_1 > 0$ $t \leqslant T$

$$|V_t^x|_q \leqslant e^{k_1 t} |x|_q + k_1 \int_0^t e^{k_1(t-s)} \left(1 + |Z_s^x|_{\gamma q}^\gamma \right) ds \quad \mathbb{P} - a.s. \tag{12.42}$$

Proof (a) Let $p \in (0,1)$ be such that $2q\gamma < p^* < \theta$. Then we may find $\alpha \in \left(0, \frac{1}{4}\right)$ satisfying

$$\frac{q\gamma - 2}{4q\gamma} + 1 - \frac{1}{p} < \alpha < \frac{1}{4}.$$

Since for a certain $\sigma > \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right)$

$$\|S_t\|_{L^2 \to L^q} \leqslant \frac{c}{t^\sigma}, \quad t \in (0,1],$$

we obtain

$$\int_0^T t^{(\alpha-1)p} \|S_t\|_{L^2 \to L^q}^p \, dt < \infty. \tag{12.43}$$

It is known that $\left\| S_t G\left(X_t^x\right) Q^{1/2} \right\|_{HS} \leqslant c_T t^{-1/4}$ for $t \in (0,T)$, where by (12.39) the constant c_T does not depend on $x \in E$; hence the standard factorization argument yields

$$\sup_{x \in B_r} \mathbb{E} \sup_{t \leqslant T} |Z_t^x|_{\gamma q}^\theta$$

$$\leqslant c \left(\int_0^T t^{(\alpha-1)p} \|S_t\|_{L^2 \to L^q}^p \, dt \right)^{1/p} \left(\mathbb{E} \int_0^T \frac{\left\| S_{t-s} G\left(X_s^x\right) Q^{1/2} \right\|_{HS}^2}{(t-s)^{2\alpha}} ds \right)^{\theta/2} < \infty, \tag{12.44}$$

which completes the proof.
(b) Proceeding as in the proof of Theorem 2.1 in [30] and invoking the Hölder inequality we obtain

$$\mathbb{E} \sup_{t \leqslant T} \left| \int_0^T S_{t-s} \left(G(\zeta_s) - G(\chi_s) \right) Q^{1/2} dW_s \right|_q^\theta \leqslant c \int_0^T \mathbb{E} \left(\int_0^t \frac{\|S_{t-s}\|_{HS}^2}{(t-s)^{2\alpha}} |\zeta_s - \chi_s|_q^2 \, ds \right)^{\theta/2}$$

$$\leqslant c \left(\int_0^T (t-s)^{-\left(2\alpha + \frac{1}{2}\right)\frac{\theta}{\theta - 2}} ds \right)^{\frac{\theta-2}{2}} \mathbb{E} \int_0^T |\zeta_s - \chi_s|_q^\theta \, ds, \tag{12.45}$$

which yields (12.41) for θ large enough.
(c) See the proof of Lemma 3.2 in [30]. It follows easily from the (a) and (c) of the above lemma that

$$\sup_{x \in B_r} \mathbb{P} \left(\sup_{t \leqslant T} |X_t^x|_q > M \right) \stackrel{M \to \infty}{\longrightarrow} 0, \tag{12.46}$$

for each $r > 0$. Let

$$F_n(x) = \begin{cases} F(x) & \text{if} \quad |x|_q \leqslant n, \\ F\left(\frac{nx}{|x|_q}\right) & \text{if} \quad |x| > n, \end{cases}$$

and consider an approximating sequence of equations (12.18) with $G_n = G$. Obviously, $F_n : E \to E$ is Lipschitz, (12.19) follows from part (b) of Lemma 12.2 and (12.21) from (12.46) and therefore Hypothesis 12.2 holds true.

If $f = 0$, then the strong Feller property and irreducibility follow from [32] (cf. also Theorem 11.2.4 of [9]). This fact together with (a) and (b) of Lemma 12.2 enables us to use Theorem 3.1 of [22] to obtain both strong Feller property and irreducibility in the case of $f \neq 0$ and thereby Hypothesis 12.1 is verified. Let us note that in fact a minor modification of Theorem 3.1 in [22] is needed: our $F : E \to E$ is not continuous but this condition has been used in [22] only to prove convergence of the Yosida approximations which are not needed here as the semigroup (S_t) is analytic. Note also that the growth condition on F in [22] must be slightly modified.

To obtain the ultimate boundedness (12.4) or (12.5) the growth condition (12.38) must be strengthened. We will assume that

$$f(\zeta + \eta)\text{sign}(\zeta) \leqslant -k_1|\zeta|^{1+\epsilon} + k_2|\eta|^\gamma + k_3, \quad \zeta, \eta \in \mathbb{R}, \tag{12.47}$$

for some $k_1, k_2, k_3 > 0$, $\gamma, \epsilon \geqslant 0$. Since $\partial|x|_q = \left\{|x|_q^{1-q}|x|^{q-2}x\right\}$, (12.47) implies

$$\langle F(x+y), x^* \rangle_{E,E^*} \leqslant -\tilde{k}_1|x|_q^{1+\epsilon} + \tilde{k}_2|y|_{q\gamma}^\gamma + \tilde{k}_3, \quad x \in \text{dom}\,(A)\,,\ y \in L^{q\gamma},\ x^* \in \partial|x|_q, \tag{12.48}$$

for some $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3 > 0$ (for $\epsilon > 0$ the constant k_1 depends on the norm of the continuous embedding $L^{q+\epsilon} \to L^q$). By analyticity of the semigroup (S_t) we find that $V_t^x \in \text{dom}(A)$ for $t > 0$ and

$$\frac{d}{dt} V_t^x = AV_t^x + F\left(V_t^x + Z_t^x\right), \quad x \in E, \quad t \in [t_0, \tau]\,, \quad 0 < t_0 < \tau, \tag{12.49}$$

hence

$$\frac{d^-}{dt}\left|V_t^x\right|_q \leqslant -\tilde{k}_1\left|V_t^x\right|_q^{1+\epsilon} + \tilde{k}_2\left|Z_t^x\right|_{\gamma q}^\gamma + \tilde{k}_3, \quad x \in E, \quad t \in [t_0, \tau]\,. \tag{12.50}$$

Let $\epsilon = 0$. Using a simple comparison argument on $[t_0, \tau]$ and taking $t_0 \to 0$ we obtain

$$\left|X_t^x\right|_q \leqslant \left|Z_t^x\right|_q + \left|V_t^x\right|_q \leqslant \left|Z_t^x\right|_q + e^{-\tilde{k}_1 t}|x|_q + \int_0^t e^{-k_1(t-s)}\left(\tilde{k}_2\left|Z_s^x\right|_{\gamma q}^\gamma + \tilde{k}_3\right) ds \tag{12.51}$$

$\mathbb{P} - a.s.$ Note that for $\alpha \in \left(0, \frac{1}{2}\right)$ and $t > 0$

$$Z_t^x = \int_0^t (t-s)^{\alpha-1} S_{t-s} I_\alpha(s) ds,$$

where

$$I_\alpha(t) = \int_0^t (t-s)^{-\alpha} S_{t-s} G\left(X_s^x\right) Q^{1/2} dW_s,$$

and the semigroup (S_t) is exponentially stable on H, so that taking α, p, and θ as above we obtain for some $\omega, \tilde{\omega} > 0$ and a universal constant C (which may differ from line to line)

$$\sup_{t>0, x\in E} \mathbb{E}\left|Z_t^x\right|_{\gamma q}^\gamma$$

$$\leqslant C\left(\int_0^t (t-s)^{p(\alpha-1)}\left\|S_{\frac{t-s}{2}}\right\|_{L^2 \to L^{q\gamma}}^p ds\right)^{\theta/p} \sup_{t>0, x\in E} \mathbb{E}\int_0^t e^{-\omega\theta(t-s)/2}\left|I_\alpha(s)\right|^\theta ds$$

$$\leqslant C \sup_{t>0, x\in E} \int_0^t e^{-\omega\theta(t-s)/2} \mathbb{E}\left|I_\alpha(s)\right|^\theta ds$$

$$\leqslant C \sup_{t>0, x\in E} \mathbb{E} \int_0^\infty e^{-\omega\theta(t-s)/2} \left(\int_0^s (s-r)^{-2\alpha} \left\|S_{s-r} G\left(Z_r^x\right) Q^{1/2}\right\|_{HS}^2 dr\right)^{\theta/2} ds$$

$$\leqslant C \int_0^\infty e^{-\omega\theta(t-s)/2} \left(\int_0^\infty r^{-2\alpha-\frac{1}{2}} e^{-2\tilde{\omega} r} dr\right)^{\theta/2} ds < \infty. \tag{12.52}$$

Therefore if $\epsilon = 0$, then the ultimate boundedness condition (12.4) is satisfied with $p = 1$ in view of (12.51). If $\epsilon > 0$, then we may proceed similarly, obtaining from (12.50) by virtue of the Jensen inequality

$$\mathbb{E}\left|V_t^x\right|_q \leqslant \mathbb{E}\left|V_{t_0}^x\right|_q - \tilde{k}_1 \int_{t_0}^t \left(\mathbb{E}\left|V_s^x\right|_q\right)^{1+\epsilon} ds + C\left(t - t_0\right), \quad 0 < t_0 < t, \tag{12.53}$$

for a suitable constant C, which yields $\mathbb{E}\left|V_t^x\right|_q \leqslant \psi(t)$, where ψ solves the differential equation

$$\dot{\psi}(t) = -\tilde{k}_1(\psi(t))^{1+\epsilon} + C, \quad \psi(0) = |x|_q,$$

so the uniform ultimate boundedness condition (12.5) follows from (12.52) and all the assumptions of Theorem 12.1 are verified.

Let us assume now that $g = 1$. Then the results of Section 12.3 are applicable with the choice of $H = L^2(0,1)$, and $E = L^q(0,1)$. Condition (12.29) follows with $\beta = \frac{1}{2}$ from a standard controllability argument (cf. [8]) and (12.28) holds with $\alpha < \frac{1}{2}$, so that Hypothesis 12.5 holds true. Hypotheses 12.3 and 12.4 are trivially satisfied or were verified above. By virtue of (12.36) we have

$$|F(x)|^2 \leqslant \int_0^1 k^2\left(1 + |x(\zeta)|^\gamma\right)^2 d\zeta \leqslant c\left(1 + |x|_{2\gamma}^{2\gamma}\right), \quad x \in L^{2\gamma},$$

for a constant $c > 0$ and taking $q > 2\gamma$ (we may always enlarge q in the general model, if necessary) the polynomial bound (12.30) is obtained. Similarly, we obtain continuity of the mapping $\tilde{F} = Q^{-1/2}F : E \to H$. On the other hand, the mapping $F : E \to E$ is not Lipschitz on bounded sets as required in Hypothesis 12.6(a). However, given that equation 12.35 has a unique mild solution, Lipschitz continuity is not needed. So we may conclude that if the growth condition (12.47) is satisfied, then we obtain V-uniform ergodicity ($\epsilon > 0$) with some explicit bounds on the convergence rates stated in Theorem 12.5. A similar example is given in [12], where however $E = C(0,1)$.

12.5.2 Multidimensional equation

Consider the equation

$$\left\{\begin{array}{ll} \frac{\partial u}{\partial t}(t,\zeta) = \Delta u(t,\zeta) + f(u(t,\zeta)) + \frac{\partial^2 W}{\partial t \partial \zeta}(t,\zeta), & (t,\zeta) \in (0,\infty)\times \mathrm{O}, \\ u(0,\chi) = x(\zeta),\ \zeta \in \mathrm{O}, & (t,\zeta) \in (0,\infty)\times \partial\mathrm{O}, \end{array}\right. \tag{12.54}$$

where $\mathrm{O} \subset \mathbb{R}^d$, $d \leqslant 3$, is a bounded domain with smooth boundary. The function $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and satisfies the growth condition (12.47). We will consider the operator Δ with the domain $\mathrm{dom}(\Delta) = H^2(\mathrm{O}) \cap H_0^1(\mathrm{O})$ in the space $L^2(\mathrm{O})$. Then Δ is self-adjoint and negative definite. Let $\{e_n : n \geqslant 1\}$ be its set of eigenvectors and $\{\alpha_n\, n \geqslant 1\}$ the corresponding set of eigenvalues. It is well known that $e_n \in C_0(\mathrm{O})$. We will assume that there exists $C > 0$ such that

$$\sup_n |e_n|_\infty < C \quad \text{and} \quad \sup_n |\nabla e_n|_\infty \leqslant C\sqrt{\alpha_n}.$$

The noise $\frac{\partial^2 W}{\partial t \partial \zeta}$ is modeled as in the previous example and we assume that $Qe_n = \lambda_n e_n$, $n \geqslant 1$, where $0 < \lambda_n < \lambda_0$. The present example has been studied in [11] (the second part of Example 3) and we summarize the conclusions here (only the case $\epsilon > 0$ is discussed in [11], the case $\epsilon = 0$ being a simple modification). We assume that

$$\sum_{n=1}^{\infty} \frac{\lambda_n}{\alpha_n^{1-\gamma}} < \infty \tag{12.55}$$

holds for some $\gamma > 0$ and

$$\sup_{n \geqslant 1} \left(\frac{\alpha_n}{\lambda_n} \left(1 - e^{-2\alpha_n t}\right)^{-1} \right)^{1/2} \in L^1(0,1). \tag{12.56}$$

Conditions (12.47) and (12.55) assure that the problem (12.54) is well posed in the space $C(\overline{\mathrm{O}})$ and (12.55) implies that for any $p > 0$

$$\sup_{t>0} \mathbb{E} \, |Z_t|^p < \infty,$$

cf. Theorems 5.2.9 and 11.3.1 in [9] ($Z_t = Z_t^x$ does not depend on x in this case). Hence the ultimate boundedness (for $\epsilon = 0$) or uniform ultimate boundedness (for $\epsilon > 0$) may be verified similarly as in the previous example (cf. also [11]). Condition (12.56) yields the strong Feller property (by virtue of a suitable smoothing property of the corresponding backward Kolmogorov equation, see [8], while irreducibility follows from [21], Proposition 2.8 and Remark 2.9, since $\|S_t\|_{H \to E} \in L^1(0,T)$ for $d \leqslant 3$. Hypothesis 12.2 has been verified in Lemma 2.2 of [11]). Hence, we may conclude that if (12.55), (12.56) and the growth condition (12.47) are satisfied then the E-valued Markov process defined by equation (12.54) is V-uniformly ergodic with $V(x) = |x|_E + 1$ (for $\epsilon = 0$) or uniformly exponentially ergodic (for $\epsilon > 0$). Note that both (12.55) and (12.56) are satisfied if

$$c_1 n^{-2a/d} \leqslant \lambda_n \leqslant c_2 n^{-2b/d}, \quad n \geqslant 1,$$

for some $c_1, c_2 > 0$, where $\frac{d}{2} - 1 < b \leqslant a < 1$.

Acknowledgment

The authors are grateful to Jan Seidler for his valuable comments and suggestions.

References

[1] S. Aida, Uniform positivity improving property, Sobolev inequalities, and spectral gaps, [illegible] 158 (1998), 152-185.

[2] S. Aida and I. Shigekawa, Logarithmic Sobolev inequalities and spectral gaps: perturbation theory, [illegible] 126 (1994), 448-475.

[3] S. Albeverio, Y.G. Kondratiev and M. Röckner, Ergodicity of L^2-semigroups and extremality of Gibbs states, [illegible] 144 (1997), 394-423.

[4] Z. Brzezniak, On stochastic convolution in Banach spaces and applications, [illegible] 61 (1997), 245-295.

[5] Z. Brzezniak and S. Peszat, Maximal inequalities and exponential estimates for stochastic convolutions in Banach spaces, [illegible] 28 (2000), 55-64.

[6] A. Chojnowska-Michalik, Transition semigroups for stochastic semilinear equations on Hilbert spaces, 396 (2001).

[7] R. Curtain and T. Pritchard, Infinite Dimensional Linear Systems Theory. 8, Springer-Verlag, Berlin, 1978.

[8] G. Da Prato and J. Zabczyk, Stochastic Equations in Inifnite Dimensions, Cambridge University Press, Cambridge, 1992.

[9] G. Da Prato and J. Zabczyk, Ergodicity for Infinite Dimensional Systems, Cambridge University Press, Cambridge, 1996.

[10] E.B. Davies and B. Simon, L^1-Properties of intrinsic Schrödinger operators, 65 (1986), 126-146.

[11] B. Goldys and B. Maslowski, Uniform exponential ergodicity of stochastic dissipative systems, 51(126) (2001), 745-762.

[12] B. Goldys and B. Maslowski, Lower estimates of transition densities and bounds on exponential ergodicity for stochastic PDE's, 2 (2004), School of Mathematics, The University of New South Wales, http://www.maths.unsw.edu.au/statistics/preprints/2004/index.html.

[13] B. Goldys and B. Maslowski, Exponential ergodicity for stochastic Burgers and 2D Navier-Stokes equations, 4 (2004), School of Mathematics, The University of New South Wales, http://www.maths.unsw.edu.au/statistics/pubs/statspreprints2004.html.

[14] F. Gong, M. Röckner and L. Wu, Poincaré inequality for weighted first order Sobolev spaces on loop spaces, 185 (2001), 527-563.

[15] M. Hairer, Exponential mixing properties of stochastic PDEs through asymptotic coupling, 124 (2002), 345-380.

[16] M. Hairer, Exponential mixing for a PDE driven by a degenerate noise, 15 (2002), 271-279.

[17] M. Hino, Exponential decay of positivity preserving semigroups on L^p, 37 (2000), 603-624.

[18] N. Jain and B. Jamison, Contributions to Doeblin's theory of Markov processes. 8 (1967), 19-40.

[19] S. Jacquot and G. Royer, Ergodicité d'une classe d'equations aux derivées partielles stochastiques, 320 (1995), 231-236.

[20] F. Masiero, Semilinear Kolmogorov Equations and Applications to Stochastic Optimal Control, PhD Thesis, Dipartimento di Matematica, Università di Milano, 2003.

[21] B. Maslowski, On probability distributions of solutions of semilinear stochastic evolution equations, 45 (1993), 17-44.

[22] B. Maslowski and J. Seidler, Probabilistic approach to the strong Feller property, 118 (2000), 187-210.

[23] B. Maslowski and J. Seidler, Invariant measures for nonlinear SPDE's: Uniqueness and stability. 34 (1998), 153-172.

[24] B. Maslowski and J. Seidler, On sequentially weakly Feller solutions to SPDE's. [illegible], 10 (1999), 69-78.

[25] B. Maslowski and I. Simão, Asymptotic properties of stochastic semilinear equations by the method of lower measures, [illegible] 72 (1997), 147-171.

[26] J.C. Mattingly, Exponential convergence for the stochastically forced Navier-Stokes equations and other partially dissipative dynamics, [illegible] 230 (2002), 421-462.

[27] S.P. Meyn and R.L. Tweedie, Markov Chains and Stochastic Stability, Springer-Verlag, Berlin, 1993.

[28] S.P. Meyn and R.L. Tweedie, Computable bounds for geometric convergence rates of Markov chains, [illegible] 4 (1994), 981-1011.

[29] M. Ondreját, Brownian representation of cylindrical local martingales, martingale problem and strong Markov property of weak solutions of SPDEs in Banach spaces. [illegible], to appear.

[30] S. Peszat, Existence and uniqueness of the solutions for stochastic equations on Banach spaces, [illegible] 55 (1995), 167-193.

[31] S. Peszat and J. Seidler, Maximal inequalities and space-time regularity of stochastic convolutions, [illegible] 123 (1998), 7-32.

[32] S. Peszat and J. Zabczyk, Strong Feller property and irreducibility for diffusions on Hilbert spaces, [illegible] 23 (1995), 157-172.

[33] G. Roberts and J. Rosenthal, Geometric ergodicity and hybrid Markov chains, [illegible] 2 (1997), 13-25.

[34] T. Shardlow, Geometric ergodicity for stochastic PDEs, [illegible], 17 (1999), 857-869.

13 Stochastic Optimal Control of Delay Equations Arising in Advertising Models

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13.1 Introduction

In this chapter we consider a class of stochastic optimal control problems where the state equation is a stochastic delay differential equations (SDDEs). Such problems arise for instance in modeling optimal advertising under uncertainty for the introduction of a new product to the market, generalizing classical work of Nerlove and Arrow [30]. The main novelty in our model is that we deal also with delays in the control: this is interesting from the applied point of view and introduces new mathematical difficulties in the problem.

Dynamic models in marketing have a long history, that begins at least with the seminal papers of Vidale and Wolfe [33] and Nerlove and Arrow [30]. Since then a considerable amount of work has been devoted to the extension of these models and to their application to problems of optimal advertising, both in the monopolistic and the competitive settings, mainly under deterministic assumptions. Models with uncertainty have received instead relatively less attention (see Feichtinger, Hartl, and Sethi [9] for a review of the existing work until 1994, Prasad and Sethi [31] for a Vidale and Wolf-like model in the competitive setting, and, e.g., Grosset and Viscolani [21], Marinelli [29] for recent work on the case of a monopolistic firm). Moreover, it has been advocated in the literature (see, e.g., Hartl [22], Feichtinger et al. [9] and references therein), as it reasonable to assume, that more realistic dynamic models of goodwill should allow for lags in the effect of advertisement. Namely, it is natural to assume that there will be a time lag between advertising expenditure and the corresponding effect on the goodwill level. More generally, a further lag structure has been considered, allowing a distribution of the forgetting time.

In this work we incorporate both lag structures mentioned above. We formulate a stochastic optimal control problem aimed at maximizing the goodwill level at a given time horizon $T > 0$, while minimizing the cumulative cost of advertising expenditure until T.

This optimization problem is studied using techniques of stochastic optimal control in infinite dimension. In particular, we extend to the stochastic case a representation result, proved by Vinter and Kwong [34] in the deterministic setting, that allows to associate to a controlled SDDE with delays both in the state and in the control a stochastic differential equation (without delays) in a suitable Hilbert space. This in turn allows us to associate to the original control problem for the SDDE, an equivalent (infinite-dimensional) control problem for the "lifted" stochastic equation.

We deal with the resulting infinite-dimensional optimal control problem through the dynamic programming approach, i.e., through the study of the associated Hamilton–Jacobi–Bellman (HJB) equation. The HJB equation that arises in this case is an infinite-dimensional second order semilinear partial differential equation (PDE) that does not seem to fall into the ones treated in the existing literature (see Section 13.5 for details). Here we give some preliminary results for this equation. First, we consider the particular case (but still interesting from the applied point of view) when the delay does not enter the control term. In this case the L^2 approach of Goldys and Gozzi [13] and the forward–backward stochastic differential equation (SDE) approach of Fuhrman and Tessitore [10], [11], [12] apply. We show how to apply the former in Section 13.4).

Moreover, we consider the general case of delays in the state and in the control: since we do not know whether a regular solution exists, the natural approach would be the one of viscosity solutions. We leave the treatment of the viscosity solution approach to a subsequent work. Here we concentrate on the special case where regular solutions exist. In this case we prove a verification theorem and the existence of optimal feedbacks. Finally, we show through a simple example (but nevertheless still interesting in applications) that, possibly in special cases only, smooth solutions may exist, allowing us to prove verification theorems and the existence of optimal feedback controls.

Some further steps are still needed to build a satisfactory theory: concerning the viscosity solutions theory, it would be important to get an existence and uniqueness result and a verification theorem; concerning regular solutions, other cases where further regularity may arise should be studied. These issues are part of work in progress.

For references on viscosity solutions of HJB equations in infinite dimensions, and their connection with stochastic control and applications, we refer to, e.g., Lions [26], [27], [28], Święch [32], Gozzi, Rouy, and Święch [17], Gozzi and Święch [19], Gozzi, Sritharan, and Święch [18].

Other approaches to optimal control problems for systems described by SDDEs without infinite-dimensional reformulation have been proposed in the literature: for instance, see Elsanosi [6] and Larssen [24] for a more direct application of the dynamic programming principle without appealing to infinite-dimensional analysis, and Kolmanovskiĭ and Shaĭkhet [23] for the linear-quadratic case. See also Elsanosi, Øksendal, and Sulem [8] for some solvable control problems of optimal harvesting, and Elsanosi and Larssen [7] for an application in financial mathematics.

The chapter is organized as follows: in Section 13.2 we formulate the optimal advertising problem as an optimal control problem for an SDDE with delays both in the state and the control. In Section 13.3 we prove a representation result allowing us to "lift" this non-Markovian optimization problem to an infinite-dimensional Markovian control problem. In Section 13.4 we study the simpler case of a controlled SDDE with delays in the state only, for which known results apply. Section 13.5 deals with the general case of delays in the state and in the control, for which we only give the verification result. Finally, in Section 13.6 we construct a simple example of a controlled SDDE with delay in the state and in the control, whose corresponding HJB equation admits a smooth solution; hence there exists an optimal control in feedback form for the control problem.

13.2 The advertising model

Our general reference model for the dynamics of the stock of advertising goodwill $y(s)$, $0 \leq s \leq T$, of the product to be launched is given by the following controlled SDDE, where z models the intensity of advertising spending

$$\begin{cases} dy(s) = \left[a_0 y(s) + \int_{-r}^{0} a_1(\xi) y(s+\xi)\, d\xi + b_0 z(s) + \int_{-r}^{0} b_1(\xi) z(s+\xi)\, d\xi\right] ds \\ \qquad\qquad +\sigma\, dW_0(s), \quad 0 \leq s \leq T \\ y(0) = \eta^0; \quad y(\xi) = \eta(\xi),\ z(\xi) = \delta(\xi)\ \ \forall \xi \in [-r, 0], \end{cases} \tag{13.1}$$

where the Brownian motion W_0 is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_s)_{s\geq 0}, \mathbb{P})$, with $\mathbb{F}$ being the completion of the filtration generated by W_0, and z belongs to $\mathcal{U} := L^2_{\mathbb{F}}([0,T];\mathbb{R}^+)$, the space of square integrable nonnegative processes adapted to $\mathbb{F}$. Moreover, we shall assume that the following conditions are verified:

(i) $a_0 \leq 0$;

(ii) $a_1(\cdot) \in L^2([-r,0];\mathbb{R})$;

(iii) $b_0 \geq 0$;

(iv) $b_1(\cdot) \in L^2([-r,0];\mathbb{R}_+)$;

(v) $\eta^0 \geq 0$;

(vi) $\eta(\cdot) \geq 0$, with $\eta(0) = \eta^0$;

(vii) $\delta(\cdot) \geq 0$.

Here a_0 is a constant factor of image deterioration in absence of advertising, b_0 is a constant advertising effectiveness factor, $a_1(\cdot)$ is the distribution of the forgetting time, and $b_1(\cdot)$ is the density function of the time lag between the advertising expenditure z and the corresponding effect on the goodwill level. Moreover, η^0 is the level of goodwill at the beginning of the advertising campaign, $\eta(\cdot)$ and $\delta(\cdot)$ are the histories of the goodwill level and of the advertising expenditure, respectively, before time zero (one can assume $\eta(\cdot) = \delta(\cdot) = 0$, for instance). Note that when $a_1(\cdot)$, $b_1(\cdot)$, and σ are identically zero, equation (13.1) reduces to the classical model of Nerlove and Arrow [30].

Finally, we define the objective functional

$$J(z(\cdot)) = \mathbb{E}\left[\varphi_0(y(T)) - \int_0^T h_0(z(s))\,ds\right], \tag{13.2}$$

where φ_0 is a concave utility function with polynomial growth at infinity, and h_0 is a convex cost function which is superlinear at infinity, i.e.

$$\lim_{z\to+\infty} \frac{h(z)}{z} = +\infty,$$

and the dynamics of y is determined by (13.1). The problem we will deal with is the maximization of the objective functional J over all admissible controls in $\mathcal{U}$.

Remark 13.1 Note that in the general framework of delay equations the functions a_1 and b_1 are measures. Here we do not consider such general framework for two reasons: first taking a_1 and b_1 to be L^2 already captures the applied idea of the problem; second, taking a_1 and b_1 to be measures would introduce some technical difficulties that we do not want to treat here. More precisely this would create some problems in the infinite-dimensional reformulation bringing unbounded terms into the state equation. Indeed, if $b_1 \equiv 0$, the case where a_1 is a measure can be easily treated by a different standard reformulation. This fact allows us to treat the case of point delays in the state with no delays in the control. This will be the subject of Section 13.4.

13.3 Reformulation in Hilbert space

Throughout the chapter, X will be the Hilbert space defined as

$$X = \mathbb{R} \times L^2([-r, 0]; \mathbb{R}),$$

with inner product

$$\langle x, y \rangle = x_0 y_0 + \int_{-r}^{0} x_1(\xi) y_1(\xi)\, d\xi$$

and norm

$$|x| = \left(|x_0|^2 + \int_{-r}^{0} |x_1(\xi)|^2\, d\xi \right)^{1/2},$$

where x_0 and $x_1(\cdot)$ denote the $\mathbb{R}$-valued and the $L^2([-r, 0]; \mathbb{R})$-valued components, respectively, of the generic element x of X.

In this section we shall adapt the approach of Vinter and Kwong [34] to the stochastic case to recast the SDDE (13.1) as an abstract SDE on the Hilbert space X and so to reformulate the optimal control problem.

We start by introducing an operator $A : D(A) \subset X \to X$ as follows:

$$A : (x_0, x_1(\xi)) \quad \mapsto \quad \left(a_0 x_0 + x_1(0), a_1(\xi) x_0 - \frac{dx_1(\xi)}{d\xi} \right) \quad \text{a.e. } \xi \in [-r, 0].$$

The domain of A is given by

$$D(A) = \left\{ (x_0, x_1(\cdot)) \in X : x_1(\cdot) \in W^{1,2}([-r, 0]; \mathbb{R}),\ x_1(-r) = 0 \right\}.$$

The operator A is the adjoint of the operator $A^* : D(A^*) \subset X \to X$ defined as

$$A^* : (x_0, x_1(\cdot)) \mapsto \left(a_0 x_0 + \int_{-r}^{0} a_1(\xi) x_1(\xi)\, d\xi, x_1'(\cdot) \right) \tag{13.3}$$

on

$$D(A^*) = \left\{ (x_0, x_1(\cdot)) \in \mathbb{R} \times W^{1,2}([-r, 0]; \mathbb{R}) : x_0 = x_1(0) \right\}.$$

It is well known that A^* is the infinitesimal generator of a strongly continuous semigroup (see, e.g., Chojnowska–Michalik [2] or Da Prato and Zabczyk [4]); therefore the same is true for A.

We also need to define the bounded linear control operator $B : U \to X$ as

$$B : u \mapsto \left(b_0 u, b_1(\cdot) u \right), \tag{13.4}$$

where $U := \mathbb{R}_+$. Sometimes we shall identify the operator B with the element $(b_0, b_1(\cdot)) \in X$.

We introduce now an abstract SDE on the Hilbert space X that is equivalent, in the sense made precise by Proposition 13.1, to the SDDE (13.1):

$$\begin{cases} dY(s) = (AY(s) + Bz(s))\, ds + G\, dW_0(s) \\ Y(0) = x \in X, \end{cases} \tag{13.5}$$

where $G : \mathbb{R} \to X$ is defined by

$$G : x_0 \to (\sigma x_0, 0),$$

and $z(\cdot) \in \mathcal{U}$. Using Theorems 5.4 and 5.9 in Da Prato and Zabczyk [3], we have that equation (13.5) admits a unique weak solution (in the sense of [3], p. 119) with continuous paths given by the variation of constants formula

$$Y(s) = e^{sA}x + \int_0^s e^{(s-\tau)A} Bz(\tau)\, d\tau + \int_0^s e^{(s-\tau)A} G\, dW_0(\tau).$$

In order to state equivalence results between the SDDE (13.1) and the abstract evolution equation (13.5), we need to introduce the operator $M : X \times L^2([-r, 0]; \mathbb{R}) \to X$ defined as follows:

$$M : (x_0, x_1(\cdot), v(\cdot)) \mapsto (x_0, m(\cdot)),$$

where

$$m(\xi) := \int_{-r}^{\xi} a_1(\zeta) x_1(\zeta - \xi)\, d\zeta + \int_{-r}^{\xi} b_1(\zeta) v(\zeta - \xi)\, d\zeta.$$

The following result is a generalization of Theorems 5.1 and 5.2 of Vinter and Kwong [34].

Proposition 13.1 $Y(\cdot)$ ( ) $x \in X$ $z(\cdot) \in \mathcal{U}$ $t \geq r$ $\mathbb{P}$ ( )

$$Y(t) = M(Y_0(t), Y_0(t + \cdot), z(t + \cdot)).$$

$\{y(t),\ t \geq -r\}$ ( ) $Y(\cdot)$ ( )

$$x = M(\eta^0, \eta(\cdot), \delta(\cdot)).$$

$t \geq 0$ $\mathbb{P}$

$$Y(t) = M(y(t), y(t + \cdot), z(t + \cdot)),$$

$y(t) = Y_0(t)$ $\mathbb{P}$ $t \geq 0$

Proof Let $x = (x_0, x_1) \in D(A)$ (for general x the same result will follow by standard density arguments — see, e.g., Vinter and Kwong [34]). Equation (13.5) can be written as

$$\begin{cases} dY_0(t) = \Big(a_0 Y_0(t) + Y_1(t)(0) + b_0 z(t)\Big)\, dt + \sigma\, dW_0(t) \\ dY_1(t)(\xi) = \Big(a_1(\xi) Y_0(t) - \dfrac{d}{d\xi} Y_1(t)(\xi) + b_1(\xi) z(t)\Big)\, dt, \\ Y_0(0) = x_0, \quad Y_1(0)(\cdot) = x_1(\cdot), \end{cases} \tag{13.6}$$

therefore, $\mathbb{P}$-a.s.,

$$\begin{aligned} Y_0(t) &= e^{ta_0} x_0 + \int_0^t e^{(t-s)a_0} Y_1(s)(0)\, ds + \int_0^t e^{(t-s)a_0} b_0 z(s)\, ds + \int_0^t e^{(t-s)a_0} \sigma\, dW_0(s) \\ Y_1(t)(\xi) &= [\Phi(t)x_1](\xi) + \int_0^t [\Phi(t-s) a_1(\cdot) Y_0(s)](\xi)\, ds + \int_0^t [\Phi(t-s) b_1(\cdot) z(s)](\xi)\, ds, \end{aligned} \tag{13.7}$$

where $\Phi(t)$ is the semigroup of truncated right shifts defined as

$$[\Phi(t) f(\cdot)](\xi) = \begin{cases} f(\xi - t), & -r \leq \xi - t \leq 0, \\ 0, & \text{otherwise} \end{cases}$$

for all $f \in L^2$. Then (13.7) for $t \geq r$ can be rewritten, using the definition of Φ and recalling that $x_1(-r) = 0$, since $x \in D(A)$, as ($\mathbb{P}$-a.s.)

$$Y_1(t)(\xi) = \int_{-r}^{\xi} a_1(\alpha) Y_0(t + \alpha - \xi)\, d\alpha + \int_{-r}^{\xi} b_1(\alpha) z(t + \alpha - \xi)\, d\alpha, \tag{13.8}$$

which is equivalent to

$$Y(t) = M(Y_0(t), Y_0(t + \cdot), z(t + \cdot)),$$

as claimed.

Let us now prove the second claim: the L^2-valued component of the weak solution of the evolution equation (13.5) with initial data $Y(0) = x = M(\eta^0, \eta(\cdot), \delta(\cdot))$ satisfies, for $\xi - t \in [-r, 0]$, $\xi \in [-r, 0]$, $t \geq 0$

$$Y_1(0)(\xi - t) = \int_{-r}^{\xi - t} a_1(\alpha)\eta(t + \alpha - \xi)\, d\alpha + \int_{-r}^{\xi - t} b_1(\alpha)\delta(t + \alpha - \xi)\, d\alpha,$$

$\mathbb{P}$-a.s., as follows from (13.7). We assume here $\eta(0) = \eta^0$, without loss of generality (the general case follows by density arguments, as in [34]). Again by (13.7) and some calculations we obtain, $\mathbb{P}$-a.s.

$$Y_1(t)(\xi) = \int_{-r}^{\xi} a_1(\alpha)\tilde{Y}_0(t + \alpha - \xi)\, d\alpha + \int_{-r}^{\xi} b_1(\alpha) z(t + \alpha - \xi)\, d\alpha,$$

where

$$\tilde{Y}_0(\xi) = \begin{cases} \eta(\xi), & \xi \in [-r, 0], \\ Y_0(\xi), & \xi \geq 0. \end{cases}$$

Observe that the definition of $\tilde{Y}$ is well posed because of the assumption $\eta(0) = \eta^0$, and because $\eta^0 = Y_0(0)$ by the definition of the operator M. In order to finish, we need to prove that $Y_0(\cdot)$ satisfies the same integral equation (in the mild sense) as the solution $y(\cdot)$ to the SDDE (13.1), i.e., that the following holds for all $t \geq 0$:

$$\int_0^t e^{(t-s)a_0} Y_1(s)(0)\, ds = \int_0^t e^{(t-s)a_0} \left[\int_{-r}^0 a_1(\xi) Y_0(s + \xi)\, d\xi + \int_{-r}^0 b_1(\xi) z(s + \xi)\, d\xi \right] ds.$$

But this follows immediately from (13.8) with $\xi = 0$:

$$Y_1(t)(0) = \int_{-r}^0 a_1(\xi) Y_0(t + \xi)\, d\xi + \int_{-r}^0 b_1(\xi) z(t + \xi)\, d\xi,$$

which proves the claim. The fact that $y(t) = Y_0(t)$, $\mathbb{P}$-a.s., for all $t \geq 0$, easily follows. ∎

Using Proposition 13.1, we can now give a Hilbert space formulation of our problem. Since we want to use the dynamic programming approach, from now on we let the initial time vary, calling it t with $0 \leq t \leq T$.

The state space is $X = \mathbb{R} \times L^2([-r, 0]; \mathbb{R})$, the control space is $U = \mathbb{R}_+$, and the control strategy is $z(\cdot) \in \mathcal{U}$. The state equation is (13.5) with initial condition at t, i.e.

$$\begin{cases} dY(s) = (AY(s) + Bz(s))\, ds + G\, dW_0(s) \\ Y(t) = x \in X, \end{cases} \tag{13.9}$$

and its unique weak solution, given the initial data (t, x) and the control strategy $z(\cdot)$, will be denoted by $Y(\cdot; t, x, z(\cdot))$. The objective functional is

$$J(t, x; z(\cdot)) = \mathbb{E}^{t,x}\left[\varphi(Y(T, t, x; z(\cdot))) + \int_t^T h(z(s))\, ds\right], \tag{13.10}$$

where the functions $h : U \to \mathbb{R}$ and $\varphi : X \to \mathbb{R}$ are defined as

$$\begin{aligned} h(z) &= -h_0(z) \\ \varphi(x_0, x_1(\cdot)) &= \varphi_0(x_0). \end{aligned}$$

The problem is to maximize the objective function $J(t, y; z(\cdot))$ over all $z(\cdot) \in \mathcal{U}$. We also define the value function V for this problem as

$$V(t, x) = \inf_{z(\cdot)\in\mathcal{U}} J(t, x; z(\cdot)).$$

Moreover, we shall say that $z^*(\cdot) \in \mathcal{U}$ is an optimal control if it is such that

$$V(t, x) = J(t, x; z^*(\cdot)).$$

Following the dynamic programming approach we would like first to characterize the value function as the unique solution (in a suitable sense) of the following HJB equation

$$\begin{cases} v_t + \dfrac{1}{2}\operatorname{Tr}(Qv_{xx}) + \langle Ax, v_x\rangle + H_0(v_x) = 0 \\ v(T) = \varphi, \end{cases} \tag{13.11}$$

where $Q = G^*G$ and

$$H_0(p) = \sup_{z\in U}(\langle Bz, p\rangle + h(z)).$$

Moreover, we would like to find a sufficient condition for optimality given in terms of V (a so-called verification theorem) and a feedback formula for the optimal control z^*.

13.4 The case with no delay in the control

In a model for the dynamics of goodwill with distributed forgetting factor, but without lags in the effect of advertising expenditure, i.e., with $b_1(\cdot) = 0$ in (13.1), it is possible to apply both the approach of HJB equations in L^2 spaces developed by Goldys and Gozzi [13], and the backward SDE approach of Fuhrman and Tessitore [11]. We follow here the first approach, showing that both the value function and the optimal advertising policy can be characterized in terms of the solution of a HJB equation in infinite dimension. In fact we treat a somewhat different case assuming that the distribution of the forgetting factor is concentrated on a point. This can be treated by a different standard reformulation and with simpler computations and interpretations. In particular, we consider the case where the goodwill evolves according to the following equation:

$$\begin{cases} dy(s) = [a_0 y(s) + a_1 y(s-r) + b_0 z_0(s)]\, ds + \sigma\, dW_0(s), \quad 0 \le s \le T \\ y(0) = \eta^0; \quad y(\xi) = \eta(\xi) \ \ \forall \xi \in [-r, 0]. \end{cases} \tag{13.12}$$

By the representation theorems for solutions of stochastic delay equations of Chojnowska–Michalik [2], one can associate to (13.12) an evolution equation on the Hilbert space X of the type

$$\begin{cases} dY(s) = (AY(s) + \sqrt{Q}z(s))\, ds + \sqrt{Q}\, dW(s), \\ Y(0) = y, \end{cases} \tag{13.13}$$

where $A : D(A) \subset X \to X$ is defined as

$$A : (x_0, x_1(\cdot)) \mapsto (a_0 x_0 + a_1 x_1(-r), x_1'(\cdot))$$

on its domain

$$D(A) = \left\{(x_0, x_1(\cdot)) \in (\mathbb{R} \times W^{1,2}([-r,0];\mathbb{R}) : x_0 = x_1(0)\right\};$$

moreover, $z = (\sigma^{-1} b_0 z_0, z_1(\cdot)) \in \mathbb{R}_+ \times L^2([-r,0];\mathbb{R})$, with $z_1(\cdot)$ a fictitious control; $Q : X \to X$ is defined as

$$Q : (x_0, x_1(\cdot)) \mapsto (\sigma^2 x_0, 0);$$

W is an X-valued cylindrical Wiener process with $W = (W_0, W_1)$, and W_1 is a (fictitious) cylindrical Wiener process taking values in $L^2([-r,0];\mathbb{R})$. Finally, $y = (\eta^0, \eta(\cdot))$.

Remark 13.2 Note that the operator A just introduced does not coincide with the A introduced in Section 13.3. In fact, A here is exactly the A^* defined there. Similarly, the initial datum of this section differs from that of Section 13.3. Note also that the reformulation carried out in this section does not extend to the more general case of delay in the control, explaining why the more elaborate construction of the previous section is needed.

We also note that the insertion of the fictitious control z_1 is not necessary here. We do it to keep the control space U equal to the state–space X so the formulation falls into the results contained in Goldys and Gozzi [13]. However, it can be easily proved that the results of [13] still hold when the weaker condition $B(U) \subset Q^{\frac{1}{2}}(X)$ is satisfied.

The operator A is the infinitesimal generator of a strongly continuous semigroup $\{S(s), s \geq 0\}$ (see again Chojnowska–Michalik [2]). More precisely, one has

$$S(s)(x_0, x_1(\cdot)) = (y(s), y(s+\xi)|_{\xi \in [-r,0]}),$$

where $y(\cdot)$ is the solution of the deterministic delay equation

$$\begin{cases} \dfrac{dy(s)}{dt} = a_0 y(s) + a_1 y(s-r), \quad 0 \leq s \leq T \\ y(0) = x_0; \quad y(\xi) = x_1(\xi) \;\; \forall \xi \in [-r,0]. \end{cases} \tag{13.14}$$

Moreover, the X-valued mild solution $Y(\cdot) = (Y_0(\cdot), Y_1(\cdot))$ of (13.13) is such that $Y_0(\cdot)$ solves the original stochastic delay equation (13.12).

As in Section 13.3, we now consider an associated stochastic control problem letting the initial time t vary in $[0,T]$. The state equation is (13.13) with initial condition at t, i.e.

$$\begin{cases} dY(s) = (AY(s) + \sqrt{Q} z(s))\, ds + \sqrt{Q}\, dW(s), \\ Y(t) = x, \end{cases} \tag{13.15}$$

and the objective function is

$$J(t,x;z_0(\cdot)) = \mathbb{E}^{t,x}\left[\varphi_0(y(T)) - \int_t^T h_0(z_0(s))\, ds\right],$$

with $y(\cdot)$ obeying the SDDE (13.12). Defining

$$\begin{aligned} h(z_0, z_1(\cdot)) &= -h_0(z_0) \\ \varphi(x_0, x_1(\cdot)) &= \varphi_0(x_0), \end{aligned}$$

thanks to the above mentioned equivalence between the SDDE (13.12) and the abstract SDE (13.13), we are led to the equivalent problem of maximizing the objective function

$$J(t,x;z(\cdot)) = \mathbb{E}^{t,x}\left[\varphi(Y(T)) + \int_t^T h(z(s))\,ds\right], \tag{13.16}$$

with Y subject to (13.13).

Before doing so, however, following [13], we need to assume conditions on the coefficients of (13.12) such that the uncontrolled version of (13.13), i.e.

$$\begin{cases} dZ(t) = AZ(t)\,dt + \sqrt{Q}\,dW(t), \\ Z(0) = x, \end{cases} \tag{13.17}$$

admits an invariant measure. It is known (see Da Prato and Zabczyk [4]) that

$$a_0 < 1, \quad a_0 < -a_1 < \sqrt{\gamma^2 + a_0^2}, \tag{13.18}$$

where $\gamma \coth\gamma = a_0$, $\gamma \in]0,\pi[$, ensures that (13.17) admits a unique nondegenerate invariant measure μ.

Remark 13.3 The deterioration factor a_0 is always assumed to be negative; hence the first condition in (13.18) is not a real constraint. In general, however, it is not clear what sign a_1 should have. If a_1 is also negative, i.e., it can again be interpreted as a deterioration factor, condition (13.18) says that a_1 cannot be "much more negative" than a_0. On the other hand, if a_1 is positive, then the second condition in (13.18) implies that the improvement effect as measured by a_1 cannot exceed the deterioration effect as measured by $|a_0|$.

Let us now define the Hamiltonian $H_0 : X \to \mathbb{R}$ as

$$H_0(p) = \sup_{z\in U}\left(\langle\sqrt{Q}z, p\rangle_X + h(z)\right)$$

and write the HJB equation associated to the control problem (13.16):

$$\begin{cases} v_t + \frac{1}{2}\operatorname{Tr}(Qv_{xx}) + \langle Ax, v_x\rangle + H_0(v_x) = 0 \\ v(T,x) = \varphi(x). \end{cases} \tag{13.19}$$

If the Hamiltonian H_0 is Lipschitz (which follows from the hypothesis on h_0), $\varphi \in L^2(X,\mu)$ (which follows from the hypothesis on φ), and the operator A satisfies (13.18), then (13.19) admits a unique mild solution v in the space $L^2(0,T;W_Q^{1,2}(X,\mu))$, as follows from Theorem 3.7 of [13]. Moreover, Theorem 5.7 of [13] guarantees that v coincides (μ-a.e.) with the value function

$$V(t,x) = \inf_{z\in\mathcal{U}} \mathbb{E}^{t,x}\left[\varphi(Y(T)) + \int_t^T h(z(s))\,ds\right]$$

(by $z \in \mathcal{U}$ we mean, with a slight abuse of notation, $z_0 \in \mathcal{U}$), and that there exists a unique optimal control z^*, i.e.

$$V(t,x) = J(t,x;z^*(\cdot))$$

with

$$z^*(t) = DH(\widetilde{D}_Q v(t,Y^*(t))), \tag{13.20}$$

and Y^* is the solution of the closed-loop equation

$$\begin{cases} dY(s) = [AY(s) + \sqrt{Q}DH(\widetilde{D}_Q v(s, Y(s))]\, ds + \sqrt{Q}\, dW(t) \\ Y(t) = x. \end{cases} \tag{13.21}$$

The gradient operator $\widetilde{D}_Q$ is, roughly speaking, a "weakly closable" extension of the Malliavin derivative $Q^{1/2}D$ acting on $W_Q^{1,2}(X, \mu)$. For the exact definition and construction of $\widetilde{D}_Q$ we refer the reader to [13].

Remark 13.4 The HJB equation (13.19) is "genuinely" infinite dimensional; i.e., it reduces to a finite-dimensional one only in very special cases. For example, by the results of Larssen and Risebro [25], (13.19) reduces to a finite-dimensional PDE if and only if $a_0 = -a_1$. However, under this assumption, we cannot guarantee the existence of a nondegenerate invariant measure for the Ornstein–Uhlenbeck semigroup associated to (13.17). Even more extreme would be the situation of distributed forgetting time: in this case the HJB is finite dimensional only if the term accounting for distributed forgetting vanishes altogether!

13.5 Delays in the state and in the control

We now consider the case when also delays in the control are present. The optimal control problem is the one described in Section 13.2 with $a_1(\cdot) \neq 0$, $b_1(\cdot) \neq 0$.

The HJB equation associated to the problem is

$$\begin{cases} v_t + \dfrac{1}{2}\operatorname{Tr}(Qv_{xx}) + \langle Ax, v_x\rangle + H_0(v_x) = 0 \\ v(T) = \varphi, \end{cases} \tag{13.22}$$

where $Q = G^*G$ and

$$H_0(p) = \sup_{z\in U}(\langle Bz, p\rangle + h(z)).$$

Unfortunately, it is not possible, in general, to obtain regular solutions of the HJB equation (13.22) using the existing theory based on perturbations of Ornstein–Uhlenbeck semigroups (see, e.g., Barbu and Da Prato [1], Da Prato and Zabczyk [5], and Gozzi [15], [16]). The main problem is the lack of regularity properties of a suitable Ornstein–Uhlenbeck semigroup associated to the problem: in particular, the associated gradient operator is not closable and the semigroup is not strongly Feller (see Goldys and Gozzi [13] and Goldys, Gozzi, and van Neerven [14]).

As mentioned in Section 13.4, if there is no lag in the effect of advertisement spending on the goodwill, i.e., if $b_1(\cdot) = 0$, then both the L^2 approach of Goldys and Gozzi [13] and the backward SDE approach of Fuhrman and Tessitore [11] can be applied, obtaining a characterization of the value function and of the optimal advertising policy in terms of the (unique) solution to (13.22). Both approaches, however, fail in this more general case, since they require, roughly speaking, that the image of the control is included in the image of the noise, i.e., that $B(U) \subseteq G(X)$, which is clearly not true.

The only approach that seems to work in the general case of delays in the state and in the control is, to the best of our knowledge, the framework of viscosity solutions (see Lions [26], [27], [28]). However, while this approach gives a characterization of the value function in terms of the unique (viscosity) solution of the HJB equation (13.22), this solution is only guaranteed to be continuous; hence one can construct from it an optimal control only in a rather weak form, through the so-called viscosity verification theorems (see Gozzi, Swiech, and Zhou [20]). The study of this problem in the framework of viscosity solutions is the subject of a forthcoming publication.

Here we want to prove a verification theorem in the case when regular solutions of the HJB equation are available.

Definition 13.1 v

- classical solution ( ) $v \in C^{1,2}([0,T] \times X)$ v fi ( )
- integral solution $v \in C^{0,2}([0,T] \times X)$ $t \in [0,T]$ $x \in D(A)$

$$\varphi(x) - v(t,x) + \int_t^T \left[\frac{1}{2} \operatorname{Tr}(Q v_{xx}(s,x)) + \langle Ax, v_x(s,x) \rangle + H_0(v_x(s,x)) \right] ds = 0. \tag{13.23}$$

Theorem 13.1 (Verification Theorem) v ( ) V

( ) $v \geq V$ $[0,T] \times X$

( ) $z \in \mathcal{U}$ (t,x)

$$H_0(v_x(s,Y(s))) = \sup_{z \in U} \{ \langle Bz, v_x(s,Y(s)) \rangle + h(z) \} = \langle Bz(s), v_x(s,Y(s)) \rangle + h(z(s))$$

$s \in [t,T]$ $\mathbb{P}$ $v(t,x) = V(t,x)$

( ) $V = v$ ( ) (ffi)

Although there is a standard way to prove such results, this version of the verification theorem is not contained in the existing literature.

We give an idea of the method by sketching the proof in the case of bounded A. The case of unbounded A can be treated by approximating A with its Yosida approximations A_n, and then passing to the limit as $n \to +\infty$ (see, e.g., Barbu and Da Prato [1]).

Proof Let A and B be bounded operators. The core of the job is to prove that, for every $(t,x) \in [0,T] \times X$ and any $z \in \mathcal{U}$ the following fundamental identity holds:

$$v\,(t,x) = J\,(t,x;z\,(\cdot)) + \int_t^T \left[H_0(v_x(s,Y(s))) - \langle Bz(s), v_x(s,Y(s)) \rangle - h(z(s)) \right] ds, \tag{13.24}$$

where $Y(s) := Y(s;t,x;z(\cdot))$. Once this is proved, the three claims of the theorem follow as straightforward consequences of the definitions of the Hamiltonian H_0, of value function and of optimal strategy.

Let us then prove (13.24). We first approximate v by a sequence of smooth function v_n that solve suitable approximating equations and that are such that

$$v_n \longrightarrow v, \qquad v_{nx} \longrightarrow v_x.$$

This is possible, e.g., using the same ideas of Goldys and Gozzi [13], Section 4. Then for a.e. $s \in ([t,T] \cap \mathbb{R})$, one may show that

$$\frac{d}{ds} v_n(s,Y(s)) = v_{nt}(s,Y(s)) + \langle y'(s), v_{nx}(s,Y(s)) \rangle \tag{13.25}$$

$$= v_{nt}(s,Y(s)) + \langle AY(s) + Bz(s), v_{nx}(s,Y(s)) \rangle. \tag{13.26}$$

Since v_n is a classical solution of a suitable approximating HJB equation (see again [13]), we have

$$v_{nt}(s, Y(s)) + \langle AY(s), v_{ns}(s, Y(s))\rangle = -H_0(B^* v_{nx}(s, Y(s))) - g_n(s, Y(s)),$$

where g_n is a term appearing in the approximating HJB such that $g_n \to 0$ as $n \to \infty$ (see again [13]). Substituting in (13.26) and then adding and subtracting $h(s, z(s))$, we obtain

$$\begin{aligned}
\frac{d}{ds} v_n(s, Y(s)) &= \langle z(s), B^* v_{nx}(s, Y(s))\rangle - H_0(B^* v_{nx}(s, Y(s))) - g_n(s, Y(s)) \\
&= \langle z(s), B^* v_{nx}(s, Y(s))\rangle + h(z(s)) \\
&\quad - H_0(B^* v_{nx}(s, Y(s))) - g_n(s, Y(s)) - h(z(s)).
\end{aligned}$$

Integrating between t and T we get

$$\begin{aligned}
& v_n(T, Y(T)) - v_n(t, x) + \int_t^T [g_n(s, Y(s)) + h(z(s))]\, ds \\
= & \int_t^T [\langle z(s), B^*(v_{nx}(s, Y(s)))\rangle + h(z(s)) - H_0(B^* v_{nx}(s, Y(s)))]\, ds,
\end{aligned}$$

which gives the desired equality (13.24) after passing to the limit as $n \to \infty$. ■

About feedback maps, the following theorem holds true.

Theorem 13.2 [illegible] $\Gamma : ([0, T] \cap \mathbb{R}) \times X \to U$ [illegible] $\Gamma(t, p)$ [illegible] $z \mapsto \langle z, B^* p\rangle + h(z)$ [illegible] $(t, p) \in ([0, T] \cap \mathbb{R}) \times X$ [illegible] (t, x) [illegible] $Y^*(\cdot)$ [illegible]

$$\begin{cases}
dY^*(\tau) = AY^*(\tau) + B\Gamma(\tau, v_x(\tau, Y^*(\tau))) + G\, dW(t), & \tau \in [t, T] \cap \mathbb{R}; \\
Y^*(t) = x, & x \in X,
\end{cases}$$

[illegible] $\tau \mapsto \Gamma(\tau, v_x(\tau, Y^*(\tau)))$ [illegible] $\mathcal{U}$ [illegible] $(Y^*(\cdot), z^*(\cdot))$ [illegible]

$$z^*(\tau) = \Gamma(\tau, v_x(\tau, Y^*(\tau))).$$

Proof This is a straightforward consequence of (ii) in the verification theorem, as the control generated by the feedback relation satisfies (ii). ■

13.6 An example with explicit solution

Here we restrict ourselves to some less general specification of the objective function, but still meaningful, for which the HJB equation admits a smooth solution, and therefore the value function and the optimal control can be completely characterized.

Let us assume

$$h(z) = -\beta z_0^2$$

and

$$\varphi(x) = \gamma x_0,$$

with β, $\gamma > 0$. Then we have

$$H_{CV}(p,z) = \langle Bz, p\rangle + h(z) = \langle B, p\rangle z - \beta z^2,$$

and

$$H_0(p) = \sup_{z\in U} H_{CV}(z,p) = \begin{cases} \dfrac{\langle B, p\rangle^2}{4\beta}, & \langle B, p\rangle \geq 0 \\ 0, & \langle B, p\rangle < 0, \end{cases}$$

or equivalently, in more compact notation,

$$H_0(p) = \frac{(\langle B, p\rangle^+)^2}{4\beta}.$$

We conjecture a solution of the HJB equation (13.22) of the form

$$v(t,x) = \langle w(t), x\rangle + c(t), \qquad t \in [0,T],\ x \in X,$$

where $w(\cdot) : [0,T] \to X$ and $c(\cdot) : [0,T] \to \mathbb{R}$ are given functions whose properties we will study below. Hence for $t \in [0,T]$ and $x \in X$ we have, assuming that all objects are well defined

$$\begin{aligned} v_t(t,x) &= \langle w'(t), x\rangle + c'(t) \\ v_x(t,x) &= w(t) \\ v_{xx} &= 0, \end{aligned}$$

and

$$\begin{cases} \langle w'(t), x\rangle + c'(t) + \langle Ax, w\rangle + \dfrac{(\langle B, w(t)\rangle^+)^2}{4\beta} = 0, & t \in [0,T),\ x \in X, \\ \langle w(T), x\rangle + c(T) = \gamma x_0, & x \in X. \end{cases} \tag{13.27}$$

It is clear that, if $A^* w(t)$ is well defined for all $t \in [0,T]$, (13.27) is equivalent to

$$\begin{cases} \langle w'(t), x\rangle + \langle A^* w(t), x\rangle = 0, & t \in [0,T[\\ w(T) = (\gamma, 0) \end{cases} \tag{13.28}$$

and

$$\begin{cases} c'(t) + \dfrac{(\langle B, w(t)\rangle^+)^2}{4\beta} = 0, & t \in [0,T[\\ c(T) = 0. \end{cases} \tag{13.29}$$

Since (13.28) must hold for all x, then it implies

$$\begin{cases} w'(t) + A^* w(t) = 0, & t \in [0,T[\\ w(T) = (\gamma, 0). \end{cases}$$

Recalling (13.3), we obtain that (13.28) is equivalent to

$$\begin{cases} w_0'(t) + a_0 w_0(t) + \displaystyle\int_{-r}^{0} a_1(\xi) w_1(t,\xi)\, d\xi = 0, & t \in [0,T[\\ w_0(T) = \gamma \end{cases} \tag{13.30}$$

and

$$\begin{cases} \dfrac{\partial w_1}{\partial t}(t,\xi) + \dfrac{\partial w_1}{\partial \xi}(t,\xi) = 0, & t \in [0,T[,\ \xi \in [-r,0[\\ w_1(T,\xi) = 0, & \xi \in [-r,0[\\ w_1(t,0) = w_0(t), & t \in [0,T]. \end{cases} \tag{13.31}$$

The solution of (13.31) is given by

$$w_1(t,\xi) = w_0(t-\xi)\mathbb{I}_{\{t-\xi\in[0,T]\}}, \tag{13.32}$$

from which one can solve equation (13.30), obtaining $w_0(\cdot)$. Note that, unfortunately, the function w is never in $D(A^*)$, except for the case when it is 0 everywhere. However, this does not exclude that the candidate $v(t,x) = \langle w(t), x\rangle + c(t)$ solves the HJB equation (13.22) in some sense. Indeed it is an integral solution of (13.22) in the sense of Definition (13.1), as it follows from the above calculations. Note that $v \in C([0,T]\times X)$ and that it is twice differentiable in the x variable, i.e., it satisfies the hypotheses of the verification Theorem 13.1. Since a maximizer of the current-value Hamiltonian is given by $z^* = \langle B, p\rangle^+/(2\beta)$, then it is immediately seen that the control

$$z^*(t) = \frac{\langle B, v_x(t)\rangle^+}{2\beta} = \frac{\langle B, w(t)\rangle^+}{2\beta}, \qquad t \in [0,T],$$

which does not depend on $Y^*(t)$, is admissible; hence (ii) of the verification Theorem 13.1 holds, and $z^*(\cdot)$ is optimal.

References

[1] V. Barbu and G. Da Prato, [illegible]. Pitman, London, 1983.

[2] A. Chojnowska-Michalik, Representation theorem for general stochastic delay equations. [illegible], 26(7):635–642, 1978.

[3] G. Da Prato and J. Zabczyk, [illegible]. Cambridge UP, Cambridge, 1992.

[4] G. Da Prato and J. Zabczyk, [illegible]. Cambridge UP, Cambridge, 1996.

[5] G. Da Prato and J. Zabczyk, [illegible]. Cambridge UP, Cambridge, 2002.

[6] I. Elsanosi, Stochastic control for systems with memory. Dr. Sci. thesis, University of Oslo, 2000.

[7] I. Elsanosi and B. Larssen, Optimal consumption under partial observation for a stochastic system with delay. Preprint, University of Oslo, 2001.

[8] I. Elsanosi, B. Øksendal, and A. Sulem, Some solvable stochastic control problems with delay. [illegible], 71(1-2):69–89, 2000.

[9] G. Feichtinger, R. Hartl, and S. Sethi, Dynamical Optimal Control Models in Advertising: Recent Developments. [illegible], 40:195–226, 1994.

[10] M. Fuhrman and G. Tessitore, Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. [illegible], 30(3):1397–1465, 2002.

[11] M. Fuhrman and G. Tessitore, Generalized directional gradients, backward stochastic differential equations and mild solutions of semilinear parabolic equations. Preprint, Politecnico di Milano, 2003.

[12] M. Fuhrman and G. Tessitore, Infinite horizon backward stochastic differential equations and elliptic equations in Hilbert spaces. [illegible], 32(1B):607–660, 2004.

[13] B. Goldys and F. Gozzi, Second order parabolic HJ equations in Hilbert spaces and stochastic control: L^2 approach. Working paper.

[14] B. Goldys, F. Gozzi, and J.M.A.M. van Neerven, On closability of directional gradients. [illegible], 18(4):289–310, 2003.

[15] F. Gozzi, Regularity of solutions of a second order Hamilton-Jacobi equation and application to a control problem. [illegible], 20(5-6):775–826, 1995.

[16] F. Gozzi, Global regular solutions of second order Hamilton-Jacobi equations in Hilbert spaces with locally Lipschitz nonlinearities. [illegible], 198(2):399–443, 1996.

[17] F. Gozzi, E. Rouy, and A. Święch, Second order Hamilton-Jacobi equations in Hilbert spaces and stochastic boundary control. [illegible], 38(2):400–430 (electronic), 2000. ISSN 1095-7138.

[18] F. Gozzi, S.S. Sritharan, and A. Święch, Bellman equations associated to the optimal feedback control of stochastic Navier-Stokes equations. [illegible], 58(5):671-700, 2005.

[19] F. Gozzi and A. Święch, Hamilton-Jacobi-Bellman equations for the optimal control of the Duncan-Mortensen-Zakai equation. [illegible], 172(2):466–510, 2000.

[20] F. Gozzi, A. Swiech, and X. Y. Zhou, A corrected proof of the stochastic verification theorem within the framework of viscosity solutions. [illegible], to appear.

[21] L. Grosset and B. Viscolani, Advertising for a new product introduction: a stochastic approach. Preprint, Università di Padova, 2003.

[22] R.F. Hartl, Optimal dynamic advertising policies for hereditary processes. [illegible], 43(1):51–72, 1984.

[23] V.B. Kolmanovskiĭ and L.E. Shaĭkhet, [illegible]. AMS, Providence, RI, 1996.

[24] B. Larssen, Dynamic programming in stochastic control of systems with delay. [illegible], 74(3-4):651–673, 2002.

[25] B. Larssen and N.H. Risebro, When are HJB-equations in stochastic control of delay systems finite dimensional? [illegible], 21(3):643–671, 2003.

[26] P.-L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. I. The case of bounded stochastic evolutions. [illegible], 161(3-4):243–278, 1988.

[27] P.-L. Lions, Viscosity solutions of fully nonlinear second order equations and optimal stochastic control in infinite dimensions. II. Optimal control of Zakai's equation. In [illegible], volume 1390 of [illegible], pages 147–170. Springer, Berlin, 1989.

[28] P.-L. Lions, Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. III. Uniqueness of viscosity solutions for general second-order equations. [illegible], 86(1):1–18, 1989.

[29] C. Marinelli, The stochastic goodwill problem. arXiv:math.OC/0310316, 2003.

[30] M. Nerlove and J.K. Arrow, Optimal advertising policy under dynamic conditions. [illegible], 29:129–142, 1962.

[31] A. Prasad and S.P. Sethi, Dynamic optimization of an oligopoly model of advertising. Preprint, 2003.

[32] A. Święch, "Unbounded" second order partial differential equations in infinite-dimensional Hilbert spaces. [illegible], 19(11-12):1999–2036, 1994.

[33] M.L. Vidale and H.B. Wolfe, An operations-research study of sales response to advertising. [illegible], 5:370–381, 1957.

[34] R.B. Vinter and R.H. Kwong, The infinite time quadratic control problem for linear systems with state and control delays: an evolution equation approach. [illegible], 19(1):139–153, 1981.

14 On Acceleration of Approximation Methods

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14.1 Introduction

We consider for every $\tau \in (0,1]$ a pair of equations

$$w(\tau) = \varphi(\tau) + \sum_{k=1}^{m} A_k(\tau)\Theta_k(\tau)(L_k(\tau)w(\tau) + f_k(\tau)), \tag{14.1}$$

$$v(\tau) = \varphi(\tau) + A_0(\tau)\Theta_0(\tau)(L(\tau)v(\tau) + f(\tau))$$

for $w = w(\tau)$ and $v = v(\tau)$ in a Banach space W, where $\varphi, f_k, f \in W$, and L, L_k, Θ_k, A_k are linear operators, L, L_k are possibly unbounded. We assume that

$$L = L_1 + \cdots + L_m, \quad f = f_1 + \cdots + f_m. \tag{14.2}$$

For brevity of notation we suppress τ in some arguments.

We are interested in the dependence of the difference of the solutions, $w - v$, on the parameter τ. For example, for each τ we can consider the equations

$$w(t) = v_0 + \sum_{k=1}^{m} \int_{(0,t]} (\Theta_k L_k w(s) + \Theta_k f_k(s))\, da_k(s), \quad t \in [0,T], \tag{14.3}$$

$$v(t) = v_0 + \int_{(0,t]} (\Theta_0 L(s)v + \Theta_0 f(s))\, da_0(s), \quad t \in [0,T], \tag{14.4}$$

for the functions $w(t)$ and $v(t)$, $t \in [0,T]$, taking values in some separable Banach space V. Here, for each τ and $k = 0,1,2,...,m$, a_k is a right-continuous function on $(0,\infty)$, having finite limits from the left, and finite variation on each finite interval. The operator A_k is defined by the Bochner integral

$$(A_k v)(t) = \int_{(0,t]} v(s)\, da_k(s), \quad t \in [0,T],$$

where v is from $W = D_w([0,T],V)$, the space of V-valued functions on $[0,T]$, having weak limits from the left and from the right. The operators L_k in this example are defined by

$$(L_k u)(t) = L_k(t)u(t), \quad t \in [0,T],$$

*The work of the first author was partially supported by EU Network HARP.

†The work of the second author was partially supported by NSF Grant DMS-0140405.

where $L_k(t)$ is a linear, possibly unbounded, operator on V for every $t \in [0,T]$. Furthermore, $f_k \in W$, and Θ_k is a bounded linear operator on W. The main example of Θ_k, from the point of view of applications we have, is defined as follows: $(\Theta_k v)(0) = 0$, and

$$(\Theta_k v)(t) = \vartheta_k v(t) + (1-\vartheta_k)v(t-), \quad t \in (0,T], \quad v \in W \tag{14.5}$$

for some real number ϑ_k. More specifically, we can think of $L_k(t)$ as differential operators, V as a Sobolev space of functions on $\mathbb{R}^d$, Θ_0 as the identity operator, and $a_0(t) = t$. Then equation (14.4) can be cast in the form

$$\frac{d}{dt}v(t) = L(t)v(t) + f(t), \quad t \in (0,T], \quad v(0) = v_0, \tag{14.6}$$

and by equation (14.3) we want to represent approximations for (14.6). Namely, we view equation (14.3) as a numerical method applied to equation (14.6) on the grid

$$T_\tau := \{i\tau : i = 0,1,...\} \cap [0,T],$$

by taking suitable operators Θ_k, L_k, and functions $a_k(\tau,t) = \tau H_k(t/\tau)$, with appropriate right-continuous functions H_1,...,H_m over $\mathbb{R}$, which have finite variation over finite intervals, such that the measures dH_1,..., dH_m are periodic with period 1. Let us take, for example, $H_1(t) = \cdots = H_m(t) = [t]$, the integer part of t, and Θ_k defined by (14.5) with $\vartheta_1 = \cdots = \vartheta_m = \vartheta$. Then equation (14.3) represents the Θ-*method*, which is the *implicit Euler method* for $\vartheta = 1$, the *explicit Euler method* for $\vartheta = 0$, and is often called the *Crank–Nicolson method* for $\vartheta = 1/2$. Using an idea from [1] we can also easily represent splitting-up approximations for equation (14.6) by an appropriate choice of the functions $a_1, ..., a_m$ in equation (14.3). To explain this let us assume that L_k, f_k in (14.3) do not depend on s. Then the splitting-up approximation u, corresponding to the splitting (14.2), is defined by

$$u(t) = \big(P_m(\tau) \cdot \cdots \cdot P_1(\tau)\big)^{t/\tau} v_0, \quad t \in T_\tau,$$

where $P_k(t)\varphi$, $k = 1,2,...,m$, denotes the solution at time t of the equation

$$\frac{d}{dt}v(t) = L_k v(t) + f_k, \quad t > 0 \quad v(0) = \varphi. \tag{14.7}$$

This means that to get the approximation $u(t_i)$ for $0 < t_i \in T_\tau$ from $u(t_{i-1})$, we solve first equation (14.7) with $k = 1$ on $[0,\tau]$ with initial value $u(t_{i-1})$; then we solve (14.7) with $k = 2$ on $[0,\tau]$ again, with initial value $P_1 u(t_{i-1})$, and so on, finally we solve (14.7) with $k = m$ on $[0,\tau]$, taking the solution at τ of the previous equation as the initial value. Borrowing an idea from [1], instead of going back and forth in time when solving equations (14.7) one after another on the same interval $[0,\tau]$, we "stretch out the time" and rearrange solving these equations in forward time. We achieve this by introducing the equation

$$\bar{u}(t) = \varphi + \int_0^t \sum_{k=1}^m (L_k \bar{u}(s) + f_k)\dot{h}_k(s/\tau)\,ds,$$

where $\dot{h}_k$ is a function of period m, defined by $\dot{h}_k(t) = 1_{[k-1,k)}(t)$ for $t \in [0,m]$, $k = 1,2,...,m$. Then clearly $\bar{u}(mt) = u(t)$ for all $t \in T_\tau$, and it is easy to see that $w(t) := \bar{u}(mt)$, $t \in [0,T]$, satisfies the equation

$$w(t) = \varphi + \int_0^t \sum_{k=1}^m (L_k w(s) + f_k)\,da_k(s) \tag{14.8}$$

with $a_k(t) := \tau H_k(t/\tau)$, $H_k(t) := \int_0^{mt} \dot{h}_k(s)\, ds$. Notice that dH_k are periodic with period 1, and equation (14.8) is of the type (14.3). A more general type of splitting-up approximation, called ***fractional step approximation***, is defined by

$$u(t) = \left(P_{k_p}(s_p\tau) \cdot \cdots \cdot P_{k_1}(s_1\tau)\right)^{t/\tau} v_0, \quad t \in T_\tau \tag{14.9}$$

for some integer $p \geq m$, $k_1, ..., k_p \in \{1, ..., m\}$, and real numbers $s_1, ..., s_p$, such that

$$\sum_{j=1}^{p} s_j \delta_{rk_j} = 1, \quad \text{for every} \quad r = 1, 2, ..., m,$$

where $\delta_{rk_j} = 1$ for $r = k_j$ and 0 otherwise. Notice that these approximations are described by equation (14.8) with the functions

$$a_k(t) := \tau H_k(t/\tau), \quad H_k(t) := \int_0^{pt} \dot{\kappa}_k(s)\, ds, \tag{14.10}$$

where $\dot{\kappa}_r$ is a function of period p, such that

$$\dot{\kappa}_r(t) = \sum_{j=1}^{p} s_j \delta_{rk_j} 1_{[j-1,j)}(t), \quad t \in [0, p), \quad r = 1, 2, ..., m.$$

We can combine the fractional step method with the Θ-method. Namely, we can consider the approximations obtained by solving (14.7) by the Θ-method in each ‘fractional’ step. It is not difficult to see that these approximations are described by (14.3) with $a_k(t) := \tau[H_k(t/\tau)]$, where H_k is defined in (14.10).

The above examples illustrate the variety of numerical methods which can be implemented by means of equation (14.1). They motivate our general setting, and our main results, Theorems 14.1, 14.2 and 14.3, given in Section 14.2. Theorem 14.1 presents an expansion of $w - v$ in powers of τ, which implies Theorem 14.2, an implementation of Richardson’s idea for accelerating numerical methods. Theorem 14.1 follows from Theorem 14.3. To keep the chapter down to a reasonable size, we will prove Theorem 14.3 elsewhere. We apply the general results to the splitting-up approximations of nonlinear ordinary differential equations (ODEs) in Section 14.3.

Splitting-up approximations have been studied extensively in numerical analysis. It is known, see, e.g., [4], [10], that if a fractional step method has accuracy higher than τ^2, then at least one of the numbers $s_1, ..., s_p$ in (14.9) is negative. This means that such methods cannot be applied to parabolic partial differential equations (PDEs), and it is considered to be a challenging problem to work out effective methods of an order higher than 2 for parabolic PDEs (see the survey article [3].) From our results it follows that for any integer $k \geq 0$ there exist universal numbers $\lambda_0, \lambda_1, ..., \lambda_k$, such that the linear combination of appropriate splitting-up approximations with these coefficients has the accuracy of order $k + 1$ for a large class of differential equations, containing parabolic PDEs and systems of parabolic, possibly degenerate, PDEs, in particular, symmetric systems of first order hyperbolic PDEs.

14.2 General setting and an illustration

In this section we present three theorems in a very abstract setting. In order not to lose connection to real things and give the reader some justification of our assumptions we interrupt a few times the main stream of the section with discussions of a simple looking example.

It is probably hard to appreciate Theorems 14.1 and 14.2 looking only at Example 14.1. We reiterate that the goal of this example is to give the reader a feeling of what is behind quite abstract assumptions and objects. Later we will see a much more serious application of our abstract results to nonlinear ODEs. We will present other applications elsewhere.

Fix an integer $l \geq 1$ and assume that we have a sequence of Banach spaces

$$W_0, W_1, W_2, ..., W_l,$$

such that W_i is continuously embedded into W_{i-1}, for every $i = 1, 2, ..., l$. We also assume that W_1 is dense in W_0. By $\| \cdot \|_i$ we mean the norm in W_i. For each number $\tau \in (0, 1]$ we consider a pair of equations

$$v = \varphi + A_0 \Theta_0 (Lv + f) \tag{14.11}$$

$$w = \varphi + \sum_{k=1}^{m} A_k \Theta_k (L_k w + f_k), \tag{14.12}$$

for $v = v(\tau)$ and $w = w(\tau)$, respectively, where $L = L(\tau)$, $L_r = L_r(\tau)$, $A_k = A_k(\tau)$, $\Theta_k = \Theta_k(\tau)$ are certain linear operators and $f = f(\tau)$, $f_r = f_r(\tau)$, $\varphi = \varphi(\tau)$ are elements from W_l, for all $k = 0, 1, ..., m$ and $r = 1, 2, ..., m$. Almost everywhere below in the chapter we drop the argument τ.

Let K be a finite constant, independent of τ.

Assumption 14.1 *(i) For all $i = 0, ..., l$ the operators A_k, Θ_k are bounded operators from W_i to W_i and for $k = 0, ..., m$ and $u \in W_i$*

$$\|\Theta_k u\|_i \leq K \|u\|_i, \quad \|A_k u\|_i \leq K \|u\|_i.$$

(ii) For all $i = 0, ..., l-1$ the operators L, L_k are bounded operators from W_{i+1} to W_i and for $k = 1, ..., m$ and $u \in W_{i+1}$

$$\|Lu\|_i \leq K \|u\|_{i+1}, \quad \|L_k u\|_i \leq K \|u\|_{i+1}.$$

(iii) For all $i = 1, 2, ..., l$ and $k = 1, 2, ..., m$ we have

$$\|\varphi\|_i \leq K, \quad \|f_k\|_i \leq K.$$

(iv)

$$L = \sum_{k=1}^{m} L_k, \quad f = \sum_{k=1}^{m} f_k.$$

Example 14.1 Let $d \geq 1$ be an integer, $T \in (0, \infty)$,

$$W_0 = ... = W_l = D([0, T], \mathbb{R}^d)$$

be the space of $\mathbb{R}^d$-valued bounded functions on $[0, T]$ having right limits on $[0, T)$ and left limits on $(0, T]$. We provide these spaces with the uniform norm. Let $m = 1$, $a_0(t) = t$, $a_1(t) = \tau[t/\tau]$, and define the operators A_k, $k = 0, 1$, by

$$(A_k u)(t) = \int_{(0,t]} u(s)\, da_k(s). \tag{14.13}$$

Next, take a $d \times d$-matrix valued càdlàg function $L(t)$, $t \in [0, T]$, and define the operators L, L_1 by

$$(Lu)(t) = (L_1 u)(t) = L(t) u(t).$$

Finally, take a $\varphi \in \mathbb{R}^d$ and consider two equations

$$v(t) = \varphi + \int_0^t L(s)v(s)\,ds, \tag{14.14}$$

$$w(t) = \varphi + \int_{(0,t]} L(s-)w(s-)\,da_1(s), \tag{14.15}$$

which in our notation are written as (14.11) and (14.12), respectively, i.e.

$$v = \varphi + A_0\Theta_0 Lv, \quad w = \varphi + A_1\Theta_1 Lw,$$

where Θ_0 is the unit operator and Θ_1 is the operator defined by

$$(\Theta_1 u)(t) = u(t-) \quad t \in (0, T], \quad (\Theta_1 u)(0) = 0. \tag{14.16}$$

Our goal is to compare w and v.

Assumption 14.2 *For each $k = 0, ..., m$ there is a bounded linear operator $\mathcal{R}_k : W_0 \to W_0$ such that*
(i) We have $\mathcal{R}_k : W_i \to W_i$ for all $i = 0, ..., l$ and

$$\|\mathcal{R}_k g\|_i \le K\|g\|_i, \quad g \in W_i, i = 0, ..., l;$$

(ii) (Existence) for any $g \in W_1$ the function $u = \mathcal{R}_k g$ satisfies

$$u = A_0\Theta_0 Lu + A_k g; \tag{14.17}$$

(iii) (Uniqueness) if $g_k \in W_0$, $k = 0, ..., m$, $u \in W_1$ and

$$u = A_0\Theta_0 Lu + \sum_{k=0}^m A_k g_k, \quad \text{then} \quad u = \sum_{k=0}^m \mathcal{R}_k g_k.$$

Remark 14.1 Assumption 14.2 is satisfied in Example 14.1. To see this it suffices to notice that for $\bar{u} = u - A_k g$ equation (14.17) becomes

$$\bar{u} = A_0(L\bar{u} + h), \quad \frac{d\bar{u}}{dt} = L\bar{u} + h,$$

where $h = LA_k g$.

We assume in the future that equations (14.11) and (14.12) have a solution $v \in W_l$ and $w \in W_l$, respectively, and we want to expand the difference $w - v$ in a kind of power series in τ. To this end we need to introduce some further objects and to formulate further assumptions.

We call a sequence of numbers $\alpha = \alpha_1\alpha_2, ..., \alpha_i$ a multinumber of length $|\alpha| := i$, if $\alpha_j \in \{0, 1, 2, ..., m\}$. For each $\tau \in (0, 1]$ and $\alpha \in \mathcal{N}$ let $b^+_\alpha = b^+_\alpha(\tau)$, $b^-_\alpha = b^-_\alpha(\tau)$ be some linear operators and let $c_\alpha = c_\alpha(\tau)$ be a real number. Let B_α be a linear operator introduced by

$$\tau B_\alpha = A_\alpha\Theta_\alpha - A_0\Theta_0, \quad |\alpha| = 1,$$
$$\tau B_{\alpha k} = A_k b^-_\alpha \Theta_k - c_{\alpha k} A_0\Theta_0, \quad k = 0, ..., m. \tag{14.18}$$

We impose the following assumptions, in which K_α, $\alpha \in \mathcal{N}$, are some fixed finite constants, independent of τ.

Assumption 14.3 *For all $i = 0, ..., l$, the operators b_α^+, b_α^- are bounded operators from W_i to W_i and*

$$\|b_\alpha^+ u\|_i \le K_\alpha \|u\|_i, \quad \|b_\alpha^- u\|_i \le K_\alpha \|u\|_i$$

for all $\alpha \in \mathcal{N}$ and $u \in W_i$.

Assumption 14.4 *For any $\alpha \in \mathcal{N}$ and $k = 0, ..., m$*

$$B_\alpha A_k = b_\alpha^+ A_k - A_k b_\alpha^-, \quad A_0 \Theta_0 b_\alpha^+ = A_0 b_\alpha^- \Theta_0. \tag{14.19}$$

Assumption 14.5 *For any $\alpha \in \mathcal{N}$ and $k = 1, ..., m$ and $r = 0, ..., m$*

$$L_k \Theta_r = \Theta_r L_k, \quad L_k b_\alpha^\pm = b_\alpha^\pm L_k, \quad A_r L_k = L_k A_r,$$

$$B_\alpha \varphi = b_\alpha^+ \varphi, \quad B_\alpha f_k = b_\alpha^+ f_k.$$

Remark 14.2 Since $L = \sum_k L_k$, the operator L commutes with Θ_r, $b_\alpha^\pm$, and A_r as well. Also it follows from the definition of B_α and Assumption 14.5 that, B_α commutes with L, L_k for all α and k.

Remark 14.3 In Example 14.1 the requirement that $A_0 L = L A_0$ means that $L(t)$ is independent of t. We want to show how to introduce $b_\alpha^\pm$ and B_α in this example and do this by formulas ready for use later on. In more general situations along with Θ_k we also need operators $\bar{\Theta}_k$ and $\bar{\Theta}_\alpha$, which we set in Example 14.1 to be identity operators. So we let k vary in $\{0, 1\}$ and for $\alpha \in \mathcal{N}$ define recursively

$$\begin{gathered} b_k(t) = \frac{1}{\tau}[a_k(t) - a_0(t)], \quad c_{\alpha k} = \frac{1}{\tau} \int_{(0,\tau]} \bar{\Theta}_\alpha b_\alpha(s)\, da_k(s), \\ b_{\alpha k}(t) = \frac{1}{\tau}\Big(\int_{(0,t]} \bar{\Theta}_\alpha b_\alpha(s)\, da_k(s) - c_{\alpha k} a_0(t) \Big). \end{gathered} \tag{14.20}$$

It is easy to prove (see, however, Lemma 14.1 in a more general setting) that c_α are independent of τ, $b_\alpha(t)$ are τ-periodic in t, and $b_\alpha(i\tau) = 0$ for integers $i \ge 0$.

Next, introduce $b_\alpha^\pm$ as the operator of multiplying by the function b_α and B_α by the formula

$$(B_\alpha u)(t) = \int_{(0,t]} u(s-)\, db_\alpha(s).$$

These definitions are consistent with what is done in the general scheme. Indeed, (14.18) holds obviously as well as the the second relation in (14.19). The first relation is a consequence of the well-known fact that for two right-continuous functions of bounded variation

$$d(b(t)a(t)) = a(t-)\, db(t) + b(t)\, da(t), \tag{14.21}$$

so that

$$d\Big(b_\alpha(t) \int_{(0,t]} u(s)\, da_k(s)\Big) = \Big(\int_{(0,t)} u(s)\, da_k(s)\Big)\, db_\alpha(t) + b_\alpha(t) u(t)\, da_k(t).$$

Remark 14.4 If we modify the definition of Θ_1 in (14.16) as

$$(\Theta_1 u)(t) = \vartheta u(t) + (1 - \vartheta) u(t-), \tag{14.22}$$

with a fixed constant $\vartheta \in \mathbb{R}$, then equation $b_\alpha^+ = b_\alpha^-$ may no longer hold, and one sees the necessity to use $b_\alpha^+ \ne b_\alpha^-$. We show this in the following modification of Example 14.1.

Example 14.2 Consider Example 14.1 with L independent of t, and with Θ_1 defined by (14.22) in place of (14.16), so that if $\vartheta = 0$ we just have the same situation as in Example 14.1. Interestingly enough, even if below $\vartheta = 0$, this time we take the operators $\bar{\Theta}_\alpha$ different from identity. As in Example 14.1 we set Θ_0 to be the identity operator and introduce the operators A_k as before by (14.13). Then clearly Assumptions 14.1 and 14.2 remain to hold. In order to make further notation ready for future use, we set $\vartheta_0 = 1$ and $\vartheta_1 = \vartheta$ and let k vary in $\{0, 1\}$. Define the operators $\bar{\Theta}_k$ by

$$(\bar{\Theta}_k u)(t) = (1 - \vartheta_k)u(t) + \vartheta_k u(t-),$$

and set for $\alpha = \alpha_1, ..., \alpha_j \in \mathcal{N}$

$$\Theta_\alpha = \Theta_{\alpha_j}, \quad \bar{\Theta}_\alpha = \bar{\Theta}_{\alpha_j}.$$

Notice that for right-continuous functions of bounded variation, say a and b, we have by (14.21) that

$$d(b(t)a(t)) = \Theta_\alpha a(t)\, db(t) + \bar{\Theta}_\alpha b(t)\, da(t). \tag{14.23}$$

Next use formulas (14.20) to define the functions b_α and numbers c_α. Observe that by Lemma 14.1 below the numbers c_α do not depend on τ. Define for every $\alpha \in \mathcal{N}$ the operator B_α by

$$(B_\alpha u)(t) = \int_{(0,t]} \Theta_\alpha u(s)\, db_\alpha(s),$$

and let b_α^- be the operator of multiplying by the function $\bar{\Theta}_\alpha b_\alpha$. Then this definition of the operator B_α reads as the general definition of B_α given by (14.18), by virtue of the above definition of b_α. Using (14.23) with $b = b_\alpha$ and $a = A_k u$ we get

$$d\big(b_\alpha(t) \int_{(0,t]} u(s)\, da_k(s)\big) = d(B_\alpha A_k u)(t) + d(A_k b_\alpha^- u)(t).$$

Thus, defining the operator b_α^+ as the multiplication by b_α, we have

$$b_\alpha^+ A_k = B_\alpha A_k + A_k b_\alpha^-,$$

i.e., the first identity in Assumption 14.4. Notice that $b_\alpha^+ \neq b_\alpha^-$ if $\vartheta \neq 0$ in (14.22). Clearly, the second identity in Assumption 14.4 and Assumption 14.5 hold also for this example.

Next we formulate a lemma, which implies that for a large class of examples of the general scheme, the numbers c_α are independent of the parameter τ. To this end let $H_0, H_1, ..., H_m$ be right-continuous real functions on $\mathbb{R}$ which have finite variation on every finite interval. Assume that

$$H_r(0) = 0, \quad H_r(t+1) - H_r(t) = H_r(1) = 1, \quad \forall t \in \mathcal{R}, \quad r = 0, 1, ..., m.$$

For each $\tau \in (0, 1]$ we define the functions

$$a_r(t) = \tau H_r(t/\tau), \quad t \geq 0, \quad r = 0, 1, ..., m.$$

For each τ and $\alpha \in \mathcal{N}$, let $\Lambda_\alpha(\tau)$ be an operator mapping $B_\tau(\mathbb{R}_+)$ into itself, where $B_\tau(\mathbb{R}_+)$ denotes the class of τ-periodic bounded functions on $\mathbb{R}_+$, having left and right limits at every $t \in (0, \infty)$. We assume that $(\Lambda_\alpha(\tau)u(\cdot))(\tau t)$, $t \in \mathbb{R}_+$, is independent of τ for each $\alpha \in \mathcal{N}$ and every $u \in B_\tau(\mathbb{R})$. For every $\alpha \in \mathcal{N}$ we define a function $b_\alpha : [0, \infty) \to \mathbb{R}$ and a number c_α recursively, as follows:

$$b_\gamma = \tau^{-1}(a_\gamma - a_0), \quad c_\gamma = 0 \text{ for } \gamma = 0, 1, 2, ..., m.$$

If for every multinumber $\beta = \beta_1, ..., \beta_i$ of length i the function B_β and the number c_β are defined, then

$$c_{\alpha\gamma} := \frac{1}{\tau} \int_0^\tau \Lambda_\alpha b_\alpha(t)\, da_\gamma(t), \tag{14.24}$$

$$b_{\alpha\gamma}(t) := \frac{1}{\tau}\Big(\int_0^t \Lambda_\alpha b_\alpha(s)\, da_\gamma(s) - c_{\alpha\gamma} a_0(t)\Big). \tag{14.25}$$

Lemma 14.1 *For every $\alpha \in \mathcal{N}$ the function b_α is τ-periodic, and $b_\alpha(i\tau) = 0$ for all integers $i \geq 0$. Moreover, the numbers c_α, the functions $C_\alpha(t) := b_\alpha(\tau t)$, and*

$$\sup_{t\geq 0} |b_\alpha(t)| = \sup_{t \geq 0} |C_\alpha(t)|$$

are finite and do not depend on τ.

Proof One can easily prove this lemma by induction on the length of $\alpha \in \mathcal{N}$ and by the change of variable $t = s\tau$ in (14.24) and (14.25).

Theorem 14.1 *Let Assumptions 14.1, 14.2, 14.3, 14.4 and 14.5 hold with $l \geq 2$, and take an integer $k \geq 0$ such that $2k+2 \leq l$. Assume that (for a given $\tau \in (0,1]$) equations (14.11) and (14.12) have a solution $v \in W_l$ and $w \in W_l$, respectively, such that $\|w\|_l \leq K$. Then for any continuous linear functional w^* on W_0 such that $w^* b_\alpha^+ = 0$ for all $\alpha \in \mathcal{N}$, it holds that*

$$\langle w^*, w\rangle = \sum_{i=0}^k \tau^i \langle w^*, v_i\rangle + O(\tau^{k+1}), \tag{14.26}$$

where $v_0 = v, v_i \in W_0$ are uniquely determined by A_0, Θ_0, L_r, f_r, and c_α, and

$$|O(\tau^{k+1})| \leq N\tau^{k+1} \|w^*\|,$$

where N depends only on K_α, K, and l.

Theorem 14.1 is a straightforward consequence of Theorem 14.3 below.

Generally, the solutions of (14.11) and (14.12) depend on τ: $w = w(\tau)$, $v = v(\tau)$. However, if A_0, Θ_0, L_r, f_r, and c_α are independent of τ, then v and other v_i's in (14.26) are independent of τ as well (since they are uniquely determined by A_0, Θ_0, L_r, f_r, and c_α). In this situation we have the following result about "acceleration."

Theorem 14.2 *Let A_0, Θ_0, L_r, f_r, and c_α be independent of τ, and assume that equation (14.11) has a solution v. Let Assumptions 14.1, 14.2, 14.3, 14.4, and 14.5 hold with $l \geq 2$, and take an integer $k \geq 0$ such that $2k + 2 \leq l$. Let $\tau_0 \in (0, 1]$, and suppose that for each $j = 0, 1, ..., k$ equation (14.12) with $\tau = \tau_j := \tau_0 2^{-j}$ has a solution $w = w_j$, such that $\|w_j\|_l \leq K$. Assume that a $w^* \in W_0^*$ satisfies*

$$w^* b_\alpha^+(\tau_j) = 0, \quad \forall \alpha \in \mathcal{N},\ j = 0, 1, ...k.$$

Then, for some constants $\lambda_0, ..., \lambda_k$, depending only on k, we have

$$\Big|\sum_{j=0}^k \lambda_j \langle w^*, w_j\rangle - \langle w^*, v\rangle\Big| \leq N\tau_0^{k+1} \|w^*\|,$$

where N depends only on K_α, K, and l.

The proof of this theorem is based on elementary algebra once we define $(\lambda_0, \lambda_1, ..., \lambda_k) = (1, 0, 0, ..., 0)V^{-1}$, where V is the square matrix with entries $V^{ij} := 2^{-(i-1)(j-1)}$, $i, j = 1, ..., k+1$.

Remark 14.5 In Example 14.1 assume that $L(t)$ is independent of t. Then by Remark 14.3 the assumptions of Theorem 14.1 are satisfied for any k with appropriate l, K, and K_α. Also since $b_\alpha(j\tau) = 0$ for all $j = 0, 1, ...$, as a w^* in Theorem 14.1 one can take the restriction of elements in $D([0,T], \mathbb{R}^d)$ to any of the times in

$$T_\tau := \{j\tau : j = 0, 1, ...\} \cap [0, T].$$

From Theorem 14.1 we now conclude that there exist $\mathbb{R}^d$-valued functions $v_i(t)$, $i = 0, 1, ...$, $t \in [0, T]$ independent of τ, with $v_0 = v$ such that

$$\sup_{t \in T_\tau} |w(\tau, t) - \sum_{i=0}^{k} \tau^i v_i(t)| \le N\tau^{k+1}, \tag{14.27}$$

where N depends only on T, k, $|L|$, and $|\varphi|$.

Clearly, under the above time independence assumption we have $v(t) = e^{Lt}\varphi$. Also equation (14.15) amounts to saying that

$$w(0) = \varphi, \quad w(t) = w(j\tau) \ \text{ for } \ t \in [j\tau, (j+1)\tau),$$

$$w((j+1)\tau) = w(j\tau) + Lw(j\tau)\tau, \quad j = 0, 1, ...,$$

which is just Euler's scheme for equation (14.14). It is also an explicit finite-difference scheme for the equation $v' = Lv$. It follows that

$$w(t) = w(j\tau) = (1 + \tau L)^j \varphi \ \text{ for } \ t \in [j\tau, (j+1)\tau), \quad j = 0, 1, ... \tag{14.28}$$

Hence (14.27) means that

$$\max_{j : j\tau \le T} |(1 + \tau L)^j \varphi - \sum_{i=0}^{k} \tau^i v_i(j\tau)| \le N\tau^{k+1},$$

with N depending only on T, k, $|\varphi|$, and L, and v_i independent of τ with $v_0 = v$. In particular, for $\tau = 1/n, T = 1, j = n$ we get that as $n \to \infty$

$$(1 + L/n)^n \varphi = e^L \varphi + \sum_{i=1}^{k} \frac{v_i}{n^i} + O(n^{-(k+1)}), \tag{14.29}$$

where v_i are some vectors. Theorem 14.2 applied to Example 14.1 says that, as $\tau \downarrow 0$

$$\max_{j : j\tau \le T} |\sum_{i=0}^{k} \lambda_i (1 + \tau 2^{-i} L)^{2^i j} \varphi - e^{Lj\tau}\varphi| = O(\tau^{k+1}).$$

We illustrate some directions of further applications in the following example.

Example 14.3 (*Splitting-up combined with finite differences*) For a $d \times d$-matrix L we want to approximate the solution, $v(t) = e^{Lt}\varphi$, of the equation

$$\frac{d}{dt} v(t) = Lv(t), \quad v(0) = \varphi \in \mathbb{R}^d \tag{14.30}$$

on the grid $T_\tau = \{t = j\tau : j = 0, 1, 2, ..., \} \cap [0, T]$, by splitting-up the equation into m equations

$$\frac{d}{dt}v(t) = L_k v(t), \quad k = 1, 2, \cdots, m, \quad L = L_1 + L_2 + ... + L_m,$$

and solving them numerically on each fixed interval $[j\tau, (j+1)\tau]$, consecutively, by finite differences. Namely, for each k we take some $\vartheta_k \in \mathbb{R}$ and approximate the equation $dv(t) = L_k v(t)\, dt$ on each $[j\tau, (j+1)\tau)$ by the θ-*method* with $\theta = \bar{\vartheta}_k := 1 - \vartheta_k$, i.e., for its numerical solution u we take

$$u(t) = u(j\tau), \quad \text{for} \quad t \in [j\tau, (j+1)\tau),$$

$$u((j+1)\tau) = u(j\tau) + \tau\bar{\vartheta}_k L_k u(j\tau) + \tau\vartheta_k L_k u((j+1)\tau).$$

Thus, assuming that the matrix $I - \tau\vartheta_k L_k$ is invertible, we have the recursion

$$u((j+1)\tau) = (I - \tau\vartheta_k L_k)^{-1}(I + \tau\bar{\vartheta}_k L_k)u(j\tau).$$

Using this recursion for each $k = 1, 2, ..., m$ consecutively on every interval $[j\tau, (j+1)\tau)$, for $j = 0, 1, 2, ..., i-1$, we get the approximation

$$w(t_i) = \left(\Pi_{k=1}^m (I - \tau\vartheta_k L_k)^{-1}(I + \tau\bar{\vartheta}_k L_k)\right)^i \varphi \tag{14.31}$$

for $v(t_i) = e^{t_i L}\varphi$, when $t_i = i\tau$. Now we describe this approximation in terms of the general setting. In order to express the splitting-up algorithm, we introduce the absolutely continuous functions h_1, ..., h_m on $\mathbb{R}$, whose derivatives are periodic with period m, such that

$$\dot{h}_k(t) := 1_{[k-1,k)}(t) \quad t \in [0, m).$$

We define for each $\tau \in [0, 1)$ the nondecreasing right-continuous functions

$$a_k(t) = \tau[h_k(mt/\tau)], \quad t \geq 0, \quad k = 1, 2, ..., m,$$

where, as before, $[c]$ denotes the integer part of c. Then the approximation w given by (14.31) coincides with the solution of the equation

$$dw(t) = \sum_{k=1}^m L_k \Theta_k w(t)\, da_k(t), \quad w(0) = \varphi,$$

at the points $t_i := i\tau \in T_\tau$, where

$$(\Theta_k w)(t) := \vartheta_k w(t) + (1 - \vartheta_k)w(t-).$$

Clearly, this equation can be written as

$$w = \varphi + \sum_{k=1}^m A_k \Theta_k L_k w,$$

and equation (14.30) has the form

$$v = \varphi + A_0 \Theta_0 L v,$$

where Θ_0 is the identity and A_0, A_1, ..., A_m are the integral operators on the spaces

$$W_0 = \cdots = W_l = D([0, T], \mathbb{R}^d)$$

defined as usual by (14.13) for $k = 0, 1, ..., m$ with $a_0(t) \equiv t$.

Now introduce the operators Θ_α and $\bar{\Theta}_\alpha$, then the functions b_α, and numbers c_α and, finally, the operators $b_\alpha^{\pm}$ and B_α by the same formulas which were used in Example 14.2 allowing there k to vary in $\{0, 1, ..., m\}$.

Notice that by Lemma 14.1 the numbers c_α do not depend on τ and as in Example 14.2, it is easy to check that all assumptions of the general scheme are satisfied. Furthermore, we have $b_\alpha(j\tau) = 0$ for all integers $j \geq 0$. Therefore we can apply Theorem 14.1 with w^*, the restriction of functions $u \in D([0,T], \mathbb{R}^d)$ to any $t_j \in T_\tau$. Then we obtain that there exist $v_0, v_1, ..., v_k \in D([0,T], \mathbb{R}^d)$, independent of τ, with $v_0 = v$, such that

$$\max_{t\in T_\tau} |w(\tau, t) - \sum_{i=0}^{k} v_i(t)\tau^i| \leq N\tau^{k+1} \quad \text{for} \quad \tau \in (0, 1],$$

where $w(\tau, \cdot) := w$ is the approximation defined by (14.31), and N is a constant depending only on T, k, m, $|L|$, $|\varphi|$, and $\vartheta_1, ..., \vartheta_m$. From Theorem 14.2 we get

$$\max_{t\in T_\tau} |e^{Lt}\varphi - \sum_{i=0}^{k} \lambda_i w(2^{-i}\tau, t)| = O(\tau^{k+1}).$$

We will see that Theorem 14.1 follows from an expansion of w into a power series with respect to τ. To state the corresponding result we need more notation.

For $\gamma \in \mathcal{N}$ we define $f_\gamma \in W_0$ and a linear operator L_γ as follows:

$$L_0 = 0,\ f_0 = 0,\ L_\gamma = L_r,\ f_\gamma = f_r \quad \text{for} \quad \gamma = r \in \{1, 2, ..., m\},$$

$$L_{\gamma 0} = LL_\gamma, \quad L_{\gamma r} = -L_\gamma L_r$$
$$f_{\gamma 0} = Lf_\gamma, \qquad f_{\gamma r} = -L_\gamma f_r$$

for $r = 1, 2, ..., m$, $\gamma \in \mathcal{N}$. Notice that L_α is a bounded linear operator from W_j into $W_{j-|\alpha|}$ if $|\alpha| \leq j$, and $f_\alpha \in W_{l-|\alpha|+1}$ if $|\alpha| \leq l + 1$.

Let $\mathcal{M}$ be the set of multinumbers $\gamma_1\gamma_2, ..., \gamma_i$ with $\gamma_j \in \{1, 2, ..., m\}$, $j = 1, 2, ..., i$, and integers $i \geq 1$. Note that $\mathcal{M} \subset \mathcal{N}$ and, in contrast with $\mathcal{N}$, the entries in $\gamma \in \mathcal{M}$ are different from zero. We introduce sequences $\sigma = (\beta_1, \beta_2, ..., \beta_i)$ of multinumbers $\beta_j \in \mathcal{M}$, where $i \geq 1$ is any integer, and set

$$|\sigma| = |\beta_1| + |\beta_2| + \cdots + |\beta_i|.$$

The set of these sequences, together with the "empty sequence" e of length $|e| = 0$ is denoted by $\mathcal{J}$. For $\sigma = (\beta_1, \beta_2, ..., \beta_i)$, $i \geq 1$, we define

$$S_\sigma = \mathcal{R}L_{\beta_1} \cdot \cdots \cdot \mathcal{R}L_{\beta_i}, \quad \text{where} \quad \mathcal{R} = \mathcal{R}_0\Theta_0,$$

and for $\sigma = e$, we set $S_e = \mathcal{R}$. Notice that S_σ is welldefined as a bounded linear operator from $W_{j+|\sigma|}$ to W_j if $j + |\sigma| \leq l$. If we have a collection of $g_\nu \in W_0$ indexed by a parameter ν taking values in a set A, then we use the notation $\sum^*_{\nu\in A} g_\nu$ for any linear combination of g_ν with coefficients depending only on c_α, A, and ν. For instance

$$\sum_{A}^{*} S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A}^{*} S_\sigma w_\gamma = \sum_{(\sigma,\gamma)\in A} c(\sigma, \gamma) S_\sigma w_\gamma,$$

where $c(\sigma, \gamma)$ are certain functions of c_α, $\alpha \in \mathcal{N}$, and $(\sigma, \gamma) \in A$. These functions are allowed to change from one occurrence to another.

For $\mu = 0, ..., l$, $\kappa \geq 0$, and functions $u = u_\alpha(\tau)$ depending on the parameters $\alpha \in \mathcal{N}$ and τ we write

$$u = O_\mu(\tau^\kappa) \ \text{ if } \ \|u_\alpha(\tau)\|_\mu \leq N\tau^\kappa,$$

where the constant $N < \infty$ depends only on $\alpha, K_\beta, \beta \in \mathcal{N}$, μ, l, and K. Finally, set

$$A(i) = \{(\sigma, \beta) : \sigma \in \mathcal{J}, \beta \in \mathcal{M}, |\sigma| + |\beta| \leq i\},$$

$$B^*(i, j) = \{(\alpha, \beta) : \alpha \in \mathcal{N}, \beta \in \mathcal{M}, |\alpha| \leq i, |\beta| \leq j\},$$

and $v_\beta = L_\beta v + f_\beta$, $w_\beta = L_\beta w + f_\beta$.

Theorem 14.3 *Under the assumptions of Theorem 14.1 we have*

$$w = v + \sum_{i=1}^{k} \tau^i \sum\nolimits^*_{A(2i)} S_\sigma v_\beta + \sum_{i=1}^{k} \tau^i \sum\nolimits^*_{B^*(i,i+j)} b^+_{\alpha_1} w_{\beta_1} + O_0(\tau^{k+1}), \tag{14.32}$$

Furthermore, if $k \geq 1$, then

$$\sum\nolimits^*_{A(2)} S_\sigma v_\beta = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) \mathcal{R} v_{ij}$$

in (14.32), so that it vanishes if $c_{ij} = c_{j0}$ for all $i, j = 1, ..., m$.

Remark 14.6 If the coefficient of τ in the first sum in (14.32) is zero, then to accelerate to get the order of accuracy τ^3 it suffices to mix *two* grids instead of three as in the general case. Indeed, let $\tau_0 \in (0, 1]$ and assume that equation (14.12) with τ_0 and $\tau_1 := \tau_0/2$ has a solution w_0 and w_1, respectively. Then by virtue of Theorem 14.3, under the assumptions of Theorem 14.2, if $c_{ij} = c_{j0}$ for all i, j, then we have

$$|\frac{4}{3}\langle w_1, w^*\rangle - \frac{1}{3}\langle w_0, w^*\rangle - \langle v, w^*\rangle| \leq N\tau_0^3 \|w^*\|$$

for all $w^* \in W_0^*$, satisfying $w^* b^+_\alpha(\tau_j) = 0$, for all $\alpha \in \mathcal{N}$, $j = 0, 1$.

14.3 An application to ODEs

We take two integers $m, d \geq 1$ and sufficiently smooth and bounded vector fields $b_1, ..., b_m$ on $\mathbb{R}^d$, and consider the ordinary differential equation (ODE)

$$\dot{x}_t = b_1(x_t) + \cdots + b_m(x_t) =: b(x_t), \quad t \geq 0. \tag{14.33}$$

We want to develop a splitting-up method for solving this equation on the basis of solving the equations

$$\dot{x}_t = b_k(x_t) \tag{14.34}$$

for each particular $k = 1, 2, ..., m$. More precisely, denote by $P(t)$ and $P_k(t)$ the mappings $x \to x_t$, where x_t is the solution of (14.33) and (14.34), respectively, with starting point x. Taking a parameter $\tau > 0$, we want to approximate $P(t)$ by means of the products

$$S(\tau) := P_{k_p}(s_p\tau) \cdot \cdots \cdot P_{k_1}(s_1\tau), \quad s_1, s_2, ..., s_p \in \mathbb{R}, k_1, ..., k_p \in \{1, ..., m\}. \tag{14.35}$$

Consider, for example, the product $S(\tau) = P_m(\tau) \cdot \cdots \cdot P_2(\tau)P_1(\tau)$, which corresponds to a simple splitting-up algorithm over the grid

$$T_\tau = \{t_i := i\tau : i = 0, 1, ...,\} \cap [0, T],$$

where $T \in (0, \infty)$ is fixed. Then it is wellknown that for every $x \in \mathbb{R}^d$

$$\max_{t_i \in T_\tau} |P(t_i)x - S^i(\tau)x| \leq N\tau, \tag{14.36}$$

for all $\tau > 0$, where N is a constant, which does not depend on τ. It is also known that the symmetric product

$$S(\tau) = P_1(\tau/2) \cdot \dots \cdot P_{m-1}(\tau/2) P_m(\tau) P_{m-1}(\tau/2) \cdot \dots \cdot P_1(\tau/2), \tag{14.37}$$

introduced by Strang [5] gives a better approximation. Namely, for this product estimate (14.36) holds with τ^2 in place of τ in the right-hand side.

Let $p \geq m$, $k_1, ..., k_p \in \{1, ..., m\}$ be any integers and let $s_1, ..., s_p$ be real numbers satisfying

$$\sum_{j=1}^{p} s_j \delta_{rk_j} = 1 \quad \text{for every} \quad r = 1, 2, ..., m,$$

where $\delta_{rk_j} = 1$ for $r = k_j$ and 0 otherwise. Then we say that the product (14.35) is a *fractional step method* of "length" p. It is called a method with accuracy of order q, if for each x the *fractional step approximations*, $S^i x$, satisfy estimate (14.36) with τ^q in place of τ in the right-hand side.

We characterize the product (14.35) by the absolutely continuous functions $\kappa_r = \kappa_r(t)$, $r = 1, ..., m$, whose derivatives, $\dot{\kappa}_r(t)$ are periodic functions on $\mathbb{R}$ with period p, such that

$$\kappa_r(0) = 0, \quad \text{and} \quad \dot{\kappa}_r(t) = \sum_{j=1}^{p} s_j \delta_{rk_j} 1_{[j-1,j)}(t) \quad \text{for} \quad t \in [0, p). \tag{14.38}$$

We say that (14.35) is a symmetric product if

$$\dot{\kappa}_r(p - t) = \dot{\kappa}_r(t) \quad \text{for} \quad dt\text{-almost every} \quad t \in (0, p), \quad r = 1, 2, ..., m. \tag{14.39}$$

Clearly, (14.37) is a simple example of a symmetric product.

By using Theorem 14.1 we will see in this section that for every fractional step method S, each integer $k \geq 0$, and each compact set $\mathbb{K} \subset \mathbb{R}^d$, there is an expansion

$$S^{t/\tau}(\tau)x = P(t)x + \sum_{j=1}^{k} \tau^j h_j(t, x) + R_k(\tau, t, x) \tag{14.40}$$

valid for all $t \in T_\tau$, $\tau \in (0, 1]$ and $x \in \mathbb{K}$, where $h_1, h_2, ..., h_k$ are some functions, independent of τ, and R_k is a function such that

$$\sup_{t \in T_\tau} \sup_{x \in \mathbb{K}} |R_k(\tau, t, x)| \leq N\tau^{k+1}$$

for all $\tau \in (0, 1]$ with a constant N independent of τ. We note that analyzing the functions $h_1, ..., h_j$ one can find fractional step methods of order j. In particular, by Theorem 14.3 one gets that for some functions g_{ij}

$$h_1(t, x) = \sum_{i,j=1}^{m} (c_{ij} - c_{j0}) g_{ij}(t, x)$$

with numbers c_{ij}, which are easily computable, at least for special products, like symmetric products. If $c_{ij} = c_{j0}$ for every i, j, then clearly S is a method of order 2. Thus, see Remark

14.7 below, we can easily rediscover that Strang's product (14.37) is a method of order 2, and we also find many other second order methods.

Analyzing the terms h_j for $j > 1$ in the expansion (14.40) is rather tedious. We are not going into this direction in the present chapter. For any given $k \geq 1$, the existence of methods of order k is known in the literature. This is proved by the Baker–Campbell–Hausdorff formula, or by other ways, different from our approach (see, e.g., [2], [7], [8], [9], [11].) It is also known, see [4], [10], that if the product (14.35) is a method of order $k > 2$, then at least one of the numbers $s_1, s_2, ..., s_p$ is negative, which means that we solve equation (14.34) backward in time in the corresponding steps.

Our main interest lies in accelerating any methods of order $k \geq 1$, with little extra computational effort, to get numerical methods of order $k+1$, or higher. Since we have the above expansion for any given fractional step method $S(\tau)$, we can easily accelerate it by choosing different "step-sizes" $\tau_0 = \tau$, τ_1, ..., τ_j and mixing $S(\tau_0)$, ..., $S(\tau_j)$, appropriately, as it is demonstrated in the general setup by Theorem 14.2. As an illustration we formulate first a result on the acceleration of an arbitrary method of order 2.

Theorem 14.4 *Let $S(\tau)$ be a fractional step method such that*

$$\int_0^p \kappa_i(t)\dot{\kappa}_j(t)\,dt = \frac{1}{2}, \quad \text{for} \quad 1 \leq i < j \leq m, \tag{14.41}$$

where p is the length of the product S. Then S is a method of order 2. Moreover, for any compact set $\mathbb{K} \subset \mathbb{R}^d$ there exists a constant N, such that for all $\tau > 0$ and $t \in T_\tau$, $x \in \mathbb{K}$ we have

$$|P(t)x - \lambda_0 S^{t/\tau}(\tau)x - \lambda_1 S^{2t/\tau}(2^{-1}\tau)x| \leq N\tau^3, \tag{14.42}$$

with $\lambda_0 = -1/3, \lambda_1 = 4/3$.

We will prove this theorem at the end of the chapter after a suitable adaptation of the general scheme of Section 14.2 to ordinary differential equations. We will present also a more general result at the end of this section.

Remark 14.7 All fractional step methods, which are symmetric products satisfy condition (14.41). There are infinitely many fractional step methods, which are nonsymmetric products, yet still satisfying (14.41). For example, when $m = 2$, every product of the form

$$S(\tau) = P_2((1-b)\tau)P_1((1-a)\tau)P_2(b\tau)P_1(a\tau) \tag{14.43}$$

with $a \neq 1$, and $b = \frac{1}{2(1-a)}$, satisfies (14.41). If $a = \frac{1}{2}$ then (14.43) is Strang's product with $m = 2$. For $a \neq \frac{1}{2}$ these products are not symmetric.

Proof Let S be a symmetric fractional step method of length p, and let κ_1, ..., κ_m be the functions characterizing S. Note that $\kappa_i(p) = 1, \kappa_i(t) + \kappa_i(p-t) = 1$. Then by the change of variable $t = p - s$, and by (14.39)

$$\int_0^p \kappa_i(t)\dot{\kappa}_j(t)\,dt = \int_0^p \kappa_i(p-s)\dot{\kappa}_j(p-s)\,ds$$

$$= \int_0^p (1-\kappa_i(s))\dot{\kappa}_j(s)\,ds = 1 - \int_0^p \kappa_i(s)\dot{\kappa}_j(s)\,ds,$$

which immediately implies equation (14.41). For the functions κ_1, κ_2, which characterize (14.43) we have

$$\dot{\kappa}_1(t) = a1_{[0,1)}(t) + (1-a)1_{[2,3)}(t), \quad \dot{\kappa}_2(t) = b1_{[1,2)}(t) + (1-b)1_{[3,4)}(t),$$

for $t \in (0,4)$, and

$$\int_0^4 \kappa_1(t)\dot{\kappa}_2(t)\,dt = ab + 1 - b = 1 - b(1-a) = \frac{1}{2},$$

i.e., condition (14.41) holds. If $a \neq \frac{1}{2}$, then clearly (14.43) is not symmetric. If $a = \frac{1}{2}$, then $b = 1$, and (14.43) is Strang's symmetric product with $m = 2$. The proof of the remark is complete.

Our approach to proving Theorem 14.4 is based on the observation that the solutions of (14.33) are characteristics of the corresponding PDE (14.44) below, where

$$Lu(t,x) := b^i(x)u_{x^i}(t,x) = \sum_{k=1}^m L_k u(t,x), \quad L_k u(t,x) := b_k^i(x)u_{x^i}(t,x).$$

The same approach is applicable to equations on smooth manifolds, one replaces $P(t)x$ in (14.42) with $\varphi(P(t)x)$, and time-dependent systems when one just adds one additional coordinate t.

To avoid introducing too much detail from the beginning and making our presentation slightly more general we take certain continuous functions H_1,..., H_m on $\mathbb{R}$ which have finite variation on every finite interval and such that

$$H_r(t+1) - H_r(t) = H_r(1) = 1, \quad \forall t \in \mathbb{R}, \quad r = 1, 2, ..., m.$$

For each $\tau > 0$ we define the functions

$$a_r(t) = \tau H_r(t/\tau), \quad t \geq 0, \quad r = 1, 2, ..., m.$$

We take a sufficiently regular function φ on $\mathbb{R}^d$, and consider two Cauchy problems

$$\frac{\partial v(t,x)}{\partial t} = Lv(t,x), \quad t > 0,\, x \in \mathbb{R}^d, \quad v(0) = \varphi, \tag{14.44}$$

$$dw(t,x) = L_k w(t,x)\, da_k(t), \quad t > 0,\, x \in \mathbb{R}^d, \quad w(0) = \varphi, \tag{14.45}$$

where by (14.45) we mean, of course, that

$$w(t,x) = \varphi(x) + \sum_{k=1}^m \int_0^t L_k w(s,x)\, da_k(s).$$

Fix an integer $l \geq 1$ and for $i = 0, ..., l$, let $W_i := C([0,T], C_0^i(\mathbb{R}^d))$ be the space of bounded continuous functions $u(t,x)$ on $[0,T] \times \mathbb{R}^d$ such that their i derivatives in x are also bounded and continuous and

$$\lim_{|x|\to\infty} \sup_{t\in[0,T]} |u(t,x)| = 0.$$

The latter condition is introduced to make W_0 separable and W_1 dense in W_0 with respect to the norms on W_i defined by

$$\|u\|_i = \sum_{|\alpha_1|+...+|\alpha_d|\leq i} \sup_{t,x} |D_1^{\alpha_1}, ..., D_d^{\alpha_d} u(t,x)|,$$

where

$$D_k^\alpha = \frac{\partial^\alpha}{(\partial x^k)^\alpha}.$$

In the following lemma by $x_t(x)$ we denote the solution of (14.33) starting at x, that is, $P(t)x$.

Lemma 14.2 *Let $H(t)$, $t \in [0,T]$, be a continuous function of bounded variation and $g \in W_0$. Define an operator $\mathcal{R}(H)$ by*

$$(\mathcal{R}(H)g)(t,x) = \int_0^t g(s, x_{t-s}(x))\, dH(s).$$

Then
(i) For all i, $\mathcal{R}(H) : W_i \to W_i$ is a bounded operator;
(ii) If $g \in W_1$, then $u = \mathcal{R}(H)g$ satisfies

$$u(t,x) = \int_0^t b^i(x) u_{x^i}(s,x)\, ds + \int_0^t g(s,x)\, dH(s), \quad t \in [0,T]\,, x \in \mathbb{R}^d; \tag{14.46}$$

(iii) If $g \in W_0$ and there is an $u \in W_1$ satisfying (14.46), then

$$u = \mathcal{R}(H)g.$$

Proof Assertion (i) follows from the well-known fact that $x_t(x)$ is a smooth function of x. It is also wellknown that $x_t(x)$ satisfies

$$x_{t+s}(x) = x_t(x_s(x)), \quad \dot{x}_t(x) = b^i(x) \frac{\partial}{\partial x^i} x_t(x), \quad s,t \geq 0, x \in \mathbb{R}^d. \tag{14.47}$$

Now in (ii) by expressing $g(s, x_{t-s})$ through the integral of its derivative in t and using (14.47) we obtain

$$u(t,x) = \int_0^t g(s,x)\, dH(s) + J(t,x),$$

where

$$\begin{aligned} J(t,x) &:= \int_0^t \Big(\int_0^{t-s} g_{x^i}(s, x_r(x)) b^i(x) \frac{\partial}{\partial x^i} x_r(x)\, dr \Big)\, dH(s) \\ &= b^i(x) \frac{\partial}{\partial x^i} \int_0^t \Big(\int_0^{t-s} g(s, x_r(x))\, dr \Big)\, dH(s) =: b^i(x) \frac{\partial}{\partial x^i} I(t,x). \end{aligned}$$

By changing variables and using Fubini's theorem we get

$$I(t,x) = \int_0^t \Big(\int_s^t g(s, x_{p-s}(x))\, dp \Big)\, dH(s) = \int_0^t u(p,x)\, dp.$$

This proves (ii). To prove (iii) it suffices to observe that for each fixed $t_0 \in [0,T]$, on $[0,t_0]$ we have

$$\begin{aligned} du(t, x_{t_0-t}(x)) &= b^i(x_{t_0-t}(x)) u_{x^i}(t, x_{t_0-t}(x))\, dt + g(t, x_{t_0-t}(x))\, dH(t) \\ &+ u_{x^i}(t, x_{t_0-t}(x)) \frac{\partial}{\partial t} x^i_{t_0-t}(x)\, dt = g(t, x_{t_0-t}(x))\, dH(t). \end{aligned}$$

The lemma is proved.

This lemma implies that Assumption 14.2 is satisfied with the integral operators A_0, and A_1, ..., A_k defined by the integrators $a_0(t) := t$, and $a_1, ..., a_k$, respectively, and with $\Theta_0 = \Theta_1 = \cdots = \Theta_m = I$ being the identity. We also define $\bar{\Theta}_0 = \bar{\Theta}_1 = \cdots = \bar{\Theta}_m = I$ and after that introduce b_α, c_α, B_α, $b^\pm_\alpha$ in exactly the same way as in Example 14.2. By the way, in our present situation we could have introduced ϑ_k as well, but they would not play any role because all our integrators are absolutely continuous.

It is a matter of simple calculations to check that these objects satisfy Assumptions 14.3, 14.4, and 14.5 (with $f_k = 0$). Next we show that the solution of (14.45) exists. To this end we fix a k and for $s \le t$ denote by $x_{s,t}(x)$ the unique solution of the equation

$$x_{s,t}(x) = x + \int_s^t b_k(x_{r,t}(x))\, da_k(r). \tag{14.48}$$

Then $x_{s,t}(x)$ is a smooth function of x, satisfying

$$x_{0,t}(x) = x_{0,s}(x_{s,t}(x)), \quad s \le t, \quad dx_{0,t}(x) = b^i_k(x)\frac{\partial}{\partial x^i}x_{0,t}(x)\, da_k(t).$$

Assume that $\varphi \in C_0^{l+1}(\mathbb{R}^d)$. Then φ satisfies Assumption 14.1 (iii). Moreover, it follows that $w(t,x) := \varphi(x_{0,t}(x))$ is in W_l, and satisfies (14.45). Under the same condition on φ

$$v(t,x) = \varphi(x_t(x)).$$

Therefore relying on Lemma 14.1 and Theorems 14.2, 14.3 we get that for $2k+2 \le l$, $k \ge 1$

$$\sup_{t\in T_\tau, x\in\mathbb{R}^d} |\varphi(x_{0,t}(x)) - \varphi(x_t(x)) - \tau\sum_{i,j=1}^m (c_{ij} - c_{j0})g_{ij}(t,x)$$

$$-\sum_{i=2}^k \tau^i h_i(t,x)| \le N\tau^{k+1}, \tag{14.49}$$

$$\sup_{t\in T_\tau, x\in\mathbb{R}^d} |\sum_{j=0}^k \lambda_j \varphi(x^{(j)}_{0,t}(x)) - \varphi(x_t(x))| \le N\tau^{k+1},$$

where N, g_{ij}, and h_i are independent of τ, and $x^{(j)}_{s,t}(x)$ is defined as the solution of (14.48) with

$$a_k(r) = 2^{-j}\tau H_k(r2^j/\tau).$$

Now we specify the above result for the fractional step method (14.35). Let $\kappa_r = \kappa_r(t)$, $r = 1, 2, ..., m$, be the absolutely continuous functions introduced by (14.38). Define $H_r(t) = \kappa_r(pt)$. Then it is easy to see that

$$x_{0t}(x) = S^{t/\tau}(\tau)x \ \text{ for all } \ t \in T_\tau, \quad \tau \in (0,1], \quad x \in \mathbb{R}^d. \tag{14.50}$$

Let $\mathbb{K} \subset \mathbb{R}^d$ be a compact set and let $\varphi_1, ..., \varphi_d \in C_0^\infty(\mathbb{R}^d)$ be such that

$$\varphi_i(x) = x^i, \ \text{ for all } \ x = (x^1, ..., x^d) \in \mathbb{K}. \tag{14.51}$$

Set $\kappa_0(t) = t/p$, and use the notation $W_0^d = W_0 \times \cdots \times W_0$. Then from (14.49) we get the following theorem.

Theorem 14.5 *Let $k \ge 1$ and l be integers such that $l \ge 2k+2$. Assume that the vector fields b_1, ..., b_m have l bounded and continuous derivatives. Then for any compact set $\mathbb{K} \subset \mathbb{R}^d$ there exists a constant N, such that for all $\tau \in (0,1]$ and $t \in T_\tau$, $x \in \mathbb{K}$ we have*

$$S^{t/\tau}(\tau)x = x_t(x) + \tau\sum_{i,j=1}^m (c_{ij} - c_{j0})h_{ij}(t,x) + \tau^2 h_2(t,x) + \ ...$$

$$+\tau^k h_k(t,x) + R_k(\tau,t,x), \tag{14.52}$$

where for $i = 1, 2, ..., m$, $j = 0, 1, 2, ..., m$

$$c_{ij} = \int_0^p (\kappa_i(t) - \kappa_0(t))\, d\kappa_j(t), \tag{14.53}$$

$h_{ij}, h_2, ..., h_k$ *belong to* W_0^d*, they are independent of* τ*,* $R_k(\tau, \cdot, \cdot) \in W_0^d$*, for every* $\tau \in (0, 1]$*, and*

$$\sup_{t \in T_\tau, x \in \mathbb{K}} |R_k(\tau, t, x)| \le N\tau^{k+1}.$$

Proof We get (14.52) from (14.49) by taking into account (14.50) and (14.51). By (14.20)

$$\begin{aligned} c_{ij} &= \frac{1}{\tau} \int_0^\tau (a_i(t) - a_0(t))\, da_j(t) \\ &= \int_0^1 (\kappa_i(pt) - t))\; d\kappa_j(pt) \\ &= \int_0^p (\kappa_i(t) - \kappa_0(t))\, d\kappa_j(t) \end{aligned}$$

for all $i = 1, 2, ..., m$ and $j = 0, 1, 2, ..., m$.

Remark 14.8 For every pair of integers $i, j \in [1, m]$

$$2(c_{ij} - c_{j0}) = \int_0^p \kappa_i(t)\, d\kappa_j(t) - \int_0^p \kappa_j(t)\, d\kappa_i(t). \tag{14.54}$$

In particular, $c_{ij} - c_{j0} = -(c_{ji} - c_{i0})$, so that if $c_{ij} - c_{j0} = 0$, then also $c_{ji} - c_{i0} = 0$. Furthermore, $c_{ij} = c_{j0}$ if and only if

$$\int_0^p \kappa_i(t)\, d\kappa_j(t) = \int_0^p \kappa_j(t)\, d\kappa_i(t),$$

which is equivalent to

$$\int_0^p \kappa_i(t)\, d\kappa_j(t) = \frac{1}{2}.$$

Proof By (14.53)

$$\begin{aligned} c_{ij} - c_{j0} &= \int_0^p (\kappa_i(t) - \kappa_0(t))\, d\kappa_j(t) - \int_0^p (\kappa_j(t) - \kappa_0(t))\, d\kappa_0(t) \\ &= \int_0^p \kappa_i(t)\, d\kappa_j(t) - \kappa_0(p)\kappa_j(p) + \frac{1}{2}\kappa_0^2(p) = \int_0^p \kappa_i(t)\, d\kappa_j(t) - \frac{1}{2}. \end{aligned}$$

Hence we get (14.54) by taking into account that

$$\frac{1}{2} = \frac{1}{2}\kappa_i(p)\kappa_j(p) = \frac{1}{2}\left(\int_0^p \kappa_i(t)\, d\kappa_j(t) + \int_0^p \kappa_j(t)\, d\kappa_i(t)\right),$$

and the rest of the remark is obvious.

Now Theorem 14.4 follows from Theorem 14.5 and Remarks 14.8 and 14.6. In order to generalize it we fix two integers $1 \le q \le k$ and define

$$(\lambda_0, \lambda_1, ..., \lambda_{k-q+1}) = (1, 0, ..., 0)V^{-1},$$

where V is a $(k - q + 2) \times (k - q + 2)$-matrix with entries $V_{i1} = 1$ and $V_{i,j} = 2^{-(i-1)(q+j-2)}$ for $i = 1, 2, ..., k-q+2$, $j = 2, ..., k-q+2$. Notice that V is invertible, since its determinant, a Vandermonde determinant generated by $1, 2^{-p}, 2^{-p-1}, ..., 2^{-k}$, is different from 0.

Theorem 14.6 *Let $l \geq 2k+2$ be an integer. Assume that the vector fields $b_1,\ldots,b_m$ have l bounded and continuous derivatives. Let S be a fractional step method of order q. Then for every compact set $\mathbb{K} \subset \mathbb{R}^d$ there exists a constant N, such that*

$$\max_{t\in T_\tau} \sup_{x\in\mathbb{K}} \left|P(t)x - \sum_{j=0}^{k-q+1} \lambda_j S^{2^j t/\tau}(2^{-j}\tau)x\right| \leq N\tau^{k+1}$$

for all $\tau \in (0,1]$.

Proof Since $S(\tau)$ is a fractional step method of order q, by Theorem 14.5 for any compact set $\mathbb{K}$ there is a constant N such that for all $\tau \in (0,1]$

$$S^{t/\tau}(\tau)x = x_t(x) + \sum_{j=q}^{k} \tau^j h_j(t,x) + R_k(\tau;t,x),$$

for all $t \in T_\tau$, $x \in \mathbb{K}$, where h_q, ..., h_k do not depend on τ, and

$$\max_{t\in T_\tau} \sup_{x\in\mathbb{K}} |R_k(\tau;t,x)| \leq N\tau^{k+1}$$

with a constant N independent of τ. Hence we get the theorem in the same way as Theorem 14.2 is proved.

References

[1] Gyöngy, I. and Krylov, N. (2003). On splitting up method and stochastic partial differential equations. *Ann. Probab.* **31**, 564–591.

[2] Murua, A. and Sanz-Serna, J.M. (1999). Order conditions for numerical integrators obtained by composing simpler integrators. *Phil. Trans. R. Soc. Lond. A* **357**, 1079–1100.

[3] McLachlan, R.I. and Quispel, G.R.W. (2002). Splitting methods. *Acta Numer.* **11**, 341–434.

[4] Sheng, Q. (1989), Solving linear partial differential equations by exponential splitting. *IMA J. Numer. Anal.* **9**, 199-212.

[5] Strang, W.G. (1963). Accurate partial difference methods I. Linear Cauchy problems. *Arch. Mech.* **12**, 392–402.

[6] Strang, W.G. (1964). Accurate partial difference methods II. Non-linear problems. *Numer. Math.* **13**, 37–46.

[7] Suzuki, M. (1992). General theory of higher order decomposition of exponential operators and symplectic integrators. *Phys. Lett. A* **165**, 387–395.

[8] Suzuki, M. (1996). Convergence of exponential product formulas for unbounded operators. *Rev. Math. Phys.* **8** No. 3, 487–502.

[9] Tsuboi, Z. and Suzuki, M. (1995). Determining equations for higher-order decompositions of exponential operators. *Int. J. Mod. Phys. B* **25**, 3241–3268.

[10] Victoir, N. (2002). On the non-existence of non-negative splitting method of degree greater or equal to 3. Preprint.

[11] Yosida, H. (1990). Construction of higher order symplectic integrators. *Phys. Lett. A* **165**, 387–395.

15 Stochastic Variational Equations in White-Noise Analysis

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15.1 Introduction

Our approach to random complex systems is always in line with the following idea:

Reduction $\longrightarrow$ Synthesis $\longrightarrow$ Analysis,
where the *causality* is always involved.

To make the problem concretized, we may say that our first step is to get the innovation of the random systems such as stochastic processes, random fields, and so on. For the case of a stochastic process $X(t)$ our idea is motivated by the following formal expression of the *stochastic infinitesimal equation*, due to P. Lévy (see [12]):

$$\delta X(t) = \Phi(X(s), s \leq t, Y(t), t, dt), \tag{15.1}$$

where $\delta X(t)$ stands for the variation of $X(t)$ for the infinitesimal interval $[t, t+dt)$, the Φ is a sure functional, and the $Y(t)$ is the *innovation*. Intuitively speaking, the innovation is a system of infinitesimal random variables such that each $Y(t)$ contains the same information as that newly gained by the $X(t)$ during the infinitesimal time interval $[t, t+dt)$ and $\{Y(t)\}$ is a system of independent idealized elemental random variable. If such an equation is obtained, then the pair $(\Phi, Y(t))$ can completely characterize the probabilistic structure of the given process $X(t)$.

As a generalization of the stochastic infinitesimal equation for $X(t)$, one can introduce a *stochastic variational equation* for random field $X(C)$ parameterized by an ovaloid C:

$$\delta X(C) = \Phi(X(C'), C' < C, Y(s), s \in C, C, \delta C), \tag{15.2}$$

where $C' < C$ means that $(C') \subset (C)$, where (C) denotes the domain enclosed by C. The system $\{Y(s), s \in C\}$ is the innovation which is understood in the similar sense to the case of $X(t)$.

A rigorous and general definition of the innovation has been given by the literature [10].

15.2 Gaussian systems

15.2.1

First, we discuss a Gaussian process $X(t), t \in T$, where T is an interval of R^1, say $[0, \infty)$ or the entire R. Assume that $X(t)$ is separable and has no remote past. Then, the innovations can be considered explicitly in this case. The original idea came from P. Lévy (the third Berkeley Synposium paper; see [13]). Under the assumption that the process has unit multiplicity and other mild conditions, a Gaussian process has innovation $\dot{B}(t)$ which is a white noise and is expressed in the form

$$X(t) = \int_0^t F(t,u)\dot{B}(u)du. \tag{15.3}$$

This is the so-called *canonical representation*. It might seem to be rather elementary; however, such an easy understanding is, in a sense, not quite correct. There is profound structure behind this formula and we are led to a deep insight that is applicable to a general class of Gaussian processes and even to a non-Gaussian case.

Let a Brownian motion $B(t)$ and a kernel function $G(t,u)$ of Volterra type be given. Define a Gaussian process $X(t)$ by

$$X(t) = \int_0^t G(t,u)\dot{B}(u)du. \tag{15.4}$$

Now we assume that $G(t,u)$ is a smooth function on the domain $0 \le u \le t < \infty$ and $G(t,t)$ never vanishes. Then we have the following.

Theorem 15.1 *The variation $\delta X(t)$ of the process $X(t)$ is defined and is given by*

$$\delta X(t) = G(t,t)\dot{B}(t)dt + dt\int_0^t G_t(t,u)\dot{B}(u)du, \tag{15.5}$$

where $G_t(t,u) = \frac{\partial}{\partial t}G(t,u)$. The $\dot{B}(t)$ is the innovation of $X(t)$ if and only if G(t, u) is the canonical kernel.

Proof The formula for the variation of $X(t)$ is easily obtained. If G in (15.2) is not a canonical kernel, then the sigma field $\mathbf{B}_t(X)$ is strictly smaller than $\mathbf{B}(\dot{B})$; in particular, the $\dot{B}(t)$ is not really a function of $X(s), s \le t+0$.

Note that if, in particular, $G(t,u)$ is of the form $f(t)g(u)$, then $X(t)$ is a Markov process and there is always given a canonical representation. Hence $\dot{B}(t)$ is the innovation.

Remark 15.1 In the variational equation, the two terms in the right-hand side seem to be different order as dt tend to zero, so that two terms may be discriminated. But in reality the problem like that is not as simple and even not our present concern.

We now have to pause to prepare some background to proceed calculus of functionals of $\dot{B}(t)$ in the Hilbert space $(L^2) \equiv L^2(\mu)$, μ being the white-noise measure. The so-called S-transform, which is an infinite-dimensional analogue of the Laplace transform, is introduced in order to have a representation of (L^2)-functionals and to achieve actual calculations and operations. The S-transform is a mapping

$$\varphi \longrightarrow (S\varphi)(\xi) = \exp[-\|\xi\|^2]\int \exp[< x, \xi >]\varphi(x)d\mu(x).$$

With this transform the partial differential operator $\partial_t = \frac{\partial}{\partial \dot{B}(t)}$ is defined to be

$$\partial_t = S^{-1} \frac{\delta}{\delta \xi(t)} S,$$

where $\frac{\delta}{\delta \xi(t)}$ is the Fréchet derivative.

Having obtained the innovation $\dot{B}(t)$ of the process $X(t)$ given by (15.3), one can use the partial differential operator to have the canonical kernel $F(t,u)$

$$F(t,u) = \partial_u X(t), u < t.$$

Note that it is given by the knowledge of the original process $X(s), s \le t$, since the representation (15.3) is now assumed to be canonical.

15.2.2 Gaussian random fields

Having reviewed the case of Gaussian process, we now begin the main part of the chapter. First, the innovation approach to Gaussain random fields is discussed.

To fix the idea we consider a Gaussian random field $X(C)$ parameterized by a smooth convex contour (in R^2) that runs through a certain class $\mathbf{C}$ which is topologized by the usual method using the Euclidean metric. Denote by $W(u), u \in R^2$, a two-dimensional 2D parameter white noise. Let (C) denote the domain enclosed by the contour C.

Given a Gaussian random field $X(C)$ and assume that it is expressed as a stochastic integral of the form

$$X(C) = \int_{(C)} F(C,u) W(u) du, \tag{15.6}$$

where $F(C,u)$ be a kernel function which is locally square integrable in u. For convenience we assume that $F(C,u)$ is smooth in (C,u). The integral is a *causal* representation of the $X(C)$. The canonical property can be defined as a generalization to a random field as in the case of a Gaussian process.

The stochastic variational equation for this $X(C)$ is of the form

$$\delta X(C) = \int_C F(C,s) \delta n(s) W(s) ds + \int_{(C)} \delta F(C,u) W(u) du. \tag{15.7}$$

In a similar manner to the case of a process $X(t)$, but somewhat complicated manner, we can form the innovation $\{W(s), s \in C\}$.

Remark 15.2 A sample function of W is often denoted by x. It is a generalized function in E^* on which a white-noise measure μ is supported. With this notation we may write (15.6) as

$$X(C) = X(C,x) = \int_{(C)} F(C,u) x(u) du.$$

Example 15.1 A variational equation of Langevin type.

Given a stochastic variational equation

$$\delta X(C) = -X(C) \int_C k \delta n(s) ds + X_0 \int_C v(s) \partial_s^* \delta n(s) ds, C \in \mathbf{C},$$

where $\mathbf{C}$ is taken to be a class of concentric circles, v is a given continuous function, and ∂_s^* is the adjoint operator of the differential operator ∂_s.

Applying the equation the so-called S-transform, we can solve the transformed equation by appealing to the classical theory of functional analysis. Then, applying the inverse transform S^{-1}, the solution, with a trivial boundary condition, is given

$$X(C) = X_0 \int_{(C)} \exp[-k\rho(C,u)]\partial_u^* v(u)du,$$

where ρ denotes the Euclidean distance.

More general stochastic variational equations will be discussed in Section 15.4.

Now one may ask the integrability condition of a given stochastic variational equation. This question has been discussed by Si Si [15].

Another question concerning how to obtain the innovation from a random field may be discussed by referring to the literature [9].

15.3 General innovations and their functionals

Returning to the innovation $Y(t)$ of a process $X(t)$ one can see that, in favorable cases, there is an additive process $Z(t)$ such that its derivative $\dot{Z}(t)$ is equal to the $Y(t)$, since the collection $\{Y(t)\}$ is an independent system. There is tacitly assumed that, in the system, there is no random function singular in t.

There is the Lévy decomposition of an additive process. If $Z(t)$ has stationary independent increments, then except trivial component the $Z(t)$ involves a compound Poisson process and a Brownian motion up to constant. With this remark in mind we proceed to the Poisson case.

15.3.1

After Brownian motion comes another kind of elemental additive process which is to be the Poisson process denoted by $P(t), t \geq 0$. Taking its time derivative $\dot{P}(t)$ we have a *Poisson white noise*. It is a generalized stationary stochastic process with independent value at every point. For convenience we may assume that t runs through the whole real line. In fact, it is easy to define such a noise. The characteristic functional of the *centered* Poisson white noise is of the form

$$C_P(\xi) = \exp[\int_{-\infty}^{\infty} (e^{i\xi(t)} - 1 - i\xi(t))dt], \tag{15.8}$$

where $\xi \in E$.

There is the associated measure space (E^*, μ_P), and the Hilbert space $L^2(E^*, \mu_P) = (L^2)_P$ is defined.

Many results of the analysis on $(L^2)_P$ have been obtained; however, most of them have been studied by analogy with the Gaussian case or its modifications, as far as the construction of the space of generalized functionals. Here we only note that the $(L^2)_P$ admits the direct sum decomposition of the form

$$(L^2)_P = \bigoplus_n H_{P,n}.$$

The subspace is formed by the Poisson Charlier polynomials.

However, there might occur a misunderstanding regarding the functionals of Poisson noise, even in the case of linear functional. The following example would illustrate this fact (see [2], [8]).

Let a stochastic process $X(t)$ be given by the following integral (analogous to (15.3)):

$$X(t) = \int_0^t F(t,u)\dot{P}(u)du. \tag{15.9}$$

It seems to be simply a linear functional of $P(t)$; however, there are two ways of understanding the meaning of the integral (15.9); one is defined

1. In the Hilbert space by taking $\dot{P}(t)dt$ to be a random measure.

Another way is to define the integral

2. For each sample function of $P(t)$ (the pathwise integral). This can be done if the kernel is a smooth function of u over the interval $[0,t]$.

Assume that $F(t,t)$ never vanishes and that it is not a canonical kernel; that is, it is not a kernel function of an invertible integral operator. Then, we can claim that for the integral in the first sense $X(t)$ has less information compared to $P(t)$, because there is a linear function of $P(s), s \le t$ which is orthogonal to $X(s), s \le t$. On the other hand, if $X(t)$ is defined in the second sense, we can prove the following.

Proposition 15.1 (c.f. [8]) Under the assumptions stated above, if the $X(t)$ above is defined sample function-wise, we have the following equality for sigmafields:

$$\mathbf{B}_t(X) = \mathbf{B}_t(P), t \ge 0.$$

Proof By assumption it is easy to see that $X(t)$ and $P(t)$ share the jump points, which means the information is fully transferred from $P(t)$ to $X(t)$. This proves the equality.

The above argument tells us that we are led to introduce a space $(\mathbf{P})$ of random variables that come from separable stochastic processes for which existence of variance is not expected. This sounds to be a vague statement; however, we can rigorously define it by using a Lebesgue space without atoms, and others. There the topology is defined by either the almost sure convergent or the convergence in probability, and there is no need to think of mean square topology. On the space $(\mathbf{P})$, filtering and prediction for strictly stationary process can naturally be discussed. For further ideas we may refer to the literature [18], where one can see further profound ideas of N. Wiener.

It is almost straightforward to come to an introduction of a multiparameter Poisson (white) noise (see [16]), denoted by $\{V(u)\}$, which is the generalization of $\{\dot{P}(t)\}$.

Theorem 15.2 *Let a random field $X(C)$ parameterized by a contour C be given by a stochastic integral*

$$X(C) = \int_{(C)} G(C,u)V(u)du, \tag{15.10}$$

where the kernel $G(C,u)$ is continuous in (C,u). Assume that $G(C,s)$ never vanishes on C for every C. Then, the $V(u)$ is the innovation.

Proof The variation $\delta X(C)$ of $X(C)$ in the equation (15.10) exists and we can easily prove that it involves a term of the form

$$\int_C G(C,s)\delta n(s)V(s)ds,$$

where $\{\delta n(s)\}$ determines the variation δC of C. Here the same technique is used as in the case of [9], so that the values $V(s), s \in C$, are determined by taking various δC's. This shows that the $V(s)$ is obtained by the $X(C)$ according to the infinitesimal change of C. Hence $V(s)$ is the innovation.

Here is an important remark. In the Poisson case one can see a significant difference on getting the innovation from the case of a representation of a Gaussian process. However, if one is permitted to use some nonlinear operations acting on sample functions, it is possible to form the innovation from a noncanonical representation of a Gaussian process (Si Si [16]), although the proof needs a profound property of a Brownian motion (see P. Lévy [11, Chapt. VI]).

15.3.2 Compound Poisson processes

As soon as we come to a compound Poisson process, which is a more general innovation, the second order moment may not exist, so that we have to come to the space $(\mathbf{P})$. The Lévy decomposition of an additive process, with which we are now concerned, is expressed in the form

$$Z(t) = \int (uP_{du}(t) - \frac{tu}{1+u^2}dn(u)) + \sigma B(t), \tag{15.11}$$

where $P_{du}(t)$ is a random measure of the set of Poisson processes, and where $dn(u)$ is the Lévy measure such that

$$\int \frac{u^2}{1+u^2}dn(u) < \infty.$$

The decomposition of a compound Poisson process into the individual elemental Poisson processes with different scales of jump can be carried out in the space $(\mathbf{P})$ with the use of the quasi-convergence (see [11, Chapt.V]) . We are now ready to discuss the analysis acting on sample functions of a compound Poisson process.

A generalization of the Proposition 15.1 in the last subsection to the case of compound Poisson white noise is not difficult in a formal way without paying much attention. However, we wish to pause at this moment to consider carefully about how to find a jump point of $Z(t)$ with scale u designated in advance. This question is heavily depending on the computability or measurement problem. Questions related to this problem shall be discussed in the separate paper.

15.4 Stochastic variational equations

We are going to discuss a stochastic vaiational equation of the form

$$\delta X(C) = \int_C X'(C,s)\delta n(s)ds. \tag{15.12}$$

Applying the S-transform, it is represented as

$$\delta U(C) = \int_C U'(C,s)\delta n(s)ds. \tag{15.13}$$

Letting C be represented by a function ξ, we are given

$$\delta U = \int_I f(\xi, U, s)\delta\xi(s)ds, \tag{15.14}$$

where ξ is a function defined on $I = [0, 1]$ and is a member of some suitable space $E \subset L^2(I)$.

Assume that

A.1) $f(\xi, U, s)$ is continuous in $(\xi, U, s) \in E \times R \times I$, and satisfies the integrability condition.

A.2) In some neighborhood of U_0, e.g., $|U - U_0| < C, |V - V_0| < C$, there exist constants M and K such that

(i) $\int_I f(\xi, U, s)^2 ds < M^2$.

(ii) $\int_I |f(\xi, U, s) - f(\xi, V, s)|^2 ds < K^2 |U - V|^2$.

A.3) We may write

$$\xi = \xi_0 + \lambda\eta, \ \ \|\eta\| = 1, \ \ \lambda > 0.$$

Then the solution of the variational equation exists and is unique.

For details of the proof see the literature [10., Chapt. VII].

Example 15.2 Consider a Gaussian 1-ple Markov random field expressed in the form (15.6) (see [10]). Then, the kernel is degenerated

$$F(C, u) = f(C)g(u),$$

where f never vanishes and g is locally square integrable. We assume that $g(u) = g(r, \theta), u = (r, \theta)$, is in the Sobolev space of order 2 and never vanishes. This assumption guarantees the existence of white noise parameterized by a point of an ovaloid. Set

$$Y(C) = \int_{(C)} g(u)W(u)du.$$

Then, it is a martingale. The S-transform of $Y(C)$ is of the form

$$U(C) = \int_{(C)} g(u)\xi(u)du.$$

Its variation is

$$\delta U(C) = \int_C g(s)\delta\xi(s)ds.$$

If we are allowed to apply automorphisms $C \to C$, then we should be given the innovation $g(s)x(s), s \in C$, (see the proof of Theorem 15.3), although it is not homogeneous. Obviously the variational equation can be solved and it is $Y(C)$ up to a random variable independent of C.

15.5 Reversible fields

There is a hope that the discussion in the previous section would be generalized to the case of a field parameterized by a plane curve C. As a first step we have established the following result (see [8]).

Let $x(u), u \in R^2$, denote a sample function of a 2D parameter white noise $W(u)$, and let $\{C_r, r > 0\}$ be a family of concentric circles. Denote by (C) the disc enclosed by C. Now we define a Gaussian random field $X(C_r)$ by

$$X(C_r) = f(r) \int_{(C_r)-(C_0)} g(u)x(u)du, \quad C_0 = C_{r_0}, \tag{15.15}$$

where

$$f(r) = \sqrt{\frac{(r_1^2 - r_0^2)(r_1^2 - r^2)}{r^2 - r_0^2}}$$

and

$$g(u) = (r_1^2 - |u|^2)^{-1}.$$

It is easy to prove that the covariance function $\Gamma(s,t) = E[X(C_s)X(C_t)]$ of $X(C_r)$ is given by

$$\pi\sqrt{(s^2, t^2; r_0^2, r_1^2)}. \tag{15.16}$$

Theorem 15.3 *The representation of $X(C_r)$ given above is a canonical representation in the sense that*

$$\mathbf{B}_t(U_r(X(C_r)), r \le t) = \mathbf{B}_t(W(u), |u| \le t), \tag{15.17}$$

where U_r is a unitary operator such that for $\varphi(x) \in (L^2), x \in E^*(\mu)$

$$U_r\varphi(x) = \varphi(g_r^* x), \quad g_r \in O(E), g_r\xi(r,\theta) = \xi(r, f(\theta))\sqrt{|f'(\theta)|}.$$

Proof is easy. Indeed, take the variation $\delta X(C_r)$, which is equal to $d_r X(C_r)$. It is expressed as a stochastic integral over S^1: if we note that $g\xi(r,\theta)$ with $f \in Diff(S^1)$ defining a member g of $O(E)$ spans a dense subset of $L^2(S^1)$ for any fixed r. In fact, those g's form a subgroup of $O(E)$ and the subgroup has an irreducible unitary representation on $L^2(S^1)$. These facts easily prove the theorem.

The variation of $X(C_r)$ is actually differential in the variable r, and we can discuss the martingale $X(C_r)/f(r)$. It is noted that we have to be careful about its behavior near C_{r_1}.

Proposition 15.2 Apply to $x(u)$ a conformal transformation

$$x(u) \longrightarrow x\left(\frac{r_0 r_1}{|u|^2} u\right)\frac{r_0 r_1}{|u|^2}. \tag{15.18}$$

Then, we have the same (in distribution) Gaussian random field.

Remark 15.3 Let $X(C_r)$ be the same as above. Take a diffeomorphism $\tilde{g}$ which defines a rotation g in the infinite dimensional rotation group $O(E)$. Then, the same conformal invariance holds for the field $\{X(\tilde{g}C_r)\}$. Hence, the system $\{X(gC_r)\}$ may be considered as a probabilistic expression of the fluctuation of a classical trajectory with variable C.

Remark 15.4 Various topics related to the variation of the random fields like $X(C_r)$ are discussed in the forthcoming *Lecture Notes* by the author.

15.6 Concluding remarks

1. Under the same idea that has been discussed so far, we can come to the Chern–Simon–Witten action integral. There we shall use a generalization of the Brownian bridge whose

value is a higher (actually two-) dimensional parameter. Consider a simple case, satisfying gauge invariant and abelian, etc. where the action is of the form

$$CS(A) = \frac{k}{4\pi} \int_M A \wedge dA,$$

where A is a 1-form expressed as

$$A = a_0 dx_0 + a_1 dx_1 + a_2 dx_2$$

a_i being Lie algebra-valued functions on R^3. Thus the Feynman path integral has the form

$$\frac{1}{Z} \int_{\mathcal{A}} \exp[iCS(A)]\phi(A)\mathcal{D}A.$$

The question is now to give a plausible mathematical definition of $\mathcal{D}A$. This problem has now developed to be an interesting and significant subject of stochastic analysis.

2. The next topic to be mentioned is an application of stochastic variational equations to quantum field theory aiming at the study of the Tomonaga–Schwinger equation, although we are far from the actual discussion at present. We should like to refer to groundbreaking article [17] by Tomonaga, where one can see suggestions on the integration of functionals of a manifold.

There is an additional remark as follows:

3. A Brownian motion and each Poisson process with a fixed scale of jump that is a component of a compound Poisson process seem to be elemental. Indeed, this is true in a sense. On the other hand, there are another aspects. For instance, we know that the inverse function of the maximum of a Brownian motion is a stable process, which is a compound Poisson process (see [11, Chapt. VI]). A Poisson process comes from a Brownian motion, certainly not by the L^2 method.

References

[1] L. Accardi et al. ed., Selected papers of Takeyuki Hida. World Scientific Pub. Co. Singapore. 2001.

[2] L. Accardi, T. Hida and Si Si, Innovation approach to some stochastic processes. Volterra Center Notes N. 537, 2002.

[3] T. Hida, Stationary stochastic processes. Princeton Univ. Press. Princeton, NJ. 1970.

[4] T. Hida, Analysis of Brownian functionals. Carleton Univ. Math. Notes, no. 13, 1975.

[5] T. Hida, Brownian motion. Springer-Verlag. Berlin. 1980.

[6] T. Hida et al., White Noise. An infinite dimensional calculus. Kluwer Academic Pub. Co. Dordrecht. 1993.

[7] T. Hida, White Noise Analysis: A New Frontier. Volterra Center Notes. no. 499 January 2002.

[8] T. Hida and N. Ikeda, Note on linear processes. J. Math. Kyoto Univ. 1 (1961) 75-86.

[9] T. Hida and Si Si, Innovations for random fields. Infinite Dimensional Analysis, Quantum Probab. Related Top. 1, no. 4 (1998), 499-509.

[10] T. Hida and Si Si, An innovation approach to random field: Application of white noise theory. World Scientific Pub. Co. River Edge, NJ. 2004.

[11] P. Lévy, Processus stochastiques et mouvement brownien. Gauthier-Villars. Paris. 1948; 2ème éd. 1965.

[12] P. Lévy, Random functions: General theory with special reference to Laplacian random functions. Univ. of California Publications in Statistics. I no. 12 (1953), 331-388.

[13] P. Lévy, A special problem of Brownian motion, and a general theory of Gaussian random functions. Proceedings of the third Berkeley Symposium on Mathematical Statistics and Probability, vol. II, (1956), 133-175.

[14] P. Lévy, Fonction aléatoires à corrélation linéaire. Illinois J. of Math. 1 no. 2 (1957), 217-258.

[15] Si Si, Integrability condition for stochastic variational equation. Volterra Center Pub. N. 217, Univ. di Roma Tor Vergata, Roma, 1995.

[16] Si Si, Effective determination of Poisson noise. Infinite Dimensional Analysis, Quantum Probab. Related Top. 6 (2003), 609-617.

[17] S. Tomonaga, On a relativistically invariant formulation of the quantum theory of wave field. Progr. Theor. Phys., 1 (1946), 27-42.

[18] N. Wiener, Extrapolation, interpolation and smoothing of stationary time series. The MIT Press, Cambridge, MA. 1949.

16 On the Foundation of the L_p-Theory of Stochastic Partial Differential Equations

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16.1 Introduction

The goal of this chapter is to give complete proofs of two related results from [Kr94a] and [Kr94b], which lie in the foundation of the L_p-theory of stochastic partial differential equations (SPDEs) originated about 1994. By now the literature related to this theory is quite impressive (see, for instance, recent articles [Kim], [KK], [Lo], [MR], and the references therein). At the same time the proofs in [Kr94a] and [Kr94b] are rather far from being complete. As a matter of fact recently the author went back to the proofs in [Kr94a] and [Kr94b] and noticed errors but just could not see how the arguments yield the desired result. These arguments are based on a parabolic version of Stampacchia's interpolation theorem and in the present chapter we also use a few tools from real analysis. However, unlike in [Kr94a] and [Kr94b], we prove all results from that theory which we need. This is done only for the sake of completeness, so that the reader can see that whatever is needed is quite simple and readily available. Furthermore, while discussing the auxiliary results we use the language of probability theory in order to make the chapter closer to probabilists. For deeper and somewhat more involved exposition also tilted to probability theory of similar and other powerful results from real analysis, we refer the reader to [Ga] and [St].

The author is sincerely grateful to Hongjie Dong who kindly indicated a few errors in the first version of the chapter. The work was partially supported by NSF Grant DMS-0140405.

16.2 Main result

Our motivation is as follows. Consider the simplest one-dimensional (1D) SPDE

$$du(t,x) = \frac{1}{2}u_{xx}(t,x)\,dt + g(t,x)\,dw_t \quad t>0, \quad u(0,x)=0,$$

where w_t is a 1D Wiener process. Naturally, the solution of this problem should be

$$u(t,x) = \int_0^t T_{t-s}g(s,\cdot)(x)\,dw_s,$$

where $T_t h(x) = Eh(x+w_t)$. If g is nonrandom and by Du we denote the derivative of u with respect to x, then the above integral is Gaussian and

$$E\int_0^T \|Du(t,\cdot)\|_{L_p}^p\,dt = N(p)\int_0^T \int_{\mathbb{R}}[\int_0^t |DT_{t-s}g(s,\cdot)(x)|^2\,ds]^{p/2}\,dx\,dt.$$

Hence, in order to prove that $Du\in L_p$ and thus get the L_p-theory a legitimate start, we have to estimate the right-hand side.

Observe that

$$DT_t h(x) = t^{-d/2}\phi(x/\sqrt{t}) * h(x),$$

where

$$\phi(x) = \frac{1}{(2\pi)^{d/2}} De^{-|x|^2/2} = -\frac{x}{(2\pi)^{d/2}} e^{-|x|^2/2}.$$

It is somewhat more convenient to consider convolutions with slightly more general functions. Fix a constant $K \in (0,\infty)$ and let $\psi(x)$ be a $C^1(\mathbb{R}^d)$ integrable function such that

$$\int_{\mathbb{R}^d} \psi\,dx = 0, \quad \int_{\mathbb{R}^d} \big(|\psi(x)| + |\nabla\psi(x)| + |x|\,|\psi(x)|\big)\,dx \le K,$$

Introduce

$$2\hat{\psi}(x) = \psi(x)d + (x, \nabla\psi(x))$$

and assume that there exists a continuously differentiable function $\bar{\psi}$ defined on $[0,\infty)$ such that

$$|\psi(x)| + |\nabla\psi(x)| + |\hat{\psi}(x)| \le \bar{\psi}(|x|), \quad \int_0^\infty |\bar{\psi}'(\rho)|\,d\rho \le K,$$

$$\bar{\psi}(\infty) = 0, \quad \int_r^\infty |\bar{\psi}'(\rho)|\rho^d\,d\rho \le K/r, \quad \forall r \ge 1.$$

Note that ϕ satisfies this assumptions with some K. Define

$$\Psi_t h(x) := t^{-d/2}\psi(x/\sqrt{t}\,) * h(x).$$

so that the above operator DT_t is a particular case of Ψ.

The classical Littlewood–Paley inequality (see, for instance, Chapter 1 in [Ste]) says that for any $p \in (1,\infty)$ and $f \in L_p$ it holds that

$$\int_{\mathbb{R}^d} \left[\int_0^\infty |\Psi_t f(x)|^2\,\frac{dt}{t}\right]^{p/2} dx \le N\|f\|_p^p,$$

where the constant N depends only on d, p.

Here we want to generalize this fact by proving the following result in which H is a Hilbert space, $\mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d\}$. For $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$, $t > a \ge -\infty$, and $x \in \mathbb{R}^d$ we set

$$\mathcal{G}_a f(t,x) = \Big[\int_a^t |\Psi_{t-s} f(s,\cdot)(x)|_H^2\,\frac{ds}{t-s}\Big]^{1/2}, \quad \mathcal{G} = \mathcal{G}_{-\infty}.$$

Theorem 16.1 *Let $p \in [2,\infty)$, $-\infty \le a < b \le \infty$*

$$f \in C_0^\infty((a,b)\times\mathbb{R}^d, H).$$

Then

$$\int_{\mathbb{R}^d}\int_a^b [\mathcal{G}_a f(t,x)]^p\,dt\,dx \le N\int_{\mathbb{R}^d}\int_a^b |f(t,x)|_H^p\,dt\,dx, \tag{16.1}$$

where the constant N depends only on d, p, and K.

The proof of this theorem is given in Section 16.6, after we prove some elementary properties of partitions in Section 16.3, prove deep albeit simple Fefferman–Stein theorem in Section 16.4 and study a few properties of the operator $\mathcal{G}$ in Section 16.5.

16.3 Partitions

Let F be a Banach space. For a domain $\Omega \subset \mathbb{R}^d$, by $L_p(\Omega, F)$ we denote the closure of the set of F-valued continuous functions compactly supported on Ω with respect to the norm $\|\cdot\|_{L_p(\Omega,F)}$ defined by

$$\|u\|^p_{L_p(\Omega,F)} = \int_\Omega |u(x)|^p_F \, dx.$$

We also stipulate that $L_p(\Omega) = L_p(\Omega, \mathbb{R})$, $L_p = L_p(\mathbb{R}^d)$. By $|\Omega|$ we denote the volume of Ω.

Definition 16.1 *Let $\mathbb{Z} = \{n : n = 0, \pm 1, \pm 2, ...\}$ and let $(\mathbb{Q}_n, n \in \mathbb{Z})$ be a sequence of partitions of $\mathbb{R}^d$ each consisting of disjoint bounded Borel subsets $Q \in \mathbb{Q}_n$. We call it a filtration of partitions if*

(i) The partitions become finer as n increases:

$$\inf_{Q\in\mathbb{Q}_n} |Q| \to \infty \quad \text{as} \quad n \to -\infty, \qquad \sup_{Q\in\mathbb{Q}_n} \operatorname{diam} Q \to 0 \quad \text{as} \quad n \to \infty.$$

(ii) The partitions are nested: for each n and $Q \in \mathbb{Q}_n$ there is a (unique) $Q' \in \mathbb{Q}_{n-1}$ such that $Q \subset Q'$.

(iii) The following regularity property holds: for Q and Q' as in (ii) we have

$$|Q'| \le N_0 |Q|,$$

where N_0 is a constant independent of n, Q, Q'.

Example 16.1 In the applications in this chapter we will be dealing with the filtration of parabolic dyadic cubes in

$$\mathbb{R}^{d+1} = \{(t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d\},$$

defined by

$$\mathbb{Q}_n = \{Q_n(i_0, i_1, ..., i_d), i_0, i_1, ..., i_d \in \mathbb{Z}\},$$

$$Q_n(i_0, i_1, ..., i_d) = [i_0 4^{-n}, (i_0 + 1)4^{-n}) \times Q_n(i_1, ..., i_d), \tag{16.2}$$

$$Q_n(i_1, ..., i_d) = [i_1 2^{-n}, (i_1 + 1)2^{-n}) \times \cdots \times [i_d 2^{-n}, (i_d + 1)2^{-n}). \tag{16.3}$$

Definition 16.2 *Let $\mathbb{Q}_n$, $n \in \mathbb{Z}$, be a filtration of partitions of $\mathbb{R}^d$.*

(i) Let $\tau = \tau(x)$ be a function on $\mathbb{R}^d$ with values in $\{\infty, 0, \pm 1, \pm 2, ...\}$. We call τ a stopping time *(relative to the filtration) if, for each $n \in \mathbb{Z}$, the set $\{x : \tau(x) = n\}$ is the union of some elements of $\mathbb{Q}_n$.*

(ii) For any $x \in \mathbb{R}^d$ and $n \in \mathbb{Z}$, by $Q_n(x)$ we denote the (unique) $Q \in \mathbb{Q}_n$ containing x.

(iii) For a function $f \in L_{1,loc}(\mathbb{R}^d, F)$ and $n \in \mathbb{Z}$, we denote

$$f_{|n}(x) = \fint_{Q_n(x)} f(y)\, dy \qquad \Big(\fint_\Gamma f\, dx = \frac{1}{|\Gamma|} \int_\Gamma f\, dx \quad \Big).$$

If we are also given a stopping time τ, we let $f_{|\tau}(x) = f_{|\tau(x)}(x)$ for those x for which $\tau(x) < \infty$ and $f_{|\tau}(x) = f(x)$ otherwise.

Remark 16.1 It is easy to see that in the case of real-valued functions $f \in L_2$, for each n, $f_{|n}$ provides the best approximation in L_2 of f by functions that are constant on each element of $\mathbb{Q}_n$.

We are going to use the following simple and well-known properties of the objects introduced above.

Lemma 16.1 *Let $\mathbb{Q}_n$, $n \in \mathbb{Z}$, be a filtration of partitions of $\mathbb{R}^d$.*

(i) Let $p \in [1,\infty)$, $f \in L_{1,loc}(\mathbb{R}^d, F)$, and let τ be a stopping time. Then

$$\int_{\mathbb{R}^d} |f_{|\tau}(x)|_F^p I_{\tau<\infty}\,dx \le \int_{\mathbb{R}^d} |f(x)|_F^p I_{\tau<\infty}\,dx. \tag{16.4}$$

In addition, (16.4) *becomes an equality if $f \ge 0$ and $p = 1$.*

(ii) Let $g \in L_1$, $g \ge 0$, and $\lambda > 0$. Then

$$\tau(x) := \inf\{n : g_{|n}(x) > \lambda\} \quad (\inf \emptyset := \infty) \tag{16.5}$$

is a stopping time. Furthermore, we have

$$0 \le g_{|\tau}(x) I_{\tau<\infty} \le N_0\lambda, \quad |\{x : \tau(x) < \infty\}| \le \lambda^{-1} \int_{\mathbb{R}^d} g(x) I_{\tau<\infty}\,dx. \tag{16.6}$$

Proof (i) By Hölder's inequality $|f_{|n}|_F^p \le (|f|_F^p)_{|n}$. Therefore, we may concentrate on $p = 1$ and real-valued nonnegative f. In that case notice that, for any n and set Γ which is the union of some elements $Q_i \in \mathbb{Q}_n$, obviously

$$\int_\Gamma f_{|n}\,dx = \sum_i \int_{Q_i} f_{|n}\,dx = \sum_i \int_{Q_i} f\,dx = \int_\Gamma f\,dx.$$

Hence

$$\int_{\mathbb{R}^d} f_{|\tau} I_{\tau<\infty}\,dx = \sum_{n=-\infty}^{\infty} \int_{\tau=n} f_n\,dx = \sum_{n=-\infty}^{\infty} \int_{\tau=n} f\,dx = \int_{\mathbb{R}^d} f I_{\tau<\infty}\,dx.$$

(ii) First notice that $\tau > -\infty$ since $g_{|n} \to 0$ as $n \to -\infty$ due to $g \in L_1$. Next, observe that

$$Q_n(x) \subset Q_m(x)$$

for all $m \le n$ since the partitions are nested. It follows that, if $y \in Q_n(x)$, then

$$Q_m(y) = Q_m(x), \quad g_{|m}(y) = g_{|m}(x), \quad \forall m \le n.$$

By adding that

$$\tau(x) = n \Longleftrightarrow g_{|n}(x) > \lambda, \quad g_{|m}(x) \le \lambda \quad \forall m < n,$$

we conclude that the set $\{\tau = n\}$ contains $Q_n(x)$ along with each x. Therefore, $\{\tau = n\}$ is indeed the union of some elements of $\mathbb{Q}_n$.

To prove the first relation in (16.6), it suffices to notice that, if $\tau(x) = n$, then

$$g_{|\tau}(x) = g_{|n}(x) = \frac{1}{|Q_n(x)|} \int_{Q_n(x)} g(y)\,dy \le N_0 \frac{1}{|Q_{n-1}(x)|} \int_{Q_n(x)} g(y)\,dy$$

$$\le N_0 \frac{1}{|Q_{n-1}(x)|} \int_{Q_{n-1}(x)} g(y)\,dy = N_0 g_{|n-1}(x) \le N_0\lambda.$$

The second inequality in (16.6) follows from Chebyshev's inequality and (i):

$$|\{x : \tau < \infty\}| = |\{x : g_{|\tau} I_{\tau<\infty} > \lambda\}|$$

$$\le \lambda^{-1} \int_{\mathbb{R}^d} g_{|\tau} I_{\tau<\infty}\,dx = \lambda^{-1} \int_{\mathbb{R}^d} g I_{\tau<\infty}\,dx.$$

The lemma is proved.

16.4 Maximal and sharp functions

Having proved Lemma 16.1 we derive the following:

Corollary 16.1 (maximal inequality) *Let $f \in L_{1,loc}(\mathbb{R}^d, F)$. Define the (filtering) maximal function of f by*

$$Mf(x) = \sup_{n<\infty} (|f|_F)_{|n}(x),$$

so that $Mf = M|f|_F$. Then, for nonnegative $g \in L_1$, the maximal inequality holds

$$|\{x : Mg(x) > \lambda\}| \leq \lambda^{-1} \int_{\mathbb{R}^d} g(x) I_{Mg>\lambda}\, dx, \quad \forall \lambda > 0. \tag{16.7}$$

Indeed, for τ as in (16.5), we have $\{x : Mg(x) > \lambda\} = \{x : \tau(x) < \infty\}$.

Remark 16.2 Our interest in estimating $|\{Mg > \lambda\}|$ as in Corollary 16.1 is based on the following formula valid for any $f \geq 0$:

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty |\{x : f(x) > t\}|\, dt. \tag{16.8}$$

Corollary 16.2 *Let $p \in (1, \infty)$, $g \in L_1$, $g \geq 0$. Then*

$$\|Mg\|_{L_p} \leq q \|g\|_{L_p},$$

where $q = p/(p-1)$.

Indeed, from (16.8), (16.7), and Fubini's theorem we conclude that, for any finite constant $\nu > 0$,

$$\|\nu \wedge Mg\|_{L_p}^p = \int_0^\infty |\{x : \nu \wedge Mg(x) > \lambda^{1/p}\}|\, d\lambda$$

$$= \int_0^{\nu^p} |\{x : Mg(x) > \lambda^{1/p}\}|\, d\lambda \leq \int_{\mathbb{R}^d} g\Big(\int_0^{\nu^p} \lambda^{-1/p} I_{Mg>\lambda^{1/p}}\, d\lambda \Big)\, dx$$

$$= \int_{\mathbb{R}^d} g\Big(\int_0^{(\nu\wedge Mg)^p} \lambda^{-1/p}\, d\lambda \Big)\, dx = q \int_{\mathbb{R}^d} g(\nu \wedge Mg)^{p-1}\, dx.$$

This and $g \in L_1$ imply that $\|\nu \wedge Mg\|_{L_p} < \infty$. Then upon using Hölder's inequality we get

$$\|\nu \wedge Mg\|_{L_p}^p \leq q \|g\|_{L_p} \|\nu \wedge Mg\|_{L_p}^{p-1}, \quad \|\nu \wedge Mg\|_{L_p} \leq q \|g\|_{L_p}$$

and it only remains to let $\nu \to \infty$ and use Fatou's theorem.

Theorem 16.2 *For any $p \in (1, \infty)$ and $g \in L_p(\mathbb{R}^d, F)$*

$$\|Mg\|_{L_p} \leq q \|g\|_{L_p(\mathbb{R}^d, F)}.$$

Proof Since

$$Mg = M|g|_F \quad \text{and} \quad \|g\|_{L_p(\mathbb{R}^d,F)} = \|\, |g|_F \,\|_{L_p}$$

we may concentrate on real-valued $g \in L_p$, $g \geq 0$. For $r > 0$ define $g^r(x) = g(x) I_{|x|\leq r}$. Then $g^r \in L_1$ and

$$\|Mg^r\|_{L_p} \leq q \|g^r\|_{L_p} \leq q \|g\|_{L_p}$$

by Corollary 16.2. It only remains to use Fatou's theorem along with the observation that for any x since $Q_n(x)$ is bounded, we have

$$(g^r)_{|n}(x) \to g_{|n}(x) \quad \text{as} \quad r \to \infty,$$

which implies

$$g_{|n}(x) \le \underline{\lim}_{r\to\infty} \sup_m (g^r)_{|m}(x), \quad Mg \le \underline{\lim}_{r\to\infty} Mg^r.$$

The theorem is proved.

Let $f \in L_{1,loc}(\mathbb{R}^d, F)$. Define the sharp function of f by

$$f^{\#}(x) = \sup_{n<\infty} \fint_{Q_n(x)} |f(y) - f_{|n}(y)|_F \, dy.$$

Obviously, $f^{\#}(x) \le 2Mf(x)$. It turns out that f and hence Mf are also controlled by $f^{\#}$.

Lemma 16.2 *For $\alpha = (2N_0)^{-1}$, any constant $c > 0$, and $f \in L_1(\mathbb{R}^d, F)$, we have*

$$|\{x : |f(x)|_F \ge c\}| \le \frac{2}{c} \int_{\mathbb{R}^d} I_{Mf(x)>\alpha c} f^{\#}(x)\, dx.$$

Proof Define $g = |f|_F$ and

$$\tau(x) = \inf\{n : g_{|n}(x) > c\alpha\}.$$

Observe that

$$g_{|n}(x) \ge |f_{|n}(x)|_F, \quad |f(x)|_F - |f_{|n}(x)|_F \le |f(x) - f_{|n}(x)|_F.$$

Also use Lemma 16.1 (ii) and the fact that along a subsequence $n' \to \infty$, we have $f_{|n'} \to f$ almost everywhere (a.e.). Then we find that (a.e.)

$$\{x : |f(x)|_F \ge c\} = \{x : |f(x)|_F \ge c, \tau(x) < \infty\}$$
$$= \{x : |f(x)|_F \ge c, \tau(x) < \infty, g_{|\tau}(x) \le c/2\}$$
$$\subset \{x : \tau(x) < \infty, |f(x) - f_{|\tau}(x)|_F \ge c/2\} =: A.$$

Next, represent the set $\{\tau < \infty\}$ as the union $\bigcup_{n,k} Q_{nk}$ of disjoint Q_{nk}, satisfying $Q_{nk} \in \mathbb{Q}_n$ and $\tau = n$ on Q_{nk} for each n, k, and use Chebyshev's inequality to find

$$|A| \le (2/c) \int_{\mathbb{R}^d} I_{\tau(x)<\infty} |f(x) - f_{|\tau}(x)|_F \, dx$$
$$= (2/c) \sum_{n,k} \int_{Q_{nk}} |f(x) - f_{|n}(x)|_F \, dx$$
$$= (2/c) \sum_{n,k} \int_{Q_{nk}} \Big(\fint_{Q_n(z)} |f(x) - f_{|n}(x)|_F \, dx \Big) \, dz$$
$$\le (2/c) \sum_{n,k} \int_{Q_{nk}} f^{\#}(z)\, dz = (2/c) \int_{\mathbb{R}^d} I_{\tau(z)<\infty} f^{\#}(z)\, dz.$$

Now it only remains to notice that $\{\tau(x) < \infty\} = \{Mf(x) > c\alpha\}$. The lemma is proved.

Theorem 16.3 (Fefferman–Stein) *Let $p \in (1, \infty)$. Then for any $f \in L_p(\mathbb{R}^d, F)$ we have*

$$\|f\|_{L_p(\mathbb{R}^d,F)} \le N \|f^{\#}\|_{L_p},$$

where $N = (2q)^p N_0^{p-1}$.

Proof As in the proof of Corollary 16.2 we get from Lemma 16.2 that if $f \in L_1(\mathbb{R}^d, F)$, then

$$\|f\|^p_{L_p(\mathbb{R}^d,F)} \le N \int_{\mathbb{R}^d} f^{\#}(Mf)^{p-1}\,dx \le N\|f^{\#}\|_{L_p}\|Mf\|^{p-1}_{L_p}.$$

If in addition $f \in L_p(\mathbb{R}^d, F)$, then it only remains to use Theorem 16.2 and check that the resulting constant is right.

If we only have $f \in L_p(\mathbb{R}^d, F)$, then it suffices to take $f_n \in L_1(\mathbb{R}^d, F)$ converging to f in $L_p(\mathbb{R}^d, F)$ and observe that $f_n^{\#} \le (f - f_n)^{\#} + f^{\#}$ and

$$\|(f - f_n)^{\#}\|_{L_p} \le 2\|M(f - f_n)\|_{L_p} \le 2q\|f - f_n\|_{L_p(\mathbb{R}^d,F)} \to 0.$$

The theorem is proved.

Remark 16.3 By Hölder's inequality, for any $p \in [1, \infty]$

$$f^{\#}(x) \le \sup_{n<\infty} \Big(\fint_{Q_n(x)} |f(y) - f_{|n}(y)|^p_F\,dy \Big)^{1/p}.$$

The maximal function introduced in Corollary 16.1 is related to the underlying filtration of partitions. Below we are also using the following more traditional maximal function:

$$\mathbb{M}g(x) = \sup_{r>0} \fint_{B_r(x)} |g(y)|\,dy, \tag{16.9}$$

where $B_r(x)$ is the open ball of radius r centered at x. Let Mg be the maximal function associated with the filtration of dyadic cubes Q_n introduced in (16.3). It turns out that, in a sense, Mg and $\mathbb{M}g$ are comparable.

First, since $Q_n(x) \subset B_{r_n}(x)$ with $r_n = 2^{-n}\sqrt{d}$, we have $|B_{r_n}(x)| = N(d)|Q_n(x)|$,

$$\fint_{Q_n(x)} |g|\,dy \le \frac{|B_{r_n}(x)|}{|Q_n(x)|} \fint_{B_{r_n}(x)} |g|\,dy \le N(d)\mathbb{M}g(x),$$

and $Mg \le N\mathbb{M}g$.

On the other hand, we have the following.

Lemma 16.3 *There is a constant $N = N(d)$ such that if $g \in L_1$, then for any $\lambda > 0$*

$$|\{x : \mathbb{M}g(x) > N\lambda\}| \le N|\{x : Mg(x) > \lambda\}|. \tag{16.10}$$

Proof Without losing generality we may assume that $g \ge 0$. If x_0 and $\lambda > 0$ are such that $\mathbb{M}g(x_0) > \lambda$, then for an $r > 0$ we have

$$\int_{B_r(x_0)} g(y)\,dy > \lambda|B_r(x_0)|. \tag{16.11}$$

Set

$$n = -[\log_2 r], \quad i_k = [x^k 2^n], \quad \bar{x}_0 = (i_1 2^{-n}, ..., i_d 2^{-n}).$$

Then $2^{-n} \le r < 2^{-n+1}$, $|x_0^k - \bar{x}_0^k| < 2^{-n} \le r$, and $B_r(x_0)$ is covered by the union of 2^d dyadic cubes each of which has $\bar{x}_0$ as one of its vertices and they are taken from the family $\mathbb{Q}_{n-2}$. Owing to (16.11) the integral of g over at least one of these cubes Q is greater than

$$\lambda 2^{-d}|B_r(x_0)| = N\lambda r^d \ge N_*\lambda|Q|.$$

Furthermore, it is not hard to see that $x_0 \in 2Q$, where by $2Q$ we mean the twice dilated Q with the center of dilation being that of Q.

Now define $\tau(x) = \inf\{n : g_{|n}(x) > N_*\lambda\}$. Then $\tau \le n-2$ on $Q \in \mathbb{Q}_{n-2}$. Actually, it may happen that $\tau = m < n-2$ on Q. In that case $Q \subset Q' \in \mathbb{Q}_m$ and $\tau = m$ on Q'. Since $x_0 \in 2Q'$, we conclude that x_0 is in the union over j and m of twice dilated dyadic cubes Q_{jm} from the family $\mathbb{Q}_m$ composing $\{x : \tau(x) = m\}$. Hence,

$$|\{x : \mathbb{M}g(x) > \lambda\}| \le 2^d \sum_m \sum_j |Q_{jm}|$$

$$= 2^d |\{x : \tau(x) < \infty\}| = 2^d |\{x : Mg(x) > N_*\lambda\}|.$$

This proves the lemma.

Here is the classical maximal function estimate.

Theorem 16.4 *Let $p \in (1, \infty)$ and $g \in L_p$. Then $\mathbb{M}g \in L_p$ and*

$$\|\mathbb{M}g\|_{L_p} \le N\|g\|_{L_p}, \tag{16.12}$$

where N is independent of g.

Proof Without losing generality we assume that $g \ge 0$. If $g \in L_1$, then (16.12) is obtained by replacing λ with $\lambda^{1/p}$ in (16.10), integrating with respect to λ, remembering (16.8), and using Corollary 16.2.

If the additional assumption that $g \in L_1$ is not satisfied, it suffices to use the argument from the proof of Theorem 16.2. The theorem is proved.

16.5 Preliminary estimates on $\mathcal{G}$

Throughout the section f is a fixed element of $C_0^\infty(\mathbb{R}^{d+1}, H)$ and $u = \mathcal{G}f$.

Lemma 16.4 *For any $T \in (-\infty, \infty]$*

$$\|u\|_{L_2(\mathbb{R}^{d+1} \cap \{t \le T\})} \le N(d, K)\|f\|_{L_2(\mathbb{R}^{d+1} \cap \{t \le T\})}. \tag{16.13}$$

Proof Since f is smooth, its values belong to a separable subspace of H. Then by using orthonormal bases and the Fourier transform it is easy to show that the square of the left-hand side in (16.13) equals

$$\int_{\mathbb{R}^d} \int_{-\infty}^T \Big[\int_{-\infty}^t |\tilde{\psi}(\xi\sqrt{t-s}\,)|^2 |\tilde{f}(s,\xi)|_H^2 \frac{ds}{t-s} \Big] \, dt \, d\xi$$

$$= \int_{-\infty}^T \int_{\mathbb{R}^d} \Big[\int_0^{T-s} |\tilde{\psi}(\xi\sqrt{t}\,)|^2 \frac{dt}{t} \Big] |\tilde{f}(s,\xi)|_H^2 \, d\xi ds =: I.$$

Here $\tilde{\psi}(0) = 0$ and

$$|\tilde{\psi}(\xi)| \le |\xi| \sup |\nabla\tilde{\psi}| \le N(d)|\xi| \int_{\mathbb{R}^d} |x| \, |\psi(x)| \, dx,$$

$$|\xi| \, |\tilde{\psi}(\xi)| \le N(d) \int_{\mathbb{R}^d} |\nabla\psi(x)| \, dx,$$

so that with $\bar{\xi} = \xi/|\xi|$

$$\int_0^\infty |\tilde{\psi}(\xi\sqrt{t}\,)|^2 \frac{dt}{t} = \int_0^\infty |\tilde{\psi}(\bar{\xi}\sqrt{t})|^2 \frac{dt}{t} \le N(d, K),$$

$$I \leq N \int_{-\infty}^{T} \int_{\mathbb{R}^d} |\tilde{f}(s,\xi)|_H^2 \, d\xi ds,$$

where the last expression equals the right-hand side of (16.13). The lemma is proved.

To proceed further we need some notation. According to (16.9) introduce the maximal function of a real-valued function h given on $\mathbb{R}^d$ relative to balls. We denote this function $\mathbb{M}_x h$ to emphasize that this maximal function is taken with respect to x. Similarly, for functions h on $\mathbb{R}$ we introduce $\mathbb{M}_t h$ as the maximal function of h relative to symmetric intervals

$$\mathbb{M}_t h(t) = \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |h(t+r)| \, dr.$$

For a function $h(t,x)$ set

$$\mathbb{M}_x h(t,x) = \mathbb{M}_x(h(t,\cdot))(x), \quad \mathbb{M}_t h(t,x) = \mathbb{M}_t(h(\cdot,x))(t).$$

Notice the following consequence of Lemma 16.4, in which and below we denote by $B_r(x)$ the open ball of radius r centered at x and $B_r = B_r(0)$.

Corollary 16.3 *Set*

$$Q_0 = [-4,0] \times [-1,1]^d \tag{16.14}$$

and assume that $f = 0$ *outside of* $[-12,12] \times B_{3d}$. *Then for any* $(t,x) \in Q_0$

$$\int_{Q_0} |u(s,y)|^2 \, dsdy \leq N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t,x), \tag{16.15}$$

where N *depends only on* d *and* K.

Indeed, for $g := |f|_H^2$ the left-hand side is less than

$$N \int_{\mathbb{R}^{d+1}, s \leq 0} g \, dyds \leq N \int_{-12}^{0} \int_{|y| \leq 3d} g \, dyds$$

$$\leq N \int_{-12}^{0} \int_{|x-y| \leq 4d} g \, dyds \leq N \int_{-12}^{0} \mathbb{M}_x g(s,x) \, ds \leq N \mathbb{M}_t \mathbb{M}_x g(t,x).$$

Here is a generalization of Corollary 16.3.

Lemma 16.5 *Assume that* $f(t,x) = 0$ *for* $t \notin (-12,12)$. *Then* (16.15) *holds again for any* $(t,x) \in Q_0$.

Proof We take a $\zeta \in C_0^\infty(\mathbb{R}^d)$ such that $\zeta = 1$ in B_{2d} and $\zeta = 0$ outside of B_{3d}. Set $\alpha = \zeta f$ and $\beta = (1-\zeta)f$. Since $\mathcal{G} f \leq \mathcal{G}\alpha + \mathcal{G}\beta$ and $\mathcal{G}\alpha$ admits the stated estimate, it suffices to concentrate on $\mathcal{G}\beta$. In other words, in the rest of the proof we may assume that $f(t,x) = 0$ for $x \in B_{2d}$.

Introduce $\bar{f} = |f|_H$, take $0 > s > r > -12$, and write

$$|\Psi_{s-r} f(r,\cdot)(y)|_H \leq (s-r)^{-d/2} \int_{\mathbb{R}^d} \bar{\psi}(|z|/\sqrt{s-r}) \bar{f}(r, y-z) \, dz.$$

We transform the last integral by using the formula

$$\int_{R \geq |z| \geq \varepsilon} F(z) G(|z|) \, dz = -\int_{\varepsilon}^{R} G'(\rho) \Big(\int_{|z| \leq \rho} F(z) \, dz \Big) \, d\rho$$

$$+ G(R) \int_{|z| \leq R} F(z) \, dz - G(\varepsilon) \int_{|z| \leq \varepsilon} F(z) \, dz, \tag{16.16}$$

where $0 \le \varepsilon \le R \le \infty$ and F and G satisfy appropriate conditions. Formula (16.16) is easily obtained by differentiating both sides with respect to R. Also notice that if $(s, y) \in Q_0$ and $|z| \le \rho$ with a $\rho > 1$, then

$$|x - y| \le 2d =: \nu, \quad B_\rho(y) \subset B_{\nu+\rho}(x) \subset B_{\mu\rho}(x), \quad \mu = \nu + 1, \tag{16.17}$$

whereas if $|z| \le 1$, then $|y - z| \le 2d$ and $f(r, y - z) = 0$.

Then we see that for $0 > s > r > -12$ and $(s, y) \in Q_0$

$$\begin{aligned}
&|\Psi_{s-r} f(r, \cdot)(y)|_H \\
&\le (s - r)^{-(d+1)/2} \int_1^\infty |\bar\psi'(\rho/\sqrt{s - r})| \Big(\int_{|z|\le\rho} \bar f(r, y - z)\, dz \Big)\, d\rho \\
&\le (s - r)^{-(d+1)/2} \int_1^\infty |\bar\psi'(\rho/\sqrt{s - r})| \Big(\int_{B_{\mu\rho}(x)} \bar f(r, z)\, dz \Big)\, d\rho \\
&\le N \mathbb{M}_x \bar f(r, x)(s - r)^{-(d+1)/2} \int_1^\infty |\bar\psi'(\rho/\sqrt{s - r})| \rho^d\, d\rho \\
&\le N \mathbb{M}_x \bar f(r, x) \int_{(s-r)^{-1/2}}^\infty |\bar\psi'(\rho)| \rho^d\, d\rho \le N(s - r)^{1/2} \mathbb{M}_x \bar f(r, x).
\end{aligned}$$

Also observe that by Hölder's inequality $(\mathbb{M}_x \bar f)^2 \le \mathbb{M}_x \bar f^2$. Then for $(s, y) \in Q_0$, we obtain

$$|u(s, y)|^2 = \int_{-12}^s |\Psi_{s-r} f(r, \cdot)(y)|^2 \frac{dr}{s - r} \le N \int_{-12}^0 \mathbb{M}_x |f|_H^2(r, x)\, dr,$$

where the last expression is certainly less than the right-hand side of (16.15). The lemma is proved.

Lemma 16.6 *Assume that* $f(t, x) = 0$ *for* $t \ge -8$. *Then for any* $(t, x) \in Q_0$

$$\int_{Q_0} |u(s, y) - u(t, x)|^2\, ds dy \le N \mathbb{M}_t \mathbb{M}_x |f|_H^2(t, x), \tag{16.18}$$

where the constant N *depends only on* K *and* d.

Proof The left-hand side of (16.18) is certainly less than a constant times

$$\sup_{Q_0} [|D_s u|^2 + |\nabla u|^2]. \tag{16.19}$$

Fix $(s, y) \in Q_0$ and note that $s \ge -4$ and by Minkowski's inequality (the derivative of a norm is less then the norm of the derivative)

$$|\nabla u(s, y)|^2 \le \int_{-\infty}^{-8} I^2(r, s, y) \frac{dr}{s - r}.$$

where

$$\begin{aligned}
I(r, s, y) &:= |\nabla \Psi_{s-r} f(r, \cdot)(y)|_H \\
&= (s - r)^{-(d+1)/2} \Big| \int_{\mathbb{R}^d} (\nabla\psi)(z/\sqrt{s - r}) f(r, y - z)\, dz \Big|_H \\
&\le (s - r)^{-(d+1)/2} \int_{\mathbb{R}^d} \bar\psi(|z|/\sqrt{s - r}) \bar f(r, y - z)\, dz,
\end{aligned}$$

and as before $\bar f = |f|_H$.

Also use again (16.16) and (16.17). Then we see that for $s > r$

$$I(r,s,y) \le (s-r)^{-(d+2)/2} \int_0^\infty |\bar\psi'(\rho/\sqrt{s-r})| \Big(\int_{B_\rho(y)} \bar f(r,z)\,dz \Big)\,d\rho$$

$$\le N\mathbb{M}_x \bar f(r,x)(s-r)^{-(d+2)/2} \int_0^\infty |\bar\psi'(\rho/\sqrt{s-r})|(\nu+\rho)^d\,d\rho$$

$$= N\mathbb{M}_x \bar f(r,x)(s-r)^{-1/2} \int_0^\infty |\bar\psi'(\rho)|(\nu/\sqrt{s-r}+\rho)^d\,d\rho.$$

For $r \le -8$ we have $s-r \ge 4$ and we conclude

$$\int_0^\infty |\bar\psi'(\rho)|(\nu/\sqrt{s-r}+\rho)^d\,d\rho \le N, \quad I(r,s,y) \le N(s-r)^{-1/2}\mathbb{M}_x \bar f(r,x),$$

$$|\nabla u(s,y)|^2 \le N \int_{-\infty}^{-8} \mathbb{M}_x \bar f^2(r,x) \frac{dr}{(4-r)^2}.$$

We transform the last integral integrating by parts or using (16.16) to find

$$|\nabla u(s,y)|^2 \le N \int_{-\infty}^{-8} \frac{1}{(4-r)^3} \Big(\int_r^0 \mathbb{M}_x \bar f^2(p,x)\,dp \Big)\,dr$$

$$\le N\mathbb{M}_t\mathbb{M}_x \bar f^2(t,x) \int_{-\infty}^{-8} \frac{|r|}{(4-r)^3}\,dr = N\mathbb{M}_t\mathbb{M}_x \bar f^2(t,x).$$

We thus have estimated part of (16.19).

To estimate $D_s u$, we proceed similarly

$$|D_s u(s,y)| \le \int_{-\infty}^{-8} |D_s \Psi_{s-r} f(r,y)|_H^2 \,\frac{dr}{s-r} = \int_{-\infty}^{-8} J^2(r,s,y)\,\frac{dr}{s-r},$$

where

$$J(r,s,y) = |D_s \Psi_{s-r} f(r,y)|_H$$

$$= (s-r)^{-(d+2)/2} \Big| \int_{\mathbb{R}^d} \hat\psi(z/\sqrt{s-r}) f(r,y-z)\,dz \Big|_H,$$

$$\le (s-r)^{-(d+2)/2} \int_{\mathbb{R}^d} \bar\psi(|z|/\sqrt{s-r}) \bar f(r,y-z)\,dz.$$

For $r \le -8$ we may further write

$$J(r,s,y) \le N(s-r)^{-(d+1)/2} \int_{\mathbb{R}^d} \bar\psi(|z|/\sqrt{s-r}) \bar f(r,y-z)\,dz$$

and then it only remains to refer to the above computations. The lemma is proved.

16.6 Proof of Theorem 16.1

First, note that for any $f \in C_0^\infty((a,b) \times \mathbb{R}^d, H)$ we have $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$ and equation (16.1) with $-\infty$ and ∞ in place of a and b, respectively, is stronger than as is. Therefore, we may assume that $a = -\infty, b = \infty$. Then our assertion is that for $f \in C_0^\infty(\mathbb{R}^{d+1}, H)$ and $u = \mathcal{G}f$ we have

$$\|u\|_{L_p(\mathbb{R}^{d+1})} \le N(d,p,K)\|f\|_{L_p(\mathbb{R}^{d+1},H)}.$$

This estimate follows from Lemma 16.4 if $p = 2$. Hence we may concentrate on $p > 2$. We start considering this case by claiming that at each point in $\mathbb{R}^{d+1}$

$$(\mathcal{G}f)^{\#} \le N(d,K)(\mathbb{M}_t\mathbb{M}_x|f|_H^2)^{1/2}, \tag{16.20}$$

where the sharp function $(\mathcal{G}f)^{\#}$ is defined relative to the parabolic dyadic cubes of type (16.2).

Remark 16.3 shows that to prove (16.20) it suffices to prove that for each $Q = Q_n(i_0,...,i_d)$ (see (16.2)) and $(t,x) \in Q$

$$⨍_Q |\mathcal{G}f - (\mathcal{G}f)_Q|^2 \, dyds \le N(d,K)\mathbb{M}_t\mathbb{M}_x|f|_H^2(t,x), \tag{16.21}$$

where

$$(\mathcal{G}f)_Q = ⨍_Q \mathcal{G}f \, dyds.$$

To prove (16.21), observe that if a constant $c \neq 0$, then $\Psi_t h(c\,\cdot)(x) = \Psi_{tc^2}h(cx)$, and

$$\mathcal{G}f(c^2\cdot, c\,\cdot)(t,x) = \Big[\int_{-\infty}^t |\Psi_{(t-s)c^2}f(c^2s,\cdot)(cx)|_H^2 \frac{ds}{t-s}\Big]^{1/2}$$

$$= \Big[\int_{-\infty}^{tc^2} |\Psi_{tc^2-s}f(s,\cdot)(cx)|_H^2 \frac{ds}{tc^2-s}\Big]^{1/2} = \mathcal{G}f(c^2t, cx).$$

This and the fact that dilations do not affect averages show that it suffices to prove (16.21) for $Q = Q_{-1}(i_0,...,i_d)$. In that case Q is just a shift of Q_0 from (16.14). Furthermore, the shift is harmless since $\mathbb{M}_x$ and $\mathbb{M}_t$ are defined in terms of balls rather than dyadic cubes.

Thus let $Q = Q_0$ and take a function $\zeta \in C_0^\infty(\mathbb{R})$ such that $\zeta = 1$ on $[-8,8]$, $\zeta = 0$ outside of $[-12,12]$, and $1 \ge \zeta \ge 0$. Set

$$\alpha = f\zeta, \quad \beta = f - \alpha.$$

Observe that

$$\Psi_{t-s}\alpha(s,\cdot) = \zeta(s)\Psi_{t-s}f(s,\cdot), \quad \mathcal{G}f \le \mathcal{G}\alpha + \mathcal{G}\beta, \quad \mathcal{G}\beta \le \mathcal{G}f.$$

It follows that for any constant c

$$|\mathcal{G}f - c| \le |\mathcal{G}\alpha| + |\mathcal{G}\beta - c|$$

and in light of Remark 16.1 the left-hand side of (16.21) is less than

$$2⨍_Q |\mathcal{G}\alpha|^2 \, dyds + 2⨍_Q |\mathcal{G}\beta - c|^2 \, dyds.$$

We finally take $c = \mathcal{G}\beta(t,x)$ and obtain (16.21) from Lemmas 16.5 and 16.6.

After having proved (16.20), by combining the Fefferman–Stein theorem with the L_q, $q > 1$, boundedness of the maximal operators we conclude (recall that $p > 2$)

$$\|u\|^p_{L_p(\mathbb{R}^{d+1})} \leq N \|(\mathbb{M}_t \mathbb{M}_x |f|^2_H)^{1/2}\|^p_{L_p(\mathbb{R}^{d+1})}$$

$$= N \int_{\mathbb{R}^d} \big(\int_{\mathbb{R}} (\mathbb{M}_t \mathbb{M}_x |f|^2_H)^{p/2} \, dt \big) \, dx \leq N \int_{\mathbb{R}^d} \big(\int_{\mathbb{R}} (\mathbb{M}_x |f|^2_H)^{p/2} \, dt \big) \, dx$$

$$= N \int_{\mathbb{R}} \big(\int_{\mathbb{R}^d} (\mathbb{M}_x |f|^2_H)^{p/2} \, dx \big) \, dt \leq N \|f\|^p_{L_p(\mathbb{R}^{d+1}, H)}.$$

This proves the theorem.

References

[Ga] A.M. Garsia, “Martingale inequalities,” Seminar notes on recent progress, Mathematics Lecture Notes Series, W.A. Benjamin, Inc., Reading, MA-London-Amsterdam, 1973.

[Kim] Kyeong-Hun Kim, *On stochastic partial di erential equations with variable coe cients in C^1 domains*, Stoch. Process. Appl. 112 (2004), No. 2, 261-283.

[Kr94a] N.V. Krylov, *A generalization of the Littlewood-Paley inequality and some other results related to stochastic partial di erential equations*, Ulam Q., Vol 2 (1994), No. 4, 16-26, http://www.ulam.usm.edu/VIEW2.4/krylov.ps.

[Kr94b] N.V. Krylov, *A parabolic Littlewood-Paley inequality with applications to parabolic equations*, Topological Methods in Nonlinear Analysis, J. Juliusz Schauder Center, Vol. 4 (1994), No. 2, 355-364.

[KK] Kyeong-Hun Kim and N.V. Krylov, *On SPDEs with variable coe cients in one space dimension*, Potential Anal., Vol. 21 (2004), No. 3, 209-239.

[Lo] S. Lototsky, *Small perturbation of stochastic parabolic equations: a power series analysis*, J. Funct. Anal., Vol. 193 (2002), No. 1, 94-115.

[MR] R. Mikulevicius and B. Rozovskii, *A note on Krylov's L_p-theory for systems of SPDEs*, Electron. J. Probab., Vol. 6 (2001), No. 12, 35 pp.

[Ste] E. Stein, “Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals,” Princeton University Press, Princeton, NJ, 1993.

[St] D.W. Stroock, *Applications of Fe erman-Stein type interpolation to probability theory and analysis*, Comm. Pure Appl. Math., Vol. 26 (1973), No. 4, 477-495.

17 Lévy Noises and Stochastic Integrals on Banach Spaces*

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17.1 Introduction

The purpose of this work is to define stochastic integrals with respect to a certain class of martingales with jumps in a Banach space. These martingales are constructed from the Poisson random measures associated with a Lévy process in a Banach space [9]. The definition of these stochastic integrals (called p-integrals) were first considered in [40] and were extended to a larger class of functions in [36]. This work is in the spirit of Pratelli [38]. In [36] we undertook a systematic study of the solutions of stochastic differential equations (SDEs) and their Markov properties and applied it to an example in finance [16] using stochastic integral with respect to (w.r.t.) Lévy process. We also extended some work of [42]. This study was done by considering stochastic integral with respect to Lévy processes to those studied in [36], [40] using Lévy–Ito decomposition in [9]. We give this, in brief, in the appendix in Section 17.5. As a further application of this idea, we derive a precise definition of Lévy white noise as a stochastic integral and relate it to the Lévy type white noise considered in [10]. In the appendix we give application to perpetuity in insurance again using stochastic integrals with respect to the Lévy process. Defining the stochastic integral in this manner allows us to use the theory developed in [36] to construct a Markov process and obtain perpetuity as an invariant measure of this process. This enables us relate the invariant measure to the infinitesimal generator of the Markov process as in the classical case using the Ito formula. Similar work was done in ad hoc manner in [18] under some additional assumptions.

17.2 Stochastic integrals w.r.t. compensated Poisson random measure on separable Banach spaces

We assume that a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le +\infty}, P)$, satisfying the "usual hypothesis," is given
(i) $\mathcal{F}_t$ contain all null sets of $\mathcal{F}$, for all t such that $0 \le t < +\infty$.
(ii) $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t}\mathcal{F}_u$, for all t such that $0 \le t < +\infty$, i.e., the filtration is right continuous.

In the whole chapter we assume that E is a separable Banach space with norm $\|\cdot\|$ and $\mathcal{B}(E)$ is the corresponding Borel σ-algebra.

*This chapter is dedicated to the 65th birthday of Sergio Albeverio.

Definition 17.1 *A process $(X_t)_{t\geq 0}$ with state–space $(E, \mathcal{B}(E))$ is an $\mathcal{F}_t$-additive process on $(\Omega, \mathcal{F}, P)$ if*
(i) $(X_t)_{t\geq 0}$ *is adapted (to $(\mathcal{F}_t)_{t\geq 0}$).*
(ii) $X_0 = 0$ *a.s. (almost surely).*
(iii) $(X_t)_{t\geq 0}$ *has increments independent of the past, i.e., $X_t - X_s$ is independent of $\mathcal{F}_s$ if $0 \leq s < t$.*
(iv) $(X_t)_{t\geq 0}$ *is stochastically continuous, i.e., $\forall \epsilon > 0$ $\lim_{s\to t} P(\|X_s - X_t\| > \epsilon) = 0$.*
(v) $(X_t)_{t\geq 0}$ *is càdlàg.*

An additive process is a Lévy process if the following condition is satisfied:
(vi) $(X_t)_{t\geq 0}$ *has stationary increments; that is, $X_t - X_s$ has the same distribution as X_{t-s}, $0 \leq s < t$.*

Let $(X_t)_{t\geq 0}$ be an additive process on $(E, \mathcal{B}(E))$ (in the sense of Definition 17.1). Set $X_{t-} := \lim_{s\uparrow t} X_s$ and $\Delta X_s := X_s - X_{s-}$.

Definition 17.2 *We denote with $N(dtdx)(\omega)$ the Poisson random measure associated to the additive process $(X_t)_{t\geq 0}$ and $\nu(dtdx)$ its compensator. We recall that $N(dtdx)(\omega)$, for each ω fixed, resp. $\nu(dtdx)$, are σ-finite measures on the σ-algebra $\mathcal{B}(I\!R_+ \times E \setminus \{0\})$, generated by the product semi-ring $\mathcal{S}(I\!R_+) \times \mathcal{B}(E \setminus \{0\})$ of product sets $(t_1, t_2] \times A$, with $0 \leq t_1 < t_2$, and $A \in \mathcal{B}(E \setminus \{0\})$, where $\mathcal{B}(E \setminus \{0\})$ is the trace σ-algebra on $E \setminus \{0\}$ of the Borel σ-algebra $\mathcal{B}(E)$ on E (see, e.g., [22], [40]). $q(dtdx)(\omega) := N(dtdx)(\omega) - \nu(dtdx)$ is the compensated Poisson random measure associated to the additive process $(X_t)_{t\geq 0}$. (We omit sometimes writing the dependence on $\omega \in \Omega$.)*

Remark 17.1 $(X_t)_{t\geq 0}$ is a Lévy process iff $\nu(dtdx) = dt\beta(dx)$, where dt denotes the Lebesgues measure on $\mathcal{B}(I\!R_+)$, and $\beta(dx)$ is a σ-finite measure on $(E \setminus \{0\}, \mathcal{B}(E \setminus \{0\}))$, and is called the Lévy measure associated to $(X_t)_{t\geq 0}$.

Given in general two σ-algebras $\mathcal{M}$ and $\mathcal{L}$, with measure m and resp. l, we denote by $\mathcal{M} \otimes \mathcal{L}$ the product σ-algebra generated by the product semiring $\mathcal{M} \times \mathcal{L}$, and by $m \otimes l$ the corresponding product measure.

Let F be a separable Banach space with norm $\|\cdot\|_F$. (When no misunderstanding is possible we write $\|\cdot\|$ instead of $\|\cdot\|_F$). Let $F_t := \mathcal{B}(I\!R_+ \times (E \setminus \{0\})) \otimes \mathcal{F}_t$ be the product σ-algebra generated by the semi-ring $\mathcal{B}(I\!R_+ \times (E \setminus \{0\})) \times \mathcal{F}_t$ of the product sets $\Lambda \times F$, $\Lambda \in \mathcal{B}(I\!R_+ \times E \setminus \{0\})$, $F \in \mathcal{F}_t$. Let $T > 0$, and

$$\begin{aligned} &M^T(E/F) := \\ &\{f : I\!R_+ \times E \setminus \{0\} \times \Omega \to F, \quad \textit{such that} \quad f \quad \textit{is} \quad F_T/\mathcal{B}(F) \quad \textit{measurable} \\ &\quad f(t,x,\omega) \quad \textit{is} \quad \mathcal{F}_t - \textit{adapted} \quad \forall x \in E \setminus \{0\}, \quad t \in (0,T]\}. \end{aligned} \tag{17.1}$$

In this section we shall introduce the stochastic integrals of random functions $f(t,x,\omega) \in M^T(E/F)$ with respect to the compensated Poisson random measures $q(dtdx)(\omega) := N(dtdx)(\omega) - \nu(dtdx)$ discussed in [36], [40]. (We omit sometimes to write the dependence on $\omega \in \Omega$.)

There is a "natural definition" of stochastic integral w.r.t. $q(dtdx)(\omega)$ on those sets $(0,T] \times \Lambda$ where the measures $N(dtdx)(\omega)$ (with ω fixed) and $\nu(dtdx)$ are finite, i.e., $0 \notin \overline{\Lambda}$. According to [40] (see also [9] for the case of deterministic functions $f(x)$, $x \in \setminus\{0\}$) we give the following definition:

Definition 17.3 *Let* $t \in (0,T]$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$, $0 \notin \overline{\Lambda}$, $f \in M^T(E/F)$. *Assume that* $f(\cdot,\cdot,\omega)$ *is Bochner integrable on* $(0,T] \times \Lambda$ *w.r.t.* ν, *for all* $\omega \in \Omega$ *fixed. The* <u>*natural integral*</u> *of* f *on* $(0,t] \times \Lambda$ *w.r.t. the compensated Poisson random measure* $q(dtdx) := N(dtdx)(\omega) - \nu(dtdx)$ *is*

$$\int_0^t \int_\Lambda f(s,x,\omega)\,(N(dsdx)(\omega) - \nu(dsdx)) :=$$
$$\sum_{0<s\le t} f(s,(\Delta X_s)(\omega),\omega)\mathbf{1}_\Lambda(\Delta X_s(\omega)) - \int_0^t \int_\Lambda f(s,x,\omega)\nu(dsdx) \;\; \omega \in \Omega \qquad (17.2)$$

where the last term is understood as a Bochner integral, (for $\omega \in \Omega$ *fixed) of* $f(s,x,\omega)$ *w.r.t. the measure* ν.

It is more difficult to define the stochastic integral on those sets $(0,T] \times \Lambda$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$, s.th. $\nu((0,T] \times \Lambda) = \infty$. For real-valued functions this problem was already discussed, e.g., in [17] and [27], [45], [47] (for general martingale measures). Different definitions of stochastic integrals were proposed. In [40] (and [9 for the case of deterministic functions $f(x)$, $x \in E \setminus \{0\}$) we introduced the definition of "strong-p-integral" (Definition 17.7). The strong-p-integral is the limit in $L_p^F(\Omega, \mathcal{F}, P)$ (the space of F-valued random variables Y, with $E[\|Y\|^p] < \infty$, defined in Definition 17.6) of the "natural integrals" (17.2) of the "simple functions" defined in (17.3). (We refer to Definition 17.7 for a precise statement.) If $p = 2$, this concept generalizes the definition in [17] of stochastic integration of real-valued functions with respect to martingales measures on $I\!R^d$, to the case of Banach space-valued functions, for the case where the martingale measures are given by compensated Poisson random measures on general separable Banach spaces. It generalizes also to the stochastic integral introduced in [45] for real-valued functions integrated w.r.t. compensated Poisson random measures associated to α-stable Lévy processes on $I\!R$. In [40] it has also been proved that the concept of strong-p-integral, with $p = 1$ or $p = 2$, is more general than the definition of stochastic integrals w.r.t. point processes introduced (for the real-valued case) in [27], Chapt. II.3. In fact, for the existence of the strong-p-integral, with $p = 1$ or $p = 2$, no predictability condition (in the sense of [27]) is needed for the integrand. This condition is, however, needed for the stochastic integrals introduced in [27], which in [40] are denoted with "simple-p-integrals." (This concept has been generalized to the Hilbert or Banach valued case in [40], too.) If the integrand is, however, left-continuous, then the strong-p-integral coincide with the simple-p-integral (we refer to [40] for a precise statement).

We recall here the definition of "strong-p-integral", $p \ge 1$, introduced in [40]. We first introduce the simple functions.

Definition 17.4 *A function* f *belongs to the set* $\Sigma(E/F)$ *of* <u>*simple functions*</u>, *if* $f \in M^T(E/F)$, $T > 0$ *and there exist* $n \in I\!N$, $m \in I\!N$, *such that*

$$f(t,x,\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} \mathbf{1}_{A_{k,l}}(x)\mathbf{1}_{F_{k,l}}(\omega)\mathbf{1}_{(t_k,t_{k+1}]}(t)a_{k,l} \qquad (17.3)$$

where $A_{k,l} \in \mathcal{B}(E \setminus \{0\})$ *and* $0 \notin \overline{A_{k,l}}$, $t_k \in (0,T]$, $t_k < t_{k+1}$, $F_{k,l} \in \mathcal{F}_{t_k}$, $a_{k,l} \in F$. *For all* $k \in 1,...,n-1$ *fixed,* $A_{k,l_1} \times F_{k,l_1} \cap A_{k,l_2} \times F_{k,l_2} = \emptyset$ *if* $l_1 \ne l_2$.

Proposition 17.1 *Let* $f \in \Sigma(E/F)$ *be of the form (17.3), then*

$$\int_0^T \int_A f(t,x,\omega)q(dtdx)(\omega) = \sum_{k=1}^{n-1} \sum_{l=1}^{m} a_{k,l}\mathbf{1}_{F_{k,l}}(\omega)q((t_k,t_{k+1}] \cap (0,T] \times A_{k,l} \cap A)(\omega), \quad (17.4)$$

for all $A \in \mathcal{B}(E \setminus \{0\})$, $T > 0$.

Remark 17.2 The random variables $\mathbf{1}_{F_{k,l}}$ in (17.4) are independent of $q((t_k, t_{k+1}] \cap (0,T] \times A_{k,l} \cap A)$ for all $k \in 1, ..., n-1$, $l \in 1, ..., m$ fixed.

Proof of Proposition 17.1: The proof is an easy consequence of the Definition 17.2 of the random measure $q(dtdx)(\omega)$. ■

We recall here the definition of strong-p-integral, $p \geq 1$, (Definition 17.7 below) given in [40] through approximation of the natural integrals of simple functions. First, we establish some properties of the functions $f \in M_\nu^{T,p}(E/F)$, where

$$M_\nu^{T,p}(E/F) := \{f \in M^T(E/F) : \int_0^T \int E[\|f(t,x,\omega)\|^p]\, \nu(dtdx) < \infty\}. \tag{17.5}$$

Theorem 17.1 *[40] Let $p \geq 1$. Suppose that the compensator $\nu(dtdx)$ of the Poisson random measure $N(dtdx)$ satisfies the following hypothesis $\mathcal{A}$.*

Hypothesis $\mathcal{A}$: ν is a product measure $\nu = \alpha \otimes \beta$ on the σ-algebra generated by the semi ring $\mathcal{S}(\mathbb{R}_+) \times \mathcal{B}(E \setminus \{0\})$, of a σ-finite measure α on $\mathcal{S}(\mathbb{R}_+)$, s.th. $\alpha([0,T]) < \infty$, $\forall T > 0$, α is absolutely continuous w.r.t the Lebesgues measure on $\mathbb{R}_+$, and a σ-finite measure β on $\mathcal{B}(E \setminus \{0\})$.

Let $T > 0$; then for all $f \in M_\nu^{T,p}(E/F)$ and all $\Lambda \in \mathcal{B}(E \setminus \{0\})$, there is a sequence of simple functions $\{f_n\}_{n \in \mathbb{N}}$ satisfying the following property:

Property P: $f_n \in \Sigma(E/F)\, \forall n \in \mathbb{N}$, f_n converges $\nu \otimes P$ -a.s. to f on $(0,T] \times \Lambda \times \Omega$, when $n \to \infty$, and

$$\lim_{n \to \infty} \int_0^T \int_\Lambda E[\|f_n(t,x) - f(t,x)\|^p]\, d\nu = 0\,, \tag{17.6}$$

i.e., $\|f_n - f\|$ converges to zero in $L^p((0,T] \times \Lambda \times \Omega, \nu \otimes P)$, when $n \to \infty$.

Definition 17.5 *We say that a a sequence of functions f_n are L^p-approximating f if these satisfy property P, i.e., f_n converge $\nu \otimes P$-a.s. to f on $(0,T] \times \Lambda \times \Omega$, when $n \to \infty$, and satisfy (17.6).*

Definition 17.6 *Let $p \geq 1$; $L_p^F(\Omega, \mathcal{F}, P)$ is the space of F-valued random variables, such that $E\|Y\|^p = \int \|Y\|^p dP < \infty$. We denote by $\|\cdot\|_p$ the norm given by $\|Y\|_p = (E\|Y\|^p)^{1/p}$. Given $(Y_n)_{n \in \mathbb{N}}, Y \in L_p^F(\Omega, \mathcal{F}, P)$, we write $\lim^p_{n \to \infty} Y_n = Y$ if $\lim_{n \to \infty} \|Y_n - Y\|_p = 0$.*

In [40] we introduce the following.

Definition 17.7 *Let $p \geq 1$, $t > 0$. We say that f is strong-p-integrable on $(0,t] \times \Lambda$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$, if there is a sequence $\{f_n\}_{n \in \mathbb{N}} \in \Sigma(E/F)$, which satisfies the property P in Theorem 17.1, and such that the limit of the natural integrals of f_n w.r.t. $q(dtdx)$ exists in $L_p^F(\Omega, \mathcal{F}, P)$ for $n \to \infty$, i.e.*

$$\int_0^t \int_\Lambda f(t,x,\omega) q(dtdx)(\omega) := \overset{p}{\lim_{n \to \infty}} \int_0^t \int_\Lambda f_n(t,x,\omega) q(dtdx)(\omega) \tag{17.7}$$

exists. Moreover, the limit (17.7) does not depend on the sequence $\{f_n\}_{n \in \mathbb{N}} \in \Sigma(E/F)$, for which property P and (17.7) hold.

We now give sufficient conditions for the existence of the strong-p-integrals, when $p = 1$, or $p = 2$. In the whole chapter we assume that hypothesis $\mathcal{A}$ in Theorem 17.1 is satisfied.

Theorem 17.2 *[40] Let $f \in M_\nu^{T,1}(E/F)$; then f is strong-1-integrable w.r.t. $q(dt, dx)$ on $(0, t] \times \Lambda$, for any $0 < t \le T$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$. Moreover*

$$E[\| \int_0^t \int_\Lambda f(s, x, \omega) q(dsdx)(\omega)\|] \le 2 \int_0^t \int_\Lambda E[\|f(s, x, \omega)\|]\nu(dsdx)(\omega). \tag{17.8}$$

Theorem 17.3 *[40] Suppose $(F, \mathcal{B}(F)) := (H, \mathcal{B}(H))$ is a separable Hilbert space. Let $f \in M_\nu^{T,2}(E/H)$; then f is strong 2-integrable w.r.t. $q(dtdx)$ on $(0, t] \times \Lambda$, for any $0 < t \le T$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$. Moreover*

$$E[\| \int_0^t \int_\Lambda f(s, x, \omega) q(dsdx)(\omega)\|^2] = \int_0^t \int_\Lambda E[\|f(s, x, \omega)\|^2]\nu(dsdx). \tag{17.9}$$

The following Theorem 17.4 was proved in [40] for the case of deterministic functions on type-2 Banach spaces, and on M-type-2 spaces for functions which do not depend on the random variable x, in [36] for the general case.

Theorem 17.4 *[36] Suppose that F is a separable Banach space of M-type 2. Let $f \in M_\nu^{T,2}(E/F)$; then f is strong 2-integrable w.r.t. $q(dtdx)$ on $(0, t] \times \Lambda$, for any $0 < t \le T$, $\Lambda \in \mathcal{B}(E \setminus \{0\})$. Moreover*

$$E[\| \int_0^t \int_\Lambda f(s, x, \omega) q(dsdx)(\omega)\|^2] \le K_2^2 \int_0^t \int_\Lambda E[\|f(s, x, \omega)\|^2]\nu(dsdx)\,. \tag{17.10}$$

where K_2 is the constant in the Definition 17.8 of M-type-2 Banach spaces.

Theorem 17.5 *[40] Suppose that F is a separable Banach space of type 2. Let $f \in M_\nu^{T,2}(E/F)$, and f be a deterministic function, i.e., $f(t, x, \omega) = f(t, x)$; then f is strong 2-integrable w.r.t. $q(dtdx)$ on $(0, t] \times A$, for any $0 < t \le T$, $A \in \mathcal{B}(E \setminus \{0\})$. Moreover, inequality (17.10) holds, with K_2 being the constant in the Definition 17.10 of type-2 Banach spaces.*

We recall here the definition of M-type-2 and type-2 separable Banach space (see, e.g., [33]).

Definition 17.8 *A separable Banach space F, with norm $\| \cdot \|$, is of M-type 2, if there is a constant K_2, such that for any F-valued martingale $(M_k)_{k \in 1,..,n}$ the following inequality holds:*

$$E[\|M_n\|^2] \le K_2 \sum_{k=1}^n E[\|M_k - M_{k-1}\|^2]\,, \tag{17.11}$$

with the convention that $M_{-1} = 0$.

We remark that a separable Hilbert space is in particular a separable Banach space of M-type 2. In fact, any 2-uniformly-smooth separable Banach space is of M-type 2 [37], [48]. We recall here the definition of 2-uniformly-smooth separable Banach space.

Definition 17.9 *A separable Banach space F, with norm $\| \cdot \|$, is 2-uniformly-smooth if there is a constant $K_2 > 0$, s.th. for all $x, y \in F$*

$$\|x + y\|^2 + \|x - y\|^2 \le 2\|x\|^2 + K_2\|y\|^2$$

.

Definition 17.10 *A separable Banach space F is of type 2, if there is a constant K_2, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent F-valued random variables, such that $E[\|X_i\|^2] < \infty$, then*

$$E[\|\sum_{i=1}^n X_i\|^2] \leq K_2 \sum_{i=1}^n E[\|X_i\|^2]. \tag{17.12}$$

We remark that any separable Banach space of M-type 2 is a separable Banach space of type 2.

Proposition 17.2 *[36], [40] Let f satisfy the hypothesis of Theorem 17.2, or 17.4. Then $\int_0^t \int_\Lambda f(s,x,\omega)q(dsdx)(\omega)$, $t \in [0,T]$ is an $\mathcal{F}_t$-martingale with mean zero and is càdlàg.*

Remark 17.3 *[36]*Let f satisfy the hypothesis of Theorem 17.2, or 17.4. From Doob's inequality it follows that for any sequence $f_n \in \Sigma(E/F)$, which is L^p-approximating f, the convergence of $\int_0^t \int_\Lambda f_n(s,x,\omega)q(dsdx)(\omega)$ to $\int_0^t \int_\Lambda f(s,x,\omega)q(dsdx)(\omega)$ holds also in the following sense:

$$\lim_{n\to\infty} P\left(\sup_{t\in[0,T]} \| \int_0^t \int_A f(s,x,\omega)q(dsdx) - \int_0^t \int_A f_n(s,x,\omega)q(dsdx)\| > \epsilon\right) = 0 \tag{17.13}$$

It follows that there is a subsequence such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \| \int_0^t \int_A f(s,x,\omega)q(dsdx)(\omega) - \int_0^t \int_A f_n(s,x,\omega)(\omega)q(dsdx)\| = 0 \quad P-a.s. \tag{17.14}$$

Definition 17.11 *We call $\int_0^t \int_A f(s,x,\omega)q(dsdx)(\omega)$ the strong-p-integral of f w.r.t. $q(dtdx)$ on $(0,t] \times \Lambda$, if it is obtained from the limit in (17.14).*

17.3 Integration w.r.t. strong-p-integrals

We use the same notations as in the previous section. Moreover, we denote also with G a separable Banach space and with $\mathcal{L}(F/G)$ the Banach space of linear bounded operators from F to G, with the usual supremum norm.

With $\|\cdot\|_K$ we denote the norm of a Banach space K, so that, in particular, $\|\mathcal{F}\|_{\mathcal{L}(F/G)} = \sup_{y\in K} \frac{\|\mathcal{F}(y)\|_G}{\|y\|_F}$, if $\mathcal{F} \in \mathcal{L}(F/G)$. When no misunderstanding is possible, we shall write simply $\|\cdot\|$, leaving the subscript which denotes the Banach space. In the above example we would, e.g., write $\|\mathcal{F}\|$ instead of $\|\mathcal{F}\|_{\mathcal{L}(F/G)}$, and $\|y\|$, resp. $\|\mathcal{F}(y)\|$, instead of $\|y\|_F$, resp. $\|\mathcal{F}(y)\|_G$.

Let $M^T(\mathbb{R}_+/\mathcal{L}(F/G))$ be the set of progressive measurable processes $(H_t)_{t\in[0,T]}$ with values on $\mathcal{L}(F/G))$.

Definition 17.12 *We denote with* $E^T(I\!R_+/\mathcal{L}(F/G))$ *the set of elementary processes* $(H(t,\omega))_{t\in[0,T]}$, *i.e., those which are in* $M^T(I\!R_+/\mathcal{L}(F/G))$, *are uniformly bounded and are of the form*

$$H(t,\omega) = \sum_{i=1}^{r-1} \mathbf{1}_{(t_i,t_i+1]}(t)H_i(\omega)\,, \tag{17.15}$$

with $H_i(\omega)$ $\mathcal{F}_{t_i}$*-adapted,* $0 < t_i < t_{i+1} \le T$.

In the usual way we introduce the stochastic integral of elementary processes w.r.t. martingales.

Definition 17.13 *Let* $(M_t)_{t\in[0,T]}$ *be an* $\mathcal{F}_t$*-adapted martingale with values on* F. *Let* $(H(t,\omega))_{t\in[0,T]} \in E^T(I\!R_+/\mathcal{L}(F/G))$, $(H(t,\omega))_{t\in[0,T]}$ *be of the form (17.15). The stochastic integral* $(H\cdot M)_t$, $t\in[0,T]$, *of* $(H(t,\omega))_{t\in[0,T]}$ *w.r.t.* $(M_t(\omega))_{t\in[0,T]}$ *is defined with*

$$(H\cdot M)_t(\omega) := \int_0^t H(s,\omega)dM_s(\omega) := \sum_{i=1}^{r-1} H_i(\omega)[M_{t_{i+1}\wedge t}(\omega) - M_{t_i\wedge t}(\omega)] \tag{17.16}$$

(we sometimes omit writing the dependence on ω*).*

Proposition 17.3 *Let* $(H(t,\omega))_{t\in[0,T]} \in E^T(I\!R_+/\mathcal{L}(F/G))$. *Let* $f \in \Sigma(E/F)$ *(defined in Definition 17.4). Let*

$$M_t(\omega) := \int_0^t \int_A f(s,x,\omega)q(dsdx)(\omega) \tag{17.17}$$

then

$$\begin{aligned}(H\cdot M)_t(\omega) &:= \textstyle\int_0^t H(s,\omega)dM_s(\omega)\\ &= \textstyle\int_0^t\int_A H(s,\omega)(f(s,x,\omega)q(dsdx)(\omega) \quad \forall\omega\in\Omega\,,\ t\in(0,T]\end{aligned} \tag{17.18}$$

i.e., $(H\cdot M)_t(\omega)$ *coincides with the integral of* $H(f(s,x,\omega))$ *w.r.t.* $q(dsdx)(\omega)$.

Proof of Proposition 17.3

Let $(H(t,\omega))_{t\in[0,T]}$ be of the form (17.15) and $f\in\Sigma(E/F)$ be of the form (17.3). By linearity it follows

$$\begin{aligned}&\textstyle\sum_{i=1}^{r-1} H_i(\omega)[M_{t_{i+1}\wedge t}(\omega) - M_{t_i\wedge t}(\omega)]\\ &= \textstyle\sum_{i=1}^{r-1}\sum_{k=1}^{n-1}\sum_{l=1}^{m} H_i(\omega)(a_{k,l})\mathbf{1}_{F_{k,l}}(\omega)q((t_k,t_{k+1}]\cap(0,T]\times A_{k,l}\cap A)(\omega)\\ &= \textstyle\int_0^t\int_A H(s,\omega)((f(s,x,\omega))q(dsdx)(\omega).\end{aligned} \tag{17.19}$$

■

Let $p=1$ or $p=2$. Let E be a separable Banach space. Let F and G be separable Banach spaces. If $p=2$, we suppose also that G is an M-type-2 Banach space. Let $f\in M_\nu^{T,p}(E/F)$ (defined in (17.5)). We define

$$M_{f,\nu}^{T,p}(I\!R_+/\mathcal{L}(F/G)) := \tag{17.20}$$
$$\{(H(t,\omega))_{t\in[0,T]} \in M^T(I\!R_+/\mathcal{L}(F/G)),\ s.th.\ \textstyle\int_0^T\int_{E\setminus\{0\}} E[\|H(s)\|^p\|f(s,x)\|^p]\nu(dsdx) < \infty\}.$$

Remark 17.4 If $(H(t,\omega))_{t\in[0,T]} \in M^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$, then there exists a sequence of elementary processes $(H_n(t,\omega))_{t\in[0,T]} \in E^T(I\!R_+/\mathcal{L}(F/G))$ such that (s.th.)

$$\lim_{n\to\infty} \int_0^T \int_{E\setminus\{0\}} E[\|H_n(s)-H(s)\|^p\|f(s,x)\|^p]\nu(dsdx)=0. \tag{17.21}$$

This can be proved, e.g., with the analogous techniques used in STEP 1–STEP 4 in the proof of Theorem 17.1 in [40].

We denote with $\mathcal{M}^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$ the set of $dt\otimes dP$ equivalence classes in $M^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$. $\mathcal{M}^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$ is a separable Banach spaces.

Proposition 17.4 *Let $p=1$ or $p=2$. If $p=2$, suppose that F and G are separable Banach spaces of M-type 2. Let $(H(t,\omega))_{t\in[0,T]} \in E^T(I\!R_+/\mathcal{L}(F/G))$, $f\in M^{T,p}_{\nu}(E/F)$. Let $M_t(\omega)$ be like in (17.17), then*

$$P\left(\int_0^t H(s,\omega)dM_s(\omega)=\int_0^t\int_A H(f(s,x,\omega))q(dsdx)(\omega)\,,\quad \forall t\in[0,T]\right)=1. \tag{17.22}$$

Proof of Proposition 17.4

We first prove that there is a set $\Gamma\in\mathcal{F}_T\otimes\mathcal{B}([0,T])$, s.th. $dt\otimes dP(\Gamma)=1$ and $\forall t,\omega\in\Gamma$

$$\int_0^t H(s,\omega)dM_s(\omega)=\int_0^t\int_A H(f(s,x,\omega))q(dsdx)(\omega)\,, \tag{17.23}$$

i.e., the right-hand side (r.h.s.) and left-hand side (l.h.s.) coincide in $\mathcal{M}^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$. Suppose that f_n is a sequence which is L^p-approximating f. We have

$$\int_0^t H(s,\omega)dM^n_s(\omega)=\int_0^t\int_A H(f_n(s,x,\omega))q(dsdx)(\omega)\quad\forall\omega\in\Omega\,, \tag{17.24}$$

where M^n_t is the strong-p-integral of f_n. Hence

$$\begin{aligned}\textstyle\int_0^t H(s,\omega)dM_s(\omega) &= \textstyle\lim^p_{n\to\infty}\int_0^t H(s,\omega)dM^n_s(\omega)\\ &= \textstyle\lim^p_{n\to\infty}\int_0^t\int_A H(f_n(s,x,\omega)q(dsdx)(\omega)\\ &\textstyle\int_0^t\int_A H(f(s,x,\omega)q(dsdx)(\omega)\end{aligned} \tag{17.25}$$

where we used that for any $0\le\tau<t\le T$

$$\lim^p_{n\to\infty}(M^n_t-M^n_\tau)=(M_t-M_\tau) \tag{17.26}$$

and

$$\begin{aligned}&\textstyle E[\|\int_0^t\int_A(H(f_n(s,x))-H(f(s,x)))q(dsdx)\|^p]\\ &\textstyle\le C\int_0^t\int_A E[\|H(f_n(s,x))-H(f(s,x))\|^p]\nu(dsdx)\end{aligned} \tag{17.27}$$

for $C=2$ or $C=K_2^2$. That equation (17.22) holds, is a consequence of inequality (17.27) and Doob's inequality (see also Remark 17.3). In fact, given any $\epsilon>0$,

$$\begin{aligned}&P\left(\sup_{t\in[0,T]}\|\textstyle\int_0^t\int_A(H(f_n(s,x))-H(f(s,x)))q(dsdx)\|>\epsilon\right)\\ &\le\frac{E[\|\int_0^t\int_A(H(f_n(s,x))-H(f(s,x)))q(dsdx)\|^p]}{\epsilon^p}\\ &\le C\frac{\int_0^t\int_A E[\|H(f_n(s,x))-H(f(s,x))\|^p]\nu(dsdx)}{\epsilon^p}\to_{n\to\infty}0.\end{aligned} \tag{17.28}$$

■

We now introduce the stochastic integral $(H \cdot M)_t$ for $(H(t,\omega))_{t\in[0,T]} \in M^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$ w.r.t. the martingale $M_t(\omega) = \int_0^t \int_A f(s,x,\omega)q(dsdx)(\omega)$.

Theorem 17.6 *Let $p = 1$ or $p = 2$. If $p = 2$ suppose that F and G are separable Banach spaces of M-type 2. Let $(H(t,\omega))_{t\in[0,T]} \in M^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$. There is a unique element $(H \cdot M)_t \in \mathcal{M}^{T,p}_{f,\nu}(I\!R_+/\mathcal{L}(F/G))$, such that*

$$(H \cdot M)_t = \lim_{n\to\infty}^{p} (H_n \cdot M)_t = \lim_{n\to\infty}^{p} \int_0^t H_n dM_s \tag{17.29}$$

for any sequence of elementary processes $(H_n(t,\omega))_{t\in[0,T]} \in E^T(I\!R_+/\mathcal{L}(F/G))$, for which (17.21) holds.

Moreover, the following properties hold:

1. The convergence (17.29) holds also in the following sense:

$$P(\sup_{[0,T]} \|(H_n \cdot M)_t - (H \cdot M)_t\| > \epsilon) \to_{n\to\infty} 0. \tag{17.30}$$

It follows that there is a subsequence such that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|(H_n \cdot M)_t - (H \cdot M)_t\| = 0 \quad P-a.s. \tag{17.31}$$

2. $(H \cdot M)_t$ coincides with the strong-p-integral of $H(f)$, i.e.

$$P\left((H \cdot M)_t = \int_0^t \int_A H(f(s,x,\omega))q(dsdx)(\omega) \quad \forall t \in [0,T]\right) = 1. \tag{17.32}$$

3. $(H \cdot M)_t$ is an $\mathcal{F}_t$-martingale.

Definition 17.14 *We call $\int_0^t H(s,\omega)dM_s(\omega) := (H \cdot M)_s(\omega)$ the stochastic integral of $(H(t,\omega))_{t\in[0,T]}$ w.r.t. $(M_t(\omega))_{t\in[0,T]}$, if it is obtained from the limit in (17.31).*

Remark 17.5 If f is a deterministic function and $p = 2$ then in Proposition 17.4 and Theorem 17.6 it is sufficient that F is a separable Banach spaces of type 2. If $H(s,\omega)= H(s)$, i.e., H is deterministic, too, then G too can be of type 2.

Proof of Theorem 17.6 Let $H_n(t,\omega)\in E^T(I\!R_+/\mathcal{L}(F/G))$ be a sequence for which (17.21) holds. Then

$$\begin{aligned} &P\left(\sup_{t\in[0,T]} \|\int_0^t \int_A (H_n(f(s,x)) - H(f(s,x)))q(dsdx)\| > \epsilon\right) \\ &\quad\le \frac{E[\|\int_0^t \int_A (H_n(f(s,x))-H(f(s,x)))q(dsdx)\|^p]}{\epsilon^p} \\ &\quad\le C\frac{\int_0^t \int_A E[\|H_n(f(s,x))-H(f(s,x))\|^p]\nu(dsdx)}{\epsilon^p} \end{aligned} \tag{17.33}$$

for $C = 2$ or $C = K_2^2$, which together with Proposition 17.4 proves that all statements in the theorem hold. ■

Remark 17.6 Let $F = H$, with H a separable Hilbert space with scalar product $< \cdot, \cdot >$, and $G = I\!R$. For this particular case Theorem 17.6 shows that the definition of Ito integral, that we usually use for L^2 bounded continuous martingales (see, e.g., [39]), can be extended also to martingales with jumps, if these are strong-p-integrals. In this case the following identity holds:

$$\int_0^t H(s,\omega) dM_s(\omega) = \int_0^t \int_A < H(s,\omega), f(s,x,\omega) > q(dsdx)(\omega). \tag{17.34}$$

Moreover, it shows that the Ito integral w.r.t. strong-p-integrals can be extended to separable Banach spaces, too.

Remark 17.7 Let $f \in M_\nu^{T,2}(E/F)$, and $f(s,x,\omega) := f(s,x)$, i.e., f be deterministic and F and G separable Banach spaces of M-type 2 (in fact F might be also of type 2). According to the definition of [38], the strong 2 -integrable of f , is for any $0 < t \leq T$ controlled by (the deterministic function)

$$A_t := \int_0^t \int_\Lambda \|f(s,x)\|^2 \nu(dsdx) \tag{17.35}$$

as

$$\begin{aligned} E[\|M_t - M_s\|^2/\mathcal{F}_s] &= E[\|M_t - M_s\|^2] \\ &\leq A_t - A_s = E[\|A_t - A_s\|/\mathcal{F}_s]. \end{aligned} \tag{17.36}$$

For controlled Banach-valued martingales the possibility of defining the Ito-integral $(H \cdot M)_t$, $H \in M^T(I\!R_+/\mathcal{L}(F/G))$ was already mentioned in [38]. We did such extension in this section for all martingales being strong-p-integrals. Moreover, we identified here such Ito-integrals with the strong-p-integrals of $H(f)$ w.r.t. the compensated Poisson random measure. The relation of these two integrals is obviously fundamental in several applications where stochastic calculus w.r.t. the strong-p-integrals appears, as, e.g., by analyzing SDEs with non-Gaussian additive noise. In the appendix we make some example to show how the relation of the Ito integral and strong-p-integral analyzed in this section is important to analyze the solution of certain SDEs coming from problems of finance and insurance.

17.4 Lévy-type white noise constructed by stochastic integration of Lévy white noise

In [3, 4, 5, 46] a program was started of constructing Euclidean covariant vector Markov random fields as solutions of stochastic partial differential equations (SPDEs) driven by Lévy-type white noise where by Lévy-type white noise we mean a generalized infinite divisible random field determined by a Lévy–Khinchine function [10] (see Definition 17.18 below). The relation between Lévy white noises and Lévy-type white noises has been discussed in great generality, e.g., in [6], [7], [43], [44], [1], [2]. We refer to [10] for an overview of such results. A concrete computation of the Lévy-type white noise through stochastic integration of Lévy noise has, however, been given in terms of a chaos expansion in [20]. We prove in this section how the Lévy-type white noise can be obtained by a direct computation of strong-2-integrals without doing a chaos expansion.

We start by recalling some well-known definition and result on Lévy measures on $I\!R$ (see, e.g., [25]).

Definition 17.15 *A σ-finite positive measure β on $(I\!R^d \setminus \{0\}, \mathcal{B}(I\!R \setminus \{0\}))$ is a "Lévy measure," if there is a probability measure μ such that the Fourier transform $\hat{\mu}(z)$, $z \in I\!R^d$ satisfies*

$$\hat{\mu}(z) = \exp(\psi(z)) \tag{17.37}$$

with

$$\psi(z) = \exp\left(\int_{I\!R\setminus\{0\}} (e^{iy\cdot z} - 1 - \mathbf{1}_{|y|<1} iy \cdot z)\beta(dy)\right). \tag{17.38}$$

We call μ the "Poisson-type measure" (associated) with (the) "Lévy measure" β and ψ the corresponding Lévy–Khinchine (or "characteristic") function.

The following theorems are well known (see, e.g., [42], or Remark 2.2. in [9]).

Theorem 17.7 *Given a Lévy measure β on $(I\!R^d \setminus \{0\}, \mathcal{B}(I\!R^d \setminus \{0\}))$, there exists a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le+\infty}, P)$ and a Lévy process $(L_t)_{t\ge 0}$ with compensator $dt\beta(dx)$.*

Theorem 17.8 *A σ-finite positive measure β on $(I\!R\setminus\{0\}, \mathcal{B}(I\!R\setminus\{0\}))$ is a "Lévy measure" if and only if the Lévy condition holds, i.e.*

$$\int_{|y|\le 1} y^2 \beta(dy) < \infty\,. \tag{17.39}$$

(Theorem 17.8 does not hold for Lévy measure defined on general separable Banach spaces [12], [34].)

Let E be a separable Banach space. We define the a Lévy noise given according to [10], [14].

Definition 17.16 *Let $(M, \mathcal{M}, \mu)$ be a σ-finite measure space. By a Lévy noise $L(dx, \omega)$ on $(M, \mathcal{M}, \mu)$ defined on the probability space $(\Omega, \mathcal{F}, P)$, we mean a random measure L : $(M, \mathcal{M}, \mu) \times (\Omega, \mathcal{F}, P) \to [0, \infty)$ having the following expression:*

$$L(A, \omega) := \int_A \int_{E\setminus\{0\}} a(x, y)q(dxdy)(\omega) + \int_A \int_{E\setminus\{0\}} b(x, y)\nu(dxdy) \tag{17.40}$$

where $a, b : E \times M \to I\!R$ are measurable, and $q(dx, dy)(\omega)$ is a compensated Poisson random measure on $M \times E, \mathcal{M} \otimes \mathcal{B}(E)$ with compensator $\nu(dxdy)$.

We call $L(dx, \omega)$ a Lévy white noise if $a(x, y)$ is linear in y and $a(x, y)= b(x, y)$.

(In Definition 17.16 we consider only the pure jump Lévy noise part, as this is the topic of this article and the Gaussian white noise was well studied in the past. We refer to [14], [10] for the general definition).

Definition 17.17 *Let $L(dx, \omega)$ be the Lévy noise on a separable Banach space F defined in (17.40). We define*

$$\begin{aligned} &\int f(x)L(dx, \omega) \\ &:= \int_{F\setminus\{0\}} \int_{E\setminus\{0\}} f(x)a(x, y)q(dxdy)(\omega) + \int_{F\setminus\{0\}} \int_{E\setminus\{0\}} f(x)b(x, y)\nu(dxdy)\,, \end{aligned} \tag{17.41}$$

where the stochastic integral w.r.t. $q(dxdy)$ is a strong-p-integral.

Remark 17.8 In case $F = I\!R$ and $\nu(dxdy) = dx\beta(dy)$ it follows from the results in Section 17.3 that

$$\int f(x)L(dx,\omega) = \int f(x)dM(x)$$

with

$$M(x)(\omega) := \int_0^x \int_{E\setminus\{0\}} a(x,y)q(dxdy)(\omega) + \int_0^x \int_{E\setminus\{0\}} b(x,y)\nu(dxdy)$$

Let us recall the definition of Lévy-type noise given in [10] and then establish a relation between these two kind of noises.

Let $1 \leq d \in I\!N$ be an arbitrarily fixed space–time dimension. Let $\mathcal{S}(I\!R^d)$ be the Schwartz space of rapidly decreasing (real) C^∞-functions on $I\!R^d$ with the Schwartz topology. Let $\mathcal{S}'(I\!R^d)$ be the topological dual of $\mathcal{S}(I\!R^d)$. We denote by $< \cdot, \cdot >$ the natural dual pairing between $\mathcal{S}(I\!R^d)$ and $\mathcal{S}'(I\!R^d)$. Let $\mathcal{B}$ be the σ-algebra generated by cylinder sets of $\mathcal{S}'(I\!R^d)$. Then $(\mathcal{S}'(I\!R^d), \mathcal{B})$ is a measurable space. By a characteristic functional on $\mathcal{S}(I\!R^d)$, we mean a functional $C : \mathcal{S}(I\!R^d) \to \mathbb{C}$ with the following three properties: (1) C is continuous on $\mathcal{S}(I\!R^d)$; (2) C is positive–definite; and (3) $C(0) = 1$. By the well-known Bochner–Minlos theorem (see, e.g., [23]) there exists a one to one correspondence between characteristic functionals C and probability measures P on $(\mathcal{S}'(I\!R^d), \mathcal{B})$ given by the following relation:

$$C(f) = \int_{\mathcal{S}'(I\!R^d)} e^{i<f,\omega>} dP(\omega), \quad f \in \mathcal{S}(I\!R^d).$$

The following result follows as a direct consequence of the Bochner–Minlos theorem (see, e.g., [10], [20])

Proposition 17.5 *Let ψ be a Lévy–Khinchine function. Then there exists a unique probability measure P_ψ on $(\mathcal{S}'(I\!R^d), \mathcal{B})$ such that*

$$\int_{\mathcal{S}'(I\!R^d)} e^{i<f,\omega>} dP_\psi(\omega) = \exp\left\{\int_{I\!R^d} \psi(f(x))dx\right\}, \quad f \in \mathcal{S}(I\!R^d). \tag{17.42}$$

In [10] we give the following.

Definition 17.18 *We call P_ψ in the above proposition a Lévy white-noise measure with Lévy–Khinchine function ψ, and $(\mathcal{S}'(I\!R^d), \mathcal{B}, P_\psi)$ the Lévy white-noise space associated with ψ. The associated coordinate process*

$$F : \mathcal{S}(I\!R^d) \times (\mathcal{S}'(I\!R^d), \mathcal{B}, P_\psi) \to I\!R \tag{17.43}$$

defined by

$$F(f,\omega) =< f, \omega >, \quad f \in \mathcal{S}(I\!R^d),\, \omega \in \mathcal{S}'(I\!R^d) \tag{17.44}$$

is a random field on $(\mathcal{S}'(I\!R^d), \mathcal{B}, P_\psi)$ with parameter space $\mathcal{S}(I\!R^d)$. We call it **Lévy-type white noise**.

F in (17.43), (17.44), can be extended to the parameter space $L^2(I\!R^d)$ from $\mathcal{S}(I\!R^d)$. (cf. [11], see also [10]).

Definition 17.19 *Given a Lévy-type white noise F on $(\mathcal{S}'(I\!R^d), \mathcal{B}, P_\psi)$, we say that there is a Lévy noise $L(\cdot,\omega)$ on $(I\!R^d, \mathcal{B}(I\!R^d))$ associated to F, if there is a Lévy noise $L(dx,\omega)$ on $(I\!R^d, \mathcal{B}(I\!R^d))$ defined on the probability space $(\mathcal{S}'(I\!R^d), \mathcal{B}, P_\psi)$ and*

$$F(f,\omega) = \int f(x)L(dx,\omega) \quad \forall f \in L^2(I\!R^d). \tag{17.45}$$

Theorem 17.9 *Let $\beta(dy)$ be a Lévy measure on $I\!R$, and ψ be the corresponding Lévy–Khinchine function. Then there is a Lévy noise $L(dx,\omega)$ on $(I\!R^d, \mathcal{B}(I\!R^d))$ with compensator $dxds\beta(dy)$, s.th*

$$L(A,\omega) := \int_A \int_0^1 \int_{0<|y|\le 1} yq(dxdsdy)(\omega) + \int_A \int_0^1 \int_{|y|>1} ydxds\beta(dy) \quad \forall A \in \mathcal{B}(I\!R^d) \tag{17.46}$$

and such that $L(\cdot,\omega)$ is associated to the Lévy-type white noise with Lévy–Khinchine function ψ.

Proof of Theorem 17.9:
First, we remark that $dxds\beta(dy)$ is a Lévy measure on $I\!R^{d+1} \times I\!R_+$, as

$$\int_{0<|x|\le 1} \int_0^1 \int_{0<|y|\le 1} s^2 |x|^2 y^2 dx\beta(dy) < \infty.$$

There is therefore a Lévy process $(L_t)_{t\ge 0}$ with compensator $dsdx\beta(dy)$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le +\infty}, P)$. (17.46) is therefore well defined on $(\Omega, \mathcal{F}, P)$. Moreover

$$L_t(A,\omega) := \int_A \int_0^t \int_{0<|y|\le 1} yq(dxdsdy)(\omega) + \int_A \int_0^t \int_{|y|>1} ydxds\beta(dy) \quad \forall A \in \mathcal{B}(I\!R^d) \tag{17.47}$$

defines a Lévy white noise, too. Let $f \in L^2(I\!R^d)$; then from Theorem 17.4 it follows that $f(x)y$ is strong-2-integrable w.r.t. $q(dxdsdy)(\omega)$ and

$$\begin{aligned} &\int f(x) L_t(dx,\omega) \\ &= \int_{R^d\setminus\{0\}} \int_0^t \int_{0<|y|\le 1} f(x) yq(dxdsdy)(\omega) + \int_{R^d\setminus\{0\}} \int_0^t \int_{|y|>1} f(x) ydxds\beta(dy) \end{aligned} \tag{17.48}$$

is a martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le +\infty}, P)$.

We apply the Ito formula proved for strong-p-integrals in [17], [45] (see [41] for the case of Banach valued strong-p-integrals and Banach valued functions). We recall here the Ito formula on R^d.

The Ito formula Let $q(dsdx)$ be a compensated Poisson random measure (c.P.r.m.) with compensator $ds\nu(dx)$. Let $A \in \mathcal{B}(I\!R^d \setminus \{0\})$, $B \in \mathcal{B}(I\!R^d)$

$$\xi_t := \int_0^t \int_A f(s,x,\omega) q(dsdx)(\omega) + \int_0^t \int_B f(s,x,\omega))\, N(dsdx)(\omega).$$

Let $\mathcal{H}(x)$ be a real-valued function, then from [41] it follows that

$$\begin{aligned} &\mathcal{H}(\xi_t(\omega)) - \mathcal{H}(\xi_\tau(\omega)) = \\ &+ \int_\tau^t \int_A \{\mathcal{H}(\xi_{s_-}(\omega) + f(s,x,\omega)) - \mathcal{H}(\xi_{s_-}(\omega))\}\, q(dsdx)(\omega) \\ &+ \int_\tau^t \int_A \{\mathcal{H}(\xi_{s_-}(\omega) + f(s,x,\omega)) - \mathcal{H}(\xi_{s_-}(\omega)) - \mathcal{H}'(\xi_{s_-}(\omega)) f(s,x,\omega)\}\, ds\beta(dx) \\ &+ \int_\tau^t \int_B \{\mathcal{H}(\xi_{s_-}(\omega) + f(s,x,\omega)) - \mathcal{H}(\xi_{s_-}(\omega))\}\, N(dsdx)(\omega) \\ &P - a.s. \quad for \quad any \quad 0 < \tau < t \le T. \end{aligned} \tag{17.49}$$

(The results in [41] are much more general.) Applying the Ito formula to $\xi_t := \int f(x) L_t(dx,\omega)$ and $\mathcal{H}(x) = \exp(ix)$ we obtain

$$E[\exp(i\xi_t)] = \int_0^t \int_{R^d\setminus\{0\}} \int_{0<|y|\leq 1} E[\exp(i\xi_{s_-})]\{\exp(if(x)y - 1 - i\exp(if(x)y)\} dxds\beta(dy) + \int_0^t \int_{R^d\setminus\{0\}} \int_{|y|>1} \{E[\exp(i\xi_{s_-}(\omega))] - 1\} \exp(if(x)y) dsdx\beta(dy). \tag{17.50}$$

$$\frac{d}{dt} E[\exp(i\xi_t)] = E[\exp(i\xi_{t-})] \int_{R^d\setminus\{0\}} \psi(f(x))dx. \tag{17.51}$$

As $\xi_0 = 0$ P -a.s., it follows

$$E[\exp(i\xi_t)] = E[\exp(i\int f(x)L_t(dx))] = t\int_{R^d\setminus\{0\}} \psi(f(x))dx \tag{17.52}$$

and hence

$$E[\exp(i\int f(x)L(dx))] = \int_{R^d\setminus\{0\}} \psi(f(x))dx\,. \tag{17.53}$$

As

$$L(dx,\omega) \in S'(I\!R^d) \quad P - a.s. \tag{17.54}$$

it follows from Proposition 17.5 and (17.53) that $L(dx,\omega)$ can be defined on $(S'(I\!R^d), \mathcal{B})$. ■

17.5 Appendix: Existence and uniqueness of stochastic integral equations driven by non-Gaussian noise. Applications

In [36] we showed an example for applications of the results there to a model for volatility in finance [16]. Also for the case where the state–space is the real line, we obtained some results in a more direct way than [29] and [42] (see Remark 17.9 below). We report in brief these applications of the theory done in [36] and then show a second application of the mentioned theory to a problem coming from insurance.

Let $p = 1$, or $p = 2$ and $(F, \mathcal{B}(F))$ be a separable Banach space of type 2. We assume that

$$\int_{0<\|x\|\leq 1} \|x\|^p \beta(dx) < \infty. \tag{17.55}$$

In [36] we analyzed

$$d\eta_t(\omega) = -a\eta_t(\omega)dt + d\xi_t(\omega) \tag{17.56}$$

where

$$\xi_t(\omega) := \int_0^t \int_{0<\|x\|\leq 1} xq(dsdx)(\omega) + \int_0^t \int_{\|x\|>1} xN(dsdx)(\omega)\,, \tag{17.57}$$

$a > 0$, $\nu(dsdx) = ds\beta(dx)$ is the compensator of $q(dsdx)(\omega)$, and with the initial condition η_0 being independent of the filtration $(\mathcal{F}_t)_{t\geq 0}$ of $(\xi_t)_{t\in[0,T]}$.

$$d\xi_t(\omega) = \int_{0<\|x\|\leq 1} xq(dsdx)(\omega) + \int_{\|x\|>1} xN(dsdx)(\omega) \tag{17.58}$$

(17.58) was given in [36] as a definition. We now know from Theorem 17.6 that the r.h.s. is really the differential of the l.h.s in (17.58).

From the results in [36] (Theorem 4.10 and Theorem 5.2) proved in [35] we know that for every $T > 0$ there is a unique pathwise solution $(\xi_t)_{t\in[0,T]}$ of (17.56) with initial condition η_0. Moreover if $\eta_0 = x$, $x \in F$, then $(\xi_t)_{t\in[0,T]}$ is Markov [36]. Using the Ito formula proved in [41], we showed in [36] that the solution is

$$\eta_t(\omega) = e^{-at}(\eta_0(\omega) + \int_0^t e^{as} d\xi_s(\omega)). \tag{17.59}$$

(In [36] we also discussed properties of the invariant measure of (17.59)).

Remark 17.9 The existence of the solution of (17.56) has been proved before [36] in Paragr. 17 of [42], only for the case where $(\xi_t)_{t\in[0,T]}$ is of bounded variation, (i.e., has big jumps), while it has been proved on the real line in [29] in an ad hoc way, by defining the stochastic integral of e^{-as} w.r.t. $d\xi_s$ by assuming that an integration by part formula holds, which follows now as a consequence of the Ito formula proved in [41] and Theorem 17.6.

We shall now consider another example [18] where the use of SDEs with respect to non-Gaussian additive noise play a role in finance and insurance. Let $(\xi_t)_{t\in\mathbb{R}_+}$ be a real-valued Lévy process with Lévy measure β, like in (17.57). Using Ito formula (17.49) for $\mathcal{H}(\xi_s) = e^{\xi_s}$ and $\beta(dx) = \nu(dx)$ we get an SDE for e^{ξ_t} as follows:

$$\begin{aligned}(e^{\xi_t} - 1) = & \int_0^t \int_{0<\|x\|\le 1} e^{\xi_{s-}} \{e^x - 1\} q(dsdx)\\ + & \int_0^t \int_{0<\|x\|\le 1} e^{\xi_{s-}} \{e^x - 1 - x\} ds\beta(dx)\\ + & \int_0^t \int_{\|x\|>1} e^{\xi_{s-}} \{e^x - 1\} N(dsdx).\end{aligned} \tag{17.60}$$

We note that with $a(s,y) = e^y \int_{0<\|x\|\le 1} \{e^x - 1 - x\}\beta(dx)$, $f(s,y,x)= e^y(e^x - 1)$ this equation is of the form studied in Skorohod ([45], page 45). Because $a(s,y)$, $f(s,y,x)$ satisfy Lipshitz conditions given there, we get e^{ξ_t} is a unique solution of (17.60) and is Markov. (See [36] for generalizations of the results of [45] to the Banach-valued case.) Let $(\eta_t)_{t\in\mathbb{R}_+}$ be another pure jump Lévy process. Then we can prove that

$$\zeta_t = e^{\xi_t}\left(\zeta_0 + \int_0^t e^{-\xi_s} d\eta_s\right) \tag{17.61}$$

is a solution of the equation

$$d\zeta_t = de^{\xi_t} + d\eta_t \tag{17.62}$$

with ζ_0 independent of $\{\xi_t, \eta_t, t \ge 0\}$. In fact (using Theorem 17.6) we get

$$\begin{aligned}de^{\xi_t} &= e^{\xi_{t-}} \int_{0<\|x\|\le 1} (e^x - 1) q(dtdx)\\ &\quad + e^{\xi_{t-}} \int_{\|x\|>1} (e^x - 1) N(dtdx)\\ &\quad + e^{\xi_{t-}} \int_{0<\|x\|\le 1} (e^x - 1 - x)\beta(dx)\end{aligned} \tag{17.63}$$

$$
\begin{aligned}
&(de^{\xi_t})\left(\zeta_0 + \textstyle\int_0^t e^{\xi_{s-}} d\eta_s\right)\\
&= e^{\xi_{t-}} \left(\zeta_0 + \textstyle\int_0^t e^{\xi_{s-}} d\eta_s\right) \times\\
&[\textstyle\int_{0<\|x\|\leq 1}(e^x - 1)q(dtdx)\\
&+ \textstyle\int_{\|x\|>1}(e^x - 1)N(dtdx) + \int_{0<\|x\|\leq 1}(e^x - 1 - x)\beta(dx)]\\
&= \zeta_{t-}[\textstyle\int_{0<\|x\|\leq 1}(e^x - 1)q(dtdx)\\
&+ \textstyle\int_{\|x\|>1}(e^x - 1)N(dtdx) + \int_{0<\|x\|\leq 1}(e^x - 1 - x)\beta(dx)]
\end{aligned}
\tag{17.64}
$$

giving

$$
\begin{aligned}
(d\zeta_t) &= \zeta_{t-} \textstyle\int_{0<\|x\|\leq 1} (e^x - 1)q(dtdx)\\
&+ \zeta_{t-} \textstyle\int_{\|x\|>1} (e^x - 1)N(dtdx)\\
&+ \zeta_{t-} \textstyle\int_{0<\|x\|\leq 1} (e^x - 1 - x)\beta(dx)\\
&+ d\eta_t .
\end{aligned}
\tag{17.65}
$$

The SDE (17.65) has a unique Markov solution [45] with initial condition $\zeta_0 = 0$. The corresponding transition functions are constant in x.

Hence $(\zeta_t)_{t\geq 0}$ is a Markov process and because of continuous dependence on initial condition [45], we get that the transition semigroup is Feller. Further using stationary independent increment property of ξ_t and η_t we get

$$
\mathcal{L}\left(\int_0^t e^{\xi_t - \xi_s} d\eta_s\right) = \mathcal{L}(Q_t)\,, \tag{17.66}
$$

where

$$
Q_t := \int_0^t e^{\xi_s} d\eta_s\,. \tag{17.67}
$$

Let us denote with $P_t(x, A) := E[\zeta_t \in A | \zeta_0 = x]$, $A \in \mathcal{B}(I\!R)$ and with P_{X_t} the distribution of a process $(X_t)_{t\in I\!R_+}$ at time t, then

$$
P_t(x, A) = P_{e^{\xi_t} x + \int_0^t e^{\xi}_s d\eta_s}. \tag{17.68}
$$

If we assume that $P_{e^{\xi_t} x} \to \delta_0$, when $t \to \infty$, then we get that if the invariant measure μ exists for ζ_t then $\mu = P_{\int_0^\infty e^{\xi_s} d\eta_s}$. In particular, if $\eta_s = s$ for all $s \in I\!R_+$, we get that $\mu = \mathcal{L}\left(\int_0^\infty e^{\xi_s} ds\right)$ which is called perpetuity as the invariant measure of the solution of

$$
\begin{aligned}
d\zeta_t &= de^{\xi_t} + dt\\
\zeta_o &= x\,.
\end{aligned}
\tag{17.69}
$$

The computation of the infinitesimal generator $\mathcal{A}$ of

$$
\zeta_t = e^{\xi_t}\left(\xi_0 + \int_0^t e^{-\xi_s} ds\right) \tag{17.70}
$$

is useful, e.g., to compute the invariant measure analytically. To compute $\mathcal{A}$ we apply the Ito formula [41] to ζ_t. Let F a real-valued function, $F \in C^2(I\!R)$.

$$\begin{aligned}
F(\zeta_t) - F(\zeta_0) = \int_0^t \int_{0<\|x\|\le 1}[F(\zeta_{s_-} + \zeta_{s_-}(e^x-1)) - F(\zeta_{s_-})]q(dsdx) \\
+ \int_0^t \int_{\|x\|>1}[F(\zeta_{s_-} + \zeta_{s_-}(e^x-1)) - F(\zeta_{s_-})]N(dsdx) \\
+ \int_0^t \int_{0<\|x\|\le 1}[F(\zeta_{s_-} + \zeta_{s_-}(e^x-1)) - F(\zeta_{s_-}) - F'(\zeta_{s_-})\zeta_{s_-}(e^x-1))]ds\beta(dx) \\
+ \int_0^t F'(\zeta_{s_-})[\zeta_{s_-} \int_{0<\|x\|\le 1}(e^x - 1 - x)\beta(dx) + 1]ds. \qquad (17.71)
\end{aligned}$$

$$\begin{aligned}
E[F(\zeta_t) - F(\zeta_0)] \qquad &= \int_0^t \int_{0<\|x\|\le 1}[F(\zeta_{s_-} + \zeta_{s_-}(e^x-1)) - F(\zeta_{s_-}) \\
&\quad -F'(\zeta_{s_-})\zeta_{s_-}(e^x-1)]\beta(dx)ds \\
&\quad + \int_0^t F'(\zeta_{s_-})\left[\zeta_{s_-} \int_{0<\|x\|\le 1}(e^x - 1 - x)\beta(dx) + 1\right] ds. \qquad (17.72)
\end{aligned}$$

It follows that the infinitesimal generator $\mathcal{A}$ acts on the functions $F \in C^2(I\!R)$ as

$$\begin{aligned}
\mathcal{A}F(y) = \quad & \int_{0<\|x\|\le 1}[F(y + y(e^x-1)) - F(y) - F'(y)(e^x-1)y]\beta(dx) \\
& + F'(y)[y \int_{0<\|x\|\le 1}(e^x - 1 - x)\beta(dx) + 1] \qquad (17.73)
\end{aligned}$$

$$\begin{aligned}
\mathcal{A}F(y) = \quad & \int_{0<\|x\|\le 1}[F(e^x y) - F(y) - F'(y)(e^x-1)y]\beta(dx) \\
& + yF'(y) \int_{0<\|x\|\le 1}(e^x - 1 - x)\beta(dx) + F'(y). \qquad (17.74)
\end{aligned}$$

Suppose that

$$\int_{0<\|x\|\le 1} \|x\|\beta(dx) < \infty \qquad (17.75)$$

then

$$\begin{aligned}
\mathcal{A}F(y) = \quad & \int_{0<\|x\|\le 1}[F(e^x y) - F(y)]\beta(dx) \\
& - yF'(y) \int_{0<\|x\|\le 1} x\beta(dx) + F'(y). \qquad (17.76)
\end{aligned}$$

Acknowledgments

We thank Sergio Albeverio for useful discussions related to this work.

References

[1] S. Albeverio, H. Gottschalk, J.-L. Wu, Partly divisible probability distributions. Forum Math. **10** (1998), 687–697.

[2] S. Albeverio, H. Gottschalk, J.-L. Wu, Partly divisible probability measures on locally compact Abelian groups. Math. Nachrichten **213** (2000).

[3] S. Albeverio, R. Høegh-Krohn, Euclidean Markov fields and relativistic quantum fields from stochastic partial differential equations. Phys. Lett. **B177** (1986), 175–179.

[4] S. Albeverio, R. Høegh-Krohn, Quaternionic non–abelian relativistic quantum fields in four space–time dimensions. Phys. Lett. **B189** (1987), 329–336.

[5] S. Albeverio, R. Høegh-Krohn, Construction of interacting local relativistic quantum fields in four space–time dimensions. Phys. Lett. **B200** (1988), 108–114; with erratum in ibid. **B202** (1988), 621.

[6] S. Albeverio, R. Høegh-Krohn, H. Holden, Stochastic multiplicative measures, generalized Markov semigroups and group valued stochastic processes and fields. J. Funct. Anal. **78** (1988), 154–184.

[7] S. Albeverio, R. Høegh-Krohn, H. Holden, T. Kolsrud, Representation and construction of multiplicative noise. J. Funct. Anal. **87** (1989), 250–272.

[8] S. Albeverio, V. Mandrekar, B. Rüdiger, Existence of weak solutions for SDEs and semilinear equations with non Gaussian noise, in preparation.

[9] S. Albeverio, B. Rüdiger, Stochastic integrals and the Lévy-Ito decomposition theorem on separable Banach spaces. Stoch. Anal. Appl. **23**, No. 2 (2005), 217-253.

[10] S. Albeverio, B. Rüdiger, J.-L. Wu, Analytic and Probabilistic Aspects of Lévy Processes and Fields in Quantum Theory. *Lévy Processes:Theory and Applications*, eds. O. Barndorff-Nielsen, T. Mikosch, S.I. Resnick, Birkhäuser Verlag, Basel (2001).

[11] S. Albeverio, J.-L. Wu, Euclidean random fields obtained by convolution from generalized white noise. J. Math. Phys. **36** (1995), 5217–5245.

[12] A. Araujo, E. Giné, The central limit theorem for real and Banach valued random variables. Wiley series in probability and mathematical statistics. New York, Chichester, Brisbane, Toronto (1980).

[13] A. Araujo, E. Giné, Type, coptype and Lévy measures in Banach spaces. Ann. Probab. **6**, No. 4, 637–643 (1978).

[14] D. Applebaum, J.-L. Wu, Stochastic partial differential equations driven by Lévy space-time white noise. Random Operators Stoch. Equations **8** (2000), No. 3.

[15] H. Bauer, Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie. 2 edition de Gruyter Lehrbuch. Walter de Gruyter-Berlin-New York (1974).

[16] O. Barndorff-Nielsen, N. Shepard, Modelling by Lévy Processes for Financial Econometrics *Lévy Processes:Theory and Applications*, eds. O. Barndorff-Nielsen, T. Mikosch, S.I. Resnick, Birkhäuser Verlag, Basel (2001).

[17] A. Bensoussan, J.-L. Lions, Contróle impulsionnel et inquations quasi variationnelles. (French) [Impulse control and quasivariational inequalities] Méthodes Mathématiques de l'Informatique [Mathematical Methods of Information Science], 11. Gauthier-Villars, Paris, 1982.

[18] P. Carmona, F. Petit, M. Yor, Exponential Functionals of Lévy processes, *Lévy Processes:Theory and Applications*, eds. O. Barndorff-Nielsen, T. Mikosch, S.I. Resnick, Birkhäuser Verlag, Basel (2001).

[19] E. Dettweiler, Banach space valued processes with independent increments and stochastic integrals, pp. 54–83 in Probability in Banach spaces IV, Proc., Oberwolfach 1982, Springer, Berlin (1982).

[20] G. Di Nunno, B. Øksendal, F. Proske, White noise analysis for Lévy processes. J. Funct. Anal. **206** (2004), 109–148.

[21] E.B. Dynkin, *Die Grundlagen der Theorie der Markoffschen Prozesse.* Berlin-Göttingen-Heidelberg-New York, Springer Verlag (1982).

[22] I.I. Gihman, A.V. Skorohod, The theory of stochastic processes II. Springer, Berlin-Heidelberg-New York 1975.

[23] I.M. Gelfand, N.Ya. Vilenkin, *Generalized Functions, IV. Some Applications of Harmonic Analysis.* Academic Press, New York-London 1964.

[24] E. Hille, Lectures on ordinary differental equations, Addison-Wesley, Reading, MA (1969).

[25] K. Ito, On stochastic processes I (infinitely divisible laws of probability). Jpn. J. Math. **18**, 261–301 (1942).

[26] K. Ito, Continuous additive $\mathcal{S}'$-processes. Lecture Notes in Control and Inform. Sci. **25**, p. 36–46, Springer, New York (1980).

[27] N. Ikeda, S. Watanabe, Stochastic Differential Equations and Diffusion Processes (second edition), North-Holland Mathematical Library, Vol. 24, North Holland Publishing Company, Amsterdam, Oxford, New York (1989).

[28] Z.J. Jurek, J.D. Mason *Operator-Limit Distributions in Probability Theory* Wiley-Interscience Publication, John Wiley and Sons, Inc., New York-Chichester-Brisbane-Toronto-Singapore (1993).

[29] Z.J. Jurek, W. Vervaat, An integral representation for selfdecomposable Banach space valued random variables. Z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 247–262 (1983).

[30] A.N. Kolmogorov, S.V. Fomin, Elementi di teoria delle funzioni e di analisi funzionale. Edizione MIR, Moscau (1980). Translation from the Russian: Elementy teorii funktsij i funktisianal'nogo Copyright by NAUKA, Moscau.

[31] S. Kwapién, Isomorphic characterisations of inner product spaces by orthogonal series with vector valued coefficients. Studia Math. **44**, 583–595 (1972).

[32] W. Linde, Infinitely Divisible and Stable Measures on Banach spaces. Teubner -Texte zur Mathematik, Band 58 (1983).

[33] G. Pisier, Probabilistic Methods in the Geometry of Banach Spaces, in Probability and Analysis , Varenna (Como) 1985, eds. G. Letta, M. Pratelli, LNM 1206, edited by A. Dold, B. Eckmann, Springer Verlag.

[34] V. Mandrekar, Characterisation of Banach spaces through validity of Bochner theorem, pp. 314–326 in Vector Space Measures and Applications I, Proc. Dublin 1977, ed. R.M. Arou, S. Dirren, Lect. N. Maths. 644, Springer Verlag, Berlin (1978).

[35] V. Mandrekar, B. Rüdiger. Existence and uniqueness of path wise solutions for stochastic integral equations driven by non Gaussian noise on separable Banach spaces, preprint SFB 611, Universität Bonn, March (2003).

[36] V. Mandrekar, B. Rüdiger, Existence and uniqueness of path wise solutions for stochastic integral equations driven by non Gaussian noise on separable Banach spaces, submitted (preprint SFB 611, Universität Bonn, November (2004)).

[37] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. **20**, 326–350 (1975).

[38] M. Pratelli, Intégration stochastique et géométrie des espaces de Banach, Séminaire de Probabilités, XXII, 129–137, Lecture Notes in Math. 1321, Springer, Berlin (1988).

[39] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion. Grundlehren der mathematischen Wissenschaft 293. A Series of Comprehensive Studies in Mathematics, Springer Verlag, Berlin 1991.

[40] B. Rüdiger, Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces, Stoch. Stoch. Rep. **76**, No. 3, 213–242, (2004).

[41] B. Rüdiger, G. Ziglio, Ito formula for stochastic integrals with respect to compensated Poisson random measures on separable Banach spaces, submitted (preprint SFB 611, Bonn, November 04).

[42] K.I. Sato, Lévy Processes and Infinitely Divisible Distributions. Cambridge studies in advanced mathematics 68. Cambridge University Press, Cambridge (1999).

[43] J. Schäfer, White noise on vector bundles and local functionals. J. Funct. Anal. **134** (1995), 1–32.

[44] J. Schäfer, Generalized random fields on manifolds and Markov properties. Dissertation, Bochum 1993.

[45] A. V. Skorohod, Studies in the theory of random processes. Addison-Wesley Publishing Company, Inc, Reading, MA (1965). Translated from the Russian by Scripta Technica, Inc.

[46] D. Surgailis, On covariant stochastic differential equations and Markov property of their solutions. Università di Roma preprint, Fisica, 1979.

[47] J.B. Walsh, An introduction to stochastic partial differential equations, pp. 266–439 in École d'Été de Probabilités de Saint-Flour XIV-1984, Editor P.L. Hennequin, LNM 1180, Springer Verlag, Berlin-Heidelberg-New York-Tokyo (1986).

[48] W.A. Woyczynski, Geometry and martingales in Banach spaces. Probability-Winter School, Proceedings of the 4th Winter School on Probability, Karpacz, Poland, January 1975 (Z. Ciesielski, K. Urbanik, and W.A. Woyczynski, eds.); LNM 472, pp. 229–275, Springer Verlag, Berlin (1975).

[49] W.A. Woyczynski, Geometry and martingales in Banach spaces-Part II: independent increments. Probability on Banach spaces, pp. 267–517, Ed. J. Kuelbs, Advances in Probability and Related Topics 4, Marcel Dekker, New York (1978).

[50] M. Yor, Existence et unicité de diffusions à valeurs dans un espace de Hilbert. Calcul des Probabilités et Statistique, **X**, n. 1, pp. 55–88, (1974).

[51] K. Yosida, Functional analysis. Classics in Mathematics, Springer Verlag, Berlin-New York (1980 sixth edition).

18 A Stabilization Phenomenon for a Class of Stochastic Partial Differential Equations

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18.1 Introduction

In this short chapter we consider real, parabolic, stochastic initial-Neumann boundary value problems of the form

$$\begin{aligned} du(x,t) &= (\triangle u(x,t) + g(u(x,t)))\,dt + h(u(x,t))W(x,dt), \\ (x,t) &\in D \times \mathbb{R}^+, \\ u(x,0) &= \varphi(x), \quad x \in D, \\ \frac{\partial u(x,t)}{\partial n(x)} &= 0, \quad (x,t) \in \partial D \times \mathbb{R}^+ \end{aligned} \tag{18.1}$$

on a bounded domain $D \subset \mathbb{R}^d$ with a smooth boundary ∂D and satisfying the cone property, where $d \in \mathbb{N}^+$ and $W(.,t)$ is an $L^2(D)$-valued Wiener process to be described below. Our aim is to investigate the long-time behavior of solutions to (18.1), assuming that the nonlinearities g and h are real-valued, twice continuously differentiable functions on a finite interval $[u_0, u_1]$, vanish at the boundary points and are strictly positive on (u_0, u_1); moreover, we also assume that g is concave. Our main result is that for any $d \in \mathbb{N}^+$, the solutions to (18.1) converge to an asymptotic random variable which takes values in the set $\{u_0, u_1\}$ provided the ratio $\frac{h}{g}$ remains bounded; moreover, if $d = 1$ or $d = 2$, and if the derivative $g'(u_0)$ is small enough, we show that the asymptotic state is equal to u_1 almost surely. Also, if we replace g by $-g$ in (18.1) and if $|g'(u_1)|$ is small enough, the asymptotic state is equal to u_0 almost surely. In the second part of the article we consider the case of a one-dimensional (1D) heat equation driven by a space–time white noise. Under similar conditions as those outlined above, we also prove the convergence of solutions to a random variable which takes values in $\{u_0, u_1\}$, but this time a different method of proof is necessary.

The case of (18.1) driven by finite-dimensional Brownian motions was considered in [3] for equations interpreted in Stratonovich's sense, and in [2], [4] for equations interpreted in Itô's sense. However, the methods of proof used there were strongly dependent on the fact that the Brownian motions were finite dimensional; in this chapter we interpret (18.1) in Itô's sense and cannot apply all the methods of [2] and [4]; rather, the analysis of (18.1) requires different techniques which we outline below.

18.2 The case of a noise with nuclear covariance

We fix once and for all an orthonormal basis $(e_i)_{i \in \mathbb{N}^+}$ in $L^2(D)$ such that $e_i \in L^\infty(D)$ for each i, and denote the norms in $L^2(D)$ and $L^\infty(D)$ by $\|.\|_2$ and $\|.\|_\infty$, respectively. Suppose that $W(.,t)$ is an $L^2(D)$-valued Wiener process of the form

$$W(.,t) := \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} e_i(.) B_i(t)$$

where the coefficients $\lambda_i > 0$ and the e_i's are related by the condition $\alpha := \sum_{i=1}^{+\infty} \lambda_i \|e_i\|_\infty^2 < +\infty$, and where the $B_i(t)$'s are a sequence of independent, 1D, standard Brownian motions starting at the origin and defined on a stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ where $\mathcal{F}_t$ is the increasing family of σ-fields generated by the Wiener process $W(.,t)$ and the $\mathbb{P}$-null sets. We consider Problem 18.1 with the following hypotheses:

(G) We have $g \in \mathcal{C}^2([u_0, u_1])$, $g(u_0) = g(u_1) = 0$, $g(u) > 0$ for every $u \in (u_0, u_1)$, $g'(u_0) > 0$, $g'(u_1) < 0$ and g is concave.

(H) We have $h \in \mathcal{C}^2([u_0, u_1])$, $h(u_0) = h(u_1) = 0$, and $h(u) > 0$ for every $u \in (u_0, u_1)$.

(I) We have $\varphi \in L^2(D)$ and φ takes its values in the interval (u_0, u_1).

In the sequel we denote by $G_t(x,y)$ the parabolic Green's function associated with the principal part of (18.1) and define

$$(G_t f)(x) := \int_D G_t(x,y) f(y) dy$$

along with

$$Qf := \frac{1}{|D|} \int_D f(y) dy$$

for any $f \in L^2(D)$; we note that $QG_t f = Qf$ by virtue of Neumann's boundary conditions. Then, under the above conditions, we know that there exists an $L^2(D)$-valued adapted random field $u(.,t)$ of the form

$$\begin{aligned} u(x,t) &= (G_t \varphi)(x) \\ &+ \int_0^t \int_D G_{t-s}(x,y) g(u(y,s)) dy ds \\ &+ \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(\int_D G_{t-s}(x,y) h(u(y,s)) e_i(y) dy \right) B_i(ds) \end{aligned} \tag{18.2}$$

which satisfies equation (18.1) in a weak sense (see, for instance, [6]). Therefore, by applying the operator Q to both sides of (18.2) we obtain

$$\begin{aligned} Qu(.,t) = Q\varphi + \int_0^t Qg(u(.,s)) ds \qquad\qquad (18.3) \\ + \frac{1}{|D|} \sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t \left(\int_D h(u(y,s)) e_i(y) dy \right) B_i(ds). \end{aligned}$$

Furthermore, by taking into account the properties of the nonlinearities g, h and of the initial condition φ we have $u_0 \le u(x,t) \le u_1$ for all $(x,t) \in D \times \mathbb{R}^+$ almost surely (a.s.) by the comparison principle (see, for instance, [2]), and hence $u_0 \le Qu(.,t) \le u_1$ for all time a.s. As a consequence $Qu(.,t)$ is a bounded submartingale, so that by the martingale convergence theorem we get the existence of

$$\lim_{t \to +\infty} Qu(.,t) := \widetilde{u} \tag{18.4}$$

a.s. Our first asymptotic result for the solutions to (18.1) is the following.

Theorem 18.1 *Assume that (G), (H), and (I) hold, and that*

$$\max_{u_0<u<u_1} \frac{h(u)}{g(u)} < +\infty$$

which implies

$$\beta := \alpha \max_{u_0<u<u_1} \frac{h^2(u)}{g^2(u)} < +\infty.$$

Then, the random variable $\widetilde{u}$ *takes values in the set* $\{u_0, u_1\}$ *and we have*

$$\lim_{t\to+\infty} \|u(.,t) - \widetilde{u}\|_2 = 0$$

a.s.

Proof On the one hand, by taking the mathematical expectation on both sides of (18.3) and by using the fact that $Qu(.,t)$ and $Q\varphi$ are uniformly bounded in t we obtain

$$\mathbb{E}\int_0^{+\infty} Qg(u(.,t))dt < +\infty \tag{18.5}$$

since $g \geq 0$. On the other hand, from Itô's formula we have

$$\begin{aligned}
\|u(.,t)\|_2^2 = \|\varphi\|_2^2 &- 2\int_0^t \int_D (\nabla u(x,s), \nabla u(x,s))_{\mathbb{R}^d}\, dxds \\
&+ 2\int_0^t \int_D u(x,s)g(u(x,s))dxds \\
&+ 2\sum_{i=1}^{+\infty} \lambda_i^{\frac{1}{2}} \int_0^t u(x,s)\left(\int_D u(x,s)h(u(x,s))e_i(x)dx\right) B_i(ds) \\
&+ \sum_{i=1}^{+\infty} \lambda_i \int_0^t \left(\int_D h^2(u(x,s))e_i^2(x)dx\right) ds.
\end{aligned}$$

Using the above hypotheses and taking expectations on both sides of the preceding equality we then get

$$\begin{aligned}
2\mathbb{E}\int_0^t \|\nabla u(.,s)\|_2^2\, ds &\leq \|\varphi\|_2^2 + 2\mathbb{E}\int_0^t \int_D u(x,s)g(u(x,s))dxds \\
&\quad + \beta \int_0^t \left(\int_D \mathbb{E}g^2(u(x,s))dx\right) ds \\
&\leq \|\varphi\|_2^2 + c\mathbb{E}\int_0^{+\infty} Qg(u(.,s))ds < +\infty.
\end{aligned}$$

Therefore, by invoking the Poincaré–Wirtinger inequality we infer that

$$\mathbb{E}\int_0^{+\infty} \|u(.,t) - Qu(.,t)\|_2^2\, dt < +\infty. \tag{18.6}$$

Estimates (18.5) and (18.6) now imply the existence of an increasing sequence of positive numbers t_n such that $\lim_{n\to+\infty} t_n = +\infty$ and

$$\lim_{n\to+\infty} \mathbb{E}Qg(u(.,t_n)) = 0,$$

along with

$$\lim_{n\to+\infty} \mathbb{E}\left\|u(.,t_n) - Qu(.,t_n)\right\|_2^2 = 0.$$

As a consequence, we may write

$$\begin{aligned}0 &\leq \mathbb{E}Qg(\widetilde{u}) \leq \left|\mathbb{E}Qg(\widetilde{u}) - \mathbb{E}g(Qu(.,t_n))\right| \\ &+ \left|\mathbb{E}g(Qu(.,t_n)) - \mathbb{E}Qg(u(.,t_n))\right| \\ &+ \mathbb{E}Qg(u(.,t_n)) \\ &\leq c\mathbb{E}\left|\widetilde{u} - Qu(.,t_n)\right| \\ &+ c\mathbb{E}\left\|u(.,t_n) - Qu(.,t_n)\right\|_2 \\ &+ \mathbb{E}Qg(u(.,t_n)) \to 0\end{aligned}$$

as $n \to +\infty$. Therefore $g(\widetilde{u}) = 0$ a.s. which implies that $\widetilde{u} \in \{u_0, u_1\}$ a.s.. Now define the auxiliary function

$$a(u) := (u - u_0)(u_1 - u).$$

Then

$$\begin{aligned}|D|^{-1}\left\|u(.,t) - Qu(.,t)\right\|_2^2 &= a(Qu(.,t)) - Qa(u(.,t)) \\ &\leq a(Qu(.,t)) \to a(\widetilde{u}) = 0\end{aligned}$$

a.s. as $t \to +\infty$, which implies the desired result by virtue of (18.4). ■

While the preceding result holds for every $d \in \mathbb{N}^+$, we now prove that under an additional condition and with $d = 1$ or $d = 2$, we have either $\widetilde{u} = u_1$ a.s. or $\widetilde{u} = u_0$ a.s. We do not know, however, whether the following result holds for $d \geq 3$.

Theorem 18.2 *The hypotheses are the same as in Theorem 18.1; assume furthermore that $d = 1$ or $d = 2$; then the following conclusions hold:*

(1) If $1 - \frac{1}{2}g'(u_0)\beta > 0$, we have $\widetilde{u} = u_1$ a.s.

(2) If we replace g by $-g$ in (18.1) and if $1 + \frac{1}{2}g'(u_1)\beta > 0$, we have $\widetilde{u} = u_0$ a.s.

Proof We only prove (1), the proof of (2) being identical; let $\mathcal{G}$ be any primitive of $u \mapsto \frac{1}{g(u)}$ on (u_0, u_1); then, by Itô's formula we have

$$\begin{aligned}\mathcal{G}(Qu(.,t)) = \mathcal{G}(Q\varphi) &+ \int_0^t \frac{Qg(u(.,s))}{g(Qu(.,s))} ds \\ &- \frac{1}{2|D|^2}\sum_{i=1}^{+\infty}\lambda_i \int_0^t \frac{g'(Qu(.,s))}{g^2(Qu(.,s))}\left|\int_D h(u(y,s))e_i(y)dy\right|^2 ds \\ &+ \int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds)\end{aligned} \tag{18.7}$$

where

$$\begin{aligned}&\int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \\ &:= \frac{1}{|D|}\sum_{i=1}^{+\infty}\lambda_i^{\frac{1}{2}} \int_0^t \frac{1}{g(Qu(.,s))}\left(\int_D h(u(y,s))e_i(y)dy\right) B_i(ds).\end{aligned} \tag{18.8}$$

From our hypotheses we can easily infer that the above integrals are well defined and finite. Now, define the process

$$\gamma(t) := \frac{1}{|D|^2} \frac{1}{g(Qu(.,t))Qg(u(.,t))} \sum_{i=1}^{+\infty} \lambda_i \left| \int_D h(u(y,t)) e_i(y) dy \right|^2 . \tag{18.9}$$

Owing to (18.9) and to the above hypotheses we have

$$\begin{aligned} \gamma(t) &\leq \frac{1}{|D|^2} \frac{\alpha}{g(Qu(.,t))Qg(u(.,t))} \left(\int_D h(u(y,t)) dy \right)^2 \\ &\leq \beta \frac{Qg(u(.,t))}{g(Qu(.,t))} \end{aligned} \tag{18.10}$$

a.s. for every $t \in \mathbb{R}^+$. Furthermore, in terms of this process we can rewrite (18.7) as

$$\begin{aligned} \mathcal{G}(Qu(.,t)) = \mathcal{G}(Q\varphi) &+ \int_0^t \frac{Qg(u(.,s))}{g(Qu(.,s))} \left(1 - \frac{1}{2} g'(Qu(.,s))\gamma(s) \right) ds \\ &+ \int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \end{aligned} \tag{18.11}$$

and we now analyze more closely the second and third terms of (18.11); we first claim that for every $\delta \in \mathbb{R}^+$, there exists a sequence of positive numbers t_n with $\lim_{n\to+\infty} t_n = +\infty$ such that the relation

$$\lim_{n \to +\infty} t_n^{-\frac{1}{2}-\delta} \int_0^{t_n} \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) = 0 \tag{18.12}$$

holds a.s. In order to show (18.12), it is sufficient to prove that

$$\lim_{t \to +\infty} t^{-1-2\delta} \mathbb{E} \left| \int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \right|^2 = 0. \tag{18.13}$$

But since g is concave, by Jensen's inequality we have

$$Qg(u(.,t)) \leq g(Qu(.,t)) \tag{18.14}$$

so that (18.13) follows from the estimate

$$\begin{aligned} &\mathbb{E} \left| \int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \right|^2 \\ &= \frac{1}{|D|^2} \sum_{i=1}^{+\infty} \lambda_i \mathbb{E} \int_0^t \frac{1}{g^2(Qu(.,s))} \left| \int_D h(u(y,s)) e_i(y) dy \right|^2 ds \\ &\leq \frac{\beta}{|D|^2} \, \mathbb{E} \int_0^t \frac{1}{g^2(Qu(.,s))} \left| \int_D g(u(y,s)) dy \right|^2 ds \\ &= \beta \mathbb{E} \int_0^t \frac{(Qg(u(.,s)))^2}{g^2(Qu(.,s))} \, ds \leq \beta t. \end{aligned}$$

Next, we want to investigate the contribution $-\frac{1}{2} g'(Qu(.,s))\gamma(s)$ in the expression of $\mathcal{G}(Qu(.,t))$; to this end we note that there exists a constant $c \in \mathbb{R}^+$ such that the inequality

$$-\frac{1}{2} g'(Qu(.,s))\gamma(s) \geq -\frac{1}{2} g'(u_0)\beta - c \left| Qu(.,s) - u_0 \right| \tag{18.15}$$

holds a.s. Indeed, because of (18.10) and (18.14), (18.15) follows from the relations

$$
\begin{aligned}
-\frac{1}{2}g'(Qu(.,s))\gamma(s) &= -\frac{1}{2}g'(u_0)\gamma(s) - \frac{1}{2}\left(g'(Qu(.,s)) - g'(u_0)\right)\gamma(s) \\
&\geq -\frac{1}{2}g'(u_0)\beta - \frac{1}{2}c\beta\left|Qu(.,s) - u_0\right|.
\end{aligned}
$$

We then substitute (18.15) into (18.11) to obtain

$$
\begin{aligned}
\mathcal{G}(Qu(.,t)) \geq \mathcal{G}(Q\varphi) &+ \left(1 - \frac{1}{2}g'(u_0)\beta\right)t \\
&- c\int_0^t \left[\left|Qu(.,s) - u_0\right| + \left|\frac{Qg(u(.,s))}{g(Qu(.,s))} - 1\right|\right] ds \\
&+ \int_0^t \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds).
\end{aligned}
\tag{18.16}
$$

Now, to prove that $\widetilde{u} = u_1$ a.s. is equivalent to proving that

$$
\mathbb{P}\left\{\omega \in \Omega : \lim_{t\to+\infty} Qu(.,t) = u_0\right\} = 0.
$$

To this end we define the set

$$
S := \left\{\omega \in \Omega : \lim_{t\to+\infty} Qu(.,t) = u_0 \text{ and } \lim_{n\to+\infty} t_n^{-\frac{1}{2}-\delta}\int_0^{t_n} \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \ = 0\right\}
$$

where t_n is the particular sequence in (18.12), and prove that if we assume $\mathbb{P}\left\{\omega \in \Omega : \lim_{t\to+\infty} Qu(.,t) = u_0\right\} > 0$, then for every $\omega \in S$ we have

$$
\lim_{n\to+\infty} \mathcal{G}(Qu(.,t_n)) = +\infty, \tag{18.17}
$$

that is, $\lim_{n\to+\infty} Qu(.,t_n) = u_1$, a contradiction. The proof of (18.17) will be done in several steps; in what follows we fix $\omega \in S$.

Step 1. We prove that there exist $c_1(\omega) > 0$ and $n_0(\omega) \in \mathbb{N}^+$ such that for all $n \geq n_0(\omega)$ we have

$$
\int_0^{t_n} \left|Qu(.,s) - u_0\right| ds \leq c_1(\omega) + \frac{1}{2c}\left(1 - \frac{1}{2}g'(u_0)\beta\right)t_n, \tag{18.18}
$$

where c is the constant appearing in (18.16). In fact, given $\varepsilon = \frac{1}{2c}\left(1 - \frac{1}{2}g'(u_0)\beta\right)$ there exists $t_\varepsilon(\omega) > 0$ such that for all $t \geq t_\varepsilon(\omega)$ we have

$$
\left|Qu(.,t) - u_0\right| < \varepsilon.
$$

Then, it is sufficient to split the integral as

$$
\int_0^{t_n} \left|Qu(.,s) - u_0\right| ds = \int_0^{t_\varepsilon(\omega)} \left|Qu(.,s) - u_0\right| ds + \int_{t_\varepsilon(\omega)}^{t_n} \left|Qu(.,s) - u_0\right| ds,
$$

and choose $n_0(\omega)$ in such a way that $t_n > t_\varepsilon(\omega)$ for all $n \geq n_0(\omega)$, and subsequently take

$$
c_1(\omega) := \int_0^{t_\varepsilon(\omega)} \left|Qu(.,s) - u_0\right| ds.
$$

Step 2. From the convergence

$$
\lim_{n\to+\infty} t_n^{-\frac{1}{2}-\delta}\int_0^{t_n} \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \ = 0
$$

we can choose $n_1(\omega) \in \mathbb{N}^+$ such that for all $n \geq n_1(\omega)$ we have

$$-t_n^{\frac{1}{2}+\delta} \leq \int_0^{t_n} \frac{Qh(u(.,s))}{g(Qu(.,s))} W(.,ds) \leq t_n^{\frac{1}{2}+\delta}. \tag{18.19}$$

Step 3. In order to estimate (18.16) we also need the following result: there exists $s(\omega) > 0$ such that for every $s \geq s(\omega)$ we have

$$g(Qu(.,s)) \geq \frac{1}{2} g'(u_0)\left(Qu(.,s) - u_0\right). \tag{18.20}$$

In fact, notice that

$$\lim_{u \to u_0} \frac{g(u)}{u - u_0} = g'(u_0)$$

so that

$$g(u) > \frac{1}{2} g'(u_0)\left(u - u_0\right)$$

for u close enough to u_0; on the other hand, we know that $\lim_{s \to +\infty} Qu(.,s) = u_0$ and the result follows.

Step 4. We claim that for $d = 1$ and $d = 2$ there exist $c_2(\omega) > 0$ and $n_2(\omega) \in \mathbb{N}^+$ such that for all $n \geq n_2(\omega)$ we have

$$\int_0^{t_n} \left| \frac{Qg(u(.,s))}{g(Qu(.,s))} - 1 \right| ds \leq c_2(\omega) + c \int_{s(\omega)}^{t_n} \|\nabla u(.,s)\|_2^{\frac{2d}{d+2}} \, ds. \tag{18.21}$$

In order to prove (18.21) we first apply Taylor's expansion up to the second order and get

$$\begin{aligned}
\left| \frac{Qg(u(.,s))}{g(Qu(.,s))} - 1 \right| &= \frac{1}{g(Qu(.,s))} \left| Qg(u(.,s)) - g(Qu(.,s)) \right| \\
&\leq \frac{c}{2\,|D|} \frac{\|u(.,s) - Qu(.,s)\|_2^2}{g(Qu(.,s))}.
\end{aligned}$$

From this estimate we obtain, for $t_n > s(\omega)$ and by using (18.20)

$$\begin{aligned}
\int_0^{t_n} \left| \frac{Qg(u(.,s))}{g(Qu(.,s))} - 1 \right| ds &\leq \frac{c}{2\,|D|} \int_0^{t_n} \frac{\|u(.,s) - Qu(.,s)\|_2^2}{g(Qu(.,s))} ds \\
&\leq \frac{c}{2\,|D|} \int_0^{s(\omega)} \frac{\|u(.,s) - Qu(.,s)\|_2^2}{g(Qu(.,s))} ds \\
&\quad + \frac{c}{g'(u_0)\,|D|} \int_{s(\omega)}^{t_n} \frac{\|u(.,s) - Qu(.,s)\|_2^2}{Qu(.,s) - u_0} ds \\
&:= c_2(\omega) + c \int_{s(\omega)}^{t_n} \frac{\|u(.,s) - Qu(.,s)\|_2^2}{Qu(.,s) - u_0} ds.
\end{aligned}$$

Now we apply the result of Proposition 18.1 to get

$$\begin{aligned}
\|u(.,s) - Qu(.,s)\|_2^2 &\leq c\,\|u(.,s) - Qu(.,s)\|_1^{\frac{4}{d+2}} \,\|u(.,s) - Qu(.,s)\|_{1,2}^{\frac{2d}{d+2}} \\
&\leq c\,|Qu(.,s) - u_0|^{\frac{4}{d+2}} \,\|u(.,s) - Qu(.,s)\|_{1,2}^{\frac{2d}{d+2}}
\end{aligned}$$

where we have used the fact that

$$\|u(.,s) - Qu(.,s)\|_1 \leq 2|D|(Qu(.,s) - u_0).$$

From this estimate we obtain the inequalities

$$\begin{aligned}\int_{s(\omega)}^{t_n} \frac{\|u(.,s)-Qu(.,s)\|_2^2}{Qu(.,s)-u_0} ds &\leq c\int_{s(\omega)}^{t_n} |Qu(.,s)-u_0|^{\frac{2-d}{d+2}} \|u(.,s)-Qu(.,s)\|_{1,2}^{\frac{2d}{d+2}}\, ds \\ &\leq c\int_{s(\omega)}^{t_n} \|u(.,s)-Qu(.,s)\|_{1,2}^{\frac{2d}{d+2}}\, ds\end{aligned}$$

by using for the first time the fact that $d=1$ or $d=2$, and we notice that

$$\begin{aligned}\|u(.,s)-Qu(.,s)\|_{1,2}^{\frac{2d}{d+2}} &= \left(\|u(.,s)-Qu(.,s)\|_2^2 + \|\nabla u(.,s)\|_2^2\right)^{\frac{d}{d+2}} \\ &\leq c\,\|\nabla u(.,s)\|_2^{\frac{2d}{d+2}}\end{aligned}$$

because of the Poincaré–Wirtinger inequality, which completes the proof of (18.21).

Step 5. The substitution of estimates (18.18), (18.19), and (18.21) with $d=1$ into (18.16) yields for all $\omega\in S$ and all $n\geq \max(n_0(\omega), n_1(\omega), n_2(\omega))$ the inequality

$$\begin{aligned}\mathcal{G}(Qu(.,t_n)) &\geq \mathcal{G}(Q\varphi) + \frac{1}{2}\left(1-\frac{1}{2}g'(u_0)\beta\right)t_n - c(\omega) \\ &\quad - c\int_{s(\omega)}^{t_n} \|\nabla u(.,s)\|_2^{\frac{2}{3}}\, ds - t_n^{\frac{1}{2}+\delta}.\end{aligned} \tag{18.22}$$

Now define $\eta := \frac{1}{2}\left(1-\frac{1}{2}g'(u_0)\beta\right) > 0$, invoke Hölder's inequality along with the interpolated Young's inequality to write

$$\begin{aligned}c\int_{s(\omega)}^{t_n} \|\nabla u(.,s)\|_2^{\frac{2}{3}}\, ds &\leq c t_n^{\frac{2}{3}}\left(\int_{s(\omega)}^{t_n} \|\nabla u(.,s)\|_2^2\, ds\right)^{\frac{1}{3}} \\ &\leq \frac{\eta}{2}t_n + \frac{c^3}{\eta^2}\int_{s(\omega)}^{t_n} \|\nabla u(.,s)\|_2^2\, ds.\end{aligned} \tag{18.23}$$

Finally, we substitute (18.23) into (18.22) to get

$$\mathcal{G}(Qu(.,t_n)) \geq \mathcal{G}(Q\varphi) + \frac{\eta}{2}t_n - c(\omega) - \frac{c^3}{\eta^2}\int_0^{+\infty} \|\nabla u(.,s)\|_2^2\, ds - t_n^{\frac{1}{2}+\delta} \to +\infty$$

a.s. as $n\to+\infty$, provided we choose $\delta<\frac{1}{2}$. We can handle the case $d=2$ in a completely similar way. ■

18.3 The case of a space–time white noise

In this section we take $D=(0,1)$, assume that $\lambda_i=1$ for every i, and define the two-parameter process

$$\overline{W}(x,t) := \int_0^x W(y,t)dy$$

for every $(x,t)\in D\times\mathbb{R}^+$; we consider the 1D initial-boundary value problem

$$\begin{aligned}&\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial^2 x} + g(u(x,t)) + h(u(x,t))\frac{\partial^2\overline{W}(x,t)}{\partial x\partial t}, \\ &\quad (x,t)\in D\times\mathbb{R}^+, \\ &u(x,0)=\varphi(x), \quad x\in D, \\ &\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(1,t)}{\partial x} = 0, \quad t\in\mathbb{R}^+.\end{aligned} \tag{18.24}$$

Our aim is still to investigate the long-time behavior of solutions to (18.24); as we shall see momentarily the conclusions of Theorem 18.1 still hold in this case, although the methods of proof of the preceding section do not all remain valid. We introduce the following hypotheses.

(G') We have $g \in \mathcal{C}^1([u_0, u_1])$, $g(u_0) = g(u_1) = 0$, and $g(u) > 0$ for every $u \in (u_0, u_1)$.

(H') We have $h \in \mathcal{C}^1([u_0, u_1])$, $h(u_0) = h(u_1) = 0$, and $h(u) > 0$ for every $u \in (u_0, u_1)$.

(I) We have $\varphi \in L^2((0,1))$ and φ takes its values in the interval (u_0, u_1).

The parabolic Green's function $G_t(x,y)$ along with the projection operator Q are defined exactly as in the preceding section. Then, under the above conditions, we know that there exists a process $u(.,t)$ of the form

$$\begin{aligned} u(x,t) = (G_t\varphi)(x) &+ \int_0^t \int_0^1 G_{t-s}(x,y) g(u(y,s)) dy ds \\ &+ \int_0^t \int_0^1 G_{t-s}(x,y) h(u(y,s)) \overline{W}(dy,ds) \end{aligned} \tag{18.25}$$

which satisfies equation (18.24) in a weak sense (see, for instance, [7]). Consequently, by applying the operator Q to both sides of (18.25) we obtain

$$Qu(.,t) = Q\varphi + \int_0^t Qg(u(.,s))ds + \int_0^t \int_0^1 h(u(y,s))\overline{W}(dy,ds). \tag{18.26}$$

Thus, exactly as in the preceding section, by invoking the martingale convergence theorem we can conclude that the limit

$$\lim_{t\to+\infty} Qu(.,t) := \widetilde{u} \tag{18.27}$$

exists a.s., the comparison principle we use being this time from [5] rather than from [2]. Then we obtain the following asymptotic result for the solutions to (18.24).

Theorem 18.3 *Suppose that (G'), (H'), and (I) hold. Then, the random variable $\widetilde{u}$ takes values in the set $\{u_0, u_1\}$ and we have*

$$\lim_{t\to+\infty} \|u(.,t) - \widetilde{u}\|_2 = 0$$

a.s.

Proof We carry out the proof in several steps.

Step 1. By taking the mathematical expectation on both sides of (18.26), and by using the fact that $Qu(.,t)$ and $Q\varphi$ are uniformly bounded we obtain

$$\mathbb{E}\int_0^{+\infty} Qg(u(.,t))dt < +\infty \tag{18.28}$$

since $g \geq 0$. We now apply Itô's formula to (18.26) and get

$$\begin{aligned} (Qu(.,t))^2 - (Q\varphi)^2 = 2\int_0^t & Qu(.,s)Qg(u(.,s))ds \\ &+ \int_0^t \int_0^1 h^2(u(y,s))dyds \\ &+ 2\int_0^t \int_0^1 Qu(.,s)h(u(y,s))\overline{W}(dy,ds). \end{aligned}$$

As a consequence we have

$$\begin{aligned}
&\mathbb{E}\int_0^t \int_0^1 h^2(u(y,s))dyds \\
&= \mathbb{E}(Qu(.,t))^2 - (Q\varphi)^2 - 2\mathbb{E}\int_0^t Qu(.,s)Qg(u(.,s))ds \le c < +\infty
\end{aligned}$$

and thereby

$$\mathbb{E}\int_0^{+\infty} \int_0^1 h^2(u(y,s))dyds < +\infty. \tag{18.29}$$

Step 2. We claim that

$$\mathbb{E}\int_0^{+\infty} \|u(.,t) - Qu(.,t)\|_2^2 \, dt < +\infty. \tag{18.30}$$

In order to prove (18.30), let us consider the difference

$$\begin{aligned}
u(x,t) - Qu(.,t) = (G_t\varphi)(x) - Q\varphi& \\
+ \int_0^t \left[G_{t-s}g(u(.,s))(x) - Qg(u(.,s))\right] ds& \\
+ \int_0^t \int_0^1 \left(G_{t-s}(x,y) - 1\right) h(u(y,s))\overline{W}(dy,ds)&
\end{aligned}$$

and define

$$M_t(x) := \int_0^t \int_0^1 \left(G_{t-s}(x,y) - 1\right) h(u(y,s))\overline{W}(dy,ds)$$

along with

$$N_t(x) := (G_t\varphi)(x) - Q\varphi + \int_0^t \left[G_{t-s}g(u(.,s))(x) - Qg(u(.,s))\right] ds.$$

We wish to compute the expectation of the square of the $L^2(D)$-norms of these processes; on the one hand, we have

$$\begin{aligned}
\mathbb{E}\,\|M_t\|_2^2 &= \int_0^t \int_0^1 \int_0^1 \left(G_{t-s}(x,y) - 1\right)^2 \mathbb{E}h^2(u(y,s))dydxds \\
&= \int_0^t \int_0^1 \left(G_{2(t-s)}(y,y) - 1\right) \mathbb{E}h^2(u(y,s))dyds.
\end{aligned}$$

Using then the estimate $G_{2(t-s)}(y,y) \le \frac{c}{(t-s)^{\frac{1}{2}}}$ valid for some $c \in \mathbb{R}^+$, we obtain

$$\mathbb{E}\,\|M_t\|_2^2 \le \int_{(t-c^2)_+}^t \left(\frac{c}{(t-s)^{\frac{1}{2}}} - 1\right) \int_0^1 \mathbb{E}h^2(u(y,s))dyds.$$

Finally, by Fubini's theorem we get

$$\begin{aligned}
\mathbb{E}\int_0^{+\infty} \|M_t\|_2^2 \, dt &\le \int_0^{+\infty} \int_s^{s+c^2} \left(\frac{c}{(t-s)^{\frac{1}{2}}} - 1\right) \int_0^1 \mathbb{E}h^2(u(y,s))dydsdt \\
&= c^2 \int_0^{+\infty} \int_0^1 \mathbb{E}h^2(u(y,s))dyds < +\infty
\end{aligned}$$

because of (18.29). On the other hand, regarding the process N_t we can prove that

$$\|N_t\|_2^2 - \|\varphi - Q\varphi\|_2^2 = -2\int_0^t \|\nabla N_s\|_2^2\,ds + \int_0^t (N_s, g(u(.,s)) - Qg(u(.,s)))_2 ds, \tag{18.31}$$

and that $QN_t = 0$ because of Neumann's boundary conditions; but by the Poincaré–Wirtinger inequality we then have

$$\|N_t\|_2^2 \leq c\,\|\nabla N_t\|_2^2\,,$$

so that by integrating this inequality with respect to time and by using (18.31) along with interpolation we get

$$\begin{aligned}
\int_0^t \|N_s\|_2^2\,ds &\leq c\int_0^t \|\nabla N_s\|_2^2\,ds \\
&\leq \frac{c}{2}\left[\|\varphi - Q\varphi\|_2^2 + \int_0^t (N_s, g(u(.,s)) - Qg(u(.,s)))_2\,ds\right] \\
&\leq \frac{c}{2}\,\|\varphi - Q\varphi\|_2^2 + \frac{1}{2}\int_0^t \|N_s\|_2^2\,ds \\
&+ \frac{c^2}{2}\int_0^t \|g(u(.,s)) - Qg(u(.,s))\|_2^2\,ds.
\end{aligned}$$

As a consequence we have

$$\mathbb{E}\int_0^{+\infty} \|N_t\|_2^2\,dt < +\infty$$

by virtue of (18.28) and the properties of g, which, along with the preceding estimates, completes the proof of (18.30).

Step 3. The above considerations allow us to conclude that there exists an increasing sequence of positive numbers t_n with $\lim_{n\to+\infty} t_n = +\infty$ such that

$$\lim_{n\to+\infty} \mathbb{E}Qg(u(.,t_n)) = 0$$

and

$$\lim_{n\to+\infty} \mathbb{E}\,\|u(.,t_n) - Qu(.,t_n)\|_2^2 = 0.$$

Then, as above we can write

$$\begin{aligned}
0 &\leq \mathbb{E}Qg(\widetilde{u}) \leq |\mathbb{E}Qg(\widetilde{u}) - \mathbb{E}g(Qu(.,t_n))| \\
&+ |\mathbb{E}g(Qu(.,t_n)) - \mathbb{E}Qg(u(.,t_n))| \\
&+ \mathbb{E}Qg(u(.,t_n)) \\
&\leq c\mathbb{E}\,|\widetilde{u} - Qu(.,t_n)| \\
&+ c\mathbb{E}\,\|u(.,t_n) - Qu(.,t_n)\|_2 \\
&+ \mathbb{E}Qg(u(.,t_n)) \to 0
\end{aligned}$$

as $n \to +\infty$. Therefore, we have $g(\widetilde{u}) = 0$ a.s., which implies that $\widetilde{u} \in \{u_0, u_1\}$ a.s.

Step 4. As in the preceding section we define the function

$$a(u) = (u - u_0)(u_1 - u).$$

Then

$$\begin{aligned}\|u(.,t) - Qu(.,t)\|_2^2 &= a(Qu(.,t)) - Qa(u(.,t)) \\ &\leq a(Qu(.,t)) \to a(\widetilde{u}) = 0\end{aligned}$$

a.s. as $t \to +\infty$, which again implies the desired result by virtue of (18.27). ∎

It is unknown at the present time whether there exists a version of Theorem 2 for (18.24).

18.4 Appendix

Proposition 18.1 *For any $d \in \mathbb{N}^+$ there exists $c \in \mathbb{R}^+$ such that for all $f \in H^1(D)$ we have*

$$\|f\|_2 \leq c \|f\|_1^{\frac{2}{d+2}} \|f\|_{1,2}^{\frac{d}{d+2}} .$$

Proof Let $s \in (0,1)$ be such that $2s < d$. By Sobolev's theorem we have the continuous embedding $H^s(D) \to L^r(D)$ where $2 \leq r \leq \frac{2d}{d-2s}$, along with the inequalities

$$\|f\|_r \leq c \|f\|_{H^s(D)} \leq c \|f\|_2^{1-s} \|f\|_{1,2}^s$$

where the second estimate is Ehrling–Browder's inequality in the fractional order case (see, for instance, [1]). Choosing $r = \frac{2d}{d-2s}$ we get

$$\|f\|_{\frac{2d}{d-2s}} \leq \; c \|f\|_2^{1-s} \|f\|_{1,2}^s . \tag{18.32}$$

Now set $\theta := \frac{2s}{d+2s}$; then, by classical interpolation we have

$$\|f\|_2 \leq \|f\|_1^{\theta} \|f\|_{\frac{2d}{d-2s}}^{1-\theta} \tag{18.33}$$

so that by substituting (18.32) into (18.33) we obtain

$$\|f\|_2 \leq \; c \|f\|_1^{\theta} \|f\|_2^{(1-s)(1-\theta)} \|f\|_{1,2}^{s(1-\theta)} , \tag{18.34}$$

or

$$\|f\|_2^{1-(1-s)(1-\theta)} \leq \; c \|f\|_1^{\theta} \|f\|_{1,2}^{s(1-\theta)}$$

for $\|f\|_2 \neq 0$, which leads to the desired inequality by inserting the value of θ. ∎

Acknowledgments

The research of the first author was supported in part by the *Dirección General de Investigación, Ministerio de Ciencia y Tecnologia* of the Spanish government under grant BFM 2000-0598. The research of the second author was supported in part by the Centre de Recerca Matemàtica in Barcelona with funds from the program *Estancias de Investigadores Extranjeros en Régimen de Año Sabático en España* of the Spanish Ministry of Education and Culture, and by the ETH-Forschungsinstitut für Mathematik in Zürich. He would like to thank Professor M. Castellet and Professor M. Burger for their respective invitations to the above institutions and for their very kind hospitality.

References

[1] R.A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics Series **65** Academic Press, New York, 1978.

[2] B. Bergé, I.D. Chueshov, P.-A. Vuillermot, *On the Behavior of Solutions to Certain Parabolic SPDE's Driven by Wiener Processes,* Stoch. Proc. Appl. **92** (2001) 237-263.

[3] I.D. Chueshov, P.-A. Vuillermot, *Long-Time Behavior of Solutions to a Class of Stochastic Parabolic Equations with Homogeneous White Noise: Stratonovich's Case,* Probab. Theory Relat. Fields **112**, 149-202 (1998).

[4] I.D. Chueshov, P.-A. Vuillermot, *Long-Time Behavior of Solutions to a Class of Stochastic Parabolic Equations with Homogeneous White Noise: Itô's Case,* Stoch. Anal. Appl. **18**, 581-615 (2000).

[5] C. Donati-Martin, E. Pardoux, *White Noise Driven SPDEs with Reflection,* Probab. Theory Relat. Fields **95**, 1-24 (1993).

[6] M. Sanz-Solé, P.-A. Vuillermot, *Equivalence and Hölder-Sobolev Regularity of Solutions for a Class of Non Autonomous Stochastic Partial Differential Equations,* Ann. I.H. Poincaré-PR **39** (2003), 703-742.

[7] J.B. Walsh, *An Introduction to Stochastic Partial Differential Equations,* in: École d'Été de Probabilités de Saint Flour XIV, Lecture Notes in Mathematics **1180** Springer, New York, 1986, pp. 265-439.

19 Stochastic Heat and Wave Equations Driven by an Impulsive Noise

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19.1 Introduction

We are concerned with the stochastic heat equation

$$du(t) = (\Delta u(t) + f(u(t)))\, dt + b(u(t))PdZ(t), \quad u(0) = \zeta, \tag{19.1}$$

and stochastic wave equation

$$\begin{aligned} &du(t) = u(t)dt, \\ &dv(t) = (\Delta u(t) + f(u(t)))\, dt + b(u(t))PdZ(t), \quad u(0) = \zeta,\ v(0) = \eta, \end{aligned} \tag{19.2}$$

where $f, b : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous, P is a regularizing linear operator (possible identity) of convolution type, and Z is a measure-valued noise process affecting the system in an impulsive way. We consider (19.1) and (19.2) with $P = I$ either on $[0, 1]$ and $\mathbb{R}$ and with $P \neq I$ on $\mathbb{R}^d$, $d \geq 1$.

If one expects that the evolution is Markovian, then the noise process should have independent increments. Thus a convenient way to construct impulsive systems is to use the theory of Poisson random measures. Stochastic integral with respect to a Poissonian noise was introduced by K. Itô, see [Ia,Ib], where he also originated the theory of stochastic differential equations. Although a Poissonian integral can also be regarded as a stochastic integral with respect to a Lévy process taking values in a sufficiently large space, see [PZ3], our approach here to stochastic integration is based on the concept of a martingale measure as presented in [W]. The solutions to stochastic partial differential equations (SPDEs), in general, are not semimartingales, and solutions to those driven by an impulsive process, even in simple situations, do not have càdlàg modification. Instead of looking for adapted càdlàg solutions it is necessary to consider predictable solutions.

The theory of SPDEs with Poissonian noise was started around 1995. As in papers [AWZ,AW,B,HS,Mu,My] our main results are on existence and uniqueness of the solutions. In our study we use the state–space approach which leads automatically to Markovian solutions. As the state space we take a weighted L^2-space, and the interplay between the existence of solutions, the weights and the spatial intensity of jumps is one of our main concerns, see, e.g., Theorems 19.3 and 19.4. Another new feature is an investigation of equations with colored noise, that is, with impulses which are not concentrated at one point of the region, on which the equation in considered. Our Theorem 19.7 gives conditions on the spatial distribution of the impulses under which there exist solutions to equations with such noise. Finally, in contrast to [AWZ,AW,B,HS,Mu,My], we investigate also stochastic wave equations, see Theorems 19.5 and 19.6.

In Section 19.2 we introduce impulsive noise through the concept of a Poisson random measure. In the next section we are concerned with the stochastic integrals. The following sections are devoted to the existence and uniqueness theorems.

19.2 Impulsive noise processes

19.2.1 Impulsive white noise process

Let us denote by $\mathcal{P}(a)$, $a \in [0, +\infty]$ the Poisson distribution with parameter a, that is, if $a < \infty$, then $\mathcal{P}(a)(\{k\}) = \frac{a^k}{k!}e^{-a}$, $k = 0, 1, \ldots$, and $\mathcal{P}(+\infty)(\{+\infty\}) = 1$.

Let $\overline{\mathbb{Z}_+} = \{0, 1, 2, \ldots, +\infty\}$, and let $(E, \mathcal{E})$ be a measurable space. We denote by $\mathcal{P}_{\overline{\mathbb{Z}_+}}(E)$ the space of all $\overline{\mathbb{Z}_+}$-valued measures on $(E, \mathcal{E})$ considered with the σ-field generated by mappings $\mathcal{P}_{\overline{\mathbb{Z}_+}} \ni \rho \to \rho(\Gamma) \in \overline{\mathbb{Z}_+}$, $\Gamma \in \mathcal{E}$.

Definition 19.1 *Let μ be a σ-finite measure on $(E, \mathcal{E})$. A* Poisson random measure on E with intensity measure μ *is a random element π in $\mathcal{P}_{\overline{\mathbb{Z}_+}}(E)$ such that every $\Gamma \in \mathcal{E}$ the random variable $\pi(\Gamma)$ has the Poisson distribution $\mathcal{P}(\mu(\Gamma))$, and for arbitrary disjoint $\Gamma_1, \ldots, \Gamma_M$, the random variables $\pi(\Gamma_1), \ldots, \pi(\Gamma_M)$ are independent.*

In the chapter we consider SPDEs driven by a noise directly related to Poisson random measures on $E = [0, \infty) \times \mathcal{O} \times \mathbb{R}$, where $\mathcal{O} = [0, 1]$ or $\mathcal{O} = \mathbb{R}^d$. We will assume that the intensity measure μ is given by $\mu(dt, dx, d\sigma) = dt\lambda(dx)\mu(d\sigma)$, where λ and ν are σ-finite measures on $\mathcal{O}$ and $\mathbb{R}$, respectively. Note that in this case π has stationary increments with respect to t. Moreover, if λ equals Lebesgue measure, then π is also stationary in x. We call λ *intensity of jump places* and ν *intensity of jump sizes of* π. We suppose that there is a disjoint partition $\{U_n\} \subset \mathfrak{B}(\mathbb{R})$ of $\mathbb{R} \setminus \{0\}$ such that

$$\nu(U_n) \in (0, \infty) \quad \text{and} \quad \int_{U_n} \sigma\nu(d\sigma) = 0, \qquad \forall\, n. \tag{19.3}$$

We will often assume that

$$a_\nu := \int_{\mathbb{R}} \sigma^2 \nu(d\sigma) < \infty. \tag{19.4}$$

Let $\{\mathcal{O}_m\} \subset \mathfrak{B}(\mathcal{O})$ be disjoint partition of $\mathcal{O}$ such that $\lambda(\mathcal{O}_m) < \infty$ for every m. Let $(\xi_j^{(n,m)})$, $(x_j^{(n,m)})$, and $(\sigma_j^{(n,m)})$ be independent random elements, defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$ and taking values in $[0, \infty)$, $\mathcal{O}$, and $\mathbb{R}$, respectively, such that $\mathbb{P}(\xi_j^{(n,m)} > t) = e^{-\kappa_{n,m} t}$, $t \geq 0$

$$\mathbb{P}(x_j^{(n,m)} \in B) = \frac{\lambda(B \cap \mathcal{O}_m)}{\lambda(\mathcal{O}_m)}, \qquad B \in \mathfrak{B}(\mathcal{O}),$$

$$\mathbb{P}(\sigma_j^{(n,m)} \in A) = \frac{\nu(A \cap U_n)}{\nu(U_n)}, \qquad A \in \mathfrak{B}(\mathbb{R}),$$

where $\kappa_{n,m} = \nu(U_n)\,\lambda(\mathcal{O}_m)$. Note that (19.3) yields $\mathbb{E}\,\sigma_j^{(n,m)} = 0$ for all n, m, j. Set $\tau_k^{(n,m)} = \xi_1^{(n,m)} + \cdots + \xi_k^{(n,m)}$. It is easy to verified that

$$\pi(dt, dx, d\sigma) = \sum_{n,m,j} \delta_{(\tau_k^{(n,m)}, x_k^{(n,m)}, \sigma_k^{(n,m)})}(dt, dx, d\sigma) \tag{19.5}$$

is a Poisson random measure with the intensity measure $dt\lambda(dx)\nu(d\sigma)$. In fact we will assume that π is given by (19.5) with $\{\tau_k^{(n,m}\}$, $\{x_k^{(n,m}\}$, and $\{\sigma_k^{(n,m}\}$ defined as above. We will integrate with respect to the measure valued process

$$Z(t,dx) = \sum_{\tau_j^{(n,m)}\le t} \sigma_j^{(n,m)} \delta_{x_j^{(n,m)}}(dx) \tag{19.6}$$

having, by the definition of the Poisson random measure, stationary independent increments. We interpret $\sigma_j^{(n,m)}$ as an amount of energy introduced to the system at random moment $\tau_j^{(n,m)}$ and at random point $x_j^{(n,m)}$. We call the process Z the *impulsive white noise*. In the final subsection we comment on the *impulsive colored noise* in which the measure δ_x is replaced by a different measure or even by a function. Later $\mathcal{O}$ is either $[0,1]$ or $\mathbb{R}^d$ and λ is a measure absolutely continuous with respect to Lebesgue measure. As far as ν is concerned the important cases are discussed in the following two examples.

Example 19.1 *(Lévy α-stable type process)* Here $\nu(d\sigma) = |\sigma|^{-1-\alpha}d\sigma$, where $\alpha \in (0,2)$. In this case (19.4) is violated. However, we will construct a solution as a limit as $N \to \infty$ of solutions to the equations driven by noise with jumps of the size less or equal to N.

Example 19.2 *(Relativistic β-stable Lévy process)* As λ we take any nonnegative σ-finite measure on $\mathcal{O}$. The measure ν is given by

$$\left(m^2 + |y|^2\right)^{\beta/2} - m^\beta = \int_{\mathbb{R}} \left(1 - e^{-iy\sigma}\right) \nu(d\sigma), \qquad y \in \mathbb{R},$$

where $\beta \in (0,2)$. In fact ν is absolutely continuous with respect to Lebesgue measure with the density, see [B],

$$\frac{d\nu}{d\sigma}(\sigma) = \frac{\beta/2}{\Gamma(1-\beta/2)} \int_0^{+\infty} \frac{1}{\sqrt{4\pi u}} e^{-\frac{\sigma^2}{4u} - m^{2/\beta}u} \frac{1}{u^{1+\beta}} du.$$

Note that ν satisfies (19.3) and (19.4).

Before introducing the stochastic integral with respect to processes Z we look more closely at their properties. For $A \in \mathfrak{B}(\mathcal{O})$ and $t \ge 0$ write $Z(t,A) = Z(t,\cdot)(A)$. Note that $Z(\cdot,A)$ has càdlàg trajectories, $Z(0,A) = 0$, and $Z(t, A\cup B) = Z(t,A) + Z(t,B)$ for any disjoint $A,B \in \mathfrak{B}(\mathcal{O})$ and $t \ge 0$. Let $\mathfrak{F}_t^Z = \sigma\{Z(s,A) : \ s \le t, \ A \in \mathfrak{B}(\mathcal{O})\}$, and let $\mathfrak{F}_t := \overline{\mathfrak{F}^Z}_{t+}$, $t \ge 0$. Clearly, $Z(t,A) - Z(s,B)$ is independent of $\mathfrak{F}_s$. The theorem below says that under the assumption (19.4), Z is the so-called orthogonal σ-finite martingale measure on $(\mathcal{O}, \mathfrak{B}(\mathcal{O}))$ with respect to the filtration $(\mathfrak{F}_t)$, see [W] and with the covariance functional

$$\langle Z(\cdot,A), Z(\cdot,B)\rangle_t - \langle Z(\cdot,A), Z(\cdot,B)\rangle_s = (t-s)\, a_\nu\, \lambda(A\cap B).$$

Theorem 19.1 *Assume* (19.4). *(i) Let $A \in \mathfrak{B}(\mathcal{O})$ be such that $\lambda(A) < \infty$. Then $Z(t,A)$, $t \ge 0$ is a square integrable martingale with respect to $(\mathfrak{F}_t)$.*
(ii) Let $A,B \in \mathfrak{B}(\mathcal{O})$ be such that $\lambda(A) < \infty$ and $\lambda(B) < \infty$. Then the quadratic variation of $Z(\cdot,A)$ and $Z(\cdot,B)$ is given by $\langle Z(\cdot,A), Z(\cdot,B)\rangle_t = t\, a_\nu\, \lambda(A\cap B)$.

Proof Let $A \in \mathfrak{B}(\mathcal{O})$. Then

$$\mathbb{E}\,|Z(t,A)|^2 = \mathbb{E} \sum_{\tau_j^{(n,m)}\le t} \sum_{\tau_{j'}^{(n',m')}\le t} \sigma_j^{(n,m)} \sigma_j^{(n',m')} \delta_{x_j^{(n,m)}}(A) \delta_{x_{j'}^{(n',m')}}(A).$$

Since $(\tau_j^{(n,m)})$, $(x_j^{n,m})$, and $(\sigma_j^{(n,m)})$ are independent we have

$$\begin{aligned}\mathbb{E}\,|Z(t,A)|^2 \quad &= \sum_{n,m,j,n',m',j'} \mathbb{P}\left(\tau_j^{(n,m)} \le t, \tau_{j'}^{(n',m')} \le t\right) \mathbb{E}\,\sigma_j^{(n,m)} \sigma_j^{(n',m')} \\ &\qquad \times \mathbb{P}\left(x_j^{(n,m)} \in A, x_{j'}^{(n',m')} \in A\right).\end{aligned}$$

Since $\mathbb{E}\,\sigma_j^{(n,m)} = 0$ for all n,m,j, and $\sigma_j^{(n,m)}$ is independent of $\sigma_{j'}^{(n',m')}$ for $(n,m,j) \neq (n',m',j')$, we have

$$\begin{aligned}\mathbb{E}\,|Z(t,A)|^2 &= \sum_{n,m,j} \mathbb{P}\left(\tau_j^{(n,m)} \le t\right) \mathbb{E}\left(\sigma_j^{(n,m)}\right)^2 \mathbb{P}\left(x_j^{(n,m)} \in A\right) \\ &= \sum_{n,m,j} \mathbb{P}\left(\tau_j^{(n,m)} \le t\right) \int_{U_n} \sigma^2 \frac{\nu(d\sigma)}{\nu(U_n)} \frac{\lambda(A\cap \mathcal{O}_m)}{\lambda(\mathcal{O}_m)}.\end{aligned}$$

Let $\kappa_{n,m} = \nu(U_n)\lambda(\mathcal{O}_m)$. Since

$$\sum_j \mathbb{P}\left(\tau_j^{(n,m)} \le t\right) \frac{1}{\nu(U_n)\lambda(\mathcal{O}_m)} = \sum_j e^{-\kappa_{n,m} t} \frac{(\kappa_{n,m} t)^j}{j!} \frac{1}{\kappa_{n,m}} = t,$$

we have $\mathbb{E}\,|Z(t,A)|^2 = t a_\nu \lambda(A)$. In the same manner one can show that

$$\mathbb{E}\, Z(t,A)Z(s,B) = t\wedge s\, a_\nu\, \lambda(A\cap B).$$

If $\lambda(A) < \infty$, then as $Z(t,A)$, $t\ge 0$ is a centered process with independent stationary increments, it is a martingale. In order to compute its quadratic variation note that

$$\begin{aligned}&\mathbb{E}\left(Z^2(t,A) - Z^2(s,A)|\mathfrak{F}_s\right) = \mathbb{E}\left((Z(t,A) - Z(s,A)))Z(t,A) - Z(s,A))|\mathfrak{F}_s\right) \\ &= \mathbb{E}\left((Z(t,A) - Z(s,A)))(Z(t,A) - Z(s,A))\right) = (t-s)\, a_\nu\, \lambda(A).\end{aligned}$$

Thus $\langle Z(\cdot,A), Z(\cdot,A)\rangle_t = t\, a_\nu\, \lambda(A)$ and by polarization we obtain the desired form of the quadratic variance. □

19.2.2 Impulsive colored noise process

As we have already said, it is interesting to consider noise processes in the form

$$Z(t,dx) = \sum_{\tau_j^{(n,m)}\le t} \sigma_j^{(n,m)} p_{x_j^{(n,m)}}(dx) \tag{19.7}$$

where p_x is a shift of a fixed sign measure p on $\mathbb{R}^d$ by vector x. We will assume that the measure p is finite and denote its variation measure by $||p||_{\mathrm{Var}}$. Thus the impulse $\sigma\delta_x$ is replaced by $\sigma\tau_x p_{(s)}$, where $p_{(s)}(A) = p(-A)$ and τ_x is a translation operator. Equivalently, in equations (19.1) and (19.2), the operator P should be defined as $P\psi(x) = p*\psi(x)$. If the measure p has a density, which belongs to the space $U = L^2(\mathbb{R}^d, dx)$, then the process Z is a usual Lévy process and, under mild conditions, is a Hilbert space valued, square integrable martingale, with respect to which the theory of stochastic integration is well developed (see [Me]).

19.3 Stochastic integration

Assume (19.4). Our aim is to define the stochastic integral $\int_0^t X(s)dZ(s)$, $t\ge 0$ first as a real-valued and then as a vector-valued process. Before proceeding to the first goal note that the product $X(s)Z(s)$ should be interpreted as the value of a linear functional on an element from a space of measures. Thus it is reasonable to assume that $X(s)$ is a function defined on the set where the measures live, and therefore the integrants can be identified with random fields $X(\omega,s,x)$, $\omega\in\Omega$, $s\ge 0$, and $x\in\mathcal{O}$.

We define first the stochastic integral of a random field of the form

$$X(\omega, s, x) = \xi(\omega)\chi_{(a,b]}(s)\chi_A(x), \tag{19.8}$$

where $0 \le a < b < \infty$, ξ is a bounded $\mathfrak{F}_a$-measurable random variable, and $A \in \mathfrak{B}(\mathcal{O})$, $\lambda(A) < \infty$. Namely, we define

$$\int_0^t X(s)dZ(s) = \int_0^t \int_{\mathcal{O}} X(s,x)dZ(s,dx) := \xi\left(Z(b\wedge t, A) - Z(a\wedge t, A)\right),\ t \ge 0.$$

Let us extend linearly the stochastic integral to the class of *simple fields* $\mathcal{S}$, that is, the class of all linear combinations of the fields of the form (19.8). Let $X \in \mathcal{S}$. Then one can easily show the following isometric identity:

$$\mathbb{E}\left|\int_{t_1}^{t_2} X(s)dZ(s)\right|^2 = a_\nu\,\mathbb{E}\int_{t_1}^{t_2}\int_{\mathcal{O}} |X(s,x)|^2 ds\lambda(dx) \tag{19.9}$$

for all $0 \le t_1 < t_2 < \infty$. Given $T < \infty$, denote by $\mathfrak{P}_T$ the *predictable σ-field* on $\Omega \times [0,T]$, that is, the σ-field generated by $\xi(\omega)\chi_{(a,b]}$, where $0 \le a < b < T$ and ξ is $\mathfrak{F}_a$-measurable. Given a Hilbert space U we set $\mathcal{P}_{T,Z}(H) = L^2\left(\Omega\times[0,T], \mathfrak{P}_T, d\mathbb{P}\otimes dt; H\right)$. Let $L^2_\lambda := L^2(\mathcal{O}, a_\nu d\lambda)$. Note that the simple fields are dense in $\mathcal{P}_{T,Z}(L^2_\lambda)$. Take in (19.9), $t_1 = 0$ and $t_2 = T$. Then the right-hand side of (19.9) is the $\mathcal{P}_{T,Z}(L^2_\lambda)$-norm of X. Hence we can extend continuously the stochastic integral to the whole space $\mathcal{P}_{T,Z}(L^2_\lambda)$.

Remark 19.1 Clearly, given $X \in \mathcal{P}_{T,Z}(L^2_\lambda)$ the stochastic integral $\int_0^t X(s)dZ(s)$, $t \in [0,T]$ is a square integrable martingale. By (19.9) it is also stochastically continuous, and hence by Doob's theorem it has a càdlàg modification. In fact, we will always assume that the stochastic integral is càdlàg. Note that if $\Gamma \in \mathfrak{F}$ is such that $Z(t,A)(\omega) = 0$ for all $t \le T$, $A : \lambda(A) < \infty$, and $\omega \in \Gamma$, then $\int_0^t X(s)dZ(s)(\omega) = 0$ for all $t \le T$, $\omega \in \Gamma$, and $X \in \mathcal{P}_{T,Z}(L^2_\lambda)$. Next, if $X \in \mathcal{P}_{T,Z}(L^2_\lambda)$ and $\tilde{\Gamma} \in \mathfrak{F}$ are such that $X(t,x)(\omega) = 0$ for all $t \ge T$, $\omega \in \tilde{\Gamma}$, and $x \in \mathcal{O}$, then $\int_0^t X(s)dZ(s)(\omega) = 0$ for all $t \in [0,T]\times\tilde{\Gamma}$.

For our purposes we need to define the stochastic integral of a random field taking values in a real separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$. We denote by $L_{(HS)}(L^2_\lambda, H)$ the space of Hilbert–Schmidt operators acting from L^2_λ into H equipped with the Hilbert–Schmidt norm $||\cdot||_{(HS)}$ and scalar product $\langle\cdot,\cdot\rangle_{(HS)}$. We have the following result.

Theorem 19.2 *Let $X \in \mathcal{P}_{T,Z}(L_{(HS)}(L^2_\lambda, H))$. Then for any orthonormal basis $\{e_n\}$ of H, and for any $t \in [0,T]$ the series*

$$\sum_n \int_0^t \left(X(s)^* e_n\right) dZ(s) e_n$$

converges in $L^2(\Omega, \mathfrak{F}_t, \mathbb{P}; H)$. Its limit $X \circ Z(t) = \int_0^t X(s)dZ(s)$ does not depend on the basis of H. The process $(X\circ Z(t), \mathfrak{F}_t)$, $t \in [0,T)$ is a square integrable H-valued martingale. Moreover, for any $X, Y \in \mathcal{P}_{T,Z}(L_{(HS)}(L^2_\lambda, H))$,

$$\mathbb{E}\left\langle \int_0^t X(s)dZ(s), \int_0^t Y(s)dZ(s)\right\rangle_H^2 = \mathbb{E}\int_0^t \langle X(s), Y(s)\rangle^2_{(HS)} ds.$$

Proof It is enough to show the convergence. By (19.9) we have

$$\begin{aligned}\sum_n \mathbb{E}\left((X^*e_n)\circ Z(t)\right)^2 &= \sum_n \mathbb{E}\int_0^t\int_{\mathcal{O}} \left(X(s)^*e_n\right)^2(x)dsdx \\ &= \mathbb{E}\int_0^t ||X(s)||^2_{(HS)} ds.\end{aligned}$$

□

19.4 Equations driven by impulsive white noise

In this section we assume that the impulsive noise is white, or equivalently that the operator P is identity. To have a meaningful solution we have to assume, as in the Gaussian case, that the space dimension is 1. Thus we consider equations (19.1) and (19.2) on $\mathcal{O} = [0,1]$ or on $\mathcal{O} = \mathbb{R}$ and write $L^2 := L^2(\mathcal{O}, dx)$. In the first two subsections we discuss equations under additional condition (19.4). A way to extend the results to more general noises is discussed in the final subsection.

The heat and wave equations on a $[0,1]$ are considered with the Dirichlet boundary condition. We are interested whether (19.1) or (19.2) defines a Markov family on a properly chosen state space H; and whether the transition semigroup P_t, $t \geq 0$ transforms the space $C_b(H)$ of bounded continuous function on H into $C_b(H)$, that is, whether (P_t) has the Feller property. If this is the case, then we will say that (19.1) or (19.2) *defines Feller family on* H. Let H be a Hilbert space. We denote by $U(T,H)$ the class of all predictable H valued processes X such that $|X|^2_{T,H} := \sup_{t\leq T} \mathbb{E}\,|X(t)|^2_H < \infty$.

19.4.1 Heat equation

As a solution to (19.1) we understand the so-called mild solution, that is, a predictable process taking values in the state–space $L^2 = L^2(0,1)$ or $L^2(\rho) := L^2(\mathbb{R}^d, e^{-\rho|x|}dx)$ such that for every t

$$u(t) = S(t)\zeta + \int_0^t S(t-s)f(u(s))ds + \int_0^t S(t-s)b(u(s))dZ(s), \tag{19.10}$$

where S is the C_0-semigroup generated by the Laplace operator with the Dirichlet boundary condition. We interpret $S(t-s)b(u(s))$ as an operator-valued process $S(t-s)b(u(s))\psi = S(t-s)(b(u(s))\psi)$, $\psi \in L^2_\lambda$. Note that the stochastic integral in (19.10) is a well-defined square integrable process in the Hilbert space $H = L^2$ or $H = L^2(\rho)$ if

$$\mathbb{E}\int_0^t ||S(t-s)b(u(s))||^2_{(HS)}ds < \infty. \tag{19.11}$$

Our first result is concerned with the equation on $[0,1]$. We assume that the measure λ is absolutely continuous with respect to Lebesgue measure. We will denote by λ its density. We set $\lambda^\#(x) = 0$ if $\lambda(x) = 0$, and $\lambda^\#(x) = (\lambda(x))^{-1}$ if $\lambda(x) \neq 0$. Let $e_k(x) = \sqrt{2}\,sin\,(\pi k x)$, $k \in \mathbb{N}$, $x \in [0,1]$ be the orthonormal basis of eigenvectors of the Laplace operator with the Dirichlet boundary condition. Write $\lambda_k := \sup_{x\in[0,1]} \left|\sqrt{\lambda^\#(x)}e_k(x)\right|$, $k \in \mathbb{N}$.

Theorem 19.3 *Assume that* (19.4) *holds true and that*

$$\sum_{k=1}^\infty k^{-2}\lambda_k^2 < \infty. \tag{19.12}$$

Then for every $\zeta \in L^2$ and every $T < \infty$ there is a unique $u^\zeta \in U(T,L^2)$ satisfying (19.1)*. Moreover, u^ζ is mean-square continuous, and* (19.1) *defines a Feller family on L^2.*

Remark 19.2 Clearly if λ is Lebesgue measure or, more generally, for a certain $m > 0$, the density satisfies $\lambda(x) \geq m$ for every x, then obviously (19.12) holds true. Note that in this case $L^2_\lambda \hookrightarrow L^2$. Finally, the condition (19.12) is also satisfied if λ vanishes at 0 like $x^{2\alpha}$ with $\alpha < 1$.

Proof of Theorem 19.3 Given $\phi \in L^2$ we denote by M the multiplication operator $M(\phi)\psi = \phi\psi$. The theorem follows easily from Theorem 19.2, and the Banach fixed point theorem provided that we have

$$\int_0^t ||S(s)M||^2_{L(L^2, L_{(HS)}(L^2_\lambda, L^2))} ds < \infty, \qquad t > 0, \tag{19.13}$$

and we are able to show that given $u \in U(T, L^2)$ there is a predictable version of the stochastic convolution $\int_0^t S(t-s)b(u(s))dZ(s)$, $t \in [0, T]$; see [PZ1, PZ2, P]. In fact (19.13), ensures that if $u \in U(T, L^2)$, then (19.11) holds, and consequently the corresponding stochastic integral is well defined. The existence of a predictable version enables us to deal with the iterating scheme. In order to show (19.13) we make use of the following representation of S:

$$S(t)\psi = \sum_{k=1}^{\infty} e^{-\pi^2 k^2 t} \langle \psi, e_k \rangle_{L^2} e_k.$$

Now let $\{f_j\}$ be an orthonormal basis of L^2_λ. For $X \in L^2$ we have

$$\begin{aligned}
&||S(s)M(X)||^2_{(HS)} = \sum_j |S(s)(Xf_j)|^2_{L^2} = \sum_{j,k} \langle S(s)(Xf_j), e_k \rangle^2_{L^2} \\
= &\sum_{j,k} e^{-2\pi^2 k^2 s} \langle Xf_j, e_k \rangle^2_{L^2} = \sum_{j,k} a_\nu^{-1}\, e^{-2\pi^2 k^2 s} \left\langle f_j, X\lambda^{\#} e_k \right\rangle^2_{L^2_\lambda} \\
= &\sum_k a_\nu^{-1}\, e^{-2\pi^2 k^2 s} \left| X\lambda^{\#} e_k \right|^2_{L^2_\lambda} = \sum_k e^{-2\pi^2 k^2 s} \left| X\sqrt{\lambda^{\#}} e_k \right|^2_{L^2} \\
\leq &\ |X|^2_{L^2} \sum_k e^{-2\pi^2 k^2 s} \lambda_k^2.
\end{aligned}$$

Since

$$\int_0^t \sum_k e^{-2\pi^2 k^2 s} \lambda_k^2 \, ds < \infty$$

for $t > 0$ we have (19.13). Note that continuity in probability of an adapted process guarantees the existence of its predictable version, see [DZ]. Hence in order to show that for every $u \in U(T, L^2)$, there is a predictable version of

$$I(t, u) := \int_0^t S(t-s)b(u(s))dZ(s)$$

it suffices to show its mean square continuity. This follows from (19.13) and the isometric formula from Theorem 19.2. For let $0 \leq t' < t \leq T$ set

$$\delta(t, t') := \mathbb{E}\, |I(t, u) - I(t', u)|^2_{L^2}.$$

Then $\delta(t, t') = \delta_1(t, t') + \delta_2(t, t')$, where

$$\delta_1(t, t') := \mathbb{E} \int_{t'}^t ||S(t-s)b(u(s))||^2_{(HS)} ds$$

and

$$\delta_2(t', t) := \mathbb{E} \int_0^{t'} ||(S(t-s) - S(t'-s))b(u(s))||^2_{(HS)} ds.$$

By the Lipschitz continuity of b there is a constant C such that $|b(x)|^2 \le C(1+|x|^2)$ for $x \in \mathbb{R}$. Then

$$\delta_1(t,t') \le C(1+|u|^2_{T,L^2}) \int_{t'}^{t} ||S(s)M||^2_{L(L^2,L_{(HS)}(L^2_\lambda,L^2))} ds \to 0 \qquad \text{as} \quad t' \to t.$$

Next

$$\delta_2(t,t') \le C(1+|u|^2_{T,L^2}) \int_0^{t'} ||(S(t-t'+s)-S(s))M||^2_{L(L^2,L_{(HS)}(L^2_\lambda,L^2))} ds.$$

Taking into account the established estimate for $||S(s)M(X)||^2_{(HS)}$ one can easily show that $\delta_2(t,t') \to 0$ as $t' \to t$. □

Let us consider now the stochastic heat equation (19.1) on the whole space. Thus Z is defined on $\mathbb{R}$. As a state space for the solution we take the space $L^2(\rho)$ with a fixed $\rho \in \mathbb{R}$. Then the heat semigroup is of the form

$$S(t)\psi(x) = \int_{\mathbb{R}} G_t(x-y)\psi(y)dy, \qquad \psi \in L^2, \quad t \ge 0,$$

where

$$G_t(x-y) := \frac{1}{\sqrt{2\pi t}} e^{\frac{|x-y|^2}{2t}}, \qquad x,y \in \mathbb{R},\ t > 0.$$

Theorem 19.4 *Assume* (19.4) *and assume that λ is absolute continuous with respect to Lebesgue measure and $\lambda^{\#}$ is bounded from above. If $\rho \le 0$, we assume also that $f(0) = 0 = b(0)$. Then for any $\zeta \in L^2(\rho)$ there is a unique $u \in U(T, L^2(\rho))$ satisfying* (19.1)*. Moreover, u^ζ is mean-square continuous and* (19.1) *defines a Feller family on $L^2(\rho)$.*

Proof Let us denote by $||\cdot||_{(HS)}$ the Hilbert–Schmidt norm on $L_{(HS)}\left(L_\lambda, L^2(\rho)\right)$. Let $\{f_j\}$ be an orthonormal basis of L^2_λ. Let $\phi \in L^2(\rho)$ and $t > 0$. Let $M(\phi)$ be a multiplication operator; $M(\phi)\psi = \phi\psi$. Then

$$\begin{aligned}
||S(t)M(\phi)||^2_{(HS)} &= \sum_j |S(t)\left(\phi f_j\right)|^2_{L^2(\rho)} \\
&= \sum_j \int_{\mathbb{R}} \left| \int_{\mathbb{R}} G_t(x-y) f_j(y)\phi(y)dy \right|^2 e^{-\rho|x|}dx \\
&= \sum_j \int_{\mathbb{R}} \left| \int_{\mathbb{R}} G_t(x-y)\lambda^{\#}(y) f_j(y)\phi(y)\lambda(y)dy \right|^2 e^{-\rho|x|}dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \left|G_t(x-y)\lambda^{\#}(y)\phi(y)\right|^2 e^{-\rho|x|}dydx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} G_t^2(x-y)\left(\lambda^{\#}(y)\right)^2 e^{-\rho|x|+\rho|y|}|\phi(y)|^2 e^{-\rho|y|}dydx \\
&\le |\phi|^2_{L^2(\rho)} \int_{\mathbb{R}} G_t^2(z)R_\rho(z)dz,
\end{aligned}$$

where

$$R_\rho(z) := \sup_{y\in\mathbb{R}} \left(\lambda^{\#}(y)\right)^2 e^{-\rho|z+y|+\rho|y|} \le e^{c(|z|+1)}$$

with a certain $c > 0$. Thus

$$\int_{\mathbb{R}} G_t^2(z)R_\rho(z)dz \le \int_{\mathbb{R}} G_t^2(z)e^{c(|z|^r+1)}dz \le \frac{1}{2\pi t}\int_{\mathbb{R}} e^{-\frac{|z|^2}{2t}} e^{-\frac{|z|^2}{2T}+c(|z|+1)}dz.$$

Since

$$\sup_{z\in\mathbb{R}}\left(-\frac{|z|^2}{2T}+c\left(|z|+1\right)\right)<\infty,$$

for any $T<\infty$ there is a constant C_T such that

$$||S(t)M(u)||^2_{(HS)}\le \frac{C_T}{\sqrt{t}}\,|u|^2_{L^2(\rho)},\qquad t\in(0,T].$$

Since $t\to 1/\sqrt{t}$ is integrable, one can use the standard arguments leading to the existence and uniqueness of a solution, see the proof of Theorem 19.3 and [PZ1,PZ2,P]. □

19.5 Wave equation

Let $e_k(x)=\sqrt{2}\sin(\pi kx)$, and let $H=L^2\times H^{-1}$, where

$$H^{-1}:=\left\{h=\sum_k h_k e_k:\ |h|^2_{H^{-1}}:=\sum_k h_k^2 k^{-2}<\infty\right\}.$$

The space H will be the state space for the wave equation on $[0,1]$. For the equation on $\mathbb{R}$ we take

$$H:=\left\{(\zeta,\eta)^T:\ \zeta\in L^2(\mathbb{R}^d,\vartheta_\rho(x)dx),\ \eta\in H^{-1}_\rho\right\},\tag{19.14}$$

where

$$H^{-1}_\rho:=\left\{\eta\in S'(\mathbb{R}^d):\ ||\eta||_{H^{-1}_\rho}:=\left\|\left(1+|\cdot|^2\right)^{-1/2}\mathcal{F}(\vartheta_\rho^{1/2}\eta)\right\|_{L^2(\mathbb{R}^d,dx)}<\infty\right\}.$$

$S'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$, $\mathcal{F}$ denotes the Fourier transform, and $\vartheta_\rho\in\mathcal{S}(\mathbb{R}^d)$ is a strictly nonnegative and $\vartheta_\rho(x)=e^{-\rho|x|}$, $|x|\ge 1$.

We write (19.2) in the form

$$dX=\left(\mathcal{A}X+\mathbf{F}(X)\right)dt+\mathbf{B}(X)dZ,\qquad X(0)=(\zeta,\eta)^T,$$

where

$$\mathcal{A}=\begin{pmatrix}0 & I\\ A & 0\end{pmatrix},\qquad \mathbf{B}\left(\begin{pmatrix}u\\ v\end{pmatrix}\right)[\psi](x)=\begin{pmatrix}0\\ b(u(x))\psi(x)\end{pmatrix},$$

$$\mathbf{F}\left(\begin{pmatrix}u\\ v\end{pmatrix}\right)(x)=\begin{pmatrix}0\\ f(u(x))\end{pmatrix}.$$

Then, see, e.g., [P], $\mathcal{A}$ generates C_0-semigroup U on H defined at the beginning of this subsection as the state space for the equation on $[0,1]$ and on $\mathbb{R}$. Obviously here in the definition of H^{-1}_ρ we take $d=1$. Thus we have the following integral form of (19.2):

$$X(t)=U(t)\left(\zeta,\eta\right)^T+\int_0^t U(t-s)\mathbf{F}(X(s))ds+\int_0^t U(t-s)\mathbf{B}(X(s))dZ(s).\tag{19.15}$$

Define

$$\mathbf{M}(X)\psi=\begin{pmatrix}0\\ \zeta\psi\end{pmatrix},\qquad X=(\zeta,\eta)^T.$$

Theorem 19.5 *Assume that the measure λ is absolutely continuous with respect to Lebesgue measure and satisfies* (19.12). *Then for any $X(0)\in H$ there is a unique solution X to* (19.2) *such that for all $T<\infty$, $X\in U(T,H)$. Moreover, X is mean-square continuous and* (19.2) *defines a Feller family.*

Proof Let T be fixed, and let $||\cdot||_{(HS)}$ be the norm on $L_{(HS)}(L^2_\lambda, H)$. It is enough to show that $\mathbf{M}(X) \in L_{(HS)}(L^2_\lambda, H)$, $X \in H$ and there is a constant C_1 such that

$$||\mathbf{M}(X)||^2_{(HS)} \le C_1 |X|^2_H, \qquad X \in H, \tag{19.16}$$

see [P]. To this end, note that for $X = (\zeta, \eta)^T \in H$ we have

$$\begin{aligned} ||\mathbf{M}(X)||^2_{(HS)} &= \sum_k \left|\mathbf{M}(X) e_k \sqrt{\lambda^\#}\right|^2_H = \sum_k \left|\zeta e_k \sqrt{\lambda^\#}\right|^2_{H^{-1}} \\ &= \sum_{k,n} n^{-2} \left\langle \zeta e_k \sqrt{\lambda^\#}, e_n \right\rangle^2_{L^2} = \sum_{k,n} n^{-2} \left\langle e_k, \zeta \sqrt{\lambda^\#} e_n \right\rangle^2_{L^2} \\ &= \sum_n n^{-2} \left|\zeta \sqrt{\lambda^\#} e_n\right|^2_{L^2} \le \sum_n n^{-2} \lambda_n^2 |\zeta|^2_{L^2}. \end{aligned}$$

Hence (19.16) holds true with $C_1 = \sum_n n^{-2}\lambda_n^2 < \infty$. □

Now we consider (19.2) on the whole space.

Theorem 19.6 *Let H be given by* (19.14). *If $\rho \le 0$, we assume that $f(0) = 0 = b(0)$. Assume that the measure λ is absolutely continuous with respect to Lebesgue measure and the density, which we denote by λ is such that $\lambda^\#$ is bounded from above. Then for any $(\zeta, \eta)^{\mathrm{T}} \in H$ there is a unique X satisfying* (19.2) *such that for all $T < \infty$, $X \in U(T, H)$. Moreover, X is mean-square continuous and* (19.2) *defines a Feller family on H.*

Proof As in the proof of Theorem 19.5 it is enough to show (19.16) with H given by (19.14). Let $X = (\zeta, \eta)^{\mathrm{T}} \in H$ and let $\{f_n\}$ be an orthonormal basis on L^2_λ. Then

$$\begin{aligned} ||\mathbf{M}(X)||^2_{(HS)} &= \sum_n |\mathbf{M}(X)(f_n)|^2_H = \sum_n \left\| \left(1 + |\cdot|^2\right)^{-1/2} \mathcal{F}\left(\vartheta_\rho^{1/2} \zeta f_n\right) \right\|^2_{L^2} \\ &= \int_{\mathbb{R}} \left(1 + |x|^2\right)^{-1} \sum_n \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta f_n\right)\right|^2 (x) dx. \end{aligned}$$

Now

$$\begin{aligned} \sum_n \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta f_n\right)\right|^2 (x) &= \sum_n \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta \left(\sqrt{\lambda}\sqrt{\lambda^\#} f_n\right)\right)\right|^2 (x) \\ &= \sum_n \left|\int_{\mathbb{R}} \mathcal{F}\left(\vartheta_\rho^{1/2} \zeta \sqrt{\lambda^\#}\right)(x-y)\, \mathcal{F}\left(\sqrt{\lambda} f_n\right)(y) dy\right|^2 \\ &\le \int_{\mathbb{R}} \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta \sqrt{\lambda^\#}\right)(x-y)\right|^2 dy \\ &\le \int_{\mathbb{R}} \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta \sqrt{\lambda^\#}\right)(y)\right|^2 dy. \end{aligned}$$

Hence

$$\begin{aligned} ||\mathbf{M}(X)||^2_{(HS)} &= \int_{\mathbb{R}} \left(1 + |x|^2\right)^{-1} \int_{\mathbb{R}} \left|\mathcal{F}\left(\vartheta_\rho^{1/2} \zeta \sqrt{\lambda^\#}\right)(y)\right|^2 dy dx \\ &= \int_{\mathbb{R}} \left(1 + |x|^2\right)^{-1} dx \int_{\mathbb{R}} |\zeta(y)|^2 \lambda^\#(y) \vartheta_\rho(y) dy \\ &\le C\, |\zeta|^2_{L^2(\mathbb{R}, \vartheta_\rho(y)dy)} \le C\, |X|^2_H, \end{aligned}$$

where

$$C = \int_{\mathbb{R}} \frac{1}{1 + |x|^2} dx \sup_{y \in \mathbb{R}} \lambda^\#(y) < \infty.$$

□

19.5.1 Equations driven by general white noise

We consider now equations driven by the noise for which (19.4) is not satisfied. To simplify the presentation we restrict our considerations to the stochastic heat and wave equations on $[0,1]$ or $\mathbb{R}$ driven by the process Z introduced in Example 19.1. We will also assume that the measure λ is the Lebesgue measure dx. Assume, that Z is given by (19.6). Let

$$Y(t) = \sum_{\tau_j^{(n,m)} \le t} \sigma_j^{(n,m)}, \qquad t \ge 0.$$

Given $N \in \mathbb{N}$ and $T < \infty$ define

$$R_N = \inf\left\{t\colon\ |Y(t-) - Y(t)| \ge N\right\},$$

$$Z_N(t,dx) = \sum_{\tau_j^{(n,m)} \le t,\ |\sigma_j^{(n,m)}| \le N} \sigma_j^{(n,m)} \delta_{x_j^{(n,m)}}(dx)$$

and

$$\Gamma_{N,T} = \left\{\omega \in \Omega\colon \left|\sigma_j^{(n,m)}(\omega)\right| \le N \text{ for all } n,m,j\colon\ \tau_j^{(n,m)} \le T\right\}.$$

Finally, let $\nu_N(d\sigma) = |\sigma|^{-1-\alpha}\chi_{\{|\sigma|\le N\}} d\sigma$ be the restriction of the measure ν to $[-N,N]$. Note that $Z_N(t)(\omega) = Z(t)(\omega)$ for $(t,\omega) \in [0,T] \times \Gamma_{N,T}$, R_N is a Markov stopping time with respect to the filtration defined by Z_N, $\Gamma_{N,T} = \{R_N \ge T\}$, and that

$$\mathbb{P}\left(\Gamma_{N,T}\right) = \exp\left\{-T\nu(\mathbb{R}\setminus[-N,N])\right\} = \exp\left\{\frac{-2TN^{-\alpha}}{\alpha}\right\} \to 1 \text{ as } N \to \infty.$$

Moreover, Z_N corresponds to the Poisson random measure with the intensity measure equal to $dt\lambda(dx)\nu_N(d\sigma)$. Thus, since ν_N satisfies (19.3) and (19.4), there is a solution X_N to the heat (or wave) equation driven by Z_N. We will show that

$$\forall\, N \le M \qquad X_N = X_M \qquad \text{on the set } [0,T]\times\Gamma_{N,T}. \tag{19.17}$$

Then as a solution X we take X_N on $[0,T]\times\Gamma_{N,T}$. We will show (19.17) only for the heat equation. However, the same arguments work for the wave equation. Let S be the heat semigroup. Then, since $Z_N = Z_M$ on $[0,T]\times\Gamma_{N,T}$ we have, see Remark 19.1,

$$\begin{aligned} X_N(t) - X_M(t) \quad &= \quad \int_0^t S(t-s)\left(f(X_N(s)) - f(X_M(s))\right) ds \\ &\qquad + \int_0^t S(t-s)\left(b(X_N(s)) - b(X_M(s))\right) dZ_N(s) \end{aligned}$$

on $[0,T]\times\Gamma_{N,T}$. Next, since on $[0,T]\times\Gamma_{N,T}$ we have

$$\left(b(X_N(s)) - b(X_M(s))\right)\chi_{\{R_N>s\}} = \left(b(X_N(s)) - b(X_M(s))\right),$$

then, see Remark 19.1, we have

$$\begin{aligned} &\left(X_N(t) - X_M(t)\right)\chi_{\{t<R_N\}} = \int_0^t S(t-s)\left(f(X_N(s)) - f(X_M(s))\right)\chi_{\{s<R_N\}} ds \\ &\quad + \int_0^t S(t-s)\left(b(X_N(s)) - b(X_M(s))\right)\chi_{\{s<R_N\}} dZ_N(s), \end{aligned}$$

which by the Lipschitz continuity of f and b, the isometric property of stochastic integral, and Gronwall's lemma lead easily to the desired conclusion.

19.6 Equations driven by a colored noise

In this final section we assume that λ is equal to Lebesgue measure on $\mathbb{R}^d$, and that operator P appearing in (19.1) and (19.2) is given by $P\psi(x) = p * \psi(x)$, where p is a sign measure on $\mathbb{R}^d$ with finite variation $||p||_{\mathrm{Var}}$; see Section 19.2.2. We consider equations (19.1) and (19.2) on $\mathbb{R}^d$. As the state spaces we take the $\mathbb{R}^d$-analogs of the spaces constructed for the heat and wave equations on $\mathbb{R}$. These spaces will be generically denoted by H. As a solution we understand a solution to the integral equation obtained from (19.10), or (19.15) by replacing dZ by PdZ.

Theorem 19.7 *Assume that*

$$\begin{array}{ll} \int_{\{|z-r|\le 1\}} \log\left(|z-r|^{-1}\right) ||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr) < \infty & \text{for } d = 2, \\ \int_{\{|z-r|\le 1\}} |z-r|^{-d+2}||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr) < \infty & \text{for } d \neq 2. \end{array} \tag{19.18}$$

In the case of $\rho \le 0$ we assume additionally that $f(0) = 0 = b(0)$. Then there is a unique solution to (19.1) *and* (19.2) *from the class $U(T,H)$ for $T > 0$. Moreover, the solution is mean-square continuous and* (19.1) *and* (19.2) *define Feller families on corresponding state spaces.*

Proof First, we consider the case of the heat equation. Let $M(\phi)\psi = \phi\psi$, and let S be the heat semigroup with the kernel $G_t(x-y)$. Let $\{f_j\}$ be an orthonormal basis of L^2, and let $\phi \in L^2(\rho)$. Then

$$\begin{aligned} ||S(t)M(\phi)P||^2_{(HS)} &= \sum_j |S(t)\left(\phi p * f_j\right)|^2_{L^2(\rho)} \\ &= \sum_j \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} G_t(x-y)(p * f_j)(y)\phi(y)dy \right|^2 e^{-\rho|x|}dx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} G_t(x-y-z)\phi(y+z)p(dz)\right|^2 e^{-\rho|x|}dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_t(x-y-z)G_t(x-y-r)\phi(y+r)\phi(y+z) \\ &\qquad \times p(dz)p(dr)e^{-\rho|x|}dydx \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_t(v+z)G_t(v+r)\phi(v+x+r)\phi(v+x+z) \\ &\qquad \times p(dz)p(dr)e^{-\rho|x|}dvdx. \end{aligned}$$

Since

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} \phi(v+x+r)\phi(v+x+z)e^{-\rho|x|}dx \right| \\ &\quad \le \left(\int_{\mathbb{R}^d} \phi^2(v+x+r)e^{-\rho|x|}dx \right)^{1/2} \left(\int_{\mathbb{R}^d} \phi^2(v+x+z)e^{-\rho|x|}dx \right)^{1/2} \\ &\quad \le |\phi|^2_{L^2(\rho)} e^{1/2|\rho|(|v+r|+|v+z|)}, \end{aligned}$$

we have

$$\begin{aligned} &||S(t)M(\phi)P||^2_{(HS)} \\ \le\ & |\phi|^2_{L^2(\rho)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_t(v+z)G_t(v+r)e^{1/2|\rho|(|v+r|+|v+z|)}||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr)dv. \end{aligned}$$

Since

$$G_t(x)e^{1/2|\rho||x|} \le 2^{-d/2} G_{t/2}(x) e^{-\frac{|x|^2}{t}+\frac{|\rho||x|}{2}},$$

for each $T<\infty$ there is a constant $C(T,\rho)$ such that for all $t\in[0,T]$ and $x\in\mathbb{R}^d$, we have $G_t(x)e^{1/2|\rho||x|}\le C(T,\rho)\,G_{t/2}(x)$. Hence, for $t\le T$,

$$\begin{aligned}&||S(t)M(\phi)P||^2_{(HS)}\\ \le\;& C^2(T,\rho)\,|\phi|^2_{L^2(\rho)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_{t/2}(v+z)G_{t/2}(v+r)||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr)dv.\end{aligned}$$

Since

$$\int_{\mathbb{R}^d} G_{t/2}(v+z)G_{t/2}(v+r)dv = G_t(z-r)$$

we eventually have

$$||S(t)M(\phi)P||^2_{(HS)} \le C^2(T,\rho)\,|\phi|^2_{L^2(\rho)}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G_t(z-r)||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr).$$

Let $(S(t)M\otimes P)(\phi)\psi := S(t)(M(\phi)P\psi)) = S(t)(\phi P\psi)$, and let $t\le T$. Then, by the estimate above we have

$$\begin{aligned}&\int_0^t ||S(t)M\otimes P||^2_{L(L^2(\rho),L_{(HS)}(L^2,L^2(\rho)))}\\ &\qquad\le C^2(T,\rho)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^T G_s(z-r)ds||p||_{\mathrm{Var}}(dz)||p||_{\mathrm{Var}}(dr).\end{aligned}$$

Since $I(x):=\int_0^T G_s(x)ds$ is a bounded function if $d=1$ and is bounded from above by $C(1+0\vee\log|x|^{-1})$ if $d=2$, and by $C(|z-r|^{d-2}+1)$ if $d>2$, (19.18) ensures that

$$\int_0^t ||S(t)M\otimes P||^2_{L(L^2(\rho),L_{(HS)}(L^2,L^2(\rho)))}<\infty$$

for every $t<\infty$, and consequently one can use the standard arguments leading to the existence and uniqueness, see [PZ1,PZ2,P].

We consider now the case of the wave equation, where H is given by (19.14). First, note that (19.18) yields

$$C(p) := \sup_{z\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\mathcal{F}p(y)|^2\,dy}{1+|y+z|^2}<\infty.$$

Now let $X=(\zeta,\eta)^T\in H$, and let $\mathbb{M}((\zeta,\eta)^T)\psi=(0,\zeta\psi)^T$. Let $\{f_n\}$ be an orthonormal basis on L^2. Then, see the proof of Theorem 19.6,

$$\begin{aligned}||\mathbf{M}(X)P||^2_{(HS)} &= \int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-1}\sum_n\left|\mathcal{F}\left(\vartheta_\rho^{1/2}\zeta p * f_n\right)\right|^2(x)dx\\ &= \int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-1}\sum_n\left|\int_{\mathbb{R}^d}\mathcal{F}\left(\vartheta_\rho^{1/2}\zeta\right)(x-y)\,\mathcal{F}\left(p*f_n\right)(y)dy\right|^2dx\\ &= \int_{\mathbb{R}^d}\left(1+|x|^2\right)^{-1}\int_{\mathbb{R}^d}\left|\mathcal{F}\left(\vartheta_\rho^{1/2}\zeta\right)(x-y)(\mathcal{F}p)(y)\right|^2dydx\\ &\le C(p)\,|\zeta|^2_{L^2(\mathbb{R}^d,\vartheta_\rho(y)dy)}\le C(p)\,|X|^2_H,\end{aligned}$$

which gives the desired conclusion; see the proof of Theorem 19.6 and [PZ1,PZ2,P]. □

References

[AWZ] S. Albeverio, J.L. Wu, and T.S. Zhang, *Parabolic SPDEs driven by Poisson white noise*, Stoch. Processes Appl. 74 (1998), 21–36.

[AW] D. Applebaum and J.L. Wu, *Stochastic partial differential equations driven by Lévy space-time white noise*, Random Operators Stoch. Equations 8 (2000), 245–261.

[B] E.S.L. Bié, *Étude d'une EDPS conduite par un bruit poissonian*, Probab. Theory Related Fields 111 (1998), 287–321.

[DZ] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge, 1992.

[HS] E. Hausenblas and J. Seidler, *Maximal inequality for stochastic convolutions driven by martingales and random measures*, preprint.

[Ia] K. Itô, *Stochatic integral*, Proc. Imp. Acad. Tokyo 20 (1944), 519–524.

[Ib] K. Itô, *On a stochatic integral equation*, Proc. Imp. Acad. Tokyo 22 (1946), 32–35.

[Me] M. Métivier, *Semimartingales*, Walter de Gruyter, Berlin-New York 1982.

[Mu] C. Mueller, *The heat equation with Lévy noise*, Stoch. Processes Appl. 74 (1998), 67–82.

[My] L. Mytnik, *Stochastic partial differential equation driven by stable noise*, Probab. Theory Related Fields 123 (2002), 157–201.

[P] S. Peszat, *The Cauchy problem for a nonlinear stochastic wave equation in any dimension*, J. Evol. Equ. 2 (2002), 383–394.

[PZ1] S. Peszat and J. Zabczyk, *Stochastic evolution equations with a spatially homogeneous Wiener process*, Stoch. Processes Appl. 72 (1997), 187–204.

[PZ2] S. Peszat and J. Zabczyk, *Nonlinear stochastic wave and heat equations*, Probab. Theory Related Fields 116 (2000),421–443.

[PZ3] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations Driven by Lévy Processes*, book in preparation.

[W] J.B. Walsh, *An introduction to stochastic partial differential equations*, In: Ecole d'été de Probabilités de St. Flour XIV, pp. 266–439, Lecture Notes in Math. 1180, Springer-Verlag, Berlin, 1986.

20 Harmonic Functions for Generalized Mehler Semigroups

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20.1 Introduction

The classical Liouville theorem for the Laplace operator L states that if, for a bounded C^2-function u,

$$Lu(x) = 0, \ \ x \in \mathbb{R}^n,$$

then u is constant on $\mathbb{R}^n$. This result can be equivalently formulated in terms of the heat semigroup P_t

$$P_t u(x) = \frac{1}{\sqrt{(2\pi t)^n}} \int_{\mathbb{R}^n} u(y) e^{\frac{|x-y|^2}{2t}} dy, \ \ t > 0, \quad P_0 u(x) = u(x), \ x \in \mathbb{R}^n, \ t \geq 0;$$

i.e., if, for a bounded Borel function u, one has $P_t u(x) = u(x)$, for all $t \geq 0$, $x \in \mathbb{R}^n$, then u is constant on $\mathbb{R}^n$.

More generally, let E be a Polish space and let P_t be a Markov semigroup, acting on the space $\mathcal{B}_b(E)$ of all real Borel and bounded functions defined on E. A bounded from below function $u : E \to \mathbb{R}$ is said to be *harmonic* for P_t, if u is Borel and invariant for P_t, i.e.

$$P_t u(x) = u(x), \quad t \geq 0, \ x \in E. \tag{20.1}$$

We say that a harmonic function u is a *bounded harmonic function* (BHF) or a *positive harmonic function* (PHF) for P_t if in addition u is bounded or nonnegative. Note that if u is a BHF for P_t, then

$$Lu(x) = 0, \ \ x \in E,$$

where the operator L is defined as follows:

$$Lu(x) = \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t}, \ \ x \in E. \tag{20.2}$$

A converse statement is true as well, see Section 20.3. Preliminaries are gathered in Section 20.2.

Our main concern in the present chapter are harmonic functions for generalized Mehler semigroups introduced in [5]. They have recently received a lot of attention; see, for instance, [9], [17], [21], [28], [32], and references therein. This class includes transition semigroups determined by infinite-dimensional Ornstein–Uhlenbeck processes perturbed by a Lévy noise.

*He was partially supported by the Italian National Project MURST "Equazioni di Kolmogorov" and by the "Centre of Excellence," IM PAN-BC, Warsaw, Poland.

Those processes are solutions to the following-infinite dimensional stochastic differential equation on a Hilbert space H

$$dX_t = AX_t dt + BdW_t + CdZ_t, \quad X_0 = x \in H, \ t \geq 0. \tag{20.3}$$

Here A generates a $\mathcal{C}_0$-semigroup e^{tA} on H, B, and C are bounded linear operators from another Hilbert space U into H. Moreover, W_t and Z_t are independent processes; W_t is a U-valued Wiener process and Z_t is a U-valued Lévy process (without a Gaussian component).

One says that the transition semigroup P_t has the Liouville property if all BHFs for P_t are constant. The Liouville property has been studied for various classes of linear and nonlinear operators L on $\mathbb{R}^n$. In particular, second order elliptic operators on $\mathbb{R}^n$, or on differentiable manifolds E, have been intensively investigated; see, for instance, [1], [3], [6], [18], [23], [31], and references therein. Liouville theorems for nonlocal operators are given in [2] and [27]. The probabilistic interpretation of the Liouville property is discussed in [27]; see also [23]. A Liouville theorem for the infinite-dimensional heat semigroup has already been considered in [12]. For connections between the Liouville property and the existence of invariant ergodic measures, see also Remark 20.4.

Theorem 20.2 of Section 20.4 is our main result on the Liouville property. In the particular case of an Ornstein–Uhlenbeck process X_t perturbed by a Lévy noise, see (20.3), and under suitable assumptions, the theorem states that the corresponding transition semigroup P_t has the Liouville property *if and only if* all λ in the spectrum $\sigma(A)$ of A have a nonpositive real part. Moreover, when there exists $\lambda \in \sigma(A)$ with a positive real part, we are able to construct a nonconstant BHF for P_t. This theorem extends to infinite dimensions a result given in [27].

In Section 20.5, we prove a result concerning positive harmonic functions. Under the assumptions of Theorem 20.2, we show that all PHFs for the transition semigroup P_t associated to (20.3) are convex. This result can be regarded as a stronger version of the first part of Theorem 20.2, see also Corollary 20.1.

The final section contains two open questions.

20.2 Preliminaries

Let H be a real separable Hilbert with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. We will identify H with H^* (the topological dual of H). Let U be another separable Hilbert space. By $\mathcal{L}(U,H)$ we denote the space of all bounded linear operators from U into H. We set $\mathcal{L}(H,H) = \mathcal{L}(H)$. If $B \in \mathcal{L}(U,H)$, its adjoint operator is denoted by B^* ($B^* \in \mathcal{L}(H,U)$).

The space $\mathcal{C}_b(H)$ (resp., $\mathcal{B}_b(H)$) stands for the Banach space of all real, continuous (resp., Borel) and bounded functions $f : H \to \mathbb{R}$, endowed with the supremum norm: $\|f\|_0 = \sup_{x \in H} |f(x)|$.

The space $\mathcal{C}_b^k(H)$ is the set of all k-times differentiable functions f, whose Fréchet derivatives $D^i f$, $1 \leq i \leq k$, are continuous and bounded on H, up to the order $k \geq 1$. Moreover, we set $\mathcal{C}_b^\infty(H) = \cap_{k \geq 1} \mathcal{C}_b^k(H)$.

20.2.1 Characteristic functions

We collect some basic facts about characteristic functions in infinite dimensions. These will be used in the sequel, see [7] or [22] for more details.

A function $\psi : H \to \mathbb{C}$ is said to be *negative definite* if, for any $h_1, \ldots h_n \in H$, $c_1, \ldots, c_n \in \mathbb{C}$, verifying $\sum_{k=1}^n c_k = 0$, one has $\sum_{i,j=1}^n \psi(h_i - h_j) c_i \overline{c_j} \leq 0$.

A function $\theta : H \to \mathbb{C}$ is said to be *positive definite* if, for any $h_1, \ldots, h_n \in H$, the $n \times n$ Hermitian matrix $(\theta(h_i - h_j))_{ij}$ is positive definite. Remark that $\psi : H \to \mathbb{C}$ is negative definite if and only if the function $\exp(-t\psi(\cdot))$ is positive definite for any $t > 0$.

A mapping $g : H \to \mathbb{C}$ is said to be *Sazonov continuous* on H if it is continuous with respect to the locally convex topology on H generated by the seminorms $p(x) = |Sx|$, $x \in H$, where S ranges over the family of all Hilbert–Schmidt operators on H. Of course, any Sazonov continuous function is, in particular, continuous.

The Bochner theorem states that any function $f : H \to \mathbb{C}$ is the *characteristic function* of a probability measure μ on H, i.e.

$$\hat{\mu}(h) = \int_H e^{i\langle y,h\rangle}\mu(dy) = f(h), \quad h \in H,$$

if and only if f is positive definite, Sazonov continuous, and such that $f(0) = 1$.

Let Q be a symmetric nonnegative definite trace class operator on H; we denote by $N(x, Q)$, $x \in H$, the *Gaussian measure* on H with mean x and covariance operator Q. The trace of Q will be denoted by $\mathrm{Tr}\,(Q)$.

20.2.2 Mehler semigroups

A Lévy process Z_t with values in H is an H-valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, continuous in probability, having stationary independent increments, càdlàg trajectories, and such that $Z_0 = 0$.

One has that

$$\mathbb{E}e^{i\langle Z_t,s\rangle} = \exp(-t\psi(s)), \ s \in H, \tag{20.4}$$

where $\psi : H \to \mathbb{C}$ is a Sazonov continuous, negative definite function such that $\psi(0) = 0$. We call ψ the *exponent* of Z_t. Vice versa given ψ with the previous properties, there exists a unique in law H-valued Lévy process Z_t, such that (20.4) holds.

The exponent ψ can be expressed by the following infinite-dimensional Lévy–Khintchine formula

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_H \Big(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1 + |y|^2}\Big)M(dy), \quad s \in H, \tag{20.5}$$

where Q is a symmetric nonnegative definite trace class operator on H, $a \in H$, and M is the spectral Lévy measure on H associated to Z_t, see also [29].

A *generalized Mehler semigroup* S_t, acting on $\mathcal{B}_b(H)$, is given by

$$S_t f(x) = \int_H f(e^{tA}x + y)\mu_t(dy), \quad t \geq 0, x \in H, f \in \mathcal{B}_b(H), \tag{20.6}$$

where e^{tA} is a $\mathcal{C}_0$-semigroup on H, with generator A, μ_t, $t \geq 0$, is a family of probability measures on H, such that

$$\hat{\mu}_t(h) = \exp\Big(-\int_0^t \psi(e^{sA^*}h)ds\Big), \quad h \in H, \ t \geq 0. \tag{20.7}$$

Here $\psi : H \to \mathbb{C}$ is a continuous, negative definite function such that $\psi(0) = 0$. We call ψ the *exponent* of S_t. Note that we are not assuming that the exponent ψ is Sazonov continuous, i.e., we are not requiring that $\exp(-\psi(\cdot))$ is the characteristic function of a probability measure on H or, equivalently, that there exists an associated H-valued Lévy process.

Generalized Mehler semigroups were introduced in [5], see also [9], [17], [21] [28], and [32].

20.3 Abstract Liouville theorems

Here, combining arguments from [14] and [24], we prove an abstract result which allows us to formulate the Liouville problem in terms of generators; see in particular, Theorem 20.1. We also provide an application to an infinite-dimensional Ornstein–Uhlenbeck operator.

Let P_t be any Markov semigroup acting on $B_b(E)$, the space of all bounded Borel functions on a Polish space E. Define the subspace

$$\mathcal{B}^0(E) = \{f \in \mathcal{B}_b(E), \quad \text{such that, for any } x \in E, \text{ the map: } t \mapsto P_t f(x) \text{ is continuous on } [0,\infty)\}. \tag{20.8}$$

This space is a slight modification of the space $\mathcal{B}_b^0(E)$ introduced in [14], see also Remark 20.1. It is easy to verify that the space $\mathcal{B}^0(E)$ is invariant for P_t. Moreover, it is a closed subspace of $\mathcal{B}_b(E)$ with respect to the supremum norm. This space also satisfies the assumptions (i) and (ii) in [24, Section 5].

We consider P_t acting on $\mathcal{B}^0(E)$ and define a generator $L : D(L) \subset \mathcal{B}^0(E) \to \mathcal{B}^0(E)$ of P_t as a version of the Dynkin weak generator, by the formula:

$$\begin{aligned} D(L) \ &:= \ \Big\{u \in \mathcal{B}^0(E) : \sup_{t>0} \Big\| \frac{P_t u - u}{t} \Big\|_0 < \infty, \ \exists\, g \in \mathcal{B}^0(E) \text{ such that} \\ &\qquad \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t} = g(x), \ \forall\, x \in E\Big\}, \\ Lu(x) \ &= \ \lim_{t \to 0^+} \frac{P_t u(x) - u(x)}{t}, \quad \text{for } u \in D(L), \quad x \in E. \end{aligned} \tag{20.9}$$

We have the following characterization.

Theorem 20.1 *If $f \in \mathcal{B}_b(E)$, then*

$$f \in D(L) \text{ and } Lf = 0 \quad \Longleftrightarrow \quad f \ \textit{ is a BHF for } P_t.$$

The theorem is a direct corollary of the following proposition.

Proposition 20.1 *For any function $f \in B_b(E)$, the following statements are equivalent:*

(i) $f \in D(L)$;
(ii) there exists $g \in \mathcal{B}^0(E)$ such that

$$P_t f(x) - f(x) = \int_0^t P_s g(x) ds, \quad x \in E, \ t \geq 0. \tag{20.10}$$

Moreover, if (20.10) holds, then $Lf = g$.

Proof $(ii) \Rightarrow (i)$. By (20.10) one has that $f \in \mathcal{B}^0(E)$. Moreover, $\frac{P_t f(x) - f(x)}{t} \to g(x)$, as $t \to 0^+$, for any $x \in E$. Finally, there results

$$\sup_{t>0} \Big\| \frac{P_t f - f}{t} \Big\|_0 \leq \sup_{s>0} \|P_s g\|_0 \leq \|g\|_0.$$

$(i) \Rightarrow (ii)$. Fix $x \in E$. Note that

$$\lim_{t \to 0^+} P_s \Big(\frac{P_t f - f}{t}\Big)(x) = P_s L f(x), \quad s \geq 0.$$

Hence, there exists the right derivative $\partial_s^+ P_s f(x) = P_s L f(x)$, $s \geq 0$. Since the functions: $s \mapsto P_s f(x)$ and $s \mapsto P_s L f(x)$ are both continuous on $[0, +\infty)$, by a well-known lemma of real analysis, the function: $s \mapsto P_s f(x)$ is $\mathcal{C}^1([0, +\infty))$. This gives the assertion. ∎

Remark 20.1 Given a Markov transition semigroup P_t, acting on $\mathcal{B}_b(E)$, Dynkin introduces in [14] the space $\mathcal{B}_b^0(E) = \{f \in \mathcal{B}_b(E)$ such that $\lim_{t\to 0^+} P_t f(x) = f(x),\ \ x \in E\}$. Moreover, he defines the *weak generator* $\tilde{L}$ of P_t as in (20.9), replacing $\mathcal{B}^0(E)$ with $\mathcal{B}_b^0(E)$. It is clear that $\tilde{L}$ extends the operator L given in (20.9). However, it seems a difficult problem to clarify if $\mathcal{B}_b^0(E) = \mathcal{B}^0(E)$ holds in general. Moreover, it is not clear how to prove an analogous of Proposition 20.1 when L is replaced by $\tilde{L}$. ∎

Let us apply the previous theorem to the generator of a Gaussian Ornstein–Uhlenbeck process X_t, which solves the SDE

$$dX_t = AX_t dt + dW_t,\ \ x \in H. \tag{20.11}$$

Here W_t is a Q-Wiener process with values in H, and Q is a trace class operator on H; see also (20.5). Moreover, A generates a $\mathcal{C}_0$-semigroup e^{tA} on H.

Define $\hat{C} \subset C_b^2(H)$ as the space of all functions f such that $Df(x) \in D(A^*)$, for all $x \in H$, and the functions A^*Df and $D^2 f$ are both uniformly continuous and bounded on H.

Combining [34, Theorem 5.1] and Theorem 20.1, we get the following

Proposition 20.2 *Let us consider the Ornstein–Uhlenbeck semigroup P_t associated to the process X_t in (20.11). Then for any $f \in \hat{C}$, one has*

$$\mathcal{A}f(x) = \frac{1}{2}\mathrm{Tr}\,(QD^2 f(x)) + \langle A^*Df(x), x\rangle = 0,\ x \in H \iff f \text{ is a BHF for } P_t.$$

Proof By the Ito formula, in [34] it is showed that, for any $f \in \hat{C}$, $f \in D(L)$ if and only if $\mathcal{A}f$ is bounded. Moreover if $f \in \hat{C} \cap D(L)$, then $Lf = \mathcal{A}f$. Using this result and Theorem 20.1, we finish the proof. ∎

20.4 The Liouville theorem

If $A : D(A) \subset H \to H$ is a closed operator on H, we denote by $\sigma(A)$ its spectrum and by A^* its adjoint operator. We collect our assumptions on the generalized Mehler semigroup S_t; see (20.6) and (20.7).

Hypothesis 20.1 *(i) there exists $B_0 \in \mathcal{L}(U, H)$, where U is another Hilbert space, such that the linear nonnegative bounded operators $Q_t : H \to H$*

$$Q_t x = \int_0^t e^{sA} B_0 B_0^* e^{sA^*} x\, ds,\ x \in H, \quad \text{are trace class},\ \ t > 0. \tag{20.12}$$

(ii) $\mu_t = \nu_t * N(0, Q_t)$, where ν_t is a family of probability measures on H, such that

$$\hat{\nu}_t(h) = \exp\Big(-\int_0^t \psi_1(e^{sA^*}h)ds\Big),\ \ h \in H,\ t \geq 0, \tag{20.13}$$

with $\psi_1 : H \to \mathbb{C}$ being a continuous, negative definite function such that $\psi_1(0) = 0$.

Hypothesis 20.2 *There exists $T > 0$, such that $e^{tA}(H) \subset Q_t^{1/2}(H),\ \ t \geq T$.*

If S_t is, in particular, the Gaussian Ornstein–Uhlenbeck semigroup corresponding to (20.11), then Hypothesis 20.2 is implied by the strong Feller property of S_t. Recall that a Markov semigroup P_t, acting on $\mathcal{B}_b(H)$, is called *strong Feller* if

$$P_t(\mathcal{B}_b(H)) \subset \mathcal{C}_b(H),\ \ t > 0. \tag{20.14}$$

Hypothesis 20.3 *One has*

$$\int_H (\log |y| \vee 0)\, M(dy) < \infty. \tag{20.15}$$

Remark that if H is finite dimensional, then the previous hypotheses reduce to the assumptions in [27, Theorem 3.1].

The aim of this section is to prove the following theorem.

Theorem 20.2 *Let S_t be a generalized Mehler semigroup on H. If Hypotheses 20.1 and 20.2 hold and, moreover*

$$s(A) := \sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0, \tag{20.16}$$

then all BHFs for S_t are constant.
If Hypotheses 20.1, 20.2 and 20.3 hold and further

$$\sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} > 0,$$

then there exists a nonconstant BHF h for S_t.

Remark 20.2 As we mentioned in Introduction, a natural class of generalized Mehler semigroups which satisfy Hypotheses 20.1 and 20.2 is the one associated to the SDE

$$dX_t = AX_t dt + BdW_t + CdZ_t, \quad X_0 = x \in H, \ t \geq 0, \tag{20.17}$$

where A generates a $\mathcal{C}_0$-semigroup e^{tA} on H, B, and $C \in \mathcal{L}(U, H)$. Here W_t and Z_t are U-valued, independent Q_0-Wiener and Lévy processes (the operator Q_0 is a symmetric nonnegative trace class operator on U). Without any loss of generality, we may assume that Z_t has no Gaussian component (i.e., the exponent ψ_0 of Z_t is given by (20.5) with $Q = 0$).

It is well known that there exists a unique mild solution to (20.17), see [9] and [11]. This is given by

$$X_t^x = Y_t^x + \eta_t, \tag{20.18}$$

where

$$Y_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B dW_s, \quad \eta_t = \int_0^t e^{(t-s)A} C dZ_s.$$

The latter stochastic integral involving Z_t can be defined as a limit in probability of elementary processes. Moreover, Y_t^x is a Gaussian Ornstein–Uhlenbeck process; compare with (20.11). Clearly, setting $B_0 = B\,Q_0^{1/2}$, the operators B_0 and A satisfy condition (i) in Hypothesis 20.1.

If μ_t denotes the law of X_t^0, then it is clear that the Markov semigroup S_t associated to X_t^x is given by

$$S_t f(x) = \int_H f(e^{tA}x + y)\mu_t(dy), \ \ t \geq 0, \, x \in H, \, f \in \mathcal{B}_b(H). \tag{20.19}$$

If ν_t is the law of η_t, then we have $\mu_t = \nu_t * N(0, Q_t)$. Indeed

$$\begin{aligned}\hat{\mu}_t(h) &= \exp\Big(- \int_0^t |B_0^* e^{sA^*} h|^2 ds\Big) \, \exp\Big(- \int_0^t \psi_0(C^* e^{sA^*} h)\, ds\Big) \\ &= \widehat{N(0, Q_t)}(h)\, \hat{\nu}_t(h), \quad h \in H. \ \blacksquare\end{aligned}$$

Remark 20.3 An example of a generalized Mehler semigroup with exponent ψ which is *not* Sazonov continuous, is the one determined by the SDE

$$dY_t = AY_t dt + B_0 dW_t, \ \ Y_0 = x \in H, \ t \geq 0, \tag{20.20}$$

where $A : D(A) \subset H \to H$ generates a $\mathcal{C}_0$-semigroup e^{tA} on H, $B_0 \in \mathcal{L}(U, H)$ and the process W_t is a U-valued *cylindrical* Wiener process; see [11] for more details.

If we assume that A and B_0 verify (i) in Hypothesis 20.1, then there exists a unique H-valued process Y_t^x, which is the mild solution to (20.20)

$$Y_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B_0 dW_s, \ \ x \in H, \ t \geq 0. \tag{20.21}$$

Note that Y_t^x is a Gaussian process. The associated Ornstein–Uhlenbeck semigroup U_t is given by

$$U_t f(x) = \mathbb{E} f(Y_t^x) \ = \ \int_H f(e^{tA}x \ + \ y) \ \kappa_t(dy), \ f \in \mathcal{B}_b(H), \tag{20.22}$$

$x \in H, \ t > 0$, where $\kappa_t = N(0, Q_t)$ is the Gaussian measure on H with mean 0 and covariance operator Q_t, see (20.12). Note that the exponent ψ of U_t, i.e.

$$\psi(y) = |B_0^* y|^2, \ \ y \in H,$$

is not Sazonov continuous unless the operator B_0 is Hilbert–Schmidt. However, the associated process Y_t^x takes values in H, i.e., the function: $y \mapsto \int_0^t \psi(e^{sA^*}y)ds$ is Sazonov continuous on H, for each $t \geq 0$. ∎

Remark 20.4 One can show that the existence of an ergodic invariant probability measure with full support for a strong Feller transition semigroup implies the Liouville property. However, we are especially interested in cases in which there are no invariant probability measures. In particular, if some $\lambda \in \sigma(A)$ is purely imaginary, then there are no invariant probability measures for the Ornstein–Uhlenbeck semigroup U_t given in (20.22), see [11], but still, under Hypothesis 20.2, the Liouville theorem holds. ∎

In the proof of the first statement of Theorem 20.2, we will need assertion (1) of the next result: this lemma also extends previous results proved in [11, Section 9.4] and in [28]. Recall that I_B denotes the indicator function of a set $B \subset H$.

Lemma 20.1 *Let us assume that Hypotheses 20.1 and 20.2 hold. Then one has*
(1) $S_t(\mathcal{B}_b(H)) \subset C_b^\infty(H)$, $t \geq T$.
(2) S_t *is irreducible, i.e.,* $S_t I_O(x) > 0$, *for any* $x \in H$, $t \geq T$ *and* O *open set in* H.

Proof Take any $f \in \mathcal{B}_b(H)$. We have

$$S_t f(x) = \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) N(0, Q_t)(dy) \tag{20.23}$$

$$= \int_H \nu_t(dz) \int_H f(y + z) N(e^{tA}x, Q_t)(dy), \ t \geq 0, \ x \in H.$$

Using the Cameron–Martin formula, see [11], we can differentiate $S_t f$ in each direction $h \in H$ and get, for any $x \in H$, $t \geq T$

$$\langle DS_t f(x), h \rangle = \int_H \nu_t(dz) \int_H f(e^{tA}x \ + y + z) \langle {Q_t}^{-1/2} y, {Q_t}^{-1/2} e^{tA} h \rangle N(0, Q_t) dy. \tag{20.24}$$

Recall that the function: $y \mapsto \langle {Q_t}^{-1/2} y, {Q_t}^{-1/2} e^{tA} h \rangle$ is a Gaussian random variable on the probability space $(H, \mathcal{B}(H), N(0, Q_t))$, for any $t \geq T$, see [11] and [34].

Formulas similar to (20.24) can be easily established for higher order derivatives of $S_t f$. It is then straightforward to verify that $S_t f \in \mathcal{C}_b^\infty(H)$, $t \geq T$. This concludes the proof of the first statement.

The second statement follows since the measure $N(0, Q_t)$ has support on the whole H, for any $t \geq T$. ■

Proof of Theorem 20.2. *The first part.* Here we prove that any bounded harmonic function for S_t is constant.

By Hypothesis 20.2, the closed operators $Q_t^{-1/2} e^{tA}$ are bounded operators on H, for any $t \geq T$. They have also a control theoretic meaning; see, for instance, [10] or [33]. Note that (i) in Hypothesis 20.1 and Hypothesis 20.2 imply that the semigroup e^{tA} is compact, for any $t \geq T$. To see this, we write $e^{TA} = Q_T^{1/2} (Q_T^{-1/2} e^{TA})$ and remark that the operator $Q_T^{1/2}$ is Hilbert–Schmidt.

Thus we can apply the following result, which is proved in [25]:

$$\lim_{t \to \infty} Q_t^{-1/2} e^{tA} x = 0, \quad x \in H, \quad \textit{if and only if} \quad s(A) = \sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0. \tag{20.25}$$

Take any BHF f for S_t. We show that f is constant. By (20.24), we get the estimate

$$\|\langle Df(\cdot), h \rangle\|_0 = \|\langle DS_t f(\cdot), h \rangle\|_0$$

$$\leq \|f\|_0 \int_H \nu_t(dz) \int_H |\langle {Q_t}^{-1/2} y, {Q_t}^{-1/2} e^{tA} h \rangle| N(0, Q_t) dy \ \leq \ |{Q_t}^{-1/2} e^{tA} h| \, \|f\|_0,$$

$t \geq T$, $h \in H$. Now letting $t \to \infty$ in the last formula, we get that f is constant, using (20.25). The assertion is proved.

The second part. Here we assume that $s(A) > 0$ and construct a nonconstant BHF h for S_t. It was already noted that Hypotheses 20.1 and 20.2 imply that e^{tA} is compact, for any $t \geq T$. Hence, see [15], pages 330 and 247, the spectrum $\sigma(A)$ consists entirely of eigenvalues of finite algebraic multiplicity, is discrete, and is at most countable. Moreover, for any $r \in \mathbb{R}$, the set

$$\{\mu \in \sigma(A) \ : \ \mathrm{Re}(\mu) \geq r\} \ \text{ is finite.} \tag{20.26}$$

It follows that there exists an isolated eigenvalue μ such that $s(A) = \mathrm{Re}(\mu)$. Using this fact, the claim follows by the next result. ■

Proposition 20.3 *Let S_t be a generalized Mehler semigroup on H. Assume that there exists an isolated eigenvalue μ of A with finite algebraic multiplicity and such that $Re(\mu) > 0$. Then there exists a nonconstant BHF h for S_t.*

Proof Let D_0 be the finite-dimensional subspace of H consisting of all generalized eigenvectors of A associated to μ.

Let $P_0 : H \to D_0$ be the linear Riesz projection onto D_0 (not orthogonal in general)

$$P_0 x = \frac{1}{2\pi i} \int_\gamma (w - A)^{-1} x \, dw, \quad x \in H, \tag{20.27}$$

where γ is a circle enclosing μ in its interior and $\sigma(A)/\{\mu\}$ in its exterior; see, for instance, Lemma 2.5.7 in [10] and [15, page 245]. We have $H = D_0 \oplus D_1$, where $D_1 = (I - P_0)H$. The closed subspaces D_0 and D_1 are both invariant for e^{tA} and, moreover, $D_0 \subset D(A)$. We set $A_0 = AP_0$ and further $A_1 = A(I - P_0)$, where

$$A_0 : D_0 \to D_0, \quad A_1 : (D(A) \cap D_1) \subset D_1 \to D_1. \tag{20.28}$$

The operator A_0 generates a group e^{tA_0} on D_0 and A_1 generates a $\mathcal{C}_0$-semigroup e^{tA_1} on D_1. The projection P_0 commutes with e^{tA} and the restrictions of e^{tA} to D_0 and D_1 coincide with e^{tA_0} and e^{tA_1}, respectively. Moreover, on D_0 one has $\sigma(A_0) = \{\mu\}$. By means of P_0, let us define a generalized Mehler semigroup S_t^0 on D_0

$$S_t^0 f(a) = \int_H f(e^{tA}P_0 a + P_0 y)\mu_t(dy) = \int_{D_0} f(e^{tA_0}a + z)(P_0 \circ \mu_t)(dz),$$

where $t \geq 0$, $a \in D_0$, $f \in \mathcal{B}_b(D_0)$, and $(P_0 \circ \mu_t)$ is the probability measure on D_0 image of μ_t under P_0. Suppose that we find $g : D_0 \to \mathbb{R}$, such that

$$S_t^0 g(a) = g(a), \ \ a \in D_0, \tag{20.29}$$

i.e., g is a BHF for S_t^0. Then, defining $h(x) = g(P_0 x)$, $x \in H$, we get that h is a nonconstant BHF for S_t. Thus our aim is to construct a nonconstant BHF g for S_t^0. Note that

$$(P_0 \hat{\circ} \mu_t)(y) = \hat{\mu}_t(P_0^* y) = \exp\Big(-\int_0^t \psi(P_0^* e^{rA^*} y)dr\Big), \ \ y \in D_0.$$

Since D_0 is finite dimensional, the negative function $\psi_0 : D_0 \to \mathbb{C}$, $\psi_0(s) = \psi(P_0^* s)$, $s \in D_0$, corresponds to a Lévy process L_t with values in D_0 and defined on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The law ν_t of L_t verifies

$$\hat{\nu}_t(y) = \exp(-t\psi(P_0^* y)), \ \ y \in D_0, \ t \geq 0.$$

Let us consider the process $\tilde{X}_t^a$ on D_0

$$\tilde{X}_t^a = e^{tA_0}a + \int_0^t e^{(t-s)A_0} dL_t, \ \ t \geq 0, \ a \in D_0. \tag{20.30}$$

It is clear that the law of $\tilde{X}_t^0$ is just $(P_0 \circ \mu_t)$, $t \geq 0$. This implies that the Markov semigroup associated to $\tilde{X}_t^a$ is S_t^0.

We have reduced our initial problem of finding a nonconstant BHF for S_t to a corresponding finite-dimensional problem. Now in order to construct a nonconstant function g such that (20.29) holds, we can apply [27, Proposition 3.6]. The proof is complete. ■

Remark 20.5 Here we show a possible improvement of Hypothesis 20.3.

Let $\mathcal{F}(H)$ be the subspace of $\mathcal{L}(H)$ consisting of all finite rank operators R which commute with e^{tA}, i.e., $Re^{tA} = e^{tA}R, \ \ t \geq 0$.

For any $R \in \mathcal{F}(H)$, M^R denotes the spectral Lévy measure on $\mathrm{Im}R = R(H)$ corresponding to ψ^R through formula (20.5), where $\psi^R(s) = \psi(R^* s)$, $s \in R(H)$; note that $\psi^R : R(H) \to \mathbb{C}$ is a continuous, negative definite function such that $\psi^R(0) = 0$. Moreover, the image of μ_t under R, has characteristic function

$$(R \hat{\circ} \mu_t)(h) = \exp\Big(-\int_0^t \psi^R(e^{sA^*}h)ds\Big), \ \ h \in R(H), \ t \geq 0.$$

It is straightforward to check that the second part of Theorem 20.2 continues to hold if Hypothesis 20.3 is replaced by the following weaker assumption:

$$\int_{P(H)} (\log|y| \vee 0)\, M^P(dy) < \infty, \ \ \text{for any projection} \ \ P \in \mathcal{F}(H). \blacksquare$$

Remark 20.6 One can extend the definition of generalized Mehler semigroup and show that Theorem 20.2 holds true in this more general setting.

A *shifted generalized Mehler semigroup* P_t, acting on $\mathcal{B}_b(H)$, is given by

$$P_t f(x) = \int_H f(e^{tA}x + e^{tA}h - h + y)\mu_t(dy), \ \ t \geq 0, \, x \in H, \, f \in \mathcal{B}_b(H). \tag{20.31}$$

Compare with (20.6), where e^{tA} is a $\mathcal{C}_0$-semigroup on H, μ_t, $t \geq 0$, is a family of probability measures on H satisfying (20.7) and h is a fixed vector in H. It is straightforward to verify that P_t is a Markov semigroup acting on $\mathcal{B}_b(H)$.

An example of shifted generalized Mehler semigroup is the Markov semigroup P_t associated to the Markov process J_t^x

$$J_t^x = X_t^{x+h} - h, \ \ t \geq 0, \, x \in H,$$

where X_t^x is the mild solution to (20.17). If in addition we assume that $h \in D(A)$, then J_t^x solves

$$dJ_t = AJ_t dt + Ah dt + BdW_t + CdZ_t, \ J_0 = x \in H, \ t \geq 0,$$

under the same assumptions of Remark 20.2.

There is a one to one correspondence between BHFs for S_t given in (20.6) and BHFs for P_t. Indeed if g is a BHF for P_t, then the function f, $f(y) = g(y-h)$, $y \in H$, is a BHF for S_t. Vice versa, if u is a BHF for S_t, then the function w, $w(z) = u(z+h)$, $z \in H$, is a BHF for P_t. This shows that Theorem 20.2, with the same assumptions on e^{tA}, B, and μ_t, holds more generally when the generalized Mehler semigroup S_t is replaced by the semigroup P_t, given in (20.31), without any additional hypothesis on $h \in H$. ∎

20.5 Convexity of positive harmonic functions

In this section we prove that positive harmonic functions for generalized Mehler semigroups are convex under suitable assumptions. This result can be regarded as a stronger version of the first part of Theorem 20.2; see, in particular, Corollary 20.1.

Theorem 20.3 *Assume Hypotheses 20.1 and 20.2 and consider the generalized Mehler semigroup S_t given in (20.6). Moreover, suppose that*

$$s(A) = \sup\{Re(\lambda) \ : \ \lambda \in \sigma(A)\} \leq 0. \tag{20.32}$$

holds. Then any positive harmonic function g for S_t is convex on H.

The following lemma is an extension of a result due to S. Kwapien [19] (proved by him in the Gaussian case with a similar proof).

Lemma 20.2 *Under Hypotheses 20.1 and 20.2, for any nonnegative function $f : H \to \mathbb{R}$, there results*

$$S_t f(x+a) + S_t f(x-a) \geq 2C_t(a)\, S_t f(x), \ \ x, \, a \in H, \tag{20.33}$$

where $C_t(a) = \exp\Big(-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big)$, $t > 0$.

Proof Using the notation in (20.23), we have

$$S_t f(x) = \int_H \nu_t(dz) \int_H f(e^{tA}x + y + z) N(0, Q_t)(dy), \ \ t \geq 0.$$

By the Cameron–Martin formula, one finds

$$S_t f(x+a) = \int_H \nu_t(dz) \int_H f(e^{tA}x+y+z) \frac{dN_{e^{tA}a,Q_t}}{dN_{0,Q_t}}(y) N_{0,Q_t}(dy)$$
$$= \int_H \nu_t(dz) \int_H f(e^{tA}x+y+z) \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2 + \langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle\Big] N_{0,Q_t}(dy).$$

It follows that

$$\begin{aligned}
&\frac{1}{2}(S_t f(x+a) + S_t f(x-a)) \\
&= e^{-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2} \int_H \nu_t(dz) \int_H f(e^{tA}x+y+z) \frac{1}{2}\Big(e^{\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle} \\
&\qquad + e^{-\langle Q_t^{-1/2}e^{tA}a, Q_t^{-1/2}y\rangle}\Big) N_{0,Q_t}(dy) \\
&\geq \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] \int_H \nu_t(dz) \int_H f(e^{tA}x+y+z) N_{0,Q_t}(dy) \\
&= C_t(a)\, S_t f(x). \quad \blacksquare
\end{aligned}$$

Proof of Theorem 20.3. By the previous lemma, we have

$$\frac{1}{2}(g(x+a)+g(x-a)) = \frac{1}{2}(S_t g(x+a) + S_t g(x-a))$$
$$\geq \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] S_t g(x) = \exp\Big[-\frac{1}{2}|Q_t^{-1/2}e^{tA}a|^2\Big] g(x).$$

Passing to the limit as $t \to \infty$, we infer, see (20.25),

$$\frac{1}{2}(g(x+a)+g(x-a)) \geq g(x), \quad x,\, a \in H. \tag{20.34}$$

By a classical result due to Sierpinski, see [30], this condition together with the measurability of g implies the convexity of g. $\blacksquare$

Corollary 20.1 *Under the assumptions of Theorem 20.3, any bounded harmonic function g for S_t is constant on H.*

Proof We may assume that that $1-g$ is a nonnegative BHF (otherwise replace g by $\frac{g}{\|g\|_0}$). Using $1-g$ instead of g in (20.34), we obtain

$$\frac{1}{2}(1-g(x+a)+1-g(x-a)) = 1 - \frac{1}{2}(g(x+a)+g(x-a)) \geq 1-g(x).$$

It follows that $g(x+a)+g(x-a) \leq 2g(x)$ and so, by (20.34),

$$g(x+a)+g(x-a) = 2g(x), \quad x \in H. \tag{20.35}$$

Note that, by Lemma 20.1, g is continuous on H. Since any continuous function which satisfies identity (20.35) is affine, we have $g(x) = g(0) + \langle h, x\rangle$ for some $h \in H$. It follows that g is constant. $\blacksquare$

20.6 Open questions

Problem 20.1 It is not known, even in finite dimension and for strong Feller Gaussian Ornstein–Uhlenbeck semigroups P_t, whether the hypothesis

$$\sup\{Re(\lambda)\ :\ \lambda \in \sigma(A)\} \le 0$$

implies that all PHFs for P_t are constant (compare with Theorems 20.2 and 20.3).

A partial positive answer can be given in $\mathbb{R}^2$, see [8], and more generally in $\mathbb{R}^n$, assuming in addition that the dimension of the Jordan part of A corresponding to eigenvalues in the imaginary axis is at most two. This condition is equivalent to the recurrence of a strong Feller Gaussian Ornstein–Uhlenbeck process X_t in $\mathbb{R}^n$, see [13], [16], and [33]. Remark that for recurrent processes with strong Feller transition semigroups all positive harmonic functions, or even more generally all excessive functions, are constant, see [4].

We also mention the following related result, which has been recently proved in [18]. Let L be the Ornstein–Uhlenbeck operator on $\mathbb{R}^n$

$$Lu(x) = \frac{1}{2}\mathrm{Tr}\ (QD^2u(x)) + \langle Ax, Du(x)\rangle, \quad x \in \mathbb{R}^n,$$

where Q and A are real $n \times n$ matrices and Q is symmetric and nonnegative definite. Assume that L is hypoelliptic (or equivalently that the corresponding Ornstein–Uhlenbeck semigroup P_t is strong Feller, see, for instance, [20]). In [18] it is shown that if 0 is the only eigenvalue of A and if in addition the matrix Q is degenerate, then any nonnegative classical solution to $Lu(x) = 0$, $x \in \mathbb{R}^n$, is constant on $\mathbb{R}^n$.

Problem 20.2 Given a generalized Mehler semigroup S_t, acting on $\mathcal{B}_b(H)$, it is an open problem to find conditions on the drift operator A and on the exponent ψ in order to construct a càdlàg Markov process Y_t with values in H, having S_t as the associated Markov semigroup. In [17] such a process is constructed only on an enlarged Hilbert space E, containing H.

Acknowledgments

The authors wish to thank the Banach Centre of the Institute of Mathematics of the Polish Academy of Sciences (Warsaw) and Scuola Normale Superiore di Pisa (Italy), where parts of the chapter were written, for hospitality and nice working conditions.

References

[1] M.T. Barlow, *On the Liouville property for divergence form operators,* Can. J. Math. 50 (1998), 487-496.

[2] M.T. Barlow, R.F. Bass, C. Gui, *The Liouville property and a conjecture of De Giorgi,* Comm. Pure Appl. Math. 53 (2000), 1007-1038.

[3] M. Bertoldi, S. Fornaro, *Gradient estimates in parabolic problems with unbounded coefficients,* Studia Math. 165, No. 3 (2004), 221-254.

[4] R.M. Blumenthal, R.K. Getoor, *Markov processes and potential theory,* Pure and Applied Mathematics, Academic Press, New York-London, 1968.

[5] V.I. Bogachev, M. Röckner, B. Schmuland, *Generalized Mehler semigroups and applications,* Probab. Theory Related Fields 105 (1996), 193-225.

[6] I. Capuzzo Dolcetta, A. Cutrì, *On the Liouville property for sub-Laplacians*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 25 (1997), 239-256.

[7] S.A. Chobanyan, V.I. Tarieladze, N.N. Vakhania, *Probability Distributions on Banach Spaces,* Reidel, Dordrecht, 1987.

[8] M. Cranston, S. Orey, U. Rösler, *The Martin boundary of two dimensional Ornstein-Uhlenbeck processes,* Probability, statistics and analyis, London Mathematical Society Lecture Note Series 79, Cambridge University Press, 63-78, 1983.

[9] A. Chojnowska-Mikhalik, *On Processes of Ornstein-Uhlenbeck type in Hilbert space,* Stochastics 21 (1987), 251-286.

[10] R.F. Curtain, H.J. Zwart, *An introduction to Infinite-Dimensional Linear Systems Theory,* Text in Applied Mathematics 21, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1995.

[11] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions,* Encyclopedia of Mathematics and its Applications 44, Cambridge University Press, Cambridge, 1992.

[12] G. Da Prato, J. Zabczyk, *Second Order Partial Differential Equations in Hilbert Spaces,* London Mathematical Society Lecture Note Series 293, Cambridge University Press, Cambridge, 2002.

[13] H. Dym, *Stationary measures for the flow of linear differential equations driven by white noise,* Trans. Am. Math. Soc. 123 (1966), 130-164.

[14] E.B. Dynkin, *Markov processes, Vol. I-II,* Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

[15] K. Engel, R. Nagel, *One-parameter Semigroups for Linear Evolution Equations,* Springer Graduate Texts in Mathematics 194, 2000.

[16] R. Erickson, *Constant coefficients linear differential equations driven by white noise,* Ann. Math. Statistics 42 (1971), 820-823.

[17] M. Fuhrman, M. Röckner, *Generalized Mehler Semigroups: The Non-Gaussian Case,* Potential Anal. 12 (2000), 1-47.

[18] A. Kogoj, E. Lanconelli, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations,* Mediterranean J. Math. 1, No. 1 (2004), 51-80.

[19] S. Kwapien, *personal communications.*

[20] E. Lanconelli, S. Polidoro, *On a class of hypoelliptic evolution operators,* Rend. Sem. Mat. Univ. Pol. Torino 52 (1994), 26-63.

[21] P. Lescot, M. Röckner, *Generators of Mehler-type semigroups as pseudo-differential operators,* Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002), 297-315.

[22] K.R. Parthasarathy, *Probability measures on metric spaces,* Academic Press, New York-London, 1967.

[23] R.G. Pinsky, *Positive Harmonic Functions and Diffusion,* Cambridge Studies in Ad. Math. 45, 1995.

[24] E. Priola, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions,* Stud. Math. 136 (3) (1999), 271-295.

[25] E. Priola, J. Zabczyk, *Null controllability with vanishing energy,* SIAM J. Control Optimization 42 (2003), 1013-1032.

[26] E. Priola, J. Zabczyk, *Liouville theorems in finite and infinite dimensions,* Preprint n. 9, Scuola Normale Superiore di Pisa, 2003.

[27] E. Priola, J. Zabczyk, *Liouville theorems for non-local operators,* J. Funct. Anal. 216 (2004), 455-490.

[28] M. Röckner, F.Y. Wang, *Harnack and Functional Inequalities for Generalised Mehler Semigroups,* J. Funct. Anal. 203 (2003), 237-261.

[29] Keni-Iti Sato, *Lévy processes and infinite divisible distributions,* Cambridge University Press, Cambridge, 1999.

[30] W. Sierpinski, *Sur le fonctions convexes mesurables,* Fund. Math. 1 (1920), 125-129.

[31] F.Y. Wang, *Liouville theorem and coupling on negatively curved Riemannian manifolds,* Stoch. Process. Appl. 100 (2002), 27-39.

[32] J. Zabczyk, *Stationary distribution for linear equations driven by general noise,* Bull. Acad. Pol. Sci. 31 (1983), 197-209.

[33] J. Zabczyk, *Mathematical Control Theory: An Introduction,* Birkhauser, 1992.

[34] J. Zabczyk, *Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions,* in Lect. Notes in Math. 1715, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1999.

21 The Dynamics of the Three-Dimensional Navier–Stokes Equations

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21.1 Introduction

The analysis of the asymptotic behavior of linear and nonlinear stochastic partial differential equations (SPDEs) is well developed and fruitful. There is one important case, the *Navier–Stokes equations* in three dimensions, that remains essentially open.

The stochastic approach is not the real source of difficulties in the study of such equations. Well-posedness of the deterministic equations is a major open problem (see, for example, Fefferman [13], where the problem is introduced in relation to the *Millennium prizes* announced by the *Clay Institute*).

In this review we will focus mainly on the ergodicity of the stochastic equations (Section 21.2) and on the existence of the global attractor (Sections 21.3 and 21.4). In both cases, the analysis faces a main open problem, since the very beginning. No natural way is known in order to define the principal objects, such as dynamical systems, flows, invariant measures, and attractors, that are the subject of the study. We shall see how the selected authors have dealt with these difficulties. Every section has its own short introduction to the problem, and we refer to those for the understanding of each single subject.

The review given here is by no means complete. There are several other possible ideas that can be applied to the problem, such as the statistical approach in Vishik and Fursikov [32], the abstract limit approach of Foiaş, the set-valued trajectories in Babin and Vishik [3], or the nonstandard analysis approach of Capiński and Cutland [6], and many others.

21.1.1 Notation

Here we fix a few standard notations, that will be conveniently used throughout the chapter. Let $D \subset \mathbf{R}^3$ be a open bounded domain, with smooth boundary. Define the space

$$H = \{\, v \in L^2(D, \mathbf{R}^3) \mid \operatorname{div}(v) = 0,\ v \cdot \vec{n} = 0 \text{ on } \partial D \,\}$$

and set $V = H^1_0(D, \mathbf{R}^3) \cap H$. Moreover, let P be the projection of $L^2(D, \mathbf{R}^3)$ onto H. The operator A is defined as

$$A = \nu \mathrm{P} \nabla, \qquad D(A) = H^2(D, \mathbf{R}^3) \cap V;$$

notice that $V = D((-A)^{\frac{1}{2}})$. Finally, the nonlinear operator $B : V \times V \to V'$ is formally given as

$$B(u, v) = \mathrm{P}(u \cdot \nabla) v.$$

We denote the norm on H by $|\cdot|$ and the norm on V by $\|\cdot\|$. There is a constant λ_1 (depending only on the domain D) such that

$$\|u\| \geq \lambda_1 |u|.$$

21.2 Invariant measures

Establishing the ergodicity property for the Navier–Stokes equations is one of the main open problems for the statistical analysis of fluid dynamics. In comparison, the theory in the case of a two-dimensional fluid is far more advanced (for a review on recent results concerning the ergodicity of the Navier–Stokes equations in two dimensions, one can see Mattingly [24]).

In the first part of the section, we shall see that a small dimensional noise is sufficient for the ergodicity of the Galerkin approximations to the Navier–Stokes equations. Such a result can have a qualitative interest for the statistical behavior of an incompressible fluid. Indeed, if the Kolmogorov theory of turbulence is taken into account, one can believe that the cascade of energy, responsible of the transport of the energy through the scales, is effective in the inertial range so that at smaller scales only the dissipation ends up to be relevant. Hence the long-time statistical properties of the fluid can be sufficiently depicted by the low modes of the velocity field. In some sense, if the ultraviolet cutoff is sufficiently large, in order to capture all the important modes, the corresponding invariant measure gives the real behavior of the fluid.

In the model, the noise injects energy at the level of the large length-scales, the geometric cascade describes what happens in the inertial range, where energy is transmitted from scale to scale, and the dissipation range is neglected via the spectral approximation.

In the second part of the section we shall see the only, as far as we know, infinite-dimensional result related to the problem under examination.

21.2.1 A geometric cascade for the ergodicity of the finite-dimensional approximation of the 3D Navier–Stokes equations forced by a degenerate noise

In [25] (see also [26]) it is proved the uniqueness of the statistical steady state (i.e., the ergodicity property) for the spectral Galerkin approximation of the 3D Navier–Stokes equations, with periodic boundary condition, driven by a random force

$$du = (\nu\Delta u - (u\cdot\nabla)u - \nabla p)\,dt + dB_t,$$

with $t \geq 0$ and $x \in [0, 2\pi]^3$.

First, the equations are projected in the space of divergence-free vector fields, in order to get rid of the pressure. After, the equations are written in the Fourier components. Namely,

$$u(t,x) = \sum_{\mathbf{k}\in\mathbf{Z}^3} u_{\mathbf{k}}(t)\mathtt{e}^{\mathtt{i}\mathbf{k}\cdot x}.$$

Here we make, for the sake of simplicity, the following assumption: *the trajectories of the noise are divergence-free and the covariance is diagonal in the Fourier basis.* So we can write our equations as an infinite system of stochastic differential equations

$$du_{\mathbf{k}} = \left[-\nu|\mathbf{k}|^2 u_{\mathbf{k}} - \mathtt{i}\sum_{\mathbf{h}+\mathbf{l}=\mathbf{k}} (\mathbf{k}\cdot u_{\mathbf{h}})\mathrm{P}_{\mathbf{k}}(u_{\mathbf{l}})\right]dt + q_{\mathbf{k}}\,d\beta^{\mathbf{k}}_t,$$

with $\mathbf{k} \in \mathbf{Z}^3$, where $\mathrm{P}_{\mathbf{k}}$ is the projection of $\mathbf{R}^3$ onto the space orthogonal to the vector $\mathbf{k}$. We use a spectral Galerkin approximation of the system above. From now on, we fix a threshold N and we write the process u as

$$u(t,x) = \sum_{|\mathbf{k}|_\infty\leq N} u_{\mathbf{k}}(t)\mathtt{e}^{\mathtt{i}\mathbf{k}\cdot x},$$

so the problem reduces to a system of finite number (but quite large: the number of equations is of the order $\mathcal{O}(N^3)$) of stochastic differential equations (SDEs):

$$du_{\mathbf{k}} = F_{\mathbf{k}}(u)\,dt + q_{\mathbf{k}} \cdot d\beta_t^{\mathbf{k}}, \qquad |\mathbf{k}|_\infty \leq N \tag{21.1}$$

where $F_{\mathbf{k}}$ is the deterministic dynamics, $\beta_t^{\mathbf{k}}$ are three-dimensional Brownian motions and the $q_{\mathbf{k}}$s are 3×3 matrices. The main assumption we take on the noise is the following:

$$q_{\mathbf{k}} \equiv 0 \quad \text{unless } |\mathbf{k}|_2 = 1,$$

that means that the only forced modes are the one corresponding to the Fourier modes $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$.

Our main result is the following.

Theorem 21.1 *The system of SDE (21.1) admits a unique invariant measure. Moreover, the support of such a measure is the whole state space. The convergence to the invariant measure is exponentially fast.*

The proof of the main theorem uses some classical tools, the main one being the Doob's theorem. We shall prove the Strong Feller property by verifying that the generator of the diffusion satisfies the *Hörmander condition.* The irreducibility property, by using the support theorems of Stroock, can be checked by solving the associated control problem.

21.2.1.1 The Hörmander condition

We define the vector field $\mathfrak{F}$ corresponding to the deterministic dynamics and the vector fields $\mathfrak{X}_{\mathbf{k}}$ giving the directions where the noise is effective as

$$\mathfrak{F} = \sum_{|\mathbf{k}|_\infty \leq N} F_{\mathbf{k}}(u) \frac{\partial}{\partial u_{\mathbf{k}}}, \qquad \mathfrak{X}_{\mathbf{k}} = q_{\mathbf{k}}^r \frac{\partial}{\partial u_{\mathbf{k}}}.$$

Proving the Hörmander condition means proving that the Lie algebra generated by the vector fields $\mathfrak{F}$ and $\mathfrak{X}_{\mathbf{k}}$ is full rank when evaluated at each point of the space state. We'll see in Section 21.2.1.3 that such a condition is true by means of some algebraic considerations.

21.2.1.2 The controllability argument

The control problem associated to our equations is simply the system (21.1) where the noise is replaced by controls. The heuristic idea behind the controllability is that, if at each point of the state space the system is allowed to follow *any* direction in *any* way (only the first assumption is true in the hypoellipticity proof), then the system is controllable.

Our proof is based on some ideas from control theory (see [21]). The main point is that one defines, for each point of the state space, the set of reachable points. After, we add new vector fields from the Lie algebra defined in the previous section, to the set of vector fields $\mathfrak{F}$ and $\mathfrak{X}_{\mathbf{k}}$. A vector field can be added only if the addition does not change the reachable set. The largest set of vector fields with this property is called the *saturation set.* If the saturation set is full rank, when evaluated at each point, then the system is global controllable.

The Lie algebra and the saturation set coincide, when we deal with an odd polynomial system, since both those sets turn out to be symmetric. In the case of even polynomials, like the one we are dealing with, the positive terms do not ensure the symmetry, and some direction can be run by the system in one way only. What we prove is that in our spectral approximation the obstructions given by those terms are not effective.

21.2.1.3 The geometric cascade

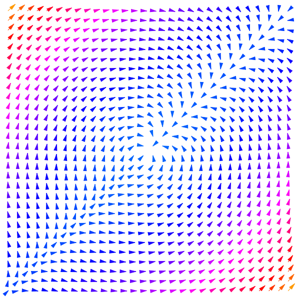
The idea of the geometric cascade is that, in order to prove that the saturation set generated by the vector fields (a similar argument works for the Lie algebra) of the dynamics is full ranked, one has to trace how the energy is transmitted through the Fourier modes. Indeed, once one knows that two modes, say $\mathbf{h}$ and $\mathbf{l}$, are indirectly forced by the noise (or, more precisely, they belong to the saturation set), then, by means of the nonlinearity, it is possible to prove that the vector fields corresponding to the mode $\mathbf{h}+\mathbf{l}$ is forced by the noise. Hence, the system can follow that direction.

21.2.1.4 A toy model

As an example, we shall examine the following toy model, where we can prove or disprove the claims of the previous sections. Consider the system of stochastic differential equations

$$\begin{cases} dx_t = -x_t + y_t^2 - x_t y_t + dB_t, \\ dy_t = -y_t + x_t^2 - x_t y_t, \end{cases}$$

where B_t is a one-dimensional Brownian motion. We can have a rough idea of the deterministic dynamics from the picture below.



It is easy to see that the system has a global in time solution (just apply Ito formula to $x_t^2 + y_t^2$). Define the following vector fields:

$$\mathfrak{F} = (-x + y^2 - xy)\frac{\partial}{\partial x} + (-y + x^2 - xy)\frac{\partial}{\partial y} \quad \text{and} \quad \mathfrak{X} = \frac{\partial}{\partial x},$$

where $\mathfrak{F}$ is the vector fields given by the deterministic dynamics (the *drift*) and $\mathfrak{X}$ is the one given by the direction of the noise. Since

$$[[\mathfrak{F}, \mathfrak{X}], \mathfrak{X}] = 2\frac{\partial}{\partial y},$$

it follows that the Lie algebra generated by $\mathfrak{F}$ and $\mathfrak{X}$ has full rank, once it is evaluated at each point of the state space $\mathbf{R}^2$. In other words, Hörmander's condition holds. On the other hand, the system is not globally controllable. Indeed, if one considers the solution at the starting point $x_0 = 0$ and $y_0 = 0$, it is easy to solve the equation for y_t (it is a linear equation with random coefficients), thus obtaining

$$y_t = \int_0^t x_s^2 \mathrm{e}^{-\int_s^t (1+x_r)\, dr}\, ds,$$

which is almost surely nonnegative.

In view of the control theory issues explained in the previous section, one can explain the above phenomenon in the following way: since the direction ∂_x is the one forced by the noise, both ∂_x and $-\partial_x$ are in the saturation set, and $\partial_y = [[\mathfrak{F}, \mathfrak{X}], \mathfrak{X}]$ as well. On the contrary, the vector field $-\partial_y$ (which is in the Lie algebra, thus ensuring the regularity of the transition probability densities) does not belong to the saturation set. Hence, the system can't follow that direction.

21.2.2 Ergodicity for the stochastic Navier–Stokes equations

The most striking result, up to now, concerning the problem of ergodicity for the three-dimensional Navier–Stokes equations has been proved in a recent paper by Da Prato and Debussche [11].

More precisely, consider the Navier–Stokes equations in an open bounded domain $D \subset \mathbf{R}^3$, with Dirichlet boundary conditions, in the abstract form (see Section 21.1.1 for notations):

$$\{\, d\, u = [Au + B(u,u)]\, dt + \sqrt{\mathrm{Q}} dW, u(0) = u_0, \tag{21.2}$$

where W is a cylindrical Wiener process on H and Q is a nonnegative symmetric linear operator on H, of trace-class. Moreover, some technical conditions of smoothness and non-degeneracy of the noise are assumed. For example, one can take $\mathrm{Q} = (-A)^\beta$ with $\beta \in (-3, -\frac{5}{2})$.

Following the same approach used for SDEs (see Stroock and Varadhan [29]), the analysis is based on the Kolmogorov equation associated to (21.2),

$$\begin{cases} \frac{dv}{dt} = \frac{1}{2}\,\mathrm{Tr}[\mathrm{Q}D^2 v] + \langle Ah + B(h,h), Dv\rangle, \\ v(0) = \varphi, \end{cases} \tag{21.3}$$

where φ is in the space $\mathcal{B}_b(H, \mathbf{R})$ of bounded measurable functions on H. The solution to the above equation is provided, formally, by the following Feynman–Kac formula:

$$v(t,h) = \mathrm{E}[\varphi(u(t,\cdot;h))],$$

where $u(t,\cdot;h)$ is the solution of (21.2) with initial condition $h \in H$. Indeed, all of the estimates of the chapter are evaluated on the Galerkin approximations of the solutions of an equation which is (21.3) modified by a potential

$$\frac{d\tilde{v}}{dt} = \frac{1}{2}\,\mathrm{Tr}[\mathrm{Q}D^2\tilde{v}] + \langle Ah + B(h,h), D\tilde{v}\rangle - K|Ah|^2\tilde{v}. \tag{21.4}$$

The constant K is large but fixed.

Theorem 21.2 (Da Prato, Debussche [11]) *There is a Markov semigroup $(P_t)_{t\geq 0}$ on $\mathcal{B}_b(D(A), \mathbf{R})$, and for every $h \in D(A)$ a martingale solution $u(t,\cdot;h)$ to the Navier–Stokes equations (21.2) on a suitable probability space, such that*

$$P_t\varphi(h) = \mathrm{E}_h\varphi(u(t,\cdot;h), \qquad \varphi \in C_b(D((-A)^\alpha, \mathbf{R}),$$

for all $\alpha < 0$. Moreover, $(P_t)_{t\geq 0}$ has a unique invariant measure which is ergodic and strongly mixing.

The proof of ergodicity follows from Doob's theorem, namely, the transition semigroup is both strong Feller and irreducible. The irreducibility follows from an argument similar to Flandoli [14], while the strong Feller property is deduced by careful and rather difficult estimates on the Galerkin approximations of (21.4).

In view of Section 21.4.3 we wish to emphasize how the transition semigroup $(P_t)_{t\geq 0}$ is selected. Consider the Galerkin approximations $(u_m)_{m\in\mathbf{N}}$ for equations (21.2); by the energy inequality one gets

$$\mathrm{E}|u_m(t,h_m)|^2 + \mathrm{E}\int_0^t |(-A)^{\frac{1}{2}}u_m(s,h_m)|^2\, ds \leq |h|^2 + t\,\mathrm{Tr}\,\mathrm{Q},$$

and it is possible to deduce that the laws $(\mathrm{L}(u_m))_{m\in\mathbf{N}}$ are tight in suitable topologies. Hence there is a subsequence, depending on the initial condition h, such that the laws converge weakly. In order to manage the dependence of the subsequence from h, they consider approximations of the solutions to the Kolmogorov equation (21.3) with nice continuity properties with respect to both the variables (t, h) and the initial condition φ.

So first they are able to find a subsequence $(m_k)_{k\in\mathbf{N}}$ such that the laws $\mathrm{L}(u_{m_k}(\cdot, h_{m_k}))$ converge for all h in a dense subset of $D(A)$. After, such a convergence holds for all $h \in D(A)$ and regular φ, due to the uniqueness of the solutions for the Galerkin approximations to (21.3) and the forementioned continuity properties.

21.3 Analysis of the path space

The approach introduced by Sell (see, for example, Sell [28]) bypasses the nonuniqueness problem in the analysis of the existence of global attractors (see also the next section) by changing the underlying state space. A new state space is introduced, where a point is a complete trajectory which is, a solution to the Navier–Stokes equations.

The dynamics on the new state space is given by the shift, forward in time, of solutions; that is, for each function f

$$\tau_t(f)(s) = f(t+s).$$

Roughly speaking, by pushing each trajectory more and more forward in time, the shift dynamics approaches the asymptotic behavior of solutions.

The *path space* approach has been applied to global attractors for the deterministic equations (Section 21.3.1.1), for the stochastic equations (Section 21.3.1.2) and for the analysis of invariant probabilities (Section 21.3.2).

21.3.1 Global attractors for the topological flow

In the first part, we aim to present some results of Sell [28] on the existence of the global attractor for the Navier–Stokes equations. In the second part, we report the existence of the random attractor given by Flandoli and Schmalfuss [18] (see also Flandoli and Schmalfuss [19] for the stochastic Navier–Stokes equations with multiplicative noise).

21.3.1.1 The deterministic theory

Let $D \subset \mathbf{R}^3$ be a bounded open set with smooth boundary, let $f \in L^2(D, \mathbf{R}^3)$, and consider the Navier–Stokes equations

$$\{\, \partial_t u + (u\cdot\nabla)u + \nabla p = \nu\Delta u + f,\ \mathrm{div}(u) = 0.$$

A global attractor for the shift semiflow $(\tau_t)_{t\geq 0}$ is a subset A of the state space such that

1. $\mathrm{A} \neq \emptyset$ is compact.
2. $\tau_t\mathrm{A} = \mathrm{A}$, for all $t \geq 0$.
3. There is a bounded neighborhood U of A such that for each neighborhood V of A, there is a time T such that $\tau(t)U \subset V$.
4. A attracts all points of the state space.

We wish to study the shift dynamics on the state space given by the trajectories of the solutions. The first step is to define a suitable state space made of solutions to the Navier–Stokes equations.

Definition 21.1 *Let $f \in L^2(D, \mathbf{R}^3)$. A vector field $u \in L^2_{loc}([0, +\infty); H)$ is a* weak solution *if*

1. $u \in L^\infty(0, +\infty; H) \cap L^2_{loc}([0, +\infty); V)$.
2. $\partial_t u \in L^{\frac{4}{3}}_{loc}([0, +\infty); V')$.
3. *For almost all t and t_0, with $t \geq t_0 \geq 0$, the following inequalities hold:*

$$|u(t)|^2 \leq \mathsf{e}^{-\nu\lambda_1(t-t_0)}|u(t_0)|^2 + C_\star|f|^2_\infty, \tag{21.5}$$
$$|u(t)|^2 + 2\nu \int_{t_0}^t \|u(s)\|^2\, ds \leq |u(t_0)|^2 + 2\int_{t_0}^t \langle f, u\rangle_H\, ds. \tag{21.6}$$

4. *For all $t \geq t_0 \geq 0$, the following equality hold for all $\phi \in V$:*

$$\langle u(t) - u(t_0), \phi\rangle + \nu \int_{t_0}^t \langle u(s), \phi\rangle_V\, ds + \int_{t_0}^t B(u(s), u(s)), \phi)\, ds = \int_{t_0}^t \langle f, \phi\rangle\, ds. \tag{21.7}$$

Denote by $\mathrm{W}_{\mathrm{WS}}(f)$ the set of all weak solutions corresponding to f. It is possible to see that $\mathrm{W}_{\mathrm{WS}}(f)$ is not empty (see Temam [30]). Notice also that, by properties *(1)* and *(4)*, it follows that every weak solution $u \in C([0, +\infty); H_{\mathrm{weak}})$. Moreover, from this fact and the lower semicontinuity of the norm, property *(3)* holds for all t and t_0.

The main problem is that the set $\mathrm{W}_{\mathrm{WS}}(f)$ of weak solutions happens to be *not* closed in $L^2_{\mathrm{loc}}([0, +\infty); H)$, hence the flow is not compact and it is not possible to apply the theory of attractors.

So, Sell introduces the notion of *generalized weak solution*, where the requirements for the solutions about $t = 0$ are relaxed.

Definition 21.2 *Let $f \in L^2(D, \mathbf{R}^3)$. A vector field $u \in L^2_{loc}([0, +\infty); H)$ is a* generalized weak solution *if*

1. $u \in L^\infty(0, 2; H) \cap L^\infty([1, +\infty); H) \cap L^2_{loc}(0, +\infty; V)$.
2. $\partial_t u \in L^{\frac{4}{3}}_{loc}(0, +\infty; V')$.
3. *For almost all t and t_0, with $t \geq t_0 > 0$, inequalities (21.5) hold.*
4. *For all $t \geq t_0 > 0$, equation (21.7) holds.*

Denote by $\mathrm{W}_{\mathrm{GWS}}(f)$ the set of all generalized weak solutions corresponding to f. Such a set ends up to be closed. Moreover, the shift dynamics $(\tau_t)_{t\geq 0}$ on this space is *compact* and *point dissipative.** By the classical theory for attractors, it follows that the semiflow $(\tau_t)_{t\geq 0}$ on $\mathrm{W}_{\mathrm{GWS}}(f)$ has a global attractor. Moreover, it is possible to show that the elements of the attractor are indeed weak solution, as given in Definition 21.1.

Theorem 21.3 (Sell [28]) *Let $f \in L^2(D, \mathbf{R}^3)$. Then there exists a global attractor* A *for the shift dynamics on $\mathrm{W}_{GWS}(f)$. Moreover,* A *attracts all bounded sets of $\mathrm{W}_{GWS}(f)$ and $\mathrm{A} \subset \mathrm{W}_{WS}(f)$.*

*A semiflow $S(t)$ is compact if $S(t)B$ is relatively compact for all $t > 0$ and all bounded sets B. It is point dissipative if there is a bounded set U such that for all points x there is a time $T = T(x)$ after which $S(t)x \in U$.

21.3.1.2 The stochastic theory

For the theory of random dynamical systems, one can refer to Arnold [1] (see also Arnold and Crauel [2]). One can see, for example, Crauel and Flandoli [10] for an introduction to random attractors.

Flandoli and Schmallfuss [18] follow the approach of Sell we have discussed in the previous section, to show that the stochastic Navier–Stokes equations with additive noise

$$du + (Au + B(u,u))\,dt = f\,dt + dW_t \tag{21.8}$$

have a unique global attractor. Here, W is a two-sided Wiener process on a probability space, with covariance of trace class in V. We assume that W is the canonical process on the space $C_0(\mathbf{R}, V)$ of functions taking value 0 at $t = 0$, so that $W(t,\omega) = \omega(t)$.

A fundamental tool for the analysis of the Navier–Stokes equations is the energy inequality (see below). In order to manage the inequality with the nondifferentiable term $\partial_t\omega$, it is proper to introduce the auxiliary linear problem

$$dz_\alpha + (A+\alpha)z_\alpha\,dt = dW_t,$$

and consider its stationary solution

$$z_\alpha(t,\omega) = \int_{-\infty}^{t} \mathsf{e}^{-(t-s)(A+\alpha)}dW_s = W_t - \int_{-\infty}^{t}(A+\alpha)\mathsf{e}^{-(t-s)(A+\alpha)}W_s\,ds$$

(the second version being more suitable for the pathwise approach), with initial condition in $D(A^{\frac{3}{8}})$. Notice that the dumping term $-\alpha z_\alpha$ has been introduced for the study of the long time behavior.

The approach is *pathwise*, that is, a weak solution solves

$$\partial_t u + Au + B(u,u) = f + \partial_t\omega \tag{21.9}$$

and the process solution to the SDE (21.8) has trajectories that are almost surely (a.s.) such weak solutions. The natural counterpart to Definition 21.2 in this framework is given as follows:

Definition 21.3 *Given $f \in L^2_{loc}(D)$, for each $\omega \in C_0([0,+\infty);V)$, a vector field $u \in L^2_{loc}([0,+\infty);H)$ is a weak solution of (21.9) corresponding to ω if*

1. *$u \in L^\infty_{loc}(0,+\infty;H) \cap L^2_{loc}(0,+\infty;V)$.*

2. *$\partial_t(u-g) \in L^{\frac{4}{3}}_{loc}(0,+\infty;V')$.*

3. *For all ω in a suitable set of full measure, all $\alpha \geq 0$ and for almost every t, t_0, with $t \geq t_0 \geq 0$*

$$V_1(u,\omega)(t) \leq V_1(u,\omega)(t_0) \qquad \textit{and} \qquad V_2(u,\omega)(t) \leq V_2(u,\omega)(t_0).$$

4. *For almost every t, t_0, with $t \geq t_0 > 0$, and for all $\varphi \in D(A)$,*

$$\langle u(t) - u(t_0), \phi\rangle + \int_{t_0}^{t}\langle A^{\frac{1}{2}}u(s), A^{\frac{1}{2}}\phi\rangle\,ds + \int_{t_0}^{t} B(u(s),u(s)),\phi)\,ds$$

$$= \int_{t_0}^{t}\langle f, u(s)\rangle\,ds + \langle\omega(t) - \omega(t_0),\phi\rangle.$$

In the above definition, the terms V_1 and V_2 substantiate the energy inequality (for the precise definition, we refer to Flandoli and Schmalfuss [18]).

Consider the set $\mathrm{W}_{\mathrm{GWS}}(\omega) \subset L^2_{\mathrm{loc}}([0,+\infty);H)$ of weak solutions corresponding to ω, and define on $\mathrm{W}_{\mathrm{GWS}}(\omega)$ the map

$$\phi(t,\omega)u = u(\cdot + t).$$

This mapping has nice dynamics properties, namely

- $\phi(t,\omega)\mathrm{W}_{\mathrm{GWS}}(\omega) \subset \mathrm{W}_{\mathrm{GWS}}(\theta_t\omega)$,
- $\phi(t+s,\omega) = \phi(s,\theta_t\omega) \circ \phi(t,\omega)$ and $\phi(0,\omega) = \mathrm{I}$,
- $u \longrightarrow \phi(t,\omega)u$ is continuous,

where θ_t is the shift of increments on $C_0(\mathbf{R},V)$.

In comparison with the theory of the following section, we see that the space of forcing terms lacks of compactness properties. Moreover, such forcings are quite irregular. So, the *forward* procedure used by Sell can't be used and Flandoli and Schmalfuss use a *pullback* procedure, where the system comes from $-\infty$ and it is observed at a finite time. In this way, the flow is compact and dissipative.

Theorem 21.4 (Flandoli and Schmalfuss [18]) *For each ω, there is a set $\mathrm{A}(\omega)$ contained in $\mathrm{W}_{GWS}(\omega)$ such that*

1. *$\mathrm{A}(\omega)$ in nonempty and compact.*
2. *It is invariant: $\phi(t,\omega)\mathrm{A}(\omega) = \mathrm{A}(\theta_t\omega)$ for all $t \geq 0$.*
3. *The map $\omega \longrightarrow \mathrm{A}(\omega)$ is a measurable multifunction.*
4. *It attracts all bounded sets, in the sense that for each functions M with subexponential growth at $-\infty$*
$$d(\phi(t,\theta_{-t}\omega)B(M(\theta_{-t}\omega),\theta_{-t}\omega),\mathrm{A}(\omega)) \longrightarrow 0.$$
5. *It is the only attractor among all measurable multifunctions that are contained in balls $B(M(\omega),\omega)$.*
6. *For any such multifunction D*
$$\mathrm{P}[d(\overline{\phi(t,\omega)D(\omega)},\mathrm{A}(\theta_t(\omega)) > \varepsilon] \longrightarrow 0 \qquad \text{as } t \to +\infty.$$

In the above theorem, for any constant M_0

$$B(M_0,\omega) = \{\, u \in \mathrm{W}_{\mathrm{GWS}}(\omega) \mid \int_0^1 |u(s)|^2\,ds \leq M_0 \,\}.$$

In a few words, the properties of attraction of $\mathrm{A}(\cdot)$ given in the theorem above say that all trajectories which started in the remote past from a point not too far, in a suitable sense, from the origin, end up to be more and more close to a set, namely, $\mathrm{A}(\cdot)$, of trajectories.

21.3.2 Stationary solutions

Here we see how the *path space* approach explained in this section can be used in the analysis of invariant measure. Indeed, the main feature of the method presented by Sell is to avoid any problem concerning the nonuniqness of the equations. Hence, a natural question to be asked is whether there are invariant measures for the time shift, and moreover which kind of objects they are, which properties they have, etc.

In order to define such objects, it is easier to look at the processes, rather than the laws; hence consider a martingale weak solution u of

$$\{ d\, u + (Au + B(u,u))\, dt = dB_t,\ \operatorname{div} u = 0.$$

The process u is stationary if the joint law of $(u(t_1+s),\dots,u(t_n+s))$ is independent of $s \geq 0$, for all n and $0 \leq t_1 \leq \cdots \leq t_n$. The stationary solution u has finite dissipation rate

$$\mathrm{E}\int_0^T \int_D |\nabla u|^2\, dx\, dt < \infty, \qquad \text{for all } T > 0,$$

and moreover satisfies a local version of the energy inequality (see Flandoli and Romito [16]) we have seen in the previous sections. The local energy inequality allows to consider the balance of energy in small space–time neighborhoods and it is aimed at the analysis of the blowup of solutions to the Navier–Stokes equations (see Caffarelli, Kohn, and Nirenberg [5]). A space–time point (t,x) is a *blowup* point, or a *singular* point, for a solution u if there is no neighborhood of (t,x) where u is bounded.

Theorem 21.5 *If u is a stationary solution with finite dissipation rate, then for all $t > 0$ the set of singular points at time t is empty for* P*-almost every trajectory of u.*

For a given stationary solution u, let μ_0 be the probability measure on H which is the law of u at a fixed time t. Notice that μ_0 is independent of t and, if the Navier–Stokes gave a proper dynamical system, μ_0 would be an invariant measure. The law of u disintegrates with respect to μ_0 and so for μ_0-a.e. (almost everywhere) initial condition u_0, it is possible to consider the martingale solution $u(\cdot; u_0)$ to the Navier–Stokes equations starting in u_0 at time $t = 0$ as a process whose law is the law of the stationary solution conditioned to the event $u(0) = u_0$. As a consequence of the previous theorem, one gets the following result.

Theorem 21.6 *For μ_0-a.e. $u_0 \in H$, there is a martingale solution to the Navier–Stokes equations starting in u_0 at time $t = 0$, having no singular points at all times.*

Such results are true both in the deterministic case (Flandoli and Romito [15]) and with a white-noise forcing (Flandoli and Romito [16]). In the former case, the above theorem is potentially trivial, in case the stationary solution is a delta measure concentrated on a time-independent solution to the equation (it is a well-known fact that such solutions are regular). If, on the other hand, we consider the stochastic equations forced by the derivative of a Brownian motion whose covariance is injective† (in the language of Section 21.2.1, the noise forces all Fourier modes), then the mixing effects of the noise avoid concentrations.

Theorem 21.7 *Assume, as above, that the Brownian motion has injective covariance. Then the measure μ_0 is fully supported in H.*

†By comparison with the results of Section 21.2.1, it is a reasonable hope that the assumption of injectivity on the covariance can be weakened.

21.4 Generalized flows

The method of Sell we have reviewed in the previous section has a drawback: in some sense the analysis loses the connection with the true evolution of the system. Indeed, the real dynamics is recovered only by means of the state space on which the topological dynamics is defined. The way to define a dynamical system from the Navier–Stokes equations seems to be still an open problem, at least if one wishes to have interesting properties (on the other hand, one can see the result of Da Prato and Debussche [11] given in Section 21.2.2).

Foiaş and Temam [20] (see also Temam [31]) do prove that there exists a global attractor, but under the (rather strong) assumption that globally defined strong solutions exist. In this section we will assume that weak solutions are continuous from $(0, +\infty)$ to H, with the strong topology[‡]. It is an assumption much weaker than the one of Foiaş and Temam. Anyway, it allows Ball [4] to prove the existence of the global attractor for the generalized semiflow associated with the Navier–Stokes equations. Similarly, Marin-Rubio and Robinson [23] have extended the result to random attractors for the stochastic equations.

21.4.1 Attractor for the generalized semiflow

Ball [4] develops an abstract theory of generalized semiflows, as a way to solve different problems, such as equations having a genuine nonuniqueness; or equations, like Navier–Stokes, where we don't know yet whether uniqueness holds; or problems with controls or parameters, where the main interest is in a global behavior, which unifies the role of the different parameters.

Given two functions φ, $\psi : [0, +\infty) \to X$, and a time $t \in [0, \infty)$ such that $\varphi(t) = \psi(0)$, define the new function $\varphi \oplus_t \psi$, obtained by *concatenation* of φ and ψ, as

$$\varphi \oplus_t \psi(s) = \begin{matrix} \varphi(s), & \qquad 0 \leq s < t, \\ \psi(s-t), & \qquad s \geq t. \end{matrix}$$

Definition 21.4 (Generalized semiflow) *A generalized semiflow* G *on a Polish space X is a family of maps $\varphi : [0, +\infty) \to X$ such that*

(S_1) *For all $x \in X$, there is a $\varphi \in \mathrm{G}$ such that $\varphi(0) = x$.*

(S_2) *If $\varphi \in \mathrm{G}$ and $t \geq 0$, then $\tau_t \varphi \in \mathrm{G}$.*

(S_3) *If φ and $\psi \in \mathrm{G}$ and $\psi(0) = \varphi(t)$, then $\varphi \oplus_t \psi \in \mathrm{G}$.*

(S_4) *If $(\varphi_j)_{j \in \mathbf{N}}$ and $\varphi_j(0) \to x$, then there is a subsequence $(\varphi_{j_n})_{n \in \mathbf{N}}$ such that $\varphi_{j_n}(t) \to \varphi(t)$ for all $t \geq 0$, with $\varphi \in \mathrm{G}$ and $\varphi(0) = x$.*

The generalized semiflow G_{NS} for the Navier–Stokes equations

$$\{\, \partial_t u + (u \cdot \nabla) u + \nabla p = \nu \Delta u + f, \ \operatorname{div} u = 0,$$

with Dirichlet boundary conditions is defined in terms of weak solutions. By usual methods, like the Galerkin approximation, etc., it is possible to show existence for the following kind of solutions.

Definition 21.5 *A function $u : [0, +\infty) \to H$ is a weak solution to the Navier–Stokes equations if*

1. *$u \in C([0, T]; H_{weak}) \cap L^2(0, T; V)$ and $u' \in L^1(0, T; V')$ for all $T > 0$.*

[‡]Indeed, it is well known that weak solutions are continuous with respect to the weak topology of H.

2. For all $\phi \in V$, for a.e. $t > 0$,

$$\langle u', \phi\rangle + \nu\langle \nabla u, \nabla\phi\rangle + \langle B(u,\phi), u\rangle = \langle u, f\rangle.$$

3. For a.a. (almost all) $s > 0$ and $s = 0$, and all $t \geq s$, $V(u)(t) \leq V(u)(s)$.

In the above definition, the term V gives the energy inequality, namely

$$V(u)(t) = \frac{1}{2}|u(t)|^2 + \nu\int_0^t \|u(s)\|^2\,ds - \int_0^t \langle f, u\rangle\,ds.$$

As such, the set G_{NS} defined above is not a generalized semiflow. The two properties (S_2) and (S_4) are not true. Here the additional (*unproved*) assumption of strong continuity of solution is *necessary and sufficient* to assure that G_{NS} is a generalized semiflow.

Once this is settled, the energy inequality shows that the semi-flow is *point-dissipative*[§] and *asymptotically compact*. Using these claims, it is possible to conclude that the attractor is unique and is the maximal compact invariant set of H.

Theorem 21.8 (Ball [4]) *There is a global attractor* A *for the generalized semiflow* G_{NS}*, that is a compact subset of H such that*

1. A *is invariant:* $\mathrm{A} = \{\,\varphi(t) \mid \varphi \in \mathrm{G},\ \varphi(0) \in \mathrm{A}\,\}$.

2. A *attracts all bounded sets: for each bounded $B \subset H$,*

$$d(\{\,\varphi(t) \mid \varphi \in \mathrm{G},\ \varphi(0) \in B\,\}, \mathrm{A}) \to 0.$$

21.4.2 Random attractor for the generalized semiflow

The approach of Ball explained in the previous section has been adapted to the stochastic setting in Marin-Rubio and Robinson [23]. On one hand, following Ball, they need to assume that weak solutions (that are understood in the sense of Flandoli and Schmallfuss, see Definition 21.3) are continuous from $(0, +\infty)$ with values in H with the strong topology. On the other hand, the global attractor attracts bounded sets in the *pullback* sense (see also Section 21.3.1.2), due to the forementioned lack of compactness of the random forcing.

Let $\Omega = C_0(\mathbf{R}, V)$ be the two-sided Wiener space. We assume, as before, that the stochastic Navier–Stokes equation are forced by a noise which is the derivative of the canonical Wiener process on the Wiener space Ω.

Definition 21.6 (stochastic generalized semiflow) *A family of pairs* $\mathrm{G} = \{\,(u,\omega) \mid \omega \in \Omega,\ u : [0,+\infty) \to X\,\}$ *is a stochastic generalized semiflow if*

(S_1) *For each $x \in X$, there is at least one $(u,\omega) \in \mathrm{G}$ such that $u(0) = x$,* P*-a.s.*

(S_2) *If $t \geq 0$ and $(u,\omega) \in \mathrm{G}$, then $(\tau_t u, \theta_t\omega) \in \mathrm{G}$.*

(S_3) *If $(u,\omega) \in \mathrm{G}$ and $(v, \theta_t\omega) \in \mathrm{G}$, then $(u \oplus_t v, \omega) \in \mathrm{G}$.*

(S_4) *If $(u_n, \omega_n) \in \mathrm{G}$ and $u_n(0) \to x$, $\omega_n \to \omega$, then there are $(u,\omega) \in \mathrm{G}$ and a subsequence such that $u_{n_k}(t) \to u(t)$ for all $t \geq 0$.*

[§]In this context, the property means that there is a bounded set B_0 such that $\varphi(t) \in B_0$ for all ϕ and t large enough.

Moreover, the flow is asymptotically compact if any sequence $(\varphi_j)_{j\in\mathbf{N}}$ with bounded values at $t = 0$ has limit points for $t \to +\infty$.

The shift θ_t for the noise is the shift of increments

$$\theta_t(\omega)(s) = \omega(t+s) - \omega(t),$$

in such a way that the Wiener measure is invariant for θ.

Let $\mathrm{G}_{\mathrm{SNS}}$ be the set of all pairs (u, ω) such that u is a weak solution to the Navier–Stokes equations corresponding to ω. Again, $\mathrm{G}_{\mathrm{SNS}}$ is a stochastic generalized semiflow if and only if every weak solution is continuous in time with respect to the weak topology of H. Under this assumption, Marin-Rubio and Robinson [23] show the existence of a random global attractor.

21.4.3 A selection principle

As we have already said, the *path space* approach explained, in all its flavors, in Section 21.3, in some sense *hides* the true dynamics of the equations, by introducing a topological dynamics, the time shift. On the other hand, the multivalued approach of the generalized semiflows probably gives an attractor which is *too large*, since, by definition, it attracts all possible solutions that, in the hypothesis of nonuniqueness, is quite large and includes nonphysical solutions as well.

In Romito [27] an attempt is made to find, by an abstract selection principle, single-valued selections of a generalized semiflow, that are themselves dynamical flows. This is, in the philosophy but not in the methods, more in the spirit of the work of Da Prato and Debussche [11] we have seen in Section 21.2.2.

We also assume that weak solutions are continuous.¶ We need to give a slightly different definition of generalized semiflow, which we call *multivalued random dynamical map* (MRDM), in order to take more precisely into account the different initial conditions.

Definition 21.7 (Multivalued random dynamical map) *The map Φ defined on $X \times \Omega$ with values in the set Comp(X) of all compact subsets of X is a multivalued random dynamical map if the following properties hold:*

(M_1) *Φ is measurable with values in Comp(X).*

(M_2) *If $x \in \Phi(x_0, \omega)$, then $\tau_t x \in \Phi(x_t, \vartheta_t \omega)$.*

(M_3) *If $x \in \Phi(x_0, \omega)$ and $y \in \Phi(x_t, \vartheta_t \omega)$. then $\pi_t x \oplus_t y \in \Phi(x_0, \omega)$.*

The weak solutions examined in this setting are those given in Definition 21.3. Under these assumptions, we define the MRDM Φ_{SNS} of solutions to the stochastic Navier–Stokes equations. The main theorem follows.

Theorem 21.9 *There exists a random dynamical system $\varphi : H \times \Omega \to H$ such that*

$$\varphi(x, \omega) \in \Phi_{SNS}(x, \omega), \qquad \text{for all } x \in X, \omega \in \Omega.$$

Moreover, the RDS φ has a global attractor.

The drawback of the selection method is that it provides a random dynamical system (RDS) which is not continuous. In particular, this means that the attractor is *not* invariant.‖ The problem seems to be quite general, as we shall see in the following example. A possibility, really a hope at this stage, is that the presence of the noise can help in solving this problem, as in the case of finite-dimensional SDEs.

¶As far as we know, such an assumption seems to be necessary for the definition of semiflows.

‖Hence, in some sense, it is not a proper attractor.

Example 21.1 We give a very simple example of *MRDM*. Let $\Omega = \mathbf{R}$, choose an arbitrary probability measure P on $\mathbf{R}$ and a Bernoulli random variable B_Ψ on Ω and set ϑ_t to be the identity on $\mathbf{R}$. Set $X = \mathbf{R}$ and consider the following ordinary differential equation:

$$\begin{aligned} &\dot{x} = \tfrac{x}{|x|} \arctan \sqrt{|x|}; \\ &x(0) \in \mathbf{R}, \end{aligned}$$

the solution is obviously global and unique if $x(0) \neq 0$. If $x(0) = 0$, we have the two maximal solutions, say $\overline{x}$ and $\underline{x}$, and every other solution starting at zero is null for an interval and then continues as $\overline{x}$ or $\underline{x}$.

Define $\Psi(x_0, \omega)$ to be the set whose only element is the unique solution of the above equation, if $x_0 \neq 0$, and $\Psi(0, \omega)$ as the set of all functions x which are 0 in a finite interval $[0, t_0]$ (where t_0 depends on x) and then continue as $\overline{x}$ if $B_\Psi(\omega) = 1$, and as $\underline{x}$ if $B_\Psi(\omega) = 0$.

It is easy to see, even without the above theorem, that there exist infinitely many *RDS* that can be selected from Ψ. Indeed, let T_λ be an exponential random variable of parameter λ and set $\psi(0, \omega)$ to be the solution which is identically 0 in the interval $[0, T_\lambda(\omega)]$ and then $\underline{x}$ or $\overline{x}$, depending on the value of B_Ψ.

Notice that none of these selections is a continuous *RDS*. Indeed, no selection of Ψ can be a continuous *RDS*.

References

[1] L. Arnold, *Random dynamical systems*, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[2] L. Arnold, H. Crauel, *Random dynamical systems*, Lyapunov exponents (Oberwolfach, 1990), 1–22, Lecture Notes in Math. **1486**, Springer, Berlin, 1991.

[3] A.V. Babin, M.I. Vishik, *Attractors of evolution equations*, Translated and revised from the 1989 Russian original by Babin. Studies in Mathematics and its Applications, 25, North-Holland Publishing Co., Amsterdam, 1992.

[4] J. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equations*, J. Nonlinear Sci. **7** (1997), no. 5, 475–502.

[5] L. Caffarelli, R. Kohn, L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771–831.

[6] M. Capiński, N.J. Cutland, *Attractors for three-dimensional Navier-Stokes equations*, Proc. R. Soc. London Ser. A **453** (1997), no. 1966, 2413–2426.

[7] T. Caraballo, J. A. Langa, J. Valero, *Global attractors for multivalued random dynamical systems*, Nonlinear Anal. **48** (2002), no. 6, Ser. A: Theory Methods, 805–829.

[8] T. Caraballo, P. Marin-Rubio, J.C. Robinson, *A comparison between two theories for multi-valued semiflows and their asymptotic behaviour*, Set-Valued Anal. **11** (2003), no. 3, 297–322.

[9] P. Constantin, C. Foiaş, R. Temam, *Attractors representing turbulent flows*, Mem. Am. Math. Soc., **53**, (1985), no. 314.

[10] H. Crauel, F. Flandoli, *Attractors for random dynamical systems*, Probab. Theory Related Fields **100** (1994), no. 3, 365–393.

[11] G. Da Prato, A. Debussche, *Ergodicity for the 3D stochastic Navier-Stokes equations*, J. Math. Pures Appl. (9) **82** (2003), no. 8, 877–947.

[12] W. E, J.C. Mattingly, *Ergodicity for the Navier-Stokes equation with degenerate random forcing: finite-dimensional approximation*, Comm. Pure Appl. Math. **54** (2001), no. 11, 1386–1402.

[13] C.L. Fefferman, *Existence and smoothness of the Navier-Stokes equation*, at the website `http://www.claymath.org/millennium/Navier-Stokes_Equations/`.

[14] F. Flandoli, *Irreducibility of the 3-D stochastic Navier-Stokes equation*, J. Funct. Anal. **149** (1997), no. 1, 160–177.

[15] F. Flandoli, M. Romito, *Statistically stationary solutions to the 3-D Navier-Stokes equation do not show singularities*, Electron. J. Probab. **6** (2001), no. 5, 15 pp. (electronic).

[16] F. Flandoli, M. Romito, *Partial regularity for the stochastic Navier-Stokes equations*, Trans. Am. Math. Soc. **354** (2002), no. 6, 2207–2241.

[17] F. Flandoli, M. Romito, *Probabilistic analysis of singularities for the 3D Navier-Stokes equations*, Math. Bohem. **127** (2002), no. 2, 211–218.

[18] F. Flandoli, B. Schmalfuss, *Weak solutions and attractors for three-dimensional Navier-Stokes equations with nonregular force*, J. Dyn. Diff. Eqns. **11** (1999), 355–398.

[19] F. Flandoli, B. Schmalfuss, *Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise*, Stoch. Stoch. Rep. **59** (1996), no. 1–2, 21–45.

[20] C. Foiaş, R. Temam, *Some analytic and geometric properties of the solutions of the evolution Navier-Stokes equations*, J. Math. Pures Appl. (9) **58** (1979), no. 3, 339–368.

[21] V. Jurdjevic, *Geometric control theory*, Cambridge Studies in Advanced Mathematics **51**, Cambridge University Press, Cambridge, 1997.

[22] N.V. Krylov, *The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 691–708.

[23] P. Marín-Rubio, J. Robinson, *Attractors for the stochastic 3D Navier-Stokes equations*, Stoch. Dyn. **3** (2003), no. 3, 279–297.

[24] J.C. Mattingly, *On recent progress for the stochastic Navier Stokes equations*, Journées "Équations aux Dérivées Partielles," Exp. No. XI, 52 pp., Univ. Nantes, Nantes, 2003.

[25] M. Romito, *Ergodicity of the finite dimensional approximation of the 3D Navier-Stokes equations forced by a degenerate noise*, J. Stat. Phys. **114** (2004), No. 112, 155–177.

[26] M. Romito, *A geometric cascade for the spectral approximation of the Navier-Stokes equations*, to appear on the IMA Volumes in Mathematics and Its Applications.

[27] M. Romito, in preparation.

[28] G. Sell, *Global attractors for the three-dimensional Navier-Stokes equation*, J. Dyn. Diff. Eqns. **8** (1996), 1–33.

[29] D.W. Stroock, S.R.S. Varadhan, *Multidimensional Diffusion Processes*, Springer, Berlin, 1979.

[30] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, Studies in Mathematics and Its Applications, **2**, North-Holland Publishing Co., Amsterdam, 1984.

[31] R. Temam, *Infinite dimensional dynamical systems in mechanics and physics*, Applied Mathematical Sciences, **68**, Springer-Verlag, New York, 1997.

[32] M.I. Vishik, A.V. Fursikov, *Mathematical Problems of Statistical Hydromechanics*, Kluwer, Dodrecht, 1988.

22 Stochastic Navier–Stokes Equations: Solvability, Control, and Filtering

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22.1 Stochastic Navier–Stokes equation: solvability

Let us begin with the abstract evolution form of the controlled stochastic Navier–Stokes equation [9] in the divergence free subspace H of square integrable vector fields which are parallel to the boundary

$$du(t) + (\nu Au(t) + B(u(t)))dt = U(t)dt + dW(t). \tag{22.1}$$

Here ν is the coefficient of kinematic viscosity, A is the Stokes operator and $B(\cdot)$ is the nonlinear inertia term with well-known properties. $U(t)$ is a distributed control with possible local support and $W(t)$ is an H-valued Wiener process with covariance operator Q. Here both the cases of degenerate noise (where Q is of trace class) and nondegenerate noise (where, for example, $Q = I$) are of importance. Moreover, flow problems in two and three-dimensional bounded, periodic as well as unbounded physical regions are of interest.

In this chapter we will denote $\|\cdot\|$ for H-norm, $\|\cdot\|_1$ for the norm of the space $V = D(A^{1/2})$ and $\|\cdot\|_{-1}$ for the norm of the dual space $V' = D(A^{-1/2})$.

Let us first consider the solvability of the stochastic Navier–Stokes equations with degenerate noise.

Theorem 22.1 (Strong Solutions [5]) *Let $(\Omega, \Sigma, \Sigma_t, m)$ be a complete filtered probability space and $W(t)$ be an H-valued Wiener process with trace-class covariance. Let the control function $U(\cdot) \in L^2(\Omega; L^2(0;T;V'))$ be adapted to Σ_t and the initial data be $u_0 \in L^2(\Omega; H)$ and measurable with respect to Σ_0. Then there exists a unique strong solution $u(\cdot) \in C([0,T]; H) \cap L^2(0,T;V)$, a.s. (almost surely) and adapted to Σ_t such that*

$$E\left[\sup_{t\in(0,T)} \|u(t)\|^2 + \nu \int_0^T \|A^{1/2}u(t)\|^2 dt\right] \le E\left[\|u_0\|^2 + \int_0^T \|U(t)\|_{-1}^2 dt\right] + TrQ.T.$$

This solution satisfies the equation (22.1) in the generalized sense and also can be conveniently described as $u(\cdot) \in L^2(\Omega; C([0,T]; H)) \cap L^2(\Omega; L^2(0,T;V))$. The proof given in [5] relies on a local monotonicity property associated with the inertia term and Minty–Browder type technique and does not utilize the usual compactness arguments used in classical Navier–Stokes theory. Because of this reason this theorem holds in two-dimensional (2D) bounded as well as unbounded regions including exterior domains and R^2.

We will now turn to the concept of weak solutions. In this case we will also consider the multiplicative noise

$$du(t) + (\nu Au(t) + B(u(t)))dt = U(t)dt + g(u)dW(t), \tag{22.2}$$

where the map $u \to g(u)$ from $H \to \mathcal{L}(H; H)$ satisfies

$$(i) \quad \|g(u)\| \le C_1\|u\| + C_2, \quad \forall u \in H,$$

$$(ii) \quad \|g(u_1) - g(u_2)\| \leq C_3\|u_1 - u_2\|, \quad \forall u_1, u_2 \in H.$$

For the path space we will take the Lusin space

$$\Omega := L^2(0,T;H) \cap D([0,T];V') \cap L^2(0,T;V)_\sigma \cap L^\infty(0,T;H)_{w^*}.$$

Here $D([0,T];V')$ is the V'-valued Skorohod space of càdlàg (right continuous, left limit) functions endowed with J-topology [7], and σ and w^* denote weak and weak-star topologies, respectively. We will take the cannonical filtration $\Sigma_t = \sigma\{u(s), s \leq t\}$. The martingale problem is to find a Radon measure P on the Borel algebra $\mathcal{B}(\Omega)$ such that

$$M_t := u(t) + \int_0^t (\nu Au(s) + B(u(s)) - U(s))\, dt$$

is an H-valued, $(\Omega, B(\Omega), \Sigma_t, P)$-martingale (i.e., a Σ_t-adapted process such that $E[M_t|\Sigma_s] = M_s$) with quadratic variation process

$$<< M >>_t := \int_0^t g(u(s))Qg^*(u(s))ds.$$

The following theorem is a simplified version of what is proved in the paper [10] which includes measure-valued relaxed controls.

Theorem 22.2 (Martingale solutions) *For 2D and 3D Navier–Stokes equation in bounded domains, there exists a martingale solution P which is carried by the subset of paths satisfying the following bounds:*

$$E^P\left[\sup_{t\in(0,T)} \|u(t)\|^2 + \nu\int_0^T \|A^{1/2}u(t)\|^2 dt\right] \leq C\left\{E\left[\|u_0\|^2 + \int_0^T \|U(t)\|_{-1}^2 dt\right] + TrQ.T\right\}.$$

Moreover, the martingale solution is unique for the 2D case.

We note finally that for the above strong and weak solutions it is possible to get the following apriori estimate for the higher order moments:

$$E^P\left[\sup_{t\in(0,T)} \|u(t)\|^{2l} + \nu\int_0^T \|u(s)\|^{2l-2}\|A^{1/2}u(t)\|^2 dt\right]$$

$$\leq C_1 E\left[\|u_0\|^{2l} + \int_0^T \|U(t)\|_{-1}^{2l} dt + C_2\left(\mathrm{Tr}Q.T\right)\right].$$

Other apriori estimates and continuous dependence theorems as well as discussions on earlier literature can be found in [5], [10].

Open Problems 22.1 The solvability of strong as well as martingale solutions are open for the nondegenerate case of $Q = I$.

22.2 Feedback control and infinite-dimensional Hamilton–Jacobi equations

Let us consider the control problem

$$J(t,u;U) := E\left\{\int_t^T \left(\|A^{1/2}u(r)\|^2 + \frac{1}{2}\|U(r)\|^2\right) dr + \|u(T)\|^2\right\} \to \inf.$$

We get insight into the nature of cost functional by noting the well-known equivalence of the integrand $\|A^{1/2}u\|^2$ and the enstrophy or the total vorticity $\|\mathrm{Curl}u\|^2$.

We will take the state equation as

$$du(t) + (\nu Au(t) + B(u(t)))dt = KU(t)dt + dW(t), \tag{22.3}$$

where $K \in \mathcal{L}(H; V)$ and the control $U(\cdot) : [0,T]\times\Omega \to \mathbf{U}$ will be taken from the set of control strategies $\mathcal{U}_t$ (for example, $\mathcal{U}_t = L^2([0,T] \times \Omega; \mathbf{U})$). The control set $\mathbf{U} = B_H(0,R) \subset H$ is the ball of radius R in H.

Let us define the value function as

$$\mathcal{V}(t,v) := \inf_{U(\cdot)\in\mathcal{U}_t} J(t,u;U(\cdot)),$$

for the initial data $u(t) = v$. Formally the value function satisfies the infinite-dimensional second-order Hamilton–Jacobi (–Bellman) equation

$$\mathcal{V}_t+\frac{1}{2}\mathrm{Tr}\left(QD^2\mathcal{V}\right)-(\nu Av+B(v), D\mathcal{V})+\|A^{1/2}v\|^2+\mathcal{H}(K^*D\mathcal{V}) = 0, \quad \text{for } (t,v) \in (0,T)\times D(A),$$

$$\mathcal{V}(T,v) = \|v\|^2, \quad \text{for } v \in H.$$

Here $\mathcal{H}(\cdot) : H \to R$ is given by

$$\mathcal{H}(Z) := \inf_{U\in\mathbf{U}} \left\{(U,Z) + \frac{1}{2}\|U\|^2\right\}.$$

More explicitly we can write

$$\mathcal{H}(Z) = \begin{cases} -\frac{1}{2}\|Z\|^2 & \text{for } \|Z\| \le R \\ -R\|Z\| + \frac{1}{2}R^2 & \text{for } \|Z\| > R. \end{cases}$$

Moreover, the optimal feedback control is given formally by

$$\tilde{U}(t) = \Upsilon\left(K^*D_v\mathcal{V}(t,u(t))\right)$$

where

$$\Upsilon(Z) := D_Z\mathcal{H}(Z) = \begin{cases} -Z & \text{for } \|Z\| \le R \\ -Z\frac{R}{\|Z\|} & \text{for } \|Z\| > R. \end{cases}$$

Let us now state a rigorous result on viscosity solutions to the above Hamilton–Jacobi equation from the paper [4] where more general cost functionals involving polynomial growth in the V-norm also treated. In [3] a semigroup treatment of this problem with nondegenerate noise is given.

Definition 22.1 (Test Functions) A function ψ is a test function of the above Hamilton–Jacobi equation if $\psi = \phi + \delta(t)(1 + \|v\|_1^2)^m$, where

(i) $\phi \in C^{1,2}((0,T) \times H)$, and $\phi, \phi_t, D\phi, D^2\phi$ are uniformly continuous on $[\epsilon, T-\epsilon] \times H$ for every $\epsilon > 0$.

(ii) $\delta \in C^1((0,T))$ is such that $\delta > 0$ on $(0,T)$ and $m \ge 1$.

Definition 22.2 (Viscosity Solution) A function $\mathcal{V} : (0,T) \times V \to R$ that is weakly sequentially upper-semicontinuous (respectively, lower-semicontinuous) on $(0,T)\times V$ is called a viscosity subsolution (respectively, supersolution) of the above Hamilton–Jacobi equation.

If for every test function ψ, whenever $\mathcal{V}-\psi$ has a global maximum (respectively, $\mathcal{V}+\psi$ has a global minimum) over $(0,T)\times V$ at (t,v), then we have $v\in D(A)$ and

$$\psi_t+\frac{1}{2}\mathrm{Tr}\left(QD^2\psi\right)-(\nu Av+B(v),D\psi)+\|A^{1/2}v\|^2+\mathcal{H}(K^*D\psi)\geq 0,$$

(respectively

$$-\psi_t-\frac{1}{2}\mathrm{Tr}\left(QD^2\psi\right)+(\nu Av+B(v),D\psi)+\|A^{1/2}v\|^2+\mathcal{H}(K^*(-D\psi))\leq 0.)$$

A function is a viscosity solution if it is both a viscosity subsolution and a supersolution.

For the 2D stochastic Navier–Stokes equation on a periodic domain (or compact manifold) with H and V degeneracies on the noise (i.e., $\mathrm{Tr}Q<\infty$ and $\mathrm{Tr}(A^{1/2}QA^{1/2})<\infty$) we can establish the following two theorems.

Theorem 22.3 (Continuity of the Value Function) *For each $r>0$, there exists a modulus of continuity ω_r such that*

$$|\mathcal{V}(t_1,v)-\mathcal{V}(t_2,z)|\leq\omega_r(|t_1-t_2|+\|v-z\|),\quad \textit{for } t_1,t_2\in[0,T] \textit{ and } \|v\|_1,\|z\|_1\leq r,$$

and

$$|\mathcal{V}(t,v)|\leq C(1+\|v\|_1^2).$$

Theorem 22.4 (Existence and Uniqueness) *The value function $\mathcal{V}$ is the unique viscosity solution for the Hamilton–Jacobi equations.*

Open Problems 22.2 Existence and uniqueness of viscosity solutions for the cases of arbitrary 2D domains (bounded and unbounded) as well as the nondegenerate noise cases (e.g., $Q=I$) are open.

22.3 Optimal stopping and infinite-dimensional variational inequality

The optimal stopping problem for the stochastic Navier–Stokes equations in 2D bounded domains has been studied in [6] and in [2] by different methods. In this section we present a slightly simplified version of the results in [2].

Consider the optimal stopping problem of characterizing the value function

$$\mathcal{V}(t,v):=\inf_{\tau}E\left\{\int_t^{\tau}\|A^{1/2}u(s)\|^2ds+k(u(\tau))\|u(\tau)\|^2\right\},$$

with state equation

$$du(t)+(\nu Au(t)+B(u(t)))dt=dW(t). \tag{22.4}$$

Formally, the value function solves the following variational inequality:

$$\mathcal{V}_t-\frac{1}{2}\mathrm{Tr}\left(QD^2\mathcal{V}\right)+(\nu Av+B(v),D\mathcal{V})\leq\|A^{1/2}v\|^2,\quad\text{for } t>0, v\in D(A),$$

$$\mathcal{V}(t,v)\leq k(v)\|v\|^2,\quad\text{for } t\geq 0, v\in H,$$
$$\mathcal{V}(0,v)=\phi_0(v),\ \text{for } v\in H,$$

and in the (continuation) set

$$\left\{(t,v)\in R^+\times H; \mathcal{V}(t,v)<k(v)\|v\|^2\right\},$$

we have equality

$$\mathcal{V}_t-\frac{1}{2}\mathrm{Tr}\left(QD^2\mathcal{V}\right)+(\nu Av+B(v),D\mathcal{V})=\|A^{1/2}v\|^2,\quad \text{for } t>0, v\in D(A).$$

This problem can be viewed as a nonlinear evolution problem with multivalued nonlinearity

$$\mathcal{W}_t-\mathcal{N}\mathcal{W}+N_K(\mathcal{W})\ni\|A^{1/2}\cdot\|^2,\quad t\in[0,T],$$

$$\mathcal{W}(0)=\phi_0.$$

Here $\mathcal{N}$ is the generator of the stochastic Navier–Stokes process (infinitesimal generator of the transition semigroup $P(t)$) and N_K is the normal cone to the closed convex subset $K\subset L^2(H,\mu)$

$$K=\left\{\phi\in L^2(H;\mu);\quad \phi\le k(\cdot)\|\cdot\|^2 \text{ on } H\right\},$$

where μ is an invariant measure for $P(t)$. In fact N_K is defined as

$$N_K(\phi)=\left\{\eta\in L^2(H;\mu);\quad \int_H\eta(v)(\psi(v)-\phi(v))\mu(dv)\le 0,\quad \forall\psi\in K\right\},\quad \phi\in K.$$

Let us use the solvability theorem (Theorem 22.1) for strong solutions to define the transition semigroup $P(t):C_b(H)\to C_b(H)$ by

$$(P(t)\psi)(v)=E\psi(u(t,v)),\quad v\in H,\forall t\ge 0,\quad \psi\in C_b(H),$$

where $u(t,v)$ is the strong solution with initial data v. Existence of invariant measure μ and its uniqueness for large ν are shown in [1]

$$\int_H(P(t)\psi)(v)\mu(dv)=\int_H\psi(v)\mu(dv),\quad \psi\in C_b(H).$$

Then $P(t)$ has an extension to a C_0-contraction semigroup on $L^2(H;\mu)$. We denote by $\mathcal{N}:D(\mathcal{N})\subset L^2(H,\mu)\to L^2(H,\mu)$ the infinitesimal generator of $P(t)$ and let $\mathcal{N}_0\subset\mathcal{N}$ be defined by

$$(\mathcal{N}_0\psi)(v)=\frac{1}{2}\mathrm{Tr}\left(QD^2\psi(v)\right)-(\nu Av+B(v),D\psi(v)),\quad \forall\psi\in\varepsilon_A(H),$$

where $\varepsilon_A(H)$ is the linear span of all functions of the form $\phi(\cdot)=\exp\left(i(h,\cdot)\right)$, $h\in D(A)$.

It is shown in [1] that if

$$\nu\ge C(\|Q\|_{\mathcal{L}(H;H)}+\ \mathrm{Tr}Q)$$

is sufficiently large and if $\mathrm{Tr}[A^\delta Q]<\infty$ for $\delta>2/3$, then $\mathcal{N}_0$ is dissipative in $L^2(H,\mu)$ and its closure $\bar{\mathcal{N}}_0$ in $L^2(H,\mu)$ coincides with $\mathcal{N}$. Moreover, from the definition of the invariant measure, taking $\psi(v)=\|v\|^2$ we have

$$\int_H(\mathcal{N}\psi)(v)\mu(dv)=0$$

which implies the integrability of enstrophy $\|curlv\|^2=\|A^2v\|^2$ with respect to the invariant measure μ

$$2\nu\int_H\|A^{1/2}v\|^2\mu(dv)=\ \mathrm{Tr}Q<\infty.$$

We will now state a slightly simplified version of the solvability theorem from [2] for the variational inequality (or the nonlinear evolution problem formulated above). The proof is based on nonlinear semigroup theory for the m-accretive operator $\mathcal{A} = -\mathcal{N} + N_K$ in $L^2(H,\mu)$.

Theorem 22.5 *Suppose $k(v)$ such that $G(v) = k(v)\|v\|^2$ satisfies $G \in C^2(H)$ and*

$$(\mathcal{N}_0 G)(v) \leq 0, \quad \forall v \in D(A).$$

Then, for each $\phi_0 \in D(\mathcal{N}) \cap K$ there exists a unique function $\phi \in W^{1,\infty}([0,T]; L^2(H,\mu))$ such that $\mathcal{N}\phi \in L^\infty(0,T; L^2(H,\mu))$ and

$$\frac{d}{dt}\phi(t) - \mathcal{N}\phi(t) + \eta(t) - \|A^{1/2}\cdot\|^2 = 0, \quad a.e.\ t \in (0,T),$$

$$\eta(t) \in N_K(\phi(t)), \quad a.e.\ t \in (0,T),$$

$$\phi(0) = \phi_0.$$

Moreover, $\phi : [0,T] \to L^2(H,\mu)$ is differentiable from the right and

$$\frac{d^+}{dt}\phi(t) - \mathcal{N}\phi(t) - \|A^{1/2}\cdot\|^2 + \mathbf{P}_{N_K(\phi(t))}(\|A^{1/2}\cdot\|^2 + \mathcal{N}\phi(t)) = 0, \quad \forall t \in [0,T),$$

where $\mathbf{P}_{N_K(\phi)}$ is the projection on the cone $N_K(\phi)$. **Remarks on Impulse Control**

In [6] impulse control problem is treated for the 2D stochastic Navier–Stokes equation with degenerate noise in bounded domains. This problem is of the form

$$du(t) + (\nu Au(t) + B(u(t)))dt = \sum_{i\geq 1} U_i\delta(t-\tau_i)dt + dW(t), \tag{22.5}$$

where the control strategy Θ consists of the set of random stopping times τ_i and the control decisions U_i

$$\Theta := \{(\tau_1, U_1); (\tau_2, U_2); , \cdots\}.$$

The goal would be to find an optimal control such that a cost functional of the following form is minimized:

$$J(v;\Theta) := E\left\{\int_0^\infty F(u(t))dt + \sum_i L(U_i)\right\}.$$

In this case we end up with quasi-variational inequalities of the following form for the value function $\mathcal{V}$:

$$\mathcal{N}\mathcal{V} \leq F, \quad \mathcal{V} \leq M(\mathcal{V}),$$

and

$$\mathcal{N}\mathcal{V} = F \text{ in the set } \{\mathcal{V} < M(\mathcal{V})\}.$$

Here the nonlinear operator M is defined as

$$M(\mathcal{V})(v) = \inf_U \{L(U) + \mathcal{V}(v)\}.$$

In [6] a "hybrid control" generalization of this problem is formulated and solvability is proved by designing a convergent sequence of stopping time problems (and a sequence of variational inequalities) to approximate the quasi-variational inequality. The proof involves a generalization of the generator $\mathcal{N}$ using the concept of weak generator and the resolvant operator of the Feller semigroup associated with the stochastic Navier–Stokes process. This is a technical construction so we omit the details and refer the interested readers to this paper.

22.4 Nonlinear filtering of stochastic Navier–Stokes equations

Let us now consider a flow field in which both the viscosity coefficient and noise may be unknown so we propose the situation to be modeled by equation (4). Let us also assume that we have sensors at specific locations measuring the flow characteristics in real time

$$z(t) = \int_0^t h(u(s))ds + W_2(t).$$

Here W_2 is a Wiener process representing uncertainty in measurements and h is called the observation vector. Depending on the type of measurement h could be finite or infinite dimensional. Moreover, the domain of $h(\cdot)$ will be H if we are making velocity measurements and $D(A^{1/2})$ if we measure the vorticity.

Let us assume that we have the back measurements $\{z(s), 0 \leq s \leq t\}$. How does the least square best estimate of a function of the velocity $f(u(t))$ evolve in time? It is well known that the best estimator is the conditional expectation of $f(u(t))$ given the back measurements (or the sigma algebra Σ_t^z generated by $\{z(s), 0 \leq s \leq t\}$). Let us denote

$$\mu_t^z[f] := E[f(u(t))|\Sigma_t^z].$$

Let us describe the special case of uncorrelated W and W_2 from the more general correlated case developed in [8]. Using martingale methods we derive the equation of evolution for $\mu_t^z[f]$ called the Fujisaki–Kallianpur–Kunita equation

$$d\mu_t^z[f] = \mu_t^z[\mathcal{N}_0 f]dt + \left(\mu_t^z[hf] - \mu_t^z[h]\mu_t^z[f]\right)\left(dz(t) - \mu_t^z[h]dt\right), \quad \text{for } f \in \varepsilon_A(H).$$

If we set

$$\vartheta_t^z[f] := \mu_t^z[f]. \exp\left\{\int_0^t \mu_s^z[h] \cdot dz(s) - \frac{1}{2}\int_0^t |\mu_s^z[h]|^2 ds\right\}.$$

We then get (using the Ito formula) a linear equation called the Duncan–Mortensen–Zakai equation

$$d\vartheta_t^z[f] = \vartheta_t^z[\mathcal{N}_0 f]dt + \vartheta_t^z[hf] \cdot dz(t), \quad \text{for } f \in \varepsilon_A(H).$$

Existence and uniqueness of measure-valued solutions to the above two evolution equations has been proved in [8] for the case of 2D periodic domains with H and V degeneracies on the noise (i.e., $\mathrm{Tr}Q < \infty$ and $\mathrm{Tr}(A^{1/2}QA^{1/2}) < \infty$).

Theorem 22.6 *Let $\mathcal{M}(H)$ and $\mathcal{P}(H)$, respectively, denote the class of positive Borel measures and Borel probability measures on H. Then there exists a unique $\mathcal{P}(H)$-valued random probability measure μ_t^z and a unique $\mathcal{M}(H)$-valued random measure ϑ_t^z, both processes being adapted to the filtration Σ_t^z such that the Fujisaki–Kallianpur–Kunita and the Zakai equations are, respectively, satisfied for the class of functions from $\varepsilon_A(H)$.*

The proof is based on the uniqueness theorem for the backward Kolmogorov equation.

Acknowledgment

This research has been supported by the Army Research Office, Probability and Statistics Program.

References

[1] V. Barbu, G. Da Prato and A. Debussche, "The Transition Semigroup of Stochastic Navier-Stokes Equations in 2-D," *Atti. Acad. Naz. Lincei.* (to appear).

[2] V. Barbu and S.S. Sritharan, "Optimal Stopping-Time Problem for Stochastic Navier-Stokes Equations and Infinite-Dimensional Variational Inequalities," preprint, 2004.

[3] G. Da Prato and A. Debussche, "Dynamic Programming of the Stochastic Navier-Stokes Equation," *Math. Model. Numer. Anal.,* **34**(2), pp. 459-475, 2000.

[4] F. Gozzi, S.S. Sritharan, and A. Swiech, "Bellman Equations Associated to the Optimal Feedback Control of Stochastic Navier-Stokes Equations," *Commun. Pure Appl. Math.*, **58**(5), pp. 671-700, 2005.

[5] J. L. Menaldi and S.S. Sritharan, "Stochastic 2-D Navier-Stokes Equation," *Appl. Math. Optimization*, **46**, pp. 31-52, (2002).

[6] J.L. Menaldi and S.S. Sritharan, "Impulse Control of Stochastic Navier-Stokes Equations," *Nonlinear Anal.*, **52**, pp. 357-381, (2003).

[7] M. Metivier, **Stochastic Partial Differential Equations in Infinite Dimensional Spaces**, Scuola Normale Superiore, Pisa, (1988).

[8] S.S. Sritharan, "Nonlinear Filtering of Stochastic Navier-Stokes Equations," in T. Funaki and W.A. Woycznski, Editors, **Nonlinear Stochastic PDEs: Burgers Turbulence and Hydrodynamic Limit**, Springer-Verlag, Berlin, 1994.

[9] S.S. Sritharan, Editor, **Optimal Control of Viscous Flow**, SIAM, Philadelphia, 1998.

[10] S.S. Sritharan, "Deterministic and Stochastic Control of Navier-Stokes Equation with Linear, Monotone and Hyperviscosities," *Appl. Math. Optimization*, **41**, pp. 255-308, 2000.

23 Stability of the Optimal Filter via Pointwise Gradient Estimates

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23.1 Introduction and main result

The purpose of this chapter is to complement the main result in [10] on the stability of the optimal filter w.r.t. (with respect to) its initial condition for a nonlinear time-dependent signal observed with independent additive noise. More precisely, consider a time-inhomogeneous signal process $(X_t)_{t\geq 0}$ given by the stochastic differential equation

$$dX_t = Q(t)\nabla_x \log\varphi(t,x)\,dt + C(t)\,dW_t\,. \tag{23.1}$$

Here, $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion, $C \in C([0,\infty);\mathbb{R}^{d\times d})$, $Q(t) = C(t)C(t)^T$, ∇_x denotes the gradient w.r.t. space variables and $\varphi \in C^{1,2}([0,\infty)\times\mathbb{R}^d)$ is strictly positive and for any $T > 0$ there exists a finite constant L_T with

$$\langle Q(t)\nabla_x \log\varphi(t,x), x\rangle \leq L_T\left(\|x\|^2+1\right)\,, (t,x)\in[0,T]\times\mathbb{R}^d\,. \tag{23.2}$$

It follows from Theorem 1 in Section V.1 of [6] that (23.1) has a unique strong solution for any initial condition $x\in\mathbb{R}^d$.

Suppose that $(X_t)_{t\geq 0}$ is observed through the p-dimensional process

$$dY_t = G(t)X_t\,dt + d\tilde{W}_t\,, Y_0 = 0\,. \tag{23.3}$$

Here

$$G : [0,\infty)\to\mathbb{R}^{p\times d} \text{ is continuously differentiable}$$

and $(\tilde{W}_t)_{t\geq 0}$ is a p-dimensional Brownian motion independent of the signal $(X_t)_{t\geq 0}$. We are interested in the stability of the conditional distribution

$$\eta_t(A) := E\left[1_A(X_t)|\mathcal{Y}_t\right]\,, A\in\mathcal{B}(\mathbb{R}^d)\,,$$

of the signal X_t, given the observation

$$\mathcal{Y}_t := \{Y_s : s\in[0,t]\}$$

up to time t, w.r.t. the initial distribution η_0 of the signal.

This is a major problem in filter theory and has been studied by many authors. It is reasonable to believe that η_t will become independent of η_0 for large t if the distribution of X_t will become independent of η_0, that is, if the process (X_t) is ergodic. This has been worked out in the time-independent setting in [7] by Ocone and Pardoux for a linear signal and for a nonlinear signal on a compact space, in [1], [2] by Atar and Zeitouni for a nonlinear signal and in [5] by Da Prato, Fuhrman, and Malliavin for a nonlinear signal on a compact Riemannian manifold. However, the linear case in [7] shows that ergodicity of the signal process is not necessary for stability.

In [9] and [10] a new variational approach has been introduced to this problem to obtain results on the stability and at the same time explicit estimates on the rate of stability for a nonlinear signal which is not necessarily ergodic. In this chapter, we provide an alternative proof of the main result in [10] on the stability for a time-dependent nonlinear signal, using pointwise gradient estimates as our main tool. For a more precise comparison with our previous result see Remark 23.1 below.

23.1.1 The Kallianpur–Striebel formula

The Kallianpur–Striebel formula provides an explicit formula for η_t: denote by $\tilde{P}$ the law of $X.$ and let

$$Z_t := \exp\left(\int_0^t G(s)X_s\,dY_s - \frac{1}{2}\int_0^t \|G(s)X_s\|^2\,ds\right).$$

Under appropriate assumptions on the coefficients of (23.1) and (23.3) (the main assumption being that the martingale problem associated to (23.1) and (23.3) is well-posed (see Chapter I of [8])), it follows that

$$E\left[f(X_t)|\mathcal{Y}_t\right] = \frac{\tilde{E}\left[f(X_t)Z_t\right]}{\tilde{E}\left[Z_t\right]}\,, f \in \mathcal{B}_b(\mathbb{R}^d)\,. \tag{23.4}$$

Proposition 23.1 gives an alternative representation for the expectation $\tilde{E}\left[f(X_t)Z_t\right]$. To state our proposition we need the following notations. Let

$$\Delta_{Q(t)}f := \text{ tr } \left(Q(t)f_{xx}\right)\,.$$

For any $y \in C([0,\infty);\mathbb{R}^p)$, $y(0)=0$, and $x \in \mathbb{R}^d$, let

$$\sigma^y(t,x) := y(t)\cdot\dot{G}(t)x - \frac{1}{2}\|C(t)^T G(t)^T y(t)\|^2 + \frac{1}{2}W(t,x)\,, \tag{23.5}$$

where

$$W(t,x) := \|G(t)x\|^2 + \frac{\Delta_{Q(t)}\varphi(t,x) + 2\partial_t\varphi(t,x)}{\varphi(t,x)}\,. \tag{23.6}$$

For a Brownian motion $(V_t)_{t\geq 0}$ on $\mathbb{R}^d$, $0 \leq s \leq t$, and $x \in \mathbb{R}^d$ let

$$\xi_t(x) := x - \int_0^t Q(r)G(r)^T y(r)\,dr + \int_0^t C(r)\,dV_r\,.$$

Define the integral kernel

$$K_t^y f(x) := E\left[f(\xi_t(x))A_t^y(x)\right], \tag{23.7}$$

where

$$A_t^y(x) := \exp\left(-\int_0^t \sigma^y(r,\xi_r(x))\,dr\right).$$

To further simplify notations in the following let:

$$M_t^y := -\int_0^t \left(\nabla_x \log\varphi(s,X_s) + y_s^T G(s)\right)C(s)\,dW_s\,.$$

Note that

$$\langle M^y_\cdot \rangle_t = \int_0^t \langle Q(s)\nabla_x \log\varphi(s,X_s), \nabla_x \log\varphi(s,X_s)\rangle\, ds$$
$$+ 2\int_0^t \langle Q(s)\nabla_x \log\varphi(s,X_s), y_s^T G(s)\rangle\, ds + \int_0^t \|C(s)^T G(s)^T y(s)\|^2\, ds\,.$$

Proposition 23.1. *Assume that*

(i) The Kallianpur–Striebel formula (23.4) holds.

(ii) For all $y \in C([0,\infty);\mathbb{R}^p)$, $y(0)=0$, the process

$$Z_t^y := \exp(M_t^y - \frac{1}{2}\langle M^y\rangle_t)$$

is a $\tilde{P}$-martingale.

Let μ_0 be the initial distribution of X_0. Then there exists a Brownian motion $(V_t)_{t\ge 0}$ on $\mathbb{R}^d$ such that

$$\tilde{E}\left[f(X_t)Z_t\right] = \int_{\mathbb{R}^d} \frac{1}{\varphi(0,\cdot)} K_t^{Y_\cdot}(f e^{G(t)^T Y_t \cdot}\varphi(t,\cdot))\, d\mu_0 \qquad a.s.$$

Proof The integration by parts formula and the time-dependent Ito formula imply that

$$\int_0^t G(s)X_s\, dY_s = G(t)X_t\cdot Y_t$$
$$- \int_0^t Y_s\cdot\Big(G(s)Q(s)\nabla_x\log\varphi(s,X_s) + \dot{G}(s)X_s\Big)\, ds - \int_0^t Y_s^T G(s)C(s)\, dW_s\,.$$

Here we used the fact that, by independence of $X_\cdot$ and $\tilde{W}_\cdot$, the covariation $\langle X,\tilde{W}\rangle_\cdot$ vanishes. It follows that

$$\begin{aligned} Z_t = \exp\left(G(t)^T Y_t\cdot X_t - \int_0^t Y_s\cdot\Big(G(s)Q(s)\nabla_x\log\varphi(s,X_s)+\dot{G}(s)X_s\Big)\, ds\right)\cdot \\ \cdot\exp\left(-\int_0^t Y_s^T G(s)C(s)\, dW_s - \frac{1}{2}\int_0^t \|G(s)X_s\|^2\, ds\right)\,. \end{aligned} \tag{23.8}$$

Using again the time-dependent Ito formula, we have that

$$\begin{aligned} &\int_0^t \frac{\left(\frac{1}{2}\Delta_{Q(s)}+\partial_t\right)\varphi(s,X_s)}{\varphi(s,X_s)}\, ds\cdot \\ &= \log\left(\frac{\varphi(t,X_t)}{\varphi(0,X_0)}\right) - \int_0^t \nabla_x\log\varphi(s,X_s)\, C(s)\, dW_s \\ &\quad - \frac{1}{2}\int_0^t \langle Q(s)\nabla_x\log\varphi(s,X_s), \nabla_x\log\varphi(s,X_s)\rangle\, ds\,. \end{aligned} \tag{23.9}$$

Combining (23.8) and (23.9) we now conclude that

$$Z_t = \frac{\varphi(t,X_t)}{\varphi(0,X_0)}\exp\left(G(t)^T Y_t\cdot X_t\right)\exp\left(-\int_0^t \sigma^{Y_\cdot}(s,X_s)\, ds\right) Z_t^{Y_\cdot}\,.$$

Fix $y \in C([0,\infty);\mathbb{R}^p)$, $y(0)=0$. By our assumption Z_t^y is an $(\mathcal{F}_t^W)_{t\ge 0}$-martingale, so that we can define the probability measure Q^y on $\mathcal{F}_t^W := \sigma\{W_s : s \in [0,t]\}$ by

$$\frac{dQ^y}{d\tilde{P}}_{|\mathcal{F}_t^W} = Z_t^y\,.$$

Girsanov's theorem implies that w.r.t. the new measure Q^y, the process

$$V_t := W_t + \int_0^t C(s)^T \left(\nabla_x \log\varphi(s,X_s) + G(s)^T y(s)\right)\, ds$$

is a Brownian motion. It follows that

$$\begin{aligned} X_t &= X_0 + \int_0^t Q(s)\nabla_x \log\varphi(s,X_s)\,ds + \int_0^t C(s)\,dW_s \\ &= X_0 - \int_0^t Q(s)G(s)^T y(s)\,ds + \int_0^t C(s)\,dV_s = \xi_t(X_0)\,, \end{aligned}$$

so that

$$\begin{aligned} E\left[f(X_t)Z_t\right] &= E^{Q^{Y\cdot}}\left[f(\xi_t(X_0))e^{G(t)^T Y_t\cdot\xi_t(X_0)}\frac{\varphi(t,\xi_t(X_0))}{\varphi(0,X_0)}A_t^{Y\cdot}(X_0)\right] \\ &= \int_{\mathbb{R}^d}\frac{1}{\varphi(0,\cdot)}K_t^{Y\cdot}\left(fe^{G(t)^T Y_t\cdot}\varphi(t,\cdot)\right)\,d\mu_0 \qquad a.s. \quad \square \end{aligned}$$

To emphasize the dependence of the conditional distribution on μ_0 we will write $E_{\mu_0}[1_A(X_t)|\mathcal{Y}_t]$ in the following. Before we state our assumptions needed for our main result let us introduce some notations. Let $C_p^{m,n}([0,\infty)\times\mathbb{R}^d)$ denote the space of all functions $f \in C^{m,n}([0,\infty)\times\mathbb{R}^d)$ for which f and all partial derivatives up to order m w.r.t. t and up to order n w.r.t. the space variables are polynomially bounded. Let $C_p^{m,n}([0,T]\times\mathbb{R}^d)$ for any $T>0$ and $C_p^n(\mathbb{R}^d)$ be defined in a similar way. For a Lipschitz continuous function f on $\mathbb{R}^d$ let $\|f\|_{Lip}$ be its Lipschitz constant. Finally, we say that a function f is *log-concave* if $f \in C^2(\mathbb{R}^d)$, $f>0$ and $-(\log f)_{xx} \ge 0$.

Assumption 23.1 W defined by (23.6) is in $C_p^{0,2}([0,\infty)\times\mathbb{R}^d)$ and

$$\exists\,\kappa_* > 0 \text{ such that } W(t,\cdot)_{xx} \ge \kappa_*^2\cdot I \text{ for any } t\ge 0\,.$$

Here, I denotes the $d\times d$-identity matrix. Note that our assumption implies, in particular, that for any T there exists a finite constant c_T such that

$$W(t,x) \ge \frac{\kappa_*^2}{2}\|x\|^2 - c_T\|x\|\,,\quad (t,x)\in[0,T]\times\mathbb{R}^d\,.$$

Assumption 23.2 For any $t\ge 0$ there exists a log-concave function $\tilde{\varphi}_t$ and some finite positive constant M_t such that

$$M_t^{-1}\tilde{\varphi}_t \le \varphi(t,\cdot) \le M_t\tilde{\varphi}_t\,.$$

Assumption 23.3 $\exists\,\kappa_- \in \mathcal{B}([0,\infty))$, $\kappa_- > 0$, $\kappa_+ \ge 1$ such that $\kappa_-(t)\cdot I \le Q(t) \le \kappa_+\cdot I$ for all $t\ge 0$.

Assumption 23.4 $Q(\cdot)$ is differentiable and

$$\chi := \int_0^\infty \|Q(t)^{-\frac12}\dot{Q}(t)Q(t)^{-\frac12}\|_{op}\,dt < \infty\,.$$

Here, $\|\cdot\|_{op}$ denotes the operator norm.

To state our main result fix a log-concave function $g_0 \in C_p^2(\mathbb{R}^d)$ such that

$$-(\log g_0)_{xx} \geq \kappa_* \cdot I\,. \tag{23.10}$$

Let

$$\nu(dx) := \varphi(0,x) g_0(x)\, dx\,.$$

We then have

Theorem 23.1. *Let μ_i, $i = 1,2$, be such that $\mu_i \ll \nu$ with Lipschitz continuous density h_i bounded from below and from above. Let $\delta > 0$ be such that $\delta \leq h_i \leq \delta^{-1}$. Then*

(i) Let $T > 0$, $y \in C([0,\infty[;\mathbb{R}^p)$, $y(0) = 0$, and $\eta_T^y(\mu_i)$ be defined by

$$\eta_T^y(\mu_i)(A) := \frac{1}{Z_T^y(\mu_i)} \int_{\mathbb{R}^d} \frac{1}{\varphi(0,\cdot)} K_T^y \left(1_A e^{G^T(T)y(T)\cdot}\varphi(T,\cdot)\right) d\mu_i\,,$$

where

$$Z_T^y(\mu_i) := \int_{\mathbb{R}^d} \frac{1}{\varphi(0,\cdot)} K_T^y \left(e^{G^T(T)y(T)\cdot}\varphi(T,\cdot)\right) d\mu_i$$

is the normalizing constant. Then

$$\|\eta_T^y(\mu_1) - \eta_T^y(\mu_2)\|_{var} \leq \sqrt{\frac{d}{2\kappa_*}\frac{M_T}{\delta^3}} \sqrt{\frac{\kappa_+ e^{\chi}}{\kappa_-(0)}} e^{-\kappa_*\kappa_+^{-\frac{1}{2}}\int_0^T \kappa_-(r)\,dr} \left(\|h_1\|_{Lip} + \|h_2\|_{Lip}\right)\,.$$

(ii) Suppose that (i) and (ii) in Proposition 23.1 hold. Let $f \in \mathcal{B}_b(\mathbb{R}^d)$. Then

$$\limsup_{t\to\infty} e^{\kappa_*\kappa_+^{-\frac{1}{2}}\int_0^t \kappa_-(r)\,dr} \left|E_{\mu_1}\left[f(X_t)|\mathcal{Y}_0^t\right] - E_{\mu_2}\left[f(X_t)|\mathcal{Y}_0^t\right]\right| < \infty \qquad a.s.$$

The proof of Theorem 23.1 is given in Section 23.3 Section 23.2 collects some facts on contraction properties of Markovian integral kernels that are needed in Section 23.3.

Remark 23.1. (i) Theorem 23.1 has been obtained with a slightly different rate in [10] under the stronger assumption that $\kappa_-(t) \geq \kappa_- > 0$ is bounded from below. On the other hand, the assumptions on the initial condition μ_i are less restrictive in [10]. Instead of Lipschitz continuity of the density $h_i = \frac{d\mu_i}{d\nu}$, only a finite energy condition is needed. Moreover, no assumption on differentiability of $Q(\cdot)$ was needed, in contrast to the present approach. The main advantage of the present proof is that existence of classical and uniqueness of weak solutions of the pathwise filter equation associated to the problem (23.1) and (23.3) are not needed in the present proof in contrast to the proof in [10].

(ii) Suppose that $C(t) = I$, $\varphi(t,\cdot) = \varphi$, and $G(t) = G$ are independent of time. Then Theorem 23.1 yields stability of $\eta_t^y(\mu)$ with exponential rate κ_* for suitable initial condition μ if

$$W(x) := \|Gx\|^2 + \frac{\Delta\varphi}{\varphi}(x)$$

is uniformly strictly convex with $W_{xx} \geq \kappa_*^2 \cdot I$. Note that W consists of two parts: the second part $\frac{\Delta\varphi}{\varphi}$ depends on the signal whereas the first part $\|Gx\|^2$ depends on our choice G how to observe the signal. Basically, the more precise our observation is, the more convex $\|Gx\|^2$. Conversely, our criterion provides a priori lower bounds on our choice G to reach a certain exponential rate κ_*. Also note that ergodic and nonergodic directions of the signal process can be "separated" in the criterion.

23.2 Contraction properties of Markovian integral operators

23.2.1 Gradient estimates for solutions of time-inhomogeneous heat-equations

Let

$$B : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d\,, \qquad C : [0,T] \to \mathbb{R}^{d\times d}$$

be continuous. Suppose that for all $R > 0$ there exists a constant L_R such that

$$\|B(t,x) - B(t,y)\| \leq L_R \|x-y\|\,; x, y \in B_R(0)\,, t \in [0,T]\,,$$

and that there exists $\kappa_* \in \mathcal{B}([0,T])_+$, such that

$$\langle B(t,x) - B(t,y), x-y\rangle \leq -\kappa_*(t)\|x-y\|^2\,; x,y \in \mathbb{R}^d\,, t\in[0,T]\,. \tag{23.11}$$

Let $Q_* : [0,T] \to \mathbb{R}^{d\times d}$ be differentiable, $Q_*(t)$ symmetric, positive definite with

$$q_*^-(t)\cdot I \leq Q_* \leq q_*^+(t)\cdot I\,, t\in[0,T]$$

for $q_*^\pm > 0$. Theorem 1 in Section V.1 of [6] implies that for all $(s,x) \in [0,T]\times\mathbb{R}^d$ there exists a unique strong solution $X_t(s,x)$, $t \in [s,T]$, of the stochastic differential equation

$$dX_t = Q_*(t)B(t,X_t)\,dt + C(t)\,dW_t \tag{23.12}$$

with initial condition $X_s(s,x) = x$. Here, $(W_t)_{t\geq 0}$ denotes a Brownian motion on $\mathbb{R}^d$.

For $h \in \mathcal{B}(\mathbb{R}^d)$, $h \geq 0$, define

$$p_{s,t}h(x) = E\left[h(X_t(s,x))\right]\,, 0 \leq s \leq t \leq T\,.$$

Consider the parabolic partial differential equation

$$\partial_t u(t,x) = -\frac{1}{2}\Delta_{Q(t)}u(t,x) - \langle Q_*(t)B(t,x), \nabla_x u(t,x)\rangle \tag{23.13}$$

where $u \in C^{1,2}([0,T]\times\mathbb{R}^d)$. Ito's formula implies for bounded u that

$$u(s,x) = E[u(t,\tilde{X}_t(s,x))]\,, 0 \leq s \leq t \leq T\,, \tag{23.14}$$

where $\tilde{X}_t(s,x)$, $t\in[s,T]$, is any weak solution of (23.12) with initial condition $\tilde{X}_s(s,x) = x$.

Proposition 23.2. *Let $f \in C_0^2(\mathbb{R}^d)$ and $u \in C^{1,2}([0,T]\times\mathbb{R}^d)$ be a bounded solution of (23.13) with terminal condition $u(T,\cdot) = f$. Then*

$$\|\nabla_x u(s,x)\|^2 \leq \frac{q_*^+(T)}{q_*^-(s)} e^{\int_s^T \|Q_*(r)^{-\frac{1}{2}}\dot{Q}_*(r)Q_*(r)^{-\frac{1}{2}}\|_{op} - 2q_*^-(r)\kappa_*(r)\,dr}\|\nabla_x f\|_\infty^2\,.$$

If in addition the weak solution of (23.12) is unique in the sense of probability law for any initial condition (s,x), $x\in\mathbb{R}^d$, then $u(s,x) = p_{s,T}f(x)$ and

$$\|\nabla_x p_{s,T}f(x)\|^2 \leq \frac{q_*^+(T)}{q_*^-(s)} e^{\int_s^T \|Q_*(r)^{-\frac{1}{2}}\dot{Q}_*(r)Q_*(r)^{-\frac{1}{2}}\|_{op} - 2q_*^-(r)\kappa_*(r)\,dr} p_{s,t}\left(\|\nabla_x f\|^2\right)(x)\,.$$

Proof We will use the coupling technique and adopt the proof from the time-homogeneous case (see Lemma 9.1 in [4]). Fix x_0, $y_0 \in \mathbb{R}^d$, $x_0 \neq y_0$, and a $2d$-dimensional Brownian motion $(V_t, \tilde{V}_t)_{t\geq 0}$. Theorem 1 in Section V.1 of [6] implies that the $2d$-dimensional stochastic differential equation

$$\begin{aligned} d\bar{X}_t &= B(t, \bar{X}_t)\, dt + \frac{1}{\sqrt{2}} C(t) \left(dV_t + d\tilde{V}_t \right) , \bar{X}_s = x_0 \, , \\ d\bar{Y}_t &= B(t, \bar{Y}_t)\, dt + \frac{1}{\sqrt{2}} C(t) \left(dV_t + d\tilde{V}_t \right) , \bar{Y}_s = y_0 \, . \end{aligned}$$

has a unique strong solution for $t \in [s, T]$. To simplify notations in the following, let:

$$\|v\|_S^2 := \langle Sv, v \rangle \, , v \in \mathbb{R}^d$$

for any symmetric $d \times d$-matrix S. Ito's formula implies that

$$\begin{aligned} \|\bar{X}_t - \bar{Y}_t\|^2_{Q_*(t)^{-1}} = \|x_0 - y_0\|^2_{Q_*(s)^{-1}} &+ \int_s^t \|\bar{X}_r - \bar{Y}_r\|^2_{Q_*(r)^{-1}\dot{Q}_*(r)Q_*(r)^{-1}}\, dr \\ &+ 2 \int_s^t \langle B(r, \bar{X}_r) - B(r, \bar{Y}_r), \bar{X}_r - \bar{Y}_r \rangle \, dr \, . \end{aligned}$$

Using the dissipativity assumption (23.11) on B, we obtain, in particular

$$\begin{aligned} \frac{d}{dt} \|\bar{X}_t - \bar{Y}_t\|^2_{Q_*(t)^{-1}} &= \|\bar{X}_r - \bar{Y}_r\|^2_{Q_*(r)^{-1}\dot{Q}_*(r)Q_*(r)^{-1}} \\ &\quad + 2\langle B(t, \bar{X}_t) - B(t, \bar{Y}_t), \bar{X}_t - \bar{Y}_t \rangle \\ &\leq \left(\|Q(t)^{-\frac{1}{2}} \dot{Q}(t) Q(t)^{-\frac{1}{2}}\|_{op} - 2q_*^-(t)\kappa_*(t) \right) \|\bar{X}_t - \bar{Y}_t\|^2_{Q_*(t)^{-1}} \, . \end{aligned}$$

Consequently

$$\|\bar{X}_t - \bar{Y}_t\|^2 \leq \frac{q_*^+(t)}{q_*^-(s)} e^{\int_s^t \|Q(r)^{-\frac{1}{2}} \dot{Q}(r) Q(r)^{-\frac{1}{2}}\|_{op} - 2q_*^-(r)\kappa_*(r)\, dr} \|x_0 - y_0\|^2 \, ,$$

and integrating the last inequality w.r.t. P yields

$$E\left[\|\bar{X}_t - \bar{Y}_t\|^2 \right] \leq \frac{q_*^+(t)}{q_*^-(s)} e^{\int_s^t \|Q(r)^{-\frac{1}{2}} \dot{Q}(r) Q(r)^{-\frac{1}{2}}\|_{op} - 2q_*^-(r)\kappa_*(r)\, dr} \|x_0 - y_0\|^2 \, .$$

Fix f and u as in the assumption. Since both, $\bar{X}_t$ and $\bar{Y}_t$, $t \in [s, T]$, are weak solutions of the stochastic differential equation (23.12) we have that

$$u(s, x_0) = E[u(T, \bar{X}_T)] = E[f(\bar{X}_T)]$$

and similarly, $u(s, y_0) = E[f(\bar{Y}_T)]$. Fix $\varepsilon > 0$. Then we can find $\delta > 0$ such that

$$\frac{|f(x) - f(y)|}{\|x - y\|} \leq \|\nabla_x f(x)\| + \varepsilon \text{ for all } \|x - y\| \in]0, \delta[\, .$$

Then

$$\begin{aligned} \frac{|u(s, x_0) - u(s, y_0)|^2}{\|x_0 - y_0\|^2} &\leq \left(E\left[\frac{\left|f(\bar{X}_T) - f(\bar{Y}_T)\right|}{\left\|\bar{X}_T - \bar{Y}_T\right\|} \cdot \frac{\left\|\bar{X}_T - \bar{Y}_T\right\|}{\|x_0 - y_0\|} 1_{\{\bar{X}_T \neq \bar{Y}_T\}} \right] \right)^2 \\ &\leq e^{-2\int_s^T \kappa_*(r)\, dr} E\left[\frac{\left|f(\bar{X}_T) - f(\bar{Y}_T)\right|^2}{\left\|\bar{X}_T - \bar{Y}_T\right\|^2} 1_{\{\bar{X}_T \neq \bar{Y}_T\}} \right] . \end{aligned} \tag{23.15}$$

Clearly

$$E\left[\frac{\left|f(\bar{X}_T)-f(\bar{Y}_T)\right|^2}{\left\|\bar{X}_T-\bar{Y}_T\right\|^2}1_{\{\bar{X}_T\neq\bar{Y}_T\}}\right]\le\|\nabla_x f\|_\infty^2$$

which implies the first assertion letting $x_0 \to y_0$. If we have in addition uniqueness in the sense of probability law of the weak solution of (23.12) for initial condition (s,x), $x \in \mathbb{R}^d$, it follows that

$$\begin{aligned}
&E\left[\frac{\left|f(\bar{X}_T)-f(\bar{Y}_T)\right|^2}{\left\|\bar{X}_T-\bar{Y}_T\right\|^2}1_{\{\bar{X}_T\neq\bar{Y}_T\}}\right]\\
&\quad\le E\left[\left(\|\nabla_x f\|(\bar{X}_T)+\varepsilon\right)^2+\|\nabla f_x\|_\infty^2 1_{\{|\bar{X}_T-\bar{Y}_T|\ge\delta\}}\right]^2 \qquad (23.16)\\
&\quad\le\left(p_{s,T}\left(\|\nabla_x f\|^2+\varepsilon\right)^2(x_0)+\frac{\|\nabla_x f\|_\infty^2}{\delta^2}E\left[\left\|\bar{X}_T-\bar{Y}_T\right\|^2\right]\right).
\end{aligned}$$

Inserting (23.16) into (23.15) and using the fact that uniqueness in the sense of probability law of the weak solution of (23.12) implies $u(s,x) = p_{s,T}f(x)$ by (23.14), we obtain that

$$\begin{aligned}
\frac{|p_{s,T}f(x_0)-p_{s,T}f(y_0)|^2}{\|x_0-y_0\|^2}&\le\frac{q_*^+(T)}{q_*^-(s)}e^{\int_s^T\|Q(r)^{-\frac12}\dot{Q}(r)Q(r)^{-\frac12}\|_{op}-2q_*^-(r)\kappa_*(r)\,dr}\\
&\quad\cdot\left(p_{s,T}\left(\|\nabla_x f\|^2+\varepsilon\right)^2(x_0)+\frac{\|\nabla_x f\|_\infty^2}{\delta^2}E\left[\left\|\bar{X}_T-\bar{Y}_T\right\|^2\right]\right).
\end{aligned}$$

Letting $x_0 \to y_0$ and then $\varepsilon \to 0$, we get the second assertion. □

23.2.2 Feynman–Kac propagators preserving log-concavity

From now on we will assume for the rest of this section that there exists $\kappa_- \in \mathcal{B}([0,T])$, $\kappa_- > 0$, and $\kappa_+ \ge 1$ such that $\kappa_-(t)\cdot I \le Q(t) \le \kappa_+ \cdot I$ for $t \in [0,T]$.

Let $\sigma \in C_p^{0,2}([0,T]\times\mathbb{R}^d)$ be such that $\sigma(t,x) \ge -\sigma_- > -\infty$ for all $(t,x) \in [0,T]\times\mathbb{R}^d$. Assume that for all $t \in [0,T]$

$$\sigma_{xx}(t,\cdot) \ge \kappa^2 \cdot I \ge 0 \text{ for some } \kappa \ge 0\,. \qquad (23.17)$$

For the rest of the section fix a d-dimensional Brownian motion $(V_t)_{t\ge0}$ and a continuous path $d : [0,T] \to \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $0 \le s \le t \le T$ define

$$X_t(s,x) := x + \int_s^t d(r)\,dr + \int_s^t C(r)\,dV_r$$

and the integral operator

$$K_{s,t}f(x) := E\left[f\left(X_t(s,x)\right)\exp\left(-\int_s^t \sigma(r, X_r(s,x))\right)\right]. \qquad (23.18)$$

Clearly, $(K_{s,t})$ is a forward propagator; that is, $K_{s,t} = K_{s,r}\circ K_{r,t}$ if $0 \le s \le r \le t \le T$. Theorem 4 in Section V.6 of [6] implies that if $f \in C_p^2(\mathbb{R}^d)$, then $u(t,x) := K_{t,T}f(x) \in C^{1,2}([0,T]\times\mathbb{R}^d)$ and u satisfies the equation

$$\partial_t u(t,x) = -\left(\frac{1}{2}\Delta_{Q(t)}u(t,x) - \sigma(t,x)u(t,x)\right). \qquad (23.19)$$

We then have the following proposition:

Proposition 23.3. *Let $f \in C_p^2(\mathbb{R}^d)$, $f > 0$, be log-concave with*

$$-(\log f)_{xx} \geq \kappa \cdot I \geq 0\,.$$

Then $u(t,x) = K_{t,T} f(x)$ is log-concave too with

$$-(\log u(t,x))_{xx} \geq \frac{\kappa}{\sqrt{\kappa_+}} \cdot I\,.$$

The proof of the Proposition 23.3 is exactly the same as the proof of Proposition 4.4 in [10]. Note that a strictly positive lower bound $\kappa_-(t) \geq \kappa_- > 0$ is not needed in the proof of Proposition 4.4.

23.2.3 Gradient estimates for Feynman–Kac propagators

For the rest of this section assume that $Q(\cdot)$ is differentiable with

$$\chi := \int_0^T \|Q(t)^{-\frac{1}{2}} \dot{Q}(t) Q(t)^{-\frac{1}{2}}\|_{op}\, dt < \infty\,.$$

Fix $m_T \in C_p^2(\mathbb{R}^d)$, such that m_T is log-concave with

$$-\left(\log m_T\right)_{xx} \geq \kappa \cdot I\,, \tag{23.20}$$

where $\kappa \geq 0$ is as in (23.17). Let

$$m_t(x) := K_{t,T} m_T(x)\,, \quad (t,x) \in [0,T] \times \mathbb{R}^d\,. \tag{23.21}$$

Define the ground state transform of $K_{s,t}$ and m_t as follows:

$$p^*_{s,t} f(x) := \frac{1}{m_s(x)} K_{s,t}\left(f m_t\right)(x)\,, \quad 0 \leq s \leq t \leq T\,. \tag{23.22}$$

Proposition 23.4. *Let $f \in C_p^2(\mathbb{R}^d)$. Then $p^*_{\cdot,T} f \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and*

$$\frac{d}{dt} p^*_{t,T} f = -\left(\frac{1}{2}\Delta_{Q(t)} p^*_{t,T} f + Q(t)\nabla_x \log m_t \cdot \nabla_x p^*_{t,T} f\right)\,. \tag{23.23}$$

Proof Let $v(t,x) := K_{t,T}(f m_T)(x)$, $0 \leq t \leq T$. Clearly, $m_t(x)$, $v(t,x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ are solutions of (23.19). Consequently, $p^*_{t,T} f(x) \in C^{1,2}([0,T] \times \mathbb{R}^d)$ and

$$\begin{aligned}
\frac{d}{dt} p^*_{t,T} f &= \frac{1}{m_t}\frac{d}{dt} v(t,\cdot) - \frac{d}{dt} m_t \cdot \frac{v(t,\cdot)}{m_t^2} \\
&= -\frac{1}{2}\left(\frac{\Delta_{Q(t)} v(t,\cdot)}{m_t} - \Delta_{Q(t)} m_t \frac{v(t,\cdot)}{m_t^2}\right) \\
&= -\frac{1}{2}\Delta_{Q(t)} p^*_{t,T} f - Q(t)\nabla_x \log m_t \cdot \nabla_x p^*_{t,T} f\,,
\end{aligned}$$

hence the assertion. □

Combining Proposition 23.2 and Proposition 23.3 we now obtain

Theorem 23.2. *Let $f \in C_0^2(\mathbb{R}^d)$ and $0 \le s \le T$. Then*

$$\|\nabla_x p^*_{s,T} f(x)\|^2 \le \frac{\kappa_+}{\kappa_0(s)} e^{\chi} e^{-2\kappa\kappa_+^{-\frac{1}{2}} \int_s^T \kappa_-(r)\,dr} \|\nabla_x f\|^2_\infty \,. \tag{23.24}$$

If, in addition, the weak solution of (23.12) with $B(t,x) = Q(t)\nabla_x \log m_t(x)$ is unique in the sense of probability law for any initial condition (s,x), $x \in \mathbb{R}^d$, then

$$\|\nabla_x p^*_{s,T} f(x)\|^2 \le \frac{\kappa_+ e^{\chi}}{\kappa_0(s)} e^{-2\kappa\kappa_+^{-\frac{1}{2}} \int_s^T \kappa_-(r)\,dr} p^*_{s,T}\left(\|\nabla_x f\|^2\right)(x) \,.$$

Proof For the proof it suffices to estimate the dissipativity constant of the vector-field $Q(t)\nabla_x \log m_t(x)$. Proposition 23.3 implies that m_t is uniformly strictly log-concave with $-(\log m_t)_{xx} \ge \frac{\kappa}{\sqrt{\kappa_+}} \cdot I$, so that $-DB(t,x) \ge \frac{\kappa}{\sqrt{\kappa_+}} \cdot I$, where $B(t,x) := \nabla_x \log m_t(x)$. Since $u(t,x) := p^*_{t,T} f(x)$ is a bounded solution in $C^{1,2}([0,T] \times \mathbb{R}^d)$ of equation (23.13) with terminal condition f, Proposition 23.2 now implies the assertion. □

Remark 23.2. Using a straightforward approximation, estimate (23.24) implies the following estimate:

$$\|p^*_{s,T} f\|^2_{Lip} \le \frac{q_*^+(T) e^{\chi}}{q_*^-(s)} e^{-2\kappa\kappa_+^{-\frac{1}{2}} \int_s^T \kappa_-(r)\,dr} \|f\|^2_{Lip} \tag{23.25}$$

for all Lipschitz continuous f.

23.3 Proof of Theorem 23.1

Fix $T > 0$ and $y \in C([0,\infty); \mathbb{R}^p)$, $y(0) = 0$. Denote by K the adjoint integral operator of K_T^y in $L^2(\mathbb{R}^d)$. Using time-reversibility of Brownian motion on $\mathbb{R}^d$ w.r.t. dx, it follows that Kf can be represented as

$$Kf(x) = E\left[f(X_T(0,x)) \exp\left(-\int_0^T \sigma(r, X_r(0,x))\,dr\right)\right]$$

where

$$X_t(s,x) = x + \int_s^t Q(T-r)G(T-r)^T y(T-r)\,dr + \int_s^t C(T-r)\,dV_r$$

for some Brownian motion $(V_t)_{t\ge 0}$ on $\mathbb{R}^d$ and

$$\sigma(t,x) := \sigma^y(T-t, x)\,, (t,x) \in [0,T] \times \mathbb{R}^d \,.$$

Assumption 23.1 implies that σ satisfies (23.17) with $\kappa = \kappa_* > 0$. Let $K_{s,t}$, $0 \le s \le t \le T$ be as in (23.18) so that, in particular, $K_{0,T} = K$. Let $m_t(x) := K_{t,T} g_0(x)$, $0 \le t \le T$, and

$$\nu_T^y(dx) := e^{G(T)^T y(T)\cdot x} \varphi(T,x) m_0(x)\,dx \,.$$

We can then write

$$\begin{aligned}\eta_T^y(\mu_i)(dx) &= \frac{1}{Z_T^y(\mu_1)} e^{G(T)^T y(T)\cdot x} \varphi(T,x) K\left(h_i g_0\right)(x) \\ &= \frac{p^*_{0,T} h_i(x)\,\nu_T^y(dx)}{\int p^*_{0,T} h_i\,d\nu_T^y}\,, i = 1,2 \,.\end{aligned}$$

Consequently

$$
\begin{aligned}
\|\eta_T^y(\mu_1) - \eta_T^y(\mu_2)\|_{var} &= \frac{1}{2}\int \left| \frac{p_{0,T}^* h_1}{\langle p_{0,T}^* h_1\rangle_{\nu_T^y}} - \frac{p_{0,T}^* h_2}{\langle p_{0,T}^* h_2\rangle_{\nu_T^y}} \right| d\nu_T^y \\
&\le \frac{1}{2\langle p_{0,T}^* h_1\rangle_{\nu_T^y}\langle p_{0,T}^* h_2\rangle_{\nu_T^y}} \cdot \\
&\quad \cdot \int\int \left| p_{0,T}^* h_1(x) p_{0,T}^* h_2(z) - p_{0,T}^* h_1(z) p_{0,T}^* h_2(x) \right| \nu_T^y(dx)\, \nu_T^y(dz)\,.
\end{aligned}
\tag{23.26}
$$

Using Theorem 23.2 and Remark 23.2, we can estimate

$$
\begin{aligned}
&\left| p_{0,T}^* h_1(x) p_{0,T}^* h_2(z) - p_{0,T}^* h_1(z) p_{0,T}^* h_2(x) \right| \\
&\quad \le p_{0,T}^* h_1(x) \left| p_{0,T}^* h_2(z) - p_{0,T}^* h_2(x) \right| \\
&\qquad + p_{0,T}^* h_2(x) \left| p_{0,T}^* h_1(x) - p_{0,T}^* h_1(z) \right| \\
&\quad \le \sqrt{\frac{\kappa_+ e^{\chi}}{\kappa_-(0)}} e^{-\kappa_* \kappa_+^{-\frac{1}{2}} \int_0^T \kappa_-(r)\, dr} \left(\|h_2\|_{Lip} \|h_1\|_\infty + \|h_1\|_{Lip} \|h_2\|_\infty \right) \|x - z\|\,.
\end{aligned}
\tag{23.27}
$$

Integrating the last inequality and using the upper and lower bound $\delta \le h_i \le \delta^{-1}$ equation (23.26) implies that

$$
\begin{aligned}
\|\eta_T^y(\mu_1) - \eta_T^y(\mu_2)\|_{var} \le \sqrt{\frac{\kappa_+ e^{\chi}}{\kappa_-(0)}} &\frac{e^{-\kappa_* \kappa_+^{-\frac{1}{2}} \int_0^T \kappa_-(r)\, dr}}{2\delta^3} \left(\|h_1\|_{Lip} + \|h_2\|_{Lip} \right) \cdot \\
&\cdot \frac{\int\int \|x - z\|\, \nu_T^y(dx)\, \nu_T^y(dz)}{\int\int \nu_T^y(dx)\, \nu_T^y(dz)}.
\end{aligned}
\tag{23.28}
$$

Denote by ϱ_T^y the density of the probability measure $\nu_T^y(\mathbb{R}^d)^{-1}\nu_T^y(dx)$ and by $\tilde{\varrho}_T^y$ the density of the probability measure $\frac{e^{G^T(T)y(T)\cdot x}\tilde{\varphi}_T(x) g_0(x)\, dx}{\int e^{G^T(T)y(T)\cdot}\tilde{\varphi}_T g_0\, dx}$, where $\tilde{\varphi}_T$ is as in Assumption 23.2. Then

$$
M_T^{-2}\tilde{\varrho}_T^y \le \varrho_T^y \le M_T^2 \tilde{\varrho}_T^y\,. \tag{23.29}
$$

Since $\tilde{\varrho}_T^y$ is uniformly strictly log-concave with

$$
-(\log \tilde{\varrho}_T^y)_{xx} = -(G^T(T)y(T)\cdot)_{xx} - (\log \tilde{\varrho}_T^y)_{xx} - (\log g_0)_{xx} \ge \kappa_* \cdot I
$$

Theorem 4.1 in [3] implies that

$$
\int \left(f - \langle f \rangle_{\tilde{\varrho}_T^y\, dx} \right)^2 \tilde{\varrho}_T^y\, dx \le \frac{1}{\kappa_*} \int |\nabla f|^2 \tilde{\varrho}_T^y\, dx\,,\, f \in C^1(\mathbb{R}^d)\,.
$$

The last estimate combined with estimate (23.29) now gives in particular

$$
\begin{aligned}
\sum_{i=1}^d \int x_i^2 - \langle x_i \rangle_{\varrho_T^y\, dx}^2 \varrho_T^y(x)\, dx &= \sum_{i=1}^d \int \left(x_i - \langle x_i \rangle_{\varrho_T^y\, dx} \right)^2 \varrho_T^y(x)\, dx \\
&\le \sum_{i=1}^d \int \left(x_i - \langle x_i \rangle_{\tilde{\varrho}_T^y\, dx} \right)^2 \varrho_T^y(x)\, dx \\
&\le \sum_{i=1}^d M_T^2 \int \left(x_i - \langle x_i \rangle_{\tilde{\varrho}_T^y\, dx} \right)^2 \tilde{\varrho}_T^y(x)\, dx \le \frac{d M_T^2}{\kappa_*}\,.
\end{aligned}
$$

It follows that

$$\int\int \|x-z\|\varrho_T^y(x)\,dx\,\varrho_T^y(z)\,dz \le \left(\int\int \|x-z\|^2\varrho_T^y(x)\,dx\,\varrho_T^y(z)\,dz\right)^{\frac{1}{2}}$$
$$= \sqrt{2}\left(\sum_{i=1}^d \int x_i^2 - \langle x_i\rangle^2_{\varrho_T^y\,dx}\varrho_T^y(x)\,dx\right)^{\frac{1}{2}} \le \sqrt{\frac{2d}{\kappa_*}}M_T\,.$$

Inserting the last estimate into (23.28) we obtain the assertion

$$\|\eta_T^y(\mu_1)-\eta_T^y(\mu_2)\|_{var} \le \sqrt{\frac{d}{2\kappa_*}}\frac{M_T}{\delta^3}\sqrt{\frac{\kappa_+e^X}{\kappa_-(0)}}e^{-\kappa_*\kappa_+^{-\frac{1}{2}}\int_0^T\kappa_-(r)\,dr}\left(\|h_1\|_{Lip}+\|h_2\|_{Lip}\right)\,.$$

The proof of Theorem 23.1 (ii) is now immediate. Indeed, part (i) implies that

$$\limsup_{t\to\infty} e^{\kappa_*\kappa_+^{-\frac{1}{2}}\int_0^t\kappa_-(r)\,dr}\left|\int f\,d\eta_t^y(\mu_1)-\int f\,d\eta_t^y(\mu_2)\right| < \infty \tag{23.30}$$

for all $y\in C([0,\infty);\mathbb{R}^p)$, $y(0)=0$, and $f\in\mathcal{B}_b(\mathbb{R}^d)$. Proposition 23.1 implies that

$$E_{\mu_i}\left[f(X_t)|\mathcal{Y}_t\right] = \int f\,d\eta_t^{Y.}(\mu_i) \qquad \text{a.s.} \tag{23.31}$$

Combining (23.30) and (23.31), we obtain that

$$\limsup_{t\to\infty} e^{\kappa_*\kappa_+^{-\frac{1}{2}}\int_0^t\kappa_-(r)\,dr}\left|E_{\mu_1}\left[f(X_t)|\mathcal{Y}_t\right]-E_{\mu_2}\left[f(X_t)|\mathcal{Y}_t\right]\right| < \infty \qquad \text{a.s.}$$

hence the assertion. □

References

[1] R. Atar, *Exponential stability for nonlinear filtering of diffusion processes in a non-compact domain*, Ann. Probab. **26** (1998), 1552-1574.

[2] R. Atar, O. Zeitouni, *Exponential stability for nonlinear filtering*, Ann. Inst. Henri Poincaré **33** (1997), 697-725.

[3] H.J. Brascamp, E.H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22** (1976), 366-389.

[4] M.F. Chen, F.Y. Wang, *Estimation of spectral gap for elliptic operators*, Trans. Am. Math. Soc., **349** (1997), 1239-1267.

[5] G. Da Prato, M. Fuhrman, P. Malliavin, *Asymptotic ergodicity of the process of conditional law in some problem of non-linear filtering*, J. Funct. Anal. **164** (1999), 356-377.

[6] N.V. Krylov, *Introduction to the Theory of Diffusion Processes*, American Mathematical Society, Providence, 1995.

[7] D. Ocone, E. Pardoux, *Asymptotic stability of the optimal filter with respect to its initial condition*, SIAM J. Control Optimization **34** (1996), 226-243.

[8] E. Pardoux, *Equations of non-linear filtering; and applications to stochastic control with partial observation*, in: Nonlinear Filtering and Stochastic Control, eds. Mitter, S.K., Moro, A., Lecture Notes in Math. **972**, Springer, Berlin, 1982.

[9] W. Stannat, *Stability of the pathwise filter equation on* $\mathbb{R}^d$, Preprint, Bielefeld (2004).

[10] W. Stannat, *Stability of the filter equation for a time-dependent signal on* $\mathbb{R}^d$, Appl. Math. Optim. **52** (2005), no. 1, 39-71.

24 Fractal Burgers' Equation Driven by Lévy Noise

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24.1 Introduction

This chapter is concerned with the initial problem for the following stochastic fractal Burgers' equation:

$$(\partial_t + \nu\Delta_\alpha u(t,x) + \lambda\partial_x(|u(t,x)|^r) = f(u)(t,x) + g(u)(t,x)F_{t,x}$$

on the given domain $[0,\infty)\times I\!R$ with $L^2(I\!R)$ initial condition, where $\nu > 0, \Delta_\alpha := -(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}$ is the fractional Laplacian on $I\!R$ with $\alpha \in (0,2], \lambda \in I\!R$ is a constant, $r \in [1,2], f, g : [0,\infty)\times I\!R\times I\!R \to I\!R$ are measurable and $F_{t,x}$ is the so-called Lévy space–time white noise consisting of Gaussian space–time white noise (i.e., a Brownian sheet on $[0,\infty)\times I\!R$) and Poisson space–time white noise (see Section 24.2 for the definition). There has recently been increasing interest in considering fractal Burgers' equations (see, e.g., [3, 4, 5] and references therein) and in studying Burgers' turbulence with non-Gaussian (random) initial data (see, e.g., [2, 10, 18] and references therein).

Stochastic Burgers' equations driven by Gaussian white noise have been studied intensively (see, e.g., [1, 6, 7, 8, 11] and references therein). As is wellknown, one of the main investigations of Burgers' equation is based on the intriguing connection between the Burgers' equation (nonlinear) and the somehow simpler linear heat equation, via the celebrated Hopf–Cole transformation. This technique can be still adapted to stochastic Burgers' equations with additive Gaussian white noise (see, e.g., [1]), but it is no longer available in the case of stochastic Burgers' equations driven by more general Gaussian white noise (for instance, multiplicative Gaussian space–time white noise). This is because the noise term cannot be written in a conservative form which then destroys the way to link the stochastic Burgers' equations with stochastic heat equations in a simple manner. Another method can be used successfully, e.g., in [6, 7, 8, 11] (here we just mention a few references), to study the mild solutions to stochastic Burgers' equations driven by Gaussian space–time white noise. Along this line, the stochastic Burgers' equation driven by Lévy space–time white noise has been considered in [16] where the initial problem for the stochastic Burgers' equation with Lévy space–time white noise is examined in the mild formulation.

In this chapter we introduce a class of stochastic fractal Burgers' equations in one space dimension driven by Lévy space–time white noise which links fractal Burgers' equations and stochastic Burgers' equations with white noises considered in the literature mentioned above. We will prove existence of a unique, local, mild solution to the initial problem for the fractal stochastic Burgers' equations we posed above.

The chapter is organized as follows. In the next section, we elucidate briefly what Lévy space–time white noise is. In Section 24.3, in order to make the problem we are considering precise, we interpret the initial problem for the stochastic fractal Burgers' equation driven by Lévy space–time white noise (weakly) as a jump type stochastic integral equation involving the convolution kernels associated with the fractional Laplacian. We present existence of a unique local L^2-solution. Namely, for any initial function from $L^2(\mathbb{R})$, we obtain a local solution with càdlàg (i.e., right continuous with left-hand limits in the time variable $t \in [0,\infty)$) trajectories in $L^2(\mathbb{R})$. Our approach is based on combining the method for solving stochastic Burgers' equations driven by Lévy noise in [16] with the techniques for solving fractal Burgers' equations developed in [3, 5].

24.2 Lévy space–time white noise

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space and $(U, \mathcal{B}(U), \nu)$ be an arbitrary σ-finite measure space. Following, e.g., [12] (cf. Theorem I.8.1), there exists a Poisson random measure on the product measure space $([0,\infty)\times \mathbb{R}\times U, \mathcal{B}([0,\infty)\times\mathbb{R})\times\mathcal{B}(U), dtdx\nu)$ associated with Lebesgue (product) measure space $([0,\infty)\times\mathbb{R}, \mathcal{B}([0,\infty)\times\mathbb{R}), dtdx)$, i.e.

$$N : \mathcal{B}([0,\infty)\times\mathbb{R})\times\mathcal{B}(U)\times\Omega \to \mathbb{N}\cup\{0\}\cup\{\infty\} \tag{24.1}$$

with mean measure $\mathbf{E}[N(A,B,\cdot)] = |A|\nu(B), A \in \mathcal{B}([0,\infty)\times\mathbb{R}), B\in\mathcal{B}(U)$. Here and in the sequel in this chapter, $|A|$ stands for Lebesgue measure for any Borel measurable set $A \in \mathcal{B}([0,\infty)\times\mathbb{R})$. In fact, N can be constructed canonically as follows:

$$N(A,B,\omega) := \sum_{n\in\mathbb{N}}\sum_{j=1}^{\eta_n(\omega)} 1_{(A\cap E_n)\times(B\cap U_n)}(\xi_j^{(n)}(\omega))1_{\{\omega\in\Omega:\eta_n(\omega)\geq 1\}}(\omega)\,,\omega\in\Omega \tag{24.2}$$

for $A \in \mathcal{B}([0,\infty)\times\mathbb{R})$ and $B\in\mathcal{B}(U)$, where

(a) $\{E_n\}_{n\in\mathbb{N}} \subset \mathcal{B}([0,\infty)\times\mathbb{R})$ is a partition of $[0,\infty)\times\mathbb{R}$ with $0<|E_n|<\infty, n\in\mathbb{N}$, and $\{U_n\}_{n\in\mathbb{N}}\subset\mathcal{B}(U)$ is a partition of U with $0<\nu(U_n)<\infty, n\in\mathbb{N}$.

(b) $\forall n,j\in\mathbb{N}, \xi_j^{(n)} : \Omega\to E_n\times U_n$ is $\mathcal{F}/\mathcal{E}_n\times\mathcal{B}(U_n)$-measurable with

$$P\{\omega\in\Omega : \xi_j^{(n)}(\omega)\in A\times B\} = \frac{|A|\nu(B)}{|E_n|\nu(U_n)}\,,\quad A\in\mathcal{E}_n\,, B\in\mathcal{B}(U_n),$$

where $\mathcal{E}_n := \mathcal{B}([0,\infty)\times\mathbb{R})\cap E_n$ and $\mathcal{B}(U_n) := \mathcal{B}(U)\cap U_n$.

(c) $\forall n\in\mathbb{N}, \eta_n : \Omega\to\mathbb{N}\cup\{0\}\cup\{\infty\}$ is a Poisson distributed random variable with

$$P\{\omega\in\Omega : \eta_n(\omega)=k\} = \frac{e^{-|E_n|\nu(U_n)}[|E_n|\nu(U_n)]^k}{k!}\,, k\in\mathbb{N}\cup\{0\}\cup\{\infty\}.$$

(d) $\xi_j^{(n)}$ and η_n are mutually independent for all $n,j\in\mathbb{N}$.

Given any σ-finite measure ν on $(U,\mathcal{B}(U))$, there is always a Poisson random measure N on the product measure space $([0,\infty)\times\mathbb{R}\times U, \mathcal{B}([0,\infty)\times\mathbb{R})\times\mathcal{B}(U), dtdx\nu)$ which can be constructed in the above manner. Such an N is called a canonical Poisson random measure associated with the product σ-finite measure $dtdx\nu$.

Let $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ be a right continuous increasing family of sub-σ-algebras of $\mathcal{F}$, each containing all P-null sets of $\mathcal{F}$, such that the canonical Poisson random measure N has the property that (i) $N([0,t]\times A,B,\cdot) : \Omega\to\mathbb{N}\cup\{0\}\cup\{\infty\}$ is $\mathcal{F}_t/\mathcal{P}(\mathbb{N}\cup\{0\}\cup\{\infty\})$-measurable $\forall(t,A,B)\in[0,\infty)\times\mathcal{B}(\mathbb{R})\times\mathcal{B}(U)$ and (ii) $\{N([0,t+s]\times A,\cdot) - N([0,t]\times A,\cdot)\}_{s>0,(A,B)\in\mathcal{B}(\mathbb{R})\times\mathcal{B}(U)}$ is independent of $\mathcal{F}_t$ for any $t\geq 0$, where $\mathcal{P}(\mathbb{N}\cup\{0\}\cup\{\infty\})$ is the power set of $\mathbb{N}\cup\{0\}\cup\{\infty\}$. (For instance, we may directly take

$$\mathcal{F}_t := \sigma(\{N([0,t]\times A,B,\cdot) : (A,B)\in\mathcal{B}(\mathbb{R})\times\mathcal{B}(U)\})\vee\mathcal{N}\,,\quad t\in[0,\infty)$$

where $\mathcal{N}$ denotes the totality of P-null sets of $\mathcal{F}$.)

Next we introduce the compensating $\{\mathcal{F}_t\}$-martingale measure

$$M(t,A,B,\omega) := N([0,t],A,B,\omega) - t|A|\nu(B) \tag{24.3}$$

for any $(t,A,B) \in [0,\infty) \times \mathcal{B}(I\!R) \times \mathcal{B}(U)$ with $|A|\nu(B) < \infty$. Obviously

$$\mathbf{E}[M(t,A,B,\cdot)] = 0$$

and

$$\mathbf{E}([M(t,A,B,\cdot)]^2) = t|A|\nu(B)\,. \tag{24.4}$$

For any $\{\mathcal{F}_t\}$-predictable integrand $f : [0,\infty) \times I\!R \times U \times \Omega \to I\!R$ which satisfies

$$\mathbf{E}\int_0^t \int_A \int_B |f(s,x,y,\cdot)| ds dx \nu(dy) < \infty\,, \quad a.s. \quad \forall t > 0$$

for some $(A,B) \in \mathcal{B}(I\!R) \times \mathcal{B}(U)$, we can define the stochastic integral

$$\begin{array}{rl} & \int_0^{t+} \int_A \int_B f(s,x,y,\omega) M(ds,dx,dy,\omega) \\ := & \int_0^{t+} \int_A \int_B f(s,x,y,\omega) N(ds,dx,dy,\omega) \\ & \quad - \int_0^{t} \int_A \int_B f(s,x,y,\omega) ds dx \nu(dy)\,. \end{array} \tag{24.5}$$

Clearly, $t \in [0,\infty) \mapsto \int_0^{t+} \int_A \int_B f(s,x,y,\cdot) M(ds,dx,dy,\cdot) \in I\!R$ is an $\{\mathcal{F}_t\}$-martingale. Moreover, stochastic integrals with respect to M are also well defined for $\{\mathcal{F}_t\}$-predictable integrands f satisfying

$$\mathbf{E}\int_0^t \int_A \int_B |f(s,x,y,\cdot)|^2 ds dx \nu(dy) < \infty\,, \quad \forall t \in [0,\infty)$$

for some $(A,B) \in \mathcal{B}(I\!R) \times \mathcal{B}(U)$ by a limit procedure (see the argument in Section II.3 of [12]) and $t \in [0,\infty) \mapsto \int_0^{t+} \int_A \int_B f(s,x,y,\cdot) M(ds,dx,dy,\cdot) \in I\!R$ is a square integrable $\{\mathcal{F}_t\}$-martingale with the quadratic variation process

$$\begin{array}{rl} & < \displaystyle\int_0^{\cdot+} \int_A \int_B f(s,x,y) M(ds,dx,dy) >_t \\ = & \displaystyle\int_0^t \int_A \int_B [f(s,x,y)]^2 ds dx \nu(dy)\,. \end{array}$$

On the other hand, it is clear that M defined by (24.3) is a worthy, orthogonal, $\{\mathcal{F}_t\}$-martingale measure in the context of Walsh [17]. Thus stochastic integrals of $\{\mathcal{F}_t\}$-predictable integrands with respect to M can be defined alternatively by the method in Section II.3 of [17].

For the Poisson random measure N and its compensating martingale measure M, we can define heuristically the Radon–Nikodym derivatives

$$N_{t,x}(B,\omega) := \frac{N(dtdx,B,\omega)}{dtdx}(t,x) \tag{24.6}$$

and

$$M_{t,x}(B,\omega) := \frac{M(dtdx,B,\omega)}{dtdx}(t,x) = N_{t,x}(B,\omega) - \nu(B) \tag{24.7}$$

for $(t,x) \in [0,\infty) \times I\!R$ and $(B,\omega) \in \mathcal{B}(U) \times \Omega$. We call $M_{t,x}$ *Poisson space–time white noise.*

A *Lévy space–time white noise* has the following heuristic structure which is similar to that of a Lévy process:

$$
\begin{aligned}
F_{t,x}(\omega) = W_{t,x}(\omega) \quad &+ \quad \int_{U_0} c_1(t,x;y) M_{t,x}(dy,\omega) \\
&+ \quad \int_{U \setminus U_0} c_2(t,x;y) N_{t,x}(dy,\omega)\,, \quad \omega \in \Omega
\end{aligned} \tag{24.8}
$$

where $c_1, c_2 : [0,\infty) \times I\!\!R \times U \to I\!\!R$ are measurable, $W_{t,x}$ is a Gaussian space–time white noise on $[0,\infty) \times I\!\!R$ used initially by Walsh [17] (formally, $W_{t,x} := \frac{\partial^2 W(t,x)}{\partial t \partial x}$, where $W(t,x)$ is a Brownian sheet on $[0,\infty) \times I\!\!R$), $M_{t,x}$ and $N_{t,x}$ are defined formally as Radon–Nikodym derivatives as in (24.7) and (24.6), respectively, and $U_0 \in \mathcal{B}(U)$ with $\nu(U \setminus U_0) < \infty$.

24.3 Fractal Burgers' equation with Lévy noise

Let $(\Omega, \mathcal{F}, P; \{\mathcal{F}_t\}_{t \in [0,\infty)})$ be given as in the previous section. In this section we will consider the Cauchy problem for the following stochastic fractal Burgers' equation:

$$
\begin{cases}
(\partial_t + \nu \Delta_\alpha u(t,x,\omega) + \lambda \partial_x (|u(t,x,\omega)|^r) = f(u)(t,x,\omega) \\
\qquad\qquad + g(u)(t,x,\omega) F_{t,x}(\omega)\,, \quad (t,x,\omega) \in (0,\infty) \times I\!\!R \times \Omega \\
u(0,x,\omega) = u_0(x,\omega)\,, \ \ (x,\omega) \in I\!\!R \times \Omega
\end{cases} \tag{24.9}
$$

where $\nu > 0, \Delta_\alpha := -(-\frac{d^2}{dx^2})^{\frac{\alpha}{2}}$ is the fractional Laplacian on $I\!\!R$ with $\alpha \in (0,2], \lambda \in I\!\!R$ is a constant, $r \in [1,2]$, $f, g : [0,\infty) \times I\!\!R \times I\!\!R \to I\!\!R$ are measurable, F is a Lévy space–time white noise as introduced in the previous section, and the initial condition u_0 is $\mathcal{F}_0$-measurable.

Following [17], let us introduce a notion of weak solution to Equation (24.9). An $L^2(I\!\!R)$-valued and $\{\mathcal{F}_t\}_{t\in[0,\infty)}$-adapted càdlàg (in the variable $t \in [0,\infty)$) process $u : [0,\infty) \times I\!\!R \times \Omega \to I\!\!R$ is a solution to (24.9) if for any $\varphi \in \mathcal{S}(I\!\!R)$, the Schwartz space of rapidly decreasing C^∞-functions on $I\!\!R$

$$
\begin{aligned}
& \int_{I\!\!R} u(t,x)\varphi(x)dx \\
= \ & \int_{I\!\!R} u_0(x)\varphi(x)dx + \int_0^t \int_{I\!\!R} u(s,x)(\Delta_\alpha \varphi)(x)dxds \\
& + \lambda \int_0^t \int_{I\!\!R} |u(s,x)|^r (\partial_x \varphi)(x)dxds + \int_0^t \int_{I\!\!R} f(s,x,u(s,x))\varphi(x)dxds \\
& + \int_0^t \int_{I\!\!R} g(s,x,u(s,x))\varphi(x) W(ds,dx) \\
& + \int_0^{t+} \int_{I\!\!R} \int_{U_0} g(s,x,u(s,x)) c_1(s,x;y)\varphi(x) M(ds,dx,dy) \\
& + \int_0^{t+} \int_{I\!\!R} \int_{U\setminus U_0} g(s,x,u(s,x)) c_2(s,x;y)\varphi(x) N(ds,dx,dy)
\end{aligned}
$$

holds for all $t \in [0,\infty)$. Moreover, based on this notion, one can present a mild formulation of Equation (24.9):

$$
u(t,x) \quad = \quad \int_{I\!\!R} G_\alpha(t,x-z) u_0(z)dz
$$

$$
\begin{aligned}
&+\lambda\int_0^t\int_{I\!R}[\partial_z G_\alpha(t-s,x-z)]|u(s,z)|^r dzds\\
&+\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)f(s,z,u(s,z))dzds\\
&+\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)g(s,z,u(s,z))W(ds,dz) \qquad (24.10)\\
&+\int_0^{t+}\int_{I\!R}\int_{U_0} G_\alpha(t-s,x-z)g(s,z,u(s-,z);y)\\
&\qquad\qquad\times c_1(s,x;y)M(ds,dz,dy)\\
&+\int_0^{t+}\int_{I\!R}\int_{U\setminus U_0} G_\alpha(t-s,x-z)g(s,z,u(s-,z);y)\\
&\qquad\qquad\times c_2(s,x;y)N(ds,dx,dy)\,,
\end{aligned}
$$

where $G_\alpha(t,x)$ stands for the fundamental solution, i.e., satisfies the following:

$$
\begin{cases} \frac{\partial}{\partial t}v = \nu\Delta_\alpha v\,, \quad (t,x)\in(0,\infty)\times I\!R \\ v(0,x)=\delta(x)\,, x\in I\!R. \end{cases} \qquad (24.11)
$$

We have in fact

$$
G_\alpha(t,x) = [\mathcal{F}^{-1}(e^{\nu t|\cdot|^\alpha})](x)\,.
$$

Moreover, by the scaling properties of Equation (24.11),

$$
G_\alpha(t,x) = (\nu s)^{-\frac{1}{\alpha}}G_\alpha(s^{-1}t,(\nu s)^{-\frac{1}{\alpha}}x)
$$

for $s,t\in(0,\infty), x\in I\!R$, or equivalently

$$
G_\alpha(t,x) = (\nu t)^{-\frac{1}{\alpha}}G_\alpha(1,(\nu t)^{-\frac{1}{\alpha}}x).
$$

Furthermore, let us list some known asymptotic estimates for G_α (cf., e.g., [14]):

(i) $\exists$ constants $0<c_{\alpha,\nu}\le C_{\alpha,\nu}$ such that $\forall (t,x)\in(0,\infty)\times I\!R$

$$
c_{\alpha,\nu}\le t^{\frac{1}{\alpha}}(\nu^{\frac{1}{\alpha}+1}+t^{-\frac{1}{\alpha}-1}|x|^{1+\alpha})G_\alpha(t,x)\le C_{\alpha,\nu} \qquad (24.12)
$$

and $\forall t\in(0,\infty)$, the following limit exists:

$$
\lim_{|x|\to\infty} t^{\frac{1}{\alpha}}(\nu^{\frac{1}{\alpha}+1}+t^{-\frac{1}{\alpha}-1}|x|^{1+\alpha})G_\alpha(t,x).
$$

(ii) $\exists$ a constant $K_{\alpha,\nu}>0$ such that $\forall (t,x)\in(0,\infty)\times I\!R$

$$
|\partial_x G_\alpha(t,x)|\le K_{\alpha,\nu}t^{-1-\frac{2}{\alpha}}|x|^\alpha(\nu^{\frac{1}{\alpha}+1}+t^{-\frac{1}{\alpha}-1}|x|^{1+\alpha})^{-2}. \qquad (24.13)
$$

Let us reformulate Equation (24.10) by the following consideration. Observing that $\nu(U\setminus U_0)<\infty$, we have

$$
\begin{aligned}
&\int_0^{t+}\int_{I\!R}\int_{U\setminus U_0} G_\alpha(t-s,x-z)g(s,z,u(s-,z))c_2(s,z;y)N(ds,dz,dy)\\
=\ &\int_0^{t+}\int_{I\!R}\int_{U\setminus U_0} G_\alpha(t-s,x-z)g(s,z,u(s-,z))c_2(s,z;y)M(ds,dz,dy)\\
&-\int_0^{t+}\int_{I\!R}\int_{U\setminus U_0}[G_\alpha(t-s,x-z)g(s,z,u(s,z))c_2(s,z;y)\nu(dy)]\,dzds\,.
\end{aligned}
$$

Thus, without loss of generality, we shall consider the equation in the following form:

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}} G_\alpha(t,x-z)u_0(z)dz \\ &\quad + \lambda\int_0^t\int_{\mathbb{R}}[\partial_z G_\alpha(t-s,x-z)]q(s,z,u(s,z,\omega))dzds \\ &\quad + \int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-z)f(s,z,u(s,z))dzds \\ &\quad + \int_0^t\int_{\mathbb{R}} G_\alpha(t-s,x-z)g(s,z,u(s,z))W(ds,dz) \\ &\quad + \int_0^{t+}\int_{\mathbb{R}}\int_U G_\alpha(t-s,x-z)h(s,z,u(s-,z);y)M(ds,dz,dy) \end{aligned} \tag{24.14}$$

where $f,g:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $h:[0,\infty)\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R}$ are measurable, and $q:[0,\infty)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is measurable and satisfies the following growth condition:

$$|q(t,x,z)| \le K_1(x) + K_2(x)|z| + C|z|^2 \tag{24.15}$$

$\forall(t,x,z)\in[0,\infty)\times\mathbb{R}\times\mathbb{R}$, for some nonnegative functions $K_1\in L^1(\mathbb{R}), K_2\in L^2(\mathbb{R})$, and for some constant $C>0$.[1] Clearly, the term containing $|u|^r$ for $r\in[1,2]$ in Equation (24.10) satisfies the above growth condition (namely, $q(t,x,z)=|z|^r$). Therefore, the condition for the coefficient q we posed above covers at least this concrete and interesting case. Also, it is obvious that $q(t,x,z)=z$ is another special case under our growth condition, which corresponds to the second term containing the linear u instead of the nonlinear $|u|^r$ on the right-hand side of Equation (24.10).

Clearly, Equation (24.14) is a mild formulation of the following (formal) equation:

$$\begin{aligned} (\partial_t+\nu\Delta_\alpha)u(t,x)+\lambda\partial_x q(u)(t,x) &= f(u)(t,x)+g(u)(t,x)W_{t,x} \\ &\quad + \int_U h(t,x,u(t,x);y)M_{t,x}(dy)\,. \end{aligned}$$

Let us now give a precise formulation of solutions for Equation (24.14). By a (global) solution of (24.14) on the setup $(\Omega,\mathcal{F},P;\{\mathcal{F}_t\}_{t\in[0,\infty)})$, we mean an $\{\mathcal{F}_t\}$-adapted function $u:[0,\infty)\times\mathbb{R}\times\Omega\to\mathbb{R}$ which is càdlàg in the variable $t\in[0,\infty)$ for all $x\in\mathbb{R}$ and for almost all $\omega\in\Omega$ such that (24.14) holds. Furthermore, we say that the solution is (pathwise) unique if whenever $u^{(1)}$ and $u^{(2)}$ are any two solutions of (24.14), then $u^{(1)}(t,x,\cdot)=u^{(2)}(t,x,\cdot)$, P-a.e., $\forall(t,x)\in[0,\infty)\times\mathbb{R}$. Moreover, one can formulate a (global) solution over a finite time interval $[0,T]$ for any $0<T<\infty$ in the same pattern. Furthermore, an $\{\mathcal{F}_t\}$-adapted function $u:[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$ which is càdlàg in $t\in[0,T]$ is called a *local solution* to Equation (24.14) if there exists an increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of stopping times such that $\forall t\in[0,T]$ and $\forall n\in\mathbb{N}$, the stopped process $u(t\wedge\tau_n,x,\omega)$ satisfies Equation (24.14) almost surely. Clearly, a local solution becomes a global solution if $\tau_\infty:=\sup_{n\in\mathbb{N}}\tau_n=T$. Moreover, a local solution to Equation (24.14) is (pathwise) unique if for any other local solution $\tilde{u}:[0,T]\times\mathbb{R}\times\Omega\to\mathbb{R}$, $u(t,x,\omega)=\tilde{u}(t,x,\omega)$ for all $(t,x,\omega)\in[0,\tau_\infty\wedge\tilde{\tau}_\infty)\times\mathbb{R}\times\Omega:=\{(t,x,\omega)\in[0,T]\times\mathbb{R}\times\Omega:0\le t<\tau_\infty(\omega)\wedge\tilde{\tau}_\infty(\omega)\}$. We have the following main result.

Theorem 24.1 *Let $\alpha\in(\frac{3}{2},2]$ and let $T>0$ be arbitrarily fixed. Assume that there exist (positive) functions $L_1,L_2,L_3\in L^1(\mathbb{R})$ such that the following growth conditions:*

$$|f(t,x,z)|^2 \le L_1(x)+C|z|^2\,, \tag{24.16}$$

[1] For simplicity, here and in the sequel the constant C is a generic positive constant whose value may vary from line to line.

$$|g(t,x,z)|^2 + \int_U |h(t,x,z;y)|^2 \nu(dy) \le L_2(x) + C|z|^2 \tag{24.17}$$

and Lipschitz conditions

$$\begin{aligned} & |q(t,x,z_1) - q(t,x,z_2)|^2 + |f(t,x,z_1) - f(t,x,z_2)|^2 \\ \le \; & [L_3(x) + C(|z_1|^2 + |z_2|^2)]|z_1 - z_2|^2 \end{aligned} \tag{24.18}$$

$$\begin{aligned} & |g(t,x,z_1) - g(t,x,z_2)|^2 + \int_U |h(t,x,z_1;y) - h(t,x,z_2;y)|^2 \nu(dy) \\ \le \; & C|z_1 - z_2|^2 \end{aligned} \tag{24.19}$$

hold for all $(t,x) \in [0,T] \times I\!R$ and $z, z_1, z_2 \in I\!R$. Then for every $\mathcal{F}_0$-measurable $u_0 : I\!R \times \Omega \to I\!R$ with $\mathbf{E}\int_{I\!R}(|u_0(x,\cdot)|^2)dx < \infty$, there exists a unique local solution u to Equation (24.14) with the following property:

$$\mathbf{E}\left(\int_{I\!R} |u(t \wedge \tau_\infty(\cdot), x, \cdot)|^2 dx\right) < \infty\,, \quad \forall t \in [0,T].$$

Remark 24.1 The assumption $\alpha \in (\frac{3}{2}, 2]$ is a sufficient condition from our proof to Theorem 24.1. It would be interesting to study the case of $\alpha \in (0, \frac{3}{2})$ as well. Apparently, our approach in the present chapter does not work for the latter case.

We need some preparation before the proof to Theorem 24.1. For any fixed $n \in I\!N$, let the mapping

$$\pi_n : L^2(I\!R) \to B_n := \{u \in L^2(I\!R) : ||u||_{L^2} := \left(\int_{I\!R} u^2(x)dx\right)^{\frac{1}{2}} \le n\}$$

be defined via

$$\pi_n(u) = \begin{cases} u, & \text{if} \quad ||u||_{L^2} \le n \\ \frac{nu}{||u||_{L^2}}, & \text{if} \quad ||u||_{L^2} > n. \end{cases}$$

Clearly, for any $n \in I\!N$, we have

$$||\pi_n(u)||_{L^2} \le n\,.$$

Moreover, it is clear that the norm

$$||\pi_n||_{L^2} := \sup_{||u||_{L^2} \le 1} ||\pi_n u||_{L^2} \le 1$$

that is, $\pi_n : L^2(I\!R) \to L^2(I\!R)$ is a contraction.

Notice that if u is a solution to Equation (24.14), then u is $L^2(I\!R)$-valued, $\{\mathcal{F}_t\}$-progressive process. Thus, by Theorem 2.1.6 in [9], $\forall n \in I\!N$

$$\tau_n(\omega) := \inf\{t \in [0,T] : \int_{I\!R} u^2(t,x,\omega)dx \ge n^2\}, \quad \omega \in \Omega$$

defines a stopping time. It is clear that $\{\tau_n\}_{n \in I\!N}$ is an increasing sequence of stopping times determined by u. Moreover, for any fixed $n \in I\!N$, the stopped process $u(t \wedge \tau_n)$ satisfies the following equation:

$$
\begin{aligned}
u(t,x,\omega) \;=\;& \int_{I\!R} G_\alpha(t,x-z)u_0(z,\omega)dz \\
&+\lambda\int_0^t\int_{I\!R}[\partial_z G_\alpha(t-s,x-z)]q(s,z,(\pi_n u)(s,z,\omega))dzds \\
&+\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)f(s,z,(\pi_n u)(s,z,\omega))dzds \qquad (24.20)\\
&+\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)g(s,z,(\pi_n u)(s,z,\omega))W(ds,dz) \\
&+\int_0^{t+}\int_{I\!R}\int_U G_\alpha(t-s,x-z) \\
&\qquad\times h(s,z,(\pi_n u)(s-,z,\omega);y)M(ds,dz,dy,\omega)\,.
\end{aligned}
$$

On the other hand, any solution to Equation (24.20) is a local solution to Equation (24.14). Therefore, the existence of a unique local solution to Equation (24.14) is equivalent to the existence of a unique solution to Equation (24.20). Hence, we will focus our attention on showing the existence of a unique solution to Equation (24.20). To that end, let us first prove the following lemma which presents some useful inequalities.

Lemma 24.1 *For $u:[0,T]\times I\!R\to I\!R$, the following estimates hold:*

$$
\begin{aligned}
&\int_{I\!R}\left(\int_0^t\int_{I\!R}[\partial_z G_\alpha(t-s,x-z)]u(s,z)dzds\right)^2 dx \\
\le\;& C\left(\int_0^t (t-s)^{-\frac{3}{2\alpha}}\int_{I\!R}|u(s,z)|dzds\right)^2 \qquad (24.21)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{I\!R}\left(\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)u(s,z)dzds\right)^2 dx \\
\le\;& C\left[\int_0^t\left(\int_{I\!R}|u(s,z)^2|dz\right)^{\frac{1}{2}} ds\right]^2, \qquad (24.22)
\end{aligned}
$$

in particular

$$
\begin{aligned}
&\left|\int_0^t\int_{I\!R} G_\alpha(t-s,x-z)u(s,z)dzds\right| \\
\le\;& C\int_0^t (t-s)^{-\frac{1}{2\alpha}}\left(\int_{I\!R}|u(s,z)|^2dz\right)^{\frac{1}{2}} ds\,. \qquad (24.23)
\end{aligned}
$$

Proof By inequality (24.13) and Minkowski inequality (cf., e.g., p. 47 of [15]) in turn, we have

$$
\begin{aligned}
&\int_{I\!R}\left(\int_0^t\int_{I\!R}[\partial_z G_\alpha(t-s,x-z)]u(s,z)dzds\right)^2 dx \\
\le\;& C\int_{I\!R}\left[\int_0^t\int_{I\!R}\frac{(t-s)|x-z|^\alpha|u(s,z)|}{\left((t-s)^{1+\frac{1}{\alpha}}+|x-z|^{1+\alpha}\right)^2}dzds\right]^2 dx
\end{aligned}
$$

$$\leq \quad C\left[\int_0^t \int_{I\!R}(t-s)|u(s,z)|\left(\int_{I\!R}\frac{|x-z|^{2\alpha}dx}{\left((t-s)^{1+\frac{1}{\alpha}}+|x-z|^{1+\alpha}\right)^4}\right)^{\frac{1}{2}}dzds\right]^2 .$$

Now, by shifting and scaling the integral variable x, we get

$$\int_{I\!R}\frac{|x-z|^{2\alpha}dx}{\left((t-s)^{1+\frac{1}{\alpha}}+|x-z|^{1+\alpha}\right)^4}=(t-s)^{-\frac{3+2\alpha}{\alpha}}\int_{I\!R}\frac{|x|^{2\alpha}dx}{(1+|x|^{1+\alpha})^4}.$$

Since

$$\begin{aligned}\int_{I\!R}\frac{|x|^{2\alpha}dx}{(1+|x|^{1+\alpha})^4} &= 2\int_0^1\frac{x^{2\alpha}dx}{(1+x^{1+\alpha})^4}+2\int_1^\infty\frac{x^{2\alpha}dx}{(1+x^{1+\alpha})^4}\\ &\leq 2\int_0^1 x^{2\alpha}dx+2\int_1^\infty x^{-4-2\alpha}dx\\ &= \frac{2}{2\alpha+1}+\frac{1}{2\alpha+3}<\infty\end{aligned}$$

which indicates that the integral $\int_{I\!R}\frac{|x|^{2\alpha}dx}{(1+|x|^{1+\alpha})^4}$ is a constant only depending on α, thus we obtain

$$\begin{aligned}&\int_{I\!R}\left(\int_0^t\int_{I\!R}[\partial_z G_\alpha(t-s,x-z)]u(s,z)dzds\right)^2 dx\\ \leq\ & C\left(\int_0^t(t-s)^{-\frac{3}{2\alpha}}\int_{I\!R}|u(s,z)|dzds\right)^2 .\end{aligned}$$

That is, inequality (24.21) is derived.

Inequality (24.22) can be proved by utilizing Fubini theorem, Minkowski inequality, and Young inequality (cf., e.g., p. 99 of [15]) in turn as follows:

$$\begin{aligned}&\int_{I\!R}\left(\int_0^t\int_{I\!R}G_\alpha(t-s,x-z)u(s,z)dzds\right)^2 dx\\ =\ &\int_{I\!R}\left(\int_0^t[\int_{I\!R}G_\alpha(t-s,x-z)u(s,z)dz]ds\right)^2 dx\\ \leq\ &\int_0^t\left[\int_{I\!R}\left(\int_{I\!R}G_\alpha(t-s,x-z)u(s,z)dz\right)^2 dx\right]^{\frac{1}{2}} ds\\ \leq\ & C\int_0^t\left(\int_{I\!R}G_\alpha(t-s,x)dx\right)\left(\int_{I\!R}|u(s,z)|^2dz\right)^{\frac{1}{2}} ds\\ \leq\ & C\left[\int_0^t\left(\int_{I\!R}|u(s,z)^2|dz\right)^{\frac{1}{2}} ds\right]^2 ,\end{aligned}$$

since

$$\int_{I\!R}G_\alpha(t-s,x)dx=1\,.$$

Finally, inequality (24.23) is a direct consequence of inequality (24.12). ∎

Proof of Theorem 24.1 We will carry out the proof by the following three steps. Notice that the assumption $\alpha \in (\frac{3}{2}, 2]$ is in force in the whole proof.

Step 1 Suppose that $u : [0,T] \times I\!R \times \Omega \to I\!R$ is an $L^2(I\!R)$-valued, $\{\mathcal{F}_t\}$-adapted, càdlàg process. For any fixed $n \in I\!N$, set

$$(\mathcal{J}u)(t,x,\omega) = \int_{I\!R} G_\alpha(t, x-z) u_0(z,\omega)dz + \sum_{k=1}^{4} (\mathcal{J}_k u)(t,x,\omega)$$

with

$$(\mathcal{J}_1 u)(t,x,\omega) = \lambda \int_0^t \int_{I\!R} [\partial_z G_\alpha(t-s, x-z)] q(s,z,(\pi_n u)(s,z,\omega))dzds$$

$$(\mathcal{J}_2 u)(t,x,\omega) = \int_0^t \int_{I\!R} G_\alpha(t-s, x-z) f(s,z,(\pi_n u)(s,z,\omega))dzds$$

$$(\mathcal{J}_3 u)(t,x,\omega) = \int_0^t \int_{I\!R} G_\alpha(t-s, x-z) g(s,z,(\pi_n u)(s,z,\omega))W(ds,dz,\omega)$$

and

$$\begin{aligned}(\mathcal{J}_4 u)(t,x,\omega) \quad &= \quad \int_0^{t+} \int_{I\!R} \int_U G_\alpha(t-s, x-z) \\ &\qquad\qquad \times h(s,z,(\pi_n u)(s,z,\omega); y) M(ds,dz,dy,\omega)\,.\end{aligned}$$

By (24.21) in Lemma 24.1, and Schwarz inequality, we have

$$\begin{aligned}
& \int_{I\!R} [(\mathcal{J}_1 u)(t,x,\omega)]^2 dx \\
\leq \;& C\left(\int_0^t (t-s)^{-\frac{3}{2\alpha}} \int_{I\!R} q(s,z,(\pi_n u)(s,z,\omega))dzds\right)^2 \\
\leq \;& C\Bigg[\int_0^t (t-s)^{-\frac{3}{2\alpha}} \Bigg(\int_{I\!R} K_1(z)dz + \int_{I\!R} K_2(z)|(\pi_n u)(s,z,\omega)|dz \\
& \qquad + C\int_{I\!R} |(\pi_n u)(s,z,\omega)|^2 dzdz\Bigg) ds\Bigg]^2 \\
\leq \;& C\int_0^t (t-s)^{-\frac{3}{2\alpha}} ds \int_0^t (t-s)^{-\frac{3}{2\alpha}} \Bigg[\left(\int_{I\!R} K_1(z)dz\right)^2 \\
& \qquad + \int_{I\!R} |K_2(z)|^2 dz \int_{I\!R} |(\pi_n u)(s,z,\omega)|^2 dz \\
& \qquad + C\left(\int_{I\!R} |(\pi_n u)(s,z,\omega)|^2 dzdz\right)^2\Bigg] ds \\
\leq \;& Ct^{1-\frac{3}{2\alpha}} \int_0^t (t-s)^{-\frac{3}{2\alpha}} \Bigg[\left(\int_{I\!R} K_1(z)dz\right)^2 \\
& \qquad + n^2 \int_{I\!R} |K_2(z)|^2 dz + Cn^4\Bigg] ds \\
\leq \;& Ct^{2-\frac{3}{\alpha}} \leq CT^{2-\frac{3}{\alpha}} < \infty\,.
\end{aligned}$$

Notice that here and after the constant C also depends on n (of course on T as well). By inequality (24.22) in Lemma 24.1 and the condition (24.15), we get

$$\int_{I\!R} [(\mathcal{J}_2 u)(t,x,\omega)]^2 dx$$

$$
\begin{aligned}
&\le C\left[\int_0^t \left(\int_{I\!R} |f(s,z,(\pi_n u)(s,z,\omega))|^2 dz\right)^{\frac{1}{2}} ds\right]^2 \\
&\le \left\{\int_0^t \left[\int_{I\!R} \left(L_1(z) + C[(\pi_n u)(s,z,\omega)]^2\right) dz\right]^{\frac{1}{2}} ds\right\}^2 \\
&\le \left[\int_0^t \left(\int_{I\!R} L_1(z)dz + Cn^2\right)^{\frac{1}{2}} ds\right]^2 \\
&\le Ct^2 \le CT^2 < \infty\,.
\end{aligned}
$$

On the other hand, by Itô's isometry property for stochastic integrals with respect to (both continuous and càdlàg) martingales, we have

$$
\begin{aligned}
&\mathbf{E}\left[\int_{I\!R} |(\mathcal{J}_3 u)(t,x,\cdot)|^2 dx\right] \\
= \;&\mathbf{E}\left[\int_{I\!R}\left(\int_0^t \int_{I\!R} G_\alpha^2(t-s,x-z) g^2(s,z,(\pi_n u)(s,z,\cdot)) dz ds\right) dx\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}\left[\int_{I\!R} |(\mathcal{J}_4 u)(t,x,\cdot)|^2 dx\right] \\
= \;&\mathbf{E}\left\{\int_{I\!R}\left[\int_0^t \int_{I\!R} G_\alpha^2(t-s,x-z)\right.\right. \\
&\qquad \left.\left.\times \left(\int_U h^2(s,z,(\pi_n u)(s,z,\cdot);y)\nu(dy)\right) dz ds\right] dx\right\}.
\end{aligned}
$$

Thus, by inequality (24.12) and the condition (24.17), we get

$$
\begin{aligned}
&\mathbf{E}\left[\int_{I\!R} |(\mathcal{J}_3 u)(t,x,\cdot)|^2 dx + \int_{I\!R} |(\mathcal{J}_4 u)(t,x,\cdot)|^2 dx\right] \\
\le \;& C\mathbf{E}\left[\int_0^t (t-s)^{-\frac{1}{\alpha}} \int_{I\!R}\left(\int_{I\!R} g^2(s,z,(\pi_n u)(s,z,\cdot))\right.\right. \\
&\qquad \left.\left. + \int_U h^2(s,z,(\pi_n u)(s,z,\cdot);y)\nu(dy)\right) dz ds\right] \\
\le \;& C\int_0^t (t-s)^{-\frac{1}{\alpha}} \left(\int_{I\!R} L_2(z)dz + C\int_{I\!R} \mathbf{E}[(\pi_n u)(s,z,\cdot)]^2 dz\right) ds \\
\le \;& Ct^{1-\frac{1}{\alpha}}\left(\int_{I\!R} L_2(z)dz + Cn^2\right) \\
\le \;& Ct^{1-\frac{1}{\alpha}} \le CT^{1-\frac{1}{\alpha}} < \infty\,.
\end{aligned}
$$

Therefore, we obtain that

$$
\mathbf{E}\left(\int_{I\!R} |(\mathcal{J} u)(t,x,\cdot)|^2 dx\right) \le C(T^{2-\frac{3}{\alpha}} + T^2 + T^{1-\frac{1}{\alpha}}) < \infty
$$

for any fixed $t \in [0,T]$.

Step 2 Now let $\theta > 0$ be arbitarily fixed. For any $L^2(I\!R)$-valued, $\{\mathcal{F}_t\}$-adapted, càdlàg process $u : [0,T] \times I\!R \times \Omega \to I\!R$ with initial condition $u(0,x,\omega) = u_0(x,\omega)$, we define

$$
||u||_\theta^2 := \int_0^T e^{-\theta t}\mathbf{E}\left(\int_{I\!R} u^2(t,x,\cdot)dx\right) dt\,.
$$

Clearly, $||\cdot||_\theta$ is a norm. Let B denote the collection of all $L^2(I\!R)$-valued, $\{\mathcal{F}_t\}$-adapted, càdlàg process $u:[0,T]\times I\!R\times\Omega\to I\!R$ with initial condition $u(0,x,\omega)=u_0(x,\omega)$, such that

$$||u||_\theta^2=\int_0^T e^{-\theta t}\mathbf{E}\left(\int_{I\!R}u^2(t,x,\cdot)dx\right)dt<\infty\,.$$

Then $(B,||\cdot||_\theta)$ is a Banach space. Now $\forall u\in B, \mathcal{J}u$ is well defined and for any fixed $t\in[0,T]$

$$\mathbf{E}\left(\int_{I\!R}|(\mathcal{J}u)(t,x,\cdot)|^2dx\right)\le C(T^{2-\frac{3}{\alpha}}+T^2+T^{1-\frac{1}{\alpha}})<\infty\,.$$

Thus, by the following Laplace transform formula:

$$\int_0^\infty t^{r-1}e^{-st}=\Gamma(r)s^{-r}\,,\quad \forall r\in(0,\infty)$$

we get

$$\begin{aligned}
||\mathcal{J}u||_\theta^2 &= \int_0^T e^{-\theta t}\mathbf{E}\left(\int_{I\!R}(\mathcal{J}u)^2(t,x,\cdot)dx\right)dt\\
&\le C\int_0^\infty(t^{2-\frac{3}{\alpha}}+t^2+t^{1-\frac{1}{\alpha}})e^{-\theta t}dt\\
&= C[\Gamma(3-\frac{3}{\alpha})\theta^{-(3-\frac{3}{\alpha})}+2\theta^{-3}+\Gamma(2-\frac{1}{\alpha})\theta^{-(2-\frac{1}{\alpha})}]\\
&\le C[\theta^{-(3-\frac{3}{\alpha})}+\theta^{-3}+\theta^{-(2-\frac{1}{\alpha})}]\\
&< \infty,
\end{aligned}$$

that is, $\mathcal{J}u\in B$, which implies that $\mathcal{J}:B\to B$.

Step 3 Now $\forall u,v\in B$, by (24.21), (24.22) and (24.23) in Lemma 24.1, Lipschitz condition (24.18), the estimate (24.12), together with Fubini theorem, Young and Schwarz inqualities, we get for any $t\in[0,T]$

$$\begin{aligned}
&\mathbf{E}\left(\int_{I\!R}|(\mathcal{J}_1u)(t,x,\cdot)-(\mathcal{J}_1v)(t,x,\cdot)|^2dx\right)\\
=\ &\mathbf{E}\int_{I\!R}\left\{\int_0^t\int_{I\!R}[\frac{\partial}{\partial z}G_\alpha(t-s,x-z)]\,(q(s,z,(\pi_nu)(s,z,\cdot))\right.\\
&\qquad\left.-q(s,z,(\pi_nv)(s,z,\cdot)))\,dzds\right\}^2dx\\
\le\ &C\mathbf{E}\left(\int_0^t(t-s)^{-\frac{3}{2\alpha}}\int_{I\!R}|q(s,z,(\pi_nu)(s,z,\cdot))\right.\\
&\qquad\left.-q(s,z,(\pi_nv)(s,z,\cdot))|dzds\right)^2\\
\le\ &C\mathbf{E}\left[\int_0^t(t-s)^{-\frac{3}{2\alpha}}\int_{I\!R}\big(L_3(z)+C[(\pi_nu)^2(s,z,\cdot))\right.\\
&\qquad\left.+(\pi_nv)^2(s,z,\cdot)]\big)^{\frac{1}{2}}|(\pi_nu)(s,z,\cdot))-(\pi_nv)(s,z,\cdot)|dzds\right]^2\\
\le\ &C\int_0^t(t-s)^{-\frac{3}{2\alpha}}ds\mathbf{E}\int_0^t(t-s)^{-\frac{3}{2\alpha}}\left[\int_{I\!R}\big(L_3(z)+C[(\pi_nu)^2(s,z,\cdot)\right.\\
&\qquad\left.+(\pi_nv)^2(s,z,\cdot)]\big)^{\frac{1}{2}}|(\pi_nu)(s,z,\cdot))-(\pi_nv)(s,z,\cdot)|dz\right]^2ds\\
\le\ &Ct^{\frac{1}{4}}\mathbf{E}\int_0^t(t-s)^{-\frac{3}{2\alpha}}\int_{I\!R}\big(L_3(z)+C[(\pi_nu)^2(s,z,\cdot)
\end{aligned}$$

$$
\begin{aligned}
&\quad +(\pi_n v)^2(s,z,\cdot)]\big)\,dz \int_{I\!R} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)|^2 dzds \\
\leq\ & CT^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{2\alpha}} \mathbf{E}\left(\int_{I\!R} |u(s,z,\cdot)-v(s,z,\cdot)|^2 dz\right) ds \\
\leq\ & C \int_0^t (t-s)^{-\frac{3}{2\alpha}} \mathbf{E}\left(\int_{I\!R} |u(s,z,\cdot)-v(s,z,\cdot)|^2 dz\right) ds\,,
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{E}\left(\int_{I\!R} |(\mathcal{J}_2 u)(t,x,\cdot)-(\mathcal{J}_2 v)(t,x,\cdot)|^2 dx\right) \\
=\ & \mathbf{E}\int_{I\!R}\left[\int_0^t \int_{I\!R} G_\alpha(t-s,x-z)\,(f(s,z,(\pi_n u)(s,z,\cdot)) \right. \\
&\quad \left. -f(s,z,(\pi_n v)(s,z,\cdot)))\,dzds\right]^2 dx \\
\leq\ & \mathbf{E}\int_{I\!R}\left\{\int_0^t \int_{I\!R} G_\alpha(t-s,x-z)\left[L_3(z)+C\left((\pi_n u)^2(s,z,\cdot)\right.\right.\right. \\
&\quad \left.\left.\left. +(\pi_n v)^2(s,z,\cdot)\right)\right]^{\frac{1}{2}} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)| dzds\right\}^2 dx \\
\leq\ & t\mathbf{E}\int_{I\!R}\int_0^t \left\{\int_{I\!R} G_\alpha(t-s,x-z)\left[L_3(z)+C\left((\pi_n u)^2(s,z,\cdot)\right.\right.\right. \\
&\quad \left.\left.\left. +(\pi_n v)^2(s,z,\cdot)\right)\right]^{\frac{1}{2}} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)| dz\right\}^2 dsdx \\
=\ & t\mathbf{E}\int_0^t \int_{I\!R} \left\{\int_{I\!R} G_\alpha(t-s,x-z)\left[L_3(z)+C\left((\pi_n u)^2(s,z,\cdot)\right.\right.\right. \\
&\quad \left.\left.\left. +(\pi_n v)^2(s,z,\cdot)\right)\right]^{\frac{1}{2}} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)| dz\right\}^2 dxds \\
\leq\ & t\mathbf{E}\int_0^t \int_{I\!R} G_\alpha^2(t-s,x-z)dz \left\{\int_{I\!R} \left[L_3(z)+C\left((\pi_n u)^2(s,z,\cdot)\right.\right.\right. \\
&\quad \left.\left.\left. +(\pi_n v)^2(s,z,\cdot)\right)\right]^{\frac{1}{2}} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)| dz\right\}^2 ds \\
\leq\ & (2\pi)^{-\frac{1}{\alpha}} t\mathbf{E}\int_0^t (t-s)^{-\frac{1}{2}} \int_{I\!R} \left[L_3(z)+C\left((\pi_n u)^2(s,z,\cdot)\right.\right. \\
&\quad \left.\left. +(\pi_n v)^2(s,z,\cdot)\right)\right] dz \int_{I\!R} |(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)|^2 dzds \\
\leq\ & C\int_0^t (t-s)^{-\frac{1}{\alpha}} \mathbf{E}\int_{I\!R} |u(s,z,\cdot)-v(s,z,\cdot)|^2 dzds
\end{aligned}
$$

and by Itô's isometry for both stochastic integrals with respect to $W(ds,dz)$ and $M(ds,dz,dy)$, we have

$$
\begin{aligned}
& \mathbf{E}\int_{I\!R} |(\mathcal{J}_3 u)(t,x,\cdot)+(\mathcal{J}_4 u)(t,x,\cdot)-(\mathcal{J}_3 v)(t,x,\cdot)-(\mathcal{J}_4 v)(t,x,\cdot)|^2 dx \\
=\ & \mathbf{E}\int_{I\!R}\left[\int_0^t \int_{I\!R} G_\alpha^2(t-s,x-z)\left(|g(\pi_n u)(s,z,\cdot)-g(\pi_n v)(s,z,\cdot)|^2\right.\right. \\
&\quad \left.\left. +\int_U |h(\pi_n u)(s,z,\cdot;y)-h((\pi_n v)(s,z,\cdot;y)|^2 \nu(dy)\right) dsdz\right] dx \\
\leq\ & C\mathbf{E}\int_{I\!R}\left(\int_0^t \int_{I\!R} G_\alpha^2(t-s,x-z)|(\pi_n u)(s,z,\cdot)-(\pi_n v)(s,z,\cdot)|^2 dsdz\right) dx
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; C\mathbf{E}\int_{I\!R}\left(\int_0^t\int_{I\!R} G_\alpha^2(t-s,x-z)|u(s,z,\cdot)-v(s,z,\cdot)|^2dzds\right)dx \\
&= \; C\mathbf{E}\int_0^t\int_{I\!R}\left(\int_{I\!R} G_\alpha^2(t-s,x-z)dx\right)|u(s,z,\cdot)-v(s,z,\cdot)|^2dzds \\
&\leq \; C\mathbf{E}\int_0^t (t-s)^{-\frac{1}{\alpha}}\left(\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dz\right)ds \\
&= \; C\int_0^t (t-s)^{-\frac{1}{\alpha}}\mathbf{E}\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dzds\,.
\end{aligned}
$$

Combining all above together, we obtain

$$
\begin{aligned}
||\mathcal{J}(u-v)||_\theta^2 &= ||\mathcal{J}u-\mathcal{J}v||_\theta^2 \\
&= \int_0^T e^{-\theta t}\mathbf{E}\left(\int_{I\!R}|(\mathcal{J}u)(t,x,\cdot)-(\mathcal{J}v)(t,x,\cdot)|^2dx\right)dt \\
&\leq C\int_0^T e^{-\theta t}\int_0^t\left((t-s)^{-\frac{3}{2\alpha}}+(t-s)^{-\frac{1}{\alpha}}\right) \\
&\qquad\qquad \times\mathbf{E}\left(\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dz\right)dsdt \\
&= C\int_0^T\left[\int_s^T e^{-\theta t}\left((t-s)^{-\frac{3}{2\alpha}}+(t-s)^{-\frac{1}{\alpha}}\right)dt\right] \\
&\qquad\qquad \times\mathbf{E}\left(\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dz\right)ds \\
&\leq C\int_0^T\left(\int_s^\infty e^{-\theta t}(t-s)^{-\frac{3}{2\alpha}}dt+\int_s^\infty e^{-\theta t}(t-s)^{-\frac{1}{\alpha}}dt\right) \\
&\qquad\qquad \times\mathbf{E}\left(\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dz\right)ds \\
&= C\left(\int_0^\infty e^{-\theta t}\theta^{-\frac{3}{2\alpha}}d\theta+\int_0^\infty e^{-\theta t}\theta^{-\frac{1}{\alpha}}d\theta\right) \\
&\qquad\qquad \times\int_0^T e^{-\theta s}\mathbf{E}\left(\int_{I\!R}|u(s,z,\cdot)-v(s,z,\cdot)|^2dz\right)ds \\
&\leq C\left(\frac{\Gamma(1-\frac{3}{2\alpha})}{\theta^{1-\frac{3}{2\alpha}}}+\frac{\Gamma(1-\frac{1}{\alpha})}{\theta^{1-\frac{1}{\alpha}}}\right)||u-v||_\theta^2 \\
&\leq C\left(\frac{1}{\theta^{1-\frac{3}{2\alpha}}}+\frac{1}{\theta^{1-\frac{1}{\alpha}}}\right)||u-v||_\theta^2\,.
\end{aligned}
$$

Now let us take θ large enough so that

$$
C\left(\frac{1}{\theta^{1-\frac{3}{2\alpha}}}+\frac{1}{\theta^{1-\frac{1}{\alpha}}}\right)<1
$$

which implies that $\mathcal{J}: B\to B$ is a contraction. Therefore there must be a unique fixed point in B for $\mathcal{J}$ and this fixed point is the unique solution for Equation (24.20). To see that this gives us a local solution to Equation (24.14), let us denote by u_n the unique solution of Equation (24.20) for each $n\in I\!N$. For this u_n, let us set the stopping time

$$
\tau_n(\omega):=\inf\{t\in[0,T]:\int_{I\!R}u_n^2(t,x,\omega)dx\geq n^2\},\quad \omega\in\Omega.
$$

Clearly by the contraction property of $\mathcal{J}$, we have for all $j \geq n$ and for almost all $\omega \in \Omega$

$$u_j(t,\cdot,\omega) = u_n(t,\cdot,\omega), \quad \forall (t,\omega) \in [0,\tau_n) \times \Omega\,.$$

Therefore we define

$$u(t,x,\omega) = u_n(t,x,\omega)$$

for any $(t,x,\omega) \in [0,\tau_n) \times I\!\!R \times \Omega$ and

$$\tau_\infty(\omega) := \sup_{n \in I\!\!N} \tau_n(\omega)\,.$$

Then

$$\{u(t,x,\omega) : (t,x,\omega) \in [0,\tau_\infty) \times I\!\!R \times \Omega\}$$

is a local solution to Equation (24.14).

Finally, for the uniqueness of the local solution to Equation (24.14), suppose that there are two local solutions u and v to Equation (24.14). Then u and v must satisfy Equation (24.20) for any fixed $n \in I\!\!N$. On the other hand, by the uniqueness of solution to Equation (24.20), we get

$$u(t,x,\omega) = v(t,x,\omega), \quad \forall (t,x,\omega) \in [0,\tau_n) \times I\!\!R \times \Omega\,.$$

Now let $n \to \infty$, we deduce

$$u(t,x,\omega) = v(t,x,\omega), \quad \forall (t,x,\omega) \in [0,\tau_\infty) \times I\!\!R \times \Omega\,.$$

Hence we obtain the uniqueness. ■

Remark 24.2 In the case $M \equiv 0$, Equation (24.14) becomes a Burgers' equation with Gaussian (space–time) white noise. Unique global L^2 solutions are obtained by Gyöngy and Nualart in [11] in the whole space line and by Da Prato, Debussche, and Teman in [6] (see also [8]) in bounded space intervals. Their methods depended critically on some uniform estimates which employed Burkholder's inequalities for continuous martingales. We observe that we are unable to follow this route herein as the corresponding inequalities for càdlàg martingales (see, e.g., [13]) do not behave so nicely.

Acknowledgments

We are grateful to Professors G. Da Prato and L. Tubaro for their invitation to these proceedings.

References

[1] S. Assing, A pregenerator for Burgers' equations forced by conservative noise, *Commun. Math. Phys.*, **225** (2002), 611–632.

[2] J. Bertoin, Structure of shocks in Burgers' turbulence with stable noise initial data, *Commun. Math. Phys.*, **203** (1999), 729–741.

[3] P. Biler, T. Funaki, and W.A. Woyczynski, Fractal Burgers' equations, *J. Differential Equations*, **148** (1998), 9–46.

[4] P. Biler, G. Karch, and W.A. Woyczynski, Asymptotics for multifractal conservation laws, *Studia Math.*, **135** (1999), 231–252.

[5] P. Biler, G. Karch, and W.A. Woyczynski, Critical nonlinear exponent and self-similar asymptotics for Lévy conservation laws, *Ann. I. H. Poincaré–AN*, **18** (2001), 613–637.

[6] G. Da Prato, A. Debussche, and R. Teman, Stochastic Burgers' equation, *NoDEA*, **1** (1994), 389–402.

[7] G. Da Prato and D. Gatarek, Stochastic Burgers' equation with correlated noise, *Stoch. Stoch. Rep.*, **52** (1995), 29–41.

[8] G. Da Prato and J. Zabczyk, *Ergodicity for Infinite Dimensional Systems.* LMS Lect. Notes **229**, Cambridge Univ. Press, Cambridge, 1996.

[9] S.N. Ethier and T.G. Kurtz, *Markov Processes: Characterization and Convergence.* John Wiley and Sons, New York, 1986.

[10] C. Giraud, On regular points in Burgers turbulence with stable noise initial data, *Ann. I. H. Poincaré–Probab. Statist.*, **38** (2002), 229–251.

[11] I. Gyöngy and D. Nualart, On the stochastic Burgers' equation in the real line, *Ann. Prob.*, **27** (1999), 782–802.

[12] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Kodansha, 1981.

[13] J. Jacod and A.N. Shiryaev, *Limit Theorems for Stochastic Processes.* Springer-Verlag, Berlin, 1987.

[14] T. Komatsu, Uniform estimates for fundamental solutions associated with non-local Dirichlet forms, *Osaka J. Math.*, **32** (1995), 833–860.

[15] E.H. Lieb and M. Loss, *Analysis* (2nd Ed.), Graduate Studies in Mathematics **14**, AMS, Providence, RI, 2001.

[16] A. Truman and J.-L. Wu, Stochastic Burgers' equation with Lévy space-time white noise, in *Probabilistic Methods in Fluid* (I.M. Davies et al. eds.), pp. 298–323, World Scientific, Singapore, 2003.

[17] J.B. Walsh, An introduction to stochastic partial differential equations, In: *Ecole d'été de Probabilités de St. Flour XIV*, pp. 266–439, Lecture Notes in Math. **1180**, Springer-Verlag, Berlin, 1986.

[18] M. Winkel, Burgers' turbulence initialized by a regenerative impulse, *Stoch. Proc. Appl.*, **93** (2001), 241–268.

25 Qualitative Properties of Solutions to Stochastic Burgers' System of Equations

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25.1 Introduction

Let $U = U(t)$ represent the *primary* velocity of the fluid, parallel to the walls of the channel and $v = v(t,x)$ describe the *secondary* velocity of the turbulent motion. Let P, ρ, and μ be constants representing, respectively, an exterior force, analogous to the mean pressure gradient in the hydrodynamic case, the density of the fluid, and its viscosity. Set $\nu = \frac{\mu}{\rho} > 0$. The functions $U(t)$, $v(t,\cdot)$, $t \geq 0$, should satisfy (see [1]) the following system of equations:

$$\frac{dU(t)}{dt} = P - \nu U(t) - \int_0^1 v^2(t,x)\,dx \qquad \text{for } t > 0, \tag{25.1}$$

$$\frac{\partial v(t,x)}{\partial t} = \nu \frac{\partial^2 v(t,x)}{\partial x^2} + U(t)\,v(t,x) - \frac{\partial}{\partial x}\left(v^2(t,x)\right) \tag{25.2}$$

with the initial and boundary conditions

$$U(0) = U_0,\ v(0,x) = v_0(x),\ v(t,0) = v(t,1) = 0,\ x \in (0,1),\ t > 0. \tag{25.3}$$

The existence and uniqueness for the global solution of the deterministic system was examined by Dłotko in [2], using the Galerkin method.

But system (25.1)–(25.3) does not display any chaotic phenomena and therefore stochastic perturbations of (25.1)–(25.3) are proposed as a better model

$$dU(t) = (P - \nu U(t) - \|v(t)\|^2)dt + g_0(U(t), v(t))dW_0(t) \tag{25.4}$$

$$\begin{aligned} dv(t,x) = {} & (\nu \frac{\partial^2 v(t,x)}{\partial x^2} + U(t)\,v(t,x) \\ & - \frac{\partial}{\partial x}\left(v^2(t,x)\right))dt + g_1(U(t), v(t,x))dW_1(t,x) \end{aligned} \tag{25.5}$$

for $t > 0$, with the initial and boundary conditions (25.3). The existence and uniqueness of the solution of the system without the term

$$g_0(U(t), v(t,x))dW_0(t)$$

in (25.4) were given in [5] and [6].

In the present chapter we establish existence and uniqueness of the global solution to the general system (25.4)–(25.5). Under appropriate conditions on the coefficients we establish the irreducibility property of the corresponding transition semigroup. In the proofs we adapt the methods from [3] and [6]. In the forthcoming paper [7] strong Feller property, as well as existence and uniqueness of invariant measure, are treated.

25.2 Preliminaries

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space on which an increasing and right-continuous family $(\mathcal{F}_t)_{t\in[0,T]}$ of sub-σ-algebras of $\mathcal{F}$ is defined such that $\mathcal{F}_0$ contains all P-null sets in $\mathcal{F}$. We consider the one-dimensional (1D) Wiener process $W_0(t)$ and the cylindrical Wiener process $W_1(t,\cdot)$ such that

$$W_1(t,x) = \sum_{k=1}^{\infty} W_k(t)e_k(x). \tag{25.6}$$

Here (e_k) is an orthonormal basis of $L^2 = L^2(0,1)$

$$e_k(x) = \sqrt{\frac{2}{\pi}}\sin k\pi x,\ \ x\in(0,1),\ \ k=1,2,.... \tag{25.7}$$

The scalar product in L^2 is denoted by $(\cdot,\cdot)$ and the usual norm by $\|\cdot\|$.

Let $S(t)$, $t\geq 0$, be the classical heat semigroup on L^2. It is well known that the generator A of the semigroup $S(t)$, $t\geq 0$, is identical with the second derivative operator $\frac{\partial^2}{\partial x^2}$ on the domain $D(A)$ consisting of functions v such that v, $\frac{\partial v}{\partial x}$ are absolutely continuous with $\frac{\partial^2 v}{\partial x^2}\in L^2$, $v(0)=v(1)=0$. In some places $S(t)$, $t\geq 0$, will be denoted by e^{At}, $t\geq 0$.

Definition 25.1 *A pair of continuous adapted processes* $\begin{pmatrix} U \\ v \end{pmatrix}$ *with values in* $\mathbb{R}^1$ *and* L^2, *respectively, is said to be an integral solution to problem (25.4) and (25.5) if*

$$\begin{aligned} U(t) = e^{-\nu t}U_0 &+ \int_0^t e^{-\nu(t-s)}(P-\|v(s)\|^2)ds \\ &+ \int_0^t e^{-\nu(t-s)}g_0(U(s),v(s))dW_0(s), \end{aligned} \tag{25.8}$$

$$\begin{aligned} v(t) &= S(t)v_0 + \int_0^t S(t-s)U\,(s)\,v(s)ds \\ -\int_0^t S(t-s)\frac{\partial}{\partial x}v^2(s)ds &+ \int_0^t S(t-s)g_1(U(s),v(s))dW_1(s). \end{aligned} \tag{25.9}$$

In the integral $\int_0^t S(t-s)\frac{\partial}{\partial x}v^2(s)ds$, $t>0$, we use the extension of the operators $S(t-s)$ to $L^1(0,1)$ described in [5] and [6].

It is not difficult to prove (see [5], [6]) that integral solution is the same as the weak solution of (25.4) and (25.5).

Let Z_T^p, $p>1$, denote the space of all continuous adapted processes $X(t)=\begin{pmatrix} U(t) \\ v(t) \end{pmatrix}$ on $[0,T]$ with values on $H=\mathbb{R}^1\times L^2$ such that

$$\begin{aligned} \| \quad X\|_{Z_T^p} &= \|\begin{pmatrix} U \\ v \end{pmatrix}\|_T \\ &= (E(\sup_{t\in[0,T]}\mid U(t)\mid^p))^{1/p} + (E(\sup_{t\in[0,T]}\|v(t)\|^p))^{1/p} < \infty \end{aligned} \tag{25.10}$$

with the fixed initial conditions $U(0)=U_0$, $v(0)=v_0$. We define

$$\|\begin{pmatrix} U \\ v \end{pmatrix}\|_T = \|U\|_{1,T} + \|v\|_{2,T}. \tag{25.11}$$

Let $\pi_{n,1} : \mathbb{R}^1 \to B_1(0,n)$ be the projection onto the interval $B_1(0,n) = \{U \in \mathbb{R}^1 :\mid U \mid\leq n\}$ and let $\pi_{n,2} : L^2 \to B_2(0,n)$ be the projection onto the ball $B_2(0,n) = \{v \in L^2 : \|v\| \leq n\}$, where

$$\pi_{n,1}(U) = \begin{cases} U \text{ if } \mid U \mid\leq n, \\ \frac{nU}{|U|} \text{ if } \mid U \mid> n \end{cases} \text{ and } \pi_{n,2}(v) = \begin{cases} v \text{ if } \|v\| \leq n, \\ \frac{nv}{|v|} \text{ if } \|v\| > n \end{cases}. \tag{25.12}$$

25.3 Existence and uniqueness of solution

It was shown in [6], Theorem 1, that system (25.4) and (25.5) but without the term $g_0(U(t), v(t,x))dW_0(t)$ in (25.4), has a unique weak solution. In the present section we extend this result to the general system (25.4) and (25.5).

Theorem 25.1 *If the functions $g_0 : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ and $g_1 : \mathbb{R}^1 \times L^2 \to \mathbb{R}^1$ are bounded and Lipschitz continuous, then system (25.4) and (25.5) has a unique integral solution.*

To show the existence of local solution it is enough to prove the following.

Proposition 25.1 *For arbitrary $p > 4$, $T > 0$, and each $n = 1, 2, ...$ the following system of equations*

$$U(t) = e^{-\nu t}U_0 + \int_0^t e^{-\nu(t-s)}(P - \|\pi_{n,2}v(s)\|^2)ds \tag{25.13}$$
$$+ \int_0^t e^{-\nu(t-s)}g_0(U(s), v(s))dW_0(s),$$

$$v(t) = S(t)v_0 + \int_0^t S(t-s)\pi_{n,1}U\,(s)\,\pi_{n,2}v(s)ds \tag{25.14}$$
$$- \int_0^t S(t-s)\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2 ds + \int_0^t S(t-s)g_1(U(s), v(s))dW_1(s),$$

$t \in [0,T]$, *has a unique weak solution in the space Z_T^p.*

Proof of Proposition 25.1 Similarly as in [6] we introduce some nonlinear operators F_n, G, H_n, I_n, and additionally operator K acting on processes $U(t)$, $t \in [0,T]$, and $v(t)$, $t \in [0,T]$, according to the following formulae:

$$F_n(U,v)(t) = e^{-\nu t}U_0 + \int_0^t e^{-\nu(t-s)}(P - \|\pi_{n,2}v(s)\|^2)ds, \tag{25.15}$$
$$G(U,v)(t) = \int_0^t S(t-s)g_1(U(s), v(s))dW_1(s), \tag{25.16}$$
$$H_n(U,v)(t) = \int_0^t S(t-s)\frac{\partial}{\partial x}(\pi_{n,2}v(s))^2 ds, \tag{25.17}$$
$$I_n(U,v)(t) = S(t)v_0 + \int_0^t S(t-s)\pi_{n,1}U\,(s)\,\pi_{n,2}v(s)ds \tag{25.18}$$

and

$$K(U,v)(t) = \int_0^t e^{-\nu(t-s)}g_0(U(s), v(s))dW_0(s). \tag{25.19}$$

Observe that system (25.15)–(25.19) is equivalent to the fixed point problem

$$U = F_n(U,v) + K(U,v), \tag{25.20}$$

$$v = G(U,v) + H_n(U,v) + I_n(U,v). \tag{25.21}$$

We have to show that for arbitrary n the mapping

$$\begin{pmatrix} U \\ v \end{pmatrix} \to \begin{pmatrix} F_n(U,v) + K(U,v) \\ G(u,v) + H_n(U,v) + I_n(U,v) \end{pmatrix} \tag{25.22}$$

is a contraction in the space $Z^p_{T_n}$, for properly chosen T_n. We estimate only the new term. We get using the Burkholder inequality

$$\begin{aligned} \| \quad & K(U,v) - K(\overline{U},\overline{v})\|_{1,T} = [E \sup_{t\in[0,T]} \mid K(U,v)(t) - K(\overline{U},\overline{v})(t) \mid^p]^{\frac{1}{p}} \\ = \quad & [E \sup_{t\in[0,T]} \mid \int_0^t e^{-\nu(t-s)}(g_0(U(s),v(s)) - g_0(\overline{U}(s),\overline{v}(s)))dW_0(s) \mid^p]^{\frac{1}{p}} \\ \leq \quad & [(\frac{p}{p-1})^p E(\int_0^T \mid g_0(U(s),v(s)) - g_0(\overline{U}(s),\overline{v}(s)))ds \mid^2)^{\frac{p}{2}}]^{\frac{1}{p}} \\ \leq \quad & [(\frac{p}{p-1})^p E(\|g_0\|^2_{Lip} \mid \begin{pmatrix} U(s) \\ v(s) \end{pmatrix} - \begin{pmatrix} \overline{U}(s) \\ \overline{v}(s) \end{pmatrix} \mid^2 \int_0^T ds)^{\frac{p}{2}}]^{\frac{1}{p}} \\ \leq \quad & (\frac{p}{p-1})\|g_0\|^{\frac{2}{p}}_{Lip} T^{\frac{1}{2}}[E \sup_{s\in[0,T]} (\mid U(s) - \overline{U}(s) \mid^2 + \|v(s) - \overline{v}(s)\|^2)^{\frac{p}{2}}]^{\frac{1}{p}}. \end{aligned}$$

Further we get using the inequalities $(a+b)^{\frac{p}{2}} \leq 2^{\frac{p}{2}-1}(a^{\frac{p}{2}} + b^{\frac{p}{2}})$ for $a,b \geq 0$, and $(a+b)^\alpha \leq a^\alpha + b^\alpha$ for $a,b \geq 0$, $0 < \alpha \leq 1$:

$$\begin{aligned} \| \quad & K(U,v) - K(\overline{U},\overline{v})\|_{1,T} \leq 2^{\frac{p}{2}-1}(\frac{p}{p-1})\|g_0\|^{\frac{p}{2}}_{Lip} \\ \times T^{\frac{1}{2}}[E \sup_{s\in[0,T]} (\quad \mid \quad & U(s) - \overline{U}(s) \mid^p + \|v(s) - \overline{v}(s)\|^p)]^{\frac{1}{p}} \\ \leq \quad & 2^{\frac{p}{2}-1}(\frac{p}{p-1})\|g_0\|^{\frac{p}{2}}_{Lip} T^{\frac{1}{2}}\{[E \sup_{s\in[0,T]} (\mid U(s) - \overline{U}(s) \mid^p]^{\frac{1}{p}} \\ + [E \sup_{s\in[0,T]} \quad \| \quad & v(s) - \overline{v}(s)\|^p]^{\frac{1}{p}}\} \\ \leq \quad & 2^{\frac{p}{2}-1}(\frac{p}{p-1})\|g_0\|^{\frac{p}{2}}_{Lip} T^{\frac{1}{2}} \|\begin{pmatrix} U(s) \\ v(s) \end{pmatrix} - \begin{pmatrix} \overline{U}(s) \\ \overline{v}(s) \end{pmatrix}\|_T. \end{aligned} \tag{25.23}$$

We put

$$C^5_{T,n} = 2^{\frac{p}{2}-1}(\frac{p}{p-1})\|g_0\|^{\frac{p}{2}}_{Lip} T^{\frac{1}{2}}. \tag{25.24}$$

It was shown in [6] that the operators F_n, G, H_n, and I_n are Lipschitz transformations with the constants

$$\begin{aligned} C^1_{T,n} \quad &= \quad 2^{1-\frac{1}{p}} 2nT,\ C^2_{T,n} = T^{\frac{1}{p}}(\frac{p}{p-1})(\frac{\sin \pi\gamma}{\gamma})\frac{2}{\pi}\|g_1\|^2_{Lip}(a_\gamma)^{\frac{1}{2}}, \\ C^3_{T,n} \quad &= \quad 2CnT^{\frac{1}{4}},\ C^4_{T,n} = Tn2^{\frac{p-1}{p}}, \end{aligned}$$

respectively, where γ is any number from the interval $(\frac{1}{p}, \frac{1}{4})$ and

$$a_\gamma = \int_0^\infty s^{-2\gamma}(\sum_{k=1}^\infty e^{-2\frac{\pi^2}{\nu}k^2 s})ds.$$

It is clear that there exists T_n such that

$$C_n = \max\{C^i_{T_n,n},\ i = 1,...,5\} < 1. \tag{25.25}$$

By Banach fixed point theorem there exists a unique fixed point of operator (25.22) in the space $Z^p_{T_n},$; hence there exists a unique solution $\begin{pmatrix} U_n(t) \\ v_n(t) \end{pmatrix}$ to problems (25.13) and (25.14) on the interval $[0, T_n]$.

By a standard iteration procedure there exists a unique solution to problem (25.13) and (25.14) on arbitrary time interval $[0, T]$. $\square$

Now let $X_n(t) = \begin{pmatrix} U_n(t) \\ v_n(t) \end{pmatrix}$, $t \geq 0$, be the solution to problems (25.4) and (25.5). Define

$$\tau_n = \min[\inf\{t \geq 0 : | U_n(t) |^2 \geq n^2\}, \inf\{t \geq 0 : \|v_n(t)\|^2 \geq n^2\}]. \tag{25.26}$$

Notice that $X_n(t) = X_m(t)$ for $m \geq n$ and $t \leq \tau_n$. Therefore, we can set $X(t) = X_n(t)$ if $t \leq \tau_n$ and this is a solution to problems (25.4) and (25.5) on the time interval $[0, \tau_\infty)$, where $\tau_\infty = \lim_{n\to\infty} \tau_n$.

It is enough to show that $\tau_\infty = +\infty$. For this we modify the proof of Theorem 1 from [6]. Let us set

$$\begin{aligned} V_0(t) &= U(t) - Z_0(t), \quad U(t) = V_0(t) + Z_0(t), && (25.27) \\ V_1(t) &= v(t) - Z_1(t), \quad v(t) = V_1(t) + Z_1(t),\ t < \tau_\infty, && (25.28) \end{aligned}$$

where

$$\begin{aligned} Z_0(t) &= \int_0^t e^{-\nu(t-s)} g_0(U(s), v(s))\chi_{s<\tau_\infty} dW_0(s), \\ Z_1(t) &= \int_0^t e^{A(t-s)} g_1(U(s), v(s))\chi_{s<\tau_\infty} dW_1(s) \end{aligned}$$

and we assume that $\begin{pmatrix} U(t) \\ v(t) \end{pmatrix}$ is the solution to problems (25.4) and (25.5).

Proposition 25.2 *Assume that for deterministic functions V_0, $Z_0 \in C([0,T], \mathbb{R}^1)$, $V_1 \in C([0,T], L^2)$, and $Z_1 \in L^\infty([0,T], L^4(0,1))$ the following equations are satisfied in the weak sense:*

$$\frac{d}{dt} V_0(t) = P - \nu V_0(t) - \|V_1(t) + Z_1(t)\|^2, \tag{25.29}$$

$$\frac{\partial V_1(t)}{\partial t} = \nu \frac{\partial^2 V_1(t)}{\partial x^2} + (V_0(t) + Z_0(t))(V_1(t) + Z_1(t)) - \frac{\partial}{\partial x}(V_1(t) + Z_1(t))^2. \tag{25.30}$$

Then there exists a constant C such that for all $t \in [0,T]$

$$\|V_1(t)\|^2 + V_0^2(t) \leq e^{C(\mu+2)t}(V_1^2(0) + V_0^2(0) + (\mu+1)T], \tag{25.31}$$

where $\mu = \sup_{t\in[0,T]}(\|Z_1(t)\|^4_{L^4} + (Z_0(t))^4)$.

Proof of Proposition 25.2 First, our aim is to estimate V_1. We have

$$(\frac{\partial V_1}{\partial t}, V_1) = \nu(\frac{\partial^2 V_1}{\partial x^2}, V_1) + U(V_1 + Z_1, V_1) - (\frac{\partial}{\partial x}(V_1 + Z_1)^2, V_1)$$

and from this

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \quad \| \quad & V_1\|^2 + \nu \|V_1\|^2_{H^1_0} = U\|V_1\|^2 + U(V_1, Z_1) \\ & +2\int_0^1 V_1 Z_1 \frac{\partial V_1}{\partial x} dx + \int_0^1 Z_1^2 \frac{\partial V_1}{\partial x} dx. \end{aligned}$$

It is estimated in Lemma 2 in [6]

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|V_1\|^2+\nu\|V_1\|^2_{H^1_0}&\le U\|V_1\|^2+U(V_1,Z_1)\\ +2\{C[\frac{1}{4}\|V_1\|^2\|Z_1\|^4_{L^4}&+\frac{5}{8}\alpha^2\|V_1\|^2_{H^1_0}+\frac{1}{8\alpha^2}\|V_1\|^2]\}\\ +\frac{\varepsilon}{2}\|V_1\|^2_{H^1_0}&+\frac{1}{2\varepsilon}\|Z_1\|^4_{L^4}.\end{aligned}\tag{25.32}$$

Now we consider equation similarly obtained from (25.29)

$$\frac{1}{2}\frac{dV_0^2(t)}{dt}+\nu V_0^2(t)=V_0(t)(P-\|V_1(t)+Z_1(t)\|^2).\tag{25.33}$$

Adding (25.32) and (25.33) we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|V_1\|^2+V_0^2]&+\nu\|V_1\|^2_{H^1_0}+\nu V_0^2\\ &\le U\|V_1\|^2+U(V_1,Z_1)+2\{C[\frac{1}{4}\|V_1\|^2\|Z_1\|^4_{L^4}+\frac{5}{8}\alpha^2\|V_1\|^2_{H^1_0}\\ +\frac{1}{8\alpha^2}\|V_1\|^2]\}&+\frac{\varepsilon}{2}\|V_1\|^2_{H^1_0}+\frac{1}{2\varepsilon}\|Z_1\|^4_{L^4}+V_0(P-\|V_1+Z_1\|^2).\end{aligned}$$

But we have

$$\begin{aligned}U\|V_1\|^2&=(V_0+Z_0)\|V_1\|^2=V_0\|V_1\|^2+Z_0\|V_1\|^2,\\ U(V_1,Z_1)&=(V_0+Z_0)(V_1,Z_1)=V_0(V_1,Z_1)+Z_0(V_1,Z_1).\end{aligned}$$

Since

$$\begin{aligned}V_0(P-\|V_1+Z_1\|^2)&=V_0(P-\|V_1\|^2-2(V_1,Z_1)-\|Z_1\|^2)\\ &=V_0P-V_0\|V_1\|^2-2V_0(V_1,Z_1)-V_0\|Z_1\|^2,\end{aligned}$$

from the Young inequality we get

$$\begin{aligned}U\|V_1\|^2&+U(V_1,Z_1)+V_0P-V_0\|V_1\|^2-2V_0(V_1,Z_1)-V_0\|Z_1\|^2\\ &=Z_0\|V_1\|^2+V_0P-V_0(V_1,Z_1)+Z_0(V_1,Z_1)-V_0\|Z_1\|^2\\ &\le Z_0\|V_1\|^2+\frac{V_0^2}{2}+\frac{1}{2}P^2+\mid(V_1,Z_1V_0)\mid+\mid(V_1,Z_1Z_0)\mid\\ +\frac{V_0^2}{2}+\frac{1}{2}\|Z_1\|^4_{L^4}&\le Z_0\|V_1\|^2+\frac{V_0^2}{2}+\frac{1}{2}P^2+\|V_1\|\mid\mid V_0\mid\|Z_1\|\\ +\|V_1\|\mid\mid Z_0\mid\|Z_1\|&+\frac{V_0^2}{2}+\frac{1}{2}\|Z_1\|^4_{L^4}\le Z_0\|V_1\|^2+\frac{V_0^2}{2}+\frac{1}{2}P^2\\ +\frac{1}{2}\|V_1\|^2&+\frac{1}{2}V_0^2\|Z_1\|^2+\frac{1}{2}\|V_1\|^2+\frac{1}{2}Z_0^2\|Z_1\|^2+\frac{V_0^2}{2}+\frac{1}{2}\|Z_1\|^4_{L^4}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}[\|V_1\|^2+V_0^2]&\le\frac{1}{2}\frac{d}{dt}[\|V_1\|^2+V_0^2]+\nu\|V_1\|^2_{H^1_0}+\nu V_0^2\\ &\le Z_0\|V_1\|^2+\frac{V_0^2}{2}+\frac{1}{2}P^2+\frac{1}{2}\|V_1\|^2+\frac{1}{2}V_0^2\|Z_1\|^2+\frac{1}{2}\|V_1\|^2\\ +\frac{1}{2}Z_0^2\|Z_1\|^2&+\frac{V_0^2}{2}+\frac{1}{2}\|Z_1\|^4_{L^4}+2C\{\frac{1}{4}\|V_1\|^2\|Z_1\|^4_{L^4}+\frac{5}{8}\alpha^2\|V_1\|^2_{H^1_0}\\ +\frac{1}{8\alpha^2}\|V_1\|^2\}&+\frac{\varepsilon}{2}\|V_1\|^2_{H^1_0}+\frac{1}{2\varepsilon}\|Z_1\|^4_{L^4}.\end{aligned}$$

Now we choose α and ε to get $\nu = \frac{\varepsilon}{2} + \frac{5}{8} \cdot 2C\alpha^2$. But for arbitrary $Z_1 \in L^4$ we have

$$\|Z_1\| \leq \|Z_1\|_{L^4}, \qquad \|Z_1\|_{L^4}^2 \leq \|Z_1\|_{L^4}^4 + 1;$$

therefore, we arrive at

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}[\quad \| & \quad V_1\|^2 + V_0^2] \leq \|V_1\|^2[| \ Z_0 \ | + \frac{C}{2}\|Z_1\|_{L^4}^4 + \frac{1}{4\alpha^2}] \\ + V_0^2[1 + \frac{1}{2}(\quad \| & \quad Z_1\|_{L^4}^4 + 1)] + \frac{1}{2}P^2 + \frac{1}{4}Z_0^4 + \frac{1}{4}\|Z_1\|_{L^4}^4 + \frac{1}{2}\|Z_1\|_{L^4}^4 + \frac{1}{2}\|Z_1\|_{L^4}^4 \\ \leq & \quad C\{(\|V_1\|^2 + V_0^2)[| \ Z_0 \ | + \|Z_1\|_{L^4}^4 + 1] + (1 + Z_0^4 + \|Z_1\|_{L^4}^4)\}. \end{aligned}$$

From the Gronwall lemma we get
$\|V_1(t)\|^2 + V_0^2(t) \leq e^{2C\int_0^t(|Z_0(s)| + \|Z_1(s)\|_{L^4}^4 + 2)ds}[(V_1(0)^2 + V_0^2(0))$

$$+ \int_0^t [1 + (Z_0(t))^4 + \|Z_1(s)\|_{L^4}^4]ds.$$

Thus

$$\|V_1(t)\|^2 + V_0^2(t) \leq e^{C(\mu+2)t}[V_1^2(0) + V_0^2(0) + (\mu+1)t]. \qquad \Box$$

Proof of Theorem 25.1 Now let $X_n(t) = \begin{pmatrix} U_n(t) \\ v_n(t) \end{pmatrix}$ be a, possibly exploding, solution to problems (25.4) and (25.5), where $U_n(t)$ is the solution to (25.8) and $v_n(t)$ is the solution to (25.9).

By Proposition 5, p. 1658 in [6] we deduce that $E[\sup_{t\in[0,T]} \|Z_1(t)\|_{L^4}^4] < \infty$. By the Burkholder inequality we deduce that $E[\sup_{t\in[0,T]} | \ Z_0(t) \ |] < \infty$; therefore $E\mu < \infty$. From (25.31) there exists a constant $C_1 \geq 1$ such that

$$\begin{aligned} \| & \quad V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1 \leq C_1[V_1^2(0) + V_0^2(0) + (\mu+1)T]e^{(C\mu+2)t} + 1 \\ \leq & \quad C_1[V_1^2(0) + V_0^2(0) + (\mu+1)T + 1]e^{(C\mu+2)t} \end{aligned}$$

so

$$\begin{aligned} \log (\quad \| & \quad V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1) \\ \leq & \quad \log C_1 + \log[V_1^2(0) + V_0^2(0) + (\mu+1)T + 1] + C(\mu+2)T. \end{aligned}$$

By Jensen inequality it follows that

$$\begin{aligned} E[\sup_{t\in[0,T]} \log(\quad \| & \quad V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1)] \\ \leq & \quad E[\log \sup_{t\leq T}(\|V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1)] \leq \log C_1 \\ & \quad + \log[V_1^2(0) + V_0^2(0) + E(\mu+1)T + 1] + C(E\mu+2)T \\ = & \quad K_T. \end{aligned}$$

Since by the Chebyshev inequality

$$\begin{aligned} P(\tau_n \quad \leq & \quad T) = P(\sup_{t\in[0,T]} \log (\|V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1) \geq \log (n+1)) \\ \leq & \quad \frac{E(\sup_{t\in[0,T]} \log (\|V_1(t)\|^2 + | \ V_0(t) \ |^2 + 1)}{\log(n+1)} \end{aligned}$$

we get, for a constant K'_T

$$P(\tau_n \leq T) \leq \frac{K'_T}{\log (n+1)} \to 0 \qquad (25.34)$$

as $n \to \infty$. Hence $\tau_\infty = \infty$. $\Box$

25.4 Irreducibility

Our aim is to prove the following irreducibility result for solutions of the Burgers' system (25.4) and (25.5).

Theorem 25.2 *Assume that functions* $g_0 : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ *and* $g_1 : \mathbb{R}^1 \times L^2 \to \mathbb{R}^1$ *are bounded and Lipschitz continuous; and, moreover, there exists a constant* $C > 0$ *such that*

$$| g_0(x,y) | \geq C \text{ for } (x,y) \in \mathbb{R}^2 \text{ and } | g_1(x,v) \geq C \text{ for } x \in \mathbb{R}^1, v \in L^2.$$

Then the semigroup corresponding to system (25.4) and (25.5) is irreducible.

As in [4] and [7], the proof will be a combination of control theoretic and stochastic methods.

We start from results on irreducibility for general equations

$$dX = [AX + F(X)]dt + G(X)dW(t), \quad X(0) = x, \; x \in H. \tag{25.35}$$

Proposition 25.3 *Assume that for arbitrary* $x \in H$ *equation (25.35) has a unique weak solution, the operators* $G(x) \in L(H,H)$ *are invertible, and*

$$\sup_x \|G^{-1}(x)\|_{L(H,H)} < +\infty.$$

Assume, moreover, that for arbitrary $T > 0$, $x, y \in H$, $r > 0$ *there exists a bounded adapted process* $\varphi(t)$, $t \in [0,T]$ *such that for the solution* Y *of*

$$dY = [AY + F(Y) + \varphi(t)]dt + G(Y)dW(t), \quad Y(0) = x, \tag{25.36}$$

one has

$$P(\|Y(T) - y\| < r) > 0.$$

Then the semigroup $P(t)$, $t \geq 0$, *corresponding to (25.35) is irreducible.*

Proof Define

$$\alpha(t) = -G^{-1}(Y(t))\varphi(t), \; t \geq 0,$$

and a new probability measure P^* such that

$$dP^* = e^{\int_0^T (\alpha(s), dW(s)) - \frac{1}{2}\int_0^T |\alpha(s)|^2 ds} dP.$$

Then

$$W^*(t) = W(t) - \int_0^t \alpha(s)ds, \; t \geq 0$$

is a P^* Wiener process and P^* is equivalent to P. Notice that

$$\begin{aligned} dY(t) &= [AY + F(Y) + \varphi(t)]dt + G(Y(t))dW^*(t) + G(Y(t))\alpha(t)dt \\ &= [AY + F(Y) + \varphi(t)]dt + G(Y(t))dW^*(t) \\ &\quad - G(Y(t))G^{-1}(Y(t))\varphi(t)dt \\ &= [AY(t) + F(Y))dt + G(Y(t))dW^*(t). \end{aligned}$$

By our assumptions the law of Y under P^* and the law of X under P are identical and the result follows. □

We shall also need some results on controllability of the following deterministic system:

$$y' = Ay + u, \quad y(0) = a \in H, \tag{25.37}$$

on a Hilbert space H, where A generates a C_0-semigroup $S(t)$, $t \geq 0$, such that $\|S(t)\|_{L(H,H)} \leq Me^{\omega t}$, $\omega \geq 0$, $M > 0$.

Lemma 25.1 *(i) Assume that $b \in D(A)$. For arbitrary $a \in H$ and $T > 0$ the control*

$$u(t) = \frac{1}{T}S(t)(b-a) - Ab, \; t \in [0,T] \tag{25.38}$$

steers a to b and the following formulae hold:

$$y(t) \;\; = \;\; b + (\frac{t}{T} - 1)S(t)(b-a), \tag{25.39}$$

$$\int_0^t S(t-\sigma)u(\sigma)d\sigma \;\; = \;\; -\frac{t}{T}S(t)a + (\frac{t}{T} - 1)S(t)b + b, \; t \in [0,T]. \tag{25.40}$$

(ii) Assume in addition that for some $c > 0$, $\|\sigma AS(\sigma)\|_H \leq c$, $\sigma > 0$. Then

$$\sup_{0\leq t\leq T} \| \; A[\int_0^t S(t-\sigma)u(\sigma)d\sigma\|_H \tag{25.41}$$

$$\leq \;\; c\|S(T)a\|_H + 3Me^{\omega T}\|Ab\|_H \leq Me^{\omega T}(c\|a\|_H + 3\|Ab\|_H).$$

Proof The result follows by direct computation:

$$\begin{aligned} y(t) &= S(t)a + \int_0^t S(t-\sigma)[\frac{1}{T}S(\sigma)(b-a) - Ab]d\sigma \\ &= S(t)a + \frac{1}{T}\int_0^t S(t)(b-a)d\sigma - \int_0^t \frac{d}{d\sigma}S(\sigma)bd\sigma \\ &= S(t)a + \frac{t}{T}S(t)(b-a) - S(T)b + b = b + (\frac{t}{T} - 1)S(t)(b-a). \end{aligned}$$

If $b \in D(A)$, then for the control u given by (25.38)

$$\begin{aligned} \sup_{0\leq t\leq T} \| \; & A[\int_0^t S(t-\sigma)u(\sigma)d\sigma\|_H \\ \leq \; & \sup_{0\leq t\leq T} \| \mid \frac{t}{T}AS(t)a\|_H + 3 \sup_{0\leq t\leq T}\|S(t)Ab\|_H \\ \leq \; & \sup_{0\leq t\leq T} \|\frac{t}{T}AS(\frac{t}{T})S(T)a\|_H + 3 \sup_{0\leq t\leq T}\|S(t)Ab\|_H. \end{aligned}$$

If $S_1(t)$, $t \geq 0$ is the heat semigroup on L^2 corresponding to the Dirichlet boundary condition, then $\omega = 0$ and $M = 1$ and the domain of its infinitesimal generator A_1 is

$$D(A) = \{x \in L^2 : x, x' \text{ are absolutely continuous, } x'' \in L^2, x(0) = x(1) = 0\}.$$

One checks easily that for arbitrary $p \geq 1$ and $x \in D(A_1)$

$$\|x\|_{L^p(0,1)} \leq \|x\|_{L^\infty(0,1)} \leq 2\|A_1\|. \tag{25.42}$$

Since for some $c > 0$, $\|\sigma A_1 S(\sigma)\| \leq c$, $\sigma > 0$, we have the following proposition.

Proposition 25.4 *Let $S(\cdot)$ be a C_0 semigroup on $H = \mathbb{R}^1 \times L^2$ acting on the first coordinate as multiplication by $S_0(t) = e^{\nu t}$ and on the second coordinate as the heat semigroup $S_1(t)$, and let $u(t) = (u_0(t), u_1(t))$ be the control from the proposition which transfers state $a = (a_0, a_1)$ to $b = (b_0, b_1)$, $b_1 \in D(A_1)$. Then*

$$\sup_{0\leq t\leq T} \|\int_0^t S_1(t-\sigma)u_1(\sigma)d\sigma\|_{L^4} \tag{25.43}$$

$$\leq 2\sup_{0\leq t\leq T}\|A_1\int_0^t S_1(t-\sigma)u_1(\sigma)d\sigma\| \leq 2(c\|a_1\| + 3\|A_1 b_1\|$$

and

$$\sup_{0\le t\le T} \| \int_0^t S_0(t-\sigma)u_0(\sigma)d\sigma\|_{L^4} \le 2(c \mid a_0 \mid +3\nu \mid b_0 \mid). \tag{25.44}$$

Denote the control u described in Proposition 25.4 by

$$v(t,T;a,b),\ t\in[0,T],\ a\in H,\ b=(b_0,b_1),\ b_1\in D(A).$$

Corollary 25.1 *For arbitrary $R>0$ and $T>0$*

$$\sup_{t\le T}\sup_{\|a\|\le R}\|v(t,T;a,b)\|<+\infty.$$

The proof of the following proposition is the same as Proposition 5, p. 1658 in [6].

Proposition 25.5 *Assume that $S_1(t), t\ge 0$, is a heat semigroup on L^2, ψ is an adapted $L(L^2,L^2)$-valued process and W_1 is a cylindrical Wiener process on L^2. Then there exists a constant $c>0$ such that for all $T\ge 0$*

$$E(\sup_{t\le T}\| \int_0^t S_1(t-s)\psi(s)dW_1(s)\|^4_{L^4(0,1)} \le cE(\int_0^T \|\psi(s)\|^4_{L(L^2,L^2)}ds). \tag{25.45}$$

Proof of Theorem 25.2 We use Proposition 25.2. In the present situation

$$\begin{aligned}
X &= \begin{pmatrix} U \\ v \end{pmatrix},\ F\begin{pmatrix} U \\ v \end{pmatrix} = \begin{pmatrix} F_0(U,v) \\ F_1(U,v) \end{pmatrix} \\
S(t) &= \begin{pmatrix} e^{-\nu t} & 0 \\ 0 & S_1(t) \end{pmatrix},\ S_1(t),\ t\ge 0, \text{ is the heat semigroup,} \\
F_0(U,v) &= P-\|v\|^2,\ F_1(U,v)=Uv-\frac{\partial}{\partial x}(v^2),
\end{aligned}$$

where $-\frac{\partial}{\partial x}(v^2)$ Burgers' nonlinearity.

We show that for arbitrary $T>0$, $r>0$, and $x,y\in H$ there exists a uniformly bounded process φ with values in H such that for the solution Y of equation (25.36) we have $P(\|Y(T)-y\|_H<r)>0$. Since the set $D(A)$ is dense in H one can assume that $y\in D(A)$.

For each $s\in[0,T]$ and $R>0$ we define

$$\varphi_{s,R}(t)=\begin{cases} 0 \text{ if } \|X(s)\|_H>R, \\ v(t-s,T-s;x(s),y) \text{ if } t\in[s,T]. \end{cases}$$

Then $\varphi_{s,R}$ is a uniformly bounded, adapted process and for $t\in[s,T]$

$$\begin{aligned}
Y(t) &= S(t-s)X(s)+\int_s^t S(t-\sigma)F(Y(\sigma))d\sigma \\
&\quad +\int_s^t S(t-\sigma)\varphi_{s,R}(\sigma)d\sigma+\int_s^t S(t-\sigma)G(Y(\sigma))dW(\sigma).
\end{aligned} \tag{25.46}$$

In particular

$$\begin{aligned}
Y(T) &= S(T-s)X(s)+\int_s^T S(T-\sigma)F(Y(\sigma))d\sigma \\
&\quad +\int_s^T S(T-\sigma)\varphi_{s,R}(\sigma)d\sigma+\int_s^T S(T-\sigma)G(Y(\sigma))dW(\sigma).
\end{aligned}$$

Thus if

$$\|X(s)\|_H \le R, \tag{25.47}$$

then

$$Y(T) = y + \int_s^T S(T-\sigma)F(Y(\sigma))d\sigma + \int_s^T S(T-\sigma)G(Y(\sigma))dW(\sigma). \tag{25.48}$$

By taking R sufficiently large the event (25.47) holds for all $s \in [0,T]$ with probability arbitrarily close to 1. Moreover, if s is sufficiently close to T, then the stochastic integral in (25.48) can be smaller in norm than a given number with probability arbitrarily close to 1. One has to show that one can find $s \in [0,T]$ such that with probability close to 1 the following events:

$$\| \int_s^T S(T-\sigma)F(Y(\sigma))d\sigma\|_H < \frac{r}{2}$$

hold.

Denote the first and second coordinates of the process Y by Y_0 and Y_1. Then the first and the second coordinates of the stochastic integral

$$\int_s^T S(T-\sigma)F(Y(\sigma))d\sigma$$

are equal, respectively

$$I_1 = \int_s^t e^{-\nu(t-s)}(P - \|Y_1(\sigma)\|)d\sigma$$

and

$$I_2 + I_3 = \int_s^t S_1(T-\sigma)Y_0(\sigma)Y_1(\sigma)d\sigma + \int_s^t S_1(T-\sigma)\frac{\partial}{\partial x}(Y_1(\sigma))^2 d\sigma.$$

Note that

$$\|I_2\| \le (T-s)(\sup_{s\le\sigma\le T} \mid Y_0(\sigma) \mid)(\sup_{s\le\sigma\le T} \|Y_1(\sigma)\|).$$

Moreover, from the definition of the process Y_0

$$\begin{aligned} \sup_{s\le t\le T} \quad &\mid \quad Y_0(t) \mid \le \mid Y_0(s) \mid + (T-s)(\sup_{s\le t\le T} \|Y_1(t)\|) \\ + \sup_{s\le t\le T} \quad &\mid \quad \int_s^t e^{-\nu(t-s)} g_0(Y(\sigma))dW_0(\sigma) \mid . \end{aligned} \tag{25.49}$$

By Burkholder inequality the stochastic integral in the above expression is bounded by a sufficiently large number with probability close to 1. Moreover, by Proposition 5 in Appendix in [6]:

$$\begin{aligned} &\| \quad \int_s^T S_1(T-\sigma)\frac{\partial}{\partial x}(v^2)(Y_1(\sigma))d\sigma\| \le \\ &\le \quad \int_s^T \|S_1(T-\sigma)\frac{\partial}{\partial x}(v^2)(Y_1(\sigma))\|d\sigma \le C(T-s)^{\frac{1}{4}} \sup_{s\le\sigma\le T} \|Y_1(\sigma)\|^2. \end{aligned} \tag{25.50}$$

Taking into account the formula for I_1 and the obtained estimate for I_2, it is enough to show that with a probability arbitrarily close to 1

$$\sup_{s\le\sigma\le T} \|Y(\sigma)\|^2 \tag{25.51}$$

is bounded by a deterministic number. To show this, fix $s \in [0,T]$ and introduce processes Z_0, Z_1 on $[s,T]$ given by

$$\begin{aligned} Z_0(t) &= \int_s^t S_0(t-\sigma)\varphi^0_{s,R}(\sigma)d\sigma + \int_s^t S_0(t-\sigma)g_0(Y(\sigma))dW_0(\sigma), \\ Z_1(t) &= \int_s^t S_1(t-\sigma)\varphi^1_{s,R}(\sigma)d\sigma + \int_s^t S_1(t-\sigma)g_1(Y(\sigma))dW_1(\sigma), \end{aligned}$$

where $\varphi^0_{s,R}(\sigma)$, $\varphi^1_{s,R}(\sigma)$ are the first and the second coordinates of $\varphi_{s,R}(\sigma)$. Then

$$\begin{aligned} Y(t) &= S_1(t-s)Y_1(s) + \int_s^t S_1(T-\sigma)Y_0(\sigma)Y_1(\sigma)d\sigma \\ &\quad - \int_s^t S_1(T-\sigma)\frac{\partial}{\partial x}(Y_1(\sigma))^2 d\sigma + Z_1(t) \end{aligned}$$

and the processes

$$V_0(t) = Y_0(t) - Z_0(t),\ V_1(t) = Y_1(t) - Z_1(t),\ t \in [s,T]$$

satisfy conditions of Proposition 25.2. Consequently

$$\|V_1(t)\|^2 + V_0^2(t) \le e^{C(\mu_s+2)t}(V_1^2(s) + V_0^2(s) + C(T-s)(\mu_s+1)], \tag{25.52}$$

where $\mu_s = \sup_{t\in[s,T]}(\|Z_1(s)\|^4_{L^4} + (Z_0(s))^4)$.

From the estimates on the steering control from Proposition 25.4 on the supremum of the stochastic convolution given by Proposition 25.5 the random variable μ_s can be estimated by a sufficiently large constant uniformly in $u \in [s,T]$ with probability arbitrarily close to 1. Taking into account that $Y_1(t) = V_1(t) + Z(t)$, $t \in [s,T[$, the required estimate by a large constant of (25.51) with a probability close to 1 follows. This finishes the proof of the irreducibility. □

References

[1] J.M. Burgers, *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion,* Verh. Kon. Nerderl. Akad. Weten-Schappen Amsterdam, Afdeel Natuurkunde, 17, No. 2 (1939), 1-53.

[2] T. Dłotko , *The one-dimensional Burgers' equation; existence, uniqueness and stability*, Zeszyty Naukowe UJ, Prace Mat., 23 (1982), 157-172.

[3] G. Da Prato and D. Gątarek, *Stochastic Burgers' equation with correlated noise*, Stoch. Stoch. Rep., 52 (1995), 29-41.

[4] S. Peszat, J. Zabczyk, *Strong Feller property and irreducibility for diffusions on Hilbert spaces,* Ann. Probab., 23, No. 1 (1995), 157-172.

[5] K. Twardowska, J. Zabczyk, *A note on stochastic Burgers' system of equations,* Preprint No. 646, Polish Academy of Sciences, Inst. of Math., Warsaw, 2003, 1-32.

[6] K. Twardowska, J. Zabczyk, *A note on stochastic Burgers' system of equations,* Stoch. Anal. Appl., 22, No. 6 (2004), 1641-1670.

[7] K. Twardowska, J. Zabczyk, *On invariant measures for Burgers' system of equations*, in preparation.

26 On the Stochastic Fubini Theorem in Infinite Dimensions

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26.1 Introduction

The stochastic Fubini theorem and stochastic processes indexed by a parameter have been studied by many authors, cf. [1, 4, 5, 8, 11, 18, 21]. A general version of the stochastic Fubini theorem, valid for real-valued semimartingales as integrators, is due to Doléans-Dade [8] and Jacod [11, Théorème 5.44]. Roughly speaking it can be formulated as follows. Let (S, Σ, μ) be a σ-finite measure space, let $(\Omega, \mathrm{F}, \mathbb{P})$ be a probability space, and let $\phi : S \times [0,T] \times \Omega \to \mathbb{R}$ be $\Sigma \otimes \mathrm{B}([0,T]) \otimes \mathrm{F}$-measurable. If $(t, \omega) \mapsto \phi_s(t,\omega) := \phi(s,t,\omega)$ is integrable with respect to a semimartingale X for all $s \in S$, if the process $(t,\omega) \mapsto \int_S \phi_s(t)\, d\mu(s)$ is well defined and integrable with respect to X, and if

$$\int_S \Big| \int_0^T \phi_s(t)\, dX(t) \Big|\, d\mu(s) < \infty \quad \text{almost surely} \tag{26.1}$$

then, almost surely,

$$\int_S \int_0^T \phi_s(t)\, dX(t)\, d\mu(s) = \int_0^T \int_S \phi_s(t)\, d\mu(s)\, dX(t).$$

Motivated by applications to stochastic differential equations in infinite dimensions, it is desirable to have a version of the stochastic Fubini theorem for integrals of operator-valued processes with respect to cylindrical Hilbert space-valued semimartingales. Generalizing an earlier result of Chojnowska-Michalik [4], a stochastic Fubini theorem for $\mathrm{L}(H,H')$-valued processes with respect to H-cylindrical Brownian motions W_H was proved by Da Prato and Zabczyk [5]. Here H and H' are separable real Hilbert spaces. In this result the condition (26.1) is replaced by the condition

$$\phi \in L^1(S; L^2((0,T) \times \Omega; S; \mathrm{L}_2(H,H'))), \tag{26.2}$$

where $\mathrm{L}_2(H,H')$ denotes the Hilbert–Schmidt operators from H into H'.

The purpose of this chapter is to prove a stochastic Fubini theorem for integration of $\mathrm{L}(H,E)$-valued processes with respect H-cylindrical Brownian motions under assumptions analogous to (26.1) but which may be easier to verify in concrete applications. Here, E is assumed to be a real Banach space. Since the special case $E = \mathbb{R}$ already exhibits all main ideas, we have written our results in detail for H-valued processes only; here we identify

*Support came from the VIDI subsidie 639.032.201 in the Vernieuwingsimpuls programme of the Netherlands Organization for Scientific Research (NWO) and the Research Training Network HPRN-CT-2002-00281.

$\mathrm{L}(H,\mathbb{R})$ with H. The extension to $\mathrm{L}(H,E)$-valued processes is sketched at the end of the chapter. It turns out that condition (26.2) may be weakened to

$$\phi \in L^1(S; L^2(0,T;\mathrm{L}_2(H,H'))) \text{ almost surely.}$$

Our approach to the stochastic Fubini theorem is based on a straightforward extension of the theory of stochastic integration developed recently by Lutz Weis and the authors in [14] to processes with values in a UMD$^-$ space, together with the basic fact that L^1-spaces possess the UMD$^-$ property. The idea is to interpret the stochastic integral parametrized by S as a stochastic integral in the Banach space $L^1(S)$. The essence of the stochastic Fubini theorem is then nothing but the statement that a bounded linear functional may be moved into the stochastic integral

$$\begin{aligned}\int_S \int_0^T \phi_s(t)\,dW_H(t)\,d\mu(s) &= \Big\langle \int_0^T \phi_{(\cdot)}(t)\,dW_H(t), \mathbf{1}\Big\rangle_{\langle L^1(S),L^\infty(S)\rangle} \\ &= \int_0^T \langle \phi_{(\cdot)}(t), \mathbf{1}\rangle_{\langle L^1(S;H),L^\infty(S)\rangle}\,dW_H(t) = \int_0^T \int_S \phi_s(t)\,d\mu(s)\,dW_H(t).\end{aligned}$$

In order to develop this simple idea in a rigorous way, some measurability problems have to be overcome. The main difficulty consists of lifting measurability properties of ϕ that hold pointwise in s to the corresponding $L^1(S)$-valued functions. This problem is discussed in Section 26.2. The main results of the chapter are contained in Section 26.3.

In a forthcoming paper, the results of this chapter will be applied to study stochastic evolution equations.

26.2 Measure theoretical preliminaries

Let (S,Σ) be a measurable space and let (Y,d) be a complete metric space. A function $\phi: S\to Y$ is called Σ-*simple* if it is of the form $\phi = \sum_{n=1}^N \mathbf{1}_{A_n}\otimes y_n$ with $A_n\in\Sigma$ and $y_n\in Y$ for $n=1,\dots,N$. countably valued Σ-simple functions are defined similarly. A function $\phi: S\to Y$ is called *strongly* Σ-*measurable* if it is the pointwise limit in Y of a sequence of Σ-simple functions. It is wellknown [17, Lemma V-2-4] that a function $\phi: S\to Y$ is strongly Σ-measurable if and only if the following two conditions are satisfied:

(i) The range of ϕ is separable.

(ii) We have $\phi^{-1}(B)\in\Sigma$ for all Borel sets B in Y.

This implies that the pointwise limit of a sequence of strongly M-measurable functions is strongly M-measurable again.

By covering the range of a strongly M-measurable function ϕ with countably many balls B_j^n with radius $\frac{1}{n}$ and center y_j^n, and defining ϕ_n to have the constant value y_j^n on the set $\phi^{-1}(B_k^n\setminus\bigcup_{j<k}B_j^n)$ we obtain a countably valued Σ-simple function $\phi_n: S\to Y$ such that $\sup_{s\in S} d(\phi_n(s),\phi(s))\leqslant \frac{1}{n}$. Thus every strongly Σ-measurable function is the uniform limit of a sequence of countably valued Σ-simple functions.

As was mentioned in the introduction, it will be important to lift measurability properties of a process indexed by a parameter s to the corresponding $L^1(S)$-valued process. This problem is easily reduced to the following abstract question:

> If (Ω,F) is a measurable space, E is a Banach space, and $\phi: S\times\Omega\to E$ is a $\Sigma\otimes\mathrm{F}$-measurable function with the property that all sections ϕ_s are strongly G-measurable, where G is some sub-σ-algebra of F, does it follows that ϕ is strongly $\Sigma\otimes\mathrm{G}$-measurable?

In general the answer is negative even for indicator functions (cf. the example below). On the other hand, the answer is almost positive if $(\Omega, \mathrm{F}, \nu)$ is a σ-finite measure space, in the sense that ϕ has a modification which does have the required properties. We call two functions $\phi : S \times \Omega \to E$ and $\tilde{\phi} : S \times \Omega \to E$ *modifications* of each other if for all $s \in S$ we have $\phi(s, \omega) = \tilde{\phi}(s, \omega)$ for ν-almost all $\omega \in \Omega$.

Proposition 26.1 *Let (S, Σ) and $(\Omega, \mathrm{F}, \nu)$ be as above, let G be a sub-σ-algebra of F, and let E be a Banach space. If $\phi : S \times \Omega \to E$ is a strongly $\Sigma \otimes \mathrm{F}$-measurable function with the property that for all $s \in S$ the function ϕ_s is strongly G-measurable, then ϕ admits a strongly $\Sigma \otimes \mathrm{G}$-measurable modification.*

We may assume without loss of generality that $\nu(\Omega) < \infty$. In fact, if $\nu(\Omega) = \infty$, pick sets $\Omega_n \in \mathrm{F}$ such that $\mu(\Omega_n) > 0$ is strictly increasing with n and $\bigcup_{n\geqslant 1} \Omega_n = \Omega$. Put $Y_1 := \Omega_1$ and $Y_{n+1} = \Omega_{n+1} \setminus \Omega_n$ for $n \geqslant 1$, and define

$$\tilde{\nu}(A) := \sum_{n\geqslant 1} \frac{1}{2^n} \frac{\nu(A \cap Y_n)}{\nu(Y_n)}, \quad A \in \mathrm{F}.$$

Then $\tilde{\nu}$ is a probability measure on (Ω, F) which has the same null sets as ν, and therefore we may replace ν by $\tilde{\nu}$ in Proposition 26.1.

From now on we assume that $\nu(\Omega) < \infty$. We denote by $L^0(\Omega; E)$ the space of strongly F-measurable functions, identifying functions that are equal ν-almost everywhere. This is a complete metric space with respect to the translation invariant metric $\|\cdot\|_0$ defined by

$$\|f\|_0 := \int_\Omega \|f(\omega)\| \wedge 1 \, d\nu(\omega).$$

A sequence in $L^0(\Omega; E)$ converges in the metric $\|\cdot\|_0$ if and only if it converges in ν-measure. If G is sub-σ-algebra of F, we denote by $L^0(\Omega, \mathrm{G}; E)$ the closed subspace of $L^0(\Omega; E)$ consisting of all strongly G-measurable functions, identifying again functions that are equal ν-almost everywhere.

For a sequence $(f_n)_{n\geqslant 1}$ in $L^0(\Omega; E)$ and $a := (a_n)_{n\geqslant 1} \in l^1$, we make the following observation: if $\|f_n\|_0 \leqslant a_n$ for all $n \geqslant 1$, then $\lim_{n\to\infty} f_n = 0$ ν-almost everywhere. Indeed, define $g : \Omega \to [0, \infty]$ by $g(\omega) := \sum_{n\geqslant 1} \|f_n(\omega)\| \wedge 1$. We have

$$\int_\Omega g(\omega) \, d\nu(\omega) = \sum_{n\geqslant 1} \|f_n\|_0 = \|a\|_{l^1} < \infty.$$

Hence g is ν-almost everywhere finite and the claim follows. The proof of the proposition follows the proof of the celebrated result of Dellacherie and Meyer on the existence of a progressively measurable version of adapted measurable processes [6, Theorem IV.30] with some simplifications due to the absence of a filtration, and is included for the reader's convenience.

Proof of Proposition 26.1 Assume that $\nu(\Omega) < \infty$. It follows from the Fubini theorem that for all $s \in S$, $\phi(s, \cdot)$ is a strongly F-measurable function, so we may define $\psi : S \to L^0(\Omega; E)$ as $(\psi(s))(\omega) := \phi(s, \omega)$. We claim that ψ is strongly Σ-measurable. By a monotone class argument we can find a sequence of $\Sigma \otimes \mathrm{F}$-simple functions $\phi_n : S \times \Omega \to E$, each of which is a finite linear combination of functions of the form $\mathbf{1}_{A\times F} \otimes x$ with $A \in \Sigma$, $F \in \mathrm{F}$, $x \in E$, such that $\phi = \lim_{n\to\infty} \phi_n$ pointwise on $S \times \Omega$. Define $\psi_n : S \to L^0(\Omega; E)$ as $(\psi_n(s))(\omega) := \phi_n(s, \omega)$. Then each ψ_n is a Σ-simple function and for all $s \in S$ we have $\psi(s) = \lim_{n\to\infty} \psi_n(s)$ in $L^0(\Omega; E)$. This proves the claim.

Choose a sequence of countably valued Σ-simple functions $\eta_n : S \to L^0(\Omega; E)$, say

$$\eta_n(s) = \sum_k \mathbf{1}_{A_k^n}(s) h_k^n$$

with $A_k^n \in \Sigma$ and $h_k^n : \Omega \to E$ strongly F-measurable, such that for all $s \in S$ we have

$$\|\psi(s) - \eta_n(s)\|_0 \leqslant 2^{-n}.$$

For $n, k \geqslant 1$ let $s_k^n \in A_k^n$ be arbitrary and fixed. Then $\|\psi(s_k^n) - h_k^n\|_0 \leqslant 2^{-n}$. Put

$$\tilde{\phi}_n(s,\omega) := \sum_k \mathbf{1}_{A_k^n}(s)\phi(s_k^n, \omega).$$

By the G-measurability assumption on the sections of ϕ, we obtain a countably valued Σ-simple function $\tilde{\psi}_n : S \to L^0(\Omega, \mathrm{G}; E)$ by

$$(\tilde{\psi}_n(s))(\omega) := \tilde{\phi}_n(s,\omega),$$

and for all $s \in S$ we have

$$\|\psi(s) - \tilde{\psi}_n(s)\|_0 \leqslant \|\psi(s) - \eta_n(s)\|_0 + \|\eta_n(s) - \tilde{\psi}_n(s)\|_0 \leqslant 2^{-n+1}.$$

By the observation preceding the proof, for all $s \in S$ we have

$$\phi(s,\omega) = (\psi(s))(\omega) = \lim_{n\to\infty} (\tilde{\psi}_n(s))(\omega) = \lim_{n\to\infty} \tilde{\phi}_n(s,\omega) \quad \text{for } \nu\text{-almost all } \omega \in \Omega.$$

Let C be the set of all $(s,\omega) \in S \times \Omega$ for which the sequence $(\tilde{\phi}_n(s,\omega))$ converges. Then the function

$$\tilde{\phi}(s,\omega) := \lim_{n\to\infty} \mathbf{1}_C(s,\omega)\tilde{\phi}_n(s,\omega).$$

is a $\Sigma \otimes \mathrm{G}$-measurable modification of ϕ. $\square$

The following example was communicated to us by Klaas Pieter Hart. It shows that in general the strong G-measurability of the sections ϕ_s of a jointly measurable function ϕ does not imply the strong $\Sigma \otimes \mathrm{G}$-measurability of ϕ.

Example 26.1 Let $(S,\Sigma) = (\Omega, \mathrm{F}) = (\omega_1, \mathrm{P})$, where ω_1 is the first uncountable ordinal and $\mathrm{P} = \mathrm{P}(\omega_1)$ is its power set. Let G be the sub-σ-algebra of P consisting of all sets that are either countable or have countable complement. Let

$$A := \{(\alpha, \beta) \in \omega_1 \times \omega_1 : \ \alpha < \beta\}.$$

It is well known that $\mathrm{P} \otimes \mathrm{P} = \mathrm{P}(\omega_1 \times \omega_1)$ [20], see also [12, Theorem 12.5], and therefore $A \in \mathrm{P} \otimes \mathrm{P}$. Moreover, for all $\alpha \in \omega_1$ the section $A_\alpha := \{\beta \in \omega_1 : (\alpha,\beta) \in A\}$ belongs to G. We will show that $A \notin \mathrm{P} \otimes \mathrm{G}$. The example announced above is obtained by taking for ϕ the indicator function of A.

Define an increasing family of collections of subsets $(\mathrm{C}_\beta)_{\beta\in\omega_1}$ as follows. Let C_0 denote the collection of all measurable rectangles in $\mathrm{P} \otimes \mathrm{G}$. If $\beta \in \omega_1$ is a successor ordinal, say $\beta = \alpha + 1$, let C_β be the collection of all sets obtained from C_α by taking complements, intersections, and countable unions. If $\beta \in \omega_1$ is a limit ordinal, let $\mathrm{C}_\beta := \bigcup_{\alpha<\beta} \mathrm{C}_\alpha$. Note that $\mathrm{P} \otimes \mathrm{G} = \bigcup_{\beta\in\omega_1} \mathrm{C}_\beta$. With induction on β it is seen that every $C \in \mathrm{C}_\beta$ belongs to a σ-algebra generated by a countable family of measurable rectangles in $\mathrm{P} \otimes \mathrm{G}$.

Suppose now, for a contradiction, that $A \in \mathrm{P} \otimes \mathrm{G}$. Since there is a first ordinal $\beta \in \omega_1$ such that $A \in \mathrm{C}_\beta$, there exists a countable collection of measurable rectangles $P_n \times G_n \in \mathrm{P}\otimes\mathrm{G}$ such that $A \in \mathrm{P}_A := \sigma(P_n \times G_n;\ n = 1, 2, \ldots)$. Choose $\alpha_1 \in \omega_1$ such that $G_n \subseteq [0, \alpha_1)$

for all countable G_n and $G_n \supseteq [\alpha_1, \omega_1)$ for all G_n whose complement is countable. For each n, $(F_n \times G_n) \cap (\omega_1 \times [\alpha_1, \omega_1))$ equals either $\varnothing$ or $F_n \times [\alpha_1, \omega)$. Hence if $B \in \mathrm{P}_A$, then $B \cap (\omega_1 \times [\alpha_1, \omega_1)) = P_B \times [\alpha_1, \omega)$ for some $P_B \subseteq \omega_1$. But obviously there exists no set $P_A \subseteq \omega_1$ such that $A \cap (\omega_1 \times [\alpha_1, \omega_1)) = P_A \times [\alpha_1, \omega_1)$. Thus $A \notin \mathrm{P}_A$, a contradiction.

Remark 26.1 Let ν be the probability measure on (ω_1, G) defined by

$$\nu(P) := \begin{cases} 0, & \text{if } P \text{ is countable,} \\ 1, & \text{if } P \text{ is uncountable.} \end{cases}$$

Any modification of the indicator function $\mathbf{1}_A$ fails to be $\mathrm{P} \otimes \mathrm{G}$-measurable equally well. This does not contradict Proposition 26.1, since ν cannot be extended to a measure on (ω_1, P).

We continue with a measurability result for functions having L^p-sections.

Proposition 26.2 *Let (S, Σ) and $(\Omega, \mathrm{F}, \nu)$ be as before, let $1 \leqslant p < \infty$, and let E be a Banach space. If $\phi : S \times \Omega \to E$ is a strongly $\Sigma \otimes \mathrm{F}$-measurable function such that for all $s \in S$ we have $\phi_s \in L^p(\Omega; E)$, then the function $\psi : S \to L^p(\Omega; E)$ defined by $\psi(s) := \phi_s$ is strongly Σ-measurable.*

Proof Let (Ω_k) be an increasing sequence of sets in F with $\nu(\Omega_k) < \infty$ and $\bigcup_k \Omega_k = \Omega$. By approximating ϕ with the functions $\mathbf{1}_{\Omega_k \cap \{\|\phi\| \leqslant k\}} \phi$ and recalling that pointwise limits of strongly Σ-measurable functions are strongly Σ-measurable, we may assume that $\nu(\Omega) < \infty$ and that ϕ is uniformly bounded.

Choose a sequence of Σ-simple functions $\phi_n : S \times \Omega \to E$, each of which is a finite linear combination of functions of the form $\mathbf{1}_{A \times F} \otimes x$ with $A \in \Sigma$, $F \in \mathrm{F}$, $x \in E$, such that $\phi = \lim_{n\to\infty} \phi_n$ pointwise on $S \times \Omega$. These functions may be chosen in such a way that in addition we have $\|\phi_n\|_\infty \leqslant 2\|\phi\|_\infty$. By the dominated convergence theorem, for all $s \in S$ we have $\phi(s, \cdot) = \lim_{n\to\infty} \phi_n(s, \cdot)$ in $L^p(\Omega; E)$. Define the Σ-simple functions $\psi_n : S \to L^p(\Omega; E)$ by $\psi_n(s) := \phi_n(s, \cdot)$. Then for all $s \in S$

$$\psi(s) = \phi(s, \cdot) = \lim_{n\to\infty} \phi_n(s, \cdot) = \lim_{n\to\infty} \psi_n(s) \text{ in } L^p(\Omega; E).$$

This shows that ψ is strongly Σ-measurable. $\square$

Note that we did not assume $L^p(\Omega; E)$ to be separable. If this is the case, the above proof can be simplified somewhat by using the Pettis measurability theorem.

By repeated application of Proposition 26.2 we obtain the following.

Proposition 26.3 *Let (S, Σ) be as before, let $(\Omega, \mathrm{F}, \nu)$ and $(\tilde\Omega, \tilde{\mathrm{F}}, \tilde\nu)$ be σ-finite measure spaces, let $1 \leqslant p, \tilde p < \infty$, and let E be a Banach space. If $\phi : S \times \Omega \times \tilde\Omega \to E$ is a strongly $\Sigma \otimes \mathrm{F} \otimes \tilde{\mathrm{F}}$-measurable function such that for all $s \in S$ we have $\phi_s \in L^p(\Omega; L^{\tilde p}(\tilde\Omega; E))$, then the function $\psi : S \to L^p(\Omega; L^{\tilde p}(\tilde\Omega; E))$ defined by $\psi(s) := \phi_s$ is strongly Σ-measurable.*

26.3 The stochastic Fubini theorem

Let H be a separable real Hilbert space. A family $W_H = \{W_H(t)\}_{t\in[0,T]}$ of bounded linear operators from H to $L^2(\Omega)$ is called a *H-cylindrical Brownian motion* if $W_H h = \{W_H(t)h\}_{t\in[0,T]}$ is an real Brownian motion for each $h \in H$ and

$$\mathbb{E}(W_H(s)g \cdot W_H(t)h) = (s \wedge t)\,[g, h]_H \qquad s, t \in [0, T],\ g, h \in H.$$

Let E be a real Banach space. A function $\Phi : [0, T] \to \mathrm{L}(H, E)$ *belongs to $L^2(0, T; H)$ scalarly* if for all $x^* \in E^*$ the function $t \mapsto \Phi^*(t)x^*$ belongs to $L^2(0, T; H)$. Note that by the separability of H and the Pettis measurability theorem [7], the strong measurability of $t \mapsto \Phi^*(t)x^*$ is equivalent to its weak measurability.

Definition 26.1 *A function* $\Phi : [0,T] \to \mathrm{L}(H,E)$ *is called* stochastically integrable *with respect to* W_H *if* Φ *belongs to* $L^2(0,T;H)$ *scalarly and there exists a sequence* (Φ_n) *of step functions such that*

(i) For all $x^* \in E^*$ *we have* $\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^*$ *in* $L^2(0,T;H)$.

(ii) There exists a strongly measurable random variable $Y : \Omega \to E$ *such that*

$$Y = \lim_{n\to\infty} \int_0^T \Phi_n(t)\, dW_H(t) \quad \text{in } L^0(\Omega;E). \tag{26.3}$$

We then write $Y =: \int_0^T \Phi(t)\, dW_H(t)$.

Note that (i) and (ii) imply that for all $x^* \in E^*$

$$\Big\langle \int_0^T \Phi(t)\, dW_H(t), x^* \Big\rangle = \int_0^T \Phi^*(t)x^*\, dW_H(t) \quad \text{almost surely.}$$

The stochastic integral for $L^2(0,T;H)$-functions on the right-hand side of (26.3) is defined in the usual way: for step functions we put

$$\int_0^T \sum_{n=1}^N \mathbf{1}_{(t_{n-1},t_n]} \otimes h_n\, dW_H(t) := \sum_{n=1}^N W_H(t_n)h_n - W_H(t_{n-1})h_n$$

and this definition is extended to arbitrary $L^2(0,T;H)$-functions by approximation and using the Itô isometry.

It was shown in [15] that Φ is stochastically integrable with respect to W_H if and only if Φ belongs to $L^2(0,T;H)$ scalarly and there exists a γ-radonifying operator $I_\Phi : L^2(0,T;H) \to E$ such that

$$\langle I_\Phi g, x^* \rangle = \int_0^T [g(t), \Phi^*(t)x^*]_H\, dt, \quad g \in L^2(0,T;H),\ x^* \in E^*. \tag{26.4}$$

If (26.4) holds, we shall say that Φ *represents* the operator I_Φ. Recall that a bounded operator S from a separable real Hilbert space H into E is called γ*-radonifying* if for some (every) Gaussian sequence (γ_n) and some (every) orthonormal basis (h_n) of H the sum $\sum_n \gamma_n\, Sh_n$ converges in the L^2 sense. The vector space of all γ-radonifying operators from H to E is denoted by $\gamma(\mathrm{H},E)$. It is a Banach space with respect to the norm $\|\cdot\|_{\gamma(\mathrm{H},E)}$

$$\|S\|_{\gamma(\mathrm{H},E)}^2 = \mathbb{E}\Big\| \sum_n \gamma_n\, Sh_n \Big\|^2.$$

If $\Phi : [0,T] \to \mathrm{L}(H,E)$ is stochastically integrable, then for the operator I_Φ from (26.4) we have

$$\|I_\Phi\|_{\gamma(L^2(0,T;H),E)}^2 = \mathbb{E}\Big\| \int_0^T \Phi(t)\, dW_H(t) \Big\|^2.$$

Let (S,Σ,μ) be a σ-finite measure space and fix an arbitrary $1 \leqslant p < \infty$. In the next two lemmas we consider a strongly $\Sigma \otimes \mathrm{B}([0,T])$-measurable function $\phi : S \times [0,T] \to H$ which has the property that for all $t \in [0,T]$ and $h \in H$, the function $s \mapsto [\phi(s,t),h]_H$ belongs to $L^p(S)$. We then define $\Phi : [0,T] \to \mathrm{L}(H, L^p(S))$ by

$$(\Phi(t)h)(s) := [\phi(s,t),h]_H. \tag{26.5}$$

As an application of Proposition 26.2 we have:

Lemma 26.1 *Let the function* $\Phi : [0,T] \to \mathrm{L}(H, L^p(S))$ *defined by* (26.5). *For all* $h \in H$ *the function* $\Phi h : [0,T] \to L^p(S)$ *defined by* $(\Phi h)(t) := \Phi(t)h$ *is strongly* $\mathrm{B}([0,T])$-*measurable.*

The following lemma gives a necessary and sufficient condition for the stochastic integrability of the function Φ. It is a special case of [14, Proposition 4.1], which generalizes the case $H = \mathbb{R}$ considered in [15, Corollary 2.10]. See also [19, Corollary 4.3] and [2, Theorem 2.3] for related results.

Lemma 26.2 *The function* $\Phi : [0,T] \to \mathrm{L}(H, L^p(S))$ *defined by* (26.5) *is stochastically integrable in* $L^p(S)$ *with respect to an* H*-cylindrical Brownian motion* W_H *if and only if* ϕ *defines an element of* $L^p(S; L^2(0,T;H))$. *In this case we have*

$$\mathbb{E}\Big\| \int_0^T \Phi(t)\, dW_H(t) \Big\|^2 = \|\Phi\|^2_{\gamma(L^2(0,T;H),L^p(S))} \eqsim_p \|\phi\|^2_{L^p(S;L^2(0,T;H))}.$$

Here $\eqsim_p$ means that we have a two-sided estimate with constants depending only on p.

In order to extend the notions introduced above to processes $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ we need to introduce some terminology. Throughout, $\mathbb{F} = (\mathrm{F}_t)_{t\in[0,T]}$ denotes a filtration satisfying the usual conditions. We assume that the H-cylindrical Brownian motion W_H is adapted to $\mathbb{F}$, by which we mean that for all $h \in H$ the real-valued Brownian motion $W_H h$ is adapted to $\mathbb{F}$.

A process $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ *belongs to* $L^0(\Omega; L^2(0,T;H))$ *scalarly* if for all x^* the process $\Phi^* x^*$ belong to $L^0(\Omega; L^2(0,T;H))$. Such a process is said to *represent* an element X_Φ of $L^0(\Omega; \gamma(L^2(0,T;H);E))$ if for all $f \in L^2(0,T;H)$ and $x^* \in E^*$ we have

$$\langle X_\Phi f, x^* \rangle = \int_0^T [f(t), \Phi^*(t)x^*]_H \, dt \quad \text{almost surely.}$$

A process $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ is called *scalarly progressively measurable* with respect to $\mathbb{F}$ if for all $h \in H$ and $x^* \in E^*$ the process $\Phi^* x^*$ is progressively measurable with respect to $\mathbb{F}$. By the Pettis measurability theorem, this happens if and only if for all $h \in H$ and $x^* \in E^*$ the process $\langle \Phi h, x^* \rangle$ is progressively measurable with respect to $\mathbb{F}$.

A process $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ is called *elementary progressive* with respect to $\mathbb{F}$ if it is of the form

$$\Phi(t,\omega) = \sum_{n=1}^{N} \sum_{m=1}^{M} 1_{(t_{n-1},t_n]\times A_{mn}}(t,\omega) \sum_{k=1}^{K} h_k \otimes x_{kmn},$$

where $0 \leqslant t_0 < \cdots < t_N \leqslant T$, $A_{mn} \in \mathrm{F}_{t_{n-1}}$, $x_{knm} \in E$, and $(h_k)_{k\geqslant 1}$ is a fixed orthonormal basis for H. Clearly, every elementary progressive process is scalarly progressive and represents an element of $L^0(\Omega; \gamma(L^2(0,T;H);E))$.

Definition 26.2 *A process* $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ *is called* stochastically integrable *with respect to* W_H *if* Φ *belongs to* $L^0(\Omega; L^2(0,T;H))$ *scalarly and there exists a sequence* (Φ_n) *of elementary progressive processes such that*

(i) For all $x^* \in E^*$ *we have* $\lim_{n\to\infty} \Phi_n^* x^* = \Phi^* x^*$ *in* $L^0(\Omega; L^2(0,T;H))$.

(ii) There exists a strongly measurable random variable $Y : \Omega \to E$ *such that*

$$Y = \lim_{n\to\infty} \int_0^T \Phi_n(t)\, dW_H(t) \quad \text{in } L^0(\Omega; E).$$

We then write $Y =: \int_0^T \Phi(t)\, dW_H(t)$.

It is easy to check that if Φ is stochastically integrable, then Φ is scalarly progressively measurable and for all $x^* \in E^*$ we have

$$\Big\langle \int_0^T \Phi(t)\, dW_H(t), x^* \Big\rangle = \int_0^T \Phi^*(t) x^*\, dW_H(t) \quad \text{almost surely.} \tag{26.6}$$

Remark 26.2 In [14] a slightly narrower definition of stochastic integrability is used and a correspondingly stronger version of Proposition 26.5 is proved. Since the proposition is used only as a technical tool in the proof of Theorem 26.1, where it is applied to elementary progressive processes Φ_n, the simpler definition given above is sufficient for our present purposes. We refer to [14] for a fuller explanation on this point.

Let $(r_n)_{n\geqslant 1}$ be a Rademacher sequence. A Banach space E is called a *UMD*$^-$ *space* if for some (every) $1 < p < \infty$ there exists a constant β_p such that for every finite E-valued martingale difference sequence $(d_n)_{n=1}^N$ independent of $(r_n)_{n\geqslant 1}$, we have

$$\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p \leqslant \beta_p^p\, \mathbb{E}\Big\|\sum_{n=1}^N r_n d_n\Big\|^p.$$

The class of *UMD*$^+$ *spaces* is defined by reversing the estimate. By a simple randomization argument it is seen that a Banach space is a UMD space if and only if it is both UMD$^-$ and UMD$^+$. The classes of UMD$^-$ and UMD$^+$ space were introduced by Garling [10] who proved among other things:

- If E is a UMD$^+$ space, then its dual E^* is a UMD$^-$ space. If E^* is a UMD$^-$ space, then its predual E is a UMD$^+$ space.
- Every UMD$^-$ space has finite cotype. Every UMD$^+$ space is superreflexive.
- E is a UMD space if and only if E is both UMD$^-$ and UMD$^+$.

For the theory of UMD spaces we refer to the review article by Burkholder [3] and the references given therein.

By [10, Theorem 2] and the Lévy–Octaviani inequalities one easily sees that a Banach space E is a UMD$^-$ space if and only if for some (every) $p \in [1, \infty)$ there exists a constant $\tilde\beta_{p,E}^- \geqslant 0$ such that for all E-valued martingale difference sequences $(d_n)_{n=1}^N$ we have

$$\mathbb{E} \sup_{1\leqslant n\leqslant N}\Big\|\sum_{k=1}^n d_k\Big\|^p \leqslant (\tilde\beta_{p,E}^-)^p\, \mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{k=1}^N \tilde r_k d_k\Big\|^p.$$

This may be used to prove the following proposition.

Proposition 26.4 *If* (S, Σ, μ) *is* σ*-finite and* E *is a UMD*$^-$ *space, then for all* $p \in [1, \infty)$ *the space* $L^p(S; E)$ *is a UMD*$^-$ *space.*

The fact that $L^1(S)$, and more generally every space which is finitely representable in l^1, is a UMD$^-$ space is proved in [10, Theorem 3]. Apart from the trivial case where (S, Σ, μ) consists of finitely many atoms, the space $L^1(S)$ is an example of a UMD$^-$ space that is not a UMD space.

The following proposition is proved in the same way as [14, Theorem 3.7] and generalizes a result of McConnell [13] for $H = \mathbb{R}$ and UMD spaces E. It uses an obvious one-sided generalization of [9, Theorem 2$'$] to UMD$^-$ spaces.

Proposition 26.5 *Let E be a UMD$^-$ space and let $\Phi : [0,T] \times \Omega \to \mathrm{L}(H,E)$ be a scalarly progressively measurable process. If Φ represents an element X_Φ of $L^0(\Omega;\gamma(L^2(0,T;H),E))$, then Φ is stochastically integrable with respect to W_H, and there exists a sequence of elementary progressive processes $\Phi_n : [0,T] \times \Omega \to \mathrm{L}(H,E)$ such that*

(i) $\lim_{n\to\infty} X_{\Phi_n} = X_{\Phi_n}$ *in* $L^0(\Omega;\gamma(L^2(0,T;H),E))$.

(ii) $\lim_{n\to\infty} \int_0^T \Phi_n(t)\,dW_H(t) = \int_0^T \Phi(t)\,dW_H(t)$ *in* $L^0(\Omega;E)$.

Below we shall apply the proposition to the space $E = L^1(S)$. By Lemma 26.2, the space $L^0(\Omega;\gamma(L^2(0,T;H),L^1(S)))$ can be identified with $L^0(\Omega;L^1(S;L^2(0,T;H)))$ isomorphically.

After these preparations we are in a position to state and prove our first main result.

Theorem 26.1 (Stochastic Fubini theorem, first version) *Let $\phi : S\times[0,T]\times\Omega \to H$ be a process satisfying the following assumptions:*

(i) *ϕ is strongly $\Sigma \otimes \mathrm{B}([0,T]) \otimes \mathrm{F}$-measurable.*

(ii) *For all $s \in S$, the section ϕ_s is progressively measurable.*

(iii) *For almost all $\omega \in \Omega$, $(s,t) \mapsto \phi(s,t,\omega)$ belongs to $L^1(S;L^2(0,T;H))$.*

Then

1. *For almost all $s \in S$, the process ϕ_s is stochastically integrable with respect to W_H.*
2. *For almost all $(t,\omega) \in [0,T] \times \Omega$, $s \mapsto \phi_s(t,\omega)$ defines an element of $L^1(S;H)$ and there exists a progressively measurable process $\langle \phi, \mu\rangle : [0,T] \times \Omega \to H$, stochastically integrable with respect to W_H, such that*
$$\langle \phi, \mu\rangle(t,\omega) = \int_S \phi_s(t,\omega)\,d\mu(s)$$
for almost all $(t,\omega) \in [0,T] \times \Omega$;
3. *For almost all $\omega \in \Omega$, $s \mapsto \big(\int_0^T \phi_s(t)\,dW_H(t)\big)(\omega)$ belongs to $L^1(S)$ and*
$$\int_S \Big(\int_0^T \phi_s(t)\,dW_H(t)\Big)(\omega)\,d\mu(s) = \Big(\int_0^T \langle \phi, \mu\rangle(t)\,dW_H(t)\Big)(\omega).$$

If in (iii) we make the stronger assumption that $\phi \in L^p(\Omega;L^1(S;L^2(0,T;H)))$ for some $p \in [1,\infty)$, then it follows from similar estimates as in [14] that
$$\mathbb{E}\Big|\int_0^T \langle \phi, \mu\rangle(t)\,dW_H(t)\Big|^p \leqslant C_p \mathbb{E}\|\phi\|^p_{L^1(S;L^2(0,T;H))},$$
for some universal constant C_p, and the equality in (26.3) may be interpreted in $L^p(\Omega)$.

Proof By Proposition 26.1 (where we replace Ω by $[0,T] \times \Omega$ and for G we take the progressive σ-algebra P of $[0,T]\times\Omega$) we may choose a version of ϕ which is $\Sigma\otimes$P-measurable.

(1) For almost all $\omega \in \Omega$ we have $\phi(s,\cdot,\omega) \in L^2(0,T;H)$ for almost all $s \in S$. Hence by Fubini's theorem, for almost all $s \in S$ the process $(t,\omega) \mapsto \phi(s,t,\omega)$ has trajectories in $L^2(0,T;H)$ almost surely. By standard results it follows that for almost all $s \in S$ the process ϕ_s is stochastically integrable with respect to W_H.

(2) Using the embedding

$$L^1(S; L^2(0,T;H)) \hookrightarrow L^1(S; L^1(0,T;H)) \eqsim L^1(0,T; L^1(S,H))$$

and the Fubini theorem, (iii) implies that for almost all $(t,\omega) \in [0,T]\times\Omega$ the function $s \mapsto \phi(s,t,\omega)$ defines an element of $L^1(S;H)$. The exceptional set N being progressively measurable, we may redefine $\phi(\cdot,t,\omega)$ to be 0 for $(t,\omega) \in N$ and thereby assume that $\phi(\cdot,t,\omega)$ defines an element of $L^1(S;H)$ for all $(t,\omega) \in [0,T]\times\Omega$. Now define an operator-valued process $\Phi : [0,T]\times\Omega \to \mathrm{L}(H, L^1(S))$ by

$$(\Phi(t,\omega)h)(s) := [\phi(s,t,\omega), h]_H.$$

Since by (ii) the process $(t,\omega) \mapsto [\phi(s,t,\omega),h]_H$ is progressively measurable for all $s \in S$, it follows by Proposition 26.2 that Φh is strongly progressively measurable for all $h \in H$. In particular, Φ is scalarly progressively measurable.

By Proposition 26.3, the random variable $\omega \mapsto \phi(\cdot,\cdot,\omega)$ is strongly F-measurable from Ω to $L^1(S;L^2(0,T;H))$. Thus ϕ defines an element of $L^0(\Omega; L^1(S; L^2(0,T;H)))$. By Proposition 26.5 and the remark following it, Φ is stochastically integrable with respect to W_H.

Identifying integration with respect to μ with a bounded linear operator T_μ acting from $\mathrm{L}(H, L^1(S))$ to H in the canonical way, we have $\langle \phi, \mu\rangle = T_\mu \circ \Phi$. Since $T_\mu\circ\Phi$ is stochastically integrable with respect to W_H the result follows.

(3) By what has been proved in Step 2, Φ is scalarly progressive and represents an element of $L^0(\Omega; \gamma(L^2(0,T;H), L^1(S)))$. Hence by Proposition 26.5 there exists a sequence of elementary progressive processes $\Phi_n : [0,T]\times\Omega \to \mathrm{L}(H, L^1(S))$ such that $\lim_{n\to\infty} X_{\Phi_n} = X_\Phi$ in $L^0(\Omega;\gamma(L^2(0,T;H),E))$. Upon passing to a subsequence we may assume that

$$\begin{aligned}\Big(\Big(\int_0^T \Phi(t)\,dW_H(t)\Big)(\omega)\Big)(s) &= \lim_{n\to\infty}\Big(\Big(\int_0^T \Phi_n(t)\,dW_H(t)\Big)(\omega)\Big)(s)\\ &= \lim_{n\to\infty}\Big(\int_0^T (\Phi_n(t))(s)\,dW_H(t)\Big)(\omega) \qquad (26.7)\\ &\qquad \text{for almost all } (s,\omega) \in S\times\Omega.\end{aligned}$$

For each n, let ϕ_n be the element of $L^0(\Omega; (L^1(S; L^2(0,T;H))))$ corresponding to the process Φ_n. By passing to a further subsequence we may also assume that

$$\lim_{n\to\infty} \phi_n(s,\cdot,\omega) = \phi(s,\cdot,\omega) \ \text{ in } L^2(0,T;H) \text{ for almost all } (s,\omega)\in S\times\Omega. \qquad (26.8)$$

Defining $\phi_{n,s}(t,\omega) := \phi_n(s,t,\omega)$, by (26.8) and the Fubini theorem for almost all $s \in S$ we have $\phi_s(\cdot,\omega) = \lim_{n\to\infty}\phi_{n,s}(\cdot,\omega)$ in $L^2(0,T;H)$ for almost all $\omega\in\Omega$. This implies that $\phi_s(\cdot) = \lim_{n\to\infty}\phi_{n,s}(\cdot)$ in $L^2(0,T;H)$ in probability. By standard results on stochastic integration, from this it follows that for almost all $s \in S$

$$\int_0^T \phi_s(t)\,dW_H(t) = \lim_{n\to\infty}\int_0^T \phi_{n,s}(t)\,dW_H(t) \ \text{ in probability.} \qquad (26.9)$$

Comparing limits in (26.7) and (26.9), for almost all $s\in S$ we obtain

$$\Big(\int_0^T \phi_s(t)\,dW_H(t)\Big)(\omega) = \Big(\Big(\int_0^T \Phi(t)\,dW_H(t)\Big)(\omega)\Big)(s) \ \text{ for almost all } \omega\in\Omega.$$

But then by the Fubini theorem, for almost all $\omega\in\Omega$ we have

$$\Big(\int_0^T \phi_s(t)\,dW_H(t)\Big)(\omega) = \Big(\Big(\int_0^T \Phi(t)\,dW_H(t)\Big)(\omega)\Big)(s) \ \text{ for almost all } s\in S. \qquad (26.10)$$

Since $\int_0^T \Phi(t)\,dW_H(t)$ is a random variable with values in $L^1(S)$, this proves the μ-integrability assertion. The final identity follows by integrating (26.10) with respect to μ. This gives, for almost all $\omega \in \Omega$

$$\begin{aligned}\int_S \Big(\int_0^T \phi_s(t)\,dW_H(t)\Big)(\omega)\,d\mu(s) &= \int_S \Big(\Big(\int_0^T \Phi(t)\,dW_H(t)\Big)(\omega)\Big)(s)\,d\mu(s)\\ &\overset{\text{(i)}}{=} \Big\langle \Big(\int_0^T \Phi(t)\,dW_H(t)\Big)(\omega), \mathbf{1}\Big\rangle\\ &\overset{\text{(ii)}}{=} \Big(\int_0^T \Phi^*(t)\mathbf{1}\,dW_H(t)\Big)(\omega)\\ &\overset{\text{(iii)}}{=} \Big(\int_0^T \langle \phi, \mu\rangle(t)\,dW_H(t)\Big)(\omega).\end{aligned}$$

In (i) the brackets denote the duality between $L^1(S)$ and $L^\infty(S)$, in (ii) we used the identity (26.6), and in (iii) we used (2) and the Fubini theorem to the effect that for almost all $t \in [0,T]$ we have, almost surely

$$[\Phi^*(t)\mathbf{1}, h]_H = \int_S [\phi(s,t,\cdot), h]_H\,d\mu(s) = [\langle \phi, \mu\rangle(t), h]_H \quad \text{for all } h \in H.$$

□

Theorem 26.1 can easily be extended to the more general situation where ϕ is a process with values in $\mathrm{L}(H, H')$. In this way, a generalization of the result by Da Prato and Zabczyk [5] as stated in the Introduction is obtained. More generally, one can replace the role of H' by an arbitrary real Banach space E. The condition from [5] that ϕ should take values in $L^0(\Omega; L^2(0,T; \mathrm{L}_2(H,H')))$ is then replaced by the condition that ϕ should take values in $L^0(\Omega; \gamma(L^2(0,T;H), E))$. The latter condition reduces to the former if $E = H'$ since $\gamma(L^2(0,T;H), H') = L^2(0,T; \mathrm{L}_2(H,H'))$ isometrically. In order to be able to give a precise statement of the theorem we need to introduce some notations from [14].

Every functional $x^* \in E^*$ induces a bounded operator $x^* : \gamma(L^2(0,T;H), E)) \to L^2(0,T;H)$ by

$$x^*(S) := S^* x^*.$$

We shall write $\langle S, x^*\rangle$ instead of $x^*(S)$. Applying this operator pointwise, we obtain an operator $x^* : L^0(\Omega; \gamma(L^2(0,T;H), E)) \to L^0(\Omega; L^2(0,T;H))$ by

$$(x^*(X))(\omega) := X^*(\omega)x^*.$$

In what follows we shall write $\langle X, x^*\rangle$ for $x^*(X)$. Let $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H), E))$ denote the closed subspace of all of $L^0(\Omega; \gamma(L^2(0,T;H), E))$ of all elements X such that for all $x^* \in E^*$, $\langle X, x^*\rangle$ is a progressively measurable as an H-valued process with respect to the filtration $\mathbb{F}$. Here, we identify the elements $\langle X, x^*\rangle \in L^0(\Omega; L^2(0,T;H))$ with processes $\langle X, x^*\rangle : [0,T] \times \Omega \to H$. Note that if $X = X_\Phi$ is represented by a process Φ, then $\langle X, x^*\rangle = \Phi^* x^*$.

Since every elementary progressive process is representable, the subspace of representable elements is dense in $L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H), E))$. It is shown as in [14] that the linear operator $X_\Phi \mapsto \int_0^T \Phi(t)\,dW_H(t)$, which is well defined for representable processes by Proposition 26.5, has a unique extension to a continuous linear operator

$$I^{W_H} : L^0_{\mathbb{F}}(\Omega; \gamma(L^2(0,T;H), E)) \to L^0(\Omega; E).$$

We shall write $\int_0^T X\,dW_H$ for Itô(X).

The proof of Theorem 26.1 can be adapted to obtain the following result.

Theorem 26.2 (Stochastic Fubini theorem, second version) *Let E be a UMD$^-$ space and let $\phi : S \times [0,T] \times \Omega \to \mathrm{L}(H,E)$ be a process satisfying the following assumptions:*

(i) For all $h \in H$, ϕh is strongly $\Sigma \otimes \mathrm{B}([0,T]) \otimes \mathrm{F}$-measurable.

(ii) For all $s \in S$, the section ϕ_s is progressively measurable for all $h \in H$.

(iii) For almost all $(s,\omega) \in S \times \Omega$, the function $t \mapsto \phi(s,t,\omega)$ represents an element $U_{s,\omega} \in \gamma(L^2(0,T;H),E)$, and for almost all $\omega \in \Omega$, $s \mapsto U_{s,\omega}$ defines an element of $L^1(S;\gamma(L^2(0,T;H),E))$.

Then

1. For almost all $s \in S$, ϕ_s is stochastically integrable with respect to W_H.

2. For all $x^* \in E^*$, $s \mapsto \phi_s^*(t,\omega)x^*$ defines an element of $L^1(S;H)$ for almost all $(t,\omega) \in [0,T] \times \Omega$, and there exists an element $\langle \phi, \mu \rangle \in L^0_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$ such that for all $x^* \in E^*$ we have

$$\big\langle \langle \phi, \mu \rangle, x^* \big\rangle(t,\omega) = \int_S \phi_s^*(t,\omega)x^* \, d\mu(s)$$

for almost all $(t,\omega) \in [0,T] \times \Omega$.

3. For almost all $\omega \in \Omega$, $s \mapsto \big(\int_0^T \phi_s(t)\, dW_H(t)\big)(\omega)$ belongs to $L^1(S;E)$ and we have

$$\int_S \Big(\int_0^T \phi_s(t)\, dW_H(t) \Big)(\omega)\, d\mu(s) = \Big(\int_0^T \langle \phi, \mu \rangle(t)\, dW_H(t) \Big)(\omega).$$

If E has type 2, we have a continuous embedding $L^2(0,T;\gamma(H,E)) \hookrightarrow \gamma(L^2(0,T;H),E)$, cf. [16]. Condition (iii) is then implied by the stronger condition.

(iii)$'$ *For almost all $\omega \in \Omega$, $(s,t) \mapsto \phi(s,t,\omega)$ defines an element of $L^1(S;L^2(0,T;\gamma(H,E)))$.*

If E has cotype 2 we have a continuous embedding $\gamma(L^2(0,T;H),E) \hookrightarrow L^2(0,T;\gamma(H,E))$, cf. [16]. Because of this, every $X \in L^0_{\mathbb{F}}(\Omega;\gamma(L^2(0,T;H),E))$ can be identified with a progressively measurable process in $L^0(\Omega;L^2(0,T;\gamma(H,E)))$ and the use of the abstract Itô operator can be avoided. Moreover, it can be shown that in this situation, (2) can be strengthened as follows:

(2)$'$ *For almost all $(t,\omega) \in [0,T] \times \Omega$, $s \mapsto \phi(s,t,\omega)h$ belongs to $L^1(S;E)$ for all $h \in H$ and there exists a process $\langle \phi, \mu \rangle : [0,T] \times \Omega \to \mathrm{L}(H,E)$, stochastically integrable with respect to W_H, such that for almost all $(t,\omega) \in [0,T] \times \Omega$ we have*

$$\langle \phi, \mu \rangle(t,\omega)h = \int_S \phi(s,t,\omega)h \, d\mu(s)$$

for all $h \in H$.

Both remarks apply if $E = H'$ is a Hilbert space, in which case we have $\gamma(H,E) = \gamma(H,H') = \mathrm{L}_2(H,H')$.

Finally, in (iii) we may replace the almost sure conditions by moment conditions to obtain random variables with finite moments in (3). For example, in the case $E = H'$ we could assume that $\phi \in L^p(\Omega;L^1(S;L^2(0,T;\mathrm{L}_2(H,H'))))$ for some $p \in [1,\infty)$, in which case we obtain

$$\mathbb{E}\Big\| \int_0^T \langle \phi, \mu \rangle(t)\, dW_H(t) \Big\|^p \leqslant C_p \mathbb{E}\|\phi\|^p_{L^1(S;L^2(0,T;\mathrm{L}_2(H,H')))},$$

and the equality in (3) may be interpreted in $L^p(\Omega;E)$.

References

[1] K. Bichteler and S.J. Lin, On the stochastic Fubini theorem, *Stoch. Stoch. Rep.* **54** (1995), 271–279.

[2] Z. Brzeźniak and J.M.A.M. van Neerven, Space-time regularity for linear stochastic evolution equations driven by spatially homogeneous noise, *J. Math. Kyoto Univ.* **43** (2003), 261–303.

[3] D.L. Burkholder, Martingales and singular integrals in Banach spaces, in: "Handbook of the Geometry of Banach Spaces," Vol. I, pp. 233–269, North-Holland, Amsterdam, 2001.

[4] A. Chojnowska-Michalik, "Stochastic Differential Equations in Hilbert Spaces and Some of Their Applications," Ph.D. thesis, Institute of Mathematics, Polish Academy of Sciences, 1977.

[5] G. Da Prato and J. Zabczyk, "Stochastic Equations in Infinite Dimensions," Encyclopedia of Mathematics and Its Applications, Vol. 44, Cambridge University Press, Cambridge, 1992.

[6] C. Dellacherie and P.A. Meyer, "Probabilities and Potential," North-Holland Mathematics Studies, Vol. 29, North-Holland, Amsterdam-New York, 1978.

[7] J. Diestel and J.J. Uhl, "Vector Measures," Math. Surveys, Vol. 15, Amer. Math. Soc., Providence, RI, 1977.

[8] C. Doléans-Dade, Intégrales stochastiques dépendant d'un paramètre, *Publ. Inst. Stat. Univ. Paris* **16** (1967), 23–24.

[9] D.J.H. Garling, Brownian motion and UMD-spaces, in: "Probability and Banach Spaces" (Zaragoza, 1985), 36–49, Lecture Notes in Math. 1221, Springer-Verlag, Berlin, 1986.

[10] D.J.H. Garling, Random martingale transform inequalities, in: "Probability in Banach Spaces VI" (Sandbjerg, 1986), 101–119, Progr. Probab., Vol. 20, Birkhäuser, Boston, 1990.

[11] J. Jacod, "Calcul Stochastique et Problèmes de Martingales," Lecture Notes in Math. 714, Springer-Verlag, Berlin, 1979.

[12] K. Kunen, "Inaccessibility Properties of Cardinals," Ph.D. thesis, Stanford University, 1968.

[13] T.R. McConnell, Decoupling and stochastic integration in UMD Banach spaces, *Probab. Math. Statist.* **10** (1989), 283–295.

[14] J.M.A.M. van Neerven, M.C. Veraar and L. Weis, Stochastic integration in UMD Banach spaces, submitted for publication.

[15] J.M.A.M. van Neerven and L. Weis, Stochastic integration of functions with values in a Banach space, *Studia Math.* **166** (2005), 131–170.

[16] J.M.A.M. van Neerven and L. Weis, Weak limits and integrals of Gaussian covariances in Banach spaces, to appear in *Probab. Math. Statist.*

[17] J. Neveu, "Discrete-Parameter Martingales," North-Holland, Amsterdam, 1975.

[18] Ph.E. Protter, "Stochastic Integration and Differential Equations," 2nd edition, Springer-Verlag, Berlin, 2004.

[19] J. Rosiński and Z. Suchanecki, On the space of vector-valued functions integrable with respect to the white noise, *Colloq. Math.* **43** (1980), 183–201.

[20] F. Rothberger, On families of real functions with a denumerable base, *Ann. Math.* **45** (1944), 397–406.

[21] C. Stricker and M. Yor, Calcul stochastique dépendant d'un paramètre, *Z. Wahrsch. Verw. Gebiete* **45** (1978), 109–133.

27 Itô–Tanaka's Formula for Stochastic Partial Differential Equations Driven by Additive Space–Time White Noise

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27.1 Introduction

Let $(u_t(x) : t \geq 0,\, x \in [0,1])$ be the unique solution of the stochastic partial differential equation (SPDE) driven by space–time white noise

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2} + f(u) + \sigma(u)\,\dfrac{\partial^2 W}{\partial t \partial x} \\ u_t(0) = u_t(1) = 0, \\ u_0 \;=\; \overline{u} \end{cases} \tag{27.1}$$

where $f, \sigma : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz, $\overline{u} : [0,1] \mapsto \mathbb{R}$ is continuous deterministic with $\overline{u}(0) = \overline{u}(1) = 0$, and $(W(t,x) : t \geq 0, x \in [0,1])$ is a Brownian sheet.

It is well known that the unique solution u of the SPDE (27.1) is almost surely continuous in t and x (see, e.g., [8]). In particular, one can fix a space point $x \in (0,1)$ and consider the process $t \mapsto u_t(x)$, $t \geq 0$. The following natural question arises: can one develop a stochastic calculus for this process?

For a finite-dimensional stochastic differential equations (SDEs), the Itô–Tanaka formula is a fundamental tool, studying nonlinear functions of the solution and in several cases computing explicitly their law: this is, for instance, the case of the Bessel processes of integer dimension, which is defined as the solution of a nonlinear one-dimensional (1D) SDE and turns out to be equal in law to the norm of a multidimensional Brownian motion (see, e.g., [7]).

In the case of an SDE, the equation gives also the semimartingale decomposition: the noise term corresponds to the martingale part and the drift term, to the bounded-variation part.

In (27.1), both the drift and noise terms are ill-defined: indeed, it is well known that (27.1) is a formal expression which makes sense only after multiplication by a smooth test function $\ell(x)$ and after integration (by parts) in x (and t). In particular, the process $t \mapsto u_t(x)$, $t \geq 0$ is not a semimartingale.

Therefore, the classical stochastic calculus does not apply for the process we are interested in. In particular, if $\varphi : \mathbb{R} \mapsto \mathbb{R}$ is a smooth or convex function, it is a natural idea that $v := \varphi(u)$ satisfies some kind of SPDE, but it is not easy to guess (and prove) the precise form of such SPDE.

The general solution of this problem is still to be found. In this chapter we want to make a first step, writing an Itô–Tanaka formula for the solution $(u_t(x))_{t\geq 0,\, x\in[0,1]}$ of the easiest version of (27.1), namely

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 u_t}{\partial x^2} + \dfrac{\partial^2 W}{\partial t \partial x} \\ u_t(0) = u_t(1) = 0, \\ u_0 = 0. \end{cases} \tag{27.2}$$

In this case the solution u has an explicit representation, called the stochastic convolution (see [8] and [3])

$$u_t(x) = \int_0^t \int_0^1 g_{t-s}(x,y)\, W(ds,dy), \qquad t \geq 0,\, x \in [0,1], \tag{27.3}$$

where $(g_t(x,y) : t \geq 0,\, x, y \in [0,1])$ is the fundamental solution of the heat equation with homogeneous Dirichlet boundary condition

$$\begin{cases} \dfrac{\partial g}{\partial t} = \dfrac{1}{2}\dfrac{\partial^2 g}{\partial x^2} \\ g_t(0,y) = g_t(1,y) = 0 \\ g_0(x,y) = \delta(x-y). \end{cases}$$

Now, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function. The Itô formula for $\varphi(u)$ is the following (see Theorem 27.1 below for a precise statement):

$$\frac{\partial}{\partial t}\varphi(u_t) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\varphi(u_t) - \frac{1}{2}\varphi''(u_t) : \left|\frac{\partial u_t}{\partial x}\right|^2 : + \varphi'(u_t)\frac{\partial^2 W}{\partial t \partial x}, \tag{27.4}$$

where, formally

$$: \left|\frac{\partial u_t}{\partial x}\right|^2 : = \left|\frac{\partial u_t}{\partial x}\right|^2 - \infty \stackrel{\text{def}}{=} \lim_{\epsilon \searrow 0}\left[\left|\frac{\partial(g_\epsilon * u_t)}{\partial x}\right|^2 - \|g_\epsilon(x,\cdot)\|^2_{L^2(0,1)}\right].$$

Recall that $u_t(\cdot)$ is known to be a.s. at most $C^\alpha(0,1)$ for all $\alpha < 1/2$. In particular, $\partial u/\partial x$ is not a function but a distribution and a diverging constant has to be subtracted from $|\partial u_t/\partial x|^2$ in order to have a limit.

In the literature one can find heuristic arguments where this formula or generalizations of it are implicit: for instance, in [2] a nonlinear function of the unique solution of a linear SPDE is interpreted as a solution of a KPZ equation, although this interpretation remains at an informal level. The main novelty of the present chapter is to give a rigorous meaning to each term of (27.4) and to give an explicit representation of the nontrivial term containing $:|\partial u/\partial x|^2:$ (see (27.8) below). Further research along this line should lead to a more precise mathematical treatment of KPZ equations.

Letting now $\varphi = \varphi_\epsilon \to |\cdot - a|$ and $\varphi_\epsilon'' \to 2\,\delta_a$ in (27.4), we can write a Tanaka formula (see Theorem 27.2 below for a precise statement)

$$\frac{\partial}{\partial t}|u_t - a| = \frac{1}{2}\frac{\partial^2}{\partial x^2}|u_t - a| - : \left|\frac{\partial u_t}{\partial x}\right|^2 : \frac{\partial L_t^a(x)}{\partial t} + \operatorname{sign}(u_t - a)\frac{\partial^2 W}{\partial t \partial x},$$

where $(L_t^a(x) : t \geq 0, a \in \mathbb{R})$ is a family of local times of the process $(u_t(x) : t \geq 0)$ with $x \in (0,1)$ fixed, defined in terms of an occupation times formula

$$\int_0^t f(u_s(x))\, ds = \int_{\mathbb{R}} f(a)\, L_t^a(x)\, da.$$

The techniques of this chapter apply, at least for smooth φ, also to more general cases than (27.2), i.e., to versions of (27.1) where σ and f are more general coefficients. On the other hand, the formulae become more complicated and obscure. Moreover, the Tanaka formula presents significant difficulties. For these reasons, it seems reasonable to present first the easiest case (27.2) in full detail and then treat more general cases in a future work.

I would like to remark that a recent paper [5] studies the same problem using a development of u in Fourier series and obtains a different description of the terms appearing in the Itô–Tanaka formula.

Moreover, the results of this chapter, and, in particular, the possibility of renormalizing $|\partial u_t/\partial x|^2$ by subtracting a diverging constant, are related with the results of [9], where integration by parts formulae on the Wiener space along a class of non-Cameron Martin vector fields are computed.

Finally, I would like to thank the organizers of two conferences on *Stochastic Partial Differential Equations*, held respectively in Banff (Canada) in September 2003 and in Levico (Italy) in January 2004, where I had the opportunity to present preliminary versions of these results.

27.2 Itô's formula

We need some notation. Let $\langle \cdot, \cdot \rangle$ denote the canonical scalar product in $L^2(0,1)$ and set

$$O(t) \;:=\; [0,t] \times [0,1], \qquad t \geq 0.$$

We shall denote the space–time white noise by $dW_{t,x}$, $t \geq 0$ and $x \geq 0$, or by $\langle \cdot, dW_t \rangle$, $t \geq 0$. For instance, the stochastic convolution (27.3) can be written equivalently

$$u_t(x) \;=\; \int_{O(t)} g_{t-s}(x,y)\; dW_{s,y} \;=\; \int_0^t \langle g_{t-s}(x,\cdot), dW_s \rangle, \qquad t \geq 0,\, x \in [0,1],$$

see [8] and [3].

We shall use the Malliavin calculus and the Skorohod integration w.r.t. the space–time white noise, using the notation introduced above and referring to [6] for the general theory. In particular, we suppose that the Brownian sheet W is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathcal{F} = \sigma(W)$ and we denote by $DF = (D_{s,y}F \,:\, s \geq 0, y \in [0,1])$ the Malliavin derivative of a random variable $F : \Omega \mapsto \mathbb{R}$, by $\mathbb{D}^{k,2}$ the Hilbert space of random variables with $k = 1, 2$ square-integrable Malliavin derivatives, and by $\mathrm{Dom}(\delta_T) \subset L^2(O(T) \times \Omega)$ the domain of the Skorohod integral δ_T over $O(T)$

$$v = (v_s(y))_{(s,y) \in O(T)} \;\mapsto\; \delta_T(v) \;=\; \int_{O(T)} v_s(y)\; dW_{s,y} \;=\; \int_0^T \langle v_s, dW_s \rangle. \tag{27.5}$$

Recall that for all $\ell \in C_c^2(0,1)$, the set of twice continuously differentiable functions with compact support in $(0,1)$

$$\langle u_T, \ell \rangle \;=\; \langle u_0, \ell \rangle \,+\, \frac{1}{2} \int_0^T \langle u_s, \ell'' \rangle\, ds \,+\, \int_0^T \langle \ell, dW_s \rangle, \qquad T \geq 0. \tag{27.6}$$

Finally, we shall use the following regularization of u for $\epsilon > 0$:

$$u_t^\epsilon(x) \;:=\; \int_0^1 g_\epsilon(x,y)\, u_t(y)\, dy \;=\; \int_{O(t)} g_{\epsilon+t-s}(x,y)\; dW_{s,y}, \quad t \geq 0,\; x \in [0,1].$$

We want to prove the following theorem.

Theorem 27.1. *Let $\varphi \in C^4(\mathbb{R})$ with bounded i-th derivative, $i = 1, \ldots, 4$. Then for all $T \geq 0$ and $\ell \in C_c^2(0,1)$*

$$\begin{aligned}\langle \varphi(u_T), \ell\rangle \quad &= \quad \langle \varphi(u_0), \ell\rangle + \frac{1}{2}\int_0^T \langle \ell'', \varphi(u_s)\rangle\, ds + \int_0^T \langle \varphi'(u_s)\,\ell,\ dW_s\rangle \\ &\quad - \frac{1}{2}\int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : \varphi''(u_s)\,\rangle\, ds \end{aligned} \tag{27.7}$$

where, setting $C_\epsilon(x) := \int_0^1 g_\epsilon^2(x,y)\,dy$

$$\begin{aligned}&\int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : \varphi''(u_s)\,\rangle\, ds \stackrel{\text{def}}{=} \lim_{\epsilon\searrow 0}\int_0^T \langle \ell, \left[\left|\frac{\partial u_s^\epsilon}{\partial x}\right|^2 - C_\epsilon\right] \varphi''(u_s^\epsilon)\rangle\, ds \\ &= -\int_{O(T)} \ell(x)\,\varphi''(u_t(x))\left[\int_0^1 |g_t(x,y)|^2\,dy\right] dx\,dt \\ &+ 2\int_{O(T)}\left\{\int_{O(s)}\left[\int_s^T \langle \partial_x g_{t-s}(\cdot,y)\ \partial_x g_{t-r}(\cdot,z)\ \varphi''(u_t), \ell\rangle\, dt\right] dW_{r,z}\right\} dW_{s,y}\end{aligned} \tag{27.8}$$

and the stochastic integrals in the last term are Skorohod integrals.

The assumption $\varphi \in C^4(\mathbb{R})$ will be removed in Corollary 27.1 below, after proving the Tanaka formula (27.13).

Proof of Theorem 27.1 Notice that $u_t^\epsilon \in C^2(0,1)$ for all $t \geq 0$. Moreover for all $x \in (0,1)$, by (27.6) with $\ell = g_\epsilon(x,\cdot)$, we have that $t \mapsto u_t^\epsilon(x)$ is a semimartingale and

$$du_t^\epsilon(x) \;=\; \frac{1}{2}\,\partial_x^2 u_t^\epsilon(x)\,dt \;+\; dW_t^\epsilon(x)$$

with the notation $\partial_x := \partial/\partial x$, and $(W_t^\epsilon(x) : t \geq 0)$ is the martingale

$$W_t^\epsilon(x) \;:=\; \int_0^1 g_\epsilon(x,y)\,W(t,dy)$$

with quadratic variation

$$< W^\epsilon(x) >_t = \; t\int_0^1 g_\epsilon^2(x,y)\,dy \;=\; t\,C_\epsilon(x).$$

Recall that $C_\epsilon(x) \sim c(x)\epsilon^{-1/2}$, as $\epsilon \searrow 0$. Then we have the Itô formula for $t \mapsto u_t^\epsilon(x)$

$$d\varphi(u^\epsilon) = \varphi'(u^\epsilon)\left(\frac{1}{2}\,\partial_x^2 u^\epsilon\,dt + dW_t^\epsilon\right) + \;\frac{1}{2}\,\varphi''(u^\epsilon)\,C_\epsilon\,dt.$$

Since $u_t(\cdot) \in C^2(\mathbb{R})$ we can compute

$$\partial_x^2\left[\varphi(u^\epsilon)\right] \;=\; \varphi'(u^\epsilon)\,\partial_x^2 u^\epsilon \;+\; \varphi''(u^\epsilon)\;|\partial_x u^\epsilon|^2\,.$$

Then

$$d\varphi(u^\epsilon) = \frac{1}{2}\,\partial_x^2\left[\varphi(u^\epsilon)\right]\,dt \;+\; \varphi'(u^\epsilon)\,dW_t^\epsilon - \;\frac{1}{2}\,\varphi''(u^\epsilon)\left(|\partial_x u^\epsilon|^2 - C_\epsilon\right)dt.$$

In particular, multiplying by $\ell \in C_c^2(0,1)$ and integrating over $O(T)$

$$\begin{aligned}\langle \varphi(u_T^\epsilon), \ell\rangle &= \langle \varphi(u_0^\epsilon), \ell\rangle + \frac{1}{2}\int_0^T \langle \ell'', \varphi(u_s^\epsilon)\rangle\, ds + \int_0^T \langle \varphi'(u_s^\epsilon)\,\ell,\ dW_s\rangle \\ &\quad - \frac{1}{2}\int_0^T \langle \ell, \varphi''(u_t^\epsilon)\left(|\partial_x u_t^\epsilon|^2 - C_\epsilon\right)\rangle\, dt.\end{aligned}$$

It is easy to see that

$$\langle \varphi(u_T^\epsilon), \ell\rangle \overset{\epsilon\searrow 0}{\to} \langle \varphi(u_T), \ell\rangle, \qquad \int_0^T \langle \ell'', \varphi(u_s^\epsilon)\rangle\, ds \overset{\epsilon\searrow 0}{\to} \int_0^T \langle \ell'', \varphi(u_s)\rangle\, ds,$$

$$\int_0^T \langle \varphi'(u_s^\epsilon)\,\ell,\ dW_s\rangle \overset{\epsilon\searrow 0}{\to} \int_0^T \langle \varphi'(u_s)\,\ell,\ dW_s\rangle$$

a.s. and in L^2. Then the process

$$J_T^\epsilon := \int_0^1 \ell(x)\int_0^T \varphi''(u_t^\epsilon(x))\left(|\partial_x u_t^\epsilon(x)|^2 - C_\epsilon(x)\right) dt\, dx, \quad T \geq 0,$$

converges a.s. and in L^2 as $\epsilon \searrow 0$ to a continuous process $(J_T)_{T\geq 0}$. We want to identify J. We fix $t \in [0,T]$ and set $G_s^\epsilon(x,y) := \partial_x g_{\epsilon+s}(x,y)$ and

$$M_s := \int_{O(s)} G_{t-r}^\epsilon(x,y)\ dW_{r,y}, \qquad s \in [0,t].$$

Then $(M_s : s \in [0,t])$ is an $(\mathcal{F}_s)$-martingale, where $\mathcal{F}_s := \sigma(W(r,y) : r \in [0,s], y \in [0,1])$. Moreover

$$\partial_x u_t^\epsilon(x) = \int_{O(t)} G_{t-s}^\epsilon(x,y)\, dW_{s,y} = M_t.$$

Applying the Itô formula to M^2 we obtain

$$M_t^2 = 2\int_0^t M_s\, dM_s + < M >_t$$

$$\text{i.e.,} \qquad |\partial_x u_t^\epsilon(x)|^2 = 2\int_0^t M_s\, dM_s + \int_0^t\int_0^1 |G_s^\epsilon(x,y)|^2\, ds\, dy.$$

Therefore $J_T^\epsilon = K_T^\epsilon + 2\, H_T^\epsilon$, where

$$K_T^\epsilon := \int_0^1 \ell(x)\int_0^T \varphi''(u_t^\epsilon(x))\left(\int_0^t\int_0^1 |G^\epsilon(x,y)|^2\, ds\, dy - C_\epsilon(x)\right) dt\, dx,$$

$$H_T^\epsilon := \int_0^1 \ell(x)\int_0^T \varphi''(u_t^\epsilon(x))\left[\int_0^t M_s\, dM_s\right] dt\, dx.$$

We consider first K_T^ϵ. We have

$$\begin{aligned}&\int_0^1 |g_{t+\epsilon}(x,y)|^2\, dy - \int_0^1 |g_\epsilon(x,y)|^2\, dy = \int_\epsilon^{t+\epsilon} \frac{d}{ds}\int_0^1 |g_s(x,y)|^2\, dy\ ds \\ &= \int_\epsilon^{t+\epsilon}\int_0^1 2g_s(x,y)\,\frac{\partial}{\partial s}\, g_s(x,y)\, dy\ ds = \int_\epsilon^{t+\epsilon}\int_0^1 g_s(x,y)\,\frac{\partial^2}{\partial y^2}\, g_s(x,y)\, dy\ ds \\ &= -\int_\epsilon^{t+\epsilon}\int_0^1 |\partial_y g_s(x,y)|^2\, dy\ ds = -\int_0^t\int_0^1 |G^\epsilon(x,y)|^2\, ds\, dy,\end{aligned}$$

so that almost surely and in L^2

$$K_T^\epsilon \overset{\epsilon\searrow 0}{\to} -\int_0^1 \ell(x)\int_0^T \varphi''(u_t(x))\left[\int_0^1 |g_t(x,y)|^2 dy\right]\, dt\, dx.$$

Since K_T^ϵ and J_T^ϵ converge a.s. and in L^2 as $\epsilon \searrow 0$, then also $H_T^\epsilon = (J_T^\epsilon - K_T^\epsilon)/2$ does and we only have to identify the limit. We claim that

$$\lim_{\epsilon\searrow 0} H_T^\epsilon = \int_{O_T}\left\{\int_{O_s}\left[\int_s^T \langle \partial_x g_{t-s}\ \partial_x g_{t-r}\ \varphi''(u_t), \ell\rangle\, dt\right] dW_{r,z}\right\} dW_{s,y}, \tag{27.9}$$

where the stochastic integrals in the right-hand side (r.h.s.) are Skorohod integrals.

In order to prove (27.9) we want to commute the deterministic integral in dt and the double stochastic Itô integral which appear in $M_s dM_s$ in the definition of H_T^ϵ. However, since $t > s > r$, the double stochastic integral $\varphi''(u_t(x))\, dW_{r,z}\, dW_{s,y}$ is anticipative and therefore has to be interpreted in the Skorohod sense.

Recall the following commutation property of the Skorohod integral (p. 40 of [6]): if $F \in \mathbb{D}^{1,2}$, $v \in \mathrm{Dom}(\delta_T)$, and $\mathbb{E}[F^2 \int_{O(T)} v^2] < \infty$, then:

$$F\int_{O(T)} v\ dW_{s,y} = \int_{O(T)} (F\, v)\ dW_{s,y} + \int_{O(T)} (D_{s,y}F)\, v\, ds\, dy \tag{27.10}$$

where D is the Malliavin derivative. Recalling that $\varphi \in C^4(\mathbb{R})$ we have

$$\begin{aligned}\varphi''(u_t^\epsilon(x))\int_{O(t)} M_s\, G_{t-s}^\epsilon(x,y)\, dW_{s,y} &= \int_{O(t)} \varphi''(u_t^\epsilon(x))\, M_s\, G_{t-s}^\epsilon(x,y)\, dW_{s,y}\\ &+ \int_{O(t)} D_{s,y}\left[\varphi''(u_t^\epsilon)\right] G_{t-s}^\epsilon(x,y)\, dy\ M_s^\epsilon\, ds =: A_1 + A_2.\end{aligned}$$

Now by the chain rule

$$D_{s,y}\left[\varphi''(u_t^\epsilon(x))\right] = \varphi'''(u_t^\epsilon(x))\ D_{s,y}u_t^\epsilon(x) = \varphi'''(u_t^\epsilon(x))\ g_{\epsilon+t-s}(x,y),$$

see (1.46) at page 38 of [6], and it follows that

$$\int_0^1 D_{s,y}\left[\varphi''(u_t^\epsilon)\right] G_{t-s}^\epsilon(x,y)\, dy = \varphi'''(u_t^\epsilon)\int_0^1 \frac{1}{2}\,\partial_y\left[g_{\epsilon+t-s}(x,y)\right]^2\, dy = 0$$

by the boundary conditions at $y = 0, 1$ so that $A_2 = 0$. Arguing analogously we obtain

$$\begin{aligned}A_1 &= \int_{O(t)}\left[\varphi''(u_t^\epsilon(x))\int_{O(s)} G_{t-r}^\epsilon(x,z)\, dW_{r,z}\right] G_{t-s}^\epsilon(x,y)\, dW_{s,y}\\ &= \int_{O(t)}\left[\int_{O(s)} \varphi''(u_t^\epsilon(x))\ G_{t-r}^\epsilon(x,z)\, dW_{r,z}\right] G_{t-s}^\epsilon(x,y)\, dW_{s,y},\end{aligned}$$

where in the r.h.s. we have two Skorohod integrals. Therefore

$$H_T^\epsilon = \int_{O(T)}\left\{\int_{O(s)}\left[\int_s^T \langle G_{t-s}^\epsilon(\cdot,y)\, G_{t-r}^\epsilon(\cdot,z)\, \varphi''(u_t^\epsilon), \ell\rangle\, dt\right] dW_{r,z}\right\} dW_{s,y}.$$

Let $F \in \mathbb{D}^{2,2}$. Denoting $\zeta := (s,y,r,z) \in O_T \times O_T$, we compute

$$\begin{aligned}
\mathbb{E}\left[H_T^\epsilon\, F\right] &= \mathbb{E}\left[\int_{O_T} D_{s,y}F \left[\int_{O_s} dW_{r,z} \int_s^T \langle G_{t-s}^\epsilon\, G_{t-r}^\epsilon\, \varphi''(u_t^\epsilon), \ell\rangle\, dt\right] ds\, dy\right] \\
&= \mathbb{E}\left[\int_{O_T}\int_{O_s} D_{s,y}D_{r,z}F \left[\int_s^T \langle G_{t-s}^\epsilon\, G_{t-r}^\epsilon\, \varphi''(u_t^\epsilon), \ell\rangle\, dt\right] d\zeta\right] \\
&= \mathbb{E}\left[\int_{O_T}\int_{O_s} [D_{s,y}D_{r,z}F]\, \Phi^\epsilon(\zeta)\, d\zeta\right],
\end{aligned}$$

where for all $\epsilon > 0$ we set

$$\Phi^\epsilon(\zeta) := \int_0^1 \ell(x) \left[\int_{s+\epsilon}^{T+\epsilon} \partial_y g_{t-s}(x,y)\, \partial_z g_{t-r}(x,z)\, \varphi''(u_{t-\epsilon}^\epsilon(x))\, dt\right] dx.$$

We claim that Φ^ϵ converges weakly in $L^2(O_T \times O_T)$, with a bound in $L^2(O_T \times O_T)$ uniform in $\omega \in \Omega$: from this (27.9) follows easily by dominated convergence. First, we estimate the norm of Φ^ϵ in $L^2(O_T \times O_T)$

$$\|\Phi^\epsilon\|_{L^2}^2 = 2\int_{O_T}\int_{O_s} \left(\int_{s+\epsilon}^{T+\epsilon} \langle \ell, \partial_y g_{t-s}(\cdot,y)\, \partial_z g_{t-r}(\cdot,z)\, \varphi''(u_{t-\epsilon}^\epsilon)\rangle\, dt\right)^2 d\zeta$$

$$\begin{aligned}
= \;& 2\int_\epsilon^{T+\epsilon} dt_1 \int_\epsilon^{T+\epsilon} dt_2 \int_0^1 dx_1\, \ell(x_1) \int_0^1 dx_2\, \ell(x_2)\, \varphi''(u_{t_1-\epsilon}^\epsilon(x_1))\, \varphi''(u_{t_2-\epsilon}^\epsilon(x_2)) \\
& \times \int_{O_{t_1\wedge t_2}}\int_{O_s} \partial_y g_{t_1-s}(x_1,y)\, \partial_z g_{t_1-r}(x_1,z)\; \partial_y g_{t_2-s}(x_2,y)\, \partial_z g_{t_2-r}(x_2,z)\, d\zeta.
\end{aligned}$$

Now we integrate by parts w.r.t. y and z

$$\begin{aligned}
& \int_{O_{t_1\wedge t_2}}\int_{O_s} \partial_y g_{t_1-s}(x_1,y)\, \partial_z g_{t_1-r}(x_1,z)\; \partial_y g_{t_2-s}(x_2,y)\, \partial_z g_{t_2-r}(x_2,z)\, d\zeta \\
= \;& \int_{O_{t_1\wedge t_2}}\int_{O_s} 2\,\frac{\partial g_{t_1-s}}{\partial t_1}(x_1,y)\, g_{t_2-s}(x_2,y)\, 2\,\frac{\partial g_{t_2-r}}{\partial t_2}(x_2,z)\, g_{t_1-r}(x_1,z)\, d\zeta \\
= \;& \int_0^{t_1\wedge t_2}\int_0^s 2\,\frac{\partial g_{t_1+t_2-2s}}{\partial t_1}(x_1,x_2)\, 2\,\frac{\partial g_{t_1+t_2-2r}}{\partial t_2}(x_2,x_1)\, dr\, ds \\
= \;& \frac{1}{2}\left(g_{|t_1-t_2|}(x_1,x_2) - g_{t_1+t_2}(x_1,x_2)\right)^2,
\end{aligned}$$

obtaining

$$\sup_{\epsilon>0} \|\Phi^\epsilon\|_{L^2}^2 \le C\int_0^1 dx_1 \int_0^1 dx_2 \int_0^{4T} dt\, (g_t(x_1,x_2))^2 < \infty.$$

By this estimate, we can conclude the proof of the claim by considering h smooth with compact support in $O_T \times O_T$ and computing

$$\begin{aligned}
\int_{O_T\times O_T} \Phi^\epsilon \; h \; d\zeta &= \int_{O_T\times O_T} \frac{\partial^2 h}{\partial y \partial z} \left[\int_{s+\epsilon}^{T+\epsilon} \langle g_{t-s}\, g_{t-r}\, \varphi''(u^\epsilon_{t-\epsilon}), \ell\rangle \, dt \right] d\zeta \\
&\overset{\epsilon\searrow 0}{\to} \int_{O_T\times O_T} \frac{\partial^2 h}{\partial y \partial z} \left[\int_{s}^{T} \langle g_{t-s}\, g_{t-r}\, \varphi''(u_t), \ell\rangle \, dt \right] d\zeta \\
&= \int_{O_T\times O_T} h \left[\int_{s}^{T} \partial_x \langle g_{t-s}\, \partial_x g_{t-r}\, \varphi''(u_t), \ell\rangle \, dt \right] d\zeta. \quad \square
\end{aligned}$$

27.3 Tanaka's formula

Recall that the Itô formula (27.7) has been proved under the assumption $\varphi \in C^4$. The reason is that the double Skorohod integral in (27.8) might require two Malliavin derivatives of $\varphi''(u_t(x))$. We are going to prove that in fact this is not the case, i.e., (27.7) makes sense also for convex φ.

First, we need existence of local times for the 1D process $t \mapsto u_t(x)$, i.e., the following.

Lemma 27.1. *There exists a jointly measurable process $(L^a_t(x) : t \geq 0, a \geq 0, x \in (0,1))$ such that for all bounded Borel $f : \mathbb{R} \mapsto \mathbb{R}$ and $x \in (0,1)$*

$$\int_0^t f(u_s(x))\, ds = \int_{\mathbb{R}} f(a)\, L^a_t(x)\, da, \qquad t \geq 0. \tag{27.11}$$

Proof Fix $x \in (0,1)$ and denote by $\Delta(s,t)$ the variance of the r.v. $u_t(x) - u_s(x)$: since u is a Gaussian process, by Theorem 22.1 of [4], a process $(L^a_t(x) : t \geq 0, a \geq 0)$ satisfying (27.11) exists if

$$\sup_{s\in[0,T]} \int_0^T (\Delta(s,t))^{-1/2} dt < \infty, \qquad \forall\, T \geq 0. \tag{27.12}$$

In order to estimate the integral in (27.12), we write for $s \leq t$

$$u_t(x) = \int_0^1 g_{t-s}(x,y)\, u_s(y)\, dy + \int_s^t \int_0^1 g_{t-r}(x,z)\, dW_{r,z},$$

so that, setting $\mathcal{F}_s := \sigma(W(r,y) : r \in [0,s], y \in [0,1])$

$$\begin{aligned}
&\mathbb{E}\left[(u_t(x) - u_s(x))^2 \,\middle|\, \mathcal{F}_s \right] \\
&= \left(\int_0^1 g_{t-s}(x,y)\, [u_s(y) - u_s(x)]\, dy \right)^2 + \int_s^t \int_0^1 g^2_{t-r}(x,z)\, dz\, dr
\end{aligned}$$

and therefore

$$\begin{aligned}
\Delta(s,t) &= \mathbb{E}\left[(u_t(x) - u_s(x))^2 \right] \geq \int_s^t \int_0^1 g^2_{t-r}(x,z)\, dz\, dr \\
&= \int_s^t g_{2(t-r)}(x,x)\, dr = \int_0^{t-s} g_{2r}(x,x)\, dr.
\end{aligned}$$

By standard estimates on heat kernels (see, e.g., p. 268 of [1]) we have for all $x \in (0,1)$: $g_{2r}(x,x) \geq c\, r^{-1/2}$ for all $r > 0$, where $c > 0$ is a constant possibly depending on x. In particular, for all $x \in (0,1)$ and $s \leq t$

$$\Delta(s,t) \;\geq\; \int_0^{t-s} g_{2r}(x,x)\, dr \;\geq\; \frac{c}{2}\,(t-s)^{1/2}$$

and (27.11) is proved. The joint measurability follows from the continuity of u. □

To the increasing process $t \mapsto L_t^a(x)$ we associate as usual a random measure $dL_t^a(x)$ on $[0,\infty)$. Notice that $(dL_t^a(x) : a \in \mathbb{R},\, x \in (0,1))$ is a measurable kernel. Moreover, if $(\rho_\epsilon)_{\epsilon>0}$ is a family of smooth mollifiers, i.e.

$$\rho_\epsilon(x) = \rho(x/\epsilon)/\epsilon, \quad x \in \mathbb{R}, \qquad \rho \in C_c^\infty(\mathbb{R}), \quad \rho \geq 0, \quad \int_{\mathbb{R}} \rho = 1,$$

then for all Borel bounded $f : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, $x \in (0,1)$, $\in \mathbb{R}$

$$\int_0^T f(s, u_s(x))\, \rho_\epsilon(u_s(x) - a)\, ds \;\overset{\epsilon \searrow 0}{\to}\; \int_0^T f(s,a)\, dL_s^a(x).$$

Then the Tanaka formula is the following.

Theorem 27.2. *For all $T \geq 0$ and $\ell \in C_c^2(0,1)$*

$$\begin{aligned} \langle |u_T - a|, \ell \rangle \;&=\; \langle |u_0 - a|, \ell \rangle + \frac{1}{2}\int_0^T \langle \ell'', |u_s - a| \rangle\, ds \\ &\quad + \int_0^T \langle \mathrm{sign}(u_s - a)\, \ell,\, dW_s \rangle - \int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : dL_s^a \rangle \end{aligned} \tag{27.13}$$

where, for any family of smooth mollifiers $(\rho_\epsilon)_{\epsilon>0}$

$$\begin{aligned} &\int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : dL_s^a \rangle \;\overset{\mathrm{def}}{=}\; \lim_{\epsilon \searrow 0} \int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : \rho_\epsilon(u_s - a) \rangle\, ds \\ &= -\int_{O(T)} \ell(x) \left[\int_0^1 |g_t(x,y)|^2\, dy\right] dL_t^a(x)\; dx \\ &\quad + 2\int_{O(T)} \left\{ \int_{O(s)} \left[\int_s^T \langle \partial_x g_{t-s}(\cdot,y)\; \partial_x g_{t-r}(\cdot,z)\, \ell, dL_t^a \rangle \right] dW_{r,z} \right\} dW_{s,y}, \end{aligned} \tag{27.14}$$

and the stochastic integrals in the last term are Skorohod integrals.

Proof Let $\Phi_\epsilon \in C^2(\mathbb{R})$ such that $\Phi_\epsilon'' = 2\,\rho_\epsilon(\cdot - a)$ and $\Phi_\epsilon \to |\cdot - a\,|$ as $\epsilon \searrow 0$. By (27.7) we have

$$\begin{aligned} \langle \Phi_\epsilon(u_T), \ell \rangle \;&=\; \langle \Phi_\epsilon(u_0), \ell \rangle + \frac{1}{2}\int_0^T \langle \ell'', \Phi_\epsilon(u_s) \rangle\, ds + \int_0^T \langle \Phi_\epsilon'(u_s)\, \ell,\, dW_s \rangle \\ &\quad - \frac{1}{2}\int_0^T \langle \ell, :\left|\frac{\partial u_s}{\partial x}\right|^2 : \Phi_\epsilon''(u_s) \rangle\, ds. \end{aligned} \tag{27.15}$$

It is easy to see that all terms in the first line of the last formula converge almost surely and in L^2 as $\epsilon \searrow 0$: this yields the convergence of the remaining term. Using (27.8) we can now identify the limit. First, by the occupation time formula

$$
\begin{aligned}
K_T^\epsilon \quad &:= \quad \int_{O(T)} \ell(x)\, \Phi_\epsilon''(u_t(x)) \left[\int_0^1 |g_t(x,y)|^2\, dy\right] dx\, dt \\
&= \quad \int_{\mathbb{R}} \Phi_\epsilon''(b) \left[\int_{O(T)} \ell(x) \left[\int_0^1 |g_t(x,y)|^2\, dy\right] dL_t^b(x)\, dx\right] db \\
&\overset{\epsilon\searrow 0}{\to} \quad 2\int_{O(T)} \ell(x) \left[\int_0^1 |g_t(x,y)|^2\, dy\right] dL_t^a(x)\, dx \;=:\; K_T
\end{aligned}
$$

almost surely and in L^2. In particular, also

$$
H_T^\epsilon := \int_{O(T)} \left\{ \int_{O(s)} \left[\int_s^T \langle \partial_x g_{t-s}(\cdot,y)\; \partial_x g_{t-r}(\cdot,z)\; \Phi_\epsilon''(u_t), \ell\rangle\, dt \right] dW_{r,z} \right\} dW_{s,y}
$$

converges almost surely as $\epsilon \searrow 0$ and in L^2 to a continuous process $(H_T)_{T\geq 0}$.

It remains to identify H as the double Skorohod integral appearing in the last term of (27.14). On one hand, by the occupation time formula

$$
\begin{aligned}
\gamma^\epsilon(\zeta) \quad &:= \quad \int_s^T \langle \partial_x g_{t-s}(\cdot,y)\; \partial_x g_{t-r}(\cdot,z)\; \Phi_\epsilon''(u_t), \ell\rangle\, dt \\
&\qquad\qquad \overset{\epsilon\searrow 0}{\to} \; 2\int_s^T \langle \partial_x g_{t-s}(\cdot,y)\; \partial_x g_{t-r}(\cdot,z)\, \ell, dL_t^a\rangle \;=:\; \gamma(\zeta),
\end{aligned}
$$

with $\zeta := (s,y,r,z)$. On the other hand, one has to prove that the last term of (27.14) is well defined, i.e., that

$$
\gamma(s,y,\cdot,\cdot) \in \mathrm{Dom}(\delta_s), \qquad (\delta_s(\gamma(s,y,\cdot,\cdot)))_{(s,y)\in O(T)} \in \mathrm{Dom}(\delta_T), \tag{27.16}
$$

(see (27.5 above). To this aim, notice first that, arguing like in the proof of Theorem 27.1, for all $F \in \mathbb{D}^{2,2}$, we can compute

$$
\begin{aligned}
&\mathbb{E}\left[F \times \int_{O_T}\int_{O_s} \gamma^\epsilon(\zeta)\; dW_{r,z}\; dW_{s,y}\right] \;=\; \mathbb{E}\left[\int_{O_T}\int_{O_s} [D_{s,y}D_{r,z}F]\; \gamma^\epsilon(\zeta)\, d\zeta\right] \\
&\overset{\epsilon\searrow 0}{\to} \; \mathbb{E}\left[\int_{O_T}\int_{O_s} [D_{s,y}D_{r,z}F]\; \gamma(\zeta)\, d\zeta\right] \;=:\; \Gamma(F).
\end{aligned}
$$

On the other hand, since $H_T^\epsilon \to H_T$ in L^2, then $\Gamma(F) = \mathbb{E}\,[F \times H_T]$, so that

$$
|\Gamma(F)| \;\leq\; \|F\|_{L^2}\, \|H_T\|_{L^2}, \qquad \forall F \in \mathbb{D}^{2,2}.
$$

Therefore (27.16) holds and we can integrate by parts twice in the expectation

$$
\Gamma(F) \;=\; \mathbb{E}\left[F \times \int_{O_T}\int_{O_s} \left[2\int_s^T \langle \partial_x g_{t-s}\, \partial_x g_{t-r},\, \ell\, dL_t^a\rangle\right] dW_{r,z}\; dW_{s,y}\right]
$$

and the Theorem follows. $\square$

Corollary 27.1. *The Itô formula (27.7) holds for φ linear combination of convex functions, with*

$$\int_0^T \langle \ell, : \left| \frac{\partial u_s}{\partial x} \right|^2 : \varphi''(u_s) \rangle \, ds$$

$$\stackrel{\text{def}}{=} -\int_{\mathbb{R}} \varphi''(da) \int_{O(T)} \ell(x) \left[\int_0^1 |g_t(x,y)|^2 \, dy \right] \, dL_t^a(x) \, dx$$

$$+ 2 \int_{\mathbb{R}} \varphi''(da) \int_{O(T)} \int_{O(s)} \int_s^T \langle \partial_x g_{t-s}(\cdot, y) \; \partial_x g_{t-r}(\cdot, z) \, \ell, dL_t^a \rangle \; dW_{r,z} \; dW_{s,y}.$$

References

[1] M. van den Berg, *Gaussian bounds for the Dirichlet heat kernel*, J. Funct. Anal. **88** (1990), 267–278.

[2] L. Bertini, G. Giacomin, *Stochastic Burgers and KPZ equations from particle systems*, Comm. Math. Phys. **183** (1997), no. 3, 571–607.

[3] G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, Encyclopedia of Mathematics and Its Applications **44** (1992), Cambridge University Press, Cambridge.

[4] D. Geman, J. Horowitz, *Occupation densities*, Ann. Probab. **8** (1980), no. 1, 1–67.

[5] M. Gradinaru, I. Nourdin, S. Tindel, *Itô's and Tanaka's type formulae for the stochastic heat equation*, preprint, (2004).

[6] D. Nualart, *The Malliavin Calculus and Related Topics*, Springer-Verlag, Berlin (1995).

[7] D. Revuz, M. Yor, *Continuous Martingales and Brownian Motion*, Springer-Verlag, Berlin (1991).

[8] J.B. Walsh, *An introduction to stochastic partial differential equations*, École d'été de probabilités de Saint-Flour, XIV 1984, LNM **1180**, Springer-Verlag, Berlin (1986), 236–439.

[9] L. Zambotti, *Integration by parts on the law of the reflecting Brownian motion*, J. Funct. Anal. **223** (2005), no. 1, 147–178.