SPECTRAL AND PSEUDO-SPECTRAL METHODS FOR PARABOLIC PROBLEMS WITH NON PERIODIC BOUNDARY CONDITIONS

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ABSTRACT - The advection-diffusion equation is approximated by Chebyshev and Legendre spectral and pseudo-spectral methods. Stability results in the energy norm and error estimates in terms of the discretization parameter and of the regularity of the solution in weighted Sobolev norms are presented.

Introduction.

In this paper we analyze spectral and pseudo-spectral methods for the one-dimensional advection-diffusion equation

$$(*) \qquad u_t - \nu u_{xx} + (b\,(x)\,u)_x + b_0\,(x)\,u = f\,(t, x)$$

submitted to Dirichlet boundary conditions in the interval $I=(-1, 1)$. Approximations based on Chebyshev and Legendre polynomial expansions are considered. The pseudo-spectral schemes are essentially collocation methods at the nodes of the Gauss-Lobatto integration formulas related to the Chebyshev and Legendre weights.

Spectral approximations of problem $(*)$ have been first developed by GOTTLIEB and ORSZAG [5] in the case of constant coefficients, and by GOTTLIEB [4] when $b=b_0=0$ and ν depends on x. They establish stability in some kind of L^2 norm, and convergence when the solution is infinitely smooth.

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The aim of this paper is to provide stability results in the energy norm, and error estimates in terms of the discretization parameter (the polynomial degree) and of the regularity of u in weighted Sobolev norms. Essential tools of our analysis are the equivalence between discrete and continuous Sobolev norms for polynomials, together with an asymptotic estimate of the error between exact and discrete Gauss-Lobatto integration. This is obtained using some interpolation and orthogonal projection operators whose approximation properties have been investigated in [1, 8].

The limit case $\nu \to 0$ is not explicitely investigated, even if we emphasize the dependence of the error on ν.

We recall that equation $(*)$ submitted to periodic boundary conditions can be approximated by Fourier methods involving trigonometric polynomials (see e. g. [5, 7]). The techniques employed in this paper can be successfully applied to the theoretical analysis of these methods.

For a given weight function $\omega > 0$ over I we define

$$L^2_\omega(I) = \{\phi\colon I \to \mathbf{R} \mid \phi \text{ is measurable and } (\phi, \phi)_\omega < +\infty\},$$

with $(\phi, \psi)_\omega = \int_I \phi(x)\, \psi(x)\, \omega(x)\, dx$, and $\|\phi\|^2_{0,\omega} = (\phi, \phi)_\omega$. For any integer $k \geq 0$ we set

$$H^k_\omega(I) = \{\phi \in L^2_\omega(I) \mid d^m\phi/dx^m \in L^2_\omega(I), \quad 0 \leq m \leq k\},$$

with the norm

$$\|\phi\|^2_{k,\omega} = \sum_{m=0}^{k} \|d^m\phi/dx^m\|^2_{0,\omega}\,.$$

The space $H^s_\omega(I)$ is defined by interpolation for non integral s. Finally we set $H^1_{0,\omega}(I) = \{\phi \in H^1_\omega(I) \mid \phi(-1) = \phi(1) = 0\}$.

Throughout this paper C will denote a generic constant, positive and independent of the discretization parameter N and of u.

1. Some results concerning the continuous problems.

We recall hereafter some known results about parabolic problems in the framework of weighted Sobolev spaces. They will be used in next sections; we add the proofs for convenience of the reader.

Let $\nu > 0$ be a real number, $b \in W^{1,\infty}(I)$ and $b_0 \in L^\infty(I)$ be given; for any $f = f(t,x)$ and $u_0 = u_0(x)$ we consider the parabolic problem $(T > 0)$

$$
(1.1) \qquad \begin{cases} u_t - \nu u_{xx} + (bu)_x + b_0\, u = f & \text{in }]0, T] \times I \\ u\,(t, x) = 0 & \text{on }]0, T] \times \Gamma \\ u\,(0, x) = u_0\,(x) & \text{in } I. \end{cases}
$$

Throughout this paper we shall consider two different weight functions over I: $\omega \equiv 1$, i. e., the Legendre weight, and $\omega\,(x) = (1-x^2)^{-1/2}$, i. e., the Chebyshev weight of the first kind. We assume that $u \in L^2\,(H^1{}_\omega)$ and $u_t \in L^2\,(L^2{}_\omega)$. In addition we assume (for simplicity) that

$$
(1.2) \qquad \frac{1}{2}\, b_x + b_0 - \frac{1}{2}\, b\, \frac{\omega_x}{\omega} > 0 \quad \text{in } I.
$$

We set (formally)

$$
a_\omega\colon H^1{}_\omega\,(I) \times H^1{}_\omega\,(I) \to \mathbf{R}, \quad a_\omega\,(u, v) = \int_I u_x\,(v\omega)_x\, dx.
$$

LEMMA 1.1. *There exist three positive constants β, γ, δ such that for any $v \in H^1{}_{0,\omega}\,(I)$ and for any $u \in H^1{}_\omega\,(I)$ we have*

$$
(1.3) \qquad \|v\|_{0,\omega} \leq \beta\, \|v_x\|_{0,\omega} \qquad \textit{(Poincaré inequality)},
$$

$$
(1.4) \qquad a_\omega\,(v, v) \geq \gamma\, \|v\|^2{}_{1,\omega},
$$

$$
(1.5) \qquad |a_\omega\,(u, v)| \leq \delta\, \|u_x\|_{0,\omega}\, \|v_x\|_{0,\omega}\,.
$$

PROOF. (i) If $\omega \equiv 1$ these results are well known, as a_ω is the normal inner product of $H^1{}_0\,(I)$.

(ii) Consider now the weight $\omega\,(x) = (1-x^2)^{-1/2}$.

a - The property (1.3) follows immediately from the forthcoming result:

LEMMA 1.1. *There exists a positive constant α such that*

$$
(1.6) \qquad \forall v \in H^1{}_{0,\omega}\,(I) \quad \int_I v^2\, \omega^5\, dx \leq \alpha\, \|v_x\|^2{}_{0,\omega}.
$$

PROOF. We note that

$$\int_0^1 v^2\,\omega\,dx \le \int_0^1 v^2 \frac{1}{\sqrt{1-x}}\,dx \le \sqrt{2}\int_0^1 v^2\,\omega\,dx,$$

so it is sufficient to prove that

$$\int_0^1 \left|\frac{v(x)}{x}\right|^2 \frac{1}{\sqrt{x}}\,dx \le C\int_0^1 v_x^2(x)\frac{1}{\sqrt{x}}\,dx \qquad \text{if} \quad v(0)=0.$$

Setting $v(x)=\int_0^\infty v_\xi(\xi)\,d\xi$, we have

$$\int_0^1 \left|\frac{v(x)}{x}\right|^2 \frac{1}{\sqrt{x}}\,dx = \int_0^1 \left|\left(\frac{1}{x}\int_0^1 v_\xi(\xi)\,d\xi\right)\ x^{-1/4}\right|^2 dx.$$

Then, by an inequality of Hardy's type (see [6, Lemma 10.1]), we can bound the last term by $C\int_0^1 (v_x(x)\,x^{-1/4})^2\,dx$ and (1.6) holds. $\square$

b - To prove (1.4) we note that for any $v\in H^1_{0,\omega}(I)$ we have:

$$\begin{aligned}(1.7)\qquad a_\omega(v,v) &= -\int_I v_{xx}\,v\,\omega\,dx = \int_I v_x^2\,\omega\,dx + \int_I v_x\,v\,\omega_x\,dx = \\ &= \int_I v_x^2\,\omega\,dx + \frac{1}{2}\int_I (v^2)_x\,\omega_x\,dx = \\ &= \int_I (v_x)^2\,\omega\,dx - \frac{1}{2}\int_I v^2\,\omega_{xx}\,dx.\end{aligned}$$

On the other hand, as $\omega_{xx}=(1+2x^2)/(1-x^2)^{5/2}$, using the identity

$$\omega^5-\omega_{xx}+2\,(\omega_x)^2\,\omega^{-1}=0$$

we also have

$$(1.8)\qquad a_\omega(v,v)=\int_I |(v\,\omega)_x|^2\,\omega^{-1}\,dx+\frac{1}{2}\int_I v^2\,\omega^5\,dx.$$

Then

$$\int_I v^2 \omega_{xx}\, dx \leq 3 \int_I v^2 \omega^5\, dx$$

and from (1.8) it follows

$$\int_I v^2 \omega_{xx}\, dx \leq 6\, a_\omega (v, v).$$

By (1.7) we obtain

$$a_\omega (v, v) \geq \int_I (v_x)^2 \omega\, dx - 3 a_\omega (v, v),$$

and finally

$$a_\omega (v, v) \geq \frac{1}{4} \|v_x\|^2_{0,\omega},$$

whence (1.4) by (1.3).

c - We have for any $u \in H^1_\omega (I)$ and $v \in H^1_{0,\omega} (I)$

$$\left| \int_I u_x v_x \omega\, dx \right| \leq \left(\int_I u_x^2 \omega\, dx \right)^{1/2} \cdot \left(\int_I v_x^2 \omega\, dx \right)^{1/2} = \|u_x\|_{0,\omega} \|v_x\|_{0,\omega};$$

$$\left| \int_I u_x v \omega_x\, dx \right| = \left| \int_I u_x (v \omega_x \omega^{-1}) \omega\, dx \right| \leq \left(\int_I u_x^2 \omega\, dx \right)^{1/2} \cdot \left(\int_I v^2 (\omega_x \omega^{-1})^2 \omega\, dx \right)^{1/2}.$$

Since $\omega_x \omega^{-1} = x \omega^2$, the continuity property (1.5) holds due to (1.6). □

Throughout this paper β, γ, δ will denote the constants defined by (1.3), (1.4) and (1.5) respectively.

Setting $V = H^1_{0,\omega} (I)$, by lemma 1.1 we deduce that a_ω is continuous and coercive in V. We consider the following weak formulation of (1.1):

$$(1.9) \quad \begin{cases} u(t) \in V, \quad t\text{-a. e.}, \quad u(0) = u_0, \\ (u_t, \phi)_\omega + \nu a_\omega (u, \phi) + ((bu)_x + b_0 u, \phi)_\omega = (f, \phi)_\omega, \quad \forall \phi \in V, \quad t\text{-a. e.} \end{cases}$$

Proposition 1.1. *The following a priori estimate holds*

(1.10) $$\|u\|_{L^\infty(L^2_\omega)}+\sqrt{\nu}\,\|u\|_{L^2(H^1_\omega)}\leq C\,(\|u_0\|_{0,\omega}+\|f\|_{L^2(L^2_\omega)})$$

where C is a positive constant independent of ν.

PROOF. Setting $\phi=u$ in (1.9) and using (1.4) we have

(1.11) $$\frac{1}{2}\frac{d}{dt}\|u\|^2_{0,\omega}+\gamma\nu\,\|u\|^2_{1,\omega}+((bu)_x+b_0\,u,\,u)_\omega\leq\|f\|_{0,\omega}\,\|u\|_{0,\omega}\,.$$

Integration by parts shows that

(1.12) $$((bu)_x+b_0\,u,\,u)_\omega=\int_I\left(\frac{1}{2}\,b_x+b_0-\frac{1}{2}\,b\,\frac{\omega_x}{\omega}\right)u^2\,\omega\,dx>0$$

where the last inequality holds by (1.2). Hence (1.10) holds by (1.11) and the Gronwall lemma. □

REMARK 1.1. Condition (1.2) is unnecessary to get (1.10); however, if (1.2) is violated, the constant C appearing in (1.10) depends on ν. On the other hand, (1.2) is not a restrictive condition; there exists a suitable $\lambda\in\mathbf{R}_+$ (see [2]) such that (1.2) can be achieved by the classical change of variable $u(t)\rightarrow e^{\lambda t}\,u(t)$. □

2. Spectral methods to approximate (1.1).

Let $\{p_n\}^\infty_{n=0}$ denote the family of polynomials which are orthonormal with respect to the inner product $(\cdot,\cdot)_\omega$, i. e.,

(2.1) $$(p_n,\,p_m)_\omega=\delta_{n,m}\,.$$

It is well known that

(2.2) $$\forall u\in L^2_\omega(I)\quad u=\sum_{n=0}^{\infty}\hat{u}_n\,p_n,\quad \hat{u}_n=(u,\,p_n)_\omega\,.$$

If $\omega\equiv 1$ then $p_n=\lambda_n\,L_n$, with $\lambda_n=\sqrt{\frac{2n+1}{2}}$ and L_n is the n-th degree Legendre polynomial which satisfies $L_n(1)=1$. If $\omega(x)=(1-x^2)^{-1/2}$ then $p_n=\tau_n\,T_n$, with $\tau_0=1/\sqrt{\pi}$, $\tau_n=\sqrt{2/\pi}\;(n\geq 1)$, and T_n is the n-th degree Chebyshev polynomial of the first kind such that $T_n(1)=1$ (see, e. g., DAVIS and RABINOWITZ [3]). For any integer $N\geq 0$ the set spanned by $\{p_n\}^N_{n=0}$ coincides with the space $\mathbf{P}_N$ of polynomials of degree $\leq N$ over I. Define

$$V_N = \{\phi \in \mathbf{P}_N \mid \phi(-1) = \phi(1) = 0\}. \tag{2.3}$$

Let $u_{0N} \in V_N$ be a suitable approximation of u_0. The spectral approximation of (1.1) is the following:

$$\begin{cases} \textit{find } u_N \in H^1(V_N) \textit{ such that} \\ (u_{N,t}, \phi)_\omega + \nu\, a_\omega(u_N, \phi) + ((bu_N)_x + b_0 u_N, \phi)_\omega = (f, \phi)_\omega \quad \forall \phi \in V_N, \quad t\text{-a. e.} \\ u_N(0) = u_{0N}. \end{cases} \tag{2.4}$$

Arguing as in the proof of proposition 1.1 we can state the following stability result.

PROPOSITION 2.1. *We have*

$$\|u_N\|_{L^\infty(L^2_\omega)} + \sqrt{\nu}\, \|u_N\|_{L^2(H^1_\omega)} \leq C\,(\|u_{0N}\|_{0,\omega} + \|f\|_{L^2(L^2_\omega)}). \qquad \square \tag{2.5}$$

REMARK 2.1. Consider for instance the Legendre case. A set of basis functions for V_N is given by $\{L^0_n\}^N_{n=2}$, with

$$L^0_n = L_n - \begin{cases} L_0 & \text{if } n \text{ is even} \\ L_1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.6}$$

This basis is no more orthogonal; as a matter of fact, setting

$$K_{nm} = \lambda_n \lambda_m \delta_{nm} - \begin{cases} 0 & \text{if } n+m \text{ is odd} \\ \lambda_0^2 & \text{if } n \text{ and } m \text{ are even} \\ \lambda_1^2 & \text{if } n \text{ and } m \text{ are odd,} \end{cases}$$

it follows easily from (2.1) that

$$(L^0_n, L^0_m)_\omega = K_{nm} \quad 2 \leq n,\ m \leq N. \tag{2.7}$$

Defining

$$P^0_N \colon L^2_\omega(I) \to V_N, \qquad (u - P^0_N u, \phi)_\omega = 0 \quad \forall \phi \in V_N, \tag{2.8}$$

a simple calculation shows that

$$(2.9)\qquad P^0_N u=\sum_{j=2}^{N} U_j L^0_j,\quad \sum_{=2}^{N} U_j K_{jm}=\hat{u}_m-\begin{cases}\hat{u}_0\lambda_0^2 & \text{if } m \text{ is even}\\ \hat{u}_1\lambda_1^2 & \text{if } m \text{ is odd}\end{cases},\quad 2\le m\le N.$$

Similar arguments hold for the Chebyshev case. Then, by the help of the projection operator P^0_N, (2.4) can be written equivalently as follows:

$$u_{N,t}+P^0_N\,Lu_N=P^0_N\,f,\quad u_N(0)=u_{0,N}$$

where

$$Lu_N=-\nu u_{N,xx}+(bu_N)_x+b_0\,u_N\,.$$

Define now

$$(2.10)\qquad \Pi_N\colon V\rightarrow V_N,\quad a_\omega(u-\Pi_N u,\phi)=0\quad \forall\phi\in V_N.$$

We have the estimate (cfr. MADAY and QUARTERONI [8])

$$(2.11)\qquad \forall u\in H^\sigma_\omega(I)\cap V,\quad \sigma\ge 1,\quad \|u-\Pi_N u\|_{\mu,\omega}\le C\,\|u\|_{\sigma,\omega}\cdot$$

$$\cdot\begin{cases}N^{\mu-\sigma} & 0\le\mu\le 1\\ N^{(3\mu-1)/2-\sigma}, & 1<\mu\le\min(2,\sigma)\end{cases}$$

in both Legendre and Chebyshev cases.

By classical techniques we can prove the following theorems; in the proofs we shall use the notations: $\tilde{u}=\Pi_N u$, $e=\tilde{u}-u_N$, $\rho=\tilde{u}-u$.

THEOREM 2.1. *Assume* $u\in L^\infty(H^\sigma_\omega)$, $u_t\in L^2(H^\sigma_\omega)$ *for some* $\sigma\ge 1$, *and take* $u_{0N}=\Pi_N u_0$. *Then*

$$(2.12)\qquad \|u-u_N\|_{L^\infty(L^2_\omega)}\le CN^{-\sigma}\{\|u\|_{L^\infty(H^\sigma_\omega)}+\|u_t\|_{L^2(H^\sigma_\omega)}\}.$$

PROOF. By (1.9) and (2.10) it follows that

$$(2.13)\qquad (\tilde{u}_t,\phi)_\omega+\nu a_\omega(\tilde{u},\phi)+((b\tilde{u})_x+b_0\,\tilde{u},\phi)_\omega=(f,\phi)_\omega+$$

$$+(\rho_t+(b\rho)_x+b_0\,\rho,\phi)_\omega\quad \forall\phi\in V_N,\quad t\text{ - a. e.}$$

and so by comparison with (2.4) we have

$$\text{(2.14)}\quad \begin{cases} (e_t, \phi)_\omega + \nu a_\omega (e, \phi) + ((be)_x + b_0 e, \phi)_\omega = (\rho_t + (b\rho)_x + b_0 \rho, \phi)_\omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \phi \in V_N, \quad t\text{-a. e.} \\ e(0) = 0. \end{cases}$$

Set now $\phi = e$ in (2.14); we have

$$\text{(2.15)}\qquad ((b\rho)_x, e)_\omega \leq \frac{2\delta^2}{\nu\gamma} \|b\rho\|^2_{0,\omega} + \frac{\nu\gamma}{2} \|e_x\|^2_{0,\omega}\,.$$

As a matter of fact, by (1.5) applied with $u = \int_{-1}^{x} (b\rho)(\xi)\, d\xi$ and $v = e$ we have

$$((b\rho)_x, e)_\omega = -\int_I b\rho\, (e\omega)_x\, dx \leq \delta\, \|b\rho\|_{0,\omega}\, \|e_x\|_{0,\omega}\,.$$

Then (2.15) holds. By (2.14) and (2.15) we get

$$\frac{d}{dt}\|e\|^2_{0,\omega} + \nu\gamma\, \|e_x\|^2_{0,\omega} \leq \left\{ \|\rho_t\|^2_{0,\omega} + \frac{C}{\nu} \|b\rho\|^2_{0,\omega} + \|b_0\, \rho\|^2_{0,\omega} \right\} + 2\, \|e\|^2_{0,\omega}$$

so by the Gronwall lemma we deduce

$$\text{(2.16)}\qquad \|e\|_{0,\omega} + \sqrt{\nu} \left(\int_0^t \|e_x\|^2_{0,\omega}\, d\tau \right)^{1/2} \leq C_1 \left(\frac{1}{\sqrt{\nu}} \right) \sqrt{1 + \exp T}$$

$$\times (\|\rho_t\|_{L^2(L^2_\omega)} + \|\rho\|_{L^2(L^2_\omega)}) \quad \forall t \in [0, T].$$

By the error estimate (2.11) and the triangular inequality

$$\|u - u_N\|_{L^\infty(L^2_\omega)} \leq \|\rho\|_{L^\infty(L^2_\omega)} + \|e\|_{L^\infty(L^2_\omega)}$$

we obtain (2.12). □

THEOREM 2.2. *Under the same hypotheses of the previous theorem we have*

$$\sqrt{\nu}\,\|(u-u_N)_x\|_{L^2(L^2_\omega)} \le C\,N^{1-\sigma}\{\|u\|_{L^2(H^\sigma_\omega)} + \|u_t\|_{L^2(H_\omega^{\sigma-1})}\}; \tag{2.17}$$

if $\omega\equiv 1$ *we have in addition*

$$\sqrt{\nu}\,\|(u-u_N)_x\|_{L^\infty(L^2)} \le C\,N^{1-\sigma}\{\|u\|_{L^\infty(H^\sigma)} + \|u_t\|_{L^2(H^\sigma)}\}. \tag{2.18}$$

PROOF.

(i) The estimate (2.17) is an immediate consequence of (2.16), (2.11) and the following triangular inequality

$$\|(u-u_N)_x\|_{L^2(L^2_\omega)} \le \|\rho\|_{L^2(H^1_\omega)} + \left(\int_0^T \|e_x\|^2_{0,\omega}\,dt\right)^{1/2}.$$

(ii) Setting $\phi=e_t$ in (2.14) we obtain

$$\|e_t\|^2_0 + \frac{\nu}{2}\frac{d}{dt}a_\omega(e,e) \le (-(be)_x - b_0\,e + \rho_t + (b\rho)_x + b_0\,\rho,\, e_t) \le$$

$$\le \frac{1}{2}\|e_t\|^2_0 + \frac{1}{2}\{\|b\|_{W^{1,\infty}(I)}(\|e_x\|^2_0 + \|\rho_x\|^2_0) +$$

$$+\|b_0\|^2_{L^\infty(I)}(\|e\|^2_0 + \|\rho\|^2_0) + \|\rho_t\|^2_0\}.$$

Ignoring the term $\frac{1}{2}\|e_t\|^2_0 \ge 0$, by (2.11) and (2.12) it follows that

$$\nu\frac{d}{dt}a_\omega(e,e) \le C_1\,N^{1-\sigma}(\|u\|_{L^2(H^\sigma)} + \|u_t\|_{L^2(H^\sigma)}) + C_2\|e_x\|^2_0\,.$$

The Gronwall lemma and the triangular inequality lead to (2.18) □

3. Pseudo-spectral methods to approximate (1.1).

We denote by $\{x_j,\omega_j\}^N_{j=0}$ the nodes and the weights of the Gauss-Lobatto integration formula related to the weight ω. With the notations of the previous section we recall that the nodes $\{x_j\}$ are the roots of the polynomial $p=p_{N+1}-p_{N-1}$, so $x_0=-1$ and $x_N=1$ $\left(\text{in the Chebyshev case we have } x_j=\cos\left(-\pi+\frac{\pi j}{N}\right)\right)$. It is well known (cfr., e. g., DAVIS and RABINOWITZ [3]) that

$$(3.1)\qquad \int_I g(x)\,\omega(x)\,dx=\sum_{j=0}^{N} g(x_j)\,\omega_j \quad \forall g\in \mathbf{P}_{2N-1}\,.$$

We define an interpolation operator P_c by

$$(3.2)\qquad P_c\colon C^0(\bar{I})\to \mathbf{P}_N,\quad P_c\,u(x_j)=u(x_j)\quad 0\le j\le N;$$

it satisfies (see CANUTO and QUARTERONI [1])

$$(3.3)\qquad \forall u\in H^{\sigma}_{\omega}(I),\quad \sigma>\frac{1}{2},\quad 0\le\mu\le\sigma$$

$$\|u-P_c\,u\|_{\mu,\omega}\le C\,\|u\|_{\sigma,\omega}\times\begin{cases} N^{2\mu-\sigma} & \text{if } \omega(x)=(1-x^2)^{-1/2}\\ N^{2\mu+1/2-\sigma} & \text{if } \omega\equiv 1.\end{cases}$$

Setting

$$(3.4)\qquad \forall\phi,\psi\in C^0(\bar{I})\quad (\phi,\psi)_{N,\omega}=\sum_{j=0}^{N}\phi(x_j)\,\psi(x_j)\,\omega_j$$

by (3.1) it follows that

$$(3.5)\qquad (\phi,\psi)_{N,\omega}=(\phi,\psi)_{\omega}\quad \text{if}\quad \phi\cdot\psi\in\mathbf{P}_{2N-1};$$

moreover we have

$$(3.6)\qquad (\phi,\psi)_{N,\omega}=(P_c\,\phi,\psi)_{N,\omega}\quad \forall\phi,\psi\in C^0(\bar{I}).$$

Then

$$P_c\,u=\sum_{k=0}^{N}\tilde{u}_k\,p_k,\quad \tilde{u}_N=(u,p_N)_{N,\omega}\cdot(\|p_N\|_{N,\omega})^{-1},\quad \tilde{u}_k=(u,p_k)_{N,\omega}\quad 0\le k\le N-1$$

where we set

$$(3.7)\qquad \|\phi\|_{N,\omega}=(\phi,\phi)_{N,\omega}^{1/2}.$$

From now on we assume that $b_0\in C^0(\bar{I})$, $b\in C^1(\bar{I})$, $u_0\in C^0(\bar{I})$, $f\in C^0([0,T]\times\bar{I})$.

The pseudo-spectral approximation of (1.1) is given by the collocation problem

$$(3.8)\qquad \begin{cases} \text{find } u_c \in C^1(\mathbf{P}_N) \text{ such that} \\ u_{c,t}(x_j) - \nu u_{c,xx}(x_j) + (bu_c)_x(x_j) + b_0 u_c(x_j) = f(x_j) \\ \qquad\qquad\qquad\qquad\qquad\qquad j=1,\dots,N-1, \quad t\text{- a. e.} \\ u_c(x_0) = u_c(x_N) = 0, \\ u_c(0, x_j) = u_0(x_j) \qquad\qquad j=0,\dots,N. \end{cases}$$

By the help of (3.4) and (3.6), the problem (3.8) can be written equivalently as follows:

$$(3.9)\qquad \begin{cases} \text{find } u_c \in C^1(V_N) \text{ such that} \\ (u_{c,t}, \phi)_{N,\omega} - \nu\,(u_{c,xx}, \phi)_{N,\omega} + ((bu_c)_x + b_0 u_c, \phi)_{N,\omega} = (f, \phi)_{N,\omega} \\ \qquad\qquad\qquad\qquad\qquad\qquad \forall \phi \in V_N, \quad t\text{- a. e.} \\ u_c(0) = P_c u_0 . \end{cases}$$

LEMMA 3.1. *For both Legendre and Chebyshev weights the norms* $\|\cdot\|_{N,\omega}$ *and* $\|\cdot\|_{0,\omega}$ *are uniformly equivalent over* $\mathbf{P}_N$, *i. e.*

$$(3.10)\qquad C_1\|\phi\|_{N,\omega} \le \|\phi\|_{0,\omega} \le C_2\|\phi\|_{N,\omega} \quad \forall \phi \in \mathbf{P}_N$$

and C_1, C_2 *are two positive constants independent of* N. □

This result has been proved by the authors in [1].

THEOREM 3.1. *The pseudo-spectral problem* (3.9) *is stable; namely we have*

$$(3.11)\qquad \|u_c\|_{L^\infty(L^2_\omega)} + \sqrt{\nu}\,\|u_{c,x}\|_{L^2(L^2_\omega)} \le C\Big(\|u_0\|_{N,\omega} + \Big(\int_0^T \|f(t)\|^2_{N,\omega}\,dt\Big)^{1/2}\Big).$$

PROOF. We set $\phi = u_c$ in (3.9) and we obtain

$$(3.12)\qquad \frac{1}{2}\frac{d}{dt}\|u_c\|^2_{N,\omega} - \nu\,(u_{c,xx}, u_c)_{N,\omega} = (f, u_c)_{N,\omega} - ((bu_c)_x + b_0 u_c, u_c)_{N,\omega} \quad t\text{- a. e.}$$

By (3.5) and (1.4) we get

$$(3.13)\qquad -\nu\,(u_{c,xx}, u_c)_{N,\omega} = -\nu\,(u_{c,xx}, u_c)_{0,\omega} = a_\omega(u_c, u_c) \ge \nu\gamma\,\|u_{c,x}\|^2_{0,\omega} .$$

Furthermore we have

$$|(f, u_c)_{N,\omega}| \leq \frac{1}{2}(||f||^2_{N,\omega} + ||u_c||^2_{N,\omega})$$

(3.14) $$|(b_0\, u_c, u_c)_{N,\omega}| \leq ||b_0||_{L^\infty(I)}\, ||u_c||^2_{N,\omega}$$

$$|((bu_c)_x, u_c)_{N,\omega}| \leq ||b||_{W^{1,\infty}(I)}\, ||u_c||^2_{N,\omega} + \frac{1}{2\nu}\, ||b||^2_{L^\infty(I)}\, ||u_c||^2_{N,\omega} + \frac{\nu}{2}||u_{c,x}||^2_{N,\omega}.$$

Finally, noting that by (3.10) $C\frac{\nu}{12}||u_{c,x}||^2_{N,\omega} \leq \frac{\nu}{2}\, ||u_{c,x}||^2_{0,\omega}$, by (3.12), (3.13) and (3.14) it follows that

$$\frac{d}{dt}\, ||u_c||^2_{N,\omega} + \nu\, ||u_{c,x}||^2_{0,\omega} \leq ||f||^2_{N,\omega} + 2\, (||b_0||_{L^\infty(I)} +$$

$$+ ||b||_{W^{1,\infty}(I)} + \frac{1}{2\nu}\, ||b||^2_{L^\infty(I)})\, ||u_c||^2_{N,\omega}.$$

Now the inequality (3.11) follows by integration in t and the Gronwall lemma. We have $C \leq C_1\, (\exp(\overline{1/\nu}))$, where C_1 is independent of ν. □

We note that the right hand side of (3.11) can be uniformly bounded by $C\,(||u_0||_{L^\infty(I)} + ||f||_{L^2(L^\infty(I))})$.

In view of the discussion about the convergence of the pseudo-spectral solution u_c to u we introduce the L^2_ω-projection operator

(3.15) $$P_N\colon L^2_\omega(I) \to \mathbf{P}_N, \quad (u - P_N\, u, \phi)_\omega = 0 \qquad \forall \phi \in \mathbf{P}_N.$$

The following error estimate has been proved by the authors in [1]:

(3.16) $$\forall u \in H^\sigma_\omega(I),\ ||u - P_N\, u||_{\mu,\omega} \leq C\, ||u||_{\sigma,\omega} \begin{cases} N^{(3/2)\mu-\sigma} & 0 \leq \mu \leq \min(1, \sigma) \\ N^{2\mu-1/2-\sigma} & 1 \leq \mu \leq \sigma. \end{cases}$$

LEMMA 3.2. *For any* $\phi = \sum_{N=0}^{\infty} \hat{\phi}_N\, p_N \in L^2_\omega(I)$ *we have*

(3.17) $$|\hat{\phi}_N| \leq ||\phi - P_{N-1}\, \phi||_{0,\omega} \qquad \forall N \geq 1.$$

Moreover, setting

(3.18) $$E_\omega(\phi, \psi) = (\phi, \psi)_{N,\omega} - (\phi, \psi)_\omega \qquad \forall \phi, \psi \in C^0(\bar{I}),$$

we get

$$(3.19)\qquad |E_\omega(\phi,\psi)|\le 2\,|\hat{\phi}_N|\,|\hat{\psi}_N|\quad \forall\phi,\psi\in\mathbf{P}_N.$$

$$(3.20)\qquad |E_\omega(g,\phi)|\le C\,(\|g-P_c\,g\|^2_{0,\omega}+\|g-P_{N-1}\,g\|^2_{0,\omega}+\|\phi\|^2_{0,\omega})$$

$$\forall g\in C^0(\bar{I}),\quad \forall\phi\in\mathbf{P}_N.$$

PROOF. (i) (3.17) is obvious by Parseval equality.

(ii) From [1] we obtain

$$E_\omega(\phi,\psi)=\gamma_N\,\hat{\phi}_N\,\hat{\psi}_N,\qquad \gamma_N=\begin{cases}1 \text{ if } \omega(x)=(1-x^2)^{-1/2}\\ \dfrac{N+1}{N} \text{ if } \omega\equiv 1\end{cases}\qquad \forall\phi,\psi\in\mathbf{P}_N$$

which implies (3.19).

(iii) By (3.6) and (3.15) we get

$$\forall\phi\in\mathbf{P}_N(I),\ |E_\omega(g,\phi)|=|(P_c\,g,\phi)_{N,\omega}-(P_N\,g,\phi)_\omega|=$$

$$=|E_\omega(P_N\,g,\phi)+((P_c-P_N)\,g,\phi)_{N,\omega}|\le$$

$$\le|E_\omega(P_N\,g,\phi)|+\|(P_c-P_N)\,g\|_{N,\omega}\,\|\phi\|_{N,\omega}.$$

By (3.17) and (3.19) we have

$$|E_\omega(P_N\,g,\phi)|\le 2\,|\hat{g}_N|\,|\hat{\phi}_N|\le\|g-P_{N-1}\,g\|^2_{0,\omega}+\|\phi\|^2_{0,\omega}.$$

By (3.10) we get

$$\|(P_c-P_N)\,g\|_{N,\omega}\,\|\phi\|_{N,\omega}\le\frac{1}{2}C_1^{-1}\,(\|(P_c-P_N)\,g\|^2_{0,\omega}+\|\phi\|^2_{0,\omega})\le$$

$$\le\frac{1}{2}\,C_1^{-1}\,(\|g-P_c\,g\|^2_{0,\omega}+\|g-P_N\,g\|^2_{0,\omega}+\|\phi\|^2_{0,\omega}).$$

Then (3.20) holds since $\|g-P_N\,g\|_{0,\omega}\le\|g-P_{N-1}\,g\|_{0,\omega}$. □

In the following, *Id* will denote the identity operator.

THEOREM 3.2. *We have*

$$\|u-u_c\|_{L^\infty(L^2_\omega)}+\sqrt{\nu}\,\|(u-u_c)_x\|_{L^2(L^2_\omega)}\leq C\{\|u_0-\Pi_N u_0\|_{0,\omega}+ \tag{3.21}$$

$$+\|u_0-P_c u_0\|_{0,\omega}+\|u-\Pi_N u\|_{L^\infty(L^2_\omega)}+\|u-\Pi_N u\|_{L^2(H^1_\omega)}+$$

$$+\|u_t-\Pi_N u_t\|_{L^2(L^2_\omega)}+\|u_t-P_{N-1}u_t\|_{L^2(L^2_\omega)}+\|f-P_c f\|_{L^2(L^2_\omega)}+$$

$$+\|f-P_{N-1}f\|_{L^2(L^2_\omega)}+R(u)\}$$

where

$$R(u)=\|u-P_N u\|_{L^2(L^2_\omega)}+\|(bu)_x-P_{N-1}(bu)_x\|_{L^2(L^2_\omega)}\quad \text{if } b_0\in\mathbf{R},\ b\in\mathbf{P}_1, \tag{3.22}$$

$$R(u)=\|(b_0 u)-P_{N-1}(b_0 u)\|_{L^2(L^2_\omega)}+\|(b_0 u)-P_c(b_0 u)\|_{L^2(L^2_\omega)}+ \tag{3.23}$$

$$+\|(Id-P_c)[b_0(u-\Pi_N u)]\|_{L^2(L^2_\omega)}+\|(bu)_x-P_{N-1}(bu)_x\|_{L^2(L^2_\omega)}+$$

$$+\|(bu)_x-P_c(bu)_x\|_{L^2(L^2_\omega)}+\|(Id-P_c)[b(u-\Pi_N u)]_x\|_{L^2(L^2_\omega)}$$

otherwise.

PROOF. Let us set $\tilde{u}=\Pi_N u$, $e=\tilde{u}-u_c$, $\rho=\tilde{u}-u$, and

$$G(\rho)=\rho_t+(b\rho)_x+b_0\rho.$$

We note that by (3.5)

$$-\nu(u_{c,xx},\phi)_{N,\omega}=\nu a_\omega(u_c,\phi)\quad \forall\phi\in\mathbf{P}_N,$$

hence by (3.9) we get

$$(u_{c,t},\phi)_{N,\omega}+\nu a_\omega(u_c,\phi)+((bu_c)_x+b_0 u_c,\phi)_{N,\omega}=(f,\phi)_{N,\omega}.$$

Using (2.13) we obtain

$$\begin{cases}(e_t,\phi)_{N,\omega}+\nu a_\omega(e,\phi)+((be)_x+b_0 e,\phi)_{N,\omega}=\\ =(G(\rho),\phi)_\omega+E_\omega(\tilde{u}_t,\phi)+E_\omega(b_0\tilde{u}+(b\tilde{u})_x,\phi)-E_\omega(f,\phi)\quad \forall\phi\in V_N,\ t\text{-a. e.}\\ e(0)=(\Pi_N-P_c)u_0.\end{cases} \tag{3.24}$$

To provide an upper bound for the right hand side, we consider separately the different terms.

(i) *Evaluation of* $\|G(\rho)\|_{0,\omega}$.

Noting that Π_N commutes with time differentiation, we get

$$\|G(\rho)\|_{0,\omega} \leq \|u_t - \Pi_N u_t\|_{0,\omega} + C\,\|u - \Pi_N u\|_{1,\omega} \tag{3.25}$$

with C depending on $\|b\|_{W^{1,\infty}(I)}$ and on $\|b_0\|_{L^\infty(I)}$.

(ii) *Evaluation of* $E_\omega(f, \phi)$.

By (3.20) it follows that

$$|E_\omega(f, \phi)| \leq C\,(\|f - P_c f\|^2_{0,\omega} + \|f - P_{N-1} f\|^2_{0,\omega} + \|\phi\|^2_{0,\omega}) \tag{3.26}$$

(iii) *Evaluation of* $E_\omega(\tilde{u}_t, \phi)$.

Since $\tilde{u}_t \in \mathbf{P}_N$ by (3.19) we get

$$|E_\omega(\tilde{u}_t, \phi)| \leq 2\,|\hat{\chi}_N|\,|\hat{\phi}_N|,\quad \hat{\chi}_N = (\tilde{u}_t, p_N)_\omega$$

and by (3.17)

$$|\hat{\chi}_N| = |(\hat{u}_t - u_t, p_N)_\omega + (u_t, p_N)_\omega| \leq \|u_t - \Pi_N u_t\|_{0,\omega} + \|u_t - P_{N-1} u_t\|_{0,\omega}.$$

Then we have

$$|E_\omega(\tilde{u}_t, \phi)| \leq \|u_t - \Pi_N u_t\|^2_{0,\omega} + \|u_t - P_{N-1} u_t\|^2_{0,\omega} + \|\phi\|^2_{0,\omega}. \tag{3.27}$$

(iv) *Evaluation of* $E_\omega(b_0\tilde{u} + (b\tilde{u})_x, \phi)$.

Assume first that $b_0 \in \mathbf{R}$ *and* $b \in \mathbf{P}_1$. In this case $\tilde{b_0 u} \in \mathbf{P}_N$ and $(bu)_x \in \mathbf{P}_N$, so by (3.19) we get

$$|E_\omega(b_0\tilde{u} + (b\tilde{u})_x, \phi)| \leq 2\,|\hat{Y}_N - \hat{Z}_N|\,|\hat{\phi}_N|$$

$$\hat{Y}_N = (b_0\tilde{u}, p_N)_\omega,\quad \hat{Z}_N = ((b\tilde{u})_x, p_N)_\omega.$$

By (3.17) we have.

$$|\hat{Y}_N| \leq |b_0| \, (||u - \Pi_N u||_{0,\omega} + ||u - P_{N-1} u||_{0,\omega})$$

$$|\hat{Z}_N| \leq ||(Id - P_{N-1}) \, (b \, \Pi_N u)_x||_{0,\omega} \leq$$

$$||(Id - P_{N-1}) \, (bu)_x||_{0,\omega} + ||(Id - P_{N-1}) \, (bu - b \, \Pi_N u)_x||_{0,\omega} \leq$$

$$||(Id - P_{N-1}) \, (bu)_x||_{0,\omega} + C \, ||bu - b \, \Pi_N u||_{1,\omega} \, ,$$

by (3.16) with $\mu = \sigma = 0$) whence

$$(3.28) \qquad |E_\omega \, (b_0 \, \widetilde{u} + (b\widetilde{u})_x, \phi)| \leq C \, (||u - \Pi_N u||_{1,\omega} + ||u - P_{N-1} u||_{0,\omega} + ||(Id - P_{N-1}) \, (bu)_x||_{0,\omega}).$$

C is a constant depending on $||b||_{W^{1,\infty}}$ and $|b_0|$.

Consider now the *general case* in which no assumption is made on the coefficients b_0 and b. By (3.20) we get

$$(3.29) \qquad |E_\omega \, (b_0 \, \widetilde{u} + (b\widetilde{u})_x, \phi)| \leq C \, (||Id - P_c) \, b_0 \, \widehat{u}||^2_{0,\omega} + ||(Id - P_c) \, (b\widetilde{u})_x||^2_{0,\omega} + + ||(Id - P_{N-1}) \, b_0 \, \widehat{u}||^2_{0,\omega} + ||(Id - P_{N-1}) \, (b\widetilde{u})_x||^2_{0,\omega} + ||\phi||^2_{0,\omega});$$

by (3.16) we obtain

$$(3.30) \qquad ||(Id - P_{N-1}) \, b_0 \, \widetilde{u}||_{0,\omega} \leq ||(Id - P_{N-1}) \, b_0 \, u||_{0,\omega} + ||b_0||_{L^\infty(I)} \, ||u - \Pi_N u||_{0,\omega}$$

$$(3.31) \qquad ||(Id - P_{N-1}) \, (b\widetilde{u})_x||_{0,\omega} \leq ||(Id - P_{N-1}) \, (bu)_x||_{0,\omega} + ||b||_{W^{1,\infty}(I)} \, ||u - \Pi_N u||_{1,\omega} \, .$$

Furthermore we get

$$(3.32) \qquad ||(Id - P_c) \, b_0 \, \widehat{u}||_{0,\omega} \leq ||(Id - P_c) \, b_0 \, u||_{0,\omega} + ||(Id - P_c) \, (b_0 \, (u - \Pi_N u))||_{0,\omega}$$

$$(3.33) \qquad ||(Id - P_c) \, (b\widetilde{u})_x||_{0,\omega} \leq ||(Id - P_c) \, (bu)_x||_{0,\omega} + ||(Id - P_c) \, (b \, (u - \Pi_N u))_x||_{0,\omega} \, .$$

(v) Set now $\phi = e$ in (3.24); by (3.24), ..., (3.33) it follows that

$$(3.34) \qquad \frac{1}{2} \frac{d}{dt} ||e||^2_{N,\omega} + \nu a_\omega \, (e, e) + (b_0 \, e + (be)_x, e)_{N,\omega} \leq$$

$$\leq C\{\|u_t-P_{N-1}u_t\|^2_{0,\omega}+\|u_t-\Pi_N u_t\|^2_{0,\omega}+\|u-\Pi_N u\|^2_{1,\omega}+$$

$$+\|f-P_c f\|^2_{0,\omega}+\|f-P_{N-1}f\|^2_{0,\omega}+H^2(t)\}+\|e\|^2_{0,\omega},$$

where

$$(3.35)\quad H(t)\equiv\begin{cases}\|u-\Pi_N u\|_{1,\omega}+\|u-P_{N-1}u\|_{0,\omega}+\|(Id-P_{N-1})(bu)_x\|_{0,\omega} \\ \qquad\qquad\qquad\text{if } b_0\in\mathbf{R},\ b\in\mathbf{P}_1, \\ \|(Id-P_{N-1})b_0 u\|_{0,\omega}+\|(Id-P_{N-1})(bu)_x\|_{0,\omega}+ \\ +\|(Id-P_c)b_0 u\|_{0,\omega}+\|(Id-P_c)(b_0(u-\Pi_N u))\|_{0,\omega}+ \\ +\|u-\Pi_N u\|_{1,\omega}+\|(Id-P_c)(bu)_x\|_{0,\omega}+\|(Id-P_c)(b(u-\Pi_N u))_x\|_{0,\omega} \\ \qquad\qquad\qquad\text{otherwise.}\end{cases}$$

By the stability result (3.11) applied to e, and by the equality $u-u_c=(u-\Pi_N u)+e$, we get the estimate

$$(3.36)\qquad \|u-u_c\|_{L^\infty(L^2_\omega)}+\sqrt{\nu}\,\|(u-u_c)_x\|_{L^2(L^2_\omega)}\leq C\{\|u_0-\Pi_N u_0\|_{0,\omega}+$$

$$+\|u_0-P_c u_0\|_{0,\omega}+\|u-\Pi_N u\|_{L^\infty(L^2_\omega)}+\|u-\Pi_N u\|_{L^2(H^1_\omega)}+$$

$$+\|u_t-\Pi_N u_t\|_{L^2(L^2_\omega)}+\|u_t-P_{N-1}u_t\|_{L^2(L^2_\omega)}+\|f-P_c f\|_{L^2(L^2_\omega)}+$$

$$+\|f-P_{N-1}f\|_{L^2(L^2_\omega)}+\|H\|_{L^2(0,T)}\}.$$

Finally we note that $\|w-P_N w\|_{0,\omega}\leq\|w-P_{N-1}w\|_{0,\omega}$ for any $w\in L^2_\omega(I)$. Then the theorem is completely proved due to (3.36) and the definition (3.35). □

THEOREM 3.3. *Assume that for a suitable $\sigma>1$ we have:*

$b_0\in W^{\sigma,\infty}(I)$, $b\in W^{\sigma,\infty}(I)$, $u_0\in H_\omega^{\sigma-1}(I)$, $f\in L^2(H_\omega^{\sigma-1})$,

$u\in L^\infty(H_\omega^{\sigma-1})\cap L^2(H_\omega^\sigma)$, $u_t\in L^2(H_\omega^{\sigma-1})$. *Then*

(i) *if $\omega(x)=(1-x^2)^{-1/2}$ we have the estimate* $(\forall\varepsilon>0)$

(3.37) $$\|u-u_c\|_{L^\infty(L^2_\omega)}+\sqrt{\nu}\,\|(u-u_c)_x\|_{L^2(L^2_\omega)}\leq$$

$$\leq C\times\begin{cases} N^{1-\sigma}K(u_0,f,u,u_t) & \text{if } b_0\in\mathbf{R},\ b\in\mathbf{P}_1\\ N^{1-\sigma}K(u_0,f,u,u_t)+N^{5/4+\varepsilon-\sigma}\|u\|_{L^2(H^\sigma_\omega)} & \text{otherwise}\end{cases}$$

(ii) *if* $\omega\equiv1$ *we have the estimate* $(\forall\varepsilon>0)$

(3.38) $$\|u-u_c\|_{L^\infty(L^2)}+\sqrt{\nu}\,\|(u-u_c)_x\|_{L^2(L^2)}\leq$$

$$\leq C\times\begin{cases} N^{3/2-\sigma}K(u_0,f,u,u_t) & \text{if } b_0\in\mathbf{R},\ b\in\mathbf{P}_1\\ N^{3/2-\sigma}K(u_0,f,u,u_t)+N^{7/4+\varepsilon-\sigma}\|u\|_{L^2(H^\sigma)} & \text{otherwise,}\end{cases}$$

where

(3.39) $$K(u_0,f,u,u_t)=\|u_0\|_{\sigma-1,\omega}+\|f\|_{L^2(H_\omega^{\sigma-1})}+\|u\|_{L^\infty(H_\omega^{\sigma-1})}+ + \|u\|_{L^2(H^\sigma_\omega)}+\|u_t\|_{L^2(H_\omega^{\sigma-1})}\,.$$

C is a suitable constant depending on ε, $\|b_0\|_{W^{\sigma,\infty}(I)}$ *and* $\|b\|_{W^{\sigma,\infty}(I)}$.

PROOF. The error estimates (3.37) and (3.38) can be achieved by previous theorem and results (2.11), (3.3), (3.16) concerning the approximation properties of the operators Π_N, P_c and P_N. We must only be careful to estimate the term $R(u)$ which appears in (3.22) and (3.23). If $b_0\in\mathbf{R}$ and $b\in\mathbf{P}_1$ we get:

(3.40) $$R(u)\leq C\{N^{-\sigma}\|u\|_{L^2(H^\sigma_\omega)}+\|b\|_{W^{\sigma,\infty}(I)}N^{1-\sigma}\|u\|_{L^2(H^\sigma_\omega)}\}.$$

For evaluating (3.23) we set now for convenience $e(\omega)=0$ if $\omega(x)=(1-x^2)^{-1/2}$, and $e(\omega)=1/2$ if $\omega\equiv1$. We have

$$\|b_0u-P_N(b_0u)\|_{L^2(L^2_\omega)}\leq C\|b_0\|_{W^{\sigma,\infty}(I)}N^{-\sigma}\|u\|_{L^2(H^\sigma_\omega)};$$

$$\|b_0u-P_c(b_0u)\|_{L^2(L^2_\omega)}\leq C\|b_0\|_{W^{\sigma,\infty}(I)}N^{e(\omega)-\sigma}\|u\|_{L^2(H^\sigma_\omega)};$$

$$\|(Id-P_c)\,(b_0\,(u-\Pi_N\,u))\|_{L^2(L^2_\omega)} \le C\,N^{e(\omega)-1/2-\varepsilon}\,\|b_0\,(u-\Pi_N\,u)\|_{L^2(H_\omega^{1/2+\varepsilon})} \le$$

$$\le C\,\|b_0\|_{W^{1/2+\varepsilon,\infty}(I)}\,N^{e(\omega)-\sigma}\,\|u\|_{L^\infty(H^\sigma_\omega)};$$

$$\|(bu)_x-P_N\,(bu)_x\|_{L^2(L^2_\omega)} \le C\,\|b\|_{W^{\sigma,\infty}(I)}\,N^{1-\sigma}\,\|u\|_{L^2(H^\sigma_\omega)};$$

$$\|(bu)_x-P_c\,(bu)_x\|_{L^2(L^2_\omega)} \le C\,\|b\|_{W^{\sigma,\infty}(I)}\,N^{e(\omega)+1-\sigma}\,\|u\|_{L^2(H^\sigma_\omega)};$$

$$\|(Id-P_c)\,(b\,(u-\Pi_N\,u))_x\|_{L^2(L^2_\omega)} \le C\,N^{e(\omega)-1/2-\varepsilon}\,\|b\,(u-\Pi_N\,u)\|_{L^2(H_\omega^{3/2+\varepsilon})} \le$$

$$\le C\,\|b\|_{W^{3/2+\varepsilon,\infty}(I)}\,N^{e(\omega)-1/2-\varepsilon}\,N^{7/4+3\varepsilon/2-\sigma}\,\|u\|_{L^2(H^\sigma_\omega)} \le$$

$$\le C\,\|b\|_{W^{3/2+\varepsilon,\infty}(I)}\,N^{e(\omega)+5/4+\varepsilon-\sigma}\,\|u\|_{L^2(H^\sigma_\omega)}.$$

This completes the proof of the theorem. □

ADDED IN PROOF. Instead of (3.8), one can consider the following pseudo-spectral problem:

$$(3.8)' \qquad \begin{cases} \text{find } u_c \in C^1(P_N) \text{ such that} \\ u_{c,t}(x_j) - \nu u_{c,xx}(x_j) + [P_c\,(bu_c)]_x\,(x_j) + b_0\,u_c(x_j) = f(x_j) \\ \qquad\qquad\qquad\qquad j=1,\dots,N-1, \quad t\text{-}a.\ e., \\ u_c(x_0) = u_c(x_N) = 0, \\ u_c(0,x_j) = u_0(x_j) \qquad\qquad j=0,\dots,N. \end{cases}$$

In this scheme, differentiation in the first-order term is done via the Fast Fourier Transform algorithm, and one does not need to know the derivative of b at the nodes.The analysis of stability and convergence for (3.8)′ can be carried out by the arguments of Sect. 3. However, one can take advantage from the fact that the first-order term is now a polynomial of degree $N-1$. Then the proof can be considerably simplified due to the relation (3.5) which can be used in different points of it. The new error estimate is as follows:

$$\|u-u_c\|_{L^\infty(L^2_\omega)} + \sqrt{\nu}\,\|(u-u_c)_x\|_{L^2(L^2_\omega)} \le CN^{1-\sigma}\,K(u_0, f, u, u_t),$$

with $K(u_0, f, u, u_t)$ defined in (3.39), for both Chebyshev and Legendre points.

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