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CONVERGENCE ESTIMATES FOR SEMIDISCRETE APPROXIMATIONS OF NONSELFADJOINT PARABOLIC EQUATIONS*

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Abstract. Semidiscrete Galerkin approximations of the second order nonselfadjoint parabolic equations are considered. L_2-error estimates for L_2-initial conditions are derived by use of semigroup methods and of analyticity of the underlined semigroup. The scheme here presented can be also freely applied to nonconforming elements.

1. Introduction. We are concerned with a semidiscrete Galerkin approximation scheme of the second order parabolic equation:

$$\frac{dy}{dt}+A(x,\partial)y=f \quad \text{in } \Omega,$$
$$y|_\Gamma=0 \quad \left(\text{or } \frac{\partial y}{\partial \eta}\bigg|_\Gamma=0\right), \tag{1.1}$$
$$y(0)=y_0,$$

where Ω is a bounded domain in R^n with sufficiently smooth boundary Γ, and $A(x,\partial)$ is a uniformly elliptic, second order differential operator:

$$A(x,\partial)y=-\sum_{j,k=1}^{n}\frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial y}{\partial x_k}\right)+\sum_{j=1}^{n}b_j(x)\frac{\partial y}{\partial x_j}+c(x)y. \tag{1.2}$$

The semidiscrete approximate of (1.1) is defined by the equation:

$$\frac{dy_n}{dt}+A_hy_h=f_h, \qquad y_h(0)=y_{0h}, \tag{1.3}$$

where $y_h, f_h, y_{0h}\in V_h$, V_h being a finite dimensional approximating subspace of order $r\geqq 2$, with h, the parameter of discretization, tending to zero. Moreover $A_h: V_h\to V_h$ denotes the approximation (in a sense to be defined later) of the operator A which is defined by $A(x,\partial)$ with corresponding homogeneous boundary conditions.

The semidiscrete Galerkin approximations of parabolic equations have been considered by a number of authors. In particular, when the operator A is selfadjoint the following error estimates were obtained in [B3] for the homogeneous case (i.e. $f\equiv 0$): L_2-error estimates of order $O(h^r)$ for $y_0\in D(A^{r/2})$ and of order $O(h^r/t^{r/2})$ for $y_0\in L_2(\Omega)$.

Very recently, three independent groups of authors: Huang and Thomee [H1], Sammon [S1], and Luskin and Rannacher [L1] have extended the results of [B3] to include the time-dependent case, and the case of A nonselfadjoint. However, in all these cited works, the various authors invariably introduce assumptions that *bound* the *degree* of *nonselfadjointness* of A.

More precisely, in [L1] it is assumed that $a_{jk}=a_{kj}$ in (1.2), and in [S1] and [H1] the authors assume more generally that

$$|(A_h^{-1}f,g)-(f,A_h^{-1}g)|\leqq C(f,A_h^{-1}f)^{1/2}\|A_h^{-1}g\| \quad \text{for } f,g\in V_h. \tag{1.4}$$

However this assumption again is satisfied in practice when $a_{jk}=a_{kj}$.

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The purpose of this paper is to derive similar error estimates but without restricting the degree of *nonselfadjointness*, i.e. without assuming (1.4). This fact may be of particular use in case of low regularity of the coefficients of $A(x, \sigma)$; that does not allow transformation of the principal part in (1.2) into formally selfadjoint form. The basic assumptions which we shall make are: (i) analyticity of the original generator and (ii) coercivity and boundedness in H' topology of the discrete generator.

As a consequence, we shall derive L_2-error estimates of order $O(h^\gamma)$; $\gamma \leqq r$ and $O(h^\gamma/t^{\gamma/2})$ for "rough data". We mention also that our techniques are essentially based on a semigroup approach and are different from the "energy" methods used in [H1], [L1].

It should be finally emphasized that our techniques do not require that the finite dimensional subspaces satisfy zero boundary conditions (in the Dirichlet case). Therefore, they can be freely applied to nonconforming elements. The paper is organized as follows. Section 2 contains some preliminary results from semigroup theory which are needed in the sequel. Section 3 is devoted to the formulation and proof of the major estimates under some abstract hypothesis concerning the approximation A_h.

In § 4 we show that a number of standard approximation schemes do satisfy our hypotheses in § 3.

The following notation is used in the paper:

$\|\cdot\|$, $|\cdot|$, norms in $L_2(\Omega)$, and $L_2(\Gamma)$, respectively;

$\|\cdot\|_s$, $|\cdot|_s$, norms in $H^s(\Omega)$, and $H^s(\Gamma)$, respectively;

$(\cdot), \langle\cdot\rangle$, scalar products in $L_2(\Omega)$, $L_2(\Gamma)$, respectively;

$H_0^s(\Omega)$, Sobolev space of order s with zero boundary conditions;

$\beta(p, q) = \int_0^t (1-z)^{1-p} z^{1-q}\, dz$, beta function with $p > 0$, $q > 0$.

2. Preliminaries and statement of results. Our approach is based on a semigroup representation of the solution to (1.1) and (1.3). The error of approximation will be also expressed via a convolution operator involving discrete and continuous semigroups. In order to define the corresponding semigroup we need to introduce an unbounded operator $A: L_2(\Omega) \to L_2(\Omega)$ by the formula

$$Af = A(\xi, \sigma)f \quad \text{for } g \in \mathcal{D}(A),$$

where

$$\mathcal{D}(A) = \left\{ f \in L_2(\Omega), Af \in L_2(\Omega); f|_\Gamma = 0 \text{ or } \left(\frac{\partial f}{\partial \eta}\right)\bigg|_\Gamma \neq 0 \right\}$$

where $\partial f/\partial n$ stands for the normal derivative.

It is a well-known fact that $-A$ generates on $L_2(\Omega)$ a strongly continuous, analytic semigroup $S(t) = e^{-At}$ such that

$$\|S(t)\| \leqq C e^{wt}. \tag{2.1}$$

Without loss of generality, we can assume that the spectrum of A is on the right of the complex plane, so that the fractional powers of A are well defined (otherwise one can consider $\bar{A} = A - \lambda I$ for a suitable positive λ). In this paper we will use crucially the analyticity of the semigroup $S(t)$, which combined with interpolation theory gives (see [P1])

$$\|A^\alpha S(t)x\| \leqq \frac{C\|x\|}{t^\alpha}, \qquad t>0, 0 \leqq \alpha \leqq 1. \tag{2.2}$$

Another property that we will use constantly is the commutativity of the semigroup with all fractional powers of the generator, i.e.

$$A^\alpha S(t)x = S(t)A^\alpha x \quad \text{for } x \in D(A^\alpha) \text{ and any real } \alpha. \tag{2.3}$$

Notice also that due to the analyticity of $S(t)$ (hence differentiability) we have

$$A^n S(t)x \in L_2(\Omega) \quad \text{for all } x \in L_2(\Omega),\ n>0 \text{ and } t>0. \tag{2.4}$$

We have $H_0^s(\Omega) = H^s(\Omega)$ for $s<\frac{1}{2}$ with topology given by:

$$\|x\|_{H^s(\Omega)} = \|A^{s/2}x\|_{L_2(\Omega)}, \qquad s<\tfrac{1}{2}. \tag{2.5}$$

In the sequel we will use the following semigroup representation for the solution to (1.1):

$$y(t) = S(t)y_0 + \int_0^t S(t-z)f(z)\,dz. \tag{2.6}$$

Now let us introduce approximating subspaces. For a sequence of discretization parameters $h>0$ tending to zero, let $V_h \subset H^k(\Omega)$ for $k \geqq 1$ be finite dimensional subspaces such that

$$V_h \text{ is a } S_h^{r,k}(\Omega) \text{ system with } r>k \quad \text{[B1]}, \tag{2.7}$$

V_h satisfies the inverse assumption property, i.e.

$$\|v_h\|_\beta \leqq Ch^{-\beta+s}\|v_h\|_s, \qquad 0 \leqq \beta \leqq 1, \quad s-\beta \leqq 0, \quad v_h \in V_h. \tag{2.8}$$

(C will stand for a generic constant independent on h.) As a consequence of (2.7) and (2.8) we have

$$\|R_h y - y\|_\beta \leqq Ch^{s-\beta}\|y\|_s, \qquad 0 \leqq \beta \leqq k, \quad 0 \leqq s \leqq r, \quad s-\beta \geqq 0 \tag{2.9}$$

where R_h stands for an L_2-orthogonal projection of $L_2(\Omega)$ on V_h. Let $A_h: V_h \to V_h$ be a family of positive approximation of A such that

$$\|(R_h A^{-1} - A_h^{-1}R_h)x\| \leqq Ch^{\alpha+2}\|A^{\alpha/2}x\| \quad \text{for } \alpha+2 \leqq r, \quad \alpha \geqq 0, \tag{H1}$$

$$\|A_h^\beta x_h\| \leqq Ch^{-2\beta}\|x_h\|, \qquad x_h \in V_h, \quad 0 \leqq \beta \leqq 1, \tag{H2}$$

and *one* of the following conditions is satisfied:

(i) $((A_h - A_h^*)x_h, y_h) \leqq C(A_h x_h, x_h)^{1/2}\|y_h\|$ for $x_h, y_h \in V_h$;

(ii) A_h^2 is positive;

(H3) (iii) $(A_h x_h, x_h) \geqq \sigma\|x_h\|_1^2$,

$(A_h x_h, y_h) \leqq C\|x_h\|_1\|y_h\|_1$ for $x_h, y_h \in V_h$

for some positive constants $\sigma, C>0$ independent of h.

Remarks.

1) Notice that conditions (i) and (ii) are satisfied for A being selfadjoint. Condition (i) is equivalent to the one required in [H1], [S1].

2) If one assumes (iii) then (H2) follows via the inverse assumption.

3) Notice that with the exception of (H1) all the other requirements imposed so far on the coefficients a_{ij} do allow them to be only in $L_\infty(\Omega)$. However, in order to satisfy the estimate of the error of the corresponding elliptic part for arbitrarily *large* α, more regularity is needed of the coefficients. Thus this point is the only one where the degree of "nonsmoothness" of the coefficients must be a constraint. In the case of nonsmooth coefficients—say piecewise smooth—one may use more refined subspaces $S_h^{t,k}$ (see [B1, Chap. 8] and references cited there) in order to obtain the estimate (H1).

3. Statement of results. Let $S_h(t)=e^{-A_ht}$ be the semigroup corresponding to $-A_h$. Now we are in a position to formulate our results.

THEOREM 1. *For all* $x\in D(A^{s/2})$ *with* $0\leqq s<r$ *we have*

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq Ch^s\|A^{s/2}x\| \quad \textit{and}$$

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq Ch^r\ln(h)\|A^{r/2}x\|.$$

For "rough data" we have the following estimates.

THEOREM 2.

(i) *For all* $x\in D(A^{\alpha/2+\varepsilon})$ *with* $\varepsilon\in(0,1)$ *and* $\alpha+2\leqq r$, $\alpha\geqq 0$, *we have*

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq \frac{Ch^{2+\alpha-2\varepsilon}}{t^{1-2\varepsilon}}\|A^{\alpha/2+\varepsilon}x\|;$$

(ii) *Assuming additionally that conditions* (H1)–(H3) *are also satisfied for* A^* *and the corresponding* A_h^*, *and assuming that* $r>2$, *we have*

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq \frac{Ch^s}{t^{s/2}}\|x\|,\quad 0\leqq s<r \quad \textit{and}$$

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq C\frac{h^r\ln(h)}{t^{r/2}}\|x\|.$$

For the nonhomogeneous case ($f\neq 0$) we obtain

THEOREM 3.

(i) *For* $f(t)\in D(A^{\alpha/2+\varepsilon})$ *with* $\alpha+2\leqq r$; $\varepsilon\in(0,1)$

$$\|(y-y_h)(t)\|\leqq Ch^{2+\alpha-2\varepsilon}\|f\|_{L_\infty[0t;D(A^{\alpha/2+\varepsilon})]}.$$

(ii) *Assuming additionally that* (H1)–(H3) *are also satisfied for* A^* *and* A_h^*, *and* $r>2$, *we have*

$$\|(y-y_h)(t)\|\leqq Ch^2\ln h\,\|f\|_{L_\infty[0,t;L_2(\Omega)]}$$

where y *and* y_h *are the solutions corresponding to* (1.1) *and* (1.3) *respectively* (*with* $y_0=0$).

Remark. Notice that the estimates of Theorem 1 and Theorem 2, 3(ii) do not give optimal results for $s=r$ (term $\ln h$ and requirement $r>2$ in Theorem 2(ii)). This is due to the time-domain technique of the proof which is given below. Its advantage however (although it is nonoptimal for $s=r$) is that it allows for direct generalizations to time-dependent problems (i.e., where $a_{ij}(x,t)$ depend on t). In fact, it is just enough to replace the analytic semigroup with the corresponding analytic evolution operator; appropriate modifications in the proof given below are reasonably straightforward. On the other hand, the results for Theorems 1, 2, 3 for time-independent problems can be also proved by using a different (Laplace transform, i.e.) λ domain technique. By doing this one can show that the results of Theorems 1, 2, 3 *give the optimal rates* also for $s=r$ and without the restriction $r>2$. This proof was given in [L2], and for the sake of completeness is reproduced in the Appendix of the present paper. The negative side of the λ-approach is that it may not allow for direct generalizations to time-independent problems. This justifies in our view the choice of the t-domain technique which, although nonoptimal for $s=r$, can be easily extended to time-dependent problems. It should be also pointed out that for slightly nonselfadjoint operators A, the energy methods used in [L1], [H1], [S1] give optimal results also for $s=r$.

4. Error estimates. Proofs of main results. We shall start by formulating and proving several lemmas that are basic for our subsequent error analysis.

LEMMA 4.1. *For all $x \in L_2(\Omega)$ we have the following identity*:

$$S_h(t)R_hx - R_hS(t)x = \int_0^t A_hS_h(t-z)[R_hA^{-1} - A_h^{-1}R_h]AS(z)x\,dz.$$

Proof.

$$\left(\frac{d}{dt}S_h(t)R_hx, v_h\right) + (A_hS_h(t)R_hx, v_h) = 0, \qquad v_h \in V_h. \tag{4.1}$$

On the other hand, by the definition of R_h

$$\left(\frac{d}{dt}R_hS(t)x, v_h\right) + (A_hR_hS(t)x, v_h) = ((A_hR_hS(t)x - AS(t)x), v_h), \qquad v_h \in V_h. \tag{4.2}$$

By subtracting (4.2) from (4.1) we obtain

$$\begin{aligned}\left(\frac{d}{dt}(S_h(t)R_hx - R_hS(t)x), v_h\right) &+ (A_h(S_h(t)R_hx - R_hS(t)x), v_h)\\ &= (AS(t)x - A_hR_hS(t)x, v_h),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&S_h(t)R_hx - R_hS(t)x\\ &\quad= \int_0^t S_h(t-z)R_h[AS(z)x - A_hR_hS(z)x]\,dz \quad (\text{since } R_hA_h = A_h)\\ &\quad= \int_0^t A_hS_h(t-z)[A_h^{-1}R_hAS(z)x - R_hS(z)x]\,dz\\ &\quad= \int_0^t A_hS_h(t-z)[A_h^{-1}R_h - R_hA^{-1}]AS(z)\,dz.\end{aligned}$$

□

LEMMA 4.2.

$$\|A_h^\beta S_h(t)x_h\| \leqq \frac{C}{t^\beta}\|x_h\| \quad \textit{for } 0 \leqq \beta \leqq 1,$$

where C does not depend on h and t for $t<T$, for some $T>0$.

Proof. Using interpolation it is enough to prove that

$$\|S_h(t)x_h\| \leqq C\|x_h\|, \qquad x_h \in V_h \tag{4.3}$$

and

$$\|A_hS_h(t)x_h\| \leqq \frac{C}{t}\|x_h\|, \qquad x_h \in V_h. \tag{4.4}$$

By the semigroup property

$$\frac{dS_h(t)}{dt}x_h + A_hS_h(t)x_h = 0. \tag{4.5}$$

Taking the inner product of (4.5) with $S_h(t)x_h$ we obtain

$$\frac{1}{2}\frac{d}{dt}\|S_h(t)x_h\|^2 + (A_hS_h(t)x_h, S_h(t)x_h) = 0.$$

After integrating the above from 0 to t we have

$$\tfrac{1}{2}\|S_h(t)x_h\|^2 + \int_0^t (A_hS_h(z)x_h, S_h(z)x_h)\,dz = \tfrac{1}{2}\|x_h\|^2, \tag{4.6}$$

which in view of the positivity of A_h shows (4.3). To prove (4.4) we differentiate (4.5) in t and take the inner product with $(d/dt)S_h(t)x_h$, to get

$$\frac{1}{2}\frac{d}{dt}\|A_hS_h(t)x_h\|^2 + \left(A_h\frac{d}{dt}S_h(t)x_h, \frac{d}{dt}S_h(t)x_h\right) = 0. \tag{4.7}$$

Multiply (4.7) by $2t^2$ and integrate from 0 to t to get

$$\begin{aligned} t^2\|A_hS_h(t)x_h\|^2 + 2\int_0^t z^2(A_h^2S_h(z)x_h, A_hS_h(z)x_h)\,dz \\ = 2\int_0^t z\|A_hS_h(z)x_h\|^2\,dz. \end{aligned} \tag{4.8}$$

Next, take an inner product of (4.5) with $t(d/dt)S_h(t)x_h$ to obtain

$$t\|A_hS_h(t)x_h\|^2 + t\left(A_hS_h(t)x_h, \frac{d}{dt}S_h(t)x_h\right) = 0.$$

The last expression can be rewritten equivalently as

$$\begin{aligned} t\|A_hS_h(t)x_h\|^2 + \frac{d}{dt}(tA_hS_h(t)x_h, S_h(t)x_h) + t(A_h^2S_h(t)x_h, S_h(t)x_h) \\ = (A_hS_h(t)x_h, S_h(t)x_h). \end{aligned} \tag{4.9}$$

After integrating the above from 0 to t we obtain

$$\begin{aligned} \int_0^t z\|A_hS_h(z)x_h\|^2\,dz + t(A_hS_h(t)x_h, S_h(t)x_h) + \int_0^t z(A_h^2S_h(z)x_h, S_h(z)x_h)\,dz \\ = \int_0^t (A_hS_h(z)x_h, S_h(z)x_h)\,dz. \end{aligned} \tag{4.10}$$

From this point on the proof we will proceed separately for each of the cases (H3)(i), (ii), (iii).

First let us assume that (H3)(ii) is satisfied. Then by using positivity of A_h, and A_h^2 in (4.10) via (4.6) we obtain

$$\int_0^t z\|A_hS_h(z)x_h\|^2\,dz \leqq \tfrac{1}{2}\|x_h\|^2, \tag{4.11}$$

which combined with (4.8) yields (4.4).

If (H3)(i) is satisfied,[1] then we rewrite (4.10) as

$$\int_0^t z\|A_hS_h(z)x_h\|^2\,dz+t(A_hS_h(t)x_h,S_h(t)x_h)+\int_0^t z\|A_hS_h(z)x_h\|^2\,dz$$
$$=\int_0^t (z(A_h-A_h^*)S_h(z)x_h,A_hS_h(z)x_h)\,dz+\int_0^t (A_hS_h(z)x_h,S_h(z)x_h)\,dz.$$

By the positivity of A_h, by (H3)(i) and by (4.6) we obtain

$$\begin{aligned}\int_0^t z\|A_hS_h(z)x_h\|^2\,dz&\\ &\leqq C\int_0^t z(A_hS_h(z)x_h,S_h(z)x_h)^{1/2}\|A_hS_h(z)x_h\|\,dz+\tfrac12\|x_h\|^2\\ &\leqq C\varepsilon\int_0^t z\|A_hS_h(z)x_h\|^2\,dz\\ &\quad+\frac{C}{\varepsilon}\int_0^t z(A_hS_h(z)x_h,S_h(z)x_h)\,dz+\tfrac12\|x_h\|^2.\end{aligned}$$

By selecting a suitable small ε and making use once more of (4.6), we obtain

$$\int_0^t z\|A_hS_h(z)x_h\|^2\,dz\leqq C\|x_h\|^2,$$

which combined again with (4.8) yields (4.4).

Our proof will be completed as soon as we handle the third case (i.e. (H3)(iii)). Taking (4.8) as a starting point, and using (H3)(iii) we get

$$\begin{aligned}&t^2\|A_hS_h(t)x_h\|^2+2\int_0^t z^2\|A_hS_h(t)x_h\|_1^2\,dz\\ &\qquad\leqq 2\int_0^t z\|A_hS_h(z)x_h\|^2\,dz.\end{aligned}\tag{4.12}$$

On the other hand by (H3)(iii) (second part)

$$\begin{aligned}\int_0^t z\|A_hS_h(z)x_h\|^2\,dz&=\int_0^t z(A_hS_h(z)x_h,A_hS_h(z)x_h)\,dz\\ &\leqq C\int_0^t z\|A_hS_h(z)x_h\|_1\,\|S_h(z)x_h\|_1\,dz\\ &\leqq C\varepsilon\int_0^t z^2\|A_hS_h(z)x_h\|_1^2\,dz+\frac{C}{\varepsilon}\int_0^t \|S_h(z)x_h\|_1^2\,dz.\end{aligned}$$

After plugging the last expression into (4.12) and selecting a suitable ε we arrive at

$$t^2\|A_hS_h(t)x_h\|^2\leqq C\int_0^t \|S_h(z)x_h\|_1^2\,dz.\tag{4.13}$$

On the other hand from (4.6) via (H3)(iii) we have

$$\int_0^t \|S_h(z)x_h\|^2\,dz\leqq C\|x_h\|^2,$$

which combined with (4.13) completes the proof. □

[1] The proof in the case (H3)(i) is conceptually similar to the proof of [L1, (2.19)].

Next, we prove the following simple lemma

LEMMA 4.3. *Hypothesis* (H1) *combined with* (H2) *imply*

$$\text{(H1}')\qquad \|A_h^\beta[R_hA^{-1}-A_h^{-1}R_h]x\| \leqq Ch^{2+\alpha-2\beta}\|A^{\alpha/2}x\|$$

for all $x \in D(A^{\alpha/2})$, $0 \leqq \beta \leqq 1$ *and* $\alpha + 2 \leqq r$.

Proof. By (H2) we obtain

$$\|A_h^\beta[R_hA^{-1}-A_h^{-1}R_h]x\| \leqq Ch^{-\beta}\|(R_hA^{-1}-A_h^{-1}R_h)x\|,$$

and by (H1)

$$\leqq Ch^{-2\beta}h^{2+\alpha}\|A^{\alpha/2}x\|. \qquad \square$$

We are finally in a position to demonstrate the validity of our first theorem.

Proof of Theorem 1. By virtue of Lemma 4.1 we can write

$$\begin{aligned}
&\|S_h(t)R_hx - R_hS(t)x\| \\
&\quad \leqq \int_0^t \|A_hS_h(t-z)[R_hA^{-1}-A_h^{-1}R_h]AS(z)x\|\,dz \\
&\qquad\qquad \text{(by Lemma 4.2 with } \beta = 1-\varepsilon\text{: } \varepsilon \in (0,1)) \\
&\quad \leqq C\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\|A_h^\varepsilon[R_hA^{-1}-A_h^{-1}R_h]AS(z)x\|\,dz \\
&\qquad\qquad \text{(by H1}') \\
&\quad \leqq Ch^{2+\alpha-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\|A^{\alpha/2}AS(z)x\|\,dz \\
&\quad = Ch^{2+\alpha-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\|A^\varepsilon S(z)A^{\alpha/2-\varepsilon+1}x\|\,dz \\
&\qquad\qquad \text{(by (2.2))} \\
&\quad \leqq Ch^{2+\alpha-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\cdot\frac{1}{z^\varepsilon}\,dz\,\|A^{\alpha/2-\varepsilon+1}x\| \\
&\quad = Ch^{2+\alpha-2\varepsilon}\|A^{1+\alpha/2-\varepsilon}x\|\beta(1-\varepsilon,\varepsilon).
\end{aligned}$$

By letting $\varepsilon \in (0,1)$ we arrive at the desired result for $s < r$. In order to prove our estimate for $s = r$ we write

$$\begin{aligned}
&\|S_h(t)R_hx - R_hS(t)x\| \\
&\quad \leqq \int_0^{t-h^2} \|A_hS_h(t-z)[R_hA^{-1}-A_h^{-1}R_h]AS(z)x\|\,dz \\
&\qquad + \int_{t-h^2}^t \|A_hS_h(t-z)[R_hA^{-1}-A_h^{-1}R_h]AS(z)x\|\,dz \\
&\qquad\qquad \text{(by (H1}'\text{) and Lemma 4.2)} \\
&\quad \leqq Ch^r\|A^{r/2}x\|\int_0^{t-h^2}\frac{1}{t-z}\,dz + Ch^{r-2}\|A^{r/2}x\|\int_{t-h^2}^t dz \\
&\quad \leqq Ch^r\|A^{r/2}x\|[\ln h + 1].
\end{aligned}$$

Remark. Notice that in the proof of Theorem 1, instead of assuming the inverse approximation property and H1, one may just assume that

$$\|A_h^\beta[R_hA^{-1}-A_h^{-1}R_h]\| \leqq Ch^{\alpha+2-2\beta}\|A^{\alpha/2}x\| \quad \text{for } 0\leqq\beta\leqq\tfrac{1}{2}, \quad \alpha+2\leqq r,$$

which assumption in fact holds for most schemes.

Now we proceed with the proof of Theorem 2. Theorem 2(i) will follow through arguments similar to those used in the proof of Theorem 1.

Proof of Theorem 2(i). As before we have

$$\begin{aligned}
&\|S_h(t)R_hx-R_hS(t)x\| \\
&\leqq C\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\|A_h^\varepsilon(R_hA^{-1}-A_h^{-1}R_h)AS(z)x\|\,dz \\
&\leqq Ch^{\alpha+2-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\|A^{1-\varepsilon}S(z)A^{\alpha/2+\varepsilon}x\|\,dz \\
&\leqq Ch^{\alpha+2-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-\varepsilon}}\frac{1}{z^{1-\varepsilon}}\,dz\,\|A^{\alpha/2+\varepsilon}x\| \\
&\qquad\qquad\qquad\qquad\text{(after the change of variables } z/t=z'\text{)} \\
&=\frac{Ch^{\alpha+2-2\varepsilon}}{t^{1-2\varepsilon}}\|A^{\alpha/2+\varepsilon}x\|\beta(\varepsilon,\varepsilon),
\end{aligned}$$

which completes the proof of Theorem 2(i). □

In order to prove Theorem 2(ii) we will need several preliminary results formulated in lemmas below.

LEMMA 4.4. *Let $r>2$. Then for all $x\in L_2(\Omega)$ and $2>\alpha>0$, $\alpha+2<r$ we have*

$$\|S_h(t)R_hx-R_hS(t)x\|\leqq\frac{Ch^\alpha}{t^{\alpha/2}}\|x\|.$$

Remark. Notice that the above result does not follow from Theorem 2(i), as we are not allowed to take $\varepsilon=0$.

Proof of Lemma 4.4. By integrating by parts the identity in Lemma 4.1 we obtain

$$\begin{aligned}
S_h(t)R_hx-R_hS(t)x&=-A_h[R_hA^{-1}-A_h^{-1}R_h]S(t)x \\
&\quad+A_hS_h(t)[R_hA^{-1}-A_h^{-1}R_h]x \\
&\quad+\int_0^t A_h^2S_h(t-z)[R_hA^{-1}-A_h^{-1}R_h]S(z)x\,dz.
\end{aligned} \tag{4.14}$$

We shall estimate each term on the right-hand side of (4.14).

To estimate the first term we use (H1′) and an analytic estimate for $S(t)$ (2.2) to obtain

$$\begin{aligned}
\|A_h[R_hA^{-1}-A_h^{-1}R_h]S(t)x\|&\leqq Ch^{2+\alpha-2}\|A^{\alpha/2}S(t)x\| \\
&\leqq\frac{Ch^\alpha}{t^{\alpha/2}}\|x\|.
\end{aligned} \tag{4.15}$$

As for the second term on the right-hand side of (4.14), via Lemmas 4.2 and 4.3 applied with $\beta=1-\alpha/2$ and $\bar{\alpha}=0$,

$$\begin{aligned}
&\|A_hS_h(t)[R_hA^{-1}-A_h^{-1}R_h]x\| \\
&=\|A_h^{\alpha/2}S_h(t)A_h^{1-\alpha/2}[R_hA^{-1}-A_h^{-1}R_h]x\|\leqq\frac{C}{t^{\alpha/2}}h^\alpha\|x\|.
\end{aligned} \tag{4.16}$$

To estimate the third term on the right-hand side of (4.14) we proceed as follows. Let $0<\beta<1$,

$$\int_0^t \|A_h^2 S_h(t-z)[R_h A^{-1} - A_h^{-1} R_h] S(z)x\| \, dz$$

$$= \int_0^t \|A_h^{\beta} A_h^{1-\beta} S_h(t-z) A_h [R_h A^{-1} - A_h^{-1} R_h] S(z)x\| \, dz$$

(by assumption (H2))

$$\leqq Ch^{-2\beta} \int_0^t \|A_h^{1-\beta} S_h(t-z) A_h [R_h A^{-1} - A_h^{-1} R_h] S(z)x\| \, dz$$

(by Lemma 4.2 and (H1′) with $0<\bar{\alpha}<2$)

$$\leqq Ch^{-2\beta+\bar{\alpha}} \int_0^t \frac{1}{(t-z)^{1-\beta}} \|A^{\bar{\alpha}/2} S(z)x\| \, dz$$

(by (2.2))

$$\leqq Ch^{\bar{\alpha}-2\beta} \|x\| \int_0^t \frac{1}{(t-z)^{1-\beta}} \frac{1}{z^{\bar{\alpha}/2}} \, dz$$

(for $\beta \in (0,1)$, $0<\bar{\alpha}<2$ after changing variables $z/t = z$)

$$= \frac{Ch^{\bar{\alpha}-2\beta}}{t^{\bar{\alpha}/2-\beta}} \|x\| \beta\left(\beta, 1-\beta-\frac{\bar{\alpha}}{2}\right). \tag{4.17}$$

Now select $\beta>0$ and $\bar{\alpha}$ such that $\beta<1-\bar{\alpha}/2$ and $\alpha=\bar{\alpha}-2\beta$. By combining (4.15), (4.16), (4.17) and (4.14) we arrive at the desired result. □

LEMMA 4.5.

$$\|A^{-s}(S_h(t)-S(t))R_h x\| \leqq Ch^s \|x\|$$

for all $x \in L_2(\Omega)$ *and* $0 \leqq s < r$; *and*

$$\|A^{-r} S_h(t) - S(t) R_h x\| \leqq Ch^r \ln(h) \|x\|.$$

Proof. By a standard "duality trick" we have

$$\|A^{-s}(S_h(t)-S(t))R_h x\|$$

$$= \sup_{\phi \in D(A^s)} \frac{|((S_h(t)-S(t))R_h x, \phi)|}{\|A^s \phi\|}$$

$$= \sup_{\phi \in D(A^s)} \frac{|(R_h x, (S_h^*(t)R_h - S^*(t))\phi)|}{\|A^s \phi\|}.$$

In view of the fact that A^* and A_h^* satisfy conditions (H1)–(H3), we are in a position to apply the results of Theorem 1 to the adjoint problem. Therefore Theorem 1 applied to A^* and A_h^* combined with (2.9) yields

$$\leqq \sup_{\phi \in D(A^s)} \frac{\|x\| h^s \|A^s \phi\|}{\|A^s \phi\|} \quad \text{for } s<r$$

$$\leqq \|x\| h^r \ln h,$$

which completes the proof of the lemma.

Proof of Theorem 2(ii). By using the semigroup property for $S(t)$ and $S_h(t)$ let us write: for all $0<z<t$

$$
\begin{aligned}
&(S_h(t)R_hx-R_hS(t)x,\,y)\\
&\quad=(S_h(t-z)S_h(z)R_hx-R_hS(t-z)S(z)x,\,R_hy)\\
&\quad=((S_h(z)-S(z))R_hx,\,S^*(t-z)R_hy)\\
&\qquad+((S_h(z)-S(z))R_hx,\,(S_h^*(t-z)-S^*(t-z))R_hy)\\
&\qquad+((S_h^*(t-z)-S^*(t-z))R_hy,\,S(z)R_hx)\\
&\qquad+(S(z)(R_h-I)x,\,S^*(t-z)R_hy).
\end{aligned}
\tag{4.18}
$$

By applying (4.18) with $z=t/2$ we get

$$
\begin{aligned}
&|(S_h(t)R_hx-R_hS(t)x,\,y)|\\
&\quad\leqq\|A^{-1-\alpha/2}(S_h(t/2)-S(t/2))R_hx\|\,\|A^{*1+\alpha/2}S^*(t/2)R_hy\|\\
&\qquad+\|(S_h(t/2)-S(t/2))R_hx\|\,\|(S_h^*(t/2)-S^*(t/2))R_hy\|\\
&\qquad+\|A^{*-1-\alpha/2}(S_h^*(t/2)-S^*(t/2))R_hy\|\,\|A^{1+\alpha/2}S(t/2)R_hx\|\\
&\qquad+\|A^{-1-\alpha/2}(R_h-I)x\|\,\|A^{*1+\alpha/2}S^*(t/2)R_hy\|.
\end{aligned}
\tag{4.19}
$$

Notice that since A^* and A_h^* satisfy (H1)–(H3), the estimates of Lemmas 4.4 and 4.5 remain valid with A and A_h replaced by their adjoints. Therefore by invoking the results of Lemmas 4.4, 4.5 and by applying (2.2), (2.9) after noticing that R_h is selfadjoint, we obtain from (4.19)

$$
\begin{aligned}
&|(S_h(t)R_hx-R_hS(t)x,\,y)|\\
&\quad\leqq Ch^{2+\alpha}\|x\|\frac{1}{t^{1+\alpha/2}}\|y\|+\frac{Ch^\alpha}{t^{\alpha/2}}\|y\|\left\|\left(S_h\left(\frac{t}{2}\right)-S\left(\frac{t}{2}\right)\right)R_hx\right\|.
\end{aligned}
$$

Hence for $\alpha+2<r$, $\alpha>0$

$$
\begin{aligned}
&\|S_h(t)R_hx-R_hS(t)x\|\\
&\quad\leqq\frac{Ch^{2+\alpha}}{t^{1+\alpha/2}}\|x\|+\frac{Ch^\alpha}{t^{\alpha/2}}\left\|\left(S_h\left(\frac{t}{2}\right)-S\left(\frac{t}{2}\right)\right)R_hx\right\|.
\end{aligned}
\tag{4.20}
$$

Repeating the same procedure with $S_h(t/2)-S(t/2)$, we obtain

$$
\left\|\left(S_h\left(\frac{t}{2}\right)-S\left(\frac{t}{2}\right)\right)R_hx\right\|\leqq\frac{Ch^\gamma}{t^{\gamma/2}}\|x\|+\frac{Ch^\alpha}{t^{\alpha/2}}\left\|\left(S_h\left(\frac{t}{4}\right)-S\left(\frac{t}{4}\right)\right)R_hx\right\|
$$

for any $0\leqq\gamma<r$.

Therefore by selecting suitable γ, applying the same procedure $E(1+\alpha/2)$ times and finally applying interpolation argument we arrive at the result claimed in Theorem 2 for $s<r$. For $s=r$ we apply the same procedure as in Theorem 1.

We consider now nonhomogeneous case, i.e. $f\neq 0$. Without loss of generality, assume $y_0=0$.

Proof of Theorem 3. To prove Theorem 3 we write

$$
\begin{aligned}
y(t)-y_h(t) &= \int_0^t S(t-z)f(z)\,dz - \int_0^t S_h(t-z)R_hf(z)\,dz \\
&= \int_0^t (R_hS(t-z)-S_h(t-z)R_h)f(z)\,dz \\
&\quad + \int_0^t (S(t-z)-R_hS(t-z))f(z)\,dz.
\end{aligned} \tag{4.21}
$$

By Theorem 2(i) and by (2.9) we have

$$
\begin{aligned}
\|y(t)-y_h(t)\| &\leqq Ch^{2+\alpha-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-2\varepsilon}}\|A^{\alpha/2+\varepsilon}f(z)\|\,dz \\
&\quad + Ch^{2+\alpha-2\varepsilon}\int_0^t \|A^{1+\alpha/2-\varepsilon}S(t-z)f(z)\|\,dz \\
&\leqq Ch^{2+\alpha-2\varepsilon}\int_0^t \frac{1}{(t-z)^{1-2\varepsilon}}\|A^{\alpha/2+\varepsilon}f(z)\|\,dz \\
&\leqq Ch^{2+\alpha-2\varepsilon}\|A^{\alpha/2+\varepsilon}f\|_{L_\infty[0,t,L_2(\Omega)]},
\end{aligned}
$$

which via interpolation completes the proof of (i). To prove (ii) proceed as follows: after applying Theorem 2(ii) to the right-hand side of (4.21) we have for $s<r$

$$
\begin{aligned}
\|y(t)-y_h(t)\| &\leqq Ch^s\int_0^t \frac{1}{(t-z)^{s/2}}\|f(z)\|\,dz \\
&\leqq Ch^s\|f\|_{L_\infty[0,t;L_2(\Omega)]}.
\end{aligned}
$$

If we take $s=2$ (since $r>2$), then we have

$$
\begin{aligned}
\|y(t)-y_h(t)\| &\leqq Ch^2\int_0^{t-h^2}\frac{1}{(t-z)}\|f(z)\|\,dz + C\int_{t-h^2}^t \|f(z)\|\,dz \\
&\leqq \left(Ch^2\ln\frac{1}{h}+Ch^2\right)\|f\|_{L_\infty[0,t;L_2(\Omega)]}. \qquad \square
\end{aligned}
$$

5. Examples. In this section we shall consider several different choices of approximations A_h of the elliptic problem and show the validity of our hypotheses for these.

5.1. The standard Galerkin method. Let $V_h \subset H_0^1(\Omega)$ be such that

$$\inf_{v_h\in V_h}\|v-v_h\|+h\|v-v_h\|_1 \leqq Ch^s\|v\|_s \quad \text{for } 1\leqq s\leqq r \tag{5.1}$$

and

$$\|v_h\|_1 \leqq Ch^{-1}\|v_h\|, \qquad v_h\in V_h. \tag{5.2}$$

Define $A_h: V_h \to V_h$ by the following formula:

$$(A_hy_h, v_h) = a(y_h, v_h), \qquad y_h, v_h \in V_h, \tag{5.3}$$

where $a(y,v)$ is a bilinear form associated with the elliptic operator $A(x,\partial)$.

By the coercivity of bilinear form, A_h is clearly positive. Condition (H1) for smooth coefficients[2] is a result of a well-known standard Galerkin error estimate [B1]

$$\|(A^{-1}-A_h^{-1}R_h)x\| \leqq Ch^{\alpha+2}\|A^{\alpha/2}x\|, \qquad \alpha+2 \leqq r \tag{5.4}$$

combined with (2.9) after making use of

$$\|x\|_s \leqq C\|A^{s/2}x\| \quad \text{for } x \in D(A^{s/2}), \quad s \geqq 0 \tag{5.5}$$

(see [F1]).

We now turn to conditions (H2) and (H3). In view of Remark 2 in § 2 it is enough to demonstrate that (H3)(iii) is satisfied. In fact, by the coercivity and H^{-1}-boundedness of the bilinear form $a(u, v)$ we have

$$(A_h x_h, x_h) = a(x_h, x_h) \geqq \sigma \|x_h\|_1^2$$

and

$$(A_h x_h, y_h) = a(x_h, y_h) \leqq C\|x_h\|_1 \|y_h\|_1.$$

Recall that A_h^* is simply the operator corresponding to A^* and consequently to $a^*(x, y) = a(y, x)$, so that all hypotheses for an adjoint problem follow as for A_h. Thus the results claimed in Theorems 1, 2, 3 apply to the standard Galerkin method.

5.2. Babuška method: The method of interpolated boundary conditions. For the detailed description of the above methods we refer to [B2], [H1] and [S2]. The important point is that both methods are defined through the same relation as the Galerkin method, i.e. (5.3). Therefore the same arguments (combined with elliptic error estimates) as before apply.

5.3. Nitsche's method. The advantage of Nitsche's method is that it does not require finite dimensional subspaces V_h to satisfy zero boundary conditions. In fact, let $V_h \subset H^1(\Omega)$ be such that

$$\begin{aligned} &V_h|_\Gamma \subset H^1(\Gamma), \\ &\inf_{v_h \in V_h} \{\|v-v_h\| + h\|v-v_h\|_1 + h^{1/2}|v-v_h| + h^{3/2}|v-v_h|_1\} \\ &\qquad \leqq Ch^s\|v\|_s, \qquad 2 \leqq s \leqq r, \end{aligned} \tag{5.6}$$

and moreover

$$\begin{aligned} &\text{a)} \qquad |v_h| \leqq Ch^{1/2}\|v_h\|_1, \\ &\text{b)} \qquad \left|\frac{\partial v_h}{\partial \eta}\right| \leqq Ch^{-1/2}\|v_h\|_1, \\ &\text{c)} \qquad \|v_h\|_1 \leqq Ch^{-1}\|v_h\|. \end{aligned} \tag{5.7}$$

Let us define $A_h: V_h \to V_h$ by

$$(A_h x_h, y_h) = a(x_h, y_h) - \left\langle \frac{\partial x_h}{\partial \eta}, y_h \right\rangle - \left\langle x_h, \frac{\partial}{\partial \eta} y_h \right\rangle + \beta h^{-1}\langle x_h, y_h\rangle, \tag{5.8}$$

for all $x_h, y_h \in V_h$ and for a suitable positive constant β. In order to verify (H1), similarly

[2] For less smooth coefficients—say piecewise smooth—we can use more refined subspaces V_h which treat the singularities arising on the interfaces (see [B1, Chap. 8]).

as before, we quote the error estimate from [N1],

$$\|(A^{-1}-A_h^{-1}R_h)x\| \leqq Ch^{2+\alpha}\|A^{\alpha/2}x\|, \qquad \alpha+2 \leqq r, \quad \alpha \geqq 0 \tag{5.9}$$

which combined with (2.9) and (5.5) yields (H1). Now we turn to the verification of condition (H3)(iii) which via Remark 2 implies (H2). To this end, let us write:

$$\begin{aligned}(A_h x_h, x_h) &= a(x_h, x_h) - 2\left\langle x_h, \frac{\partial}{\partial \eta} x_h\right\rangle + \beta h^{-1}|x_h|^2 \\ &\geqq \sigma\|x_h\|_1^2 + \beta h^{-1}|x_h|^2 - 2|x_h|\left|\frac{\partial}{\partial \eta} x_h\right| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (5.7b))} \\ &\geqq \sigma\|x_h\|_1^2 + \beta h^{-1}|x_h|^2 - Ch^{-1/2}|x_h|\,\|x_h\|_1 \\ &\geqq (\sigma-\varepsilon)\|x_h\|_1^2 + \left(\beta - \frac{C}{\varepsilon}\right) h^{-1}|x_h|^2.\end{aligned}$$

Hence for suitable ε and β large enough we have $(A_h x_h, x_h) \geqq \tilde{\sigma}\|x_h\|_1^2$ for some $\tilde{\sigma} > 0$.

As for the second part of condition (H3)(iii) we have

$$\begin{aligned}(A_h x_h, y_h) &\leqq C\|x_h\|_1\|y_h\|_1 + |x_h|\left|\frac{\partial}{\partial \eta} y_h\right|_\Gamma + \left|\frac{\partial}{\partial \eta} x_h\right||y_h| + \beta h^{-1}|x_h||y_h| \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (5.7a, b))} \\ &\leqq C\|x_h\|_1\|y_h\|_1 + Ch^{-1/2}h^{1/2}\|x_h\|_1\|y_h\|_1 + \beta h^{-1}h^{1/2}\|x_h\|_1 h^{1/2}\|y_h\|_1 \\ &\leqq C\|x_h\|_1\|y_h\|_1.\end{aligned} \tag{5.10}$$

Remark. If one does not assume (5.7a) then, in order to show the validity of (H3)(iii), instead of the H^1-norm one should use the seminorm defined by

$$|x|_I^2 = \|x\|_1^2 + C_1 h^{-}|x|^2 + C_2 h\left[\left|\frac{\partial}{\partial n} x\right|^2 + |x|_1^2\right]$$

(see [S1]).

Remark. Similar examples of approximations were considered in [H1], [S1], [L1]. However, in all of these cited works, the authors assumed that $a_{ij} = a_{ji}$ in the bilinear form $a(u, v)$, hence restricting the class of considered problems. Instead, we do not bind "symmetricity" of the bilinear form.

Appendix. Below we prove that the results of Theorems 1 and 2 are also optimal for $s = r$ and without the restriction $r > 2$. We start with Lemma 4.2, whose Laplace transform version is the following.

LEMMA A.1. *The spectra of A_h and A are essentially contained in the sector*

$$\Sigma = \{\lambda : \operatorname{Re}\lambda \leqq -b|\operatorname{Im}\lambda|\}$$

for some positive constant b independent of h; and

$$\|A^\beta R(\lambda, A)\| \leqq \frac{C}{|\lambda|^{1-\beta}} \quad \textit{for } \lambda \in \Gamma, \tag{A.1}$$

where Γ denotes the contour of Σ, and $R(\lambda, A)$ stands for the resolvent of A.

The proof of Lemma A.1 is an easy consequence of our previous Lemma 4.2, see [B4]. Next we have

LEMMA A.2. *For all* $\lambda \in \Gamma$ *and* $0 \leqq s \leqq r$

(i) $$\|R(\lambda, A_h) - R(\lambda, A_h)R_h\| \leqq Ch^2,$$

(ii) $$\|R_h R(\lambda, A)x - R(\lambda, A_h)R_h x\| \leqq \frac{Ch^s \|A^{s/2}x\|}{|\lambda|}.$$

Proof. By resolvent identity we have

$$R(\lambda, A) - R(\lambda, A_h)R_h = (A^{-1} - A_h^{-1}R_h)A_h R(\lambda, A_h) + \lambda A^{-1}[R(\lambda, A) - R(\lambda, A_h)R_h].$$

Since λ is not in the spectrum of A, $I - \lambda A^{-1}$ is injective and we obtain

$$R(\lambda, A) - R(\lambda, A_h)R_h = \lambda R(\lambda, A)[A^{-1}A_h^{-1}R_h]A_h R(\lambda, A_h). \tag{A.2}$$

From assumption (H1), using the same "duality trick" as in Lemma 4.5, we obtain

$$\|A^{-\alpha/2}(R_h A^{-1} - A_h^{-1}R_h)\| \leqq Ch^{2+\alpha}, \qquad s = \alpha + 2 \leqq r, \quad \alpha \geqq 0. \tag{A.3}$$

Equations (2.2), (2.9), Assumptions (A.1), (A.2) and (A.3) yield

$$\|R(\lambda, A) - R(\lambda, A_h)R_h\| \leqq Ch^2,$$

which proves Lemma A.2(i).

As for Lemma A.2(ii) we use the dual representation to (A.2), i.e.

$$R_h R(\lambda, A) - R(\lambda, A_h)R_h = \lambda R(\lambda, A_h)[R_h A^{-1} - A_h^{-1}R_h]AR(\lambda, A).$$

By (2.9) and (A.3) similarly we obtain

$$\begin{aligned}
&\|R_h R(\lambda, A) - R(\lambda, A_h)R_h\| \\
&\quad = \|\lambda R(\lambda, A_h)A^{-\alpha/2}[R_h A^{-1} - A_h^{-1}R_h]A^{\alpha/2}AR(\lambda, A)\| \\
&\quad \leqq Ch^{2+\alpha}\|A^{1+\alpha/2}R(\lambda, A)\| \leqq \frac{Ch^{2+\alpha}}{|\lambda|}\|A^{1+\alpha/2}x\| = \frac{Ch^s}{|\lambda|}\|A^{s/2}x\|,
\end{aligned}$$

which completes the proof of Lemma A.2.

Now we are in a position to prove our theorems.

To prove Theorem 1 we use Lemma A.2(ii) which yields

$$\begin{aligned}
\|R_h S(t)x - S_h(t)R_h x\| &= \left\| \int_\Gamma e^{\lambda t}[R_h R(\lambda, A) - R(\lambda, A_h)R_h]x \, d\lambda \right\| \\
&\leqq Ch^s \|A^{s/2}x\| \int_\Gamma \frac{|e^{\lambda t}|}{|\lambda|} d\lambda \leqq Ch^s \|A^{s/2}x\|
\end{aligned}$$

thus completing proof of Theorem 1 for an arbitrary $0 \leqq s \leqq r$.

As for the proof of Theorem 2(ii) we use Lemma A.2(i) to obtain

$$\|S(t) - S_h(t)R_h\| = \left\| \int_\Gamma e^{\lambda t}[R(\lambda, A) - R(\lambda, A_h)R_h] \, d\lambda \right\| \leqq Ch^2 \int_\Gamma |e^{\lambda t}| \, d\lambda \leqq \frac{Ch^2}{t},$$

which proves Theorem 2(ii) with $s = r = 2$. For $s > 2$, we apply the same procedure as in (4.18), (4.19). □

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