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A spectral-based numerical method for Kolmogorov equations in Hilbert spaces

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We propose a numerical solution for the solution of the Fokker–Planck–Kolmogorov (FPK) equations associated with stochastic partial differential equations in Hilbert spaces. The method is based on the spectral decomposition of the Ornstein–Uhlenbeck semigroup associated to the Kolmogorov equation. This allows us to write the solution of the Kolmogorov equation as a deterministic version of the Wiener–Chaos Expansion. By using this expansion we reformulate the Kolmogorov equation as an infinite system of ordinary differential equations, and by truncating it we set a linear finite system of differential equations. The solution of such system allow us to build an approximation to the solution of the Kolmogorov equations. We test the numerical method with the Kolmogorov equations associated with a stochastic diffusion equation, a Fisher–KPP stochastic equation and a stochastic Burgers equation in dimension 1.

Keywords: Fokker–Planck–Kolmogorov equations; SPDEs; spectral methods; numerical solution.

AMS Subject Classification: 60H10, 65C20, 35Q84

1. Introduction

Stochastic Partial Differential Equations (SPDEs) are important tools in modeling complex phenomena, they arise in many fields of knowledge like physics, biology, economy, finance, etc. Develop efficient numerical methods for simulating SPDEs is very important but also very difficult and challenging.

There exists in literature several approaches in order to solve numerically a SPDE. Among them there exists Monte Carlo simulations, Karhunen–Loeve expansion, Wiener Chaos expansion, stochastic Taylor approximations for SPDEs, etc.

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In order to solve numerically an SPDE, one can apply one of this methods. Here we will mention some of them but our list of references is far away to be exhaustive.

The Monte Carlo (MC) simulations for SPDEs have been explored intensively in the last 20 years.[21,28] The basis idea of MC is to sample the randomness in the SPDEs and solve the stochastic equations by realization, this is because for each given realization of the randomness, the SPDEs becomes deterministic and can be solved by the usual deterministic numerical methods. the disadvatage is that many of that "samples" are required for suffcient accuracy, causing suboptimal efficiency even if optimal algebraic solvers are used; to overcome this issue, Giles[13,14] has introduce a modificacion of MC for the numerical solution of Itô stochastic ordinary differential equations, following basic ideas in earlier work by Heinrich[16] on numerical quadrature. This method is the so-called Multilevel Monte Carlo (MLMC).

The main idea of MLMC methods is to apply the MC method for a nested sequence of stepsizes while balancing the number of samples according to the stepsize. MLMC allows to significantly speed up to classical MC methods, thanks to this hierarchical sampling; however this method can still have limitations for SPDEs.

Other approach is to use spectral methods, in particular use the Karhunen–Loeve expansion (KLE) and the Wiener–Chaos expansions for solving SPDEs. For the former one can study the theory developed in Ref. 22; in this work they proposed several methods to solve SPDEs and this methods are later applied to solve the stochastic Navier–Stokes equations. Nevertheless, this method can have limitations since in this approach the source of randomness is usually represented by a fixed number of random variables and if we consider, for instance, stochastic equations arising in fluid dynamics with a random forcing white in time which has a divergent Karhunen–Loeve expansion, then it is not possible to apply the KLE to this kind of equations.

In 1938 Norbert Wiener using Hermite polynomials constructed an orthonormal random basis for expanding homogeneous chaos depending on white noise. Following Wiener's idea Cameron and Martin[3] developed an explicit discretization of the white noise process through its Fourier expansion. This approach is much easier and intuitive, so it's more convenient to use. This Fourier–Hermite expansion was commonly called Wiener chaos expansions.

Hou *et al.*[17] (see also Ref. 26) propose a numerical method based on Wiener–Chaos expansion and apply it to solve the stochastic Burgers and Navier–Stokes equations driven by Brownian motion. They consider an SPDE with Brownian motion forcing and since a Brownian motion can be expanded as a linear combination of independent Gaussian random variables, they expand the solution of the SPDEs as a Fourier–Hermite series of those Gaussian random variables, this is a version of the Cameron–Martin decomposition.

There is another approach that involves stochastic Taylor approximations for stochastic partial differential equations (see Ref. 20). These Taylor expansions are based on an iterated application of the Itô formula. However, for the solutions of stochastic partial differential equations in Hilbert (or Banach spaces) there is no way to define directly the Itô formula. Nevertheless, it can be constructed by taking advantage of the mild form representation of the solutions.

The Fokker–Planck–Kolmogorov (FPK) equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, it is a kind of continuity equation for densities. Citing Ref. 9 "parabolic equations on Hilbert spaces appear in mathematical physics to model systems with infinitely many degrees of freedom. Typical examples are provided by spin configurations in statistical mechanics and by crystals in solid state theory. Infinite-dimensional parabolic equations provide an analytic description of infinite dimensional diffusion processes in such branches of applied mathematics as population biology, fluid dynamics, and mathematical finance". This kind of equations have been deeply studied in the last years, see for instance Refs. 2, 7, 11, and the references therein.

Numerical methods for FPK equations associated with SPDEs have been studied, up to our knowledge, just in a few papers, here we will mention just one. Schwab and Suli[29] have formulated a space-time variational method to approximate solution of Kolmogorov-type equations in infinite dimensions. They consider an infinite-dimensional Hilbert space $\mathcal{H}$, a Gaussian measure μ with trace class covariance operator Q on $\mathcal{H}$ and the space $L^2(H,\mu)$ of functions on $\mathcal{H}$ which are square-integrable with respect to the measure μ. They showed the well-posedness of Fokker–Planck equations and Ornstein–Uhlenbeck equations on $L^2(H,\mu)$. Moreover, they constructed sequences of finite-dimensional approximations that attain the best possible convergence rates afforded by best N-terms approximations of the solution. They used a spectral method based on Wiener–Hermite polynomial chaos expansions in terms of a sequence of independent Gaussian random variables on $\mathcal{H}$ and a wavelet type Riesz basis with respect to the time variable. The use of the spectral basis of Wiener–Hermite polynomial chaos allow them to avoid meshing the infinite-dimensional "domain" $\mathcal{H}$ of solutions of the Kolmogorov-type equations. However, they do not present numerical examples and the questions about the feasibility of their method are open.

In this paper, we introduce a novel numerical method that can have some similitude with the one proposed by Schwab and Suli but also have substantial differences. Indeed, our method is also based on spectral methods for the variable on $\mathcal{H}$, but we use a deterministic version of the Wiener–Chaos expansion on the infinite-dimensional "domain" $\mathcal{H}$ instead the classical Wiener–Chaos expansion with the use of a sequence of Gaussian random variables, this allows us to avoid meshing the space $\mathcal{H}$ but we also avoid the so-called curse of dimensionality: the associated computational cost grows exponentially as a function of the number of random variables defining the underlying probability space of the problem (see Ref. 12 for instance).

The second difference is with respect to the time variable, where, instead of using wavelet type Riesz basis we set up a finite system of coupled ordinary differential

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equations and by solving it we fix the coefficients as a time function. We have applied the method to three SPDEs: a stochastic diffusion, a stochastic Fisher–KPP equation and a stochastic burgers equation and the results show that the behavior of the method is good. However, since the method is analogue to the classical deterministic spectral method, it can be extended to improve its performance. This is the subject of a future research.

This paper is organized as follows. In Sec. 2, we review the Fokker–Planck–Kolmogorov equation associated with SPDEs in a separable Hilbert space. In Sec. 3, we study the spectral decomposition of the Ornstein–Uhlenbeck semigroup associated to the Kolmogorov equation which will be used to do the numerical approximation to the solution of the FPK equation, this is done in Sec. 4. In Sec. 5, we prove a theorem on the well posedness and convergence of the numerical approximation. Results on the application of the proposed method are presented in Sec. 6, where we have applied the method to a linear stochastic diffusion equation, a Fisher–KPP stochastic equation and a stochastic Burgers equation in dimension 1.

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2. Fokker–Planck–Kolmogorov Equation

In a separable infinite-dimensional Hilbert space $\mathcal{H}$ with inner product $\langle , \rangle_{\mathcal{H}}$ we define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $\mathrm{Tr}(\Lambda) < +\infty$. We focus on the stochastic differential equation in $\mathcal{H}$

$$dX_t = AX_t dt + B(X_t)dt + \sqrt{Q}dW_t, \tag{2.1}$$

where the operator $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$, Q is a bounded operator from another Hilbert space $\mathcal{U}$ to $\mathcal{H}$ and $B : \mathcal{D}(B) \subset \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping.

Equation (2.1) can be associated to a Kolmogorov equation in the following, we define

$$u(t, x) = \mathbb{E}[u_0(X_t^x)], \tag{2.2}$$

where $u_0 : \mathcal{H} \to \mathbb{R}$ and X_t^x is the solution to (2.1) with initial conditions $X_0 = x$ where $x \in \mathcal{H}$. Then u satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{Tr}(QD^2u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \quad x \in D(A). \tag{2.3}$$

Several authors have proved results on the existence and uniqueness of the solution of the Kolmogorov equations, see for instance Da Prato[7] for a survey, Da Prato–Debussche[8] for the Burgers equation, Barbu–Da Prato[1] for the 2D Navier–Stokes stochastic flow in a channel.

3. On the Ornstein–Uhlenbeck Semigroup

Following Ref. 6 (see also Ref. 5), in $\mathcal{H}$ we define a Gaussian measure μ with mean zero and nuclear covariance operator Λ with $\mathrm{Tr}(\Lambda) < +\infty$ and since $\Lambda : \mathcal{H} \mapsto \mathcal{H}$

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is a positive definite, self-adjoint operator then its square-root operator $\Lambda^{1/2}$ is a positive definite, self-adjoint Hilbert–Schmidt operator on $\mathcal{H}$.

Define the inner product

$$\langle g, h\rangle_0 := \langle \Lambda^{-1/2} g, \Lambda^{-1/2} h\rangle_{\mathcal{H}}, \quad \text{for } g, h \in \Lambda^{1/2}\mathcal{H}.$$

Let $\mathcal{H}_0$ denote the Hilbert subspace of $\mathcal{H}$, which is the completion of $\Lambda^{1/2}\mathcal{H}$ with respect to the norm $\|g\|_0 := \langle g, g\rangle_0^{1/2}$. Then $\mathcal{H}_0$ is dense in $\mathcal{H}$ and the inclusion map $i : \mathcal{H}_0 \hookrightarrow \mathcal{H}$ is compact. The triple $(i, \mathcal{H}_0, \mathcal{H})$ forms an abstract Wiener space.

Let $\mathbb{H} = L^2(\mathcal{H}, \mu)$ denote the Hilbert space of Borel measurable functionals on the probability space with inner product

$$\langle \Phi, \Psi\rangle_{\mathbb{H}} := \int_{\mathcal{H}} \Phi(v)\Psi(v)\mu(dv), \quad \text{for } \Phi, \Psi \in \mathbb{H},$$

and norm $\|\Phi\|_{\mathbb{H}} := \langle \Phi, \Phi\rangle_{\mathbb{H}}^{1/2}$. In $\mathbb{H}$ we choose a basis system $\{\varphi_k\}$ such that $\varphi_k \in \mathcal{H}$.

A functional $\Phi : \mathcal{H} \mapsto \mathbb{R}$, is said to be a smooth simple functional (or a cylinder functional) if there exists a C^∞-function ϕ on $\mathbb{R}^n$ and n-continuous linear functional $l_1, \ldots, l_n$ on $\mathcal{H}$ such that for $h \in \mathcal{H}$

$$\Phi(h) = \phi(h_1, \ldots, h_n) \quad \text{where } h_i = l_i(h), \quad i = 1, \ldots, n.$$

The set of all such functionals will be denoted by $\mathcal{S}(\mathbb{H})$.

Denote by $P_k(x)$ the Hermite polynomial of degree k taking values in $\mathbb{R}$. Then, $P_k(x)$ is given by the following formula

$$P_k(x) = \frac{(-1)^k}{(k!)^{1/2}} e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{-\frac{x^2}{2}}$$

with $P_0 = 1$. It is well known that $\{P_k(\cdot)\}_{k\in\mathbb{N}}$ is a complete orthonormal system for $L^2(\mathbb{R}, \mu_1(dx))$ with $\mu_1(dx) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$.

Define the set of infinite multi-index as

$$\mathcal{J} = \left\{ \boldsymbol{\alpha} = (\alpha_i, i \geq 1) \,|\, \alpha_i \in \mathbb{N} \cup \{0\},\ |\boldsymbol{\alpha}| := \sum_{i=1}^{\infty} \alpha_i < +\infty \right\}.$$

For $\mathbf{n} \in \mathcal{J}$ define the *Hermite polynomial functionals* on $\mathcal{H}$ by

$$H_{\mathbf{n}}(h) = \prod_{i=1}^{\infty} P_{n_i}(l_i(h)), \quad h \in \mathcal{H}_0,\ \ \mathbf{n} \in \mathcal{J}, \tag{3.1}$$

where

$$l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i\rangle_{\mathcal{H}}, \quad i = 1, 2, \ldots$$

and $P_n(\xi)$ is the usual Hermite polynomial for $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 3.1. Notice that $l_i(h)$ is defined only for $h \in \mathcal{H}_0$. However, regarding h as a μ-random variable in $\mathcal{H}$, we have $\mathbb{E}(l_i(h)) = \|\varphi_i\|^2 = 1$ and then $l_k(h)$ can be defined μ-a.e. $h \in \mathcal{H}$, similar to defining a stochastic integral.

It is possible to identify the Hermite polynomial functionals defined in (3.1), for $h \in \mathcal{H}_0$, as a deterministic version of the Wick polynomials defined on the canonical Wiener space. In fact, in the classical book of Paul Malliavin (see Sec. 3.4

in Ref. 27) it is named as a *numerical model*. For further details see Sec. 1.2 in Ref. 18 for instance.

We have the following result (see Theorems 9.1.5 and 9.1.7 in Da Prato–Zabczyk[9] or Lemma 3.1 in Chap. 9 from Chow[6]).

Lemma 3.2. *For $h \in \mathcal{H}$ let $l_i(h) = \langle h, \Lambda^{-1/2}\varphi_i\rangle_{\mathcal{H}}, i = 1, 2, \ldots$. Then the set $\{H_{\mathbf{n}}\}$ of all Hermite polynomials on $\mathcal{H}$ forms a complete orthonormal system for $\mathbb{H}$. Hence the set of all functionals are dense in $\mathbb{H}$. Moreover, we have the direct sum decomposition*:

$$\mathbb{H} = \bigoplus_{j=0}^{\infty} K_j,$$

where K_j is the subspace of $\mathbb{H}$ spanned by $\{H_{\mathbf{n}} : |\mathbf{n}| = j\}$. □

3.1. *Spectral decomposition of the Ornstein–Uhlenbeck semigroup*

Consider the linear stochastic equation

$$\begin{aligned} du_t &= Au_t dt + dW_t, \\ u_0 &= h \in \mathcal{H}. \end{aligned} \tag{3.2}$$

Here, as before $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is the infinitesimal generator of a strongly continuous semigroup e^{tA} in $\mathcal{H}$. W_t is a Q-Wiener process in $\mathcal{H}$.

Chow in Lemma 9.4.1 of Ref. 6 has shown the following result.

Lemma 3.3. *Suppose that A and Q satisfy the following*:

(1) *$A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{H}$ is self-adjoint and there is $\beta > 0$ such that*

$$\langle Av, v\rangle_{\mathcal{H}} \leq -\beta\|v\|_{\mathcal{H}} \quad \forall\, v \in \mathcal{H}.$$

(2) *A commutes with Q in $\mathcal{D}(A) \subset \mathcal{H}$.*

Then (3.2) *has a unique invariant measure μ which is a Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator $\Lambda = \frac{1}{2}Q(-A)^{-1} = \frac{1}{2}(-A)^{-1}Q$.* □

We define the operator

$$\mathcal{A}_0 u = \frac{1}{2}\mathrm{Tr}(QD^2u) + \langle Ax, Du\rangle_{\mathcal{H}}, \quad x \in \mathcal{H} \tag{3.3}$$

and suppose that $-A$ and Q have the same eigenfunctions e_k with eigenvalues λ_k and ρ_k respectively.

Then the operator $\mathcal{A}_0$ satisfies the following result.

Lemma 3.4. *Let $H_{\mathbf{n}}(h)$ be a Hermite polynomial functional given by* (3.1)*. Then the following holds*

$$\mathcal{A}_0 H_{\mathbf{n}}(h) = -\lambda_{\mathbf{n}} H_{\mathbf{n}}(h), \tag{3.4}$$

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for any $\mathbf{n} \in \mathcal{J}$ *and* $h \in \mathcal{H}$ *and where*

$$\lambda_{\mathbf{n}} := \sum_{k=1}^{\infty} n_k \lambda_k. \qquad \square$$

The proof can be found in Lemma 9.4.3 of Ref. 6 or Ref. 4.

Using Lemmas 3.4 and 3.2, ($\{H_{\mathbf{n}}\}$ forms a complete orthonormal system for $L^2(\mathcal{H}, \mu)$) we can write

$$u(t, x) = \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) H_{\mathbf{n}}(x), \quad x \in \mathcal{H}, \quad t \in [0, T], \tag{3.5}$$

where $u_{\mathbf{n}} : [0, T] \mapsto \mathbb{R}$ and $H_{\mathbf{n}}(x)$ are the Hermite functionals.

Remark 3.5. The decomposition given in (3.5) is a deterministic version to the Wiener–Chaos expansion (WCe), also known as a Fourier–Hermite series. The WCe has been used to prove several results in stochastic analysis and also it has been applied to solve numerically stochastic partial differential equations (see for instance Lototsky and Rozovskii,[25] Lototsky,[24] Hou *et al.*[17]).

Notice that the Kolmogorov equation can be writen as

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2}\mathrm{Tr}(QD^2 u) + \langle Ax, Du \rangle_{\mathcal{H}} + \langle B(x), Du \rangle_{\mathcal{H}} \\ &= \mathcal{A}_0 u + \langle B(x), Du \rangle_{\mathcal{H}}. \end{aligned} \tag{3.6}$$

Using (3.5), we calculate

$$\frac{\partial u}{\partial t} = \sum_{\mathbf{n} \in \mathcal{J}} \dot{u}_{\mathbf{n}}(t) H_{\mathbf{n}}(x)$$

$$\begin{aligned} \mathcal{A}_0 u &= \mathcal{A}_0 \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) H_{\mathbf{n}}(x) = \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) \mathcal{A}_0 H_{\mathbf{n}}(x) \\ &= -\sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) \lambda_{\mathbf{n}} H_{\mathbf{n}}(x), \end{aligned}$$

where in the last equality we have used Lemma 3.4.

For the last term in (3.6) we have

$$\begin{aligned} \langle B(x), Du \rangle_{\mathcal{H}} &= \left\langle B(x), D_x \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) H_{\mathbf{n}}(x) \right\rangle_{\mathcal{H}} \\ &= \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) \langle B(x), D_x H_{\mathbf{n}}(x) \rangle_{\mathcal{H}}, \end{aligned}$$

where D_x is the Fréchet derivative.

Therefore the Kolmogorov equation becomes

$$\sum_{\mathbf{n} \in \mathcal{J}} \dot{u}_{\mathbf{n}}(t) H_{\mathbf{n}}(x) = -\sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) \lambda_{\mathbf{n}} H_{\mathbf{n}}(x) + \sum_{\mathbf{n} \in \mathcal{J}} u_{\mathbf{n}}(t) \langle B(x), D_x H_{\mathbf{n}}(x) \rangle_{\mathcal{H}}.$$

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Multiplying by $H_{\mathbf{m}}(x)$, $\mathbf{m} \in \mathcal{J}$ and integrating in $\mathcal{H}$ w.r.t. $\mu(dx)$ we have

$$\sum_{\mathbf{n}\in\mathcal{J}} \dot{u}_{\mathbf{n}}(t) \int_{\mathcal{H}} H_{\mathbf{m}}(x) H_{\mathbf{n}}(x)\mu(dx) = -\sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t)\lambda_{\mathbf{n}} \int_{\mathcal{H}} H_{\mathbf{m}}(x)H_{\mathbf{n}}(x)\mu(dx)$$
$$+ \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t) \int_{\mathcal{H}} H_{\mathbf{m}}(x)\langle B(x), D_x H_{\mathbf{n}}(x)\rangle_{\mathcal{H}} \mu(dx).$$

From this, and using the orthogonality of the system $\{H_{\mathbf{m}}(x)\}$ we get the infinite system of coupled ordinary differential equations

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t) C_{\mathbf{n},\mathbf{m}}, \quad \mathbf{n}, \mathbf{m} \in \mathcal{J}, \tag{3.7}$$

where $C_{\mathbf{n},\mathbf{m}}$ is given by

$$C_{\mathbf{n},\mathbf{m}} := \int_{\mathcal{H}} \langle B(x), D_x H_{\mathbf{n}}(x)\rangle_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx). \tag{3.8}$$

We focus on $C_{\mathbf{n},\mathbf{m}}$. Since $H_{\mathbf{n}}(x) = \prod_{i=1}^{\infty} P_{n_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}})$, we get

$$D_x H_{\mathbf{n}}(x) = \sum_{k=1}^{\infty} \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}}) P'_{n_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}) \Lambda^{-\frac{1}{2}} e_k,$$

then

$$\langle B(x), D_x H_{\mathbf{n}}(x)\rangle_{\mathcal{H}} = \sum_{k=1}^{\infty} \langle B(x), \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}} \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}}) P'_{n_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}).$$

Thus,

$$C_{\mathbf{n},\mathbf{m}} = \int_{\mathcal{H}} \sum_{k=1}^{\infty} \langle B(x), \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}} \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}}) P'_{n_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}) H_{\mathbf{m}}(x)\mu(dx).$$

3.1.1. *A technical result*

The following result is important for the numerical simulation since it will allow us to use the evaluation functional on the Hilbert space $\mathcal{H}$.

Lemma 3.6. (i) *The Gaussian measure μ on $\mathcal{H} = L^2(0,1)$ with covariance $\Lambda = \frac{1}{2}(-A)^{-1}$ is supported on $C([0,1])$.*

(ii) *Let $\xi_0 \in [0,1]$ be given. Let $u_0 : C([0,1]) \to \mathbb{R}$ be defined as $u_0(x) = x(\xi_0)$. Then*

$$\int_{\mathcal{H}} u_0^2(x)\mu(dx) < \infty$$

(*and therefore* $\sum_m (u_m^0)^2 < \infty$).

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Proof. Recall that $Af = f''$, $D(A) = H^2(0,1) \cap H_0^1(0,1)$. By solving the two-point boundary value problem $f'' = g$, $f(0) = f(1) = 0$, after several manipulations one can show that

$$(\Lambda h)(\xi) = \int_0^1 \lambda(\xi, \xi') h(\xi') d\xi', \quad h \in \mathcal{H}$$

where

$$\lambda(\xi, \xi') = \frac{1}{2}[(\xi(1 - \xi')) - (\xi - \xi')1_{\xi' \leq \xi}].$$

The reader may more easily get convinced that this is correct *a posteriori*, by showing that $\frac{d^2}{d\xi^2} \int_0^1 \lambda(\xi, \xi') f(\xi') d\xi' = -\frac{1}{2} f(\xi)$ and that $(\Lambda h)(0) = (\Lambda h)(1) = 0$.

Consider the canonical process $(X_\xi)_{\xi \in [0,1]} : (\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined for a.e. $\xi \in [0,1]$ as $X_\xi(x) = x(\xi)$ and denote by E the mathematical expectation on $(\mathcal{H}, \mathcal{B}(\mathcal{H}), \mu)$. The process X has zero mean. One can prove that

$$Cov(X_\xi, X_{\xi'}) = q(\xi, \xi'), \quad \text{a.e. } \xi, \xi' \in [0,1].$$

Indeed, since $\langle \Lambda h, k \rangle_{\mathcal{H}} = \int_{\mathcal{H}} \langle x, h \rangle_{\mathcal{H}} \langle x, k \rangle_{\mathcal{H}} \mu(dx)$, we have

$$\begin{aligned}
\int_0^1 \int_0^1 q(\xi, \xi') h(\xi') k(\xi) d\xi d\xi' &= \int_{\mathcal{H}} \int_0^1 x(\xi') h(\xi') d\xi' \int_0^1 x(\xi) k(\xi) d\xi \mu(dx) \\
&= \int_0^1 \int_0^1 \left(\int_{\mathcal{H}} x(\xi') x(\xi) \mu(dx) \right) h(\xi') k(\xi) d\xi d\xi' \\
&= \int_0^1 \int_0^1 E[X_{\xi'} X_\xi] h(\xi') k(\xi) d\xi d\xi'
\end{aligned}$$

and the formula for $Cov(X_\xi, X_{\xi'})$ follows from the arbitrarity of h and k.

The paths of the process X are obviously of class $L^2(0,1)$; there is a continuous modification of X if and only if μ is supported on $C([0,1])$. If we check the condition

$$E[|X_\xi - X_{\xi'}|^2] \leq C|\xi - \xi'|^\alpha \tag{3.9}$$

for some $\alpha, C > 0$, then, by gaussianity,

$$E[|X_\xi - X_{\xi'}|^p] \leq C_p |\xi - \xi'|^{\alpha p/2}$$

for every $p \geq 1$ and for a suitable constant $C_p > 0$, hence there is a continuous modification by Kolmogorov criterion. But

$$\begin{aligned}
E[|X_\xi - X_{\xi'}|^2] &= E[X_\xi^2] + E[X_{\xi'}^2] - 2E[X_\xi X_{\xi'}] \\
&= q(\xi, \xi) + q(\xi', \xi') - 2q(\xi, \xi') \\
&= q(\xi, \xi) - q(\xi, \xi') + q(\xi', \xi') - q(\xi, \xi').
\end{aligned}$$

We have

$$q(\xi, \xi) - q(\xi, \xi') = \frac{1}{2}\xi(1 - \xi) - \frac{1}{2}[(\xi(1 - \xi')) - (\xi - \xi')1_{\xi' \leq \xi}] \leq C|\xi - \xi'|$$

and similarly $q(\xi', \xi') - q(\xi, \xi') \leq C|\xi - \xi'|$. Hence condition (3.9) is satisfied with $\alpha = 1$. We have proved that μ is supported on $C([0,1])$.

We also have

$$\int_{\mathcal{H}} u_0^2(x)\mu(dx) = \int_{\mathcal{H}} x^2(\xi_0)\mu(dx) = E[X_{\xi_0}^2] = q(\xi_0, \xi_0) < \infty.$$

The proof is complete. □

Remark 3.7. To convince ourselves, a more concise but a little formal proof of the claim $\int_{\mathcal{H}} u_0^2(x)\mu(dx) < \infty$ is

$$\int_{\mathcal{H}} u_0^2(x)\mu(dx) = \int_{\mathcal{H}} \langle x, \delta_{\xi_0}\rangle_{\mathcal{H}}^2 \mu(dx) = \langle \Lambda\delta_{\xi_0}, \delta_{\xi_0}\rangle_{\mathcal{H}} = \frac{1}{2}\|(-A)^{-1/2}\delta_{\xi_0}\|_{\mathcal{H}}^2 < \infty$$

because $(-A)^{-1/2}\delta_{\xi_0} \in L^2(0,1)$, since by duality

$$\begin{aligned}\langle (-A)^{-1/2}\delta_{\xi_0}, f\rangle_{\mathcal{H}} &= \langle \delta_{\xi_0}, (-A)^{-1/2} f\rangle_{\mathcal{H}} = ((-A)^{-1/2} f)(\xi_0) \\ &\le \|(-A)^{-1/2} f\|_{L^\infty} \le C\|(-A)^{-1/2} f\|_{H^1} \le C\|f\|_{L^2},\end{aligned}$$

where we have used Sobolev embedding $H^1 \subset L^\infty$ and the fact that $(-A)^{-1/2}$ maps L^2 into H^1.

4. Numerical Approximation

Define the set of *finite* multi-index $J^{M,N}$ as

$$\mathcal{J}^{M,N} = \{\boldsymbol{\alpha} = (\alpha_i, 1 \le i \le M) \,|\, \alpha_i \in \{0, 1, 2, \ldots, N\}\}$$

this is the set of M-tuple which can take values in the set $\{0, 1, 2, \ldots, N\}$.

We approximate the solutions of the Kolmogorov equation by the following expression

$$\hat{u}_N(t, x) = \sum_{\mathbf{n} \in \mathcal{J}^{M,N}} u_{\mathbf{n}}(t) H_{\mathbf{n}}(x), \quad x \in \mathcal{H}, \quad t \in [0, T]. \tag{4.1}$$

Notice the use of the finite M-tuple in opposition to the infinite multi-index $\mathcal{J}$ as in (3.5).

We truncate the infinite system (3.7) in the following sense. Consider the same value M as in $J^{M,N}$ and $\mathbf{m}_1, \ldots, \mathbf{m}_M \in \mathcal{J}^{M,N}$ and define the finite system of equations

$$\dot{u}_{\mathbf{m}_i}(t) = -u_{\mathbf{m}_i}(t)\lambda_{\mathbf{m}_i} + \sum_{j=1}^{M} u_{\mathbf{n}_j}(t) C_{\mathbf{n}_j, \mathbf{m}_i}, \quad 1 \le i \le M. \tag{4.2}$$

Set the vectors

$$\begin{aligned} U^M(t) &= (u_{\mathbf{m}_1}(t), u_{\mathbf{m}_2}(t), \ldots, u_{\mathbf{m}_M}(t))^T, \\ \dot{U}^M(t) &= (\dot{u}_{\mathbf{m}_1}(t), \dot{u}_{\mathbf{m}_2}(t), \ldots, \dot{u}_{\mathbf{m}_M}(t))^T \end{aligned}$$

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and the matrix

$$A = \begin{pmatrix} -\lambda_1 + C_{1,1} & C_{2,1} & \cdots & C_{M-1,1} & C_{M,1} \\ C_{1,2} & -\lambda_2 + C_{2,2} & \cdots & C_{M-1,2} & C_{M,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,M-1} & C_{2,M-1} & \cdots & -\lambda_{M-1} + C_{M-1,M-1} & C_{M,M-1} \\ C_{1,M} & C_{2,M} & \cdots & C_{M-1,M} & -\lambda_M + C_{M,M} \end{pmatrix},$$

where $\lambda_i = \lambda_{\mathbf{m}_i}$ and $C_{i,j} = C_{\mathbf{n}_i,\mathbf{m}_j}$ for $1 \leq i, j \leq M$. Notice that, given the expression (3.8), in general the matrix A is not symmetric. We can now write the system (4.2) as a matrix differential equation:

$$\dot{U}^M(t) = AU^M(t). \tag{4.3}$$

Then, if A has M real and distint eigenvalues η_i and M eigenvectors $\mathbf{V}_i$, then the solution to (4.3) is given by

$$U^M(t) = \sum_{i=1}^{M} c_i \mathbf{V}_i e^{\eta_i t}. \tag{4.4}$$

In the case when some of the eigenvalues and eigenvectors, or at least one of them, take values in the complex field we can still have real solutions. Indeed, suppose that we have the case with one complex eigenvalue and eigenvector then it is known that we will have $M-2$ real eigenvalues but we can obtain two real solutions from the complex eigenvalue (see Ref. 15 for instance).

Let us write one of the complex eigenvalues and eigenvectors as

$$\mathbf{V} = \mathbf{a} + i\mathbf{b},$$

$$\eta = \gamma + i\mu,$$

then we can write two *real* solutions as follows:

$$e^{\gamma t}(\mathbf{a}\cos(\mu t) - \mathbf{b}\sin(\mu t)), \quad e^{\gamma t}(\mathbf{a}\sin(\mu t) + \mathbf{b}\cos(\mu t)).$$

4.1. *Initial conditions*

In contrast to several types of differential equations, whether ordinary or partial, deterministic or stochastic, for FPK equations there is no standard way to determine the initial conditions. This is because in this type of equations we must choose a functional that acts on the initial condition, this implies that depending on the functional chosen we must adapt the method. Here we present the method for two examples of functionals.

We will consider two cases:

$$u_0^{z_0}(g) := g(z_0), \quad \text{for fixed } z_0 \in [0, 1]$$

and

$$u_0(g) := \int_0^1 g(z)dz.$$

For the first functional, define the set points in the set $[a,b]$ as $\{z_i\}$, $i=0,\dots,P$, such that $z_0=a$ and $z_P=b$. Then for each point z_i we have that $X_0(z_i)=X(0,z_i)$, and for each z_i set $u_0(x)$ as the evaluation functional $z_i \mapsto X_t^x(z_i)$ then from $u(t,x)=\mathbb{E}(u_0(X_t^x))$ we obtain

$$u(0,x)=\mathbb{E}(u_0^{z_i}(X_0^x))=X^x(0,z_i)=x(z_i),$$

and on the other hand

$$u(0,x)=\sum_{\mathbf{n}\in\mathcal{J}^{M,N}} u_{\mathbf{n}}(0)H_{\mathbf{n}}(x),$$

then for each z_i

$$x(z_i)=u(0,x)=\sum_{\mathbf{n}\in\mathcal{J}^{M,N}} u_{\mathbf{n}}(0)H_{\mathbf{n}}(x).$$

Then, multiplying by $H_{\mathbf{m}}(x)$ and integrating in the Hilbert space $L^2(\mathcal{H},\mu)$ we have

$$u_{\mathbf{m}}(0)=\int_{\mathcal{H}} x(z_i)H_{\mathbf{m}}(x)\mu(dx).$$

Here the value of the initial condition $u_{\mathbf{m}}(0)$ depends on z_i, i.e. $u_{\mathbf{m}}(0)=u_{\mathbf{m}}^{z_i}(0)$.

Notice that in the direction of the eigenfunction e_k the expression x can be written as $\langle x,e_k\rangle_{\mathcal{H}}e_k$ and then we can write $H_{\mathbf{m}}(x)x(z_i)$ in the direction e_k as $P_{m_k}(\xi_k)\langle x,e_k\rangle_{\mathcal{H}}e_k(z_i)$ with $\xi_k=\langle x,\Lambda^{-1/2}e_k\rangle_{\mathcal{H}}$. Furthermore, $\xi_k=\langle x,\Lambda^{-1/2}e_k\rangle_{\mathcal{H}}=|\lambda_k|\langle x,e_k\rangle_{\mathcal{H}}$ then we have

$$\begin{aligned}
u_{\mathbf{m}}^{z_i}(0)&=\int_{\mathcal{H}} x(z_i)H_{\mathbf{m}}(x)\mu(dx)\\
&=\int_{\mathbb{R}^{\mathbb{N}}}\sum_{k=1}^{\infty} e_k(z_i)\langle x,e_k\rangle_{\mathcal{H}}P_{m_k}(\xi_k)\mu(d\xi_1,d\xi_2,\cdots)e_k\\
&=\int_{\mathbb{R}^{\mathbb{N}}}\sum_{k=1}^{\infty} e_k(z_i)\frac{\xi_k}{\lambda_k}P_{m_k}(\xi_k)\mu(d\xi_1,d\xi_2,\cdots)e_k\\
&=\sum_{k=1}^{\infty}\frac{e_k(z_i)}{\lambda_k}\int_{\mathbb{R}} P_{m_k}(\xi_k)\xi_k\mu(d\xi_k)\\
&\approx\sum_{k=1}^{M}\frac{e_k(z_i)}{\lambda_k}\int_{\mathbb{R}} P_{m_k}(\xi_k)\xi_k\mu(d\xi_k). \qquad (4.5)
\end{aligned}$$

Notice that the general solution to each $u_{\mathbf{m}}^{z_i}(0)$ is given by the expression

$$\begin{pmatrix} u_1(t)\\ u_2(t)\\ \vdots\\ u_{M-1}(t)\\ u_M(t)\end{pmatrix}=\begin{pmatrix}\mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_{M-1} & \mathbf{V}_M\end{pmatrix}\begin{pmatrix} c_1e^{\lambda_1 t}\\ c_2e^{\lambda_2 t}\\ \vdots\\ c_{M-1}e^{\lambda_{M-1}t}\\ c_Me^{\lambda_M t}\end{pmatrix},$$

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where $\mathbf{V}_j$ and λ_j are the eigenvector and eigenvalue of the matrix A and we are denoting $u_j(t) = u^{z_i}_{\mathbf{m}_j}(t)$, $1 \leq j \leq M$. Evaluating in $t = 0$ we have

$$\begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_{M-1}(0) \\ u_M(0) \end{pmatrix} = (\mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_{M-1} \quad \mathbf{V}_M) \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{M-1} \\ c_M \end{pmatrix},$$

and therefore

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{M-1} \\ c_M \end{pmatrix} = (\mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_{M-1} \quad \mathbf{V}_M)^{-1} \begin{pmatrix} u_1(0) \\ u_2(0) \\ \vdots \\ u_{M-1}(0) \\ u_M(0) \end{pmatrix},$$

with $u_j(t) = u^{z_i}_{\mathbf{m}_j}(t)$ given by the expression (4.5). Now we are able to fix the value of the initial conditions for the first case. Notice also that the contants c_j depend on the value z_i, i.e. $c_j = c^{z_i}_j$.

For the second functional, from $u(t, x) = \mathbb{E}(u_0(X^x_t))$ we obtain

$$u(0, x) = \mathbb{E}(u_0(X^x_0)) = \int_0^1 x(z)dz,$$

on the other hand

$$u(0, x) = \sum_{\mathbf{n}\in\mathcal{J}^{M,N}} u_{\mathbf{n}}(0)H_{\mathbf{n}}(x),$$

then

$$\int_0^1 x(z)dz = \sum_{\mathbf{n}\in\mathcal{J}^{M,N}} u_{\mathbf{n}}(0)H_{\mathbf{n}}(x).$$

Multiplying by $H_{\mathbf{m}}(x)$ and integrating in the Hilbert space $L^2(\mathcal{H}, \mu)$ and by using Fubini we have

$$u_{\mathbf{m}}(0) = \int_{\mathcal{H}} \int_0^1 x(z)dzH_{\mathbf{m}}(x)\mu(dx) = \int_0^1 \left(\int_{\mathcal{H}} x(z)H_{\mathbf{m}}(x)\mu(dx) \right) dz.$$

We focus on the integral on $\mathcal{H}$. By following the steps given for the first functional (just replacing z_i by z) we arrive at the following expression:

$$\int_{\mathcal{H}} x(z)H_{\mathbf{m}}(x)\mu(dx) \approx \prod_{k=1}^{M} \frac{e_k(z)}{\lambda_k} \int_{\mathbb{R}} P_{m_k}(\xi_k)\xi_k\mu(d\xi_k),$$

thus

$$u_{\mathbf{m}}(0) \approx \int_0^1 \prod_{k=1}^{M} \frac{e_k(z)}{\lambda_k} \left(\int_{\mathbb{R}} P_{m_k}(\xi_k)\xi_k \mu(d\xi_k) \right) dz$$
$$= \prod_{k=1}^{M} \int_{\mathbb{R}} P_{m_k}(\xi_k)\xi_k \mu(d\xi_k) \int_0^1 \frac{e_k(z)}{\lambda_k} dz. \tag{4.6}$$

From here and by following the procedure for the first functional we are able to fix the initial conditions.

5. Well Posedness and Convergence

Let $\mathcal{J}$ be a countable set, $\{\lambda_m; m \in \mathcal{J}\}$ a sequence of positive real numbers diverging to infinity and $\{C_{nm}; n, m \in \mathcal{J}\}$ a sequence of real numbers. Consider the infinite system of equations

$$u_m'(t) = -\lambda_m u_m(t) + \sum_{n \in \mathcal{J}} C_{nm} u_n(t), \quad t \geq 0$$
$$u_m(0) = u_m^0, \quad m \in \mathcal{J}$$

with given initial condition $\{u_m^0; m \in \mathcal{J}\}$. We always assume

$$\sum_{m \in \mathcal{J}} (u_m^0)^2 < \infty.$$

Definition 5.1. A solution is a sequence $\{u_m(\cdot); m \in \mathcal{J}\}$ of continuous functions on $[0, T]$ such that:

(i)
$$\sup_{t \in [0,T]} \sum_{m \in \mathcal{J}} u_m^2(t) + \int_0^T \sum_{m \in \mathcal{J}} \lambda_m u_m^2(s) ds < \infty.$$

(ii) The series $\sum_{n \in \mathcal{J}} C_{nm} u_n(t)$ converges, for a.e. t, to an integrable functions on $[0, T]$ and

(iii)
$$u_m(t) = u_m^0 - \int_0^t \lambda_m u_m(s) ds + \int_0^t \sum_{n \in \mathcal{J}} C_{nm} u_n(s) ds$$

for all $m \in \mathcal{J}$ and $t \in [0, T]$.

Consider also, for any finite subset $\widetilde{\mathcal{J}} \subset \mathcal{J}$, the finite system

$$\widetilde{u}_m'(t) = -\lambda_m \widetilde{u}_m(t) + \sum_{n \in \widetilde{\mathcal{J}}} C_{nm} \widetilde{u}_n(t), \quad t \geq 0$$
$$\widetilde{u}_m(0) = u_m^0, \quad m \in \widetilde{\mathcal{J}}.$$

The definition of solution for this finite system is obvious and the existence and uniqueness is well known.

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Theorem 5.2. *Assume that the family $\{C_{nm}; n, m \in \mathcal{J}\}$ satisfies, for some constant $C > 0$,*

$$\sum_{n,m\in\mathcal{J}} C_{nm}\alpha_n\beta_m \le C\left(\sum_{n\in\mathcal{J}}\lambda_n\alpha_n^2\right)^{1/2}\left(\sum_{m\in\mathcal{J}}\beta_m^2\right)^{1/2} \quad \text{for all sequences } \{\alpha_n, \beta_n; n \in \mathcal{J}\}. \tag{5.1}$$

Then there exists a unique solution. Moreover,

$$\sup_{t\in[0,T]}\sum_{m\in\widetilde{\mathcal{J}}}(u_m(t)-\widetilde{u}_m(t))^2 + \int_0^T\sum_{m\in\widetilde{\mathcal{J}}}\lambda_m(u_m(s)-\widetilde{u}_m(s))^2ds \le C_1\int_0^T\sum_{m\in\widetilde{\mathcal{J}}^c}\lambda_m u_m^2(s)ds$$

for some $C_1 > 0$ independent of $\widetilde{\mathcal{J}}$; where the term $\int_0^T\sum_{m\in\widetilde{\mathcal{J}}^c}\lambda_m u_m^2(s)ds$ converges to zero as $\widetilde{\mathcal{J}}$ converges to $\mathcal{J}$.

Remark 5.3. Under assumption (5.1), given $m_0 \in \mathcal{J}$ and $s \in [0, T]$, choose $\alpha_n = u_n(s)$ and β_n equal to zero except for $\beta_{m_0} = 1$; then

$$\left|\sum_{n\in\mathcal{J}} C_{nm_0}u_n(s)\right| = \left|\sum_{n,m\in\mathcal{J}} C_{nm}u_n(s)\beta_m\right| \le C\left(\sum_{n\in\mathcal{J}}\lambda_n u_n^2(s)\right)^{1/2} \le C\left(1+\sum_{n\in\mathcal{J}}\lambda_n u_n^2(s)\right)$$

hence, in Definition 5.1, condition (i) implies (ii).

Proof. Step 1 (Existence and uniqueness). Let H, V be the real separable Hilbert spaces of sequences $\alpha = \{\alpha_n; n \in \mathcal{J}\}$ such that, respectively $\|\alpha\|_H^2 := \sum_{n\in\mathcal{J}}\alpha_n^2 < \infty$, $\|\alpha\|_V^2 := \sum_{n\in\mathcal{J}}\lambda_n\alpha_n^2 < \infty$, with norms $\|\alpha\|_H^2$ and $\|\alpha\|_V^2$ respectively; let $\langle\cdot,\cdot\rangle_H$ denote the inner product in H. Since we have assumed at the beginning that $\{\lambda_m; m \in \mathcal{J}\}$ diverges to infinity, we have $V \subset H$ and there exists a constant $C_{H,V}$ such that $\|\alpha\|_H^2 \le C_{H,V}\|\alpha\|_V^2$ for all $\alpha \in V$. Let V' be the dual space of V, with norm $\|\cdot\|_{V'}^2$. We identify H with its dual H' so that $V \subset H \subset V'$ and denote by $\langle\cdot,\cdot\rangle$ the dual pairing between V and V', which extends $\langle\cdot,\cdot\rangle_H$.

Let $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ be the bilinear map defined as

$$a(\alpha,\beta) = \sum_{n\in\mathcal{J}}\lambda_n\alpha_n\beta_n - \sum_{n,m\in\mathcal{J}} C_{nm}\alpha_n\beta_m.$$

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It holds

$$
\begin{aligned}
|a(\alpha,\beta)| &\le \sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2 + \sum_{n\in\mathcal{J}} \lambda_n \beta_n^2 + C\left(\sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2\right)^{1/2}\left(\sum_{n\in\mathcal{J}} \beta_n^2\right)^{1/2}\\
&= (1+C)\|\alpha\|_V^2 + \|\beta\|_V^2 + C\|\beta\|_H^2
\end{aligned}
$$

hence $a(\cdot,\cdot)$ is well-defined and continuous on $V\times V$. Moreover, since

$$
C\left(\sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2\right)^{1/2}\left(\sum_{n\in\mathcal{J}} \beta_n^2\right)^{1/2} \le \frac{1}{2}\sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2 + 2C^2 \sum_{n\in\mathcal{J}} \beta_n^2
$$

we get

$$
a(\alpha,\alpha) = \sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2 - \sum_{n,m\in\mathcal{J}} C_{nm}\alpha_n\alpha_m \ge \frac{1}{2}\sum_{n\in\mathcal{J}} \lambda_n \alpha_n^2 - 2C^2\|\alpha\|_H^2
$$

hence $a(\cdot,\cdot)$ is coercive on $V\times V$. Consider the equation

$$
\langle u(t),\phi\rangle_H + \int_0^t a(u(s),\phi)ds = \langle u^0,\phi\rangle_H + \int_0^t \langle f(s),\phi\rangle ds
$$

with $\phi\in V, u^0\in H$, $f\in L^2(0,T;H)$ (one can also treat $f\in L^2(0,T;V')$ but this is not important here). By solution we mean a function $u\in L^\infty(0,T;H)\cap L^2(0,T;V)$ which satisfies this equation for all $\phi\in V$ and all $t\in[0,T]$. By a well-known theorem (see Ref. 23), there exists a unique solution of this equation, with

$$
\sup_{t\in[0,T]} \|u(t)\|_H^2 + \int_0^T \|u(s)\|_V^2 ds < \infty.
$$

This proves the existence and uniqueness of a solution of the infinite system above, in the sense of Definition 5.1.

Step 2 (Convergence) Let us prove the estimate between the finite and infinite systems. We have

$$
u_m(t) = u_m^0 - \int_0^t \lambda_m u_m(s)ds + \int_0^t \sum_{n\in\widetilde{\mathcal{J}}} C_{nm}u_n(s)ds + \int_0^t R_m^{\widetilde{\mathcal{J}}}(s)ds,
$$

where $R_m^{\widetilde{\mathcal{J}}}(s) = \sum_{n\in\widetilde{\mathcal{J}}^c} C_{nm}u_n(s)$; we know that $R_m^{\widetilde{\mathcal{J}}}$ is an integrable function, by definition of solution. Then, for the new variable $v_m(t) := u_m(t) - \widetilde{u}_m(t)$, we have

$$
v_m(t) = -\int_0^t \lambda_m v_m(s)ds + \int_0^t \sum_{n\in\widetilde{\mathcal{J}}} C_{nm}v_n(s)ds + \int_0^t R_m^{\widetilde{\mathcal{J}}}(s)ds.
$$

It follows that the family $\{v_m; m\in\widetilde{\mathcal{J}}\}$ satisfies the finite system

$$
\begin{aligned}
v_m'(t) &= -\lambda_m v_m(t) + \sum_{n\in\widetilde{\mathcal{J}}} C_{nm}v_n(t) + R_m^{\widetilde{\mathcal{J}}}(t), \quad t\ge 0\\
v_m(0) &= 0, \quad m\in\widetilde{\mathcal{J}}.
\end{aligned}
$$

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We have

$$\begin{aligned}\sum_{m\in\widetilde{\mathcal{J}}} v_m R_m^{\widetilde{\mathcal{J}}} &= \sum_{m\in\widetilde{\mathcal{J}}}\sum_{n\in\widetilde{\mathcal{J}}^c} C_{nm}u_n v_m \le C\left(\sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2\right)^{1/2}\left(\sum_{n\in\widetilde{\mathcal{J}}} v_n^2\right)^{1/2}\\ &\le C^2\sum_{n\in\widetilde{\mathcal{J}}} v_n^2 + \sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2\end{aligned}$$

and thus

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m\in\widetilde{\mathcal{J}}} v_m^2 + \sum_{m\in\widetilde{\mathcal{J}}}\lambda_m v_m^2\\ &\quad = \sum_{n,m\in\widetilde{\mathcal{J}}} C_{nm}v_n v_m + \sum_{m\in\widetilde{\mathcal{J}}} v_m R_m^{\widetilde{\mathcal{J}}}\\ &\quad \le C\left(\sum_{n\in\mathcal{J}}\lambda_n v_n^2\right)^{1/2}\left(\sum_{n\in\mathcal{J}} v_n^2\right)^{1/2} + C^2\sum_{n\in\widetilde{\mathcal{J}}} v_n^2 + \sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2\\ &\quad \le \frac{1}{2}\sum_{m\in\widetilde{\mathcal{J}}}\lambda_m v_m^2 + 3C^2\sum_{m\in\widetilde{\mathcal{J}}} v_m^2 + \sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2\end{aligned}$$

hence (renaming the constant C)

$$\frac{1}{2}\frac{d}{dt}\sum_{m\in\widetilde{\mathcal{J}}} v_m^2 + \frac{1}{2}\sum_{m\in\widetilde{\mathcal{J}}}\lambda_m v_m^2 \le 3C^2\sum_{m\in\widetilde{\mathcal{J}}} v_m^2 + \sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2$$

which, by Gronwall lemma, easily implies that there exists a constant $C_1 > 0$, independent of the finite subset $\widetilde{\mathcal{J}}$, such that

$$\sup_{t\in[0,T]}\sum_{m\in\widetilde{\mathcal{J}}} v_m^2(t) + \int_0^T\sum_{m\in\widetilde{\mathcal{J}}}\lambda_m v_m^2(s)ds \le C_1\int_0^T\sum_{n\in\widetilde{\mathcal{J}}^c}\lambda_n u_n^2(s)ds.$$

The proof is complete. □

Proposition 5.4. *Let $B : \mathcal{H} \to \mathcal{H}$ be bounded measurable and let C_{nm} be given by*

$$C_{nm} = \int_{\mathcal{H}}\langle B(x), D_x H_n(x)\rangle_{\mathcal{H}} H_m(x)\mu(dx).$$

If

$$\int_{\mathcal{H}}|D_x\varphi(x)|_{\mathcal{H}}^2\mu(dx) \le 2\sum_{n\in\mathcal{J}}\lambda_n\varphi_n^2 \tag{5.2}$$

for every function $\varphi(x)$ of the form $\varphi(x) = \sum_{n\in\mathcal{J}}\varphi_n H_n(x)$, then condition (5.1) *holds.*

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Proof. Given two sequences $\{\alpha_n, \beta_n; n \in \mathcal{J}\}$, setting

$$\varphi(x) = \sum_{n\in\mathcal{J}} \alpha_n H_n(x), \quad \psi(x) = \sum_{m\in\mathcal{J}} \beta_m H_m(x)$$

one simply has

$$\begin{aligned}
\sum_{n,m\in\mathcal{J}} C_{nm}\alpha_n\beta_m &= \int_{\mathcal{H}} \langle B(x), D_x\varphi(x)\rangle_{\mathcal{H}}\psi(x)\mu(dx) \\
&\le \|B\|_\infty \int_{\mathcal{H}} |D_x\varphi(x)|_{\mathcal{H}}|\psi(x)|\mu(dx) \\
&\le \|B\|_\infty \left(\int_{\mathcal{H}} |D_x\varphi(x)|^2_{\mathcal{H}}\mu(dx)\right)^{1/2} \left(\int_{\mathcal{H}} |\psi(x)|^2\mu(dx)\right)^{1/2} \\
&\le \|B\|_\infty \left(2\sum_{n\in\mathcal{J}} \lambda_n\alpha_n^2\right)^{1/2} \left(\sum_{m\in\mathcal{J}} \beta_m^2\right)^{1/2}. \qquad \square
\end{aligned}$$

Now, we will prove that (5.2) is satisfied in our case. Assume the conditions in Lemma 3.4 holds. Then, for any $\Phi, \Psi \in \mathcal{S}(\mathbb{H})$[a], the following Green's formula holds (for a proof see Lemma 4.4 in Ref. 6 for instance)

$$-\frac{1}{2}\int_{\mathcal{H}} \langle QD_x\Phi, D_x\Psi\rangle_{\mathcal{H}}\mu(dx) = \int_{\mathcal{H}} (\mathcal{A}_0\Phi)\Psi\mu(dx) = \int_{\mathcal{H}} \Phi(\mathcal{A}_0\Psi)\mu(dx).$$

By taking $\Psi = \Phi = \varphi$ and $Q = \mathrm{Id}$ we have

$$\int_{\mathcal{H}} |D_x\varphi|^2_{\mathcal{H}}\mu(dx) = \int_{\mathcal{H}} \langle D_x\varphi, D_x\varphi\rangle_{\mathcal{H}}\mu(dx) = -2\int_{\mathcal{H}} (\mathcal{A}_0\varphi)\varphi\mu(dx).$$

If $\varphi(x) = \sum_{n\in\mathcal{J}} \varphi_n H_n(x)$, then

$$\begin{aligned}
-\int_{\mathcal{H}} (\mathcal{A}_0\varphi)\varphi\mu(dx) &= \int_{\mathcal{H}} \left(-\mathcal{A}_0 \sum_{n\in\mathcal{J}} \varphi_n H_n(x)\right) \sum_{m\in\mathcal{J}} \varphi_m H_m(x)\mu(dx) \\
&= \sum_{m\in\mathcal{J}} \int_{\mathcal{H}} \left(\sum_{n\in\mathcal{J}} \varphi_n[-\mathcal{A}_0 H_n(x)]\right) \varphi_m H_m(x)\mu(dx) \\
&= \sum_{m\in\mathcal{J}} \int_{\mathcal{H}} \sum_{n\in\mathcal{J}} \varphi_n\lambda_n H_n(x)\varphi_m H_m(x)\mu(dx) \\
&= \sum_{m\in\mathcal{J}} \sum_{n\in\mathcal{J}} \varphi_n\varphi_m\lambda_n \int_{\mathcal{H}} H_n(x)H_m(x)\mu(dx) \\
&= \sum_{n\in\mathcal{J}} \lambda_n\varphi_n^2.
\end{aligned}$$

[a]Recall that $\mathcal{S}(\mathbb{H})$ is the set of all cylinder functionals on $\mathcal{H}$.

Where in the last step we will use that $H_n(x)$ is an orthonormal basis for $\mathcal{H}$. Then, we have

$$\begin{aligned}\int_{\mathcal{H}} |D_x\varphi|^2_{\mathcal{H}}\mu(dx) &= -2\int_{\mathcal{H}} (\mathcal{A}_0\varphi)\varphi\mu(dx)\\ &= 2\sum_{n\in\mathcal{J}} \lambda_n\varphi_n^2.\end{aligned}$$

6. Numerical Results

6.1. *Algorithm description*

In this subsection we describe the algorithm we follow to get the simulations for the Kolmogorov equations associated with three stochastic partial differential equations whose results we show in the next subsections.

(1) Choose the algorithm's parameters:
   (a) The space $\mathcal{H}$ where the SPDE will be defined.
   (b) The operator A and its eigenfunctions λ_k and eigenvalues $e_k(\cdot)$.
   (c) The functional $u_0 : \mathcal{H} \to \mathbb{R}$.
   (d) N, M and then fix the set $J^{N,M}$.
   (e) The time step Δt and Δx in the physical space.
(2) Compute the quantities $\bar{C}_{\mathbf{n},\mathbf{m}}$, for each $\mathbf{n},\mathbf{m} \in J^{N,M}$, to approximate (3.8).
(3) Set the finite system of coupled ordinary differential equation (4.2).
(4) Rewriting the system (4.2) as a matrix differential equations and by solving it numerically we obtain, up to a set of constants, the time-functions $u_{\mathbf{n}}(t)$, for each $\mathbf{n} \in J^{N,M}$.
(5) By using the functional u_0 the constants in the last step are fixed.
(6) We then define the space-time approximation for the Kolmogorov equation as

$$\begin{aligned}u_N(t,x) &= \sum_{j=1}^{N} u_j(t)H_j(x)\\ &\approx \sum_{j\geq 1} u_j(t)H_j(x) = u(t,x).\end{aligned}$$

Remark 6.1. • Given the operator A, we choose its eigenvalues as the basis for the Hilbert space $\mathcal{H}$ and we have to find its eigenvalues λ_k.

- The choice of the functional u_0 will change the way we determine the initial condition of the Kolmogorov equation, then it will necessaring adapt the method for each u_0.
- the quantities $\bar{C}_{\mathbf{n},\mathbf{m}}$ are those that require more computing resources because we have to compute and approximate several integrals for each $\mathbf{n},\mathbf{m} \in J^{N,M}$. In our examples these quantities are given by the expressions (6.5), (6.14) and (6.21).

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6.2. *Stochastic heat equation in an interval*

As a first application consider the stochastic diffusion in dimension 1.

Let $\mathcal{H} = L^2([0,1]), Q = \mathrm{Id}$, and A be given by $Ax = \nu \triangle_\xi x$, $x \in D(A)$ with $D(A) = H^2(0,1) \cap H_0^1(0,1)$ (where $H^2(0,1)$ is the Sobolev spaces and $H_0^1(0,1)$ is the subspace of $H^1(0,1)$ of all functions vanishing at $0,1$).

Consider the heat equation in $[0,1]$

$$\frac{\partial X(t,\xi)}{\partial t} = \nu \frac{\partial^2 X(t,\xi)}{\partial \xi^2} + f(\xi) + \frac{\partial^2 W}{\partial t \partial \xi}, \quad \xi \in [0,1] \tag{6.1}$$

$$X(t,\xi)\mid_{t=0} = X_0(\xi), \quad X_0 \in \mathcal{H},$$

$$X(t,\xi) = 0, \quad t \geq 0, \quad \xi = 0,1,$$

where $t \in [0,T]$, $f(\xi) = \xi^3$, $X_0(\xi) = \sin(\pi x)$. W is a cylindrical Wiener process on $\mathcal{H}$, associated to a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. ν denotes the thermal diffusivity.

The complete orthonormal system of eigenfunctions e_k is defined as

$$e_k(\xi) = \sqrt{2}\sin(k\pi\xi), \quad \xi \in [0,1], \quad k \in \mathbb{N}.$$

A is self-adjoint negative operator and $Ae_k = -\nu k^2 \pi^2 e_k$, $k \in \mathbb{N}$.

We rewrite Eq. (6.1) as an abstract differential equation on $\mathcal{H}$. Set $B = f$, then

$$dX = [AX + B(X)]dt + dW_t,$$

$$X(0) = x, \quad x \in \mathcal{H}.$$

Define $u(t,x) = \mathbb{E}[u_0(X_t^x)]$ and then $u(t,x)$ satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{Tr}(QD^2u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \quad x \in D(A).$$

We will consider two cases:

$$u_0^{\xi_0}(g) := g(\xi_0), \quad \text{for fixed } \xi_0 \in (0,1)$$

and

$$u_0(g) := \int_0^1 g(\xi)d\xi.$$

As before we write the solution as

$$u(t,x) = \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t)H_{\mathbf{n}}(x), \quad x \in \mathcal{H}, \quad t \in [0,T], \tag{6.2}$$

where $u_{\mathbf{n}} : [0,T] \mapsto \mathbb{R}$ and $H_{\mathbf{n}}(x)$ are the Hermite functionals. Following the last procedure we set the infinite system of coupled ordinary differential equations.

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t)C_{\mathbf{n},\mathbf{m}}, \quad \mathbf{n},\mathbf{m} \in \mathcal{J} \tag{6.3}$$

where $C_{\mathbf{n},\mathbf{m}}$ is given by

$$C_{\mathbf{n},\mathbf{m}} := \int_{\mathcal{H}} \langle B(x), D_x H_{\mathbf{n}}(x)\rangle_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx). \tag{6.4}$$

We apply now the numerical method for this case. We have that $\Lambda = \frac{1}{2}(-A)^{-1}$ have eigenvalues $1/(2\nu\pi^2|k|^2)$, then the operator Λ^{-1} is well-defined and have eigenvalues $2\nu\pi^2|k|^2$, and $\Lambda^{-\frac{1}{2}}$ can also be befined having eigenvalues $\sqrt{2\nu}\pi|k|$, then

$$\langle B(x), \Lambda^{-\frac{1}{2}} e_k \rangle_{L^2([0,1])} = \sqrt{2\nu}\pi|k|\langle f, e_k\rangle_{L^2([0,1])}.$$

Notice that $H_{\mathbf{n}} = \prod_\alpha P_{n_\alpha}(\xi_\alpha)$, $H_{\mathbf{m}} = \prod_\alpha P_{m_\alpha}(\xi_\alpha)$ and $P'_{m_k}(\xi_k) = m_k^{1/2} \times P_{m_k-1}(\xi_k)$. Then, we rewrite $C_{\mathbf{n},\mathbf{m}}$ as follows.

$$\begin{aligned} C_{\mathbf{n},\mathbf{m}} &= \sum_{k=1}^{\infty} \int_{\mathcal{H}} \langle B(x), \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}} \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i} \\ &\quad \cdot (\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}}) P'_{n_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}) H_{\mathbf{m}}(x)\mu(dx) \\ &= \sum_{k=1}^{\infty} \int_{\mathcal{H}} \lambda_k \langle f, e_k\rangle_{\mathcal{H}} P_{m_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}) P'_{n_k}(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}) \\ &\quad \cdot \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}}) P_{m_i}(\langle x, \Lambda^{-\frac{1}{2}} e_i\rangle_{\mathcal{H}})\mu(dx). \end{aligned}$$

Writing the measure $\mu(dx)$ in the direction e_k as $\mu(dx)e_k = \frac{1}{\lambda_k}\mu(d(\langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}})) = \frac{1}{\lambda_k}\mu(d\xi_k)$ with $\xi_k = \langle x, \Lambda^{-\frac{1}{2}} e_k\rangle_{\mathcal{H}}$, then we approximate $C_{\mathbf{n},\mathbf{m}}$ as

$$\begin{aligned} C_{\mathbf{n},\mathbf{m}} &= \sum_{k=1}^{\infty} \lambda_k \int_0^1 f(\xi)e_k(\xi)d\xi \int_{\mathbb{R}} n_k^{1/2} P_{m_k}(\xi_k) P_{n_k-1}(\xi_k) \frac{1}{\lambda_k}\mu(d\xi_k) \\ &\quad \cdot \int_{\mathbb{R}^{\mathbb{N}}} \prod_{\substack{i=1\\ i\neq k}}^{\infty} P_{n_i}(\xi_i) P_{m_i}(\xi_i) \frac{1}{\lambda_i}\mu(d\xi_i) \\ &\approx \sum_{k=1}^{M} \int_0^1 f(\xi)e_k(\xi)d\xi \int_{\mathbb{R}} n_k^{1/2} P_{m_k}(\xi_k) P_{n_k-1}(\xi_k)\mu(d\xi_k) \\ &\quad \cdot \int_{\mathbb{R}^{M-1}} \prod_{\substack{i=1\\ i\neq k}}^{M} P_{n_i}(\xi_i) P_{m_i}(\xi_i) \frac{1}{\lambda_i}\mu(d\xi_i) \\ &= \sum_{k=1}^{M} n_k^{1/2} \int_0^1 f(\xi)e_k(\xi)d\xi \int_{\mathbb{R}} P_{m_k}(\xi_k) P_{n_k-1}(\xi_k)\mu(d\xi_k) \\ &\quad \cdot \prod_{\substack{i=1\\ i\neq k}}^{M} \frac{1}{\lambda_i} \int_{\mathbb{R}} P_{n_i}(\xi_i) P_{m_i}(\xi_i)\mu(d\xi_i). \end{aligned}$$

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For $N_1 \in \mathbb{N}$ define the set $S_{N_1} = \{\mathbf{n}_1, \mathbf{n}_2, \dots, \mathbf{n}_{N_1} : \mathbf{n}_i \in J^{M,N}, i = 1, \dots, N_1\}$. Moreover, for $\mathbf{n}, \mathbf{m} \in S_M$ define

$$\begin{aligned} \bar{C}_{\mathbf{n},\mathbf{m}} := \sum_{k=1}^{M} \sqrt{2\nu}\pi |k| n_k^{1/2} \int_0^1 f(\xi) e_k(\xi) d\xi \int_{\mathbb{R}} P_{m_k}(\xi_k) P_{n_k - 1}(\xi_k) \mu(d\xi_k) \\ \cdot \prod_{\substack{i=1 \\ i \neq k}}^{M} \int_{\mathbb{R}} P_{n_i}(\xi_i) P_{m_i}(\xi_i) \mu(d\xi_i), \end{aligned} \tag{6.5}$$

and the finite system of ordinary differential equations:

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n} \in S_M} u_{\mathbf{n}}(t) \bar{C}_{\mathbf{n},\mathbf{m}}, \quad \text{for each } \mathbf{m} \in S_M \quad \text{and} \quad \mathbf{n} \in S_M. \tag{6.6}$$

Then (6.6) approximates to the infinite system of ordinary differential equations (6.3) when $N, M \to \infty$. We use the system (6.6) to approximate the solution of the FPK equation associated with the Diffusion equation.

We need to evaluate the integrals and do the finite sum on k, to do this we use a Gauss–Hermite quadrature to approximate the value of the integrals

$$\int_0^1 f(\xi) e_k(\xi) d\xi, \quad \int_{\mathbb{R}} P_{m_k}(\xi_k) P_{n_k - 1}(\xi_k) \mu(d\xi_k), \quad \int_{\mathbb{R}} P_{n_i}(\xi_i) P_{m_i}(\xi_i) \mu(d\xi_i).$$

When the constants $C_{\mathbf{n},\mathbf{m}}$ are fixed we solve the matrix differential equation (4.3):

$$\dot{U}^M(t) = AU^M(t). \tag{6.7}$$

with

$$A = \begin{pmatrix} -\lambda_1 + C_{1,1} & C_{2,1} & \cdots & C_{M-1,1} & C_{M,1} \\ C_{1,2} & -\lambda_2 + C_{2,2} & \cdots & C_{M-1,2} & C_{M,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ C_{1,M-1} & C_{2,M-1} & \cdots & -\lambda_{M-1} + C_{M-1,M-1} & C_{M,M-1} \\ C_{1,M} & C_{2,M} & \cdots & C_{M-1,M} & -\lambda_M + C_{M,M} \end{pmatrix},$$

$\lambda_i = \lambda_{\mathbf{m}_i}$ and $C_{i,j} = C_{\mathbf{n}_i, \mathbf{m}_j}$ for $1 \le i, j \le M$, and

$$U^M(t) = (u_{\mathbf{m}_1}(t), u_{\mathbf{m}_2}(t), \dots, u_{\mathbf{m}_M}(t))^T,$$
$$\dot{U}^M(t) = (\dot{u}_{\mathbf{m}_1}(t), \dot{u}_{\mathbf{m}_2}(t), \dots, \dot{u}_{\mathbf{m}_M}(t))^T.$$

From this, the general solution of (6.7) is given by

$$\begin{pmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_{M-1}(t) \\ u_M(t) \end{pmatrix} = (\mathbf{V}_1 \quad \mathbf{V}_2 \quad \cdots \quad \mathbf{V}_{M-1} \quad \mathbf{V}_M) \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_{M-1} e^{\lambda_{M-1} t} \\ c_M e^{\lambda_M t} \end{pmatrix}, \tag{6.8}$$

where $\mathbf{V}_i$ and λ_i are the eigenvector and eigenvalue of the matrix A. It remains to fix the set of constants $\{c_i, 1 \leq i \leq M\}$ which are determined by using the initial conditions given in Sec. 4.1.

Initial conditions

We define the set points in the set $[0, 1]$ as $\{\xi_i\}$, $i = 0, \ldots, P$, such that $\xi_0 = 0$ and $\xi_P = 1$. Then by using (4.5) we fix the values of the constants c_i

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_M \end{pmatrix} = \left[\begin{pmatrix} H_1(x_0) & H_2(x_0) & \cdots & H_m(x_0) \\ H_1(x_1) & H_2(x_1) & \cdots & H_m(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ H_1(x_{P-1}) & H_2(x_{P-1}) & \cdots & H_m(x_{P-1}) \\ H_1(x_P) & H_2(x_P) & \cdots & H_m(x_P) \end{pmatrix} \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \\ \vdots \\ \mathbf{V}_M \end{pmatrix}^T \right]^{-1} \begin{pmatrix} X_0(\xi_0) \\ X_0(\xi_1) \\ \vdots \\ X_0(\xi_P) \end{pmatrix}. \tag{6.9}$$

With this, we have now completed the process to build the approximation for the solution.

6.2.1. *Deterministic equation associated with the stochastic diffusion* (6.1)

Set

$$y(t, \xi) = \mathbb{E}[X_t(\xi)],$$

then $y(t, \xi)$ solves the differential equation

$$\begin{aligned} \frac{\partial y}{\partial t} &= \nu \frac{\partial^2 y}{\partial \xi^2} + f, \\ y|_{t=0} &= \mathbb{E}(X_0). \end{aligned} \tag{6.10}$$

We solve numerically this equation by using the Matlab library *pdepe* and we compare our results by using the spectral method with the one obtained with the pdepe Matlab library.

Results on the simulation

The results of the simulations are presented in Fig. 1. The method have been applied with different values of $J^{N,M}$, $N = 7, 8$. We make a comparison with the solution of the deterministic equation, as was described in Sec. 6.2.1, by using the matlab library *pdepe*.

First we show the result on the simulation for the evaluation functional. The second group of graphs shows the simulation for the second functional. The results were obtained with the coefficient $\nu = 0.1$.

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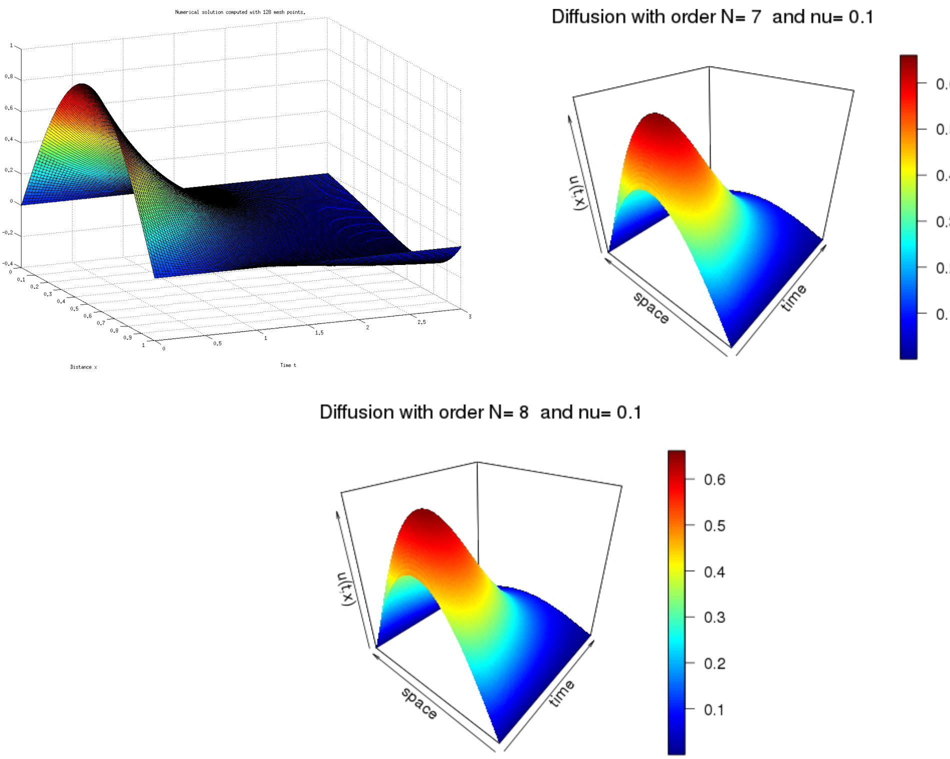

Fig. 1. Simulations for the diffusion equation with the spectral method, for $N = 7, 8$ and $\nu = 0.1$ with $u_0^{\xi_0}(g) = g(\xi_0)$.

6.3. *Stochastic Fisher–KPP equation in an interval*

Set $\mathcal{H} = L^2(0,1)$. We consider the stochastic Fisher–KPP equation in the interval $[0,1]$:

$$\begin{aligned} dX(t,\xi) &= [\nu\partial_\xi^2 X(t,\xi) + X(t,\xi)(1 - X(t,\xi))]dt + dW_t(t,\xi), \quad t > 0, \quad \xi \in (0,1) \\ X(t,0) &= X(t,1) = 0, \quad t > 0, \\ X(0,\xi) &= x(\xi), \quad x \in \mathcal{H}. \end{aligned} \tag{6.11}$$

W is a cylindrical Wiener process on $\mathcal{H}$, associated to a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. ν is the viscosity coefficient. We will consider the initial condition $X(0,\xi) = \mathrm{sech}^2(5(\xi - 0.5))$.

We rewrite the Fisher–KPP equation as an abstract differential equation in $\mathcal{H}$. Set $A = \nu\partial_\xi^2$ and $B(x) = x(1-x)$, $x \in \mathcal{H}$, with domains $D(A) = H^2(0,1) \cap H_0^1(0,1)$ and $D(B) = H_0^1(0,1)$, respectively. Then, (6.11) can be rewriten as

$$\begin{aligned} dX &= [AX + B(X)]dt + dW_t, \\ X(0) &= x, \quad x \in \mathcal{H}. \end{aligned} \tag{6.12}$$

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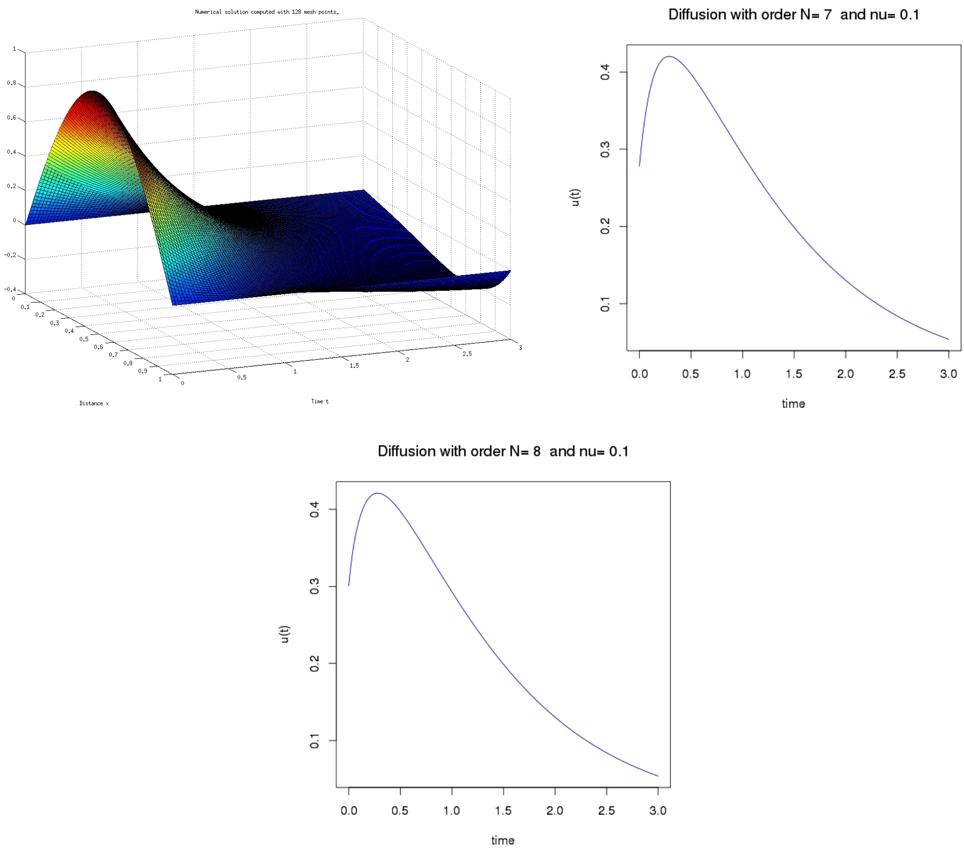

Fig. 2. Simulations for the diffusion equation with the spectral method, for $N = 7, 8$, $\nu = 0.1$ and $u_0(g) = \int_0^1 g(\xi)d\xi)$.

The operator A is self-adjoint with a complete orthonormal system of eigenfunctions in $\mathcal{H}$ given by

$$e_k(\xi) = \sqrt{2}\sin(k\pi\xi), \quad \xi \in [0, 1], \quad k \in \mathbb{N}.$$

Moreover, A satisfies $Ae_k = -\nu\pi^2k^2e_k$, for $k \in \mathbb{N}$.

As before we define $u(t, x) = \mathbb{E}[u_0(X_t^x)]$ and then $u(t, x)$ satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{Tr}(QD^2u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \quad x \in D(A).$$

Results on the existence and uniqueness of the solution to the Kolmogorov equation can be found, for instance, in Chap. 4 of Ref. 7.

About the functional $u_0 : \mathcal{H} \to \mathbb{R}$ we will consider two cases:

$$u_0^{\xi_0}(g) := g(\xi_0). \quad \text{for fixed } \xi_0 \in (0, 1)$$

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and

$$u_0(g) := \int_0^1 g(\xi)d\xi.$$

We now apply the numerical method. We write the solution u as

$$u(t,x) = \sum_{\mathbf{n}} u_{\mathbf{n}}(t)H_{\mathbf{n}}(x)$$

and by following the procedure done before we arrive at an infinite system of ordinary differential equations:

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t)C_{\mathbf{n},\mathbf{m}}, \qquad \mathbf{n},\mathbf{m}\in\mathcal{J}, \tag{6.13}$$

where $C_{\mathbf{n},\mathbf{m}}$ is given by

$$C_{\mathbf{n},\mathbf{m}} = \int_{\mathcal{H}} \langle B(x), D_xH_{\mathbf{n}}(x)\rangle_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx)$$

we need to calculate the value of the constants $C_{\mathbf{n},\mathbf{m}}$, then we need to calculate expressions such as $B(x), D_xH_{\mathbf{n}}(x)$.

Focus on the term $B(x) = x(1-x)$. By writing $x = \sum_k \beta_k e_k$, with $\beta_k := \langle x, e_k\rangle_{\mathcal{H}}$ we have

$$B(x) = \left(\sum_k \beta_k e_k\right)\left(1 - \sum_k \beta_k e_k\right) = \sum_k \beta_k e_k - \sum_k\sum_l \beta_l\beta_k e_l e_k.$$

For the expression $D_xH_{\mathbf{n}}(x)$ we have

$$D_xH_{\mathbf{n}}(x) = \sum_{j=1}^{\infty}\prod_{\substack{i=1\\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}})P'_{n_j}(\langle x, \Lambda^{-1/2}e_j\rangle_{\mathcal{H}})\Lambda^{-1/2}e_j.$$

Setting $\Lambda = (-A)^{-1}$ and by recalling that $Ae_j = -\nu\pi^2 j^2 e_j$ we have $\Lambda^{-1/2}e_j = \sqrt{2\nu}\pi|j|e_j$, and by using the last expression we have,

$$\begin{aligned}
C_{\mathbf{n},\mathbf{m}} &= \int_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx)\sum_{j=1}^{\infty}\prod_{\substack{i=1\\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}})P'_{n_j}(\langle x, \Lambda^{-1/2}e_j\rangle_{\mathcal{H}})\sqrt{2\nu}\pi|j| \\
&\quad\cdot\left[\sum_k \beta_k\langle e_k, e_j\rangle_{\mathcal{H}} - \sum_l\sum_k \beta_l\beta_k\langle e_l e_k, e_j\rangle_{\mathcal{H}}\right] \\
&= \int_{\mathcal{H}} \mu(dx)\sum_{j=1}^{\infty}\sqrt{2\nu}\pi|j|P_{m_j}(\langle x, \Lambda^{-1/2}e_j\rangle_{\mathcal{H}})P'_{n_j}(\langle x, \Lambda^{-1/2}e_j\rangle_{\mathcal{H}}) \\
&\quad\cdot\prod_{\substack{i=1\\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}})P_{m_i}(\langle x, \Lambda^{-1/2}e_i\rangle_{\mathcal{H}})\left[\beta_j - \sum_l\sum_k \beta_l\beta_k\langle e_l e_k, e_j\rangle_{\mathcal{H}}\right].
\end{aligned}$$

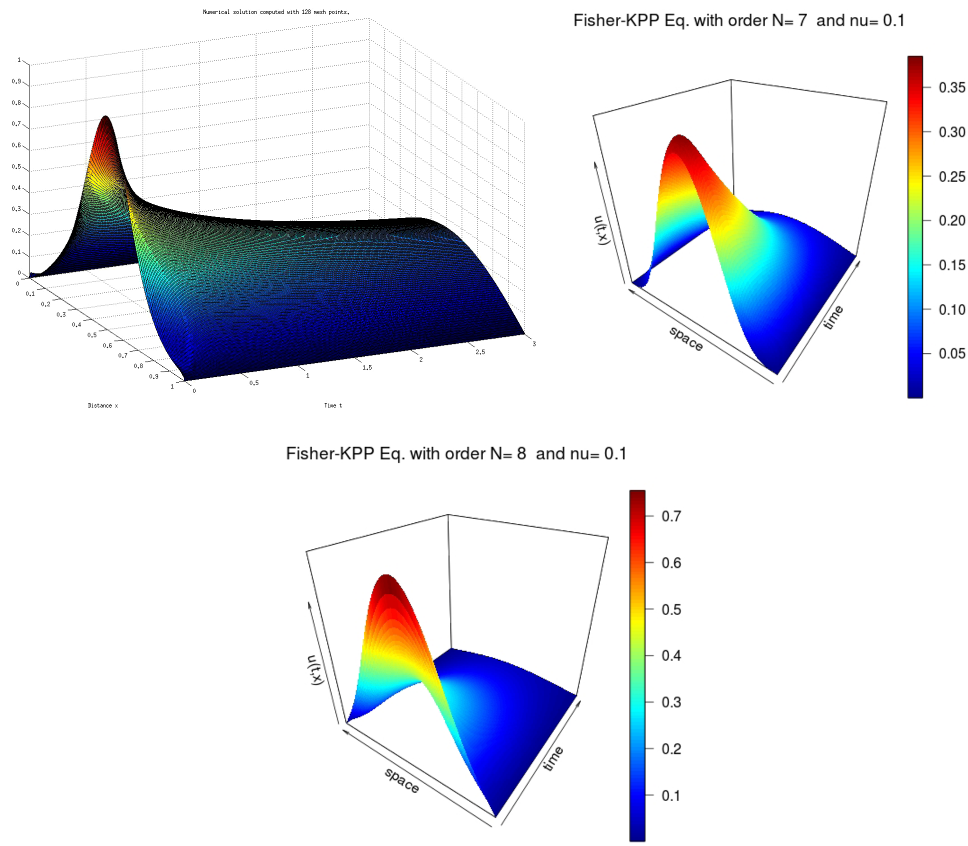

Fig. 3. Simulations for the Fisher–KPP equation with the Matlab library *pdepe* and with the spectral method for $N = 7, 8$, $u_0^{\xi_0}(g) = g(\xi_0)$.

For $N_1 \in \mathbb{N}$ define as before the set $S_{N_1} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{N_1} : \mathbf{n}_i \in J^{M,N}, i = 1, \ldots, N_1\}$. Moreover, for $\mathbf{n}, \mathbf{m} \in S_M$ define

$$\bar{C}_{\mathbf{n},\mathbf{m}} := \sum_{j=1}^{M} \int_{\mathbb{R}^M} P_{m_j}(\xi_j) P'_{n_j}(\xi_j) \mu(d\xi_j) \cdot \prod_{\substack{i=1 \\ i \neq j}}^{M} P_{m_i}(\xi_i) P_{n_i}(\xi_i) \frac{\mu(d\xi_i)}{\lambda_i} \left[\beta_j - \sum_{l=1}^{M} \sum_{k=1}^{M} \beta_l \beta_k \langle e_l e_k, e_j \rangle_{\mathcal{H}} \right] \tag{6.14}$$

and the finite system of ordinary differential equations:

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n} \in S_M} u_{\mathbf{n}}(t) \bar{C}_{\mathbf{n},\mathbf{m}}, \quad \text{for each } \mathbf{m} \in S_M \text{ and } \mathbf{n} \in S_M. \tag{6.15}$$

Then (6.15) approximates to the infinite system of ordinary differential equations (6.13) when $N, M \to \infty$. We use the system (6.15) to approximate the solution of the FPK equation associated to the Fisher–KPP equation.

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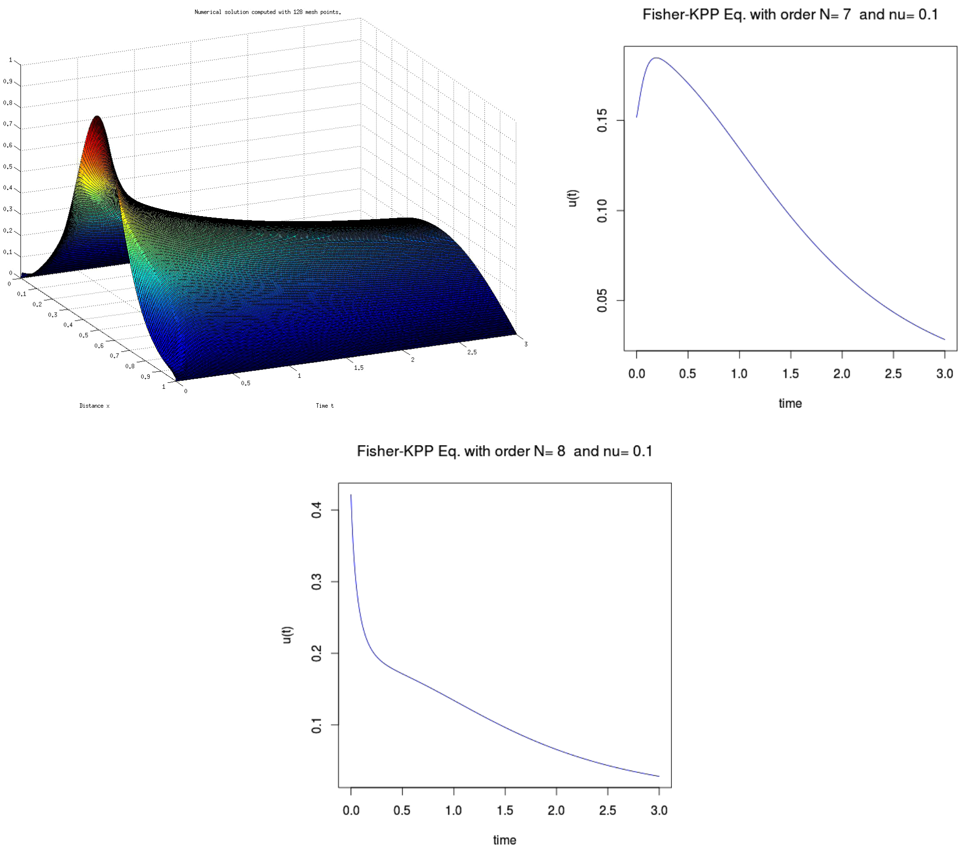

Fig. 4. Simulations for the Fisher–KPP equation with the Matlab library *pdepe* and with the spectral method for $N = 7, 8$, $u_0(g) = \int_0^1 g(\xi)d\xi$.

6.3.1. *Deterministic equation associated with the stochastic Fisher–KPP equation*

Set

$$y(t,\xi) = \mathbb{E}[X_t(\xi)],$$

then $y(t,\xi)$ solves the differential equation

$$\begin{aligned} \frac{\partial y}{\partial t} &= \nu \frac{\partial^2 y}{\partial \xi^2} + y(t,\xi)[1 - y(t,\xi)], \\ y|_{t=0} &= \mathbb{E}(X_0). \end{aligned} \tag{6.16}$$

We solve numerically this equation by using the Matlab library *pdepe* and we compare our results by using the spectral method with the one obtained with the pdepe Matlab library.

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Results on the simulation

We have the following graphs of simulations using the proposed method with diferents values of $J^{N,M}$, $N = 4, 5$. We make a comparison with the solution of the deterministic equation, as was described in Sec. 6.3.1, by using the matlab library *pdepe*.

We show the results on the simulation for the evaluation functional. The second graph shows the simulation for the second functional. The results were obtained with the coefficient $\nu = 0.1$.

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6.4. *Stochastic Burgers equation in an interval*

Set $\mathcal{H} = L^2(0, 1)$. We consider the stochastic Burgers equation in the interval $[0, 1]$:

$$dX(t,\xi) = \left[\nu\partial_\xi^2 X(t,\xi) + \frac{1}{2}\partial_\xi(X^2(t,\xi))\right]dt + dW_t(t,\xi), \quad t > 0,\ \ \xi \in (0,1) \tag{6.17}$$

$$X(t,0) = X(t,1) = 0, \quad t > 0,$$

$$X(0,\xi) = x(\xi), \quad x \in \mathcal{H} \tag{6.18}$$

W is a cylindrical Wiener process on $\mathcal{H}$, associated to a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$. ν is the viscosity coefficient.

We rewrite the Burgers equation as an abstract differential equation in $\mathcal{H}$. Set $A = \nu\partial_\xi^2$ and $B(x) = \frac{1}{2}\partial_\xi(x^2)$, $x \in \mathcal{H}$, with domains $D(A) = H^2(0, 1) \cap H_0^1(0, 1)$ and $D(B) = H_0^1(0, 1)$, respectively. Then, (6.17) can be rewriten as

$$\begin{aligned} dX &= [AX + B(X)]dt + dW_t, \\ X(0) &= x, \quad x \in \mathcal{H}. \end{aligned} \tag{6.19}$$

The operator A is self-adjoint with a complete orthonormal system of eigenfunctions in $\mathcal{H}$ given by

$$e_k(\xi) = \sqrt{2}\sin(k\pi\xi), \quad \xi \in [0,1], \quad k \in \mathbb{N}.$$

Moreover A satisfies $Ae_k = -\nu\pi^2 k^2 e_k$, for $k \in \mathbb{N}$.

As before we define $u(t, x) = \mathbb{E}[u_0(X_t^x)]$ and then $u(t, x)$ satisfies the Kolmogorov equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\mathrm{Tr}(QD^2u) + \langle Ax, Du\rangle_{\mathcal{H}} + \langle B(x), Du\rangle_{\mathcal{H}}, \quad x \in D(A).$$

Results on the existence and uniqueness of the solution to the Kolmogorov equation can be found, for instance, in Chap. 5 of Ref. 7.

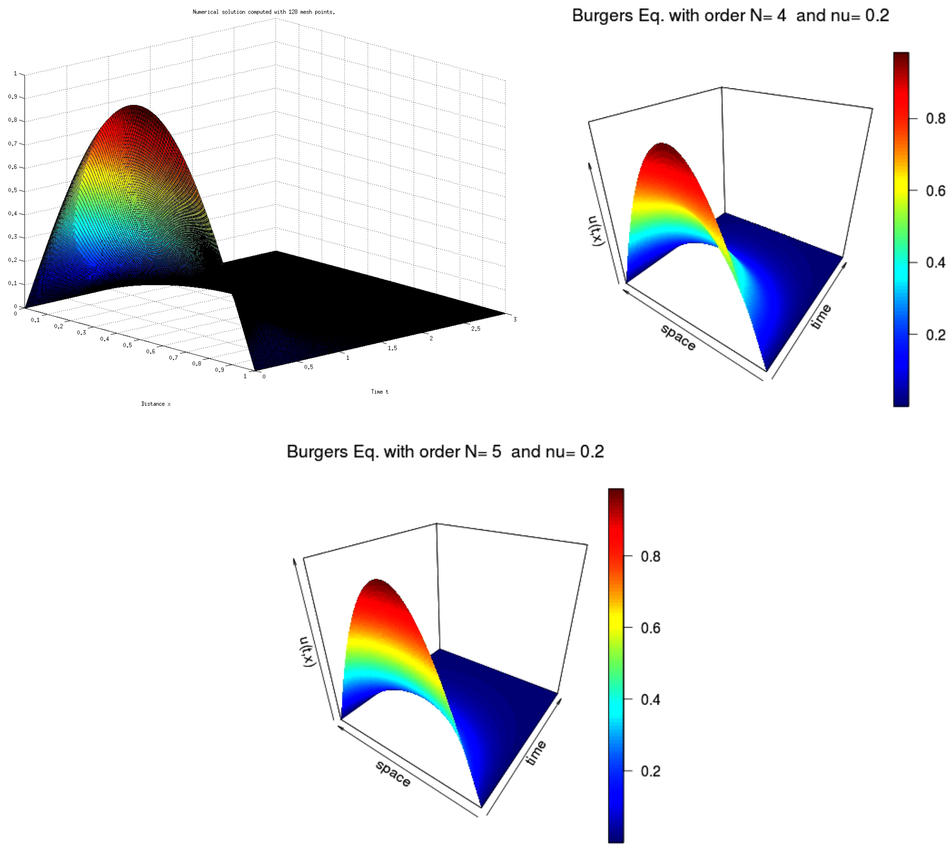

Fig. 5. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5, u_0^{\xi_0}(g) = g(\xi_0)$.

We will consider again two types of functionals:

$$u_0^{\xi_0}(g) := g(\xi_0). \quad \text{for fixed } \xi_0 \in (0,1)$$

and

$$u_0(g) := \int_0^1 g(\xi)d\xi.$$

We now apply the numerical method. We write the solution u as

$$u(t,x) = \sum_{\mathbf{n}} u_{\mathbf{n}}(t)H_{\mathbf{n}}(x)$$

and by following the procedure done before we arrive at an infinite system of ordinary differential equations:

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n}\in\mathcal{J}} u_{\mathbf{n}}(t)C_{\mathbf{n},\mathbf{m}}, \quad \mathbf{n},\mathbf{m} \in \mathcal{J} \tag{6.20}$$

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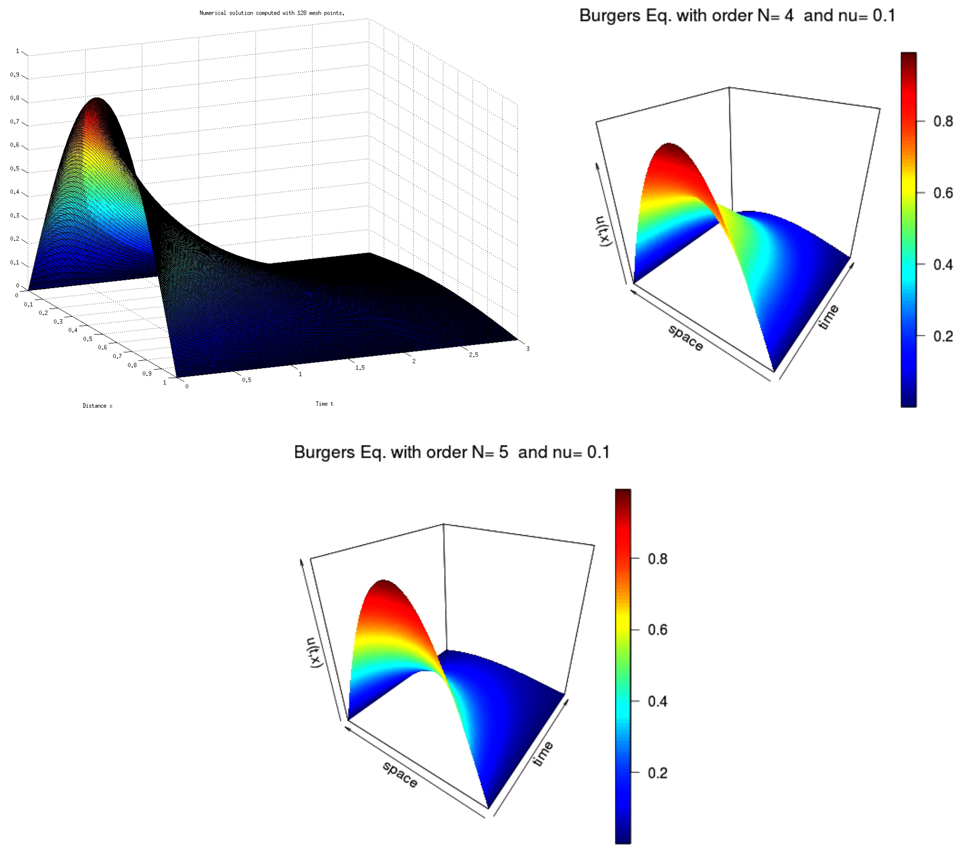

Fig. 6. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5$, $u_0^{\xi_0}(g) = g(\xi_0)$.

where $C_{\mathbf{n},\mathbf{m}}$ is given by

$$C_{\mathbf{n},\mathbf{m}} = \int_{\mathcal{H}} \langle B(x), D_x H_{\mathbf{n}}(x)\rangle_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx)$$

we need to calculate the value of the constants $C_{\mathbf{n},\mathbf{m}}$, then we need to calculate expressions such as $B(x), D_x H_{\mathbf{n}}(x)$.

Focus on the term $B(x) = \frac{1}{2}\partial_\xi(x^2)$. By writing $x = \sum_k \beta_k e_k$, with $\beta_k := \langle x, e_k\rangle_{\mathcal{H}}$, we have

$$B(x) = \frac{1}{2}\partial_\xi \left(\sum_k \beta_k e_k\right)^2 = \frac{1}{2}\partial_\xi \left[\sum_l \sum_k \beta_l \beta_k e_l e_k\right] = \frac{1}{2}\sum_l \sum_k \beta_l \beta_k (e_l e_k' + e_l' e_k).$$

For the expression $D_x H_{\mathbf{n}}(x)$ we have

$$D_x H_{\mathbf{n}}(x) = \sum_{j=1}^{\infty} \prod_{\substack{i=1 \\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}) P_{n_j}'(\langle x, \Lambda^{-1/2} e_j\rangle_{\mathcal{H}})\Lambda^{-1/2} e_j.$$

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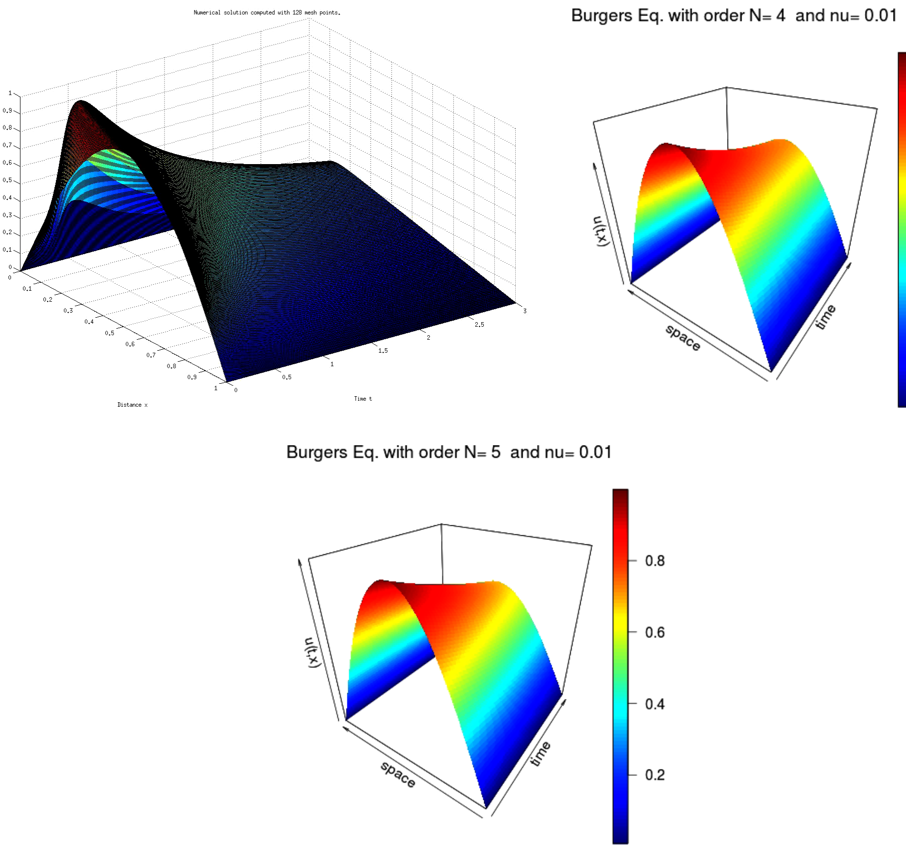

Fig. 7. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5$, $u_0^{\xi_0}(g) = g(\xi_0)$.

Setting $\Lambda = (-A)^{-1}$ and by recalling that $Ae_j = -\nu\pi^2 j^2 e_j$ we have $\Lambda^{-1/2} e_j = \sqrt{2\nu}\pi |j| e_j$, and by using the last expression we have,

$$C_{\mathbf{n},\mathbf{m}} = \frac{1}{2}\int_{\mathcal{H}} H_{\mathbf{m}}(x)\mu(dx) \sum_{j=1}^{\infty} \prod_{\substack{i=1\\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}) P'_{n_j}(\langle x, \Lambda^{-1/2} e_j\rangle_{\mathcal{H}}) \sqrt{2\nu}\pi |j|$$

$$\cdot \sum_l \sum_k \beta_l \beta_k \langle e_l e'_k + e'_l e_k, e_j\rangle_{\mathcal{H}}$$

$$= \frac{1}{2}\int_{\mathcal{H}} \mu(dx) \sum_{j=1}^{\infty} \sqrt{2\nu}\pi |j| P_{m_j}(\langle x, \Lambda^{-1/2} e_j\rangle_{\mathcal{H}}) P'_{n_j}(\langle x, \Lambda^{-1/2} e_j\rangle_{\mathcal{H}})$$

$$\cdot \prod_{\substack{i=1\\ i\neq j}}^{\infty} P_{n_i}(\langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}}) P_{m_i}(\langle x, \Lambda^{-1/2} e_i\rangle_{\mathcal{H}})$$

$$\cdot \sum_l \sum_k \beta_l \beta_k \langle e_l e'_k + e'_l e_k, e_j\rangle_{\mathcal{H}}.$$

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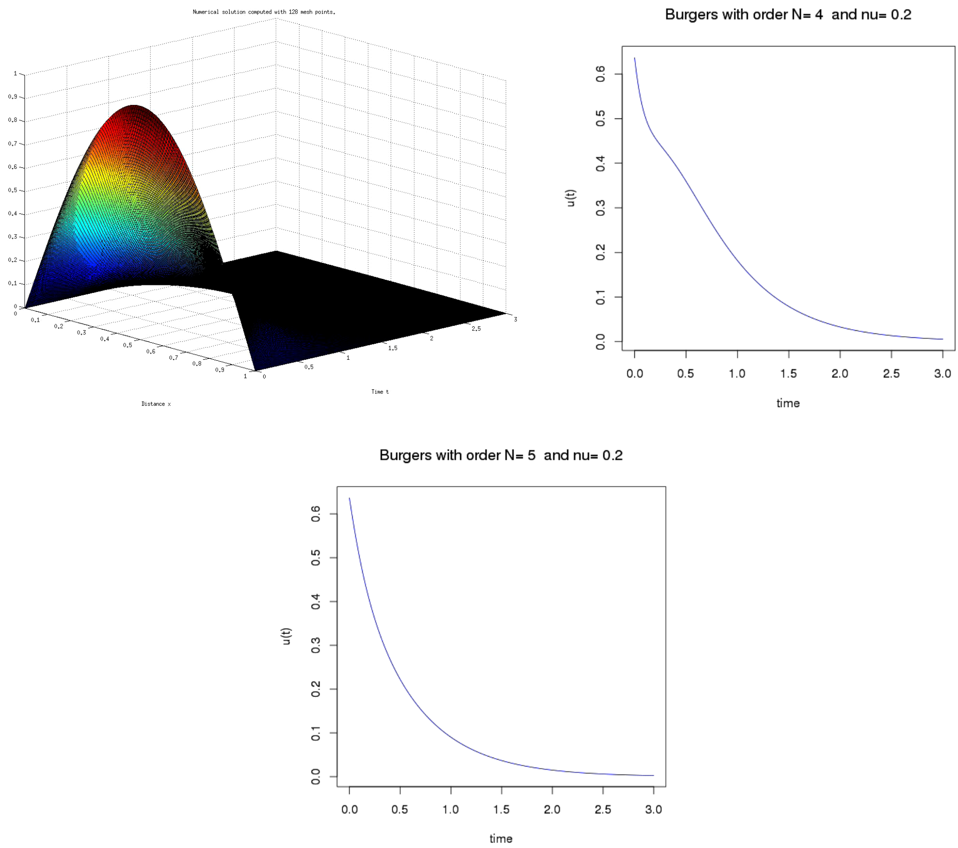

Fig. 8. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5$, $u_0(g) = \int_0^1 g(\xi)d\xi$.

For $N_1 \in \mathbb{N}$ define as before the set $S_{N_1} = \{\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{N_1} : \mathbf{n}_i \in J^{M,N}, i = 1, \ldots, N_1\}$. Moreover, for $\mathbf{n}, \mathbf{m} \in S_M$ define

$$\bar{C}_{\mathbf{n},\mathbf{m}} := \frac{1}{2}\sum_{j=1}^{M} \sqrt{2\nu}\pi|j| \int_{\mathbb{R}^M} P_{m_j}(\xi_j)P'_{n_j}(\xi_j)\mu(d\xi_j) \cdot \prod_{\substack{i=1\\ i\neq j}}^{M} P_{m_i}(\xi_i)P_{n_i}(\xi_i)\mu(d\xi_i) \sum_{l=1}^{M}\sum_{k=1}^{M} \beta_l\beta_k\langle e_l e'_k + e'_l e_k, e_j\rangle_{\mathcal{H}} \tag{6.21}$$

and the finite system of ordinary differential equations:

$$\dot{u}_{\mathbf{m}}(t) = -u_{\mathbf{m}}(t)\lambda_{\mathbf{m}} + \sum_{\mathbf{n}\in S_M} u_{\mathbf{n}}(t)\bar{C}_{\mathbf{n},\mathbf{m}}, \quad \text{for each } \mathbf{m} \in S_M \quad \text{and} \quad \mathbf{n} \in S_M. \tag{6.22}$$

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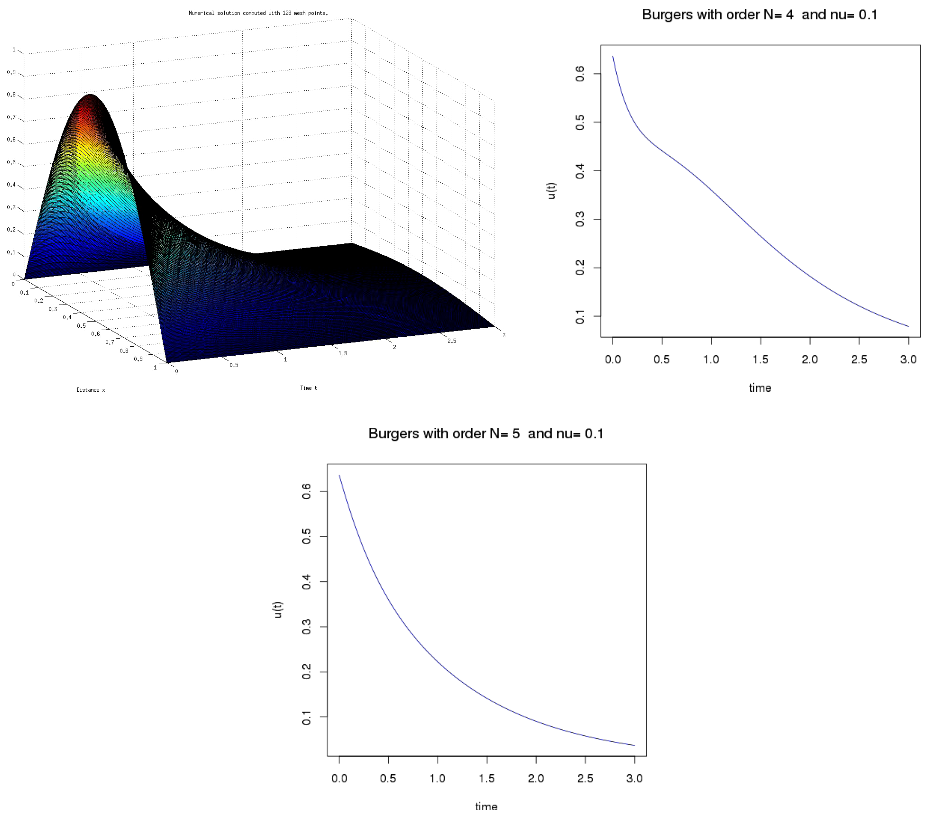

Fig. 9. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5$, $u_0(g) = \int_0^1 g(\xi)d\xi$.

Then (6.22) approximates to the infinite system of ordinary differential equations (6.20) when $N, M \to \infty$. We use the system (6.22) to approximate the solution of the FPK equation associated with the Burgers equation.

6.4.1. *Deterministic equation associated with the stochastic Burgers equation*

Set

$$y(t,\xi) = \mathbb{E}[X_t(\xi)],$$

then $y(t,\xi)$ solves the differential equation

$$\frac{\partial y}{\partial t} = \nu\frac{\partial^2 y}{\partial \xi^2} + \frac{1}{2}\partial_\xi(y^2(t,\xi)), \tag{6.23}$$

$$y|_{t=0} = \mathbb{E}(X_0).$$

We solve numerically this equation by using the Matlab library *pdepe* and we compare our results by using the spectral method with the one obtained with the pdepe Matlab library.

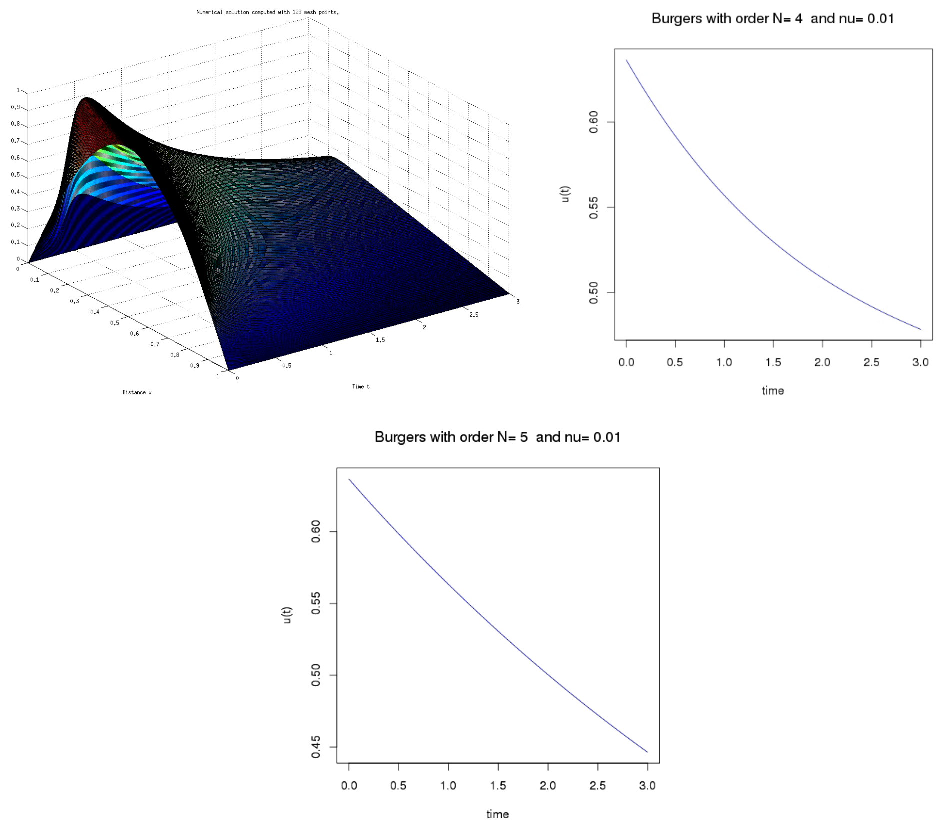

Fig. 10. Simulations for the Burgers equation with the Matlab library *pdepe* and with the spectral method for $N = 4, 5, u_0(g) = \int_0^1 g(\xi)d\xi$.

Results on the simulation

Figures 5–10 show simulations by using the proposed method with differents values of $J^{N,M}$, $N = 4, 5$. We make a comparison with the solution of the deterministic equation, as was described in Sec. 6.4.1, by using the matlab library *pdepe*.

The results on the simulation for the evaluation functional are in the first group of graphs. The second graph shows the simulation for the second functional. The results were obtained with the coefficient $\nu = 0.2, 0.1, 0.01$.

7. Conclusions

In this paper we introduced a numerical method to solve Fokker–Plank–Kolmogorov equations and we tested this method by applying it to the Kolmogorov equations associated to three stochastic partial differential equations: a stochastic diffusion, a Fisher–KPP stochastic equation and a stochastic Burgers equation in 1D, in a simple domain in the three cases. The results obtained are really promising. However, there are a few limitations. The first is that the noise in the SPDE is

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restricted to the additive case and to cover the multiplicative case seems unfeasible at this moment. Indeed, even if one is able to prove the existence and uniqueness of an invariant measure ν for the Ornstein–Uhlenbeck semigroup associated with the SPDE, there would remain the fully characterize of the measure and to find a basis for the Hilbert space $L^2(\mathcal{H}, \nu)$. Another issue is that we have applied the method to very simple domains, However, to cover the cases with complex domains one can use ideas of domain decomposition techniques similar to those used in spectral element methods. This is part of a forthcoming paper.

The method can be adapted to cover the Fokker–Planck equations associated with SPDEs, this will be studied in a subsequent work.

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