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Legendre and Chebyshev Spectral Approximations of Burgers' Equation

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Summary. We give error estimates for both Legendre and Chebyshev spectral approximations of the steady state Burgers' problem

$$-u_{xx}+\lambda(u u_x-f)=0 \quad \text{in } I=(-1,1); \quad u(-1)=u(1)=0.$$

To do that we prove some abstract approximation properties of orthogonal projection operators in some weighted Sobolev spaces $H^s_\omega(I)$ for both Legendre and Chebyshev weights.

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Introduction

The aim of this paper is to present stability results and error estimates for the approximation of the one dimensional, stationary Burgers' equation by Legendre and Chebyshev spectral methods.

For that we need inclusion and compactness properties of some weighted Sobolev spaces $H^s_\omega(I)$. Relative to the Legendre approximation these results are well known (actually the weight is $\omega\equiv 1$ and $H^s_\omega(I)$ reduce to the usual Sobolev spaces $H^s(I)$). For Chebyshev approximation (the weight is $\omega(x)=(1-x^2)^{-\frac{1}{2}}$) the above quoted properties are proved in Sect. 1.1. In Sect. 1.2 we introduce, in $H^1_\omega(I)$ and $H^1_{0,\omega}(I)$, the orthogonal projection operators upon the space S_N of polynomials of degree $\leq N$ over $I=(-1,1)$. Optimal approximation results are proved.

(For the case $\omega\equiv 1$ some results in this direction have been recently obtained by Babuska, Szabo and Katz [2] in a paper devoted to the analysis of the "P-version" of the finite element method.)

In section two we define the spectral approximations of Burgers' problem

$$-u_{xx}+\lambda(u u_x-f)=0 \quad \text{in } I; \quad u(-1)=u(1)=0 \tag{0.1}$$

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(λ is a positive parameter and f is a given function). The analysis is carried out essentially by means of an abstract result of Brezzi, Rappaz and Raviart [4], concerned with approximations of a large class of non-linear problems, and a generalisation of it exposed in [12].

We state the following result: for any compact subset Λ of $\mathbb{R}^+$ assume that there exists a non-singular solution of (0.1) which belongs to $H^\sigma_\omega(I)$, for some $\sigma \geqq 1$. Then the (Legendre or Chebyshev) spectral approximation $u_N \in S_N$ of u satisfies

$$\forall \lambda \in \Lambda \qquad \|u_N(\lambda)-u(\lambda)\|_{H^1_{0,\omega}(I)} + N\,\|u_N(\lambda)-u(\lambda)\|_{L^2_\omega(I)} = 0(N^{1-\sigma}).$$

Throughout this paper C will denote a generic positive constant independent of N.

1. Approximation Properties of Some Projection Operators in Sobolev Spaces

Let $I=(-1,1)$; we are given a weight function ω on I satisfying $\omega \in L^1(I)$, $\omega(x) \geqq \omega_0 > 0$ for any $x \in I$. Set

$$L^2_\omega(I)=\{\phi\colon I \to \mathbb{R} \mid \phi \text{ is measurable and } (\phi,\phi)_\omega < \infty\} \tag{1.1}$$

equipped with the inner product $(\phi,\psi)_\omega = \int_I \phi(x)\,\psi(x)\,\omega(x)\,dx$. For any integer $s \geqq 0$ we set

$$H^s_\omega(I)=\{\phi \in L^2_\omega(I) \mid \|\phi\|_{s,\omega} < \infty\} \tag{1.2}$$

where

$$\|\phi\|^2_{s,\omega} = \sum_{k=0}^{s} \left(\frac{d^k\phi}{dx^k}, \frac{d^k\phi}{dx^k}\right)_\omega.$$

Clearly $L^2_\omega(I) \equiv H^0_\omega(I)$. For real $s>0$ we define $H^s_\omega(I)$ by complex interpolation between the spaces $H^{\bar{s}}_\omega(I)$ and $H^{\bar{s}+1}_\omega(I)$, where $\bar{s}$ denotes the integral part of s (see e.g. [3, Chap. 4]). If $s \geqq 0$ is an integer, we define $H^s_{0,\omega}(I)$ to be the closure of $\mathscr{D}(I)$ in $H^s_\omega(I)$ (for $s=0$ we still get $L^2_\omega(I)$); for real $s>0$, $H^s_{0,\omega}(I)$ is again defined by complex interpolation between $H^{\bar{s}}_{0,\omega}(I)$ and $H^{\bar{s}+1}_{0,\omega}(I)$.

For $\omega \equiv 1$, the spaces $H^s_\omega(I)$ (resp. $H^s_{0,\omega}(I)$) coincide with the usual Sobolev spaces $H^s(I)$ (resp. $H^s_0(I)$, provided $s \notin \mathbb{N}+\frac{1}{2}$) (cf. e.g. [3, 10]), so the index ω will be omitted.

Remark 1.1. Let us notice that for $\omega \equiv 1$ or $\omega(x)=(1-x^2)^{-\frac{1}{2}}$ we have:

$$H^1_{0,\omega}(I)=\{\phi \in H^1_\omega(I) \mid \phi(-1)=\phi(1)=0\}. \tag{1.3}$$

This is standard in the case $\omega \equiv 1$ and has been proved in [8] for $\omega(x)=(1-x^2)^{-\frac{1}{2}}$. □

1.1. Some Properties of Spaces $H^s_\omega(I)$ for $\omega(x)=(1-x^2)^{-\frac{1}{2}}$

It follows from the theory of interpolation spaces that $H^\mu_\omega(I)$ is topologically imbedded in $H^\nu_\omega(I)$ if $\mu>\nu\geqq 0$ (cf. e.g. [3, Thm. 4.2.1]). Let us first prove that in the case of the Chebyshev weight $\omega(x)=(1-x^2)^{-\frac{1}{2}}$ this imbedding is compact. We begin with

Lemma 1.1. *The imbedding $H^1_{0,\omega}(I)\subset L^2_\omega(I)$ is compact.*

Proof. Since $H^1_0(I)\subset L^2(I)$ with compact imbedding it is sufficient to prove that

$$\begin{aligned} &u\in L^2_\omega(I)\to u\,\omega^{\frac{1}{2}}\in L^2(I) && \text{is an isomorphism, and}\\ &u\in H^1_{0,\omega}(I)\to u\,\omega^{\frac{1}{2}}\in H^1_0(I) && \text{is a continuous mapping.}\end{aligned} \tag{1.4}$$

The first property holds trivially. Next, let $u\in H^1_{0,\omega}(I)$; we note that

$$(u\,\omega^{\frac{1}{2}})_x=u_x\,\omega^{\frac{1}{2}}+x\,u\,\omega^{\frac{5}{2}}.$$

Clearly $u_x\,\omega^{\frac{1}{2}}\in L^2(I)$; moreover by [7] there exists $\alpha>0$ such that

$$\forall\, v\in H^1_{0,\omega}(I),\qquad \int_I v^2\,\omega^5\,dx\leqq \alpha\,\|v\|^2_{1,\omega}. \tag{1.5}$$

Hence $u\,\omega^{\frac{5}{2}}\in L^2(I)$, so that $u\,\omega^{\frac{1}{2}}\in H^1(I)$.

Now, using the density of $\mathscr{D}(I)$ into $H^1_{0,\omega}(I)$, there exists a sequence (ϕ_n) of elements of $\mathscr{D}(I)$ which converges to u in $H^1_\omega(I)$. Then $u\,\omega^{\frac{1}{2}}$ is the limit in $H^1(I)$ of a sequence $(\phi_n\,\omega^{\frac{1}{2}})$ of function which vanish at $x=\pm 1$. Therefore $u\,\omega^{\frac{1}{2}}$ belongs to $H^1_{0,\omega}(I)$ and (1.4) follows. $\square$

Lemma 1.2. *The imbedding $H^1_\omega(I)\subset L^2_\omega(I)$ is compact.*

Proof. Let (u_n) be a bounded sequence of $H^1_\omega(I)$; using (1.3) there exists, for all n, $u^0_n\in H^1_{0,\omega}$ and $v_n\in S_1$ such that

$$u_n=u^0_n+v_n\qquad \|u^0_n\|_{1,\omega}+\|v_n\|_{L^\infty(I)}\leqq C\,\|u_n\|_{1,\omega}$$

where S_n is the space of polynomials of degree $\leqq n$ over I. Using Lemma 1.1 and the equivalence of norms on S_1 we can extract from $(\{u^0_n, v_n\})$ a subsequence $(\{u^0_{n,k}, v_{n,k}\})$ which is convergent to $\{u^0, v\}$ in $L^2_\omega(I)\times L^\infty(I)$.

Then $(u_{n,k}=u^0_{n,k}+v_{n,k})$ converges to u^0+v in $L^2_\omega(I)$, and this proves the lemma. $\square$

Theorem 1.1. *The imbedding $H^\mu_\omega(I)\subset H^\nu_\omega(I)$ is compact for any real μ and ν satisfying $0\leqq\nu<\mu$.*

Proof. Using Lemma 1.2 one can prove by an inductive procedure that for any integer $n\geqq 0$ $H^{n+1}_\omega(I)\subset H^n_\omega(I)$ with compact imbedding. For real μ and ν the result holds by interpolation (cf. e.g. [3, Thms. 3.8.2 and 4.7.1]). $\square$

Theorem 1.2. *We have*

$$\begin{aligned} &\text{(i) } H^s_\omega(I)\subset L^\infty(I) && \text{if } s>\tfrac{1}{2}\\ &\text{(ii) } H^s_\omega(I) \text{ is an algebra} && \text{if } s\geqq 1.\end{aligned}$$

Proof. (i) Since $\omega(x)>1$ for all $x\in I$, we have $H^s_\omega(I)\subset H^s(I)$ for any integer $s\geqq 0$, and therefore for any real $s>0$ by means of interpolation ([3, Thm. 4.4.1]). Now the first result follows by the well known Sobolev inequality $H^s(I)\subset L^\infty(I)$, $s>\frac{1}{2}$.

(ii) To prove (ii) we only need to verify that

$$\forall u, v\in H^s_\omega(I), \qquad s\geqq 1, \ \|uv\|_{s,\omega}\leqq C\|u\|_{s,\omega}\|v\|_{s,\omega}.$$

We prove it first for an integer $s\geqq 1$, then the result can be extended to real s by interpolation [3, Thm. 4.4.1]. Denoting by D the derivative operator and using the result proved in (i) we get

$$\begin{aligned}\|D^s(uv)\|_{0,\omega}=\Big(\int_I [D^s(uv)]^2\,\omega\,dx\Big)^{\frac{1}{2}}&\leqq \sum_{k=0}^{s-1} C_{k,s}\|u\|_{k+1,\omega}\|D^{s-k}v\|_{0,\omega}\\ &+\|D^s u\|_{0,\omega}\|v\|_{1,\omega}\leqq C\|u\|_{s,\omega}\|v\|_{s,\omega}. \qquad \square\end{aligned}$$

We conclude this section by showing some results concerning the elliptic operator $T=\left(-\dfrac{d^2}{dx^2}\right)^{-1}$ which we are going to define. First, we introduce the bilinear form c: $H^1_\omega(I)\times H^1_{0,\omega}(I)\to\mathbb{R}$ defined by

$$c(u,v)=\int_I u_x(v\omega)_x\,dx. \tag{1.6}$$

We have (see [7])

Lemma 1.3. *There exist three positive constants β,γ,δ such that*

$$\forall v\in H^1_{0,\omega}(I) \qquad \|v\|_{0,\omega}\leqq\beta\|v_x\|_{0,\omega} \ \textit{(Poincaré inequality)} \tag{1.7}$$

$$\forall v\in H^1_{0,\omega}(I) \qquad c(v,v)\geqq\gamma\|v\|^2_{1,\omega} \tag{1.8}$$

$$\forall v\in H^1_{0,\omega}(I), \quad \forall u\in H^1_\omega(I) \qquad c(u,v)\leqq\delta\|u_x\|_{0,\omega}\|v_x\|_{0,\omega}. \quad \square \tag{1.9}$$

We define the linear operator T: $(H^1_{0,\omega}(I))'\to H^1_{0,\omega}(I)$ by

$$c(Tg,\phi)=\langle g,\phi\rangle \qquad \forall\phi\in H^1_{0,\omega}(I) \tag{1.10}$$

where $\langle\,,\rangle$ denotes the duality pairing. It follows from (1.8) that T is a bounded operator whose norm is $\leqq 1/\gamma$. In addition, we have

Theorem 1.4. *For any $s\geqq 0$ T is a linear, bounded continuous operator from $H^s_\omega(I)$ into $H^1_{0,\omega}(I)\cap H^{s+2}_\omega(I)$. For any $s\in[-1,0[$ T is continuous from $(H^{-s}_{0,\omega}(I))'$ into $H^{2+s}_\omega(I)\cap H^1_{0,\omega}(I)$.*

Proof. If $s\geqq 0$ is an integer, integrating by parts in (1.10) it gives

$$(-Tg)_{xx}=g \text{ in } I,$$

so that

$$\|Tg\|_{s+2,\omega}\leqq(1+1/\gamma)\|g\|_{s,\omega}.$$

For non integral $s>0$ the same result follows by interpolation.

If $-1 \leqq s < 0$, the result can be obtained as in [3, Corollary 4.5.2]. □

Remark 1.2. If $\omega \equiv 1$ we can still define T: $H^{-1}(I) \to H_0^1(I)$ as in (1.10), and $c(u,v) = \int_I u_x v_x \, dx$. It is well known that T is a linear and continuous operator from $H^s(I)$ into $H_0^1(I) \cap H^{s+2}(I)$ for any $s \geqq -1$. □

Consider now the following problem. Given $g \in (H_\omega^1(I))'$, find $u \in H_\omega^1(I)$ such that

$$(u_x, v_x)_\omega + (u, v)_\omega = \langle g, v \rangle \qquad \forall\, v \in H_\omega^1(I). \tag{1.11}$$

Existence and uniqueness of u is assured from the Lax-Milgram Lemma.

Theorem 1.5. *Assume that g belongs to $L_\omega^2(I)$. Then the solution u of (1.11) belongs to $H_\omega^2(I)$ and there exists a positive constant C such that*

$$\|u\|_{2,\omega} \leqq C \|g\|_{0,\omega}. \tag{1.12}$$

Proof. Taking $v \in \mathscr{D}(I)$ in (1.11) implies that

$$(u_x \omega)_x = (u-g)\omega \tag{1.13}$$

in the distribution sense.

Since $(u-g) \in L_\omega^2(I)$, $(u-g)\omega \in L_{\omega^{-1}}^2(I)$. Then $u_x \omega$ belongs to $H_{\omega^{-1}}^1(I)$. An easy calculation shows that

$$\begin{aligned} \forall\, x_1, x_2 \in I, \quad & |u_x \omega(x_1) - u_x \omega(x_2)| \\ & \leqq \|u_x \omega\|_{1,\omega^{-1}} |\arccos x_2 - \arccos x_1|^{\frac{1}{2}} \end{aligned} \tag{1.14}$$

so that $(u_x \omega)(\pm 1)$ make sense.

Now, multiplying (1.13) by $v \in H_\omega^1(I)$ and integrating by parts, we obtain

$$\int_{-1}^{1} u_x \omega v_x \, dx + \int_{-1}^{1} u v \omega \, dx - \int_{-1}^{1} g v \omega \, dx = \int_{-1}^{1} (u_x \omega v)_x \, dx.$$

By (1.11) the right hand side vanishes, then

$$u_x \omega(1) = u_x \omega(-1) = 0. \tag{1.15}$$

Let us prove now that

$$\int_{-1}^{1} u_x^2 \omega^5 \, dx \leqq C \|g\|_{0,\omega}^2. \tag{1.16}$$

By (1.15), (1.13) and the Cauchy-Schwarz inequality we get

$$\begin{aligned} \int_{-1}^{0} u_x^2 \omega^5 \, dx &= \int_{-1}^{0} \left(\int_{-1}^{t} (u_x \omega)_x \, dx \right)^2 \omega^3 \, dt \\ &\leqq \int_{-1}^{0} \left(\int_{-1}^{t} (u-g)\omega \, dx \right)^2 \omega^3 \, dt \leqq \int_{-1}^{0} \left(\int_{-1}^{t} (u-g)^2 \omega \, dx \right) \left(\int_{-1}^{t} \omega \, dx \right) \omega^3 \, dt. \end{aligned}$$

Since for $-1 \leqq t \leqq 0$ $1-t^2 \geqq 1+t$ it follows that

$$\int_{-1}^{0} u_x^2 \omega^5 \, dx \leqq 2 \int_{-1}^{0} (1+t)^{-1} \left(\int_{-1}^{t} (u-g)^2 \omega \, dx \right) dt$$

$$\leqq 2 \, \|(u-g)^2 \omega\|_{L^1(I)} = 2 \, \|u-g\|_{0,\omega}^2.$$

The same estimate can be established for $\int_0^1 u_x^2 \omega^5 \, dx$, hence (1.16) holds.

Next it is an easy matter to check that $|u_x \omega_x|^2 \omega^{-1} \leqq 2 u_x^2 \omega^5$. Then from (1.13) and (1.16) it follows that $u_{xx} \omega \in L^2_{\omega^{-1}}(I)$ and $\|u_{xx} \omega\|_{0,\omega^{-1}} \leqq C \, \|g\|_{0,\omega}$, so (1.12) holds. □

1.2. Error Estimates for L^2_ω and H^1_ω Projection Operators

For any integer $s \geqq 0$ we set

$$V_N = \{\phi \in S_N \,|\, \phi(-1) = \phi(1) = 0\}. \tag{1.17}$$

In this section, we refer constantly to both cases $\omega = 1$ and $\omega(x) = (1-x^2)^{-\frac{1}{2}}$. Let $\Pi_{0,N}\colon L^2_\omega(I) \to S_N$ be the L^2_ω projection operator upon S_N, i.e.

$$(u - \Pi_{0,N} u, \phi)_\omega = 0 \qquad \forall \, \phi \in S_N. \tag{1.18}$$

For any $\sigma \geqq 0$ and for any $u \in H^\sigma_\omega(I)$ we have the estimate (cf. [5, 6])

$$\|u - \Pi_{0,N} u\|_{\mu,\omega} \leqq C \, \|u\|_{\sigma,\omega} \begin{cases} N^{\frac{3\mu}{2} - \sigma}, & 0 \leqq \mu \leqq \min(\sigma, 1) \\ N^{2\mu - \frac{1}{2} - \sigma}, & 1 \leqq \mu \leqq \sigma. \end{cases} \tag{1.19}$$

Note that, thanks to (1.3), $V_N \subset H^1_{0,\omega}(I)$; due to Lemma 1.3 we can define a projection operator $\Pi^0_{1,N}\colon H^1_{0,\omega}(I) \to V_N$ by

$$c(u - \Pi^0_{1,N} u, \phi) = 0 \qquad \forall \, \phi \in V_N. \tag{1.20}$$

Theorem 1.6. *Let $u \in H^1_{0,\omega}(I) \cap H^\sigma_\omega(I)$, $\sigma \geqq 1$; then*

$$\|u - \Pi^0_{1,N} u\|_{\mu,\omega} \leqq C N^{\mu - \sigma} \, \|u\|_{\sigma,\omega}, \qquad 0 \leqq \mu \leqq 1. \tag{1.21}$$

Proof. In order to prove (1.21) we follow a strategy that will be adopted in other situations later on in this section: we prove first the result for $\mu = 1$, then for $\mu = 0$ by a duality technique, finally for $0 < \mu < 1$ by interpolation.

(i) We start with the case $\omega \equiv 1$. From (1.12) and (1.20) we obtain respectively

$$(u_x - \Pi_{0,N-1} u_x, \psi) = 0 \qquad \forall \, \psi \in S_{N-1}, \tag{1.22}$$

$$(u_x - (\Pi^0_{1,N} u)_x, \phi_x) = 0 \qquad \forall \, \phi \in V_N. \tag{1.23}$$

Taking $\psi = \phi_x$ and substrating (1.22) from (1.23), it follows that

$$((\Pi^0_{1,N} u)_x - \Pi_{0,N-1} u_x, \phi_x) = 0 \qquad \forall \, \phi \in V_N. \tag{1.24}$$

Since u vanishes at $x=\pm 1$, from (1.22) we deduce that $\Pi_{0,N-1}u_x$ has zero average in I, and so $\phi_N(\xi)=\int_{-1}^{\xi}\Pi_{0,N-1}u_x\,dx\in V_N$. Taking $\phi=\Pi^0_{1,N}u-\phi_N$ in (1.24) we get immediately

$$(\Pi^0_{1,N}u)_x=\Pi_{0,N-1}u_x. \tag{1.25}$$

Finally, applying (1.19) to u_x for $\mu=0$ and using the Poincaré inequality we get

$$\|u-\Pi^0_{1,N}u\|_1 \leqq CN^{1-\sigma}\|u\|_\sigma, \quad \forall\,\sigma\geqq 1. \tag{1.26}$$

To obtain (1.21) for $\mu=0$ we identify $L^2(I)$ and its dual space; we have

$$\|u-\Pi^0_{1,N}u\|_0=\sup_{\psi\in L^2(I)}\frac{(u-\Pi^0_{1,N}u,\psi)}{\|\psi\|_0}.$$

Using (1.10) and (1.20), we obtain

$$\|u-\Pi^0_{1,N}u\|_0=\sup_{\psi\in L^2(I)}\frac{c(u-\Pi^0_{1,N}u,T\psi-\Pi^0_{1,N}T\psi)}{\|\psi\|_0}$$

and by (1.9) and (1.26)

$$\|u-\Pi^0_{1,N}u\|_0\leqq CN^{-1}\sup\frac{\|T\psi\|_2}{\|\psi\|_0}\|u-\Pi^0_{1,N}u\|_1.$$

Then, by Theorem 1.4, the estimate (1.26) and Remark 1.2 we get

$$\|u-\Pi^0_{1,N}u\|_0\leqq CN^{-\sigma}\|u\|_\sigma, \quad \forall\,\sigma\geqq 1. \tag{1.27}$$

Finally, (1.21) holds for $\mu\in]0,1[$ by interpolation between (1.26) and (1.27) (cf. [3, Thm. 4.1.2]).

(ii) Consider now the case $\omega(x)=(1-x^2)^{-\frac{1}{2}}$. The relation (1.25) does not hold, so we define an auxiliary operator $\tilde{\Pi}^0_{1,N}$: $H^1_{0,\omega}(I)\to V_N$ by

$$((u-\tilde{\Pi}^0_{1,N}u)_x,\phi_x)_\omega=0 \quad \forall\,\phi\in V_N. \tag{1.28}$$

We set for any $\xi\in\bar{I}$

$$H_N(\xi)=\int_{-1}^{\xi}\Pi_{0,N-1}u_x\,dx. \tag{1.29}$$

By the Cauchy-Schwarz inequality we get

$$\forall\,\xi\in\bar{I},\quad |u(\xi)-H_N(\xi)|=\left|\int_{-1}^{\xi}(u_x-\Pi_{0,N-1}u_x)\,dx\right|$$
$$\leqq\left(\int_{-1}^{\xi}\omega^{-1}dx\right)^{\frac{1}{2}}\left(\int_{-1}^{\xi}|u_x-\Pi_{0,N-1}u_x|^2\omega\,dx\right)^{\frac{1}{2}}\leqq C\|u_x-\Pi_{0,N-1}u_x\|_{0,\omega}$$

so by (1.19) we obtain

$$\forall\,\xi\in\bar{I}\quad |u(\xi)-H_N(\xi)|\leqq CN^{1-\sigma}\|u\|_{\sigma,\omega}\quad (\sigma\geqq 1). \tag{1.30}$$

As in (i) we get (thanks to (1.18) and (1.28))

$$(\Pi_{0,N-1} u_x - (\tilde{\Pi}^0_{1,N} u)_x, \phi_x)_\omega = 0 \quad \forall \phi \in V_N. \tag{1.31}$$

We define

$$R_N(\xi) = \int_{-1}^{\xi} (\Pi_{0,N-1} u_x - H_N(1)/2)\, dx;$$

$R_N \in V_N$, and thanks to (1.31) we get

$$|((\tilde{\Pi}^0_{1,N} u - R_N)_x, \phi_x)_\omega| = \tfrac{1}{2} |(H_N(1), \phi_x)_\omega| \quad \forall \phi \in V_N.$$

Then by the Poincaré inequality (1.7) and the estimate (1.30) (with $\xi = 1$) we get (for any $\sigma \geqq 1$)

$$\|\tilde{\Pi}^0_{1,N} u - R_N\|_{1,\omega} \leqq C\, |H_N(1)| \leqq C N^{1-\sigma} \|u\|_{\sigma,\omega}.$$

Using (1.7), (1.19) and (1.30) (with $\xi = 1$) we get

$$\|u - \tilde{\Pi}^0_{1,N} u\|_{1,\omega} \leqq C N^{1-\sigma} \|u\|_{\sigma,\omega}. \tag{1.32}$$

Using (1.8), (1.9) and going back to $\Pi^0_{1,N}$ we get

$$\begin{aligned} \|u - \Pi^0_{1,N} u\|^2_{1,\omega} &\leqq \gamma^{-1} c(u - \Pi^0_{1,N} u, u - \Pi^0_{1,N} u) = \gamma^{-1} c(u - \Pi^0_{1,N} u, u - \tilde{\Pi}^0_{1,N} u) \\ &\leqq \gamma^{-1} \delta \|u - \Pi^0_{1,N} u\|_{1,\omega} \|u - \tilde{\Pi}^0_{1,N} u\|_{1,\omega}, \end{aligned}$$

and for $\mu = 1$ the result (1.21) is achieved thanks to (1.32). For $\mu \in [0, 1[$ we proceed as in (i). □

Define now the operator $\Pi_{1,N}$: $H^1_\omega(I) \to S_N$ by

$$((u - \Pi_{1,N} u)_x, \phi_x)_\omega + (u - \Pi_{1,N} u, \phi)_\omega = 0 \quad \forall \phi \in S_N. \tag{1.33}$$

Theorem 1.7. *Let $u \in H^\sigma_\omega(I)$, $\sigma \geqq 1$; then*

$$\|u - \Pi_{1,N} u\|_{\mu,\omega} \leqq C N^{\mu-\sigma} \|u\|_{\sigma,\omega}, \quad 0 \leqq \mu \leqq 1. \tag{1.34}$$

Proof. Let $v \in S_1$ satisfy $v(\pm 1) = u(\pm 1)$. We have $u - v \in H^1_{0,\omega}(I)$, so that by (1.21) we get

$$\begin{aligned} \|u - \Pi_{1,N} u\|_{1,\omega} &= \inf_{\phi \in S_N} \|u - \phi\|_{1,\omega} \leqq \inf_{\phi \in V_N} \|(u - v) - \phi\|_{1,\omega} \\ &\leqq C N^{1-\sigma} \|u - v\|_{\sigma,\omega}. \end{aligned}$$

Thanks to the result (i) of Theorem 1.2, and to the definition of v, we have

$$\|v\|_{\sigma,\omega} \leqq C_1 \|v\|_{L^\infty(I)} \leqq C_2 \|u\|_{L^\infty(I)} \leqq C \|u\|_{\sigma,\omega},$$

then (1.34) holds for $\mu = 1$. Now, using (1.12) and the classical duality argument, we get (1.34) for $\mu = 0$. Finally the result for $\mu \in]0, 1[$ is obtained by interpolation. □

In some applications (see e.g. [7]) it is useful to deal with an L^2_ω-projection operator taking into account the boundary conditions. To this end we define the operator $\Pi^0_{0,N}$: $L^2_\omega(I) \to V_N$ by

$$(u - \Pi^0_{0,N} u, \phi) = 0 \qquad \forall\, \phi \in V_N. \tag{1.35}$$

We set $\mathscr{H}^\sigma_\omega = H^\sigma_{0,\omega}(I)$ if $\sigma \leqq 1$, and $\mathscr{H}^\sigma_\omega = H^1_{0,\omega}(I) \cap H^\sigma_\omega(I)$ if $\sigma > 1$.

Theorem 1.8. *Let $u \in \mathscr{H}^\sigma_\omega$; we have the estimate*

$$\|u - \Pi^0_{0,N} u\|_{0,\omega} \leqq C N^{-\sigma} \|u\|_{\sigma,\omega}. \tag{1.36}$$

Proof. By definition of $\Pi^0_{0,N}$ and by (1.21) we have, if $u \in \mathscr{H}^\sigma_\omega$, $\sigma \geqq 1$

$$\|u - \Pi^0_{0,N} u\|_{0,\omega} \leqq C \|u\|_{0,\omega}$$

and

$$\|u - \Pi^0_{0,N} u\|_{0,\omega} = \inf_{\phi \in V_N} \|u - \phi\|_{0,\omega} \leqq C N^{-\sigma} \|u\|_{\sigma,\omega}.$$

Denoting hereafter by Id the identity operator, we deduce that the linear mapping $\mathrm{Id} - \Pi^0_{0,N}$ is continuous from L^2_ω in L^2_ω with a norm $\leqq C$, and from $\mathscr{H}^\sigma_\omega$ in L^2_ω with a norm $\leqq C N^{-\sigma}$. Then (1.36) can be achieved by interpolation (see [3, Thm. 4.4.1]). □

2. Approximations of Burgers' Equation by Spectral Methods

Let ε be a positive real number, and f be a given function; we consider the problem

$$-\varepsilon u_{xx} + u u_x = f \quad \text{in } I, \qquad u(-1) = u(1) = 0. \tag{2.1}$$

As we are interested in the spectral approximation of (2.1) by Legendre and by Chebyshev polynomials, we look for a variational formulation of the above problem relative to two different weight functions: $\omega \equiv 1$ (Legendre's weight), and $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$ (Chebyshev's weight of the first kind). For convenience of exposition we adopt a unified notation for the two weights. We set:

$$V = H^1_{0,\omega}(I), \qquad W = L^2_\omega(I), \quad \lambda = \varepsilon^{-1}. \tag{2.2}$$

The mapping G: $\mathbb{R} \times V \to W$ defined by

$$G(\lambda, u) = \lambda(u u_x - f) \tag{2.3}$$

is infinitely differentiable, and $D^2 G$ is bounded on any bounded subset of $\mathbb{R} \times V$. Moreover, since $V \subset L^\infty(I)$ we get

$$\|G(\lambda, u)\|_{0,\omega} \leqq C |\lambda| (\|u\|_{L^\infty(I)} \|u\|_{1,\omega} + \|f\|_{0,\omega}) \leqq C |\lambda| (\|u\|^2_{1,\omega} + \|f\|_{0,\omega}). \tag{2.4}$$

Let T be the operator defined by (1.10); the following result is an easy consequence of Theorem 1.4, Remark 1.2 and Theorem 1.1.

Lemma 2.1. *T is a compact operator from W into V.* □

Setting F: $\mathbb{R} \times V \to V$

$$F(\lambda, u) = u + T G(\lambda, u), \tag{2.5}$$

problem (2.1) can be written equivalently as follows: find $u \in V$ such that

$$F(\lambda, u)=0. \tag{2.6}$$

Let Λ be any compact interval of $\mathbb{R}^+$; using the techniques of [9], for instance, one can prove that there exists a branch of solutions of problem (2.6), $\{(\lambda, u(\lambda)), \lambda \in \Lambda\}$, non singular in the sense that there exists $\delta > 0$ such that

$$\forall \lambda \in \Lambda, \ \forall v \in V \quad \|(\mathrm{Id}+TD_u G[\lambda, u(\lambda)])v\|_{1,\omega} \geqq \delta \|v\|_{1,\omega}. \tag{2.7}$$

Hereafter $D_u G[\lambda_0, u_0]$ denotes the Frechet derivative with respect to u of $G(\lambda, u)$, computed at the point $\{\lambda_0, \mathrm{u}_0\}$.

In this paper we are interested in the approximation of the above branch of solutions.

For any positive integer N let V_N be defined by (1.17), and for convenience of notation, let us denote by Π_N the projection operator $\Pi^0_{1,N}$ defined by (1.20). We define T_N: $V' \to V_N$ by

$$T_N = \Pi_N \circ T. \tag{2.8}$$

Thanks to (1.10) and (1.20) for all $g \in V'$ it follows that

$$c(T_N g, \phi) = \langle g, \phi \rangle \qquad \forall \phi \in V_N. \tag{2.9}$$

Setting F_N: $\Lambda \times V_N \to V_N$,

$$F_N(\lambda, v) = v + T_N G(\lambda, v) \tag{2.10}$$

the spectral approximation of problem (2.6) is given by: find $u_N \in V_N$ such that

$$F_N(\lambda, u_N) = 0. \tag{2.11}$$

For the reader's convenience let us recall the following results:

Lemma 2.2 (see [4, Thm. 6]). *Let $m \geqq 1$ be an integer; assume that G is a C^{m+1} mapping from $\Lambda \times V$ into W, and that $D^{m+1}G$ is bounded over any bounded subset of $\Lambda \times V$. Let Π_N: $V \to V_N$ be a continuous operator satisfying*

$$\forall v \in V \quad \lim \|\Pi_N v - v\|_{1,\omega} = 0. \tag{2.12}$$

Moreover assume that $T_N \in \mathscr{L}(V'; V_N)$ satisfies

$$\lim_{N \to \infty} \|T - T_N\|_{\mathscr{L}(W;V)} = 0. \tag{2.13}$$

Then there exist a neighborhood θ of the origin in V and, for $N \geqq N_0$ large enough, a unique C^{m+1} mapping $\lambda \in \Lambda \to u_N(\lambda) \in V_N$ such that

$$\forall \lambda \in \Lambda \quad F_N(\lambda, u_N(\lambda)) = 0, \qquad u_N(\lambda) - u(\lambda) \in \theta. \tag{2.14}$$

Furthermore, there exists a positive constant K, independent of λ and N, such that the following estimate holds

$$\begin{aligned} \forall \lambda \in \Lambda \quad & \|u_N(\lambda) - u(\lambda)\|_{1,\omega} \\ & \leqq K\{\|u(\lambda) - \Pi_N u(\lambda)\|_{1,\omega} + \|(T_N - T)G(\lambda, u(\lambda))\|_{1,\omega}\}. \quad \square \end{aligned} \tag{2.15}$$

Lemma 2.3 (see [12, Corollary 1.1]). *Assume that the hypotheses of Lemma* 2.2 *hold; moreover assume that:*

$$\text{the mapping } v\in V \to D_u G[\lambda, v] \in \mathscr{L}(L^2_\omega(I); V') \text{ is continuous;} \tag{2.16}$$

$$\lim_{N\to\infty} \|T - T_N\|_{\mathscr{L}(V'; L^2_\omega(I))} = 0; \tag{2.17}$$

$$\text{if } v \in L^2_\omega(I) \text{ satisfies } v + T D_u G[\lambda, u(\lambda)]\, v = 0 \text{ then } v \in V. \tag{2.18}$$

Then for $N \geqq N_0$ *large enough we get the estimate*

$$\forall\, \lambda\in\Lambda \qquad \|u_N(\lambda) - u(\lambda)\|_{L^2_\omega(I)} \leqq C\, \|F_N(\lambda, u(\lambda))\|_{L^2_\omega(I)}. \quad \square \tag{2.19}$$

Using the two previous lemmas we can state the following

Theorem 2.1. *There exists a neighborhood* θ *of the origin in* V *and, for* $N \geqq N_0$ *large enough, a unique* C^∞*-mapping* $\lambda\in\Lambda \to u_N(\lambda) \in V_N$ *such that for any* $\lambda\in\Lambda$, $u_N(\lambda)$ *solves* (2.10) *and* $u_N(\lambda) - u(\lambda) \in \theta$. *Furthermore, assuming that the mapping* $\lambda\in\Lambda \to u(\lambda) \in V \cap H^\sigma_\omega(I)$ *is continuous for a suitable* $\sigma \geqq 1$, *for any* $\lambda\in\Lambda$ *there exists a constant* $C(\lambda)$ *depending on* $\|u(\lambda)\|_{\sigma,\omega}$ *such that*

$$\|u_N(\lambda) - u(\lambda)\|_{1,\omega} + N\, \|u_N(\lambda) - u(\lambda)\|_{0,\omega} \leqq C(\lambda) N^{1-\sigma}. \tag{2.20}$$

Proof. By (1.21) and a classical density argument it follows that (2.12) holds. To prove (2.13) let us note that by (2.8) for any $g \in W$ we have

$$(T - T_N)\, g = (\mathrm{Id} - \Pi_N)\, T g;$$

then by (1.21), Theorem 1.4 and Remark 1.2, for any $g \in W$ we get

$$\|(T - T_N) g\|_{1,\omega} \leqq C N^{-1}\, \|g\|_{0,\omega}.$$

This proves (2.13).

Applying Lemma 2.2 we can state the first part of this theorem together with the estimate (2.15). As, by definition

$$(T - T_N)\, G(\lambda, u(\lambda)) = -\Pi_N\, u(\lambda)\, u(\lambda)$$

from (1.21) we obtain the estimate

$$\forall\, \lambda\in\Lambda \qquad \|u_N(\lambda) - u(\lambda)\|_{1,\omega} \leqq C N^{1-\sigma}\, \|u(\lambda)\|_{\sigma,\omega}. \tag{2.21}$$

To obtain the L^2_ω-estimate let us check the hypotheses of Lemma 2.3. From (2.3) it follows that $D_u G[\lambda, v]\, w = \frac{\lambda}{2}(v w)_x$, then (2.16) is a consequence of the property (i) of Theorem 1.2. By (1.21) and Theorem 1.4 we deduce (2.17). Finally (2.18) is a consequence of (2.16) and Theorem 1.4. Then the estimate (2.19) holds; since

$$F_N(\lambda, u(\lambda)) = (T_N - T)\, G(\lambda, u(\lambda)) = \Pi_N\, u(\lambda) - u(\lambda)$$

by (1.21) it follows that

$$\forall\, \lambda\in\Lambda \qquad \|u(\lambda) - u_N(\lambda)\|_{L^2_\omega(I)} \leqq C N^{-\sigma}\, \|u(\lambda)\|_{\sigma,\omega}. \tag{2.22}$$

Then (2.20) follows from (2.21) and (2.22). □

Remark 2.1. Following [12, Sect. 1.3], we propose a Newton method to solve the approximate problem: for a given $v^0 \in V_N$ find $(v^n)_{n \geqq 1}$ satisfying

$$v^n \in V_N, \ \forall \lambda \in \Lambda \qquad D_u F_N[\lambda, v^{n-1}]\, v^n = D_u F_N[\lambda, v^{n-1}]\, v^{n-1} - F_N(\lambda, v^{n-1}). \tag{2.23}$$

Applying [12, Thm. 1.4] it is easy to check that a suitable choice of v^0 allows to get that v^n converges quadratically to u_N for any $\lambda \in \Lambda$. □

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