3.1.

$$S'_N(x_j) = \begin{cases} 0 & \text{si } j \equiv 0 \bmod (N) \\ \frac{1}{2}(-1)^j \cot\left(\frac{jh}{2}\right) & \text{si } j \not\equiv 0 \bmod (N) \end{cases}$$

Para $N = 2$, entonces $h = \frac{2\pi}{N} = \frac{2\pi}{2} = \pi$

$$D_2 = \begin{bmatrix} S_2(x_2) & S_2(x_1) \\ S_2(x_1) & S_2(x_2) \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2}\cot\left(\frac{\pi}{2}\right) \\ -\frac{1}{2}\cot\left(\frac{\pi}{2}\right) & 0 \end{bmatrix} = O_{2\times 2}$$

$$\Rightarrow D_2^2 = O_{2\times 2}$$

$$S''_N(x_j) = \begin{cases} -\frac{\pi^2}{3h^2} - \frac{1}{6} & j \equiv 0 \bmod (N) \\ -\frac{(-1)^j}{2\operatorname{sen}^2\left(\frac{jh}{2}\right)} & \text{si } j \not\equiv 0 \bmod (N) \end{cases} \qquad \frac{1}{\operatorname{sen}^2(x)} = \csc^2(x)$$

$$D_2^{(2)} = \begin{bmatrix} S''_1(x_0) & S''_2(x_1) \\ S''_1(x_1) & S''_2(x_2) \end{bmatrix} = \begin{bmatrix} -\frac{\pi^2}{3\pi^2} - \frac{1}{6} & \frac{1}{2\operatorname{sen}^2\left(\frac{\pi}{2}\right)} \\ \frac{1}{2\operatorname{sen}^2\left(\frac{\pi}{2}\right)} & -\frac{\pi^2}{3\pi^2} - \frac{1}{6} \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

Por lo tanto $D_2^{(2)} \neq D_2^2$

Para $N = 4$, $h = \frac{2\pi}{N} = \frac{\pi}{2}$

$$D_4 = \begin{bmatrix} S'_4(x_4) & S'_4(x_3) & S'_4(x_2) & S'_4(x_1) \\ S'_4(x_1) & S'_4(x_4) & S'_4(x_3) & S'_4(x_2) \\ S'_4(x_2) & S'_4(x_1) & S'_4(x_4) & S'_4(x_3) \\ S'_4(x_3) & S'_4(x_2) & S'_4(x_1) & S'_4(x_4) \end{bmatrix} \qquad S'_4(x_j) = \begin{cases} 0 & \text{si: } j \equiv 0 \bmod (4) \\ -\frac{1}{2}\cot\left(\frac{jh}{2}\right) & \text{si } j \not\equiv 0 \bmod (4) \end{cases}$$

$$= \begin{bmatrix} 0 & -\frac{1}{2}\cot\left(\frac{3\pi}{4}\right) & \frac{1}{2}\cot\left(\frac{\pi}{2}\right) & -\frac{1}{2}\cot\left(\frac{\pi}{4}\right) \\ -\frac{1}{2}\cot\left(\frac{\pi}{4}\right) & 0 & -\frac{1}{2}\cot\left(\frac{3\pi}{4}\right) & \frac{1}{2}\cot\left(\frac{\pi}{2}\right) \\ \frac{1}{2}\cot\left(\frac{\pi}{2}\right) & -\frac{1}{2}\cot\left(\frac{\pi}{4}\right) & 0 & -\frac{1}{2}\cot\left(\frac{3\pi}{4}\right) \\ -\frac{1}{2}\cot\left(\frac{3\pi}{4}\right) & \frac{1}{2}\cot\left(\frac{\pi}{2}\right) & -\frac{1}{2}\cot\left(\frac{\pi}{4}\right) & 0 \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0 & 1/2 & 0 & -1/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \end{bmatrix} \qquad D_4^2 = \begin{bmatrix} -1/2 & 0 & 1/2 & 0 \\ 0 & -1/2 & 0 & 1/2 \\ 1/2 & 0 & -1/2 & 0 \\ 0 & 1/2 & 0 & -1/2 \end{bmatrix}$$

De la misma manera construimos a $D_4^{(2)}$ con:

Con $N = 4$

$h = \frac{2\pi}{4} = \frac{\pi}{2}$

$h^2 = \frac{\pi^2}{4}$

$$S_4''(x_j) = \begin{cases} -\frac{\pi^2}{3h^2} - \frac{1}{6} & \text{si } j \equiv 0 \bmod (4) \\ -\frac{1}{2}\csc^2\left(\frac{jh}{2}\right)\cdot(-1)^j & \text{o.c.} \end{cases}$$

$$D_4^{(2)} = \begin{bmatrix} -\frac{4}{3} - \frac{1}{6} & \frac{1}{2}\csc^2\left(\frac{3\pi}{4}\right) & -\frac{1}{2}\csc^2\left(\frac{\pi}{2}\right) & \frac{1}{2}\csc^2\left(\frac{\pi}{4}\right) \\ \frac{1}{2}\csc^2\left(\frac{\pi}{4}\right) & -\frac{4}{3} - \frac{1}{6} & \frac{1}{2}\csc^2\left(\frac{3\pi}{4}\right) & -\frac{1}{2}\csc^2\left(\frac{\pi}{2}\right) \\ -\frac{1}{2}\csc^2\left(\frac{\pi}{2}\right) & \frac{1}{2}\csc^2\left(\frac{\pi}{4}\right) & -\frac{4}{3} - \frac{1}{6} & \frac{1}{2}\csc^2\left(\frac{3\pi}{4}\right) \\ \frac{1}{2}\csc^2\left(\frac{3\pi}{4}\right) & -\frac{1}{2}\csc^2\left(\frac{\pi}{2}\right) & \frac{1}{2}\csc^2\left(\frac{\pi}{4}\right) & -\frac{4}{3} - \frac{1}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & 1 & -1/2 & 1 \\ 1 & -3/2 & 1 & -1/2 \\ -1/2 & 1 & -3/2 & 1 \\ 1 & -1/2 & 1 & -3/2 \end{bmatrix}$$

Por lo tanto $D_4^{(2)} \neq D_4^2$

3.2. Derive (3.9) and (3.11)

$$S'_N(x_j) = \begin{cases} 0 & \text{si } j \equiv 0 \bmod(N) \\ \frac{1}{2}(-1)^j \cot\left(\frac{jh}{2}\right) & \text{si } j \not\equiv 0 \bmod(N) \end{cases}$$

Tenemos que $S_N(x) = \dfrac{\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\left(\frac{2\pi}{h}\right)\tan\left(\frac{x}{2}\right)}$ (1)

Con $h = \frac{2\pi}{N}$ y $x_j = j \cdot h = \frac{2\pi}{N} \cdot j$ $j = 1, \dots, N$

Entonces:

$$S'_N(x) = \frac{h}{2\pi} \cdot \left[\frac{\frac{\pi}{h}\cos\left(\frac{\pi x}{h}\right)\tan\left(\frac{x}{2}\right) - \frac{1}{2}\operatorname{Sen}\left(\frac{\pi x}{h}\right)\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)} \right] \quad (2)$$

$$= \frac{h}{2\pi}\left[\frac{\pi}{h} \cdot \frac{\cos\left(\frac{\pi x}{h}\right)}{\tan\left(\frac{x}{2}\right)} - \frac{1}{2} \cdot \frac{\operatorname{sen}\left(\frac{\pi x}{h}\right)\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)} \right]$$

$$= \frac{h}{2\pi} \cdot \left[\frac{\pi}{h} \cdot \frac{\cos\left(\frac{\pi x}{h}\right)\cos\left(\frac{x}{2}\right)}{\operatorname{Sen}\left(\frac{x}{2}\right)} - \frac{1}{2} \cdot \frac{\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}^2\left(\frac{x}{2}\right)} \right]$$

Substituyendo h y $x = x_j$

$$S'_N(x_j) = \frac{1}{N}\left[\frac{N}{2} \cdot \frac{\cos(j\pi)\cos\left(\frac{j\pi}{N}\right)}{\operatorname{Sen}\left(\frac{j\pi}{N}\right)} - \frac{1}{2} \cdot \frac{\operatorname{Sen}(j\pi)}{\operatorname{Sen}^2\left(\frac{j\pi}{N}\right)} \right]$$

Como $\operatorname{Sen}(j\pi) = 0$ y $\operatorname{Sen}\left(\frac{j\pi}{N}\right) \neq 0$ pues $\frac{j}{N} < 1$

Ademas $\cos(j\pi) = (-1)^j$ y $\dfrac{\cos\left(\frac{j\pi}{N}\right)}{\operatorname{Sen}\left(\frac{j\pi}{N}\right)} = \cot\left(\frac{j\pi}{N}\right)$

$$\Rightarrow S'_N(x_j) = \frac{1}{2}(-1)^j \cot\left(\frac{j\pi}{N}\right), \quad \text{si } N = \frac{2\pi}{h}$$

$$\Rightarrow S'_N(x_j) = \frac{1}{2}(-1)^j \cot\left(\frac{j\pi}{2}\right); \; j = 1, \dots, N-1$$

Si $\quad x = 0$

$$S_N'(0) = \frac{1}{N}\lim_{x\to 0}\left[\frac{\overbrace{\left(\frac{N}{2}\right)\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Cos}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right) - \frac{N}{4}\cdot\operatorname{Sen}\left(\frac{Nx}{2}\right)}^{f(x)}}{\underbrace{\frac{N}{2}\cdot\operatorname{Sen}^2\left(\frac{x}{2}\right)}_{g(x)}}\right]$$

$$f'(x) = \frac{N}{2}\cdot\left[\frac{1}{2}\operatorname{Cos}^2\left(\frac{x}{2}\right)\operatorname{Cos}\left(\frac{Nx}{2}\right) + \frac{N}{2}\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right)\right.$$
$$\left. - \frac{1}{2}\operatorname{Sen}^2\left(\frac{x}{2}\right)\operatorname{Cos}\left(\frac{Nx}{2}\right) - \frac{N}{4}\operatorname{Cos}\left(\frac{Nx}{2}\right)\right]$$

$$= \frac{N}{4}\cdot\left[\operatorname{Cos}\left(\frac{Nx}{2}\right)\cdot\left[\operatorname{Cos}^2\left(\frac{x}{2}\right) - \operatorname{Sen}^2\left(\frac{x}{2}\right)\right] - N\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right)\right.$$
$$\left. - \frac{N}{2}\cos\left(\frac{Nx}{2}\right)\right]$$

$$= \frac{N}{4}\cdot\left[2\operatorname{Cos}\left(\frac{Nx}{2}\right)\operatorname{Cos}^2\left(\frac{x}{2}\right) - \operatorname{Cos}\left(\frac{Nx}{2}\right) - \frac{N}{2}\operatorname{Cos}\left(\frac{Nx}{2}\right) - N\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right)\right]$$

$$f''(x) = \frac{N}{4}\left[-\operatorname{Sen}\left(\frac{Nx}{2}\right)\operatorname{Cos}^2\left(\frac{x}{2}\right) - 2\operatorname{Cos}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{x}{2}\right)\right.$$
$$+ \frac{1}{2}\operatorname{Sen}\left(\frac{Nx}{2}\right) + \frac{N}{4}\operatorname{Sen}\left(\frac{Nx}{2}\right) - \frac{N}{2}\operatorname{Cos}^2\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{Nx}{2}\right)$$
$$\left. - \frac{N^2}{2}\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Cos}\left(\frac{Nx}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right) + \frac{N}{2}\operatorname{Sen}^2\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{Nx}{2}\right)\right]$$

$$g'(x) = \frac{N}{2}\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{Cos}\left(\frac{x}{2}\right)$$
$$g''(x) = \frac{N}{2}\left[\operatorname{Cos}^2\left(\frac{x}{2}\right) - \operatorname{Sen}^2\left(\frac{x}{2}\right)\right]$$

$$\Rightarrow \quad S_N'(0) = \frac{1}{N}\cdot\lim_{x\to 0}\frac{f''(x)}{g''(x)} = \frac{1}{N}\left[\frac{0}{\frac{N}{2}[1-0]}\right] = 0$$

Por lo tanto

$$S_N'(x_j) = \begin{cases} 0 & \text{si } j \equiv 0 \bmod(N) \\ \frac{1}{2}\cdot(-1)^j \operatorname{Cot}\left(\frac{jh}{2}\right) & \text{si } j \not\equiv 0 \bmod(N) \end{cases}$$

$$S''_N(x_j) = \begin{cases} -\dfrac{\pi^2}{3h^2} - \dfrac{1}{6}, & j \equiv 0 \bmod N \\ -\dfrac{(-1)^j}{2\,\mathrm{sen}^2\left(\frac{jh}{2}\right)}, & j \equiv 0 \bmod N \end{cases} \qquad (3.11)$$

Derivando la expresión (2)

$$S'_N(x) = \underbrace{\frac{1}{2}\left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\tan\left(\frac{x}{2}\right)}\right]}_{f(x)} - \underbrace{\frac{h}{4\pi}\left[\frac{\mathrm{sen}\left(\frac{\pi x}{h}\right)}{\mathrm{sen}^2\left(\frac{x}{2}\right)}\right]}_{g(x)}$$

$$f'(x) = \frac{1}{2}\left[\frac{-\frac{\pi}{h}\,\mathrm{sen}\left(\frac{\pi x}{h}\right)\tan\left(\frac{x}{2}\right) - \frac{1}{2}\sec^2\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right)}{\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{-\frac{\pi}{2h}\,\mathrm{sen}\left(\frac{\pi x}{h}\right)}{\tan\left(\frac{x}{2}\right)} - \frac{1}{4}\cdot\left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\mathrm{sen}^2\left(\frac{x}{2}\right)}\right]$$

$$f'(x_j) = -\frac{N}{4}\left[\frac{\mathrm{sen}(j\pi)}{\tan\left(\frac{j\pi}{N}\right)}\right] - \frac{1}{4}\left[\frac{\cos(jh)}{\mathrm{sen}^2\left(\frac{j\pi}{N}\right)}\right] = -\frac{1}{4}\cdot\frac{(-1)^j}{\mathrm{sen}^2\left(\frac{jh}{2}\right)}$$

$$g'(x) = -\frac{h}{4\pi}\left[\frac{\frac{\pi}{h}\cos\left(\frac{\pi x}{h}\right)\mathrm{sen}^2\left(\frac{x}{2}\right) - \mathrm{sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\mathrm{sen}\left(\frac{\pi x}{h}\right)}{\mathrm{sen}^4\left(\frac{x}{2}\right) \neq 0 \Rightarrow \left\{\frac{h\cdot j}{2} = \frac{\pi}{N}\cdot j \pm k\pi\right\}}\right]$$

$$g'(x_j) = -\frac{1}{4}\,\frac{\cos(j\pi)}{\mathrm{sen}^2\left(\frac{jh}{2}\right)} + 0 = -\frac{1}{4}\cdot\frac{(-1)^j}{\mathrm{sen}^2\left(\frac{jh}{2}\right)}$$

$$\Rightarrow S''_N(x_j) = f'(x_j) + g'(x_j) = -\frac{1}{2}\cdot\frac{(-1)^j}{\mathrm{sen}^2\left(\frac{jh}{2}\right)} \qquad j = 1, \ldots, N-1$$

Si $x = 0$, $S_N''(0) = \lim_{x \to 0} \left[f'(x) + g'(x) \right]$

$$\lim_{x \to 0} f'(x) = \frac{-\pi}{2h}\left(2\frac{\pi}{h}\right) \lim_{x \to 0} \left[\frac{\cos\left(\frac{\pi}{h}x\right)}{\sec^2\left(\frac{x}{2}\right)} \right] + \frac{1}{4}\left(\frac{\pi}{h}\right) \lim_{x \to 0} \left[\frac{\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \right]$$

$$= -\frac{\pi^2}{h^2} + \frac{1}{4}\left(\frac{\pi^2}{h^2}\right) \lim_{x \to 0} \left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\frac{1}{2}\left[\cos^2\left(\frac{x}{2}\right) - \operatorname{sen}^2\left(\frac{x}{2}\right)\right]} \right]$$

$$= -\left(\frac{\pi}{h}\right)^2 + \frac{1}{4}\left(\frac{\pi}{h}\right)^2 = -\frac{3}{4}\left(\frac{\pi}{h}\right)^2$$

$$\lim_{x \to 0} g'(x) = -\frac{1}{4} \cdot \left(\frac{\pi}{h}\right) \cdot 2 \cdot \lim_{x \to 0} \left[\frac{-\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{2\operatorname{Sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \right] + \lim_{x \to 0} h(x)$$

$$= -\frac{1}{4}\left(\frac{\pi}{h}\right)^2 \lim_{x \to 0} \left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\frac{1}{2}\left[\cos^2\left(\frac{x}{2}\right) - \operatorname{Sen}^2\left(\frac{x}{2}\right)\right]} \right] + \quad ''$$

$$= \frac{1}{2}\left(\frac{\pi}{h}\right)^2 + \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0} h(x) = \frac{h}{4\pi} \lim_{x \to 0} \left[\frac{-\frac{1}{2}\operatorname{Sen}\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right) + \frac{\pi}{h}\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right)}{\frac{3}{2}\left[\operatorname{Sen}\left(\frac{x}{2}\right)\right]^2 \cos\left(\frac{x}{2}\right)} \right]$$

$$= \frac{h}{4\pi} \lim_{x \to 0} \left[a(x) + b(x) \right]$$

$$\lim_{x \to 0} a(x) = -\frac{1}{3}\left[\frac{\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \right] = -\frac{1}{3}\left(2\frac{\pi}{h}\right)$$

$$\lim_{x \to 0} b(x) = \frac{2\pi}{3h} \lim_{x \to 0} \left[\frac{\cos\left(\pi x / h\right)}{\operatorname{Sen}\left(\frac{x}{2}\right)} \right] = 0$$

$$\Rightarrow S_N''(0) = -\frac{3}{4}\left(\frac{\pi}{h}\right)^2 + \frac{1}{2}\left(\frac{\pi}{h}\right)^2 - \frac{h}{4\pi}\left(\frac{2\pi}{3h}\right) = -\frac{1}{4}\left(\frac{\pi}{h}\right)^2 - \frac{1}{6}$$

3.3. Derive $D_N^{(3)}$

De la expresión del ejercicio anterior tenemos que:

$$S''_N(x) = f'(x) + g'(x)$$

Entonces $S'''_N(x) = f''(x) + g''(x)$

Donde, $f'(x) = -\frac{\pi}{2h}\left[\frac{\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\tan\left(\frac{x}{2}\right)}\right] - \frac{1}{4}\cdot\left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^2\left(\frac{x}{2}\right)}\right] = W(x) + h(x)$

y $g'(x) = -\frac{1}{4}\left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^2\left(\frac{x}{2}\right)}\right] - \frac{h}{4\pi}\cdot\left[\frac{\cos\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^3\left(\frac{x}{2}\right)}\right] = h(x) + L(x)$

$$W'(x) = -\frac{\pi}{2h}\cdot\left[\frac{\frac{\pi}{h}\cdot\cos\left(\frac{\pi x}{h}\right)\tan\left(\frac{x}{2}\right) - \frac{1}{2}\operatorname{sen}\left(\frac{\pi x}{h}\right)\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= -\frac{\pi^2}{2h^2}\cdot\left[\frac{\cos\left(\frac{\pi x}{h}\right)}{\tan\left(\frac{x}{2}\right)}\right] + \frac{\pi}{4h}\cdot\left[\frac{\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^2\left(\frac{x}{2}\right)}\right] \quad ; \; h = \frac{2\pi}{N} \quad h^2 = \frac{4\pi^2}{N^2}$$

$$W'(x_j) = -\frac{N^2}{8}\cdot\left[\frac{\cos(j\pi)}{\operatorname{sen}\left(\frac{jh}{2}\right)}\cdot\cos\left(\frac{jh}{2}\right)\right] + \frac{N}{8}\cdot\left[\frac{\operatorname{sen}(j\pi)}{\operatorname{sen}^2\left(\frac{jh}{2}\right)}\right]$$

$$h'(x) = -\frac{1}{4}\cdot\left[\frac{-\frac{\pi}{h}\operatorname{sen}\left(\frac{\pi x}{h}\right)\operatorname{sen}^2\left(\frac{x}{2}\right) - \operatorname{sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^4\left(\frac{x}{2}\right)}\right]$$

$$= \frac{\pi}{4h}\left[\frac{\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^2\left(\frac{x}{2}\right)}\right] + \frac{1}{4}\cdot\left[\frac{\cos\left(\frac{x}{2}\right)\cdot\cos\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^3\left(\frac{x}{2}\right)}\right]$$

$$h(x_j) = \frac{N}{8}\cdot\left[\frac{\operatorname{sen}(j\pi)}{\operatorname{sen}^2\left(\frac{jh}{2}\right)}\right] + \frac{1}{4}\cdot\left[\frac{\cos\left(\frac{jh}{2}\right)\cdot\cos(j\pi)}{\operatorname{sen}^3\left(\frac{jh}{2}\right)}\right]$$

$$L'(x) = -\frac{h}{4\pi}\cdot\left[\frac{\left[-\frac{1}{2}\operatorname{sen}\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right) + \frac{\pi}{h}\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right)\right]\operatorname{sen}^3\left(\frac{x}{2}\right) - \frac{3}{2}\operatorname{sen}^2\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^6\left(\frac{x}{2}\right)}\right]$$

$$= \frac{h}{8\pi}\cdot\left[\frac{\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^2\left(\frac{x}{2}\right)}\right] - \frac{1}{4}\cdot\left[\frac{\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^3\left(\frac{x}{2}\right)}\right] + \frac{3h}{8\pi}\cdot\left[\frac{\cos^2\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right)}{\operatorname{sen}^4\left(\frac{x}{2}\right)}\right]$$

$$L(x_j) = -\frac{1}{4}\cdot\left[\frac{\cos\left(\frac{jh}{2}\right)\cos(j\pi)}{\operatorname{Sen}^3\left(\frac{jh}{2}\right)}\right] + \frac{6}{8N}\cdot\left[\frac{\cos^2\left(\frac{jh}{2}\right)\cos(j\pi)}{\operatorname{Sen}^4\left(\frac{jh}{2}\right)}\right]$$

$$W'(x_j) = -\frac{N^2}{8}\cdot(-1)^j\cot\left(\frac{jh}{2}\right)$$

$$h'(x_j) = \frac{1}{4}(-1)^j\frac{\cot\left(\frac{jh}{2}\right)}{\operatorname{Sen}^2\left(\frac{jh}{2}\right)}$$

$$L'(x_j) = -\frac{(-1)^j}{4}\cdot\frac{\cot\left(jh/2\right)}{\operatorname{Sen}^2\left(jh/2\right)} + \frac{6}{8N}\cdot(-1)^j\cot^2\left(\frac{jh}{2}\right)\cdot\csc^2\left(\frac{jh}{2}\right)$$

$$\Rightarrow S'''_N(x_j) = W'(x_j) + 2h(x_j) + L'(x_j)$$

$$= -\frac{N^2}{8}(-1)^j\cot\left(\frac{jh}{2}\right) + \frac{1}{4}(-1)^j\cot\left(\frac{jh}{2}\right)\csc^2\left(\frac{jh}{2}\right)$$

$$+\frac{6}{8N}(-1)^j\cot^2\left(\frac{jh}{2}\right)\cdot\csc^2\left(\frac{jh}{2}\right)$$

Para $x=0$ $\quad \lim_{x\to 0} S'''_N(x) = \lim_{x\to 0}\left[W'(x) + 2h'(x) + L'(x)\right]$

$$\lim_{x\to 0} W'(x) = 0$$

$$\lim_{x\to 0} h'(x) = 0 + \frac{1}{4}\lim_{x\to 0}\left[\frac{\cos\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}^3\left(\frac{x}{2}\right)}\right]$$

$$= \frac{1}{4}\left(\frac{2}{3}\right)\lim_{x\to 0}\left[\frac{-\frac{1}{2}\operatorname{Sen}\left(\frac{x}{2}\right)\cos\left(\frac{\pi x}{h}\right) - \frac{\pi}{h}\cos\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}^2\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right]$$

$$= \frac{1}{4}\left(\frac{2}{3}\right)\lim_{x\to 0}\left[-\frac{1}{2}\frac{\cos\left(\frac{\pi x}{2}\right)}{\operatorname{Sen}\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} - \frac{\pi}{h}\cdot\frac{\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}^2\left(\frac{x}{2}\right)}\right]$$

$$= 0$$

$$\lim_{x\to 0} L'(x) = 0 + \lim_{x\to 0}\frac{3h}{8\pi}\left[\frac{\cos^2\left(\frac{x}{2}\right)\operatorname{Sen}\left(\frac{\pi x}{h}\right)}{\operatorname{Sen}^4\left(\frac{x}{2}\right)}\right]$$

$$\lim_{x \to 0} L'(x) = \frac{3h}{8\pi} \lim_{x\to 0} \left[\frac{-\cos\left(\frac{x}{2}\right) \operatorname{sen}^5\left(\frac{x}{2}\right) \operatorname{sen}\left(\frac{\pi x}{h}\right) - 2\operatorname{sen}^3\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right)}{4\operatorname{sen}^3\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \right]$$

$$= \frac{3h}{8\pi} \lim_{x\to 0} \left[-\frac{1}{4} \cdot \operatorname{sen}^2\left(\frac{x}{2}\right)\operatorname{sen}\left(\frac{\pi x}{h}\right) - \frac{1}{2} \operatorname{sen}\left(\frac{\pi x}{2}\right) \right] = 0$$

Por lo tanto ;

$$\lim_{x \to 0} S_N'''(x) = 0$$

Así,

$$S_N'''(x_j) = \begin{cases} 0 & \text{si } j \equiv 0 \bmod (N) \\ -\frac{N^2}{8N}(-1)^j \cot\left(\frac{jh}{2}\right) + \frac{1}{4}(-1)^j \cot\left(\frac{jh}{2}\right)\csc^2\left(\frac{jh}{2}\right) & \\ + \frac{6}{8N}(-1)^j \cot^2\left(\frac{jh}{2}\right) \cdot \csc^2\left(\frac{jh}{2}\right) & \text{si } j \not\equiv 0 \bmod (N) \end{cases}$$

Obtener de (3.3) a (3.4)

$$v_j = \frac{1}{2\pi} \sum_{k=N/2+1}^{N/2} e^{ikx_j} \hat{v}_k \ ; \ j = 1, \dots, N \qquad (3.3)$$

Notemos que si $\hat{v}_{N/2} = \hat{v}_{-N/2}$, Entonces para hacer simétrico el intervalo:

$$v_j = \frac{1}{2\pi}\left[\frac{1}{2} e^{-i\frac{N}{2}x_j} \hat{v}_{-N/2} + \frac{1}{2} e^{i\frac{N}{2}x_j} \hat{v}_{\frac{N}{2}} + \frac{1}{2} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} e^{ikx_j} \hat{v}_k + \frac{1}{2} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}-1} e^{ikx_j} \hat{v}_k \right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{2} \sum_{k=-\frac{N}{2}}^{N/2-1} e^{ikx_j} \cdot \hat{v}_k + \frac{1}{2} \sum_{k=-\frac{N}{2}+1}^{N/2} e^{ikx_j} \cdot \hat{v}_k \right] \quad (3.3.1)$$

$$= \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{N/2}{}' e^{ikx_j} \cdot \hat{v}_k \qquad (3.4)$$

El problema con (3.3) es que el termino $i\left(\frac{N}{2}\right) e^{i\frac{N}{2}x}$ es un complejo, y como esta es la derivada debería de ser cero.

Pero con (3.3.1) podemos ver que:

$$-i\left(\frac{N}{2}\right) \cdot e^{-i\frac{N}{2}\cdot x} \cdot \hat{v}_{-\frac{N}{2}} + i\left(\frac{N}{2}\right) e^{i\frac{N}{2}\cdot x} \cdot v_{\frac{N}{2}} = 0$$

Se anulan si $\hat{v}_{-N/2} = \hat{v}_{N/2}$ tal como lo queríamos.