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METHODS BASED ON THE WIENER–HOPF TECHNIQUE

for the solution of partial differential equations

by

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CONTENTS

		PAGE
Preface		vii
Some basic notation and results from Chapter I		x

I. COMPLEX VARIABLE AND FOURIER TRANSFORMS

1.1	Introduction	1
1.2	Complex variable theory	5
1.3	Analytic functions defined by integrals	11
1.4	The Fourier integral	21
1.5	The wave equation	27
1.6	Contour integrals of a certain type	31
1.7	The Wiener–Hopf procedure	36
	Miscellaneous examples and results I	38

II. BASIC PROCEDURES: HALF-PLANE PROBLEMS

2.1	Introduction	48
2.2	Jones's method	52
2.3	A dual integral equation method	58
2.4	Integral equation formulations	61
2.5	Solution of the integral equations	67
2.6	Discussion of the solution	72
2.7	Comparison of methods	76
2.8	Boundary conditions specified by general functions	77
2.9	Radiation-type boundary conditions	83
	Miscellaneous examples and results II	86

III. FURTHER WAVE PROBLEMS

3.1	Introduction	98
3.2	A plane wave incident on two semi-infinite parallel planes	100
3.3	Radiation from two parallel semi-infinite plates	105
3.4	Radiation from a cylindrical pipe	110
3.5	Semi-infinite strips parallel to the walls of a duct	118
3.6	A strip across a duct	122
	Miscellaneous examples and results III	125

IV. EXTENSIONS AND LIMITATIONS OF THE METHOD

4.1	Introduction	141
4.2	The Hilbert problem	141
4.3	General considerations	147
4.4	Simultaneous Wiener–Hopf equations	153
4.5	Approximate factorization	160
4.6	Laplace's equation in polar co-ordinates	164
	Miscellaneous examples and results IV	167

V. SOME APPROXIMATE METHODS

5.1 Introduction 178
5.2 Some problems which cannot be solved exactly 180
5.3 General theory of a special equation 184
5.4 Diffraction by a thick semi-infinite strip 187
5.5 General theory of another special equation 196
5.6 Diffraction by strips and slits of finite width 203
Miscellaneous examples and results V 207

VI. THE GENERAL SOLUTION OF THE BASIC WIENER–HOPF PROBLEM

6.1 Introduction 220
6.2 The exact solution of certain dual integral equations 222
Miscellaneous examples and results VI 228

Bibliography 237
Index 243

PREFACE

The methods described in this book solve certain boundary-value problems of practical importance involving partial differential equations. A typical problem requires solution of the steady-state wave equation in free space when semi-infinite boundaries are present. Examples are given from electromagnetic theory, acoustics, hydrodynamics, elasticity and potential theory.

The twin aims of this book are: to take the student from ordinary degree studies into the research field covered by the Wiener–Hopf technique, and to provide the research worker with a reasonably comprehensive summary of what can and what cannot be done at the moment by the technique. The reader's attention is drawn particularly to the various methods for *approximate* solution of problems. One of the remarkable features is the range of apparently unrelated topics covered by ramifications of the technique. It is hoped that some of the comments in the text and in examples may suggest suitable lines for further research.

The Wiener–Hopf technique was invented about 1931 to solve an integral equation of a special type. During the war it was noted by J. Schwinger (and independently by E. T. Copson) that problems involving diffraction by semi-infinite planes could be formulated in terms of integral equations which could be solved by the Wiener–Hopf technique. The solution depends on the use of Fourier integrals to obtain a complex variable equation which is solved by analytic continuation. Most of this book is based on a different but equivalent approach due to D. S. Jones. Fourier transforms are applied directly to the partial differential equation and the complex variable equation is obtained directly without the necessity for formulation of an integral equation. From this point of view the Wiener–Hopf technique provides a significant and natural extension of the range of problems that can be solved by the use of Fourier, Laplace and Mellin transforms. I started this book with the intention of running the integral equation and Jones's method alongside each other, but as the writing progressed it seemed pointless and confusing to elaborate two equivalent methods. Jones's method seems simpler to me and I have included only sufficient details of the integral equation method to enable the reader to follow the literature.

The material in this book should be accessible to anyone who is familiar with the Laplace transform, its complex inversion formula, and integration in the complex plane. The first chapter is intended to supplement the usual undergraduate course in complex variable theory, and to familiarize the reader with the use of the Fourier transform in the complex plane. As this book has been written for workers whose interests are primarily in applications of the theory rather than in the theory itself, the standard of rigour may not satisfy the pure mathematician though it should suffice for practical purposes.

It is important to emphasize that from the point of view adopted in this book the essence of the Wiener–Hopf technique is that it can be used to obtain numerical values for physical quantities. For various reasons I have decided to omit numerical tables although as far as possible results are given in a form suitable for numerical computation and references are given to the few existing sets of tabulated values. Practically no discussion is given of the physical implications of results. For electromagnetic theory this gap should be filled by the volume in this series by D. S. Jones.

In the examples treated in the text I have carried the analysis far enough in each case to obtain at least one result of physical significance in simple form. This is partly to encourage the beginner who might think that complicated formulae can be interpreted only with the aid of an electronic computer.

The stimulus to write this book came originally from a course of post-graduate lectures suggested by Prof. D. C. Pack. Among other things I am grateful to Prof. Pack for organizing ideal working conditions in his department at the Royal College of Science and Technology, Glasgow. I am also grateful to Prof. I. N. Sneddon for asking me to write this volume for his series, and for helpful suggestions in connexion with shortening a manuscript that was much longer than originally envisaged. It is perhaps worth mentioning that I have had to omit a chapter dealing with applications of Bessel function dual integral equations to disk problems. A referee for the *Proc. Camb. Phil. Soc.* (whose name cannot now be traced) suggested to me that these problems are best tackled by the Wiener–Hopf method. Having written this book I am not entirely convinced that the suggestion is correct, but in a sense the referee's comment provoked the book! I am indebted to Prof. D. S. Jones for various references and correspondence and to Dr. W. E. Williams who carefully checked Chapter V. My thanks are due to the printers for accurate work on a difficult manuscript.

B. N.

31. 12. 57.

SOME BASIC NOTATION AND RESULTS FROM CHAPTER 1

The following may prove useful for reference. A time factor $\exp(-i\omega t)$ is used throughout this book.

$$\alpha = \sigma + i\tau \quad : \quad k = k_1 + ik_2, \qquad (k_1 > 0, k_2 > 0).$$

$$\gamma = (\alpha^2 - k^2)^{1/2} = -i(k^2 - \alpha^2)^{1/2}. \qquad [(1.14)]$$

$$(k - \alpha)^{1/2} = i(\alpha - k)^{1/2} \quad : \quad (-k - \alpha)^{1/2} = -i(\alpha + k)^{1/2}. \qquad [(1.12)]$$

$$\gamma = -ik \quad \text{if} \quad \alpha = 0 \quad : \quad \gamma \approx |\alpha| \quad \text{if} \quad \alpha \text{ is real and large.}$$

$$\Phi(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x,y)e^{i\alpha x}\,dx = \Phi_+(\alpha,y) + \Phi_-(\alpha,y),$$

where
$$\Phi_+(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x,y)e^{i\alpha x}\,dx,$$

$$\Phi_-(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi(x,y)e^{i\alpha x}\,dx.$$

We write, for instance,

$$\Phi_+(\alpha,y) \equiv \Phi_+(\alpha) \equiv \Phi_+(y) \equiv \Phi_+,$$

where any of these forms may be used according to convenience provided that there is no risk of confusion.

$$\left.\begin{array}{l} \text{If } |\phi(x)| < A \exp(\tau_- x) \text{ as } x \to +\infty,\ \Phi_+(\alpha) \text{ is regular in } \tau > \tau_- \\ \text{If } |\phi(x)| < B \exp(\tau_+ x) \text{ as } x \to -\infty,\ \Phi_-(\alpha) \text{ is regular in } \tau < \tau_+ \end{array}\right\}$$

[§1.3]

$$\left.\begin{array}{l} \text{If } \phi(x) \sim x^\eta \text{ as } x \to +0, \text{ then } \Phi_+(\alpha) \sim \alpha^{-\eta-1} \text{ as } \alpha \to \infty \text{ in } \tau > \tau_- \\ \text{If } \phi(x) \sim x^\eta \text{ as } x \to -0, \text{ then } \Phi_-(\alpha) \sim \alpha^{-\eta-1} \text{ as } \alpha \to \infty \text{ in } \tau < \tau_+ \end{array}\right\}$$

[Cf. (1.74)]

($f(x) \sim g(x)$ as $x \to a$ means that $f(x) = g(x) + h(x)$ where $h/g \to 0$ as $x \to a$. The number a may be infinite. Occasionally as in the last paragraph we use $f \sim g$ to mean $f = Cg + h$ for some constant C, if the value of C is not important.)

CHAPTER I

COMPLEX VARIABLE AND FOURIER TRANSFORMS

1.1 Introduction

One of the remarkable features of the mathematical description of natural phenomena by means of partial differential equations is the comparative ease with which solutions can be obtained for certain geometrical shapes, such as circles and infinite strips, by the method of separation of variables: in contrast, considerable difficulty is usually encountered in finding solutions for shapes not covered by the method of separation of variables. The Wiener–Hopf technique provides a significant extension of the range of problem that can be solved by Fourier, Laplace and Mellin integrals.

To illustrate these remarks and to remind the reader of the real-variable Fourier integral consider three problems connected with the steady-state wave equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\phi=0. \tag{1.1}$$

Suppose we wish to find a solution of this equation in the semi-infinite region $-\infty < x < \infty$, $y \geqslant 0$, such that ϕ represents an outgoing wave at infinity in each of three separate cases

(i) $\quad \phi = f(x) \quad \text{on} \quad y=0, \quad -\infty < x < \infty;$

(ii) $\quad \partial\phi/\partial y = g(x) \quad \text{on} \quad y=0, \quad -\infty < x < \infty;$

(iii)
$$\left.\begin{aligned}\phi &= f(x) \quad \text{on} \quad y=0, \quad 0 < x < \infty,\\ \partial\phi/\partial y &= g(x) \quad \text{on} \quad y=0, \quad -\infty < x < 0.\end{aligned}\right\} \tag{1.2}$$

Separation-of-variables solutions exist for (1.1) in the form $\phi = X(x)Y(y)$ with

$$X(x)=e^{\pm i\alpha x} \quad : \quad Y(y)=e^{\pm\gamma y}, \qquad \gamma=(\alpha^2-k^2)^{1/2},$$

where α is a parameter. Together with the fact that the range of x is infinite this suggests use of the Fourier integral in $-\infty < x < \infty$, and in fact we show that the first two problems can be solved exactly by Fourier integrals: the third leads to equations which can be solved by the Wiener–Hopf technique.

Although it will appear that we must use the Fourier integral in the complex plane, consider in this section the ordinary form

$$\Phi(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x,y)e^{i\alpha x}\,dx$$
$$\phi(x,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \Phi(\alpha,y)e^{-i\alpha x}\,d\alpha, \tag{1.3}$$

where α is real. We use the method of Fourier transforms (I. N. Sneddon [1], C. J. Tranter [1], or E. C. Titchmarsh [1] Chapter X, "formal solutions"). Multiply both sides of (1.1) by $(2\pi)^{-1/2}\exp(i\alpha x)$ and integrate throughout with respect to x from $-\infty$ to ∞:

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{\partial^2\phi}{\partial x^2} e^{i\alpha x}\,dx + \frac{d^2\Phi}{dy^2} + k^2\Phi = 0. \tag{1.4}$$

Integrate the following expression by parts:

$$\int_{-A}^{A} \frac{\partial^2\phi}{\partial x^2} e^{i\alpha x}\,dx = \left[\frac{\partial\phi}{\partial x} e^{i\alpha x}\right]_{-A}^{A} - i\alpha\left[\phi e^{i\alpha x}\right]_{-A}^{A} - \alpha^2 \int_{-A}^{A} \phi e^{i\alpha x}\,dx.$$

Let $A \to \infty$ and assume that contributions from the bracketed terms at upper and lower limits tend to zero. (This is connected with the condition that ϕ represents an outgoing wave at infinity and is investigated further in §1.5.) Equation (1.4) then becomes

$$\frac{d^2\Phi}{dy^2} - \gamma^2\Phi = 0, \quad \text{where} \quad \gamma^2 = (\alpha^2 - k^2). \tag{1.5}$$

Define
$$\gamma = +(\alpha^2 - k^2)^{1/2}, \qquad \alpha > k,$$

where in this section we assume that k is real. A difficulty arises since we need to define γ for $\alpha < k$ and it is not clear, for example, whether to take the upper or lower sign in the formula $\gamma = \pm i(k^2 - \alpha^2)^{1/2}$, $|\alpha| < k$. This question is examined in §1.5, by using analytic continuation arguments developed in §1.2. Assuming that the definition of γ has been settled, the solution of (1.5) which must be used is

$$\Phi = A(\alpha)e^{-\gamma y}, \tag{1.6}$$

since it will appear that $\gamma = +(\alpha^2 - k^2)^{1/2}$ for $\alpha < -k$ so that this solution is bounded for all α as y tends to plus infinity whereas the solution in $\exp(+\gamma y)$ increases exponentially as y tends to infinity for $|\alpha| > k$. The function $A(\alpha)$ is an arbitrary function determined from the boundary condition on $y = 0$.

Now consider problems (i)–(iii) in turn.

(i) Application of the boundary condition on $y = 0$ to (1.6) gives

$$(\Phi)_{y=0} = A(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(\xi) e^{i\alpha\xi}\, d\xi.$$

Substitute this value for $A(\alpha)$ in (1.6) and use the Fourier inversion formula (1.3). This gives the solution

$$\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\alpha x - \gamma y} \int_{-\infty}^{\infty} f(\xi) e^{i\alpha\xi}\, d\xi\, d\alpha.$$

(ii) In an exactly similar way the second problem gives

$$\left(\frac{\partial\Phi}{\partial y}\right)_{y=0} = -\gamma A(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} g(\xi) e^{i\alpha\xi}\, d\xi,$$

$$\phi = -\frac{1}{2} \quad {}_{-\infty}^{\infty} \qquad\qquad {}_{-\infty}^{\infty} \tag{1.7}$$

(iii) In this case there are three methods of procedure which are basically identical but deserve separate mention.

A. In (1.2) extend $f(x)$ to denote the (unknown) value of ϕ on $y = 0$, $x < 0$, and $g(x)$ to denote the (unknown) value of $\partial\phi/\partial y$ on $y = 0$, $x > 0$. Define

$$\Phi_{+}(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi(x,y) e^{i\alpha x}\, dx,$$

$$\Phi_{-}(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi(x,y) e^{i\alpha x}\, dx.$$

Then $\Phi_{+}(\alpha,0)$, $\Phi_{-}(\alpha,0)$ are the corresponding integrals of $\phi(x,0) = f(x)$. Use a dash to denote differentiation with respect to y, so that

$$\Phi'_{+}(\alpha,0) = \{\partial\Phi_{+}(\alpha,y)/\partial y\}_{y=0} = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} g(x) e^{i\alpha x}\, dx,$$

with a corresponding definition for $\Phi'_-(\alpha,0)$. Tho boundary conditions now yield

$$\Phi_+(\alpha,0) + \Phi_-(\alpha,0) = A(\alpha) \quad : \quad \Phi'_+(\alpha,0) + \Phi'_-(\alpha,0) = -\gamma A(\alpha).$$

Eliminate $A(\alpha)$ from these equations:

$$\Phi'_+(\alpha,0) + \Phi'_-(\alpha,0) = -\gamma\{\Phi_+(\alpha,0) + \Phi_-(\alpha,0)\}. \qquad (1.8)$$

The functions Φ'_-, Φ_+ are known but there are two unknown functions, Φ'_+ and Φ_-. It will appear that if α is taken as a complex variable in the original Fourier transform (1.3), a process involving analytic continuation and Liouville's theorem can be used to determine the unknown functions in (1.8). This process, which is described in §1.7, is the "Wiener–Hopf technique".

B. Next consider an integral equation formulation of the problem. In (1.7) interchange orders of integration, let y tend to zero, introduce boundary condition (1.2) for $x > 0$, split the range of integration in ξ into $(-\infty,0)$, $(0,\infty)$ and rearrange:

$$\int_0^\infty K(x-\xi)g(\xi)\,d\xi = f(x) - \int_{-\infty}^0 g(\xi)K(x-\xi)\,d\xi, \qquad (x>0),$$

where $$K(x-\xi) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \gamma^{-1}e^{i\alpha(\xi-x)}\,d\alpha,$$

and the quantities on the right-hand side of the equation are known. This is an integral equation for the unknown function $g(\xi)$, $\xi > 0$. The important feature from the present point of view is that the kernel $K(x-\xi)$ is a function of $(x-\xi)$. Such integral equations can be solved by the Wiener–Hopf technique. In the literature the usual procedure is to obtain this type of equation by a Green's function technique, and then to reduce the integral equation to (1.8) by Fourier transforms. A more detailed discussion is given in §§2.4, 2.5.

C. Finally consider the formulation in terms of dual integral equations. From (1.6),

$$\phi = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} A(\alpha)e^{-i\alpha x-\gamma y}\,d\alpha.$$

The boundary conditions give

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} A(\alpha) e^{-i\alpha x}\, d\alpha = f(x), \qquad (x > 0), \tag{1.9a}$$

$$-\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \gamma A(\alpha) e^{-i\alpha x}\, d\alpha = g(x), \qquad (x < 0). \tag{1.9b}$$

These are dual integral equations for the unknown function $A(\alpha)$. It will be shown in §2.2 that these equations can be solved directly by a procedure depending on the essential step in the Wiener–Hopf technique.

This completes our introductory discussion. In order to solve problem (iii) above by any of the methods A, B, C, it is necessary to consider complex α. This requires a discussion of certain topics in complex variable and Fourier transform theory which are examined in §§1.2–1.5. Various questions which have been left unanswered in the above treatment are solved in §1.5 with the help of §§1.2, 1.3, 1.4.

The equation which replaces (1.8) when complex α is considered is stated in §1.7, where a summary of the Wiener–Hopf method for solution of the equation is given.

The reader who is prepared to refer back to this chapter can proceed directly to Chapter II where a problem equivalent to (iii) above is solved in detail.

1.2 Complex variable theory

We start with a brief summary of complex variable theory required for succeeding developments. Our references will be the standard texts of E. T. Copson [1], and E. C. Titchmarsh [2]. Greek letters will be used to denote complex variables, e.g. $\zeta = \xi + i\eta$, $\alpha = \sigma + i\tau$. When the complex variable is associated with the Fourier transform we invariably use $\alpha = \sigma + i\tau$. Latin letters a, b, k, etc., will be used for constants. It will be clear from the context whether these are to be regarded as real or complex. We recall the following definitions and results.

If to each point ζ in a certain region R there correspond one or more complex numbers, denoted by χ, then we write $\chi = f(\zeta)$ and say that χ is a function of the complex variable ζ. If the function has a uniquely defined value at each point of the region R it is said to be single-valued in R. The crucial property possessed by useful functions is that at most points of R they are differentiable, i.e.

$$f'(\zeta) = \lim_{\delta \to 0} \frac{f(\zeta + \delta) - f(\zeta)}{\delta}$$

exists and is independent of the direction along which the complex number δ tends to zero. The function $\chi = f(\zeta)$ is said to be *analytic* at the point ζ when it is single-valued and differentiable at this point. The function f is said to be *regular in a region* R if it is analytic at every point of R. The phrase "$f(\zeta)$ is an analytic function in a region R" means that the function is analytic at every point of a region except for a certain number of exceptional points: this will be defined more precisely later in connexion with analytic continuation. Points at which the function is not analytic are called *singularities*. The singularities of a function are very important since they characterize the function.

The next idea required is that of a line integral. The central result concerning line integrals is *Cauchy's theorem* of which we quote the following form (E. T. Copson [1], p. 61): If $f(\zeta)$ is an analytic function, continuous within and on the simple closed rectifiable curve C, and if $f'(\zeta)$ exists at each point within C, then

$$\int_C f(\zeta)\, d\zeta = 0.$$

From this can be deduced *Cauchy's integral formula:* If $f(\zeta)$ obeys the same conditions as for Cauchy's theorem and if α is any point within C, then

$$f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - \alpha}\, d\zeta.$$

Some familiarity is assumed with the application of these theorems to evaluation of contour integrals by residues and shifting of contours in the complex plane, particularly when branch points are present.

An analytic function which is regular in every finite region of the ζ-plane is called an *integral function*, e.g. a polynomial in ζ is an integral function: also $\exp \zeta$ is an integral function. *Liouville's theorem* states that if $f(\zeta)$ is an integral function such that $|f(\zeta)| \leqslant M$ for all ζ, M being a constant, then $f(\zeta)$ is a constant. It is easy to extend this result to the following: If $f(\zeta)$ is an integral function such that $|f(\zeta)| \leqslant M|\zeta|^p$ as $|\zeta| \to \infty$ where M, p are constants, then $f(\zeta)$ is a polynomial of degree less than or equal to $[p]$ where $[p]$ is the integral part of p.

Taylor's theorem states that if $f(\zeta)$ is an analytic function regular in the neighbourhood $|\zeta - a| < R$ of the point $\zeta = a$, it can be expressed in this neighbourhood as a convergent power series of the form

$$f(\zeta) = f(a) + \sum_{r=1}^{\infty} a_r \frac{(\zeta - a)^r}{r!}, \qquad a_r = \left[\frac{d^r f}{d\zeta^r}\right]_{\zeta=a}.$$

A *zero* of an analytic function $f(\zeta)$ is a value of ζ such that $f(\zeta) = 0$. It can be deduced from Taylor's theorem that the zeros of an analytic function are isolated points, i.e. if $f(\zeta)$ is regular in a region including $\zeta = a$ then there is a region $|\zeta - a| < \rho$, $(\rho > 0)$, inside which $f(\zeta)$ has no zeros except possibly $\zeta = \alpha$ itself. If a singularity is isolated it is possible to specify an annulus in which the function is analytic and to deduce a *Laurent expansion.* If this expansion is of the form

$$f(\zeta) = \sum_{r=-n}^{\infty} a_r(\zeta - a)^r, \qquad (n > 0),$$

then the function is said to have a *pole of order n* at the point a.

The remainder of this section is devoted to analytic continuation, which is important for later developments. Since we shall need analytic continuation mainly in connexion with the Fourier integral we use the complex variable $\alpha = \sigma + i\tau$.

It often happens that a representation of a function of a complex variable is valid only for restricted α. Thus the series

$$f(\alpha) = 1 + \alpha + \alpha^2 + \ldots$$

converges only for $|\alpha| < 1$. However for $|\alpha| < 1$ we have

$$f(\alpha) = (1 - \alpha)^{-1}.$$

The extension of the definition of $f(\alpha)$ by identifying $f(\alpha)$ with $(1 - \alpha)^{-1}$ for $|\alpha| > 1$ is called analytic continuation. It is possible to carry out analytic continuation systematically by means of power series but we do not go into details. We assume merely that the functions $f(\alpha)$ with which we deal are defined in such a way that if we start at any point $\alpha = a$ in the complex plane and draw a continuous curve to another point say $\alpha = b$ in such a way that no singularities of the function lie on the curve, then the values of $f(\alpha)$ vary continuously along the curve and can be determined from the definition of $f(\alpha)$. The expression "the analytic function $f(\alpha)$" can now be defined as the totality of all values of $f(\alpha)$ which can be obtained by analytic continuation as just described, starting at a given point $\alpha = a$.

The natural question which arises is whether a function which is continued along two different curves from $\alpha = a$ to $\alpha = b$ will have the same final value for the two ways. This question is partly answered by the following theorem (E. C. Titchmarsh [2], p. 145): If we continue an analytic function $f(\alpha)$ along two different routes from a to b and obtain two different values of $f(b)$ then $f(\alpha)$ must have a singularity between the two routes. Of course the converse is not true, that if there is a singularity between the two routes we necessarily obtain two different values of $f(b)$: it needs to be a special type of singularity, namely a *branch-point* to produce a difference in value.

If the values of a function found by analytic continuation are unique, independent of the paths of continuation, then the function is called *single-valued*. Otherwise the function is called *many-valued*. A branch point a is a singular point such that there exists no neighbourhood $|\alpha - a| < \varepsilon$ in which $f(\alpha)$ is single-valued. By inserting certain lines in the complex plane and stating that paths of analytic continuation must not cross these lines it is possible to specify a *branch* of a many-valued function which in itself is single-valued. Such lines are called *branch-lines* or *cuts*. Branch points always occur in pairs, and branch lines join branch points.

A simple example of a many valued function is $\chi = \alpha^{1/2}$. One branch point is $\alpha = 0$. The point at infinity is also a branch point as can be seen by setting $\alpha = \zeta^{-1}$. A branch line in this case is any line joining these two points. For instance draw the branch line along the negative real axis, $-\infty < \sigma \leqslant 0$. In order to specify the branches consider the behaviour of the function at only one point say $\alpha = p$, where p is a positive real quantity. Adopt the convention that $p^{1/2}$ refers to the positive square root of p. Then the two branches of $\chi = \alpha^{1/2}$ correspond to analytic continuation from the two values $\chi = \pm p^{1/2}$ at $\alpha = p$. If we choose the branch specified by $\chi = p^{1/2}$ then on the upper side of the negative real axis, $\alpha = r\exp(i\pi)$, $\chi = ir^{1/2}$, and on the lower side, $\alpha = r\exp(-i\pi)$, $\chi = -ir^{1/2}$.

Suppose next that we cut the α-plane by a straight line from the origin to infinity in the upper half-plane and define two functions $\chi = \alpha^{1/2}$, and $\psi = (-\alpha)^{1/2}$, where branches are chosen so that if p is a positive real number, $\chi = +p^{1/2}$ when $\alpha = p$ and $\psi = +p^{1/2}$ when $\alpha = -p$. Then by analytic continuation $\chi = -ip^{1/2}$ when $\alpha = -p$ and $\psi = +ip^{1/2}$ when $\alpha = +p$. Thus on any segment of the real axis,

$$\chi = -i\psi, \quad \text{or} \quad (\alpha)^{1/2} = -i(-\alpha)^{1/2}. \tag{1.10}$$

There is an important theorem which states that if two functions are equal along any line of finite length in the complex plane they are equal in any region into which they can both be continued along a common line, starting from any point at which they have the same value. Thus (1.10) holds at any point in the cut plane, with the branches as previously defined. We must remember that the second equation in (1.10) is not an ordinary algebraic equation, e.g. if we change the sign of α on both sides we obtain $(-\alpha)^{1/2} = -i(\alpha)^{1/2}$ which seems to contradict the original equation in (1.10). The

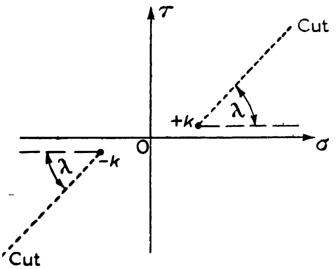

FIG. 1.1.

reason for the apparent contradiction is of course that if we change the sign of α we also change the position of the cut.

Now define

$$\left.\begin{aligned} \chi_1 &= (\alpha - k)^{1/2}, & \chi_2 &= (\alpha + k)^{1/2} \\ \chi_3 &= (k - \alpha)^{1/2}, & \chi_4 &= (-k - \alpha)^{1/2} \end{aligned}\right\} \quad \begin{aligned} k &= k_1 + ik_2, \\ k_1, k_2 &> 0, \end{aligned} \tag{1.11}$$

where the α-plane is cut as in Fig. 1.1. Define the branches of χ_1 and χ_2 so that both functions tend to $+\sigma^{1/2}$ as $\alpha \to +\infty$ along the positive real axis, and define the branches of χ_3, χ_4 so that both functions tend to $(-\sigma)^{1/2}$ as $\alpha \to -\infty$ along the negative real axis. Then we readily prove by the method of the last two paragraphs that

$$\begin{aligned} \chi_1 &= -ik^{1/2}, & \alpha &= 0 &:\quad \chi_1 &\to -i(-\sigma)^{1/2}, & \alpha &= \sigma \to -\infty; \\ \chi_2 &= k^{1/2}, & \alpha &= 0 &:\quad \chi_2 &\to +i(-\sigma)^{1/2}, & \alpha &= \sigma \to -\infty; \\ \chi_3 &= k^{1/2}, & \alpha &= 0 &:\quad \chi_3 &\to +i\sigma^{1/2}, & \alpha &= \sigma \to +\infty; \\ \chi_4 &= -ik^{1/2}, & \alpha &= 0 &:\quad \chi_4 &\to -i\sigma^{1/2}, & \alpha &= \sigma \to +\infty. \end{aligned}$$

We see that with the specified branches, $\chi_3 = i\chi_1$, $\chi_4 = -i\chi_2$, everywhere in the cut plane, i.e.

$$(k - \alpha)^{1/2} = i(\alpha - k)^{1/2} \quad : \quad (-k - \alpha)^{1/2} = -i(\alpha + k)^{1/2}. \tag{1.12}$$

We noted that it was not permissible to change the sign of α on both sides of the equation (1.10). In defining the χ_i care has been taken to choose the branches so that it is permissible to change the sign of α in any equation involving the χ_i. Thus in (1.12), changing the sign of α merely changes one equation into the other. On the other hand we cannot change the sign of k in these equations, and if we let $k \to 0$ in (1.12) we obtain the apparently contradictory results: $(-\alpha)^{1/2} = i(\alpha)^{1/2}$, $(-\alpha)^{1/2} = -i(\alpha)^{1/2}$.

Later we shall meet expressions like $(k \cos \Theta - k)^{1/2}$ where $0 < \Theta < \pi$. This could be obtained by writing $\alpha = k \cos \Theta$ in χ_1 or $\alpha = -k \cos \Theta$ in χ_4. With the branches specified, the reader will readily prove that these are identical, and that in fact

$$(k \cos \Theta - k)^{1/2} = -i(k - k \cos \Theta)^{1/2} = -i(2k)^{1/2} \sin \tfrac{1}{2}\Theta. \quad (1.13a)$$

Similarly

$$(-k \cos \Theta - k)^{1/2} = -i(k + k \cos \Theta)^{1/2} = -i(2k)^{1/2} \cos \tfrac{1}{2}\Theta. \quad (1.13b)$$

The most important example of all from our point of view is the function $\gamma = (\alpha$
$\gamma \to \sigma$ if $\alpha = \sigma \to +\infty$. We can write, in notation (1.11), $\gamma = (\alpha - k)^{1/2}(\alpha + k)^{1/2} = \chi_1\chi_2 = \chi_3\chi_4$ and the results obtained above give $\gamma = -ik$ when $\alpha = 0$ and $\gamma \to |\sigma|$ when $\alpha = \sigma \to -\infty$. Similarly we can consider $\kappa = (k^2 - \alpha^2)^{1/2}$ where we choose the branch such that $\kappa = k$ if $\alpha = 0$. Then we can write $\kappa = \chi_2\chi_3 = -\chi_1\chi_4$ and $\kappa \to +i\sigma$ if $\alpha = \sigma \to +\infty$; $\kappa \to i|\sigma|$ if $\alpha = \sigma \to -\infty$. Equations (1.12) now give the important results

$$\gamma = -i\kappa \quad : \quad \kappa = i\gamma, \quad (1.14)$$

everywhere in the plane cut as in Fig 1.1, the branches being defined as above.

As a special case suppose that k is real. Then on the real axis in the α-plane

$$\begin{aligned} \gamma &= +(\sigma^2 - k^2)^{1/2} &:\quad \kappa &= +i(\sigma^2 - k^2)^{1/2}, &&(\sigma > k); \\ &= -i(k^2 - \sigma^2)^{1/2} &:\quad &= +(k^2 - \sigma^2)^{1/2}, &&(-k < \sigma < k); \quad (1.15) \\ &= +(\sigma^2 - k^2)^{1/2} &:\quad &= +i(\sigma^2 - k^2)^{1/2}, &&(\sigma < -k). \end{aligned}$$

If the angle λ in Fig. 1.1 tends to zero, the branch lines lie on the real axis. Results (1.15) still hold providing that σ is taken on the *upper* side of the branch cut $\sigma < -k$, and on the *lower* side of the branch cut $\sigma > +k$.

When reading literature on the Wiener–Hopf technique it is essential to start off by checking the conventions used. Some authors take $k = k_1 - ik_2$, $(k_1, k_2 > 0)$, which means that branch cuts have to be chosen in a different way. Others use Laplace transforms instead of Fourier transforms which means that α is replaced by (is) and instead of γ defined above, it is necessary to consider the function $(s^2 + k^2)^{1/2}$. Some examples to clarify the situation are given at the end of the chapter.

1.3 Analytic functions defined by integrals

We shall often meet functions defined by integrals of the type

$$G(\alpha) = \int_C g(\alpha,\zeta)\, d\zeta, \tag{1.16}$$

where $g(\alpha,\zeta)$ is a function of the complex variables α and ζ, and C is a contour in the complex ζ-plane. The variable α will be assumed to lie inside a region R, i.e. the boundary of R, if any, is excluded. The contour C is assumed to be smooth, i.e. it is possible to specify position on the contour by means of a parameter t such that $\zeta = \xi(t) + i\eta(t)$, $t_0 \leqslant t \leqslant t_1$, and $\xi'(t)$, $\eta'(t)$ exist and are continuous.

Before stating conditions for $G(\alpha)$ to be regular we make some general remarks. Any line integral like (1.16) can be reduced to real integrals and we shall assume that these are Riemann integrals. It would be convenient in some ways to assume knowledge of Lebesgue integration. Theorems would then be expressed more concisely and elegantly. On the other hand the Riemann integral requires conditions which correspond more directly to the properties we intuitively associate with the functions occurring in physics. Also most books on complex variable and Laplace transform are written in terms of Riemann integrals. Where conditions of validity are given below, they are usually sufficient but not necessary.

We now state conditions under which $G(\alpha)$ is regular (cf. E. C. Titchmarsh [2], pp. 99, 100):

Theorem A. *Let $g(\alpha,\zeta) = f(\zeta)h(\alpha,\zeta)$ satisfy the conditions*

(i) *$h(\alpha,\zeta)$ is a continuous function of the complex variables α and ζ where α lies inside a region R and ζ lies on a contour C.*

(ii) *$h(\alpha,\zeta)$ is a regular function of α in R for every ζ on C.*

(iii) *$f(\zeta)$ has only a finite number of finite discontinuities on C and a finite number of maxima and minima on any finite part of C.*

(iv) *$f(\zeta)$ is bounded except at a finite number of points. If ζ_0 is such a point, so that $g(\alpha,\zeta) \to \infty$ as $\zeta \to \zeta_0$, then*

$$\int_C g(\alpha,\zeta)\, d\zeta = \lim_{\delta\to 0} \int_{C-\delta} g(\alpha,\zeta)\, d\zeta$$

exists where the notation $(C - \delta)$ denotes the contour C apart from a small length δ surrounding ζ_0, and lim $(\delta \to 0)$ *denotes the limit as this excluded length tends to zero. The limit must be approached uniformly when α lies in any closed domain R' within R.*

(v) *If C goes to infinity then any bounded part of C must be smooth and conditions (i) and (ii) must be satisfied for any bounded part* *of C. The infinite integral defining $G(\alpha)$ must be uniformly convergent when α lies in any closed domain R' within R.*

Then $G(\alpha)$ defined by (1.16) *is a regular function of α in R.*

As a special case ζ may be real, say $\zeta = \xi$, and the contour may consist of the portion $a \leqslant \xi \leqslant b$ of the real axis. Then (1.16) is an ordinary real integral:

$$G(\alpha) = \int_a^b g(\alpha,\xi)\, d\xi.$$

We shall generally prove uniform convergence by using the comparison test in the following form. Suppose that $g(\alpha,\zeta)$ satisfies the conditions of the above theorem, and $|g(\alpha,\zeta)| \leqslant M(t)$ for any α in R where position on the contour C is specified by a parameter t, $\zeta = \xi(t) + i\eta(t)$, $a \leqslant t \leqslant b$, and $\int_a^b M(t)|\xi'(t) + i\eta'(t)|\, dt$ converges, then $\int_C g(\alpha,\zeta)\, d\zeta$ is uniformly and absolutely convergent in R'.

We can now deduce some important results. In the following σ_+, σ_-, τ_+, τ_- are real constants.

(1) If

$$F_+(\alpha) = \int_0^\infty f(x)e^{i\alpha x}\, dx,$$

where $\alpha = \sigma + i\tau$, $f(x)$ satisfies conditions (iii), (iv) above, and $|f(x)| < A \exp(\tau_- x)$ as $x \to +\infty$, then $F(\alpha)$ is regular in the upper half-plane $\tau > \tau_-$. Similarly if $f(x)$ satisfies (iii), (iv) and $|f(x)| < B \exp(\tau_+ x)$ as $x \to -\infty$ then

$$F_-(\alpha) = \int_{-\infty}^0 f(x)e^{i\alpha x}\, dx$$

is regular in the lower half-plane $\tau < \tau_+$. These statements follow immediately from the theorem since $\exp(i\alpha x)$ obviously satisfies (i), (ii) and the restrictions as $x \to \pm\infty$ ensure uniform convergence.

(2) If

$$F_P(s) = \int_0^1 f(\rho)\rho^{s-1}\, d\rho, \qquad F_N(s) = \int_1^\infty f(\rho)\rho^{s-1}\, d\rho, \qquad s = \sigma + i\tau,$$

where $f(\rho)$ satisfies conditions (iii), (iv) and $|f(\rho)| < A\rho^{-\sigma_-}$ as $\rho \to 0$, $|f(\rho)| < B\rho^{-\sigma_+}$ as $\rho \to +\infty$, then $F_P(s)$ is regular in a right-hand half-plane $\sigma > \sigma_-$, and $F_N(s)$ is regular in a left-hand half-plane $\sigma < \sigma_+$.

(3) If

$$F(\alpha) = \int_{-\infty+ic}^{\infty+ic} \frac{f(\zeta)}{\zeta - \alpha}\, d\zeta = \int_{-\infty}^{\infty} \frac{f(\xi + ic)}{(\xi - \sigma) + i(c - \tau)}\, d\xi,$$

where $\alpha = \sigma + i\tau$, $\zeta = \xi + ic$, c is a given constant, $f(\xi + ic)$ regarded as a function of ξ satisfies (iii), (iv) and $|f(\xi + ic)| < C|\xi|^{-k}$, $k > 0$, for $|\xi| > M$, say, then $F(\alpha)$ is a regular function of α in $\tau > c$ and it is also a (different) regular function of α in $\tau < c$. Again the result follows from the theorem since under the stated conditions $(\zeta - \alpha)^{-1}$ satisfies (i), (ii). Also if we consider the region R' of (v) such that $c + \varepsilon \leqslant \tau \leqslant K$, $a \leqslant \sigma \leqslant b$, we have

$$|F(\alpha)| \leqslant \int_{-\infty}^{\infty} \frac{|f(\xi + ic)|}{\{(\xi - \sigma)^2 + (c - \tau)^2\}^{1/2}}\, d\xi,$$

$$\leqslant \int_{-\infty}^{\infty} \frac{|f(u + \sigma + ic)|}{(u^2 + \varepsilon^2)^{1/2}}\, du,$$

on introducing $u = \xi - \sigma$. Divide the range of u into $(-\infty, A)$, (A, B), (B, ∞) where the finite number of discontinuities that f is allowed from (iii), (iv) lie in (A, B), and $A < -(M + b)$, $B > (M - a)$. Under these conditions $|f(u + \sigma + ic)| < C|u + \sigma|^{-k}$, $k > 0$, in $(-\infty, A)$ and (B, ∞) and the integral is absolutely convergent, independent of the position of α in R'.

Next consider

THEOREM B. *Let $f(\alpha)$ be an analytic function of $\alpha = \sigma + i\tau$, regular in the strip $\tau_- < \tau < \tau_+$, such that $|f(\sigma + i\tau)| < C|\sigma|^{-p}$, $p > 0$, for $|\sigma| \to \infty$, the inequality holding uniformly for all τ in the strip $\tau_- + \varepsilon \leqslant \tau \leqslant \tau_+ - \varepsilon$, $\varepsilon > 0$. Then for $\tau_- < c < \tau < d < \tau_+$,*

$$f(\alpha) = f_+(\alpha) + f_-(\alpha),$$

$$f_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty+ic}^{\infty+ic} \frac{f(\zeta)}{\zeta - \alpha}\, d\zeta \quad ; \quad f_-(\alpha) = -\frac{1}{2\pi i} \int_{-\infty+id}^{\infty+id} \frac{f(\zeta)}{\zeta - \alpha}\, d\zeta, \tag{1.17}$$

where $f_+(\alpha)$ is regular for all $\tau > \tau_-$, and $f_-(\alpha)$ is regular for all $\tau < \tau_+$.

The statements regarding regularity are proved as in (3) above. To prove (1.17) apply Cauchy's integral theorem to the rectangle with vertices $\pm a + ic$, $\pm a + id$. From our assumption as regards the behaviour of $f(\alpha)$ as $|\sigma| \to \infty$ in the strip, the integrals on $\sigma = \pm a$ tend to zero as $a \to \infty$ and we are left with the required equation.

As an example consider

$$f(\alpha) = \frac{1}{(\alpha - k \cos \theta)(\alpha + k)^{1/2}}, \qquad k = k_1 + ik_2, \qquad (k_1, k_2 > 0),$$

$$\tau_- = -k_2, \qquad \tau_+ = k_2 \cos \theta, \tag{1.18}$$

where it is assumed that $-\pi < \theta < \pi$. $f(\alpha)$ has a simple pole at $\alpha = k \cos \theta$ and a branch point at $\alpha = -k$. Cut the plane by a line running from $-k$ to infinity in the lower half-plane, choose the point α so that $-k_2 < \tau < k_2 \cos \theta$, and complete the contours for both $f_+(\alpha)$ and $f_-(\alpha)$ by semi-circles in the upper half-plane. The integrand for $f_+(\alpha)$ has simple poles at $\zeta = \alpha$ and $\zeta = k \cos \theta$ giving

$$f_+(\alpha) = \frac{1}{(\alpha - k \cos \theta)} \left\{ \frac{1}{(\alpha + k)^{1/2}} - \frac{1}{(k + k \cos \theta)^{1/2}} \right\},$$

regular in $\tau > -k_2$. The integral for $f_-(\alpha)$ has only a simple pole at $\zeta = k \cos \theta$ giving

$$f_-(\alpha) = \frac{1}{(\alpha - k \cos \theta)(k + k \cos \theta)^{1/2}}, \qquad \text{regular in } \tau < k \cos \theta.$$

These results are obviously correct and of course they could have been guessed without using the theorem. (See ex. 1.7 for a generalization.)

Next consider the decomposition of a function $K(\alpha)$ in the form of a product, $K(\alpha) = K_+(\alpha)K_-(\alpha)$, where K_+ and K_- are regular and non-zero in upper and lower half-planes $\tau > \tau_-$, $\tau < \tau_+$ respectively, where $\tau_- < \tau_+$. Sometimes this decomposition can be guessed, e.g. if $K(\alpha) = (\alpha^2 - k^2)^{1/2}$, $k = k_1 + ik_2$, $(k_1, k_2 > 0)$, with the α-plane cut as in Fig. 1.1 we can write $K_+(\alpha) = (\alpha + k)^{1/2}$ regular and non-zero in $\tau > -k_2$, and $K_-(\alpha) = (\alpha - k)^{1/2}$ regular and non-zero in $\tau < k_2$. Obviously any decomposition of this type is not unique since we can multiply K_+ by any integral non-zero function providing that we divide K_- by the same factor. This remark will be important later. Another kind of example where the answer can be guessed is given in ex. 1.11.

If $K(\alpha)$ is an integral function which can be expressed as an infinite product (see ex. 1.9 below) then the decomposition is immediate. The important case from our point of view occurs when $K(\alpha)$ is an even function of α. Then the roots occur in pairs, say $\alpha = \pm\alpha_n$, and $K'(0) = 0$. From ex. 1.9 we can write

$$K(\alpha) = K(0) \prod_{n=1}^{\infty} \{1 - (\alpha/\alpha_n)^2\}.$$

This can be decomposed in the form

$$K_{\pm}(\alpha) = \{K(0)\}^{1/2} e^{\mp\chi(\alpha)} \prod_{n=1}^{\infty} \{1 \pm (\alpha/\alpha_n)\} e^{\mp(\alpha/\beta_n)}, \tag{1.19}$$

where upper and lower signs go together and the terms have been arranged so that all the zeros of $K_+(\alpha)$ lie in the lower half-plane and vice-versa. Hence $K_+(\alpha)$ is regular and non-zero in the upper half-plane $(\text{Im}\,\alpha) > -(\text{Im}\,\alpha_1)$. The function $\chi(\alpha)$ is arbitrary and can be chosen to ensure that K_+, K_- have suitable behaviour as $\alpha \to \infty$ in appropriate half-planes. The infinite product will in general have exponential behaviour as $\alpha \to \infty$ whereas the functions which need to be decomposed later have algebraic behaviour at infinity due to additional terms multiplying the infinite product. These facilitate the choice of $\chi(\alpha)$ and it is convenient to postpone further discussion until we require the decomposition of concrete examples in §3.2 (see exs. 3.3–3.6). It is emphasized that the correct choice of $\chi(\alpha)$ is crucial for the successful application of the Wiener–Hopf technique.

When the function has a branch point the infinite product method will break down. We now give a theorem which proves the existence of the decomposition $K(\alpha) = K_+(\alpha)K_-(\alpha)$ for a general class of $K(\alpha)$. The theorem also provides a practical method for performing the decomposition in complicated cases.

THEOREM C. *If* $\ln K(\alpha)$ *satisfies the conditions of theorem* B, *which implies in particular that* $K(\alpha)$ *is regular and non-zero in a strip* $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, *and* $K(\alpha) \to +1$ *as* $\sigma \to \pm\infty$ *in the strip, then we can write* $K(\alpha) = K_+(\alpha)\, K_-(\alpha)$ *where* $K_+(\alpha)$, $K_-(\alpha)$ *are regular, bounded, and non-zero in* $\tau > \tau_-$, $\tau < \tau_+$, *respectively.*

(The conditions of the theorem are more restrictive than necessary but cover the applications needed in this book. A more general theorem is stated in ex. 1.12 so as to bring to the reader's notice certain points which must be borne in mind in more complicated examples—in particular the more general theorem shows how to deal with zeros of $K(\alpha)$ in the strip, and with cases where $K(\alpha) \to \exp(i\mu)$, $\exp(i\nu)$ as $\sigma \to +\infty, -\infty$ respectively, or $|K(\alpha)| \sim |\sigma|^p$ as $|\sigma| \to \infty$, in the strip.)

To prove theorem C, apply theorem B to $f(\alpha) = \ln K(\alpha)$.

$$\ln K(\alpha) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{\ln K(\zeta)}{\zeta - \alpha}\,d\zeta - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{\ln K(\zeta)}{\zeta - \alpha}\,d\zeta \qquad (1.20)$$

$$= f_+(\alpha) + f_-(\alpha), \text{ say,}$$

where c, d are any numbers such that $\tau_- < c < \tau < d < \tau_+$. The integral for $f_+(\alpha)$ is convergent for all α such that $\tau > c$. Hence $f_+(\alpha)$ is bounded and regular in $\tau > \tau_-$ since we can choose c as near as we please to τ_-. Similarly $f_-(\alpha)$ is bounded and regular in $\tau < \tau_+$. If we set

$$K_+(\alpha) = \exp\{f_+(\alpha)\} \quad : \quad K_-(\alpha) = \exp\{f_-(\alpha)\}, \qquad (1.21)$$

then

$$\ln K_+(\alpha) + \ln K_-(\alpha) = \ln K(\alpha), \quad \text{i.e.} \quad K_+(\alpha)\,K_-(\alpha) = K(\alpha).$$

From the properties of $f_+(\alpha)$, it is seen that $K_+(\alpha)$ is regular, bounded, *and non-zero* in $\tau > \tau_-$. Similarly $K_-(\alpha)$ is regular, bounded, and non-zero in $\tau < \tau_+$. Therefore the theorem is proved since $K_+(\alpha)$, $K_-(\alpha)$ have been constructed which satisfy the necessary conditions.

The presence of the logarithms in (1.20) often makes the integrations difficult. Sometimes simpler integrals are obtained as follows. We have $\ln K(\alpha) = \ln K_+(\alpha) + \ln K_-(\alpha)$. Differentiate with respect to α:

$$\frac{K'(\alpha)}{K(\alpha)} = \frac{K'_+(\alpha)}{K_+(\alpha)} + \frac{K'_-(\alpha)}{K_-(\alpha)}. \qquad (1.22)$$

If the expression on the left-hand side is decomposed by theorem B we can set

$$K'_+(\alpha)/K_+(\alpha) = f_+(\alpha) \quad : \quad K'_-(\alpha)/K_-(\alpha) = f_-(\alpha),$$

and an integration with respect to α will give $\ln K_+(\alpha)$, $\ln K_-(\alpha)$. An example is given in ex. 2.10.

It is clear that in the practical application of theorems B and C the chief difficulty is the evaluation of the contour integrals (1.17), (1.20). When the integrals have simple poles and no branch points the situation is comparatively easy since the integrals can be evaluated by residues, but in this case the reduction or separation can usually be carried out more directly by simpler methods, e.g. in the case of theorem B we pointed out that (1.18) could be decomposed by inspection, and in the case of theorem C the reader will readily verify that evaluation by residues is equivalent to the infinite product method given above.

We now derive two results which are useful when evaluating integrals of type (1.17), (1.20). Suppose that $f(\alpha)$ is an even function of α, regular in $-k_2 < \tau < k_2$, satisfying the conditions of theorem B. Then for $-k_2 < d < k_2$

$$f_-(-\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{f(\zeta)}{\zeta+\alpha}\,d\zeta = \frac{1}{2\pi i}\int_{-id-\infty}^{-id+\infty} \frac{f(-\zeta)}{\zeta-\alpha}\,d\zeta = f_+(\alpha),$$

i.e. $$f_-(-\alpha) = f_+(\alpha) \quad : \quad f_+(-\alpha) = f_-(+\alpha). \tag{1.23}$$

Instead of evaluating the integrals in (1.20) directly it is usually convenient to introduce a quantity $g(\alpha)$ in the following way. Suppose that $K(\alpha)$ is even, regular in $-k_2 < \tau < k_2$, and satisfies the conditions of theorem C. We have

$$f_+(\alpha) + f_-(\alpha) = f(\alpha), \qquad [f(\alpha) = \ln K(\alpha)].$$

Set $$f_+(\alpha) - f_-(\alpha) = g(\alpha), \quad \text{say.} \tag{1.24}$$

Then $$f_+(\alpha) = \tfrac{1}{2}f(\alpha) + \tfrac{1}{2}g(\alpha) \quad : \quad f_-(\alpha) = \tfrac{1}{2}f(\alpha) - \tfrac{1}{2}g(\alpha). \tag{1.25}$$

Since $f_-(\alpha) = f_+(-\alpha)$ we have $g(\alpha) = -g(-\alpha)$. Introducing $K(\alpha)$ into (1.25), we obtain:

$$\ln K_+(\alpha) = \tfrac{1}{2}\ln K(\alpha) + \tfrac{1}{2}g(\alpha) \quad : \quad \ln K_-(\alpha) = \tfrac{1}{2}\ln K(\alpha) - \tfrac{1}{2}g(\alpha),$$

i.e. $$K_+(\alpha) = \{K(\alpha)\}^{1/2}e^{\frac{1}{2}g(\alpha)} \quad : \quad K_-(\alpha) = \{K(\alpha)\}^{1/2}e^{-\frac{1}{2}g(\alpha)}. \tag{1.26}$$

It is sometimes easier to evaluate the integral for $g(\alpha)$ rather than the integrals for $f_+(\alpha)$, $f_-(\alpha)$ separately. If α is real, say $\alpha = \xi$, and the contour in (1.20) can be taken as the real axis ($c \to -0$, $d \to +0$) then on using the result in ex. 1.24 we find

$$f_+(\xi) = \tfrac{1}{2}f(\xi) + \tfrac{1}{2}g(\xi) \quad : \quad f_-(\xi) = \tfrac{1}{2}f(\xi) - \tfrac{1}{2}g(\xi), \tag{1.27a}$$

with $$g(\xi) = \frac{1}{\pi i}\,P\int_{-\infty}^{\infty} \frac{f(x)}{x-\xi}\,d\xi, \tag{1.27b}$$

where P denotes a Cauchy principal value.

The most important branch points in our applications appear when integrands are functions of $\gamma = (\alpha^2 - k^2)^{1/2}$. To illustrate the procedure suppose that $f(\alpha)$ in (1.17) is a function only of $\gamma a = (\alpha^2 - k^2)^{1/2}a$, say $f(\alpha) = F(\gamma a)$, where the constant a has been added for convenience. Consider the integral for $f_-(\alpha)$ and examine three methods of procedure in turn. We have

$$f_-(\alpha) = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{F\{(\zeta^2 - k^2)^{1/2}a\}}{\zeta - \alpha}\,d\zeta, \tag{1.28}$$

where $\tau < d$ and $-k_2 < d < k_2$. Assume that the contour can be deformed into an upper half-plane if necessary and that in any such deformation no poles are crossed. (Contributions from poles can be taken into account in the usual way.) Similarly contributions due to poles at $\alpha = \pm k$ will be ignored. The three procedures to be examined correspond to different ways of choosing branch cuts, as shown in Fig. 1.2. It is assumed that α, k are real and that $-k < \alpha < k$. A convenient method of finding the positions of indentations is to imagine the diagrams for the general case $\operatorname{Im} \alpha < 0$, $\operatorname{Im} k > 0$, and then let $\operatorname{Im} \alpha \to -0$, $\operatorname{Im} k \to +0$. If α lies outside the range $(0,k)$ the indentations and contributions from indentations must be altered accordingly.

(a) First of all consider simply the limiting case of (1.28) when $k_2 \to 0$, the complex plane is cut from $+k$ to $+\infty$, $-k$ to $-\infty$, along the positive and negative real axes respectively, and the contour becomes the real axis, passing above the branch point at $\zeta = -k$, below the branch point at $\zeta = +k$ and above the indentation at $\zeta = \alpha$. The values of $\arg \gamma$ are shown in Fig. 1.2(a). For clarity set $\alpha = \xi$, a real variable. As in the argument leading to (1.27),

$$f_+(\xi) = \tfrac{1}{2}F\{(\xi^2 - k^2)^{1/2}a\} + \tfrac{1}{2}g(\xi) \quad : \quad f_-(\xi) = \tfrac{1}{2}F\{(\xi^2 - k^2)^{1/2}a\} - \tfrac{1}{2}g(\xi), \tag{1.29}$$

where

$$g(\xi) = \frac{1}{\pi i} P \int_{-k}^{k} F\{(x^2 - k^2)^{1/2}a\} \frac{dx}{x - \xi} +$$
$$+ \frac{1}{\pi i} \int_{-\infty}^{-k} F\{(x^2 - k^2)^{1/2}a\} \frac{dx}{x - \xi} + \frac{1}{\pi i} \int_{k}^{\infty} F\{(x^2 - k^2)^{1/2}a\} \frac{dx}{x - \xi}.$$

Change x to $(-x)$ in the second integral and combine with the third. Split the first integral into integrals over $(-k,0)$, $(0,k)$ and combine into one integral over the range $(0,k)$. Change variable to u defined by $(x^2 - k^2)a^2 = u^2$, $(k^2 - x^2)a^2 = u^2$ in the integrals over (k, ∞), $(0,k)$ respectively. This gives

$$g(\xi) = \frac{2\xi a}{\pi i} P \int_0^{ka} \frac{F(-iu)}{\{a^2(k^2 - \xi^2) - u^2\}} \cdot \frac{u\,du}{(k^2a^2 - u^2)^{1/2}} +$$
$$+ \frac{2\xi a}{\pi i} \int_0^{\infty} \frac{F(u)}{\{u^2 + a^2(k^2 - \xi^2)\}} \cdot \frac{u\,du}{(k^2a^2 + u^2)^{1/2}}. \tag{1.30}$$

(b) In Fig. 1.2(b) the contour is deformed along two sides of a branch cut which consists of the real axis of ζ from $+k$ to $+\infty$. If $\zeta = x + iy$, then on the lower side of the cut $(\zeta^2 - k^2)^{1/2} = +(x^2 - k^2)^{1/2}$, and on the upper side $(\zeta^2 - k^2)^{1/2} = -(x^2 - k^2)^{1/2}$. Hence for $\xi < k$,

$$f_-(\xi) = -\frac{1}{2\pi i} \int_k^{\infty} [F\{(x^2 - k^2)^{1/2}a\} - F\{-(x^2 - k^2)^{1/2}a\}] \frac{dx}{x - \xi}. \tag{1.31}$$

As in (1.23) $f_+(\xi) = f_-(-\xi)$ and as in (1.24), (1.25) it is convenient to consider $g(\xi) = f_+(\xi) - f_-(\xi)$. If we change variable to u defined by $(x^2 - k^2)a^2 = u^2$, we find

$$g(\xi) = \frac{\xi a}{\pi i} \int_0^\infty \frac{F(u) - F(-u)}{\{u^2 + a^2(k^2 - \xi^2)\}} \cdot \frac{u\, du}{(u^2 + k^2a^2)^{1/2}}. \tag{1.32}$$

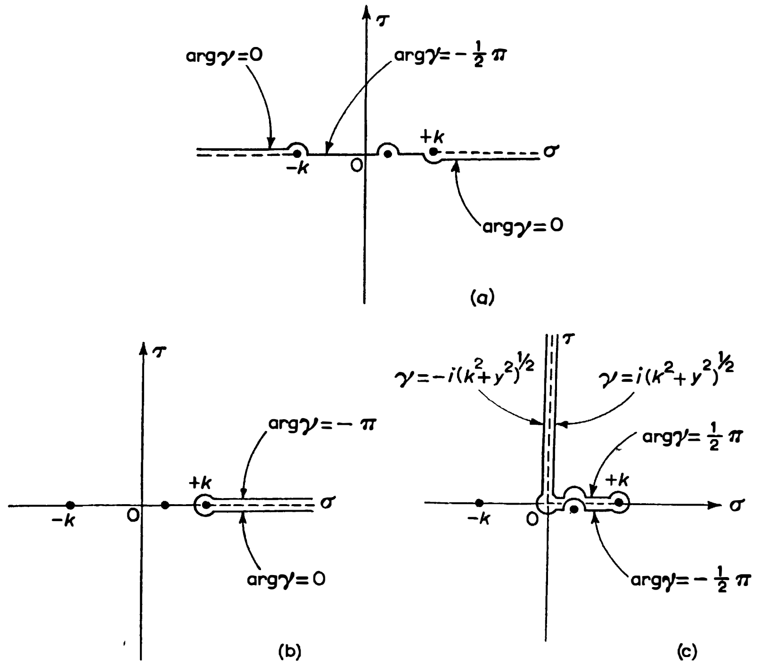

FIG. 1.2. – – – Branch cut. ——— Contour.

(c) Finally consider the contour in Fig. 1.2(c). On combining the integrals along the two sides of the branch cut from 0 to k in x and 0 to ∞ in y,

$$f_-(\xi) = \tfrac{1}{2}[F\{-i(k^2 - \xi^2)^{1/2}a\} - F\{i(k^2 - \xi^2)^{1/2}a\}] -$$

$$- \frac{1}{2\pi} \int_0^\infty [F\{i(k^2 + y^2)^{1/2}a\} - F\{-i(k^2 + y^2)^{1/2}a\}] \frac{dy}{iy - \xi} +$$

$$+ \frac{1}{2\pi i} P \int_0^k [F\{i(k^2 - x^2)^{1/2}a\} - F\{-i(k^2 - x^2)^{1/2}a\}] \frac{dx}{x - \xi}.$$

If $-k < \xi < 0$ the first term on the right does not occur and the integral is not a principal value. On following a procedure similar to that used in (b) we obtain

$$g(\xi) = \frac{1}{2}\frac{\xi}{|\xi|}[F\{i(k^2 - \xi^2)^{1/2}a\} - F\{-i(k^2\,\xi^2 -)^{1/2}a\}] -$$

$$-\frac{\xi}{\pi}\int_0^\infty [F\{i(k^2 + y^2)^{1/2}a\} - F\{-i(k^2 + y^2)^{1/2}a\}]\frac{dy}{y^2+\xi^2} -$$

$$-\frac{\xi a}{\pi i}P\int_0^{ka}\frac{\{F(iu) - F(-iu)\}}{\{a^2(k^2 - \xi^2) - u^2\}}\frac{u\,du}{(k^2a^2 - u^2)^{1/2}}. \qquad (1.33)$$

The form in (a) is the simplest to obtain: in (b) and (c) care has to be exercised regarding contributions from poles. The form in (b) may be convenient if $\{F(u) - F(-u)\}$, (u real), is simple: also no principal value occurs if $|\xi| < k$. The form in (c) may be convenient if $\{F(iu) - F(-iu)\}$ is simple: also no principal value occurs if $|\xi| > k$. Instructive examples of the use of integrals of the above type to perform factorizations are given, for instance, in H. Levine and J. Schwinger [1], L. Vajnshtejn [1], [3], and W. Chester [1].

In the general case $k = k_1 + ik_2$, $k_2 > 0$, there are various forms of contour which reduce to the forms used above when $k_2 \to 0$. Thus we can take the contour from $+k$ to infinity as (i) a straight line from $k = k_1 + ik_2$ to $\infty + ik_2$ parallel to the x-axis, (ii) a continuation of the straight line joining the origin to k, (iii) the part of the hyperbola through k defined by $\xi\eta = k_1k_2$, ($\zeta = \xi + i\eta$), on which $\gamma = (\zeta^2 - k^2)^{1/2}$ is real.

As an example of case (b) above consider

$$K(\alpha) = e^{-\gamma d} \quad : \quad \ln K(\alpha) = -\gamma d = -(\alpha^2 - k^2)^{1/2}d. \qquad (1.34)$$

We cannot apply theorem B to $\ln K(\alpha)$ as it stands, since it increases as $\alpha \to \infty$ but we can write

$$\ln K(\alpha) = d(k^2 - \alpha^2)(\alpha^2 - k^2)^{-1/2}.$$

The factor $(k^2 - \alpha^2)$ is an integral function and we can apply theorem B to $(\alpha^2 - k^2)^{-1/2} = \gamma^{-1}$. Then (1.32) with $F(u) = u^{-1}$, $a = 1$, gives

$$g(\xi) = \frac{2\xi}{\pi i}\int_0^\infty \frac{1}{\{u^2 + (k^2 - \xi^2)\}}\cdot\frac{du}{(u^2+k^2)^{1/2}}.$$

A standard integral is

$$\int \frac{du}{(u^2+p)(u^2+q)^{1/2}} = \frac{1}{p^{1/2}(q-p)^{1/2}}\arctan\left\{\frac{u(q-p)^{1/2}}{p^{1/2}(u^2+q)^{1/2}}\right\}, \; (q > p).$$

Hence for $0 \leqslant \xi \leqslant k$.

$$g(\xi) = f_+(\xi) - f_-(\xi) = \frac{2}{\pi i(k^2 - \xi^2)^{1/2}} \text{arc tan} \frac{\xi}{(k^2 - \xi^2)^{1/2}},$$

But $\quad f_+(\xi) + f_-(\xi) = (\xi^2 - k^2)^{-1/2}$ and $i(k^2 - \xi^2)^{1/2} = -(\xi^2 - k^2)^{1/2}$.

Hence

$$\begin{aligned} f_+(\xi) &= \pi^{-1}(\xi^2 - k^2)^{-1/2}[\tfrac{1}{2}\pi - \text{arc tan}\,\{\xi(k^2 - \xi^2)^{-1/2}\}] \\ &= \pi^{-1}(\xi^2 - k^2)^{-1/2} \text{arc cos}\,(\xi/k) \\ &= \frac{2}{\pi}(\xi^2 - k^2)^{-1/2} \text{arc tan} \left(\frac{k - \xi}{k + \xi}\right)^{1/2}, \end{aligned}$$

where the branch of the inverse cosine is chosen so that when $\xi = 0$, arc cos $0 = \frac{1}{2}\pi$. By analytic continuation we can now define $f_+(\alpha)$ by these expressions with ξ replaced by α, provided that branches are defined to agree with the above values when $\alpha = \xi$, ξ real, $0 \leqslant \xi \leqslant k$. In this case

$$\text{arc cos}\,(\alpha/k) = +i \ln \{(\alpha + \gamma)/k\}, \qquad \gamma = (\alpha^2 - k^2)^{1/2}.$$

By analytic continuation

$$\begin{aligned} \text{arc cos}\,(\alpha/k) &= -i \ln (2\alpha/k) + 0(|\alpha|^{-2}), \\ (\alpha^2 - k^2)^{1/2} &= -\alpha + 0(|\alpha|^{-2}), \end{aligned}$$

as $\alpha \to \infty$ in an upper half-plane. Returning to (1.34) we have therefore the decomposition

$$\begin{aligned} &K_+(\alpha) = \exp\{-T_+(\alpha)\}; && K_-(\alpha) = \exp\{-T_-(\alpha)\}; \\ &T_+(\alpha) = \pi^{-1}d(\alpha^2 - k^2)^{1/2} \text{arc cos}\,(\alpha/k); && T_-(\alpha) = T_+(-\alpha); \qquad (1.35) \\ &T_+(\alpha) = (id\alpha/\pi) \ln (2\alpha/k) + 0(|\alpha|^{-1}), \end{aligned}$$

as $\alpha \to \infty$ in an upper half-plane. An alternative method for obtaining these results is given in ex. 1.25.

1.4 The Fourier integral

At the outset is is necessary to decide whether to use Fourier or Laplace transforms. Both have been used in the literature in connexion with the Wiener–Hopf technique and in the complex plane the two transforms are completely equivalent as will be clear later in this section. In this book we use Fourier transforms for no special reason apart from the need to standardize on one or the other.

The Fourier integral can be stated in the form

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x}\, dx \quad : \quad f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x}\, d\alpha, \tag{1.36}$$

where α is usually taken to be real. The Laplace integral is usually stated in this form

$$\mathscr{F}(s) = \int_{0}^{\infty} f(x) e^{-sx}\, dx \quad : \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathscr{F}(s) e^{sx}\, ds, \qquad (c > c_0), \tag{1.37}$$

where the second integral gives automatically that $f(x) = 0$ for $x < 0$. The Laplace integral corresponding to (1.36) is the two-sided Laplace integral

$$\mathscr{F}(s) = \int_{-\infty}^{\infty} f(x) e^{-sx}\, dx \quad : \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathscr{F}(s) e^{sx}\, ds, \quad (c_1 < c < c_2). \tag{1.38}$$

Corresponding to (1.37) we have the Fourier integral

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} f(x) e^{i\alpha x}\, dx \quad : \quad f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x}\, d\alpha, \tag{1.39}$$

where the second integral gives automatically that $f(x) = 0$ for $x < 0$. This form of the Fourier integral is not often used in the literature.

It is sometimes stated that it is preferable to use the Laplace integral instead of the Fourier integral since the Laplace integral can deal with more general situations. The example that is usually quoted is $f(x) = 1$, $(x > 0)$: $f(x) = 0$, $(x < 0)$. The Laplace transform of this function, from (1.37), is s^{-1} whereas the first integral in (1.39) does not converge for this $f(x)$, if α is real. This is true if we are dealing with real α and real s, but if α and s are both complex we have merely the equivalent statements: if $f(x) = 1$, $(x > 0)$: 0, $(x < 0)$, then the Laplace transform of $f(x)$ is s^{-1} provided that $\mathrm{Re}\, s > 0$; the Fourier transform of $f(x)$ is $i\alpha^{-1}$ provided that $\mathrm{Im}\, \alpha > 0$. This complete equivalence of Fourier and Laplace transforms will be emphasized later in this section. One of the advantages of using the Laplace transform would be that we are accustomed to considering the parameter s as complex: thus the limits on the integral and the statement $c > c_0$ in (1.37) automatically imply that s is complex.

We shall start by assuming (1.36) for real α and show how the formulae are modified when α is allowed to take complex values. It is not necessary from our point of view to state the exact conditions which must be satisfied by $f(x)$. We content ourselves with stating that integrals will be understood in the Riemann sense, and that $f(x)$ may have a finite number of infinite discontinuities and need not be absolutely integrable in the infinite range, e.g. we have Fourier transforms of the following types:

$$\left\{\begin{array}{ll} f(x) = (a^2 - x^2)^{-1/2}, \quad (|x| < a) & : \quad 0, (|x| > a); \\ F(\alpha) = (\tfrac{1}{2}\pi)^{1/2} J_0(a\alpha). & \end{array}\right\}$$

$$\left\{\begin{array}{ll} f(x) = x^{\nu-1}, \quad (0 < \nu < 1, x > 0) & : \quad 0, (x < 0); \\ F(\alpha) = (2\pi)^{-1/2}\Gamma(\nu)e^{\frac{1}{2}i\pi\nu}\alpha^{-\nu}. & \end{array}\right\}$$

(It is worth noting that the original work of N. Wiener and E. Hopf is rigorous and uses Lebesgue integration. There is a temptation to use the Plancherel theory as starting point because of the elegance and generality of the results. It was felt that any such attempt at rigour would be out of character with the remainder of this book.)

Now suppose that α is a complex variable, $\alpha = \sigma + i\tau$. We state and prove the following:

THEOREM A. *Let $f(x)$ be a function of the real variable x such that $|f(x)| \leqslant A \exp(\tau_- x)$ as $x \to +\infty$ and $|f(x)| \leqslant B \exp(\tau_+ x)$ as $x \to -\infty$, with $\tau_- < \tau_+$: suppose that for some given τ_0, $\tau_- < \tau_0 < \tau_+$, Fourier's theorem for real variables as given by* (1.36) *applies to the function $f(x) \exp(-\tau_0 x)$. Then if we define*

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\, dx, \qquad \alpha = \sigma + i\tau, \tag{1.40}$$

$F(\alpha)$ is an analytic function of α, regular in $\tau_- < \tau < \tau_+$, and

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{i\tau-\infty}^{i\tau+\infty} F(\alpha)e^{-i\alpha x}\, d\alpha, \quad \textit{for any } \tau,\ \tau_- < \tau < \tau_+. \tag{1.41}$$

The statement regarding the analytic nature of $F(\alpha)$ follows from §1.3, theorem B, example (1). Substitute (1.40) in the right-hand side of (1.41), which becomes

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\infty+i\tau}^{\infty+i\tau} \int_{-\infty}^{\infty} f(\xi)e^{i\alpha\xi}\, d\xi e^{-i\alpha x}\, d\alpha \\ &= \frac{1}{2\pi} e^{\tau x} \int_{-\infty}^{\infty} e^{-i\sigma x} \int_{-\infty}^{\infty} \{f(\xi)e^{-\tau\xi}\}e^{i\sigma\xi}\, d\xi\, d\sigma \\ &= e^{\tau x}\{f(x)e^{-\tau x}\} = f(x). \end{aligned}$$

In the second line we changed the variable by setting $\alpha - i\tau = \sigma$, $d\alpha = d\sigma$. The third line is obtained from the Fourier integral (1.36), since if a single τ_0 exists as postulated in the theorem, Fourier's integral applies to $f(x) \exp(-\tau x)$ for any τ in $\tau_- < \tau < \tau_+$ because of the assumed behaviour of $f(x)$ as $x \to \pm\infty$.

As an example, if $f(x) = 1$, $(x > 0)$, $f(x) = e^x$, $(x < 0)$, then $\tau_- = 0$, $\tau_+ = 1$, and $F(\alpha)$ should be regular in $0 < \tau < 1$. In fact

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}}\left\{\int_0^\infty e^{i\alpha x}\,dx + \int_{-\infty}^0 e^{(1+i\alpha)x}\,dx\right\}$$

$$= \frac{1}{(2\pi)^{1/2}}\left\{-\frac{1}{i\alpha} + \frac{1}{1+i\alpha}\right\}, \tag{1.42}$$

where the first integral can be evaluated only for $\tau > 0$, and the second only for $\tau < 1$. $F(\alpha)$ has singularities on the lines $\tau = 0$ and $\tau = 1$ as we should expect.

The converse of theorem A is given in

THEOREM B. *Suppose that $F(\alpha)$, $\alpha = \sigma + i\tau$, is regular in the strip $\tau_- < \tau < \tau_+$, and $|F(\alpha)| \to 0$ uniformly as $|\sigma| \to \infty$ in the strip $\tau_- + \varepsilon \leqslant \tau \leqslant \tau_+ - \varepsilon$ where ε is an arbitrary positive number. If we define a function $f(x)$ by*

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{i\tau-\infty}^{i\tau+\infty} F(\alpha)e^{-i\alpha x}\,d\alpha \tag{1.43}$$

for a given τ, $\tau_- < \tau < \tau_+$, and any given real x, then

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x)e^{i\alpha x}\,dx. \tag{1.44}$$

Also $|f(x)| < \exp(\tau_- + \delta)x$ as $x \to +\infty$, and $|f(x)| < \exp(\tau_+ - \delta)x$ as $x \to -\infty$ where δ is an arbitrarily small positive number. The function $f(x)$ defined by (1.43) *can be regarded as the solution of the integral equation* (1.44).

To prove this theorem choose c, d such that $\tau_- < c < \tau < d < \tau_+$ where τ is the imaginary part of α in (1.43). Substitute (1.43) in the right-hand side of (1.44):

$$I = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\alpha x}\,dx \int_{i\tau-\infty}^{i\tau+\infty} e^{-i\beta x}F(\beta)\,d\beta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{i\alpha x}\,dx \int_{id-\infty}^{id+\infty} e^{-i\beta x}F(\beta)\,d\beta + \frac{1}{2\pi}\int_{0}^{\infty} e^{i\alpha x}\,dx \int_{ic-\infty}^{ic+\infty} e^{-i\beta x}F(\beta)\,d\beta.$$

The splitting of the integral and the use of different though equivalent τ-values has been arranged so that when orders of integration are interchanged the inner integrals are convergent since $\text{Im}\,(\alpha - \beta) < 0$, $x < 0$ in the first integral and $\text{Im}\,(\alpha - \beta) > 0$, $x > 0$ in the second. On interchanging orders of integration and evaluating the inner integrals, we find

$$I = -\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{F(\beta)}{\beta - \alpha}\,d\beta + \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{F(\beta)}{\beta - \alpha}\,d\beta,$$

$$= \frac{1}{2\pi i}\int_C \frac{F(\beta)}{\beta - \alpha}\,d\beta,$$

where C is a closed contour consisting of the limit of the rectangle joining the points $(\pm A, c)$, $(\pm A, d)$ as $A \to \infty$, the rectangle being traversed in a positive direction. The integrals along the vertical sides of the rectangle vanish in the limit because of the assumption that $F(\alpha)$ tends to zero uniformly as $|\sigma| \to \infty$ in the strip. Cauchy's integral theorem then gives $I = F(\alpha)$ as required. If $|f(x)| < \exp\,(px)$ as $x \to +\infty$ and $f(x) < \exp\,(qx)$ as $x \to -\infty$ then from (1.44) $F(\alpha)$ would be regular in $p < \tau < q$. But we are given that $F(\alpha)$ is regular in $\tau_- < \tau < \tau_+$. Hence by *reductio ad absurdum* we can take $p = \tau_- + \delta$, $q = \tau_+ - \delta$, where δ is an arbitrarily small positive number.

A special case of the above theorem, namely when either τ_- or τ_+ is infinite, is sufficiently important to warrant statement as a separate theorem:

THEOREM C. *Suppose that* $F(\alpha)$, $\alpha = \sigma + i\tau$, *is regular in* $\tau > \tau_-$ *and* $F(\alpha) \to 0$ *as* $r \to \infty$ *where* r *is defined by* $\alpha - i\tau_- - i\varepsilon = r \exp\,(i\theta)$ *for arbitrary* $\varepsilon > 0$ *and the limit is approached uniformly for* $0 \leqslant \theta \leqslant \pi$. *If we define* $f(x)$ *as in* (1.43) *for any given* $\tau > \tau_-$ *and any given real* x *then*

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{i\alpha x} f(x)\,dx, \tag{1.45}$$

and the function $f(x)$ *defined by* (1.43) *can be regarded as a solution of the integral equation* (1.45). *Also* $f(x) = 0$ *for* $x < 0$, *and* $|f(x)| < \exp\,(\tau_- + \delta)x$ *as* $x \to +\infty$, *where* δ *is an arbitrarily small positive number.*

The proof is left to the reader. If follows exactly the same lines as before except that the contour for the application of Cauchy's theorem is now completed by an infinite semi-circle in the upper half-plane. A similar situation arises when $F(\alpha)$ is regular in some lower half-plane $\tau < \tau_+$. Then under appropriate conditions $f(x)$ defined by (1.43) satisfies an integral equation similar to (1.45) except that the limits $(0,\infty)$ are replaced by $(-\infty,0)$.

Next consider the relation of theorems A and B to the two-sided Laplace transform. In the integrals (1.40), (1.41) make the change of variable $i\alpha = -s$ where $s = \sigma' + i\tau'$, another complex variable. Since $\alpha = \sigma + i\tau$, then $\sigma' = +\tau$, $\tau' = -\sigma$. Define $\mathscr{F}(s) \equiv (2\pi)^{-1/2}F(\alpha)$ and find, for some constant value of σ' which we denote by b,

$$\mathscr{F}(s) = \int_{-\infty}^{\infty} f(x)e^{-sx}\,dx \quad : \quad f(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \mathscr{F}(s)e^{sx}\,ds. \tag{1.46}$$

This is the two-sided Laplace transform. To make clearer the relation between α and s the paths of integration and strips of regularity in the two complex planes are plotted in Fig. 1.3.

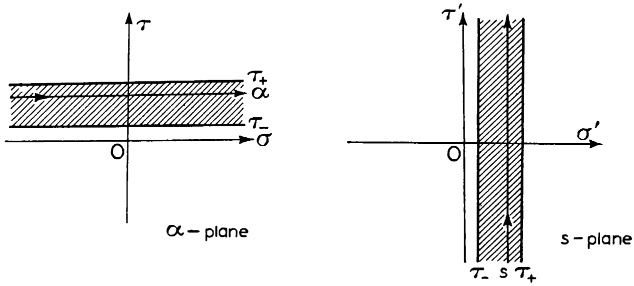

FIG. 1.3.

Similarly the form given in theorem C, with the same change of variable $i\alpha = -s$, reduces to the standard Laplace transform

$$\mathscr{F}(s) = \int_{0}^{\infty} f(x)e^{-sx}\,dx \quad : \quad f(x) = \frac{1}{2\pi i}\int_{b-i\infty}^{b+i\infty} \mathscr{F}(s)e^{sx}\,ds. \tag{1.47}$$

As already mentioned some of the literature on the Wiener–Hopf technique has been written in terms of the Laplace transform and in order to translate the notation of this book into terms of the Laplace integral and vice-versa it is important that the reader should understand the complete equivalence of Fourier and Laplace integrals in the complex plane. As an example, the Laplace integral corresponding to (1.42) gives

$$\mathscr{F}(s) = \left\{\frac{1}{s} + \frac{1}{1-s}\right\}.$$

The statement that (1.42) represents a function which is regular in

$0 < \tau < 1$ is equivalent to stating that the corresponding Laplace result is regular in $0 < \sigma' < 1$.

The reader will find a great deal of material on the subject matter of this section in books on the Laplace transform, e.g. the standard texts of Doetsch, Van der Pol and Bremmer, Widder. An elementary treatment of some topics is given in R. V. Churchill, Modern operational mathematics in engineering, McGraw-Hill (1944), Chapters V, VI.

Finally consider the relation of the above results to the Mellin transform. Consider the substitution

$$\rho = e^x \quad : \quad dx = d\rho/\rho \quad : \quad f(x) \equiv g(\rho).$$

If we use this in (1.40), (1.41) we find that reciprocal formulae are:

$$F(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty g(\rho)\rho^{i\alpha-1}\, d\rho \quad : \quad g(\rho) = \frac{1}{(2\pi)^{1/2}} \int_{i\tau-\infty}^{i\tau+\infty} F(\alpha)\rho^{-i\alpha}\, d\alpha. \tag{1.48}$$

Similarly in (1.46) make the substitution $\rho = \exp(-x)$. This gives

$$\mathscr{F}(s) = \int_0^\infty g(\rho)\rho^{s-1}\, d\rho \quad : \quad g(\rho) = \frac{1}{2\pi i} \int_{\tau-i\infty}^{\tau+i\infty} \mathscr{F}(s)\rho^{-s}\, ds. \tag{1.49}$$

These two forms are completely equivalent. The second is the well-known Mellin transform and we use it in this book. Perhaps it would be slightly more consistent to use the form (1.48) since we have standardized on the Fourier transform instead of the Laplace transform. We shall have to talk of right and left half-planes when using the Mellin transform instead of upper and lower half-planes for the Fourier transform. However the reader will see that this causes no confusion in applications. There is of course a theorem for Mellin transforms corresponding to theorem C above (ex. 1.14).

In the applications of the Mellin transform made later it would be possible to apply the substitution $\rho = \exp(-x)$ to the original partial differential equation, and then apply the Fourier transform to the new equation, so that theoretically we need never use the Mellin transform at all. But in practice it is convenient to use the Mellin transform directly.

1.5 The wave equation

In this section we wish to clarify several points concerning the wave equation. For concreteness consider the two-dimensional equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0. \tag{1.50}$$

For reasons which will become apparent presently it is desirable to treat this equation as the limiting case of the equation with damping.

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\frac{\varepsilon}{c^2}\frac{\partial\psi}{\partial t}=0, \tag{1.51}$$

where ε is a damping factor greater than zero. We consider two types of problem:

(i) Steady-state problems in which $\psi(x,y,t)=\phi(x,y)\exp(\pm i\omega t)$;
(ii) Transient problems in which it is necessary to apply a Fourier transform in time.

A point which has not been mentioned previously arises in connexion with the Fourier transform, namely, that we can choose one of two forms:

$$\Psi=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\psi e^{\pm i\omega t}\,dt \qquad : \qquad \psi=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\Psi e^{\mp i\omega t}\,d\omega, \tag{1.52}$$

where upper and lower signs go together, and it is assumed that ω is real. In previous sections we have tacitly assumed the upper sign. On applying each of these to (1.51) in the usual way, we find

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\left\{\frac{\omega^2\pm i\varepsilon\omega}{c^2}\right\}\Psi=0, \tag{1.53}$$

where the upper and lower signs correspond to those in (1.52). Consider next a steady-state problem with ψ proportional to $\exp(\pm i\omega t)$. On substituting in (1.51) we see that for $\exp(+i\omega t)$ we obtain an equation corresponding to the *lower* sign in (1.52) and (1.53), and for $\exp(-i\omega t)$ the equation corresponds to the *upper* sign in (1.52) and (1.53).

There is no *necessity* to use the same sign in the exponential for Fourier transforms in space and time, or to choose a steady-state time factor which gives an equation corresponding to a particular sign of the Fourier transform used for space variables, but for convenience we standardize on the upper sign in (1.52) for all Fourier transforms. This has been used in previous sections, e.g. equation (1.3). Corresponding to this we use the steady-state time factor $\exp(\times i\omega t)$. This means that when we deal with the steady-state equation in the form

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\phi=0, \tag{1.54a}$$

then k^2, when damping is present, has a positive imaginary part (cf. (1.53), $k^2 = (\omega^2 + i\varepsilon\omega)/c^2$). We write

$$k = k_1 + ik_2, \qquad k_2 \geqslant 0. \tag{1.54b}$$

Next consider application of a Fourier transform in a space variable to (1.54a). Assume that (Im k) $= k_2 > 0$. For definiteness consider problem (i) of §1.1: find a solution of (1.54a) in $-\infty < x < \infty$, $y \geqslant 0$ such that $\phi = f(x)$ on $y = 0$, $-\infty < x < \infty$. To solve this problem apply a Fourier transform in x. Define

$$\Phi(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(x,y)e^{i\alpha x}\,dx.$$

From (1.54a), as in §1.1,

$$d^2\Phi(\alpha,y)/dy^2 - (\alpha^2 - k^2)\Phi(\alpha,y) = 0; \tag{1.55a}$$

$$\Phi(\alpha,y) = A(\alpha)e^{-\lambda y} + B(\alpha)e^{\lambda y}, \qquad \lambda = (\alpha^2 - k^2)^{1/2}. \tag{1.55b}$$

As in §1.1 the following question arises: In this solution there are branch points at $\alpha = \pm k$. How should the cuts be arranged in the α-plane so that (1.55b) represents a solution of (1.55a) which can be inverted to give $\phi(x,y)$?

We have already implicitly assumed:

(a) For any given y, $\Phi(\alpha,y)$ exists in a certain strip $c < \tau < d$, $-\infty < \sigma < \infty$ of the α-plane.

We make the further assumption:

(b) $\Phi(\alpha,y)$ is bounded as $y \to +\infty$ for all α in the strip $c < \tau < d$. (In our applications $\Phi(\alpha,y)$ in fact tends to zero exponentially as $y \to \infty$ but the weaker assumption just given is easy to verify when conditions on ϕ are specified and it is sufficient for our purpose.)

If we cut the α-plane by straight lines from $\alpha = \pm k$ to infinity, both of the straight lines going to infinity in the lower half-plane, then it will be necessary to invert (1.55b) for a value of τ, say τ_0, such that $\tau_0 > k_2$. But in this case it can be proved by analytic continuation, as in §1.2, that if we choose the branch of λ such that $\lambda \to |\alpha|$ as $\sigma \to +\infty$, $\alpha = \sigma + i\tau_0$, then $\lambda \to -|\alpha|$ as $\sigma \to -\infty$. Thus in (1.55b) it will not be possible to choose A and B such that Φ is bounded as $y \to \infty$ for all α on the line $\alpha = \sigma + i\tau_0$. A similar argument holds if we cut the plane by two lines both of which go to infinity in the upper half-plane or if we cut the plane by a straight line joining $+k$ to $-k$ (cf. ex. 1.4). The only remaining possibility is to cut the α-plane by a line from $+k$ to infinity in the upper half-plane and $-k$ to infinity in the lower half-plane as in Fig. 1.1. By analytic continuation we have that $\lambda \to +|\alpha|$ as $\sigma \to \pm\infty$ so that

(1.55b) represents a solution satisfying condition (b) if we take $B(\alpha) = 0$. In fact we can identify λ with γ defined at the end of §1.2 and

$$\Phi(\alpha,y) = A(\alpha)e^{-\gamma y}.$$

(This argument will not be repeated in detail again.) To determine $A(\alpha)$ we set $y = 0$ and use the boundary condition on $y = 0$. This gives

$$\Phi(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(u)e^{i\alpha u}\, du\, e^{-\gamma y}.$$

$\phi(x,y)$ is obtained by inversion. It is clear that a third condition is needed:

(c) The strips $c < \tau < d$ and $-k_2 < \tau < k_2$ overlap.

In any particular case it is necessary to verify that ϕ is such that conditions (a)–(c) are satisfied (cf. §2.2).

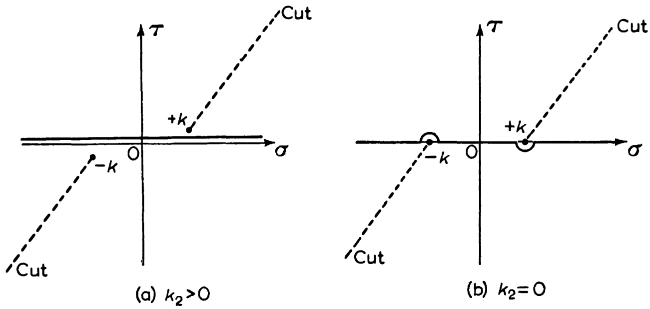

FIG. 1.4. – – – Branch cut. ——— Contour.

Finally consider what happens when $k_2 \to 0$. It will be remembered that k_2 arises from the damping factor in (1.51) and $k_2 = 0$ means that the damping is zero. When $k_2 > 0$ the contours in the α-plane are shown in Fig. 1.4a.

When $k_2 = 0$, the contour is along the real axis but by performing the limiting process $k_2 \to 0$, we see that the contour has to be indented as in Fig. 1.4b, i.e. *above* at $-k$, and *below* at $+k$. If we had used exp $(+i\omega t)$ as the steady-state time factor, or used the lower sign in the Fourier transform (1.52) then k wou.d have a negative imaginary part and by drawing diagrams similar to Fig. 1.4, it is clear that when $k_2 \to 0$ in this case, the contour along the real axis has to be indented *below* at $\alpha = -k$ and *above* at $\alpha = +k$.

The case of zero damping factor requires modifications in our conditions (a)–(c) above. Physically when $k_2 > 0$, ϕ behaves exponentially at infinity because of the finite damping—outgoing waves must decrease exponentially as we go to infinity. If no damping is present this is no longer true. Instead we need to assume some condition such as Sommerfeld's "radiation condition at infinity". In the two dimensional case this states: as $r \to \infty$, if ϕ represents an outgoing wave at infinity, and if the time factor is taken as $\exp(-i\omega t)$ then $r^{1/2}\{(\partial\phi/\partial r) - ik\phi\} \to 0$; if the time factor is $\exp(+i\omega t)$ then $r^{1/2}\{(\partial\phi/\partial r) + ik\phi\} \to 0$. These conditions must be satisfied uniformly with respect to θ.

In this monograph we shall always work with finite k_2 and obtain the case of zero damping by letting k_2 tend to zero. The Sommerfeld radiation condition will not be used directly although it always provides a useful check on the final solutions.

1.6 Contour integrals of a certain type

Consider a type of integral which occurs frequently later,

$$I = \int_{ia-\infty}^{ia+\infty} f(\alpha)e^{-i\alpha x-\gamma|y|}\,d\alpha, \tag{1.56}$$

where $\gamma = (\alpha^2 - k^2)^{1/2}$ as defined at the end of §1.2, and $-\text{Im}\, k < a < \text{Im}\, k$. $f(\alpha)$ is an analytic function such that the integral converges. Further necessary restrictions will be clear from the discussion below.

First consider a useful shift of contour in the α-plane which enables us to evaluate two special integrals explicitly. Suppose that $x = r\cos\theta$, $|y| = r\sin\theta$ where we assume without loss of generality that $0 \leqslant \theta \leqslant \pi$. Consider the contour

$$\alpha = -k\cos(\theta + it), \qquad (-\infty < t < \infty). \tag{1.57}$$

The reader will easily prove that this represents one half of a hyperbola passing through $\alpha = -k\cos\theta$ with symmetry about the line joining $-k$ to $+k$. A typical case is shown in Fig. 1.5. An easy way of visualizing the picture is to note that if k_2 is small then

$$\sigma \doteqdot -k_1\cos\theta\cosh t \quad : \quad \tau \doteqdot k_1\sin\theta\sinh t.$$

For $0 < \theta < \pi/2$, (1.57) represents the half of the hyperbola in the left-hand half-plane, and for $\pi/2 < \theta < \pi$, the half in the right-hand half-plane. If t goes from $-\infty$ to ∞, the direction in which the contour is traversed is shown by arrows in the figure. If the line joining $-k$ to $+k$ makes an angle λ with the σ-axis then the asymptotes make angles $(\lambda + \theta)$ and $(\lambda - \theta)$ with the σ-axis. We have

$$\gamma = -i(k^2 - \alpha^2)^{1/2} = -ik\sin(\theta + it), \qquad (0 \leqslant \theta \leqslant \pi), \tag{1.58}$$

where the sign of the last term is settled by analytic continuation since when $\theta = \pi/2$, $t = 0$, i.e. $\alpha = 0$, then $\gamma = -ik$. Therefore

$$-i\alpha x - \gamma|y| = ikr\{\cos\theta\cos(\theta + it) + \sin\theta\sin(\theta + it)\} = ikr\cosh t.$$

Now suppose that the contour in (1.56) can be deformed on to the appropriate half of the hyperbola. In the usual way connect the two contours by arcs at infinity. For $0 < \theta < \pi/2$ these are shown by DE and ABC in Fig. 1.5. For $\pi/2 < \theta < \pi$ we need to use the right-hand half of the hyperbola and the arcs are then obtained by extending DE and contracting ABC. Assume that the behaviour of the integrand at infinity is such that the contributions from the arcs at infinity are zero (see ex. 1.6). Assume also that no singularities of $f(\alpha)$ fall inside the

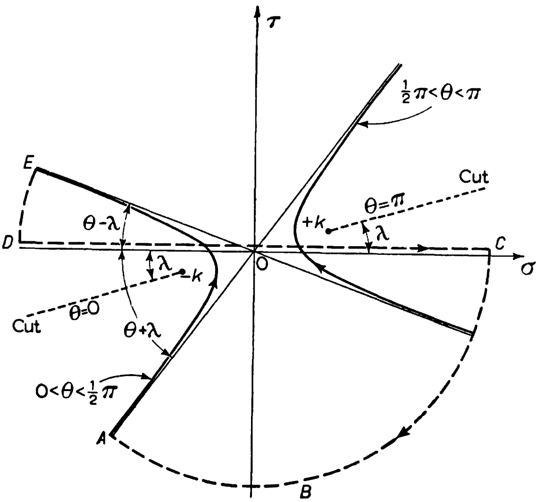

FIG. 1.5.

closed contour. (As shown in example (2) below it is easy to deal with poles inside the contour.)

Finally therefore, since $d\alpha = ik \sin(\theta + it)\,dt$, (1.56) becomes

$$I = -ik \int_{-\infty}^{\infty} f\{-k \cos(\theta + it)\} e^{ikr \cosh t} \sin(\theta + it)\,dt, \qquad (1.59)$$

where $r^2 = x^2 + y^2$.

Consider two examples

(1) $$I = \int_{ia-\infty}^{ia+\infty} \gamma^{-1} e^{-i\alpha x - \gamma|y|}\,d\alpha, \qquad (-k_2 < a < k_2). \qquad (1.60)$$

No poles are enclosed on shifting the contour. Equation (1.59) reduces to a well known representation of a Bessel function (G. N. Watson [1], p. 180, (10)):

$$I = \int_{-\infty}^{\infty} e^{ikr \cosh t}\,dt = \pi i H_0^{(1)}(kr) = \pi i\{J_0(kr) + iY_0(kr)\}. \quad (1.61)$$

Note that if k is taken to have a negative imaginary part then the integral represents $-\pi i H_0^{(2)}(kr)$.

$$\text{(2)} \qquad I = (k\cos\Theta - k)^{1/2} \int_{ia-\infty}^{ia+\infty} \frac{\exp(-i\alpha x - \gamma|y|)}{(\alpha - k)^{1/2}(\alpha - k\cos\Theta)}\, d\alpha, \tag{1.62}$$

where Θ is a constant angle, $0 < \Theta < \pi$, $-k_2 < a < k_2 \cos\Theta$, and $(\alpha - k)^{1/2}$ is defined as the branch which tends to $-ik^{1/2}$ as α tends to zero. Similarly $(k\cos\Theta - k)^{1/2} = -i(2k)^{1/2} \sin\frac{1}{2}\Theta$ as in (1.13a). The integrand has a pole at $\alpha = k\cos\Theta$. When the contour is shifted by (1.57) the pole is inside the contour when $-1 < \cos\Theta < -\cos\theta$ for any θ, $0 < \theta < \pi$, i.e. $0 < \pi - \Theta < \theta$. We find

$$I = \begin{cases} J + 2\pi i \exp\{-ikr\cos(\theta + \Theta)\}, & 0 < \pi - \Theta < \theta < \pi, \\ J & 0 < \theta < \pi - \Theta < \pi, \end{cases} \tag{1.63}$$

$$\text{where} \qquad J = 2i \sin\tfrac{1}{2}\Theta \int_{-\infty}^{\infty} \frac{\exp(ikr\cosh t)\sin\frac{1}{2}(\theta + it)}{\cos(\theta + it) + \cos\Theta}\, dt. \tag{1.64}$$

From the results in ex. 1.22 it can be shown that (1.63), (1.64) give

$$I = 2\pi^{1/2} e^{i\pi/4} [-e^{-ikr\cos(\theta-\Theta)} F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta - \Theta)\} + \\ + e^{-ikr\cos(\theta+\Theta)} F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta + \Theta)\}], \tag{1.65}$$

for all values of θ, Θ, $0 \leqslant \theta \leqslant \pi$, $0 < \Theta < \pi$, where

$$F(v) = \int_v^\infty e^{iu^2}\, du. \tag{1.66}$$

Thus the integral has been evaluated in terms of the complex Fresnel function which has been extensively tabulated.

Next consider a change of variable by means of which (1.56) can be considered from a more general point of view. For simplicity suppose that k is real and positive. The transformation $\alpha = -k\cos\beta$ where $\alpha = \sigma + i\tau$, $\beta = \mu + i\nu$ transforms the α-plane cut along the real axis from $-\infty$ to $-k$ and from $+k$ to $+\infty$ into the region $0 \leqslant \mu \leqslant \pi$, $-\infty < \nu < \infty$ in the β-plane, where we have chosen one of the infinitely many strips in the β-plane into which the α-plane is mapped. The situation is depicted in Fig. 1.6 where corresponding points bear the same letter. The contour $EBAD$ along the real axis lying above the cut $(-\infty, -k)$ and below the cut (k, ∞) transforms into the path $E'B'A'D'$. The half-hyperbola defined by $\alpha = -k\cos(\theta + it)$, $-\infty < t < \infty$, with θ a constant such that $0 < \theta < \pi$ (cf. equation (1.57) above) transforms into the line $\mu = \theta =$ constant, in the β-plane $(Q'P'R')$.

We have $\gamma = (\alpha^2 - k^2)^{1/2} = -ik\sin\beta$ and therefore the change of variable $\alpha = -k\cos\beta$ in the integral (1.56) gives

$$I = k \int_C f(-k\cos\beta) e^{ikr\cos(\beta-\theta)} \sin\beta\, d\beta, \tag{1.67}$$

where the contour C must be indented appropriately if there are indentations in the contour in the α-plane. As an example, (1.60) becomes

$$I = i \int_C e^{ikr\cos(\beta-\theta)}\, d\beta = \pi i H_0^{(1)}(kr). \tag{1.68}$$

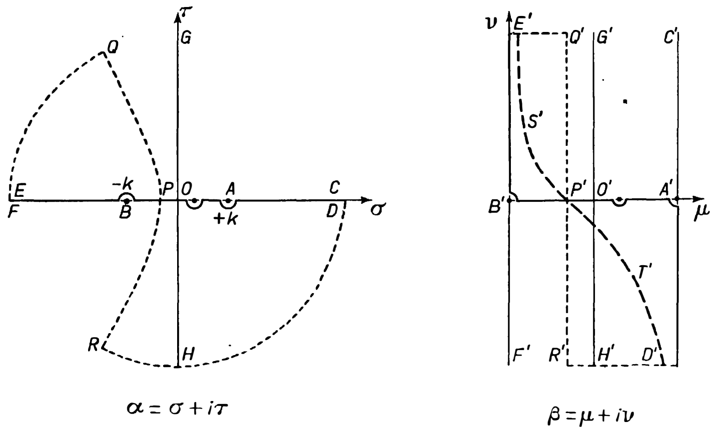

FIG. 1.6. $\alpha = -k\cos\beta$.

We now apply the method of steepest descent to (1.67). The idea of this method for approximate evaluation of the contour integral

$$J = \int_C G(\beta)\exp\{zg(\beta)\}\, d\beta, \tag{1.69}$$

is that we should deform the contour C so that it passes through a point, say ζ, such that the major part of the integral is given by the integration over the part of the deformed contour near ζ, with $G(\beta)$ slowly varying around $\beta = \zeta$. We write approximately

$$J \simeq G(\zeta) \int_D \exp\{zg(\beta)\}\, d\beta, \tag{1.70}$$

where D is the new contour into which C has been deformed: then try to integrate the new contour integral exactly or approximately.

It turns out that the criterion which fixes ζ is $dg(\beta)/d\beta = 0$. For (1.67) this gives $\sin(\beta - \theta) = 0$, i.e. $\beta = \theta$. If the real part of $g(\beta) = i\cos(\beta - \theta)$ is plotted in the neighbourhood of $\beta = \theta$ it will be found that there is a "path of steepest descent". On this path $\operatorname{Re} g(\beta)$ is a maximum at $\theta = \beta$ and decreases rapidly on either side of the maximum. A suitable path is shown by the dashed line $E'S'P'T'D'$ in Fig. 1.6 provided that the tangent to $S'P'T'$ at P' makes an angle of 135° with the μ-axis. Assume then that the contour C in (1.67), namely $E'B'A'D'$ in Fig. 1.6 can be deformed into the contour $E'S'P'T'D'$ which we denote by D. Contributions from poles can be included in the usual way and are ignored here. Then (1.70) gives

$$I \simeq kf(-k\cos\theta)\sin\theta \int_D e^{ikr\cos(\beta-\theta)}\,d\beta.$$

The contour D in this integral can now be deformed back into the original contour C in which case the integral becomes identical with (1.68) i.e.

$$I \simeq \pi kf(-k\cos\theta)\sin\theta H_0^{(1)}(kr).$$

This approximation is valid only when r is large since it is only then that a pronounced saddle point exists. Thus we can introduce the asymptotic expansion of the Hankel function to obtain the final result that as $r \to \infty$

$$I \sim (2k\pi)^{1/2}e^{-\frac{1}{4}i\pi}f(-k\cos\theta)\sin\theta r^{-1/2}e^{ikr}. \tag{1.71}$$

(A general reference on asymptotic expansions is A. Erdelyi [1]. It is often convenient to obtain a complete asymptotic expansion for integrals of the above type, in the form of a series, by using the method of steepest descent in conjunction with Watson's lemma (G. N. Watson [1], p. 235) cf. ex. 1.23(i).)

As an example apply (1.71) to (1.62). Then as $r \to \infty$, for $0 < \theta < \pi - \Theta$,

$$I \sim -2(2\pi/k)^{1/2}e^{-\frac{1}{4}i\pi}r^{-1/2}e^{ikr}\,\frac{\sin\frac{1}{2}\theta\sin\frac{1}{2}\Theta}{\cos\theta+\cos\Theta}. \tag{1.72}$$

However it is clear that this result is valid only when $(\cos\theta + \cos\Theta)$ is appreciably different from zero. From the point of view of the method of steepest descent the reason for the breakdown of the asymptotic formula (1.71) is clear, for when $(\cos\theta + \cos\Theta)$ is nearly zero, the pole at $\alpha = k\cos\Theta$ in the original integral (1.62) is very near the saddle point $\alpha = -k\cos\theta$.

We mention two methods for dealing with this situation. Suppose that we can write (1.69) as

$$J = \int_C H(\beta)h(\beta)\exp\{zg(\beta)\}\,d\beta,$$

where $H(\beta)$ varies slowly near the saddle point $\beta = \zeta$, and $h(\beta)$ has a simple pole at $\beta = \beta_0$ which is the cause of the difficulty. Suppose that

$$L_r = \int_C (\beta-\zeta)^r h(\beta)\exp\{zg(\beta)\}\,d\beta$$

can be evaluated exactly. The two methods of procedure are

(a) Set

$$J = \int_C \{H(\beta) - H(\beta_0)\}h(\beta)\exp\{zg(\beta)\}\,d\beta + H(\beta_0)L_0.$$

If $\{H(\beta) - H(\beta_0)\}h(\beta)$ varies slowly near the saddle point we can write

$$J = \{H(\zeta) - H(\beta_0)\}h(\zeta)\int_C \exp\{zg(\beta)\}\,d\beta + H(\beta_0)L_0, \qquad (1.73)$$

where the first integral can now be evaluated by steepest descents.

(b) Suppose that $H(\beta)$ can be expanded in a Taylor series about $\beta = \zeta$. Then

$$J = H(\zeta)L_0 + H'(\zeta)L_1 + \dots .$$

Under certain circumstances this gives an asymptotic series of functions of z instead of the usual asymptotic series in inverse powers of z.

The first method is the one usually used in connexion with the Wiener–Hopf technique (see also F. Oberhettinger, *J. Math. Phys.* **34** (1955), 245–255). An application is given in §5.4. The second method is discussed, for instance, by P. C. Clemmow, *Quart. J. Mech. and Applied Math.* **3** (1950), 241–256.

We conclude this section by quoting certain well-known results concerning asymptotic relations between functions and their Fourier and Laplace transforms. These will be used frequently later. Define

$$F_+(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty f(x)e^{i\alpha x}\,dx \quad ; \quad \mathscr{F}(s) = \int_0^\infty f(x)e^{-sx}\,dx.$$

Then if, for $-1 < \eta < 0$,

$$f(x) \sim Ax^\eta, \qquad \left.\begin{matrix}(x \to +0)\\(x \to \infty)\end{matrix}\right\},$$

we have

$$F_+(\alpha) \sim A(2\pi)^{-1/2}\Gamma(\eta + 1)e^{\frac{1}{2}\pi i(\eta+1)}\alpha^{-\eta-1}, \qquad \left.\begin{matrix}(\alpha \to \infty)\\(\alpha \to +0)\end{matrix}\right\}, \qquad (1.74)$$

$$\mathscr{F}_+(s) \sim A\Gamma(\eta + 1)s^{-\eta-1}, \qquad \left.\begin{matrix}(s \to \infty)\\(s \to 0)\end{matrix}\right\},$$

where upper and lower limiting processes go together. α tends to zero or infinity along paths in the upper half-plane, $(\operatorname{Im}\alpha) > 0$. s tends to zero or infinity along paths in the right half-plane, $(\operatorname{Re} s) > 0$. (Cf. theorems in A. Erdelyi [1], Chap. II, G. N. Watson [1], pp. 230, 235, and the Abelian theorems given in most books on the Laplace transform e.g. Doetsch, Van der Pol and Bremmer, Widder.)

1.7 The Wiener–Hopf procedure

The practical details of applying Fourier transforms in the examples considered later tend to obscure the essential simplicity of the complex variable procedure, which is therefore summarized in this section. The typical problem obtained by applying Fourier transforms to partial differential equations is the following. Find unknown functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ satisfying

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0, \tag{1.75}$$

where this equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ of the complex α-plane, $\Phi_+(\alpha)$ is regular in the half-plane $\tau > \tau_-$, $\Psi_-(\alpha)$ is regular in $\tau < \tau_+$, and certain information which will be specified later is available regarding the behaviour of these functions as α tends to infinity in appropriate half-planes. The functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ are given functions of α, regular in the strip. For simplicity we assume that A, B are also non-zero in the strip.

The fundamental step in the Wiener–Hopf procedure for solution of this equation is to find $K_+(\alpha)$ regular and non-zero in $\tau > \tau_-$, $K_-(\alpha)$ regular and non-zero in $\tau < \tau_+$, such that

$$A(\alpha)/B(\alpha) = K_+(\alpha)/K_-(\alpha). \tag{1.76}$$

Sometimes K_+, K_- can be found by inspection but in any case, for the A, B which occur in our applications, they can always be found with the help of theorem C of §1.3. (The precise details will become clear when specific examples are considered later.) Use (1.76) to rearrange (1.75) as

$$K_+(\alpha)\Phi_+(\alpha) + K_-(\alpha)\Psi_-(\alpha) + K_-(\alpha)C(\alpha)/B(\alpha) = 0. \tag{1.77}$$

Decompose $K_-(\alpha)C(\alpha)/B(\alpha)$ in the form

$$K_-(\alpha)C(\alpha)/B(\alpha) = C_+(\alpha) + C_-(\alpha), \tag{1.78}$$

where $C_+(\alpha)$ is regular in $\tau > \tau_-$, $C_-(\alpha)$ is regular in $\tau < \tau_+$. In the general case this can be done by using theorem B of §1.3. With the help of (1.78) rearrange (1.77) so as to define a function $J(\alpha)$ by

$$J(\alpha) = K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = -K_-(\alpha)\Psi_-(\alpha) - C_-(\alpha). \tag{1.79}$$

So far this equation defines $J(\alpha)$ only in the strip $\tau_- < \tau < \tau_+$. But the second part of the equation is defined and is regular in $\tau > \tau_-$, and the third part is defined and is regular in $\tau < \tau_+$. Hence by analytic continuation we can define $J(\alpha)$ over the whole α-plane and $J(\alpha)$ is regular in the whole α-plane. Now suppose that it can be shown that

$$\begin{aligned} |K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha)| &< |\alpha|^p \quad \text{as} \quad \alpha \to \infty,\ \tau > \tau_-; \\ |K_-(\alpha)\Psi_-(\alpha) + C_-(\alpha)| &< |\alpha|^q \quad \text{as} \quad \alpha \to \infty,\ \tau < \tau_+. \end{aligned} \tag{1.80}$$

Then by the extended form of Liouville's theorem $J(\alpha)$ is a polynomial $P(\alpha)$ of degree less than or equal to the integral part of min (p, q), i.e.

$$K_+(\alpha)\Phi_+(\alpha) + C_+(\alpha) = P(\alpha),$$
$$K_-(\alpha)\Psi_-(\alpha) + C_-(\alpha) = -P(\alpha). \tag{1.81}$$

These equations determine $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ to within the arbitrary polynomial $P(\alpha)$, i.e. to within a finite number of arbitrary constants which must be determined otherwise.

The reader who desires a concrete example can work through the solution of equation (2.24) in the next chapter, as given in equations (2.28), (2.30).

The crucial step is the finding of $K_+(\alpha)$, $K_-(\alpha)$ to satisfy (1.76). The methods for solution of partial differential equations described in this book do not all involve an equation of form (1.75) explicitly. But if we say that a method is "based on the Wiener–Hopf technique" we imply that at some stage of the solution a decomposition of form (1.76) is involved.

Miscellaneous Examples and Results I

Unless otherwise stated,

$\alpha = \sigma + i\tau$, $\gamma = (\alpha^2 - k^2)^{1/2}$, $k = k_1 + ik_2$, $(k_1, k_2 > 0)$, *as in* §1.2.

1.1 If $\gamma_1 = (\alpha^2 - k^2)^{1/2}$, $\kappa_1 = (k^2 - \alpha^2)^{1/2}$, where $k = k_1 - ik_2$, $(k_1, k_2 > 0)$, and the α-plane is cut by lines going to infinity in the upper and lower half-planes for $\alpha = -k, +k$, respectively, show that if we choose branches such that $\gamma_1 = +ik$, $\kappa_1 = k$, when $\alpha = 0$, then $\gamma_1 = +i\kappa_1$ everywhere in the cut plane (cf. (1.14) and Fig. 1.1). Show also that if the branches of the four functions $(\pm\alpha \pm k)^{1/2}$ are defined as in connexion with (1.11) then the equations corresponding to (1.12) are

$$(k - \alpha)^{1/2} = -i(\alpha - k)^{1/2} \quad : \quad (-k - \alpha)^{1/2} = +i(\alpha + k)^{1/2}.$$

1.2 Define a complex variable $s = \sigma + i\tau$, and let $k = k_1 - ik_2$, $(k_1, k_2 > 0)$. Cut the s-plane by straight lines from ik to infinity in the right half-plane ($\sigma > 0$) and from $-ik$ to infinity in the left half-plane ($\sigma < 0$). Let $k^{1/2}$ be the square root which tends to $+k_1^{1/2}$ when k_2 tends to zero. Define

$$\zeta_1 = (k + is)^{1/2} \quad : \quad \zeta_2 = (k - is)^{1/2}, \quad \zeta_1, \zeta_2 = +k^{1/2} \text{ when } s = 0.$$
$$\zeta_3 = (-k - is)^{1/2} \quad : \quad \zeta_4 = (-k + is)^{1/2}, \quad \zeta_3, \zeta_4 = ik^{1/2} \text{ when } s = 0.$$

Show that everywhere in the cut plane $\zeta_3 = i\zeta_1$, $\zeta_4 = i\zeta_2$, and if we define the branch of $(s^2 + k^2)^{1/2}$ so that $(s^2 + k^2)^{1/2} = k$ if $s = 0$, then

$$(s^2 + k^2)^{1/2} = \zeta_1\zeta_2 = -\zeta_3\zeta_4 = -i\zeta_1\zeta_4 = -i\zeta_2\zeta_3.$$

Show that if s lies on the imaginary axis, $s = i\tau$, and k is real, then

$$(s^2 + k^2)^{1/2} = (k^2 - \tau^2)^{1/2}, \qquad (|\tau| < k),$$
$$(s^2 + k^2)^{1/2} = -i(\tau^2 - k^2)^{1/2}, \qquad (|\tau| > k).$$

1.3 Prove (i) γ always has a positive real part when α lies in the strip $-k_2 < \tau < k_2$: (ii) if $(k^2 - \sigma^2)^{1/2} = p + iq$ then $|q| \geqslant k_2$, $|p| \leqslant k_1$.

1.4 Show that if $\gamma_2 = (\alpha^2 - k^2)^{1/2}$ where the α-plane is cut by a line joining $+k$ to $-k$ and $\gamma_2 \to +\sigma$ as $\alpha = \sigma \to +\infty$, then $\gamma_2 \to -|\sigma|$ as $\alpha = \sigma \to -\infty$.

1.5 Choose the branch of $f(\alpha) = \ln(\alpha - c)$, $c = a + ib$, such that $f(\alpha) \to \ln \sigma$ if $\sigma \to +\infty$, $\tau = 0$. Cut the α-plane by the line $\alpha - c = r\exp(i\theta)$, $0 \leqslant r < \infty$. Show that as $\sigma \to -\infty$, $\tau = 0$, (i) if $0 < \theta < \pi$ then $f(\alpha) \to -i\pi + \ln|\sigma|$: (ii) if $-\pi < \theta < 0$, then $f(\alpha) \to i\pi + \ln|\sigma|$.

1.6 In connexion with the discussion of integral (1.56) at the beginning of §1.6 suppose that any point on the arc ABC in Fig. 1.5 is given by $\alpha = R\exp(i\Theta)$, $0 \geqslant \Theta > -\pi + \theta + \lambda$, $0 < \theta < \pi$. By definition $\gamma \to \alpha$ as $\alpha \to \infty$ in this sector so that as $R \to \infty$

$$\begin{aligned} -i\alpha x - \gamma|y| &\to -Re^{i\Theta}(ir\cos\theta + r\sin\theta) \\ &= Rr\sin(\Theta - \theta) - iRr\cos(\Theta - \theta). \end{aligned}$$

But the restriction on Θ just given ensures that the real part of this expression is negative. Hence on ABC the exponential factor in (1.56) goes to zero exponentially as the radius of the arc tends to infinity. Similarly on DE, but we must remember that if $R \to \infty$ in $\lambda < \Theta < \pi + \lambda$ then $\gamma \to R\exp i(\Theta - \pi)$.

1.7 Let $f(\alpha)$ be an analytic function regular in the strip $\tau_- < \tau < \tau_+$, the only singularities of $f(\alpha)$ in the upper half-plane $\tau > \tau_-$ being a finite number of poles $a_1, \ldots, a_n$. Show by inspection that

$$f(\alpha) = f_+(\alpha) + \sum_{r=1}^{n} \frac{b_r}{\alpha - a_r} = f_+(\alpha) + f_-(\alpha), \text{ say,}$$

where $f_-(\alpha)$ is defined by the series and is regular in $\tau < \tau_+$, and then $f_+(\alpha)$ is defined by $\{f(\alpha) - f_-(\alpha)\}$ and is regular in $\tau > \tau_-$. Similarly if the only singularity of $f(\alpha)$ in the upper half-plane is a double pole at $\alpha = a$, show that

$$f(\alpha) = f_+(\alpha) + f_-(\alpha) \quad ; \quad f_-(\alpha) = \frac{b}{\alpha - a} + \frac{c}{(\alpha - a)^2}.$$

A similar decomposition can be applied to meromorphic functions (ex. 1.8).

1.8 *Expansion of meromorphic functions in partial fractions* (E. C. Titchmarsh [2], p. 110). A function is meromorphic in a region if it is regular in the region except for a finite number of poles. Let $f(\alpha)$ be a function whose only singularities except possibly at infinity are poles. For simplicity suppose that all the poles are simple. Let them be $\alpha_1, \alpha_2, \ldots$, where $0 < |\alpha_1| \leqslant |\alpha_2| \leqslant \ldots$, and let the residues at the poles be $a_1, a_2, \ldots$, respectively. Suppose that there exists an increasing sequence of numbers R_m such that $R_m \to \infty$ as $m \to \infty$ and such that the circles C_m with equations $|\alpha| = R_m$ pass through no pole of $f(\alpha)$ for any m. Suppose that $f(\alpha)$ is bounded on C_m for all m. Then

$$f(\alpha) = f(0) + \sum_{n=1}^{\infty} a_n \left(\frac{1}{\alpha - \alpha_n} + \frac{1}{\alpha_n}\right)$$

for all α except the poles. As an example

$$\operatorname{cosec} \alpha - \frac{1}{\alpha} = \sum_{n=-\infty}^{\infty}{}' (-1)^n \left\{\frac{1}{\alpha - n\pi} + \frac{1}{n\pi}\right\},$$

where the dash means that the term $n = 0$ is omitted.

1.9 *The infinite product theorem* (E. C. Titchmarsh [2], p. 113). If $f(\alpha)$ is an integral function of α with simple zeros at $\alpha_1, \alpha_2, \ldots$, then it can be shown that $f'(\alpha)/f(\alpha)$ is a meromorphic function of α which can be expanded in partial fractions as in ex. 1.8. On integrating this expansion the following infinite product representation of $f(\alpha)$ is found

$$f(\alpha) = f(0) \exp \{\alpha f'(0)/f(0)\} \prod_{n=1}^{\infty} \left(1 - \frac{\alpha}{\alpha_n}\right) e^{\alpha/\alpha_n}.$$

In this form the exponential factors are necessary to ensure convergence, since $\alpha_n \sim an + b$ as $n \to \infty$ in the examples considered below. If $f(\alpha)$ is an even function of α, the roots occur in pairs, $\pm\alpha_n$, and $f'(0) = 0$, so that we can write

$$f(\alpha) = f(0) \prod_{n=-\infty}^{\infty}{}' \left\{1 - \frac{\alpha}{\alpha_n}\right\} e^{\alpha/\alpha_n} = f(0) \prod_{n=1}^{\infty} \left\{1 - \frac{\alpha}{\alpha_n}\right\}^2,$$

where $\alpha_{-n} = -\alpha_n$ and the dash denotes that the term $n = 0$ is omitted. As examples of this type we have:

$$(\alpha a)^{-1} \sin \alpha a, \qquad \alpha_n a = n\pi \quad ; \qquad \cos \alpha a, \qquad \alpha_n a = (n - \tfrac{1}{2})\pi;$$
$$(\gamma a)^{-1} \sinh \gamma a = (\kappa a)^{-1} \sin \kappa a, \qquad \alpha_n a = (k^2a^2 - n^2\pi^2)^{1/2};$$
$$\cosh \gamma a = \cos \kappa a, \qquad \alpha_n a = \{k^2a^2 - (n - \tfrac{1}{2})^2\pi^2\}^{1/2}.$$

1.10 *The gamma function.* We frequently require certain well-known properties of the gamma function which it is convenient to state here for reference. $\Gamma(\alpha)$ is an analytic function, regular except when

$\alpha = 0, -1, -2, \ldots$. At these points there are simple poles, the residue at $\alpha = -n$ being $(-1)^n/(n!)$.

$$\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}\,dt \quad : \quad \Gamma(\alpha+1) = \alpha\Gamma(\alpha).$$

$$\pi\alpha/\sinh\alpha = \Gamma(1-i\alpha)\Gamma(1+i\alpha) \quad : \quad \pi/\cosh\pi\alpha = \Gamma(\tfrac{1}{2}+i\alpha)\Gamma(\tfrac{1}{2}-i\alpha).$$

$$\{\Gamma(\alpha)\}^{-1} = \alpha e^{C\alpha}\prod_{n=1}^{\infty}\{1+(\alpha/n)\}e^{-\alpha/n},$$

where $C = 0{\cdot}5772\ldots$, the Euler constant. Stirling's formula states that

$$\Gamma(\alpha) \sim e^{-\alpha}\alpha^{\alpha-\frac{1}{2}}(2\pi)^{1/2}\{1+(12\alpha)^{-1}+\ldots\} \quad \text{as} \quad \alpha\to\infty,\ |\arg\alpha|<\pi.$$

$$\Gamma(\alpha+b)/\Gamma(\alpha) \sim \alpha^b \quad \text{as} \quad \alpha\to\infty,\ |\arg\alpha|<\pi.$$

$$\begin{aligned}\Gamma(\alpha)/\Gamma(2\alpha) &= 2e^{C\alpha}\prod_{n=1}^{\infty}\left\{1+\frac{\alpha}{(n-\frac{1}{2})}\right\}e^{-\alpha/(n-\frac{1}{2})},\\ &= \pi^{1/2}2^{1-2\alpha}/\Gamma(\alpha+\tfrac{1}{2}) \sim 2^{\frac{1}{2}-2\alpha}e^{\alpha+\frac{1}{2}}\alpha^{-\alpha}, \quad \text{as} \quad \alpha\to\infty.\end{aligned}$$

$$\prod_{n=1}^{\infty}\left\{1+\frac{\alpha}{an+b}\right\}e^{-\alpha/an} = e^{-C\alpha/a}\Gamma\left(\frac{b}{a}+1\right)\Big/\Gamma\left(\frac{\alpha}{a}+\frac{b}{a}+1\right).$$

$$\sum_{n=1}^{\infty}\left\{\frac{1}{n}-\frac{1}{n+z}\right\} = C+\Psi(z+1) \quad : \quad \Psi(z) = \Gamma'(z)/\Gamma(z).$$

$$\Gamma'(1) = -C \quad : \quad \Gamma'(\tfrac{1}{2}) = -\pi^{1/2}(C+2\ln 2).$$

In all these expressions C is Euler's constant $0{\cdot}5772\ldots$.

1.11 The product decomposition of certain functions can be written down directly in terms of gamma functions e.g.

$$K(\alpha) = K_+(\alpha)\,K_-(\alpha) = (\pi\alpha)\coth\pi\alpha,$$

$$K_+(\alpha) = \pi^{1/2}\Gamma(1-i\alpha)/\Gamma(\tfrac{1}{2}-i\alpha) \quad : \quad K_-(\alpha) = K_+(-\alpha).$$

$K_+(\alpha)$ is regular and non-zero in $(\mathrm{Im}\,\alpha) > -\frac{1}{2}$, and $|K_+(\alpha)| \sim |\alpha|^{1/2}$ as $\alpha\to\infty$ in an upper half-plane.

1.12 The following is a more general form of theorem C of §1.3, based on the original theorem of N. Wiener and E. Hopf [1]. (See also E. C. Titchmarsh [1], p. 339.) Let $K(\alpha)$ be an analytic function, regular in the strip $\tau_- < \tau < \tau_+$ such that $K(\alpha)\to\exp(i\mu)$ as $\sigma\to+\infty$, and $K(\alpha)\to\exp(i\nu)$ as $\sigma\to-\infty$ in the strip (μ,ν real); $|1-|K(\alpha)|| < C|\sigma|^{-p}$, $p>0$, as $|\sigma|\to\infty$ in the strip, and the inequality holds uniformly in any interior strip (c,d), where $\tau_- < c < \tau < d < \tau_+$. Then in any such interior strip $K(\alpha)$ has only a finite number of zeros. If those are $\alpha_1, \alpha_2, \ldots, \alpha_n$ we can write

$$K(\alpha) = K_+(\alpha)K_-(\alpha)(\alpha-\alpha_1)\ldots(\alpha-\alpha_n), \tag{a}$$

where $K_+(\alpha)$, $K_-(\alpha)$ are regular and free from zeros in $\tau \geqslant c$, $\tau \leqslant d$ respectively, and in their respective half-planes of regularity

$$C_1|\alpha|^{-\frac{1}{2}n-\lambda} < |K_+(\alpha)| < C_2|\alpha|^{-\frac{1}{2}n-\lambda}, \qquad \alpha \to \infty,\ \tau > c;$$
$$C_3|\alpha|^{-\frac{1}{2}n+\lambda} < |K_-(\alpha)| < C_4|\alpha|^{-\frac{1}{2}n+\lambda}, \qquad \alpha \to \infty,\ \tau < d,$$

where C_1 to C_4 are non-zero constants and $\lambda = (2\pi)^{-1}(\mu - \nu)$.

The proof is left to the reader with the hint that one considers the function

$$F(\alpha) = e^{-i\mu}K(\alpha)\,\frac{(\alpha - it_-)^{\frac{1}{2}n+\lambda}(\alpha - it_+)^{\frac{1}{2}n-\lambda}}{(\alpha - \alpha_1)\ldots(\alpha - \alpha_n)},$$

where $t_- \leqslant \tau_-$, $t_+ \geqslant \tau_+$. The factor on the bottom has been inserted to ensure that $F(\alpha)$ has no zeros in the strip and the powers in n ensure that $|F(\alpha)| \to 1$ as $|\sigma| \to \infty$ in the strip. If the α-plane is cut from it_+ to $i\infty$ and it_- to $-i\infty$ and λ is chosen as $(2\pi)^{-1}(\mu - \nu)$ then the branches can be arranged so that $F(\alpha) \to 1 = \exp(i0)$ as $\sigma \to \pm\infty$ in the strip (cf. ex. 1.5).

In theorem C of §1.3 and above, it is assumed that $|K(\alpha)| \to 1$ as $\sigma \to \pm\infty$ in the strip. If $|K(\alpha)| \sim |\sigma|^p$ as $\sigma \to \pm\infty$ in the strip the simplest procedure is to apply the theorems to

$$L(\alpha) = K(\alpha)(\alpha - it_-)^{-p+\eta}(\alpha - it_+)^{-\eta},$$

where $t_- \leqslant \tau_-$, $t_+ \geqslant \tau_+$, and η is a suitable constant. If K is even it is convenient to choose $\eta = \frac{1}{2}p$ and then $|K_+(\alpha)| \sim |\alpha|^{\frac{1}{2}p}$ as $\alpha \to \infty$ in $\tau > \tau_-$. It is also possible to apply the decomposition theorems to such $K(\alpha)$ directly but the integrals which occur must be understood in the sense

$$\lim_{A\to\infty} \int_{ia-A}^{ia+A} \frac{\ln K(\zeta)}{\zeta - \alpha}\,d\zeta.$$

(Cf. H. Levine and J. Schwinger [1], §V and Appendix B.)

1.13 *The generalized Fourier integral.* The following theorem overcomes the restriction $\tau_- < \tau_+$ of theorem A, §1.4 (E. C. Titchmarsh [1], p. 4). Assume that

$$F_+(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty f(x)e^{i\alpha x}\,dx \quad : \quad F_-(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^0 f(x)e^{i\alpha x}\,dx,$$

exist, the former for $\tau > \tau_-$, the latter for $\tau < \tau_+$ where now the only restriction is that τ_-, τ_+ are finite. Then for $a > \tau_-$, $b < \tau_+$,

$$f(x) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty+ia}^{\infty+ia} F_+(\alpha)e^{-i\alpha x}\,d\alpha + \frac{1}{(2\pi)^{1/2}}\int_{-\infty+ib}^{\infty+ib} F_-(\alpha)e^{-i\alpha x}\,d\alpha.$$

The first integral equals $f(x)$ for $x > 0$ and is zero for $x < 0$: the second is zero for $x > 0$ and equals $f(x)$ for $x < 0$. This theorem can be regarded as a superposition of two cases of theorem C, §1.4.

1.14 Prove the Mellin transform analogue of theorem C, §1.4: Suppose that $F(s)$, $s = \sigma + i\tau$, is regular in $\sigma > \sigma_-$ and $s^{-1}F(s) \to 0$ as $r \to \infty$ where r is defined by $s - \sigma_- - \varepsilon = r \exp(i\theta)$, for arbitrary $\varepsilon > 0$, the limit being approached uniformly in $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Then the solution of the integral equation

$$F(s) = \int_0^1 f(\rho)\rho^{s-1}\,d\rho$$

is
$$f(\rho) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)\rho^{-s}\,ds, \qquad (\sigma > \sigma_-). \tag{a}$$

Similarly if $F(s)$ is regular in $\sigma < \sigma_+$ and satisfies suitable conditions at infinity, the solution of

$$F(s) = \int_1^\infty f(\rho)\rho^{s-1}\,d\rho$$

is (a) with $\sigma < \sigma_+$.

1.15 If

$$\frac{d^2\Phi(y)}{dy^2} - \gamma^2\Phi(y) = f(y), \qquad \gamma = (\alpha^2 - k^2)^{1/2},$$

then (i) the solution in $(-\infty < y < \infty)$ such that $\Phi \to 0$ as $y \to \pm\infty$ is:

$$\Phi(y) = -\frac{1}{2\gamma}\int_{-\infty}^{\infty} f(\eta)e^{-\gamma|y-\eta|}\,d\eta:$$

(ii) solutions in $(0 \leqslant y < \infty)$ such that $\Phi \to 0$ as $y \to +\infty$ are:

$$\Phi(y) = -\frac{1}{2\gamma}\int_0^\infty f(\eta)\{e^{-\gamma|\eta-y|} \pm e^{-\gamma|\eta+y|}\}\,d\eta,$$

where the upper sign gives a solution with $d\Phi/dy = 0$ on $y = 0$, and the lower sign gives $\Phi = 0$ on $y = 0$.

1.16 For convenience we occasionally use the Dirac delta-function, $\delta(x - \xi)$. The fundamental property of this function is

$$\int_a^b f(x)\delta(x-\xi)\,dx = \begin{cases} f(\xi) & \text{if} \quad a < \xi < b, \\ 0 \quad \text{if} \quad a, b > \xi & \text{or} \quad a, b < \xi. \end{cases}$$

As an example we find a solution of

$$\phi_{xx} + \phi_{yy} + k^2\phi = -4\pi\delta(x - x_0)\delta(y - y_0)$$

in free space, such that ϕ represents an outgoing wave at infinity. The expression on the right-hand side represents a line source at (x_0,y_0). A transform in x gives

$$d^2\Phi/dy^2 - \gamma^2\Phi = -2(2\pi)^{1/2}e^{i\alpha x_0}\delta(y - y_0).$$

From ex. 1.15 or from first principles the solution such that $\Phi \to 0$ as $y \to \pm\infty$ is

$$\Phi = (2\pi)^{1/2}\gamma^{-1}e^{i\alpha x_0 - \gamma|y-y_0|}.$$

Inversion and (1.61) gives

$$\phi(x,y) = \pi i H_0^{(1)}(kR), \qquad R^2 = (x - x_0)^2 + (y - y_0)^2.$$

1.17 Show that if in ex. 1.16 we assume that k has a negative imaginary part then the solution is

$$\phi(x,y) = -\pi i H_0^{(2)}(kR) = -\pi i\{J_0(kR) - iY_0(kR)\}.$$

1.18 Consider, in free space,

$$\psi_{xx} + \psi_{yy} - c^{-2}\psi_{tt} - \varepsilon c^{-2}\psi_t = -4\pi s(x,y,t),$$

where the expression on the right represents a source function. Apply a triple Fourier transform in x, y, t (all limits from $-\infty$ to $+\infty$),

$$\Psi = \frac{1}{(2\pi)^{3/2}}\iiint \psi e^{i(\alpha x+\beta y+\omega t)}\, dx\, dy\, dt.$$

In the usual way we find

$$\psi = \frac{4\pi}{(2\pi)^{3/2}}\iiint \frac{S(\alpha,\beta,\omega)e^{-i(\alpha x+\beta y+\omega t)}}{\alpha^2 + \beta^2 - c^{-2}(\omega^2 + i\varepsilon\omega)}\, d\alpha\, d\beta\, d\omega.$$

If we wish to evaluate the integral in α, the poles are given by

$$\alpha = \pm c^{-1}\{(\omega^2 - \beta^2c^2) + i\varepsilon\omega\}^{1/2}.$$

In the limit as ε tends to zero, if $\omega^2 > \beta^2c^2$ so that these roots lie on the real axis, the contour must be indented as in Fig. 1.4b. But if we wish to evaluate the integral in ω instead of the integral in α, the integrand has poles when

$$\omega^2 + i\varepsilon\omega - c^2(\alpha^2 + \beta^2) = 0, \quad \text{i.e.} \quad \omega = -\tfrac{1}{2}i\varepsilon \pm \{c^2(\alpha^2 + \beta^2) - \tfrac{1}{4}\varepsilon^2\}^{1/2}.$$

For $\varepsilon > 0$ both of these lie in the lower half-plane. If $\varepsilon \to 0$ the contour in the ω-plane is along the real axis indented by semi-circles passing *above* both the points $\omega = \pm c(\alpha^2 + \beta^2)^{1/2}$.

1.19 Show that if in (1.56) the integrand contains $\exp(i\alpha x)$ instead of $\exp(-i\alpha x)$ then the appropriate equations corresponding to (1.57), (1.58) are

$$\alpha = k\cos(\theta + it) \quad : \quad \gamma = -ik\sin(\theta + it).$$

1.20 Show that if $k = k_1 - ik_2$, $k_2 > 0$, in (1.56) then the equations corresponding to (1.57), (1.58) are

$$\alpha = k\cos(\theta + it) \quad : \quad \gamma = +ik\sin(\theta + it).$$

In Fig. 1.5 the cuts will lie in the second and fourth quadrants and the arcs at infinity will be drawn differently. Show that the equation corresponding to (1.71) is

$$I \sim (2k\pi)^{1/2} e^{\frac{1}{4}i\pi} f(k\cos\theta)\sin\theta r^{-1/2} e^{ikr}.$$

1.21 The asymptotic behaviours of the Hankel functions as $r \to \infty$ are

$$H_0^{(1)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr - \frac{1}{4}\pi)} \quad : \quad H_0^{(2)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} e^{-i(kr-\frac{1}{4}\pi)}.$$

Hence $H_0^{(1)}(kr)\exp(-i\omega t)$, $H_0^{(2)}(kr)\exp(+i\omega t)$ represent outgoing waves at infinity.

1.22 In order to evaluate (1.64) note that

$$\frac{\sin\frac{1}{2}(\theta + it)\sin\frac{1}{2}\Theta}{\cos(\theta + it) + \cos\Theta} = \frac{1}{4}\left\{\frac{1}{\cos\frac{1}{2}(it + \theta + \Theta)} - \frac{1}{\cos\frac{1}{2}(it + \theta - \Theta)}\right\},$$

so that

$$J = -2iH(\theta - \Theta) + 2iH(\theta + \Theta), \tag{a}$$

where

$$H(\lambda) = \frac{1}{4}\int_{-\infty}^{\infty} \frac{\exp(ikr\cosh t)}{\cos\frac{1}{2}(it + \lambda)}\,dt = \int_0^{\infty} \frac{\cos\frac{1}{2}\lambda\cosh\frac{1}{2}t\exp(ikr\cosh t)}{\cos\lambda + \cosh t}\,dt.$$

Make the substitution $\tau = \sinh\frac{1}{2}t$, $\cosh t = 1 + 2\tau^2$. This gives

$$H(\lambda) = e^{-ikr\cos\lambda}\cos\tfrac{1}{2}\lambda\int_0^{\infty}\frac{\exp\{2ikr(\tau^2 + \cos^2\frac{1}{2}\lambda)\}}{\tau^2 + \cos^2\frac{1}{2}\lambda}\,d\tau.$$

To evaluate the integral write

$$\frac{d}{dr}\{e^{ikr\cos\lambda}H(\lambda)\} = e^{3i\pi/4}\cos\tfrac{1}{2}\lambda(\pi k/2r)^{1/2}\exp(2ikr\cos^2\tfrac{1}{2}\lambda),$$

where the integral that occurs on differentiation has been evaluated by the well-known result

$$\int_0^{\infty} e^{ia\tau^2}d\tau = \tfrac{1}{2}(\pi/a)^{1/2}e^{i\pi/4}.$$

Integration gives

$$H(\lambda) = \pi^{1/2}e^{-i\pi/4}e^{-ikr\cos\lambda}G(\lambda),$$

where

$$G(\lambda) = \begin{cases} F\{(2kr)^{1/2}\cos\frac{1}{2}\lambda\}, & \text{if } \cos\frac{1}{2}\lambda > 0, \text{ e.g. } -\pi < \lambda < \pi, \\ -F\{-(2kr)^{1/2}\cos\frac{1}{2}\lambda\}, & \text{if } \cos\frac{1}{2}\lambda < 0, \text{ e.g. } \pi < \lambda < 2\pi, \end{cases}$$

$$F(v) = \int_v^{\infty} e^{iu^2}\,du.$$

The constants of integration have been arranged so that $G \to 0$ as $r \to \infty$. Note that

$$F(v) + F(-v) = \pi^{1/2}e^{i\pi/4}. \tag{b}$$

Also $0 \leqslant \theta \leqslant \pi$, $0 < \Theta < \pi$, so that $-\pi < \theta - \Theta < \pi$ in all cases. For $(\theta + \Theta)$ there are two cases depending on whether $(\theta + \Theta)$ is greater or less than π. If $(\theta + \Theta) > \pi$, on using (b), it can be shown that

$$H(\theta + \Theta) = -\pi e^{-ikr\cos(\theta+\Theta)} + \\ + \pi^{1/2}e^{-i\pi/4}e^{-ikr\cos(\theta+\Theta)}F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta + \Theta)\}.$$

Equation (1.65) can be obtained from these results on using (1.63) and (a) above.

From the standard result $F(v) \sim i(2v)^{-1}\exp(iv^2)$ as $v \to \infty$ we can show that

$$H(\lambda) \sim \tfrac{1}{2}(\pi/2kr)^{1/2}(\cos\tfrac{1}{2}\lambda)^{-1}e^{i(kr+\frac{1}{4}\pi)}, \qquad \text{as } r \to \infty.$$

Then the result (1.72) can be deduced from the exact evaluation (1.65).

1.23 Two other methods for deducing the asymptotic behaviour of (1.56) may be useful. We assume that $g(x) \sim Ax^{2\mu-1}$ as $x \to 0$.

(i) In conjunction with the method of steepest descent we can use the result that under certain conditions

$$\int_0^\infty g(x)e^{-px^2}\,dx \sim \tfrac{1}{2}A\Gamma(\mu)p^{-\mu}, \qquad \text{as } p \to \infty.$$

This can be applied to (1.67) if we make the substitution $\cos(\beta - \theta) = 1 + i\tau^2$.

(ii) In conjunction with the method of stationary phase we can use the lemma that under certain conditions

$$\int_0^\infty g(x)e^{iqx^2}\,dx \sim \tfrac{1}{2}A\Gamma(\mu)q^{-\mu}e^{i\mu\pi/2}.$$

This can be applied to (1.59) with the substitution $\sinh\frac{1}{2}t = \tau$.

In each case we can verify that the formula (1.71) is obtained for the asymptotic expansion of (1.56). (Cf. A. Erdelyi [1], Chap. II, G. N. Watson [1], pp. 230, 235, and the Abelian theorems at the end of §1.6).

1.24 Show that

$$\int_C \frac{f(\alpha)}{\alpha - \xi}\,d\alpha = \pm\pi i f(\xi) + P\int_a^b \frac{f(\sigma)}{\sigma - \xi}\,d\sigma,$$

where a, b, ξ are real, $a < \xi < b$, and the contour C goes from a to b along the real axis and avoids the pole at $\alpha = \xi$ by an indentation below the real axis for the plus sign and above the real axis for the minus sign. The integral on the right-hand side is a Cauchy principal value:

$$P\int_a^b \frac{f(\sigma)}{\sigma-\xi}\,d\sigma = \lim_{\varepsilon\to 0}\left\{\int_{\xi+\varepsilon}^{b} \frac{f(\sigma)}{\sigma-\xi}\,d\sigma + \int_a^{\xi-\varepsilon} \frac{f(\sigma)}{\sigma-\xi}\,d\sigma\right\}.$$

The above is a special case of the Plemelj formulae.

1.25 An ingenious suggestion has been made by W. S. Ament [1] for factorization of a function $K(\alpha)$ in the form $K_+(\alpha)K_-(\alpha)$. We have seen in §1.3 that if the only singularities of $K(\alpha)$ are poles the factorization is comparatively easy, but if branch points are present we need to use a general method which may involve complicated integrations. If, however, we can invent a transformation of the α-plane, say $\alpha = \chi(\beta)$ such that the *transformed function has no branch points in the β-plane*, then it may be possible to decompose the function in the β-plane by elementary methods. Consider an example treated in §1.3, $K(\alpha) = \exp(\gamma a)$. This was reduced to the decomposition of $g(\alpha) = (\alpha^2 - k^2)^{-1/2}$ into the *sum* of two functions regular in upper and lower half-planes. Assume k real and use the transformation $\alpha = -k\cos\beta$ considered in §1.6 in connexion with (1.67). Then $g(\alpha) = +i(k\sin\beta)^{-1}$. The transformed plane is shown in Fig. 1.6 where the contour is indented below at $\alpha = k$, i.e. $\beta = \pi$, and above at $\alpha = -k$, i.e. $\beta = 0$. There are now only simple poles at these points and the factorization can be carried out by inspection (cf. ex. 1.7):

$$\frac{1}{\sin\beta} = \frac{\beta}{\pi\sin\beta} + \frac{\pi-\beta}{\pi\sin\beta}.$$

If $\beta = \mu + i\nu$, the second function on the right-hand side has no singularities in $0 \leqslant \mu \leqslant \pi$, $\nu \geqslant 0$ above and on the contour. The first function has no singularities in $0 \leqslant \mu \leqslant \pi$, $\nu \leqslant 0$, below and on the contour. Transform back into the α-plane.

$$\gamma^{-1} = (\pi\gamma)^{-1}\operatorname{arc\,cos}(-\alpha/k) + (\pi\gamma)^{-1}\operatorname{arc\,cos}(\alpha/k),$$

which agrees with the result at the end of §1.3.

In some cases it may be useful to consider poles in the whole of the β-plane instead of only the strip into which the α-plane transforms. W. S. Ament [1] has indicated in a general way how the method can be applied to

$$K(\alpha) = \{\gamma^{-1} + a(b^2 - \alpha^2)^{-1}\}, \qquad (a,\ b \text{ are constants});$$
$$K(\alpha) = \{c(\alpha^2 - k^2)^{1/2} + (\alpha^2 - l^2)^{1/2}\}, \quad (c,\ k,\ l \text{ are constants}).$$

This last case requires a transformation involving elliptic functions.

CHAPTER II

BASIC PROCEDURES: HALF-PLANE PROBLEMS

2.1 Introduction

As already stated at the end of §1.7, when we say that a method of solution of a partial differential equation is "based on the Wiener–Hopf technique" we mean that at some stage of the solution a function $K(\alpha)$, $\alpha = \sigma + i\tau$, is given which is regular and non-zero in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, of the complex α-plane. The method of solution requires that this function be decomposed in the form

$$K(\alpha) = K_+(\alpha)K_-(\alpha), \tag{2.1}$$

where K_+ is regular and non-zero in $\tau > \tau_-$, K_- is regular and non-zero in $\tau < \tau_+$.

In order to illustrate some basic procedures for using (2.1) to solve problems involving partial differential equations, this chapter begins with an extended discussion of the Sommerfeld half-plane diffraction problem. The solution will be obtained by three methods, all depending on the Wiener–Hopf technique. In §2.2 the problem is solved by a straightforward method due to D. S. Jones [3]. In §2.3 the problem is solved by a dual integral equation approach. In §2.4 the problem is formulated as an integral equation which is solved in §2.5.

We give pride of place to Jones's method because of its simplicity. It provides a routine procedure for problems which can be solved exactly by the Wiener–Hopf technique. An appropriate transform is applied directly to the partial differential equation. No integral equation is formulated. The method will be used extensively in this book for both exact and approximate solutions of various problems.

A dual integral equation method which will also be used later is described in §2.3. It will be particularly useful in connexion with problems involving general boundary conditions, and also in certain approximate methods of solution. Although it depends on an apparently special method of procedure, it will be found that this is no limitation in this book. The method does not seem to have been used previously.

The integral equation method of §§2.4, 2.5 has been used extensively in the literature. The Wiener–Hopf technique was originally invented to solve an integral equation of the type

$$\int_0^\infty f(\xi)K(x-\xi)\,d\xi = g(x), \qquad (0 < x < \infty), \tag{2.2}$$

where K and g are given, f is to be found. The problems encountered in this chapter and the next can be formulated in terms of integral equations of this type and it was in fact for this reason that Schwinger and Copson noted that these problems could be solved by the Wiener–Hopf technique. Much of the literature is written in terms of the integral equation approach, the integral equation being formulated by a Green's function method. We shall give a detailed description of the formulation and solution of the Sommerfeld half-plane diffraction problem from this point of view which should enable the reader to follow the method as used in the literature. Our treatment is also designed to illustrate the connexion between transform and Green's function methods. However, we shall not make much use of the Green's function-integral equation method in this book since Jones's method is equivalent and more mechanical.

This book could have been written in several different ways. We could have regarded the above Green's function-integral equation method as fundamental; or we could have made much more use of a different dual integral equation approach associated with work of Vajnshtejn, Karp and Clemmow (§4.3); or we could have based the whole discussion on the "Hilbert problem" instead of the Wiener–Hopf technique, which has certain advantages (§4.2). However the point we wish to emphasize here is that the principal method used in this book is Jones's method described in §2.2. Once the reader has mastered this section he should be in a position to read most of the book, providing that the relevant parts of Chapter I have been understood.

Now consider the half-plane problem which we are to study in detail. Steady-state waves with time factor $\exp(-i\omega t)$ exist in two-dimensional (x,y) space. There is a rigid boundary along the negative x-axis and waves

$$\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta), \qquad 0 < \Theta < \pi, \tag{2.3}$$

are incident on the screen (Fig. 2.1). Denote the total velocity potential at any point by ϕ_t and define ϕ by the equation

$$\phi_t = \phi + \phi_i. \tag{2.4}$$

Then ϕ satisfies

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+k^2\phi=0, \tag{2.5}$$

where k has a positive imaginary part (§1.5). The following conditions apply:

(i) $\partial\phi_t/\partial y=0$ on $y=0$, $-\infty<x\leqslant 0$, so that

$$\partial\phi/\partial y=ik\sin\Theta\exp(-ikx\cos\Theta),\quad y=0,\quad -\infty<x\leqslant 0. \tag{2.6}$$

(ii) $\partial\phi_t/\partial y$ and therefore $\partial\phi/\partial y$ are continuous on $y=0$, $-\infty<x<\infty$.

(iii) ϕ_t and therefore ϕ are continuous on $y=0$, $0<x<\infty$.

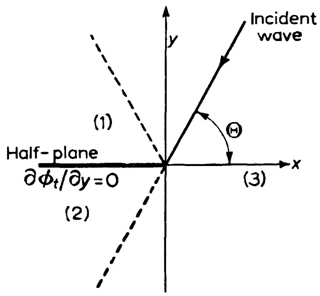

FIG. 2.1.

In addition to these conditions we need certain assumptions concerning the behaviour of ϕ at infinity and near the edge of the screen at the origin. When considering ϕ we need to subtract out the incident wave by definition (2.4). If $x=r\cos\theta$, $y=r\sin\theta$ then as r tends to infinity the (x,y) plane can be divided into three regions as shown in Fig. 2.1:

(1) Region (1) in which ϕ consists of a diffracted wave and a reflected wave.

(2) Region (2) in which ϕ consists of a diffracted wave minus an incident wave.

(3) Region (3) in which ϕ consists of only a diffracted wave.

As r tends to infinity, since the diffracted wave can be regarded as produced by line sources in the screen, the diffracted waves behave, for any fixed θ $(-\pi<\theta<\pi)$, as (ex. 1.21):

$$\lim_{r\to\infty} C_1H_0^{(1)}(kr)\sim C_2r^{-1/2}e^{+ik_1r}e^{-k_2r},$$

$$(k=k_1+ik_2,\qquad k_1>0,\qquad k_2>0;\qquad C_1,\,C_2 \text{ constants}).$$

In region (1) the reflected wave is given by $\exp(-ikx\cos\Theta + iky\sin\Theta)$. From these statements we deduce:

(iv) For any fixed y, $y \geqslant 0$ or $y \leqslant 0$

(a) $|\phi| < C_3 \exp(k_2 x \cos\Theta - k_2|y|\sin\Theta)$ for $-\infty < x < -|y|\cot\Theta$.

(b) $|\phi| < C_4 \exp\{-k_2(x^2+y^2)^{1/2}\}$ for $-|y|\cot\Theta < x < \infty$.

Finally near the edge of the screen at the origin we assume

$$\begin{array}{llllll}
\text{(v)} & (\partial\phi_t/\partial y) \to C_5 x^{-1/2} & \text{as} & x \to +0 & \text{on} & y = 0, \\
& \phi_t \to C_6 & \text{as} & x \to +0 & \text{on} & y = 0, \\
& \phi_t \to C_7 & \text{as} & x \to -0 & \text{on} & y = +0, \\
& \phi_t \to C_8 & \text{as} & x \to -0 & \text{on} & y = -0.
\end{array}$$

(For convenience we assume more information than is strictly necessary. cf. §2.6.) In these conditions, the expression $x \to -0$, $y = +0$, for example, means that x tends to zero through negative values of x on the upper side of the screen. The C_i are constants which need not be known explicitly. It can be considered that the "edge conditions" (v) are given by the physics of the problem. They play an important rôle in the solution since they are concerned with the uniqueness of the result. A discussion in general terms is given in §2.6.

Many methods are available for solution of the problem we have just stated. One virtue of the Wiener–Hopf technique is that the method used to solve this simple problem generalizes immediately to deal with problems involving more complicated conditions on the half-planes, and more complicated equations. (The method also generalizes to deal with parallel half-planes: this is discussed in Chapter III.)

Connected with the problem which has just been stated we have the following variants and generalizations.

(a) The "absorbent" half-plane with boundary condition $\phi_t = 0$ instead of $\partial\phi_t/\partial y = 0$ on the screen. (See ex. 2.3.)

(b) The imperfectly reflecting or imperfectly absorbent half-plane with $p\,\partial\phi_t/\partial y + q\phi_t = 0$ on the screen where p and q are suitable constants. (See §2.9.)

(c) The half-plane between two different media. Suppose that the potential is ϕ_1 in $x \geqslant 0$ and ϕ_2 in $x \leqslant 0$, where ϕ_1 and ϕ_2 satisfy (2.5) and

$$\partial\phi_1/\partial y = \partial\phi_2/\partial y = 0 \text{ on } y = 0, \quad 0 \leqslant x < \infty, \tag{2.7a}$$

$$\left.\begin{aligned} p_1\,\partial\phi_1/\partial y &= p_2\,\partial\phi_2/\partial y \\ \phi_1 &= \phi_2 \end{aligned}\right\} \text{ on } y = 0, \quad -\infty < x < 0. \tag{2.7b}$$

It is required to find potentials satisfying these conditions with an incident wave (2.3). (See ex. 2.12(B).)

Another type of generalization occurs when the incident plane wave (2.3) is replaced by a source distribution. For example our original problem would become: Find a solution of the equation

$$\phi_{xx} + \phi_{yy} + k^2\phi = s(x,y), \qquad -\infty < x,\ y < \infty, \tag{2.8}$$

where $\partial\phi/\partial y = 0$ on $y = 0$, $-\infty < x \leqslant 0$, and ϕ represents an outgoing wave at infinity. (See ex. 2.2.)

We shall show that the Sommerfeld problem stated in detail at the beginning of this discussion is equivalent to the following: Find a solution of the two-dimensional steady-state wave equation in $y \geqslant 0$, $-\infty < x < \infty$ such that

$$\left.\begin{aligned} \partial\phi/\partial y &= ik \sin\Theta \exp(-ikx\cos\Theta), & y &= 0, & -\infty < x \leqslant 0 \\ \phi &= 0 \qquad\qquad\qquad\qquad\qquad , & y &= 0, & 0 < x < \infty \end{aligned}\right\},$$

and ϕ represents an outgoing wave at infinity.

Connected with this formulation there are two generalizations: Find a solution of (2.5) in $y \geqslant 0$, $-\infty < x < \infty$ under the following boundary conditions.

A. No incident wave, but

$$\left.\begin{aligned} \phi &= f(x), & y &= 0, & 0 < x < \infty, \\ \partial\phi/\partial y &= g(x), & y &= 0, & -\infty < x < 0. \end{aligned}\right\} \tag{2.9}$$

This was considered in §1.1 and will be solved in §2.8.

B. Incident wave (2.3) with

$$\left.\begin{aligned} p(\partial\phi/\partial y) - q\phi &= 0, & y &= 0, & 0 < x < \infty, \\ r(\partial\phi/\partial y) - s\phi &= 0, & y &= 0, & -\infty < x < 0. \end{aligned}\right\} \tag{2.10}$$

(See ex. 2.12(A). cf. §2.9.)

Another variant occurs when the equation is different from the two-dimensional steady-state wave equation. Some equations similar to (2.5) are mentioned in ex. 2.12(C)(D)(E). Diffraction of an electromagnetic wave in three dimensions by a half-plane is considered in ex. 2.14. Diffraction of a wave by a half-plane in an elastic medium and a viscous medium are mentioned in exs. 2.15, 2.16 respectively.

All the problems considered in this chapter involve only the half planes $-\infty < x < 0$, $0 < x < \infty$, for $y = 0$. Generalizations involving more complicated geometry are considered in Chapter III.

2.2 Jones's method

In this section the Sommerfeld half-plane diffraction problem formulated in §2.1 is solved by means of a straightforward method due to D. S. Jones [3]. (Our conventions differ from those of Jones since he uses the Laplace transform, and assumes that k has a negative imaginary part. Also in [3] he solves the problem $\phi_t = 0$ on $x > 0$, $y = 0$. But these are details: see exs. 2.1, 2.3.)

We introduce the Fourier transforms

$$\Phi(\alpha,y) = \Phi_+(\alpha,y) + \Phi_-(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi e^{i\alpha x}\, dx, \quad \alpha = \sigma + i\tau,$$

$$\Phi_+(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} \phi e^{i\alpha x}\, dx \quad : \quad \Phi_-(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi e^{i\alpha x}\, dx. \tag{2.11}$$

From condition (iv) in §2.1, for a given y, $|\phi| < D_1 \exp(-k_2 x)$ as $x \to +\infty$ and $|\phi| < D_2 \exp(k_2 \cos \Theta x)$ as $x \to -\infty$ where D_1, D_2

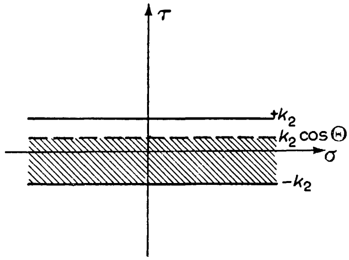

FIG. 2.2.

are constants. Therefore as in deduction (1) from theorem A, §1.3, Φ_+ is analytic for $\tau > -k_2$, Φ_- is analytic for $\tau < k_2 \cos \Theta$, and Φ is analytic in the strip $-k_2 < \tau < k_2 \cos \Theta$ (cf. Fig. 2.2). Also

$$|\Phi(\alpha,y)| \leqslant \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} |\phi e^{-\tau x}|\, dx = |\Phi_1| + |\Phi_2|,$$

where, from conditions (iv) (a), (b), §2.1, we can set

$$|\Phi_1| < K_1 \int_{-\infty}^{-|y|\cot\Theta} \exp\{(k_2 x \cos\Theta - k_2|y| \sin\Theta) - \tau_1 x\}\, dx,$$

$$|\Phi_2| < K_2 \int_{-\infty}^{\infty} \exp\{-k_2(x^2 + y^2)^{1/2} - \tau_2 x\}\, dx.$$

It is easily proved that $|\Phi_1|$ is bounded as $|y| \to \infty$ if $\tau_1 < k_2 \cos \Theta$ and $|\Phi_2|$ is bounded if $-k_2 < \tau_2 < k_2$. Hence $\Phi(\alpha,y)$ is bounded as $|y| \to \infty$ if τ lies in the strip $-k_2 < \tau < k_2 \cos \Theta$.

If now we apply a Fourier transform in x to (2.5) we find, as in §1.5 by the argument following equation (1.55), that

$$d^2\Phi(\alpha,y)/dy^2 - \gamma^2\Phi(\alpha,y) = 0, \qquad \gamma = (\alpha^2 - k^2)^{1/2}, \tag{2.12}$$

where γ has been defined at the end of §1.2. This equation has solutions

$$\begin{aligned}\Phi(\alpha,y) &= A_1(\alpha)e^{-\gamma y} + B_1(\alpha)e^{\gamma y}, \qquad (y \geqslant 0),\\ &= A_2(\alpha)e^{-\gamma y} + B_2(\alpha)e^{\gamma y}, \qquad (y \leqslant 0),\end{aligned} \tag{2.13}$$

where A_1, B_1, A_2, B_2 are functions of α only. There are two forms of Φ since ϕ and hence Φ is discontinuous across $y = 0$.

In equation (2.12) the real part of γ is always positive in $-k_2 < \tau < k_2$ (ex. 1.3), and therefore in (2.13) we must take $B_1 = A_2 = 0$. From (ii) in §2.1, $\partial\phi/\partial y$ is continuous across $y = 0$. Hence $d\Phi/dy$ is continuous across $y = 0$ and we can set

$$A_1(\alpha) = -B_2(\alpha) = A(\alpha), \quad \text{say.}$$

Hence

$$\Phi(\alpha,y) = \begin{cases} A(\alpha)e^{-\gamma y}, & (y \geqslant 0),\\ -A(\alpha)e^{\gamma y}, & (y \leqslant 0).\end{cases} \tag{2.14}$$

For brevity we shall sometimes write $\Phi(\alpha)$ or $\Phi(y)$ instead of $\Phi(\alpha,y)$ when there is no risk of confusion. Similarly for $\Phi_+(\alpha)$, $\Phi_-(y)$ etc. The shortened notation helps to concentrate attention on the important variable. An expression like $\Phi_+(0)$ will always refer to the value of $\Phi_+(\alpha,y)$ for $y = 0$. When a transform is discontinuous across $y = 0$ we extend the notation:

$$\Phi_-(\pm 0) = \Phi_-(\alpha,\pm 0) = \lim_{y\to\pm 0} \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \phi e^{i\alpha x}\, dx, \tag{2.15}$$

where in the usual way (lim, $y \to +0$) means the limit as y tends to zero approached from positive values of y, etc. Similarly for $\Phi_+(\pm 0)$, though in this case because of (iii), §2.1,

$$\Phi_+(+0) = \Phi_+(-0) = \Phi_+(0), \quad \text{say.} \tag{2.16}$$

We also write

$$\Phi'_+(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} \frac{\partial\phi}{\partial y} e^{i\alpha x}\, dx \quad : \quad \Phi'_-(\alpha,y) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} \frac{\partial\phi}{\partial y} e^{i\alpha x}\, dx. \tag{2.17}$$

From (ii) in §2.1, $\Phi'_+(\alpha,+0) = \Phi'_+(\alpha,-0) = \Phi'_+(0)$, and similarly for Φ'_-.

On applying these definitions to (2.14) we find

$$\Phi_+(0) + \Phi_-(+0) = A(\alpha), \tag{2.18a}$$

$$\Phi_+(0) + \Phi_-(-0) = -A(\alpha), \tag{2.18b}$$

$$\Phi'_+(0) + \Phi'_-(0) = -\gamma A(\alpha). \tag{2.18c}$$

The procedure at this point is of general application and it is important to understand clearly the logic behind it. We wish to deal with equations which contain only functions whose regions of regularity are known. Hence eliminate $A(\alpha)$ from the above three equations. (2.18a) and (2.18b) are obviously analogous. Add these two equations to find

$$2\Phi_+(0) = -\Phi_-(+0) - \Phi_-(-0). \tag{2.19}$$

Next subtract (2.18b) from (2.18a) and eliminate $A(\alpha)$ between the resulting equation and (2.18c). Then

$$\Phi'_+(0) + \Phi'_-(0) = -\tfrac{1}{2}\gamma\{\Phi_-(+0) - \Phi_-(-0)\}. \tag{2.20}$$

In this equation $\Phi'_-(0)$ is known. In fact from (i), §2.1,

$$\Phi'_-(0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} e^{i\alpha x}(ik \sin \Theta e^{-ikx\cos\Theta})\, dx = \frac{k \sin \Theta}{(2\pi)^{1/2}(\alpha - k \cos \Theta)}. \tag{2.21}$$

For simplicity write

$$\Phi_-(+0) - \Phi_-(-0) = 2D_- \quad : \quad \Phi_-(+0) + \Phi_-(-0) = 2S_-, \tag{2.22}$$

where D_- is short for $D_-(\alpha)$, S_- for $S_-(\alpha)$, and the letters D and S are used merely to remind us that these are derived from the differences and sum of two functions. Both D_- and S_- are regular in $\tau < k_2 \cos \Theta$.

Equations (2.19) and (2.20) now become

$$\Phi_+(0) = -S_-, \tag{2.23}$$

$$\Phi'_+(0) + \frac{k \sin \Theta}{(2\pi)^{1/2}(\alpha - k \cos \Theta)} = -\gamma D_-. \tag{2.24}$$

In these two equations there are four unknown functions Φ_+, Φ'_+, S_-, D_-. Each equation holds in the strip $-k_2 < \tau < k_2 \cos \Theta$.

Both of these equations are in the standard Wiener–Hopf form (cf. §1.7, (1.75)):

$$R(\alpha)\Phi_+(\alpha) + S(\alpha)\Psi_-(\alpha) + T(\alpha) = 0, \tag{2.25}$$

where R, S, T are given, Φ_+, Ψ_- are unknown, Φ_+ is regular in an upper half-plane, $\tau > \tau_-$, Φ_- in a lower half-plane $\tau < \tau_+$, and $R.S$

are regular in the strip $\tau_- < \tau < \tau_+$. The object of the elimination of $A(\alpha)$ from (2.18) was to obtain equations of this type and it is easy to see that the above procedure is the only correct one. If for example we eliminate (α) from A(2.18b) and (2.18c),

$$\Phi'_+(0) + \Phi'_-(0) = \gamma\{\Phi_+(0) + \Phi_-(0)\}. \tag{2.26}$$

In this equation $\Phi'_+(0)$ and $\Phi_+(0)$ are both unknown, and both regular in the upper half-plane $\tau > -k_2$. Therefore (2.26) is *not* of form (2.25). From a slightly different point of view, in (2.18c) $\Phi'_-(0)$ is known and $\Phi'_+(0)$ is unknown. If $\Phi_+(0)$ is also unknown it is useless to try to obtain an equation involving both Φ_+ and Φ'_+ as this will certainly be of type (2.26). Since $\Phi'_-(0)$ is known we shall need at least one equation involving $\Phi'_+(0)$. If in obtaining such an equation we wish to use other equations which involve $\Phi_+(0)$, we must first of all eliminate $\Phi_+(0)$ between these other equations as we did above.

Now return to the solution of the problem. First deal with (2.24). We have $\gamma = (\alpha^2 - k^2)^{1/2} = (\alpha - k)^{1/2}(\alpha + k)^{1/2}$, where the branch of each factor is defined so that $(\alpha \pm k)^{1/2} \to \alpha^{1/2}$ as $\sigma \to +\infty$ in the strip $-k_2 < \tau < k_2$ (§1.2). The factor $(\alpha + k)^{1/2}$ is regular and non-zero in $\tau > -k_2$. Divide (2.24) by $(\alpha + k)^{1/2}$:

$$\frac{\Phi'_+(0)}{(\alpha + k)^{1/2}} + \frac{k \sin \Theta}{(2\pi)^{1/2}(\alpha + k)^{1/2}(\alpha - k \cos \Theta)} = -(\alpha - k)^{1/2}D_-. \tag{2.27}$$

The first term on the left-hand side is regular in $\tau > -k_2$: the term on the right-hand side is regular in $\tau < k_2 \cos \Theta$. The remaining term is regular only in the strip $-k_2 < \tau < k_2 \cos \Theta$, but we can write it in the following way (cf. (1.18) onwards)

$$\begin{aligned}
&\frac{k \sin \Theta}{(2\pi)^{1/2}(\alpha + k)^{1/2}(\alpha - k \cos \Theta)} \\
&\quad = \frac{k \sin \Theta}{(2\pi)^{1/2}(\alpha - k \cos \Theta)}\left\{\frac{1}{(\alpha + k)^{1/2}} - \frac{1}{(k + k \cos \Theta)^{1/2}}\right\} + \\
&\qquad + \frac{k \sin \Theta}{(2\pi)^{1/2}(k + k \cos \Theta)^{1/2}(\alpha - k \cos \Theta)} \\
&\quad = H_+(\alpha) + H_-(\alpha), \quad \text{say,}
\end{aligned} \tag{2.28}$$

where H_+ is regular in $\tau > -k_2$, H_- is regular in $\tau < k_2 \cos \Theta$. Insert in (2.27) and rearrange to find

$$J(\alpha) = (\alpha + k)^{-1/2}\Phi'_+(0) + H_+(\alpha) = -(\alpha - k)^{1/2}D_- - H_-(\alpha). \tag{2.29}$$

In this form the equation defines a function $J(\alpha)$ which is regular in $\tau > -k_2$, and also regular in $\tau < k_2 \cos \Theta$ i.e. in the whole of the α-plane, since these two half-planes overlap.

Providing that $J(\alpha)$ has algebraic behaviour as α tends to infinity we can use the extended form of Liouville's theorem (§1.2) to determine the exact form of $J(\alpha)$. Thus we proceed to examine the behaviour of the functions in (2.29) as α tends to infinity.

From (v) in §2.1 and the Abelian theorem at the end of §1.6,

$$|\Phi_-(+0)| < c_1|\alpha|^{-1} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau < k_2 \cos \Theta,$$
$$|\Phi'_+(0)| < c_2|\alpha|^{-1/2} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau > -k_2.$$

Also from definition (2.28),

$$|H_-(\alpha)| < c_3|\alpha|^{-1} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau < k_2 \cos \Theta,$$
$$|H_+(\alpha)| < c_4|\alpha|^{-1} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau > -k_2.$$

Hence from (2.29)

$$|J(\alpha)| < c_5|\alpha|^{-1/2} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau < k_2 \cos \Theta,$$
$$< c_6|\alpha|^{-1} \quad \text{as} \quad \alpha \to \infty \quad \text{in} \quad \tau > -k_2.$$

i.e. $J(\alpha)$ is regular in the whole of the α-plane and tends to zero as α tends to infinity in any direction. Hence from Liouville's theorem $J(\alpha)$ must be identically zero, i.e.

$$\Phi'_+(0) = -(\alpha + k)^{1/2} H_+(\alpha), \tag{2.30a}$$

$$D_- = -(\alpha - k)^{-1/2} H_-(\alpha). \tag{2.30b}$$

To complete the solution note first of all that from (2.18c), (2.21), (2.28), (2.30a),

$$A(\alpha) = -\frac{1}{(2\pi)^{1/2}} \frac{k \sin \Theta}{(k + k \cos \Theta)^{1/2}(\alpha - k)^{1/2}(\alpha - k \cos \Theta)}. \tag{2.31}$$

Use (2.14) and the Fourier inversion formula to find

$$\phi = \mp \frac{1}{2\pi} (k - k \cos \Theta)^{1/2} \int_{-\infty + ia}^{\infty + ia} \frac{e^{-i\alpha x \mp \gamma y}}{(\alpha - k)^{1/2}(\alpha - k \cos \Theta)} \, d\alpha, \tag{2.32a}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{2\pi} (k - k \cos \Theta)^{1/2} \int_{-\infty + ia}^{\infty + ia} \frac{(\alpha + k)^{1/2} e^{-i\alpha x \mp \gamma y}}{(\alpha - k \cos \Theta)} \, d\alpha, \tag{2.32b}$$

where $-k_2 < a < k_2 \cos \Theta$ and the upper sign refers to $y \geqslant 0$, the lower to $y \leqslant 0$. This solution is discussed in §2.6.

We finish our treatment by Jones's method by noting that the remaining equation, (2.23), can be solved in exactly the same way as (2.24). The two sides of the equation together define a function which is regular in the whole α-plane and from the behaviour of the two sides as α tends to infinity this function must be identically zero. Thus $\Phi_+(0) = 0$. Also $S_- = 0$ or from definition (2.22), $\Phi_-(+0) = -\Phi_-(-0)$. The result $\Phi_+(0) = 0$ means that $\phi = 0$ or $\phi_t = \exp(-ik \cos \Theta)$ on $y = 0$, $x > 0$.

Alternatively we see immediately from (2.32a) that $\phi = 0$ on $y = 0$, $x > 0$ by completing the contour in the lower half-plane. Hence $\Phi_+(0) = 0$ and from (2.18a, b) we then have $\Phi_-(+0) = -\Phi_-(-0)$. Thus (2.23) is, strictly speaking, superfluous and the complete solution can be obtained from (2.24).

2.3 A dual integral equation method

In this section we consider another method of procedure. Apply Fourier transforms to the partial differential equation as in §2.2 and obtain (2.14). Invert by the Fourier inversion formula and find

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} \pm \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} A(\alpha) e^{\mp\gamma y - i\alpha x}\, d\alpha, \qquad (2.33)$$

where $-k_2 < a < k_2 \cos \Theta$ and the upper sign applies for $y \geqslant 0$, the lower for $y \leqslant 0$. Since ϕ_t is continuous on $y = 0$, $x > 0$ we have

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} A(\alpha) e^{-i\alpha x}\, d\alpha = 0, \qquad (x > 0). \qquad (2.34)$$

(Note that the deduction $\phi_t = \exp(-ikx \cos \Theta)$ on $y = 0$, $x > 0$ is immediate. In Jones's method this was established either from the final solution or from an analytic continuation argument. Cf. the end of §2.2.) Since $\partial\phi_t/\partial y = 0$ on $y = 0$, $x < 0$,

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \gamma A(\alpha) e^{-i\alpha x}\, d\alpha = -ik \sin \Theta e^{-ikx\sin\Theta}, \qquad (x < 0). \qquad (2.35)$$

Equations (2.34) and (2.35) are dual integral equations from which the unknown function $A(\alpha)$ can be found by the following procedure. Replace x by $(x + \xi)$, $x > 0$, $\xi > 0$, in (2.34), multiply throughout by some function $\mathscr{N}_1(\xi)$ to be determined and integrate with respect to ξ from 0 to ∞. Assume that orders of integration can be interchanged.

We find

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \int_{0}^{\infty} \mathscr{N}_1(\xi)e^{-i\alpha\xi}\,d\xi\, A(\alpha)e^{-i\alpha\xi}\,d\alpha = 0, \qquad x > 0. \qquad (2.36)$$

Similarly in equation (2.35) replace x by $(x - \xi)$, $x < 0$, $\xi > 0$, multiply through by some function $\mathscr{N}_2(\xi)$ to be determined, and integrate with respect to ξ from 0 to infinity. Interchange orders of integration. This gives

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \int_{0}^{\infty} \mathscr{N}_2(\xi)e^{i\alpha\xi}\,d\xi\, \gamma A(\alpha)e^{-i\alpha x}\,d\alpha$$

$$= -ik \sin \Theta e^{-ikx\cos\Theta} \int_{0}^{\infty} \mathscr{N}_2(\xi)e^{ik\xi\cos\Theta}\,d\xi, \qquad (x < 0). \qquad (2.37)$$

Introduce functions $N_+(\alpha)$, $N_-(\alpha)$ defined by

$$\int_{0}^{\infty} \mathscr{N}_1(\xi)e^{-i\alpha\xi}\,d\xi = N_-(\alpha) \quad : \quad \int_{0}^{\infty} \mathscr{N}_2(\xi)e^{i\alpha\xi}\,d\xi = N_+(\alpha). \qquad (2.38)$$

The functions N_- and N_+ are regular in some lower and upper half-planes respectively (Example (1) following theorem A of §1.3). If we could choose $\mathscr{N}_1$ and $\mathscr{N}_2$ so that

$$N_-(\alpha) = N_+(\alpha)\gamma = N_+(\alpha)(\alpha^2 - k^2)^{1/2} = N(\alpha), \quad \text{say}, \qquad (2.39)$$

then the left-hand sides of (2.36) and (2.37) would become identical. In fact we should have

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} N(\alpha)A(\alpha)e^{-i\alpha x}\,d\alpha$$

$$= \begin{cases} 0 \qquad , & (x > 0), \\ -ik \sin \Theta e^{-ikx\cos\Theta} \displaystyle\int_{0}^{\infty} \mathscr{N}_2(\xi)e^{ikx\cos\Theta}\,d\xi, & (x < 0). \end{cases}$$

Then $A(\alpha)$ could be found directly by the Fourier inversion formula. From (2.39) the obvious choice for N_+, N_- is $N_+(\alpha) = (\alpha + k)^{-1/2}$, $N_-(\alpha) = (\alpha - k)^{1/2}$, where we have used the crucial Wiener–Hopf decomposition of $\gamma = (\alpha^2 - k^2)^{1/2}$ into functions regular in upper and lower half-planes. The choice of N_- is not suitable since from its definition (2.38), N_- must tend to zero as α tends to infinity in a lower half-plane. However we shall see that it is suitable to choose

$N_- = (\alpha - k)^{-1/2}$. The reason for this will be clear later (see also the treatment of the general case in §6.2). Theoretically, having chosen N_+, N_-, we could regard (2.38) as integral equations for $\mathcal{N}_1$, $\mathcal{N}_2$, which can be solved generally by theorem C, §1.4. However in this case it is sufficient to use the elementary results of ex. 2.4 and we find

$$\mathcal{N}_1(\xi) = \pi^{-1/2}\xi^{-1/2}e^{i\xi k + i\pi/4} \quad : \quad \mathcal{N}_2(\xi) = \pi^{-1/2}\xi^{-1/2}e^{i\xi k - i\pi/4}. \tag{2.40}$$

Equations (2.36), (2.37) become

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{-1/2} A(\alpha) e^{-i\alpha x}\, d\alpha = 0, \qquad (x > 0), \tag{2.41a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{1/2} A(\alpha) e^{-i\alpha x}\, d\alpha \tag{2.41b}$$

$$= -\pi^{-1/2}e^{i\pi/4}\, k \sin \Theta e^{-ikx\cos\Theta} \int_0^\infty \xi^{-1/2} e^{i\xi(k + k\cos\Theta)}\, d\xi, \quad (x < 0).$$

The integral on the right can be evaluated (ex. 2.4). If we multiply (2.41a) by $\exp(ikx)$ and differentiate with respect to x, the left-hand sides of the resulting equation and (2.41b) are identical. These steps give

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{1/2} A(\alpha) e^{-i\alpha x}\, d\alpha$$
$$= \begin{cases} 0, & (x > 0), \\ -ik \sin\Theta (k + k\cos\Theta)^{-1/2} e^{-ikx\sin\Theta}, & (x < 0). \end{cases}$$

The Fourier inversion formula then yields

$$A(\alpha) = -\frac{ik\sin\Theta}{(2\pi)^{1/2}(k + k\cos\Theta)^{1/2}(\alpha - k)^{1/2}} \int_{-\infty}^{0} e^{i(\alpha - k\cos\Theta)x}\, dx,$$
$$= -\frac{k\sin\Theta}{(2\pi)^{1/2}(k + k\cos\Theta)^{1/2}(\alpha - k)^{1/2}(\alpha - k\cos\Theta)}.$$

This is identical with (2.31) found by Jones's method.

One of the pleasing features of this method is that analytic continuation is not involved. The crucial step of decomposing a function in the form $K(\alpha) = K_+(\alpha)\, K_-(\alpha)$ is still present. It is this step which gives the clue to the correct N_+, N_- and hence $\mathcal{N}_1$, $\mathcal{N}_2$. In the above case $\mathcal{N}_1$, $\mathcal{N}_2$ can be found explicitly and the solution can be carried through formally with only nominal reference to the Wiener–Hopf technique. We shall see later that this use of explicit $\mathcal{N}_1$, $\mathcal{N}_2$ and consequent avoidance of the machinery of the Wiener–Hopf technique is possible in several important cases (Chapter VI). The real utility of the method will appear in connexion with specific applications later.

It may seem to the reader that the method of this section does not involve so much analysis of detail as Jones's method. This is partly illusory since we have assumed various results that were proved in §2.2 in connexion with Jones's method, e.g. the analysis leading to (2.33). The edge conditions (v) of §2.1 do not occur explicitly in this section but they are assumed implicitly in that we assume that certain orders of integration can be interchanged and certain integrals are convergent. Some further comments are given in §2.6 below.

2.4 Integral equation formulations

In this section we consider the formulation of the Sommerfeld half-plane diffraction problem of §2.1 in terms of integral equations. This is carried out in more detail than is strictly necessary in order to illustrate some points of general interest. We shall compare Green's function and transform methods of approach and clarify the reasons for the occurrence of divergent integrals in some cases. We first of all consider the Green's function approach and then show that the same results can be obtained by transform methods.

A Green's function $G(x,y:x_0,y_0)$ for the two-dimensional steady-state wave equation (2.5) is defined as a solution representing the potential at a point (x,y) caused by a line source of unit strength at the point (x_0,y_0) in a region of any shape with given boundary conditions. Thus G can be found by solving the equation

$$G_{xx} + G_{yy} + k^2G = -4\pi\delta(x - x_0)\delta(y - y_0). \tag{2.42}$$

For brevity represent the point (x,y) by $\mathbf{r}$ and (x_0,y_0) by $\mathbf{r}_0$. The basic equation used to establish integral equations by a Green's function method is that if ϕ is a solution of $\phi_{xx} + \phi_{yy} + k^2\phi = 0$ in a region R, then

$$\phi(\mathbf{r}_0) = \frac{1}{4\pi}\int_{S^0}\left\{G(\mathbf{r}|\mathbf{r}_0)\,\frac{\partial\phi(\mathbf{r}_0)}{\partial n_0} - \frac{\partial G(\mathbf{r}|\mathbf{r}_0)}{\partial n_0}\,\phi(\mathbf{r}_0)\right\}dS_0, \tag{2.43}$$

where the point $\mathbf{r}_0$ i.e. (x_0,y_0) lies on the boundary S_0 of the region R, and the integral is taken over this boundary. $\partial/\partial n_0$ refers to differentiation along the outward normal at the point (x_0,y_0) on S_0. The region R must be contained in the region D in which the Green's function is defined. R and D are often taken to be the same. The boundary conditions on any finite part of the boundary of D are usually taken as homogeneous conditions of the form $a\partial G/\partial n_0 + bG = 0$ on S_0. If D extends to infinity the appropriate condition is that G represents an outgoing wave at infinity. For further details the reader is referred to P. M. Morse and H. Feshbach [1], Chapter 7.

Now consider formulation of integral equations for the Sommerfeld half-plane problem of §2.1:

(1a) *Greens' function method: unknown function* $\partial\phi_t/\partial y = h(x)$ *on* $y = 0$, $x > 0$. Consider separately the regions $y \geqslant 0$, $y \leqslant 0$. In $y \geqslant 0$ apply (2.43) to the region bounded by the x-axis and a semi-circle at infinity. Then

$$\phi(x,y) = -\frac{1}{4\pi}\int_{-\infty}^{\infty} G(x,y:x_0,0)\,\frac{\partial\phi(x_0,0)}{\partial y_0}\,dx_0 + \\ + \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\partial G(x,y:x_0,0)}{\partial y_0}\,\phi(x_0,0)\,dx_0, \quad (2.44)$$

where there is an additional minus sign since $\partial/\partial n_0 = -\partial/\partial y_0$, and the contribution from a semi-circle at infinity is zero if G represents outgoing waves at infinity. We wish to formulate an integral equation in terms of $\partial\phi_t/\partial y$. (2.44) involves the two unknown functions $\partial\phi/\partial y_0$ and ϕ. The term involving ϕ will disappear if we choose a G such that $\partial G/\partial y_0 = 0$ on $y_0 = 0$. A suitable choice for G is obviously (cf. ex. 1.16 and (2.42))

$$G(x,y:x_0,y_0) = \pi i\,\{H_0^{(1)}(kR) + H_0^{(1)}(kR')\}, \quad (2.45)$$

where

$$R = \{(x-x_0)^2 + (y-y_0)^2\}^{1/2} \quad : \quad R' = \{(x-x_0)^2 + (y+y_0)^2\}^{1/2}.$$

We cannot apply (2.44) to ϕ_t since this contains a wave incident from infinity. But define ϕ by

$$\phi_t(x,y) = e^{-ikx\cos\Theta - iky\sin\Theta} + e^{-ikx\cos\Theta + iky\sin\Theta} + \phi, \quad (2.46)$$

where we have removed both the incident and reflected waves. The object of this is to make $\partial\phi/\partial y = \partial\phi_t/\partial y$ on $y = 0$. Then (2.44), (2.45) and (2.46) give for $y \geqslant 0$

$$\phi_t(x,y) = e^{-ikx\cos\Theta - iky\sin\Theta} + \\ + e^{-ikx\cos\Theta + iky\sin\Theta} - \tfrac{1}{2}i\int_0^{\infty} H_0^{(1)}(kR_0)h(\xi)\,d\xi, \quad (2.47)$$

where

$$R_0 = \{(x-\xi)^2 + y^2\}^{1/2}.$$

Green's theorem can be applied directly to ϕ_t in $y \leqslant 0$. In this case, on $y = 0$, $\partial/\partial n_0 = +\partial/\partial y_0$, and we find for $y \leqslant 0$,

$$\phi_t(x,y) = \tfrac{1}{2}i \int_0^\infty H_0^{(1)}(kR_0)h(\xi)\, d\xi. \tag{2.48}$$

But ϕ_t is continuous on $y = 0$, $x > 0$ and on letting $y \to 0$ in (2.47), (2.48) and equating the results on $y = 0$, $x > 0$ we obtain the integral equation

$$i \int_0^\infty H_0^{(1)}(k|x - \xi|)h(\xi)\, d\xi = 2e^{-ikx\cos\Theta}. \tag{2.49}$$

(1b) *Green's function method: unknown function* $2e(x) = \phi_t(x,+0) - \phi_t(x,-0)$ *for* $x < 0$. Adopt the procedure in (1a) but now eliminate $\partial\phi/\partial y_0$ instead of ϕ from (2.44) by choosing G so that $G = 0$ on $y_0 = 0$ i.e. choose (cf. 2.45)

$$G(x,y : x_0,y_0) = \pi i\,\{H_0^{(1)}(kR) - H_0^{(1)}(kR')\}. \tag{2.50}$$

Write (cf. 2.46)

$$\phi_t(x,y) = e^{-ikx\cos\Theta - iky\sin\Theta} - e^{-ikx\cos\Theta + iky\sin\Theta} + \phi,\ (y \geqslant 0), \tag{2.51}$$

where we have removed the incident wave and inserted an extra term so that $\phi_t = \phi$ on $y = 0$. Apply Green's theorem (2.44) to ϕ in $y \geqslant 0$ and to ϕ_t in $y \leqslant 0$. This gives

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} - e^{-ikx\cos\Theta + iky\sin\Theta} +$$
$$+ \frac{1}{4\pi}\int_{-\infty}^{0}\left(\frac{\partial G}{\partial y_0}\right)_0 f(x_0,+0)\, dx_0 + \frac{1}{4\pi}\int_0^\infty \left(\frac{\partial G}{\partial y_0}\right)_0 f(x_0)\, dx_0,\ (y \geqslant 0). \tag{2.52a}$$

$$\phi_t = -\frac{1}{4\pi}\int_{-\infty}^{0}\left(\frac{\partial G}{\partial y_0}\right)_0 f(x_0,-0)\, dx_0 - \frac{1}{4\pi}\int_0^\infty \left(\frac{\partial G}{\partial y_0}\right)_0 f(x_0)\, dx_0,\ (y \leqslant 0), \tag{2.52b}$$

where $\partial G/\partial y_0$ is to be taken on $y_0 = 0$, and we have denoted the value of $\phi_t = \phi$ on $y = 0$, $x > 0$ by $f(x)$, and on $y = \pm 0$, $x < 0$ by $f(x,\pm 0)$ respectively. We now use the fact that $\partial\phi_t/\partial y = 0$ on $y = 0$, $x < 0$ to obtain an integral equation. Differentiate the above equations with respect to y and let y tend to $+0$ and -0 respectively, $x < 0$. We have

$$\lim_{y_0 \to +0} \frac{\partial G}{\partial y_0} = \lim_{y_0 \to -0} \frac{\partial G}{\partial y_0} = 2\pi i \lim_{y_0 \to 0} \frac{\partial}{\partial y_0}\{H_0^{(1)}(kR)\}.$$

Eliminate terms involving $f(x)$ by adding (2.52) and introduce $e(x)$ defined by

$$f(x,+0) - f(x,-0) = 2e(x). \tag{2.53}$$

Then

$$\lim_{y,y_0\to 0} i \int_{-\infty}^{0} \frac{\partial^2}{\partial y \partial y_0} \{H_0^{(1)}(kR)\} e(x_0)\, dx_0 = 2ik \sin \Theta e^{-ikx\cos\Theta}, \qquad x < 0. \tag{2.54}$$

We cannot go to the limit explicitly since the differentiations under the integral sign give a kernel such that the integral would be divergent. We have

$$\begin{aligned}\lim_{y,y_0\to 0} \frac{\partial^2}{\partial y \partial y_0} H_0^{(1)}(kR) &= -\lim_{y\to 0} \frac{\partial^2}{\partial y^2} H_0^{(1)}(kR_0) \\ &= \left(\frac{\partial^2}{\partial x^2} + k^2\right) H_0^{(1)}(k|x - x_0|),\end{aligned}$$

where $R_0^2 = (x - x_0)^2 + y^2$. Hence (2.54) can also be written

$$-\lim_{y\to 0} i \frac{\partial^2}{\partial y^2} \int_{-\infty}^{0} H_0^{(1)}[k\{(x - \xi)^2 + y^2\}^{1/2}] e(\xi)\, d\xi = 2ik \sin \Theta e^{-ikx\cos\Theta}, \qquad (x < 0); \tag{2.55}$$

or

$$i \left(\frac{\partial^2}{\partial x^2} + k^2\right) \int_{-\infty}^{0} H_0^{(1)}(k|x - \xi|) e(\xi)\, d\xi = 2ik \sin \Theta e^{-ikx\cos\Theta}, \qquad (x < 0). \tag{2.56}$$

We shall see in §2.5 that either of these forms can be solved by the Wiener–Hopf technique.

If we interchange orders of differentiation and integration in (2.56) the resulting integral is divergent. The divergency obviously comes from the region $x \simeq \xi$ where

$$H_0^{(1)}(k|x - \xi|) \simeq C \ln |x - \xi| \;:\; \partial^2/\partial x^2\{H_0^{(1)}(k|x - \xi|)\} \simeq -C(x - \xi)^{-2}.$$

Thus the divergent integral is of the type

$$\int_{-\infty}^{0} (x - \xi)^{-2} F(\xi)\, d\xi, \qquad (-\infty < x < 0).$$

This is exactly the kind of integral involved in Hadamard's theory of the finite part of an infinite integral invented in connexion with hyperbolic equations. However it is not necessary to invoke any such theory here since it is possible to apply Fourier transforms directly to integral equations of type (2.55), (2.56), as we see in the next section.

In (1a), (1b) different but equivalent integral equations have been obtained, one from a Green's function such that $G = 0$ on finite boundaries, the other such that $\partial G/\partial n = 0$ on boundaries. There are an infinite number of intermediate integral equations which can be obtained by using a radiation condition $p\partial G/\partial n + qG = 0$. It will appear that although the Green's function is to some extent arbitrary and the integral equations may seem different, all integral equations lead to exactly the same Wiener–Hopf problem. The function $K(\alpha)$ which has to be decomposed in the form $K_+(\alpha)\ K_-(\alpha)$ is the same in all cases.

We next show that results equivalent to the above can be obtained by transform methods. (2a) and (2b) correspond to (1a) and (1b) above.

(2a). *Transform method: unknown function* $\partial\phi_t/\partial y = h(x)$ *on* $y = 0$, $x > 0$. As in (1a) introduce ϕ defined by (2.46). A Fourier transform can be applied to ϕ and we find in the usual way $\Phi(\alpha) = A(\alpha)\exp(-\gamma y)$. An expression for $A(\alpha)$ can be obtained from the boundary condition:

$$\partial\phi/\partial y = 0,\quad y = 0,\quad x < 0 \quad : \quad \partial\phi/\partial y = h(x),\quad y = 0,\quad x > 0,$$

where $h(x)$ is unknown. Hence

$$A(\alpha) = -\frac{1}{(2\pi)^{1/2}\gamma}\int_0^\infty h(\xi)e^{i\alpha\xi}\,d\xi.$$

Apply the Fourier inversion formula and find

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} + e^{-ikx\cos\Theta + iky\sin\Theta} -$$
$$-\frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib}\gamma^{-1}e^{-i\alpha x-\gamma y}\int_0^\infty h(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha, \qquad (y \geqslant 0). \qquad (2.57)$$

An analysis of the behaviour of ϕ_t as $x \to \pm\infty$ shows that this is valid for $k_2\cos\Theta < b < k_2$. In $y \leqslant 0$ transforms can be applied directly to ϕ_t and we find

$$\phi_t = \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib}\gamma^{-1}e^{-i\alpha x+\gamma y}\int_0^\infty h(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha, \qquad (y \leqslant 0). \qquad (2.58)$$

But ϕ_t is continuous on $y = 0$, $x > 0$ and by letting $y \to 0$ in (2.57) (2.58), and subtracting the two equations find the following integral equation for $h(\xi)$:

$$\frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib}\gamma^{-1}e^{-i\alpha x}\int_0^\infty h(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha = 2e^{-ikx\cos\Theta}, \qquad (x > 0). \qquad (2.59)$$

In this equation it is permissible to interchange orders of integration. If we use the result (1.60), (1.61) namely

$$\frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib} \gamma^{-1}e^{i\alpha(\xi-x)}\,d\alpha = iH_0^{(1)}(k|x-\xi|), \qquad -k_2 < \tau < k_2, \quad (2.60)$$

then (2.59) reduces to (2.49) found by a Green's function method.

(2b). *Transform method: unknown function* $\phi_t(x,+0) - \phi_t(x,-0)$ *on* $y = 0$, $x < 0$. As in (2b) introduce ϕ defined by (2.51). Fourier transforms can be applied to ϕ and we find $\Phi = A(\alpha)\exp(-\gamma y)$,

$$A(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{0} f(\xi,+0)e^{i\alpha\xi}\,d\xi + \frac{1}{(2\pi)^{1/2}}\int_{0}^{\infty} f(\xi)e^{i\alpha\xi}\,d\xi,$$

where $\phi = f(x,+0)$ on $y = +0$, $x < 0$, $\phi = f(x)$ on $y = 0$, $x > 0$ as defined in connexion with (2.52). Hence

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} - e^{-ikx\cos\Theta + iky\sin\Theta} +$$
$$+\frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} e^{-i\alpha x-\gamma y}\left\{\int_{-\infty}^{0} f(\xi,+0)e^{i\alpha\xi}\,d\xi + \int_{0}^{\infty} f(\xi)e^{i\alpha\xi}\,d\xi\right\}d\alpha, \quad (y \geqslant 0). \qquad (2.61)$$

A Fourier transform can be applied directly to ϕ_t in $y \leqslant 0$. This gives

$$\phi_t = \frac{1}{2\pi}\int_{-\infty+ib}^{\infty+ib} e^{-i\alpha x-\gamma y}\left\{\int_{-\infty}^{0} f(\xi,-0)e^{i\alpha\xi}\,d\xi + \int_{0}^{\infty} f(\xi)e^{i\alpha\xi}\,d\xi\right\}d\alpha, \quad (y \leqslant 0). \qquad (2.62)$$

Apply the condition $\partial\phi_t/\partial y = 0$ on $y = 0, x < 0$ to (2.61), (2.62), and eliminate $f(\xi)$, $\xi > 0$ by adding the resulting equations. We find, using (2.53),

$$\frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib} \gamma e^{-i\alpha x}\int_{-\infty}^{0} e(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha = -2ik\sin\Theta e^{-ikx\cos\Theta}, \qquad (x < 0). \qquad (2.63)$$

This is an integral equation for $e(\xi)$. It is important to note that we cannot now interchange orders of integration since the inner integral would then diverge (cf. (2.59)). This integral equation can be solved directly by the Wiener–Hopf technique (§2.5 below) or we can convert into (2.55) or (2.56) as follows. Write (2.63) as

$$\lim_{y\to 0}\frac{\partial^2}{\partial y^2}\frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib}\gamma^{-1}e^{-i\alpha x-\gamma y}\int_{-\infty}^{0}e(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha$$
$$= -2ik\sin\Theta e^{-ikx\cos\Theta},\qquad (x<0),\qquad (2.64)$$

or

$$\left(\frac{\partial^2}{\partial x^2}+k^2\right)\frac{1}{\pi}\int_{-\infty+ib}^{\infty+ib}\gamma^{-1}e^{-i\alpha x}\int_{-\infty}^{0}e(\xi)e^{i\alpha\xi}\,d\xi\,d\alpha$$
$$= 2ik\sin\Theta e^{-ikx\cos\Theta},\qquad (x<0).\qquad (2.65)$$

We can now interchange orders of integration and evaluate the inner integral by (1.60), (1.61) (cf. 2.60) to find (2.55), (2.56).

Further comments on the formulation of problems in terms of integral equations are given in exs. 2.6, 2.7, 2.8.

2.5 Solution of the integral equations

In §2.4 we have obtained integral equations of several different types which we analyse in turn. A fundamental result we shall need is the Fourier inverse of (1.60), (1.61):

$$\tfrac{1}{2}i\int_{-\infty}^{\infty}H_0^{(1)}\{k(x^2+y^2)^{1/2}\}e^{i\alpha x}\,dx=\gamma^{-1}e^{-\gamma|y|},\qquad (-k_2<\tau<k_2).\ (2.66)$$

First of all consider the integral equation (cf. 2.49)

$$\int_0^{\infty}k(x-\xi)h(\xi)\,d\xi=f(x),\qquad (0\leqslant x<\infty),\qquad (2.67)$$

where k, f are given, g is unknown. Extend this equation by writing

$$\int_0^{\infty}k(x-\xi)h(\xi)\,d\xi=e(x),\qquad (-\infty<x<0),\qquad (2.68)$$

where $e(x)$ is an unknown function defined by this equation. Apply a Fourier transform to (2.67), (2.68): multiply throughout by $(2\pi)^{-1/2}\exp(i\alpha x)$ and integrate with respect to x from $-\infty$ to ∞. Then

$$F_+(\alpha)+E_-(\alpha)=\frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}e^{i\alpha x}\int_0^{\infty}k(x-\xi)h(\xi)\,d\xi\,dx$$
$$=\frac{1}{(2\pi)^{1/2}}\int_0^{\infty}h(\xi)\int_{-\infty}^{\infty}k(x-\xi)e^{i\alpha x}\,dx\,d\xi,$$

where

$$F_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty f(x) e^{i\alpha x}\, dx \quad : \quad E_-(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^0 e(x) e^{i\alpha x}\, dx. \tag{2.69}$$

In the inner integral change variable to $y = (x - \xi)$. The double integral separates into two independent integrals and we find

$$F_+(\alpha) + E_-(\alpha) = H_+(\alpha) K(\alpha), \tag{2.70}$$

where

$$H_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty h(\xi) e^{i\alpha\xi}\, d\xi \quad : \quad K(\alpha) = \int_{-\infty}^\infty k(y) e^{i\alpha y}\, dy. \tag{2.71}$$

Note that $K(\alpha)$ has been defined in such a way that there is no factor $(2\pi)^{-1/2}$ in front of the integral, in order to simplify the form of (2.70). This equation is of the Wiener–Hopf type (2.25) if it holds in a strip $\tau_- < \tau < \tau_+$ of the α-plane. In the concrete example we are considering, namely (2.49),

$$F_+(\alpha) = \frac{2}{(2\pi)^{1/2}} \int_0^\infty e^{ix(\alpha - k\cos\Theta)}\, dx$$

$$= \frac{2}{(2\pi)^{1/2}} \cdot \frac{i}{(\alpha - k\cos\Theta)}, \qquad (\tau > k_2 \cos\Theta), \tag{2.72a}$$

$$K(\alpha) = i \int_{-\infty}^\infty H_0^{(1)}(k|x|) e^{i\alpha x}\, dx = 2\gamma^{-1}, \qquad (-k_2 < \tau < k_2). \tag{2.72b}$$

Hence (2.70) becomes

$$\frac{2}{(2\pi)^{1/2}} \cdot \frac{i}{(\alpha - k\cos\Theta)} + E_-(\alpha) = \frac{2}{\gamma} H_+(\alpha). \tag{2.73}$$

This equation corresponds to (2.24) in Jones's method. By definition $h(x) = \partial\phi_t/\partial y$ on $y = 0$, $x > 0$. Since $|\phi_t| < \exp(k_2 x \cos\Theta)$ as $x \to +\infty$, H_+ is regular in $\tau > k_2 \cos\Theta$. Also, as $x \to -\infty$

$$e(x) = i \int_0^\infty H_0^{(1)}(k|x - \xi|) h(\xi)\, d\xi \sim C_1 \int_0^\infty (\xi - x)^{-1/2} e^{ik(\xi - x)} h(\xi)\, d\xi;$$

$$|e(x)| < C_2 e^{k_2 x} \int_0^\infty e^{-k_2\xi} h(\xi)\, d\xi, \qquad (x \to -\infty). \tag{2.74}$$

i.e. $E_-(\alpha)$ is regular in $\tau < k_2$. On using these results and inspecting (2.73) we see that this equation holds in $k_2 \cos\Theta < \tau < k_2$. Deal with (2.73) as with (2.24) in Jones's method. Multiply by $(\alpha - k)^{1/2}$ and rewrite as

$$P(\alpha) = \frac{2}{(2\pi)^{1/2}} \cdot \frac{i}{\alpha - k\cos\Theta} \{(\alpha - k)^{1/2} - (k\cos\Theta - k)^{1/2}\} + E_-(\alpha)$$
$$= \frac{2}{(\alpha + k)^{1/2}} H_+(\alpha) - \frac{2}{(2\pi)^{1/2}} \cdot \frac{i(k\cos\Theta - k)^{1/2}}{(\alpha - k\cos\Theta)} . \tag{2.75}$$

This defines a function which is regular in the whole of the α-plane.

From the physics of the problem it is known that $h(x) \sim x^{-1/2}$ as $x \to +0$. Hence $H_+(\alpha) \sim \alpha^{-1/2}$ as $\alpha \to \infty$ in an upper half-plane. From the definition (2.74), $e(x) \sim C$ as $x \to -0$, where C is some constant, and so $E_-(\alpha) \sim \alpha^{-1}$ as $\alpha \to \infty$ in a lower half-plane. Hence from (2.75), $P(\alpha) \to 0$ as $\alpha \to \infty$ in any direction in the α-plane and from Liouville's theorem $P(\alpha)$ is identically zero. Thus

$$H_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \cdot \frac{i(k\cos\Theta - k)^{1/2}(\alpha + k)^{1/2}}{(\alpha - k\cos\Theta)}, \quad (k_2\cos\Theta < \tau < k_2). \tag{2.76}$$

To obtain the potential at any point of space, return to (2.44), (2.45), (2.46):

$$\phi_t = q(x,y) \mp \tfrac{1}{2}i \int_0^\infty H_0^{(1)}[k\{(x-\xi)^2 + y^2\}^{1/2}]h(\xi)\,d\xi, \tag{2.77}$$

where the upper sign applies for $y \geqslant 0$, the lower for $y \leqslant 0$, and

$$q(x,y) = \begin{cases} e^{-ikx\cos\Theta - iky\sin\Theta} + e^{-ikx\cos\Theta + iky\sin\Theta}, & (y \geqslant 0), \\ 0 \qquad , & (y \leqslant 0). \end{cases}$$

To complete the solution substitute the following expression in (2.77):

$$h(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ib}^{\infty+ib} H_+(\alpha)e^{-i\alpha\xi}\,d\alpha, \qquad k_2\cos\Theta < b < k_2,$$

where this integral is automatically zero for $\xi < 0$. The limits of integration in (2.77) can now be extended to $(-\infty,\infty)$. Interchange orders of integration and change variable in the inner integral by setting $y = (x - \xi)$. Evaluate the inner integral by (2.66). We find

$$\phi_t = q(x,y) \mp \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ib}^{\infty+ib} \gamma^{-1}e^{-i\alpha x \mp \gamma y}H_+(\alpha)\,d\alpha.$$

(Alternatively this could be obtained by applying a convolution theorem to (2.77).) Insert $H_+(\alpha)$ from (2.76) and use the result (1.13a) i.e. $(k \cos \Theta - k)^{1/2} = -i(k - k \cos \Theta)^{1/2}$. This gives

$$\phi_t = q(x,y) \mp \frac{1}{2\pi} (k - k \cos \Theta)^{1/2} \int_{-\infty+ib}^{\infty+ib} \frac{\exp(-i\alpha x \mp \gamma y)}{(\alpha - k)^{1/2}(\alpha - k \cos \Theta)}\, d\alpha. \tag{2.78}$$

Shift the contour parallel to itself to a line $\tau = a$ where $-k_2 < a < k_2 \cos \Theta$. We cross a pole at $\alpha = k \cos \Theta$ and the contribution from this pole produces a term $-\exp(-ikx \cos \Theta + iky \sin \Theta)$ in $y \geqslant 0$, and $+\exp(-ikx \cos \Theta - iky \sin \Theta)$ in $y \leqslant 0$. The solution is then identical with that found by Jones's method i.e. $\phi_t = \phi_i + \phi$ where $\phi_i = \exp(-ikx \cos \Theta - iky \sin \Theta)$, the incident wave, and ϕ is given by (2.32a).

Further details regarding solution of this type of integral equation will be found in E. T. Copson [2], B. B. Baker and E. T. Copson [1].

We treat the other integral equations of §2.4 in less detail. Consider next (cf. (2.56))

$$\left(\frac{\partial^2}{\partial x^2} + k^2\right) \int_{-\infty}^{0} k(x - \xi)e(\xi)\, d\xi = \begin{cases} g(x), & (x < 0), \\ h(x), & (x > 0), \end{cases} \tag{2.79}$$

where the integral equation is given in the first line, $e(\xi)$ being unknown, and $h(x)$ is defined by the second line as the value of the left-hand side when $x > 0$. Apply a Fourier transform in x: multiply both sides by $\exp(i\alpha x)$ and integrate with respect to x in $(-\infty,\infty)$. On the left-hand side the double differentiation with respect to x can be dealt with by integration by parts. Interchange orders of integration and change variable as before. These steps give

$$G_-(\alpha) + H_+(\alpha) = (k^2 - \alpha^2)E_-(\alpha)K(\alpha). \tag{2.80}$$

In the particular case (2.56)

$$G_-(\alpha) = \frac{2ik \sin \Theta}{(2\pi)^{1/2}} \int_{-\infty}^{0} e^{ix(\alpha - k \cos \Theta)}\, dx = \frac{2k \sin \Theta}{(2\pi)^{1/2}(\alpha - k \cos \Theta)},$$

and $K(\alpha)$ is given by (2.72b). Thus

$$k \sin \Theta (2\pi)^{-1/2}(\alpha - k \cos \Theta)^{-1} + \tfrac{1}{2}H_+(\alpha) = -(\alpha^2 - k^2)^{1/2}E_-(\alpha). \tag{2.81}$$

A detailed analysis as before shows that this equation holds in $-k_2 < \tau < k_2 \cos \Theta$. Apart from this, the equation is almost the same as (2.24) obtained in Jones's method, since $E_-(\alpha)$ is the same as

$D_-(\alpha)$ and $h(x)$ is proportional to $\partial\phi/\partial y$ on $y = 0$. (See the derivation of the integral equations in §2.4.) There is therefore no need for further analysis of this equation. We have incidentally illustrated a point that we mention again later. In Jones's method the unknown functions (e.g. $H_+(\alpha)$ and $E_-(\alpha)$ in (2.81)) are treated symmetrically, and their physical significance is evident by definition. In the integral equation the symmetry is not quite so self-evident though it appears when transforms are applied.

Finally consider (2.55),

$$-\lim_{y\to 0}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{0} k\{(x-\xi):y\}e(\xi)\,d\xi = g(x), \qquad (x < 0), \qquad (2.82)$$

where we have written the kernel as $k\{(x - \xi) : y\}$ to emphasize that this is given as a function of y although we shall eventually let y tend to zero. As before extend this to general x by defining the left-hand side for $x > 0$ as a function $h(x)$. Apply a Fourier transform as before and assume that orders of the limiting process, differentiation, and integration, can be interchanged. This can be justified by uniform convergence. We find, using an obvious notation,

$$-\left\{\lim_{y\to 0}\frac{\partial^2 K(\alpha:y)}{\partial y^2}\right\} E_-(\alpha) = G_-(\alpha) + H_+(\alpha). \qquad (2.83)$$

In the cases in which we are interested it will be found that we can now perform the differentiation and go to the limit; e.g. in (2.55) we find

$$K(\alpha:y) = K(\alpha)e^{-\gamma y} \quad : \quad -\lim\,\{\partial^2 K(\alpha:y)/\partial y^2\} = (k^2 - \alpha^2)K(\alpha).$$

Hence (2.83) is identical with (2.80).

We have deliberately avoided solving the integral equations (2.59), (2.63) obtained by transform methods, which are equivalent to the integral equations solved above. It will be clear to the reader that if we apply Fourier transforms to those equations we shall obtain results equivalent to those obtained above. But if we do this we are merely "unwinding" the process by which the integral equations were obtained. There is no reason for obtaining integral equations by transforms and then applying transforms in reverse to undo the process. This is the point of Jones's method, that the Wiener–Hopf equations are obtained by working directly with functions obtained by applying transforms to the partial differential equation, without the intermediate step of formulating integral equations.

2.6 Discussion of the solution

With some care it may be possible to provide a rigorous justification of the methods described in the preceding sections. We should then know that the solution obtained is the *only* solution satisfying the conditions of the problem. (The original work of N. Wiener and E. Hopf on the solution of integral equations was of this type.) When dealing with partial differential equations it may be easier to obtain a solution by a more or less formal procedure. It is then necessary to verify that the formal solution satisfies the conditions of the problem. *If we know that the problem has a unique solution* we can conclude that our solution is correct.

The solution obtained by the Wiener–Hopf technique will usually be in the form of a contour integral which cannot be evaluated explicitly. In order to show how a result of this type can be verified consider the contour integral solution of the Sommerfeld problem (cf. (2.32a)):

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} \pm$$
$$\mp \frac{1}{2\pi}(k - k\cos\Theta)^{1/2} \int_{-\infty+ia}^{\infty+ia} \frac{e^{-i\alpha x-\gamma|y|}\,d\alpha}{(\alpha - k)^{1/2}(\alpha - k\cos\Theta)}, \tag{2.84a}$$

where $-k_2 < a < k_2 \cos\Theta$. First of all we prove that this solution satisfies the steady-state wave equation. We use standard theorems dealing with the uniform convergence of infinite integrals, and the conditions under which such integrals can be differentiated under the integral sign (e.g. H. S. Carslaw, The theory of Fourier integrals and series, Macmillan (1930), pp. 196–201).

For $|y| > 0$ the integrand in (2.84a) decreases exponentially as $\mathrm{Re}\,\alpha \to \pm\infty$ in the strip $-k_2 < \tau < k_2 \cos\Theta$. The integrals obtained by applying the operators $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ under the integral sign converge uniformly for $|y| > 0$. Hence when forming second derivatives of the integral it is permissible to interchange orders of differentiation and integration for $|y| > 0$, $-\infty < x < \infty$. Similarly if the α-plane is cut by lines parallel to the y-axis from $k = k_1 + ik_2$ to $k_1 + i\infty$ and $-k$ to $-k_1 - i\infty$, and the contour is deformed on to the two sides of the appropriate cut depending on whether $x > 0$ or $x < 0$, it will be found that it is permissible to differentiate under the integral sign for $|x| > 0$, $0 \leqslant y < \infty$, $-\infty < y \leqslant 0$. We can also show that for $x > 0$, $\partial^2\phi/\partial x^2$, $\partial^2\phi/\partial y^2$ are continuous across $y = 0$. (In fact for this special case the integral and its even order derivatives are identically zero on $y = 0$.) As a consequence of these results it is easily verified that (2.84a) satisfies the steady-state wave equation in the whole of the x–y plane cut along the line $y = 0$, $-\infty < x \leqslant 0$.

Consider next

$$\frac{\partial\phi}{\partial y} = \frac{1}{2\pi}(k - k\cos\Theta)^{1/2}\int_{-\infty+ia}^{\infty+ia}\frac{(\alpha+k)^{1/2}e^{-i\alpha x-\gamma|y|}}{\alpha - k\cos\Theta}\,d\alpha, \qquad (2.84\text{b})$$

obtained by differentiating the integral in (2.84a). The integral is uniformly convergent for all x, y such that $(x^2 + y^2)^{1/2} \geqslant \varepsilon$ where ε is arbitrarily small. For $y = 0$, $x < 0$ we can complete the contour by an infinite semi-circle in the upper half-plane. The only singularity of the integral is a simple pole at $\alpha = k\cos\Theta$ and this gives exactly (2.6) in condition (i) of §2.1. Similarly we can verify that conditions (ii) and (iii) of §2.1 are satisfied. From theorem B, §1.4, since the integrand of the integral in (2.84a) is regular in $-k_2 < \tau < k_2\cos\Theta$ we have that $|\phi| < \exp(-k_2 + \delta)x$ as $x \to +\infty$, and $|\phi| < \exp(k_2\cos\Theta - \delta)x$ as $x \to -\infty$ for arbitrarily small δ (cf. condition (iv), §2.1).

Finally consider condition (v), §2.1. If $y = 0$, $x > 0$ in (2.84b) the contour can be deformed in the lower half-plane. Cut the α-plane by a line parallel to the imaginary axis from $-k = -k_1 - ik_2$ to $-k_1 - i\infty$. (2.84b) becomes

$$\frac{\partial\phi}{\partial y} = \frac{1}{\pi}(k - k\cos\Theta)^{1/2}e^{ikx - i\pi/4}\int_0^\infty\frac{u^{1/2}e^{-ux}\,du}{u - i(k + k\cos\Theta)}.$$

Hence from (1.74), $\partial\phi/\partial y \sim Cx^{-1/2}$ as $x \to +0$. The other conditions in (v) §2.1 can be verified in a similar way.

(A detailed verification of a formal solution obtained by a Wiener–Hopf technique is given in J. Bazer and S. N. Karp [1].)

For completeness we note that (2.84a) can be evaluated by results given in §1.6 to give a well-known expression in terms of Fresnel functions. If we compare (2.84a) with (1.62), remembering that $(k - k\cos\Theta)^{1/2} = i(k\cos\Theta - k)^{1/2}$, we find

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} \mp i(2\pi)^{-1}I, \qquad (2.85)$$

where I is given in terms of Fresnel integrals by (1.65), and the minus sign holds for $y \geqslant 0$, the plus for $y \leqslant 0$. On using the result $F(v) + F(-v) = \pi^{1/2}\exp(i\pi/4)$ we find for $y \geqslant 0$ i.e. $0 \leqslant \theta \leqslant \pi$,

$$\phi_t = \pi^{-1/2}e^{-i\pi/4}[e^{-ikr\cos(\theta-\Theta)}F\{-(2kr)^{1/2}\cos\tfrac{1}{2}(\theta - \Theta)\} + \\ + e^{-ikr\cos(\theta+\Theta)}F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta + \Theta)\}], \qquad (2.86)$$

where $F(v)$ is the Fresnel integral defined by (1.66). For $y \leqslant 0$ we must remember that (2.84a) and (1.64) are expressed in terms of $|y|$ so that if θ is used to denote the *actual* angle that the radius vector makes with the line $y = 0$, $x > 0$, θ being negative for $y < 0$, then θ in (1.65) must be replaced by $(-\theta)$ if $y < 0$. We must also take the lower sign in (2.85) and it is found that the combination of these two factors means that ϕ_t is again given by (2.86) for $y \leqslant 0$ i.e. $-\pi \leqslant \theta \leqslant 0$. The total field in the whole space is given by (2.86) provided that we take $-\pi \leqslant \theta \leqslant \pi$.

The edge condition (v) of §2.1 is the condition which ensures uniqueness of the solution obtained by the Wiener–Hopf technique in the problem we have just considered. We wish to solve (2.29), namely

$$J(\alpha) = (\alpha + k)^{-1/2}\Phi'_+(0) + H_+(\alpha) = -(\alpha - k)^{1/2}D_- - H_-(\alpha), \tag{2.87}$$

where both H_+ and $H_- \sim \alpha^{-1}$ as $\alpha \to \infty$ in appropriate half-planes. Instead of assuming (v) in §2.1 suppose quite generally that $\partial\phi/\partial y \sim x^\mu$ as $x \to +0$ and $\phi \sim x^\nu$ as $x \to -0$ on $y = 0$. Then

$$(\alpha + k)^{-1/2}\Phi'_+(0) \sim \alpha^{-(3/2)-\mu} \quad : \quad (\alpha - k)^{1/2}D_- \sim \alpha^{-(1/2)-\nu}$$

as $\alpha \to \infty$ in appropriate half-planes. By the extended form of Liouville's theorem in §1.2 we deduce from (2.87) that $J(\alpha) = P_n(\alpha)$ where $P_n(\alpha)$ is a polynomial of degree less than or equal to n where n is the integral part of the *lesser* of $(-\frac{3}{2} - \mu)$, $(-\frac{1}{2} - \nu)$, providing that this quantity is greater than or equal to zero. ($P_0(\alpha) = C =$ constant.) Thus to prove $J(\alpha) = 0$ it is *sufficient* to assume *either* ν finite and $\mu > -\frac{3}{2}$ i.e. $\partial\phi/\partial y < C_1 x^{-(3/2)+\varepsilon}$ as $x \to 0$ on $y = 0$ *or* μ finite, $\nu > -\frac{1}{2}$ i.e. $\phi < C_2 x^{-(1/2)+\varepsilon}$ as $x \to -0$ on $y = \pm 0$ (ε is an arbitrarily small positive quantity). We have in fact assumed $\mu = -\frac{1}{2}$, $\nu = 0$ i.e. we have assumed conditions on *both* ϕ and $\partial\phi/\partial y$ which are amply sufficient to prove $J(\alpha) = 0$. Also we assumed $\phi \sim C$ as $x \to -0$ (since $\phi_t \sim C$, $\phi_i \sim C$) whereas $\phi \sim x^{1/2}$ as we can prove from the final solution. This is not important so long as we do not relax the condition too much as discussed in the next paragraph. It would be possible to provide a more rigorous procedure by working in terms of inequalities e.g. to write $|\phi| < C$ instead of $\phi \sim C$. But this is unnecessary here.

If we relax the conditions on ϕ and $\partial\phi/\partial y$ we no longer obtain a unique solution for the Sommerfeld problem. Thus suppose we assume $\partial\phi/\partial y \sim C_1 x^{-3/2}$ as $x \to +0$ on $y = 0$, and $\phi \sim C_2 x^{-1/2}$ as $x \to -0$ on $y = \pm 0$. Then Liouville's theorem gives $J(\alpha) = C$ where C is a constant. As in the analysis leading to (2.32a) we find

$$\Phi'_+(0) = -(\alpha + k)^{1/2}H_+(\alpha) + C(\alpha + k)^{1/2},$$

$$\phi = \phi_0 \mp \frac{C}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \frac{\exp(-i\alpha x - \gamma|y|)}{(\alpha - k)^{1/2}}\, d\alpha,$$

where ϕ_0 is given by (2.32a) and upper and lower signs refer to $y \geqslant 0$, $y \leqslant 0$ respectively. On $y = 0$, using ex. 2.4, we find

$$\phi = \begin{cases} \phi_0 \mp Ci\pi^{-1/2}(-x)^{-1/2}e^{ikx}, & (x < 0), \\ \phi_0 \qquad\qquad\qquad\qquad\quad , & (x > 0). \end{cases}$$

If we impose our previous condition $\phi \sim$ const. as $x \to -0$ then of course we must take $C = 0$.

A different kind of non-unique solution has appeared in an elasticity problem investigated by W. T. Koiter [2], pp. 369–370, where the additional term corresponds to a concentrated force at the corner of a wedge.

At first sight it might seem that we can obtain a non-unique solution by a procedure of the following type. Multiply (2.87) by $(\alpha + p)$ where p is some complex constant. Liouville's theorem gives $(\alpha + p)J(\alpha) = \text{constant} = C$, say, and

$$\Phi'_+(0) = -(\alpha + k)^{1/2}H_+(\alpha) + C(\alpha + k)^{1/2}(\alpha + p)^{-1}.$$

But since $\Phi'_+(0)$ is regular in $\tau > -k_2$ we must have $\operatorname{Im} p \leqslant -k_2$. Similarly by examining D_- we find that $\operatorname{Im} p \geqslant k_2 \cos \Theta$. Hence $C = 0$ and we arrive back at our previous solution.

The purpose of the foregoing discussion is to show the part played by the edge conditions (v) of §2.1 in determining the uniqueness of the solution when using the Wiener–Hopf technique. From a more general point of view the question of the necessary edge conditions for uniqueness of solution is connected with existence theorems. The subject has a considerable literature. We mention papers by:

C. J. Bouwkamp [1], p. 45 (a summary); *Philips Res. Rep.* **5** (1950), 401.
D. S. Jones [1]; *Proc. Lond. Math. Soc.* (3) **2** (1952), 440; *Quart. J. Mech. Appl. Math.* **5** (1952), 363.
A. W. Maue, *Z. Phys.* **126** (1949), 601.
J. Meixner, *Ann. Phys.* **6** (1949), 2.

It would take us too far afield to reproduce the arguments of any of these authors in detail. They all lead to essentially the same results from different viewpoints.

In scalar problems it would seem to be sufficient from our point of view to assume $k = 0$ in the immediate neighbourhood of edges and consider the corresponding problem for Laplace's equation. For the problem considered above we have $\partial\phi_t/\partial y = 0$ on $y = 0$, $x < 0$ and it is well known that in the static case $\partial\phi_t/\partial y \sim x^{-1/2}$ on $y = 0$, $x \to +0$, and ϕ_t is finite near the edge. (Physically we can consider the flow of incompressible fluid past the edge of a sheet on which the normal velocity is zero.) Similarly if we had $\phi_t = 0$ on $y = 0$,

$x < 0$ we should have $\phi_t \sim x^{1/2}$ on $y = 0$, $x \to +0$, and $\partial\phi_t/\partial y \sim (-x)^{-1/2}$ on $y = 0$, $x \to -0$. (Physically we can consider the electrostatic field and charge near the sharp edge of a perfectly conducting sheet at zero potential.)

In electromagnetic problems convenient edge conditions are that the component of current density normal to the edge vanishes at the edge as $r^{1/2}$ and the component tangential to the edge vanishes as $r^{-1/2}$, where r is the distance from the edge. The charge density varies as $r^{-1/2}$ near the edge. Discontinuities in field components are related to current and charge densities by

$$\mathbf{n} \times (\mathbf{B}_+ - \mathbf{B}_-) = \Omega\mathbf{k},$$

$$\mathbf{n} \cdot (\mathbf{E}_+ - \mathbf{E}_-) = \omega,$$

where Ω is the magnitude of the current, $\mathbf{k}$ is a unit vector giving its direction, and ω is the charge density. These edge conditions are extremely important since it is possible to produce solutions of problems which satisfy all the boundary conditions except that they have the wrong edge singularities.

2.7 Comparison of methods

We first of all compare Jones's method and the integral equation method.

The integral equation method needs: choice of Green's functions, formulation of the integral equation, application of transforms. Jones's method is more direct since the Wiener–Hopf equation is obtained directly from transforms applied to the partial differential equation.

The Green's function method for formulating integral equations is cumbersome. It is sometimes not completely obvious which Green's function should be chosen; examples have occurred in the literature where this has caused confusion or made problems seem more complicated than need be. Also functions may be introduced whose Fourier transforms are required e.g. the Hankel functions as in (2.66); these complicated functions are avoided altogether in Jones's method—the required transforms are obtained in the process of solution.

In each Wiener–Hopf equation of type (2.25) there are two unknown functions. In Jones's method these appear in a completely symmetrical way and the physical significance of each of the unknown functions is immediately obvious. In the integral equation approach the symmetry is lost in the integral equation itself, though it reappears when the integral equation is transformed.

The main advantage of the integral equation method of approach seems to be that it is very easy to recognize problems that can be solved by the Wiener–Hopf technique since the corresponding integral equations have a semi-infinite range and the kernels are of the form $k(x - \xi)$ i.e. a function of $(x - \xi)$ only (cf. (2.49), (2.55), (2.56)). Sometimes when using Jones's method in complicated problems it may not be so immediately obvious that the transform equations can be reduced to the Wiener–Hopf form.

It may be possible that a solution can be obtained by means of the Wiener–Hopf technique from an integral equation formulated by a Green's function procedure, whereas it may not be possible to obtain the same solution by Jones's method. But no such case has so far occurred in connexion with partial differential equations, to my knowledge.

We have probably mentioned a sufficient number of points to show the reader why we have used Jones's method in most of this book.

At first sight Jones's method and the dual integral equation method of §2.3 seem quite dissimilar except for the use of the fundamental factorization (2.1). Jones's method uses analytic continuation to solve an equation of type (2.25) valid in a strip in the α-plane. The dual integral equation method does not use analytic continuation, and it uses only functions specified on a contour in the α-plane. The relationship between the two methods will be clarified in later sections (e.g. §§4.3, 6.1).

We content ourselves with stating that in this book the relative advantages of the two methods are: Jones's method enables us to give a careful discussion in a routine manner of the regions in which functions are regular, the effect on the solution of edge conditions, and so on. The dual integral equation method will enable us to solve easily problems involving general boundary conditions, e.g. when $\partial\phi_t/\partial y = f(x)$ instead of $\partial\phi_t/\partial y = 0$ on $y = 0$, $x < 0$ (cf. §2.8) but questions of uniqueness and rigour will not be so obvious as in Jones's method.

2.8 Boundary conditions specified by general functions

We have now completed our discussion of basic procedures applied to the Sommerfeld problem. In this section and the next we consider two of the generalizations mentioned in §2.1. These will be sufficient to enable us to deal with the other cases mentioned, some of which are outlined in exercises at the end of this chapter.

At the end of §2.2 it was proved that in the problem considered, $\phi = 0$ on $y = 0$, $x > 0$. Hence the problem solved in §2.2 is equivalent to the following: find a solution of $\nabla^2\phi + k^2\phi = 0$ in $y \geqslant 0$, $-\infty < x < \infty$ such that ϕ represents outgoing waves at infinity; also on $y = 0$ (cf. condition (i) of §2.1),

$$\phi = 0, \qquad x > 0 \quad : \quad \partial\phi/\partial y = ik \sin\Theta \exp(-ikx\cos\Theta).$$

An obvious generalization is: find a solution of $\nabla^2\phi + k^2\phi = 0$ such that ϕ represents outgoing waves at infinity; also on $y = 0$

$$\phi = f(x), \qquad (x > 0) \quad : \quad \partial\phi/\partial y = g(x), \qquad (x < 0). \tag{2.88}$$

Assume that $|f(x)| < C_1 \exp(\tau_- x)$ as $x \to +\infty$, $|g(x)| < C_2 \exp(\tau_+ x)$ as $x \to -\infty$, with $-k_2 \leqslant \tau_- < \tau_+ \leqslant k_2$. There are no sources or incident waves in $y \geqslant 0$. We give three methods of solution.

Perhaps the simplest method is to use the dual integral equation approach of §2.3. As in §§2.2, 2.3 apply a Fourier transform in x and find in $y \geqslant 0$,

$$\Phi = A(\alpha)e^{-\gamma y} \quad : \quad \phi = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} A(\alpha)e^{-i\alpha x-\gamma y}\, d\alpha, \tag{2.89}$$

where $\tau_- < a < \tau_+$. Application of the boundary conditions on $y = 0$ gives the dual integral equations

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} A(\alpha)e^{-i\alpha x}\, d\alpha = f(x), \qquad (x > 0), \tag{2.90a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \gamma A(\alpha)e^{-i\alpha x}\, d\alpha = -g(x), \qquad (x < 0). \tag{2.90b}$$

As in §2.3 replace x by $(x + \xi)$ in the first equation, and by $(x - \xi)$ in the second. Multiply respectively by $\mathcal{N}_1(\xi)$, $\mathcal{N}_2(\xi)$ given by (2.40) and integrate with respect to ξ from 0 to infinity. On using (2.38) with $N_+(\alpha) = (\alpha + k)^{-1/2}$, $N_-(\alpha) = (\alpha - k)^{-1/2}$ as in §2.3, we find

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{-1/2}A(\alpha)e^{-i\alpha x}\, d\alpha$$
$$= \pi^{-1/2}e^{i\pi/4} \int_0^\infty \xi^{-1/2}e^{i\xi k}f(x + \xi)\, d\xi, \qquad (x > 0), \tag{2.91a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{1/2}A(\alpha)e^{-i\alpha x}\, d\alpha$$
$$= -\pi^{-1/2}e^{-i\pi/4} \int_0^\infty \xi^{-1/2}e^{i\xi k}g(x - \xi)\, d\xi, \qquad (x < 0). \tag{2.91b}$$

Multiply the first equation by $\exp(ikx)$ and differentiate with respect to x. Then, for $x > 0$,

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} (\alpha - k)^{1/2}A(\alpha)e^{-i\alpha x}\, d\alpha$$
$$= \pi^{-1/2}ie^{i\pi/4}e^{-ikx} \frac{d}{dx} \int_0^\infty \xi^{-1/2}e^{ik(x+\xi)}f(x + \xi)\, d\xi. \tag{2.92}$$

The left-hand sides of (2.91b), (2.92) are identical. Hence we can use the Fourier inversion theorem to find

$$A(\alpha) = 2^{-1/2}\pi^{-1}(\alpha - k)^{-1/2}e^{3i\pi/4}\left\{\int_0^\infty e^{i(\alpha-k)u}\,du\right.$$

$$\left.\times\frac{d}{du}\int_0^\infty \xi^{-1/2}e^{ik(u+\xi)}f(u+\xi)\,d\xi + \int_{-\infty}^0 e^{i\alpha u}\,du\int_0^\infty \xi^{-1/2}e^{ik\xi}g(u-\xi)\,d\xi\right\}. \tag{2.93}$$

The value of ϕ at any point is obtained by substituting in (2.89). This gives

$$\phi = \tfrac{1}{2}\pi^{-3/2}e^{3i\pi/4}\left\{\int_0^\infty M(u,x,y)e^{-iku}\,du\frac{d}{du}\int_0^\infty \xi^{-1/2}e^{ik(u+\xi)}f(u+\xi)\,d\xi +\right.$$

$$\left.+\int_{-\infty}^0 M(u,x,y)\,du\int_0^\infty \xi^{-1/2}e^{ik\xi}g(u-\xi)\,d\xi\right\}, \tag{2.94a}$$

where

$$M(u,x,y) = \int_{-\infty+ia}^{\infty+ia}(\alpha - k)^{-1/2}e^{i\alpha(u-x)-\gamma y}\,d\alpha. \tag{2.94b}$$

The solution simplifies slightly if $y = 0$ since then (ex. 2.4)

$$M(u,x,0) = \begin{cases} 2\pi^{1/2}e^{i\pi/4}(u-x)^{-1/2}e^{ik(u-x)}, & (u-x) > 0,\\ 0, & (u-x) < 0.\end{cases} \tag{2.95}$$

The nature of the solution will be clarified by discussion of more general equations in Chapter VI.

We next obtain the same solution by Jones's method. As in §2.2 write

$$\Phi_+(0) + \Phi_-(0) = A(\alpha), \tag{2.96a}$$

$$\Phi'_+(0) + \Phi'_-(0) = -\gamma A(\alpha). \tag{2.96b}$$

Eliminate $A(\alpha)$:

$$\Phi'_+(0) + \Phi'_-(0) = -\gamma\{\Phi_+(0) + \Phi_-(0)\}. \tag{2.97}$$

The functions Φ'_+, Φ_- are unknown; Φ'_-, Φ_+ are known. In fact

$$\Phi_+(0) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty f(\xi)e^{i\alpha\xi}\,d\xi \quad : \quad \Phi'_-(0) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^0 g(\xi)e^{i\alpha\xi}\,d\xi. \tag{2.98}$$

Rearrange (2.97) by writing (cf. (2.29))

$$J(\alpha) = (\alpha + k)^{-1/2}\Phi'_+(0) + H_+(\alpha) = -(\alpha - k)^{1/2}\Phi_-(0) - H_-(\alpha),$$

where H_+ and H_- are obtained from

$$H_+(\alpha) + H_-(\alpha) = \{(\alpha + k)^{-1/2}\Phi'_-(0) + (\alpha - k)^{1/2}\Phi_+(0)\},$$

by using theorem B, §1.3 to decompose the function on the right-hand side into the sum of two functions regular in suitable upper and lower half-planes. For example

$$H_-(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+ia}^{\infty+ia}\{(\zeta + k)^{-1/2}\Phi'_-(\zeta,0) + \\ + (\zeta - k)^{1/2}\Phi_+(\zeta,0)\}\frac{d\zeta}{\zeta - \alpha},$$

where $\tau_- < a < \tau_+$ and $(\text{Im } \alpha) = \tau < a$. As in §2.2 we assume that sufficient conditions are given to prove that $J(\alpha)$ is an analytic function regular over the whole α-plane vanishing at infinity so that it must be identically zero. Then $\Phi_-(0) = -(\alpha - k)^{-1/2}H_-(\alpha)$, and

$$A(\alpha) = \Phi_+(0) + (2\pi i)^{-1}(\alpha - k)^{-1/2}\int_{-\infty+ia}^{\infty+ia}\{(\zeta + k)^{-1/2}\Phi'_-(\zeta,0) + \\ + (\zeta - k)^{1/2}\Phi_+(\zeta,0)\}\frac{d\zeta}{\zeta - \alpha}. \quad (2.99)$$

Since $\text{Im } (\zeta - \alpha) > 0$ we can write the first half of the integral as

$$-(2\pi)^{-3/2}(\alpha - k)^{-1/2}\int_{-\infty+ia}^{\infty+ia}(\zeta + k)^{-1/2}\,d\zeta\int_{-\infty}^{0}g(\xi)e^{i\zeta\xi}\,d\xi\int_{0}^{\infty}e^{iu(\zeta-\alpha)}\,du. \quad (2.100)$$

The integral in ζ is (ex. 2.4(d))

$$\int_{-\infty+ia}^{\infty+ia}(\zeta + k)^{-1/2}e^{i\zeta(\xi+u)}\,d\zeta \\ = \begin{cases} 0, & (\xi + u) > 0, \\ 2\pi^{1/2}e^{-i\pi/4}(-\xi - u)^{-1/2}e^{-i(\xi+u)k}, & (\xi + u) < 0, \end{cases}$$

when $\text{Im } (\zeta + k) > 0$, a condition which is assumed to be satisfied in the above analysis. Thus (2.100) reduces to

$$-2^{-1/2}\pi^{-1}(\alpha - k)^{-1/2}e^{-i\pi/4}\int_0^\infty e^{-iu(k+\alpha)}\,du\int_{-\infty}^{-u}g(\xi)e^{-ik\xi}(-\xi - u)^{-1/2}\,d\xi,$$

which is readily proved to be equivalent to the term involving g in (2.93). The remaining part of (2.99) is

$$\Phi_+(0) + (2\pi i)^{-1}(\alpha - k)^{-1/2} \int_{-\infty+ia}^{\infty+ia} (\zeta - k)^{1/2}\Phi_+(\zeta,0)\,\frac{d\zeta}{\zeta-\alpha}, \qquad \operatorname{Im}(\zeta-\alpha) > 0,$$

$$= (2\pi i)^{-1}(\alpha - k)^{-1/2} \int_{-\infty+ib}^{\infty+ib} (\zeta - k)^{1/2}\Phi_+(\zeta,0)\,\frac{d\zeta}{\zeta-\alpha}, \qquad \operatorname{Im}(\zeta-\alpha) < 0, \quad (2.101)$$

where we have shifted the contour parallel to itself over the pole at $\zeta = \alpha$. This is written (cf. (2.100))

$$(2\pi)^{-3/2}(\alpha - k)^{-1/2} \int_{-\infty+ib}^{\infty+ib} (\zeta - k)^{1/2}\,d\zeta \int_0^\infty f(\xi)e^{i\alpha\xi}\,d\xi \int_0^\infty e^{-iu(\zeta-\alpha)}\,du.$$

In this case we cannot repeat our previous procedure exactly since the integral in ζ obtained by changing orders of integration is divergent. However we can write the integral as

$$i(2\pi)^{-3/2}(\alpha - k)^{-1/2} \int_0^\infty e^{i(\alpha-k)u}\,\frac{d}{du}\,e^{iku} \int_0^\infty f(\xi) \times \int_{-\infty+ib}^{\infty+ib} (\zeta - k)^{-1/2}\,e^{i\zeta(\xi-u)}\,d\zeta\,d\xi\,du. \quad (2.102)$$

The integral in ζ is (ex. 2.4c)

$$\int_{-\infty+ib}^{\infty+ib} (\zeta - k)^{-1/2}e^{i\zeta(\xi-u)}\,d\zeta = \begin{cases} 2\pi^{1/2}e^{i\pi/4}(\xi - u)^{-1/2}e^{ik(\xi-u)}, & (\xi - u) > 0, \\ 0\,, & (\xi - u) < 0. \end{cases}$$

The result of substituting this in (2.102) is readily proved to be identical with the term in (2.93) involving f.

We finally obtain the same results by a "superposition method" which is practically the same as the second method just described, except that it does not make explicit use of theorem B, §1.3. We have

$$\Phi_+(\zeta,0) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} f(\xi) e^{i\zeta\xi}\, d\xi$$

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ib}^{\infty+ib} \Phi_+(\zeta,0) e^{-i\zeta x}\, d\zeta, \tag{2.103a}$$

$$\Phi'_-(\zeta,0) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} g(\xi) e^{i\zeta\xi}\, d\xi$$

$$g(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \Phi'_-(\zeta,0) e^{-i\zeta x}\, d\zeta, \tag{2.103b}$$

where in the first equation $b > \tau_-$, and in the second $a < \tau_+$. Solve two separate problems:

(a) $\partial\phi/\partial y = \exp(-i\zeta x)$ on $y = 0$, $x < 0$: $\phi = 0$ on $y = 0$, $x > 0$. If we use Jones's method exactly as in §2.2, we find

$$\Phi_-(0) = A_1(\alpha,\zeta), \text{ say} \quad : \quad \Phi'_+(0) = i(2\pi)^{-1/2}(\alpha-\zeta)^{-1} - \gamma A_1(\alpha,\zeta),$$

where

$$A_1(\alpha,\zeta) = \frac{i}{(2\pi)^{1/2}} \cdot \frac{1}{(\alpha-k)^{1/2}(\zeta+k)^{1/2}(\alpha-\zeta)}, \ \mathrm{Im}\,(\alpha-\zeta) < 0.$$

(b) $\partial\phi/\partial y = 0$ on $y = 0$, $x < 0$: $\phi = \exp(-i\zeta x)$ on $y = 0$, $x > 0$. Jones's method gives

$$\Phi'_+(0) = -\gamma A_2(\alpha,\zeta), \quad \text{say} \quad :$$

$$\Phi_-(0) = -i(2\pi)^{1/2}(\alpha-\zeta)^{-1} + A_2(\alpha,\zeta),$$

where

$$A_2(\alpha,\zeta) = \frac{i}{(2\pi)^{1/2}} \cdot \frac{(\zeta-k)^{1/2}}{(\alpha-k)^{1/2}(\alpha-\zeta)}, \qquad \mathrm{Im}\,(\alpha-\zeta)<0.$$

Now superimpose solutions using (2.103). This yields

$$A(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ia}^{\infty+ia} \Phi'_+(\zeta,0) A_1(\alpha,\zeta)\, d\zeta + \frac{1}{(2\pi)^{1/2}} \int_{-\infty+ib}^{\infty+ib} \Phi_+(\zeta,0) A_2(\alpha,\zeta)\, d\zeta,$$

where $\tau_- < b < \tau < a < \tau_+$. On substituting our expressions for A_1, A_2 we find that this is identical with (2.99) (remember result (2.101)).

2.9 Radiation-type boundary conditions

We deal briefly with one example of problems where radiation-type boundary conditions occur. Examples of a similar type, with references, are given in ex. 2.12. The problem treated here is equivalent to diffraction of an electromagnetic wave by a semi-infinite metallic sheet of finite conductivity which has been treated in detail by T. B. A. Senior [1]. We choose our conventions so that our results are directly comparable with those of Senior. (See the end of this section. Note that we take the plane to lie in $y = 0$, $x > 0$, not $y = 0$, $x < 0$ as in the first part of this chapter.)

We wish to solve $\nabla^2\phi_t + k^2\phi_t = 0$ in $-\infty < x$, $y < \infty$ with incident wave $\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta)$. Suppose that an imperfectly absorbent sheet lies in the plane $y = 0$, $x > 0$, the conditions on the sheet being

$$\phi_t = \pm i\delta(\partial\phi_t/\partial y), \qquad y = \pm 0, \quad (0 \leqslant x < \infty). \tag{2.104}$$

Write $\phi_t = \phi_i + \phi$. A Fourier transform can be applied to ϕ. As in §2.2 we have $\Phi = A(\alpha)\exp(-\gamma y)$, $y \geqslant 0$, and $\Phi = B(\alpha)\exp(\gamma y)$, $y \leqslant 0$. In the usual notation these give

$$\Phi_+(+0) + \Phi_-(0) = A(\alpha), \tag{2.105a}$$

$$\Phi'_+(+0) + \Phi'_-(0) = -\gamma A(\alpha), \tag{2.105b}$$

$$\Phi_+(-0) + \Phi_-(0) = B(\alpha), \tag{2.106a}$$

$$\Phi'_+(-0) + \Phi'_-(0) = \gamma B(\alpha). \tag{2.106b}$$

Elimination of $A(\alpha)$, $B(\alpha)$ gives

$$\Phi'_+(+0) + \Phi'_-(0) = -\gamma\{\Phi_+(+0) + \Phi_-(0)\}, \tag{2.107a}$$

$$\Phi'_+(-0) + \Phi'_-(0) = \gamma\{\Phi_+(-0) + \Phi_-(0)\}. \tag{2.107b}$$

If we write $\phi_t = \phi_i + \phi$ in (2.104) and apply a Fourier transform we find

$$i(2\pi)^{-1/2}(1 - \delta k\sin\Theta)(\alpha - k\cos\Theta)^{-1} + \Phi_+(+0) - i\delta\Phi'_+(+0) = 0, \tag{2.108a}$$

$$i(2\pi)^{-1/2}(1 + \delta k\sin\Theta)(\alpha - k\cos\Theta)^{-1} + \Phi_+(-0) + i\delta\Phi'_+(-0) = 0. \tag{2.108b}$$

Eliminate $\Phi_+(+0)$ between (2.107a) and (2.108a): eliminate $\Phi_+(-0)$ between (2.107b) and (2.108b). Then

$$\Phi'_-(0) + \gamma\Phi_-(0) = -(1 + i\delta\gamma)\Phi'_+(+0) + \\ + i(2\pi)^{-1/2}\gamma(1 - \delta k\sin\Theta)(\alpha - k\cos\Theta)^{-1},$$

$$\Phi'_-(0) - \gamma\Phi_-(0) = -(1 + i\delta\gamma)\Phi'_+(-0) - \\ - i(2\pi)^{-1/2}\gamma(1 + \delta k\sin\Theta)(\alpha - k\cos\Theta)^{-1}.$$

Two independent Wiener–Hopf equations are obtained by adding and subtracting these equations:

$$2\Phi'_-(0) = -(1 + i\delta\gamma)\{\Phi'_+(+0) + \Phi'_+(-0)\} - \\ - 2i(2\pi)^{-1/2}\delta\gamma k \sin\Theta(\alpha - k\cos\Theta)^{-1}, \qquad (2.109a)$$

$$2\gamma\Phi_-(0) = -(1 + i\delta\gamma)\{\Phi'_+(+0) - \Phi'_+(-0)\} + \\ + 2i(2\pi)^{-1/2}\gamma(\alpha - k\cos\Theta)^{-1}. \qquad (2.109b)$$

The logic of the procedure by which (2.109a, b) are obtained should be clear. The writing down of (2.105), (2.106) is routine. We wish to obtain equations involving only functions whose regions of regularity are known, so we eliminate $A(\alpha)$, $B(\alpha)$. Next use the boundary conditions: these lead to (2.108). In (2.107), (2.108) we have four equations involving six unknown functions $\Phi_-(0)$, $\Phi'_-(0)$, $\Phi_+(\pm 0)$, $\Phi'_+(\pm 0)$. (2.108) involve only $\Phi_+(\pm 0)$, $\Phi'_+(\pm 0)$. So clearly we must try to obtain two Wiener–Hopf equations involving $\Phi_-(0)$, $\Phi'_-(0)$ and two "plus" functions, using (2.108) to eliminate two of the four "plus" functions from (2.107). There are many ways of carrying out the manipulations, but the reader will readily show that the final Wiener–Hopf equations are always equivalent to (2.109).

The main object of this section is to show in detail how the Wiener–Hopf equations (2.109) are obtained. Now that this has been accomplished we merely summarize the remaining steps in the solution. Both of the equations in (2.109) are similar to:

$$p(\alpha - k\cos\Theta)^{-1} + F_-(\alpha) = K(\alpha)G_+(\alpha), \qquad (2.110)$$

where p is some constant. The procedure is:

(a) Prove that the equation holds in a strip $k_2 \cos\Theta < \tau < k_2$. (In the Sommerfeld problem of §2.1 the strip was $-k_2 < \tau < k_2\cos\Theta$ because in that case the sheet was chosen in $y = 0$, $x < 0$.)

(b) Factorize $K(\alpha) = 1 + i\delta\gamma$ in the form $K_+(\alpha)K_-(\alpha)$ (ex. 2.10).

(c) Rewrite (2.110) as

$$P(\alpha) = \frac{p}{\alpha - k\cos\Theta}\left\{\frac{1}{K_-(\alpha)} - \frac{1}{K_-(k\cos\Theta)}\right\} + \frac{F_-(\alpha)}{K_-(\alpha)}$$
$$= K_+(\alpha)G_+(\alpha) - \frac{p}{(\alpha - k\cos\Theta)}\cdot\frac{1}{K_-(k\cos\Theta)}\cdot \qquad (2.111)$$

(d) Investigate the behaviour of (2.111) as α tends to infinity in appropriate half-planes so that Liouville's theorem can be applied to determine $P(\alpha)$.

One important difference between this problem and the other problems considered in this chapter is that previously the factorization step (b) could be carried out by inspection (namely $(\alpha^2 - k^2)^{1/2} = (\alpha - k)^{1/2}(\alpha + k)^{1/2}$) whereas in the present case the factorization of

$$K(\alpha) = 1 + i\delta(\alpha^2 - k^2)^{1/2} = K_+(\alpha)K_-(\alpha)$$

is much more difficult. Details are given in ex. 2.10.

A detailed examination of (2.109a, b) along these lines would take us too far afield. We shall examine several similar cases in Chapter III, and details for the present case will be found in T. B. A. Senior [1], who gives explicit expressions for the distant field. Senior's integral equation approach is of course more cumbersome than Jones's method.

Finally we compare our notation with that used by Senior. He uses the terminology of electromagnetic theory as explained in ex. 2.11, the results of which we now assume. The problem treated here is equivalent to the case of E-polarization. In the following μ is the permeability, ε the dielectric constant. We have $\phi_t \equiv E_z$, $\partial\phi_t/\partial y \equiv i\mu\omega H_x$. The boundary condition used by Senior, corresponding to (2.104), is $E_z = \mp\eta Z H_x$ on $y = \pm 0$, where η is the conductivity and $Z = \mu^{1/2}\varepsilon^{-1/2}$. Thus on comparing our notation with that of Senior,

$$-\eta Z(i\mu\omega)^{-1} \equiv i\delta \quad \text{i.e.} \quad \delta = \eta/k, \quad \text{where} \quad k = \omega(\mu\varepsilon)^{1/2},$$

and k is the constant in the wave equation $\nabla^2\phi + k^2\phi = 0$. Senior also writes

$$I_1(x) = (E_z)_{y=+0} - (E_z)_{y=-0}, \quad \text{i.e.} \quad \bar{I}_1(\alpha) \equiv \Phi_+(+0) - \Phi_+(-0),$$

where we have used a bar to denote the transform of I_1. Also

$$I_2(x) = (H_x)_{y=+0} - (H_x)_{y=-0},$$
$$\text{i.e.} \quad \bar{I}_2(\alpha) \equiv \frac{1}{i\mu\omega}\{\Phi'_+(+0) - \Phi'_+(-0)\}.$$

Noting that $\alpha^S \equiv \Theta$, $\zeta^S \equiv -\alpha$ where the superscript 'S' refers to Senior's notation, it will be found that (2.108), (2.109) are identical with Senior's (27) and (14).

Although many of the problems in this chapter can be solved by other means, some of the virtues of methods based on the Wiener–Hopf technique should already be apparent. In particular the procedures used to solve half-plane problems with general boundary conditions and with radiation type boundary conditions are direct extensions of the method for the much easier Sommerfeld problem. In the next chapter we consider examples involving more complicated geometrical configurations.

Miscellaneous Examples and Results II

2.1 The conventions used in this book are to some extent arbitrary and in the following various other possibilities are illustrated in connexion with the Sommerfeld half-plane problem. For convenience we quote two equations from §2.2:

$$\Phi'_+(0) + \Phi'_-(0) = -\tfrac{1}{2}\gamma\{\Phi_-(+0) - \Phi_-(-0)\},$$
$$-k_2 < \tau < k_2 \cos\Theta, \quad [(2.20)]$$
$$\Phi'_-(0) = (2\pi)^{-1/2} k \sin\Theta(\alpha - k\cos\Theta)^{-1}. \quad [(2.21)]$$

Most of the details below are left to the reader.

(i) If the half-plane on which $\partial\phi_t/\partial y = 0$ lies in $y = 0$, $0 \leqslant x < \infty$ instead of $-\infty < x \leqslant 0$, then (2.20), (2.21) become

$$\Phi'_+(0) + \Phi'_-(0) = -\tfrac{1}{2}\gamma\{\Phi_+(+0) - \Phi_+(-0)\}, \qquad k_2\cos\Theta < \tau < k_2,$$
$$\Phi'_+(0) = -(2\pi)^{-1/2} k\sin\Theta(\alpha - k\cos\Theta)^{-1}.$$

(ii) If the half-plane lies in $x = 0$, $-\infty < y \leqslant 0$, with $\partial\phi_t/\partial x = 0$, a Fourier transform in y is used. The incident wave is given by (2.3) but $-\tfrac{1}{2}\pi < \Theta < +\tfrac{1}{2}\pi$. (2.20) is unchanged in form but holds in $-k_2 < \tau < k_2 \sin\Theta$. (2.21) becomes

$$\Phi'_-(0) = (2\pi)^{-1/2} k\cos\Theta(\alpha - k\sin\Theta)^{-1}.$$

(iii) If the time factor is $\exp(+i\omega t)$ then k has a negative imaginary part and $\gamma = (\alpha^2 - k^2)^{1/2} = +i(k^2 - \alpha^2)^{1/2}$. The incident wave, if situated as in Fig. 2.1, is $\phi_i = \exp(ikx\cos\Theta + iky\sin\Theta)$, $0 < \Theta < \pi$. (2.20) is unchanged but

$$\Phi'_-(0) = -(2\pi)^{-1/2} k\sin\Theta(\alpha + k\cos\Theta)^{-1}.$$

The result is, essentially, to change the sign of k (or more exactly to change the sign of k_1 but not of k_2). If $k_2 \to 0$ then the contour of integration in the inversion formula is the real axis, indented *below* at $\alpha = -k$, $\alpha = -k\cos\Theta$, and *above* at $\alpha = +k$.

(iv) A wave incident from the lower half-plane can be obtained by taking $\pi < \Theta < 2\pi$, or by setting $\phi_i = \exp(ikx\cos\Theta + iky\sin\Theta)$, $0 < \Theta < \pi$.

(v) The sign of α can be reversed in the Fourier transform (i.e. we take the lower sign in (1.52), §1.5). Define

$$\Phi(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi e^{-i\alpha x}\, dx; \qquad \Phi_-(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{0}^{\infty} \phi e^{-i\alpha x}\, dx, \text{ etc.}$$

The result is essentially to interchange upper and lower halves of the α-plane.

(vi) The Laplace transform can be used instead of the Fourier transform. Define

$$\Phi(s,y) = \int_{-\infty}^{\infty} \phi e^{-sx}\, dx \quad : \quad \Phi_p(s) = \int_{0}^{\infty} \phi e^{-sx}\, dx,$$

and similarly for Φ_N where Φ_P, Φ_N are valid in right and left half-planes respectively. The partial differential equation gives

$$\Phi = A(s)\exp(i\kappa y), \qquad (y \geqslant 0); \qquad -A(s)\exp(-i\kappa y), \qquad (y \leqslant 0),$$

where $\kappa = (s^2 + k^2)^{1/2}$ with $\kappa = k$ when $s = 0$. The results for Fourier and Laplace transforms will be found to correspond exactly if $\alpha \equiv is$, $\gamma \equiv -i\kappa$, $(\mathrm{Im}\,\alpha) \equiv (\mathrm{Re}\,s)$, $\Phi_+(\alpha) \equiv (2\pi)^{-1/2}\Phi_P(s)$, etc. for $k_2 > 0$.

2.2 Suppose we wish to solve $(\phi_t)_{xx} + (\phi_t)_{yy} + k^2\phi_t = s(x,y)$ with a half-plane in $y = 0$, $-\infty < x \leqslant 0$ on which $\partial\phi_t/\partial y = 0$. The term $s(x,y)$ represents an arbitrary source distribution and there is now no wave incident from infinity. We use the method of superposition. Set

$$S(\beta,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s(x,y)e^{i(\beta x+\zeta y)}\,dx\,dy$$

$$s(x,y) = \frac{1}{2\pi}\int_{ic-\infty}^{ic+\infty}\int_{id-\infty}^{id+\infty} S(\beta,\zeta)e^{-i(\beta x+\zeta y)}\,d\beta\,d\zeta.$$

First of all solve the special case

$$(\psi_t)_{xx} + (\psi_t)_{yy} + k^2\psi_t = \exp\{-i(\beta x + \zeta y)\}.$$

Introduce ψ defined by $\psi_t = \psi + (k^2 - \beta^2 - \zeta^2)^{-1}\exp\{-i(\beta x + \zeta y)\}$. Then we have to solve $\psi_{xx} + \psi_{yy} + k^2\psi = 0$ with boundary condition $\partial\psi/\partial y = i\zeta(k^2 - \beta^2 - \zeta^2)^{-1}\exp(-i\beta x)$ on $y = 0, x \leqslant 0$. From (2.6), (2.31), if $-k_2 < \mathrm{Im}\,\alpha < \mathrm{Im}\,\beta < k_2$, and $\lambda = +1 (y > 0)$, $-1 (y < 0)$,

$$\psi = \frac{-\lambda\zeta}{2\pi(k+\beta)^{1/2}(k^2-\beta^2-\zeta^2)}\int_{ia-\infty}^{ia+\infty}\frac{e^{-i\alpha x-\gamma|y|}}{(\alpha-k)^{1/2}(\alpha-\beta)}\,d\alpha.$$

By superposition we can set $\phi_t = \phi_i + \phi$, where

$$\phi_i = \frac{1}{2\pi}\int_{ic-\infty}^{ic+\infty} d\beta\int_{id-\infty}^{id+\infty} d\zeta\,\frac{S(\beta,\zeta)}{k^2-\beta^2-\zeta^2}\,e^{-i(\beta x+\zeta y)}, \tag{a}$$

$$\phi = \frac{-\lambda}{4\pi^2}\int_{ic-\infty}^{ic+\infty} d\beta\int_{id-\infty}^{id+\infty} d\zeta\,\frac{\zeta S(\beta,\zeta)}{(k+\beta)^{1/2}(k^2-\beta^2-\zeta^2)} \times \int_{ia-\infty}^{ia+\infty}\frac{e^{-i\alpha x-\gamma|y|}}{(\alpha-k)^{1/2}(\alpha-\beta)}\,d\alpha. \tag{b}$$

This gives the answer to the problem.

In the special case of a line source, $s(x,y) = -4\pi\delta(x - x_0)\delta(y - y_0)$. Hence $S(\beta,\zeta) = -2\exp(i\beta x_0 + i\zeta y_0)$. The integral in ζ in (a) and (b) can be evaluated by residues and the integral obtained from (a) can then be evaluated from (1.60), (1.61). This gives

$$\phi_i = \pi i H_0^{(1)}(kR), \qquad R^2 = (x - x_0)^2 + (y - y_0)^2.$$

$$\phi = -\frac{\lambda\lambda_0}{2\pi} \int_{ic-\infty}^{ic+\infty} d\beta \int_{ia-\infty}^{ia+\infty} d\alpha \, \frac{e^{-i\alpha x - \gamma|y|} e^{-i\beta x_0 - \mu|y_0|}}{(\alpha - k)^{1/2}(\beta - k)^{1/2}(\alpha + \beta)},$$

where $-k_2 < a < -c < k_2$, $\mu = (\beta^2 - k^2)^{1/2}$ and we have changed the sign of β. This is symmetrical in (x,y), (x_0,y_0) and in α, β as we should expect. Other applications of a similar type of analysis are given by P. C. Clemmow [2] who deals with reduction of the double integral to a single integral, and with asymptotic evaluation of more general double integrals. The half-plane problem for a line source has also been treated by R. F. Harrington [1].

2.3 Suppose that instead of $\partial\phi_t/\partial y = 0$ on $y = 0$, $x < 0$, we have $\phi_t = 0$ for the Sommerfeld half-plane problem of §2.1. If $\phi_t = \phi_i + \phi$, $\Phi_-(0) = i(2\pi)^{-1/2}(\alpha - k\cos\Theta)^{-1}$. The complex variable equation corresponding to (2.24) turns out to be

$$\Phi_+(0) + i(2\pi)^{-1/2}(\alpha - k\cos\Theta)^{-1} = -\gamma^{-1}S'_-(0), \quad (-k_2 < \tau < k_2\cos\Theta),$$

where $S'_-(0) = \Phi'_-(+0) - \Phi'_-(-0)$. The solution corresponding to (2.31) is

$$A(\alpha) = \frac{i}{(2\pi)^{1/2}(\alpha - k\cos\Theta)} \cdot \frac{(k + k\cos\Theta)^{1/2}}{(\alpha + k)^{1/2}}, \quad (-k_2 < \tau < k_2\cos\Theta).$$

2.4
$$\int_0^\infty t^{q-1} e^{\pm i\alpha t}\, dt = \Gamma(q)\alpha^{-q} e^{\pm iq\pi/2}, \qquad (0 < \operatorname{Re} q < 1),$$

$$\int_{-\infty+ia}^{\infty+ia} \alpha^{-q} e^{\mp i\alpha t}\, d\alpha = \begin{cases} 2\pi\{\Gamma(q)\}^{-1} t^{q-1} e^{\mp iq\pi/2}, & (t > 0), \\ 0, & (t < 0), \end{cases}$$

where upper and lower signs go together, $(\operatorname{Im}\alpha) > 0$ for the upper sign and $(\operatorname{Im}\alpha) < 0$ for the lower. The first is a standard result and the second is its Fourier inverse. We can deduce the following useful special cases:

$$\int_0^\infty \xi^{-1/2} e^{\pm i\xi(\alpha\pm k)}\, d\xi = e^{\pm i\pi/4}\, \pi^{1/2}(\alpha \pm k)^{-1/2}, \qquad \operatorname{Im}(k \pm \alpha) > 0, \text{(a)}$$

$$\int_{-\infty}^0 (-\xi)^{-1/2} e^{\mp i\xi(\alpha\pm k)}\, d\xi = e^{\pm i\pi/4}\, \pi^{1/2}(\alpha \pm k)^{-1/2}, \qquad \operatorname{Im}(k \pm \alpha) > 0, \text{(b)}$$

$$\int_{ia-\infty}^{ia+\infty} (\alpha \pm k)^{-1/2} e^{\mp i\alpha\xi}\, d\alpha = \left.\begin{matrix} 2\pi^{1/2} e^{\mp i\pi/4} \xi^{-1/2} e^{+i\xi k}, & \xi > 0 \\ 0, & \xi < 0 \end{matrix}\right\},$$

$$\operatorname{Im}(k \pm \alpha) > 0, \qquad \text{(c)}$$

$$\int_{ia-\infty}^{ia+\infty} (\alpha \pm k)^{-1/2} e^{\pm i\alpha\xi}\, d\alpha = \left.\begin{array}{ll} 0, & \xi > 0 \\ 2\pi^{1/2} e^{\mp i\pi/4}(-\xi)^{-1/2} e^{-i\xi k}, & \xi < 0 \end{array}\right\},$$

$$\operatorname{Im}(k \pm \alpha) > 0, \quad \text{(d)}$$

where upper and lower signs go together.

2.5 Another method of solving the general equation (2.97) is the following. Multiply through by $(\alpha + k)^{-1/2}$ and take the inverse Fourier transform, replacing α by say β for convenience:

$$-\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \{(\beta + k)^{-1/2}\Phi'_-(0) + (\beta - k)^{1/2}\Phi_+(0)\} e^{-i\beta x}\, d\beta$$

$$= \begin{cases} \dfrac{1}{(2\pi)^{1/2}} \displaystyle\int_{ia-\infty}^{ia+\infty} (\beta + k)^{-1/2}\Phi'_+(0) e^{-i\beta x}\, d\beta, & (x > 0), \\[2ex] \dfrac{1}{(2\pi)^{1/2}} \displaystyle\int_{ia-\infty}^{ia+\infty} (\beta - k)^{1/2}\Phi_-(0) e^{-i\beta x}\, d\beta, & (x < 0), \end{cases}$$

where the integrals involving Φ'_+, Φ_- are identically zero when $x < 0$, $x > 0$ respectively. If we multiply through by $(2\pi)^{-1/2}\exp(i\alpha x)$ and integrate with respect to x from 0 to ∞, the right-hand side becomes $(\alpha + k)^{-1/2}\Phi'_+(0)$: if the integration is in $(-\infty, 0)$, the right-hand side becomes $(\alpha - k)^{+1/2}\Phi_-(0)$. The solution in §2.8 can be derived from these results.

The basis of the method just described is the following procedure for writing $F(\alpha)$ in the form $F_+(\alpha) + F_-(\alpha)$. Define $f(x)$ by

$$f(x) = \frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} F(\alpha) e^{-i\alpha x}\, d\alpha.$$

Then

$$F_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^{\infty} f(x) e^{i\alpha x}\, dx \quad : \quad F_-(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{0} f(x) e^{i\alpha x}\, dx.$$

2.6 When integral equations were established by the Green's function method in §2.4, the space was divided into the two regions $0 \leqslant y < \infty$ and $-\infty < y \leqslant 0$. Alternatively it is possible to apply Green's theorem to the whole space, excluding the line $-\infty < x < 0$ occupied by the half-plane, e.g. we can set $\phi_t = \phi_i + \phi$, and apply Green's theorem to obtain an expression for ϕ, by integrating over $R\exp(i\theta)$, $-\pi \leqslant \theta \leqslant \pi$ for some large R; $-R < x < r$, $y = +0$ (i.e. $\theta = +\pi$); $r\exp(i\theta)$, $-\pi \leqslant \theta \leqslant \pi$ for some small r; $-r > x > -R$, $y = -0$ (i.e. $\theta = -\pi$). Then let $R \to \infty$, $r \to 0$. This will obviously give an integral equation in terms of the discontinuity in ϕ across $y = 0$, $-\infty < x < 0$, for the Sommerfeld problem of §2.1.

2.7 It is sometimes possible to obtain the integral equation for a problem in a direct way by a physical argument. Consider a half-plane in $y = 0$, $-\infty < x \leqslant 0$ under two conditions.

(i) If there is a charge distribution $u(x)$ on $y = 0$, $x < 0$, the potential at (x,y) due to a unit line charge at $(\xi,0)$ is $\pi i H_0^{(1)}(kR)$, $R^2 = (x-\xi)^2 + y^2$, and by superposition the total potential at (x,y) is

$$\phi(x,y) = \pi i \int_{-\infty}^{0} H_0^{(1)}(kR)u(\xi)\, d\xi.$$

The potential is continuous on crossing $y = 0$, $x < 0$ but the derivative of the potential in the y-direction is discontinuous.

(ii) If there is a distribution $v(x)$ of y-oriented dipoles on $y = 0$, $x < 0$, by superposition the total potential at (x,y) is

$$\phi(x,y) = \lim_{\eta\to 0} \pi i \frac{\partial}{\partial \eta} \int_{-\infty}^{0} H_0^{(1)}(kR)v(\xi)\, d\xi, \qquad R^2 = (x-\xi)^2 + (y-\eta)^2.$$

The potential is discontinuous across the layer but the y-derivative of the potential is continuous.

As an example consider the Sommerfeld problem of §2.1. Using the notation $\phi_t = \phi_i + \phi$, $\partial\phi/\partial y$ is continuous across $y = 0$ and we can use representation (ii) above. This gives

$$\left(\frac{\partial \phi}{\partial y}\right)_{y=0} = \lim_{y\to 0}\lim_{\eta\to 0} \pi i \frac{\partial^2}{\partial y\,\partial \eta} \int_{-\infty}^{0} H_0^{(1)}(kR)v(\xi)\, d\xi = ik \sin \Theta e^{-ikx\cos\Theta}.$$

This agrees with (2.54), apart from a factor of proportionality relating v and e. Similarly for the absorbent half-plane on which $\phi_t = 0$ we should use (i) above.

2.8 It is possible to obtain the integral equations of §2.4 by applying transforms in y instead of x. As an example consider case (2b) of §2.4. Set $\phi_t = \phi_i + \phi$ and apply a transform in y. On integrating by parts, since ϕ is discontinuous on $y = 0$,

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{\partial^2 \phi}{\partial y^2} e^{i\alpha y}\, dy = i\alpha(2\pi)^{-1/2} 2e(x) - \alpha^2 \Phi(\alpha),$$

where $2e(x) = \phi_t(x,+0) - \phi_t(x,-0)$, $-\infty < x < 0$, and of course $e(x) = 0$ for $x > 0$. On solving the ordinary differential equation for Φ by ex. 1.15, and inverting,

$$\phi = \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} i\alpha\gamma^{-1} e^{-i\alpha y} \int_{-\infty}^{0} e(\xi) e^{-\gamma|x-\xi|}\, d\xi\, d\alpha.$$

If we use the boundary condition $\partial\phi/\partial y = ik \sin \Theta \exp(-ikx \cos \Theta)$, we find, by an obvious manipulation,

$$-ik \sin \Theta e^{-ikx \cos \Theta} = \lim_{y \to 0} \frac{1}{2\pi} \frac{d^2}{dy^2} \int_{-\infty}^{0} e(\xi) \int_{ia-\infty}^{ia+\infty} \gamma^{-1} e^{-\gamma|x-\xi| - i\alpha y} \, d\alpha \, d\xi, \quad (x < 0).$$

This is equivalent to (2.64), by (1.60), (1.61).

2.9 The half-plane problems of this chapter (mixed boundary conditions on $y = 0$, $-\infty < x < 0$, $0 < x < \infty$) can also be solved by introducing unknown functions on $x = 0$, $-\infty < y < 0, 0 < y < \infty$.

2.10 In order to perform the factorization (2.112) write

$$\ln K_+(\alpha) + \ln K_-(\alpha) = \ln \{1 + i\delta(\alpha^2 - k^2)^{1/2}\},$$

$$\frac{d}{d\alpha} \ln K_+(\alpha) + \frac{d}{d\alpha} \ln K_-(\alpha) = i\delta\alpha\{1 + i\delta(\alpha^2 - k^2)^{1/2}\}^{-1}(\alpha^2 - k^2)^{-1/2}$$

$$= \frac{1}{2(\alpha - id)} + \frac{1}{2(\alpha + id)} + \frac{i}{2\delta}\left(\frac{1}{\alpha - id} + \frac{1}{\alpha + id}\right)\frac{1}{(\alpha^2 - k^2)^{1/2}} \tag{a}$$

where $d^2 = \{(1/\delta^2) - k^2\}$. This is obtained by multiplying the fraction in the previous line top and bottom by $\{1 - i\delta(\alpha^2 - k^2)^{1/2}\}$ and separating into partial fractions. Now use the result obtained at the end of §1.3, that if

$$f_\pm(\alpha) = \pi^{-1}(\alpha^2 - k^2)^{-1/2} \operatorname{arc\,cos}(\pm\alpha/k),$$

then $f_+(\alpha), f_-(\alpha)$ are regular in $\tau > -k_2$, $\tau < k_2$ respectively, and

$$f_+(\alpha) + f_-(\alpha) = (\alpha^2 - k^2)^{-1/2}.$$

From this we deduce by inspection that if p is any complex number and

$$F_+(\alpha) = \frac{1}{\alpha - p}\{f_+(\alpha) \mp f_\pm(p)\} \quad : \quad F_-(\alpha) = \frac{1}{\alpha - p}\{f_-(\alpha) \pm f_\pm(p)\},$$

where the upper sign holds for p in the upper half-plane and the lower sign for p in the lower half-plane, then $F_+(\alpha)$, $F_-(\alpha)$ are regular in upper and lower half-planes respectively, and

$$F_+(\alpha) + F_-(\alpha) = (\alpha - p)^{-1}(\alpha^2 - k^2)^{-1/2}.$$

From (a), using the fact that if $\alpha = -id$ then $(\alpha^2 - k^2)^{1/2} = (-i/\delta)$, we can show that

$$\frac{d}{d\alpha} \ln K_+(\alpha) \doteq \frac{i}{2\delta}\left\{\frac{f_+(\alpha) - f_+(id)}{\alpha - id} + \frac{f_+(\alpha) - f_+(-id)}{\alpha + id}\right\}. \tag{b}$$

A similar procedure can be used for

$$K(\alpha) = 1 + i\varepsilon(\alpha^2 - k^2)^{-1/2}.$$

The results can be deduced from the above or we can proceed independently to find

$$\frac{d}{d\alpha}\ln K_+(\alpha) = -\frac{1}{2(\alpha+k)} - \frac{i\varepsilon}{2}\left\{\frac{f_+(\alpha)-f_+(p)}{\alpha-p} + \frac{f_+(\alpha)-f_+(-p)}{\alpha+p}\right\}, \quad \text{(c)}$$

where $p = (k^2 - \varepsilon^2)^{1/2}$.

In order to find $K_+(\alpha)$ expressions (b) and (c) have to be integrated. It is not possible to do this explicitly but formulae which are practical for numerical work may be obtained by the substitution $\theta = \text{arc cos}\,(\alpha/k)$ which gives, for example,

$$\int^{\alpha} \frac{f_+(\alpha)}{\alpha - id}\,d\alpha = \frac{1}{\pi i}\int^{\text{arc cos}\,(\alpha/k)} \frac{\theta\,d\theta}{d + ik\cos\theta}, \qquad d^2 = (\delta^{-2} - k^2). \quad \text{(d)}$$

The denominator can be expanded as a series in $(ik/d)\cos\theta$ if d is large.

Similar formulae are given by T. B. A. Senior [1] and A. E. Heins and H. Feshbach [2]. Senior's formulae can be obtained by writing

$$\frac{2}{\alpha^2+d^2} = \delta^2\left\{\frac{1}{1-\delta(k^2-\alpha^2)^{1/2}} + \frac{1}{1+\delta(k^2-\alpha^2)^{1/2}}\right\},$$

$$\int^{\alpha} \frac{f_+(\alpha)\,d\alpha}{1-\delta(k^2-\alpha^2)^{1/2}} = \frac{1}{\pi i}\int^{\text{arc cos}\,(\alpha/k)} \frac{\theta\,d\theta}{1-\delta k\sin\theta}.$$

2.11 We remind the reader of the connexion between the scalar wave equation and certain two-dimensional cases of the electromagnetic equations

$$\text{curl}\,\mathbf{E} = i\mu\omega\mathbf{H} \quad : \quad \text{curl}\,\mathbf{H} = -i\varepsilon\omega\,\mathbf{E},$$

where ε is the dielectric constant and μ is the permeability. If the fields are independent of the z-co-ordinate then these separate into two independent sets:

(a) *TM* or *transverse magnetic* waves when $H_z = E_x = E_y = 0$ and

$$\frac{\partial E_z}{\partial y} = i\mu\omega H_x \quad : \quad \frac{\partial E_z}{\partial x} = -i\mu\omega H_y \quad : \quad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -i\varepsilon\omega E_z.$$

Then E_z satisfies

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k^2 E_z = 0, \qquad k^2 = \varepsilon\mu\omega^2.$$

The boundary condition on a perfectly conducting plate lying in a plane parallel to the z-axis is $E_z = 0$. When the plate lies in a plane $a < x < b$, $y = 0$, $-\infty < z < \infty$, currents are induced in the z-direction. The current is given by

$$I_z = (H_x)_{+0} - (H_x)_{-0} = \frac{1}{i\mu\omega}\left\{\left(\frac{\partial E_z}{\partial y}\right)_{+0} - \left(\frac{\partial E_z}{\partial y}\right)_{-0}\right\}, \qquad a < x < b,$$

where the symbols ± 0 refer to $y = \pm 0$, i.e. the two sides of the conducting plane in $y = 0$. This case is sometimes referred to as *parallel polarization* or *E-polarization.*

(b) TE or *transverse electric* waves when $E_z = H_x = H_y = 0$ and

$$\frac{\partial H_z}{\partial y} = -i\varepsilon\omega E_x \quad : \quad \frac{\partial H_z}{\partial x} = i\varepsilon\omega E_y \quad : \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\mu\omega H_z$$

Then H_z satisfies the two-dimensional steady-state wave equation with $k^2 = \varepsilon\mu\omega^2$. When a perfect conductor lies in, say, $a < x < b$, $y = 0$, $-\infty < z < \infty$ the boundary condition is $E_x = 0$ i.e. $\partial H_z/\partial y = 0$.

Electric currents occur in the conductor in the x-direction and we have

$$I_x = (H_z)_{+0} - (H_z)_{-0}.$$

It is often useful to represent these fields in terms of a scalar ϕ which is a solution of the two-dimensional wave equation, in the following way:

$$H_z = (\partial\phi/\partial y) \quad : \quad E_y = (i\varepsilon\omega)^{-1}(\partial^2\phi/\partial x\,\partial y):$$

$$E_x = -\frac{1}{i\varepsilon\omega}\frac{\partial^2\phi}{\partial y^2} = \frac{1}{i\varepsilon\omega}\left(\frac{\partial^2\phi}{\partial x^2} + k^2\phi\right).$$

This case is sometimes referred to as *perpendicular polarization* or *H-polarization.*

On using the above terminology we see that the half-plane problem of §2.1 is equivalent to diffraction of an H-polarized wave falling on a perfectly conducting half-plane. Similarly the problem of ex. 2.3 is equivalent to diffraction of an E-polarized wave falling on a perfectly conducting half-plane.

2.12 We summarize below some of the cases considered in the literature. The following headings are used: (i) the equation solved, (ii) the boundary conditions, (iii) the function $K(\alpha)$ which has to be factorized, (iv) references.

(A) (i) $$\nabla^2\phi_t + k^2\phi_t = 0, \qquad y \geqslant 0, \qquad -\infty < x < \infty.$$

(ii) $$p(\partial\phi_t/\partial y) - q\phi_t = 0, \qquad y = 0, \qquad 0 < x < \infty,$$

$$r(\partial\phi_t/\partial y) - s\phi_t = 0, \qquad y = 0, \qquad -\infty < x < 0.$$

$$\phi_t = \phi_i + \phi, \quad \phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta).$$

(iii) $K(\alpha) = (p\gamma + q)(r\gamma + s)^{-1}$. (See ex. 2.10).

(iv) The special case $q = 0$ or $s = 0$ is considered by T. B. A. Senior [1] (cf. §2.9). Application of the general case to propagation of radio waves over a land–sea interface has been made by P. C. Clemmow [2] (see note in (B) (iv) below). A. E. Heins and H. Feshbach [2] have considered the corresponding problem in acoustics where the boundary conditions are due to partially sound-absorbent materials.

(B) (i) $\nabla^2\phi_t + k^2\phi_t = 0, \quad y > 0, \quad -\infty < x < \infty,$

$\nabla^2\phi_t + K^2\phi_t = 0, \quad y < 0, \quad -\infty < x < \infty.$

(ii) $\partial\phi_t/\partial y = 0, \quad y = 0, \quad 0 < x < \infty.$

$$\left.\begin{array}{l}(\partial\phi_t/\partial y)_{+0} = p(\partial\phi_t/\partial y)_{-0} \\ \phi_t \text{ continuous}\end{array}\right\}, \quad y = 0, \quad -\infty < x < 0.$$

$$\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta).$$

(iii) $K(\alpha) = (\alpha^2 - k^2)^{1/2} + p(\alpha^2 - K^2)^{1/2}.$

(iv) P. C. Clemmow [2] discusses an application to the propagation of radio waves past a land–sea interface. Clemmow shows that in this case, if we are interested only in the field at sea-level due to a transmitter on the ground, the far-field can be expressed in terms of $K(\alpha)$ alone and we do not need $K_+(\alpha)$, $K_-(\alpha)$. This holds also if we apply (A) above to the problem.

(C) (i) $\nabla^2\phi - k^2\phi = 0, \quad y \leqslant 0, \quad -\infty < x < \infty.$

(ii) $\partial\phi/\partial y = 0, \quad y = 0, \quad 0 < x < \infty,$

$\partial\phi/\partial y = p\phi, \quad y = 0, \quad -\infty < x < 0.$

(iii) $K(\alpha) = 1 - p(\alpha^2 + k^2)^{-1/2}.$

(iv) T. R. Greene and A. E. Heins [1] and A. E. Heins [10] discuss an application to surface waves on water.

(D) (i) $\nabla^2\phi + i\varepsilon\phi = 0, \quad y \leqslant 0, \quad -\infty < x < \infty.$

(ii) $\phi = 1, \quad (y = 0, \quad x < 0) \; : \; \partial\phi/\partial y - i\phi = 0, \quad (y = 0, \quad x > 0).$

(iii) $K(\alpha) = 1 + i(\alpha^2 - i\varepsilon)^{1/2}.$

(iv) G. F. Carrier and W. H. Munk [1] discuss the diffusion of tides in permeable rock. There is a periodic tidal fluctuation of pressure on $y = 0$, $x < 0$ and the problem is to find the fluctuation of the free surface level of the water in the rock, where the free surface is assumed to lie in $y = 0$, $x > 0$, approximately.

(E) (i) $\nabla^2\phi - 2n(\partial\phi/\partial x) = 0, \quad y \geqslant 0, \quad -\infty < x < \infty.$

(ii) $\phi = 1, \quad y = 0, \quad x > 0 \quad : \quad \partial\phi/\partial y = 0, \quad y = 0, \quad x < 0.$

(iii) $K(\alpha) = \alpha^{1/2}(\alpha - 2in)^{1/2}.$

(iv) These equations describe convection of heat from a flat plate in $y = 0$, $x > 0$, at constant temperature, in an inviscid stream of constant velocity flowing in the x-direction. If we set $\psi = \phi\exp(-nx)$ then ψ satisfies $\psi_{xx} + \psi_{yy} - n^2\psi = 0.$

2.13 The oscillating quarter-plane airfoil at supersonic speeds has been considered by J. W. Miles [1]. The problem can be reduced to solution of

$$\phi_{yy} + \phi_{zz} = \phi_{xx} + k^2\phi, \qquad -\infty < x, y < \infty, \qquad z \geqslant 0,$$

with boundary conditions, on $Z = 0$

$$\begin{aligned} \partial\phi/\partial z &= f(x,y), \text{ (known)}, & x \geqslant 0, \quad & y \geqslant 0,\\ \partial\phi/\partial x - ip\phi &= 0, & x \geqslant 0, \quad & y < 0,\\ \phi &= 0, & x < 0, \quad & -\infty < y < \infty. \end{aligned}$$

If we apply a Fourier transform

$$\Phi(\alpha,\beta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi e^{i(\alpha x+\beta y)}\, dx\, dy,$$

it appears that if $\Phi \to 0$ as $z \to \infty$ the α and β planes must be cut so that $\gamma = (\beta^2 + k^2 - \alpha^2)^{1/2}$ has a positive real part for all α, β considered. On using the arguments in §1.5 and ex. 1.18 this means that if we set $\beta_0 = (\alpha^2 - k^2)^{1/2}$, $\operatorname{Im}\beta_0 > 0$, the β plane must be cut from $+\beta_0$ to infinity in the upper half-plane and $-\beta_0$ to infinity in the lower half-plane. But the α-plane must be cut from $\pm(\beta^2 + k^2)^{1/2}$ to infinity in the lower half-plane. When the cuts are fixed the solution is straightforward. There are easier methods for solving this particular problem but the Wiener–Hopf technique may be useful when considering generalizations.

2.14 Consider diffraction of the three-dimensional electromagnetic wave $\mathbf{E}_i = (\lambda,\mu,\nu)\exp\{-ik(ax + by + cz)\}$ by a semi-infinite perfectly conducting plane in $y = 0$, $x < 0$, $-\infty < z < \infty$. The notation means that the x, y, z components of $\mathbf{E}_i$ are given by multiplying the exponential factor by λ, μ, ν. These constants must be chosen so that the wave is plane. Also $a = \sin\theta_0\cos\phi_0$, $b = \sin\theta_0\sin\phi_0$, $c = \cos\theta_0$. All quantities are proportional to $\exp(-ikcz)$ and this is the only way in which they depend on z. If we write $\mathbf{E}_t = \mathbf{E}_i + \mathbf{E}$, $\mathbf{E} = (E_x, E_y, E_z)\exp(-ikcz)$, each of the components of $\mathbf{E}$ satisfies $\phi_{xx} + \phi_{yy} + k^2\sin^2\theta_0\phi = 0$. Denote the transform of E_x with respect to x in $(0,\infty)$ by $\mathscr{E}_x^+$, etc. The method of §2.2 gives the simultaneous Wiener–Hopf equations

$$\Gamma\mathscr{D}_x^- = -\gamma^2\mathscr{E}_z^+ + \alpha kc\mathscr{E}_x^+ + i(2\pi)^{-1/2}(\alpha - ka)^{-1}\{\lambda\alpha kc - \nu\gamma^2\},$$

$$-\Gamma\mathscr{D}_z^- = k^2(1-c^2)\mathscr{E}_x^+ + \alpha kc\mathscr{E}_z^+ + i(2\pi)^{-1/2}(\alpha - ka)^{-1}\{\lambda k^2(1-c^2) + \nu\alpha kc\},$$

where

$$\Gamma = (\alpha^2 - k^2\sin^2\theta_0)^{1/2} \quad : \quad 2\mathscr{D}_u^- = i\mu\omega\{(\mathscr{H}_u)_{+0} - (\mathscr{H}_u)_{-0}\}, \quad u = x, z.$$

These equations can be solved exactly. Appropriate references are D. S. Jones [1], [3], cf. E. T. Copson [4].

2.15 Consider diffraction of waves in an elastic medium by a half-plane. The strains and stresses can be expressed in terms of two potentials ϕ, ψ satisfying the steady-state equations (cf. I. N. Sneddon [1], p. 444):

$$\phi_{xx}+\phi_{yy}+k^2\phi=0,\quad (k^2=\omega^2/c^2) \quad : \quad \psi_{xx}+\psi_{yy}+K^2\psi=0,\quad (K^2=\omega^2/C^2),$$

where $c^2 = (\lambda + 2\mu)/\rho$, $C^2 = \mu/\rho$; λ, μ are Lamé's elastic constants and ρ is the density. The strains and stresses are given by

$$\begin{aligned} u &= \phi_x + \psi_y \quad &:\quad \sigma_{xx} &= -k^2\lambda\phi + 2\mu(\phi_{xx} + \psi_{xy}),\\ v &= \phi_y - \psi_x \quad &:\quad \sigma_{yy} &= -k^2\lambda\phi + 2\mu(\phi_{yy} - \psi_{xy}),\\ & & \sigma_{xy} &= \mu(2\phi_{xy} - \psi_{xx} + \psi_{yy}). \end{aligned}$$

Suppose that a plane wave in an infinite elastic medium falls on the half-plane $-\infty < x < 0$, $y = 0$ on which $\sigma_{yy} = \sigma_{xy} = 0$. Write $\phi_t = \phi_i + \phi$, $\psi_t = \psi_i + \psi$, where $\phi_i = a \exp(-ikx \cos\Theta - iky \sin\Theta)$, $\psi_i = b \exp(-iKx \cos\Theta - iKy \sin\Theta)$. The method of §2.2 gives

$$\begin{aligned} \Sigma_{yy}^{+}(0) + \Sigma_{yy}^{-}(0) &= 2\mu(\gamma K^2)^{-1}\{(\alpha^2 - \tfrac{1}{2}K^2)^2 - \alpha^2\gamma\Gamma\}F_-,\\ \Sigma_{xy}^{+}(0) + \Sigma_{xy}^{-}(0) &= 2\mu(\Gamma K^2)^{-1}\{(\alpha^2 - \tfrac{1}{2}K^2)^2 - \alpha^2\gamma\Gamma\}E_-, \end{aligned}$$

where

$$\gamma = (\alpha^2 - k^2)^{1/2} \quad : \quad \Gamma = (\alpha^2 - K^2)^{1/2} \quad :$$

$$U_-(+0) - U_-(-0) = E_- \quad : \quad V_-(+0) - V_-(-0) = F_-.$$

In the above equations $\Sigma_{yy}^{-}(0)$ and $\Sigma_{xy}^{-}(0)$ are known. The difficult step in the solution is the factorization of

$$K(\alpha) = (\alpha^2 - \tfrac{1}{2}K^2)^2 - \alpha^2(\alpha^2 - k^2)^{1/2}(\alpha^2 - K^2)^{1/2}. \tag{a}$$

The α-plane can be cut from $+k$ to $+K$ and from $-k$ to $-K$. Details are given in A. W. Maue [1] who has obtained the above results by an integral equation method. (Cf. A. W. Maue, *Z. Angew. Math. Mech.*, **34** (1954), pp. 1–12, where a transient problem involving a half-plane is solved by the method of conical flows and reduction to a Hilbert problem. The factorization which occurs in the Hilbert problem is identical with that required in (a) above.)

The problem of diffraction by a half-plane on which $u = v = 0$ can be formulated in a similar way.

2.16 The diffraction of waves by a half-plane in a viscous medium has been considered by J. B. Alblas [1], using the linearized Navier–Stokes equations. These equations are effectively of fourth order but the solutions can be expressed in terms of two potentials satisfying second order equations (cf. ex. 2.15). The factorization required is

$$K(\alpha) = \alpha^2 + (\alpha^2 - k^2)^{1/2}(\alpha^2 - il^2)^{1/2}, \qquad k \text{ complex}, \qquad l \text{ real}.$$

Alblas discusses the far-field and the field near the half-plane (the boundary layer).

2.17 The boundary-layer problem for two-dimensional flow of a viscous fluid past a plate lying in $y = 0$, $x > 0$ has been considered by J. A. Lewis and G. F. Carrier [1]. The flow as $x \to -\infty$ is assumed to be uniform with velocity U in the positive x-direction. Oseen's linearized equations can be converted into the following equation for a stream function ψ:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x}\right)\psi = 0. \tag{a}$$

This equation is of Laplace type and in order to apply the Wiener–Hopf technique it is convenient to introduce a parameter k (real and positive) and to write the Fourier transform of (a) as

$$\left(\frac{d^2}{dy^2} - \xi^2\right)\left(\frac{d^2}{dy^2} - \eta^2\right)\Psi = 0,$$

where $\quad \xi^2 = (\alpha^2 + k^2)^{1/2}, \qquad \eta^2 = (\alpha - i)(\alpha + ik).$

The parameter k is set equal to zero in the final solution. The results are of the correct form but disagree with the Blasius solution near the half-plane by a factor of proportionality. A semi-empirical correction factor is suggested in the paper quoted.

2.18 All the problems solved in this chapter have been steady-state, with time factor $\exp(-i\omega t)$. Theoretically it is easy to derive the solution of transient problems since a Laplace transform in t (parameter s) applied to the wave equation produces the steady-state wave equation with $k^2 = -(s^2/c^2)$. Hence if we solve a suitable steady-state problem, replace k by (is/c), and take the inverse Laplace transform, we obtain the solution of a transient problem. So far as I know, the only Wiener–Hopf example where this has been carried through in detail is given by D. S. Jones [5]. Transient problems involving the heat conduction equation can be solved in a similar way.

CHAPTER III

FURTHER WAVE PROBLEMS

3.1 Introduction

The main features of various methods of exact solution using the Wiener–Hopf technique have been explained in some detail in connexion with the examples solved in the last chapter. In this chapter we apply the same basic technique to solve problems with more complicated geometry. One of the main objects is to illustrate practical difficulties by examining concrete examples in detail. As far as possible the end-results are presented in a form suitable for numerical computation, although numerical results are not quoted.

For simplicity we use only one of the three techniques described in Chapter II, namely Jones's method. The reader should be able to follow the examples in this chapter after reading §§2.1, 2.2 and certain background material from Chapter I. The corresponding solutions by the integral equation method are developed in some of the exercises and in most of the references.

A feature common to most of the problems considered in this chapter is the presence of a duct or waveguide. We remind the reader of some results concerning the steady-state wave equation in $-\infty < x < \infty$, $-\infty < z < \infty$, $-b \leqslant y \leqslant b$, when it is assumed that wave propagation is in the x-direction and is independent of z. The problem is two-dimensional and ϕ satisfies

$$\phi_{xx} + \phi_{yy} + k^2\phi = 0, \tag{3.1}$$

where we first of all assume that k is real. If the boundary condition on $y = \pm b$ is $\phi = 0$ then the only permissible solutions of the equation are

$$\phi = f_n(z) \sin (n\pi/2b)(y - b), \qquad n = 1, 2, 3 \ldots. \tag{3.2}$$

Substituting in the wave equation, we find

$$f_n(x) = A_n e^{\gamma_n x} + B_n e^{-\gamma_n x}, \tag{3.3}$$

where A_n, B_n are constants and

$$\gamma_n = \{(n\pi/2b)^2 - k^2\}^{1/2}, \tag{3.4}$$

If $(n\pi/2b) > k$ then γ_n is real, and one wave is exponentially increasing, the other exponentially decreasing, as x increases. These are called *attenuated* modes. If $(n\pi/2b) < k$ then γ_n is imaginary. Since a time factor $\exp(-i\omega t)$ is assumed we write

$$\kappa_n = -i\gamma_n = \{k^2 - (n\pi/2b)^2\}^{1/2}, \tag{3.5}$$

where κ_n is real and positive. The corresponding waves are called *progressive* or *travelling* modes:

$$\phi = \{A_n \exp(-i\kappa_n x) + B_n \exp(i\kappa_n x)\} \sin(n\pi/2b)(y - b).$$

The term in B_n represents a wave travelling to the right in the duct and the term in A_n represents a wave to the left. The coefficients A_n and B_n are called the amplitudes of the travelling waves. For a given k, only a finite number of progressive modes exist. Thus if $0 \leqslant k \leqslant (\pi/2b)$, no travelling waves can exist: if $\pi/2b < k < \pi/b$ only one type of travelling wave can exist, corresponding to κ_1 in the above notation: and so on.

In a similar way, when the conditions on the wall of the duct are $\partial\phi/\partial y = 0$, $y = \pm b$, the only permissible solutions are of the form

$$\phi = g_n(x) \cos(n\pi/2b)(y - b), \qquad n = 0, 1, 2 \ldots. \tag{3.6}$$

We always have the progressive solution

$$g_0(x) = A_0 e^{-ikx} + B_0 e^{ikx}. \tag{3.7}$$

The other solutions correspond exactly with (3.3), (3.4) and may be either attenuated or progressive modes.

Next consider propagation in the rectangular duct $-c \leqslant z \leqslant c$, $-b \leqslant y \leqslant b$, $-\infty < x < \infty$. Suppose that ϕ satisfies

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + K^2\phi = 0. \tag{3.8}$$

Corresponding to (3.2) and (3.6) we have

$$\phi = h_{np}(x) \begin{matrix}\sin\\ \cos\end{matrix} (n\pi/2b)(y - b) \begin{matrix}\sin\\ \cos\end{matrix} (p\pi/2c)(z - c).$$

On substituting in the wave equation we find

$$h_{np}(x) = A_{np} \exp(\gamma_{np} x) + B_{np} \exp(-\gamma_{np} x),$$

where now

$$\gamma_{np} = \{(n\pi/2b)^2 + (p\pi/2c)^2 - K^2\}^{1/2}.$$

Again a finite number of the waves may be progressive: the remainder will be attenuated. Suppose next that in the problem considered the waves have all the same z-factor. Then in the course of the solution we can forget about the z-factor providing that instead of K^2 we take $k^2 = \{K^2 - (p\pi/2c)^2\}$ and consider the corresponding two-dimensional problem involving (3.1) instead of (3.8). Once the solution of this problem is obtained, the solution of the original problem is found by multiplying by the factor in z and expressing k in terms of K.

The above comments hold for real k, K. It will be necessary later to assume that k has a small positive imaginary part. In this case the strict distinction between attenuated and progressive waves no longer holds—all progressive waves are attenuated to some extent. However if the imaginary part of k is small and we define γ_n by (3.4) as above, there will be a sharp difference of degree at a certain value of n—above this value the modes will be strongly attenuated and will have little progressive character: below this value waves will exist which have only small attenuation. We shall still be justified in referring to these as attenuated and progressive modes respectively. In this connexion we remind the reader of a result in ex. 1.3. If $k = k_1 + ik_2$, $k_1 > 0$, $k_2 > 0$ and ξ is real, then

$$\operatorname{Im} (k^2 - \xi^2)^{1/2} \geqslant k_2. \tag{3.9}$$

The physical meaning of this is that if k is complex then all modes are attenuated at least as rapidly as $\exp(-k_2|x|)$ in the appropriate x-direction.

3.2 A plane wave incident on two semi-infinite parallel planes

Suppose that the plane wave $\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta)$ is incident on two semi-infinite parallel planes of zero thickness in $y = \pm b$, $x \leqslant 0$. (See Fig. 3.1.) The boundary condition on the

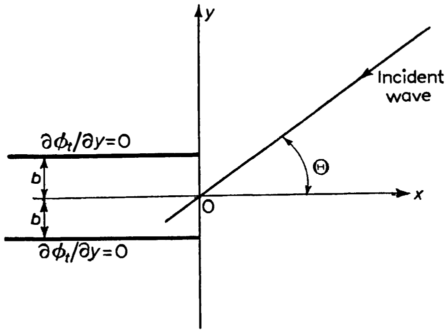

FIG. 3.1.

plates is assumed to be $\partial\phi_t/\partial y = 0$, $y = \pm b$, $x < 0$. If we set $\phi_t = \phi_i + \phi$ we find (cf. §2.1)

(i) $\partial\phi/\partial y = ik\sin\Theta\exp(-ikx\cos\Theta \mp ikb\sin\Theta)$, $y = \pm b$, $x < 0$.

(ii) $\partial\phi/\partial y$ is continuous on $y = \pm b$, $-\infty < x < \infty$.

(iii) ϕ is continuous on $y = \pm b$, $x > 0$.

(iv) $\phi = 0(1)$, $\partial\phi/\partial y = 0(r^{-1/2})$ as $r \to 0$, where r is the distance from (x,b) to $(0,b)$ or $(x,-b)$ to $(0,-b)$, respectively, with $x > 0$.

(v) For any fixed y, $-\infty < y < \infty$, $|\phi| < C_1\exp(-k_2x)$ as $x \to +\infty$ due to the presence of a diffracted wave. For any fixed y,

$|\phi| < C_2 \exp(k_2 \cos \Theta x)$ as $x \to -\infty$. If $y \geqslant b$ this is due to the presence of a reflected wave. In $y \leqslant -b$, in the shadow, only a diffracted wave is present, i.e. $|\phi_t| < C_3 \exp(k_2 x)$ as $x \to -\infty$. But $\phi = \phi_t - \phi_i$ and therefore ϕ follows the behaviour of $-\phi_i$ which gives the desired result. Similarly in $-b \leqslant y \leqslant b$, $|\phi_t| < C_4 \exp(k_2 x)$ due to a progressive wave $C_5 \exp(-ikx)$ inside the duct. From (3.9) all other waves are more strongly attenuated, and as in $y \leqslant -b$, ϕ follows the behaviour of $(-\phi_i)$.

An application of a Fourier transform in x to the partial differential equation gives (cf. §2.2):

$$\Phi = Ae^{-\gamma y}, \qquad (y \geqslant b) \qquad : \qquad De^{\gamma y}, \qquad (y \leqslant -b): \tag{3.10a}$$

$$= Be^{-\gamma y} + Ce^{\gamma y}, \qquad (-b \leqslant y \leqslant b). \tag{3.10b}$$

As in §2.2 an essential step in the procedure is the elimination of the unknown functions A–D in favour of functions whose regions of regularity are known. In complicated examples the best method for carrying out this elimination to obtain the equations of Wiener–Hopf type may not be immediately obvious. Although the present example is relatively straightforward we give two methods of procedure. In the first, the boundary conditions given in the problems are introduced at an early state to help in the elimination of A–D: in the second, the functions A–D are eliminated and then the boundary conditions are introduced afterwards. The first method is shorter and is used below: the second method is more systematic and is explained in ex. 3.1.

Since $\partial\phi/\partial y$ is continuous on $y = \pm b$ so is $d\Phi/dy$ and therefore from (3.10),

$$A = B - Ce^{2\gamma b} \qquad : \qquad D = -Be^{2\gamma b} + C. \tag{3.11}$$

Use notation corresponding to (2.15). Equations (3.10) give

$$\Phi_+(b+0) + \Phi_-(b+0) = Be^{-\gamma b} - Ce^{\gamma b}, \tag{3.12a}$$

$$\Phi_+(b-0) + \Phi_-(b-0) = Be^{-\gamma b} + Ce^{\gamma b}, \tag{3.12b}$$

$$\Phi_+(-b+0) + \Phi_-(-b+0) = Be^{\gamma b} + Ce^{-\gamma b}, \tag{3.12c}$$

$$\Phi_+(-b-0) + \Phi_-(-b-0) = -Be^{\gamma b} + Ce^{-\gamma b}, \tag{3.12d}$$

From (iii), $\Phi_+(b+0) = \Phi_+(b-0)$, $\Phi_+(-b+0) = \Phi_+(-b-0)$. Subtract (3.12b) from (3.12a) and (3.12c) from (3.12d). Write

$$\Phi_-(b+0) - \Phi_-(b-0) = 2F_-(b), \tag{3.13a}$$

$$\Phi_-(-b-0) - \Phi_-(-b+0) = 2F_-(-b), \tag{3.13b}$$

where from (v) the F_- are functions regular in $\sigma < k_2 \cos \Theta$. We find

$$F_-(b) = -Ce^{\gamma b} \quad : \quad F_-(-b) = -Be^{\gamma b}. \tag{3.14}$$

We have also, on taking the derivative of (3.10b) and setting $y = \pm b$,

$$\Phi'_+(b) + \Phi'_-(b) = \gamma(-Be^{-\gamma b} + Ce^{\gamma b}), \tag{3.15a}$$

$$\Phi'_+(-b) + \Phi'_-(-b) = \gamma(-Be^{\gamma b} + Ce^{-\gamma b}). \tag{3.15b}$$

From (i)

$$\Phi'_-(\pm b) = k \sin \Theta e^{\mp ikb \sin \Theta}(2\pi)^{-1/2}(\alpha - k \cos \Theta)^{-1}. \tag{3.16}$$

Define

$$\Phi'_+(b) + \Phi'_+(-b) = S'_+ \quad : \quad \Phi'_+(b) - \Phi'_+(-b) = D'_+, \tag{3.17a}$$

$$F_-(b) - F_-(-b) = D_- \quad : \quad F_-(b) + F_-(-b) = S_-. \tag{3.17b}$$

Add and subtract the two equations in (3.14) and the two equations in (3.15). Introduce notation (3.17) and use result (3.16). Eliminate $(B + C)$ and $(B - C)$ between the resulting equations. This gives

$$S'_+ + \frac{2k \sin \Theta \cos (kb \sin \Theta)}{(2\pi)^{1/2}(\alpha - k \cos \Theta)} = -\gamma(1 + e^{-2\gamma b})\, D_-, \tag{3.18a}$$

$$D'_+ - \frac{2ik \sin \Theta \sin (kb \sin \Theta)}{(2\pi)^{1/2}(\alpha - k \cos \Theta)} = -\gamma(1 - e^{-2\gamma b})\, S_-. \tag{3.18b}$$

These are equations to which the standard Wiener–Hopf technique can be applied. Suppose we can write

$$\tfrac{1}{2}(1 + e^{-2\gamma b}) = e^{-\gamma b} \cosh \gamma b = K(\alpha) = K_+(\alpha)K_-(\alpha), \tag{3.19a}$$

$$(2\gamma b)^{-1}(1 - e^{-2\gamma b}) = (\gamma b)^{-1}e^{-\gamma b} \sinh \gamma b = L(\alpha) = L_+(\alpha)L_-(\alpha), \tag{3.19b}$$

where K_+, L_+ are regular in $\tau > -k_2$; K_-, L_- are regular in $\tau < k_2$; $|K_+|$, $|K_-|$ are asymptotic to constants, and $|L_+|$, $|L_-|$ are asymptotic to $|\alpha|^{-1/2}$ as α tends to infinity in appropriate half-planes. Explicit expressions for these quantities are given below. Rewrite (3.18a) as

$$\frac{S'_+}{(\alpha + k)^{1/2}K_+(\alpha)} + \frac{2k \sin \Theta \cos (kb \sin \Theta)}{(2\pi)^{1/2}(\alpha - k \cos \Theta)} \left\{ \frac{1}{(\alpha + k)^{1/2}K_+(\alpha)} - \frac{1}{(k \cos \Theta + k)^{1/2}K_+(k \cos \Theta)} \right\}$$

$$= -2(\alpha - k)^{1/2}K_-(\alpha)D_- - \frac{2k \sin \Theta \cos (kb \sin \Theta)}{(2\pi)^{1/2}(\alpha - k \cos \Theta)(k \cos \Theta + k)^{1/2}K_+(k \cos \Theta)}.$$

From (iv), $D_- = 0(|\alpha|^{-1})$ and $S'_+ = 0(|\alpha|^{-1/2})$ as $\alpha \to \infty$ in appropriate half-planes. Hence all terms in the above equation tend to zero as $\alpha \to \infty$ in appropriate half-planes. There is a common strip of regularity, $-k_2 < \tau < k_2 \cos \Theta$. On applying Liouville's theorem, each side of the equation equals zero. Hence

$$D_- = -\frac{k \sin \Theta \cos (kb \sin \Theta)}{(2\pi)^{1/2}(k + k \cos \Theta)^{1/2} K_+(k \cos \Theta)(\alpha - k)^{1/2} K_-(\alpha)(\alpha - k \cos \Theta)}. \tag{3.20}$$

In an exactly similar way, from (3.18b),

$$S_- = \frac{ik \sin \Theta \sin (kb \sin \Theta)}{(2\pi)^{1/2} b(k + k \cos \Theta) L_+(k \cos \Theta)(\alpha - k) L_-(\alpha)(\alpha - k \cos \Theta)}. \tag{3.21}$$

We next obtain explicit factorizations. (Cf. J. F. Carlson and A. E. Heins [1], A. E. Heins [1].) Consider first K_+, K_- defined in (3.19a). The factor $\cosh \gamma b$ is an integral function to which the infinite product theory of ex. 1.9 and §1.3 can be applied.

From the general theory (cf. ex. 1.9)

$$\cosh \gamma b = \cos kb \prod_{n=1}^{\infty} \{1 - (\alpha/\alpha_{n-\frac{1}{2}})^2\}$$

$$\cos kb = \prod_{n=1}^{\infty} (1 - k^2 b^2_{n-\frac{1}{2}}),$$

where

$$\alpha_{n-\frac{1}{2}} = i\{(1/b^2_{n-\frac{1}{2}}) - k^2\}^{1/2} \quad : \quad b_{n-\frac{1}{2}} = b/\{(n - \tfrac{1}{2})\pi\}.$$

Combine these infinite products:

$$\cosh \gamma b = \prod_{n=1}^{\infty} \{(1 - k^2 b^2_{n-\frac{1}{2}}) + \alpha^2 b^2_{n-\frac{1}{2}}\} = H(\alpha), \quad \text{say}.$$

Decompose $H(\alpha)$ in the form

$$H_\pm(\alpha) = \prod_{n=1}^{\infty} \{(1 - k^2 b^2_{n-\frac{1}{2}})^{1/2} \mp i\alpha b_{n-\frac{1}{2}}\} \, e^{\pm i\alpha b_{n-\frac{1}{2}}}.$$

Combine this result with the decomposition of $\exp(-\gamma b)$ in (1.35):

$$\exp(-\gamma b) = [\exp\{-T_+(\alpha)\}][\exp\{-T_-(\alpha)\}],$$

$$T_+(\alpha) = \pi^{-1} b\gamma \arccos (\alpha/k) \quad : \quad T_-(\alpha) = T_+(-\alpha):$$

$$T_+(\alpha) = (ib\alpha/\pi) \ln (2\alpha/k) + 0(\alpha^{-1}),$$

as $\alpha \to \infty$ in an upper half-plane. These results give

$$K_\pm(\alpha) = \exp\{\mp\chi_1(\alpha) - T_\pm(\alpha)\} \times \prod_{n=1}^{\infty} \{(1 - k^2 b^2_{n-\frac{1}{2}})^{1/2} \mp i\alpha b_{n-\frac{1}{2}}\} e^{\pm i\alpha b_{n-\frac{1}{2}}}, \tag{3.22a}$$

where $\chi_1(\alpha)$ is an arbitrary function which will now be chosen so that K_+ and K_- have polynomial behaviour at infinity. As in exs. 3.4, 3.6 when $|\alpha| \to \infty$ the behaviour of K_+, K_- is independent of the term $(k^2 b_{n-\frac{1}{2}}^2)$ in the infinite product. Thus, using results in ex. 1.10, as $|\alpha| \to \infty$ in a lower half-plane,

$$K_-(\alpha) \sim \exp\{\chi_1(\alpha) + ib\alpha\pi^{-1} \ln(-2\alpha/k)\} \prod_{n=1}^{\infty} (1 + i\alpha b_{n-\frac{1}{2}}) e^{-i\alpha b_{n-\frac{1}{2}}},$$

$$\sim A \exp[\chi_1(\alpha) + ib\alpha\pi^{-1}\{1 - C + \ln(\pi/2bk) + i\tfrac{1}{2}\pi\}],$$

where A is a constant independent of α, and $C = 0{\cdot}5772\ldots$, the Euler constant. Choose

$$\chi_1(\alpha) = -ib\alpha\pi^{-1}\{1 - C + \ln(\pi/2bk)\} + \tfrac{1}{2}\alpha b, \tag{3.22b}$$

and then both K_+ and K_- are asymptotic to constants as $|\alpha| \to \infty$ in upper and lower half-planes respectively.

In a similar way we can decompose (3.19b) in the form

$$L_\pm(\alpha) = \exp\{\mp \chi_2(\alpha) - T_\pm(\alpha)\} \prod_{n=1}^{\infty} \{1 - k^2 b_n^2)^{1/2} \mp i\alpha b_n\} e^{\pm i\alpha b_n}, \tag{3.23a}$$

where

$$b_n = (b/n\pi) \quad : \quad \chi_2(\alpha) = -ib\alpha\pi^{-1}\{1 - C + \ln(2\pi/bk)\} + \tfrac{1}{2}\alpha b. \tag{3.23b}$$

In this case $|L_+|$, $|L_-| \sim |\alpha|^{-1/2}$ as $|\alpha| \to \infty$ in appropriate half-planes. Note that

$$K_+(-\alpha) = K_-(+\alpha) \quad : \quad L_+(-\alpha) = L_-(+\alpha). \tag{3.24}$$

Now return to the solution (3.20), (3.21). From (3.14), (3.17b),

$$B = -\tfrac{1}{2}(S_- - D_-)e^{-\gamma b} \quad : \quad C = -\tfrac{1}{2}(S_- + D_-)e^{-\gamma b},$$

and we readily deduce

$$\phi = -\frac{1}{(2\pi)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} (S_- \cosh \gamma y + D_- \sinh \gamma y) e^{-\gamma b - i\alpha x}\, d\alpha, \quad (-b \leqslant y \leqslant b), \tag{3.25a}$$

$$= \frac{1}{(2\pi)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} (S_- \sinh \gamma b \pm D_- \cosh \gamma b) e^{-\gamma|y| - i\alpha x}\, d\alpha, \quad (|y| \geqslant b), \tag{3.25b}$$

where the upper sign refers to $y \geqslant b$, the lower to $y \leqslant -b$.

In the region $-b \leqslant y \leqslant b$, $x < 0$ close the contour in the upper half-plane. For simplicity assume that $0 < k < (\pi/2b)$. Then S_- and D_- have poles at $\alpha = k \cos \Theta$ and S_- has a pole at $\alpha = k$. These are the only real poles. On evaluating the residues, the pole at $\alpha = k \cos \Theta$ gives a contribution $-\exp(-ikx \cos \Theta - iky \sin \Theta)$ which cancels the incident wave. The pole at $\alpha = k$ gives the travelling wave in the duct:

$$\frac{\sin(kb \sin \Theta)}{kb \sin \Theta \; L_+(k \cos \Theta) L_-(k)} e^{-ikx}. \tag{3.26}$$

All other poles give terms which decrease exponentially as $x \to -\infty$. There are no branch points for this region of the x-y plane.

The far field can be evaluated by the method given in §1.6. Use cylindrical co-ordinates (r,θ). Then, as an example, in the region $0 \leqslant \theta < \pi - \Theta$, if θ is not too near $(\pi - \Theta)$, we find on using (1.71) that the field as $r \to \infty$ is given by

$$\begin{aligned} \phi \sim\; & 2^{1/2}(k\pi)^{-1/2} e^{\frac{1}{4}i\pi} \sin \tfrac{1}{2}\Theta \sin \tfrac{1}{2}\theta (\cos \theta + \cos \Theta)^{-1} \\ & \times \left\{ \frac{\cos(kb \sin \Theta) \cos(kb \sin \theta)}{K_+(k \cos \Theta) K_+(k \cos \theta)} + \right. \\ & \left. + \frac{i \sin(kb \sin \Theta) \sin(kb \sin \theta).}{2kb \cos \frac{1}{2}\Theta \cos \frac{1}{2}\theta L_+(k \cos \Theta) L_+(k \cos \theta)} \right\}. \end{aligned} \tag{3.27}$$

This is symmetrical in Θ and θ as we should expect.

Formulae for the practical computation of the K and L factors occurring in (3.26) and (3.27) are given in ex. 3.3.

3.3 Radiation from two parallel semi-infinite plates

In §3.2 we discussed the progressive waves excited in a duct formed by two parallel semi-infinite plates by a plane wave incident from outside the duct. In this section we reverse the situation and discuss the waves radiated from a pair of plates by progressive waves travelling towards the open end from inside the duct. As in §3.2 assume that the plates lie in $y = \pm b$, $x \leqslant 0$. Consider two cases together.

(a) $\partial \phi_t / \partial y = 0$ on the plates. Assume that the progressive wave

$$\phi_i = \exp(iKx) \cos(N\pi/2b)(y - b), \tag{3.28}$$

$$K = \{k^2 - (N\pi/2b)^2\}^{1/2}, \qquad k > (N\pi/2b), \text{ for some given N,}$$

is the only wave travelling towards the open end of the duct. N may be any of the numbers 0, 1, 2, . . . , with $N < 2bk/\pi$.

(b) $\phi_t = 0$ on the plates. Assume that the only progressive wave inside the duct, travelling towards the open end of the duct, is

$$\phi_i = \exp(iKx) \sin(N\pi/2b)(y - b), \tag{3.29}$$

where K is defined as before and again N is a given number but this time $N = 1, 2, 3 \ldots$, with $N < 2bk/\pi$, i.e. zero is excluded.

In the regions $y \geqslant b$, $y \leqslant -b$, only radiated waves exist and we can set

$$\Phi_t = Ae^{-\gamma y}, \qquad y \geqslant b \quad : \quad = De^{\gamma y}, \qquad y \leqslant -b.$$

In $-b \leqslant y \leqslant b$ set $\phi_t = \phi_i + \phi$ where ϕ can be represented by a Fourier transform and

$$\Phi = Be^{-\gamma y} + Ce^{\gamma y}, \qquad -b \leqslant y \leqslant b.$$

We have removed the incident wave which is the only wave which increases exponentially as $x \to -\infty$ inside the duct. All other waves inside the duct are attenuated at least as rapidly as exp $(-k_2|x|)$ as $x \to -\infty$ (cf. (3.9)). Outside the duct only radiated waves exist and as in §2.2 these are attenuated as exp $(-k_2|x|)$ as $|x| \to \infty$. Hence in $-k_2 < \tau < k_2$,

$$\Phi_{t+}(b+0) + \Phi_{t-}(b+0) = Ae^{-\gamma b}, \tag{3.30a}$$

$$\Phi_{+}(b-0) + \Phi_{-}(b-0) = Be^{-\gamma b} + Ce^{\gamma b}, \tag{3.30b}$$

$$\Phi'_{t+}(b+0) + \Phi'_{t-}(b+0) = -\gamma Ae^{-\gamma b}, \tag{3.30c}$$

$$\Phi'_{+}(b-0) + \Phi'_{-}(b-0) = -\gamma(Be^{-\gamma b} - Ce^{\gamma b}), \tag{3.30d}$$

with similar equations for $y = -b$. Consider cases (a) and (b) separately.

(a) If $\partial\phi_t/\partial y = 0$ on the plates with ϕ_i given by (3.28), then $\partial\phi_t/\partial y = \partial\phi/\partial y$ on $y = \pm b$, $-\infty < x < \infty$, $A = B - Ce^{2\gamma b}$, and

$$\Phi'_{t+}(b+0) = \Phi'_{+}(b-0) = \Phi'_{+}(b), \quad \text{say}$$
$$\Phi'_{t-}(b+0) = \Phi'_{-}(b-0) = 0$$
$$\Phi_{t+}(b+0) - \Phi_{+}(b-0) = i(2\pi)^{-1/2}(\alpha + K)^{-1}.$$

Introduce the unknown function,

$$\Phi_{t-}(b+0) - \Phi_{-}(b-0) = 2F_{-}(b).$$

Subtract (3.30b) from (3.30c), and write down the common form of (3.30c, d).

$$\tfrac{1}{2}i(2\pi)^{-1/2}(\alpha + K)^{-1} + F_{-}(b) = -Ce^{\gamma b}, \tag{3.31a}$$

$$\Phi'_{+}(b) = -\gamma(Be^{-\gamma b} - Ce^{\gamma b}). \tag{3.31b}$$

In an exactly similar way, conditions at $y = -b$ give

$$\tfrac{1}{2}i \cos N\pi(2\pi)^{-1/2}(\alpha + K)^{-1} + F_{-}(-b) = -Be^{\gamma b}, \tag{3.31c}$$

$$\Phi'_{+}(-b) = -\gamma(Be^{\gamma b} - Ce^{-\gamma b}),$$

where (3.31d)

$$2F_{-}(-b) = \Phi_{t-}(-b-0) - \Phi_{-}(-b+0).$$

Use notation (3.17) and follow exactly the same procedure as in §3.2. Then (3.31a–d) give

$$S'_+ = -\gamma(1 + e^{-2\gamma b})\{D_- + \tfrac{1}{2}i(2\pi)^{-1/2}(\alpha + K)^{-1}(1 - \cos N\pi)\}, \tag{3.32a}$$

$$D'_+ = -\gamma(1 - e^{-2\gamma b})\{S_- + \tfrac{1}{2}i(2\pi)^{-1/2}(\alpha + K)^{-1}(1 + \cos N\pi)\}. \tag{3.32b}$$

Thus there are two subdivisions

(i) If N is even ϕ has even symmetry about $y = 0$ and $(1 - \cos N\pi) = 0$. By Liouville's theorem we can prove from the first of the above equations that S'_+, D_- are zero. The remaining equation gives, on writing $(2\gamma b)^{-1}\{1 - \exp(-2\gamma b)\} = L_+(\alpha)L_-(\alpha)$ defined in §3.2, and rearranging in the usual way,

$$\begin{aligned}(\alpha + k)^{-1}\{L_+(\alpha)\}^{-1}D'_+(\alpha) - i2b(2\pi)^{-1/2}(\alpha + K)^{-1}(K + k)L_-(-K)\\ = -2b(\alpha - k)L_-(\alpha)S_-(\alpha) - i2b(2\pi)^{-1/2}(\alpha + K)^{-1}\{(\alpha - k)L_-(\alpha) +\\ + (K + k)L_-(-K)\}.\end{aligned}$$

By the usual arguments both sides are zero, and

$$D'_+(\alpha) = i2b(2\pi)^{-1/2}(K + k)L_-(-K)(\alpha + k)(\alpha + K)^{-1}L_+(\alpha). \tag{3.33}$$

It has already been noted that $S'_+(\alpha) = 0$. Hence from the definitions (3.17a), $\Phi'_+(b) = -\Phi'_+(-b) = \tfrac{1}{2}D'_+$. From the equations at the beginning of this section we can deduce A–D (e.g. (3.30c, d) and the corresponding equations for $y = -b$). It is found that

$$\phi = \frac{ib}{2\pi}(K + k)L_-(-K)\int_{-\infty+i\tau}^{\infty+i\tau} \frac{(\alpha + k)L_+(\alpha)}{\gamma(\alpha + K)} \cdot \frac{\cosh \gamma y}{\sinh \gamma b}\, e^{-i\alpha x}\, d\alpha, \quad (-b \leqslant y \leqslant b), \tag{3.34}$$

$$\phi_t = -\frac{ib}{2\pi}(K + k)L_-(-K)\int_{-\infty+i\tau}^{\infty+i\tau} \frac{(\alpha + k)L_+(\alpha)}{\gamma(\alpha + K)}\, e^{\gamma(b-|y|)-i\alpha x}\, d\alpha, \quad (|y| \geqslant b).$$

(ii) If N is odd, ϕ has odd symmetry about $y = 0$ and $(1 + \cos N\pi) = 0$. Corresponding to (3.33) — (3.34) we find

$$S'_+(\alpha) = -2(2\pi)^{-1/2}(K + k)^{1/2}K_-(-K)(\alpha + k)^{1/2}K_+(\alpha)(\alpha + K)^{-1}, \tag{3.35a}$$

$$D'_+(\alpha) = 0, \tag{3.35b}$$

$$\phi = -\frac{1}{2\pi}(K+k)^{1/2}K_-(-K)$$
$$\times \int_{-\infty+i\tau}^{\infty+i\tau} \frac{(\alpha+k)^{1/2}K_+(\alpha)}{\gamma(\alpha+K)} \cdot \frac{\sinh \gamma y}{\cosh \gamma b} e^{-i\alpha x}\, d\alpha, \qquad (-b \leqslant y \leqslant b),$$

$$\phi_t = \pm \frac{1}{2\pi}(K+k)^{1/2}K_-(-K) \int_{-\infty+i\tau}^{\infty+i\tau} \frac{(\alpha+k)^{1/2}K_+(\alpha)}{\gamma(\alpha+K)} e^{\gamma(b\mp y)-i\alpha x}\, d\alpha, \tag{3.36}$$

where the upper sign refers to $y \geqslant b$, the lower to $y \leqslant -b$.

(b) The argument in this case is very similar. Instead of (3.32) we find

$$S_+ = -\gamma^{-1}(1+e^{-2\gamma b})$$
$$\times \{D'_- + (N\pi/4b)(2\pi)^{-1/2}(\alpha+K)^{-1}i(1-\cos N\pi)\}, \tag{3.37a}$$

$$D_+ = -\gamma^{-1}(1-e^{-2\gamma b})$$
$$\times \{S'_- + (N\pi/4b)(2\pi)^{-1/2}(\alpha+K)^{-1}i(1+\cos N\pi)\}. \tag{3.37b}$$

Again there are two cases:

(i) N even, odd symmetry in ϕ about $y = 0$. We find

$$D_+(\alpha) = -iN\pi(2\pi)^{-1/2}L_-(-K)L_+(\alpha)(\alpha+K)^{-1}, \tag{3.38a}$$

$$S_+(\alpha) = 0, \tag{3.38b}$$

$$\phi = \frac{N}{4i}L_-(-K)\int_{-\infty+i\tau}^{\infty+i\tau} \frac{L_+(\alpha)}{(\alpha+K)} \frac{\sinh \gamma y}{\sinh \gamma b} e^{-i\alpha x}\, d\alpha, \qquad (-b \leqslant y \leqslant b),$$

$$\phi_t = \pm \frac{N}{4i}L_-(-K)\int_{-\infty+i\tau}^{\infty+i\tau} \frac{L_+(\alpha)}{(\alpha+K)} e^{\gamma(b-|y|)-i\alpha x}\, d\alpha, \tag{3.39}$$

where the upper sign refers to $y \geqslant b$, the lower to $y \leqslant -b$.

(ii) N odd, even symmetry in ϕ about $y = 0$. We find

$$S_+(\alpha) = \frac{N\pi}{b} \cdot \frac{K_-(-K)}{(2\pi)^{1/2}(K+k)^{1/2}} \cdot \frac{K_+(\alpha)}{(\alpha+K)(\alpha+k)^{1/2}}, \tag{3.40a}$$

$$D_+(\alpha) = 0, \tag{3.40b}$$

$$\phi = \frac{N}{4b} \cdot \frac{K_-(-K)}{(K+k)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{K_+(\alpha)}{(\alpha+K)(\alpha+k)^{1/2}} \cdot \frac{\cosh \gamma y}{\cosh \gamma b} e^{-i\alpha x}\, d\alpha,$$
$$(-b \leqslant y \leqslant b), \qquad (3.41)$$

$$\phi_t = \frac{N}{4b} \cdot \frac{K_-(-K)}{(K+k)^{1/2}} \int_{-\infty+i\tau}^{\infty+i\tau} \frac{K_+(\alpha)}{(\alpha+K)(\alpha+k)^{1/2}} e^{\gamma(b-|y|)-i\alpha x}\, d\alpha,$$
$$(|y| > b).$$

A similar analysis can be applied to the results of all four cases, i.e. equations (3.34), (3.36), (3.39), (3.41). These are of the form

$$\phi_t \quad \text{or} \quad \phi = \int_{-\infty+i\tau}^{\infty+i\tau} f(\alpha) e^{-\gamma z - i\alpha x}\, d\alpha,$$

where $-k_2 < \tau < k_2$, and z is some constant, $z \geqslant 0$.

(A) In the region $|y| \geqslant b$ we use the theory of §1.6. The apparent pole at $\alpha = -K$ in the explicit formulae is also a zero of the numerator in each case since $L_+(-K) = 0$ if N is even and $K_+(-K) = 0$ if N is odd. Hence there are no poles. We have for the distant field, considering $y > b$, $r \to \infty$ in direction θ where polar co-ordinates are taken from $(0,b)$ as origin (from equation (1.71))

$$\phi_t \sim 2(k\pi)^{1/2} e^{-\frac{1}{4}i\pi} r^{-1/2} e^{ikr} f(-k\cos\theta)\sin\theta. \qquad (3.42)$$

(B) In the region $-b \leqslant y \leqslant b$, $x \to +\infty$, close the contour in a lower half-plane. It will be found that the pole at $\alpha = -K$ gives in each case $-\exp(iKx)\{\sin \text{ or } \cos\}(n\pi/2b)(y-b)$ i.e. a term which exactly cancels the incident wave. The remaining integral has a branch point at $\alpha = -k$ and the distant field can be dealt with by the theory of §1.6, though this is unnecessary in practice since the field as $x \to +\infty$ must be continuous with that found in (A).

(C) In the region $-b \leqslant y \leqslant b$, $x < 0$, we can close the contour in the upper half-plane. It is found that there are no branch points in this region and an infinite number of poles at the roots of $L_-(\alpha) = 0$ or $K_-(\alpha) = 0$. ($L_+(\alpha)(\sinh \gamma b)^{-1} = \{\gamma b L_-(\alpha)\}^{-1} \exp(-\gamma b)$, etc.) In case (a) (i) there is also a root at $\alpha = k$. Denote the roots by $\alpha = \alpha_n$. Then

$$\phi = 2\pi i \sum_n \lim_{\alpha \to \alpha_n} (\alpha - \alpha_n) f(\alpha)\, e^{-i\alpha_n x}. \qquad (3.43)$$

If we let $k_2 \to 0$ it will be found that a finite number of the α_n are real, the remainder imaginary. Thus there are a finite number of propagated modes and an infinite number of attenuated modes. Of course it will be found that the propagated waves travel to the left in the duct and the attenuated modes tend to zero exponentially as $x \to -\infty$ in the duct.

We consider one result in more detail, namely when $\partial\phi_t/\partial y = 0$ on the walls of the duct and the only progressive waves are $\exp(\pm ikx)$. This is case (a) (i) with $N = 0$, $K = k$. The solution is (3.34) and in conjunction with (3.43) this gives the reflected wave

$$\phi_r = -\{L_+(k)\}^2 e^{-ikx}, \qquad (-b \leqslant y \leqslant b, \quad x < 0).$$

The coefficient of $\exp(-ikx)$ is the reflection coefficient R and we find (ex. 3.3)

$$R = -|R|e^{2ikl}, \quad \text{where} \quad |R| = e^{-bk}, \tag{3.44a}$$

$$(l/b) = \pi^{-1}\{1 - C + \ln(2\pi/bk)\} - \\ - (bk)^{-1}\sum_{n=1}^{\infty}\{\sin^{-1}(bk/n\pi) - (bk/n\pi)\}. \tag{3.44b}$$

The distant field, from (3.42), is

$$\phi_t \sim (2k\pi)^{-1/2} e^{-\frac{1}{4}i\pi}\, 2kb L_+(k)L_+(-k\cos\theta) r^{-1/2} e^{ikr}.$$
$$(x = r\cos\theta, \qquad |y| - b = r\sin\theta). \tag{3.45}$$

3.4 Radiation from a cylindrical pipe

Use cylindrical co-ordinates (ρ,z) and suppose that there is a rigid circular pipe in $\rho = a$, $-\infty < z \leqslant 0$. ϕ_t satisfies the steady-state wave equation in cylindrical co-ordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\phi_t}{\partial\rho} + \frac{\partial^2\phi_t}{\partial z^2} + k^2\phi_t = 0, \tag{3.46}$$

where for simplicity we assume symmetry in the radial direction. Consider the waves that can exist in $0 \leqslant \rho \leqslant a, z < 0$. As in §3.1 this is equivalent to considering waves in a tube $0 \leqslant \rho \leqslant a$, $-\infty < z < \infty$. By separation of variables we set $\phi_t = f(\rho)g(z)$ and find

$$\frac{d^2g(z)}{dz^2} - (\xi^2 - k^2)g(z) = 0 \quad : \quad \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{df(\rho)}{d\rho} + \xi^2 f(\rho) = 0,$$

i.e. (3.47)

$$g(z) = \exp\{\pm(\xi^2 - k^2)^{1/2}z\} \quad : \quad f(\rho) = AJ_0(\xi\rho) + BY_0(\xi\rho).$$

In these equations ξ is the separation parameter. Since the pipe is rigid, $\partial\phi_t/\partial\rho = 0$ on $\rho = a$. Also ϕ_t must be finite on $\rho = 0$. Hence we must have $B = 0$ and

$$\{dJ_0(\xi\rho)/d\rho\}_{\rho=a} = -\xi J_1(\xi a) = 0. \tag{3.48}$$

Suppose that the roots of $J_1(\mu) = 0$, apart from the root $\mu = 0$, are given by $\mu_i(i = 1,2,3 \ldots)$ where $\mu_1 = 3{\cdot}832$, $\mu_2 = 7{\cdot}016, \ldots$ etc. Then any wave in the tube can be expressed in the form

$$\phi_t = a_0 e^{ikz} + \sum_{i=1}^{\infty} a_i J_0(\mu_i \rho/a) \exp\{-(\mu_i^2 - k^2a^2)^{1/2}(z/a)\} +$$
$$+ b_0 e^{-ikz} + \sum_{i=1}^{\infty} b_i J_0(\mu_i \rho/a) \exp\{+(\mu_i^2 - k^2a^2)^{1/2}(z/a)\}. \quad (3.49)$$

Return now to the problem of the semi-infinite pipe. Suppose that a wave $\phi_i = \exp(ikz)$ is incident on the mouth of the pipe in $0 \leqslant \rho \leqslant a$ from $z = -\infty$, where $0 < ka < \mu_1 = 3{\cdot}832 \ldots$, so that only the fundamental wave propagates. Thus inside the pipe we must have (cf. (3.49))

$$\phi_t = e^{ikz} + Re^{-ikz} + \sum_{i=1}^{\infty} b_i J_0(\mu_i \rho/a) \exp\{(\mu_i^2 - k^2a^2)^{1/2}(z/a)\}. \quad (3.50)$$

Also as $z \to +\infty$ in $0 \leqslant \rho \leqslant a$, $\phi_t \sim z^{-1}\exp(ikz)$. Hence if for $0 \leqslant \rho \leqslant a$ we write $\phi_t = \phi_i + \phi$, then $\phi(\rho,\alpha)$ the Fourier transform of ϕ with respect to z in $0 \leqslant \rho \leqslant a$, is regular in $-k_2 < \tau < k_2$. For $\rho \geqslant a$, using spherical polar co-ordinates,

$$\phi_t \sim f(\theta) r^{-1} e^{ikr}, \qquad \text{as } r \to \infty \text{ outside the pipe.} \quad (3.51)$$

Hence $\Phi_t(\rho,\alpha)$, the Fourier transform of ϕ_t with respect to z in $\rho \geqslant a$ is regular in $-k_2 < \tau < k_2$. The remaining boundary condition we require is

$$\partial\phi_t/\partial\rho = \partial\phi/\partial\rho = 0, \qquad \rho = a, \qquad -\infty < z \leqslant 0. \quad (3.52)$$

ϕ satisfies the same equation as ϕ_t, namely (3.46). If we apply a Fourier transform in z in the usual way we find

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\Phi_t}{d\rho} - \gamma^2\Phi_t = 0, \qquad (\rho \geqslant a) \quad : \quad \frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\Phi}{d\rho} - \gamma^2\Phi = 0,$$
$$(0 \leqslant \rho \leqslant a), \quad (3.53)$$

where $\gamma = (\alpha^2 - k^2)^{1/2}$. Suitable solutions are

$$\Phi_t = A(\alpha)K_0(\gamma\rho), \qquad (\rho \geqslant a), \quad (3.54a)$$

$$\Phi = B(\alpha)I_0(\gamma\rho), \qquad (0 \leqslant \rho \leqslant a). \quad (3.54b)$$

where in (3.54b) we choose the solution which is finite as $\rho \to 0$ and in (3.54a) the solution which tends to zero or represents an outgoing wave at infinity. Note that

$$K_0(-i\kappa\rho) = \tfrac{1}{2}\pi i H_0^{(1)}(\kappa\rho) \quad : \quad I_0(-i\kappa\rho) = J_0(\kappa\rho), \quad (3.55)$$

where $\gamma = (\alpha^2 - k^2)^{1/2} = -i(k^2 - \alpha^2)^{1/2} = -i\kappa$, in our standard notation. By Jones's procedure, (3.54) give

$$\Phi_{t+}(a) + \Phi_{t-}(a) = A(\alpha)K_0(\gamma a), \tag{3.56a}$$

$$\Phi_{+}(a) + \Phi_{-}(a) = B(\alpha)I_0(\gamma a), \tag{3.56b}$$

$$\Phi'_{+}(a) + \Phi'_{-}(a) = \gamma A(\alpha)K'_0(\gamma a) = \gamma B(\alpha)I'_0(\gamma a), \tag{3.56c}$$

where in the third equation we have used the fact that $\partial\phi_t/\partial\rho = \partial\phi/\partial\rho$ on $\rho = a$, $-\infty < z < \infty$, and we have written

$$K'_0(u) = dK_0(u)/du = -K_1(u) \quad : \quad I'_0(u) = dI_0(u)/du = I_1(u).$$

Note that

$$K_1(-i\kappa\rho) = -\tfrac{1}{2}\pi H_1^{(1)}(\kappa\rho) \quad : \quad I_1(-i\kappa\rho) = -iJ_1(\kappa\rho), \tag{3.57a}$$

$$I_\nu(u)K'_\nu(u) - K_\nu(u)I'_\nu(u) = -u^{-1}. \tag{3.57b}$$

Eliminate $A(\alpha)$ and $B(\alpha)$ from (3.56):

$$\Phi_{t+}(a) + \Phi_{t-}(a) = K_0(\gamma a)\gamma^{-1}\{K'_0(\gamma a)\}^{-1}\{\Phi'_{+}(a) + \Phi'_{-}(a)\},$$

$$\Phi_{+}(a) + \Phi_{-}(a) = I_0(\gamma a)\gamma^{-1}\{I'_0(\gamma a)\}^{-1}\{\Phi'_{+}(a) + \Phi'_{-}(a)\}.$$

Subtract these equations. Use the results $\Phi'_{-}(a) = 0$ and

$$\Phi_{t+}(a) - \Phi_{+}(a) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty e^{i(\alpha+k)x}\,dx = \frac{1}{(2\pi)^{1/2}}\cdot\frac{i}{\alpha+k}.$$

Set

$$\Phi_{t-}(a) - \Phi_{-}(a) = F_{-}.$$

Then

$$\frac{1}{(2\pi)^{1/2}}\cdot\frac{i}{\alpha+k} + F_{-} = \frac{1}{\gamma}\left\{\frac{K_0(\gamma a)I'_0(\gamma a) - I_0(\gamma a)K'_0(\gamma a)}{I'_0(\gamma a)K'_0(\gamma a)}\right\}\Phi'_{+}(a)$$

$$= \{\gamma^2 a K'_0(\gamma a)I'_0(\gamma a)\}^{-1}\Phi'_{+}(a). \tag{3.58}$$

Set

$$K(\alpha) = -2K'_0(\gamma a)I'_0(\gamma a) = 2K_1(\gamma a)I_1(\gamma a)$$

$$= \pi i H_1^{(1)}(\kappa a)\,J_1^{(1)}(\kappa a), \tag{3.59}$$

where the numerical factors have been inserted for convenience. Decompose this function in the form $K(\alpha) = K_{+}(\alpha)K_{-}(\alpha)$ by the theory of §1.3, theorem C. Then for $-k_2 < c < k_2$, $\gamma = (\zeta^2 - k^2)^{1/2}$,

$$\ln K_{+}(\alpha) = \frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{\ln\{2K_1(\gamma a)I_1(\gamma a)\}}{\zeta-\alpha}\,d\zeta, \qquad (\mathrm{Im}\,\alpha > c),$$

$$\ln K_{-}(\alpha) = -\frac{1}{2\pi i}\int_{-\infty+ic}^{\infty+ic}\frac{\ln\{2K_1(\gamma a)I_1(\gamma a)\}}{\zeta-\alpha}\,d\zeta, \qquad (\mathrm{Im}\,\alpha < c).$$

The following results will be useful later

$$K_-(\alpha) = K_+(-\alpha) \quad : \quad K_+(k)K_-(k) = 1. \tag{3.60}$$

Note that although $\ln\{2K_1(\gamma a)I_1(\gamma a)\} \sim -\ln|\alpha|$ as $\alpha \to \infty$ in the strip, the integrals are convergent if taken in the sense

$$\lim_{T\to\infty} \int_{-T+i\tau}^{T+i\tau} \{\quad\} \, d\alpha.$$

We could insert an extra factor γ so that the logarithm tends to zero at infinity, but this is not necessary.

Let k_2 tend to zero and let α tend to a real number ξ, $-k \leqslant \xi \leqslant k$. For simplicity consider only procedure (a) of §1.3, following theorem C. Then, on setting $(k^2 - \xi^2)^{1/2} = \kappa$,

$$\ln K_+(\alpha) = \tfrac{1}{2}\ln\{\pi i H_1^{(1)}(\kappa a)J_1(\kappa a)\} + \tfrac{1}{2}g(\xi), \tag{3.61a}$$

$$\ln K_-(\alpha) = \tfrac{1}{2}\ln\{\pi i H_1^{(1)}(\kappa a)J_1(\kappa a)\} - \tfrac{1}{2}g(\xi), \tag{3.61b}$$

where, on using (1.30) with

$$F(z) = \ln\{2K_1(z)I_1(z)\} \quad : \quad F(-iz) = \ln\{\pi i H_1^{(1)}(z)I_1(z)\},$$

we have

$$g(\xi) = \frac{2\xi a}{\pi i}\int_0^\infty \frac{\ln\{2K_1(z)I_1(z)\}}{a^2(k^2-\xi^2)+z^2}\cdot\frac{z\,dz}{(k^2a^2+z^2)^{1/2}} + $$

$$+\frac{2\xi a}{\pi i}P\int_0^{ka}\frac{\ln\{\pi i H_1^{(1)}(z)J_1(z)\}}{a^2(k^2-\xi^2)-z^2}\cdot\frac{z\,dz}{(k^2a^2-z^2)^{1/2}}.$$

For convenience set $\xi = k\cos\Theta$. Then

$$g(k\cos\Theta) = p(k\cos\Theta) + iq(k\cos\Theta), \quad \text{say,} \tag{3.62}$$

where p and q are real and

$$p(k\cos\Theta) = -\frac{2ka\cos\Theta}{\pi}P\int_0^{ka}\frac{\arctan\{-J_1(z)/Y_1(z)\}}{z^2-k^2a^2\sin^2\Theta}\cdot\frac{z\,dz}{(k^2a^2-z^2)^{1/2}}, \tag{3.63a}$$

$$q(k\cos\Theta) = \frac{2ka\cos\Theta}{\pi}\int_0^\infty\frac{\ln\{1/2K_1(z)I_1(z)\}}{z^2+k^2a^2\sin^2\Theta}\cdot\frac{z\,dz}{(k^2a^2+z^2)^{1/2}} +$$

$$+\frac{2ka\cos\Theta}{\pi}P\int_0^{ka}\frac{\ln\{\pi J_1(z)[J_1^2(z)+Y_1^2(z)]^{1/2}\}}{z^2-k^2a^2\sin^2\Theta}\cdot\frac{z\,dz}{(k^2a^2-z^2)^{1/2}}. \tag{3.63b}$$

Return to the solution of (3.58) which we now write as

$$\frac{1}{(2\pi)^{1/2}}\frac{i}{(\alpha+k)}+F_-=-\frac{2}{(\alpha^2-k^2)aK_+(\alpha)K_-(\alpha)}\Phi'_+(a).$$

In the usual way this can be rewritten in the form

$$(2\pi)^{-1/2}i(\alpha+k)^{-1}\{(\alpha-k)K_-(\alpha)+2kK_-(-k)\}+(\alpha-k)K_-(\alpha)F_-$$
$$=-\frac{2}{(\alpha+k)aK_+(\alpha)}\Phi'_+(a)+\frac{i}{(2\pi)^{1/2}}\cdot\frac{2kK_-(-k)}{(\alpha+k)}.$$

From the general theory (ex. (1.12)) we have $|K_\pm(\alpha)|\sim|\alpha|^{-1/2}$ as α tends to infinity in appropriate half-planes. Assuming the usual edge conditions, $|F_-|\sim|\alpha|^{-1}$ and $|\Phi'_+|\sim|\alpha|^{-1/2}$ as $\alpha\to\infty$ in appropriate half-planes. Hence the above equation defines an integral function which by Liouville's theorem is identically zero. Thus

$$\Phi'_+(\alpha)=ika(2\pi)^{-1/2}K_+(\alpha)K_-(-k).$$

On using (3.54), (3.56c) we have finally

$$\phi_t=-\frac{ika}{2\pi}K_-(-k)\int_{-\infty+ia}^{\infty+ia}K_+(\alpha)\frac{K_0(\gamma\rho)}{\gamma K_1(\gamma a)}e^{-i\alpha z}\,d\alpha,\qquad(\rho\geqslant a),\tag{3.64a}$$

$$\phi=\frac{ika}{2\pi}K_-(-k)\int_{-\infty+ia}^{\infty+ia}K_+(\alpha)\frac{I_0(\gamma\rho)}{\gamma I_1(\gamma a)}e^{-i\alpha z}\,d\alpha,\qquad(0\leqslant\rho\leqslant a).\tag{3.64b}$$

There are three regions to consider

(a) $0\leqslant\rho\leqslant a$, $z\to+\infty$. Deform the contour into the lower half-plane. There is a branch point at $\alpha=-k$. The small semi-circle round this branch point gives a contribution $-\exp(ikx)$ which exactly cancels the incident wave. The asymptotic behaviour of ϕ_t in this region is most easily obtained as a limiting case of (c) below.

(b) $0\leqslant\rho\leqslant a$, $z\to-\infty$. Close the contour in the upper half-plane. There are no branch points and only simple poles at the zeros of $\gamma I_1(\gamma a)$. The zero at $\alpha=k$ gives a contribution

$$-K_-(-k)K_+(k)e^{-ikz}=-\{K_+(k)\}^2e^{-ikz}=Re^{-ikz},\tag{3.65}$$

since we have already defined the reflection coefficient R by (3.50) as the coefficient of $\exp(-ikz)$ as $z\to-\infty$ in $0\leqslant\rho\leqslant a$. Write $R=-|R|\exp(2ikl)$ where $|R|$ is the magnitude of the reflection coefficient and l is the "end correction". In (3.61a), if $\xi \to k$ the first term on the right hand side vanishes and we have

$$K_+(k) = \exp\{\tfrac{1}{2}g(k)\} = \exp\{\tfrac{1}{2}p(k) + \tfrac{1}{2}iq(k)\},$$

where p and q are real and defined by (3.63) with $\Theta = 0$. In this case there is no need for the principal value sign and

$$|R| = \exp\left\{-\frac{2ka}{\pi}\int_0^{ka}\frac{\arc\tan\{-J_1(z)/Y_1(z)\}}{z(k^2a^2 - z^2)^{1/2}}\,dz\right\}, \tag{3.66}$$

$$\frac{l}{a} = \frac{1}{\pi}\int_0^{\infty}\frac{\ln\{1/2K_1(z)I_1(z)\}}{z(k^2a^2 + z^2)^{1/2}}\,dz + $$

$$+ \frac{1}{\pi}\int_0^{ka}\frac{\ln[\pi J_1(z)\{J_1^2(z) + Y_1^2(z)\}^{1/2}]}{z(k^2a^2 - z^2)^{1/2}}\,dz.$$

(c) $\rho > a$. We consider the far field as $(\rho^2 + z^2)^{1/2} \to \infty$. Write $\rho = r\sin\theta$, $z = r\cos\theta$ where (r,θ) are spherical polar co-ordinates. Let $r \to \infty$ in (3.64a) and use the asymptotic expansion $K_0(\gamma\rho) \sim (\pi/2\gamma\rho)^{1/2}\exp(-\gamma\rho)$ which holds as $r \to \infty$ for any fixed θ. Then (3.64a) takes the form (1.56) considered in §1.6. Use of the asymptotic expansion (1.71) gives

$$\begin{aligned}\phi_t &\sim -\frac{aK_-(-k)K_+(-k\cos\theta)}{\pi\sin\theta\, H_1^{(1)}(ka\sin\theta)}\cdot\frac{e^{ikr}}{r}\\ &= -\frac{aiJ_1(ka\sin\theta)}{\sin\theta}\cdot\frac{K_+(k)}{K_+(k\cos\theta)}\cdot\frac{e^{ikr}}{r} = f(\theta)\frac{e^{ikr}}{r},\end{aligned} \tag{3.67}$$

where we have used the results (3.59), (3.60) and the function $f(\theta)$ has been defined in (3.51). This gives the variation of the far-field with θ. We easily find (remember (3.59), (3.65))

$$f(0) = -\tfrac{1}{2}ika^2 \quad : \quad f(\pi) = \tfrac{1}{2}ika^2R. \tag{3.68}$$

These equations differ in sign from H. Levine and J. Schwinger [1], equation (III, 12, 13), and P. M. Morse and H. Feshbach [1], equation (11.4.33), but the sign in the references seems to be in error.

This completes our discussion of the radiation from a circular pipe. Problems involving $k > 3{\cdot}832$ when more than one progressive mode can exist in the pipe, and problems when the incident mode is not axially symmetrical can be handled by a straightforward extension of the above analysis (cf. the general case for the two-dimensional duct in §3.3 and L. A. Vajnshtejn [3], [4], [5]).

Next consider a plane wave

$$\phi_i = \exp\{-ikr(\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\psi-\psi_0))\}, \tag{3.69}$$

incident on a circular pipe in $\rho = a$, $z \leqslant 0$ from direction (θ_0,ψ_0) where we use (r,θ,ψ) for spherical polar co-ordinates, and we shall use (ρ,ψ,z) for cylindrical polars, axial symmetry being no longer present. We use the well-known result

$$e^{-iv\cos\Theta} = \sum_{m=-\infty}^{\infty} (-i)^m e^{-im\Theta} J_m(v) \tag{3.70}$$

to write (3.69) in the form

$$\phi_i = e^{-ikz\cos\theta_0} \sum_{m=-\infty}^{\infty} (-i)^m e^{-im(\psi-\psi_0)} J_m(k\rho\sin\theta_0). \tag{3.71}$$

Write $\phi_t = \phi_i + \phi$, where ϕ satisfies

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\psi^2} + \frac{\partial^2\phi}{\partial z^2} + k^2\phi = 0. \tag{3.72}$$

The condition $\partial\phi_t/\partial\rho = 0$ on the wall of the pipe gives

$$\begin{aligned}\frac{\partial\phi}{\partial\rho} &= -\frac{\partial\phi_i}{\partial\rho}\\ &= -k\sin\theta_0 e^{-ikz\cos\theta_0} \sum_{m=-\infty}^{\infty} (-i)^m e^{-im(\psi-\psi_0)} J'_m(k\rho\sin\theta_0)\end{aligned}$$

on $\rho = a$, $-\infty < z \leqslant 0$. Apply a finite transform in ψ and an infinite transform in z:

$$\bar{\phi}(n) = \int_0^{2\pi} \phi e^{in\psi}\,d\psi \quad : \quad \phi = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \bar{\phi}(n) e^{-in\psi},$$

$$\Phi(n,\alpha) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} \bar{\phi}(n) e^{i\alpha z}\,dz \ : \ \bar{\phi}(n) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty+ib}^{\infty+ib} \Phi(n,\alpha) e^{-i\alpha z}\,d\alpha. \tag{3.73}$$

The equation for $\Phi(n,\alpha)$ is

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d\Phi(n,\alpha)}{d\rho} - \left(\frac{n^2}{\rho^2} + \gamma^2\right)\Phi(n,\alpha) = 0.$$

This gives

$$\Phi(n,\alpha) = A_n(\alpha) K_n(\gamma\rho), \qquad (\rho \geqslant a), \tag{3.74a}$$

$$\qquad = B_n(\alpha) I_n(\gamma\rho), \qquad (0 \leqslant \rho \leqslant a). \tag{3.74b}$$

The boundary conditions on $\rho = a$ give

$$\Phi_+(n, a+0) + \Phi_-(n, a+0) = A_n(\alpha) K_n(\gamma a), \tag{3.75a}$$

$$\Phi_+(n, a-0) + \Phi_-(n, a-0) = B_n(\alpha) I_n(\gamma a), \tag{3.75b}$$

$$\Phi'_+(n) + \Phi'_-(n) = \gamma A_n(\alpha) K'_n(\gamma a) = \gamma B_n(\alpha) I'_n(\gamma a). \tag{3.75c}$$

Introduce

$$\Phi_-(n, a+0) - \Phi_-(n, a-0) = D_-(n),$$

and use the result

$$\Phi'_-(n) = -\frac{k \sin \theta_0}{(2\pi)^{1/2}} \int_0^{2\pi} e^{in\psi}\, d\psi \int_{-\infty}^{0} e^{iz(\alpha - k\cos\theta_0)}\, dz \times \sum_{m=-\infty}^{\infty} (-i)^m e^{-im(\psi-\psi_0)} J'_m(ka \sin \theta_0)$$

$$= (-1)^n i^{n+1} (2\pi)^{1/2} k \sin \theta_0\, J'_n(ka \sin \theta_0)(\alpha - k \cos \theta_0)^{-1} e^{in\psi_0}.$$

Subtract (3.75b) from (3.75a) and use (3.75c) to find

$$D_-(n) = \frac{1}{\gamma^2 a I'_n(\gamma a) K'_n(\gamma \alpha)} \times \left\{\Phi'_+(n) + (-1)^n i^{n+1} (2\pi)^{1/2} \frac{k \sin \theta_0\, J'_n(ka \sin \theta_0)}{\alpha - k \cos \theta_0} e^{in\psi_0}\right\}.$$

This equation can be solved by the standard Wiener–Hopf technique. We deal with the case $n = 0$ in detail. The equation holds in $-k_2 < \tau < k_2 \cos \theta_0$. Introduce $K(\alpha)$ defined in (3.59). Then rearrange and separate in the usual way. This gives

$$D_-(0) = \frac{2i(2\pi)^{1/2} k \sin \theta_0\, J_1(ka \sin \theta_0)}{a(k + k \cos \theta_0) K_+(k \cos \theta_0)} \cdot \frac{1}{(\alpha - k \cos \theta_0)(\alpha - k) K_-(\alpha)}. \tag{3.76}$$

From (3.73), (3.74) we have

$$\phi_t = \phi_i + \frac{1}{(2\pi)^{3/2}} \sum_{n=-\infty}^{\infty} e^{-in\psi} \int_{-\infty+ib}^{\infty+ib} e^{-i\alpha z} B_n(\alpha) I_n(\gamma \rho)\, d\alpha, \quad (0 \leqslant \rho \leqslant a), \tag{3.77}$$

where $-k_2 < b < k_2 \cos \theta_0$. We can use (3.75a, b) to express $B_n(\alpha)$ in terms of $D_-(n)$. The term in the infinite series with $n = 0$ is

$$-\frac{1}{(2\pi)^{3/2}} \int_{-\infty+ib}^{\infty+ib} \gamma a K_1(\gamma a) I_0(\gamma \rho) D_-(0) e^{-i\alpha z}\, d\alpha$$

$$= -\frac{i}{\pi} \cdot \frac{k \sin \theta_0\, J_1(ka \sin \theta_0)}{(k + k \cos \theta_0) K_+(k \cos \theta_0)} \times \int_{-\infty+ib}^{\infty+ib} \frac{\gamma K_1(\gamma a) I_0(\gamma \rho) e^{-i\alpha z}}{(\alpha - k \cos \theta_0)(\alpha - k) K_-(\alpha)}\, d\alpha,$$

where we have used (3.76). As $z \to -\infty$ in $0 \leqslant \rho \leqslant a$, close the contour in the upper half-plane. The pole at $\alpha = k \cos \theta_0$ gives $-J_0(ka \sin \theta_0) \exp(-ikz \cos \theta_0)$ which exactly cancels the term in $n = 0$ in the incident wave. The pole at $\alpha = k$ gives a contribution

$$\frac{2J_1(ka \sin \theta_0)K_+(k)}{ka \sin \theta_0 K_+(k \cos \theta_0)} e^{-ikz} = \frac{f(\theta_0)}{f(0)} e^{-ikz},$$

where $f(\theta)$ is the angle distribution factor defined in (3.51) and found explicitly in (3.67). The agreement between these formulae is an expression of the reciprocity principle that the amplitude of the progressive wave produced in a tube by an incident wave of unit intensity at angle θ with the axis of the tube is proportional to the wave radiated at angle θ by a progressive wave of unit amplitude incident on the open end of the tube from inside the tube.

The remaining terms in the infinite series (3.77) can be found in exactly the same way but we do not pursue this here.

3.5 Semi-infinite strips parallel to the walls of a duct

The problems to be examined in this section and the next differ from those in previous sections in that they will be concerned with only the *interior* of a duct $0 \leqslant y \leqslant 2b$, $-\infty < z < \infty$. One important result is that the integrands of integrals expressing the potential at any point will possess only poles and no branch points.

Consider the duct $0 \leqslant y \leqslant 2b$, $-\infty < z < \infty$ with $\partial\phi_t/\partial y = 0$ on $y = 0, 2b$, and a strip in $y = c$, $0 \leqslant z < \infty$ with $\phi_t = 0$ on the strip. Suppose there is a wave $\phi_i = \exp(ikz)$ incident from $z = -\infty$. Write $\phi_t = \phi_i + \phi$. Apply a Fourier transform to the equation for ϕ and use the condition that $d\Phi/dy = 0$ on $y = 0, 2b$. In the usual way this gives

$$\begin{aligned} \Phi(y) &= A(\alpha) \cosh \gamma y \quad , && (0 \leqslant y \leqslant c), \\ &= B(\alpha) \cosh \gamma(2b - y), && (c \leqslant y \leqslant b). \end{aligned}$$

On $y = c$ we have that $\Phi_\pm(y)$, $\Phi'_-(y)$ are continuous but $\Phi'_+(y)$ is discontinuous. Also

$$\Phi_+(c) = -\frac{1}{(2\pi)^{1/2}} \int_0^\infty e^{iz(\alpha+k)}\, dz = -\frac{i}{(2\pi)^{1/2}} \cdot \frac{1}{(\alpha + k)} \cdot$$

Thus

$$-i(2\pi)^{-1/2}(\alpha + k)^{-1} + \Phi_-(c) = A(\alpha) \cosh \gamma c = B(\alpha) \cosh \gamma(2b - c), \tag{3.78a}$$

$$\Phi'_+(c - 0) + \Phi'_-(c) = \gamma A(\alpha) \sinh \gamma c, \tag{3.78b}$$

$$\Phi'_+(c + 0) + \Phi'_-(c) = -\gamma B(\alpha) \sinh \gamma(2b - c). \tag{3.78c}$$

Introduce

$$\Phi'_+(c+0) - \Phi'_+(c-0) = D'_+. \tag{3.79a}$$

Subtract (3.78b) from (3.78c) and eliminate A, B by means of (3.78a). We find

$$D'_+ = -\frac{\gamma \sinh 2\gamma b}{\cosh \gamma c \cosh \gamma(2b-c)}\left\{\Phi_-(c) - \frac{i}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha+k)}\right\}.$$

Define (3.79b)

$$K(\alpha) = 2\gamma b \cosh \gamma c \cosh \gamma(2b-c)\,\{\sinh 2\gamma b\}^{-1} = K_+(\alpha)K_-(\alpha), \tag{3.79c}$$

where $|K_+|$, $|K_-| \sim |\alpha|^{1/2}$ as $\alpha \to \infty$ in appropriate half-planes. Explicit formulae are easily obtained by the infinite product theory of §1.3. (See (3.96) below for the case $b = c$). Rearrange (3.79b) as:

$$\frac{K_+(\alpha)D'_+}{2b(\alpha+k)} + \frac{i}{(2\pi)^{1/2}}\cdot\frac{2k}{(\alpha+k)K_-(-k)}$$
$$= -\frac{(\alpha-k)\Phi_-(c)}{K_-(\alpha)} + \frac{i}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha+k)}\left\{\frac{(\alpha-k)}{K_-(\alpha)} + \frac{2k}{K_-(-k)}\right\}.$$

Apply the Wiener–Hopf technique in the usual way. Then

$$\Phi_-(c) = \frac{i}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha+k)} + \frac{i}{(2\pi)^{1/2}}\cdot\frac{2k}{(\alpha^2-k^2)}\cdot\frac{K_-(\alpha)}{K_-(-k)}.$$

Hence from (3.78a),

$$A(\alpha)\cosh \gamma c = B(\alpha)\cosh \gamma(2b-c) = \frac{i}{(2\pi)^{1/2}}\cdot\frac{2k}{(\alpha^2-k^2)}\cdot\frac{K_-(\alpha)}{K_-(-k)} \tag{3.80a}$$

$$= \frac{i}{(2\pi)^{1/2}}\cdot\frac{4kb}{\gamma}\cdot\frac{\cosh \gamma c \cosh \gamma(2b-c)}{\sinh 2\gamma b}\cdot\frac{1}{K_+(\alpha)K_-(-k)} \tag{3.80b}$$

The potential at any point is given by

$$\phi_t = e^{ikz} + \frac{1}{(2\pi)^{1/2}}\int_{-\infty+ib}^{\infty+ib} A(\alpha)\cosh \gamma y e^{-i\alpha z}\,d\alpha, \qquad (0 \leqslant y \leqslant c),$$

(3.81)

$$= e^{ikz} + \frac{1}{(2\pi)^{1/2}}\int_{-\infty+ib}^{\infty+ib} B(\alpha)\cosh \gamma(2b-y)e^{-i\alpha z}\,d\alpha, \quad (c \leqslant y \leqslant 2b).$$

If $z < 0$ complete the contour in the upper half-plane and use forms given by (3.80b) for A, B. The only singularities are simple poles at $\alpha = k$, and values of α in the upper half-plane defined by $2\gamma b = m\pi i (m = 1, 2, 3 \ldots)$ i.e.

$$\alpha = i\{(m\pi/2b)^2 - k^2\}^{1/2} = i\gamma_m, \quad \text{say}, \tag{3.82}$$

where we assume $k < (\pi/2b)$ so that γ_m is real and positive. Then either of the integrals in (3.81) gives for $z < 0$, $0 \leqslant y \leqslant 2b$,

$$\phi_t = e^{ikz} - \frac{1}{\{K_+(k)\}^2} e^{-ikz} - \\ - \frac{2k}{K_+(k)} \sum_{n=1}^{\infty} \frac{1}{\gamma_n K_+(i\gamma_n)} \cos\frac{n\pi c}{2b} \cos\frac{n\pi y}{2b} e^{\gamma_n z}. \tag{3.83}$$

If $z > 0$ we close the contour in the lower half-plane and use forms given by (3.80a) for A, B. For $0 \leqslant y \leqslant c$, $c \leqslant y \leqslant 2b$ there are simple poles at $\alpha = -k$ and values of α in the lower half-plane given by $\gamma c = (n + \frac{1}{2})\pi i$ and $\gamma(2b - c) = (n + \frac{1}{2})\pi i$ respectively. The pole at $\alpha = -k$ exactly cancels the incident wave and the remaining terms give eigenfunction expansions in terms of $\cos (n + \frac{1}{2})\pi y/c$ and $\cos (n + \frac{1}{2})\pi(2b - y)/(2b - c)$, respectively, which are appropriate for boundary conditions $\partial\phi_t/\partial y = 0$ on $y = 0, 2b$ and $\phi_t = 0$ on $y = c$.

We next assume that the strip is resistive so that instead of $\phi_t = 0$ on the strip we have $\phi_t = \pm i\delta\partial\phi_t/\partial y$ on $y = c \pm 0$ respectively. This gives, on writing $\phi_t = \phi_i + \phi$ as before,

$$\Phi_+(c \pm 0) \mp i\delta\Phi'_+(c \pm 0) = -i(2\pi)^{-1/2}(\alpha + k)^{-1}. \tag{3.84}$$

Instead of (3.78) we have

$$\begin{aligned}
\Phi_+(c - 0) + \Phi_-(c) &= A(\alpha) \cosh \gamma c, \\
\Phi'_+(c - 0) + \Phi'_-(c) &= \gamma A(\alpha) \sinh \gamma c, \\
\Phi_+(c + 0) + \Phi_-(c) &= B(\alpha) \cosh \gamma(2b - c), \\
\Phi'_+(c + 0) + \Phi'_-(c) &= -\gamma B(\alpha) \sinh \gamma(2b - c).
\end{aligned}$$

Elimination of A, B and use of (3.84) (cf. §2.9) gives

$$\Phi_-(c) - \gamma^{-1} \coth \gamma c \Phi'_-(c) = i(2\pi)^{-1/2}(\alpha + k)^{-1} + \\ + L(\alpha : c)\{\gamma \sinh \gamma c\}^{-1}\Phi'_+(c - 0), \tag{3.85a}$$

$$\Phi_-(c) + \gamma^{-1} \coth \gamma(2b - c)\Phi'_-(c) = i(2\pi)^{-1/2}(\alpha + k)^{-1} - \\ - L(\alpha : 2b - c)\{\gamma \sinh \gamma(2b - c)\}^{-1}\Phi'_+(c + 0), \tag{3.85b}$$

where

$$L(\alpha : d) = \cosh \gamma d + i\delta\gamma \sinh \gamma d.$$

These are *simultaneous* Wiener–Hopf equations which cannot be solved exactly in the general case by the standard procedure (cf. §4.4). However if $c = b$, (3.85) can be reduced to two independent Wiener–Hopf equations. On subtracting (3.85b) from (3.85a) we can readily prove that in this special case $\Phi'_-(b) = 0$, $\Phi'_+(b+0) = -\Phi'_+(b-0)$. These results are obvious physically from the symmetry of the problem. If we next add (3.85a, b) with $c = b$, we find, using notation (3.79a),

$$D'_+ = -\frac{2\gamma \sinh \gamma b}{\cosh \gamma b + i\delta\gamma \sinh \gamma b}\left\{\Phi_-(b) - \frac{i}{(2\pi)^{1/2}}\frac{1}{\alpha + k}\right\}. \quad (3.86)$$

An equation which is similar to this, apart from notation, is considered in ex. 3.13. If $\delta = 0$, (3.86) reduces to the special case of (3.79b) with $c = b$.

A similar analysis holds for an infinite circular duct containing a semi-infinite circular tube. Consider the case where there are no resistive losses. Suppose that the infinite duct occupies $0 \leqslant \rho \leqslant a$, $-\infty < z < \infty$ with $\partial\phi_t/\partial\rho = 0$ on $\rho = a$; suppose that $\phi_t = 0$ on a cylinder lying in $\rho = b$, $0 \leqslant z < \infty$. Assume a wave $\phi_i = \exp(ikz)$ incident from $z = -\infty$ and write $\phi_t = \phi_i + \phi$. If we apply a Fourier transform in z we find (cf. §3.4)

$$\begin{aligned}\Phi(\alpha) &= A(\alpha)I_0(\gamma\rho), && (0 \leqslant \rho \leqslant b),\\ &= B(\alpha)\{I_0(\gamma\rho)K_0'(\gamma a) - K_0(\gamma\rho)I_0'(\gamma a)\}, && (b \leqslant \rho \leqslant a),\end{aligned}$$

where we have used the boundary conditions: Φ finite on $\rho = 0$, and $d\Phi/d\rho = 0$ on $\rho = a$. The boundary conditions on $\rho = b$ give

$$\begin{aligned}-i(2\pi)^{-1/2}(\alpha + k)^{-1} + \Phi_-(\alpha) &= A(\alpha)I_0(\gamma b)\\ &= B(\alpha)\{I_0(\gamma b)K_0'(\gamma a) - K_0(\gamma b)I_0'(\gamma a)\},\\ \Phi'_+(b+0) + \Phi'_-(b) &= \gamma B(\alpha)\{I_0'(\gamma b)K_0'(\gamma a) - K_0'(\gamma b)I_0'(\gamma a)\},\\ \Phi'_+(b-0) + \Phi'_-(b) &= \gamma A(\alpha)I_0'(\gamma b).\end{aligned}$$

Eliminate $\Phi'_-(b)$ by subtracting the last two equations, and set

$$\Phi'_+(b+0) - \Phi'_+(b-0) = D'_+(\alpha).$$

Eliminate A and B, and use (3.57b). Then

$$K(\alpha)D'_+(\alpha) = \{\Phi_-(\alpha) - i(2\pi)^{-1/2}(\alpha + k)^{-1}\}, \quad (3.87)$$

where

$$K(\alpha) = bI_0(\gamma b)\{I_0'(\gamma a)\}^{-1}\{I_0(\gamma b)K_0'(\gamma a) - K_0(\gamma b)I_0'(\gamma a)\}.$$

This is now in standard form (cf. (3.79b)) and the solution proceeds as before. It might seem that $K(\alpha)$ has branch points at $\alpha = \pm k$

because of the presence of $K_0'(\gamma a)$, $K_0(\gamma b)$ but this is not the case since

$$\gamma K_1(\gamma a) = \gamma I_1(\gamma a) \ln(\tfrac{1}{2}\gamma) + \text{a function with no branch points,}$$
$$K_0(\gamma b) = -I_0(\gamma b) \ln(\tfrac{1}{2}\gamma) + \text{a function with no branch points.}$$

Hence the logarithmic terms in $K(\alpha)$ cancel, and $K(\alpha)$ has only simple zeros and poles. The factorization can therefore be performed by an infinite product decomposition. (See L. L. Bailin [1], N. Marcuvitz [1]. These references contain numerical tables.)

3.6 A strip across a duct

Consider the duct $0 \leqslant y \leqslant 2b$, $-\infty < z < \infty$ with a strip $0 \leqslant y \leqslant b$ at $z = 0$. (Finite width in the x-direction can be dealt with as already indicated in §3.1.) Suppose that the boundary condition on all boundaries is $\partial\phi_t/\partial n = 0$ and that a wave $\phi_i = \exp(ikz)$ is incident from $z = -\infty$. Set $\phi_t = \phi_i + \phi$. Then $\partial\phi/\partial z = -ik$ on $z = 0$, $0 \leqslant y \leqslant b$.

Apply a Fourier transform in z to the equation for ϕ in the region $b \leqslant y \leqslant 2b$. Using the condition that $\partial\phi/\partial y = 0$ on $y = 2b$ we find $\Phi(y,\alpha) = A(\alpha) \cosh \gamma(2b - y)$. Hence on $y = b$, in the usual notation,

$$\Phi_+(b,\alpha) + \Phi_-(b,\alpha) = A(\alpha) \cosh \gamma b, \tag{3.88a}$$

$$\Phi'_+(b,\alpha) + \Phi'_-(b,\alpha) = -\gamma A(\alpha) \sinh \gamma b. \tag{3.88b}$$

Elimination of $A(\alpha)$ gives

$$\Phi_+(\alpha) + \Phi_-(\alpha) = -\gamma^{-1} \coth \gamma b\{\Phi'_+(\alpha) + \Phi'_-(\alpha)\}, \tag{3.89}$$

where we have simplified the notation slightly by omitting reference to 'b'.

Apply a Fourier transform in z in the region $0 \leqslant y \leqslant b$, $z \geqslant 0$, noting that

$$\frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{\partial^2\phi}{\partial z^2} e^{i\alpha z}\, dz \doteq -\frac{1}{(2\pi)^{1/2}} \left(\frac{\partial\phi}{\partial z}\right)_0 + \frac{i\alpha}{(2\pi)^{1/2}} (\phi)_0 - \alpha^2\Phi_+(y,\alpha),$$

where the suffix *zero* refers to the value on $z = 0$. We know that $(\partial\phi/\partial z)_0 = -ik$ but $(\phi)_0$ is unknown. Denote $(\phi)_0$ by $f(y)$. Then the partial differential equation becomes

$$\frac{d^2\Phi_+(y,\alpha)}{dy^2} - \gamma^2\Phi_+(y,\alpha) = -\frac{ik}{(2\pi)^{1/2}} - \frac{i\alpha f(y)}{(2\pi)^{1/2}}\cdot \tag{3.90}$$

To eliminate the unknown function $f(y)$, change the sign of α and add the resulting equation to (3.90). This gives

$$d^2\{\Phi_+(y,\alpha) + \Phi_+(y,-\alpha)\}/dy^2 - \\ - \gamma^2\{\Phi_+(y,\alpha) + \Phi_+(y,-\alpha)\} = -(2\pi)^{-1/2}2ik.$$

The required solution of this equation, with $d\Phi_+/dy = 0$ on $y = 0$, is

$$\Phi_+(y,\alpha) + \Phi_+(y,-\alpha) = B(\alpha)\cosh\gamma y + (2\pi)^{-1/2}2ik\gamma^{-2}. \quad (3.91)$$

Differentiate with respect to y, eliminate $B(\alpha)$ between the resulting equation and (3.91), and set $y = b$. We obtain

$$\Phi_+(\alpha) + \Phi_+(-\alpha) = \gamma^{-1}\coth\gamma b\{\Phi'_+(\alpha) + \Phi'_+(-\alpha)\} + \\ + (2\pi)^{-1/2}2ik\gamma^{-2}. \quad (3.92)$$

In exactly the same way, in $0 \leqslant y \leqslant b$, $-\infty < z \leqslant 0$, we find

$$\Phi_-(\alpha) + \Phi_-(-\alpha) = \gamma^{-1}\coth\gamma b\{\Phi'_-(\alpha) + \Phi'_-(-\alpha)\} - \\ - (2\pi)^{-1/2}2ik\gamma^{-2}. \quad (3.93)$$

Equations (3.89), (3.92), (3.93) are the basic equations for the problem.

Add (3.92), (3.93):

$$\{\Phi_+(\alpha) + \Phi_-(\alpha)\} + \{\Phi_+(-\alpha) + \Phi_-(-\alpha)\} \\ = \gamma^{-1}\coth\gamma b[\{\Phi'_+(\alpha) + \Phi'_-(\alpha)\} + \{\Phi'_+(-\alpha) + \Phi'_-(-\alpha)\}].$$

From (3.89):

$$\{\Phi_+(\alpha) + \Phi_-(\alpha)\} + \{\Phi_+(-\alpha) + \Phi_-(-\alpha)\} \\ = -\gamma^{-1}\coth\gamma b[\{\Phi'_+(\alpha) + \Phi'_-(\alpha)\} + \{\Phi'_+(-\alpha) + \Phi'_-(-\alpha)\}].$$

Add and subtract these two equations and apply the Wiener–Hopf technique. Then

$$\Phi_+(\alpha) = -\Phi_-(-\alpha) \quad : \quad \Phi'_+(\alpha) = -\Phi'_-(-\alpha). \quad (3.94)$$

Eliminate $\Phi_+(\alpha)$ from (3.89), (3.92) by subtraction and use (3.94). We find

$$\Phi_+(-\alpha) = \gamma^{-1}\coth\gamma b\Phi'_+(\alpha) + (2\pi)^{-1/2}ik\gamma^{-2}, \qquad (-k_2 < \tau < k_2). \quad (3.95)$$

This may be compared with (3.79b) with $b = c$. Set

$$\gamma b\coth\gamma b = K(\alpha) = K_+(\alpha)K_-(\alpha), \quad (3.96a)$$

where

$$K_\pm(\alpha) = \prod_{n=1}^{\infty}\frac{(1 - k^2b^2_{n-\frac{1}{2}})^{1/2} \mp i\alpha b_{n-\frac{1}{2}}}{(1 - k^2b_n^2)^{1/2} \mp i\alpha b_n}, \qquad b_p = \frac{b}{p\pi}. \quad (3.96b)$$

This decomposition has been obtained in ex. 3.6. We can rewrite (3.95) as

$$\frac{\alpha - k}{K_-(\alpha)}\Phi_+(-\alpha) - \frac{ik}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha+k)}\left\{\frac{1}{K_-(\alpha)} - \frac{1}{K_-(-k)}\right\}$$
$$= \frac{\Phi'_+(\alpha)K_+(\alpha)}{b(\alpha+k)} + \frac{ik}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha+k)K_-(-k)}.$$

Hence, on using the Wiener–Hopf technique and remembering that $K_+(\alpha) = K_-(-\alpha)$,

$$\Phi'_+(\alpha) = -(2\pi)^{-1/2}ikb\{K_+(k)K_+(\alpha)\}^{-1}.$$

From (3.88b), (3.94), to find the potential in $b \leqslant y \leqslant 2b$, we need

$$A(\alpha) = -\{\gamma \sinh \gamma b\}^{-1}\{\Phi'_+(\alpha) - \Phi'_+(-\alpha)\}.$$

Hence

$$\phi(y,z) = \frac{ikb}{2\pi K_+(k)}\int_{-\infty+i\tau}^{\infty+i\tau}\left\{\frac{1}{K_+(\alpha)} - \frac{1}{K_+(-\alpha)}\right\}\frac{\cosh \gamma(2b-y)}{\gamma \sinh \gamma b}e^{-i\alpha z}\,d\alpha. \tag{3.97}$$

On changing the signs of z and α we see directly that, as implied by the first equation in (3.94),

$$\phi(y,-z) = -\phi(y,z). \tag{3.98}$$

To obtain ϕ for $z \geqslant 0$, $b \leqslant y \leqslant 2b$, close the contour in (3.97) in a lower half-plane and evaluate by residues. It is convenient to use (3.96a) and the result $K_+(-\alpha) = K_-(\alpha)$ to rearrange (3.97) as

$$\phi(y,z) = \frac{ik}{2\pi K_+(k)}\int_{-\infty+i\tau}^{\infty+i\tau}\left\{\frac{K_-(\alpha)}{\gamma^2\cosh\gamma b} - \frac{b}{\gamma\sinh\gamma b K_-(\alpha)}\right\}$$
$$\times \cosh\gamma(2b-y)e^{-i\alpha z}\,d\alpha.$$

Remembering that $\phi_t = \phi_i + \phi$ and $\phi(y,-z) = -\phi(y,z)$ we obtain

$$\phi_t = P_0 e^{ikz} + S(y,z), \qquad (z \geqslant 0), \tag{3.99a}$$
$$= e^{ikz} + (1 - P_0)e^{-ikz} - S(y,z), \qquad (z \leqslant 0), \tag{3.99b}$$

where, if $\gamma_p = \{(p\pi/b)^2 - k^2\}^{1/2}$,

$$S(y,z) = \sum_{n=1}^{\infty}[P_{n-\frac{1}{2}}\cos\{(n-\tfrac{1}{2})\pi y/b\}e^{-\gamma_{n-\frac{1}{2}}|z|} + P_n\cos\{n\pi y/b\}e^{-\gamma_n|z|}],$$

and

$$P_0 = \tfrac{1}{2}[1 + \{K_+(k)\}^{-2}],$$
$$P_n = (-1)^{n+1}\frac{ik}{\gamma_n K_+(k)K_+(i\gamma_n)} \tag{3.100}$$
$$P_{n-\frac{1}{2}} = (-1)^{n+1}\frac{ikK_+(i\gamma_{n-\frac{1}{2}})}{(n-\frac{1}{2})\pi\gamma_{n-\frac{1}{2}}K_+(k)}.$$

We have indicated how (3.99a, b) are derived for $b \leqslant y \leqslant 2b$. Exactly the same expressions are found in $0 \leqslant y \leqslant b$ by a similar method.

The results of this section have been obtained by G. L. Baldwin and A. E. Heins [1]. We have already noted that (3.95) and (3.79b) with $b = c$ are of similar form. The equivalence of the problems leading to these two equations has been shown by S. N. Karp and W. E. Williams [1] by means of a direct argument not involving the Wiener–Hopf technique. A connexion between the problem solved above and a set of linear simultaneous algebraic equations is indicated in ex. 4.12.

Various methods have been suggested for approximate solution of problems like those solved exactly in this Chapter. It should now be possible to check exact against approximate solutions, and perhaps to develop improved approximate methods. In connexion with diffraction problems some interesting investigations of this type are given in L. A. Vajhnstejn [1], [2], [6].

Miscellaneous Examples and Results III

3.1 In complicated examples the following procedure may be preferred to that used in §3.2. We use it to obtain (3.18). First define our standard notation. If ϕ is continuous on $y = \pm b$, $x > 0$, we define

$$S_+ = \Phi_+(b) + \Phi_+(-b) \quad : \quad D_+ = \Phi_+(b) - \Phi_+(-b). \tag{a}$$

If however ϕ is discontinuous on $y = \pm b$, $x > 0$, we define

$$\begin{aligned} S_+^{(o)} &= \Phi_+(b+0)+\Phi_+(-b-0) &:\quad D_+^{(o)} &= \Phi_+(b+0)-\Phi_+(-b-0),\\ S_+^{(i)} &= \Phi_+(b-0)+\Phi_+(-b+0) &:\quad D_+^{(i)} &= \Phi_+(b-0)-\Phi_+(-b+0),\\ S_+ &= \tfrac{1}{2}(S_+^{(o)}-S_+^{(i)}) &:\quad D_+ &= \tfrac{1}{2}(D_+^{(o)}-D_+^{(i)}). \end{aligned} \tag{b}$$

The superscript 'o' refers to the 'outer' sides of the half-planes and 'i' to the 'inner'. Similar definitions apply to dashed quantities. On $y = \pm b$, $x < 0$ the 'plus' subscript is replaced by 'minus.'

From (3.10a), using notation corresponding to (2.11), (2.15) and remembering that ϕ is discontinuous across $y = \pm b$, $x < 0$, but all other functions are continuous,

$$\begin{aligned} \Phi_+(b) + \Phi_-(b+0) &= Ae^{-\gamma b} &:\quad \Phi_+(-b) + \Phi_-(-b-0) &= De^{-\gamma b},\\ \Phi'_+(b) + \Phi'_-(b) &= -\gamma Ae^{-\gamma b} &:\quad \Phi'_+(-b) + \Phi'_-(-b) &= \gamma De^{-\gamma b}. \end{aligned}$$

Eliminate A and D:

$$\begin{aligned} \Phi'_+(b) + \Phi'_-(b) &= -\gamma\{\Phi_+(b) + \Phi_-(b+0)\} = -\gamma Ae^{-\gamma b},\\ \Phi'_+(-b) + \Phi'_-(-b) &= \gamma\{\Phi_+(-b) + \Phi_-(-b-0)\} = \gamma De^{-\gamma b}. \end{aligned}$$

Add and subtract these and introduce the above notation

$$S'_+ + S'_- = -\gamma(D_+ + D_-^{(o)}) = -\gamma(A - D)e^{-\gamma b}, \tag{c}$$
$$D'_+ + D'_- = -\gamma(S_+ + S_-^{(o)}) = -\gamma(A + D)e^{-\gamma b}. \tag{d}$$

Similarly (3.10b) give

$$\Phi_+(\pm b) + \Phi_-(\pm b \mp 0) = Be^{\mp\gamma b} + Ce^{\pm\gamma b},$$
$$\Phi'_+(\pm b) + \Phi'_-(\pm b) = -\gamma Be^{\mp\gamma b} + \gamma Ce^{\pm\gamma b},$$

where upper and lower signs go together. Add and subtract in pairs and eliminate $(B + C)$ and $(B - C)$. Introducing our standard notation, we find

$$S'_+ + S'_- = \gamma \coth \gamma b(D_+ + D_-^{(i)}) = 2\gamma(C - B)\cosh \gamma b, \qquad \text{(e)}$$
$$D'_+ + D'_- = \gamma \tanh \gamma b(S_+ + S_-^{(i)}) = 2\gamma(C + B)\sinh \gamma b. \qquad \text{(f)}$$

Clearly (c) and (e) go together. The functions S'_+, D_+, $D_-^{(o)}$, $D_-^{(i)}$ are unknown, S'_- is known. In a situation of this kind it is a general rule that D_+ should be eliminated since a Wiener–Hopf equation involves one 'plus' function and one 'minus' function, and an equation between D_+ and D_- will not be of the Wiener–Hopf type. In fact elimination of D_+ between (c) and (e) gives (3.18a), elimination of S_+ between (d) and (f) gives (3.18b).

The point of this method is that the constants A–D are eliminated first, the initial conditions are introduced afterwards.

3.2 Consider a plane wave incident on two semi-infinite parallel planes as in §3.2 (Fig. 3.1) except that $\phi_t = 0$ on the planes instead of $\partial\phi_t/\partial y = 0$. Show that corresponding to (3.18) we have

$$S_+ + (2\pi)^{-1/2}2i\cos(kb\sin\Theta)\{\alpha - k\cos\Theta\}^{-1} = -\gamma^{-1}(1 + e^{-2\gamma b})D'_-,$$
$$D_+ + (2\pi)^{-1/2}2\sin(kb\sin\Theta)\{\alpha - k\cos\Theta\}^{-1} = -\gamma^{-1}(1 - e^{-2\gamma b})S'_-.$$

Hence (cf. (3.20), (3.21))

$$D'_- = -(2\pi)^{-1/2}i\cos(kb\sin\Theta)(k + k\cos\Theta)^{1/2}(\alpha - k)^{1/2} \times \{K_+(k\cos\Theta)K_-(\alpha)(\alpha - k\cos\Theta)\}^{-1},$$
$$S'_- = -(2\pi)^{-1/2}b^{-1}\sin(kb\sin\Theta)\{L_+(k\cos\Theta)L_-(\alpha)(\alpha - k\cos\Theta)\}^{-1}.$$

The far field and the travelling waves inside the duct can be deduced as in §3.2. As an example suppose that $\frac{1}{2}\pi < kb < \pi$ and define $\kappa = \{k^2 - (\pi/2b)^2\}^{1/2}$. In $-b \leqslant y \leqslant b$, $x < 0$ close the contour for ϕ in the upper half-plane. It will be found that there are two poles at $\alpha = k\cos\Theta$, κ. The residue from the first pole exactly cancels the incident wave and the residue from the second gives the travelling wave

$$\frac{2b}{\pi}\cdot\frac{\cos(kb\sin\Theta)(k + k\cos\Theta)^{1/2}}{K_+(k\cos\Theta)}\cdot\frac{(\kappa - k)^{1/2}}{\kappa - k\cos\Theta}\lim_{\alpha\to\kappa}\left\{\frac{\alpha - \kappa}{K_-(\alpha)}\right\}e^{-i\kappa x}\cos\left(\frac{\pi y}{2b}\right).$$

The coefficient of $\exp(-i\kappa x)$ is the "transmission coefficient". The limit is easily computed by writing

$$\lim_{\alpha\to\kappa}\left\{\frac{\alpha - \kappa}{K_-(\alpha)}\right\} = K_+(\kappa)\lim_{\alpha\to\kappa}\left\{\frac{\alpha - \kappa}{K(\alpha)}\right\} = \frac{K_+(\kappa)}{K'(\kappa)} = \frac{K_+(\kappa)\pi}{2\kappa b^2 i},$$

where we have used the result that for $\alpha = \kappa$, $\gamma = -\frac{1}{2}i\pi b^{-1}$.

The solution of this problem by means of an integral equation approach is given by A. E. Heins [1].

3.3 Equations (3.22), (3.23) can be transformed into expressions which are convenient for numerical work as follows. If $0 < kd < 1$ we write

$$(1 - k^2d^2)^{1/2} - i\alpha d = \{1 - (k^2 - \alpha^2)d^2\}^{1/2} \exp(-i\psi),$$

where $\tan\psi = \alpha d/(1 - k^2d^2)^{1/2}$, or equivalently

$$\sin\psi = \alpha d/\{1 - (k^2 - \alpha^2)d^2\}^{1/2},$$

this latter form being convenient when $\alpha = k$.

If $0 < kb < \pi$ so that $kb_n = (kb/n\pi) < 1$ for all n, (3.23) give

$$L_+(\alpha) = \{(\gamma b)^{-1}\sinh\gamma b\}^{1/2} \exp[-\tfrac{1}{2}\alpha b + i\alpha b\pi^{-1}\{1 - C + \ln(2\pi/bk)\} - \\ - \gamma b\pi^{-1}\arccos(\alpha/k) + i\sum_{n=1}^{\infty}(\alpha b_n - \psi_n)],$$

where

$$\tan\psi_n = \alpha/\{(n\pi/b)^2 - k^2\}^{1/2}, \qquad b_n = (b/n\pi).$$

If α, k are real and $-k < \alpha < k$ we can set $\alpha = k\cos\lambda$ (λ real), and then $\gamma = -ik\sin\lambda$, $\arccos(\alpha/k) = \lambda$ and an elegant expression is obtained for $L_+(k\cos\lambda)$. In particular

$$L_+(k) = \exp[-\tfrac{1}{2}kb + ikb\pi^{-1}\{1 - C + \ln(2\pi/bk)\} + \\ + i\sum_{n=1}^{\infty}\{(bk/n\pi) - \sin^{-1}(bk/n\pi)\}].$$

From (3.22), with $0 < kb < \tfrac{1}{2}\pi$, we find

$$K_+(\alpha) = \{\cosh\gamma b\}^{1/2} \exp[-\tfrac{1}{2}\alpha b + i\alpha b\pi^{-1}\{1 - C + \ln(\pi/2kb)\} - \\ - \gamma b\pi^{-1}\arccos(\alpha/k) + i\sum_{n=1}^{\infty}(\alpha b_{n-\frac{1}{2}} - \psi_{n-\frac{1}{2}})],$$

where $b_{n-\frac{1}{2}}$, $\psi_{n-\frac{1}{2}}$ are defined as for b_n, ψ_n with $(n - \tfrac{1}{2})$ in place of n. If $\tfrac{1}{2}\pi < kb < \tfrac{3}{2}\pi$ and we define $\kappa = \{k^2 - (\pi/2b)^2\}^{1/2}$, it is convenient to write the first term of the infinite product in (3.22a) as $\{(1 - k^2b_{1/2}^2)^{1/2} - i\alpha b_{1/2}\} = -i(\alpha + \kappa)(2b/\pi)$ and we find

$$K_+(\alpha) = -i\{\cosh\gamma b/(\alpha^2 - \kappa^2)\}^{1/2}(\alpha + \kappa)\exp[-\tfrac{1}{2}\alpha b - \\ - \gamma b\pi^{-1}\arccos(\alpha/k) + i\alpha b\pi^{-1}\{3 - C + \ln(\pi/2kb)\} + i\sum_{n=2}^{\infty}(\alpha b_{n-\frac{1}{2}} - \psi_{n-\frac{1}{2}})].$$

The term under the square root is positive for all α in the range $0 < \alpha < k$ (α, k real).

When the Laplace transform is used, the appropriate formulae have been given by D. S. Jones [4].

3.4 Consider the asymptotic behaviour as α tends to infinity of the following expression (cf. (1.19)),

$$K_+(\alpha) = \{K(0)\}^{1/2} e^{-\chi(\alpha)} \prod_{n=1}^{\infty} \{1 + (\alpha/\alpha_n)\} e^{-\alpha/\beta_n}. \tag{a}$$

Suppose that

$$\alpha_n = an + b + O(n^{-1}) \quad : \quad \beta_n = an + c + O(n^{-1}), \quad \text{as} \quad n \to \infty. \tag{b}$$

Compare the behaviour of $K_+(\alpha)$ as α tends to infinity with that of

$$J(\alpha) = \prod_{n=1}^{\infty} \left(1 + \frac{\alpha}{an+b}\right) e^{-\alpha/an} = e^{-C\alpha/a}\, \Gamma\left(\frac{b}{a}+1\right) \Big/ \Gamma\left(\frac{\alpha}{a} + \frac{b}{a} + 1\right) \tag{c}$$

$$\text{i.e.} \quad K_+(\alpha)/J(\alpha) = \{K(0)\}^{1/2} e^{-\chi(\alpha)} \exp\left\{\alpha \sum_{n=1}^{\infty} \left(\frac{1}{an} - \frac{1}{\beta_n}\right)\right\} \times \prod_{n=1}^{\infty} \left(\frac{an+b}{\alpha_n}\right) \prod_{n=1}^{\infty} \left(\frac{\alpha_n + \alpha}{an+b+\alpha}\right). \tag{d}$$

Denote the last infinite product by $Q(\alpha)$. An algebraic rearrangement gives

$$Q(\alpha) = \prod_{n=1}^{\infty} \left\{1 + \frac{\alpha_n - an - b}{an + b - \alpha}\right\} = \prod_{n=1}^{\infty} \{1 + f_n(\alpha)\}, \quad \text{say.}$$

Assume that $an + b + \alpha$ is non-zero for all α, $(\operatorname{Im} \alpha) > -(\operatorname{Im} \alpha_1)$ where α_1 is the smallest root of $K(\alpha)$. Then in view of (b) it is easy to prove that $|f_n(\alpha)| < Cn^{-2}$ where C is a constant independent of α, for all α such that $(\operatorname{Im} \alpha) \geqslant -(\operatorname{Im} \alpha_1) + \varepsilon$. Hence the infinite product for $Q(\alpha)$ is uniformly convergent and

$$\lim_{\alpha\to\infty} Q(\alpha) = \prod_{n=1}^{\infty} \lim_{\alpha\to\infty} \{1 + f_n(\alpha)\} = 1, \qquad (\operatorname{Im} \alpha) > -(\operatorname{Im} \alpha_1) + \varepsilon.$$

(As in J. Bazer and S. N. Karp [1], if the α_n are imaginary it is possible to prove that this result holds for $-\frac{1}{2}\pi + \varepsilon' < \arg \alpha < \frac{3}{2}\pi - \varepsilon'$ but this is not necessary here.) Now return to (c). Inserting the asymptotic expression for $J(\alpha)$, we find

$$K_+(\alpha) \sim \exp\left\{-\chi(\alpha) + \frac{\alpha}{a}(1 - C) - \left(\frac{\alpha}{a} + \frac{b}{a} + \frac{1}{2}\right) \ln\left(\frac{\alpha}{a} + \frac{b}{a} + 1\right) + \right. \left. + \alpha \sum_{n=1}^{\infty} \left(\frac{1}{an} - \frac{1}{\beta_n}\right)\right\}.$$

3.5 We reduce the infinite product (a) of ex. 3.4 to a form suitable for computation for the special case $\alpha_n = (k^2 - \delta_n^2)^{1/2} = i(\delta_n^2 - k^2)^{1/2}$, $\beta_n = i\varepsilon_n$, where δ_n, ε_n are sequences of real numbers. Assume that k and α are real. α_n is real for $1 \leqslant n \leqslant N$, say, and imaginary for $n > N$.

Denote the real values by $\alpha_n = \kappa_n$, and the imaginary values by $\alpha_n = i\gamma_n$. Following an idea similar to that leading to (1.27) it is convenient to investigate

$$\frac{K_+(\alpha)}{K_-(\alpha)} = e^{-2\chi(\alpha)} \prod_{n=1}^{N} \frac{\kappa_n + \alpha}{\kappa_n - \alpha} \exp\left\{+2i \sum_{n=1}^{N} \left(\frac{\alpha}{\varepsilon_n}\right) - 2i\Psi\right\},$$

where

$$\Psi = \sum_{n=N+1}^{\infty} \{\psi_n - (\alpha/\varepsilon_n)\}, \qquad \tan \psi_n = \frac{\alpha}{\gamma_n} \quad \text{or} \quad \sin \psi_n = \frac{\alpha}{(\alpha^2 + \gamma_n^2)^{1/2}}.$$

On multiplying the above expression by $K(\alpha) = K_+(\alpha)K_-(\alpha)$ we have

$K_+(\alpha) =$

$$e^{-\chi(\alpha)}\left[K(\alpha)\Big/\left\{\prod_{n=1}^{N} (\kappa_n^2 - \alpha^2)\right\}\right]^{1/2} \prod_{n=1}^{N} (\kappa_n + \alpha) \exp\left\{i \sum_{n=1}^{N} \left(\frac{\alpha}{\varepsilon_n}\right) - i\Psi\right\}.$$

The modulus $|K_+(\alpha)|$ has a particularly simple form.

3.6 Consider the decomposition of

$$K(\alpha) = (cd/a)\gamma \sinh \gamma a\{\sin \gamma c \sinh \gamma d\}^{-1} = K_+(\alpha)K_-(\alpha)$$

where $c + d = a$. As in ex. 1.9 and §3.2

$$(\gamma b)^{-1} \sinh \gamma b = \prod_{n=1}^{\infty} \{(1 - k^2 b_n^2) + \alpha^2 b_n^2\}, \qquad b_n = b/n\pi.$$

Define

$$G_+(\alpha,b) = \prod_{n=1}^{\infty} \{(1 - k^2 b_n^2)^{1/2} - i\alpha b_n\}e^{i\alpha b_n},$$

and write

$$K_+(\alpha) = G_+(\alpha,a)\{G_+(\alpha,c)G_+(\alpha,d)\}^{-1} \exp\{-\chi_1(\alpha)\}.$$

If the three infinite products are combined into one we obtain

$$K_+(\alpha) = e^{-\chi_1(\alpha)} \prod_{n=1}^{\infty} \frac{\{(1 - k^2 a_n^2)^{1/2} - i\alpha a_n\}}{\{(1 - k^2 c_n^2)^{1/2} - i\alpha c_n\}\{(1 - k^2 d_n^2)^{1/2} - i\alpha d_n\}}. \tag{a}$$

The arbitrary function $\chi_1(\alpha)$ is chosen so that $K_+(\alpha)$ has algebraic behaviour as $\alpha \to \infty$ in an upper half-plane. We examine the asymptotic behaviour of $K_+(\alpha)$ for large α. The terms $k^2 a_n^2$ etc. can be ignored (ex. 3.4) and

$$K_+(\alpha) \sim \{\Gamma(1 - i\alpha a\pi^{-1})\}^{-1}\Gamma(1 - i\alpha c\pi^{-1})\Gamma(1 - i\alpha d\pi^{-1}) \exp\{-\chi_1(\alpha)\}.$$

On using Stirling's formula we deduce that if we choose

$$\chi_1(\alpha) = i\alpha\pi^{-1}(a \ln a - c \ln c - d \ln d),$$

then $K_+(\alpha) \sim \alpha^{1/2}$ as $\alpha \to \infty$ in an upper half-plane.

A slightly different, though equivalent, expansion is obtained by writing instead of (a),

$$K_+(\alpha) = e^{-\chi_2(\alpha)} \prod_{n=1}^{\infty} \frac{\{(1 - k^2 a_{2n-1}^2)^{1/2} - i\alpha a_{2n-1}\}\{(1 - k^2 a_{2n}^2) - i\alpha a_{2n}\}}{\{(1 - k^2 c_n^2)^{1/2} - i\alpha c_n\}\{(1 - k^2 d_n^2) - i\alpha d_n\}}.$$

It is then found that we must choose

$$\chi_2(\alpha) = i\alpha\pi^{-1}(a \ln \tfrac{1}{2}a - c \ln c - d \ln d).$$

This is particularly convenient when $c = d = \frac{1}{2}a$ since then $\chi_2(\alpha) = 0$. In this case we have $K(\alpha) = \gamma b \coth \gamma b = K_+(\alpha)K_-(\alpha)$ where

$$K_+(\alpha) = \prod_{n=1}^{\infty} \frac{\{(1 - k^2 b_{n-\frac{1}{2}}^2)^{1/2} - i\alpha b_{n-\frac{1}{2}}\}}{\{(1 - k^2 b_n^2)^{1/2} - i\alpha b_n\}}.$$

A result which is required later is that if we set $\alpha = k$ in this formula and let $k \to 0$, then

$$K_+(k) = 1 - (ikb/\pi)2 \ln 2 + O(k^2). \qquad \text{(b)}$$

3.7 The decomposition of $K(\alpha) = 2I_1(\gamma a)K_1(\gamma a) = \pi i H_1^{(1)}(\kappa a)J_1(\kappa a)$ where $\gamma = (\alpha^2 - k^2)^{1/2} = -i\kappa$, has been discussed by H. Levine and J. Schwinger [1] who use methods similar to (a), (b) at the end of §1.3, D. S. Jones [5] who uses a variant of method (b), and L. Vajnshtejn [3] who uses method (c). When these authors shift contours in methods (b) and (c) the presence of the zeros of $I_1(\gamma a)$ tends to make manipulations complicated. It would seem to be simpler to proceed as follows: Set

$$K(\alpha) = 2e^{-\gamma a}\{(\gamma a)^{-1}I_1(\gamma a)\}\{\gamma a e^{\gamma a}K_1(\gamma a)\}.$$

The factor $\exp(-\gamma a)$ has the simple decomposition given in (1.35). $(\gamma a)^{-1}I_1(\gamma a)$ can be dealt with by the infinite product theory already developed. The only factor which needs to be decomposed by the integral formula is $\gamma a \exp(\gamma a)K_1(\gamma a)$ and this function has no zeros in the complex plane. We can obtain some idea of the complexity of the integrand in the decomposition formulae (1.32), (1.33) by considering

$$F(z) = \ln K_1(z).$$

We have

$$K_1(-z) = K_1(e^{-i\pi}z) = -K_1(z) + \pi i I_1(z),$$

$$K_1(iz) = -\tfrac{1}{2}\pi H_1^{(2)}(z) \quad : \quad K_1(-iz) = -\tfrac{1}{2}\pi H_1^{(1)}(z).$$

Hence

$$F(z) - F(-z) = \tfrac{1}{2} \ln [K(z)/_1^2\{K_1^2(z) + \pi^2 I_1^2(z)\}] - \\ - \tfrac{1}{2}i\pi - i \arctan \{K_1(z)/\pi I_1(z)\},$$

$$F(iz) - F(-iz) = -2i \arctan \{Y_1(z)/J_1(z)\}.$$

The simplicity of this second result suggests the use of method (c) of §1.3 and in fact elegant formulae of this type are given by L. A. Vajnshtejn [3]. All three authors quoted above give approximate formulae.

The only published table of values for $K_+(\alpha)$ are given by D. S. Jones [5] who gives tables for $\alpha = k, \kappa_1, \kappa_2$ where $\kappa_i = (k^2 - r_i)^{1/2}$ and r_i is the ith root of the equation $J_0(r_i a) = 0$. The tables are for $ka = 0(0{\cdot}25)10$.

3.8 *Integral equation formulations.* There are broadly speaking three methods of procedure and most of the important features have already been mentioned in Chapter II. We give a brief summary to introduce the reader to the literature. General references are P. M. Morse and H. Feshbach [1] and A. E. Heins [10]. It has already been emphasized that when we use Jones's method the discussion of integral equations is unnecessary.

(i) The easiest procedure in some cases is the physical method of ex. 2.7. Thus suppose there are two semi-infinite plates in $y = \pm b$, $-\infty < x \leqslant 0$ with $\phi_t = 0$ on the two plates and incident field ϕ_i. Set $\phi_t = \phi_i + \phi$ and suppose that ϕ can be produced by a distribution of line sources $f_1(x)$, $f_2(x)$ on $y = +b, -b$ respectively. Then by superposition the total potential at (x,y) is

$$\phi_t(x,y) = \phi_i(x,y) + \pi i \int_{-\infty}^{0} \{H_0^{(1)}(kR_1)f_1(\xi) + H_0^{(1)}(kR_2)f_2(\xi)\}\, d\xi,$$

where

$$R_1^2 = (x - \xi)^2 + (y - b)^2 \quad : \quad R_2^2 = (x - \xi)^2 + (y + b)^2.$$

Let (x,y) tend to $(x, \pm b)$ and obtain two simultaneous integral equations

$$\pi i \int_{-\infty}^{0} \{H_0^{(1)}(k|x-\xi|)f_1(\xi) + H_0^{(1)}(kR)f_2(\xi)\}\, d\xi + \phi_i(x,b) = 0, \quad (-\infty < x \leqslant 0),$$

$$\pi i \int_{-\infty}^{0} \{H_0^{(1)}(kR)f_1(\xi) + H_0^{(1)}(k|x-\xi|)f_2(\xi)\}\, d\xi + \phi_i(x,-b) = 0, \quad (-\infty < x \leqslant 0),$$

where $R^2 = (x - \xi)^2 + 4b^2$. If we add and subtract these and set $f_1(\xi) + f_2(\xi) = s(\xi)$, $f_1(\xi) - f_2(\xi) = d(\xi)$ we obtain two independent integral equations, each holding in $-\infty < x \leqslant 0$:

$$\pi i \int_{-\infty}^{0} \{H_0^{(1)}(k|x - \xi|) \pm H_0^{(1)}(kR)\} \begin{matrix} s(\xi) \\ d(\xi) \end{matrix}\, d\xi + \phi_i(x,b) \pm \phi_i(x,-b) = 0.$$

These are of the Wiener–Hopf type and can be solved in the usual way, remembering the result (cf. (1.60), (1.61)):

$$i \int_{-\infty}^{\infty} H_0^{(1)}\{k(a^2 + x^2)^{1/2}\} e^{i\alpha x}\, dx = 2\gamma^{-1} e^{-\gamma|a|}.$$

A. E. Heins [1], part I, has used this approach to solve the problem of ex. 3.2. See also L. Lewin [1], L. A. Vajnshtejn [1].

(ii) Integral equations are readily established by applying Green's formula to the whole space excluding the region occupied by the thin plates (cf. ex. 2.6). This has been used by H. Levine and J. Schwinger [1] for radiation from a circular pipe, and W. Chester [1] for radiation from parallel plates.

(iii) The space can be split into subspaces and Green's formula can be applied separately to each of the subspaces. This is the procedure used in §2.4. In the problem of two parallel plates with $\phi_t = 0$ on $y = \pm b$, $-\infty < x \leqslant 0$, the space is split into three regions (a) $y \geqslant b$, (b) $-b \leqslant y \leqslant b$, (c) $y \leqslant -b$. Green's functions for (a) and (c) present no difficulty. For (b) a suitable Green's function would have $\phi = 0$ on $y = \pm b$, e.g.

$$\phi = \frac{2\pi}{b} \sum_{n=1}^{\infty} \gamma_n^{-1} \sin \frac{n\pi}{2b} (y - b) \sin \frac{n\pi}{2b} (y_0 - b) e^{-\gamma_n |x - x_0|},$$

where $\gamma_n = \{(n\pi/2b)^2 - k^2\}^{1/2}$. Although this type of Green's function is frequently quoted in the literature it is not very useful since we are interested ultimately only in the Fourier transform of this with respect to x and to compute a closed form of this result we should need to sum an infinite series. It is much easier to find the transform of the Green's function by applying a Fourier transform in x to

$$\phi_{xx} + \phi_{yy} + k^2\phi = -4\pi\delta(x - x_0)\delta(y - y_0)$$

which gives $d^2\Phi(\alpha)/dy^2 - \gamma^2\Phi(\alpha) = -4\pi(2\pi)^{-1/2}\delta(y - y_0) \exp(i\alpha x_0)$. The required transform of the Green's function is the solution of this equation such that $\Phi = 0$ on $y = \pm b$. We shall not pursue this method further.

3.9 The analysis of §3.4 has been extended to the radiation and diffraction of electromagnetic waves by circular waveguides by T. Iijima [1], J. D. Pearson [1] and L. A. Vajnshtejn [5]. The only new feature that appears will be explained in general terms. Consider the simultaneous Wiener–Hopf equations,

$$2\alpha(\alpha^2 - k^2)^{-1}\Psi_1^-(\alpha) + p\Psi_2^-(\alpha) = K(\alpha)\Phi_2^+(\alpha) + F(\alpha),$$
$$q\Phi_1^+(\alpha) + 2\alpha(\alpha^2 - k^2)^{-1}\Phi_2^+(\alpha) = L(\alpha)\Psi_1^-(\alpha) + G(\alpha),$$

where p, q are given constants. Write $K(\alpha) = K_+(\alpha)/K_-(\alpha)$ and set

$$F(\alpha)K_-(\alpha) = E_+(\alpha) + E_-(\alpha)$$

$$\frac{K_-(\alpha)\Psi_1^-(\alpha)}{\alpha + k} = \frac{\{K_-(\alpha)\Psi_1^-(\alpha) - K_1(-k)\Psi_1^-(-k)\}}{\alpha + k} + \frac{K_1(-k)\Psi_1^-(-k)}{\alpha + k}.$$

Then the first equation can be solved by the Wiener–Hopf technique in the usual way, and we find

$$K_+(\alpha)\Phi_2^+(\alpha) - K_-(-k)\Psi_1^-(-k)(\alpha + k)^{-1} + E_+(\alpha) = 0. \qquad \text{(a)}$$

Similarly the second equation can be solved to give

$$L_-(\alpha)\Psi_1^-(\alpha) - L_+(k)\Phi_2^+(k)(\alpha - k)^{-1} + H_-(\alpha) = 0, \qquad \text{(b)}$$

where $L(\alpha) = L_-(\alpha)/L_+(\alpha)$. If we set $\alpha = k$ in (a) and $\alpha = -k$ in (b) we obtain two simultaneous linear algebraic equations for the unknown constants $\Phi_2^+(k)$, $\Psi_1^-(-k)$. The solution can then be completed.

3.10 Examine the radiation from a coaxial waveguide with infinite central conductor and semi-infinite outer conductor i.e. solve the steady state wave equation in $\rho \geqslant a$, $-\infty < z < \infty$ with a cylinder in $\rho = b > a$, $-\infty < z \leqslant 0$, and wave incident from $z = -\infty$ in $a \leqslant \rho \leqslant b$ (cf. N. Marcuvitz [1], p. 208).

3.11 (i) Examine the radiation from the parallel plate duct $y = \pm b$, $-\infty < x \leqslant 0$ enclosed in parallel walls $y = \pm B$, $-\infty < x < \infty$ $(B \gg b)$.

(ii) Examine the radiation from the circular duct $\rho = a$, $-\infty < z \leqslant 0$ enclosed in the tube $\rho = A$, $-\infty < z < \infty$ $(A \gg a)$.

In the limiting case $A, B \to \infty$ we obtain the problems of §§3.3, 3.4. The case of infinite A, B is much more complicated than the case of finite A, B due to the presence of branch points. It would be interesting to know how far the results for finite (but large) A, B could be used to approximate to the results for infinite space e.g. near the mouth of the duct. (The transition from a function with poles to a function with branch points is examined from another point of view in V. Kourganoff [1], §27, where references are given to the original work of S. Chandrasekhar.)

3.12 The reflection of an electromagnetic wave falling on an infinite set of staggered semi-infinite plates lying in $y = na$, $na \cot \lambda \leqslant x < \infty$, $(n = \ldots -2, -1, 0, +1, +2 \ldots)$ has been treated by J. F. Carlson and A. E. Heins [1], A. E. Heins and J. F. Carlson [1], and A. E. Heins [11]. From the point of view of Jones's method it is natural to make use of the periodicity of the structure in y, to consider the problem in the strip $0 \leqslant y \leqslant a$, $-\infty < x < \infty$ (cf. A. E. Heins [12]). The solution involves the infinite product decomposition of

$$\cosh \gamma a - \cos (k\rho - \alpha b), \qquad (a, k, \rho, b \text{ given constants}).$$

3.13 *Sound propagation in a lined duct.* Consider transmission of sound inside the duct $0 \leqslant y \leqslant b$, $-\infty < x < \infty$ with $\partial\phi_t/\partial y = 0$ on $y = 0$, $-\infty < z < \infty$, and $y = b$, $-\infty < z \leqslant 0$, but $\partial\phi_t/\partial y = ik\eta\phi_t$ on $y = b$, $0 < z < \infty$, where η is a complex constant. Suppose that a wave $\phi_i = \exp(ikz)$ is incident from $z = -\infty$. Set $\phi_t = \phi_i + \phi$. The Wiener–Hopf equation is

$$-(2\pi)^{-1/2}k\eta(\alpha + k)^{-1} - ik\eta\Phi_-(b) = \Phi'_+(b)/K(\alpha),$$

where

$$K(\alpha) = \gamma \sinh \gamma b\{\gamma \sinh \gamma b - ik\eta \cosh \gamma b\}^{-1}.$$

Define μ_m by the equation

$$\pi\mu_m \tan \pi\mu_m = -ikb\eta.$$

As $\eta \to 0$,

$$\mu_0 \simeq \pi^{-1}(-ik\eta b)^{1/2} \quad : \quad \mu_m \sim m - ikb\eta(\pi^2 m)^{-1}, \ (m = 1, 2, \ldots).$$

The infinite product decomposition of $K(\alpha)$ gives

$$K_+(\alpha) = -\pi^2\mu_0^2(ik\eta b)^{-1}L_+(\alpha) \quad : \quad K_-(\alpha) = L_-(\alpha) \quad : \quad L_-(\alpha) = L_+(-\alpha).$$

$$L_+(\alpha) = (\alpha+k)(\alpha+k_0)^{-1}\prod_{m=1}^{\infty}[(1-k^2A_m^2)^{1/2}-i\alpha A_m][(1-k^2B_m^2)^{1/2}-i\alpha B_m]^{-1},$$

where $A_m = b/(m\pi)$, $B_m = b/(\mu_m\pi)$, $k_0^2 = k^2 - (\pi\mu_0/b)^2$. Solution of the Wiener–Hopf equation gives

$$\Phi'_+(b) = -(2\pi)^{-1/2}k\eta K_-(-k)K_+(\alpha)/(\alpha + k).$$

As $z \to -\infty$, $\phi_t \sim \exp(ikz) + R\exp(-ikz)$ and it is easily deduced that the reflection coefficient R is given by

$$\begin{aligned} R &= -ik\eta(4k^2b)^{-1}K_-(-k)K_+(k) \\ &= \frac{k-k_0}{k+k_0}\prod_{m=1}^{\infty}[(1-k^2A_m^2)^{1/2}-ikA_m]^2[(1-k^2B_m^2)^{1/2}-ikB_m]^{-2}. \end{aligned}$$

This problem has been treated in detail by an integral equation method by A. E. Heins and H. Feshbach [1]: see also P. M. Morse and H. Feshbach [1], p. 1522.

The problem of the semi-infinite resistive strip parallel to the walls of a duct, and centrally placed (§3.5, (3.86)), is equivalent to the above problem except that then we are generally interested in small $\delta = \eta^{-1}$ instead of small η. A similar problem involving a resistive strip has been considered by V. M. Papadopoulos [1] but the solution of his Wiener–Hopf equation (his equation (17)) is unnecessarily complicated.

3.14 Consider transmission of sound inside the circular duct $0 \leqslant \rho \leqslant a$, $-\infty < z < \infty$, with $\partial\phi_t/\partial\rho = 0$ on $\rho = a$, $z < 0$ and $\partial\phi_t/\partial\rho = ik\eta\phi_t$ for $\rho = a$, $z > 0$, and incident wave $\phi_i = \exp(ikz)$.

3.15 It is convenient to summarize some standard notation for radiation-type boundary conditions. For sound waves falling on partially absorbent material we have $\partial\phi/\partial n = i\delta\phi$, where (Re δ) is positive but (Im δ) may be positive or negative (P. M. Morse and H. Feshbach [1], p. 1522). The positive direction of the normal is taken *into* the material. For electromagnetic waves we generally write $\partial\phi/\partial n = i\delta_1\phi$ for E-polarization, $\phi = -i\delta_2\partial\phi/\partial n$ for H-polarization, where δ_1, δ_2 are constants with positive real and imaginary parts (cf. ex. 2.11 for terminology and T. B. A. Senior [1] for references). If a time factor $\exp(+i\omega t)$ is used then the sign of i must be changed throughout.

3.16 A. E. Heins [2], [3], [4], M. Weitz and J. B. Keller [1], have considered various cases of reflection of water waves in water of finite depth in $-b \leqslant y \leqslant 0$, $-\infty < x < \infty$, $-\infty < z < \infty$. ϕ satisfies $\phi_{xx} + \phi_{yy} - k^2\phi = 0$, ($k$ real). Typical problems are

(i) $\partial\phi/\partial y = 0$ on $y = -b$, $\quad -\infty < x < \infty$.

$\partial\phi/\partial y = p\phi$ on $y = 0$, $x \leqslant 0$ $\quad : \quad$ $\partial\phi/\partial y = q\phi$ on $y = 0$, $x > 0$.

(ii) $\partial\phi/\partial y = 0$ on $y = -b$, $\quad -\infty < x < \infty$.

$\partial\phi/\partial y = p\phi$ on $y = 0$, $\quad -\infty < x < \infty$.

$\partial\phi/\partial y = 0$ on $y = -a$, $\quad 0 \leqslant x < \infty$, $\quad (-b < -a < 0)$.

In both cases the infinite product decomposition of the following function is required:

$$\mu \sinh \mu a - A \cosh \mu a, \qquad \mu = (\alpha^2 + k^2)^{1/2}.$$

3.17 *Laplace-type equations.* We consider solution of $\nabla^2\phi = 0$ by means of the Wiener–Hopf technique. The two-dimensional examples which follow can be solved by conformal mapping. The justification for discussing the Wiener–Hopf solutions is two-fold. The same technique can be used for axially symmetrical problems: and the same procedure can be applied to more complicated equations where it may not be very obvious how to obtain the solution by conformal mapping.

(i) In some cases it is possible to solve Laplace's equation directly without considering the limiting case $k \to 0$ of a solution of $\nabla^2\phi + k^2\phi = 0$. As an example consider the solution of $\nabla^2\phi = 0$ in $0 \leqslant y \leqslant b$, $-\infty < x < \infty$ with $\phi = 0$ on $y = b$, $-\infty < z < \infty$, $\phi = V$ on $y = 0$, $0 < z < \infty$, $\partial\phi/\partial y = 0$ on $y = 0$, $-\infty < z < 0$. In order to apply the Wiener–Hopf technique replace $\phi = V$ by

$$\phi = V \exp(-\varepsilon z) \quad \text{on} \quad y = 0, \qquad 0 < z < \infty, \qquad (\varepsilon > 0).$$

We shall ultimately let $\varepsilon \to 0$. The Wiener–Hopf solution is

$$\Phi'_+(\alpha) = -(2\pi)^{-1/2} V b^{-1} K_-(-i\varepsilon) K_+(\alpha)(\varepsilon - i\alpha)^{-1},$$

where $K(\alpha) = \alpha b \coth \alpha b = K_+(\alpha)K_-(\alpha)$. This function has been factorized in ex. 1.11. We can evaluate the contour integral for ϕ by residues. We obtain, for $z > 0$,

$$\phi = V\left\{\frac{\sin \varepsilon(b - y)}{\sin \varepsilon b} e^{-\varepsilon z} + K_-(-i\varepsilon)\sum_{n=1}^{\infty} \frac{(-1)^n \sin (n\pi/b)(b - y)}{K_-(-n\pi i/b)(n\pi - \varepsilon b)} e^{-(n\pi z/b)}\right\}.$$

If we let $\varepsilon \to 0$,

$$\phi = V\left\{1 - \frac{y}{b} - \frac{1}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{n\Gamma(n + 1)} \sin \frac{n\pi y}{b} e^{-(n\pi z/b)}\right\}, \qquad z > 0.$$

(ii) Consider the solution of $\nabla^2\phi_t = 0$ in $0 \leqslant y \leqslant b$, $-\infty < z < \infty$, with $\partial\phi_t/\partial y = 0$ on $y = b$, $-\infty < z < \infty$, $\partial\phi_t/\partial y = 0$ on $y = 0$, $0 < z < \infty$, and $\phi_t = 0$ on $y = 0$, $-\infty < z < 0$, such that $\phi_t \to Az + B$ as $z \to +\infty$, where A is given but B is to be determined. We first solve the corresponding problem for $\nabla^2\phi_t + k^2\phi_t = 0$ with $\phi_t = \phi_i + \phi$, $\phi_i = C\exp(-ikz)$, a wave incident from $z = +\infty$. By means of the Wiener–Hopf technique we find

$$\phi_t = Ce^{-ikz} - \frac{2kbC}{2\pi i K_+(k)} \int_{-\infty+ia}^{\infty+ia} \frac{\cosh\gamma(b-y)}{K_-(\alpha)\gamma\sinh\gamma b} e^{-i\alpha z}\, d\alpha.$$

If we let $k \to 0$, then $\gamma \to |\alpha|$ and the integral is divergent. However the correct procedure becomes clear if we evaluate by residues. For $z > 0$,

$$\phi_t = Ce^{-ikz} - \frac{C}{\{K_+(k)\}^2} e^{ikz} + \frac{2ikC}{K_+(k)} \sum_{n=1}^{\infty} \frac{\cos(n\pi y/b)}{\gamma_n K_-(-i\gamma_n)} e^{-\gamma_n z},$$

where $\gamma_n = \{(n\pi/b)^2 - k^2\}^{1/2}$. We set $2kC = iA$ and as $k \to 0$ we adjust C so that A remains constant. Then on using (b) in ex. 3.6 we find that for $z > 0$,

$$\phi_t = A\left\{z + \frac{2b}{\pi}\ln 2 - \frac{b}{\pi^{3/2}} \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{n\Gamma(n+1)} \cos\left(\frac{n\pi y}{b}\right) e^{-(n\pi z/b)}\right\}.$$

(iii) Consider the limiting case $k \to 0$ for transmission in a duct with a strip across it (§3.6). As in (ii) the troublesome terms are those involving the progressive waves. From (3.99), (3.100),

$$\begin{aligned}\phi_t &= \tfrac{1}{2}[1 + \{K_+(k)\}^{-2}]e^{ikz} + ik\{\ldots\}, & z \geqslant 0,\\ &= e^{ikz} + \tfrac{1}{2}[1 - \{K_+(k)\}^{-2}]e^{-ikz} + ik\{\ldots\}, & z \leqslant 0,\end{aligned}$$

where $\{\ldots\}$ represents exponentially decreasing terms. We multiply by C, let $ikC = A$, and as $k \to 0$ adjust C so that A remains constant. Then on using (b) of ex. 3.6 we find

$$\begin{aligned}\phi_t &= (ik)^{-1}A\cos kz + A\{k^{-1}\sin kz + (2b/\pi)\ln 2\cos kz\}\\ &\qquad + A\{\ldots\} + 0(k), \qquad z \geqslant 0,\\ &= (ik)^{-1}A\cos kz + A\{k^{-1}\sin kz - (2b/\pi)\ln 2\cos kz\}\\ &\qquad + A\{\ldots\} + 0(k), \qquad z \leqslant 0.\end{aligned}$$

If we let $k \to 0$ the first term tends to infinity. However if we subtract $(ik)^{-1}A\cos kz$ (which satisfies $\nabla\phi + k^2\phi = 0$ and the boundary conditions on $y = \pm b$ and $z = 0$) we obtain the correct limiting behaviour:

$$\phi_t = A[z \pm (2b/\pi)\ln 2 + \{\ldots\}], \qquad z \gtrless 0.$$

(iv) Solve Laplace's equation in $0 \leqslant \rho \leqslant a$, $-\infty < z < \infty$, with $\phi = 0$ on $\rho = a$, $z < 0$: $\partial\phi/\partial\rho = 0$ on $\rho = a$, $z > 0$: and $\phi \to Az + B$ as $z \to +\infty$ where A is given, B is to be determined.

(v) Solve Laplace's equation in $0 \leqslant \rho \leqslant a$, $-\infty < z < \infty$ with $\phi = 0$ on $\rho = a$, $-\infty < z < \infty$, and a conducting cylinder at constant potential V in $\rho = b < a$, $0 < z < \infty$.

(vi) Next consider a two-dimensional example where the potential in the static case is logarithmic at infinity, namely radiation from two semi-infinite plates in $y = \pm b$, $x \leqslant 0$, with $\partial\phi_t/\partial y = 0$ on the plates (§3.3 (3.34)). For clarity suppose that

$$\phi_t = B\left\{e^{ikx} + Re^{-ikx} + \sum_{n=1}^{\infty} R_n \cos\left(\frac{n\pi y}{b}\right) e^{\gamma_n x}\right\}, \quad |y| < b, \quad x < 0, \quad \text{(a)}$$

$$= B \int_{-\infty+ia}^{\infty+ia} f(\alpha)e^{-i\alpha x-\gamma|y|}\, d\alpha, \qquad |y| > b,$$

$$\sim B(2k\pi)^{1/2}e^{-\frac{1}{4}i\pi}r^{-1/2}e^{ikr}f(-k\cos\theta)\sin\theta, \quad \text{as} \quad r \to \infty, \qquad \text{(b)}$$

where we have used the asymptotic expansion (1.71) for (1.56). In concrete examples we find that as $k \to 0$,

$$R = -1 + ik\{p \ln k + q\} + 0(k^2) \quad : \quad R_n = ikT_n + 0(k^2), \quad \text{(c)}$$

$$f(-k\cos\theta)\sin\theta = f_0 + 0(k), \quad \text{say}, \qquad \text{(d)}$$

where f_0 is a constant independent of θ. e.g. from (3.34), (3.44), (3.45):

$$f_0 = (b/\pi) \quad : \quad R = -1 - (2ikb/\pi)\{1 - C + \ln(2\pi/bk) + \tfrac{1}{2}\pi i\}, \quad \text{(e)}$$

where C is Euler's constant 0·5772 . . . . In (a), (b) set $2ikB = A$, let $k \to 0$, $B \to \infty$ in such a way that A remains constant, and use (c), (d). Then for small k,

$$\phi_t = A\{x + \tfrac{1}{2}(p \ln k + q) + \tfrac{1}{2}\sum_{n=1}^{\infty} T_n \cos(n\pi y/b)e^{n\pi x/b} + 0(k^2)\},$$

$$|y| < b, \quad x < 0. \quad \text{(f)}$$

$$\phi_t \sim -iAf_0(\pi/2)^{1/2}(kr)^{-1/2}e^{ikr}e^{-\frac{1}{4}i\pi}, \quad \text{as} \quad r \to \infty.$$

If the far field is expressed in terms of Hankel functions this gives (ex. 1.21)

$$\phi_t \sim -iAf_0(\pi/2)H_0^{(1)}(kr), \quad \text{as} \quad r \to \infty.$$

Now choose a fixed large r and let $k \to 0$. If we use the expansion of $H_0^{(1)}(kr)$ for small (kr) we have

$$\phi_t \sim Af_0\{\ln(\tfrac{1}{2}kr) + C - \tfrac{1}{2}\pi i\}. \qquad \text{(g)}$$

The velocity of flow as $r \to \infty$ is $\partial\phi_t/\partial r = Af_0/r$. The velocity of flow as $x \to -\infty$ in the duct is $\partial\phi_t/\partial x = A$. Hence the equation of continuity gives $\pi f_0 = b$. If we impose the condition $\phi_t \sim (Ab/\pi)\ln r + o(1)$ as $r \to \infty$ we must subtract $(Ab/\pi)\{\ln\tfrac{1}{2}k + C - \tfrac{1}{2}\pi i\}$ from (f) and (g). The coefficient of $\ln k$ in the modified form of (f) must then be zero which gives $p = 2f_0 = (2b/\pi)$. Then finally

$$\phi_t \to A[x + \{\tfrac{1}{2}q + (b/\pi)(\ln 2 - C + \tfrac{1}{2}\pi i)\}] \quad \text{as} \quad x \to -\infty, \qquad |y| < b.$$

In the particular case (e),

$$\phi_t \sim (Ab/\pi)\ln r, \quad \text{as} \quad r \to \infty \quad \text{outside the duct},$$

$$\phi_t \to A[x - (b/\pi)\{1 + \ln(\pi/b)\}], \quad \text{as} \quad x \to -\infty, \qquad |y| < b.$$

(vii) In contrast with the two-dimensional example in (vi), axially symmetrical problems in infinite space usually present no difficulties when we let $k \to 0$. Thus for radiation from a circular pipe (§3.4) we can deduce directly from (3.64)–(3.67) that on multiplying ϕ_t by C, setting $ikCa^2 = 2A$, and letting $k \to 0$,

$$\phi_t \sim -A/r, \quad r \to \infty \quad \text{outside the pipe},$$
$$\phi_t \to (4A/a^2)(z - l), \quad z \to -\infty \quad \text{inside the pipe},$$

where l is obtained by setting $k = 0$ in (3.66) i.e.

$$\frac{l}{a} = \frac{1}{\pi}\int_0^\infty u^{-2} \ln \{1/2K_1(u)I_1(u)\}\, du = 0{\cdot}6133 \ldots .$$

3.18 The infinities we have eliminated in ex. 3.17 (vi) are by no means peculiar to the Wiener–Hopf technique. They are inherent in the transition from the steady-state wave equation to Laplace's equation in two dimensions. Thus the solution of the two-dimensional steady-state wave equation for a line source at the origin is

$$\phi = \pi i H_0^{(1)}(kr) \approx \pi i - 2 \ln (\tfrac{1}{2}kr) - 2C, \quad \text{as} \quad k \to 0, \tag{a}$$

where C is Euler's constant and $r = (x^2 + y^2)^{1/2}$. The solution of Laplace's equation with a line source is

$$\phi = -2 \ln r. \tag{b}$$

To make (a) reduce to (b) as $k \to 0$ we need to subtract $(\pi i - 2 \ln \frac{1}{2}k - 2C)$ which is similar to the procedure used in ex. 3.18 (vi). It is sometimes convenient to avoid difficulties of this type by working in terms of the derivatives $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$ instead of ϕ.

3.19 *The biharmonic equation.* (i) Consider the determination of stresses in a two-dimensional elastic strip in $-b \leqslant y \leqslant b$, $-\infty < x < \infty$ with boundary conditions

$$\sigma_{xy} = 0, \quad y = \pm b, \quad -\infty < x < \infty. \tag{a}$$

$$\sigma_{yy} = -p, \quad -\infty < x < 0, \quad v = 0, \quad 0 < x < \infty, \quad y = \pm b. \tag{b}$$

Introduce the Airy stress function ϕ such that

$$\sigma_{xx} = \partial^2\phi/\partial y^2 \quad : \quad \sigma_{yy} = \partial^2\phi/\partial x^2 \quad : \quad \sigma_{xy} = -\partial^2\phi/\partial x\,\partial y,$$

where ϕ satisfies $\nabla^4\phi = 0$. If we apply a Fourier transform in x we obtain, on using symmetry conditions and (a) above,

$$\Phi(\alpha,y) = B(\alpha)\{\alpha y \sinh \alpha y - (1 + \alpha b \coth \alpha b) \cosh \alpha y\}.$$

If we apply a Fourier transform in x to the stress–strain relations it is found that the transform of V can be expressed in terms of the transform of ϕ as follows:

$$\alpha^2 E V(\alpha) = (1 - \nu^2)\, d^3\Phi(y)/dy^3 + \alpha^2(\nu^2 - \nu - 2)\, d\Phi(y)/dy,$$

where E, ν are constants. If we set $\sigma_{yy} = -p \exp(\varepsilon x)$ for $y = \pm b$, $-\infty < x < 0$ we obtain the Wiener–Hopf equation

$$F_+(\alpha) + \frac{p}{(2\pi)^{1/2}(\varepsilon + i\alpha)} = \frac{\alpha(\sinh \alpha b \cosh \alpha b + \alpha b)}{2 \sinh^2 \alpha b} G_-(\alpha),$$

where $F_+(\alpha)$ is the transform of σ_{yy} in $(0, \infty)$ and $G_-(\alpha) = E(1-\nu^2)^{-1}V_-(\alpha)$. The function which has to be decomposed in the form $K_+(\alpha)K_-(\alpha)$ has no branch points and the standard infinite product theory can be applied, though the zeros of the numerator are complex (cf. the approximate method of §4.5) ϕ can be expressed as a contour integral which can be evaluated by residues and the final solution is obtained by taking the limit $\varepsilon \to 0$. A similar example is considered in W. T. Koiter [2].

(ii) Consider the flow of a highly viscous fluid in $-b \leqslant y \leqslant b$, $-\infty < x < \infty$ with a semi-infinite plate in $y = 0$, $0 < x < \infty$. The velocity components can be expressed in terms of a stream function ψ_t which satisfies the biharmonic equation. The velocity components are given by $u = -\partial\psi_t/\partial y$, $v = \partial\psi_t/\partial x$. Set $\psi_t = \psi_i + \psi$ with $\psi_i = \frac{3}{2}B\{(y/b) - \frac{1}{3}(y/b)^3\}$, where $\psi_t = 0$ on $y = 0$, B on $y = b$. Then ψ satisfies the boundary conditions

$$\psi = 0 \quad \text{on} \quad y = 0, \quad \psi \doteq \partial\psi/\partial y = 0 \quad \text{on} \quad y = b, \quad -\infty < z < \infty.$$
$$\partial^2\psi/\partial y^2 = 0, \quad -\infty < z < 0, \quad \partial\psi/\partial y = -\tfrac{3}{2}(B/b), \quad 0 < z < \infty, \text{ on } y = 0.$$

This can be formulated as a Wiener–Hopf problem involving the factorization

$$K(\alpha) = \alpha(\sinh 2\alpha - 2\alpha)(\sinh^2 \alpha - \alpha^2)^{-1}.$$

An approximate solution for this example and for two similar problems involving deflexion of plates has been given by W. T. Koiter [1].

3.20 *A supersonic jet in a subsonic stream.* (The application of the Wiener–Hopf technique to this problem was suggested by Prof. D. C. Pack, the Royal College of Science and Technology, Glasgow, and the formulation given below was carried out in collaboration with S. C. Lennox, the Heriot-Watt College, Edinburgh. Relevant references, which do not however use the Wiener–Hopf technique, are: "Supersonic flow of a two-dimensional jet", S. I. Pai, *J. Aero. Sci.*, **19** (1952), pp. 61–65, and a note on this paper by E. B. Klunker and K. C. Harder, p. 427 of the same issue.) Suppose that a two-dimensional supersonic jet lies in a region $-b \leqslant y \leqslant b$, $-\infty < x < \infty$, with a subsonic stream in $|y| > b$. Assume that the jet issues from rigid walls in $y = \pm b$, $-\infty < x < 0$, and that the supersonic flow and the subsonic stream are in contact along $y = \pm b$, $x > 0$. The boundary conditions on these lines are: $\partial\phi/\partial y$ is continuous but the pressure, $\partial\phi/\partial x$, is discontinuous. More precisely we assume

$$(\partial\phi/\partial x)_{y=b+0} - m(\partial\phi/\partial x)_{y=b-0} = \delta e^{-\varepsilon x}, \qquad x > 0,$$

where m, δ are constants and ε is a positive constant. We shall ultimately let ε tend to zero. In $-b \leqslant y \leqslant b$ the potential satisfies $\phi_{xx} - \phi_{yy} = 0$, but in $|y| > b$ the equation will be assumed to be $\phi_{xx} + \phi_{yy} + k^2\phi = 0$, where we shall ultimately let $k \to 0$. To avoid logarithmic potentials at infinity we work in terms of velocities $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$ with corresponding Fourier transforms in x given by U, V. We have

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad \text{i.e.} \quad V = \frac{i}{\alpha}\frac{dU}{dy}.$$

Then

$$U_+(\alpha,y) + U_-(\alpha,y) = Ae^{-\gamma y} \quad : \quad V_+(\alpha,y) + V_-(\alpha,y) = -i\gamma\alpha^{-1}Ae^{-\gamma y}, \quad (y \geqslant b),$$

$$U_+(\alpha,y) + U_-(\alpha,y) = C\cos\alpha y \quad : \quad V_+(\alpha,y) + V_-(\alpha,y) = -iC\sin\alpha y, \quad (0 \leqslant y \leqslant b),$$

where $\gamma = (\alpha^2 - k^2)^{1/2}$. The boundary conditions are

$$V_-(\alpha, b+0) = V_-(\alpha, b-0) = 0 \quad : \quad V_+(\alpha, b+0) = V_+(\alpha, b-0)$$

$$U_+(\alpha, b+0) - mU_+(\alpha, b-0) = \delta(\varepsilon - i\alpha)^{-1}.$$

The Wiener–Hopf equation is

$$\delta(\varepsilon - i\alpha)^{-1} + \{U_-(\alpha, b+0) - mU_-(\alpha, b-0)\} = i(\alpha\gamma^{-1} - m\cot\alpha b)V_+(\alpha,b).$$

If $k \to 0$, then $(\alpha/\gamma) \to (\alpha/|\alpha|)$ which is a typical situation arising in connexion with Laplace's equation. We do not pursue the solution further. It is left to the reader to show that by application of a Fourier transform in x the following integral equation obtained by Klunker and Harder, *loc. cit.*, can be reduced to the above Wiener–Hopf problem with $\delta = 0$:

$$v(x) + v(x+2p) = -\frac{q}{\pi}\int_0^\infty \left\{\frac{v(\xi)}{x-\xi} - \frac{v(\xi)}{x+2p-\xi}\right\} d\xi, \qquad (x \geqslant 0).$$

Suppose next that there are rigid walls at $y = \pm B$, $B > b$, on which $\partial\phi/\partial y = v = 0$. The Wiener–Hopf equation is then

$$\delta(\varepsilon - i\alpha)^{-1} + \{U_-(\alpha, b+0) - mU_-(\alpha, b-0)\} = i(\alpha\gamma^{-1}\coth\gamma B - m\cot\alpha b)V_+(\alpha,b).$$

When B is finite the function to be factorized has no branch points and the infinite product method is applicable.

It is probable that the above method can be applied to the subsonic jet in a subsonic or supersonic stream, and to axially symmetrical jets.

CHAPTER IV

EXTENSIONS AND LIMITATIONS OF THE METHOD

4.1 Introduction

Chapters II and III have been concerned mainly with the solution of specific examples. In this chapter we discuss a number of more general topics.

In §4.2 it is shown that the complex variable problem solved by the Wiener–Hopf technique is a special case of the Hilbert problem. In §4.3 we examine the criteria for successful application of the Hilbert problem (or the Wiener–Hopf technique) to a typical mixed boundary value problem, involving any co-ordinate of a separable system.

§4.4 deals with a generalization of the basic Wiener–Hopf equation where there is a set of simultaneous equations for several unknown functions.

A major difficulty in applying the Wiener–Hopf technique in practice is often the fundamental factorization $K(\alpha) = K_+(\alpha)K_-(\alpha)$. §4.5 discusses the position when we can find a function $K^*(\alpha)$ with a known factorization such that $K^*(\alpha)$ and $K(\alpha)$ are approximately equal to each other in the strip in which the Wiener–Hopf equation holds.

In §4.6 it is shown that certain problems involving Laplace's equation in polar co-ordinates can be solved by the Wiener–Hopf technique in conjunction with a Mellin transform.

We draw the reader's attention to various topics in connexion with the solution of simultaneous linear algebraic equations in exs. 4.10–4.13 at the end of the chapter.

4.2 The Hilbert problem†

The fundamental equation which appears in the Wiener–Hopf method is

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0, \tag{4.1}$$

where A, B, C are known analytic functions, $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ are unknown, and the equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ in the complex α-plane. In this section we show that the

† Also called the Riemann problem, or the Riemann–Hilbert problem.

solution of (7.1) can be regarded as a special case of the Hilbert problem: to find certain analytic functions when the boundary values of these functions on a set of smooth contours and arcs of arbitrary shape satisfy

$$a(t)\Phi_+(t) + b(t)\Psi_-(t) + c(t) = 0, \tag{4.2}$$

where t is the complex number specifying any point on the contour. In this equation $\Phi_+(t)$, $\Psi_-(t)$ are the (unknown) boundary values of analytic functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ which are to be determined. The functions $a(t)$, $b(t)$, $c(t)$ are given and need not be related to analytic functions.

We first specify the non-homogeneous Hilbert problem more precisely as follows. (N. I. Muskhelishvili [1], pp. 86, 92, 235.)

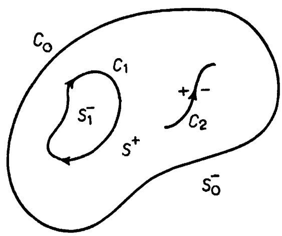

FIG. 4.1.

Consider the complex α-plane. Let S^+ be a connected region bounded by smooth contours and arcs $C_0, C_1, \ldots, C_p$, not intersecting each other, the first of which encloses all the others (see Fig. 4.1). The contour C_0 may be absent in which case S^+ is an infinite region. The contours and arcs may go to infinity but in this case some modifications are needed in the following statements. By C denote the union of $C_0, C_1, \ldots, C_p$. Denote by S^- that part of the plane which is the complement of $S^+ + C$, and by S' that part not belonging to C (i.e. $S' = S^+ + S^-$). Let $E(\alpha)$ be a function of the complex variable α such that

(i) $E(\alpha)$ is regular everywhere in S' and $E(\alpha) \sim \alpha^m$ as $\alpha \to \infty$, where m is an integer.

(ii) $E(\alpha)$ tends to a finite limiting value if α tends to a point on C from either side of C, other than the ends of any arcs.

(iii) At the end of any arc, say $\alpha = d$,

$$|E(\alpha)| < A|\alpha - d|^{-\mu}, \qquad (0 < \mu < 1).$$

Each part of C has a definite positive and negative side. For arcs these may be defined arbitrarily. For contours the positive side is the side adjoining S^+. If α lies on C denote its value by t. Denote by

$E^+(t)$, $E^-(t)$ the boundary values of the function $E(\alpha)$ as α tends to the point t on C from positive and negative sides of C respectively. We can now state the non-homogeneous Hilbert problem: find a function $E(\alpha)$ with the above properties such that

$$E^+(t) = G(t)E^-(t) + g(t), \tag{4.3}$$

where $G(t)$, $g(t)$ are functions given on C and $G(t)$ is a non-vanishing function on C satisfying everywhere on C the Hölder condition

$$|G(t_2) - G(t_1)| \leqslant A|t_2 - t_1|^\nu,$$

where A, ν are positive constants.

We now show that the solution of the Wiener–Hopf equation (4.1) can be reduced to the solution of a Hilbert problem. In (4.1) we assume

(a) The equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$. In this strip $A(\alpha)$, $B(\alpha)$, $C(\alpha)$ are regular functions of α.

(b) $\Phi_+(\alpha)$ is regular in $\tau > \tau_-$ and $|\Phi_+| < C_1|\alpha|^p$ as $\alpha \to \infty$ in this half-plane. $\Psi_-(\alpha)$ is regular in $\tau < \tau_+$ and $|\Psi_-| < C_2|\alpha|^q$ as $\alpha \to \infty$ in this half-plane.

Choose any τ_1, $\tau_- < \tau_1 < \tau_+$ such that no zeros of $A(\alpha)$, $B(\alpha)$ lie on the line $\tau = \tau_1$, $-\infty < \sigma < \infty$. Call this line the contour C and denote the complex number representing a point on C by the letter t. Then on C, (7.1) becomes

$$A(t)\Phi_+(t) + B(t)\Psi_-(t) + C(t) = 0, \tag{4.4}$$

where now $\Phi_+(t)$ is the limiting value, as α tends to the point t, of an analytic function $\Phi_+(\alpha)$ where $\Phi_+(\alpha)$ is regular for $\tau \geqslant \tau_1$, and similarly for $\Psi_-(\alpha)$. Also $A(t)$, $B(t)$ are continuous, differentiable and non-zero on C.

We see that this is a restricted form of the Hilbert problem. The two functions $\Phi_+(\alpha)$, $\Psi_-(\alpha)$ together make up the function $E(\alpha)$ defined above. The contour C of the Hilbert problem is now very simple—an infinite straight line parallel to the real axis. In comparing Wiener–Hopf and Hilbert problems we need to be careful when considering conditions at infinity. Apply a transformation, say $\alpha = (w - a)^{-1}$, to bring the point at infinity into the finite part of the plane—in this case to $w = a$. For the Wiener–Hopf problem in the w-plane we have a contour passing through $w = a$, and as $w \to a$ on one side of the contour, $|\Phi_+| < C_1|w - a|^{-p}$, and as $w \to a$ on the other side, $|\Psi_-| < C_2|w - a|^{-q}$. On the other hand for the Hilbert problem considered at the beginning of this section there is no contour through the point $w = a$ in the w-plane but $E(\alpha)$ has a pole of order m at this point.

However the essential difference between (4.3) and (4.4) is that in

(4.3), $G(t)$ need only satisfy a Hölder condition whereas in (4.4), $A(t)$ and $B(t)$ must satisfy much more restrictive conditions. In (4.4) $A(t)$ and $B(t)$ are derived from functions which are regular in a strip in the α-plane, which in turn is the basic feature used to solve the original equation (4.1) by the Wiener–Hopf method. The function $G(t)$ in (4.3) need have no connection, directly, with analytic functions. If the function $G(t)$ in (4.3) is regular in a strip of finite width including C, the Wiener–Hopf method can be used to solve the Hilbert problem. But in the general case it is necessary to use a different method.

We outline briefly the solution of the Hilbert problem for contours in order to show the connexion with the Wiener–Hopf technique. For details and a rigorous exposition the reader is referred to N. I. Muskhelishvili [1].

The major difficulty in solving (4.3) is the determination of suitable $K^+(t)$, $K^-(t)$ such that

$$K^+(t)G(t) = K^-(t), \tag{4.5}$$

where $K^+(t)$, $K^-(t)$ are limiting values of functions regular in S^+, S^- respectively. Muskhelishvili calls this the corresponding homogeneous Hilbert problem. The first steps of his solution are reminiscent of the theorem in ex. 1.12. When t vanishes over any of the closed contours C_k, $\ln G(t)$ will vary by an integral multiple of $2\pi i$ (cf. ex. 1.5). We write $G_0(t) = H^-(t)G(t)/H^+(t)$ where $H^\pm(t)$ are multiplying factors chosen so that $\ln G_0(t)$ is *one-valued* as well as satisfying a Hölder condition on C, and $H^\pm(t)$ are the limiting values of functions $H^\pm(\alpha)$ regular in $S^\pm$ respectively. Explicit forms for $H^\pm(\alpha)$ are given by N. I. Muskhelishvili [1], p. 88. Introduce $K_0^\pm(t) = H^\pm(t)K^\pm(t)$ where the upper and lower signs go together. Then (4.5) gives

$$K_0^+(t)G_0(t) = K_0^-(t), \tag{4.6}$$

where now $K_0^\pm(t)$ are unknown, $G_0(t)$ is known. Take logarithms of both sides:

$$\ln K_0^-(t) - \ln K_0^+(t) = \ln G_0(t). \tag{4.7}$$

At this stage it is not permissible to use theorem B of §1.3 to determine $K_0^\pm(t)$ which is the Wiener–Hopf procedure. Instead of Muskhelishvili uses the Plemelj formulae which can be stated as follows. Consider the Hilbert problem for contours. Define

$$Q(\alpha) = \frac{1}{2\pi i}\int_C \frac{\phi(s)}{s-\alpha}\,ds, \tag{4.8}$$

for any complex α not on C. It is assumed that $\phi(s)$ satisfies a Hölder condition on C. Then $Q(\alpha)$ is regular everywhere in the

α-plane except on C. Denote the limiting value of $Q(\alpha)$ as α tends to a point t of C from S^+ by $Q^+(t)$ and similarly for $Q^-(t)$. Define direction on C so that S^+ lies on the left of C when C is traversed in the positive direction. In (4.8) let α tend to a point t on C from S^+, and from S^-. By indenting the contour suitably we find

$$Q^+(t) = \tfrac{1}{2}\phi(t) + \frac{1}{2\pi i} P \int_C \frac{\phi(s)}{s-t}\, ds, \tag{4.9a}$$

$$Q^-(t) = -\tfrac{1}{2}\phi(t) + \frac{1}{2\pi i} P \int_C \frac{\phi(s)}{s-t}\, ds, \tag{4.9b}$$

where the P sign indicates that the integrals are now Cauchy principal values (cf. ex. 1.24). These are the Plemelj formulae. By subtraction,

$$Q^+(t) - Q^-(t) = \phi(t). \tag{4.10}$$

Hence we have the theorem: *If $\phi(t)$ is given on C and satisfies a Hölder condition then a function $Q(\alpha)$ whose limiting values on C satisfy* (4.10) *is given by* (4.8). This theorem can be used to solve the Hilbert problem (equation (4.3)) in exactly the same way as theorem B of §1.3 was used to solve the corresponding Wiener–Hopf problem. We first of all solve the subsidiary equation (4.6). In simple cases it may be possible to do this by inspection. In the general case we use the above theorem. Define

$$\ln K_0(\alpha) = \frac{1}{2\pi i} \int_C \frac{\ln G_0(s)}{s-\alpha}\, ds. \tag{4.11}$$

Then the limiting values as α tends to C from different sides of C determine $\ln K_0^-(t)$ and $\ln K_0^+(t)$. Hence we can find $K^+(t)$, $K^-(t)$. Multiply (4.3) by $K^+(t)$:

$$K^+(t)E^+(t) - K^-(t)E^-(t) = K^+(t)g(t).$$

It may be possible to solve this by inspection but in the general case we again use (4.8)–(4.10). Define

$$K(\alpha)E(\alpha) = \frac{1}{2\pi i} \int_C \frac{K^+(s)g(s)}{s-\alpha}\, ds. \tag{4.12}$$

Then $K^+(t)E^+(t)$ and $K^-(t)E^-(t)$ are the limiting values of this expression as α tends to C from opposite sides of C. This completes the solution of (4.3).

Formally the above solution is exactly what we should obtain by using the Wiener–Hopf technique in a strip of finite width and then letting the width of the strip tend to zero.

We note that the applications of the Hilbert problem made by N. I. Muskhelishvili are quite different from the applications in this book. Muskhelishvili is interested in cases where the solution of the partial differential equation in two independent variables (x,y) can be expressed directly as

$$\phi(x,y) = F(z), \qquad z = x + iy,$$

and the Hilbert problem is posed in terms of the complex variable z. In contrast to this, we are interested in cases when the solution of such an equation can be expressed in the form

$$\phi(x,y) = \int_C f(\alpha)K(x,y : \alpha)\, d\alpha,$$

and the Hilbert problem is formulated on the contour C in the α-plane.

There are three ways in which the Hilbert problem is more general than the Wiener–Hopf problem:

(a) The Hilbert problem is formulated on a contour and not in a strip of finite width. Hence it can be used to solve directly problems involving functions whose transforms exist only on a contour in the α-plane instead of in a strip. Strictly speaking nearly all the problems considered in this book are of this type. In previous chapters we have written $k = k_1 + ik_2$ where k_2 is finite, to ensure that transforms exist in a strip of finite width, although we have ultimately let k_2 tend to zero, in which case the transforms usually exist only on a contour. The solution for the case $k_2 = 0$ could have been obtained directly by solving a Hilbert problem.

(b) The function $G(t)$ occurring in the Hilbert problem need be defined only on a contour and it need satisfy only a Hölder condition on the contour. There is no need for $G(t)$ to be derived from a function of a complex variable which is analytic in a strip in the complex plane.

(c) The Hilbert problem can be solved for an arbitrary number of contours and arcs of arbitrary shape.

The Hilbert problem has been mentioned in some detail because almost certainly it will be useful in future researches. But in spite of the generality of the solution of the Hilbert problem we have not used it in the main part of this book. We have preferred the Wiener–Hopf technique because it seems more routine when used in conjunction with transforms for the particular kind of problem solved in this book. The method that we have used proceeds by a very straightforward analysis in the complex plane: branch cuts are clearly defined and we see automatically where indentations are required in the limiting case $k_2 \to 0$.

4.3 General considerations

The problems solved in previous chapters form an impressive collection, especially if we think of how few problems of this type can be solved by any other means. On the other hand the range of problem which can be solved is limited. The methods described so far for the exact solution of problems apply essentially to two-part mixed boundary value problems when the partial differential equation possesses a separation-of-variables solution $\exp(i\alpha x)$, the complex variable α being the parameter of separation. Different conditions involving ϕ and $\partial\phi/\partial n$ are specified on $-\infty < x < 0$ and $0 < x < \infty$. (The Mellin transform case is essentially the same and can be obtained by change of variable. The separation-of-variables solution is ρ^{-s} and the mixed boundary conditions are specified on $0 \leqslant \rho < a$, and $a < \rho < \infty$.)

Ideally we should like to be able to solve two-part mixed boundary value problems for *any* co-ordinate of *any* separable system. In this section we aim to clarify the relation of the Wiener–Hopf technique to this very general objective. Unfortunately it will become only too clear that we should not expect the Wiener–Hopf technique to work for any co-ordinate system.

Consider first of all a concrete example. Suppose that

$$\nabla^2\phi + k^2\phi = 0, \qquad 0 \leqslant x < \infty, \qquad 0 \leqslant y < \infty,$$

$$\partial\phi/\partial x = 0 \quad \text{on} \quad x = 0, \quad (0 \leqslant y < \infty),$$

$$\phi = 1, \quad (0 \leqslant x < a), \quad \partial\phi/\partial y = 0, \quad (a < x < \infty), \quad \text{on} \quad y = 0.$$

(A related physical problem is diffraction of a normally incident wave in infinite two-dimensional space, by a strip in $y = 0$, $-a \leqslant x \leqslant a$.) The natural transform to use is a cosine transform:

$$\Phi(\alpha,y) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \phi(x,y) \cos \alpha x \, dx \quad : \quad \phi = \left(\frac{2}{\pi}\right)^{1/2} \int_0^\infty \Phi \cos \alpha x \, dx. \tag{4.13}$$

Without going through the details (cf. §§5.2, 5.6) we state that the problem can be reduced to solution of the following equation which can be regarded as a generalization of (4.1):

$$(2/\pi)^{1/2}\alpha^{-1} \sin \alpha a + \Phi_2(\alpha) = -\gamma^{-1}\Phi_1'(\alpha), \tag{4.14}$$

where

$$\Phi_1'(\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \int_0^a \frac{\partial\phi}{\partial y} \cos \alpha x \, dx \quad : \quad \Phi_2(\alpha) = \left(\frac{2}{\pi}\right)^{1/2} \int_a^\infty \phi \cos \alpha x \, dx. \tag{4.15a}$$

ϕ and $(\partial\phi/\partial y)$ are taken on $y = 0$. $\Phi_1'(\alpha)$ is an integral function of α and $\Phi_2(\alpha)$ can be written

$$\Phi_2(\alpha) = e^{i\alpha a}\Phi_+(\alpha) + e^{-i\alpha a}\Phi_+(-\alpha), \tag{4.15b}$$

$$\Phi_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_a^\infty \phi e^{i\alpha(x-a)}\, dx.$$

Owing to the occurrence of $\Phi_+(-\alpha)$ in conjunction with $\Phi_+(\alpha)$, it is found that (4.14) is more complicated than (4.1) and cannot be solved exactly by the Wiener–Hopf technique. It can be solved approximately for large ka by the method of §§5.5, 5.6.

Another approach is to try two-dimensional polar co-ordinates

$$r\frac{\partial}{\partial r} r\frac{\partial \phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} + k^2r^2\phi = 0. \tag{4.16}$$

If $k = 0$ i.e. if we are dealing with Laplace's equation, problems of this type can be solved by a Mellin transform (§4.6). But if we apply a Mellin transform to (4.16) for $k \neq 0$ we find

$$d^2\Phi(s)/d\theta^2 + s^2\Phi(s) + k^2\Phi(s+2) = 0. \tag{4.17}$$

In this case it is the separation-of-variables solution which is at fault—the transform does not fit the partial differential equation. Further progress could be made if it were possible to solve (4.17) but this is deceptively simple.

The next step is to enquire whether there is a generalization of the Mellin transform which *does* fit the partial differential equation. This is the Lebedev–Kontorovich transform (A. Erdelyi *et al.* [1], Vol. II):

$$\Phi(\mu) = \int_0^\infty \phi K_\mu(\lambda r)\frac{dr}{r} \quad : \quad \phi = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \mu\phi(\mu) I_\mu(\lambda r)\, d\mu, \tag{4.18}$$

where $\lambda = -ik$. On applying (4.18) to (4.16) the problem can be reduced to an equation similar to (4.14) but involving the following unknown functions (cf. ex. 5.12):

$$\Phi_1'(\mu) = \int_0^a \left(\frac{\partial\phi}{\partial\theta}\right) K_\mu(\lambda r)\frac{dr}{r} \quad : \quad \Phi_2(\mu) = \int_a^\infty \phi K_\mu(\lambda r)\frac{dr}{r}, \tag{4.19}$$

where ϕ and $(\partial\phi/\partial\theta)$ are taken on $\theta = 0$. In this case $\Phi_2(\mu)$ is an integral function of μ and $\Phi_1(\mu)$ has to be split into two parts, one involving a $\Phi_+(\mu)$ and one a $\Phi_+(-\mu)$. Again it is found that the resulting equation cannot be solved by means of the Wiener–Hopf technique. An approximate solution can be obtained by the method of §5.4 (cf. ex. 5.12). The solution is suitable for small ka.

This example indicates that when we try to apply the Wiener–Hopf method to problems which cannot be solved by the straightforward Wiener–Hopf technique, various generalizations of the basic equation (4.1) appear. Some of these generalizations are investigated in Chapter V. Here we examine a different aspect of the matter: Consider any separable system. If we are given mixed boundary conditions on any bounding surface of the system, what are the prerequisites for successful application of the Wiener–Hopf technique? For convenience we carry out the discussion in terms of the Hilbert problem: the statement in the Wiener–Hopf terminology is left to the reader. The discussion given below is formal. We assume that orders of integration can be interchanged as necessary. We ignore the behaviour of functions at infinity and in general assume that contours lie in the finite part of the plane.

Suppose that by separation of variables we can find a solution of a given partial differential equation in two variables (x,y) in the form $K(t,x)L(t,y)$ where t is a separation parameter which may be complex. By superposition, a solution of the partial differential equation is

$$\phi = \int_C \mu(t)K(t,x)L(t,y)\,dt, \tag{4.20}$$

where $\mu(t)$ is an arbitrary function and C is in general some contour in the complex α-plane. We emphasize at the outset that there is no necessity here for μ, K, L to be regular or analytic functions of t. The functions need only be defined for special values of α, namely α on C, and it is for this reason that we use a separate letter t to denote these special values of α.

Suppose that by suitable choice of K, L we can satisfy all boundary conditions except on $y = y_0$ where

$$\phi = f(x), \quad (p < x \leqslant b) \quad : \quad \partial\phi/\partial y = g(x), \quad (a \leqslant x < p).$$

We define

$$\chi(t) = \mu(t)L'(t,y_0) \quad : \quad K(t) = L(t,y_0)/L'(t,y_0),$$

where

$$L'(t,y_0) = [\partial L(t,y)/\partial y]_{y=y_0}.$$

From (4.20), on applying the boundary conditions on $y = y_0$, we find the dual integral equations

$$\int_C K(t)\chi(t)K(t,x)\,dt = f(x), \qquad (p < x \leqslant b), \tag{4.21a}$$

$$\int_C \chi(t)K(t,x)\,dt = g(x), \qquad (a \leqslant x < p). \tag{4.21b}$$

We next assume that there is an inversion theorem

$$\int_C K(t,x)p(t)\,dt = p(x), \qquad (a \leqslant x \leqslant b), \tag{4.22a}$$

$$\int_a^b k(t,x)p(x)\,dx = P(t), \qquad (t \text{ on } C). \tag{4.22b}$$

Suppose that the left-hand sides of (4.21a) and (4.21b) equal $e(x)$ and $h(x)$ for $a \leqslant x < p$, $p < x \leqslant b$, respectively. Invert by (4.22). This gives

$$K(t)\chi(t) = F(t) + E(t) \quad : \quad \chi(t) = G(t) + H(t), \qquad (t \text{ on } C), \tag{4.23}$$

where

$$E(t) = \int_a^p k(t,x)e(x)\,dx \qquad : \qquad H(t) = \int_p^b k(t,x)h(x)\,dx, \tag{4.24}$$

and there are similar definitions for $F(t)$, $G(t)$. Elimination of $\chi(t)$ gives

$$E(t) = K(t)H(t) + M(t), \quad \text{where} \quad M(t) = K(t)G(t) - F(t). \tag{4.25}$$

E, H are unknown and all other functions are known. This equation is formally very similar to the equation (4.2) of the Hilbert problem or (4.1) for the equivalent Wiener–Hopf problem.

Before examining (4.25) we digress to show that the same equation can be obtained directly from the transform point of view by Jones's method. Define (cf. (4.22))

$$\Phi(t,y) = \int_a^b k(t,x)\phi(x,y)\,dx, \qquad (t \text{ on } C).$$

Apply this transform to the partial differential equation, solve the ordinary differential equation which results, and insert boundary conditions on all boundaries except $y = y_0$. Our statement in connexion with (4.20), (4.21) that K, L are chosen so that all boundary conditions are satisfied except on $y = y_0$ means that we shall find

$$\Phi(t,y) = \mu(t)L(t,y),$$

where μ is an arbitrary function and L is known. Form $d\Phi/dy$ and set $y = y_0$ in the expressions for Φ and $d\Phi/dy$. Insert the mixed boundary conditions and use the notation for E, F, G, H defined above. Elimination of $\mu(t)$ then gives (4.25).

Although both $E(t)$ and $H(t)$ are unknown in (4.25), the fact that they are defined by (4.24) provides additional information which is sufficient to specify $e(x)$, $h(x)$ completely. If we wish to try to apply the Wiener–Hopf technique, we define $E(\alpha)$, $H(\alpha)$ for the general complex variable α by (cf. (4.24))

$$E(\alpha) = \int_a^p k(\alpha,x)e(x)\,dx \quad : \quad H(\alpha) = \int_p^b k(\alpha,x)h(x)\,dx. \tag{4.26}$$

Suppose that the contour C (on which $\alpha = t$) divides the α-plane into two parts, say S^+, S^-. Then if $E(\alpha)$ is regular in say S^-, $H(\alpha)$ in S^+, we can extend our notation to write (4.25) in the form

$$E_-(t) = K(t)H_+(t) + M(t),$$

where now $E_-(t)$, $H_+(t)$ are (unknown) values of analytic functions $E_-(\alpha)$, $H_+(\alpha)$ which are regular in S^-, S^+ respectively. The problem is now to determine the analytic functions $E_-(\alpha)$, $H_+(\alpha)$, which is precisely the Hilbert problem (4.2).

However we emphasize that $E(\alpha)$ will be regular in say S^- only in very special circumstances. If we set $E(t) = E_1(t) + iE_2(t)$ where E_1 and E_2 are real, then it is well known that the specification of E_1 *alone* is sufficient to determine functions $E_-(\alpha)$, $E_+(\alpha)$ regular in S^- and S^+, and unique apart from an arbitrary constant. This is true because $E_1(t)$ defines a Dirichlet boundary value problem in say S^-. Since the problem is two-dimensional the solution of the Dirichlet problem, say $U(\sigma,\tau)$, determines a conjugate function $V(\sigma,\tau)$ such that $U(\sigma,\tau) + iV(\sigma,\tau)$ is a function of $(\sigma + i\tau)$. If (σ,τ) tends to a point on the boundary of the region then $U(\sigma,\tau)$ will tend to $E_1(t)$ but $V(\sigma,\tau)$ will not in general tend to $E_2(t)$. This means that when $E_1(t)$ is specified, $E_2(t)$ cannot be specified arbitrarily if $E_-(\alpha)$ is to exist.

In the general case it will therefore not be possible to identify $E(\alpha)$, $H(\alpha)$, defined by (4.26), with $E_-(\alpha)$, $H_+(\alpha)$ regular in S^-, S^+ respectively. It will be necessary to examine the structure of $E(\alpha)$, $H(\alpha)$ in detail. Some examples are considered in Chapter V.

Two features illustrated by this investigation are:

(i) Any two-part boundary value problem can be formulated as an equation of type (4.25). The all-important factor which determines whether this equation is equivalent to a Hilbert problem is the behaviour of the integrals (4.26) considered as functions of a complex variable.

(ii) From the Hilbert problem point of view, the function $K(t)$ need not have any direct connexion with analytic functions of a complex variable: it need be defined only on a contour in the complex plane and it need satisfy only a Hölder condition. In the Wiener–Hopf method $K(\alpha)$ must be analytic in a strip in the complex α-plane but from a general viewpoint this would seem to be accidental rather than fundamental.

In particular cases it may be possible to deduce the solution of dual integral equations almost by inspection without going through the reduction to a Hilbert or a Wiener–Hopf problem. Thus consider the solution of the following equations (see §2.3 (2.34) (2.35), cf. P. C. Clemmow [1], p. 296. Note that Clemmow takes a time factor $\exp(i\omega t)$ so that his contour is indented differently.):

$$\frac{1}{(2\pi)^{1/2}}\int_C A(t)e^{-itx}\,dt = 0, \qquad (x > 0), \tag{4.27a}$$

$$\frac{1}{(2\pi)^{1/2}}\int_C (t^2-k^2)^{1/2}A(t)e^{-itx}\,dt = -ik\sin\Theta e^{-ikx\cos\Theta}, \qquad (x < 0), \tag{4.27b}$$

where the contour C is the real axis indented above at $t = -k$ and below at $t = k\cos\Theta$ and $t = k$.

We assume that an analytic function $A(\alpha)$ can be derived such that its limiting value on C gives $A(t)$ which satisfies the above equation. As previously remarked it is not obvious that such an $A(\alpha)$ exists but we can either prove the existence of $A(\alpha)$ by general arguments as above, or merely find $A(\alpha)$ and show *a posteriori* that it satisfies the required conditions. In the first integral complete the contour by an infinite semi-circle below C. Then the first equation is satisfied by choosing

$$A(\alpha) = \Psi_-(\alpha), \qquad (\tau < 0), \tag{4.28}$$

where $\Psi_-(\alpha)$ is regular below the contour and has algebraic behaviour at infinity. In the second integral complete the contour by an infinite semi-circle above C. Then a suitable form for $A(\alpha)$ is

$$A(\alpha) = \frac{1}{(2\pi)^{1/2}(\alpha^2-k^2)^{1/2}}\cdot\frac{\Phi_+(\alpha)}{\Phi_+(k\cos\Theta)}\cdot\frac{-k\sin\Theta}{(\alpha-k\cos\Theta)}, \tag{4.29}$$

since the pole at $\alpha = k\cos\Theta$ (contour indented below) gives the right-hand side of (7.33b) and there are no other poles or branch points of $(\alpha^2-k^2)A(\alpha)$ above C. The limiting value of $A(\alpha)$ in

(7.34) as $\tau \to -0$ must be the same as in (7.35) as $\tau \to +0$. It is easily seen that we need to write $(\alpha^2 - k^2)^{1/2} = (\alpha - k)^{1/2}(\alpha + k)^{1/2}$ and choose $\Phi_+(\alpha) = (\alpha + k)^{1/2}$. Then

$$A(\alpha) = -\frac{1}{(2\pi)^{1/2}} \cdot \frac{1}{(k + k\cos\Theta)^{1/2}} \cdot \frac{1}{(\alpha - k)^{1/2}} \frac{k\sin\Theta}{(\alpha - k\cos\Theta)} .$$

This agrees with (2.31).

There are a group of methods due to L. A. Vajnshtejn ([1], [3], [4], [5]), S. N. Karp [1], and P. C. Clemmow [1], which are essentially the same as the method illustrated in the last paragraph. All these methods are similar in that they start with a pair of dual integral equations obtained by separation of variables, or solution of an integral equation or a physical argument. Instead of reduction to a Hilbert or Wiener–Hopf problem the equations are solved by "function-theoretic" methods depending on the regions of regularity, and poles, of the functions under the integral sign. Instead of working with general f, g on the right-hand sides of their equations Vajnshtejn–Karp–Clemmow work with explicit f, g so that the poles of unknown functions are known as in the last paragraph. The justification is essentially *a posteriori* (cf. J. Bazer and S. N. Karp [1].) The V–K–C methods may be convenient in particular cases where they give an elegant solution without invoking the reduction to a Hilbert or Wiener–Hopf problem but they are not essentially different from the procedures used in this book and we do not describe them in further detail.

4.4 Simultaneous Wiener–Hopf equations

In this section we consider the situation which arises when, instead of the single Wiener–Hopf equation (4.1) in two unknown functions there are n simultaneous equations in $2n$ unknown functions.

Many of the examples in Chapter III were in fact simultaneous Wiener–Hopf equations with $n = 2$ but in all cases except one it was possible to reduce these twin equations to independent single equations of the Wiener–Hopf type. The exception was (3.85a, b). These equations are of the form

$$\Psi_1^-(\alpha) - M(\alpha)\Psi_2^-(\alpha) = \{M(\alpha) + i\delta\}\Phi_1^+(\alpha) + F(\alpha), \qquad (4.30a)$$

$$\Psi_1^-(\alpha) + N(\alpha)\Psi_2^-(\alpha) = -\{N(\alpha) + i\delta\}\Phi_2^+(\alpha) + G(\alpha), \qquad (4.30b)$$

where δ is a (small) constant and $M(\alpha)$, $N(\alpha)$ possess only simple poles and no branch points. It is possible to obtain an exact solution when $M(\alpha) = N(\alpha)$, or when $\delta = 0$, but not in the general case.

In Chapter III we deliberately restricted ourselves (except in the one case we have just mentioned) to problems which reduced to single Wiener–Hopf equations. But it is easy to suggest extensions of the problems in Chapter III which lead to twin Wiener–Hopf equations. Thus consider a plane wave incident on two parallel partially reflecting half-planes (cf. §2.2). For simplicity suppose that a wave $\exp(-ikx)$ is incident on two half-planes in $y = \pm b$, $-\infty < x < 0$. Suppose that

$$\partial\phi_t/\partial y = \mp i\delta\phi_t, \quad y = b \pm 0 \quad : \quad \partial\phi_t/\partial y = \mp i\delta\phi_t, \quad y = -b \pm 0,$$

for $-\infty < x < 0$. From symmetry $\partial\phi_t/\partial y = 0$ on $y = 0$. It is left to the reader to show that the Wiener–Hopf equations are

$$\begin{aligned} \Phi'_+(b) + \delta(\alpha - k)^{-1} &\\ = \{\gamma \tanh \gamma b - i\delta\}\Phi_-(b - 0) &+ \gamma \tanh \gamma b \Phi_+(b), \\ \Phi'_+(b) - \delta(\alpha - k)^{-1} = (-\gamma + i\delta)\Phi_-(b + 0) &- \gamma\Phi_+(b). \end{aligned}$$

These are similar to (4.30) but in this case the function corresponding to $N(\alpha)$ possesses branch points.

Other generalizations of the problems of §§3.2, 3.4 which result in simultaneous Wiener–Hopf equations are: (i) A plane wave incident on two parallel staggered plates in $y = b$, $-\infty < z < -h$, and $y = -b$, $-\infty < z < +h$. (ii) A plane wave incident on three parallel plates in $y = 0, \pm \mathrm{b}$, $-\infty < z < 0$. (iii) Radiation from a coaxial guide $a \leqslant \rho \leqslant b$, $-\infty < z < 0$.

Before considering the general theory we examine the solution of the equation.

$$K(\alpha)\Phi_1^+(\alpha) + \varepsilon_1 L(\alpha)\Psi_1^-(\alpha) + M(\alpha)\Psi_2^-(\alpha) = N(\alpha), \tag{4.31}$$

where ε_1 is a parameter introduced for convenience, as will appear later. We assume that the only singularities of $M(\alpha)$, $L(\alpha)$ are simple poles, though $K(\alpha)$ may have branch points. In the usual way we assume that the equation holds in $\tau_- < \tau < \tau_+$, $-\infty < 0 < \infty$, and that all functions are regular in this strip and have suitable behaviour as $\alpha \to \infty$ in appropriate half-planes. We set

$$K(\alpha) = K_+(\alpha)K_-(\alpha) \ : \ L(\alpha) = L_+(\alpha)L_-(\alpha) \ : \ M(\alpha) = M_+(\alpha)M_-(\alpha).$$

Divide (4.31) throughout by $K_-(\alpha)M_+(\alpha)$ and suppose that

$$\frac{L(\alpha)}{M_+(\alpha)} = l_-(\alpha) + \sum_{s=1}^{\infty} \frac{l_s}{\alpha - \beta_s},$$

where β_s are the zeros of $M_+(\alpha)$ and the poles of $L(\alpha)$ in the lower half plane. We can write

$$\frac{L(\alpha)}{M_+(\alpha)} \cdot \frac{\Psi_1^-(\alpha)}{K_-(\alpha)} = A_+(\alpha) + A_-(\alpha), \tag{4.32a}$$

where

$$A_+(\alpha) = \sum_{s=1}^{\infty} \frac{l_s}{\alpha - \beta_s} \cdot \frac{\Psi_1^-(\beta_s)}{K_-(\beta_s)}, \tag{4.32b}$$

$$A_-(\alpha) = l_-(\alpha) \frac{\Psi_1^-(\alpha)}{K_-(\alpha)} + \sum_{s=1}^{\infty} \frac{l_s}{\alpha - \beta_s} \left\{ \frac{\Psi_1^-(\alpha)}{K_-(\alpha)} - \frac{\Psi_1^-(\beta_s)}{K_-(\beta_s)} \right\}. \tag{4.32c}$$

We write

$$N(\alpha)\{M_+(\alpha)K_-(\alpha)\}^{-1} = G_+(\alpha) + G_-(\alpha), \quad \text{say.}$$

Separation of (4.31) by means of the Wiener–Hopf technique then gives

$$K_+(\alpha)\Phi_1^+(\alpha) + \varepsilon_1 M_+(\alpha) \sum_{s=1}^{\infty} \frac{l_s}{\alpha - \beta_s} \cdot \frac{\Psi_1^-(\beta_s)}{K_-(\beta_s)} = M_+(\alpha)G_+(\alpha). \tag{4.33}$$

If we are given a second equation in the form

$$P(\alpha)\Psi_1^-(\alpha) + \varepsilon_2 Q(\alpha)\Phi_1^+(\alpha) + R(\alpha)\Phi_2^+(\alpha) = S(\alpha), \tag{4.34}$$

then an exactly similar procedure gives

$$P_-(\alpha)\Psi_1^-(\alpha) + \varepsilon_2 R_-(\alpha) \sum_{s=1}^{\infty} \frac{q_s}{\alpha - \theta_s} \frac{\Phi_1^+(\theta_s)}{P_+(\theta_s)} = R_-(\alpha)T_-(\alpha), \tag{4.35}$$

where

$$Q(\alpha)/R_-(\alpha) = q_+(\alpha) + \sum_{s=1}^{\infty} \frac{q_s}{\alpha - \theta_s},$$

$$S(\alpha)\{R_-(\alpha)P_+(\alpha)\}^{-1} = T_+(\alpha) + T_-(\alpha).$$

The functions $\Phi_1^+(\alpha), \Psi_1^-(\alpha)$ satisfying the simultaneous Wiener–Hopf equations (4.31), (4.34) are therefore given by (4.33), (4.35), where the constants $\Psi_1^-(\beta_s)$, $\Phi_1^+(\theta_s)$ are so far unknown. Twin sets of infinite simultaneous linear algebraic equations for these constants are obtained by setting $\alpha = \theta_r$ in (4.33) and $\alpha = \beta_r$ in (4.35), $r = 1, 2, \ldots$. The equations can be written

$$a_r x_r + \varepsilon_1 \sum_{s=1}^{\infty} \frac{y_s}{\theta_r + \eta_s} = A_r, \qquad (r = 1, 2, \ldots),$$

$$b_r y_r - \varepsilon_2 \sum_{s=1}^{\infty} \frac{x_s}{\theta_s + \eta_r} = B_r, \qquad (r = 1, 2, \ldots),$$

where

$$q_s\Phi_1^+(\theta_s)/P_+(\theta_s) = x_s \quad : \quad l_s\Psi_1^-(\beta_s)/K_-(\beta_s) = y_s,$$

with appropriate expressions for a_r, b_r, A_r, B_r. We have also set $\eta_s = -\beta_s$ so that θ_s and η_s both lie in an upper half-plane. In general it would appear that these equations must be solved numerically. If ε_1 or ε_2 is small, or if ε_1 and ε_2 are both small, it may be convenient to solve the equations iteratively.

The scope of the above method for solution of twin Wiener–Hopf equations is of course strictly limited since the functions L, M, Q, R must not possess branch points. When the method is applicable it will be advantageous to arrange that ε_1 and ε_2 are as small as possible. Thus in the case of (4.30) if we divide (4.30a) by $M(\alpha)$, (4.30b) by $N(\alpha)$, and add, we find

$$2U(\alpha)\Psi_1^-(\alpha) = i\delta V(\alpha)S^+(\alpha) + \{1 + i\delta U(\alpha)\}D^+(\alpha) + \\ + F(\alpha)/M(\alpha) + G(\alpha)/N(\alpha),$$

where

$$S^+(\alpha) = \Phi_1^+(\alpha) + \Phi_2^+(\alpha) \quad : \quad D^+(\alpha) = \Phi_1^+(\alpha) - \Phi_2^+(\alpha),$$

$$2U(\alpha) = \{M(\alpha)\}^{-1} + \{N(\alpha)\}^{-1} \ : \ 2V(\alpha) = \{M(\alpha)\}^{-1} - \{N(\alpha)\}^{-1}.$$

This is a convenient equation when δ is small. From (4.31), (4.34) the second equation must involve only $S^+(\alpha)$, $\Phi_1^-(\alpha)$, $\Psi_2^-(\alpha)$, and this can be derived from (4.30) in only one way.

We now consider some general theory. Instead of (4.1) suppose that we are given a set of n equations

$$\sum_{s=1}^{n} A_{rs}(\alpha)\Phi_s^+(\alpha) + \sum_{s=1}^{n} B_{rs}(\alpha)\Psi_s^-(\alpha) + C_r(\alpha) = 0, \ (r = 1, 2, \ldots, n), \tag{4.36}$$

where there are $2n$ unknown functions $\Phi_s^+(\alpha)$, $\Psi_s^-(\alpha)$, regular in upper and lower half-planes, $\tau > \tau_-$, $\tau < \tau_+$, respectively, with $\tau_- < \tau_+$. The equations hold in the strip $\tau_- < \tau < \tau_+$. The above equations can be written in matrix notation as

$$\mathbf{A}\boldsymbol{\Phi}^+ + \mathbf{B}\boldsymbol{\Psi}^- + \mathbf{C} = 0, \tag{4.37}$$

where $\mathbf{A}$, $\mathbf{B}$ are $(n \times n)$ square matrices and $\boldsymbol{\Phi}^+$, $\boldsymbol{\Psi}^-$, $\mathbf{C}$ are $(n \times 1)$ matrices. $\mathbf{A}$, $\mathbf{B}$ are assumed non-singular in $\tau_- < \tau < \tau_+$.

The following set of equations of type (4.37) can be solved exactly:

$$\mathbf{K}^+\boldsymbol{\Phi}^+ + \mathbf{K}^-\boldsymbol{\Psi}^- + \mathbf{L} = 0, \tag{4.38}$$

where all equations are assumed to hold in a common strip and the general elements K_{rs}^+, K_{rs}^- of the matrices $\mathbf{K}^+$, $\mathbf{K}^-$ are regular in upper and lower half-planes respectively. Then the standard Wiener–Hopf procedure can be used to solve each of the n simultaneous equations exactly, and the Φ_s^+, Ψ_s^- can then be obtained by solving two $n \times n$ sets of simultaneous linear algebraic equations.

It may be possible to reduce (4.37) to form (4.38) by premultiplying by some $n \times n$ matrix $\mathbf{D}$ whose elements are functions of α (i.e. by taking linear combinations of the equations in (4.37), multiplied by suitable functions of α). Then (4.37) would become

$$\mathbf{DA\Phi}^+ + \mathbf{DB\Psi}^- + \mathbf{DC} = 0,$$

and on comparing with (4.38) it is clear that $\mathbf{D}$ must be chosen so that

$$\mathbf{DA} = \mathbf{K}^+ \quad : \quad \mathbf{DB} = \mathbf{K}^-. \tag{4.39}$$

Suppose that $\mathbf{A}$ is a non-singular matrix so that we can write $\mathbf{D} = \mathbf{K}^+\mathbf{A}^{-1}$. Eliminate $\mathbf{D}$ from (4.39) by using this result in the second equation. Then we must have

$$\mathbf{K}^+\mathbf{A}^{-1}\mathbf{B} = \mathbf{K}^-. \tag{4.40}$$

The existence of $\mathbf{K}^+$ and $\mathbf{K}^-$ satisfying this equation is obviously a sufficient condition for solution of the original equation (4.37) providing of course that certain conditions concerning behaviour at infinity, etc. are satisfied. However it is by no means obvious that, given any equation (4.37) with general $\mathbf{A}$, $\mathbf{B}$ and non-singular $\mathbf{A}$, then matrices $\mathbf{K}^+$ and $\mathbf{K}^-$ can always be found to satisfy (4.40), so that (4.37) can always be solved in this way. However we showed in §4.2 that the Wiener–Hopf problem was a special case of the Hilbert problem: similarly the Wiener–Hopf problem for several unknown functions is a special case of the Hilbert problem for several unknown functions. This latter problem has been treated in N. I. Muskhelishvili [1], Chap. 18, where it is proved that the matrices $\mathbf{K}^+$, $\mathbf{K}^-$ exist under quite general conditions. (The references quoted by Muskhelishvili go back to Plemelj.) However although the proof of the existence of $\mathbf{K}^+$, $\mathbf{K}^-$ given by Muskhelishvili provides in theory a method for constructing $\mathbf{K}^+$, $\mathbf{K}^-$ this does not seem to have been carried through in any concrete example.

An interesting suggestion has been made by A. E. Heins [6]. If we know the eigenvalues of the matrix $\mathbf{K}$ then in general we can find a matrix $\mathbf{L}$ such that

$$\mathbf{K} = \exp\{\mathbf{L}\}.$$

The elements of $\mathbf{K}$ and $\mathbf{L}$ are functions of α. If we decompose each element of $\mathbf{L}$ additively,

$$l_{ij}(\alpha) = l_{ij}^+(\alpha) + l_{ij}^-(\alpha),$$

then in an obvious notation we can write

$$\mathbf{K}^+ = \exp\{\mathbf{L}^+\} \quad : \quad \mathbf{K}^- = \exp\{\mathbf{L}^-\}.$$

It would seem difficult to obtain concrete results for specific examples by this method.

In the remainder of this section we content ourselves with the reduction of (4.37) to a set of Fredholm equations in two different ways. There is no loss of generality in considering, instead of (4.37),

$$\mathbf{\Phi}^{+} + \mathbf{K}\mathbf{\Psi}^{-} + \mathbf{M} = 0. \tag{4.41}$$

Assume for simplicity that the strip in which this holds is such that $\tau_{-} < \tau < \tau_{+}$. As in §5.3, (5.28), if t is a real number,

$$\tfrac{1}{2}\Phi_i^{+}(t) = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \Phi_i^{+}(s) \frac{ds}{s-t}$$

$$\tfrac{1}{2}\Psi_i^{-}(t) = -\frac{1}{2\pi i} P \int_{-\infty}^{\infty} \Psi_i^{-}(s) \frac{ds}{s-t}, \qquad (i = 1 \text{ to } n).$$

Write these in matrix notation in an obvious way:

$$\begin{aligned} \tfrac{1}{2}\mathbf{\Phi}^{+}(t) &= \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \mathbf{\Phi}^{+}(s) \frac{ds}{s-t} \\ \tfrac{1}{2}\mathbf{\Psi}^{-}(t) &= -\frac{1}{2\pi i} P \int_{-\infty}^{\infty} \mathbf{\Psi}_i^{-}(s) \frac{ds}{s-t}. \end{aligned} \tag{4.42}$$

Take values of $\mathbf{\Phi}^{+}(t)$ from (4.41) for values of α on the line $\tau = 0$, $\alpha = t$ and substitute in the first equation in (4.42). Then

$$\tfrac{1}{2}\{\mathbf{K}(t)\mathbf{\Psi}^{-}(t) + \mathbf{M}(t)\} = \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \{\mathbf{K}(s)\mathbf{\Psi}^{-}(s) + \mathbf{M}(s)\} \frac{ds}{s-t}.$$

Multiply through by $[\mathbf{K}(t)]^{-1}$ and add to the second equation in (4.42). This gives

$$\begin{aligned} \mathbf{\Psi}^{-}(t) &- \frac{1}{2\pi i} \int_{-\infty}^{\infty} \{[\mathbf{K}(t)]^{-1}\mathbf{K}(s) - \mathbf{I}\}\mathbf{\Psi}^{-}(s) \frac{ds}{s-t} \\ &= [\mathbf{K}(t)]^{-1} \left\{\frac{1}{2\pi i} P \int_{-\infty}^{\infty} \mathbf{M}(s) \frac{ds}{s-t} - \tfrac{1}{2}\mathbf{M}(t)\right\} \end{aligned} \tag{4.43}$$

where the integral on the left is no longer a Cauchy principal value integral since the integrand is now well-behaved at $s = t$. This is a system of Fredholm integral equations of the second kind. The above method is based on the Plemelj treatment of the homogeneous Hilbert problem (cf. N. I. Muskhelishvili [1], p. 387).

Another method of reduction is the following. We have

$$\Phi_r^+(\alpha) + \sum_{s=1}^{n} K_{rs}(\alpha)\Psi_s^-(\alpha) + M_r(\alpha) = 0, \qquad (r = 1 \text{ to } n).$$

Suppose that $K_{rr}(\alpha) = K_{rr}^-(\alpha)/K_{rr}^+(\alpha)$. Multiply through by $K_{rr}^+(\alpha)$. Then

$$K_{rr}^+\Phi_r^+ + K_{rr}^-\Psi_r^- + K_{rr}^+\sum_{s=1}^{n}{}' K_{rs}\Psi_s^- + K_{rr}^+M_r = 0, \tag{4.44}$$

where the dash indicates that the term $r = s$ is omitted. In the following the limits $s = 1$ to n on the summation sign will be omitted. Set

$$K_{rr}^+(\alpha)\sum{}' K_{rs}(\alpha)\Psi_s^-(\alpha) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} K_{rr}^+(\zeta)\sum{}' K_{rs}(\zeta)\Psi_s^-(\zeta)\frac{d\zeta}{\zeta-\alpha}$$
$$-\frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} K_{rr}^+(\zeta)\sum{}' K_{rs}(\zeta)\Psi_s^-(\zeta)\frac{d\zeta}{\zeta-\alpha}, \tag{4.45}$$

where the first term is regular in an upper half-plane and the second in a lower. If we apply the Wiener–Hopf technique in the usual way we find from (4.44) that

$$K_{rr}^-\Psi_r^- - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} K_{rr}^+(\zeta)\sum{}' K_{rs}(\zeta)\Psi_s^-(\zeta)\frac{d\zeta}{\zeta-\alpha} + N_r^-(\alpha) = P_r(\alpha), \tag{4.46a}$$

$$K_{rr}^+\Phi_r^+ + \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} K_{rr}^+(\zeta)\sum{}' K_{rs}(\zeta)\Psi_s^-(\zeta)\frac{d\zeta}{\zeta-\alpha} + N_r^+(\alpha) = -P_r(\alpha), \tag{4.46b}$$

where we have set $K_{rr}^+M_r = N_r^+ + N_r^-$ and the $P_r(\alpha)$ are arbitrary polynomials determined by conditions at infinity. (4.46a) holds for $r = 1$ to n and this gives a set of n simultaneous integral equations for the Ψ_r^-. If these are found the Φ_r^+ are then obtained from (4.46b).

As in the analysis leading to (4.43), if we let $d = 0$, denote the real value of ζ by s and let α tend to a real number t, then (4.46a) can be reduced to the system

$$K_{rr}^-(t)\Psi_r^-(t) - \tfrac{1}{2}K_{rr}^+(t)\sum{}' K_{rs}(t)\Psi_s^-(t)$$
$$-\frac{1}{2\pi i}\int_{-\infty}^{\infty} K_{rr}^+(s)\sum{}'\{K_{rs}(s)\Psi_s^-(s) - K_{rs}(t)\Psi_s^-(t)\}\frac{ds}{s-t}$$
$$+ N_r^-(t) = P_r(t), \qquad (r = 1 \text{ to } n.)$$

If the matrix $\mathbf{K}$ is diagonal the exact solution can be obtained immediately from these equations and therefore we should expect that this form will be useful for iterative solution if the diagonal terms are predominant.

4.5 Approximate factorization

All the problems solved in this book depend on the fundamental factorization $K(\alpha) = K_+(\alpha)K_-(\alpha)$. If we have to use the integral formula of theorem C, §1.3, to perform this factorization then the practical details of finding numerical solutions tend to become complicated. For concreteness consider

$$K(\alpha)\Phi_+(\alpha) = \Psi_-(\alpha) + F(\alpha), \tag{4.47}$$

valid in $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$. It may happen that we can set

$$K(\alpha) = K^*(\alpha) + Q(\alpha), \tag{4.48}$$

where a simple factorization exists for $K^*(\alpha)$, and $K(\alpha)$ is very nearly equal to $K^*(\alpha)$ in the strip in which (4.47) is true. Assume that in this case we can find the solution of

$$K^*(\alpha)\Phi_+^*(\alpha) = \Psi_-^*(\alpha) + F(\alpha). \tag{4.49}$$

It is then natural to compare Φ_+, Φ_+^* and Ψ_-, Ψ_-^*. Two slightly different situations arise:

(a) We may wish to know whether the solution of (4.49) can be taken as an approximation to the solution of (4.47).

(b) It may be possible to write (4.48) in the form

$$K(\alpha) = K^*(\alpha) + \varepsilon q(\alpha), \tag{4.50}$$

where ε is a small parameter, and we may be interested in the perturbation of the solution of (4.49) caused by the presence of the term in ε.

In problems of type (a) it is easy to check whether $K(\alpha)$ and $K^*(\alpha)$ are approximately equal since it is necessary only to compare the numerical values of these functions on some line $\tau = \tau_1$, $-\infty < \sigma < \infty$, where $\tau_- < \tau_1 < \tau_+$. The factors $K_+(\alpha)$, $K_-(\alpha)$ and the final solution can be expressed as integrals along a contour consisting of this line. If $K(\alpha)$, $K^*(\alpha)$ are approximately equal on this line then the final solutions will be approximately equal. It is particularly important to notice that there is no need for the behaviour of $K(\alpha)$ and $K^*(\alpha)$ to be similar in the complex plane, off the line $\tau = \tau_1$, $-\infty < \sigma < \infty$. (Apart from the argument just given we recall from §4.2 that the analytic nature of $K(\alpha)$ is in a sense accidental. If we use the Hilbert problem formulation then $K(\alpha)$ need be defined only on a contour in the α-plane and need not even be defined off this contour.)

In this connexion an instructive example has been given by W. T. Koiter [1], part IIb. He compares problems involving the following two functions, both of which possess simple factorizations:

$$K(\alpha) = \alpha^{-1} \tanh \alpha \qquad : \qquad K^*(\alpha) = (\alpha^2 + 1)^{-1/2}.$$

The behaviour of these functions is similar in a narrow strip, $-a < \tau < a$, $-\infty < \sigma < \infty$. They both tend to unity as α tends to zero and to $|\alpha|^{-1}$ as α tends to infinity. A comparison of numerical values indicates that they agree to within 9% on the real axis. However the behaviour off the real axis is completely different. $K(\alpha)$ has an infinite number of poles and zeros: $K^*(\alpha)$ has no zeros or poles, but it possesses two branch points.

Consider

$$\alpha^{-1} \tanh \alpha\, \Phi_+(\alpha) = i(2\pi)^{-1/2}\alpha^{-1} + \Psi_-(\alpha), \qquad 0 < \tau < \tfrac{1}{2}\pi,$$

$$(\alpha^2 + 1)^{-1/2}\, \Phi_+^*(\alpha) = i(2\pi)^{-1/2}\alpha^{-1} + \Psi_-^*(\alpha), \qquad 0 < \tau < 1.$$

The solution of these equations is straightforward and it is left to the reader to verify that (cf. W. T. Koiter [1]):

$$\psi(x) = -\int_{-\infty}^{x} e^{\frac{1}{2}\pi\xi}(1 - e^{\pi\xi})^{-1/2}\, d\xi = \frac{2}{\pi} \operatorname{arc\,sin} (e^{\frac{1}{2}\pi x}), \qquad (x < 0), \tag{4.51a}$$

$$\phi(x) = \{1 - \exp(-\pi x)\}^{-1/2}, \qquad (x > 0), \tag{4.51b}$$

$$\psi^*(x) = 1 - E[\sqrt{(-x)}], \qquad (x < 0), \tag{4.52a}$$

$$\phi^*(x) = (\pi x)^{-1/2} \exp(-x) + E[\sqrt{x}], \qquad (x > 0), \tag{4.52b}$$

where $\quad E(s) = 2\pi^{-1/2} \int_0^s e^{-t^2}\, dt, \qquad$ the error function.

If these expressions are compared it is found that $\phi(x)$, $\phi^*(x)$ both $\sim (\pi x)^{-1/2}$ as $x \to 0$, and they both tend to unity as $x \to \infty$. The difference in their values does not exceed 3% for any x. The expressions for $\psi(x)$, $\psi^*(x)$ both vary as $\{1 - 2(-x)^{1/2}\pi^{-1/2}\}$ as $x \to -0$ but as $x \to -\infty$, $\psi(x) \sim 2\pi^{-1} \exp(\frac{1}{2}\pi x)$, $\psi^*(x) \sim (-\pi x)^{-1/2} \exp(x)$. The numerical agreement is within 5% for $0 > x > -\frac{1}{2}$ but becomes progressively worse as $x \to -\infty$ (though then $\psi(x)$ and $\psi^*(x)$ are small).

It is of interest that we can find explicit formulae for the kernels of the corresponding integral equations:

$$k(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha^{-1}\tanh\alpha e^{-i\alpha x}\,d\alpha = \frac{1}{\pi}\ln\frac{1+\exp(-\frac{1}{2}\pi|x|)}{1-\exp(-\frac{1}{2}\pi|x|)},$$

$$k^*(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\alpha^2+1)^{-1/2}e^{-i\alpha x}\,d\alpha = \frac{1}{\pi}K_0(|x|).$$

As $|x| \to \infty, k(x) \sim 2\pi^{-1}\exp(-\frac{1}{2}\pi|x|)$, $k^*(x) \sim (2\pi|x|)^{-1/2}\exp(-|x|)$.

W. T. Koiter has suggested that in order to check and improve the accuracy of any solution obtained by replacing $K(\alpha)$ by $K^*(\alpha)$, a second approximate solution should be carried out, using

$$K^{**}(\alpha) = K^*(\alpha)\frac{\alpha^4 + C\alpha^2 + D}{\alpha^4 + E\alpha^2 + D}, \tag{4.53}$$

where C, D, E are chosen so that $K^{**}(\alpha)$ and $K(\alpha)$ agree as closely as possible. It is assumed that $K(\alpha)$ is an even function of α and that $K(\alpha)$, $K^*(\alpha)$ agree as $\alpha \to 0$ and $\alpha \to \infty$, so that this is the simplest rational function which can be used. The object of choosing the correction factor as a rational function is merely that it can be factorized easily, so that the split can still be performed in a straightforward way. By this means W. T. Koiter has reduced the difference between the solutions for $K(\alpha)$ and $K^{**}(\alpha)$ to less than 1%.

W. T. Koiter [1] has also made some applications of these ideas to the approximate factorization of functions which occur in problems involving the biharmonic equation, e.g.

$$K_1(\alpha) = \frac{\sinh\alpha\cosh\alpha - \alpha}{2\alpha\sinh^2\alpha} \quad : \quad K_2(\alpha) = \frac{\sinh^2\alpha - \alpha^2}{2\alpha(\sinh\alpha\cosh\alpha - \alpha)}.$$

These can be approximated by $A(\alpha^2 + B^2)^{-1/2}$ or $a\alpha^{-1}\tanh(b\alpha)$ to within 10%, and by more complicated expressions of the form (4.53) to within 1%. But for the above functions an exact decomposition is possible by means of the infinite product theory, though the roots involved are complex. It seems to me that in this case the crude approximations provide an excellent method for obtaining a rough answer. But if an accurate answer is required it is doubtful whether a refined approximation of type (4.53) is to be preferred to a straightforward though laborious solution based on the infinite product decomposition.

We consider next problem (b) stated at the beginning of this section. Suppose that we require a solution of the following equation as a function of ε for small ε:

$$[K^*(\alpha) + \varepsilon q(\alpha)]\Phi_+(\alpha) = \Psi_-(\alpha) + A(\alpha + k\cos\Theta)^{-1}, \tag{4.54}$$

where $k_2 \cos \Theta < \tau < k_2$, say, and the solution is known for $\varepsilon = 0$. We set $K^*(\alpha) = K^*_+(\alpha)K^*_-(\alpha)$, a known decomposition, divide (4.54) throughout by $K^*_-(\alpha)$ and use theorem B of §1.3 to obtain the additive decomposition of $q(\alpha)\Phi_+(\alpha)/K^*_-(\alpha)$. Then separation by the Wiener–Hopf technique gives

$$K^*_+(\alpha)\Phi^*_+(\alpha) + \frac{\varepsilon}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{q(\zeta)\Phi_+(\zeta)}{K^*_-(\zeta)(\zeta-\alpha)}\,d\zeta = \frac{A}{(\alpha + k\cos\Theta)K^*_+(k\cos\Theta)}.$$

This is an integral equation for $\Phi_+(\alpha)$ and holds for any τ_1, $-\infty < \sigma < \infty$, with $k_2 \cos \Theta < c < \tau_1 < k_2$. Iterative solution gives successive approximations:

$$\Phi^{(1)}_+(\alpha) = A\{(\alpha + k\cos\Theta)K^*_+(k\cos\Theta)K^*_+(\alpha)\}^{-1},$$

$$\Phi^{(2)}_+(\alpha) = \Phi^{(1)}_+(\alpha) - \frac{A\varepsilon}{2\pi i K^*_+(k\cos\Theta)K^*_+(\alpha)}\int_{ic-\infty}^{ic+\infty}\frac{q(\zeta)\,d\zeta}{K^*(\zeta)(\zeta-\alpha)(\zeta+k\cos\Theta)}. \tag{4.55}$$

We examine two concrete examples

(i) $K^*(\alpha) = (\alpha^2 - k^2)^{1/2}$, $q(\alpha) = 1$. Then the integral in (4.55) is convergent and the iterative procedure is presumably satisfactory.

(ii) $K^*(\alpha) = (\alpha^2 - k^2)^{-1/2}$, $q(\alpha) = 1$. The integrand in (4.55) behaves as $|\alpha|^{-1}$ as $|\alpha| \to \infty$ in the strip and the integral is divergent. In this case the iterative solution is *not* satisfactory.

Some light is thrown on the difficulty which has arisen in this second case, by the following remarks:

(A). The exact decomposition for the above examples has been investigated in ex. 2.10. On examining case (ii) it appears that

$$K_+(\alpha) = K^*_+(\alpha) + \varepsilon \ln \varepsilon\, S_+(\alpha) + \ldots, \tag{4.56}$$

i.e. the first correction term is of order $\varepsilon \ln \varepsilon$, not of order ε.

(B). If we apply the product decomposition theorem C of §1.3 to (4.50) we have

$$\ln K_+(\alpha) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\ln[K^*(\zeta) + \varepsilon q(\zeta)]}{\zeta - \alpha}\,d\zeta.$$

In example (ii) above for finite ε, large ζ, the term $\varepsilon q(\zeta)$ is more important than $K^*_+(\zeta)$ and this suggests the presence of terms in $\ln \varepsilon$.

(C). Suppose that we can write

$$\begin{aligned} K(\alpha) &= K_+^*(\alpha)\{1 + \delta s_+(\alpha)\}K_-^*(\alpha)\{1 + \delta s_-(\alpha)\} \\ &= K^*(\alpha)\{1 + \delta(s_+(\alpha) + s_-(\alpha)) + \delta^2 s_+(\alpha)s_-(\alpha)\}, \end{aligned} \tag{4.57}$$

where δ is small. Then on comparing with (4.50) it is natural to identify δ with ε, neglect δ^2 in (4.57) and find $s_+(\alpha)$, $s_-(\alpha)$ such that

$$s_+(\alpha) + s_-(\alpha) = q(\alpha)/K^*(\alpha).$$

This can be done explicitly for both the above examples by means of (1.35). For case (ii) we find

$$K_+(\alpha) \approx (\alpha + k)^{-1/2}\{1 + \varepsilon\pi^{-1}(\alpha^2 - k^2)^{1/2} \operatorname{arc\,cos}(\alpha/k)\}. \tag{4.58}$$

But this result is wrong since the leading correction term should be of order $\varepsilon \ln \varepsilon$. We cannot assume that if $K(\alpha)$ can be expanded in a power series in ε then so can $K_+(\alpha)$, $K_-(\alpha)$. If we multiply (4.58) by the corresponding expression for $K_-(\alpha)$ we find that for a given finite ε the result does not approximate to $K(\alpha)$ as $|\alpha| \to \infty$ in the strip. This is in contrast to the examples considered in the first half of this section where the approximate $K^*(\alpha)$ agreed with the exact $K(\alpha)$ over the whole range of α in the strip.

The question of finding a valid second approximation by means of relatively simple formulae in problems involving $K(\alpha)$ of the form given in (ii) above would seem to merit further investigation. Several of the $K(\alpha)$ which have appeared in previous chapters are of this type.

4.6 Laplace's equation in polar co-ordinates

Various problems involving Laplace's equation in two-dimensional polar and spherical polar co-ordinates can be solved by the Wiener–Hopf technique in conjunction with the Mellin transform:

$$\Phi(s) = \int_0^\infty \phi r^{s-1}\,dr \quad : \quad \phi = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \Phi(s) r^{-s}\,ds, \tag{4.59}$$

where s is a complex variable, $s = \sigma + i\tau$. The basic forms of Laplace's equation are

$$r\frac{\partial}{\partial r} r\frac{\partial \phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} = 0, \quad \text{(two-dimensional polars)}, \tag{4.60a}$$

$$\frac{\partial}{\partial r} r^2 \frac{\partial \phi}{\partial r} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi}{\partial\theta} = 0, \quad \text{(spherical polars)}. \tag{4.60b}$$

From one point of view there is no need to use the Mellin transform at all. The substitution $r = \exp(u)$ converts the above equations into

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial\theta^2} = 0,$$

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial\phi}{\partial u} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\phi}{\partial\theta} = 0.$$

These can be solved by a Fourier or Laplace transform in u. (This corresponds to the fact that the Mellin and Laplace transforms are related by an exponential substitution, cf. (1.49).) However in practice it is convenient to use the Mellin transform and we proceed to show how this can be done.

We write

$$\Phi_P(s,\theta) = \int_0^1 \phi(r,\theta)r^{s-1}\,dr \quad : \quad \Phi_N(s,\theta) = \int_1^\infty \phi(r,\theta)r^{s-1}\,dr, \tag{4.61}$$

where subscripts 'P', 'N' are used instead of '+' '—' to remind the reader that Φ_P, Φ_N are regular in right- and left-hand half-planes instead of upper and lower half-planes as for the Fourier transform. We need the following results derived from example (2) following theorem A in §1.3, and from the Abelian theorem quoted at the end of §1.6 ($\ln(r) \approx (r-1)$ when $r \approx 1$):

If $|\phi| < Ar^{-\sigma_-}$ as $r \to 0$, $\Phi_P(s)$ is regular in $\sigma > \sigma_-$.
If $|\phi| < Br^{-\sigma_+}$ as $r \to \infty$, $\Phi_N(s)$ is regular in $\sigma < \sigma_+$.
If $\phi \sim (1-r)^\xi$ as $r \to 1-0$, then $|\Phi_P(s)| \sim |s|^{-\xi-1}$ as $s \to \infty$.
If $\phi \sim (r-1)^\eta$ as $r \to 1+0$, then $|\Phi_N(s)| \sim |s|^{-\eta-1}$ as $s \to \infty$.

In the last two lines $s \to \infty$ in right and left half-planes respectively.

Consider a concrete example. Suppose that there is a cone at uniform potential V in $0 \leqslant r \leqslant 1$, $\theta = \Theta$. Application of a Mellin transform to (4.60b) gives

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d\Phi(s)}{d\theta} + s(s-1)\Phi(s) = 0,$$

$$\Phi(s) = A(s)P_{s-1}(\cos\theta) + B(s)P_{s-1}(-\cos\theta).$$

The functions $P_{s-1}(\cos\theta)$, $P_{s-1}(-\cos\theta)$ are singular at $\theta = \pi$, 0, respectively. Hence we write

$$\Phi(s) = A(s)P_{s-1}(\cos\theta)\quad , \qquad 0 \leqslant \theta \leqslant \Theta, \tag{4.62a}$$

$$= B(s)P_{s-1}(-\cos\theta), \qquad \Theta \leqslant \theta \leqslant \pi. \tag{4.62b}$$

We apply boundary conditions on $\theta = \Theta$. Use notation (4.61) and denote the corresponding transforms of $\partial\phi/\partial\theta$ by $\Phi'_P(s,\theta)$, $\Phi'_N(s,\theta)$. Let $\Phi'_P(s, \Theta + 0)$ denote the value of $\Phi'_P(s,\theta)$ as $\theta \to \Theta$ through values of θ greater than Θ, etc. $\Phi(s)$ is continuous on $\theta = \Theta$ and from (4.62) we write

$$A(s)P_{s-1}(\cos\Theta) = B(s)P_{s-1}(-\cos\Theta) = C(s), \quad \text{say.}$$

Then

$$\Phi_P(s,\Theta) + \Phi_N(s,\Theta) = C(s), \tag{4.63}$$

$$\Phi'_P(s, \Theta + 0) + \Phi'_N(s,\Theta) = C(s)P'_{s-1}(-\cos\Theta)/P_{s-1}(-\cos\Theta), \tag{4.64a}$$

$$\Phi'_P(s, \Theta - 0) + \Phi'_N(s,\Theta) = C(s)P'_{s-1}(\cos\Theta)/P_{s-1}(\cos\Theta). \tag{4.64b}$$

These can be reduced by the standard procedure of §2.2 to the Wiener–Hopf equation

$$-F_P(s,\Theta) = K(s)\{\Phi_N(s,\Theta) + V/s\}, \tag{4.65}$$

where

$$K(s) = \sin\pi s\{\pi \sin\Theta P_{s-1}(\cos\Theta)P_{s-1}(-\cos\Theta)\}^{-1}, \tag{4.66}$$

$$2F_P(s,\Theta) = \Phi'_P(s, \Theta + 0) - \Phi'_P(s, \Theta - 0),$$

and we have used the results

$$P_{s-1}(-\cos\Theta)P'_{s-1}(\cos\Theta) - P_{s-1}(\cos\Theta)P'_{s-1}(-\cos\Theta) = 2\sin\pi s/\pi\sin\Theta,$$

$$\Phi_P(s,\Theta) = V\int_0^1 r^{s-1}\,dr \doteq V/s.$$

$K(\alpha)$ is an integral function of s and can be decomposed by the infinite product theory. Details are given in S. N. Karp [2]. The solution can then be completed in the usual way. A similar example is discussed in J. Bazer and S. N. Karp [1].

To illustrate the procedure we consider the special case of a disk, $0 \leqslant r \leqslant 1$, $\Theta = \pi/2$. It is known that

$$P_{s-1}(0) = \pi^{1/2}\{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}s)\Gamma(1 - \tfrac{1}{2}s)\}^{-1}.$$

Then (4.66) can be reduced to

$$K(s) = 2\Gamma(\tfrac{1}{2} + \tfrac{1}{2}s)\Gamma(1 - \tfrac{1}{2}s)\{\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s)\}^{-1}. \tag{4.67}$$

We have the additional information:

(i) As $r \to 0$ the charge density on the disk is finite. Hence $F_P(s,0)$ is regular for $\sigma > -1$.

(ii) $\Phi_P(s,0)$ exists and is regular for $\sigma > 0$.

(iii) As $r \to \infty$, $|\phi| < Cr^{-1}$. Hence $\Phi_N(s,0)$ is regular for $\sigma < 1$.

(iv) As $r \to 1 + 0$, $\phi \sim$ const. Hence $|\Phi_N(s,0)| \sim |s|^{-1}$ as $s \to \infty$ in a left half-plane.

From (i)–(iii) and (4.67) we conclude that (4.65) holds in the strip $0 < \sigma < 1$. The required factorization of (4.67) is then $K(s) = K_P(s)K_N(s)$ where

$$K_P(s) = 2^{\frac{1}{2}}\Gamma(\tfrac{1}{2} + \tfrac{1}{2}s)/\Gamma(\tfrac{1}{2}s) \quad : \quad K_N(s) = 2^{\frac{1}{2}}\Gamma(1 - \tfrac{1}{2}s)/\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s).$$

On applying the Wiener–Hopf technique, (4.65) gives the solution

$$\Phi_N(s,0) = \frac{V}{s}\left\{\frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{\pi^{1/2}\Gamma(1 - \frac{1}{2}s)} - 1\right\}.$$

Hence

$$\phi = \frac{V}{2\pi i \pi^{1/2}}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{1}{2} - \frac{1}{2}s)}{s\Gamma(1 - \frac{1}{2}s)} \cdot \frac{P_{s-1}(|\cos\theta|)}{P_{s-1}(0)} r^{-s}\, ds.$$

To obtain the charge distribution on the disk we calculate $r^{-1}(\partial\phi/\partial\theta)$ on $\theta = +0$. On using the result

$$P'_{s-1}(0) = -2\pi^{\frac{1}{2}}\{\Gamma(\tfrac{1}{2}s)\Gamma(\tfrac{1}{2} - \tfrac{1}{2}s)\}^{-1},$$

we find

$$\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right)_{\theta=\frac{1}{2}\pi-0} = \frac{V}{2\pi i \pi^{1/2} r}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}s)}{\Gamma(1 + \frac{1}{2}s)} r^{-s}\, ds = \frac{2}{\pi}(1 - r^2)^{-1/2},$$

Further details concerning application of the Wiener–Hopf technique in connection with Laplace's equation in polar co-ordinates will be found in J. Bazer and S. N. Karp [1], S. N. Karp [1], [2], and D. S. Jones [7], [8].

Miscellaneous Examples and Results IV

4.1 In this book we have been concerned with solution of partial differential equations and in general it has not been necessary to formulate problems in terms of integral equations. However a great deal of the literature is concerned with integral equations and for reference we quote the classic example considered by N. Wiener and E. Hopf [1]. This is Milne's integral equation which occurs in connexion with radiation and neutron transport problems:

$$f(x) = \int_0^\infty f(\xi)k(x - \xi)\, d\xi, \quad (x > 0) \quad : \quad k(\xi) = \tfrac{1}{2}\int_{|\xi|}^\infty \lambda^{-1}e^{-\lambda}\, d\lambda. \tag{a}$$

If we suppose that the right-hand side of this equation defines a function $g(x)$ for $x < 0$, application of a double-sided Laplace transform gives

$$F_P(s)K(s) = G_N(s), \tag{b}$$

$$K(s) = 1 - \int_{-\infty}^{\infty} k(x)e^{-sx}\,dx = 1 - \tfrac{1}{2}s^{-1}\ln\frac{1+s}{1-s}.$$

$K(s)$ has a double zero at the origin and $\ln K(s)$ has branch points at $s = \pm 1$. Equation (a) is usually deduced from

$$\mu\,\partial\psi/\partial z + \psi = \tfrac{1}{2}\psi_0, \qquad \psi = \psi(z,\mu), \tag{c}$$

$$\psi_0 = \int_{-1}^{+1} \psi(z,\mu)\,d\mu \equiv f(z) \quad \text{in (a) above.}$$

The Wiener–Hopf technique is often applied to the following equation which can also be obtained from (c):

$$F_P(s)K(s) = -\tfrac{1}{2}\int_{-1}^{0} \frac{F_P(-1/\zeta)}{1+s\zeta}\,d\zeta. \tag{d}$$

On comparing (d) and (b) it is clear that the right-hand side of (d) is a different representation of $G_N(s)$. In fact (d) can be obtained by applying a Laplace transform in $(0,\infty)$ to (a). This gives

$$F(s)K(s) = \int_0^{\infty} f(\xi)\int_{\xi}^{\infty} e^{sy}k(y)\,dy\,d\xi.$$

It is left to the reader to show that the right-hand side of this expression can be reduced to the right-hand side of (d).

There is an extensive literature, e.g. N. Wiener and E. Hopf [1], E. Hopf [1], G. Placzek [1], G. Placzek and W. Seidel [1], C. Mark [1], R. E. Marshak [1]. Brief treatments are given in E. C. Titchmarsh [1], I. N. Sneddon [1]. Two books which deal with other methods of solution of the equation in addition to the Wiener–Hopf method are V. Kourganoff [1], and B. Davison [1].

4.2 We summarize briefly the conditions for uniqueness of solution of the equation

$$f(x) = \int_0^{\infty} k(|x-y|)f(y)\,dy + g(x), \qquad (0 \leqslant x < \infty),$$

where f is unknown, g and k are known. For the homogeneous equation, i.e. $g(x) = 0$, this has been examined by N. Wiener and E. Hopf [1]. (See E. C. Titchmarsh [1], p. 340.) For the non-homogeneous case the theory has been extended by V. Fock [1], [2]. In the following remarks integrals are to be understood in the Lebesgue sense. Suppose that $k_1(x) = k(x) \exp(cx)$ is of limited variation and absolutely integrable in $(0, \infty)$ for some $c > 0$. Define

$$K(\alpha) = \int_{-\infty}^{\infty} e^{i\alpha x} k(|x|)\, dx,$$

which is even and regular in the strip $-c < \operatorname{Im} \alpha < +c$; $|\alpha K(\alpha)|$ is bounded at infinity in the strip; $\ln \{1 - K(\alpha)\}$ tends to zero as $|\alpha|^{-1}$ at infinity in the strip.

(i) *The homogeneous equation*, i.e. $g(x) = 0$.

(a) If $\{1 - K(\alpha)\} = 0$ has no real roots or singular points and if τ_0 is the imaginary part of the complex root or singular point nearest to the real axis, then the homogeneous equation has no solutions satisfying the inequality

$$|f(x)| < C \exp(\tau_0 x), \tag{A}$$

where C is a constant.

(b) If $\{1 - K(\alpha)\} = 0$ has $2n$ real roots, the homogeneous equation has exactly n solutions, say $f_r(x)$, which are linearly independent and satisfy equation (A).

(ii) *The non-homogeneous equation*, i.e. $g(x) \neq 0$, where $g(x)$ is absolutely integrable in $(0, \infty)$ and of limited variation.

(a) If $\{1 - K(\alpha)\} = 0$ has no real roots there exists exactly one solution which is bounded and tends to zero at infinity.

(b) If $\{1 - K(\alpha)\} = 0$ has $2n$ roots whose multiplicity does not exceed s, and if $g(x)$ satisfies two additional conditions, namely $x^{s-1} g(x) \ln x$ is absolutely integrable and

$$\int_0^{\infty} g(x) f_r(x)\, dx = 0, \qquad (r = 1 \text{ to } n),$$

where the $f_r(x)$ have been defined in (i)(b), then the non-homogeneous equation has exactly one solution which tends to zero at infinity.

4.3 When solving integral equations by the usual Wiener–Hopf method it is assumed that there exists a common strip of regularity for all transforms in the complex plane. It is possible to extend the theory to the case where Fourier transforms exist only on a common line parallel to the real axis in the complex plane by reducing the problem to a Hilbert problem. The normal Wiener–Hopf theory applies when the kernel $k(x)$ of the integral equation decays exponentially, $|k(x)| < C \exp(-\varepsilon|x|)$, $\varepsilon > 0$, as $|x| \to \infty$. The case considered in this example occurs when $k(x)$ decreases algebraically, $|k(x)| < C|x|^{-p}$, $p > 0$, as $|x| \to \infty$. The following is based on J. A. Sparenberg [1], [2]. Consider

$$f(x) = \mu \int_0^{\infty} \frac{f(\xi)}{a^2 + (x - \xi)^2}\, d\xi + g(x), \qquad x > 0, \qquad \pi\mu < a. \tag{a}$$

Introduce a function $h(x)$, $x < 0$, defined by the right-hand side of this equation for $x < 0$. $f(x) = g(x) = 0$ for $x < 0$. A Fourier transform in $-\infty < x < \infty$ gives

$$F_+(\alpha)K(\alpha) + H_-(\alpha) = G_+(\alpha), \tag{b}$$

$$K(\alpha) = 1 - \mu\pi a^{-1} \exp(-a|\alpha|).$$

This holds only for real α since the Fourier transform of the kernel holds for only real α. We can treat this as a Hilbert problem. Define

$$\ln K_+(\alpha) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\ln K(x)}{x - \alpha}\, dx, \qquad \operatorname{Im} \alpha > 0, \tag{c}$$

with a suitable indentation of the contour if $\operatorname{Im} \alpha \to 0$. Define $K_-(\alpha)$ by a similar formula with a minus sign and $\operatorname{Im} \alpha \leqslant 0$. Then

$$F_+(\alpha) = \frac{1}{2\pi i K_+(\alpha)} \int_{-\infty}^{\infty} \frac{G_+(x)}{K_-(x)(x - \alpha)}\, dx.$$

Complications arise if $\pi\mu > a$ since then $K(\alpha)$ has two zeros on the real axis, say $x = \pm p$. This suggests that $f(x)$ is finite and oscillatory as $x \to +\infty$ due to the presence of terms of the form $\exp(\pm ipx)$. Following this hint we introduce

$$v(x) = f(x) - Ae^{-ipx} - Be^{ipx}, \qquad (x > 0),$$

where A, B are constants which will be determined by the condition that $v(x) \to 0$ as $x \to +\infty$. Introduce into (a). For simplicity we consider the homogeneous case, $g(x) = 0$. Then for $x > 0$,

$$v(x) + Ae^{-ipx} + Be^{ipx} = \mu \int_0^{\infty} \frac{v(\xi)\, d\xi}{a^2 + (x - \xi)^2} + \mu \int_0^{\infty} \frac{Ae^{-ip\xi} + Be^{ip\xi}}{a^2 + (x - \xi)^2}\, d\xi.$$

The terms in A, B cancel as $x \to +\infty$. Suppose that the right-hand side of this equation defines a function $h(x)$ for $x < 0$. A Fourier transform gives

$$V_+(\alpha)K(\alpha) + H_-(\alpha) = -\frac{i}{(2\pi)^{1/2}} \left\{\frac{A}{\alpha - p} + \frac{B}{\alpha + p}\right\} K(\alpha).$$

In order to avoid difficulties from the zeros of $K(\alpha)$ we define

$$L(\alpha) = \frac{K(\alpha)}{\alpha^2 - p^2} \quad : \quad \ln L_+(\alpha) = \frac{\alpha}{\pi i} \int_0^{\infty} \frac{\ln L(x)}{x^2 - \alpha^2}\, dx, \qquad \operatorname{Im} \alpha > 0.$$

Suppose that the arbitrary separation polynomial is a constant C. Then separation gives

$$V_+(\alpha) = \frac{C}{(\alpha^2 - p^2)L_+(\alpha)} - \frac{Ai}{(2\pi)^{1/2}(\alpha - p)} - \frac{Bi}{(2\pi)^{1/2}(\alpha + p)}.$$

In order that $V_+(\alpha)$ should have no poles at $\alpha = \pm p$ we need to choose

$$A = -\frac{(2\pi)^{1/2}Ci}{2pL_+(p)} \quad : \quad B = \frac{(2\pi)^{1/2}Ci}{2pL_+(-p)}.$$

The solution is now determined.

4.4 J. A. Sparenberg [2] has considered integral equations of the following form

$$a_0 f(x) + a_1 f'(x) = \int_0^\infty \{b_0 f(\xi) + b_1 f'(\xi)\}k(x - \xi)\, d\xi.$$

4.5 Show that

$$\phi(\xi) = \tfrac{1}{2}\lambda \int_0^\infty \frac{\phi(\eta)}{\cosh \frac{1}{2}(\xi - \eta)}\, d\eta, \qquad (\xi > 0),$$

can be reduced to the Wiener–Hopf equation

$$\Phi_+(\alpha) + \Psi_-(\alpha) = \lambda\pi\{\cosh \pi\alpha\}^{-1}\Phi_+(\alpha).$$

The factorization required for the solution of this equation can be performed in terms of gamma functions. For details see A. E. Heins [5], correcting E. C. Titchmarsh [1], p. 343.

4.6 Show that integral equations of the form

$$f(x) + g(x) = \lambda \int_0^1 \frac{f(\xi)}{x + \xi}\, d\xi, \qquad (0 \leqslant x \leqslant 1),$$

or

$$f(x) + g(x) = \lambda \int_1^\infty \frac{f(\xi)}{x + \xi}\, d\xi, \qquad (1 \leqslant x < \infty),$$

are of the Wiener–Hopf type. (Set $x = \exp(-y)$, $\xi = \exp(-\eta)$. Cf. ex. 4.5. See E. C. Titchmarsh [1], p. 342.)

4.7 E. C. Titchmarsh [1], p. 335, has solved a particular pair of dual integral equations by a method related to various topics in this book. We use the method to solve the general equations

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ic}^{\infty+ic} K(\alpha)A(\alpha)e^{-i\alpha x}\, d\alpha = f(x), \qquad x > 0, \tag{a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{-\infty+ic}^{\infty+ic} A(\alpha)e^{-i\alpha x}\, d\alpha = g(x), \qquad x < 0, \tag{b}$$

where c is a real number, $\tau_- < c < \tau_+$. $A(\alpha)$ is regular in this strip and $K(\alpha)$ is regular and non-zero in the strip. Suppose that we can write $K(\alpha) = K_-(\alpha)/K_+(\alpha)$ in the usual way, and introduce $B(\alpha) = A(\alpha)/K_+(\alpha)$. Multiply (a) by $\exp(i\omega_1 x)$, $\operatorname{Im}\omega_1 > c$, and integrate with respect to x from 0 to ∞. Similarly multiply (b) by $\exp(i\omega_2 x)$, $\operatorname{Im}\omega_2 < c$, and integrate with respect to x from $-\infty$ to 0. The equations become

$$-\frac{i}{(2\pi)^{1/2}}\int_{-\infty+ic}^{\infty+ic} K_-(\alpha)B(\alpha)\frac{d\alpha}{\alpha-\omega_1} = \int_0^\infty e^{i\omega_1 x}f(x)\,dx = F_+(\omega_1), \quad \text{say,} \quad \text{(c)}$$

$$\frac{i}{(2\pi)^{1/2}}\int_{-\infty+ic}^{\infty+ic} K_+(\alpha)B(\alpha)\frac{d\alpha}{\alpha-\omega_2} = \int_{-\infty}^0 e^{i\omega_2 x}g(x)\,dx = G_-(\omega_2), \quad \text{say,} \quad \text{(d)}$$

Now choose some constant b such that $c < \operatorname{Im}\omega_1 < b < \tau_+$ and move the line of integration in (c) to $\tau = b$. This gives

$$-\frac{i}{(2\pi)^{1/2}}\int_{-\infty+ib}^{\infty+ib} K_-(\alpha)B(\alpha)\frac{d\alpha}{\alpha-\omega_1} = -(2\pi)^{1/2}K_-(\omega_1)B(\omega_1) + F_+(\omega_1). \quad \text{(e)}$$

The function on the left-hand side is regular in the lower half-plane $\operatorname{Im}\omega_1 < b$. Similarly choose a so that $\tau_- < a < \operatorname{Im}\omega_2 < c$, and shift the contour in (d) to $\tau = a$. Then

$$\frac{i}{(2\pi)^{1/2}}\int_{-\infty+ia}^{\infty+ia} K_+(\alpha)B(\alpha)\frac{d\alpha}{\alpha-\omega_2} = -(2\pi)^{1/2}K_+(\omega_2)B(\omega_2) + G_-(\omega_2). \quad \text{(f)}$$

The left-hand side and therefore the right-hand side is regular in the upper half-plane $\operatorname{Im}\omega_2 > a$.

From the point of view adopted in this book the most direct method of solution at this point would be to replace ω_1 and ω_2 by α, denote the left-hand side of (e) by $\Psi_-(\alpha)$, the left-hand side of (f) by $\Phi_+(\alpha)$, and then eliminate $B(\alpha)$ to obtain a Wiener–Hopf problem. Titchmarsh proceeds as follows: Since the function on the right in (f) is regular in the upper half-plane $\tau > \tau_-$, so is

$$B(\alpha) - (2\pi)^{-1/2}\{K_+(\alpha)\}^{-1}G_-(\alpha).$$

Hence

$$\int_{-\infty+ic}^{\infty+ic}\left\{B(\alpha) - \frac{G_-(\alpha)}{(2\pi)^{1/2}K_+(\alpha)}\right\}\frac{d\alpha}{\alpha-\omega} = 0, \qquad (\operatorname{Im}\omega < c). \qquad \text{(g)}$$

Similarly from (e)

$$\int_{-\infty+ic}^{\infty+ic}\left\{B(\alpha) - \frac{F_+(\alpha)}{(2\pi)^{1/2}K_-(\alpha)}\right\}\frac{d\alpha}{\alpha-\omega}$$

$$= -2\pi i\left\{B(\omega) - \frac{F_+(\omega)}{(2\pi)^{1/2}K_-(\omega)}\right\}, \qquad (\operatorname{Im}\omega < c). \qquad \text{(h)}$$

In this case there is exactly one pole, at $\alpha = \omega$, when the contour is completed in the lower half-plane. The function $B(\omega)$ is now obtained by eliminating the integral involving $B(\alpha)$ between (g) and (h). This is the required solution, and it can be reduced to the general solution in §6.2 (cf. ex. 6.1). In practice it may not be possible to carry out the procedure exactly as described because of convergence difficulties but artifices to deal with these can be invented along the lines indicated in §6.2.

4.8 The method of §4.6 has been used to solve the problem of steady flow of incompressible fluid past a thin aerofoil by D. S. Jones [7]. The method has been extended to deal with flow past an oscillating aerofoil in D. S. Jones [8]. An interesting feature is the way in which the Kutta–Joukowsky condition is applied at the trailing edge.

4.9 The method of §4.6 can be used to solve problems involving stresses in a wedge of elastic material when mixed boundary value conditions are specified on the faces of the wedge, cf. W. T. Koiter [2], and ex. 3.19.

4.10 The Wiener–Hopf technique can be used to solve the set of simultaneous linear algebraic equations:

$$\sum_{m=0}^{\infty} k_{n-m} x_m = b_n, \qquad (n = 0, 1, 2 \ldots), \tag{a}$$

where the known coefficients k_{n-m} are functions of only the difference $(n - m)$. We extend the equations by writing the left-hand side equal to the (unknown) constants c_n for $n = -1, -2, \ldots$. Multiply the equation for any given n by α^n and sum over n from $-\infty$ to $+\infty$. Interchange orders of summation on the left-hand side. It is found that

$$\left\{\sum_{r=-\infty}^{\infty} k_r \alpha^r\right\}\left\{\sum_{m=0}^{\infty} x_m \alpha^m\right\} = \sum_{n=0}^{\infty} b_n \alpha^n + \sum_{n=-\infty}^{-1} c_n \alpha^n,$$

or

$$K(\alpha) X_+(\alpha) = B_+(\alpha) + C_-(\alpha), \tag{b}$$

where the subscripts indicate that $B_+(\alpha)$, $X_+(\alpha)$ are regular inside some circle of convergence, whereas $C_-(\alpha)$ is regular outside a certain circle in the α-plane. Assume that $K(\alpha)$ is a regular function of α in $r_+ < |\alpha| < r_-$, with 'plus' functions regular in $|\alpha| < r_-$, 'minus' functions regular in $|\alpha| > r_+$. Then (b) holds in a strip in the complex plane and the Wiener–Hopf technique can be applied. We write $K(\alpha) = K_+(\alpha)K_-(\alpha)$. Divide (b) by $K_-(\alpha)$ and write

$$B_+(\alpha)/K_-(\alpha) = H_+(\alpha) + H_-(\alpha).$$

Then (b) can be written

$$J(\alpha) = K_+(\alpha)X_+(\alpha) - H_+(\alpha) = C_-(\alpha)/K_-(\alpha) + H_-(\alpha).$$

This defines a function which is regular in the whole of the α-plane and $X_+(\alpha)$ can be determined by Liouville's theorem in the usual way. The x_n can be found by expanding $X_+(\alpha)$ as a power series in α.

This solution has been obtained in a slightly different way by A. N. Fel'd [1]. As an application he considers the propagation of a wave in a waveguide $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$, $-\infty < z < \infty$, with cylindrical posts of (small) radius r, the centres lying in $y = d$, $z = 0, D, 2D, \ldots,$ each post extending over $0 \leqslant x \leqslant a$. (The solution of this example is not such a feat as might appear at first sight since once it is realized that the solution is of the form $x_m = Pp^m$ where P, p are constants, the values of P, p can be obtained by direct substitution in the original simultaneous equations. Nevertheless the example provides an instructive application of the theory.)

4.11 Some of the problems we have solved by means of the Wiener–Hopf technique can be formulated in terms of simultaneous linear algebraic equations. In such cases the linear equation formulation is often the more natural procedure. We first of all reduce a typical Wiener–Hopf problem to a set of linear simultaneous algebraic equations. Consider

$$F_-(\alpha) = K(\alpha)G_+(\alpha) + H(\alpha), \tag{a}$$

where this equation holds in $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, and $K(\alpha)$ is regular and non-zero in the strip. It is necessary to assume that $K(\alpha)$ has no branch points in the upper half-plane. Suppose that $K(\alpha)$ has simple poles $\alpha_1, \alpha_2, \ldots$ and simple zeros $\beta_1, \beta_2, \ldots$ in the upper half-plane and that these are interlaced:

$$|\alpha_1| < |\beta_1| < |\alpha_2| < |\beta_2| < \ldots$$

Suppose that $K(\alpha)$ can be expanded in the form

$$K(\alpha) = k_+(\alpha) + \sum_{n=1}^{\infty} a_n \left\{\frac{1}{\alpha - \alpha_n} + \frac{1}{\alpha_n}\right\}. \tag{b}$$

Substitute in (a), set $H(\alpha) = H_+(\alpha) + H_-(\alpha)$, and rearrange in the form

$$F_-(\alpha) - \sum_{n=1}^{\infty} \frac{a_n G_+(\alpha_n)}{\alpha - \alpha_n} - H_-(\alpha)$$

$$= k_+(\alpha)G_+(\alpha) + \sum_{n=1}^{\infty} a_n \left\{(G_+(\alpha) - G_+(\alpha_n))\frac{1}{\alpha - \alpha_n} + G_+(\alpha)\frac{1}{\alpha_n}\right\} + H_+(\alpha).$$

This defines an integral function which we assume to be zero. Then

$$\sum_{n=1}^{\infty} a_n G_+(\alpha_n)\frac{1}{\alpha - \alpha_n} = F_-(\alpha) - H_-(\alpha). \tag{c}$$

If we set $\alpha = \beta_n$ in (a), by definition $K(\beta_n) = 0$ and $G_+(\beta_n)$ is finite since β_n is in an upper half-plane. Then

$$F_-(\beta_n) = H(\beta_n), \quad \text{i.e.} \quad F_-(\beta_n) - H_-(\beta_n) = H_+(\beta_n).$$

On setting $\alpha = \beta_m$ in (c) we obtain a set of linear simultaneous algebraic equations for $G_+(\alpha_n)$:

$$\sum_{n=1}^{\infty} \frac{1}{\beta_m - \alpha_n} a_n G_+(\alpha_n) = H_+(\beta_m), \qquad (m = 1, 2, 3, \ldots).$$

The procedure can obviously be worked backwards. To solve

$$\sum_{n=1}^{\infty} \frac{1}{\beta_m - \alpha_n} x_n = c_m, \qquad (m = 1, 2, 3, \ldots), \tag{d}$$

we need to find a function $K(\alpha)$ which can be written in the form (b) and such that $K(\beta_m) = 0$. Then if $H(\alpha)$ is a function such that $H(\beta_m) = c_m$ and we can solve the Wiener–Hopf problem (a), the solution of (d) is given by $x_n = a_n G_+(\alpha_n)$.

4.12 Consider a wave $\exp(ikz)$ incident from $z = -\infty$ in a duct $0 \leqslant y \leqslant b$, $-\infty < z < \infty$ with $\partial\phi_t/\partial y = 0$ on $y = 0$, $-\infty < z < \infty$, and on $y = b$, $-\infty < z < 0$, and $\phi_t = 0$ on $y = b$, $0 < z < \infty$. If we use an eigenfunction expansion in $\cos(n\pi y/b)$ in $z \leqslant 0$, and in $\cos(n - \frac{1}{2})\pi y/b$ in $z \geqslant 0$, and equate ϕ_t and $\partial\phi_t/\partial z$ across $z = 0$, $0 \leqslant y \leqslant b$, we find

$$1 + A_0 + \sum_{n=1}^{\infty} A_n \cos n\pi y/b = \sum_{n=1}^{\infty} B_n \cos(n - \tfrac{1}{2})\pi y/b,$$

$$ik(1 - A_0) + \sum_{n=1}^{\infty} \gamma_n A_n \cos n\pi y/b = -\sum_{n=1}^{\infty} \gamma_{n-\frac{1}{2}} B_n \cos(n - \tfrac{1}{2})\pi y/b,$$

where A_n, B_n are unknown constants and $\gamma_p = \{(p\pi/b)^2 - k^2\}^{1/2}$. By eliminating say A_n show that the equations for B_n can be written in the form (d) of ex. 4.11. Hence find the B_n. ($\alpha_n = i\gamma_{n-\frac{1}{2}}$, $n = 1, 2, \ldots$: $\beta_1 = k$, $\beta_m = i\gamma_{m-1}$, $m = 2, 3, \ldots$. $K(\alpha) = \gamma \tanh \gamma b$, $H(\alpha) = 4kb\gamma^{-1} \tanh \gamma b$.) The simultaneous equations are a generalization of a set considered by E. H. Linfoot and W. M. Shepherd, *Quart. J. Math.* (Oxford) **10** (1939), 84–98. Cf. W. Magnus and F. Oberhettinger, *Comm. Pure and Applied Math.* **3** (1950), 393–410.)

An equivalent problem is that of a strip across a duct and we can use the above method to find the solution (3.99), (3.100) of §3.6.

4.13 It has been shown by W. Magnus, *Z. Phys.* **177** (1941), 168–179, that the integral equation

$$\int_0^{\infty} H_0^{(1)}(k|x - \xi|) f(\xi)\, d\xi = g(x), \qquad (0 < x < \infty), \tag{a}$$

can be solved by setting

$$\begin{aligned} g(x) &= \sum_{m=0}^{\infty} (-i)^m J_m(kx) a_m, \\ f(x) &= \tfrac{1}{2}\pi e^{i\pi/4} \sum_{m=0}^{\infty} (-i)^m (2m + 1) x^{-1} J_{m+\frac{1}{2}}(kx) C_m. \end{aligned} \tag{b}$$

The unknown C_m are related to the known a_m by

$$\varepsilon_n \sum_{m=0}^{\infty} C_m \left\{ \frac{1}{-n+m+\frac{1}{2}} + \frac{1}{n+m+\frac{1}{2}} \right\} = a_n, \quad \varepsilon_0 = 1, \quad \varepsilon_n = 2(n \geqslant 1). \tag{c}$$

We can deduce (c) and the solution of (c) from (a), (b) by a method related to the Wiener–Hopf technique. Application of a Fourier transform to (a) gives

$$-2i\gamma^{-1}F_+(\alpha) = G_+(\alpha) + H_-(\alpha), \tag{d}$$

where $F_+(\alpha)$, $H_-(\alpha)$ are unknown. Introduce (cf. §1.6) $\alpha = -k\cos\beta$, $\gamma = -ik\sin\beta$. Write $G_+(\beta)$ in place of $G_+(\alpha)$ etc. Since $H_-(\beta)$ has no singularity at $\beta = 0$ ($\alpha = -k$) an examination of cuts and symmetries in the β-plane shows that $H_-(\beta) = H_-(-\beta)$. In the β-plane (d) is

$$2(k \sin \beta)^{-1}F_+(\beta) = G_+(\beta) + H_-(\beta). \tag{e}$$

Change the sign of β and subtract the resulting equation from (e). Then

$$2(k \sin \beta)^{-1}\{F_+(\beta) + F_+(-\beta)\} = G_+(\beta) - G_+(-\beta). \tag{f}$$

The Fourier transforms of (b) give (A. Erdelyi *et al.* [1]):

$$\left.\begin{aligned} G_+(\alpha) &= \frac{i}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} \gamma^{-1}(\alpha+\gamma)^{-m}k^m a_m \\ F_+(\alpha) &= \frac{\pi i}{(2\pi)^{1/2}} \sum_{m=0}^{\infty} k^{m+\frac{1}{2}}(\alpha+\gamma)^{-m-\frac{1}{2}}C_m. \end{aligned}\right\} \tag{g}$$

Replace α by $-k\cos\beta$, remembering that by analytic continuation $(\alpha+\gamma)^{1/2} = -ik^{1/2}\exp(\frac{1}{2}i\beta)$. Then (f) gives

$$\sum_{m=0}^{\infty} (-1)^m \cos m\beta\, a_m = 2\pi \sum_{m=0}^{\infty} (-1)^m \cos(m+\tfrac{1}{2})\beta\, C_m.$$

If we multiply by $\cos n\beta$ and integrate in $(0,\pi)$ we obtain (c). But if we multiply by $\cos(n+\frac{1}{2})\beta$ and integrate in $(0,\pi)$ we obtain the solution of W. Magnus directly:

$$C_m = \frac{1}{2\pi^2} \sum_{n=0}^{\infty} a_n \left\{ \frac{1}{-n+m+\frac{1}{2}} + \frac{1}{n+m+\frac{1}{2}} \right\}.$$

One unsatisfactory feature of the above analysis is that we have had to assume the forms of the expansions for $f(x)$ and $g(x)$ (cf. (g)). It would be of great interest if one could deal with branch points by extending the above method to apply to the general Wiener–Hopf equation.

4.14 Consider the potential distribution due to a two-dimensional perfectly conducting strip in $0 \leqslant x \leqslant a$ charged to a uniform potential V. Suppose that there is a charge distribution $f(x)$ per unit length on the strip. By superposition the potential at any point (x,y) is

$$\phi = -\int_0^a f(\xi) \ln \{(x - \xi)^2 + y^2\}\, d\xi.$$

This gives the following integral equation for $f(\xi)$:

$$-2\int_0^a f(\xi) \ln (|x - \xi|)\, d\xi = V, \qquad (0 \leqslant x \leqslant a).$$

Differentiate with respect to x:

$$\int_0^a \frac{f(\xi)}{\xi - x}\, d\xi = 0, \qquad (0 \leqslant x \leqslant a). \tag{a}$$

To solve this equation let the right-hand side equal the unknown function $g(x)$ for $x > 0$. Apply a Mellin transform. Interchange orders of integration on the left and set $x = \xi\eta$. Then

$$\int_0^a f(\xi)\xi^{s-1}\, d\xi \int_0^\infty \eta^{s-1}(1 - \eta)^{-1}\, d\eta = \int_a^\infty g(x)x^{s-1}\, dx,$$

i.e.

$$F_P(s)\pi \cot \pi s = G_N(s).$$

This is a Wiener–Hopf equation which can be solved in the usual way. From the point of view adopted in this book it is unnecessary to formulate the integral equation (a). The problem can be solved by applying a Mellin transform directly to the partial differential equation in polar co-ordinates (cf. §4.6).

SOME APPROXIMATE METHODS

5.1 Introduction

All the problems considered in previous chapters have been highly idealized. In connection with the two-dimensional wave equation, for example, only parallel, infinitely thin, semi-infinite plates were considered. In this chapter we discuss some approximate methods which can be used to deal with plates of finite thickness and finite length.

From a mathematical point of view we can say that the typical two-dimensional wave-equation problem considered so far has involved two-part mixed boundary value conditions on infinite parallel boundaries, e.g. on boundaries, say, $y = \pm b$, $-\infty < x < \infty$ we have been given different conditions involving ϕ and $\partial\phi/\partial y$ on $-\infty < x < 0$, and $0 < x < \infty$. Ideally we should like to be able to solve problems involving the two-dimensional wave equation with:

(i) Boundaries along $x =$ constant as well as $y =$ constant, i.e. boundaries perpendicular to each other as well as parallel to each other.

(ii) m-part mixed boundary value problems when the boundary is divided into m parts and different conditions involving ϕ and $\partial\phi/\partial y$ are specified on alternate segments. We consider below only three-part problems.

In this chapter we make some limited progress towards these objectives. The basic Wiener–Hopf equation which has appeared in the exact solutions considered in previous chapters is

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0,$$

where A, B, C are known analytic functions, Φ_+, Ψ_- are unknown functions regular in upper and lower half-planes respectively, and the equation holds in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$ in the complex α-plane. The problems in this chapter can be reduced to an equation of the type

$$A(\alpha)\Phi_+(\alpha) + E(\alpha)\Phi_+(-\alpha) + B(\alpha)\Psi_-(\alpha) + D(\alpha)\Phi_1(\alpha) + C(\alpha) = 0, \tag{5.1}$$

where $A, \ldots E$ are known, Φ_+, Ψ_- are unknown, and Φ_1 is an unknown integral function. Equation (5.1) cannot be solved exactly by the Wiener–Hopf technique. It will be convenient in some cases to write $E(\alpha) = B(\alpha)b(\alpha)$ so that (5.1) can be written

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\{\Psi_-(\alpha) + b(\alpha)\Phi_+(-\alpha)\} + D(\alpha)\Phi_1(\alpha) + C(\alpha) = 0. \tag{5.2}$$

The nature of the function $b(\alpha)$ will play an important part in the approximate solution.

§5.2 deals with formulation of specific problems as equations of type (5.1). The remainder of this chapter is divided into two parts. One deals with "perpendicular boundaries" (case (i) above). The other deals with three-part problems (case (ii) above). The reader who is interested mainly in applications may prefer to skip §§5.3, 5.5 on a first reading, whereas the mathematically minded reader may prefer to read these first. §§5.3, 5.5 require a good understanding of the basic decomposition theorem B of §1.3: If $f(\alpha)$ is regular in $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, with suitable behaviour as $|\sigma| \to \infty$ in the strip then we can write

$$f(\alpha) = \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{f(\zeta)}{\zeta-\alpha}\,d\zeta - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{f(\zeta)}{\zeta-\alpha}\,d\zeta,$$

$$= f_+(\alpha) + f_-(\alpha), \quad \text{respectively,} \tag{5.3}$$

where $\tau_- < c < \tau < d < \tau_+$, and $f_+(\alpha)$ is regular for $\tau > \tau_-$, $f_-(\alpha)$ for $\tau < \tau_+$.

In a book of this size devoted to one particular technique it is impossible to give the reader a sense of proportion. The Wiener–Hopf technique is only one method out of many for solution of field problems, and from another point of view it is only one method out of many for solution of singular integral equations. Even though a problem can be solved exactly or approximately by the Wiener–Hopf technique there may be some alternative method of obtaining sufficiently accurate results in an easier way. These remarks apply particularly to the approximate methods discussed in this chapter.

A discussion of various methods for solving the type of problem in which we are interested will be found in P. M. Morse and H. Feshbach [1]. More specialized references are N. Marcuvitz [1], and L. Lewin [1] for waveguide problems, and C. J. Bouwkamp [1] for diffraction theory. In connexion with the topics in this chapter it would seem that the Wiener–Hopf procedure is the only technique available at the moment for the thick-plate problem of §5.4, and it is convenient for dealing with problems involving strips or apertures of large width as in §5.6. For problems like the flanged pipe (§5.2, (5.14)), the duct with a step (ex. 5.1), and diffraction by slits and strips of small width, the most natural approach is to try to find the field distribution across a finite aperture or obstacle in the system whereas the Wiener–Hopf technique concentrates attention on semi-infinite apertures or obstacles; on the other hand, in many such problems the approximate methods of this chapter will be convenient for certain ranges of parameters.

A note is necessary on the use of the word "approximate" in the title of the chapter. In the problem considered in §5.4 the solution is expressed in terms of constants which satisfy a given infinite set of simultaneous linear algebraic equations. The constants must be determined numerically by solving a finite number of the linear equations, so that the constants and hence the solution are known only approximately. For small enough wave numbers, i.e. long enough wavelengths the solution can be determined as accurately as desired by solving a large enough set of equations. On the other hand for wave numbers greater than a certain value the solution of $n \times n$ sets of equations for various n may not converge as $n \to \infty$ so that it may not be possible to obtain a satisfactory approximate solution in this way. The solutions of §§5.5–5.6 are approximate in a different sense. At an early stage of the calculation an integral is replaced by the first term of an asymptotic expansion for the integral so that the solutions are inherently approximate. It may be possible to ascribe an order of magnitude to the error. But some degree of error is inevitably present and the size of the error will increase as the wave number decreases. The essential feature distinguishing the two types of approximation involves the nature of certain functions of a complex variable which occur in the Wiener–Hopf equations. In the first case these functions possess only simple poles: in the second case branch points occur. We shall return to this point at the end of this chapter.

5.2 Some problems which cannot be solved exactly

In this section we consider the formulation of some problems which cannot be solved exactly by the Wiener–Hopf technique. Only very simple examples are considered and these are treated from various points of view in order to illustrate several ways in which equations of form (5.2) can occur.

The symbols $\Phi(\alpha)$, $\Phi_+(\alpha)$, etc. in this section sometimes refer to transforms with respect to x, and sometimes with respect to y. Similarly dashed quantities will sometimes denote derivatives with respect to y, sometimes with respect to x. The different meanings will be clear from the context.

Suppose first that the plane wave $\exp(-ikx)$ travelling from right to left parallel to the x-axis is incident normally on the end of a flat plate of finite thickness lying in $-b \leqslant y \leqslant b$, $-\infty < x \leqslant 0$. There is symmetry about $y = 0$ and we can write $\phi_t = \phi_i + \phi$. In the usual way, by applying a Fourier transform in x to the two-dimensional wave equation,

$$\Phi(\alpha) = A(\alpha)\exp(-\gamma y), \qquad y \geqslant b. \tag{5.4}$$

In $-b \leqslant y \leqslant b$, $x \geqslant 0$, on integrating by parts,

$$\int_0^\infty \frac{\partial^2\phi}{\partial x^2} e^{i\alpha x}\,dx = -\left(\frac{\partial\phi}{\partial x}\right)_0 + i\alpha(\phi)_0 - (2\pi)^{1/2}\alpha^2\Phi_+(\alpha).$$

Hence in this region the partial differential equation becomes

$$d^2\Phi_+(\alpha)/dy^2 - \gamma^2\Phi_+(\alpha) = (2\pi)^{-1/2}(\partial\phi/\partial x)_0 - (2\pi)^{-1/2}i\alpha(\phi)_0. \tag{5.5}$$

Assume that $\partial\phi_t/\partial n = 0$ on the plate so that $\partial\phi/\partial x = ik$ on $x = 0$, $-b \leqslant y \leqslant b$. In this case $(\phi)_0$ is unknown. To eliminate $(\phi)_0$ from (5.5) change the sign of α and add the resulting equation to (5.5). This gives

$$d^2\{\Phi_+(\alpha) + \Phi_+(-\alpha)\}/dy^2 - \gamma^2\{\Phi_+(\alpha) + \Phi_+(-\alpha)\} = (2\pi)^{-1/2}2ik.$$

Solve this equation and find, on using the symmetry about $y = 0$,

$$\Phi_+(\alpha) + \Phi_+(-\alpha) = -(2\pi)^{-1/2}2ik\gamma^{-2} + B\cosh\gamma y, \qquad (0 \leqslant y \leqslant b). \tag{5.6}$$

Insert $\Phi = \Phi_+ + \Phi_-$ in (5.4) with $y = b$. Differentiate (5.4) with respect to y, set $\Phi' = \Phi'_+ + \Phi'_-$ on $y = b$. Eliminate A and use the fact that $\Phi'_- = 0$ to find

$$\Phi_+(\alpha) + \Phi_-(\alpha) = -\gamma^{-1}\Phi'_+(\alpha). \tag{5.7a}$$

Similarly from (5.6), on $y = b$,

$$\Phi_+(\alpha) + \Phi_+(-\alpha) = -(2\pi)^{-1/2}2ik\gamma^{-2} + \gamma^{-1}\coth\gamma b\{\Phi'_+(\alpha) + \Phi'_+(-\alpha)\}. \tag{5.7b}$$

Eliminate $\Phi_+(\alpha)$ between (5.7a, b).

$$2\gamma^{-1}(1 - e^{-2\gamma b})^{-1}\Phi'_+(\alpha) + [\{\Phi_-(\alpha) - \Phi_+(-\alpha)\} + + \gamma^{-1}\coth\gamma b\Phi'_+(-\alpha)] - (2\pi)^{-1/2}ik\gamma^{-2} = 0. \tag{5.8}$$

This is in form (5.2) with $b(\alpha) = \gamma^{-1}\coth\gamma b$, i.e. $b(\alpha)$ has only simple poles and no branch points. Equation (5.8) contains the complete solution of the problem.

An alternative formulation can be obtained by applying transforms in the y-direction and matching functions across $x = 0$. We have in the usual way

$$\Phi(\alpha) = A(\alpha) \exp(-\gamma x), \qquad x \geqslant 0. \tag{5.9}$$

Define, on $x = 0$,

$$\Phi_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_b^\infty \phi\, e^{i\alpha(y-b)}\, dy \quad : \quad \Phi_1(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-b}^{b} \phi e^{i\alpha y}\, dy. \tag{5.10a}$$

$\Phi_1(\alpha)$ is an integral function of x and $\Phi_+(\alpha)$ has been defined so that it has polynomial behaviour as $|\alpha| \to \infty$ in an upper half-plane. From the symmetry about $y = 0$,

$$\Phi(\alpha) = e^{i\alpha b}\Phi_+(\alpha) + \Phi_1(\alpha) + e^{-i\alpha b}\Phi_+(-\alpha). \tag{5.10b}$$

Since $\partial\phi/\partial x = ik$ on $x = 0$, $-b \leqslant y \leqslant b$,

$$\Phi_1'(\alpha) = \frac{ik}{(2\pi)^{1/2}} \int_{-b}^{b} e^{i\alpha y}\, dy = \frac{2ik}{(2\pi)^{1/2}\alpha} \sin \alpha b.$$

Differentiate (5.9) and eliminate A between the resulting equation and (5.9). Set $x = 0$ and insert the above notation, which gives

$$\begin{aligned} &-\gamma\{e^{i\alpha b}\Phi_+(\alpha) + \Phi_1(\alpha) + e^{-i\alpha b}\Phi_+(-\alpha)\} \\ &\qquad = e^{i\alpha b}\Phi'_+(\alpha) + 2ik(2\pi)^{1/2}\alpha^{-1} \sin \alpha b + e^{-i\alpha b}\Phi'_+(-\alpha). \end{aligned} \tag{5.11}$$

Next consider $x \leqslant 0$, $y \geqslant b$. Apply a Fourier transform in y and eliminate the unknown value of ϕ on $y = b$ as in the analysis leading to (5.6). We find

$$\Phi_+(\alpha) + \Phi_+(-\alpha) = B \exp(\gamma x), \qquad (x \leqslant 0),$$

where Φ_+ is defined in (5.10a) . Differentiate with respect to x, set $x = 0$, and eliminate B to find

$$\gamma\{\Phi_+(\alpha) + \Phi_+)-\alpha)\} = \Phi'_+(\alpha) + \Phi'_+(-\alpha). \tag{5.12}$$

Multiply (5.11) by exp $(-i\alpha b)$ and add to (5.12) to eliminate $\Phi_+(\alpha)$ and obtain

$$\begin{aligned} &2\Phi'_+(\alpha) - \gamma(1 - e^{-2i\alpha b})\{\Phi_+(-\alpha) + i\gamma^{-1} \cot \alpha b \Phi'_+(-\alpha)\} + \\ &\qquad + \gamma e^{-i\alpha b}\Phi_1(\alpha) + (2\pi)^{-1/2} 2ik\alpha^{-1} \sin \alpha b e^{-i\alpha b} = 0. \end{aligned} \tag{5.13}$$

This is in form (5.2) with $b(\alpha) = i\gamma^{-1} \cot \alpha b$ so that in this case $b(\alpha)$ has branch points at $\alpha = \pm k$.

We might have expected on general grounds that the coefficient of $\Phi'_+(-\alpha)$ in (5.8) should have only poles whereas in (5.13) branch points should be present. In (5.8) $\Phi'_+(-\alpha)$ was introduced by a boundary of finite length whereas in (5.13) the boundary introducing $\Phi'_+(-\alpha)$ goes to infinity.

Consider next a semi-infinite duct with a flange. Suppose that the duct lies in $-b \leqslant y \leqslant b$, $-\infty < x \leqslant 0$, and the flange in $x = 0$, $|y| \geqslant b$. Assume that a wave $\exp(-ikx)$ is incident from the right on the duct, from the half-space $x \geqslant 0$, $-\infty < y < \infty$. Suppose that $\partial\phi_t/\partial x = 0$ on the flange and $\partial\phi_t/\partial y = 0$ on the walls of the duct. Proceed as above, and find

$$\Phi_+(\alpha,y) + \Phi_+(-\alpha,y) = -(2\pi)^{-1/2}\, 2ik\gamma^{-2} + A\exp(-\gamma y), \; (y \geqslant b),$$

$$\Phi_+(\alpha,y) + \Phi_-(\alpha,y) = B\cosh\gamma y, \qquad (0 \leqslant y \leqslant b).$$

Differentiate each of these equations with respect to y, set $y = b$, eliminate A, B and then $\Phi_+(\alpha)$, to obtain, on using the result $\Phi'_-(\alpha,b) = 0$,

$$2\gamma^{-1}(1 - e^{-2\gamma b})^{-1}\Phi'_+(\alpha) + \{\Phi_+(-\alpha) - \Phi_-(\alpha)\} + \\ + \gamma^{-1}\Phi'_+(-\alpha) + (2\pi)^{-1/2}2ik\gamma^{-2} = 0. \quad (5.14)$$

This is an equation of type (5.2) with $b(\alpha) = \gamma^{-1}$. Since $\Phi'_+(-\alpha)$ was introduced by a semi-infinite boundary, $b(\alpha)$ has branch points as we should expect.

To conclude this section, consider a typical three-part problem. Suppose that $\nabla^2\phi + k^2\phi = 0$ in $-\infty < y < \infty$, $x \geqslant 0$, with

$$\partial\phi/\partial x = 1, \qquad (x = 0, -b < y < b),$$

$$\phi = 0, \qquad (x = 0, -\infty < y < -b, b < y < \infty).$$

Apply a Fourier transform in y in the usual way and find $\Phi(\alpha) = A(\alpha)\exp(-\gamma x)$ as in (5.9). Differentiate with respect to x, eliminate $A(\alpha)$, and set $x = 0$. ϕ is symmetrical about $y = 0$ and we can use the notation (5.10a, b). Insertion of the boundary conditions on $x = 0$ gives

$$e^{i\alpha b}\Phi'_+(\alpha) + \gamma\Phi_1(\alpha) + e^{-i\alpha b}\Phi'_+(-\alpha) = -(2\pi)^{-1/2}2\alpha^{-1}\sin\alpha b. \quad (5.15)$$

This equation is of type (5.2) with $b(\alpha) = 0$.

We finally formulate the problem by means of transforms in the x-direction. The procedure leading to (5.6) for eliminating unknown functions on $x = 0$ gives

$$\Phi_+(\alpha,y) - \Phi_+(-\alpha,y) = Ae^{-\gamma y}, \qquad (y \geqslant b),$$

$$\Phi_+(\alpha,y) + \Phi_+(-\alpha,y) = -(2\pi)^{-1/2}2\gamma^{-2} + B\cosh\gamma y, \quad (0 \leqslant y \leqslant b).$$

Differentiate each of these equations with respect to y, set $y = b$, eliminate A, B, and then $\Phi_+(\alpha,b)$. Write $\Phi_+(\alpha)$ instead of $\Phi_+(\alpha,b)$, etc. We find

$$\gamma^{-1}(1 + \coth \gamma b)\Phi'_+(\alpha) - 2\Phi_+(-\alpha) -$$
$$- \gamma^{-1}(1 - \coth \gamma b)\Phi'_+(-\alpha) - (2\pi)^{-1/2}2\gamma^{-2} = 0. \qquad (5.16)$$

This is of form (5.2); the function $b(\alpha)$ has branch points at $\alpha = \pm k$.

Of the five formulations in this section, namely (5.8), (5.13), (5.14), (5.15), (5.16), the only cases that can be solved satisfactorily at the moment by methods based on the Wiener–Hopf technique are (5.8) for small b and (5.15) for large b. The remainder of this chapter is devoted to the corresponding special cases of (5.2). One of the objects of this section is to indicate that a practical method for solution of the general equation (5.2) should prove to be of considerable value.

5.3 General theory of a special equation

In this section we examine the following special case of (5.2) where no term involving the integral function $\Phi_1(\alpha)$ is present:

$$A(\alpha)\Phi_+(\alpha) + B(\alpha)\{\Psi_-(\alpha) + b(\alpha)\Phi_+(-\alpha)\} + C(\alpha) = 0, \quad (5.17)$$

valid in a strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, of the complex α-plane. Set

$$A(\alpha)/B(\alpha) = K_+(\alpha)/K_-(\alpha) \quad : \quad K_-(\alpha)C(\alpha)/B(\alpha) = E_+(\alpha) + E_-(\alpha). \qquad (5.18)$$

Introduce these in (5.17) and rearrange:

$$K_+(\alpha)\Phi_+(\alpha) + E_+(\alpha)$$
$$= -K_-(\alpha)\Psi_-(\alpha) - E_-(\alpha) - b(\alpha)K_-(\alpha)\Phi_+(-\alpha). \quad (5.19)$$

The last term is not regular in either an upper or a lower half-plane, in general.

In many examples $b(\alpha)$ possesses only simple poles in the lower half-plane $\tau < \tau_+$ and we deal with this special case first. $b(\alpha)$ can then be written

$$b(\alpha) = b_-(\alpha) + \sum_{s=1}^{n} \frac{b_s}{\alpha - \beta_s}, \qquad (5.20)$$

where b_s is the residue of $b(\alpha)$ at $\alpha = \beta_s$ and the sum is over all the poles in the lower half-plane (n may be infinite). Introduce (5.20) into (5.19) and add extra terms to both sides:

$$K_+(\alpha)\Phi_+(\alpha) + E_+(\alpha) + \sum_{s=1}^{n} \frac{b_s}{\alpha - \beta_s} K_-(\beta_s)\Phi_+(-\beta_s)$$
$$= -K_-(\alpha)\Psi_-(\alpha) - E_-(\alpha) - b_-(\alpha)K_-(\alpha)\Phi_+(-\alpha) -$$
$$- \sum_{s=1}^{n} \frac{b_s}{\alpha - \beta_s}\left\{K_-(\alpha)\Phi_+(-\alpha) - K_-(\beta_s)\Phi_+(-\beta_s)\right\}. \quad (5.21)$$

The additional terms ensure that the left- and right-hand sides are regular in upper and lower half-planes respectively. Apply the Wiener–Hopf technique and assume that both sides are then identically zero, from conditions at infinity. Then

$$K_+(\alpha)\Phi_+(\alpha) + E_+(\alpha) + \sum_{s=1}^{n} \frac{b_s}{\alpha - \beta_s} K_-(\beta_s)\Phi_+(-\beta_s) = 0. \quad (5.22a)$$

This equation holds for all α. Set $\alpha = -\beta_r$, $r = 1$ to n. A set of simultaneous linear algebraic equations is obtained for the unknown constants $\Phi_+(-\beta_s) = x_s$, say:

$$K_+(-\beta_r)x_r + E_+(-\beta_r) - \sum_{s=1}^{n} \frac{b_s}{\beta_r + \beta_s} K_-(\beta_s)x_s = 0, \qquad (r = 1 \text{ to } n). \quad (5.22b)$$

When the x_s are determined the solution is complete.

For reference, the analogous solution for the equation

$$A(\alpha)\{\Phi_+(\alpha) + a(\alpha)\Psi_-(-\alpha)\} + B(\alpha)\Psi_-(\alpha) + C(\alpha) = 0, \quad (5.23a)$$

$$a(\alpha) = a_+(\alpha) + \sum_{s=1}^{m} \frac{a_s}{\alpha - \alpha_s}, \qquad \alpha_s \text{ in } \tau > \tau_-, \quad (5.23b)$$

is given by the following equations in which $y_s = \Psi_-(-\alpha_s)$:

$$K_-(\alpha)\Psi_-(\alpha) + E_-(\alpha) + \sum_{s=1}^{m} \frac{a_s}{\alpha - \alpha_s} K_+(\alpha_s)y_s = 0, \quad (5.23c)$$

where

$$K_-(-\alpha_r)y_r + E_-(-\alpha_r) - \sum_{s=1}^{m} \frac{a_s}{\alpha_r + \alpha_s} K_+(\alpha_s)y_s = 0. \quad (5.23d)$$

When these results are used to solve concrete problems it can be shown that the x_r, y_r are related to physical quantities occurring in the problem (cf. ex. 5.2).

Next consider general $b(x)$ in (5.17). The standard formula (5.3) gives

$$b(\alpha)K_-(\alpha)\Phi_+(-\alpha)$$
$$= \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{b(\zeta)K_-(\zeta)\Phi_+(-\zeta)}{\zeta-\alpha}\,d\zeta - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{b(\zeta)K_-(\zeta)\Phi_+(-\zeta)}{\zeta-\alpha}\,d\zeta, \tag{5.24}$$

where $\tau_- < c < (\operatorname{Im}\alpha) < d < \tau_+$, and the first integral is regular in an upper half-plane $\tau > \tau_-$, and the second in a lower half-plane $\tau < \tau_+$. It is assumed that these integrals are convergent. Introduce (5.24) into (5.19) and separate in the usual way, assuming that each half of the equation is identically zero. Then

$$K_+(\alpha)\Phi_+(\alpha) + E_+(\alpha) + \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty} \frac{b(\zeta)K_-(\zeta)\Phi_+(-\zeta)}{\zeta-\alpha}\,d\zeta = 0, \tag{5.25a}$$

$$K_-(\alpha)\Psi_-(\alpha) + E_-(\alpha) - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty} \frac{b(\zeta)K_-(\zeta)\Phi_+(-\zeta)}{\zeta-\alpha}\,d\zeta = 0. \tag{5.25b}$$

The first equation is an integral equation for $\Phi_+(\alpha)$. When $\Phi_+(\alpha)$ is found, $\Psi_-(\alpha)$ can be found from the second equation.

If the only singularities of $b(\alpha)$ in the lower half-plane are simple poles the contour in (5.25a) can be completed in the lower half-plane and the integral can be evaluated by residues. With representation (5.20) for $b(\alpha)$ it is found that this gives exactly (5.22a) so that the integral equation method merely reproduces results obtained by the more direct procedure described at the beginning of this section.

In the general case (5.25a) can be reduced to a Fredholm integral equation in the following way. For simplicity suppose that we can take $c = 0$ and denote the value of ζ on the real axis by s. Suppose that α tends to a point on the real axis which we denote by t. Then the contour must be indented below, and on using ex. 1.24, equation (5.25a) reduces to

$$K_+(t)\Phi_+(t) + E_+(t) + \tfrac{1}{2}b(t)K_-(t)\Phi_+(-t)$$
$$+ \frac{1}{2\pi i}P\int_{-\infty}^{\infty} \frac{b(s)K_-(s)\Phi_+(-s)}{s-t}\,ds = 0, \quad (-\infty < t < \infty), \tag{5.26}$$

where P denotes that the integral must be understood as a Cauchy principal value. The singularity in the kernel can be eliminated as follows. From Cauchy's theorem

$$\Phi_+(-\alpha) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\Phi_+(-s)}{s-\alpha}\,ds, \quad (\alpha \text{ in the lower half-plane}), \tag{5.27}$$

where we assume that $\Phi_+(+\alpha)$ tends to zero uniformly as $(+\alpha)$ tends to infinity in an upper half-plane, so that the above integral can be converted into a contour integral by completing the contour in the lower half-plane. Let α tend to a real number t and indent the contour above to find that (5.27) reduces to

$$\Phi_+(-t) = \tfrac{1}{2}\Phi_+(-t) - \frac{1}{2\pi i}\,P\int_{-\infty}^{\infty}\frac{\Phi_+(-s)}{s-t}\,ds, \quad (-\infty < t < \infty). \tag{5.28}$$

Substitute for $\frac{1}{2}\Phi_+(-t)$ in (5.26) and find

$$K_+(t)\Phi_+(t) + E_+(t)$$
$$+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\{b(s)K_-(s) - b(t)K_-(t)\}}{s-t}\Phi_+(-s)\,ds = 0, \quad (-\infty < t < \infty), \tag{5.29}$$

where the kernel is now finite at $s = t$ so that the integral is no longer a Cauchy principal value.

The result in this section from (5.24) onwards will not be used in the remainder of this chapter, but have been included to indicate one way in which it may be possible to solve more complicated cases.

5.4 Diffraction by a thick semi-infinite strip

We consider a single example in detail to illustrate the theory at the beginning of the last section. Suppose that a wave $\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta)$ is incident on a plate lying in $x \leqslant 0$, $-b \leqslant y \leqslant b$, with $\partial\phi_t/\partial n = 0$ on the plate. The results given below are due to D. S. Jones [4] who treats also the case $\phi_t = 0$ on the plate. Our method of determining the basic equations is somewhat different from that of D. S. Jones. Set

$$\begin{aligned}\phi_t &= \phi + e^{-ikx\cos\Theta - iky\sin\Theta} + e^{-ikx\cos\Theta + ik(y-2b)\sin\Theta}, \qquad y \geqslant b,\\ &= \phi, \qquad -b \leqslant y \leqslant b, \qquad x \geqslant 0; \text{ and } \quad y \leqslant b,\end{aligned} \tag{5.30}$$

so that $\phi \to 0$ as $|x|$, $|y| \to \infty$, and $\partial\phi/\partial n = 0$ on the plate. Define in the usual way

$$\Phi_+(\alpha,y) = \frac{1}{(2\pi)^{1/2}}\int_0^{\infty}\phi e^{i\alpha x}\,dx \quad : \quad \Phi_-(\alpha,y) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{0}\phi e^{i\alpha x}\,dx.$$

$\Phi_+(\alpha,y)$ is regular in $\tau > k_2 \cos \Theta$, Φ_- in $\tau < k_2$, and the common strip is $k_2 \cos \Theta < \tau < k_2$.

From (5.30), Φ'_+ is continuous on $y = \pm b$, and Φ_+ is continuous on $y = -b$. On $y = +b$, with the usual notation that $\phi(x, b + 0)$ denotes the value of $\phi(x,y)$ as $y \to b$ from values of y just greater than b, etc. (5.30) gives

$$\phi(x, b + 0) + 2e^{-ikx \cos \Theta - ikb \sin \Theta} = \phi(x, b - 0), \qquad (x > 0),$$

$$\text{i.e.}\quad \Phi_+(x, b + 0) + \frac{2i}{(2\pi)^{1/2}} \frac{\exp(-ikb \sin \Theta)}{\alpha - k \cos \Theta} = \Phi_+(\alpha, b - 0). \quad (5.31)$$

Apply a Fourier transform in x to the two-dimensional steady-state wave equation. In $-b \leqslant y \leqslant b$, $x \geqslant 0$,

$$\frac{1}{(2\pi)^{1/2}} \int_0^\infty \frac{\partial^2 \phi}{\partial x^2} e^{i\alpha x}\, dx = \frac{i\alpha}{(2\pi)^{1/2}} (\phi)_0 - \alpha^2 \Phi_+(\alpha,y),$$

where $(\phi)_0$ denotes the value of ϕ on $x = 0$, $-b \leqslant y \leqslant b$. The partial differential equation, in $x \geqslant 0$, $-b \leqslant y \leqslant b$, therefore transforms into

$$d^2\Phi_+(\alpha,y)/dy^2 - \gamma^2\Phi_+(\alpha,y) = -i\alpha(2\pi)^{-1/2}(\phi)_0. \qquad (5.32)$$

Eliminate $(\phi)_0$ by changing the sign of α and adding the resulting equation to (5.32). This gives

$$\frac{d^2}{dy^2} \{\Phi_+(\alpha,y) + \Phi_+(-\alpha,y)\} - \gamma^2\{\Phi_+(\alpha,y) + \Phi_+(-\alpha,y)\} = 0,$$

$$\text{i.e.}\quad \Phi_+(\alpha,y) + \Phi_+(-\alpha,y) = Ae^{-\gamma y} + Be^{\gamma y}, \qquad -b \leqslant y \leqslant b.$$

From this we deduce

$$\Phi_+(\alpha, b - 0) + \Phi_+(-\alpha, b - 0) = Ae^{-\gamma b} + Be^{\gamma b}, \qquad (5.33a)$$

$$\Phi'_+(\alpha,b) + \Phi'_+(-\alpha, b) = -\gamma Ae^{-\gamma b} + \gamma Be^{\gamma b}, \qquad (5.33b)$$

$$\Phi_+(\alpha,-b) + \Phi_+(-\alpha,-b) = Ae^{\gamma b} + Be^{-\gamma b}, \qquad (5.34a)$$

$$\Phi'_+(\alpha,-b) + \Phi'_+(-\alpha,-b) = -\gamma Ae^{\gamma b} + \gamma Be^{-\gamma b}. \qquad (5.34b)$$

Introduce the notation

$$\begin{aligned} \Phi_+(\alpha, b - 0) + \Phi_+(\alpha,-b) &= S_+(\alpha) \\ \Phi_+(\alpha, b - 0) - \Phi_+(\alpha,-b) &= D_+(\alpha), \end{aligned} \qquad (5.35)$$

and similarly for dashed quantities. Elimination of A, B between (5.33), (5.34) then gives

$$S'_+(\alpha) + S'_+(-\alpha) = \gamma \coth \gamma b\{D_+(\alpha) + D_+(-\alpha)\}, \quad (5.36a)$$

$$D'_+(\alpha) + D'_+(-\alpha) = \gamma \tanh \gamma b\{S_+(\alpha) + S_+(-\alpha)\}. \quad (5.36b)$$

Application of transforms in $y \geqslant b$, $y \leqslant -b$, gives

$$\Phi_+(\alpha,y) + \Phi_-(\alpha,y) = Ce^{-\gamma y}, \qquad (y \geqslant b), \quad (5.37a)$$

$$= De^{\gamma y}, \qquad (y \leqslant -b). \quad (5.37b)$$

Differentiate with respect to y, eliminate C, D between the resulting equations and (5.37), and set $y = b + 0$, $y = -b$ in the appropriate equations. This gives, on remembering that $\Phi'_-(\alpha,b) = \Phi'_-(\alpha,-b) = 0$,

$$\Phi'_+(\alpha,b) = -\gamma\{\Phi_+(\alpha, b + 0) + \Phi_-(\alpha,b)\}, \quad (5.38a)$$

$$\Phi'_+(\alpha,-b) = \gamma\{\Phi_+(\alpha,-b) + \Phi_-(\alpha,-b)\}. \quad (5.38b)$$

Add and subtract, use (5.31), and introduce notation (5.35). This gives

$$S'_+(\alpha) = -\gamma\left\{D_+(\alpha) + D_-(\alpha) - \frac{2i}{(2\pi)^{1/2}}\frac{\exp(-ikb\sin\Theta)}{\alpha - k\cos\Theta}\right\}, \quad (5.39a)$$

$$D'_+(\alpha) = -\gamma\left\{S_+(\alpha) + S_-(\alpha) - \frac{2i}{(2\pi)^{1/2}}\frac{\exp(-ikb\sin\Theta)}{\alpha - \cos\Theta}\right\}. \quad (5.39b)$$

Equations (5.36), (5.39) contain the solution of the problem. These split into two independent pairs of equations. Eliminate $D_+(\alpha)$ between (5.36a), (5.39a). This gives

$$S'_+(\alpha) + S'_+(-\alpha) = \gamma \coth \gamma b \Big\{D_+(-\alpha) - D_-(\alpha) - \gamma^{-1}S'_+(\alpha) +$$

$$+ \frac{2i}{(2\pi)^{1/2}}\frac{\exp(-ikb\sin\Theta)}{\alpha - k\cos\Theta}\Big\}. \quad (5.40)$$

Set
$$e^{-\gamma b}\cosh \gamma b = K_+(\alpha)K_-(\alpha), \quad (5.41)$$

where K_+, K_- have been defined by (3.19a), (3.22). Then (5.40) can be rearranged as

$$\frac{S'_+(\alpha)}{(\alpha + k)^{1/2}K_+(\alpha)} = \frac{2i}{(2\pi)^{1/2}}\cdot\frac{\exp(-ikb\sin\Theta)}{\alpha - k\cos\Theta}(\alpha - k)^{1/2}K_-(\alpha) +$$

$$+ (\alpha - k)^{1/2}K_-(\alpha)\{D_+(-\alpha) - D_-(\alpha)\} -$$

$$- \gamma^{-1}\tanh \gamma b(\alpha - k)^{1/2}K_-(\alpha)S'_+(-\alpha). \quad (5.42)$$

This is in form (5.19) with $K_-(\alpha)$ in that equation replaced by $(\alpha - k)^{1/2}K_-(\alpha)$ and $b(\alpha) = \gamma^{-1}\tanh \gamma b$. Introduce notation

$$\gamma_r = \{(r\pi/2b)^2 - k^2\}^{1/2} \quad : \quad \gamma_0 = -ik. \tag{5.43}$$

The poles and residues of $\gamma^{-1}\tanh\gamma b$ in the lower half-plane are (cf. the notation in (5.20)):

$$\beta_r = -i\gamma_{2r+1} \quad : \quad b_s = i(\gamma_{2r+1}b)^{-1}, \qquad (r = 0 \text{ to } \infty).$$

The solution of (5.42) can now be written down from (5.22), or it is easy to work out the solution from first principles. It is left to the reader to check that the conditions at the corners $(0, \pm b)$, namely $|\phi| \sim$ const., $|\text{grad}\,\phi| \sim r^{-1/3}$, ensure that the separation polynomial is identically zero when the Wiener–Hopf technique is applied. We find

$$\frac{S'_+(\alpha)}{(\alpha+k)^{1/2}K_+(\alpha)} - \frac{2i}{(2\pi)^{1/2}}\frac{\exp(-ikb\sin\Theta)(k\cos\Theta - k)^{1/2}K_-(k\cos\Theta)}{\alpha - k\cos\Theta}$$
$$+\frac{i}{b}\sum_{s=0}^{\infty}\frac{(-i\gamma_{2s+1}-k)^{1/2}K_-(-i\gamma_{2s+1})S'_+(i\gamma_{2s+1})}{\gamma_{2s+1}(\alpha + i\gamma_{2s+1})} = 0. \tag{5.44}$$

We introduce simplifying notation which also makes the results comparable with D. S. Jones [4]. Set

$$\lambda_{2s+1} = \pi(2b)^{-1/2}(2s+1)(\gamma_{2s+1} - ik)^{-1/2}\{K_+(i\gamma_{2s+1})\}^{-1}, \tag{5.45a}$$

$$x_{2s+1} = \tfrac{1}{2}\pi^{1/2}b^{-1/2}e^{ikb\sin\Theta}$$
$$\times\, e^{-i\pi/4}(k\cos\Theta - k)^{-1/2}\{K_-(k\cos\Theta)\}^{-1}(2s+1)^{-1}S'_+(i\gamma_{2s+1}). \tag{5.45b}$$

Then (5.44) becomes

$$S'_+(\alpha) = A(\alpha)\left\{\frac{i}{\alpha - k\cos\Theta} - \frac{i\pi}{b}\sum_{s=0}^{\infty}\frac{(2s+1)^2x_{2s+1}}{\gamma_{2s+1}(\alpha+i\gamma_{2s+1})\lambda_{2s+1}}\right\}, \tag{5.46a}$$

where

$$A(\alpha) = (2/\pi)^{1/2}e^{-ikb\sin\Theta}(k\cos\Theta - k)^{1/2}K_-(k\cos\Theta)(\alpha+k)^{1/2}K_+(\alpha). \tag{5.46b}$$

An infinite set of simultaneous linear algebraic equations for the x_{2s+1} are obtained by setting $\alpha = i\gamma_{2r+1}$ $(r = 0, 1, 2, \ldots)$ in this equation:

$$\frac{2b}{\pi}\lambda_{2r+1}x_{2r+1} = \frac{1}{\gamma_{2r+1} + ik\cos\Theta} - \frac{\pi}{b}\sum_{s=0}^{\infty}\frac{(2s+1)^2x_{2s+1}}{\gamma_{2s+1}(\gamma_{2s+1}+\gamma_{2r+1})\lambda_{2s+1}}. \tag{5.47a}$$

We shall later neglect small order terms. Set

$$\gamma_{2s+1} = (2s+1)(\pi/2b)\{1 + O(k^2b^2)\}. \tag{5.47b}$$

Then (5.47a) becomes

$$\lambda_{2s+1}x_{2s+1} = \frac{1}{2s+1} - \sum_{s=0}^{\infty} \frac{(2s+1)x_{2s+1}}{(r+s+1)\lambda_{2s+1}}. \tag{5.47c}$$

Equations (5.36b), (5.39b) can be solved in exactly the same way. Eliminate $S_+(\alpha)$ and introduce a split defined in (3.19b), (3.23):

$$(\gamma b)^{-1}e^{-\gamma b} \sinh \gamma b = L_+(\alpha)L_-(\alpha).$$

In this case we find $b(\alpha) = \gamma^{-1} \coth \gamma b$ with poles and residues in the lower half-plane given by

$$\beta_0 = -k = -i\gamma_0 \quad : \quad b_0 = -(2kb)^{-1}$$

$$\beta_s = -i\gamma_{2s} \quad : \quad b_s = i(\gamma_{2s}b)^{-1}.$$

The solution is

$$\frac{D'_+(\alpha)}{b(\alpha+k)L_+(\alpha)} - \frac{2i}{(2\pi)^{1/2}} \frac{\exp(-ikb \sin\Theta)(k\cos\Theta - k)L_-(k\cos\Theta)}{\alpha - k\cos\Theta}$$

$$+ \frac{L_-(-k)D'_+(k)}{b(\alpha+k)} - i\sum_{s=1}^{\infty} \frac{(i\gamma_{2s}+k)L_-(-i\gamma_{2s})D'_+(i\gamma_{2s})}{\gamma_{2s}b(\alpha+i\gamma_{2s})} = 0. \tag{5.48}$$

Again introduce simplifying notation

$$\mu_{2s} = i2^{1/2}\pi s b^{-1}(i\gamma_{2s}+k)^{-1}\{L_+(i\gamma_{2s})\}^{-1}, \tag{5.49a}$$

$$y_{2s} = -i(4bs)^{-1}\pi^{1/2}e^{ikb\sin\Theta}(k\cos\Theta - k)^{-1}\{L_-(k\cos\Theta)\}^{-1}D'_+(i\gamma_{2s}). \tag{5.49b}$$

Then (5.48) becomes

$$D'_+(\alpha) = B(\alpha)\left\{\frac{i}{\alpha - k\cos\Theta} + \frac{2bQ}{\pi}\cdot\frac{D'_+(k)}{\alpha+k} - \right.$$

$$\left. - \frac{i}{b}\sum_{s=1}^{\infty}\frac{4\pi s^2 y_{2s}}{\gamma_{2s}(\alpha+i\gamma_{2s})\mu_{2s}}\right\}, \tag{5.50a}$$

where

$$B(\alpha) = (2/\pi)^{1/2}e^{-ikb\sin\Theta}(k\cos\Theta - k)L_-(k\cos\Theta)b(\alpha+k)L_+(\alpha), \tag{5.50b}$$

$$Q = (\pi/2)^{1/2}(\pi/2b^2)L_-(-k)e^{ikb\sin\Theta}(k - k\cos\Theta)^{-1}\{L_-(k\cos\Theta)\}^{-1}. \tag{5.50c}$$

An infinite set of simultaneous linear equations for the y_{2s} is obtained by setting $\alpha = i\gamma_{2r}$ $(r = 0, 1, 2 \ldots)$ in (5.50a). Write

$$\gamma_{2r} = (r\pi/b)\{1 + O(k^2b^2)\}, \qquad (r \geqslant 1), \tag{5.51a}$$

and neglect small order terms. Then (5.50a) gives

$$y_{2r}\mu_{2r} = \frac{1}{2r} - \frac{iQb}{r\pi} D'_+(k) - \sum_{s=1}^{\infty} \frac{2sy_{2s}}{(r+s)\mu_{2s}}, \qquad (r \geqslant 1), \tag{5.51b}$$

$$QD'_+(k)[1 + \{L_+(k)\}^{-2}]$$
$$= -\frac{i\pi k}{b} \cdot \frac{1}{k - k\cos\Theta} + 4k \sum_{s=1}^{\infty} \frac{y_{2s}}{\mu_{2s}}, \qquad (r = 0). \tag{5.51c}$$

In order to find the field in, say, $y \geqslant b$, we use (5.37a). To determine C in this equation, since we have found $S'_+(\alpha)$, $D'_+(\alpha)$, we differentiate (5.37a) with respect to y and set $y = b$. An inverse transform gives

$$\phi = -\frac{1}{2}\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \gamma^{-1}\{S'_+(\alpha) + D'_+(\alpha)\}e^{-\gamma(y-b)-i\alpha x}\, d\alpha. \tag{5.52}$$

The far field can be found by the asymptotic expansion methods of §1.6. The values of S'_+, D'_+ are given by (5.46a), (5.50a). If we set $x = r\cos\theta$, $y - b = r\sin\theta$, the saddle point occurs at $\alpha = -k\cos\theta$, but we cannot use the standard formula (1.71) since the pole at $\alpha = k\cos\Theta$ may be near the saddle point $\alpha = -k\cos\theta$ (cf. (1.72)). To overcome this difficulty, apply the method associated with (1.73), and write (5.52) in the form

$$\phi = -\frac{1}{2(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} [\gamma^{-1}\{S'_+(\alpha) + D'_+(\alpha)\} - f(\alpha)]e^{-\gamma(y-b)-i\alpha x}\, d\alpha$$
$$- \frac{1}{2(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} f(\alpha)e^{-\gamma(y-b)-i\alpha x}\, d\alpha,$$

where $$f(\alpha) = \frac{2i}{(2\pi)^{1/2}} \cdot \frac{(k\cos\Theta - k)^{1/2}e^{-ikb\sin\Theta}}{(\alpha - k)^{1/2}(\alpha - k\cos\Theta)}.$$

The first integral has now no pole near the saddle point and can be evaluated asymptotically by (1.71). The second integral is given by (1.62), (1.65). As $r \to \infty, 0 < \theta < \pi - \Theta$,

$$\phi \sim -e^{\frac{1}{4}i\pi}\left[\tfrac{1}{2}\{S'_+(-k\cos\theta) + D'_+(-k\cos\theta)\}\right.$$
$$\left. + \frac{i}{(2\pi)^{1/2}} \cdot \frac{(k\cos\Theta - k)^{1/2}\sin\theta e^{-ikb\sin\Theta}}{(k\cos\theta + k)^{1/2}(\cos\theta + \cos\Theta)}\right](kr)^{-\frac{1}{2}}e^{ikr} + G(r,\theta), \tag{5.53a}$$

where

$$G(r,\theta) = -\pi^{-1/2}e^{3i\pi/4}e^{-ikb\sin\Theta}[-e^{-ikr\cos(\theta-\Theta)}F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta-\Theta)\}$$
$$+ e^{-ikr\cos(\theta+\Theta)}F\{(2kr)^{1/2}\cos\tfrac{1}{2}(\theta+\Theta)\}], \tag{5.53b}$$

$$F(v) = \int_v^\infty \exp(iu^2)\,du. \tag{5.53c}$$

Write
$$\phi = G(r,\theta) + \phi_1 + \phi_2, \tag{5.54}$$

where $G(r,\theta)$ is proportional to the field due to diffraction by a semi-infinite plate of zero thickness, ϕ_1 is the correction term that should be added to G to give the far field due to diffraction by a semi-infinite duct, and ϕ_2 is the further contribution due to the flat end of the plate. ϕ_2 comes from the parts of S'_+, D'_+ that do not involve $(\alpha - k\cos\Theta)^{-1}$, i.e.

$$\phi_2 = \{-\pi(2b)^{-1}e^{-i\pi/4}A(-k\cos\theta)\sum_{s=0}^{\infty}\frac{(2s+1)^2x_{2s+1}}{\gamma_{2s+1}(i\gamma_{2s+1} - k\cos\theta)\lambda_{2s+1}}$$
$$- (2b)^{-1}e^{-i\pi/4}B(-k\cos\theta)\sum_{s=1}^{\infty}\frac{4\pi s^2 y_{2s}}{\gamma_{2s}(i\gamma_{2s} - k\cos\theta)\mu_{2s}} \tag{5.55}$$
$$- (b/\pi)e^{+i\pi/4}B(-k\cos\theta)QD'_+(k)(k - k\cos\theta)^{-1}\}(kr)^{-1/2}e^{ikr}.$$

We now neglect terms of small order in (kb). In addition to (5.47b), (5.51a) we need the results that for small α,

$$L_\pm(\alpha) = 1 + O(kb\ln kb) \quad : \quad K_\pm(\alpha) = 1 + O(kb\ln kb).$$

Then (5.46b), (5.50b, c), (5.51c) give, for small kb,

$$A(-k\cos\theta) = -i(2/\pi)^{1/2}2k\sin\tfrac{1}{2}\Theta\sin\tfrac{1}{2}\theta,$$
$$B(-k\cos\theta) = -(2/\pi)^{1/2}k^2b(1-\cos\Theta)(1-\cos\theta),$$
$$Q = (\pi/2)^{1/2}(\pi/2b^2)(k - k\cos\Theta)^{-1},$$
$$D'_+(k) = -ikb(2/\pi)^{1/2}.$$

The notation has been chosen so that x_{2s+1}, y_{2s}, λ_{2s+1}, μ_{2s} are purely numerical. Hence in (5.55) the first and last terms are of order (kb):

the middle term is of order (k^2b^2) and will be ignored. In the series, $(k \cos \theta)$ can be ignored compared with $i\gamma_{2s+1}$. Then (5.55) gives, for small (kb),

$$\phi_2 = \{e^{-i\pi/4}(2/\pi)^{1/2}kp \sin \tfrac{1}{2}\Theta \sin \tfrac{1}{2}\theta + e^{-i\pi/4}\tfrac{1}{2}(2/\pi)^{1/2}kb\}(kr)^{-1/2}e^{ikr}, \tag{5.56a}$$

where

$$p = (4/\pi)b \sum_{s=0}^{\infty} x_{2s+1}\lambda_{2s+1}^{-1}. \tag{5.56b}$$

The series in (5.56b) is a purely numerical constant which we now proceed to determine. From (3.22a, b), (5.45a), it is easily shown that for small (kb)

$$\lambda_m = 2^m\Gamma(\tfrac{1}{2}m + \tfrac{1}{2})m^{1/2} \exp \{\tfrac{1}{2}m(1 - \ln 2m)\}, \tag{5.57a}$$

i.e. $\lambda_1 = 2{\cdot}33 \;:\; \lambda_3 = 4{\cdot}22 \;:\; \lambda_5 = 5{\cdot}51 \;:\; \lambda_7 = 6{\cdot}55.$ (5.57b)

The first four equations in (5.47a), assuming $x_{2s+1} = 0$ for $s > 3$ are

$$\begin{aligned}
2{\cdot}759x_1 + 0{\cdot}355x_3 + 0{\cdot}302x_5 + 0{\cdot}267x_7 &= 1{\cdot}0000\\
0{\cdot}214x_1 + 4{\cdot}457x_3 + 0{\cdot}227x_5 + 0{\cdot}214x_7 &= 0{\cdot}3333\\
0{\cdot}143x_1 + 0{\cdot}178x_3 + 5{\cdot}691x_5 + 0{\cdot}178x_7 &= 0{\cdot}2000\\
0{\cdot}107x_1 + 0{\cdot}142x_3 + 0{\cdot}151x_5 + 6{\cdot}703x_7 &= 0{\cdot}1429.
\end{aligned}$$

We find an approximate solution of the infinite system by solving the first n equations in n unknowns, assuming $x_{2r+1} = 0$, $r \geqslant n$. Denote the resulting roots by $x_{2r+1}^{(n)}$. We find:

$x_1^{(1)} = 0{\cdot}3624.$

$x_1^{(2)} = 0{\cdot}3550 \;:\; x_3^{(2)} = 0{\cdot}0577.$

$x_1^{(3)} = 0{\cdot}3524 \;:\; x_3^{(3)} = 0{\cdot}0566 \;:\; x_5^{(3)} = 0{\cdot}0246.$

$x_1^{(4)} = 0{\cdot}3512 \;:\; x_3^{(4)} = 0{\cdot}0560 \;:\; x_5^{(4)} = 0{\cdot}0242 \;:\; x_7^{(4)} = 0{\cdot}0140.$

Successive approximations alter the values of the unknowns by only a few per cent. The first unknown is much larger than the others. Since $\partial\phi/\partial y \sim r^{-1/3}$ at the corner of the plate we have $|\Phi'_+(\alpha,d)| \sim |\alpha|^{-2/3}$ as $\alpha \to \infty$ in appropriate half-planes i.e. from (5.45b),

$$x_{2r+1} \sim A(2r+1)^{-5/3} \quad \text{as} \quad r \to \infty.$$

A numerical check can be obtained from the above results:

$$x_3^{(4)}/x_5^{(4)} = 2{\cdot}31 \quad : \quad (5/3)^{5/3} = 2{\cdot}34.$$
$$x_5^{(4)}/x_7^{(4)} = 1{\cdot}73 \quad : \quad (7/5)^{5/3} = 1{\cdot}75.$$

By using the asymptotic expansion of the gamma-function, (5.57a) gives $\lambda_m \approx (2\pi m)^{1/2}$ for large m. This gives $\lambda_5 \approx 5{\cdot}60$, $\lambda_7 \approx 6{\cdot}63$

which should be compared with the exact values in (5.57b). We can now examine the value of the series in (5.56b). Set

$$S_n = \sum_{s=0}^{n-1} x_{2s+1}^{(n)} \lambda_{2s+1}^{-1} \quad : \quad S = S_\infty.$$

The above numerical results give

$S_1 = 0{\cdot}155_5 \quad : \quad S_2 = 0{\cdot}166_1 \quad : \quad S_3 = 0{\cdot}169_1 \quad : \quad S_4 = 0{\cdot}170_5.$

These indicate that S_4 gives an underestimate for $S = S_\infty$. Using the previous approximations, another estimate for S is given by

$$S'_n = \sum_{s=0}^{n-2} x_{2s+1}^{(n)} \lambda_{2s+1}^{-1} + x_{2n-1}^{(n)} \lambda_{2n-1}^{-1} \sum_{s=n-1}^{\infty} (2n-1)^{13/6}(2s+1)^{-13/6}.$$

The second series can be summed by means of the result

$$\sum_{s=0}^{\infty} (2s+1)^{-z} = \zeta(z)(1-2^{-z}),$$

where $\zeta(z)$ is the Riemann zeta-function. The numerical results quoted above give

$$S'_1 = 0{\cdot}183_0 \quad : \quad S'_2 = 0{\cdot}178_6 \quad : \quad S'_3 = 0{\cdot}177_0 \quad : \quad S'_4 = 0{\cdot}176_3.$$

These indicate that S'_n gives an overestimate of $S = S_\infty$. Hence we can say with some confidence, though the argument is not rigorous, that

$$0{\cdot}171 < S < 0{\cdot}176.$$

D. S. Jones [4] has shown rigorously that $S = 0{\cdot}175 \pm 0{\cdot}014$. If we take $S = 0{\cdot}175$ then (5.56b) gives

$$p = 0{\cdot}22b. \tag{5.58}$$

We can now provide a physical interpretation of ϕ_2. Make the following substitution (equivalent to a change of origin)

$$r = r' + q \cos \theta' \quad : \quad \theta = \theta' - (q/r') \sin \theta'$$

in $G(r,\theta)$ defined by (5.53b). For large r' and small (kb), ignoring small order terms,

$$r \cos (\theta \pm \Theta) = r' \cos (\theta' \pm \Theta) + q \cos \Theta,$$

$$(2kr)^{1/2} \cos \tfrac{1}{2}(\theta \pm \Theta) = (2kr')^{1/2} \cos \tfrac{1}{2}(\theta' \pm \Theta) + \\ + \tfrac{1}{2} q (2k/r')^{1/2} \cos \tfrac{1}{2}(\theta' \mp \Theta),$$

$$F(z + \delta z) = F(z) - (\delta z) \exp (iz^2).$$

To the order to which we are working

$$G(r,\theta) = e^{ikq \cos \Theta} G(r',\theta') - \\ - (2/\pi)^{1/2} e^{-i\pi/4} kq \sin \tfrac{1}{2}\theta' \sin \tfrac{1}{2}\Theta (kr')^{-1/2} e^{ikr'}. \tag{5.59}$$

The total field is $G(r,\theta) + \phi_1 + \phi_2$, from (5.54). If we choose $q = p = 0{\cdot}22b$, the last term in (5.59) will cancel the first term in (5.56a), for ϕ_2, and we find

$$\phi = e^{ikp\cos\Theta}G(r',\theta') + \phi_1(r,\theta) + e^{-i\pi/4}\tfrac{1}{2}(2/\pi)^{1/2}\,kb(kr')^{-1/2}e^{ikr'}.$$

ϕ_1 is a small term and to our order of approximation $\phi_1(r,\theta) = \phi_1(r',\theta')$. The first two terms then represent the field of a semi-infinite duct extending from $x = -\infty$ to $x = p = 0{\cdot}22b$. The last term is equivalent to the potential produced by a line source of strength $-i(2\pi)^{-1}kb$, assuming a line source of unit strength produces a field $\pi i H_1^{(1)}(kr')$. This line source is due to the fact that there would be a wave of fundamental frequency propagated along the corresponding semi-infinite duct. For the condition $\phi_t = 0$ on the plate, for small thicknesses of plate, there will be no propagated wave and this term will not occur. These results are independent of Θ and therefore hold for any field constructed from a spectrum of plane waves.

The technique of this section has been applied to a semi-infinite solid cylindrical rod by D. S. Jones [5]. Propagation of waves in a two-dimensional duct with a step has been considered by W. E. Williams [2] (cf. exs. 5.1–5.4). The corresponding problem for a cylindrical duct has been considered by V. M. Papadopoulos [2]. (The reader may prefer the method used in these references to the method used in this section. Cf. ex. 5.2.)

From a numerical point of view the success of the method depends on whether it is practical to perform the factorizations and solve the simultaneous linear equations. One of the main reasons for discussing the numerical details in the above example is to show that in this particular case a suitable solution can be obtained by solving a comparatively small number of the simultaneous linear equations. Also the first unknown is much larger than the others and it is possible to obtain crude results by considering only the first equation and assuming that all the unknowns are zero except the first. It would appear that these remarks are often true in other cases.

5.5 General theory of another special equation

As explained in the next section by means of specific examples, the following equation occurs in connection with three-part mixed boundary value problems:

$$e^{i\alpha q}\Phi_+(\alpha) + K(\alpha)\Phi_1(\alpha) + e^{i\alpha p}\Phi_-(\alpha) = \frac{A}{(2\pi)^{1/2}}\cdot\frac{e^{i(\alpha-k\cos\Theta)q} - e^{i(\alpha-k\cos\Theta)p}}{\alpha - k\cos\Theta}, \tag{5.60}$$

where A is a constant and the equation holds in the strip $-k_2 < \tau < k_2$. $\Phi_+(\alpha)$, $\Phi_-(\alpha)$, $\Phi_1(\alpha)$ are unknown functions.

$$\begin{aligned}\Phi_+(\alpha) &= \frac{1}{(2\pi)^{1/2}} \int_q^\infty \phi(x) e^{i\alpha(x-q)}\, dx \\ \Phi_-(\alpha) &= \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^p \phi(x) e^{i\alpha(x-p)}\, dx \\ \Phi_1(\alpha) &= \frac{1}{(2\pi)^{1/2}} \int_p^q \phi(x) e^{i\alpha x}\, dx.\end{aligned} \tag{5.61}$$

$\Phi_+(\alpha)$ is assumed regular in $\tau > -k_2$, $\Phi_-(\alpha)$ in $\tau < k_2$, and $\Phi_1(\alpha)$ is an integral function.

$$e^{-i\alpha q}\Phi_1(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_p^q \phi(x) e^{i\alpha(x-q)}\, dx = \frac{1}{(2\pi)^{1/2}} \int_{p-q}^0 \phi(u+q) e^{i\alpha u}\, du,$$

so that the left-hand side is regular in a lower half-plane and has algebraic behaviour at infinity as $\alpha \to \infty$ in a lower half-plane $\tau < k_2$. Similarly $\exp(-i\alpha p)\Phi_1(\alpha)$ is regular in an upper half-plane and has algebraic behaviour as $\alpha \to \infty$ in the upper half-plane $\tau > -k_2$. It is assumed that $K(\alpha)$ is regular in $-k_2 < \tau < k_2$, and has branch points at $\alpha = \pm k$. Suppose that we can write

$$K(\alpha) = K_+(\alpha)\, K_-(\alpha) \qquad : \qquad K_+(-\alpha) = K_-(+\alpha).$$

Assume $0 < \Theta < \frac{1}{2}\pi$ so that $k_2 \cos \Theta > 0$. It will then prove convenient to rearrange (5.60) so as to apply the Wiener–Hopf technique in a strip $-k_2 < \tau < k_2 \cos \Theta$. (If $\frac{1}{2}\pi < \Theta < \pi$ then $k_2 \cos \Theta < 0$ and we should apply the Wiener–Hopf technique in a strip $k_2 \cos \Theta < \tau < k_2$. In either case the object is to obtain relations holding in a symmetrical strip $-k_2|\cos \Theta| < \tau < k_2|\cos \Theta|$. The reason for this will become clear below.)

Multiply (5.60) by $\exp(-i\alpha q)\{K_+(\alpha)\}^{-1}$ and rearrange in the form

$$\begin{aligned}\frac{\Phi_+(\alpha)}{K_+(\alpha)} - \frac{A}{(2\pi)^{1/2}} \cdot \frac{e^{-ik\cos\Theta q}}{\alpha - k\cos\Theta} \left\{\frac{1}{K_+(\alpha)} - \frac{1}{K_+(k\cos\Theta)}\right\} + \\ + U_+(\alpha) + V_+(\alpha) = -e^{-i\alpha q} K_-(\alpha)\Phi_1(\alpha) - U_-(\alpha) - V_-(\alpha) + \\ + \frac{A}{(2\pi)^{1/2}} \cdot \frac{e^{-ik\cos\Theta q}}{(\alpha - k\cos\Theta)K_+(k\cos\Theta)}.\end{aligned} \tag{5.62}$$

In this equation we have written

$$U_+(\alpha) + U_-(\alpha) = e^{i\alpha(p-q)}\Phi_-(\alpha)/K_+(\alpha),$$

$$V_+(\alpha) + V_-(\alpha) = A(2\pi)^{-1/2}e^{i\alpha(p-q)-ik\cos\Theta p}/\{(\alpha - k\cos\Theta)K_+(\alpha)\}.$$

These decompositions cannot be performed by inspection and it is necessary to use the general theorem B of §1.3 (cf. (5.3) above). In a similar way, multiply (5.60) by $\exp(-i\alpha p)\{K_-(\alpha)\}^{-1}$ and rearrange as

$$\frac{\Phi_-(\alpha)}{K_-(\alpha)} + R_-(\alpha) + \frac{A}{(2\pi)^{1/2}}\cdot\frac{e^{-ik\cos\Theta p}}{(\alpha - k\cos\Theta)K_-(\alpha)} - S_-(\alpha)$$

$$= -e^{-i\alpha p}K_+(\alpha)\Phi_1(\alpha) - R_+(\alpha) + S_+(\alpha), \tag{5.63}$$

where

$$R_+(\alpha) + R_-(\alpha) = e^{i\alpha(q-p)}\Phi_+(\alpha)/K_-(\alpha),$$

$$S_+(\alpha) + S_-(\alpha) = A(2\pi)^{-1/2}e^{i\alpha(q-p)-ik\cos\Theta q}/\{(\alpha - k\cos\Theta)K_-(\alpha)\}.$$

The left-hand side of (5.62) and the right-hand side of (5.63) are regular in $\tau > -k_2$. The other sides are regular in $\tau < k_2\cos\Theta$. Assume that behaviours at infinity are such that Liouville's theorem can be applied in the usual way to prove that each side of each equation equals zero. We are interested in the left-hand sides of these equations. It will appear that it is convenient, for brevity, to introduce the notation

$$\Phi_+(\alpha) - (2\pi)^{-1/2}Ae^{-ik\cos\Theta q}(\alpha - k\cos\Theta)^{-1} = \Psi_+^*(\alpha), \tag{5.64a}$$

$$\Phi_-(\alpha) + (2\pi)^{-1/2}Ae^{-ik\cos\Theta p}(\alpha - k\cos\Theta)^{-1} = \Psi_-(\alpha), \tag{5.64b}$$

where we use a star to remind us that Ψ_+^* has a pole at $\alpha = k\cos\Theta$ but apart from this it is regular in $\tau > -k_2$. Ψ_- is regular in $\tau < k_2\cos\Theta$. On equating the left-hand sides of (5.62), (5.63) to zero, introducing explicit expressions for the U_+, V_+, R_+, S_+ by using the general decomposition theorem (e.g. (5.3)) and using notation (5.64), we obtain

$$\frac{\Psi_+^*(\alpha)}{K_+(\alpha)} + \frac{1}{2\pi i}\int_{ic-\infty}^{ic+\infty}\frac{e^{i\zeta(p-q)}\Psi_-(\zeta)}{(\zeta-\alpha)K_+(\zeta)}\,d\zeta +$$

$$+\frac{A}{(2\pi)^{1/2}}\cdot\frac{e^{-ik\cos\Theta q}}{(\alpha - k\cos\Theta)K_+(k\cos\Theta)} = 0,$$

$$\frac{\Psi_-(\alpha)}{K_-(\alpha)} - \frac{1}{2\pi i}\int_{id-\infty}^{id+\infty}\frac{e^{i\zeta(q-p)}\Psi_+^*(\zeta)}{(\zeta-\alpha)K_-(\zeta)}\,d\zeta = 0.$$

In these equations $-k_2 < d < k_2 \cos \Theta$, $-k_2 < c < k_2 \cos \Theta$. In the first equation $\tau > c$; in the second $\tau < d$. From the assumption $0 < \Theta < \frac{1}{2}\pi$ we can choose a so that $-k_2 \cos \Theta < a < k_2 \cos \Theta$ and take $d = -c = a$. In the first equation above replace ζ by $(-\zeta)$. In the second equation replace α by $(-\alpha)$. This gives, remembering that $K_+(-\alpha) = K_-(+\alpha)$,

$$\frac{\Psi_+^*(\alpha)}{K_+(\alpha)} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\zeta(q-p)}\Psi_-(-\zeta)}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta + $$
$$+ \frac{A}{(2\pi)^{1/2}} \cdot \frac{e^{-ik\cos\Theta q}}{(\alpha - k\cos\Theta)K_+(k\cos\Theta)} = 0,$$

$$\frac{\Psi_-(-\alpha)}{K_+(\alpha)} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\zeta(q-p)}\Psi_+^*(\zeta)}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta = 0,$$

where now $\tau > -a$ in both equations. Define

$$S_+^*(\alpha) = \Psi_+^*(\alpha) + \Psi_-(-\alpha) \quad : \quad D_+^*(\alpha) = \Psi_+^*(\alpha) - \Psi_-(-\alpha), \tag{5.65}$$

where in this case the star denotes that expressions are regular in $(\mathrm{Im}\,\alpha) > -k_2 \cos \Theta$ except for simple poles at $\alpha = k \cos \Theta$. Then adding and subtracting the above equations gives

$$\frac{S_+^*(\alpha)}{K_+(\alpha)} - \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\zeta(q-p)}S_+^*(\zeta)}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta + $$
$$+ \frac{A}{(2\pi)^{1/2}} \cdot \frac{e^{-ik\cos\Theta q}}{(\alpha - k\cos\Theta)K_+(k\cos\Theta)} = 0, \tag{5.66a}$$

$$\frac{D_+^*(\alpha)}{K_+(\alpha)} + \frac{1}{2\pi i}\int_{ia-\infty}^{ia+\infty} \frac{e^{i\zeta(q-p)}D_+^*(\zeta)}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta + $$
$$+ \frac{A}{(2\pi)^{1/2}} \cdot \frac{e^{-ik\cos\Theta q}}{(\alpha - k\cos\Theta)K_+(k\cos\Theta)} = 0. \tag{5.66b}$$

Each of these equations is of the same type and we obtain an approximate solution by a method due to D. S. Jones [2]. Consider first the asymptotic evaluation of integrals of the form

$$I = \int_{ia-\infty}^{ia+\infty} \frac{1}{\zeta+\beta} H(\zeta) e^{i\zeta l}\,d\zeta, \tag{5.67a}$$

where $-(\mathrm{Im}\,\beta) < a < k_2$. Suppose that we can write

$$H(\zeta) = (\zeta - k)^{r+\frac{1}{2}} f(\zeta) \tag{5.67b}$$

$$= (\zeta - k)^{r+\frac{1}{2}}\{f(p) + (\zeta - p)f'(p) + \ldots\}, \tag{5.67c}$$

where r is one of the numbers $-1, 0, +1, +2, \ldots$, and $f(\zeta)$ can be expanded as a Taylor series about an arbitrary number p which will be chosen later. $f(p)$ is non-zero and finite. Since $-(\mathrm{Im}\,\beta) < a$, the pole at $\zeta = -\beta$ lies below the contour. On the other hand β can lie close to $\zeta = -k$ and this is the reason why the factor $(\zeta + \beta)^{-1}$ is kept separate from $H(\zeta)$: otherwise it would be sufficient to expand $(\zeta + \beta)^{-1}H(\zeta)$ in the form (5.67c). In (5.67a) it is assumed that $H(\zeta)$ possesses no branch points other than $\zeta = k$ in $(\mathrm{Im}\,\zeta) \geqslant a$ and that l is a large positive number so that the contour can be deformed into the upper ζ-plane. Cut the ζ-plane from $k = k_1 + ik_2$ to $k_1 + i\infty$ by a straight line parallel to the imaginary axis and deform the contour on to the two sides of the cut. When the contour is deformed in this way the contributions from any poles crossed in the process must be included: these are ignored here. Deform the contour and set $\zeta - k = iu$. On the right-hand side of the cut $(\zeta - k)^{1/2} = u^{1/2}\exp(i\pi/4)$ and u goes from 0 to ∞. On the left-hand side $(\zeta - k)^{1/2} = u^{1/2}\exp(-3i\pi/4)$ and u goes from $+\infty$ to 0. Then (5.67a, c) give

$$I \sim 2e^{i\pi/4}e^{ikl}i^r[f(p)W_r(z) + $$
$$+ f'(p)\{il^{-1}W_{r+1}(z) + (k - p)W_r(z)\} + \ldots]. \tag{5.68a}$$

In this equation $z = -i(k + \beta)l$ and we have used the notation

$$W_{j-\frac{1}{2}}(z) = \int_0^\infty \frac{u^j e^{-u}}{u + z}\,du = \Gamma(j + 1)e^{\frac{1}{2}z}z^{\frac{1}{2}j-\frac{1}{2}}W_{-\frac{1}{2}(j+1),\frac{1}{2}j}(z), \tag{5.68b}$$

where $W_{k,m}$ is a Whittaker function (cf. ex. 5.9).

The expansion used by D. S. Jones corresponds to the choice $p = k$ in (5.68a). Then

$$I \sim D_r W_r\{-i(k + \beta)l\} \quad : \quad D_r = 2e^{i\pi/4}e^{ikl}l^{-r-\frac{1}{2}}i^r f(k). \tag{5.69}$$

This is the expansion used in the remainder of this section.

An improved expansion is obtained by choosing p in (5.68a) so that the coefficient of $f'(p)$ vanishes, i.e.

$$p = k + il^{-1}W_{r+1}(z)/W_r(z) \quad : \quad z = -i(k + \beta)l.$$

This gives an expansion similar to (5.69) except that $f(k)$ is replaced by $f(p)$. (There is some scope for investigation of improvements on (5.69). This asymptotic formula is the factor which limits the accuracy of the method of this section. An alternative approach would be to write, in (5.67a),

$$\frac{1}{\zeta+\beta}H(\zeta) = \frac{(\zeta-k)^{r+\frac{1}{2}}}{\zeta+\beta}f(-\beta) + \frac{(\zeta-k)^{r+\frac{1}{2}}}{\zeta+\beta}\{f(\zeta)-f(-\beta)\},$$

where we have used the notation (5.67b). The second term on the right is regular at $\zeta+\beta=0$ and the first gives an integral which can be expressed in terms of the W_r function. It may also be practical to include several terms of an asymptotic expansion.)

Now return to the problem of finding an asymptotic solution of

$$\frac{F_+^*(\alpha)}{K_+(\alpha)} + \frac{\lambda}{2\pi i}\int_{ia-\infty}^{ia+\infty}\frac{F_+^*(\alpha)e^{i\zeta(q-p)}}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta$$
$$= -\frac{A}{(2\pi)^{1/2}}\cdot\frac{e^{-ik\cos\Theta q}}{(\alpha-k\cos\Theta)K_+(k\cos\Theta)}, \tag{5.70}$$

where F_+^* will be set equal to S_+^*, D_+^* in turn, so that from (5.64), (5.65), (5.66), F_+^* has the form

$$F_+^*(\alpha) = F_+(\alpha) - \frac{A}{(2\pi)^{1/2}}\frac{e^{-ik\cos\Theta q}}{\alpha-k\cos\Theta} + \frac{\lambda A}{(2\pi)^{1/2}}\cdot\frac{e^{-ik\cos\Theta p}}{(\alpha+k\cos\Theta)}. \tag{5.71}$$

λ is a constant which later we shall take as -1 or $+1$. $F_+(\alpha)$ is regular in $\tau > -k_2$. We should expect that $F_+(\alpha)$ will have a branch point at $\alpha=-k$ but for large l this is sufficiently far from the point $\alpha=+k$ to enable us to write in the above asymptotic expansion:

if $$\{K_-(\zeta)\}^{-1} \sim h_r(\zeta-k)^{r+\frac{1}{2}} \quad \text{as} \quad \zeta\to k, \tag{5.72a}$$

then $$F_+(\zeta)/K_-(\zeta) \sim F_+(k)h_r(\zeta-k)^{r+\frac{1}{2}} \quad \text{as} \quad \zeta\to k.$$

Define

$$E_r = 2e^{i\pi/4}e^{ikl}l^{-r-\frac{1}{2}}i^r h_r. \tag{5.72b}$$

(5.67a), (5.69) give, on writing $(q-p)=l$,

$$\int_{ia-\infty}^{ia+\infty}\frac{F_+(\zeta)e^{i\zeta(q-p)}}{(\zeta+\alpha)K_-(\zeta)}\,d\zeta \sim E_rW_r\{-i(k+\alpha)l\}F_+(k)$$
$$= 2\pi iT(\alpha)F_+(k), \quad \text{say}; \tag{5.72c}$$

$$\int_{ia-\infty}^{ia+\infty}\frac{e^{i\zeta(q-p)}\,d\zeta}{(\zeta+\alpha)(\zeta-k\cos\Theta)K_-(\zeta)}$$
$$\sim 2\pi i\frac{e^{ik\cos\Theta l}}{(\alpha+k\cos\Theta)K_-(k\cos\Theta)} + 2\pi iR_2(\alpha), \quad \text{say};$$

$$\int_{ia-\infty}^{ia+\infty}\frac{e^{i\zeta(q-p)}\,d\zeta}{(\zeta+\alpha)(\zeta+k\cos\Theta)K_-(\zeta)} \sim 2\pi iR_1(\alpha), \quad \text{say},$$

where

$$R_{1,2}(\alpha) = \frac{E_r[W_r\{-i(k \pm k\cos\Theta)l\} - W_r\{-i(k+\alpha)l\}]}{2\pi i(\alpha \mp k\cos\Theta)}, \quad (5.73)$$

and upper and lower signs refer to R_1, R_2 respectively. The last two results are obtained by splitting the integrand into partial fractions. On introducing (5.71) into (5.70) and using the above expressions for the resulting integrals we find:

$$F_+(\alpha)/K_+(\alpha) = (2\pi)^{-1/2}Ae^{-ik\cos\Theta q}\{P_1(\alpha) + \lambda R_2(\alpha)\} -$$
$$-(2\pi)^{-1/2}\lambda Ae^{-ik\cos\Theta p}\{P_2(\alpha) + \lambda R_1(\alpha)\} - \lambda T(\alpha)F_+(k), \quad (5.74)$$

where, in addition to our previous definitions,

$$P_{1,2}(\alpha) = \frac{1}{\alpha \mp k\cos\Theta}\left\{\frac{1}{K_+(\alpha)} - \frac{1}{K_+(\pm k\cos\Theta)}\right\}. \quad (5.75)$$

Upper and lower signs refer to P_1, P_2 respectively. The value of $F_+(k)$ can be found by setting $\alpha = k$ in (5.74). When this value of $F_+(k)$ is substituted back into (5.74) we find, on remembering that $\lambda^2 = 1$,

$$\frac{F_+(\alpha)}{K_+(\alpha)} = \frac{A}{(2\pi)^{1/2}}\{G_1(\alpha) - \lambda G_2(\alpha)\} -$$
$$-\frac{\lambda A}{(2\pi)^{1/2}}\cdot\frac{K_+(k)T(\alpha)}{1+\lambda T(k)K_+(k)}\{G_1(k) - \lambda G_2(k)\}, \quad (5.76a)$$

where

$$G_1(\alpha) = e^{-ik\cos\Theta q}P_1(\alpha) - e^{-ik\cos\Theta p}R_1(\alpha), \quad (5.76b)$$

$$G_2(\alpha) = e^{-ik\cos\Theta p}P_2(\alpha) - e^{-ik\cos\Theta q}R_2(\alpha). \quad (5.76c)$$

Next return to (5.66). From definitions (5.64), (5.65), for $S_+^*(\alpha)$ we take $\lambda = -1$, $F_+(\alpha) = \Phi_+(\alpha) + \Phi_-(-\alpha)$, and for $D_+^*(\alpha)$ we take $\lambda = +1$, $F_+(\alpha) = \Phi_+(\alpha) - \Phi_-(-\alpha)$. Substitute these values in turn into (5.76) and add the resulting equations. This gives

$$\Phi_+(\alpha) = (2\pi)^{-1/2}AG_1(\alpha)K_+(\alpha) +$$
$$+\frac{A}{(2\pi)^{1/2}}\frac{K_+(k)T(\alpha)K_+(\alpha)}{\{1 - T^2(k)K_+^2(k)\}}\{G_2(k) + T(k)K_+(k)G_1(k)\}. \quad (5.77)$$

If in this equation G_1 is replaced by G_2, and G_2 by G_1, and the sign of α is changed, the resulting expression gives $\Phi_-(\alpha)$.

In specific problems it will be possible to simplify these formulae to some extent by neglecting some terms completely and using the asymptotic forms of W_r in other terms. In any case the W_r which are of importance in applications can be expressed in terms of tabulated functions (ex. 5.9). An application of this analysis is given in the next section.

5.6 Diffraction by strips and slits of finite width

Consider diffraction of the incident wave $\phi_i = \exp(-ikx\cos\Theta - iky\sin\Theta)$ in the following four cases:

(i) By the strip $p \leqslant x \leqslant q$, $y = 0$, with $\partial\phi_t/\partial y = 0$ on the strip,
(ii) By the strip $p \leqslant x \leqslant q$, $y = 0$, with $\phi_t = 0$ on the strip,
(iii) By the slit $p \leqslant x \leqslant q$, $y = 0$ with $\partial\phi_t/\partial y = 0$ on $y = 0$, off the slit,
(iv) By the slit $p \leqslant x \leqslant q$, $y = 0$ with $\phi_t = 0$ on $y = 0$, off the slit.

In cases (i) and (ii) write $\phi_t = \phi_i + \phi$. In cases (iii) and (iv) write

$$\phi_t = e^{-ikx\cos\Theta - iky\sin\Theta} \pm e^{-ikx\cos\Theta + iky\sin\Theta} + \phi, \qquad (y \geqslant 0),$$
$$\phi_t = \phi, \qquad (y \leqslant 0),$$

with the upper sign for case (iii), the lower for case (iv). The object of these substitutions is to ensure that transforms are regular in $-k_2 < \tau < k_2$, since ϕ tends to zero as $\exp(-k_2|x|)$ as $|x| \to \infty$ in all cases.

Apply a Fourier transform in x to the partial differential equation for ϕ and find in the usual way

$$\Phi(y) = A\exp(-\gamma y), \qquad (y \geqslant 0); \qquad B\exp(\gamma y), \qquad (y \leqslant 0).$$

Differentiate with respect to y, eliminate A and B and let y tend to plus and minus zero respectively. This gives

$$\Phi'(+0) = -\gamma\Phi(+0) \qquad : \qquad \Phi'(-0) = \gamma\Phi(-0).$$

Introduce notation similar to (5.61) except that instead of $\Phi_+(\alpha)$ we write, for convenience, $\Phi_+(y)$. As usual dashes denote differentiation with respect to y. Then the above equations become

$$e^{i\alpha q}\Phi'_+(+0) + \Phi'_1(+0) + e^{i\alpha p}\Phi'_-(+0)$$
$$= -\gamma\{e^{i\alpha q}\Phi_+(+0) + \Phi_1(+0) + e^{i\alpha p}\Phi_-(+0)\}, \quad (5.78a)$$

$$e^{i\alpha q}\Phi'_+(-0) + \Phi'_1(-0) + e^{i\alpha p}\Phi'_-(-0)$$
$$= \gamma\{e^{i\alpha q}\Phi_+(-0) + \Phi_1(-0) + e^{i\alpha p}\Phi_-(-0)\}. \quad (5.78b)$$

In case (i) $\partial\phi/\partial y$ is continuous on $y = 0$ and ϕ is continuous on $y = 0$ except for $p \leqslant x \leqslant q$.

$$\Phi'_1(+0) + \Phi'_1(-0) = \frac{2ik\sin\Theta}{(2\pi)^{1/2}}\int_p^q e^{i(\alpha - k\cos\Theta)x}\,dx = 2k\sin\Theta\, G(\alpha),$$

say,

where

$$G(\alpha) = (2\pi)^{-1/2}(\alpha - k\cos\Theta)^{-1}\{e^{i(\alpha - k\cos\Theta)q} - e^{i(\alpha - k\cos\Theta)p}\}.$$

Hence the addition of (5.78), setting $\Phi_1(+0) - \Phi_1(-0) = 2\Psi_1(0)$, gives

$$e^{i\alpha q}\Phi'_+(0) + \gamma\Psi_1(0) + e^{i\alpha p}\Phi'_-(0) = -k \sin \Theta G(\alpha). \tag{5.79}$$

This is of form (5.60).

Similarly in case (ii) ϕ is continuous on $y = 0$ and $\partial\phi/\partial y$ is continuous on $y = 0$ except for $p \leqslant x \leqslant q$. Also

$$\Phi_1(+0) + \Phi_1(-0) = 2iG(\alpha).$$

Subtract (5.78), write $\Phi_1'(+0) - \Phi_1'(-0) = 2\Psi_1'(0)$, and find

$$e^{i\alpha q}\Phi_+(0) + \gamma^{-1}\Psi_1'(0) + e^{i\alpha p}\Phi_-(0) = -iG(\alpha). \tag{5.80}$$

In case (iii), $\partial\phi/\partial y$ is continuous on $y = 0$ and $\Phi'_+(+0) = \Phi'_+(-0) = 0$, $\Phi'_-(+0) = \Phi'_-(-0) = 0$. Also

$$\Phi_1(+0) - \Phi_1(-0) = 2iG(\alpha).$$

Add (5.78), write $\Phi_+(+0) - \Phi_+(-0) = 2\Psi_+(0)$ etc. to find

$$e^{i\alpha q}\Psi_+(0) + \gamma^{-1}\Phi_1'(0) + e^{i\alpha p}\Psi_-(0) = -iG(\alpha). \tag{5.81}$$

In case (iv), ϕ is continuous on $y = 0$ and $\Phi_+(+0) = \Phi_+(-0) = 0$, $\Phi_-(+0) = \Phi_-(-0) = 0$. Also

$$\Phi_1'(+0) - \Phi_1'(-0) = 2k \sin \Theta G(\alpha).$$

Subtract (5.78), write $\Phi'_+(+0) - \Phi'_+(-0) = 2\Psi'_+(0)$ etc. to find

$$e^{i\alpha q}\Psi'_+(0) + \gamma\Phi_1(0) + e^{i\alpha p}\Psi'_-(0) = -k \sin \Theta G(\alpha). \tag{5.82}$$

The equations for all four cases are of form (5.60). The similarity of (5.79), (5.82) corresponding to cases (i), (iv), and of (5.80), (5.81) corresponding to cases (ii), (iii), is obviously connected with Babinet's principle.

An approximate solution in each of the above cases can be obtained by using the theory in §5.5. Consider case (iv). Comparing the notation in (5.60) and (5.82) we have

$$\Phi_+(\alpha) \equiv \Psi'_+(0) \quad : \quad \Phi_-(\alpha) \equiv \Psi'_-(0),$$

$$K(\alpha) = (\alpha^2 - k^2)^{1/2} \quad : \quad A = -k \sin \Theta.$$

Hence

$$K_+(\alpha) = e^{-i\pi/4}(k + \alpha)^{1/2} \quad : \quad K_-(\alpha) = e^{-i\pi/4}(k - \alpha)^{1/2} = e^{i\pi/4}(\alpha - k)^{1/2}.$$

From (5.72a), $\quad r = -1 \quad : \quad h_{-1} = e^{-i\pi/4}$.

From (5.72b), $\quad E_{-1} = 2l^{1/2}e^{ikl}e^{-i\pi/2}$.

Consider the field in $y \leqslant 0$. Using the notation defined in connexion with (5.82), the transform of the potential in $y \leqslant 0$ is given by

$$\Phi(y) = B \exp \gamma y = -\gamma^{-1}\{e^{i\alpha q}\Psi'_{+}(0) + \Psi'_1(0) + e^{i\alpha p}\Psi'_{-}(0)\}e^{\gamma y}.$$

$$\therefore \qquad \phi(x,y) = -\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \gamma^{-1}\{e^{i\alpha q}\Psi'_{+}(0) + \Psi'_1(0) + e^{i\alpha p}\Psi'_{-}(0)\}\, e^{\gamma y - i\alpha x}\, d\alpha.$$

In these equations $\Psi'_1(0) = -AG(\alpha)$. $\Psi'_{+}(0)$, $\Psi'_{-}(0)$ can be computed in a straightforward way from (5.77), using definitions (5.72)–(5.76). For convenience we assume that the slit lies in $-d \leqslant x \leqslant d$ so that $q = d$, $p = -d$.

When the various approximations are substituted into the integral it is found that the potential can be split into two parts:

$$\phi(tot.) = \phi(sep.) + \phi(int.).$$

$\phi(sep.)$ is the sum of the diffracted waves produced by each half-plane separately i.e. acting as though the other half-plane were absent. This is a natural first approximation.

$$\phi(sep.) = \frac{-ik \sin \Theta}{2\pi(k + k \cos \Theta)^{1/2}} \int_{ia-\infty}^{ia+\infty} \frac{e^{i(\alpha - k \cos \Theta)d}}{(\alpha - k \cos \Theta)(k - \alpha)^{1/2}} e^{-i\alpha x - \gamma|y|}\, d\alpha$$
$$+ \frac{ik \sin \Theta}{2\pi(k - k \cos \Theta)^{1/2}} \int_{ia-\infty}^{ia+\infty} \frac{e^{-i(\alpha - k \cos \Theta)d}}{(\alpha - k \cos \Theta)(k + \alpha)^{1/2}} e^{-i\alpha x - \gamma|y|}\, d\alpha.$$

The exact value of $\phi(sep.)$ can be found in terms of Fresnel integrals by using (1.62), (1.65) and the corresponding expressions obtained by changing the signs of α, x and replacing Θ by $(\pi - \Theta)$.

$\phi(int.)$ is an interaction term. Using the notation of (5.72)–(5.76) it is given by

$$\phi(int.) = \frac{A}{2\pi} \int_{ia-\infty}^{ia+\infty} \left\{ e^{ikd \cos \Theta} R_1(\alpha) \frac{e^{i\alpha d}}{K_{-}(\alpha)} + e^{-ikd \cos \Theta} R_2(-\alpha) \frac{e^{-i\alpha d}}{K_{+}(\alpha)} - \right.$$
$$\left. - C_1 T(\alpha) \frac{e^{i\alpha d}}{K_{-}(\alpha)} - C_2 T(-\alpha) \frac{e^{-i\alpha d}}{K_{+}(\alpha)} \right\} e^{-i\alpha x - \gamma|y|}\, d\alpha,$$

where

$$C_1 = \frac{K_{+}(k)}{1 - T^2(k)K_{+}^2(k)} \{G_2(k) + T(k)K_{+}(k)G_1(k)\},$$

and C_2 is obtained by interchanging G_1 and G_2. On examining the forms of $R_1(\alpha)$, $R_2(\alpha)$, $T(\alpha)$ it is found that the far-field due to

$\phi(int.)$ can be found by a straightforward asymptotic expansion as in §1.6, (1.71). Define (r,θ) by $x = r\cos\theta$, $|y| = r\sin\theta$, so that θ is measured from the positive x-axis in a clockwise direction. (Θ is positive measured from the positive x-axis in an anti-clockwise direction.) From (1.71), as $r \to \infty$,

$$\begin{aligned}\phi(int.) \sim -k^2(2\pi)^{-1/2}\sin\Theta\sin\theta e^{-i\pi/4}(kr)^{-1/2}e^{ikr} \\ \times [\{K_+(k\cos\theta)\}^{-1}\{e^{ikd(\cos\Theta-\cos\theta)}R_1(-k\cos\theta) - \\ - C_1e^{-ikd\cos\theta}T(-k\cos\theta)\} \\ + \{K_-(k\cos\theta)\}^{-1}\{e^{-ikd(\cos\Theta-\cos\theta)}R_2(k\cos\theta) - \\ - C_2e^{ikd\cos\theta}T(k\cos\theta)\}].\end{aligned}$$

When explicit expressions are substituted in this equation it is found that the W_{-1} functions which occur can be replaced by their asymptotic expansions if (kd) is large, provided that θ, Θ are not nearly equal to 0 or π:

$$W_{-1}\{-2i(k \pm k\cos\delta)d\} \sim \pi^{1/2}\{-2i(k \pm k\cos\delta)d\}^{-1}, \qquad kd \to \infty,$$

if δ is not nearly equal to 0 or π. It is then found that considerable simplifications occur, and after some manipulation the interaction field as $r \to \infty$ turns out to be

$$\begin{aligned}\frac{i}{(2\pi)^{1/2}} \cdot \frac{1}{1 + (4\pi ikd)^{-1}e^{4ikd}} \cdot \frac{e^{2ikd}}{(4\pi kd)^{1/2}} \cdot \frac{e^{ikr}}{(kr)^{1/2}} \\ \times [\{(\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\Theta)^{-1}e^{-ikd(\cos\theta-\cos\Theta)} + (\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\Theta)^{-1}e^{ikd(\cos\theta-\cos\Theta)}\} \\ - e^{i\pi/4}(4\pi kd)^{-1/2}e^{2ikd}\{(\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\Theta)^{-1}e^{-ikd(\cos\theta+\cos\Theta)} + \\ + (\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\Theta)^{-1}e^{ikd(\cos\theta+\cos\Theta)}\}].\end{aligned}$$

This is identical with the result found by S. N. Karp and A. Russek [1] who used a completely different argument. (Note that they used a time-factor $\exp(+i\omega t)$ so that in comparisons the sign of i must be changed. Also their ϕ, ϕ_0 are related to our θ, Θ by $\theta + \phi = \frac{1}{2}\pi$, $\Theta - \phi_0 = \frac{1}{2}\pi$.) The argument used by S. N. Karp and A. Russek is essentially physical. It is assumed that the field diffracted by, say, the left-hand half-plane can be regarded as produced by a line source at the edge of the left half-plane when considering the interaction effect caused by diffraction of this field by the right-hand half-plane. There is a discussion of this physical argument, and of numerical results, in the paper of S. N. Karp and A. Russek.

The present method and the Karp–Russek method are both related to the work of K. Schwartzchild and E. N. Fox. References and some discussion can be found in B. Baker and E. T. Copson [1], and S. N. Karp and A. Russek [1].

The exposition we have given may have obscured the laborious nature of the present method, whereas the Karp–Russek method is relatively elementary and direct. On the other hand the Wiener–Hopf method can be extended to many other problems in a straightforward way. In would be interesting to know whether the Karp–Russek method could be extended to deal with more complicated problems, assuming the results for corresponding semi-infinite obstacles.

Three papers apply the analysis in the last two sections to more complicated problems. D. S. Jones [2] and W. E. Williams [1] examine diffraction by a waveguide of finite length with $\phi_t = 0$, $\partial\phi_t/\partial n = 0$, respectively, on the walls of the guide. When the separation of the plates tends to zero these reduce to the strip problems considered in this section. W. E. Williams [3] examines diffraction by a circular waveguide of finite length. Various complicating features can appear in these problems. Resonance can occur within the guide: in this case the factor $\{1 - T^2(k)K_+^2(k)\}$ in (5.77) is small. Poles may occur at $\alpha = k$ or near $\alpha = k$ in addition to the poles at $\alpha = \pm k \cos \Theta$: the expansions of $K_+(\alpha)$, $K_-(\alpha)$ may have expansions containing significant terms of the form $(\alpha - k)^s \ln (\alpha - k)$: if either of these complications occur it will be necessary to modify the general formula (5.77).

It may be possible to improve the treatment of the present section by taking further terms in the asymptotic expansion or by using a different type of asymptotic expansion. However one basic feature underlying the whole of this chapter is that if the singularities of certain critical functions are simple poles a solution can be given which is satisfactory for small wave numbers: if branch points occur an asymptotic expansion method will give a solution suitable for large wave numbers. If it is desired to find two approximate solutions, one for large and one for small wave numbers, it is at present necessary to formulate the problem in two different ways by using the possibilities outlined in §5.2 or by using some completely different method (cf. ex. 5.12 for the strip problem).

Most of the topics considered in this chapter offer considerable scope for further investigation.

Miscellaneous Examples and Results V

5.1 Suppose that a wave $\phi_i = \exp(ikz)$ is incident from $z = -\infty$ in a rigid-walled duct with a step, occupying $0 \leqslant y \leqslant D$, $-\infty < z \leqslant 0$, and $d \leqslant y \leqslant D$, $0 \leqslant z < \infty$, $D > d > 0$. Set $\phi_t = \phi_i + \phi$ so that $\partial\phi/\partial y = 0$ on walls $y = \text{const.}$, and $\partial\phi/\partial z = -ik$ on $z = 0$, $0 \leqslant y \leqslant d$.

In exs. 5.1–5.4 we consider this example in detail to show that from a theoretical point of view the analysis in §5.4 is by no means complete:

in this type of problem the basic equations can be obtained by other methods, and the problem can be formulated in terms of a different, complementary, set of equations. W. E. Williams [2] has obtained (e), (g) below by using a Laplace transform and a variant of the method of ex. 5.2. He gives a discussion of the physical interpretation of the results together with numerical values of the reflection coefficient for various (kD/π), (kd/π) including the case where more than one wave propagates in the duct.

Apply Fourier transforms in the z-direction. In $0 \leqslant y \leqslant d$, $z \leqslant 0$, the partial differential equation becomes

$$d^2\Phi_-(\alpha,y)/dy^2 - \gamma^2\Phi_-(\alpha,y) = (2\pi)^{-1/2}ik + (2\pi)^{-1/2}i\alpha(\phi)_0. \qquad \text{(a)}$$

Eliminate $(\phi)_0$ as in §5.4 to obtain

$$\Phi_-(\alpha)+\Phi_-(-\alpha) = -(2\pi)^{-1/2}2ik\gamma^{-2}+\gamma^{-1}\coth \gamma d\{\Phi'_-(\alpha)+ \Phi'_-(-\alpha)\}. \quad \text{(b)}$$

In $d \leqslant y \leqslant D$

$$\Phi_+(\alpha) + \Phi_-(\alpha) = -\gamma^{-1}\coth \gamma(D - d)\Phi'_-(\alpha). \qquad \text{(c)}$$

Eliminate $\Phi_-(\alpha)$ and set

$$-\gamma^{-1}\sinh \gamma D\{\sinh \gamma d \sinh \gamma(D - d)\}^{-1} = M_+(\alpha)M_-(\alpha). \qquad \text{(d)}$$

Separation of the resulting equations as in §5.4 gives

$$M_-(\alpha)\Phi'_-(\alpha) = -\frac{i}{(2\pi)^{1/2}}\cdot\frac{1}{(\alpha - k)M_+(k)} + \frac{1}{2kd(\alpha - k)}\cdot\frac{\Phi'_-(-k)}{M_+(k)} + $$
$$+\sum_{n=1}^{\infty}\frac{1}{i\gamma_n d(\alpha - i\gamma_n)}\cdot\frac{\Phi'_-(-i\gamma_n)}{M_+(i\gamma_n)}, \qquad \text{(e)}$$

where $\gamma_n = \{(n\pi/d)^2 - k^2\}^{1/2}$, $\gamma_0 = -ik$.

Introduce $x_n = -i(2\pi)^{1/2}\Phi'_-(-i\gamma_n)M_+(k)$, (f)

and set $\alpha = -i\gamma_m$ in (e). Then

$$M_+(i\gamma_m)x_m = \frac{1}{k + i\gamma_m} - \frac{1}{2kd(k + i\gamma_m)}\cdot\frac{x_0}{M_+(k)} +$$
$$+\frac{1}{d}\sum_{n=1}^{\infty}\frac{1}{\gamma_n(\gamma_m + \gamma_n)}\cdot\frac{x_n}{M_+(i\gamma_n)}, \qquad (m = 0, 1, 2\ldots). \quad \text{(g)}$$

If only one mode is propagated in the guide these can be reduced to a set of real equations. Introduce $J_+(\alpha) = (k + \alpha)M_+(\alpha)$ where $J_+(i\gamma_n)$ is now real, multiply by $(k + i\gamma_m)$, and separate real and imaginary parts. It will be found that the imaginary part is a series which is independent of m. The first equation of the set (i.e. $m = 0$) can be used to eliminate this imaginary part in the remaining equations. By solving the resulting (real) equations for $m = 1, 2, 3, \ldots$ we can find $x_1, x_2, \ldots$ in terms of x_0. Substitution in the first equation then gives x_0 (cf. W. E. Williams [2]).

5.2 We give an alternative method for obtaining the results of ex. 5.1 which illustrates the physical significance of the $\Phi'_-(-i\gamma_n)$. Proceed as in ex. 5.1 up to equation (a). Suppose that $(\phi)_0$ can be expanded in a cosine series ($\varepsilon'_0 = \frac{1}{2} : \varepsilon'_n = 1, \quad n \geqslant 1$):

$$(\phi)_0 = \frac{2}{d}\sum_{n=0}^{\infty} \varepsilon'_n f_n \cos(n\pi y/d) \qquad : \qquad f_n = \int_0^d (\phi)_0 \cos(n\pi y/d)\, dy. \qquad \text{(h)}$$

Introduce into (a) and solve the resulting equation with $\Phi'_-(\alpha,0) = 0$. Differentiation with respect to y and elimination of an unknown function of α then gives, on $y = d$,

$$\Phi_-(\alpha) = -\frac{ik}{(2\pi)^{1/2}\gamma^2} - \frac{i\alpha}{(2\pi)^{1/2}}\cdot\frac{2}{d}\sum_{n=0}^{\infty} \varepsilon'_n f_n \frac{\cos n\pi}{\alpha^2+\gamma_n^2} + \frac{1}{\gamma}\coth \gamma d\Phi'_-(\alpha). \quad \text{(i)}$$

This replaces (b). Eliminate $\Phi_-(\alpha)$ between (i) and (c) and separate in the usual way to find

$$M_-(\alpha)\Phi'_-(\alpha) = -\frac{i}{(2\pi)^{1/2}2(\alpha-k)M_+(k)} - \\ -\frac{i}{(2\pi)^{1/2}\, d}\sum_{n=0}^{\infty} \varepsilon'_n f_n \frac{\cos n\pi}{(\alpha - i\gamma_n)M_+(i\gamma_n)}. \qquad \text{(j)}$$

To complete the solution we need a relationship between $\Phi'_-(\alpha)$ and f_n which can be found in various ways, e.g.

(i) The left-hand side of (i) is regular in a lower half-plane. Hence the residues at $\alpha = -k,\ -i\gamma_n$, on the right-hand side must cancel, i.e.

$$f_n = (2\pi)^{1/2}\gamma_n^{-1}\cos n\pi\Phi'_-(-i\gamma_n),\ (n > 0) : f_0 = (2\pi)^{1/2}ik^{-1}\Phi'_-(-k) + d. \qquad \text{(k)}$$

(ii) The solution of (a) such that $\Phi'_-(\alpha,0) = 0$ is

$$\Phi_-(\alpha,y) = -(2\pi)^{-1/2}ik\gamma^{-2} + (2\pi)^{-1/2}\gamma^{-1}i\alpha\int_0^y (\phi)_0 \sinh\gamma(y-\xi)\, d\xi + \\ + A\cosh\gamma y. \qquad \text{(l)}$$

Differentiate with respect to y, set $y = d$, and substitute the resulting value of A into (l). The right-hand side of the equation for Φ_- which is obtained in this way must be regular in a lower half-plane, and cancellation of residues at $\alpha = -k,\ -i\gamma_n$ again gives (k).

(iii) Apply a finite cosine transform in $0 \leqslant y \leqslant d$, $z \leqslant 0$ and solve the resulting ordinary differential equation, assuming $\partial\phi/\partial y = 0$ on $y = 0$, and $\partial\phi/\partial y$ known on $y = d$. This gives, for $n \geqslant 1$, with $\partial\phi/\partial z = 0$ on $z = 0$,

$$\int_0^d (\phi)_0 \cos(n\pi y/d)\, dy = \frac{\cos n\pi}{2\gamma_n}\int_{-\infty}^0 \left(\frac{\partial\phi}{\partial y}\right)_d \{e^{-\gamma_n|z-\xi|} + e^{-\gamma_n|z+\xi|}\}\, d\xi.$$

Set $z = 0$ and this gives (k) for $n \geqslant 1$. A similar procedure gives the case $n = 0$.

If now we set $\alpha = -i\gamma_m$ in (j) and use (k), equations are obtained for the f_n. These are equivalent to (g). The advantage of the present method is that the $\Phi'_-(-i\gamma_n)$ are shown to be proportional to the coefficients in the Fourier cosine expansion of ϕ on $z = 0$, $0 \leqslant y \leqslant d$.

5.3 In ex. 5.2 $(\phi)_0$ was expanded in a series such that $\partial\phi/\partial y = 0$ on $y = 0, d$. It is logical to try an alternative expansion such that $\partial\phi/\partial y = 0$ on $y = 0$, $\phi = 0$ on $y = d$ i.e.

$$(\phi)_0 = \frac{2}{d}\sum_{n=0}^{\infty} g_n \cos\{(n+\tfrac{1}{2})\pi y/d\} \quad : \quad g_n = \int_0^d (\phi)_0 \cos\{(n+\tfrac{1}{2})\pi y/d\}\,dy. \tag{m}$$

In place of (i) we find

$$\Phi'_-(\alpha) = \frac{i\alpha}{(2\pi)^{1/2}}\cdot\frac{2\pi}{d^2}\sum_{n=0}^{\infty}(n+\tfrac{1}{2})g_n\frac{\cos n\pi}{\alpha^2+\gamma^2_{n+\frac{1}{2}}} + $$
$$+ \gamma\tanh\gamma d\left\{\Phi_-(\alpha) + \frac{ik}{(2\pi)^{1/2}\gamma^2}\right\}. \tag{n}$$

Cancellation of residues at $\alpha = -i\gamma_{n+\frac{1}{2}}$ gives (cf. (k))

$$(-1)^n g_n = (2\pi)^{1/2}\gamma^{-1}_{n+\frac{1}{2}}(n+\tfrac{1}{2})(\pi/d)\Phi_-(-i\gamma_{n+\frac{1}{2}}) - \{(n+\tfrac{1}{2})(\pi/d)\gamma_{n+\frac{1}{2}}\}^{-1}ik. \tag{o}$$

Eliminate $\Phi'_-(\alpha)$ between (n) and (c). The resulting equation can be split in the usual way to give (cf. (j))

$$N_-(\alpha)\Phi_-(\alpha) = -\frac{ik}{(2\pi)^{1/2}}G_-(\alpha) - \frac{i}{(2\pi)^{1/2}}\cdot\frac{2\pi}{d^2}\sum_{n=0}^{\infty}(-1)^n(n+\tfrac{1}{2})g_n H_-^{(n)}(\alpha), \tag{p}$$

where

$$H_+^{(n)}(\alpha) + H_-^{(n)}(\alpha) = -\alpha(\alpha^2+\gamma^2_{n+\frac{1}{2}})^{-1}\{\gamma\tanh\gamma(D-d)\}^{-1}N_+(\alpha),$$
$$G_+(\alpha) + G_-(\alpha) = -\gamma^{-2}\tanh\gamma d\{\tanh\gamma(D-d)\}^{-1}N_+(\alpha),$$
$$N_+(\alpha)/N_-(\alpha) = -\sinh\gamma D\{\cosh\gamma d\sinh\gamma(D-d)\}^{-1}.$$

On setting $\alpha = -i\gamma_{m+\frac{1}{2}}$ in (p) and using (o) we obtain a set of equations for the g_n.

Of course from a practical point of view this formulation is much more complicated than the previous one because of the splits required for $H_-^{(n)}$, G_-. The main point we wish to make, however, is that theoretically two complementary formulations are possible.

5.4 An approach which gives the two formulations in a direct and symmetrical way is the following. Proceed as in ex. 5.1 up to (b). Deal with this equation in one of two ways.

(i) Rewrite (b) as

$$\Phi_-(\alpha) - A_-(\alpha) + (2\pi)^{-1/2}i(\alpha - k)^{-1} = -\Phi_-(-\alpha) + A_+(\alpha) + (2\pi)^{-1/2}i(\alpha + k)^{-1}, \quad \text{(q)}$$

where

$$A_+(\alpha) + A_-(\alpha) = \gamma^{-1}\coth \gamma d\{\Phi'_-(\alpha) + \Phi'_-(-\alpha)\}.$$

Write

$$\gamma^{-1}\coth \gamma d\Phi'_-(-\alpha) = a_+(\alpha) + a_-(\alpha), \quad \text{say,}$$

where

$$a_-(\alpha) = \frac{1}{2kd(\alpha - k)}\Phi'_-(-k) + \sum_{n=1}^{\infty}\frac{1}{i\gamma_n d(\alpha - i\gamma_n)}\Phi'_-(-i\gamma_n).$$

Then
$$A_-(\alpha) = a_-(\alpha) + a_+(-\alpha).$$

Equation (q) can be separated, and if this value of $A_-(\alpha)$ is inserted the resulting equation is exactly (i).

(ii) Rewrite (b) as

$$\Phi'_-(\alpha) - B_-(\alpha) = -\Phi'_-(-\alpha) + B_+(\alpha), \quad \text{(r)}$$

where

$$B_+(\alpha) + B_-(\alpha) = \gamma \tanh \gamma d\{\Phi_-(\alpha) + \Phi_-(-\alpha) + (2\pi)^{-1/2}2ik\gamma^{-2}\}.$$

Write

$$\gamma \tanh \gamma d\{\Phi_-(\alpha) + (2\pi)^{-1/2}ik\gamma^{-2}\} = b_+(\alpha) + b_-(\alpha), \quad \text{say.}$$

Then
$$B_-(\alpha) = b_-(\alpha) + b_+(-\alpha).$$

Equation (r) can be separated and if this value of $B_-(\alpha)$ is inserted the resulting equation is exactly (n).

5.5 A rigid-walled duct occupying $0 \leqslant y \leqslant D$, $-\infty < z < \infty$ is filled with a medium (1) for which $\nabla^2\phi_1^t + k^2\phi_1^t = 0$ except for the space $0 \leqslant y \leqslant d$, $z \geqslant 0$ which is filled with a medium (2) for which $\nabla^2\phi_2^t + K^2\phi_2^t = 0$. Suppose that at the interface between the two media $\phi_1^t = \phi_2^t$ and $c\,\partial\phi_1^t/\partial n = \partial\phi_2^t/\partial n$. Assume a wave $\phi^i = \exp(ikz)$ incident from $z = -\infty$ and write $\phi_1^t = \phi^i + \phi_1$, $\phi_2^t = \phi^i + \phi_2$, so that $\phi_2 = \phi_1$, $\partial\phi_2/\partial n = c\,\partial\phi_1/\partial n + (c - 1)ik$ on $z = 0$, $0 \leqslant y \leqslant d$. Expand ϕ_1 and $\partial\phi_1/\partial z$ on $z = 0$, $0 \leqslant y \leqslant d$ as cosine series (cf. ex. 5.2):

$$(\phi_1)_0 = \frac{2}{d}\sum_{n=0}^{\infty}\varepsilon'_n f_n \cos(n\pi y/d) \quad : \quad (\partial\phi_1/\partial z)_0 = \frac{2}{d}\sum_{n=0}^{\infty}\varepsilon'_n g_n \cos(n\pi y/d).$$

It will be found that the procedure of ex. 5.2 gives two sets of equations corresponding to (i), (k), each involving both f_n and g_n. Linear combinations of f_n and g_n are related to $\Phi'_{1-}(-i\gamma_n)$ and $\Phi'_{2+}(i\Gamma_n)$ where $\Gamma_n^2 = \{(n\pi/d)^2 - K^2\}$. Equation (c) is still true and the equations can be manipulated to give two relations corresponding to (j). From these simultaneous linear algebraic equations for the f_n, g_n can be deduced. But the splits required for the solution seem complicated (cf. ex. 5.3).

5.6 A plane wave $\exp(ikz)$ is incident in a duct $0 \leqslant y \leqslant 2b$, $-\infty < z < \infty$ on a plate of finite thickness lying in $c \leqslant y \leqslant d$, $0 \leqslant z < \infty$, where $0 < c < d < 2b$. $\partial\phi_t/\partial n = 0$ on all bounding surfaces. Find, in particular, the effect on the reflection coefficient of the thickness of the plate when $(d - c) \ll 2b$ (cf. §3.5).

5.7 Two generalizations of the problem of the strip across a duct (§3.6) give rise to equations of type (5.17), namely (i) the strip of arbitrary width, (ii) the strip of finite resistivity. For simplicity instead of the problem in §3.6 consider the following. Find a solution of the steady-state wave equation in $0 \leqslant y \leqslant 2b$, $0 \leqslant z < \infty$ with incident wave $\phi_i = \exp(-ikz)$, $\partial\phi_t/\partial y = 0$ on $y = 0, 2b$ $(0 \leqslant z < \infty)$, and mixed boundary conditions on $z = 0$ $(0 \leqslant y \leqslant 2b)$. We set $\phi_t = \phi_i + \phi$.

(i) *The strip of arbitrary width.* $\partial\phi_t/\partial x = 0$ on $z = 0$ $(0 \leqslant y < c)$; $\phi_t = 0$ on $z = 0$ $(c < y \leqslant 2b)$. The technique in §5.2 gives

$$\Phi_+(\alpha,y) - \Phi_+(-\alpha,y) = A(\alpha)\cosh\gamma(2b-y) - (2\pi)^{-1/2}2i\alpha\gamma^{-2}, \quad (c < y \leqslant 2b), \tag{a}$$

$$\Phi_+(\alpha,y) + \Phi_+(-\alpha,y) = B(\alpha)\cosh\gamma y - (2\pi)^{-1/2}2ik\gamma^{-2}, \quad (0 \leqslant y < c). \tag{b}$$

Eliminate $A(\alpha)$, $B(\alpha)$, $\Phi_+(-\alpha,c)$ in the usual way and set $2b - c = d$. Then (cf. (5.17)):

$$2\Phi_+(\alpha,c) = K(\alpha)\{k(\alpha)\Phi'_+(\alpha,c) + \Phi'_+(-\alpha,c)\} - (2\pi)^{-1/2}2i(\alpha - k)^{-1},$$

where

$$K(\alpha) = \sinh 2\gamma b\{\gamma\sinh\gamma c\sinh\gamma d\}^{-1} \;:\; k(\alpha) = \sinh 2\gamma(b - c)\{\sinh 2\gamma b\}^{-1}.$$

If $b = c$ then $k(\alpha) = 0$ and the resulting equation has an exact solution.

(ii) *The half-width strip of finite resistivity.* $\phi_t = 0$ on $z = 0$ $(b < y \leqslant 2b)$; $\partial\phi_t/\partial z = -i\delta\phi_t$ on $z = 0$ $(0 \leqslant y < b)$. The technique of §5.2 again gives (a) above in $b < y \leqslant 2b$. In $0 \leqslant y < b$ we find

$$d^2\Phi_+(\alpha,y)/dy^2 - \gamma^2\Phi_+(\alpha,y) = (2\pi)^{-1/2}i(k - \delta) - (2\pi)^{-1/2}i(\alpha + \delta)(\phi)_0.$$

To eliminate the unknown $(\phi)_0$ multiply by $(\alpha - \delta)$, change the sign of α, and subtract. Solution of the resulting differential equation gives

$$(\alpha - \delta)\Phi_+(\alpha,y) + (\alpha + \delta)\Phi_+(-\alpha,y) = B(\alpha)\cosh\gamma y - (2\pi)^{-1/2}2i\alpha(k - \delta)\gamma^{-2}. \tag{c}$$

(a) and (c) give, in the usual way (cf. (5.17)):

$$\Phi_+(\alpha,b) = \gamma^{-1}\coth\gamma b\{-\delta\alpha^{-1}\Phi'_+(\alpha,b) + (1 + \delta\alpha^{-1})\Phi'_+(-\alpha,b)\} - (2\pi)^{-1/2}i(\alpha - k)^{-1}.$$

5.8 It may be convenient to use a sine or cosine transform to deal with perpendicular boundaries. Consider the problem leading to (5.6). Set

$$\Psi(\alpha) = \left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \phi\cos\alpha x\,dx = \{\Phi_+(\alpha) + \Phi_+(-\alpha)\}.$$

Then
$$\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty \frac{\partial^2\phi}{\partial x^2}\cos\alpha x\,dx = -\left(\frac{2}{\pi}\right)^{1/2}ik - \alpha^2\Psi(\alpha).$$

A cosine transform reduces the partial differential equation to an ordinary differential equation for $\Psi(\alpha)$, the solution of which gives (5.6) directly.

5.9 In the notation of (5.68b) it can be shown that

$$\begin{aligned} W_{j-\frac{1}{2}}(z) &\sim \Gamma(j+1)z^{-1} \quad \text{as} \quad z \to \infty, \qquad |\arg z| < \pi, \\ W_{-1}(-iy) &= 2\pi^{1/2}e^{-iy}y^{-1/2}F(\sqrt{y}), \\ W_0(-iy) &= \pi^{1/2} + 2\pi^{1/2}ie^{-iy}y^{1/2}F(\sqrt{y}), \end{aligned}$$

where

$$F(v) = \int_v^\infty e^{iu^2}\,du.$$

5.10 It will be possible to extend the methods of this chapter to diffraction of waves by a two-dimensional obstacle of finite length *and* finite width, for large length and small width (cf. the discussion in D. S. Jones [4], p. 169). A similar problem is that of a plane wave incident from any angle on a solid cylindrical rod of large length and small diameter. This is a possible method of approach to antenna problems.

5.11 Consider a plane electromagnetic wave in infinite space, falling on a semi-infinite slab of dielectric material lying in $-\infty < x \leqslant 0$, $-b \leqslant y \leqslant b$ (cf. §5.4 and ex. 5.5).

5.12 Various problems involving diffraction of waves by disks and strips can be reduced to the solution of infinite sets of simultaneous linear algebraic equations by using the Kontorovich–Lebedev transform (see A. Erdelyi *et al.* [1], Vol. II):

$$\Psi(\mu) = \int_0^\infty \psi(r)K_\mu(\lambda r)\,\frac{dr}{r} \qquad : \qquad \psi(r) = \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} \mu\Psi(\mu)I_\mu(\lambda r)\,d\mu. \tag{a}$$

For simplicity the analysis is carried out on the assumption that λ has a positive real part, but we set $\lambda = -ik$, k real, at the end of the work, corresponding to a time factor $\exp(-i\omega t)$. The analysis and results for disks are based on a paper by A. Leitner and C. P. Wells [1]; they use the Vajnshtejn–Karp–Clemmow procedure. The method used here is the same as that in §5.4. We give the analysis in some detail since this would seem to be the first case where a transform other than the related Fourier–Laplace–Mellin trio has been used to give significant results by a Wiener–Hopf method.

It is necessary to consider the properties of $I_\mu(\lambda r)$, $K_\mu(\lambda r)$ as functions of μ for fixed (λr). Asymptotically

$$I_\mu(\lambda r) \sim \{\Gamma(1+\mu)\}^{-1}(\tfrac{1}{2}\lambda r)^\mu\{1 + O(\mu^{-1})\}, \qquad |\mu/\lambda r| \gg 1. \tag{b}$$

$I_\mu(\lambda r)$ decreases rapidly as $\operatorname{Re}\mu \to +\infty$ but increases as $\operatorname{Re}\mu \to -\infty$.

$$K_\mu(\lambda r) = \tfrac{1}{2}\pi\,\frac{I_{-\mu}(\lambda r) - I_\mu(\lambda r)}{\sin \pi\mu} \sim \tfrac{1}{2}(\tfrac{1}{2}\lambda r)^{\mp\mu}\Gamma(\pm\mu), \qquad \operatorname{Re}\mu \to \pm\infty. \tag{c}$$

Hence $K_\mu(\lambda r)$ increases rapidly as $|\text{Re}\,\mu| \to \infty$ in both right and left half-planes. On the imaginary axis, $\mu = i\tau$,

$$K_\mu(\lambda r) \sim |\tau|^{-1/2} \exp(-\tfrac{1}{2}\pi|\tau|), \qquad \tau \to \pm\infty.$$

From these results it is seen that if we write

$$\Phi_P(\mu) = \Gamma(1+\mu)(\tfrac{1}{2}\lambda a)^{-\mu} \int_0^a f(r) I_\mu(\lambda r)\,\frac{dr}{r}, \tag{d}$$

then $$\Phi_P(\mu) \sim \int_0^a f(r)(r/a)^{\mu-1}\,dr, \qquad \mu \to \infty, \qquad \text{Re}\,\mu > 0, \tag{e}$$

assuming that $f(r) \sim$ const. as $r \to 0$ so that the integral converges for $\text{Re}\,\mu > 0$. $\Phi_P(\mu)$ is regular and bounded in the right-hand half-plane $\text{Re}\,\mu > 0$. In this notation

$$\int_0^a f(r) K_\mu(\lambda r)\,\frac{dr}{r} = \tfrac{1}{2}\{(\tfrac{1}{2}\lambda a)^{-\mu}\Gamma(\mu)\Phi_P(-\mu) + (\tfrac{1}{2}\lambda a)^{\mu}\Gamma(-\mu)\Phi_P(\mu)\}. \tag{f}$$

Next consider the integral

$$I = \int_a^\infty g(r) K_\mu(\lambda r)\,\frac{dr}{r}. \tag{g}$$

It is assumed that $g(r)$ is such that I is convergent at both upper and lower limits for all μ. Then I represents an integral function of μ and from (b), (c) we can write

$$(\tfrac{1}{2}\lambda a)^{\mu}\,\frac{1}{\Gamma(\mu)} \int_a^\infty g(r) K_\mu(\lambda r)\,\frac{dr}{r} = \Psi_P(\mu),$$

$$(\tfrac{1}{2}\lambda a)^{-\mu}\,\frac{1}{\Gamma(-\mu)} \int_a^\infty g(r) K_\mu(\lambda r)\,\frac{dr}{r} = \Psi_N(\mu), \tag{h}$$

where Ψ_P, Ψ_N are regular and tend to zero in right and left half-planes respectively.

For problems involving circular disks use spherical co-ordinates (r,θ,ϕ). Assume axial symmetry so that the steady-state wave equation is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial\phi}{\partial\theta} + \frac{\partial}{\partial r} r^2 \frac{\partial\phi}{\partial r} + k^2 r^2 \phi = 0.$$

Set $r^{1/2}\phi = \psi$. The equation for ψ is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\,\frac{\partial\psi}{\partial\theta} + r\frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} - \tfrac{1}{4}\psi + k^2 r^2 \psi = 0.$$

Application of transform (a) gives

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d\Psi}{d\theta} + (\mu^2 - \tfrac{1}{4})\Psi = 0,$$

$$\Psi(\mu) = AP_{\mu-\frac{1}{2}}(\cos\theta) + BP_{\mu-\frac{1}{2}}(-\cos\theta). \tag{i}$$

Consider one example in detail. A rigid disk lies in $\theta = \frac{1}{2}\pi$, $0 \leqslant r \leqslant a$. A wave $\phi_i = \exp(-ikz)$ is incident normally on the disk. Set $\phi_t = \phi_i + \phi$, $\psi = r^{1/2}\phi$, so that $\partial\psi/\partial\theta = ikr^{3/2}$ on the disk. Since $\partial\psi/\partial\theta$ is continuous for $\theta = \frac{1}{2}\pi$ and finite for $\theta = 0, \pi$, (i) gives

$$\Psi(\mu) = AP_{\mu-\frac{1}{2}}(\cos\theta),\ 0 \leqslant \theta \leqslant \tfrac{1}{2}\pi;\quad -AP_{\mu-\frac{1}{2}}(-\cos\theta),\ \tfrac{1}{2}\pi \leqslant \theta \leqslant \pi. \tag{j}$$

Differentiate these equations with respect to θ, eliminate A, and apply boundary conditions on $\theta = \frac{1}{2}\pi$. Then, if suffix zero refers to $\theta = \frac{1}{2}\pi$,

$$ik\int_0^a r^{1/2}K_\mu(\lambda r)\,dr + \int_a^\infty \left(\frac{\partial\psi}{\partial\theta}\right)_0 K_\mu(\lambda r)\frac{dr}{r} = -2K(\mu)\int_0^a f(r)K_\mu(\lambda r)\frac{dr}{r}, \tag{k}$$

where $\qquad 2f(r) = (\psi)_{+0} - (\psi)_{-0},$

so that $\qquad A = \{P_{\mu-\frac{1}{2}}(0)\}^{-1}\int_0^a f(r)K_\mu(\lambda r)\frac{dr}{r}.$

Also

$$K(\mu) = -\tfrac{1}{2}P'_{\mu-\frac{1}{2}}(0)/P_{\mu-\frac{1}{2}}(0) = L_P(\mu)/L_N(\mu),$$

$$\{P_{\mu-\frac{1}{2}}(0)\}^{-1} = \pi^{-1/2}\Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu)\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\mu) \tag{l}$$

$$\{P'_{\mu-\frac{1}{2}}(0)\}^{-1} = -\tfrac{1}{2}\pi^{-1/2}\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu)\Gamma(\tfrac{1}{4}-\tfrac{1}{2}\mu)$$

$$L_P(\mu) = \Gamma(\tfrac{3}{4}+\tfrac{1}{2}\mu)/\Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu)$$

$$L_N(\mu) = \Gamma(\tfrac{1}{4}-\tfrac{1}{2}\mu)/\Gamma(\tfrac{3}{4}-\tfrac{1}{2}\mu) \tag{m}$$

and L_P, L_N are regular and non-zero for Re $\mu > -\frac{1}{2}$, $<\frac{1}{2}$, respectively. In the following manipulations we must remember that

$$\lim_{\mu\to m}(m-\mu)\Gamma(-\mu) = \frac{(-1)^m}{m!} \quad : \quad \lim_{\mu\to 2m}(2m-\mu)\Gamma(-\tfrac{1}{2}\mu) = 2\frac{(-1)^m}{m!}.$$

In view of (f), (h) it is convenient to rewrite (k) as (cf. (5.17)):

$$ikL_N(\mu)(\tfrac{1}{2}\lambda a)^{-\mu}\{\Gamma(-\mu)\}^{-1}\int_0^a r^{1/2}K_\mu(\lambda r)\,dr + L_N(\mu)\Psi'_N(\mu)$$
$$= -L_P(\mu)[\Phi_P(\mu) + \Gamma(\mu)\{\Gamma(-\mu)\}^{-1}(\tfrac{1}{2}\lambda a)^{-2\mu}\Phi_P(-\mu)]. \tag{n}$$

Set

$$L_P(\mu)\Gamma(\mu)\{\Gamma(-\mu)\}^{-1}(\tfrac{1}{2}\lambda a)^{-2\mu}\Phi_P(-\mu)$$
$$= [L_P(\mu)\Gamma(\mu)\{\Gamma(-\mu)\}^{-1}(\tfrac{1}{2}\lambda a)^{-2\mu}\Phi_P(-\mu) - U - V] + [U+V], \tag{o}$$

where

$$U = \sum_{n=1}^{\infty} (\tfrac{1}{2}\lambda a)^{2n}\Phi_P(n)\,\frac{U_n}{\mu + n}, \qquad U_n = \frac{n}{(n!)^2}\cdot\frac{\Gamma(\frac{1}{2}n + \frac{3}{4})}{\Gamma(\frac{1}{2}n + \frac{1}{4})},$$

$$V = -2\sum_{n=0}^{\infty} (\tfrac{1}{2}\lambda a)^{4n+3}\Phi_P(2n + \tfrac{3}{2})\,\frac{V_n}{\mu + 2n + \frac{3}{2}},$$

$$V_n = \frac{(2n + \frac{3}{2})}{\{\Gamma(2n + \frac{5}{2})\}^2}\cdot\frac{\Gamma(n + \frac{3}{2})}{n!}.$$

$[U + V]$ is regular in $\operatorname{Re}\mu > -1$, and the first bracketed term on the right of (o) is regular in $\operatorname{Re}\mu < \frac{1}{2}$, assuming that $\phi \sim$ const. i.e. $f(r) \sim r^{1/2}$ as $r \to 0$. Next examine

$$J = \int_0^a r^{1/2}K_\mu(\lambda r)\,dr$$

$$= \frac{2^{1/2}}{\lambda^{3/2}}\cdot\frac{\pi}{\sin\pi\mu}\sum_{n=0}^{\infty}\frac{1}{n!}\left\{\frac{(\frac{1}{2}\lambda a)^{-\mu+2n+(3/2)}}{\Gamma(-\mu + n + 1)(-\mu + 2n + \frac{3}{2})} - \right.$$

$$\left. - \frac{(\frac{1}{2}\lambda a)^{\mu+2n+(3/2)}}{\Gamma(\mu + n + 1)(\mu + 2n + \frac{3}{2})}\right\},$$

where we have used (c), substituted the series expansion of I_μ, and integrated term by term. This is valid for $-\operatorname{Re}\mu < \frac{3}{2} < \operatorname{Re}\mu$. The series defines the analytic continuation of the integral for all μ. At first sight it would seem that there are poles at $\mu = \pm n$, but the reader will readily prove that this is not so. The only poles occur at $\mu = \pm(2n + \frac{3}{2})$ with residues

$$-2^{1/2}\lambda^{-3/2}(-1)^n(n!)^{-1}\Gamma(n + \tfrac{3}{2}).$$

Write

$$L_N(\mu)(\tfrac{1}{2}\lambda a)^{-\mu}\{\Gamma(-\mu)\}^{-1}J = [L_N(\mu)(\tfrac{1}{2}\lambda a)^{-\mu}\{\Gamma(-\mu)\}^{-1}J - W] + W, \quad \text{(p)}$$

where

$$W = -\sum_{n=0}^{\infty}(\tfrac{1}{2}\lambda a)^{2n+3/2}\,\frac{W_n}{\mu + 2n + \frac{3}{2}},$$

$$W_n = (-1)^n\,\frac{2^{1/2}}{\lambda^{3/2}}\cdot\frac{1}{\Gamma(2n + \frac{3}{2})},$$

and W is regular in $\operatorname{Re}\mu > -\frac{3}{2}$. We can now separate (n) by the Wiener–Hopf technique to give, since $\Phi_P(\mu) \sim \mu^{-1}$ as $\operatorname{Re}\mu \to +\infty$,

$$L_P(\mu)\Phi_P(\mu) + U + V + ikW = 0. \quad \text{(q)}$$

This agrees with A. Leitner and C. P. Wells [1], p. 28. Introduce

$$ik\pi^{-1/2}a^{3/2}X(\mu) = L_P(\mu)\Phi_P(\mu). \quad \text{(r)}$$

Then (q) gives

$$X(\mu) + \sum_{n=1}^{\infty} \frac{n}{(n!)^2} \cdot \frac{(\frac{1}{2}\lambda a)^{2n}}{\mu + n} X(n) -$$

$$- 2 \sum_{n=0}^{\infty} \frac{2n + \frac{3}{2}}{\{\Gamma(2n + \frac{5}{2})\}^2} \cdot \frac{(\frac{1}{2}\lambda a)^{4n+3}}{(\mu + 2n + \frac{3}{2})} X(2n + \tfrac{3}{2})$$

$$= \tfrac{1}{2}\pi^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\lambda a)^{2n}}{\Gamma(2n + \frac{3}{2})(\mu + 2n + \frac{3}{2})}. \tag{s}$$

From the form of this result it is clear that $X(\mu)$ can be expanded as a series

$$X(\mu) = \frac{1}{\mu + \frac{3}{2}} + \sum_{m=2}^{\infty} X_m(\mu)(\tfrac{1}{2}\lambda a)^m. \tag{t}$$

If this is substituted in (s) and the coefficient of each power of $(\frac{1}{2}\lambda a)$ is equated to zero, the X_m can be found explicitly by means of recursive formulae.

$$X_2(\mu) = -\frac{2}{5(\mu + 1)} - \frac{4}{15(\mu + \frac{7}{2})} \quad : \quad X_3(\mu) = \frac{16}{9\pi(\mu + \frac{3}{2})}$$

$$X_4(\mu) = \frac{7}{27(\mu + 1)} - \frac{1}{7(\mu + 2)} + \frac{16}{945(\mu + \frac{11}{2})}, \qquad \text{etc.}$$

From (a), (j), for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$,

$$r^{1/2}\phi = \psi = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \mu \int_0^a f(\rho) K_\mu(\lambda\rho) \frac{d\rho}{\rho} \frac{P_{\mu-\frac{1}{2}}(\cos\theta)}{P_{\mu-\frac{1}{2}}(0)} I_\mu(\lambda r)\, d\mu.$$

To evaluate this integral for $r > a$ it is necessary to substitute (c) for $K_\mu(\lambda\rho)$ and change the sign of μ in the part involving $I_{-\mu}(\lambda\rho)$, remembering that $P_{\mu-\frac{1}{2}}$ is an even function of μ. This gives

$$r^{1/2}\phi = \frac{1}{\pi i} \int_{c-i\infty}^{c+i\infty} \mu \int_0^a f(\rho) I_\mu(\lambda\rho) \frac{d\rho}{\rho} \frac{P_{\mu-\frac{1}{2}}(\cos\theta)}{P_{\mu-\frac{1}{2}}(0)} K_\mu(\lambda r)\, d\mu$$

$$= \pi^{-2} k a^{3/2} \int_{c-i\infty}^{c+i\infty} (\tfrac{1}{2}\lambda a)^\mu \{\Gamma(\mu)\}^{-1} \Gamma(\tfrac{3}{4} - \tfrac{1}{2}\mu)\Gamma(\tfrac{1}{4} + \tfrac{1}{2}\mu)$$
$$\times P_{\mu-\frac{1}{2}}(\cos\theta) K_\mu(\lambda r) X(\mu)\, d\mu,$$

where we have used (d), (r). The contour can be completed in a right half-plane. The only singularities are poles at $\mu = 2n + \frac{3}{2}$, and these give

$$\phi = 2\pi^{-1} k^2 a^3 \sum_{n=0}^{\infty} \frac{(-1)^n (\lambda a)^{2n}}{1.\,3.\,5.\,\ldots\,(4n + 1)} X(2n + \tfrac{3}{2})$$
$$\times P_{2n+1}(\cos\theta) \left(\frac{2\lambda}{\pi r}\right)^{1/2} K_{2n+(3/2)}(\lambda r).$$

The far field is obtained by using the asymptotic expansion

$$\{2\lambda/(\pi r)\}^{1/2}K_{2n+(3/2)}(\lambda r) \sim r^{-1}e^{-\lambda r},$$

and this term comes outside the summation sign. The field depends only on $X(2n+\frac{3}{2})$ which can be computed from (t). In order to evaluate the field to $(\lambda a)^4$ we have, for instance,

$$X\left(\frac{3}{2}\right) = \frac{1}{3} - \frac{4}{75}(\lambda a)^2 + \frac{2}{27\pi}(\lambda a)^3 + \frac{1}{5\,.\,7^2}(\lambda a)^4 + \ldots$$

$$X\left(\frac{7}{2}\right) = \frac{1}{5} - \frac{2}{3^2\,.\,7}(\lambda a)^2 + \ldots \quad : \quad X\left(\frac{11}{2}\right) = \frac{1}{7} + \ldots .$$

The scattering cross-section is given by the left-hand side of the following equation. On substituting the above expression for ϕ we find

$$\frac{2\pi}{k}\int_{-1}^{+1} \mathrm{Re}\left\{i\phi\left(\frac{\overline{\partial\phi}}{\partial r}\right)\right\} r^2\, d(\cos\theta)$$

$$= \frac{16}{27}\frac{k^4a^6}{\pi}\left[1 + \frac{8}{25}(ka)^2 + \frac{311}{6125}(ka)^4 + \ldots\right].$$

This agrees with the existing literature, e.g. C. J. Bouwkamp [1]. This concludes our discussion of the problem.

In exactly the same way a solution can be found for the case $\phi_t = 0$ instead of $\partial\phi_t/\partial n = 0$ on the disk. The method can also be used to deal with the biconical antenna—as shown by J. A. Meier and A. Leitner [1].

A similar type of analysis can be used for the two-dimensional strip. The equation is

$$r\frac{\partial}{\partial r}r\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial\theta^2} + k^2r^2\phi = 0.$$

Application of transform (a) gives

$$\frac{d^2\Phi}{d\theta^2} + \mu^2\Phi = 0 \quad : \quad \Phi = A\sin\mu\theta + B\cos\mu\theta.$$

The solution proceeds as before. The factorization corresponding to (m) can again be expressed in terms of gamma-functions but it so happens that double poles occur in the split (o). It will be found that the separation of these double poles involves differentiations of the type

$$\frac{d}{d\mu}\{\Gamma(-\mu)\}, \qquad \frac{d}{d\mu}(\lambda a)^{-2\mu}.$$

These introduce a familiar parameter of the form

$$p = C + \ln(\tfrac{1}{4}kb) - i\pi/2, \qquad C = 0{\cdot}5772\ldots .$$

We shall not pursue the analysis here.

The solutions of these problems given in the literature usually involve complicated evaluation of integrals. One of the interesting features of the present method is that once the machinery is established the solution needs little more than a few elementary properties of gamma-functions.

5.13 Consider a wave $\exp(-ikz)$ incident on a circular disk in $z = 0$, $0 \leqslant \rho \leqslant a$. By applying Fourier transforms in the z-direction and introducing unknown functions on $\rho = a$, $-\infty < z < \infty$, formulate this problem as a generalized Wiener–Hopf equation of type (5.17). This formulation should be suitable for large k. A similar formulation can be obtained for the two-dimensional strip.

5.14 If we try to generalize the method of §5.6 to a circular disk of radius a it would appear that corresponding to the Fourier transform in x we should use a Hankel transform in the radial co-ordinate ρ. To obtain a decomposition corresponding to (5.10) we should presumably write

$$J_0(\alpha\rho) = \tfrac{1}{2}\{H_0^{(1)}(\alpha\rho) + H_0^{(2)}(\alpha\rho)\}, \qquad (a < \rho < \infty).$$

5.15 The Fourier inverse of (1.60), (1.61) gives

$$\tfrac{1}{2}i \int_{-\infty}^{\infty} H_0^{(1)}\{k(x^2 + y^2)^{1/2}\} e^{i\alpha x}\, dx = \gamma^{-1} e^{-\gamma|y|}.$$

In this result replace x by $x - X$, and then set $X = \xi \cos \nu$, $|y| = \xi \sin \nu$:

$$\tfrac{1}{2}i \int_{-\infty}^{\infty} H_0^{(1)}\{k(x^2 + \xi^2 - 2x\xi \cos \nu)^{1/2}\} e^{i\alpha x}\, dx = \gamma^{-1} e^{i\xi(\alpha \cos \nu + i\gamma \sin \nu)}$$

Consider the integral equation

$$\tfrac{1}{2}i \int_{0}^{\infty} f(\xi)[H_0^{(1)}\{k|x - \xi|\} + $$
$$+ pH_0^{(1)}\{k(x^2 + \xi^2 - 2x\xi \cos \nu)\}^{1/2}]\, d\xi = g(x), \qquad x > 0,$$

where p is a constant. A Fourier transform gives

$$\gamma^{-1}\{F_+(\alpha) + pF_+(\alpha \cos \nu + i\gamma \sin \nu)\} = G_+(\alpha) + H_-(\alpha).$$

This equation can be regarded as a generalization of (5.17) which is the case $\nu = \pi$. The case $\nu = \frac{1}{2}\pi$ has been considered to some extent by R. Jost [1] in connexion with a problem in quantum mechanics. He does not obtain an explicit solution suitable for calculation. Similar integral equations appear in connexion with diffraction problems involving wedges. In this type of problem it is natural to make the substitution $\alpha = -k \cos \beta$ of §1.6.

CHAPTER VI

THE GENERAL SOLUTION OF THE BASIC WIENER–HOPF PROBLEM

6.1 Introduction

So far we have been concerned mainly with the solution of specific examples. In §6.2 we obtain the general solution of the basic Wiener–Hopf problem underlying all the examples which have been solved exactly in previous chapters.

As indicated in Chapter II, problems which can be solved exactly by the Wiener–Hopf technique can be reduced to the solution of one of three equivalent subsidiary problems:

(i) Solution of the complex variable equation

$$K(\alpha)\{H_+(\alpha) + G_-(\alpha)\} = F_+(\alpha) + E_-(\alpha), \tag{6.1}$$

where this equation holds in the strip $\tau_- < \tau < \tau_+$, $-\infty < \sigma < \infty$, and $h(x)$, $e(x)$, the transforms of $H_+(\alpha)$, $E_-(\alpha)$, are the unknown functions.

(ii) Solution of certain types of integral equation, e.g.

$$\text{(a)} \qquad \int_0^\infty k(x-\xi)h(\xi)\,d\xi = f(x), \qquad (x > 0), \tag{6.2}$$

or

$$\text{(b)} \quad \{d^2/dx^2 + k^2\}\int_0^\infty l(x-\xi)h(\xi)\,d\xi = f(x), \qquad (x > 0). \tag{6.3}$$

In these equations $h(\xi)$ is unknown.

(iii) Solution of certain dual integral equations:

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} K(\alpha)A(\alpha)e^{-i\alpha x}\,d\alpha = f(x), \qquad (x > 0), \tag{6.4a}$$

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} A(\alpha)e^{-i\alpha x}\,d\alpha = g(x), \qquad (x < 0), \tag{6.4b}$$

where $A(\alpha)$ is unknown.

Problems (i)–(iii) are equivalent under certain conditions and the notation has been chosen to indicate the relationship between the various functions in the three problems. Thus

$$F_+(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_0^\infty f(x)e^{i\alpha x}\,dx \quad : \quad G_-(\alpha) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^0 g(x)e^{i\alpha x}\,dx. \tag{6.5}$$

For convenience we have defined

$$K(\alpha) = \int_{-\infty}^{\infty} k(x)e^{i\alpha x}\,dx, \tag{6.6}$$

i.e. there is no factor $(2\pi)^{-1/2}$ in front of the integral.

To show that (iii) is equivalent to (i) suppose that the left-hand sides of (6.4a), (6.4b) are equal to the unknown functions $e(x)$, $h(x)$ for $x < 0$, $x > 0$ respectively. Then a Fourier transform gives

$$K(\alpha)A(\alpha) = F_+(\alpha) + E_-(\alpha) \quad : \quad A(\alpha) = G_-(\alpha) + H_+(\alpha). \tag{6.7}$$

Elimination of $A(\alpha)$ gives (6.1). Conversely if in (6.1) we define

$$A(\alpha) = H_+(\alpha) + G_-(\alpha), \tag{6.8a}$$

then from (6.1)

$$K(\alpha)A(\alpha) = F_+(\alpha) + E_-(\alpha). \tag{6.8b}$$

The Fourier inverses of (6.8a), (6.8b) give (6.4b), (6.4a) for $x < 0$, $x > 0$, respectively. These manipulations assume of course that the appropriate Fourier transforms exist.

In §6.2 we solve the dual integral equations (6.4) by the 'multiplying factor method' of §2.3. This is probably the easiest formal method for finding the general solution of the three problems. A direct solution of problem (i) is sketched in ex. 6.1. A result which is related to the investigation in §6.2 is that any Wiener–Hopf integral equation of type (6.2) can be decomposed into repeated Volterra integral equations (ex. 6.4).

In exs. 6.5 and 6.8 the results in §6.2 and ex. 6.4 are used to reduce certain singular integral equations of the first kind (or related equations) to Fredholm integral equations of the second kind. These techniques can be applied to approximate solution of the three-part problems considered in §§5.5, 5.6 but it would seem that this method of approach is not so useful as the complex variable methods of Chapter V. However there are some interesting and effective applications of the same basic ideas in connection with dual integral equations involving Bessel functions, applied to boundary value problems involving disks. These applications will be discussed elsewhere.

6.2 The exact solution of certain dual integral equations

In this section we solve the dual integral equations (6.4) by the method of §§2.3, 2.8. As in §2.3 replace x by $(x + \xi)$ in (6.4a) and by $(x - \xi)$ in (6.4b), with $\xi > 0$ in each case. Multiply by $\mathscr{N}_-(\xi)$, $\mathscr{N}_+(\xi)$ respectively where these functions will be defined below. Integrate each equation with respect to ξ from 0 to ∞. Introduce $N_-(\alpha)$, $N_+(\alpha)$ defined by

$$\begin{aligned} \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{N}_-(\xi) e^{-i\alpha\xi}\, d\xi &= N_-(\alpha) \\ \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{N}_+(\xi) e^{i\alpha\xi}\, d\xi &= N_+(\alpha). \end{aligned} \tag{6.9a}$$

By inversion

$$\begin{aligned} \mathscr{N}_-(\xi) &= \frac{1}{(2\pi)^{1/2}} \int_{ib-\infty}^{ib+\infty} N_-(\alpha) e^{i\alpha\xi}\, d\alpha \\ \mathscr{N}_+(\xi) &= \frac{1}{(2\pi)^{1/2}} \int_{ic-\infty}^{ic+\infty} N_+(\alpha) e^{-i\alpha\xi}\, d\alpha, \end{aligned} \tag{6.9b}$$

where these integrals are zero for $\xi < 0$. Using these results, (6.4) reduce to

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} N_-(\alpha) K(\alpha) A(\alpha) e^{-i\alpha x}\, d\alpha = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{N}_-(\xi) f(x+\xi)\, d\xi, \quad (x > 0), \tag{6.10a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} N_+(\alpha) A(\alpha) e^{-i\alpha x}\, d\alpha = \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{N}_+(\xi) g(x-\xi)\, d\xi, \quad (x < 0). \tag{6.10b}$$

The natural method of procedure would be to determine N_+, N_- in these equations so that

$$N_+(\alpha) = N_-(\alpha) K(\alpha), \tag{6.11}$$

but, as indicated in §2.3 in a special case, convergence difficulties indicate that a slight modification in this procedure is required. Consider two cases separately (i) $|K(\alpha)| \sim |\alpha|^{-1}$ as $|\alpha| \to \infty$ in the strip, (ii) $|K(\alpha)| \sim |\alpha|$ as $|\alpha| \to \infty$ in the strip.

(i) In case (i) assume that we can write $K(\alpha) = K_+(\alpha)K_-(\alpha)$ where $|K_+(\alpha)| \sim |\alpha|^{-1/2}$, $|K_-(\alpha)| \sim |\alpha|^{-1/2}$, as $\alpha \to \infty$ in appropriate half-planes. For convenience we define here several functions that we need later:

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_+(\alpha)e^{-i\alpha u}\,d\alpha = \mathscr{L}_+(u), \quad (u > 0) \quad : \quad = 0, \quad (u < 0), \tag{6.12a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_-(\alpha)e^{i\alpha u}\,d\alpha = \mathscr{L}_-(u), \quad (u > 0) \quad : \quad = 0, \quad (u < 0), \tag{6.12b}$$

$$\frac{i}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \frac{1}{K_+(\alpha)(\alpha - \alpha_1)} e^{-i\alpha u}\,d\alpha$$
$$= \mathscr{M}_+(u), \quad (u > 0) \quad : \quad = 0, \quad (u < 0), \tag{6.12c}$$

$$-\frac{i}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \frac{1}{K_-(\alpha)(\alpha - \alpha_2)} e^{i\alpha u}\,d\alpha$$
$$= \mathscr{M}_-(u), \quad (u > 0) \quad : \quad = 0, \quad (u < 0), \tag{6.12d}$$

where α_1, α_2 represent arbitrary complex constants in lower and upper half-planes respectively. Each of these equations has an inverse analogous to one of (6.9a). We use the shorthand:

$$\mathscr{D}_1^{\pm} \equiv \pm e^{\mp i\alpha_1 y} \frac{d}{dy} e^{\pm i\alpha_1 y} \quad : \quad \mathscr{D}_2^{\pm} \equiv \pm e^{\pm i\alpha_2 y} \frac{d}{dy} e^{\mp i\alpha_2 y} \tag{6.13}$$

where upper and lower signs go together and the appropriate variable to be used in place of y (namely x or η) will be obvious from the context.

Comparing the equation $K(\alpha) = K_+(\alpha)K_-(\alpha)$ with (6.11) and remembering that $|K_+|$, $|K_-| \sim |\alpha|^{-1/2}$ as $|\alpha| \to \infty$ in appropriate half-planes we see that suitable expressions for N_+, N_-, which of course do not satisfy (6.11), are

$$N_+(\alpha) = K_+(\alpha) \quad : \quad N_-(\alpha) = \{K_-(\alpha)(\alpha - \alpha_2)\}^{-1}, \tag{6.14}$$

where α_2 is an arbitrary constant in the upper half-plane, inserted to overcome convergence difficulties. Comparing (6.9b), (6.12a, d), we see that $\mathscr{N}_-(\xi) = i\mathscr{M}_-(\xi)$, $\mathscr{N}_+(\xi) = \mathscr{L}_+(\xi)$ so that (6.10) become

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} (\alpha - \alpha_2)^{-1} K_+(\alpha) A(\alpha) e^{-i\alpha x}\, d\alpha$$
$$= \frac{i}{(2\pi)^{1/2}} \int_0^\infty \mathscr{M}_-(\xi)\, f(x+\xi)\, d\xi, \quad (x > 0), \tag{6.15a}$$

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_+(\alpha) A(\alpha) e^{-i\alpha x}\, d\alpha$$
$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{L}_+(\xi) g(x-\xi)\, d\xi, \quad (x < 0). \tag{6.15b}$$

Multiply the first equation by $\exp(i\alpha_2 x)$ and differentiate with respect to x:

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_+(\alpha) A(\alpha) e^{-i\alpha x}\, d\alpha$$
$$= \frac{1}{(2\pi)^{1/2}} \mathscr{D}_2^- \int_0^\infty \mathscr{M}_-(\xi) f(x+\xi)\, d\xi, \quad (x > 0), \tag{6.16}$$

using notation (6.13). The left-hand sides of (6.15b), (6.16) are identical and we can invert to find

$$K_+(\alpha)A(\alpha) = \frac{1}{2\pi} \int_0^\infty e^{i\alpha\eta}\, \mathscr{D}_2^- \int_0^\infty \mathscr{M}_-(\xi) f(\eta+\xi)\, d\xi\, d\eta\, +$$
$$+ \frac{1}{2\pi} \int_{-\infty}^0 e^{i\alpha\eta} \int_0^\infty \mathscr{L}_+(\xi) g(\eta-\xi)\, d\xi\, d\eta. \tag{6.17}$$

This is the solution of the dual integral equations.

Quantities which are of interest are the left-hand sides of (6.4a), (6.4b) when $x < 0$, $x > 0$ respectively. Inserting (6.17) in (6.4a) and using notation (6.12) gives, for $x < 0$,

$$e(x) = \frac{1}{2\pi} \int_0^\infty \mathscr{L}_-(\eta - x)\, \mathscr{D}_2^- \int_\eta^\infty \mathscr{M}_-(\xi-\eta) f(\xi)\, d\xi\, d\eta\, +$$
$$+ \frac{1}{2\pi} \int_x^0 \mathscr{L}_-(\eta - x) \int_{-\infty}^\eta \mathscr{L}_+(\eta-\xi) g(\xi)\, d\xi\, d\eta. \tag{6.18}$$

If we insert (6.17) in (6.4b) we cannot interchange orders of integration directly. We use the following device to avoid divergence, considered for a typical case,

$$\int_{ia-\infty}^{ia+\infty} e^{-i\alpha x}\frac{d\alpha}{K_+(\alpha)}\int_0^\infty F(\eta)e^{i\alpha\eta}\,d\eta$$

$$= ie^{-i\alpha_1 x}\frac{d}{dx}\int_{ia-\infty}^{ia+\infty}\frac{e^{-i(\alpha-\alpha_1)x}}{K_+(\alpha)(\alpha-\alpha_1)}\int_0^\infty F(\eta)e^{i\alpha\eta}\,d\eta\,d\alpha$$

$$= (2\pi)^{1/2}e^{-i\alpha_1 x}\frac{d}{dx}\,e^{i\alpha_1 x}\int_0^x F(\eta)\mathscr{M}_+(x-\eta)\,d\eta, \tag{6.19}$$

where we have interchanged orders of integration and used notation (6.12c). Insertion of (6.17) in (6.4b) now gives, for $x > 0$,

$$h(x) = \frac{1}{2\pi}\mathscr{D}_1^+\int_0^x \mathscr{M}_+(x-\eta)\,\mathscr{D}_2^-\int_\eta^\infty \mathscr{M}_-(\xi-\eta)f(\xi)\,d\xi\,d\eta\, +$$

$$+\frac{1}{2\pi}\mathscr{D}_1^+\int_{-\infty}^0 \mathscr{M}_+(x-\eta)\int_{-\infty}^\eta \mathscr{L}_+(\eta-\xi)g(\xi)\,d\xi\,d\eta. \tag{6.20}$$

(ii) If $|K(\alpha)| \sim |\alpha|$ as $|\alpha| \to \infty$ in the strip, assume $|K_+(\alpha)|$, $|K_-(\alpha)| \sim |\alpha|^{1/2}$ as $|\alpha| \to \infty$ in appropriate half-planes. This time we define

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty}\frac{1}{K_+(\alpha)}e^{-i\alpha u}\,d\alpha = \mathscr{L}_+(u),\quad (u>0)\quad :\quad = 0,\quad (u<0), \tag{6.21a}$$

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty}\frac{1}{K_-(\alpha)}e^{i\alpha u}\,d\alpha = \mathscr{L}_-(u),\quad (u>0)\quad :\quad = 0,\quad (u<0), \tag{6.21b}$$

$$\frac{i}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty}\frac{K_+(\alpha)}{\alpha-\alpha_1}e^{-i\alpha u}\,d\alpha$$
$$= \mathscr{M}_+(u),\quad (u>0)\quad :\quad = 0,\quad (u<0), \tag{6.21c}$$

$$-\frac{i}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty}\frac{K_-(\alpha)}{\alpha-\alpha_2}e^{i\alpha u}\,d\alpha$$
$$= \mathscr{M}_-(u),\quad (u>0)\quad :\quad = 0,\quad (u<0). \tag{6.21d}$$

Then choose (cf. (6.14))

$$N_+(\alpha) = (\alpha - \alpha_1)^{-1}K_+(\alpha) \quad : \quad N_-(\alpha) = \{K_-(\alpha)\}^{-1}.$$

Instead of (6.15) we find

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_+(\alpha)A(\alpha)e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{(2\pi)^{1/2}} \int_0^\infty \mathscr{L}_-(\xi)f(x+\xi)\, d\xi, \qquad (x > 0), \quad (6.22a)$$

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} \frac{K_+(\alpha)}{\alpha - \alpha_1} A(\alpha)e^{-i\alpha x}\, d\alpha$$

$$= -\frac{i}{(2\pi)^{1/2}} \int_0^\infty \mathscr{M}_+(\xi)g(x-\xi)\, d\xi, \qquad (x < 0). \quad (6.22b)$$

Multiply the second equation by $\exp(i\alpha_1 x)$ and differentiate with respect to x to find

$$\frac{1}{(2\pi)^{1/2}} \int_{ia-\infty}^{ia+\infty} K_+(\alpha)A(\alpha)e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{(2\pi)^{1/2}} \mathscr{D}_1^+ \int_0^\infty \mathscr{M}_+(\xi)g(x-\xi)\, d\xi, \qquad (x < 0). \quad (6.23)$$

We can now invert (6.22a), (6.23). This gives (cf. (6.17, 18, 20))

$$K_+(\alpha)A(\alpha) = \frac{1}{2\pi} \int_0^\infty e^{i\alpha\eta} \int_0^\infty \mathscr{L}_-(\xi)f(\eta+\xi)\, d\xi\, d\eta +$$

$$+ \frac{1}{2\pi} \int_{-\infty}^0 e^{i\alpha\eta}\mathscr{D}_1^+ \int_0^\infty \mathscr{M}_+(\xi)g(\eta-\xi)\, d\xi\, d\eta. \quad (6.24a)$$

$$e(x) = \frac{1}{2\pi}\mathscr{D}_2^- \int_0^\infty \mathscr{M}_-(\eta - x) \int_\eta^\infty \mathscr{L}_-(\xi-\eta)f(\xi)\, d\xi\, d\eta +$$

$$+ \frac{1}{2\pi}\mathscr{D}_2^- \int_x^0 \mathscr{M}_-(\eta - x)\mathscr{D}_1^+ \int_{-\infty}^\eta \mathscr{M}_+(\eta-\xi)g(\xi)\, d\xi\, d\eta, \quad (x < 0),$$

(6.24b)

$$h(x) = \frac{1}{2\pi}\int_0^x \mathscr{L}_+(x-\eta)\int_\eta^\infty \mathscr{L}_-(\xi-\eta)f(\xi)\,d\xi\,d\eta +$$

$$+\frac{1}{2\pi}\int_{-\infty}^0 \mathscr{L}_+(x-\eta)\,\mathscr{D}_1^+\int_{-\infty}^\eta \mathscr{M}_+(\eta-\xi)g(\xi)\,d\xi\,d\eta, \qquad (x>0). \tag{6.24c}$$

This concludes our discussion of case (ii).

These general solutions have not been used in previous chapters for various reasons. They are unwieldy since they involve fourfold integrals. And the known functions $F_+(\alpha)$, $G_-(\alpha)$ in (6.1) (or $f(x)$, $g(x)$ in (6.4)) are usually given explicitly and have a simple form: in concrete examples it is usually easy to solve the Wiener–Hopf equation from first principles. However in several important cases the functions $\mathscr{L}_\pm(u)$, $\mathscr{M}_\pm(u)$ which occur in the general solution can be evaluated explicitly and this means that the general solution involves only twofold integrals. Thus for the Sommerfeld half-plane problem of Chapter II, with $K(\alpha) = (\alpha^2 - k^2)^{-1/2}$, we can choose

$$K_+(\alpha) = (\alpha + k)^{-1/2} : K_-(\alpha) = (\alpha - k)^{-1/2} : \alpha_1 = -k : \alpha_2 = +k.$$

This gives (ex. 2.4):

$$\mathscr{M}_+(u) = \frac{i}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} (\alpha+k)^{-1/2}e^{-i\alpha u}\,d\alpha = 2^{1/2}e^{i\pi/4}u^{-1/2}e^{iku}, \qquad (u>0), \tag{6.25a}$$

$$\mathscr{M}_-(u) = -\frac{i}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} (\alpha-k)^{-1/2}e^{+i\alpha u}\,d\alpha = 2^{1/2}e^{-i\pi/4}u^{-1/2}e^{iku}, \qquad (u>0). \tag{6.25b}$$

As an application of these results consider the solution of

$$\tfrac{1}{2}i\int_0^\infty H_0^{(1)}(k|x-\xi|)h(\xi)\,d\xi = f(x), \qquad (x>0). \tag{6.26}$$

Suppose that the left-hand side of this equation is equal to the (unknown) function $e(x)$ for $x<0$. Application of a Fourier transform as in §2.5 gives (6.1) with $G_-(\alpha) = 0$, $K(\alpha) = (\alpha^2 - k^2)^{-1/2}$.

The solution of (6.26) is therefore given by (6.20) with $g(\xi) = 0$. On using the notation (6.13) and the results (6.25) we have

$$h(x) = -\frac{1}{\pi} e^{ikx} \frac{d}{dx} \int_0^x (x - \eta)^{-1/2} e^{-2ik\eta}$$
$$\times \frac{d}{d\eta} \int_\eta^\infty f(\xi)(\xi - \eta)^{-1/2} e^{ik\xi}\, d\xi\, d\eta. \quad (6.27)$$

In §2.5 we solved (2.49), i.e. (6.26) with $f(x) = \exp(-ikx \cos \Theta)$. If this value of $f(x)$ is substituted in (6.27) it is found that the inner integral can be evaluated and the inner differentiation can then be performed. This gives

$$h(x) = 2\pi^{-1/2} e^{-i\pi/4} \tan \tfrac{1}{2}\Theta e^{ikx} \frac{d}{dx} e^{-i(k+k\cos\Theta)x} \int_0^{(2kx)^{1/2}\cos\frac{1}{2}\Theta} \exp(iy^2)\, dy.$$

It is left to the reader to show that the solution in Chapter II, which is given by inverting (2.76), is equivalent to this result.

Miscellaneous Examples and Results VI

6.1 Show that, by using the decomposition theorem B of §1.3, the function $H_+(\alpha)$ in (6.1) is given by

$$H_+(\alpha)K_+(\alpha) = U_+(\alpha) - V_+(\alpha),$$

where

$$U_+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{F_+(\zeta)}{K_-(\zeta)(\zeta - \alpha)}\, d\zeta \quad : \quad V_+(\alpha) = \frac{1}{2\pi i} \int_{ic-\infty}^{ic+\infty} \frac{K_+(\zeta)G(\zeta)}{\zeta - \alpha}\, d\zeta.$$

In these integrals $\tau_- < c < \tau < \tau_+$. Obtain a similar formula for $E_-(\alpha)$.

If we use the results

$$(\zeta - \alpha)^{-1} = -i \int_0^\infty e^{i(\zeta-\alpha)\eta}\, d\eta, \qquad \mathrm{Im}\,(\zeta - \alpha) > 0,$$

$$(\zeta - \alpha)^{-1} = i \int_0^\infty e^{-i(\zeta-\alpha)\eta}\, d\eta, \qquad \mathrm{Im}\,(\zeta - \alpha) < 0,$$

and replace $F_+(\zeta)$, $G_-(\zeta)$ by their expressions in terms of $f(x)$, $g(x)$, it will be found that the inverses of $H_+(\alpha)$, $E_-(\alpha)$ obtained above can be reduced to the results (6.18), (6.20), (6.24) for $h(x)$, $e(x)$.

6.2 By applying a Fourier transform in $(0, \infty)$ show that the solution of

$$\int_0^x \mathscr{L}_+(x - \xi)g(\xi)\,d\xi = f(x), \qquad x > 0, \tag{a}$$

is

$$g(x) = +\frac{1}{2\pi}\mathscr{D}_1^+ \int_0^x \mathscr{M}_+(x - \xi)\,f(\xi)\,d\xi,$$

where we have used the notation (6.12), (6.13), i.e. $\mathscr{M}_+(u)$ is defined by (6.12c) where $K_+(\alpha)$ is the inverse of (6.12a), namely

$$K_+(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty \mathscr{L}_+(u)e^{i\alpha u}\,du.$$

Show that the solution of

$$\int_x^\infty \mathscr{L}_-(\xi - x)g(\xi)\,d\xi = f(x), \qquad x > 0, \tag{b}$$

is

$$g(x) = \frac{1}{2\pi}\mathscr{D}_2^- \int_x^\infty \mathscr{M}_-(\xi - x)\,f(\xi)\,d\xi,$$

where $\mathscr{M}_-(u)$ is defined by (6.12d), $K_-(\alpha)$ being the inverse of (6.12b), namely

$$K_-(\alpha) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty \mathscr{L}_-(u)e^{-i\alpha u}\,du.$$

(Suppose that

$$\int_0^\infty \mathscr{L}_-(\xi - x)g(\xi)\,d\xi = e(x), \qquad x < 0, \tag{c}$$

where $e(x)$ is unknown. Apply a Fourier transform in $(-\infty, \infty)$ to (b) and (c). Then

$$G_+(\alpha) = F_+(\alpha)/K_-(\alpha) + E_-(\alpha)/K_-(\alpha).$$

If this is inverted the second term on the right will give no contribution to $g(x)$, $x > 0$.)

6.3 In case (i) of §6.2, using notation (6.12a, b), the inverse of (6.6) gives

$$k(x-\xi) = \frac{1}{2\pi}\int_{ia-\infty}^{ia+\infty} K_+(\alpha)K_-(\alpha)e^{-i\alpha(x-\xi)}\,d\alpha$$

$$= \frac{1}{(2\pi)^{3/2}}\int_{ia-\infty}^{ia+\infty} K_+(\alpha)\int_0^\infty \mathscr{L}_-(y)e^{-i\alpha y}\,dy e^{-i\alpha(x-\xi)}\,d\alpha$$

$$= \frac{1}{2\pi}\int_{\max(x,\xi)}^{\infty} \mathscr{L}_-(u-x)\mathscr{L}_+(u-\xi)\,du. \qquad \text{(a)}$$

As a special case show that

$$\tfrac{1}{2}\pi i H_0^{(1)}(k|x-\xi|) = e^{-ik(x+\xi)}\int_{\max(x,\xi)}^{\infty} \frac{e^{2iku}}{(u-x)^{1/2}(u-\xi)^{1/2}}\,du.$$

6.4 We can use the result in ex. 6.3 to show that any Wiener–Hopf type equation can be reduced to a repeated Volterra equation. Substitute (a) from ex. 6.3 in (6.2):

$$\int_0^x h(\xi)\int_x^\infty \mathscr{L}_-(u-x)\mathscr{L}_+(u-\xi)\,du\,d\xi +$$

$$+\int_x^\infty h(\xi)\int_\xi^\infty \mathscr{L}_-(u-x)\mathscr{L}_+(u-\xi)\,du\,d\xi = 2\pi f(x), \qquad x>0.$$

Interchange orders of integration in each repeated integral. It is found that then the integrals can be combined to give

$$\int_x^\infty \mathscr{L}_-(u-x)\int_0^u \mathscr{L}_+(u-\xi)h(\xi)\,d\xi\,du = 2\pi f(x), \qquad x>0.$$

But this is equivalent to the repeated Volterra equations

$$\int_0^u \mathscr{L}_+(u-\xi)h(\xi)\,d\xi = p(u), \qquad u>0,$$

$$\int_x^\infty \mathscr{L}_-(u-x)p(u)\,du = 2\pi f(x), \qquad x>0.$$

If these are solved by means of ex. 6.2 it will be found that the expression for $h(x)$, the solution of (6.2), is identical with that found in §6.2, namely (6.20) with $g(\xi) = 0$.

I was led to the above analysis by a paper of E. T. Copson [5] which deals with the problem of the electrified disk. This problem is in fact of the Wiener–Hopf type although the Wiener–Hopf technique is not used in Copson's paper. A related paper is that of D. S. Jones [6]. I have made extensive applications of this method in connection with problems involving parallel disks, disks in tubes, etc. It is not necessary to use the Wiener–Hopf technique explicitly. It is appropriate to denote by 'Copson's method' the technique of interchange of orders of integration to produce repeated Volterra integral equations (cf. I. N. Sneddon, Elements of Partial Differential Equations, McGraw-Hill (1957), p. 179).

6.5 Consider the three-part problem

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} K(\alpha)A(\alpha)e^{-i\alpha x}\,d\alpha = u(x), \qquad (x > q), \tag{a}$$

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} A(\alpha)e^{-i\alpha x}\,d\alpha = s(x), \qquad (p < x < q), \tag{b}$$

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} K(\alpha)A(\alpha)e^{-i\alpha x}\,d\alpha = v(x), \qquad (x < p). \tag{c}$$

Suppose that (b) equals the (unknown) functions $\phi(x)$, $\psi(x)$ for $x < p$ and $x > q$ respectively. Take (b) and (c) together and replace x by $(X + q)$. Then on setting $B(\alpha) = A(\alpha) \exp(-i\alpha q)$,

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} K(\alpha)B(\alpha)e^{-i\alpha X}\,d\alpha = u(X+q), \qquad (X > 0), \tag{d}$$

$$\frac{1}{(2\pi)^{1/2}}\int_{ia-\infty}^{ia+\infty} B(\alpha)e^{-i\alpha X}\,d\alpha = \begin{cases} s(X+q), & ((p-q) < X < 0), \\ \phi(X+q), & (X < (p-q)). \end{cases} \tag{e}$$

Consider case (i) of §6.2 where $|K(\alpha)| \sim |\alpha|^{-1}$ as $|\alpha| \to \infty$ in the strip. Then (6.20) gives, on interchanging orders of integration in the second repeated integral, and rearranging,

$$\psi(X+q) + \int_0^\infty \phi(p-Z)N(X,Z)\,dZ = G(X), \qquad (X > 0), \tag{f}$$

where $G(X)$ is known and

$$N(X,Z) = -\frac{1}{2\pi}\mathscr{D}_1^+ \int_{-d-Z}^{0} \mathscr{M}_+(X-\eta)\mathscr{L}_+(\eta + Z + d)\,d\eta, \quad d = (q-p). \tag{g}$$

In exactly the same way, by considering (a) and (b), we can obtain a similar relation with the rôles of ϕ and ψ interchanged. This gives two integral equations of Fredholm type for the unknown functions ϕ and ψ. It may be possible to solve these iteratively but we shall not consider this here.

The above analysis can be regarded as a generalization of a special result obtained by a different method by E. N. Fox, *Phil. Trans. Roy. Soc.* (A) **241** (1948), 71–103. To indicate the relation between the present method and that of Fox, invert (e) which is equal to $\psi(X + q)$ for $X > 0$, and insert the resulting expression for $B(\alpha)$ in (d). This gives

$$\int_0^\infty \{\psi(X + q)k(Y - X) + \phi(p - X)k(Y + X + d)\}\, dX = F(Y), \quad Y > 0, \tag{h}$$

where $F(Y)$ is a known function. Suppose that we can write

$$k(Y + X + d) = \int_0^\infty k(Y - Z)M(Z,X)\, dZ. \tag{i}$$

If we substitute this expression in (h) and interchange orders of integration, we find

$$\int_0^\infty k(Y - X)\left\{\psi(X + q) + \int_0^\infty \phi(p - Z)M(X,Z)\, dZ\right\} dX = F(Y).$$

This is now a straightforward Wiener–Hopf equation for the quantity in brackets, which is identical with the expression on the left of (f) if we have $M(X,Z) \equiv N(X,Z)$. This indicates that

$$k(Y + X + d)$$
$$= -\frac{1}{2\pi}\int_0^\infty k(Y - Z)\mathscr{D}_1^+ \int_0^{d+X} \mathscr{M}_+(Z + \eta)\mathscr{L}_+(X + d - \eta)\, d\eta\, dZ. \tag{j}$$

In the special case discussed by Fox we have

$$K(\alpha) = (\alpha^2 - k^2)^{-1/2}, \qquad k(x) = \tfrac{1}{2}iH_0^{(1)}(k|x|),$$
$$\mathscr{M}_+(u) = i\mathscr{L}_+(u) = 2^{1/2}e^{i\pi/4}u^{-1/2}e^{iku}.$$

The integral in (g) is elementary and can be evaluated explicitly. We find that (j) reduces to Fox's result:

$$H_0^{(1)}\{k(X + Y + d)\} = \frac{1}{\pi}\int_0^\infty H_0^{(1)}(k|X - Z|)\, \frac{e^{ik(Z+Y+d)}}{Z + Y + d}\left\{\frac{d + Y}{Z}\right\}^{1/2} dZ.$$

The problem considered above is of the same type as that considered in §5.5 (cf. ex. 6.6). It would appear that for the problem of diffraction by slits and strips the method of §§5.5, 5.6 is to be preferred to that given above. (The two methods are of course related.)

6.6 Show that the solution of the integral equations (a)–(c) of ex. 6.5 is equivalent to the solution of the generalized Wiener–Hopf equation

$$e^{i\alpha q}U_+(\alpha) + W(\alpha) + e^{i\alpha p}V_-(\alpha) = K(\alpha)\{e^{i\alpha q}R_+(\alpha) + S(\alpha) + e^{i\alpha p}T_-(\alpha)\},$$

where the notation is analogous to that used in (5.61), and $R_+(\alpha)$, $T_-(\alpha)$, $W(\alpha)$ are unknown.

6.7 Show that the following two problems are equivalent:
(i) Solution of an integral equation

$$\int_0^\infty \{k(x - \xi) + l(x + \xi)\}h(\xi)\, d\xi = f(x), \qquad (x > 0).$$

(ii) Solution of the generalized Wiener–Hopf equation

$$K(\alpha)H_+(\alpha) + L(\alpha)H_+(-\alpha) = F_+(\alpha) + E_-(\alpha).$$

6.8 We reduce the equation

$$\int_0^1 k(x - \xi)h(\xi)\, d\xi = f(x), \qquad (0 < x < 1), \tag{a}$$

to an integral equation of Fredholm type. For simplicity suppose that the transform of $k(x)$, i.e. $K(\alpha)$, is even, $|K(\alpha)| \sim |\alpha|^{-1}$ as $|\alpha| \to \infty$, and $K_-(\alpha) = K_+(-\alpha)$. Then in definitions (6.12), if $\alpha_2 = -\alpha_1$,

$$\mathscr{L}_+(u) = \mathscr{L}_-(u) = \mathscr{L}(u), \text{ say} \quad : \quad \mathscr{M}_+(u) = \mathscr{M}_-(u) = \mathscr{M}(u), \text{ say.}$$

From ex. 6.2 the following are inverse formulae:

$$\left.\begin{aligned} f(x) &= \int_0^x \mathscr{L}(x - \xi)g(\xi)\, d\xi \\ g(x) &= \frac{1}{2\pi} e^{i\beta x} \frac{d}{dx} e^{-i\beta x} \int_0^x \mathscr{M}(x - \xi)\, f(\xi)\, d\xi, \end{aligned}\right\} \tag{b}$$

where β is a complex constant in the upper half-plane. By simple change of variable we have also

$$\left.\begin{aligned} f(x) &= \int_x^1 \mathscr{L}(\xi - x)g(\xi)\, d\xi \\ g(x) &= -\frac{1}{2\pi} e^{-i\beta x} \frac{d}{dx} e^{i\beta x} \int_x^1 \mathscr{M}(\xi - x)\, f(\xi)\, d\xi. \end{aligned}\right\} \tag{c}$$

We now consider (a). From ex. 6.3 we can set

$$k(x-\xi) = \frac{1}{2\pi}\int_{\max(x,\xi)}^{1} \mathscr{L}(u-x)\mathscr{L}(u-\xi)\,du + \\ + \frac{1}{2\pi}\int_{1}^{\infty} \mathscr{L}(u-x)\,\mathscr{L}(u-\xi)\,du.$$

Substitution in (a) gives, for $0 < x < 1$,

$$\int_x^1 \mathscr{L}(u-x)\int_0^u \mathscr{L}(u-\xi)h(\xi)\,d\xi + \\ + \int_0^1 h(\xi)\int_1^\infty \mathscr{L}(u-x)\,\mathscr{L}(u-\xi)\,du\,d\xi = 2\pi f(x). \tag{d}$$

Set

$$\int_0^u \mathscr{L}(u-\xi)h(\xi)\,d\xi = \chi(u), \quad \text{say.} \tag{e}$$

Apply the inversion formula in (c) to (d). In the second integral in (d) replace $h(\xi)$ by its expression in terms of $\chi(u)$ obtained by inverting (e) by (b). Then

$$\chi(x) - \frac{1}{4\pi^2}\,e^{-i\beta x}\frac{d}{dx}\,e^{i\beta x}\int_x^1 \mathscr{M}(u-x)\int_1^\infty \mathscr{L}(\zeta-u)N(\zeta)\,d\zeta\,du \\ = -e^{-i\beta x}\frac{d}{dx}\,e^{i\beta x}\int_x^1 \mathscr{M}(u-x)\,f(u)\,du, \tag{f}$$

where

$$N(\zeta) = \int_0^1 e^{i\beta v}\frac{d}{dv}\left\{e^{-i\beta v}\int_0^v \mathscr{M}(v-\xi)\chi(\xi)\,d\xi\right\}\mathscr{L}(\zeta-v)\,dv \\ = \mathscr{L}(\zeta-1)\int_0^1 \mathscr{M}(1-\xi)\chi(\xi)\,d\xi - \int_0^1 \chi(\xi)I(\xi)\,d\xi, \tag{g}$$

with

$$I(\xi) = \int_\xi^1 \mathscr{M}(v-\xi)e^{-i\beta v}\frac{d}{dv}\,\{e^{i\beta v}\mathscr{L}(\zeta-v)\}\,dv \\ = \int_0^{1-\xi} \mathscr{M}(u)e^{-i\beta(u+\xi)}\frac{d}{du}\,\{e^{i\beta(u+\xi)}\mathscr{L}(\zeta-u-\xi)\}\,du.$$

From the symmetry of the expression in curly brackets we can replace d/du by $d/d\xi$. We can then verify that

$$e^{-i\beta\xi}\frac{d}{d\xi}e^{i\beta\xi}\int_0^{1-\xi}\mathscr{M}(u)\mathscr{L}(\zeta-u-\xi)\,du = -\mathscr{M}(1-\xi)\mathscr{L}(\zeta-1)+I(\xi). \quad \text{(h)}$$

Substitution of $I(\xi)$ from this expression into (g) gives, on then setting $v = \xi + u$,

$$N(\zeta) = -\int_0^1 \chi(\xi)e^{-i\beta\xi}\frac{d}{d\xi}e^{i\beta\xi}\int_\xi^1 \mathscr{M}(v-\xi)\mathscr{L}(\zeta-v)\,dv.$$

On using this result in (f) we find, for $(0 < x < 1)$,

$$\chi(x)+\int_0^1\chi(\xi)K(x,\xi)\,d\xi = -e^{-i\beta x}\frac{d}{dx}e^{i\beta x}\int_x^1\mathscr{M}(\xi-x)f(\xi)\,d\xi, \quad \text{(i)}$$

where

$$K(x,\xi) = \int_1^\infty k(x,\zeta)k(\xi,\zeta)\,d\zeta, \quad \text{(j)}$$

$$k(x,\zeta) = \frac{1}{2\pi}e^{-i\beta x}\frac{d}{dx}\left\{e^{i\beta x}\int_x^1\mathscr{M}(u-x)\mathscr{L}(\zeta-u)\,du\right\}. \quad \text{(k)}$$

It is obvious that alternatively we could have obtained a Fredholm equation for $h(\xi)$ by inverting (d) twice by (b) and (c). But the kernel of the resulting integral equation is not so elegant as (j).

As an example consider

$$\tfrac{1}{2}i\int_0^1 H_0^{(1)}(k|x-\xi|)h(\xi)\,d\xi = f(x), \qquad (0 < x < 1).$$

Then

$$K_+(\alpha) = e^{i\pi/4}(k+\alpha)^{-1/2} \;:\; K_-(\alpha) = e^{i\pi/4}(k-\alpha)^{-1/2} = e^{-i\pi/4}(\alpha-k)^{-1/2}.$$

Choose $\alpha_2 = -\alpha_1 = k$. Then (cf. (6.12), (6.25)),

$$\mathscr{L}(u) = \mathscr{M}(u) = 2^{1/2}u^{-1/2}e^{iku}.$$

The integration in (k) is elementary and we find from (j), (k), if $\beta = k$,

$$k(x,\zeta) = -\frac{1}{\pi}e^{ik(\zeta-x)}\frac{(\zeta-1)^{1/2}}{(1-x)^{1/2}(\zeta-x)},$$

$$K(x,\xi) = \frac{1}{\pi^2(1-x)^{1/2}(1-\xi)^{1/2}(x-\xi)}\{e^{ik(\xi-x)}F(\xi) - e^{ik(x-\xi)}F(x)\},$$

where
$$F(\xi) = (1 - \xi) \int_{2k(1-\xi)}^{\infty} u^{-1} e^{iu} \, du.$$

6.9 Show that the problems in exs. 6.5, 6.8 are equivalent since equations (a)–(c) of ex. 6.5 can be reduced to a single integral equation of type (a) in ex. 6.8 and conversely. Show also that the solution of the integral equation (a) in ex. 6.8 is equivalent to the solution of a generalized Wiener–Hopf equation of the type given in ex. 6.6 (cf. R. Latter [1]).

[*Note added in page proof:* Approximate methods based on the Wiener–Hopf technique can be considered under three headings:

(i) Problems which can be formulated exactly in terms of a generalized Wiener–Hopf complex variable equation (p. 178); the approximations are made when trying to solve this equation. The problems in Chapter V are of this type. Examples which have been treated in the literature from an integral equation point of view are J. B. Alblas [2], H. Levine [2], [3]. (See p. 242 for the last two references.) The integral equation method and Jones's method for obtaining the complex variable equation are equivalent and most of the comments in §2.7, p. 76, apply (though these refer to problems which can be solved exactly by means of the Wiener–Hopf technique).

(ii) Problems which are usually formulated in terms of integral equations or dual integral equations; these are converted into equivalent integral equations of a more convenient form, e.g. Fredholm integral equations of the second kind. The examples of this kind given in this book (exs. 6.5, 6.8) are of type (i) above and in my opinion they are more conveniently formulated and solved by the methods of Chapter V (cf. exs. 6.6, 6.9). But this is not necessarily true in all cases (cf. p. 221, bottom).

(iii) Problems not directly amenable to the Wiener–Hopf technique which are converted into a form suitable for application of the technique by some approximation or transformation, or by virtue of the method used in formulating the problem. There are a large number of possibilities, e.g. H. Levine [1], H. Levine and T. T. Wu [1], [2] (the first and last references are given on p. 242). No examples of this type are given in this book.

It is hoped that examples of types (ii) and (iii) will be included in a book being written by Prof. I. N. Sneddon and myself which will be devoted mainly to methods other than the Wiener–Hopf technique for the solution of mixed boundary value problems.

The above remarks were prompted by an interesting discussion with Dr. H. Levine at the International Congress of Mathematicians, Edinburgh, 1958.]

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INDEX

(The letters ff. following a page number generally denote that an example has been worked in some detail. For notation and conventions see the summary on p. x. References in the text to papers in the bibliography are indexed under the author's name.,

Abelian theorems 36, 46, 57
Absorbent boundaries (*see* Resistive boundaries)
Aerofoil, quarter-infinite 95; two-dimensional 173
ALBLAS, J. B. 96, 236
AMENT, W. J. 47
Analytic continuation 7ff., 38ff.
Antenna, biconical 218; cylindrical rod 213
Approximate factorization 160ff.
Approximate methods 178ff., 236 (*see also* Wiener–Hopf complex variable equation, generalized); for integral equations 231ff., 233ff.; for strips and slits 203ff.; for thick plates 187ff.; meaning of 'approximate' 180; miscellaneous formulations 181ff.
Asymptotic expansions, for contour integrals 33ff., 46; for transforms 36; in approximate solutions 200ff.; (*see also* Far-field)

BAILIN, L. L. 122
BAKER, B. B. and COPSON, E. T. 70, 206
BALDWIN, G. L. and HEINS, A. E. 125
BAZER, J. and KARP, S. N. 73, 128, 153, 166, 167
Biharmonic equation, (*see also* Elastic medium) approximate factorization for 162; stresses in strip 138
Boundary conditions, specified by arbitrary functions 77ff.; (*see also* Resistive boundaries)
Boundary layer on flat plate, 96, 97
BOUWKAMP, C. J. 75, 179, 218
Branch cuts 8ff.; special cases 9, 19, 38, 39
Branch points and poles (*see* Poles and branch points)

CARLSON, J. F. and HEINS, A. E. 103, 133
CARRIER, G. F. and MUNK, W. H. 94
CARSLAW, H. S. 72
CHESTER, W. 20, 132
CHURCHILL, R. V. 27
Circular waveguide (*see* Duct, cylindrical)
CLEMMOW, P. C. 36, 88, 93, 94, 152 (*see also* Vajnshtejn–Karp–Clemmow method)
Cone, Laplace's equation 165; wave equation 218
Contour integrals, for sum and product decompositions 18ff.; special type 31ff.
Convection of heat from flat plate 94
Conventions x, 10, 28, 86
COPSON, E. T. vii, 5, 6, 70, 95, 231
Copson's method 231
Cosine transform 147, 212
Cylindrical pipe (*see* Duct, cylindrical)

DAVISON, B. 168
Dielectric slab 211, 213
Diffraction, (*see* Disc, Duct, Half-planes, Strips and slits) methods other than the Wiener–Hopf for 125, 179
Dirac delta function 43
Disc, Laplace's equation 166, 231; wave equation, high frequency 219, — low frequency 213ff., 231
Divergent integrals 64
DOETSCH, G. 27, 36
Dual integral equations, exact solution of general cases 77ff., 222ff.; for boundary conditions specified arbitrarily 77ff.; formulation in terms of 4, 58, 150; for three-part problems 231; general comments 48, 77, 221, 236; half-plane problem solved by 58ff.; involving Bessel functions viii, 221; multiplying factor method for 58ff., 77ff., 222ff.; reduction to complex variable equation 150, 151, 221

Duct, cylindrical, finite length 207; infinite length, resistive wall 134, — semi-infinite tube in 121; propagated and attenuated modes 110; semi-infinite, electromagnetic waves 132, — radiation from 111ff., — radiation from coaxial 133, 154, — wave incident on 116ff.
Duct, two-dimensional, finite length 207; infinite length with semi-infinite obstacle, dielectric slab 211, — resistive wall 133, — set of posts 174, — solution via simultaneous linear algebraic equations 175, — strip parallel to wall 118ff., — strip of finite thickness parallel to wall 212; infinite length with step 196, 207ff.; infinite length with strip across 122ff., 212; propagated and attenuated modes 98ff.; semi-infinite (*see* Half-planes, two parallel)

Edge conditions 51, 74ff.; references 75
Elastic medium, (*see also* Biharmonic equation) diffraction by half-plane in 96; stresses in wedge with mixed boundary conditions 173
Electromagnetic waves, boundary conditions 134; diffraction by half-plane 95; diffraction by semi-infinite cylinder 132; edge conditions 76; terminology 85, 92
End correction, cylindrical tube 115; thick semi-infinite strip 196
ERDELYI, A. 36, 46
ERDELYI, A. *et al.* 148, 176, 213

Factorization (*see* Product factorization)
Far-field, examples of calculation of 105, 110, 115, 192, 206
FEL'D, A. N. 174
FOCK, V. 168
Fourier integral 1ff., 21ff.; in complex plane 23; generalized 42
Fourier transforms 2; basic example 53ff.
FOX, E. N. 232
Fox's integral identity 232
Fredholm integral equations, first kind (*see* Wiener–Hopf integral equations); second kind 221, 231ff., 233ff., 236
Fresnel functions 33, 45, 73, 213
Function-theoretic methods 153

Gamma functions 40; factorization by means of 41, 167, 171, 215
Green's function 61ff.
GREENE, T. R. and HEINS, A. E. 94

Half-planes, infinite set of staggered 133; single infinitely thin, basic (Sommerfeld) diffraction problem 48ff., — diffraction of electromagnetic waves by 95, — finite resistivity 83, — generalizations of basic problem 51ff., — in elastic or viscous medium 96, — solution by simultaneous linear algebraic equations 175; single of finite thickness, 181, 187ff., — dielectric material 213; three parallel 154; two parallel, radiation from 105ff., — wave falling on 100ff., 125, 126, — with flange 183; two parallel, staggered 154
Hankel functions, asymptotic behaviour 45; integral representation 32
HARRINGTON, R. F. 88
HEINS, A. E. 94, 103, 127, 131, 133, 134, 157, 171
HEINS, A. E. and CARLSON, J. F. 133
HEINS, A. E. and FESHBACH, H. 92, 93, 134
Hilbert problem 49, 141ff.; integral equation application 169
HOPF, E. 168

IIJIMA, T. 132
Imperfect conductivity (*see* Resistive boundaries)
Indentation of contour 30, 44, 86; for transform in time 44
Infinite product 15, 40, 103, 129, 130; asymptotic behaviour of 104, 128, 129; computation of 127, 128; for gamma functions 41
Institute of Mathematical Sciences, New York, reports 237
Integral equation method, general remarks vii, 49, 71, 76, 236
Integral equations (*see* Fredholm, Volterra, Wiener–Hopf)
Integral functions 6, 178, 197

Jets 139
JONES, D. S. vii, viii, 48, 52, 75, 95, 97, 127, 130, 131, 167, 173, 187, 190, 195, 196, 199, 200, 207, 231
Jones's method, detailed application to a special example 52ff.; for approximate solutions 178ff., 236; for boundary conditions specified arbitrarily 79ff.; general remarks

vii, 48, 71, 76, 98; routine applications 100ff.
JOST, RES 219

$k = k_1 \pm ik_2$, 10, 29, 38, 44, 45, 86
KARP, S. N. 166, 167 (*see also* Vajnshtejn–Karp–Clemmow method)
KARP, S. N. and RUSSEK, A. 206
KARP, S. N. and WILLIAMS, W. E. 125
KLUNKER, E. B. and HARDER, K. C. 139
KOITER, W. T. 75, 139, 161, 162, 173
KOURGANOFF, V. 133, 168

Laplace transform 10, 21, 52, 86; complete equivalence with Fourier transform 22, 26
Laplace's equation, as limit of wave equation 135ff.; in jet problem 140; in polar coordinates 164ff., 177
Laplace-type equations 135 (*see also* Biharmonic equation) Oseen's equations 97
LATTER, R. 236
Lebedev–Kontorovich transform 148, 213ff.
Lebesgue integration, reason for not using 23
LEITNER, A. and WELLS, C. P. 213, 216
LENNOX, S. C. 139
LEVINE, H. 236, 242
LEVINE, H. and SCHWINGER, J. 20, 42, 115, 130, 132
LEVINE, H. and WU, T. T. 236
LEWIN, L. 131, 179, 242
LEWIS, J. A. and CARRIER, G. F. 97
LINFOOT, E. H. and SHEPHERD, W. M. 175
Liouville's theorem 6, 38, 57, 69, 103, 114

MAGNUS, W. 175
MAGNUS, W. and OBERHETTINGER, F. 175
MARCUVITZ, N. 122, 133, 179
MARK, C. 168
MARSHAK, R. E. 168
MAUE, A. W. 75, 96
MEIER, J. A. and LEITNER, A. 218
MEIXNER, J. 75
Meromorphic functions 39
MILES, J. W. 95
Milne's integral equation 167
Mixed boundary value problem 1; solution of a specific example 77ff.; three-part 183, 203ff., 231ff.; two-part, general theory 147
MORSE, P. M. and FESHBACH, H. 62, 115, 131, 134, 179
Multiplying factor method 58ff., 78ff., 221ff.
MUSKHELISHVILI, N. I. 142, 144, 146, 158

Navier–Stokes equations, linearized 96
Notation x, 54; in certain approximate methods 182, 197
Numerical evaluation of infinite products 127, 128
Numerical tables viii, 122, 131

OBERHETTINGER, F. 36
Oseen's equations 97

PACK, D. C. viii, 139
PAI, S. I. 139
PAPADOPOULOS, V. M. 134, 196
PEARSON, J. D. 132
Perpendicular boundaries 178, 181, 187ff., 207ff.
PLACZEK, G. 168
PLACZEK, G. and SEIDEL, W. 168
Plemelj formulae 47, 145
POL, B. VAN DER and BREMMER, H. 27, 36
Polar coordinates, Laplace's equation 164ff., 177; steady-state wave equation 148, 214ff.; wedge of elastic material 173
Poles and branch points 47, 133, 207
Product factorization 14ff.; approximate 160ff.; general theorems 15, 41; involving gamma functions 41, 167, 171, 215; of $\exp(-\gamma d)$, 20; of $1 + i\delta\gamma$, $1 + i\varepsilon\gamma^{-1}$, 91, 163; of $1 \pm \exp(-2\gamma b)$, 102ff.; of $K_1(\gamma a)I_1(\gamma a)$, 112, 130; transformation $\alpha = -k \cos \beta$ for 47

Quantum mechanics, reference to problem in 219
Quarter-plane aerofoil 95

Radiation condition at infinity 2, 31
Radiation-type boundary conditions (*see* Resistive boundaries)
Radio-wave propagation over land-sea interface 93, 94
Reflection coefficient in duct 110, 114, 120, 124
Regularity, of functions defined by integrals 11ff.; of transforms in complex plane 12
Resistive boundaries, half-plane 83ff.; notation 134; strip across duct 212; strip parallel to walls of duct 120
Rod, solid cylindrical, finite length 213; semi-infinite 196

Scattering cross-section of disc 218
SCHWINGER, J. vii
Semi-infinite plane (*see* Half-planes)
SENIOR, T. B. A. 83, 85, 92, 93, 134
Separation of variables 1, 147, 149
Simultaneous linear algebraic equations, from simultaneous Wiener–Hopf equations 155; in approximate solutions 185, 190ff., 208, 217; in duct problem 175; in half-plane problem 176; solution of special sets 173, 174ff.
Simultaneous Wiener–Hopf equations, general theory 153ff.; in electromagnetic cylindrical duct problem 132; in electromagnetic half-plane problem 95; in two-dimensional duct problems 121, 154
Slits (*see* Strips and slits)
SNEDDON, I. N. viii, 2, 96, 168, 231, 236
Sommerfeld, half-plane problem 48ff.; radiation condition 31
Sound waves (*see* Duct, Half-planes, etc.)
Source distribution 44, 52, 87
SPARENBERG, J. A. 169, 171
Steady-state wave equation 28
Steepest descent 34
Strips and slits of finite thickness, semi-infinite length 181, 187ff., and finite length 213, in duct 212
Strips and slits of finite width, Laplace's equation 177; wave equation, high frequency 183, 203ff., 219, — low frequency 218
Sum decomposition, function with poles 39; general formulae for concrete cases 18; general theorem 13, — applications of 80, 179, 186, 198, 228; in approximate solutions 179, 186; of $\gamma = (\alpha^2 - k^2)^{1/2}$ 20, 47; transformation $\alpha = -k \cos \beta$ for 47
Superposition, method of 81, 87

Tables (*see* Numerical tables)
Tides in permeable rock 94
Time factor $\exp(-i\omega t)$ 28
TITCHMARSH, E. C. 2, 5, 8, 11, 39, 40, 41, 42, 168, 171
Titchmarsh's method 171
Transformation $\alpha = -k \cos \beta$, applied to diffraction by half-plane 176; for evaluation of integrals 33; for sum decompositions 47; in wedge problems 219
Transient problems 28, 97
TRANTER, C. J. 2

Uniqueness 72; and edge conditions 51, 74; of solution of integral equations 69, 168

VAJNSHTEJN, L. A. 20, 115, 125, 130, 131, 132, 153
Vajnshtejn–Karp–Clemmow method 49, 153, 213
Verification of solution 72
Viscous medium 96, 97, 139
Volterra integral equations 229, 230, 233

Water waves 94, 134
Watson, G. N. 36, 46
Wave equation 27ff.
Waveguide (*see* Duct, two-dimensional; Duct, cylindrical; Half-planes, two parallel)
Wedges, diffraction by 219; of elastic material 173
WEITZ, M. and KELLER, J. B. 134
Whittaker functions 200, 213
WIDDER, D. V. 27, 36
WIENER, N. and HOPF, E. 41, 167, 168
Wiener–Hopf complex variable equation (for exact solutions), basic procedure 36ff.; derived by Jones's method 55, 79, 84, 101, 125, 150, — from an integral equation 68, 168, — from dual integral equations 150, 151, 221; general solutions 79ff., 89, 222ff., 228; reduction to linear algebraic equations 174; solutions for simple cases 55ff., 68ff.
Wiener–Hopf complex variable equation, generalized (for approximate solutions) 178; formulations of 181ff., 187ff., 203ff., 233, 236; solutions of 184ff., 196ff.
Wiener–Hopf equations, general considerations 151
Wiener–Hopf integral equations, formulation by Green's functions 61ff., 89, 132, — by physical reasoning 90, 131, 177, — by transforms 4, 65ff.; Milne's integral equation 167; reduction of more general types to Fredholm equations of second kind 232, 233ff.; reduction to repeated Volterra equations 230; solution by the Wiener–Hopf technique 67ff., 167ff., 227, — via the Hilbert problem 169; special equations 171, 177; uniqueness of solution 69, 168
WILLIAMS, W. E. viii, 196, 207, 208