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The one-dimensional Burgers' equation; existence, uniqueness and stability

by Tomasz DŁOTKO (Katowice)

§ 1. Introduction. The existence, uniqueness and stability of the equation first investigated by J. M. Burgers [1] has been studied. This equation is derived from the theory of turbulent fluid motion and has properties similar to those of the Navier-Stokes equation, but is simpler to study. Investigations on the stability of this equation, or rather system of one differential-integral and a second partial differential equation, were reported recently [3]. It was shown that the solution $(U, v) = \left(\frac{P}{v}, 0\right)$ of (1) is locally stable for $\frac{P}{v} < v$, and

the case $\frac{P}{v} = v$ was classified as unstable.

We show that for $\frac{P}{v} < v$ the solution $\left(\frac{P}{v}, 0\right)$ is global exponential stable and for $\frac{P}{v} = v$ it is global asymptotic stable (for definitions see [2]).

First using the method of J. L. Lions [6] we show the existence and uniqueness of the solution of (1) in a weak sense, but our solution is sufficiently smooth to study the stability in a rather normal manner.

The one-dimensional Burgers' equation may be written in the following form:

$$(1.1) \quad \frac{dU(t)}{dt} = P - vU(t) - \int_{\Omega} v^2(t, x) dx \quad \text{for } t > 0,$$

$$(1.2) \quad \frac{\partial v(t, x)}{\partial t} = U(t)v(t, x) + v \frac{\partial^2 v(t, x)}{\partial x^2} - \frac{\partial}{\partial x}(v^2(t, x))$$

in the set $Q := \{(t, x) \in R^2; t > 0, x \in (0, \pi)\}$, $\Omega := (0, \pi)$, $U = U(t): [0, \infty) \rightarrow R$, $v = v(t, x): \bar{Q} \rightarrow R$, with the conditions

$$(2) \quad \begin{cases} U(0) = U_0 \\ v(0, x) = \varphi(x) & \text{for } x \in \Omega, \\ v(t, 0) = v(t, \pi) = 0 & \text{for } t \geq 0. \end{cases}$$

§ 2. Notation and definitions. The following real Banach spaces are considered (for details see [7] or [4]):

$C^m(\Omega)$ — the set of all real functions together with m -th order continuous derivatives (classical) in Ω ,

$$L^p(\Omega) = \{u: \Omega \rightarrow R; \|u\|_{L^p} = \left(\int_{\Omega} |u|^p dx \right)^{\frac{1}{p}} < \infty, u\text{-measurable}\},$$

$$L^\infty(\Omega) = \{u: \Omega \rightarrow R; \|u\|_{L^\infty} = \text{ess sup } |u| < \infty, u\text{-measurable}\},$$

$$H^m(\Omega) = \{u: \Omega \rightarrow R; D^\alpha u \in L^2(\Omega), |\alpha| \leq m, u\text{-measurable}\}, D^\alpha := \frac{\partial^\alpha}{\partial x^\alpha}, \text{ where } D^\alpha \text{ is}$$

taken to be the derivative in the distributional sense ([4], [7], [8]).

By $H_0^1(\Omega)$ we mean the completion of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ ($\mathcal{D}(\Omega)$ as in [7]).

$$H^{-1}(\Omega) = (H_0^1(\Omega))'$$

The symbol

" $f \in L^2(0, T; H_0^1(\Omega))$ " denotes that the function $[0, T] \ni t \rightarrow f(t, \cdot) \in H_0^1(\Omega)$ belongs to $L^2(0, T)$. $\mathcal{L}(X, Y)$ denote the space of linear continuous operators from the Banach space X into the Banach space Y . The symbol $\langle \cdot, \cdot \rangle$ denotes the real scalar product in $L^2(\Omega)$,

$\|\cdot\|_{H_0^1}$, $\|\cdot\|_{L^2}$ denotes the norm in $H_0^1(\Omega)$, $L^2(\Omega)$ respectively.

We also define certain other symbols

$$a(f, h) := \int_{\Omega} \frac{\partial f}{\partial x} \frac{\partial h}{\partial x} dx = \langle Af, h \rangle,$$

$$b(f, h, l) := \int_{\Omega} f \frac{\partial h}{\partial x} l dx = \langle g(f, h), l \rangle,$$

$$c(f, h, l) := \int_{\Omega} f h l dx = \langle C(f, h), l \rangle.$$

For any real fixed $T > 0$ we define the weak solution of the problem (1), (2) as follows:

Definition 1. A pair of functions (U, v) is said to be a *weak solution* of the problem (1), (2) in $Q \cap \{t < T\}$ if U satisfies (1.1) in Caratheodory's sense (as in [5] or [2]);

$$U(t) = U_0 + \int_0^t (P - v U(\tau) - \int_{\Omega} v^2(\tau, x) dx) d\tau$$

for any $t \in [0, T]$, and U is absolutely continuous, and

$$(3) \quad v \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$$

satisfies

$$(4) \quad \langle v', w \rangle + v a(v, w) + 2b(v, v, w) = c(U, v, w)$$

for any function $w \in H_0^1(\Omega)$, $v' = \frac{\partial v}{\partial t}$ (distributional with the values in the Banach space [4], [7]),

$$(5) \quad v(0, x) = \varphi(x) \equiv v_0(x) \in L^2(\Omega).$$

§ 3. We now prove certain lemmas which guarantee the correctness of the conditions (4) and (5).

LEMMA 1 (Wirtinger's inequality). *For any functions $f(x) \in H_0^1(\Omega)$*

$$(6) \quad \|f\|_{L^2(\Omega)}^2 \leq \|D_x f\|_{L^2(\Omega)}^2.$$

Proof. For any $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$

$$f(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

where

$$a_n = \frac{2}{\pi} \int_{\Omega} f(x) \sin nx dx.$$

Differentiating the series with respect to x we get

$$f'(x) = \sum_{n=1}^{\infty} n a_n \cos nx$$

where the equality is satisfied in $L^2(\Omega)$. Now the inequality

$$\int_{\Omega} \sum_{n=1}^{\infty} a_n^2 \sin^2 nx dx \leq \int_{\Omega} \sum_{n=1}^{\infty} a_n^2 n^2 \cos^2 nx dx$$

and the fact that $a_n^2 \geq 0$ finishes the proof for the dense subset of $H_0^1(\Omega)$ and so for all $H_0^1(\Omega)$.

Let us note that the equality in (6) is possible only for the functions $f(x) = \text{const.} \sin x$, because $H_0^1(\Omega) \subset C^0(\bar{\Omega})$. The inclusion below is the consequence of the Sobolev embedding theorems (see [3], [6]):

$$\begin{aligned} H_0^1(\Omega) &\subset C^0(\bar{\Omega}) \subset L^\infty(\Omega) \quad \text{for bounded } \Omega \subset R, \\ H_0^1(\Omega) &\subset L_p(\Omega) \quad \text{for } p \geq 1. \end{aligned}$$

Now we prove the following:

LEMMA 2. *For any functions $f \in H_0^1(\Omega)$, $h, l \in L^2(\Omega)$*

$$(7) \quad \left| \int_{\Omega} f h l dx \right| \leq \|f\|_{H_0^1}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2} \|l\|_{L^2}.$$

Proof. The following estimate is the consequence of the fact that $H_0^1(\Omega) \subset L^\infty(\Omega)$:

$$\left| \int_{\Omega} f h l dx \right| \leq \|f\|_{L^\infty} \|h\|_{L^2} \|l\|_{L^2}.$$

For any function $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$

$$\begin{aligned} |f(x)|^2 &= f(x)^2 = \int_0^x D_s f(s) \cdot f(s) ds + \int_\pi^x D_s f(s) \cdot f(s) ds \\ &\leq \left| \int_0^x D_s f(s) \cdot f(s) ds \right| + \left| \int_\pi^x D_s f(s) \cdot f(s) ds \right| \\ &\leq \int_{\Omega} |D_s f(s) \cdot f(s)| ds \leq \|D_x f\|_{L^2} \|f\|_{L^2} \leq \|f\|_{H_0^1} \|f\|_{L^2} \end{aligned}$$

and so (7) is satisfied for all $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$, and also for all $f \in H_0^1(\Omega)$.

For the functions $f, h \in H_0^1(\Omega)$ and $l \in L^2(\Omega)$ the following fact should be noted;

$$(8) \quad |b(f, h, l)| = \left| \int_{\Omega} f \cdot D_x h \cdot l dx \right| \leq \|f\|_{H_0^1}^{\frac{1}{2}} \|f\|_{L^2}^{\frac{1}{2}} \|D_x h\|_{L^2} \|l\|_{L^2}.$$

Remark 1. For any function v satisfying Definition 1 we have

$$g(v, v) \in L^2(0, T; H^{-1}(\Omega)).$$

The following estimate ($w \in H_0^1(\Omega)$) ensure that this is so:

$$|b(v, v, w)| = \frac{1}{2}|b(v, w, v)| \leq \frac{1}{2}\|v\|_{H_0^1}^{\frac{1}{2}} \|v\|_{L^2}^{\frac{1}{2}} \|w\|_{H_0^1} \|v\|_{L^2} \leq \frac{1}{2}\|v\|_{H_0^1} \|w\|_{H_0^1} \|v\|_{L^2}.$$

Remark 2. For any weak solution (U, v) the function

$$[0, T] \ni t \rightarrow \|v(t, x)\|_{L^2}^2$$

is measurable and belongs to $L^\infty(0, T)$. So U given by

$$(9) \quad \frac{dU}{dt} = P - vU - \|v(t, x)\|_{L^2}^2$$

$$U(0) = U_0$$

has the derivative almost everywhere, and the derivative belongs to $L^\infty(0, T)$, which is the consequence of (9).

LEMMA 3. *For any weak solution (U, v) of (1), (2) is $v' \in L^2(0, T; H^{-1}(\Omega))$.*

Proof. With our notation condition (4) may be written in the operator form:

$$(10) \quad v' = -vA(v) - 2g(v, v) + C(U, v).$$

Under our definition of the norm in $H_0^1(\Omega)$:

$$\|f\|_{H_0^1} := \sqrt{a(f, f)},$$

$$\|a(v, w)\| \leq \|v\|_{H_0^1} \|w\|_{H_0^1}$$

and the fact that $v \in L^2(0, T; H_0^1(\Omega))$ implies that

$$A(v) \in L^2(0, T; H^{-1}(\Omega)).$$

Also the following condition is satisfied

$$\left| \int_{\Omega} U(t)v(t, x)w(t, x)dx \right| \leq \|U\|_{C^0} \|v\|_{L^2} \|w\|_{L^2} \leq \|U\|_{C^0} \|v\|_{H_0^1} \|w\|_{H_0^1}$$

but $U \in C^0([0, T])$, so $C(U, v) \in L^2(0, T; H^{-1}(\Omega))$.

Hence Remark 1 finishes the proof of Lemma 3.

LEMMA 4. *Any weak solution v before the change on the set of the null measure is continuous as the function*

$$[0, T] \ni t \rightarrow v(t, \cdot) \in L^2(\Omega).$$

Proof. The proof is the consequence of Theorem 3.1, 1.3 in [7], and the fact that

$$[H_0^1(\Omega), H^{-1}(\Omega)]_{\frac{1}{2}} = L^2(\Omega)$$

(the symbol $[,]_{\frac{1}{2}}$ as in [7]).

Remark 3. Lemma 4 makes condition (5) sensible. Moreover the function $U(t)$ satisfies (1.1) in the usual sense.

§ 4. Now we start to prove the existence of the weak solution of (1), (2), using the Bubnov-Galerkin method. The proof is parallel to the proof of existence for the Navier-Stokes equation in [6].

THEOREM 1. *There exists a weak solution of the problem (1), (2) in the sense of Definition 1.*

Proof. We are looking for the function $v(t, x)$ as the limit of the approximate solutions $v_m(t, x)$ of the form

$$(11) \quad v_m(t, x) = \sum_{k=1}^m c_m^k(t) V_k(x)$$

where $\{V_k(x)\}_1^\infty$ is the “special” complete system in $H_0^1(\Omega)$. $\{V_k\}_1^\infty$ consists precisely of the own functions of the problem

$$\langle V_j, w \rangle_{H_0^1} := a(V_j, w) = \lambda_j \langle V_j, w \rangle$$

for any $w \in H_0^1(\Omega)$. The existence of such a base in H_0^1 for any dimension of the domain is a known fact (see [8], [6]).

The functions $c_m^k(t)$ and U_m are given by a system of $(m+1)$ -ordinary differential equations:

$$(12.1) \quad \left| \frac{dU_m}{dt} = P - v U_m - \|v_m\|_{L^2}^2 \right.$$

$$(12.2) \quad \left| \langle v'_m, V_l \rangle + v a(v_m, V_l) + 2b(v_m, v_m, V_l) = U_m \langle v_m, V_l \rangle, \quad l = 1, \dots, m \right.$$

with the conditions

$$(12.3) \quad \begin{cases} U_m(0) = U_{om}, & U_{om} \rightarrow U_0, \\ v_m(0) = v_{om}, & v_{om} \rightarrow v_0 \text{ in } L^2(\Omega), \end{cases}$$

where the convergence of v_{om} follows from (11).

The classical existence theorems ensure the existence of the solutions U_m , $\{c_m^l\}_1^m$ for any m . The functions c_m^k exist in the intervals $[0, t_m^k]$ and U_m exists in $[0, \min_k t_m^k]$.

Now we draw the “first a priori estimate”, which guarantees the existence of $\{c_m^k\}_1^m$ and U_m in the whole interval $[0, T]$, and the fact that

U_m is bounded in $L^\infty(0, T)$,

v_m is bounded in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

Equation (12.1) is multiplied by U_m and integrated over $[0, t]$ for any $t \in [0, T]$, equations (12.2) are multiplied by c_m^l respectively, integrated over $[0, t]$ and summed over l

$$\begin{aligned} \frac{1}{2} \int_0^t \frac{d}{d\tau} U_m^2(\tau) d\tau + v \int_0^t U_m^2(\tau) d\tau &= P \int_0^t U_m(\tau) d\tau - \int_0^t U_m(\tau) \|v_m(\tau)\|_{L^2}^2 d\tau, \\ \frac{1}{2} \int_0^t \frac{d}{d\tau} \|v_m(\tau)\|_{L^2}^2 d\tau + v \int_0^t \|v_m(\tau)\|_{H_0^1}^2 d\tau &= \int_0^t U_m(\tau) \|v_m(\tau)\|_{L^2}^2 d\tau \end{aligned}$$

summing these equalities by sides and using Young's inequality:

$$xy \leq \frac{\epsilon}{2}x^2 + \frac{1}{2\epsilon}y^2 \quad \text{for any } \epsilon > 0$$

we have

$$\begin{aligned} (13) \quad \frac{1}{2} U_m^2(t) + \frac{1}{2} \|v_m(t)\|_{L^2}^2 + v \int_0^t [U_m^2(\tau) + \|v_m(\tau)\|_{H_0^1}^2] d\tau \\ \leq \frac{1}{2} U_m^2(0) + \frac{1}{2} \|v_{om}\|_{L^2}^2 + \frac{\epsilon T}{2} P^2 + \frac{1}{2\epsilon} \int_0^t U_m^2(\tau) d\tau \end{aligned}$$

setting $v - \frac{1}{2\epsilon} > 0$ we obtain that

U_m are bounded in $L^\infty(0, T)$,

v_m are bounded in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$.

In the next part of the proof we show that v'_m are bounded in $L^2(0, T; H^{-1}(\Omega))$. Denoting as P_m the projection

$$L^2(\Omega) \rightarrow \lim(V_1, \dots, V_m)$$

such that

$$P_m h = \sum_{i=1}^m \langle h, V_i \rangle V_i,$$

multiplying (12.2) by V_l and summing over l from 1 to m we get

$$\begin{aligned} (14) \quad \sum_l^m \langle v'_m, V_l \rangle V_l &= -v \sum_l^m \int_{\Omega} \frac{\partial v_m}{\partial x} \frac{\partial V_l}{\partial x} dx V_l \\ &\quad - 2 \sum_l^m \int_{\Omega} v_m \frac{\partial v_m}{\partial x} V_l dx V_l + U_m \sum_l^m \langle v_m, V_l \rangle V_l. \end{aligned}$$

Note that $P_m = P'_m$, and

$$\sum_l^m \langle v'_m, V_l \rangle V_l = v'_m$$

because

$$v'_m = \sum_k^m \frac{dc_m^k}{dt} V_k \in \lim(V_1, \dots, V_m).$$

Multiplying (14) by any function $w \in H_0^1(\Omega)$ and integrating over Ω we obtain the operator equality

$$(15) \quad v'_m = -2P_m(g(v_m)) - v P_m A v_m + P_m C(U_m, v_m).$$

Taking in the definition of V_j ; $w = V_j$ we obtain that

$$\lambda_j = \|V_j\|_{H_0^1}^2 : \|V_j\|_{L^2}^2$$

and so by Lemma 1 it follows that $\lambda_j \geq 1$. Now

$$\left\| \sum_{j=1}^m \langle V_j, w \rangle V_j \right\|_{H_0^1} \leq \left\| \sum_{j=1}^m \frac{1}{\lambda_j} \langle w, V_j \rangle_{H_0^1} V_j \right\|_{H_0^1} \leq \|w\|_{H_0^1}$$

because $w \in H_0^1(\Omega)$, and $\frac{1}{\lambda_j} \leq 1$ for any j . So we obtain $\|P_m\|_{\mathcal{L}(H_0^1, H_0^1)} \leq 1$

and by $P_m = P'_m$

$$(16) \quad \|P_m\|_{\mathcal{L}(H^{-1}, H^{-1})} \leq 1.$$

Just as in Remark 1 and Lemma 3 $g(v_m)$ is bounded in $L^2(0, T; H^{-1}(\Omega))$, also $A v_m$ and $C(U_m, v_m)$ are bounded in $L^2(0, T; H^{-1}(\Omega))$, and so (15) and (16) imply that v'_m is bounded in $L^2(0, T; H^{-1}(\Omega))$.

Now we make use of the “theorem of compactness” [6], Theorem 5.1, 1.5, with

$$(17) \quad B_0 = H_0^1(\Omega) \subset B = L^2(\Omega) \subset B_1 = H^{-1}(\Omega).$$

In our situation Theorem 5.1 laid down that for any sequence v_m , such that the following sequences are bounded

$$v_m \in L^2(0, T; H_0^1(\Omega)) \text{ and } v'_m \in L^2(0, T; H^{-1}(\Omega))$$

a subsequence v_μ may be taken such that v_μ is convergent in $L^2(0, T; L^2(\Omega)) = L^2(Q)$.

Now using the well-known properties of bounded sets in reflexive spaces (see [4]) we may take a subsequence of v_μ such that

$$(18) \quad v_\eta \rightarrow v \quad \text{weak in } L^2(0, T; H_0^1(\Omega)),$$

$$(19) \quad v_\eta \rightarrow v \quad \text{weak star in } L^\infty(0, T; L^2(\Omega)),$$

$$(20) \quad v_\eta \rightarrow v \quad \text{in } L^2(Q), \text{ almost everywhere in } Q,$$

$$(21) \quad v_\eta \rightarrow v' \quad \text{weak in } L^2(0, T; H^{-1}(\Omega)).$$

By (18) and (20) and Lemma 1.2, 1,2 in [6]

$$\begin{aligned} v_{0\eta} &\rightarrow v_0 && \text{weak in } H^{-1}(\Omega), \text{ but from (12.3)} \\ v_{0\eta} &\rightarrow v_0 && \text{in } L^2(\Omega), \text{ and so} \\ v(0, x) &= v_0(x) && \text{in } L^2(\Omega). \end{aligned}$$

It follows from Lemma 2 that for $f \in C^1(\bar{\Omega})$, $f(0) = f(\pi) = 0$

$$(22) \quad \|f(x)\|_{L^4}^4 \leq \|f\|_{H_0^1}^2 \|f\|_{L^2}^2 \int_{\Omega} 1 dy = \pi \|f\|_{H_0^1}^2 \|f\|_{L^2}^2.$$

Using (22) and (3) we obtain that v_η^2 are bounded in $L^2(0, T; L^2(\Omega)) = L^2(Q)$ and so v_η^2 are weak convergent in $L^2(Q)$. But v_η from (20) is almost everywhere convergent in Q to v , so

$$(23) \quad v_\eta^2 \rightarrow v^2 \quad \text{weak in } L^2(Q).$$

Finally from the condition

$$\int_0^T b(v_\eta, v_\eta, V_j) \psi dt = -\frac{1}{2} \int_0^t b(v_\eta, V_j, v_\eta) \psi dt$$

for any $\psi \in L^2(0, T)$, and by (23) it follows that

$$b(v_\eta, v_\eta, V_j) \rightarrow b(v, v, V_j) \quad \text{weak in } L^2(0, T).$$

Now (18) implies that

$$a(v_\eta, V_j) \rightarrow a(v, V_j) \quad \text{weak in } L^2(0, T)$$

and (21) implies that

$$\langle v'_\eta, V_j \rangle \rightarrow \langle v', V_j \rangle \quad \text{in } \mathcal{D}'(0, T) \text{ (see [6])}.$$

It remains to study the component $U_m \langle v_\eta, V_i \rangle$. From (20) it follows that

$$\langle v_\eta, V_i \rangle \rightarrow \langle v, V_i \rangle.$$

We show now that U_m converges uniformly to U in $[0, T]$.

Remark 4. U_m converges uniformly to U in $[0, T]$.

The condition (20) implies that $\|v_\eta\|_{L^2(Q)}^2 \rightarrow \|v\|_{L^2(Q)}^2$. Let U_m satisfy (12,1)

$$U_m(t) = U_m(0) + \int_0^t [P - vU_m - \|v_m\|_{L^2}^2] d\tau$$

Then the difference $U_m - U_n$ satisfies

$$(24) \quad |U_m(t) - U_n(t)| \leq [|U_m(0) - U_n(0)| + \varepsilon_{mn}] \exp(vt),$$

where $\varepsilon_{mn} := |\|v_m\|_{L^2(Q)}^2 - \|v_n\|_{L^2(Q)}^2| \rightarrow 0$.

All the facts previously studied allow us to pass with η to infinity in (12), and the fact that $\{V_i\}_1^\infty$ is the complete system in $H_0^1(\Omega)$ shows that there exists a weak solution of (1), (2).

Now we show that under the "smoothness" initial conditions our solution exists in a better sense.

THEOREM 2. If $v_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, then for the weak solution of (1), (2) we have

$$(25) \quad v' \in L^2(0, T; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)).$$

Proof. Let now $\{V_k\}_1^\infty$ denote the base in $H_0^1(\Omega) \cap H^2(\Omega)$. Differentiating (12.2) with respect to t we obtain:

$$\langle v''_m, V_l \rangle + va(v'_m, V_l) + 2b(v'_m, v_m, V_l) + 2b(v_m, v'_m, V_l) = U'_m \langle v_m, V_l \rangle + U_m \langle v'_m, V_l \rangle \\ v_{0m} \rightarrow v_0 \quad \text{in } H_0^1(\Omega) \cap H^2(\Omega).$$

Multiplying by $c_m^{l'}$ and summing over l from 1 to m :

$$(26) \quad \langle v''_m, v'_m \rangle + va(v'_m, v'_m) + 2b(v'_m, v_m, v'_m) + \\ + 2b(v_m, v'_m, v'_m) = U'_m \langle v_m, v'_m \rangle + U_m \langle v'_m, v'_m \rangle.$$

The following estimates are now used:

$$2b(v'_m, v_m, v'_m) + 2b(v_m, v'_m, v'_m) = -4b(v'_m, v'_m, v_m) + \\ + 2b(v'_m, v'_m, v_m) = -2b(v'_m, v'_m, v_m) \\ |-2b(v'_m, v'_m, v_m)| \leq 2 \|v'_m\|_{H_0^1}^{\frac{3}{2}} \|v'_m\|_{L^2}^{\frac{1}{2}} \|v_m\|_{L^2} \\ \leq \varepsilon_1 \|v'_m\|_{H_0^1}^2 \|v_m\|_{L^2}^2 + \frac{\varepsilon_2}{2\varepsilon_1} \|v'_m\|_{L^2}^2 + \frac{1}{2\varepsilon_1 \varepsilon_2} \|v'_m\|_{H_0^1}^2 \\ |U'_m \langle v_m, v'_m \rangle| \leq \frac{1}{2} |U'_m|^2 \|v_m\|_{L^2}^2 + \frac{1}{2} \|v'_m\|_{L^2}^2$$

and $U'_m \in L^\infty(0, T)$,

$$|U'_m \langle v'_m, v'_m \rangle| \leq |U'_m| \|v'_m\|_{L^2}^2.$$

Choosing constants ε_1 and ε_2 such that

$$\beta := v - \text{ess sup} \left(\frac{1}{2\varepsilon_1 \varepsilon_2} - \varepsilon_1 \|v_m\|_{L^2}^2 \right) > 0$$

and integrating (26) over $[0, t]$ we get

$$(27) \quad \frac{1}{2} \|v'_m(t)\|_{L^2}^2 + \beta \int_0^t \|v'_m\|_{H_0^1}^2 d\tau \leq \frac{1}{2} \|v'_m(0)\|_{L^2}^2 + \frac{1}{2} \int_0^t |U'_m| \|v_m\|_{L^2}^2 d\tau + \\ + \int_0^t \|v'_m\|_{L^2}^2 \left[\frac{\varepsilon_2}{2\varepsilon_1} + \frac{1}{2} + |U'_m| \right] d\tau \leq \frac{1}{2} \left[c_1 + c_2 \int \|v'_m\|_{L^2}^2 d\tau \right]$$

so by Gronwalls' inequality

$$(28) \quad \|v'_m(t)\|_{L^2}^2 \leq c_1 \exp(c_2 T)$$

from which v'_m is bounded in $L^\infty(0, T; L^2(\Omega))$, and by (27) v'_m is bounded in $L^2(0, T; H_0^1(\Omega))$. We must show only that $\|v'_m(0)\|_{L^2}^2$ in (27) is bounded. We multiply (12.2) by $c_m'(0)$ and sum over l from 1 to m , with $t = 0$

$$(29) \quad \|v'_m(0)\|_{L^2}^2 \leq v \left| \int_{\Omega} \frac{\partial^2 v_{0m}}{\partial x^2} v'_m(0) dx \right| + \\ + 2 \|v'_m(0)\|_{L^2} \|v_{0m}\|_{H_0^1}^{\frac{3}{2}} \|v_{0m}\|_{L^2}^{\frac{1}{2}} |U_m| \|v_{0m}\|_{L^2} \times \\ \times \|v'_m(0)\|_{L^2} \leq c_3 \|v'_m(0)\|_{L^2}$$

because $v_{0m} \rightarrow v_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$. We have also used the following estimate for the functions v_{0m} converging to v_0 in $H_0^1(\Omega) \cap H^2(\Omega)$:

$$\left\| \frac{\partial^2}{\partial x^2} v_{0m} \right\| \leq \text{const.}$$

and so

$$v |a(v_{0m}, v'_m(0))| = v \left| \int_{\Omega} \frac{\partial^2}{\partial x^2} v_{0m} \cdot v'_m(0) dx \right| \leq v \text{const.} \|v'_m(0)\|_{L^2}.$$

From (29) it follows that $\|v'_m(0)\|_{L^2}^2$ is bounded.

To finish the proof of Theorem 2 it remains to note that $v_m \rightarrow v$ weak in $L^2(0, T; H_0^1(\Omega))$, and so we may pass to the limit in (12.2).

Remark 5. From (3) and (25) using Sobolev's theorem it follows that

$$(30) \quad v \in C^0([0, T]; H_0^1(\Omega)),$$

so the initial condition for v is satisfied in the sense of (30).

§ 5. Now we show the uniqueness of the weak solution (U, v) of the problem (1), (2).

THEOREM 3. *There exists at most one solution of the problem (1), (2) in the sense of Definition 1.*

Proof. Let (U_1, v_1) and (U_2, v_2) be two different solutions of (1), (2). We put

$$U = U_1 - U_2, \quad (U(0) = 0),$$

$$v = v_1 - v_2, \quad (v(0, x) = 0 \text{ in } L^2(\Omega)).$$

Then it is satisfied that

$$(31.1) \quad U' = -vU - \|v_1\|_{L^2}^2 + \|v_2\|_{L^2}^2 \quad (\text{almost everywhere, when } U' \text{ exists}),$$

$$(31.2) \quad \langle v', V \rangle = U_1 \langle v, V \rangle + U \langle v_2, V \rangle - va(v, V) - 2b(v_1, v, V) - 2b(v, v_2, V).$$

Multiplying (31.1) by U and putting in (31.2) $V = v$ (v belongs to $H_0^1(\Omega)$ for almost all $t \in [0, T]$) we get:

$$\begin{aligned} \frac{1}{2}(U^2)' &= -vU^2 - U \int_{\Omega} v(v_1 + v_2) dx \\ \langle v', v \rangle &= -v \|v\|_{H_0^1}^2 + U_1 \|v\|_{L^2}^2 + U \langle v_2, v \rangle - \\ &\quad - 2b(v_1, v, v) - 2b(v, v_2, v) \end{aligned}$$

now by

$$2b(v, v_2, v) = -4b(v, v, v_2)$$

we have

$$(32.1) \quad \frac{1}{2}(U^2)' + vU^2 \leq \frac{c_1}{2} U^2 \|v_1 + v_2\|_{L^2}^2 + \frac{1}{2c_1} \|v\|_{L^2}^2$$

$$\begin{aligned} (32.2) \quad \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + v \|v\|_{H_0^1}^2 &\leq c_2 \|v\|_{L^2}^2 + \\ &\quad + \frac{c_3}{2} \|v_2\|_{L^2}^2 U^2 + \frac{1}{2c_3} \|v\|_{L^2}^2 + 2|b(v_1, v, v)| + 4|b(v_2, v, v)| \end{aligned}$$

now

$$\begin{aligned} |b(v_i, v, v)| &\leq \|v\|_{L^2}^{\frac{1}{2}} \|v\|_{H_0^1}^{\frac{3}{2}} \|v_i\|_{L^2} \\ &\leq \frac{\varepsilon_i}{2} \|v\|_{H_0^1}^2 \|v_i\|_{L^2}^2 + \frac{\varepsilon_{ii}}{4\varepsilon_i} \|v\|_{L^2}^2 + \frac{1}{4\varepsilon_{ii}\varepsilon_i} \|v_i\|_{H_0^1}^2, \quad i = 1, 2. \end{aligned}$$

Choosing the constants in such a way that:

$$\alpha_1 = v - \text{ess sup} \left(\frac{c_1}{2} \|v_1 + v_2\|_{L^2}^2 - \frac{c_3}{2} \|v_2\|_{L^2}^2 \right) > 0$$

$$\alpha_2 = v - \text{ess sup} \left(\varepsilon_1 \|v_1\|_{L^2}^2 - 2\varepsilon_3 \|v_2\|_{L^2}^2 - \frac{1}{2\varepsilon_{11}\varepsilon_1} - \frac{1}{\varepsilon_{22}\varepsilon_2} \right) > 0$$

summing (32) together, integrating over $[0, t]$ and using the fact that $U(0) = 0, v(0, x) = 0$ in $L^2(\Omega)$ we obtain

$$\begin{aligned} \frac{1}{2} U^2(t) + \frac{1}{2} \|v(t)\|_{L^2}^2 &\leq \frac{1}{2} U^2(t) + \frac{1}{2} \|v(t)\|_{L^2}^2 + \\ &\quad + \alpha_1 \int_0^t U^2(\tau) d\tau + \alpha_2 \int_0^t \|v(\tau)\|_{H_0^1}^2 d\tau \leq \text{const.} \int_0^t \|v(\tau)\|_{L^2}^2 d\tau \end{aligned}$$

and so by Gronwall's inequality

$$U^2(t) + \|v(t, x)\|_{L^2}^2 = 0 \quad \text{in } [0, T],$$

which finishes the proof of Theorem 3.

§ 6. Finally we shall study the stability of the solution $\left(\frac{P}{v}, 0\right)$ of (1), (2).

All the definitions for this part will be found in [2].

As the solution we mean the functions U, v in the weak sense (see Remark 3). $U(t)$ satisfies (1.1) in the classical sense and the function $z(t) := \|v(t, x)\|_{L^2}^2$ is continuous. The regularity allowed us to study the following system obtained from (4) and (1.1):

$$(33) \quad \begin{cases} \frac{dU}{dt} = P - vU - z \\ \frac{1}{2}z(T_2) - \frac{1}{2}z(T_1) = \int_{T_1}^{T_2} U(\tau)z(\tau)d\tau - v \int_{T_1}^{T_2} \|v(\tau)\|_{H_0^1}^2 d\tau \end{cases}$$

with the conditions

$$U(0) = U_0, \quad z(0) = \|v_0\|_{L^2}^2 = z_0,$$

where $0 \leq T_1 \leq T_2$ -arbitrary.

Our purpose is to show the global exponential stability of the solution $\left(\frac{P}{v}, 0\right)$ of (33)

when $\frac{P}{v} < v$. Using the transformation $W(t) = U(t) - \frac{P}{v}$ we arrive at the problem of the stability of the $(0, 0)$ solution for

$$(34.1) \quad \frac{dW}{dt} = -vW - z$$

$$(34.2) \quad \frac{1}{2}(z(T_2) - z(T_1)) = \int_{T_1}^{T_2} \left(W(\tau) + \frac{P}{v}\right) z(\tau) d\tau - v \int_{T_1}^{T_2} \|v(\tau)\|_{H_0^1}^2 d\tau$$

$$W(0) = U_0 - \frac{P}{v}, \quad z(0) = z_0.$$

We start with the following;

LEMMA 5. When $W(t) > 0$, then W is strictly decreasing.

Proof. The proof is the consequence of (34.1) and the fact that $z(t) \geq 0$.

By Lemma 5 it follows that if the function W exists in all real axes, then $\limsup_{t \rightarrow \infty} W(t) \leq 0$. Also when for some $t_0 \geq 0$ is $W(t_0) = 0$, then $W(t) \leq 0$ for $t \geq t_0$.

Now we show that the solutions W and z are bounded on $[0, \infty)$ when $\frac{P}{v} \leq v$, which implies that

$$U \in C^1([0, \infty)) \quad \text{and is bounded,}$$

$$v \in C^0([0, \infty); L^2(\Omega)) \quad \text{and is bounded.}$$

THEOREM 4. When $\frac{P}{v} \leq v$ then the solution (W, z) of (34) is bounded independently on t .

Proof. Multiplying (34.1) by W , integrating over $[T_1, T_2]$ and summing with (34.2) by sides we get:

$$(35) \quad \frac{1}{2}z(T_2) + \frac{1}{2}W^2(T_2) + v \int_{T_1}^{T_2} [\|v(t)\|_{H_0^1}^2 + W^2(t)] dt = \\ = \frac{1}{2}z(T_1) + \frac{1}{2}W^2(T_1) + \frac{P}{v} \int_{T_1}^{T_2} z(t) dt.$$

By Lemma 1 and $\frac{P}{v} \leq v$ it follows that

$$(36) \quad \frac{1}{2}z(T_2) + \frac{1}{2}W^2(T_2) \leq \frac{1}{2}z(T_1) + \frac{1}{2}W^2(T_1)$$

for every $0 \leq T_1 \leq T_2$, and so

$$z(t) \leq z(0) + W^2(0) =: M, \quad W^2(t) \leq M \quad \text{for any } t \geq 0.$$

And so the proof is completed.

Additionally from (36) it follows that the function

$$t \rightarrow \frac{1}{2}z(t) + \frac{1}{2}W^2(t) =: H(t)$$

is decreasing on t . This function plays the role of the Lapunov's function for (34).

LEMMA 6. For $v - \frac{P}{v} =: \gamma > 0$ the solution $(0, 0)$ of (34) is global exponential stable (in the sense of Remark 6 below).

Proof. From (34) and Lemma 1 it follows that

$$(37) \quad H(T_2) \leq H(T_1) - \gamma \int_{T_1}^{T_2} z(\tau) d\tau - v \int_{T_1}^{T_2} W^2(\tau) d\tau.$$

The following fact occurs:

Remark 6. If for real, continuous, defined for $t \geq 0$ and non-negative function $h(t)$ for a constant $c_1 > 0$ is:

$$h(T_2) \leq h(T_1) - c_1 \int_{T_1}^{T_2} h(\tau) d\tau \quad \text{for all } 0 \leq T_1 \leq T_2$$

then $h(t) \rightarrow 0$ for $t \rightarrow \infty$.

Under our conditions we get

a) The integral on the right is convergent,

b) Every solution of the inequality is majorized by the function

$$A(t) := h(0) \exp(-c_1 t).$$

We show b). The function $A(t)$ satisfies

$$A(t) = h(0) - c_1 \int_0^t A(\tau) d\tau$$

and the difference $B(t) = h(t) - A(t)$, continuous, satisfies

$$B(t) \leq -c_1 \int_0^t B(\tau) d\tau.$$

The function B could not be positive in the interval $(0, a)$, because then the integral is negative. If then (T_1, T_2) is any interval with $B > 0$, $h(T_1) = A(T_1)$, the reasoning for (T_1, T_2) is the same as before for $(0, a)$.

From (37) and Remark 6 it follows that Lemma 6 is satisfied.

Now we show the global asymptotic stability of $(0, 0)$ when $\frac{P}{v} = v$.

THEOREM 5. When $\frac{P}{v} = v$ then the solution $(0, 0)$ of (34) is global asymptotic stable.

Proof. In the proof we consider two cases:

a) First a case consisting of two subcases:

- a.1) for any $t \geq 0$ is $W(t) \leq 0$,
- a.2) $W(0) > 0$ and $W(t_0) = 0$ for some $t_0 > 0$.

In a.2) by Lemma 5 for $t \geq t_0$ we have $W(t) \leq 0$. Denoting $t_1 := \max(0, t_0)$ we may study a.1) and a.2) together.

Using (34.2) with $\frac{P}{v} = v$

$$(38) \quad \frac{1}{2}z(T_2) = \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} (W+v)z dt - v \int_{T_1}^{T_2} \|v\|_{H_0^1}^2 dt \leq \frac{1}{2}z(T_1) + \int_{T_1}^{T_2} Wz dt$$

we conclude that for $t \geq t_1$ the function $z(t)$ is decreasing on t , and so it has a limit $\alpha \geq 0$.

If $\alpha > 0$ then from (34.1) it follows that

$$W'(t) \leq -vW(t) - \frac{\alpha}{2}$$

for sufficiently large $t > 0$, and so

$$W(t) \leq -\frac{\alpha}{4}$$

for sufficiently large t . But then (38) implies that $z(T_2) \rightarrow -\infty$ for $T_2 \rightarrow \infty$, which is impossible, because $z \geq 0$. So $\alpha = 0$, which finishes the proof of a).

b) Second case:

$$W(t) > 0 \quad \text{for } 0 \leq t < \infty.$$

By Lemma 5 the function W is strictly decreasing on t , so it has a limit $\alpha \geq 0$. If $\alpha > 0$, then by (34.1)

$$\frac{dW}{dt} < -v\alpha$$

so $W(t) \rightarrow -\infty$ for $t \rightarrow \infty$, which contradicts b). Hence $\alpha = 0$. We then show that it must also be $z(t) \rightarrow 0$ for $t \rightarrow \infty$. We show that for any $\varepsilon_0 > 0$ the following condition could not occur: there exists an ε_0 for any $T_0 > 0$ such that there exists $T_1 > T_0$ that

$$(39) \quad z(T_1) > 2\varepsilon_0.$$

For some ε_0 let the condition (39) be satisfied. In the case b) $W(t) \searrow 0$ for $t \nearrow \infty$, and so there exists a $\tau > 0$ such that

$$(40) \quad W(\tau) = \delta, \quad W(t) < \delta \quad \text{for } t > \tau,$$

and

$$\delta > 0, \quad \delta^2 < \frac{\varepsilon_0^2}{8M}.$$

We may also require that for such a τ the following condition is also satisfied:

$$(41) \quad z(\tau) < \varepsilon_0$$

because if the conditions (40) and (41) could not be satisfied together for any τ , then $z(t) > \varepsilon_0$ for sufficiently large $t > 0$, and by (34.1)

$$\frac{dW}{dt} < -vW - \varepsilon_0$$

and $W(t) < -\frac{\varepsilon_0}{2}$ for sufficiently large t , which contradicts b). And so there exists a $\tau > 0$

such that the conditions (40) and (41) are satisfied together.

Now by (39) there exists $T_\tau > \tau$ such that

$$z(T_\tau) > 2\varepsilon_0.$$

Our idea is to show that for a sufficiently long interval of $t \leq T_\tau$ we have $z(t) > \varepsilon_0$, and then by (34.1) the function $W(t)$ must decrease in this interval to a negative value, which contradicts b).

The convergence of $W(t)$ to 0 and (38) imply that

$$z(t) > \varepsilon_0$$

in some interval D with the right end in T_τ . We want to estimate the interval D . From (38) it follows that:

$$(42) \quad \frac{1}{2}z(T_\tau) \leq \int_t^{T_\tau} W(s)z(s)ds + \frac{1}{2}z(t) \leq (T_\tau - t)M\delta - \frac{1}{2}z(t),$$

where M is defined in (36). The estimate $W(t) \leq \delta$ follows from the fact that $z(\tau) < \varepsilon_0$ ((41)) and we are looking for $t \leq T_\tau$, in which $z(t) > \varepsilon_0$. And so from (40) it follows that $W(t) < \delta$ in D . Now if

$$(43) \quad \frac{1}{2}z(T_\tau) - \frac{1}{2}z(t) \leq (T_\tau - t)M\delta \leq \frac{\varepsilon_0}{2}$$

is satisfied, then the fact that $z(T_\tau) > 2\varepsilon_0$ implies that $z(t) > \varepsilon_0$. And so by (43)

$$(44) \quad D \supset \left[T_\tau - \frac{\varepsilon_0}{4M\delta}, T_\tau \right] =: D_1.$$

By (34.1) we obtain that

$$(45) \quad W(T_\tau) \leq W\left(T_\tau - \frac{\varepsilon_0}{4M\delta}\right) + \max_{D_1} W'(t) \cdot \frac{\varepsilon_0}{4M\delta}$$

and the following estimates hold good:

$$0 > \max_{D_1} W'(t) \leq \max_{D_1} (-z(t)) = -\min_{D_1} z(t) \leq -\varepsilon_0.$$

So from (45) it follows that

$$(46) \quad W(T_\tau) \leq W\left(T_\tau - \frac{\varepsilon_0}{4M\delta}\right) - \varepsilon_0 \cdot \frac{\varepsilon_0}{4M\delta} \leq \delta - \frac{\varepsilon_0^2}{4M\delta}$$

but from (40) $\delta < \frac{\varepsilon_0^2}{8M\delta}$, and so the left side of (46) is less than $-\frac{\varepsilon_0^2}{8M\delta}$, which contradicts b). So the proof of Theorem 5 is completed.

Acknowledgement

The author would like to express his gratitude to Professor Andrzej Lasota for suggesting the problem and for many very valuable discussions during the preparation of this work.

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