G-Convergence and Homogenization of Nonlinear Partial Differential Operators

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G-Convergence and Homogenization of Nonlinear Partial Differential Operators

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Table of Contents

Preface

Various applications of the homogenization theory of partial differential equations resulted in the further development of this branch of mathematics, attracting an increasing interest of both mathematicians and experts in other fields. In general, the theory deals with the following:

Let A_k be a sequence of differential operators, linear or nonlinear. We want to examine the asymptotic behaviour of solutions u_k to the equation $Au_k = f$, as $k \to \infty$, provided coefficients of A_k contain rapid oscillations. This is the case, e.g. when the coefficients are of the form $a(\varepsilon_k^{-1}x)$, where the function $a(y)$ is periodic and $\varepsilon_k \to 0$ as $k \to \infty$. Of course, many other kinds of oscillation, like almost periodic or random homogeneous, are of interest as well. It seems a good idea to find a differential operator A such that $u_k \to u$, where u is a solution of the limit equation $Au = f$. Such a limit operator is usually called the *homogenized* operator for the sequence A_k. Sometimes, the term *"averaged"* is used instead of "homogenized".

Let us look more closely what kind of convergence one can expect for u_k. Usually, we have some *a priori* bound for the solutions. However, due to the rapid oscillations of the coefficients, such a bound may be uniform with respect to k in the corresponding energy norm only. Therefore, we may have convergence of solutions only in the weak topology of the energy space. This leads to the notion of *G-convergence* of *abstract* operators in such a way that the homogenized operator A is exactly the G-limit of A_k.

However, while the notion of G-convergence seems to be natural, it is not sufficient for our purpose. Indeed, if A_k is a differential operator, then so it must be for the homogenized operator A. At the same time, the G-limit is defined, in general, only as an abstract operator. To overcome this difficulty it is natural to use an other kind of convergence which is specific for differential operators, namely, *strong G-convergence* (in French literature, the name *H-convergence* is normally used).

Thus, the present book is devoted to strong G-convergence of nonlinear divergence form elliptic and parabolic operators, and applications to homogenization problems proper for periodic, almost periodic, and random homogeneous operators of such kind.

Nevertheless, we start, in Chapter 1, with a discussion of G-convergence for abstract operators, as this theory provides useful tools for the rest of the monograph. Moreover, examination of the situation on this abstract level clarifies some basic ideas. Many results presented here are more or less well-known to experts, but they are scattered in various papers, frequently in an implicit form. It should be pointed out that, beside more or less standard situation, we consider here the case of abstract parabolic operators, which is less familiar.

The core of the book is Chapter 2, in which we study in detail strong G-convergence of nonlinear elliptic operators. We consider both the case of *monotone multivalued,* and *pseudomonotone single-valued* operators. These are treated separately in order to present different approaches. Beside general properties, like strong G-compactness, localization property, and convergence of arbitrary solutions, being essentially common for both cases under consideration, we discuss, in Section 2.4, some additional results which seem to be specific for the single-valued case only. Interesting in themselves these last results provide useful tools for the study of the almost periodic homogenization problem.

Next, in Chapter 3, we discuss nonlinear elliptic homogenization problems. First, we study the case of random homogeneous operators, both single-valued and multivalued. The results we obtain have a *statistical* character, i.e. homogenization takes place for almost all realizations (almost surely). Nevertheless, it is sufficient to get individual homogenization theorems for periodic operators, as an immediate consequence. Then, using the statistical homogenization theorem, general results on strong G-convergence, and the Bohr compactification, we derive an *individual* homogenization theorem for single-valued *almost periodic* operators. Notice that it is unclear how to extend the last result to the case of multivalued operators. Moreover, it is not even known what "almost periodic multivalued operator"means.

Chapter 4 deals with strong G-convergence and homogenization of nonlinear parabolic operators. Here we restrict our study only to the case of single-valued operators. Conceptually, our presentation here is similar to that of Chapter 2 for single-valued elliptic operators. Therefore, we sketch the proofs indicating only main differences. As for homogenization, we consider it only in the periodic setting, but for the whole range of the ratio of time and space scales. In addition, we discuss a class of filtration equations.

We supplement the main body of the monograph by two Appendices. In the first one a version of the homogenization theorem for difference schemes is outlined, while in the second we list some open problems.

Even restricting our work to the subject just described, no attempt has been made to give an exhaustive account of the field or a complete survey of the literature. For additional information we refer to the monographs [40, 47, 113, 164]. We recommend especially the book [164], in which many interesting problems are discussed including homogenization of nonlinear variational problems, and [113] containing a clear and detailed exposition of Γ-convergence. For further results in this direction see, also, papers of A. Braides, G. Dal Maso, R. De Arcangelis, and others, listed in the bibliography. On the other hand, it must be pointed out that the present volume has hardly any overlap with the books cited above.

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Notations

$\mathbf{Z}$	the integers
$\mathbf{N}$	the positive integers
$\mathbf{R}$	the real numbers
$\mathbf{C}$	the complex numbers
p'	the dual exponent, $\frac{1}{p} + \frac{1}{p'} = 1$, $p \in [1, \infty]$
$\mathrm{L}^p(Q)$	the usual Lebesgue space on Q with the exponent $p \in [1, \infty]$
$\mathrm{L}^p_{loc}(Q)$	the space of functions which are locally in $\mathrm{L}^p(Q)$
$C_0^\infty(Q)$	the space of infinitely differentiable compactly supported functions
$W^{1,p}(Q)$	the usual Sobolev space of functions in $\mathrm{L}^p(Q)$ whose first derivatives are in $\mathrm{L}^p(Q)$
$W_0^{1,p}(Q)$	the closure of $C_0^\infty(Q)$ in $W^{1,p}(Q)$
$W_{loc}^{1,p}(Q)$	the space of functions which are locally in $W^{1,p}(Q)$
$W^{-1,p'}(Q)$	the dual space to $W_0^{1,p}(Q)$
$H^1(Q)$	$W^{1,2}(Q)$
$H_0^1(Q)$	$W_0^{1,2}(Q)$
$H^{-1}(Q)$	$W^{-1,2}(Q)$
$\operatorname{supp} u$	the support of u
$\mathrm{cl}(X)$, $\overline{X}$	the closure of X
$\mathrm{gr}(A)$	the graph of a (multivalued) map A
∇	the gradient operator
∂_t	the time derivative
$\xrightarrow{G}$	G-convergence
$\stackrel{G}{\Longrightarrow}$	strong G-convergence
I	identity map
$\lVert \cdot \rVert_p$, $\lVert \cdot \rVert_{p,Q}$	the norm in $\mathrm{L}^p(Q)$
$\lvert K \rvert$	the Lebesgue measure of K.

CHAPTER 1

G-convergence of Abstract Operators

1.1 Preliminaries

1.1.1 Multivalued Monotone Operators

Here we fix notations and recall some results concerning multivalued operators.

For any members x and y of a set X we denote by (x, y) the ordered pair formed by x and y.

Let X and Y be two sets. A *multivalued map* (or, *operator*) F from X into Y is a map that associates to any $x \in X$ a subset Fx of Y. The subset Fx is called the *image* of x under the map F (or the *value* at the point x). The sets

$$D(F) = \{x \in X \ : \ Fx \neq \emptyset\}$$

and

$$R(F) = \bigcup_{x \in X} Fx$$

are called the *domain* and the *range* of F, respectively. The set

$$\mathrm{gr}(F) = \{(x, y) \in X \times Y : y \in Fx\}$$

is called the *graph* of F.

We say that F is *single-valued* if for every $x \in D(F)$ the set Fx consists of exactly one element of Y.

In the sequel, we shall identify any multivalued map with its graph in $X \times Y$. Associated to F there is an *inverse* multivalued map F^{-1} defined by

$$\mathrm{gr}(F^{-1}) = \{(y, x) \in Y \times X : (x, y) \in \mathrm{gr}(F)\}.$$

In other words, F^{-1} is the multivalued map from Y into X such that $x \in F^{-1}y$ if and only if $y \in Fx$.

Now let V be a reflexive Banach space over the field $\mathbf{R}$ of real numbers [1], V^* its topological dual space. By $(\cdot,\cdot)_V$ we denote the natural duality pairing between V^* and V. We shall shorten this notation to $(\cdot,\cdot)$ if no confusion may occur.

A subset $A \subset V \times V^*$ is called *monotone* (resp. *strictly monotone*) if

$$(y_1 - y_2, x_1 - x_2)_V \geq 0 \quad (\text{resp.} > 0)$$

for any $(x_1, y_1) \in A$, $(x_2, y_2) \in A$, $x_1 \neq x_2$.

A monotone subset $A \subset V \times V^*$ is said to be a *maximal monotone* set if it is not properly contained in any other monotone subset of $V \times V^*$, i.e. for every $(x, y) \in V \times V^*$ such that

$$(y - \eta, x - \xi)_V \geq 0 \quad \forall (\xi, \eta) \in A$$

it follows that $(x, y) \in A$.

We say that a multivalued operator $F : V \longrightarrow V^*$ is a *monotone* (resp. *strictly monotone, maximal monotone*) operator if its graph $\mathrm{gr}(F)$ is a monotone (resp. strictly monotone, maximal monotone) subset of $V \times V^*$.

Monotonicity is invariant with respect to transposition of the domain and the range of a map. Hence, F is (maximal) monotone if and only if F^{-1} has the same property. Notice that if F is a strictly monotone operator, then the operator F^{-1} is single-valued.

We note also that if F is a maximal monotone multivalued operator from V into V^*, then for any $x \in D(F)$ the value Fx is a closed convex subset of V^*.

Now we recall the concept of upper semicontinuity for multivalued maps. Let X and Y be two topological spaces, F a multivalued map from X into Y. The map F is said to be *upper semicontinuous* if for every $x_0 \in X$ and for every open neighborhood W of Fx_0 there exists a neighborhood U of $x_0 \in X$ such that $Fx \subset W$ for every $x \in U$.

The proof of the following criterion of maximal monotonicity may be found in [78].

Theorem 1.1.1 *Let V be a reflexive Banach space, V^* its dual, and F a monotone multivalued map from V into V^*. Suppose that for each $x \in V$, Fx is a non-empty closed convex subset of V^* and for each line interval in V, F is an upper semicontinuous map from the line interval into V^* endowed with its weak topology. Then F is maximal monotone.*

We recall that a line interval in V is a set of the form

$$\{x \in V : x = \lambda x_0 + (1 - \lambda)x_1,\ \lambda \in [0, 1]\}$$

[1] We shall always consider Banach spaces over $\mathbf{R}$ only.

for some $x_0,\ x_1 \in V$.

The following result is more or less well-known, at least in the Hilbert space case.

Theorem 1.1.2 *Let F be a monotone multivalued operator from a reflexive Banach space V into V^* and let J be a single-valued strictly monotone operator from V into V^*, with $D(J) = V$. Suppose that $R(F + J) = V^*$. Then F is a maximal monotone operator.*

Proof. Assume that

$$(u - v, x - y)_V \geq 0 \tag{1.1.1}$$

for any $(y, v) \in \mathrm{gr}(F)$. We have to show that $u \in Fx$. By assumption, there exists a solution $y_0 \in V$ of the inclusion

$$u + J(x) \in J(y_0) + Fy_0.$$

Then

$$v_0 = u_0 + J(x) - J(y_0) \in Fy_0.$$

Substituting $y = y_0$ and $v = v_0$ into (1.1.1) we get

$$(J(x) - J(y_0), x - x_0)_V = (v_0 - u, x - y_0)_V \leq 0.$$

Since J is a strictly monotone operator, we have $x = y_0$ and

$$u = v_0 \in Fy_0 = Fx.$$

The proof is complete. □

Later on it will be usefull the following result due to R. Rockafellar [238].

Theorem 1.1.3 *Let V be a reflexive Banach space and let F, F_0 be multivalued maximal monotone operators from V into V^*. Assume that*

$$\mathrm{int}(D(F)) \cap D(F_0) \neq \emptyset.$$

Then $F + F_0$ is a maximal monotone operator.

Now we recall that a multivalued operator $F : V \longrightarrow V^*$ is called *coercive* if there exists a real valued function c on $[0, +\infty)$ such that

$$\lim_{r \to \infty} c(r) = +\infty$$

and

$$(y, x)_V \geq c\,(\|x\|)\,\|x\| \tag{1.1.2}$$

for every $x \in V$ and $y \in Fx$.

Theorem 1.1.4 *Let V be a reflexive Banach space and $F : V \longrightarrow V^*$ a maximal monotone operator. If F is coercive, then $R(F) = V^*$.*

For the proof we refer to [230].

Example 1.1.1 Suppose $f : \mathbf{R} \longrightarrow \mathbf{R} = \mathbf{R}^*$ is a function defined by the formula

$$f(x) = \begin{cases} x+1 & \text{if } x \geq 0, \\ x-1 & \text{if } x < 0. \end{cases}$$

Then f is a single-valued monotone map. Another monotone map may be defined by

$$F_0 x = \begin{cases} \{f(x)\} & \text{if } x \neq 0, \\ \emptyset & \text{if } x = 0. \end{cases}$$

We can also define a map F as follows:

$$Fx = \begin{cases} F_0(x) & \text{if } x \neq 0, \\ [-1, 1] & \text{if } x = 0. \end{cases}$$

The map F is maximal monotone. Moreover, it is a proper extension of f and F_0.

Example 1.1.2 Let S be a space with a σ-finite complete measure, $V = \mathrm{L}^p(S)$, $p > 1$. Define a map

$$J : V \longrightarrow V^* = \mathrm{L}^{p'}(S), \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

by the formula

$$Ju = |u|^{p-2}u. \tag{1.1.3}$$

Then J is a maximal monotone (single-valued) operator. Moreover, J is strictly monotone.

Now we recall that a multivalued operator $F : V \longrightarrow V^*$ is said to be *cyclically monotone* if

$$(y_0, x_0 - x_1)_V + (y_1, x_1 - x_2)_V + \ldots + (y_n, x_n - x_0)_V \geq 0$$

for every finite set of points $(x_i, y_i) \in \mathrm{gr}(F)$ (i.e. $y_i \in Fx_i$), $i = 0, 1, \ldots, n$. The operator F is called maximal cyclically monotone if it is cyclically monotone and has no cyclically monotone proper extension in $V \times V^*$.

To state the main characterization of cyclically monotone operators we recall some concepts from convex analysis [32, 147]. Let

$$\varphi : V \longrightarrow \overline{\mathbf{R}} = (-\infty, +\infty]$$

be a convex function, i.e.

$$\operatorname{epi}\varphi = \{(x,\xi) \in V \times \mathbf{R} : \xi \geq \varphi(x)\}$$

is a convex subset of $V \times \mathbf{R}$. The function φ is called *lower semicontinuous* if the set $\operatorname{epi}\varphi$ is closed. The function φ is said to be *proper* if $\varphi \not\equiv +\infty$, i.e.

$$D(\varphi) = \{x \in V : \varphi(x) < \infty\} \neq \emptyset.$$

Let φ be a proper lower semicontinuous function on a Banach space V. The multivalued operator $\partial\varphi : V \longrightarrow V^*$ defined by

$$\partial\varphi(x) = \{y \in V^* : (y, u - x)_V + \varphi(x) \leq \varphi(u) \quad \forall u \in V\}$$

is called the *subdifferential* of φ.

Theorem 1.1.5 *Let F be a multivalued operator from a reflexive Banach space V into V^*. Then the following two statements are equivalent:*

(i) *there exists a proper lower semicontinuous convex function φ on V such that $F = \partial\varphi$;*

(ii) *F is a maximal cyclically monotone operator.*

Moreover, F determines φ uniquely up to an additive constant and given $x_0 \in D(F)$ we have

$$\varphi(x) = \sup\{\varphi(x_0) + \sum_{i=1}^{m}(y_{i-1}, x_i - x_{i-1})_V\},$$

the supremum being taken over all finite families $(x_i, y_i) \in \operatorname{gr}(F)$ (i.e. $y_i \in Fx_i$), $i = 0, 1, \ldots, m-1$, and $x_m = x$.

For the proof we refer to [43, 237].

1.1.2 Single-Valued Operators of Monotone Type

Let V be a reflexive Banach space, $A : V \longrightarrow V^*$ a single-valued operator with $D(A) = V$. We say that A is *bounded* if A takes bounded subsets of V into bounded subsets of V^*. We recall that A is said to be *hemicontinuous* if its restriction to any line interval is a continuous map from that interval into V^* equipped with the weak topology.

Now we discuss briefly the notion of pseudomonotone operator. An operator $A : V \longrightarrow V^*$ is called *pseudomonotone* if the following two conditions are fulfilled:

(i) *A is a bounded operator*;

(ii) *if* $u_k \longrightarrow u$ *weakly in* V *and*

$$\limsup (Au_k, u_k - u)_V \leq 0,$$

then for every $v \in V$

$$\liminf (Au_k, u_k - v)_V \geq (Au, u - v)_V.$$

Any pseudomonotone operator maps continuously V, with the norm topology, into V^*, with the weak topology.

Proposition 1.1.1 *Let A be a bounded hemicontinuous monotone operator. Then A is a pseudomonotone operator.*

Remark 1.1.1 Theorem 1.1.1 implies that under the assumption of Proposition 1.1.1 A is a maximal monotone operator.

Theorem 1.1.6 *Let A be a pseudomonotone operator. Assume A to be coercive, i.e.*

$$\lim_{\|u\| \to \infty} \frac{(Au, u)_V}{\|u\|} = +\infty.$$

Then A is surjective.

An important class of pseudomonotone operators consists of so-called operators of the calculus of variation. An operator $A : V \longrightarrow V^*$ is said to be an *operator of the calculus of variation* if it is of the form

$$Au = A(u, u),$$

where the operator $(u, v) \longmapsto A(u, v)$ from $V \times V$ into V^* satisfies the following conditions:

1. *for any $u \in V$ the map $v \longmapsto A(u, v)$ is a bounded hemicontinuous operator from V into V^* and*

$$(A(u, u) - A(u, v), u - v)_V \geq 0 \quad \forall v \in V;$$

2. *for any $v \in V$ the map $u \longmapsto A(u, v)$ is a bounded hemicontinuous operator from V into V^*;*

3. *if* $u_k \longrightarrow u$ *weakly in* V *and*

$$\lim \left(A(u_k, u_k) - A(u_k, u), u_k - u\right)_V = 0,$$

then $\lim A(u_k, v) = A(u, v)$ *weakly in* V^* *for any* $v \in V$;

4. *if* $u_k \to u$ *weakly in* V *and* $A(u_k, v) \to \psi$ *weakly in* V^*, *then*

$$\lim\,(A(u_k, v), u_k)_V^{\cdot} = (\psi, u)_V.$$

Any such operator is hemicontinuous.

Proposition 1.1.2 *Let A be an operator of the calculus of varition. Then A is a pseudomonotone operator.*

Now we recall some basic facts on duality operators. Assume that the norms $\|\cdot\|$ and $\|\cdot\|_*$ in V and V^*, respectively, are strictly convex, i.e. the corresponding unit balls are strictly convex. It is known [20] that any reflexive Banach space may be endowed with an equivalent norm such that this new norm and its dual are strictly convex.

Let

$$c : [0, +\infty) \longrightarrow [0, +\infty)$$

be a strictly increasing continuous function such that $c(0) = 0$ and

$$\lim_{r \to +\infty} c(r) = +\infty.$$

A map $J : V \longrightarrow V^*$ is said to be a *duality operator* if

$$(J(u), u)_V = \|J(u)\|_* \|u\|$$

and

$$\|J(u)\|_* = c(\|u\|).$$

Proposition 1.1.3 *For any $c(r)$ with above mentioned properties there exists a unique duality operator $J : V \longrightarrow V^*$. Any such operator is continuous with respect to the strong topology in V and the weak topology in V^*.*

The operator $J : \mathrm{L}^p(S) \longrightarrow \mathrm{L}^{p'}(S)$ defined by (1.1.3) furnishes us an example of duality operator.

For the proofs of these results we refer to [200].

1.1.3 Convergence in the Sense of Kuratowski

Here we recall a general concept of set convergence named Kuratowski convergence [194]. Let (X, τ) be an arbitrary topological space and let (E_k) be a sequence of subsets of X. We define the *K-lower limit* of (E_k), in symbols $K\text{-}\liminf E_k$, as the set of all points $x \in X$ with the following property :

- *for every neighborhood U of x there exists $k_0 \in \mathbf{N}$ such that $U \cap E_k \neq \emptyset$ if $k \geq k_0$.*

The *K-upper limit* of (E_k) denoted by $K\text{-}\limsup E_k$ is the set of all points $x \in X$ with the following property:

- *for every neighborhood U of x and for every $k_0 \in \mathbf{N}$ there exists $k \geq k_0$ such that $U \cap E_k \neq \emptyset$.*

If there exists a subset E of X such that

$$E = K\text{-}\liminf E_k = K\text{-}\limsup E_k,$$

we write

$$E = K\text{-}\lim E_k$$

and say that (E_k) *converges* to the set E *in the sense of Kuratowski*, or *K-converges* to E.

We need also a sequential version of such convergence. We define the *sequential K-lower* and *K-upper limits* of (E_k) by

$$K_s\text{-}\liminf E_k = \{x \in X : \exists x_k \to x,\ \exists k_0 \in \mathbf{N},\ \forall k \geq k_0 : x_k \in E_k\},$$

and

$$K_s\text{-}\limsup E_k = \{x \in X : \exists \sigma(k) \to +\infty,\ \exists x_k \to x,\ \forall k \in N : x_k \in E_{\sigma(k)}\}.$$

Now we say that the sequence (E_k) *K_s-converges* to E if

$$K_s\text{-}\liminf E_k = K_s\text{-}\limsup E_k = E.$$

Sometimes we shall write $K(\tau)$-lim, $K_s(\tau)$-lim, etc., to indicate the topology τ explicitly.

We note that K-convergence coincides with K_s-convergence if the space X satisfies the first axiom of countability.

Example 1.1.3 If $E \subset X$ and $E_k = E$, then $K\text{-}\lim E_k = \overline{E}$, the closure of E in X.

Example 1.1.4 Let (x_k) be a sequence in X. If

$$E_k = \{x_k\}, \quad k \in \mathbf{N},$$

then $K\text{-}\limsup E_k$ is the set of all cluster points of (x_k), while $K\text{-}\liminf E_k$ is the (possibly empty) set of all limits of (x_k) (we do not assume X to be a Hausdorff space). If $E_k = \{x_h : h \geq k\}$, then $K\text{-}\lim E_k$ is the set of all cluster points of (x_k).

Example 1.1.5 Let $X = \mathbf{R}^2$, $E_k = \{(1/k, y) : 0 < y < 1\}$. Then

$$K\text{-}\lim E_k = \{(0, y) : 0 \leq y \leq 1\}.$$

It is easy to see that

$$K\text{-}\liminf E_k \subset K\text{-}\limsup E_k.$$

A similar assertion holds with K replaced by K_s. Hence, $E = K\text{-}\lim E_k$ if and only if

$$K\text{-}\limsup E_k \subset E \subset K\text{-}\liminf E_k,$$

and there is a similar statement for K_s-convergence.

Proposition 1.1.4 *For any subsequence $(\sigma(k))$ we have*

$$K\text{-}\liminf E_k \subset K\text{-}\liminf E_{\sigma(k)},$$

$$K\text{-}\limsup E_k \supset K\text{-}\limsup E_{\sigma(k)}.$$

If $K\text{-}\lim E_k = E$, then $K\text{-}\lim E_{\sigma(k)} = E$.

Proposition 1.1.5 *If $E_k \subset F_k$ for every $k \in \mathbf{N}$, then*

$$K\text{-}\liminf E_k \subset K\text{-}\liminf F_k,$$

$$K\text{-}\limsup E_k \subset K\text{-}\limsup F_k.$$

Remark 1.1.2 The statements of Propositions 1.1.4 and 1.1.5 are still valid if we replace K-convergence by K_s-convergence.

Proposition 1.1.6 *The sets $K\text{-}\liminf E_k$ and $K\text{-}\limsup E_k$ are closed. We have also*

$$K\text{-}\liminf E_k = K\text{-}\liminf \overline{E}_k,$$

$$K\text{-}\limsup E_k = K\text{-}\limsup \overline{E}_k,$$

where the bar denotes the closure.

Proposition 1.1.7 *A sequence (E_k) is K_s-convergent to E if and only if any subsequence of (E_k) has a further subsequence K_s-converging to E.*

The main result on K-convergence is the following

Theorem 1.1.7 (Kuratowski compactness theorem) *Assume that X satisfies the second axiom of countability, i.e. X has a countable base. Then for every sequence (E_k) there exists a subsequence $(E_{\sigma(k)})$ such that it K-converges to a subset $E \subset X$ (maybe, empty).*

For the proofs and more details we refer to [194].

1.2 G-convergence of Monotone Operators

Throughout this section we denote by V a separable reflexive Banach space with the norm $\|\cdot\|$, and by V^* its dual space with the dual norm $\|\cdot\|_*$.

1.2.1 Classes of Operators

Here we introduce main classes of operators we deal with later on. Denote by p a fixed real number, $1 < p < +\infty$, and by p' its dual exponent, $1/p + 1/p' = 1$. Moreover, we fix two nonnegative constants m_1 and m_2, and two constants $c_1 > 0$ and $c_2 > 0$.

Definition 1.2.1 *Denote by $\hat{\mathcal{M}} = \hat{\mathcal{M}}(m_1, m_2, c_1, c_2)$ the set of all (multivalued) monotone operators A from V into V^* such that the estimates*

$$\|f\|_*^{p'} \leq m_1 + c_1 \cdot (f, u)_V, \tag{1.2.1}$$

$$\|u\|^p \leq m_2 + c_2 \cdot (f, u)_V \tag{1.2.2}$$

hold for every $u \in V$ and $f \in Au$. By $\mathcal{M} = \mathcal{M}(m_1, m_2, c_1, c_2)$ we denote the set of all maximal monotone operators from $\hat{\mathcal{M}}$.

Remark 1.2.1 Inequalities (1.2.1) and (1.2.2) imply that there exist constants $m_3 \geq 0, m_4, c_3 > 0$ and $c_4 > 0$ such that

$$\|f\|_* \leq m_3 + c_3\|u\|^{p-1}, \tag{1.2.3}$$

$$(f, u)_V \geq m_4 + c_4\|u\|^p, \tag{1.2.4}$$

for every $u \in V$ and $f \in Au$. Conversely, if the operator A satisfies inequalities (1.2.3) and (1.2.4), then (1.2.1) and (1.2.2) are valid for suitable m_1, m_2, c_1 and c_2. Additionaly, for $A \in \mathcal{M}$ we have $D(A) = V$. Indeed, A^{-1} is a maximal monotone operator. Inequality (1.2.1) implies that A^{-1} is coercive. Hence, $D(A) = R(A^{-1}) = V$.

Definition 1.2.2 *Given constants $m \geq 0, c > 0$ and α, with*

$$0 < \alpha \leq \min\left[\frac{p}{2}, (p-1)\right],$$

denote by $\mathcal{U} = \mathcal{U}(\alpha, c, m)$ the class of all operators $A \in \mathcal{M}$ such that

$$m + (f_1, u_1)_V + (f_2, u_2)_V \geq 0 \tag{1.2.5}$$

and

$$\|f_1 - f_2\|_* \leq c \cdot \Phi^{(p-1-\alpha)/p} \cdot (f_1 - f_2, u_1 - u_2)_V^{\alpha/p} \tag{1.2.6}$$

for every u_1 ,$u_2 \in V$ and $f_1 \in Au_1$, $f_2 \in Au_2$, where $\Phi = \Phi(u_1, u_2, f_1, f_2)$ stands for the left hand side of (1.2.5).

Definition 1.2.3 *Given $m \geq 0$, $c > 0$ and β, with $\beta \geq \max(p, 2)$, denote by $\mathcal{S} = \mathcal{S}(\beta, c, m)$ the class of all operators $A \in \mathcal{M}$ such that (1.2.5) and*

$$(f_1 - f_2, u_1 - u_2)_V \geq c \cdot \Phi^{(p-\beta)/p} \cdot \|u_1 - u_2\|^\beta \tag{1.2.7}$$

are valid for every u_1, $u_2 \in V$ and $f_1 \in Au_1$, $f_2 \in Au_2$.

Remark 1.2.2 Inequalities (1.2.1) and (1.2.2) imply the existence of $m \geq 0$ such that (1.2.5) holds for any $A \in \hat{\mathcal{M}}$. Moreover, it is easy that operators of the class $\mathcal{U}$ are single-valued and continuous. It is also evident that for any $A \in \mathcal{S}$ we have $D(A^{-1}) = V^*$ and A^{-1} is a single-valued continuous operator.

Proposition 1.2.1 **(i)** *If*

$$0 < \alpha' \leq \alpha \leq \min\left[\frac{p}{2}, p-1\right],$$

then

$$\mathcal{U}(\alpha, c, m) \subset \mathcal{U}(\alpha', c', m')$$

for suitable constans $m' \geq 0$ and $c' > 0$.

(ii) *If*

$$\max(p, 2) \leq \beta \leq \beta',$$

then

$$\mathcal{S}(\beta, c, m) \subset \mathcal{S}(\beta', c', m')$$

for suitable $m' \geq 0$ and $c' > 0$.

The proof is very simple and we leave it to the reader.

Definition 1.2.4 *Given $m \geq 0$, $c > 0$ and α, with $0 \leq \alpha \leq \min(1, p-1)$, denote by $\mathcal{U}^* = \mathcal{U}^*(\alpha, c, m)$ the class of all single-valued operators $A \in \mathcal{M}$ such that*

$$\|Au_1 - Au_2\|_* \leq c \cdot (m + \|u_1\| + \|u_2\|)^{p-1-\alpha} \cdot \|u_1 - u_2\|^\alpha \tag{1.2.8}$$

for every u_1, $u_2 \in V$.

Definition 1.2.5 *Given $m \geq 0$, $c > 0$ and β, with $\beta \geq \min(p, 2)$, denote by $\mathcal{S}^* = \mathcal{S}^*(\beta, c, m)$ the class of all operators $A \in \mathcal{M}$ such that*

$$(f_1 - f_2, u_1 - u_2)_V \geq c \cdot (m + \|u_1\| + \|u_2\|)^{p-\beta} \|u_1 - u_2\|^\beta \tag{1.2.9}$$

for every u_1, $u_2 \in A$ and $f_1 \in Au_1$, $f_2 \in Au_2$.

Proposition 1.2.2 **(i)** *We have*

$$\mathcal{U}(\alpha, c, m) \subset \mathcal{U}^*(\alpha', c', m') \tag{1.2.10}$$

with $\alpha' = \alpha(p-\alpha)^{-1}$ and suitable m' and c'.

(ii) *Given c', c'', m' and m''*

$$\mathcal{U}^*(\alpha, c', m') \bigcap \mathcal{S}^*(\beta, c'', m'') \subset \mathcal{U}(\alpha', c, m), \tag{1.2.11}$$

with $\alpha' = \alpha p/\beta$ and appropriate c and m.

(iii) *We have*

$$\mathcal{S}(\beta, c, m) \subset \mathcal{S}^*(\beta, c', m'), \tag{1.2.12}$$

$$\mathcal{S}^*(\beta, c, m) \subset \mathcal{S}(\beta, c'', m'') \tag{1.2.13}$$

for suitable c', c'', m' and m''.

Proof. As for (i), from (1.2.6) and (1.2.3) we deduce

$$\begin{aligned}
\|Au_1 - Au_2\|_* &\leq c\left[m + (Au_1,u_1)_V + (Au_2,u_2)_V\right]^{(p-1-\alpha)/p} \times \\
&\quad \times (Au_1 - Au_2, u_1 - u_2)_V^{\alpha/p} \leq \\
&\leq c\left[m + \|Au_1\|_*\|u_1\| + \|Au_2\|_*\|u_2\|\right]^{(p-1-\alpha)/p} \times \\
&\quad \times \|Au_1 - Au_2\|_*^{\alpha/p}\|u_1 - u_2\|^{\alpha/p} \leq \\
&\leq c'\left[m' + \|u_1\| + \|u_2\|\right]^{p-1-\alpha} \|Au_1 - Au_2\|_*^{\alpha/p} \times \\
&\quad \times \|u_1 - u_2\|^{\alpha/p}.
\end{aligned}$$

Hence,

$$\|Au_1 - Au_2\|_*^{(p-\alpha)/p} \leq c'\left[m' + \|u_1\| + \|u_2\|\right]^{p-1-\alpha} \|u_1 - u_2\|^{\alpha/p}.$$

This implies inequality (1.2.8) with c, m and α replaced by c', m' and $\alpha' = \alpha/(p-\alpha)$, respectively.

Statements (ii) and (iii) may be proved in the similar way. □

In the sequel we shall not use the operator classes $\mathcal{U}^*$ and $\mathcal{S}^*$. However, when we shall study elliptic operators, it will be helpful to consider another class of operators, $\mathcal{K}$, which is closely related to $\mathcal{U}^*$ and $\mathcal{S}^*$.

Definition 1.2.6 *Given $c_0 > 0$, $c_1 \geq 0$, $c_2 \geq 0$, $\kappa > 0$, $\theta > 0$, $\beta \geq \min(p,2)$ and s, with $0 < s \leq \min(p,p')$, denote by $\mathcal{K} = \mathcal{K}(c_0,c_1,c_2,\kappa,\theta,\beta,s)$ the set of all single-valued operators $A : V \longrightarrow V^*$ such that*

$$\|Au\|_*^{p'} \leq c_0\|u\|^p + c_1, \quad u \in V, \tag{1.2.14}$$

$$\|Au_1 - Au_2\|_*^{p'} \leq \theta \cdot (c_2 + \|u_1\|^p + \|u_2\|^p)^{1-s/p} \cdot \|u_1 - u_2\|^s, \tag{1.2.15}$$

and

$$(Au_1 - Au_2, u_1 - u_2)_V \geq \kappa \cdot (c_2 + \|u_1\|^p + \|u_2\|^p)^{1-\beta/p} \cdot \|u_1 - u_2\|^\beta, \tag{1.2.16}$$

for any u_1, $u_2 \in V$.

It is obvious that any class $\mathcal{K}$ is contained in a suitable class $\mathcal{U}^* \cap \mathcal{S}^*$ with properly choosen values of parameters. Conversely, given class $\mathcal{U}^* \cap \mathcal{S}^*$ one can determine the parameters of $\mathcal{K}$ so that $\mathcal{U}^* \cap \mathcal{S}^* \subset \mathcal{K}$.

Definition 1.2.6 does not include explicitly any coerciveness condition. However, inequality (1.2.14) and inequality (1.2.16), with $u_1 = u$, $u_2 = 0$, imply easily the following inequality

$$(Au,u)_V \geq d_0\|u\|^p - K(c_1 + c_2), \quad u \in V, \tag{1.2.17}$$

where $d_0 > 0$ and $K > 0$ do not depend on c_1 and c_2, but depend, generally, on the other paramerters.

1.2.2 G-convergence and G-compactness

Let us denote by w the weak topology of V and by ρ the strong (norm) topology of V^*.

Definition 1.2.7 *We say that a sequence (A_k) of (generally, multivalued) operators from V into V^* is G-convergent to an operator A (in symbols, $A_k \xrightarrow{G} A$) if*

$$K_s(w \times \rho)\text{-}\lim_{k\to\infty} \operatorname{gr}(A_k) = \operatorname{gr}(A). \tag{1.2.18}$$

Remark 1.2.3 Equation (1.2.18) holds if and only if both the following conditions (g) and (gg) are satisfied:

(g) *if (f_k) converges to f strongly in V^*, (u_k) converges to u weakly in V, and*

$$A_k u_k \ni f_k \tag{1.2.19}$$

for infinitely many $k \in \mathbf{N}$, then

$$Au \ni f; \tag{1.2.20}$$

(gg) *if $f \in V^*$ and $u \in V$ is a solution of* (1.2.20), *then there exist (f_k) converging to f strongly in V^* and (u_k) converging to u weakly in V such that u_k satisfies equation* (1.2.19) *for every $k \in \mathbf{N}$.*

Proposition 1.2.3 *Asume that $A_k \xrightarrow{G} A$. Then for any subsequence $(\sigma(k))$ we have $A_{\sigma(k)} \xrightarrow{G} A$.*

Proof. It follows immediately from Proposition 1.1.4. □

Remark 1.2.4 Proposition 1.1.7 implies that $A_k \xrightarrow{G} A$ if and only if any subsequence of (A_k) contains a further subsequence G-converging to A. If $A_k = A$ and A is closed in $V \times V^*$ with respect to the topology $w \times \rho$, i.e. $\operatorname{gr}(A)$ is closed, then $A_k \xrightarrow{G} A$. In particular, it is true in the case when A is a maximal monotone operator.

The following statement will be usefull later on.

Proposition 1.2.4 *Let $J : V \longrightarrow V^*$ be an operator which is continuous from V with the weak topology into V^* with the strong topology. Assume that $A_k \xrightarrow{G} A$. Then $A_k + J \xrightarrow{G} A + J$.*

Proof. Let $A_k u_k + J u_k \ni f_k$, where (f_k) converges to f strongly in V^* and (u_k) converges to u weakly in V. Then $f_k - Ju_k \longrightarrow f - Ju$ strongly in V^*. Since $A_k u_k \ni f_k - Ju_k$, we have $Au \ni f - Ju$. Hence, $Au + Ju \ni f$ and $A_k + J$ satisfies condition (g) of Remark 1.2.3.

Similarly one can prove that $A_h + J$ satisfies condition (gg) as well. □

The following example shows that the "compactness" assumption in Proposition 1.2.4 cannot be omited.

Example 1.2.1 Suppose $V = \mathrm{L}^2(0, 2\pi)$ and $V^* = \mathrm{L}^2(0, 2\pi)$. Let us define an operator $A_k : V \longrightarrow V^*$ by the formula

$$(A_k u)(x) = a(kx)u(x),$$

where a is a continuous 2π-periodic function such that $\mu_1 \geq a(y) \geq \mu_0 > 0$. It is easy that $A_k \xrightarrow{G} \hat{A}$, where

$$(\hat{A}u)(x) = \langle a^{-1} \rangle^{-1} u(x),$$

and

$$\langle b \rangle = \frac{1}{2\pi} \int_0^{2\pi} b(s)ds. \tag{1.2.21}$$

On the other hand, $A_k + \lambda I \xrightarrow{G} \hat{A}_\lambda$, where I is the identity operator and

$$(\hat{A}_\lambda u)(x) = \langle (a + \lambda)^{-1} \rangle^{-1} u(x).$$

Evidently, $\hat{A}_\lambda \neq \hat{A} + \lambda I$.

Proposition 1.2.5 *Let $(A_k) \subset \hat{\mathcal{M}}(m_1, m_2, c_1, c_2)$. Then there exist a subsequence $(\mathcal{A}_{\sigma(k)})$ and an operator $A \in \hat{\mathcal{M}}$ such that $A_{\sigma(k)} \xrightarrow{G} A$.*

Proof. Since the space V is separable and reflexive, there exists a metric d such that for any sequence (u_k) in V the following conditions are equivalent:

(j) $(u_k) \rightarrow u$ *weakly in* V;

(jj) (u_k) *is bounded in* V *and* $d(u_k, u) \rightarrow 0$

(see, e.g., [144]). We denote by τ the topology associated to the metric d on V. This topology has a countable base.

Since the topology $\tau \times \rho$ has a countable base, by the Kuratowski compactness theorem (Theorem 1.1.7), there exists a subsequence of (A_k), still denoted by (A_k), which $K_s(\tau \times \rho)$-converges to a *set* $A \subset V \times V^*$.

Now we prove that $A = K_s(w \times \rho)\text{-}\lim A_k$. With this aim it is enough to show that

$$K_s(w \times \sigma)\text{-}\limsup A_k \subset A, \tag{1.2.22}$$

and

$$A \subset K_s(w \times \sigma)\text{-}\liminf \cdot A_k. \tag{1.2.23}$$

First, let us verify (1.2.22). Suppose $(u, f) \in K_s(w \times \rho)\text{-}\limsup A_k$. Then there exist a subsequence $\sigma(k)$ and a sequence (u_k, f_k) converging to (u, f) in the topology $w \times \rho$ such that $(u_k, f_k) \in A_{\sigma(k)}$ for every $k \in \mathbf{N}$. Since (j) implies (jj), we see that (u_k, f_k) converges to (u, f) with respect to the topology $\tau \times \rho$. Hence, $(u, f) \in A$.

Now we prove (1.2.23). Let $(u, f) \in A$. Then there exists a sequence (u_k, f_k) converging to (u, f) in the topology $\tau \times \rho$ such that $(u_k, f_k) \in A_k$ for k large enough. Since (f_k) is bounded in V^*, inequality (1.2.4) implies that (u_k) is bounded in V. Then the equivalence between conditions (j) and (jj) yields weak convergence of (u_k) to u. Hence, (u_k, f_k) converges to (u, f) in the topology $w \times \rho$, which implies (1.2.23).

Finally, we prove that $A \in \hat{\mathcal{M}}$. Let $f \in Au$. By (1.2.22), there exist two sequences (u_k) and (f_k) such that (f_k) converges to f strongly in V^*, (u_k) converges to u weakly in V, and $f_k \in Au_k$ for k large enough. Since $A_k \in \hat{\mathcal{M}}$, inequality (1.2.1) for A_k implies

$$\|f_k\|_*^p \le m_1 + c_1 \cdot (f_k, u_k)_V.$$

Passing to the limit we obtain

$$\|f\|_*^p \le m_1 + c_1 \cdot (f, u)_V.$$

Hence, A satisfies inequality (1.2.1).

In the similar way one can prove inequality (1.2.2) and the monotonicity of A. □

Now we show that under suitable conditions maximal monotonicity is stable with respect to G-convergence.

Proposition 1.2.6 *Let $A_k \xrightarrow{G} A$. Then the following statements hold true:*

(i) *Assume that there exists a single-valued strictly monotone operator $J : V \longrightarrow V^*$ such that J is continuous with respect to the weak topology of V and the strong topology of V^*. If $A_k \in \mathcal{M}$, then $A \in \mathcal{M}$.*

(ii) *If $A_k \in \mathcal{U}$, then $A \in \mathcal{U}$.*

(iii) *If $A_k \in \mathcal{S}$, then $A \in \mathcal{S}$.*

Proof. (i) By Proposition 1.2.5, $A \in \hat{\mathcal{M}}$. Hence, to prove that $A \in \mathcal{M}$ we have to show that A is a maximal monotone operator.

With this aim, we prove at first that $R(A_k + \lambda J) = V^*$, $\lambda \geq 0$. Since A_k and J are maximal monotone operators, Theorem 1.1.3 implies that $A_k + \lambda J, \lambda \geq 0$, is also maximal monotone.

Since

$$(J(u), u)_V \geq (J(0), u)_V \geq -c\|u\|, \quad c \geq 0,$$

inequality (1.2.4) implies that $A_k + \lambda J$ is coercive. Hence, by Theorem 1.1.4,

$$R(A_k + \lambda J) = V^*.$$

Now we prove that $R(A + \lambda J) = V^*$, $\lambda \geq 0$. By Proposition 1.2.4,

$$A_k + \lambda J \xrightarrow{G} A + \lambda J.$$

Let $f \in V^*$. Then there exists $u_k \in V$ such that

$$A_k(u_k) + \lambda J(u_k) \ni f.$$

By (1.2.4), the sequence (u_k) is bounded in V. Therefore, it contains a subsequence converging to u weakly in V. By condition (g) for $A_k + \lambda J$, we have

$$A(u) + \lambda J(u) \ni f,$$

which yields $R(A + \lambda J) = V^*$. Hence, by Theorem 1.1.2, A is a maximal monotone operator.

(ii) As in the proof of Proposition 1.2.5, one can prove that the G-limit operator A satisfies inequality (1.2.6). So, to prove that $A \in \mathcal{U}$ we need to show that A is a maximal monotone operator. But (1.2.6) implies that A is continuous and we conclude applying Remark 1.1.1.

(iii) Since A_k satisfies inequality (1.2.7), we conclude easy that A satisfies the same inequality. As in the proof of (i), we have $R(A) = V^*$. Hence, $D(A^{-1}) = V^*$. Inequality (1.2.7) implies that A^{-1} is a single-valued continuous operator from V^* into V. Therefore, by Remark 1.1.1, A^{-1} is a maximal monotone operator. Hence, so is A and we conclude. □

As consequence, we have the following

Theorem 1.2.1 (G-compactness theorem) *Let (A_k) be a sequence of operators. Then the following statements hold true.*

(i) *If $A_k \in \mathcal{M}$, then there exist a subsequence $\sigma(k) \to \infty$ and an operator $A \in \mathcal{M}$ such that $A_{\sigma(k)} \xrightarrow{G} A$ provided there exists a single-valued strictly monotone operator $J : V \longrightarrow V^*$ which is continuous with respect to the weak topology of V and the strong topology of V^*.*

(ii) *If* $A_k \in \mathcal{U}$*, then there exist a subsequence* $\sigma(k) \to \infty$ *and an operator* $A \in \mathcal{U}$ *such that* $A_{\sigma(k)} \xrightarrow{G} A$.

(iii) *If* $A_k \in \mathcal{S}$*, then there exist a subsequence* $\sigma(k) \to \infty$ *and an operator* $A \in \mathcal{S}$ *such that* $A_{\sigma(k)} \xrightarrow{G} A$.

Now we discuss the additional condition which appears in Proposition 1.2.6(i) and Theorem 1.2.1(i), the existence of the operator J. The typical situation when this condition is fullfilled is the following. Let W be another reflexive Banach space such that $V \subset W$, with the embedding being dense and compact. Then the dual embedding $W^* \subset V^*$ is also dense and conpact. Assume that the norms in W and W^* are strictly convex. Then we can take as J any duality operator between W and W^*, restricted to V.

Finaly, let us discuss the classes $\mathcal{U}^*$ and $\mathcal{S}^*$. Given a sequence (A_k) of operators which belong to $\mathcal{U}^*$ or to $\mathcal{S}^*$ we still can extract a G-convergent subsequence. However, these classes are not closed with respect to G-convergence. Nevertheless, if $A_k \in \mathcal{S}^*(\beta, c, m)$, then, by Proposition 1.2.2 (iii), we have $A \in \mathcal{S}^*(\beta, c', m')$ for suitable c' and m', where A is the G-limit of the sequence (A_k). Moreovere, if $A_k \in \mathcal{U}^*(\alpha, c', m') \cap \mathcal{S}^*(\beta, c'', m'')$, then Proposition 1.2.2 (i) and (ii) implies that $A \in \mathcal{U}(\alpha', c, m)$, with $\alpha' = \alpha p/\beta$, and $A \in \mathcal{U}^*(\alpha'', \tilde{c}, \tilde{m})$, with $\alpha'' = \alpha/(\beta - \alpha)$. for suitable $c, \tilde{c}, m$, and $\tilde{m}$.

1.2.3 Comparision of Different Types of Operator Convergence

Sometimes G-convergence of operators may be characterized in a slightly different way. We consider here the case of operators having single-valued inverses.

Proposition 1.2.7 *Suppose* $\mathcal{F} \subset \mathcal{M}$ *is a* G*-compact set of (multivalued) operators acting from* V *into* V^**, and* $A_k \in \mathcal{F}$ $(k \in \mathrm{N})$*,* $A \in \mathcal{F}$ *are strictly monotone operators. Then the sequence* (A_k) *is* G*-convergent to* A *if and only if*

$$w\text{-}\lim A_k^{-1} f = A^{-1} f$$

for any $f \in V^*$.

Proof. Assume that $A_k \xrightarrow{G} A$. Then, by definition of G-convegence,

$$w\text{-}\lim A_k^{-1} f = A^{-1} f,$$

and the "only if" part of the statement is proved.

Let us prove the "if" part. Suppose that

$$w\text{-}\lim A_k^{-1} f = A^{-1} f$$

for any $f \in V^*$. By G-compactness of $\mathcal{F}$, there exists a subsequence $(A_{\sigma(k)})$ such that $A_{\sigma(k)} \xrightarrow{G} \hat{A} \in \mathcal{M}$. The definition of G-convergence implies that $A^{-1}f \in \hat{A}^{-1}f$ for any $f \in V^*$. Hence, $\hat{A}^{-1}$ is a monotone exension of A^{-1}. Since A is a maximal monotone operator, so is A^{-1}. Therefore, $\hat{A}^{-1} = A^{-1}$ and $\hat{A} = A$. Thus, $A_{\sigma(k)} \xrightarrow{G} A$.

This implies that for any subsequence of (A_k) there exists a further subsequence which G-converges to A. Hence, the initial sequence, (A_k), G-converges to A and we conclude. □

Many authors (see, e.g., [266]) call a sequense of operators (A_k) to be G-convergent to A if A and A_k have single-valued inverse operators and

$$A_k^{-1}f \to A^{-1}f$$

weakly in V for any $f \in V^*$. Proposition 1.2.7 (together with Theorem 1.2.1) shows us that on a reasonable class of operators the last definition of G-convergence is equivalent to that we use in the present Section.

In the case of *single-valued* operators there are also other types of operator convergence. Let us discuss the following ones provided all the operators we consider satisfy inequality (1.2.3).

1. Define the metric

$$d(A_1, A_2) = \sup_{u \in V} \frac{\|A_1u - A_2u\|_*}{1 + \|u\|^{p-1}}. \tag{1.2.24}$$

The corresponding convergence may be named the *uniform* convergence. We shall write

$$u\text{-}\lim A_k = A$$

if (A_k) converges to A with respect to the metric d.

2. *Pointwise* convergence, i.e. $\lim A_k u = Au$ for any $u \in V$.

3. *Weak* convergence, i.e. $w\text{-}\lim A_k u = Au$ for any $u \in V$.

We have

Proposition 1.2.8 (i) *Uniform convergence implies pointwise convergence.*

(ii) *Pointwise convergence implies weak convergence.*

(iii) *On the subset of $\mathcal{S}$ consisting of single-valued operators pointwise convergence implies G-convergence.*

Proof. Statements (i) and (ii) are trivial.

To prove (iii) assume that $(A_k) \subset \mathcal{S}$ and $A_k \to A$ pointwise, A_k and A are single-valued. For any $f \in V^*$, there exist a unique $u \in V$ and a unique $u_k \in V$ such that

$$Au = f \quad \text{and} \quad A_k u_k = f_k.$$

By assumption, we have

$$A_k u = f_k \to f$$

strongly in V^*. Inequality (1.2.4) implies that the sequence (u_k) is bounded. Then, by inequality (1.2.7), we have

$$\begin{aligned}(f - f_k, u_k - u)_V &= (A_k u_k - A_k u, u_k - u)_V \geq \\ &\geq \Phi^{(p-\beta)/p}(u_k, u, f_k, f) \cdot \|u_k - u\|^\beta.\end{aligned}$$

Therefore, $u_k \to u$ strongly in V and, hence, weakly in V. By Proposition 1.2.7, $A_k \xrightarrow{G} A$ and the proof is complete. □

Weak convergence and G-convergence are independent, as it follows from the example due to S. Spagnolo [258].

Example 1.2.2 Let $a(y)$, $y \in \mathbf{R}$, be a 2π-periodoc measurable function such that $\mu_1 \geq a(y) \geq \mu_0 > 0$, $V = \mathrm{H}_0^1(0, 2\pi)$, and $V^* = \mathrm{H}^{-1}(0, 2\pi)$. Consider the operator

$$A_k : V \longrightarrow V^*$$

defined by the formula

$$(A_k u)(x) = (a(kx)u'(x))', \quad u \in V,$$

where the derivative is regarded in the weak sense. It is not difficult to see that (A_k) converges weakly to the operator

$$\tilde{A}u = \langle a \rangle u'', \quad u \in V,$$

where $\langle a \rangle$ is the mean value defined by (1.2.21). On the other hand $A_k \xrightarrow{G} \hat{A}$, where

$$\hat{A}u = \langle a^{-1} \rangle^{-1} u'', \quad u \in V.$$

The following statement contains an estimate of the distance between G-limit operators provided an upper estimate of the distance between up-to-limit operators is known.

Proposition 1.2.9 *Let $A_k^i \in \mathcal{U} \cap \mathcal{S}$ ($k \in \mathbf{N}$, $i = 1, 2$) and $A_k^i \xrightarrow{G} A^i$ ($i = 1, 2$). Assume that $d(A_k^1, A_k^2) \le \delta$ for any $k \in \mathbf{N}$. Then there exists a constant $C > 0$ such that*

$$d(A^1, A^2) \le C\delta^{\alpha/((p-\alpha)(\beta-1))}$$

Proof. Let $u \in V$. Then there exist a unique $u_k \in V$ and a unique $v_k \in V$ such that $A_k^1 u_k = A^1 u$ and $A_k^2 v_k = A^2 u$. By G-convergence of (A_k^i), we have

$$w\text{-}\lim u_k = w\text{-}\lim v_k = u.$$

Moreover, inequalities (1.2.3) and (1.2.4) imply

$$\max[\|u_k\|, \|v_k\|] \le c(1 + \|u\|).$$

Now we have

$$\begin{aligned} (A^1 u - A^2 u, u_k - v_k)_V \ &\ge \ (A_k^1 u_k - A_k^1 v_k, u_k - v_k)_V - \\ &\quad - |(A_k^1 v_k - A_k^2 v_k, u_k - v_k)_V| \ge \\ &\ge \ c(1 + \|u\|)^{p-\beta}\|u_k - v_k\|^\beta - \|A_k^1 v_k - A_k^2 v_k\|_* \times \\ &\quad \times \|u_k - v_k\| \ge \\ &\ge \ c(1 + \|u\|)^{p-\beta}\|u_k - v_k\|^\beta - c_1\delta(1 + \|u\|)^{p-1} \times \\ &\quad \times \|u_k - v_k\|. \end{aligned}$$

To estimate the second term in the right-hand side of the last inequality we notice that, by the Young inequality,

$$c_1\delta \cdot (1 + \|u\|)^{-1}\|u_k - v_k\| \le \varepsilon(1 + \|u\|)^{-\beta}\|u_k - v_k\|^\beta + c_2(\varepsilon)\delta^{\beta/(\beta-1)}.$$

Hence,

$$(A^1 u - A^2 u, u_k - v_k)_V \ge c'(1 + \|u\|)^{p-\beta}\|u_k - v_k\|^\beta - c_2(1 + \|u\|)^p\delta^{\beta/(\beta-1)}.$$

Passing to the limit we have, evidently,

$$\limsup \|u_k - v_k\| \le c\delta^{1/(\beta-1)}(1 + \|u\|). \tag{1.2.25}$$

By Proposition 1.2.5 (i), $A_k^i \in \mathcal{U}^*(\alpha', c', m')$, where $\alpha' = \alpha/(p - \alpha)$. Hence, using (1.2.8), we have

$$\begin{aligned} \|A^1 u - A^2 u\|_* \ &= \ \|A_k^1 u_k - A_k^2 v_k\|_* \le \\ &\le \ \|A_k^1 u_k - A_k^2 u_k\|_* + \|A_k^2 u_k - A_k^2 v_k\|_* \le \\ &\le \ C\delta(1 + \|u\|)^{p-1} + C(1 + \|u\|)^{p-1-\alpha'}\|u_k - v_k\|^{\alpha'}. \end{aligned}$$

By (1.2.25),

$$\|A^1 u - A^2 u\|_* \le C(1 + \|u\|)^{p-1}(\delta + \delta^{\alpha/((p-\alpha)(\beta-1))}).$$

Since the set $\mathcal{U} \cap \mathcal{S}$ is bounded with respect to the metric $d(\cdot,\cdot)$ and

$$\frac{\alpha}{(p-\alpha)(\beta-1)} \leq 1,$$

we obtain the required. □

As an immediate consequence of Proposition 1.2.9, we see that G-convergence commutes with uniform convergence. More precisely, we have

Corollary 1.2.1 *Let A_k^n, $A^n \in \mathcal{U} \cap \mathcal{S}$ $(n, k \in \mathbf{N})$,*

$$u\text{-}\lim_{n\to\infty} A_k^n = A_k$$

uniformly with respect to $k \in \mathbf{N}$, and

$$u\text{-}\lim_{n\to\infty} A^n = A.$$

Assume that $A_k^n \xrightarrow{G} A^n$ as $k \to \infty$ for any $n \in \mathbf{N}$. Then $A_k \xrightarrow{G} A$.

Now making use of Proposition 1.2.7 we discuss G-convergence of operators from the classes $\mathcal{K}$ (see Definition 1.2.6).

Proposition 1.2.10 *Let $(A_k) \subset \mathcal{K}(c_0, c_1, c_2, \kappa, \theta, \beta, s)$ and $A_k \xrightarrow{G} A$. Then $A \in \mathcal{K}(\overline{c}_0, \overline{c}_1, \overline{c}_2, \overline{\kappa}, \overline{\theta}, \beta, \overline{s})$, where*

$$\overline{c}_1 = \overline{c}_2 = K\left(c_1 + c_2\right), \quad \overline{s} = \frac{ps}{p\beta - ps + s},$$

$\overline{c}_0$, $\overline{\kappa}$, $\overline{\theta}$ and K do not depended on c_1 and c_2.

Proof. Since $A_k \in \mathcal{U} \cap \mathcal{S}$, with appropriate values of parameters, we get, by Theorem 1.2.1, that $A \in \mathcal{U} \cap \mathcal{S}$. Hence, all the operators we consider are strictly monotone and, in particular, have single-valued inverse operators.

By (1.2.17) for A_k, we have

$$(A_k u, u)_V \geq d_0 \|u\|^p - K \cdot (c_1 + c_2).$$

Then for any $f \in V^*$

$$\left(f, A_k^{-1} f\right)_V \geq d_0 \|A_k^{-1} f\|^p - K \cdot (c_1 + c_2). \tag{1.2.26}$$

This implies immediately that

$$\|A_k^{-1} f\|^p \leq K \cdot \left(\|f\|_*^{p'} + c_1 + c_2\right). \tag{1.2.27}$$

Taking into account inequality (1.2.14) for A_k we derive from (1.2.26) the inequality

$$\left(f, A_k^{-1} f\right)_V \geq c_0^{-1} d_0 \|f\|_*^{p'} - c_0^{-1} c_1 - K(c_1 + c_2).$$

By Proposition 1.2.7, $A_k^{-1} f \to A^{-1} f$ weakly in V. Passing to the limit we have

$$(f, A^{-1} f)_V \geq c_0^{-1} d_0 \|f\|_*^{p'} - K(c_1 + c_2),$$

of course, with a new constant K. This and the Young inequality implies, after the substitution $u = A^{-1} f$, the inequality

$$\|Au\|_*^{p'} \leq \bar{c}_0 \|u\|^p + K(c_1 + c_2). \tag{1.2.28}$$

Now let us introduce the following notations:

$$H(u, v) = c_2 + \|u\|^p + \|v\|^p$$

and

$$H_1(u, v) = H(u, v) + c_1.$$

Given $v, w \in V$ we set

$$v_k = A_k^{-1} A v$$

and

$$w_k = A_k^{-1} A w.$$

We have $v_k \to v$, $w_k \to w$ weakly in V. By (1.2.27),

$$H(v_k, w_k) \leq K \cdot H_1(v, w).$$

Inequality (1.2.16) for A_k and the identities

$$Av = A_k v_k$$

and

$$Aw = A_k w_k$$

imply

$$\begin{aligned}(Av - Aw, v_k - w_k)_V &\geq \kappa H(v_k, w_k)^{1-\beta/p} \|v_k - w_k\|^\beta \geq \\ &\geq \bar{\kappa} H_1(v, w)^{1-\beta/p} \|v_k - w_k\|^\beta. \end{aligned} \tag{1.2.29}$$

Passing to the limit we get

$$(Av - Aw, v - w)_V \geq \bar{\kappa} H_1(v, w)^{1-\beta/p} \|v - w\|^\beta. \tag{1.2.30}$$

Finally, by inequality (1.2.15) for A_k and definitions of v_k and w_k, we have

$$\begin{aligned}\|Av - Aw\|_*^{p'} &\le \theta \cdot \dot{H}(v_k, w_k)^{1-s/p}\|v_k - w_k\|^s \le \\ &\le \overline{\theta} \cdot H_1(v, w)^{1-s/p}\|v_k - w_k\|^s\end{aligned}$$

Estimating the right-hand side by means of (1.2.30) and passing to the limit we get

$$\begin{aligned}\|Av - Aw\|_*^{p'} &\le \overline{\theta} \cdot H_1(v, w)^{1-s/\beta}(Av - Aw, v - w)_V^{s/\beta} \le \\ &\le \overline{\theta} \cdot H_1(v, w)^{1-s/\beta}\|Av - Aw\|_*^{s/\beta}\|v - w\|^{s/\beta}.\end{aligned}$$

This implies immediatly the inequality

$$\|Av - Aw\|_*^{p'} \le \overline{\theta} \cdot H_1(v, w)^{1-\overline{s}/p}\|v - w\|^{\overline{s}}. \tag{1.2.31}$$

The proof is complete. □

1.2.4 Some Special Propeties of G-convergence

Here we consider some operator properties which are stable with respect to G-convergence.

Proposition 1.2.11 *Assume that $A_k \xrightarrow{G} A$ and A_k is a cyclically monotone operator for any $k \in \mathbf{N}$. Then A is cyclically monotone as well.*

Proof. We have to prove that for any finite set $(u_i, f_i) \in \mathrm{gr}(A)$, $i = 0, 1, \ldots, n$,

$$(f_0, u_0 - u_1)_V + \ldots + (f_n, u_n - u_0)_V \ge 0.$$

By condition (gg) of Remark 1.2.3, for any $i = 0, 1, \ldots, n$ there exist a sequence (f_{ik}) converging to f_i strongly in V^* and a sequence (u_{ik}) converging to u_i weakly in V such that $(u_{ik}, f_{ik}) \in \mathrm{gr}(A_k)$ for every $k \in \mathbf{N}$. Since A_k is cyclically monotone, we have

$$(f_{0k}, u_{0k} - u_{ik})_V + \ldots + (f_{nk}, u_{nk} - u_{0k})_V \ge 0.$$

Passing to the limit we obtain the required. □

Corollary 1.2.2 *Let $(A_k) \in \mathcal{M}$ and $A_k \xrightarrow{G} A$. If for every $k \in \mathbf{N}$, A_k is a subdifferential of a proper lower semicontinuous convex function, then A is also a subdifferential of a proper lower semicontinuous convex function.*

The class of convex functions, φ, such that $\partial\varphi \in \mathcal{M}$ may be described in the following way. If φ is a proper lower semicontinuous convex function satisfying the inequality

$$\mu_0\|u\|^p - \mu_1 \leq \varphi(u) \leq \mu(1 + \|u\|^p), \tag{1.2.32}$$

with $0 < \mu_0 < \mu$ and $\mu_1 \geq 0$, then $\partial\varphi \in \mathcal{M}$ for suitable constants m_1, m_2, c_1 and c_2.

Vise versa, by Theorem 1.1.5, any cyclically monotone operator $A \in \mathcal{M}$ is of the form $A = \partial\varphi$, where φ is a proper lower semicontinuous convex function satisfying (1.2.32) and $\varphi(0) = 0$.

Moreover, for functions defined on V there is a special kind of convergence, named Γ-convergence, such that G-convergence of subdifferentials is connected with Γ-convergence of corresponding convex functions. We do not discuss the concept of Γ-convergence here and refer the reader to the very interesting book [113].

Finally, we recall that an operator A is said to be *odd* if

$$A(-u) = -Au$$

for any $u \in V$. A is called *positively homogeneous of degree* r if

$$A(\lambda u) = \lambda^r Au$$

for any $u \in V$ and $\lambda > 0$.

We have the following simple statement.

Proposition 1.2.12 *Let $A_k \xrightarrow{G} A$. If A_k is odd (resp. homogeneous) for every $k \in \mathbf{N}$, then A is odd (resp. homogeneous) as well.*

1.3 G-convergence of Abstract Parabolic Operators

1.3.1 Abstract Parabolic Operators

Let V be a separable reflexive Banach space, V^* its dual space, and H a Hilbert space identified with its dual, $H^* = H$. Assume that

$$V \subset H \subset V^*, \tag{1.3.1}$$

with the embedding being dense and continuous. We have $(u, v)_V = (u, v)_H$ for $u, v \in H$, where $(\cdot,\cdot)_V$ is the natural duality pairing on $V \times V^*$ and $(\cdot,\cdot)_H$ is the inner product in H. For simplicity of notations we suppress the superscripts and write $(\cdot,\cdot)$ for both these bilinear forms. Moreover, we denote by $|\cdot|$ the norm in H.

Let T be a constant, $0 < T < \infty$. For any Banach space E we denote by $L^r(0,T;E)$, $1 \leq r < \infty$, the space of all E-valued measurable functions on $[0,T]$ such that the norm

$$\|u\|_{L^r(0,T;E)} = \left(\int_0^T \|u(t)\|_E^r dt\right)^{1/r}$$

is finite. In the case $r = \infty$ the norm is defined by

$$\|u\|_{L^\infty(0,T;E)} = \operatorname{ess}\sup_{t\in[0,T]} \|u(t)\|_E.$$

It is well-known that $L^r(0,T;E)$, $1 \leq r \leq \infty$, is a Banach space.
Now fixed $p \in (1,\infty)$ we set

$$\mathcal{V} = L^p(0,T;V),$$

$$\mathcal{H} = L^2(0,T;H).$$

Then $\mathcal{H}$ is a Hilbert space and $\mathcal{V}$ is a reflexive Banach space with the dual

$$\mathcal{V}^* = L^{p'}(0,T;V^*), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

The duality pairing on $\mathcal{V}^* \times \mathcal{V}$ and the inner product in $\mathcal{H}$ is defined by

$$\langle f, g\rangle = \int_0^T (f(t), g(t))dt.$$

Induced by (1.3.1), there are continuous and dense embeddings

$$\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}^* \tag{1.3.2}$$

provided $p \geq 2$.

Denote by $C([0,T];E)$ the Banach space of all continuous E-valued functions on $[0,T]$, endowed with the usual supremum norm, and by $C^\infty([0,T];E)$ the space of all infinitely differentiable E-valued functions on $[0,T]$. Let $C_0^\infty(0,T;E)$ be the space of all compactly supported infinitely differentiable functions defined on $(0,T)$.

Definition 1.3.1 *A function $w(t) \in \mathcal{V}^*$ is said to be generalized derivative of $u(t) \in \mathcal{V}$ if*

$$\langle \varphi', u\rangle = -\langle w, \varphi\rangle, \quad \forall \varphi \in C_0^\infty(0,T;V).$$

We shall denote $w(t)$ by $u'(t)$ or $\partial_t u$.

We introduce now the space

$$\mathcal{W} = \{u \in \mathcal{V} \ : \ u' \in \mathcal{V}^*\};$$

it is a reflexive Banach space endowed with the norm

$$\|u\|_{\mathcal{W}} = \|u\|_{\mathcal{V}} + \|u'\|_{\mathcal{V}^*}.$$

Recall some well-known properties of the space $\mathcal{W}$ (see, e.g., [200]).

Proposition 1.3.1 **(i)** *The space $C^\infty([0,T];V)$ is dense in $\mathcal{W}$.*

(ii) *The space $\mathcal{W}$ is embedded continuously into $C([0,T];H)$. More precisely, for any $u \in \mathcal{W}$ there exists a function $\tilde{u}(t) \in C([0,T];H)$ such that $u(t) = \tilde{u}(t)$ almost everywhere on $[0,T]$ and*

$$\|\tilde{u}\|_{C(0,T;H)} \leq k\|u\|_{\mathcal{W}},$$

where $k > 0$ is independent on u.

(iii) *The embedding $\mathcal{W} \subset L^p(0,T;H)$ is compact provided $V \subset H$ is compact.*

In what follows we shall identify $u \in \mathcal{W}$ with the function $\tilde{u} \in C([0,T];H)$. For any $u, v \in \mathcal{W}$ the identity

$$\langle u', v\rangle + \langle v', u\rangle = (u(T), v(T)) - (u(0), v(0)) \tag{1.3.3}$$

holds true. This identity is obviously valid for $u, v \in C^\infty([0,T];V)$. The general case may be covered by trivial passage to the limit.

Set

$$\mathcal{W}_0 = \{u \in \mathcal{W} \ : \ u(0) = 0\}.$$

The subspace $\mathcal{W}_0$ is dense in $\mathcal{V}$. Identity (1.3.3) implies that

$$\langle u', u\rangle \ \geq 0, \ u \in \mathcal{W}_0.$$

Now let us introduce the main class of abstract parabolic operators we shall consider. We fix nonnegative functions $m, m_1, m_2 \in L^1(0,T)$, constants $c_1 > 0$, $c_2 > 0$, $c_3 > 0$, $c_4 > 0$ and constants α and β such that

$$0 < \alpha \leq \min\left[\frac{p}{2}, p-1\right]$$

and

$$\beta \geq \max[p, 2].$$

Let $A(t) : V \longrightarrow V^*, t \in [0,T]$, be a family of operators satisfying the Carathéodory condition:

(Car) *$A(\cdot)u$ is a measurable V^*-valued function on $[0,T]$ for any $u \in V$ and $A(t)$ is a continuous operator from V into V^* for almost all $t \in [0,T]$.*

Suppose that, for almost all $t \in [0,T]$, the following inequalities hold true:

$$\|A(t)u\|_*^{p'} \le m_1(t) + c_1(A(t)u, u), \tag{1.3.4}$$

$$(A(t)u, u) \ge c_2\|u\|^p - m_2(t), \tag{1.3.5}$$

$$\|A(t)u_1 - A(t)u_2\|_* \le c_3\Phi^{(p-1-\alpha)/p}(A(t)u_1 - A(t)u_2, u_1 - u_2)^{\alpha/p}, \tag{1.3.6}$$

$$(A(t)u_1 - A(t)u_2, u_1 - u_2) \ge c_4\Phi^{(p-\beta)/p}\|u_1 - u_2\|^\beta, \tag{1.3.7}$$

for any $u, u_1, u_2 \in V$, where

$$\Phi = \Phi(u_1, u_2) = m(t) + (A(t)u_1, u_1) + (A(t)u_2, u_2). \tag{1.3.8}$$

It is assumed here that

$$m(t) \ge \min[2m_2(t), 2c_1^{-1}m_1(t)],$$

which, by (1.3.4) and (1.3.5), makes Φ to be nonnegative.

Lemma 1.3.1 *Under conditions (Car) and (1.3.4) – (1.3.7) the operator $\mathcal{A}$ defined by*

$$(\mathcal{A}u)(t) = A(t)u(t) \tag{1.3.9}$$

acts from $\mathcal{V}$ into $\mathcal{V}^*$. Moreover, for $u, u_1, u_2 \in \mathcal{V}$ the following inequalities hold true

$$\|\mathcal{A}u\|_{\mathcal{V}^*}^{p'} \le \overline{m}_1 + c_1\langle \mathcal{A}u, u\rangle, \tag{1.3.10}$$

$$\langle \mathcal{A}u, u\rangle \ \ge c_2\|u\|_{\mathcal{V}}^p - \overline{m}_2, \tag{1.3.11}$$

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{\mathcal{V}^*} \le c_3\overline{\Phi}^{(p-1-\alpha)/p}\langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle^{\alpha/p}, \tag{1.3.12}$$

$$\langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle \ \ge c_4\overline{\Phi}^{(p-\beta)/p}\|u_1 - u_2\|_{\mathcal{V}}^\beta. \tag{1.3.13}$$

Here

$$\overline{\Phi} = \overline{\Phi}(u_1, u_2) = \overline{m} + \langle \mathcal{A}u_1, u_1\rangle + \langle \mathcal{A}u_2, u_2\rangle, \tag{1.3.14}$$

and

$$\overline{m} = \int_0^T m(t)dt, \quad \overline{m}_i = \int_0^T m_i(t)dt, \quad i = 1, 2. \tag{1.3.15}$$

Proof. As it is well-known (see, e.g., [200]), condition (Car) implies that $A(t)u(t)$ is a measurable V^*-valued function provided $u(t)$ is a measurable V-valued function. Inequalities (1.3.10) – (1.3.13) follow from corresponding inequalities (1.3.4) – (1.3.7) by integration. □

Lemma 1.3.1 implies that the operator $\mathcal{A}$ belongs to a suitable class $\mathcal{U} \cap \mathcal{S}$ of operators acting from $\mathcal{V}$ into $\mathcal{V}^*$.

Definition 1.3.2 *We denote by $\mathcal{P}$ the class of operators of the form*

$$Lu = u' + \mathcal{A}u, \tag{1.3.16}$$

where $\mathcal{A}$ is defined by (1.3.9) and $A(t)$ satisfies conditions (Car) and (1.3.4) – (1.3.7).

We shall consider L as an operator acting from $\mathcal{W}_0$ into $\mathcal{V}^*$ (or, sometimes, from $\mathcal{W}$ into $\mathcal{V}^*$).

The following result is well-known (see, e.g., [200]).

Theorem 1.3.1 *Let $L \in \mathcal{P}$. For any $f \in \mathcal{V}^*$ the equation*

$$Lu = f \tag{1.3.17}$$

has a unique solution $u \in \mathcal{W}_0$. Moreover, we have

$$\|u\|_{\mathcal{W}} \leq k_1 \|f\|_{\mathcal{V}^*}^{p'-1} + k_2, \tag{1.3.18}$$

where the constants $k_1 > 0$ and $k_2 \geq 0$ depend on c_1, c_2, $\overline{m}_1$ and $\overline{m}_2$ only.

As consequence, for any $L \in \mathcal{P}$ there exists a single-valued inverse operator $L^{-1} : \mathcal{V}^* \longrightarrow \mathcal{W}_0$.

Remark 1.3.1 The result of Theorem 1.3.1 is still valid for any operator L of the form (1.3.16), where $\mathcal{A}$ acts from $\mathcal{V}$ into $\mathcal{V}^*$ and satisfies inequalities (1.3.10) – (1.3.13).

Lemma 1.3.2 *Let $f_i \in \mathcal{V}^*$, $i = 1, 2$. Then for any $L \in \mathcal{P}$ we have*

$$\|L^{-1}f_1 - L^{-1}f_2\|_{\mathcal{W}} \leq k \left(1 + \|f_1\|^{p'} + \|f_2\|^{p'}\right)^{(1-\gamma)/p'} \|f_1 - f_2\|_{\mathcal{V}^*}^{\gamma},$$

where the constant $k > 0$ depends only on $\overline{m}$, $\overline{m}_1$, $\overline{m}_2$, c_1, c_2, c_3 and c_4, and

$$\gamma = \frac{\alpha}{(p - \alpha)(\beta - 1)}.$$

Proof. Set $u_i = \mathcal{P} f_i$. Then we have

$$(u_1 - u_2)' + \mathcal{A}u_1 - \mathcal{A}u_2 = f_1 - f_2.$$

Multiplying by $(u_1 - u_2)$ and integrating, we obtaine

$$\frac{1}{2}|u_1(T) - u_2(T)|^2 + \langle \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2\rangle \;=\; \langle f_1 - f_2, u_1 - u_2\rangle.$$

This and (1.3.13) imply

$$c_4 \overline{\Phi}^{(p-\beta)/p} \|u_1 - u_2\|_{\mathcal{V}}^{\beta} \le \|f_1 - f_2\|_{\mathcal{V}^*} \|u_1 - u_2\|_{\mathcal{V}}. \tag{1.3.19}$$

Since $\mathcal{A} \in \mathcal{U} \cap \mathcal{S}$, we have (see (1.2.3))

$$\|\mathcal{A}u\|_{\mathcal{V}^*} \le k\left(1 + \|u\|_{\mathcal{V}}^{p-1}\right).$$

Hence, using the definition of $\overline{\Phi}$, (1.3.10), and (1.3.18), we derive from (1.3.19) the inequality

$$\|u_1 - u_2\|_{\mathcal{V}} \le k\left(1 + \|f_1\|^{p'} + \|f_2\|^{p'}\right)^{(\beta-p)/(p(\beta-1))} \|f_1 - f_2\|^{1/(\beta-1)} \tag{1.3.20}$$

(here it is essential that $\beta \ge p$).

Now we have

$$\|u_1' - u_2'\|_{\mathcal{V}^*} \le \|\mathcal{A}u_1 - \mathcal{A}u_2\|_{\mathcal{V}^*} + \|f_1 - f_2\|_{\mathcal{V}^*}.$$

By Proposition 1.2.2 (i),

$$\|\mathcal{A}u_1 - \mathcal{A}u_2\|_{\mathcal{V}_*} \le k\left(\overline{m} + \|u_1\| + \|u_2\|\right)^{p-1-\alpha/(p-\alpha)} \|u_1 - u_2\|^{\alpha/(p-\alpha)}.$$

Using (1.3.18) and (1.3.20) we obtain

$$\begin{aligned}
\|u_1' - u_2'\|_{\mathcal{V}^*} \;&\le\; k\left(1 + \|f_1\|^{p'} + \|f_2\|^{p'}\right)^{1-\alpha/(p'(p-\alpha)(\beta-1))} \times \\
&\qquad \times \|f_1 - f_2\|^{\alpha/((p-\alpha)(\beta-1))} + \|f_1 - f_2\|_{\mathcal{V}^*} \le \\
&\le\; k\left(1 + \|f_1\|^{p'} + \|f_2\|^{p'}\right)^{(1-\gamma)/p'} \|f_1 - f_2\|^{\gamma}.
\end{aligned} \tag{1.3.21}$$

Since $\frac{\alpha}{p-\alpha} \le 1$, we have $\gamma \le \frac{1}{\beta-1}$. Combining (1.3.20) and (1.3.21) we obtain the required. □

In the rest of this section we impose the following assumption:

$$\textit{the embedding } V \subset H \textit{ is compact.} \tag{1.3.22}$$

Lemma 1.3.3 *Let*

$$B = \{u \in \mathcal{W} \;:\; \|u\|_{\mathcal{W}} \le C\}.$$

Then for any $\varphi \in H$ *the family* $\{(u(t), \varphi), u \in B\}$ *is equicontinuous on* $[0, T]$.

Proof. Since, by Proposition 1.3.1 (ii), B is bounded in $C([0,T];H)$ and V is dense in H, it is sufficient to prove the assertion assumimg $\varphi \in V$. In this case we have

$$\begin{aligned}|(u(t+\Delta t)-u(t),\varphi)| &\leq \left|\int_t^{t+\Delta t}(u'(\tau),\varphi)d\tau\right| \leq \\ &\leq \int_t^{t+\Delta t}\|u'(\tau)\|_*\|\varphi\|_V d\tau \leq \\ &\leq \|\varphi\|\cdot\|u\|_{\mathcal{W}}(\Delta t)^{1/p} \leq C\cdot\|\varphi\|\cdot(\Delta t)^{1/p}.\end{aligned}$$

This implies the required. □

Lemma 1.3.4 *Let F be a precompact subset of $\mathcal{V}^*$ and*

$$\mathcal{R} = \{u \in \mathcal{W}_0 \ : \ u = L^{-1}f, \ L \in \mathcal{P} \text{ and } f \in F\}.$$

Then $\mathcal{R}$ is precompact in $C([0,T];H)$.

Proof. It is sufficient to prove that the set

$$\mathcal{R}_f = \{u \in \mathcal{W}_0 \ : \ u = L^{-1}f, \ L \in \mathcal{P}\}$$

is precompact in $C([0,T];H)$ for any $f \in \mathcal{V}^*$. Indeed, if it is so, we see that, by Lemma 1.3.2, for any finite ε-net $\{f_1,\dots,f_n\}$ in F the set

$$\bigcup_{k=1}^{n}\bigcup_{L\in\mathcal{P}}\mathcal{R}_{f_k}$$

is a precompact $\delta(\varepsilon)$-net for $\mathcal{R}$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, since $C^\infty([0,T];H)$ is dense in $\mathcal{V}^*$, we can assume that $f \in C^\infty([0,T];H)$.

To prove that $\mathcal{R}_f$ is precompact, let us consider a sequence $(u_k) \subset \mathcal{R}_f$. By Theorem 1.3.1, (u_k) is bounded in $\mathcal{W}_0$. Hence, passing to a subsequence we may assume that (u_k) converges weakly in $\mathcal{W}_0$ to a function $u \in \mathcal{W}_0$. By Proposition 1.3.1 (iii), we may also assume that $u_k \to u$ in $\mathrm{L}^p(0,T;H)$ and almost everywhere on $[0,T]$.

Since, by Proposition 1.3.1 (iii), $u_k(t)$ is bounded in H uniformly with respect to k and $t \in [0,T]$, and the space H is separable, we may assume, by Lemma 1.3.3 and Arzellà-Ascoli Theorem, that

$$(u_k(t),\varphi) \to (v(t),\varphi)$$

uniformly in $t \in [0,T]$ for any $\varphi \in H$. This implies that $u_k \to v$ weakly in $\mathcal{H}$. Hence, $v(t) = u(t)$ for almost all $t \in [0,T]$. But $u(t) \in C([0,T];H)$ and $v(t)$ is a weakly continuous H-valued function. Therefore, $u(t) = v(t)$ for any $t \in [0,T]$.

Let us show that (u_k) converges to u in the space $C([0,T];H)$. If not, then there exist $\varepsilon > 0$ and $t_k \to t_0$ such that

$$|u_k(t_k) - u(t_k)| > \varepsilon.$$

Since u is continuous,

$$|u_k(t_k) - u(t_0)| > \frac{\varepsilon}{2} \tag{1.3.23}$$

for k being large enough. Equicontinuity of $(u_k(t), \varphi), \varphi \in H$, implies that $u_k(t_k) \to u(t_0)$ weakly in H. By weak lower semicontinuity of the norm, we have (passing to a subsequence if it is needed)

$$\lim |u_k(t_k)| = a \geq |u(t_0)|.$$

By (1.3.23),

$$a^2 = |u(t_0)|^2 + \varepsilon_1,$$

where $\varepsilon_1 > 0$.

Now we prove that $t_0 \neq 0$. Assume the contrary, i.e. $t_0 = 0$. Integrating the equation

$$(u_k', u_k) + (\mathcal{A}_k u_k, u_k) = (f, u_k) \tag{1.3.24}$$

over the interval $[0, t_k]$ and using (1.3.5) we obtain

$$\begin{aligned} |u_k(t_k)|^2 &\leq 2\int_0^{t_k} m_2(t)dt + 2\int_0^{t_k} (f, u_k)dt \leq \\ &\leq 2\int_0^{t_k} m_2(t)dt + 2\|f\|_{C(0,T;V^*)}\|u_k\|_V t_k^{1/p'}. \end{aligned}$$

Since $m_2 \in \mathrm{L}^1(0,T)$, we have $|u_k(t_k)|^2 \to 0$, a contradiction to the positivity of a. Thus, we have proved that $t_0 > 0$.

Integrating (1.3.24) over the interval $[t_{k-\eta}, t_k]$ we have

$$\begin{aligned} |u_k(t_k)|^2 - |u_k(t_k - \eta)|^2 &\leq 2\int_{t_k-\eta}^{t_k} m_2(t)dt + 2\int_{t_k-\eta}^{t_k} (f, u_k)dt \leq \\ &\leq 2\int_{t_k-\eta}^{t_k} m_2(t)dt + 2\eta^{1/p'}\|f\|_{C(0,T;V^*)}\|u_k\|_V. \end{aligned}$$

This implies

$$\begin{aligned} |u_k(t_k - \eta)|^2 &\geq |u_k(t_k)|^2 - \frac{\varepsilon_1}{4} \geq \\ &\geq a^2 - \frac{\varepsilon_1}{2} \geq |u(t_0)|^2 + \frac{\varepsilon_1}{2}, \end{aligned} \tag{1.3.25}$$

for $0 \leq \eta \leq \delta_1$, with k being large enough and $\delta_1 > 0$ being small enough. Since $u \in C([0,T]; H)$, we have

$$|u(t_0)|^2 \geq |u(t_k - \eta)|^2 - \frac{\varepsilon_1}{4}$$

whenever $\eta \in [0, \delta_2]$ and $|t_k - t_0| \leq \delta_2$, where $\delta_2 > 0$ is sufficiently small. Hence, (1.3.25) implies

$$|u_k(t_k - \eta)|^2 \geq |u(t_k - \eta)|^2 + \frac{\varepsilon_1}{4},$$

where $\eta \in [0, \delta_0]$ and $\delta_0 = \min(\delta_1, \delta_2)$. Therefore, for

$$t \in \left[t_0 - \frac{3\delta_0}{4}, t_0 - \frac{\delta_0}{4}\right]$$

and sufficiently large k, we have $\eta = t_k - t \in [0, \delta_0]$ and

$$|u_k(t)|^2 \geq |u(t)|^2 + \frac{\varepsilon_1}{4}.$$

On the other hand, $u_k(t) \to u(t)$ in H almost everywhere on $[0, T]$ and we get the contradiction. $\square$

Remark 1.3.2 In the proof of Lemma 1.3.4 we have used the condition $u_k \in \mathcal{W}_0$ only to show that $t_0 \neq 0$. Therefore, the following statement holds true. Let

$$L_k u_k = f_k \in \mathcal{V}^*, \quad u_k \in \mathcal{W},$$

where $L_k \in \mathcal{P}$. If (u_k) is bounded in $\mathcal{W}$ and (f_k) is precompact in $\mathcal{V}^*$, then the sequence (u_k) is precompact in $C([\delta, T]; H)$ for any $\delta > 0$.

1.3.2 G-compactness

We begin with the definition of G-convergence of parabolic operators suggested by Theorem 1.3.1 (cf. n^0 1.2.3). In the contrast to the case of elliptic operators, parabolic operators do not act from a Banach space into its dual; they acts from the dense subspace $\mathcal{W}_0 \subset \mathcal{V}$ into $\mathcal{V}^*$. This is the reason to consider the $\mathcal{W}_0$-weak convergence of solutions.

Definition 1.3.3 *Let $(L_h) \subset \mathcal{P}$. The sequence (L_k) is called G-convergent to $L \in \mathcal{P}$ if for any $f \in \mathcal{V}^*$ we have $L_k^{-1} f \to L^{-1} f$ weakly in $\mathcal{W}_0$. In this case we write $L_h \xrightarrow{G} L$.*

Notice that Definition 1.3.3 makes sense for a more wide class of operators L of the form (1.3.16), where the operator $\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}^*$ satisfies inequalities (1.3.10) – (1.3.13). We denote by $\mathcal{P}'$ the class of all such operators L.

The main result on G-convergence of parabolic operators is the following.

Theorem 1.3.2 (*G*-compactness theorem) *Let $(L_k) \subset \mathcal{P}$.Then there exist an operator $L \in \mathcal{P}$ and a subsequence $\sigma(k) \to +\infty$ such that $L_{\sigma(k)} \xrightarrow{G} L$.*

As the first step of the proof of Theorem 1.3.2 we state the existence of a G-limit point which belongs to $\mathcal{P}'$ and next we prove that this G-limit point lies really in $\mathcal{P}$. Thus, we start with the following

Proposition 1.3.2 *Let $(L_k) \subset \mathcal{P}$. Then there exist an operator $L \in \mathcal{P}'$ and a subsequence $\sigma(k) \to +\infty$ such that $L_{\sigma(k)} \xrightarrow{G} L$.*

Proof. Let $B_k = L_k^{-1}$. By Theorem 1.3.1, $(B_k f)$ is bounded in $\mathcal{W}_0$ for any $f \in \mathcal{V}^*$. Hence, passing to a subsequence (still denoted by k) we may assume that $(B_k f)$ is weakly convergent for any f which belongs to a countable dense subset of $\mathcal{V}^*$. Since, by Lemma 1.3.2, the family of operators (B_k) is equicontinuous, the last statement takes place really for any $f \in \mathcal{V}^*$. Hence, there exists an operator $B : \mathcal{V}^* \longrightarrow \mathcal{W}_0$ such that $B_k f \to Bf$ weakly in $\mathcal{W}_0$ for any $f \in V^*$. Moreover, by Lemma 1.3.4, $B_h f \to Bf$ strongly in $C([0,T]; H)$.

Now let us show that the operator B is injective. Let $f_i \in V^*$ and $u_{ik} = B_k f_i$, $i = 1, 2$. Then

$$u'_{ik} + \mathcal{A}_k u_{ik} = f_i,\ i = 1, 2.$$

Hence,

$$\mathcal{A}_k u_{ik} = f_i - u'_{ik} \to f_i - (Bf_i)'$$

weakly in $\mathcal{V}^*$. Moreover, by (1.3.12),

$$\begin{aligned}
\|\mathcal{A}_k u_{1k} - \mathcal{A}_k u_{2k}\|_{\mathcal{V}^*} &\le c_3(\overline{m} + \langle \mathcal{A}_k u_{1k}, u_{2k}\rangle + \\
&+\langle \mathcal{A}_k u_{2k}, u_{2k}\rangle)^{(p-1-\alpha)/p} \times \langle \mathcal{A}_k u_{1k} - \mathcal{A}_k u_{2k}, u_{1k} - u_{2k}\rangle^{\alpha/p} = \\
&= c_3(\overline{m} + \langle \mathcal{A}_k u_{1k}, u_{1k}\rangle + \langle \mathcal{A}_{2k} u_{2k}, u_{2k}\rangle)^{(p-1-\alpha)/p} \times \\
&\times \left[\langle f_1 - f_2, u_{1k} - u_{2k}\rangle - \frac{1}{2}|u_{1k}(T) - u_{2k}(T)|^2\right]^{\alpha/p}
\end{aligned}$$

Passing to the limit and using weak lower semicontinuity of the norm of $\mathcal{V}^*$ we get

$$\begin{aligned}
&\|f_1 - f_2 - [(Bf_1)' - (Bf_2)']\|_{\mathcal{V}^*} \le \\
&\le c_3 \left(\overline{m} + \langle f_1 - (Bf_1)', Bf_1\rangle + \langle f_2 - (Bf_2)', Bf_2\rangle\right)^{(p-1-\alpha)/p} \times \\
&\times \left[\langle f_1 - f_2, Bf_1 - Bf_2\rangle - \frac{1}{2}|(Bf_1)(T) - (Bf_2(T)|^2\right]^{\alpha/p} \qquad (1.3.26)
\end{aligned}$$

This implies that $f_1 = f_2$ provided $Bf_1 = Bf_2$.

Next we show that the image of B is dense in $\mathcal{V}$. Similarly to (1.3.26), inequality (1.3.13) implies

$$\langle f_1 - f_2, Bf_1 - Bf_2\rangle - \frac{1}{2}|(Bf_1)(T) - (Bf_2)(T)|^2 \geq$$

$$\geq c_4\left(\overline{m} + \langle f_1 - (Bf_1)', Bf_1\rangle + \langle f_2 - (Bf_2)', Bf_2\rangle\right)^{(p-\beta)/p} \times$$

$$\times \|Bf_1 - Bf_2\|^{\beta}_{\mathcal{V}}. \tag{1.3.27}$$

Assume that the image of B is not dense in $\mathcal{V}$. Then there exists $\varphi \in \mathcal{V}^*$ such that $\varphi \neq 0$ and $\langle \varphi, Bf\rangle = 0$ for any $f \in \mathcal{V}^*$. Inequality (1.3.27), with $f_1 = \varphi$, $f_2 = 0$, gives rise to

$$\langle \varphi, B\varphi - B(0)\rangle \geq c_4\left(\overline{m} + \langle \varphi - (B\varphi)', \varphi\rangle\right)^{(p-\beta)/p}\|B\varphi - B(0)\|^{\beta}_{\mathcal{V}}.$$

Hence, $B\varphi = B(0)$. Then $\varphi = 0$ and we get a contradition.

Now we define the operator $\mathcal{A}_0$ acting from $B(\mathcal{V}^*)$ into $\mathcal{V}^*$ by the formula

$$\mathcal{A}_0(B(f)) = f - B(f)'.$$

On $D(\mathcal{A}_0) = B(\mathcal{V}^*)$, the operator $\mathcal{A}_0$ satisfies inequalities (1.3.10) – (1.3.13). Indeed, since

$$\frac{1}{2}|(Bf_1)(T) - (Bf_2)(T)|^2 = \langle (Bf_1 - Bf_2)', Bf_1 - Bf_2\rangle,$$

inequalities (1.3.26) and (1.3.27) imply (1.3.12) and (1.3.13), respectively. As for (1.3.10) and (1.3.11), they may be proved in the similar manner. By continuity (see inequality (1.3.12)), one can extend the operator $\mathcal{A}_0$ to an operator $\mathcal{A}$ which acts from $\mathcal{V}$ into $\mathcal{V}^*$ and satisfies inequalities (1.3.10) – (1.3.13).

Let

$$Lu = u' + \mathcal{A}u.$$

By Remark 1.3.1, for any $f \in \mathcal{V}^*$ there exists a unique $u \in \mathcal{W}_0$ such that $Lu = f$. By construction, Bf is a solution of the same equation and hence, $u = Bf$. Therefore, $B = L^{-1}$. Now it is easy to see that $L_k \xrightarrow{G} L$ and the proof is complete. □

The proof of Theorem 1.3.2 will be given in the next subsection, after some properties of G-convergence of parabolic operators.

1.3.3 Properties of G-convergence

First of all we show that the Cauchy problem appeared in the definition of G-convergence does not play any special role. More precisely, we have the following result on convergence of arbitrary solutions.

Theorem 1.3.3 *Let $L_k \in \mathcal{P}$, $L_k \xrightarrow{G} L \in \mathcal{P}'$, and $u_k \in \mathcal{W}$. Assume that $L_k u_k = f_k \to f$ in $\mathcal{V}^*$ and $u_k \to u$ weakly in $\mathcal{W}$. Then $Lu = f$ and $\mathcal{A}_k u_k \to \mathcal{A}u$ weakly in $\mathcal{V}^*$.*

Proof. Since $u_k \to u$ weakly in $\mathcal{W}$, we have

$$\mathcal{A}_k u_k = f_k - u_k' \to g = f - u'$$

weakly in $\mathcal{V}^*$.

Now we show that, for any $v \in \mathcal{V}$, the inequality

$$\langle g - \mathcal{A}v, u - v\rangle \geq 0 \tag{1.3.28}$$

holds true. At first, since $\mathcal{W}_0$ is dense in $\mathcal{V}$ and $\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}^*$ is continuous, without loss of generality we may assume that $v \in \mathcal{W}_0$. Moreover, to state (1.3.28) it is sufficient to prove that

$$\int_\delta^T (g - \mathcal{A}v, u - v)dt \geq 0$$

for any $v \in \mathcal{W}_0$ and $\delta \in (0, T)$

By Theorem 1.3.1, given $v \in \mathcal{W}_0$ there exists $v_k \in \mathcal{W}_0$ such that $L_k v_k = Lv$. By definition of G-convergence, $v_k \to v$ weakly in $\mathcal{W}_0$. Monotonicity of $\mathcal{A}_k(t)$ implies that

$$\chi_k = \int_\delta^T (\mathcal{A}_k u_k - \mathcal{A}_k v_k, u_k - v_k)dt \geq 0,$$

and it is sufficient to show that

$$\lim \chi_k = \int_\delta^T (g - \mathcal{A}v, u - v)dt.$$

We rewrite χ_k as follows

$$\begin{aligned}
\chi_k &= \int_\delta^T (f_k - u' - \mathcal{A}v, u_k - v_k)dt - \int_\delta^T ((u_k - v_k)', u_k - v_k)\, dt + \\
&\quad + \int_\delta^T ((u - v)', u_k - v_k)\, dt = \\
&= \int_\delta^T (f_k - u' - \mathcal{A}v, u_k - v_k)dt + \mu_k.
\end{aligned}$$

By Remark 1.3.2, the sequences (u_k) and (v_k) are precompact in $C([\delta, T]; H)$. Hence, $u_k \longrightarrow u$ and $v_k \longrightarrow V$ strongly in $C([\delta, T]; H)$. Therefore,

$$\begin{aligned} \mu_k &= \int_\delta^T ((u-v)', u_k - v_k)\, dt - \\ &\quad - \frac{1}{2}\left(|u_k(T) - v_k(T)|^2 - |u_k(\delta) - v_k(\delta)|^2\right) \longrightarrow \\ &\longrightarrow \int_\delta^T ((u-v)', u - v) dt - \\ &\quad - \frac{1}{2}\left(|u(T) - v(T)|^2 - |u(\delta) - v(\delta)|^2\right) = \\ &= 0. \end{aligned}$$

Thus, inequality (1.3.28) is proved.

Now inequality (1.3.28), with $v = u - \lambda w$, $w \in \mathcal{V}$, $\lambda > 0$, gives rise to

$$\langle g - \mathcal{A}(u - \lambda w), w\rangle \geq 0.$$

Since $\mathcal{A}$ is continuous and w is an arbitrary member of $\mathcal{V}$, letting $\lambda \longrightarrow 0$ we have $g = \mathcal{A}u$. Hence, $\mathcal{A}_k u_k \longrightarrow \mathcal{A}u$ weakly in $\mathcal{V}^*$. This implies that $Lu = f$ and the proof is complete. □

Now let us consider the localization property for G-limit operators. We say that $\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}^*$ is a *local operator* if for any subinterval $[a, b] \subset [0, T]$ the following property holds true:

$$\text{if } u_{|[a,b]} = v_{|[a,b]}, \quad \text{then } (\mathcal{A}u)_{|[a,b]} = (\mathcal{A}v)_{|[a,b]}.$$

If $\mathcal{A}$ is a local operator, then the corresponding operator

$$L = \partial_t + \mathcal{A}$$

is also said to be local. It is easy that any operator $L \in \mathcal{P}$ is local, but it is not true for $L \in \mathcal{P}'$ in general.

For any local operator $\mathcal{A}$ we can define easily the rectriction $\mathcal{A}_{|[a,b]}$ which acts from $\mathcal{V}(a, b) = \mathrm{L}^p(a, b; V)$ into $\mathcal{V}^*(a, b) = \mathrm{L}^{p'}(a, b; V^*)$. Hence, if L is a local parabolic operator, then one can define the restriction

$$L_{|[a,b]} = \partial_t + \mathcal{A}_{|[a,b]}.$$

The last operator is considered as an operator from $\mathcal{W}_0(a, b)$ (or $\mathcal{W}(a, b)$) into $\mathcal{V}^*(a, b)$, where

$$\mathcal{W}(a, b) = \{u \in \mathcal{V}(a, b) \ : \ u' \in \mathcal{V}^*(a, b)\},$$

$$\mathcal{W}_0(a, b) = \{u \in \mathcal{W}(a, b) \ : \ u(0) = 0\}.$$

Of course, the concept of G-convergence makes sense for parabolic operators defined on any interval $[a, b]$. Moreover, all the results we have just stated are still valid with $[0, T]$ replaced by an arbitrary interval $[a, b]$.

Theorem 1.3.4 *Let $L_k \in \mathcal{P}$ and $L_k \xrightarrow{G} L \in \mathcal{P}'$. Then L is a local operator and*

$$L_{k|[a,b]} \xrightarrow{G} L_{|[a,b]} \quad for\ any\ [a,b] \subset [0,T].$$

Proof. Let $u \in \mathcal{W}_0$ and $Lu = f \in \mathcal{V}^*$. Then there exists $u \in \mathcal{W}_0$ such that

$$L_k u_k = Lu = f.$$

By definition of G-convergence, $u_k \longrightarrow u$ weakly in $\mathcal{W}_0$.

By Proposition 1.3.2, passing to a subsequence we can assume that $L_{k|[a,b]} \xrightarrow{G} \hat{L}$. But $u_{k|[a,b]} \longrightarrow u_{|[a,b]}$ weakly in $\mathcal{W}(a,b)$ and

$$L_k(u_{k|[a,b]}) = (L_k u)_{|[a,b]} = f_{|[a,b]}.$$

By Theorem 1.3.3, we have

$$\hat{L}(u_{|[a,b]}) = f_{|[a,b]} = (Lu)_{|[a,b]}.$$

Hence, $(Lu)_{|[a,b]}$ depends only on $u_{|[a,b]}$. Thus L is a local operator and $\hat{L} = L_{|[a,b]}$. The proof is complete. □

Now we are able to prove Theorem 1.3.2.

Proof of Theorem 1.3.2. By Proposition 1.3.2, there are a subsequence $\sigma(k) \longrightarrow \infty$ and an operator

$$L = \partial_t + \mathcal{A} \in \mathcal{P}'$$

such that $L_{\sigma(k)} \xrightarrow{G} L$.

In order to prove the theorem we have to show that the operator $\mathcal{A}$ is of the form

$$(\mathcal{A}u)(t) = A(t)u(t), \tag{1.3.29}$$

where $A(t) : V \longrightarrow V^*$ satisfies condition (Car) and inequalities (1.3.4) – (1.3.7).

Let us define the family of operators $A(t)$ by the formula

$$A(t)\varphi = (\mathcal{A}\varphi)(t), \quad \varphi \in V,$$

where $\varphi \in V$ is viewed as a constant function. It is easy that $A(t)\varphi$ is a measurable function for any $\varphi \in V$.

Now we prove inequality (1.3.4). Let t_0 be a common Lebesgue point of the functions $m_1(t)$ and $A(t)\varphi$. Denote by $\mathcal{O}_\varepsilon$ the ε-neighborhood of the point t_0. By Theorem 1.3.4, the operator

$$\mathcal{A}_{|\mathcal{O}_\varepsilon} : \mathcal{V}(\mathcal{O}_\varepsilon) \longrightarrow \mathcal{V}^*(\mathcal{O}_\varepsilon)$$

satisfies inequality (1.3.10), with

$$\overline{m}_1 = \int_{\mathcal{O}_\varepsilon} m_1(t)dt,$$

i.e.

$$\int_{\mathcal{O}_\varepsilon} \|A(t)\varphi\|_*^{p'} dt \leq \overline{m}_1 + \int_{\mathcal{O}_\varepsilon} (A(t)\varphi, \varphi) dt.$$

Dividing this inequality by $|\mathcal{O}_\varepsilon| = 2\varepsilon$ and passing to the limit as $\varepsilon \longrightarrow 0$ we get

$$\|A(t_0)\varphi\|_*^{p'} dt \leq m_1(t_0) + (A(t_0)\varphi, \varphi).$$

Since common Lebesgue points of $m_1(t)$ and $A(t)\varphi$ form a set of full Lebesgue measure, we have proved (1.3.4).

Inequalities (1.3.5) – (1.3.7) may be establised in the similar manner.

Now to complete the proof we need to prove (1.3.29). Let $\tilde{\mathcal{A}} : \mathcal{V} \longrightarrow \mathcal{V}^*$ be an operator defined by

$$(\tilde{\mathcal{A}}u)(t) = A(t)u(t).$$

We have to show that $\mathcal{A}u = \tilde{\mathcal{A}}u$ for any $u \in \mathcal{V}$. Since $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are local operators, we see that this is so if u is a picewise constant function. Hence, $\mathcal{A}$ coincides with $\tilde{\mathcal{A}}$ on a dense subset of $\mathcal{V}$. But both the operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are continuous. Therefore, $\mathcal{A} = \tilde{\mathcal{A}}$. Thus, the proof is complete. □

We now discuss a relation between G-convergence of parabolic operators and their stationary parts. However, before to do this some preliminaries are in order. We recall that on the class $\mathcal{U} \cap \mathcal{S}$ of operators acting from $\mathcal{V}$ into $\mathcal{V}^*$ there is a metric defined by

$$d(\mathcal{A}_1, \mathcal{A}_2) = \sup_{u \in \mathcal{V}} \frac{\|\mathcal{A}_1 u - \mathcal{A}_2 u\|_{\mathcal{V}^*}}{1 + \|u\|_{\mathcal{V}}^{p-1}} \tag{1.3.30}$$

(see n^0 1.2.3). The corresponding convergence is called uniform convergence and we write $\mathcal{A} = u\text{-}\lim \mathcal{A}^n$ if $\mathcal{A}^n$ converges to $\mathcal{A}$ with respect to this metric.

It is not difficult to see that uniform convergence preserves localization, i.e. if $\mathcal{A} = u\text{-}\lim \mathcal{A}^n$ and $\mathcal{A}^n$ is local for any $n \in \mathbf{N}$, then $\mathcal{A}$ is local as well.

The following result is similar to Proposition 1.2.9.

Proposition 1.3.3 *Let*

$$L_k^i = \partial_t + \mathcal{A}_k^i, \quad k \in \mathbf{N},\ i = 1, 2,$$

be an operator from the class $\mathcal{P}$ and

$$L_k^i \xrightarrow{G} L^i = \partial_t + \mathcal{A}^i.$$

Assume that $d(\mathcal{A}_k^1, \mathcal{A}_k^2) \leq \delta$ for any $k \in \mathbf{N}$. Then there exists a constant $C > 0$ such that

$$d(\mathcal{A}^1, \mathcal{A}^2) \leq C\delta^{\alpha/((p-\alpha)(\beta-1))}.$$

The proof is analogous to that of Proposition 1.2.9 and we leave it to the reader.

Corollary 1.3.1 *Let*

$$L_k^n = \partial_t + \mathcal{A}_k^n, \quad L^n = \partial_t + \mathcal{A}^n, \quad n, k \in \mathbf{N},$$

be operators from the class $\mathcal{P}$,

$$u\text{-}\lim_{n\to\infty} \mathcal{A}_k^n = \mathcal{A}_k$$

uniformly with respect to $k \in \mathbf{N}$, and

$$u\text{-}\lim_{n\to\infty} \mathcal{A}^n = \mathcal{A}.$$

Assume that $L_k^n \xrightarrow{G} L^n$ as $k \to \infty$ for any $n \in \mathbf{N}$. Then

$$L_k = \partial_t + \mathcal{A}_k \xrightarrow{G} L = \partial_t + \mathcal{A}.$$

Now we are able to prove the following result.

Theorem 1.3.5 *Let*

$$L_k = \partial_t + \mathcal{A}_k(t), \quad k \in \mathbf{N},$$

be an operator from the class $\mathcal{P}$. Assume that

$$\sup_{k\in\mathbf{N}} \sup_{u\in V} \frac{\|A_k(t+\Delta t)u - A_k(t)u\|_*}{1 + \|u\|^{p-1}} \to 0, \tag{1.3.31}$$

as $\Delta t \to 0$, uniformly with respect to $t \in [0,T]$. If $A_k(t) \xrightarrow{G} A(t)$ for every $t \in [0,T]$, then

$$L_k \xrightarrow{G} L = \partial_t + A(t).$$

Proof. The case $A_k(t) \equiv A_k$. By Theorem 1.3.2, passing to a subsequence we may assume that $L_k \xrightarrow{G} \hat{L}$, where

$$\hat{L} = \partial_t + \hat{A}(t)$$

belongs to $\mathcal{P}$. Hence, we need only to verify that $\hat{A}(t) \equiv A$. Obviously, this implies that the previous passage to a subsequence is realy superfluous.

Given $u \in V$ there exists a unique $u_k \in V$ such that $A_k u_k = Au$. Since $A_k \xrightarrow{G} A$, we have $u_k \to u$ weakly in V. We can consider u_k and u as constant functions on $[0,T]$. Then it is evident that $u_k \to u$ weakly in $\mathcal{W}$. Since

$$L_k u_k = Au$$

and $u_k \to u$ weakly in $\mathcal{W}$, Theorem 1.3.3 implies that

$$\hat{L}u = \hat{A}(t)u = Au.$$

Hence, in this case the theorem is proved.

General case. For any $n \in \mathbf{N}$, let $t_h = hT/n, h = 0, 1, \ldots, n$. We define the piecewise constant operator functions $A_k^n(t)$ and $A^n(t)$ by the formulae

$$A_k^n(t) = A_k(t_h) \quad \text{if } t_h \leq t < t_{h+1},\ h = 0, \ldots, n-1,$$

$$A^n = A(t_h) \quad \text{if } t_h \leq t < t_{h+1},\ h = 0, \ldots, n-1.$$

We have, obviously, $A_k^n(t) \xrightarrow{G} A^n(t)$, as $k \to \infty$, for every $t \in [0, T]$ and $n \in \mathbf{N}$. Since the theorem is already proved for the special case of time independent operators, Theorem 1.3.4 implies that our statement is valid for piecewise constant stationary parts as well. Thus, we have

$$L_k^n \xrightarrow{G} L^n,$$

as $k \to \infty$, for every $n \in \mathbf{N}$. Now assumption (1.3.31) implies directly that

$$u\text{-lim}\, \mathcal{A}_k^n = \mathcal{A}$$

uniformly with respect to $k \in \mathbf{N}$ and

$$u\text{-lim}\, \mathcal{A}^n = \mathcal{A}.$$

Taking into account Proposition 1.3.3 we complete the proof. $\square$

1.3.4 Time Homogenization of Abstract Parabolic Operators

Consider a simple but, in a sense, typical application of previous results. Let

$$A(\tau) : V \longrightarrow V^*, \quad \tau \in \mathbf{R},$$

be a 1-periodic operator valued function. Suppose that condition (Car) and inequalities (1.3.4) – (1.3.7) are fulfilled. Moreover, we assume here that m, m_1 and m_2 contained in these inequalities are constants. We consider a family of operators

$$L_\varepsilon = \partial_t + A(\varepsilon^{-1}t), \quad \varepsilon > 0,$$

defined on an interval $(0, T)$. We intend here to prove that the family L_ε is G-convergent, as $\varepsilon \to 0$, and to find its G-limit. To do this we need to introduce some additional notations. By $\langle f \rangle$ we denote the mean value

$$\langle f \rangle = \int_0^1 f(\tau)d\tau$$

of a periodic function $f(\tau)$. We define the mean value $\langle A\rangle$ of an operator valued function $A(\tau)$ by the formula

$$\langle A\rangle u = \langle Au\rangle = \int_0^1 A(\tau)u d\tau, \quad u \in V.$$

It is easy that $\langle A\rangle$ is a well-defined operator acting from V into V^*. Moreover, $\langle A\rangle$ satisfies inequalities (1.3.4) - (1.3.7).

Let

$$\mathcal{V}_{per} = \{v : v \in \mathrm{L}^p_{loc}(\mathbf{R}; V), v \text{ is 1-periodic and } \langle v\rangle = 0\}.$$

Endowed with the norm

$$\|v\|_{\mathcal{V}_{per}} = \langle \|v\|^p_V\rangle^{1/p},$$

$\mathcal{V}_{per}$ is a reflexive Banach space. Its dual, $\mathcal{V}^*_{per}$, may be identified with the space

$$\{f : f \in \mathrm{L}^{p'}_{loc}(\mathbf{R}, V^*), f \text{ is 1-periodic and } \langle f\rangle = 0\},$$

and the canonical bilinear pairing on $\mathcal{V}^*_{per} \times \mathcal{V}_{per}$ is given by the formula

$$\langle f, v\rangle_{per} = \langle (f, v)\rangle.$$

Let

$$\mathcal{W}_{per} = \{v \in \mathcal{V}_{per} : v' \in \mathcal{V}^*_{per}\},$$

where v' is regarded in the sense of distributions. Equipped with the natural norm, $\mathcal{W}_{per}$ is a reflexive Banach space. Moreover,

$$\langle u', v\rangle_{per} = -\langle v', u\rangle_{per}, \quad u, v \in \mathcal{W}_{per}.$$

Theorem 1.3.6 *Assume $A(\tau) : V \longrightarrow V^*, \tau \in \mathbf{R}$, to be a 1-periodic operator valued function satisfying (Car) and (1.3.4) - (1.3.7), where m, m_1 and m_2 are constants. Then for any interval $(0,T)$*

$$L_\varepsilon \xrightarrow{G} \hat{L} = \partial_t + \hat{A},$$

as $\varepsilon \to 0$, where $\hat{A} = \langle A\rangle$.

Proof. By Theorem 1.3.2, there exist a parabolic operator L and a subsequence still denoted by ε such that

$$L_\varepsilon \xrightarrow{G} L = \partial_t + A(t).$$

To prove the theorem it is now sufficient to show that $A(t)u = \hat{A}u$ for every $u \in V$ and almost all $t \in [0,T]$. To do this we consider the following identity

$$\varepsilon\partial_t w_\delta(t/\varepsilon) + A(t/\varepsilon)(u + \varepsilon w_\delta(t/\varepsilon)) = \hat{A}u + \varphi_{\varepsilon,\delta} + \psi_{\varepsilon,\delta}, \tag{1.3.32}$$

where

$$\varphi_{\varepsilon,\delta} = \varepsilon\partial_t w_\delta(t/\varepsilon) + A(t/\varepsilon)u - \hat{A}u \tag{1.3.33}$$

and

$$\psi_{\varepsilon,\delta} = A(t/\varepsilon)(u + \varepsilon w_\delta(t/\varepsilon)) - A(t/\varepsilon)u. \tag{1.3.34}$$

The function w_δ will be specified later on.

Consider ∂_τ as a closed linear operator from $\mathcal{V}_{per}$ into $\mathcal{V}^*_{per}$ with the domain $\mathcal{W}_{per}$. Since the kernel of ∂_τ is trivial, its image is dense in $\mathcal{V}^*_{per}$. Therefore, for any $\delta > 0$ there exist $w_\delta(\tau) \in \mathcal{W}_{per}$, $b^\delta(\tau) \in \mathcal{V}^*_{per}$ and $c^\delta(\tau) \in \mathcal{V}^*_{per}$ such that

$$\hat{A}u - A(\tau)u = b^\delta(\tau) - c^\delta(\tau),$$

$$\partial_\tau w_\delta = b^\delta,$$

and

$$\|c^\delta\|_{\mathcal{V}^*_{per}} \le \delta.$$

Now we have

$$\begin{aligned} \|w_\delta(t/\varepsilon)\|^p_{\mathcal{V}} &= \int_0^T \|w_\delta(t/\varepsilon)\|^p_V dt = \int_0^{T/\varepsilon} \varepsilon\|w_\delta(\tau)\|^p_V d\tau \le \\ &\le \varepsilon\left([T/\varepsilon]+1\right)\int_0^1 \|wq_\delta(\tau)\|^p_V d\tau \le \\ &\le T_1\|w_\delta(\tau)\|^p_{\mathcal{V}_{per}}, \end{aligned} \tag{1.3.35}$$

where $[\alpha]$ is the integer part of α. In the similar way, equation

$$\partial_\tau w_\delta = b^\delta$$

implies

$$\|\partial_t[\varepsilon w_\delta(t/\varepsilon)]\|^{p'}_{\mathcal{V}'} \le T_2\|b^\delta(\tau)\|_{\mathcal{V}'_{per}} \le C.$$

Thus, for any fixed $\delta > 0$ the sequence $(\varepsilon w_\delta(t/\varepsilon))$ is bounded in $\mathcal{W}$. By (1.3.35), $\varepsilon w_\delta(t/\varepsilon) \to 0$ strongly in $\mathcal{V}$. Hence, $\varepsilon w_\delta(t/\varepsilon) \to 0$ weakly in $\mathcal{W}$. Moreover, inequality (1.3.6) implies that $\psi_{\varepsilon,\delta} \to 0$ in $\mathcal{V}^*$. Evidently, we have $\varphi_{\varepsilon,\delta} = c^\delta(t/\varepsilon)$. Hence, $\|\varphi_{\varepsilon,\delta}\|_{\mathcal{V}^*} \to 0$, as $\delta \to 0$, uniformly with respect to ε.

Now for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) \to 0$ such that $\varepsilon w_{\delta(\varepsilon)}(t/\varepsilon) \to 0$ weakly in $\mathcal{W}$, while $\varphi_{\varepsilon,\delta(\varepsilon)} \to 0$ and $\psi_{\varepsilon,\delta(\varepsilon)} \to 0$ strongly in $\mathcal{V}^*$ as $\varepsilon \to 0$. By Theorem 1.3.3, we have, using (1.3.32) – (1.3.34),

$$\partial_t 0 + A(t)u = \hat{A}u.$$

Hence $A(t)u = \hat{A}u$, and the proof is complete. □

Comments

Detailed accounts of the theory of monotone operators, convex analysis, and multivalued analysis may be found in [32, 33, 43, 75, 147]. As for Kuratowski convergence, we refer to [113] and [194].

In the case of linear invertible operators the notion of G-convergence was introduced by S. Spagnolo [258, 259]. General definition was given by A. Ambrosetti and C. Sbordone [10]. There are well-known results on G-compactness of various classes of abstract operators (see, e.g., [10, 266, 164]). Our presentation seems to be new, although many results we discuss here are hidden in various papers on G-convergence of differential operators ([98, 221, 226] and others).

The results of n^0 1.3.1 are well-known (see, e.g., [200]). G-convergence of linear abstract parabolic operators was investigated in [267]. The results on G-convergence of nonlinear abstract parabolic operators presented here are taken from [191, 192] (see, also, [207, 208]). As for time homogenization, or averating, of abstract parabolic operators see also [199, 246, 247, 265]. Those results supply more information, but for semilinear operators only.

CHAPTER 2

Strong G-convergence of Nonlinear Elliptic Operators

2.1 Nonlinear Elliptic Operators

2.1.1 Measurable Multivalued Functions

Let $(X, \mathcal{T})$ be a measurable space, i.e. X is a set and $\mathcal{T}$ is a σ-field of subsets of X, $F : X \longrightarrow \mathbf{R}^n$ a multivalued map with non-empty values. For any subset $B \subset \mathbf{R}^n$ we define the inverse image of B under F as

$$F^{-1}(B) = \{x \in X : B \cap Fx \neq \emptyset\}.$$

Recall that a selection, σ, of F is a map $\sigma : X \longrightarrow \mathbf{R}^n$ such that $\sigma(x) \in Fx$ for every $x \in X$.

Denote by $\mathcal{B}(\mathbf{R}^n)$ the σ-field of all Borel subset of $\mathbf{R}^n$. We recall that the tensor product, $\mathcal{T}_1 \otimes \mathcal{T}_2$, of two σ-fields $\mathcal{T}_1$ and $\mathcal{T}_2$ is the σ-field generated by all sets of the form $B_1 \times B_2$, where $B_1 \in \mathcal{T}_1$ and $B_2 \in \mathcal{T}_2$.

To understand what does it mean measurable multivalued function, we consider the following measurability conditions:

(1) $F^{-1}(B) \in \mathcal{T}$ *for any* $B \in \mathcal{B}(\mathbf{R})^n$;

(2) $F^{-1}(C) \in \mathcal{T}$ *for any closed subset* $C \subset \mathbf{R}^n$;

(3) $F^{-1}(U) \in \mathcal{T}$ *for any open subset* $U \subset \mathbf{R}^n$;

(4) *there exists a countable family* (σ_k) *of measurable selections such that*

$$F(x) = \mathrm{cl}\{\sigma_k(x) : k \in \mathbf{N}\}$$

for every $x \in X$;

A. Pankov, *G-Convergence and Homogenization of Nonlinear Partial Differential Operators*

(5) $\mathrm{gr}(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$.

A multivalued function $F : X \longrightarrow \mathbf{R}^n$ is said to be *measurable* (with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbf{R}^n)$) if condition (2) above is fulfilled. The following theorem describes the interrelations between conditions (1) – (5).

Theorem 2.1.1 *Let $(X, \mathcal{T})$ be a measurable space, $F : X \longrightarrow \mathbf{R}^n$ a multivalued map with non-empty closed values. Then*

(i) *(1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5);*

(ii) *if there exists a complete σ-finite measure μ defined on $\mathcal{T}$, then conditions (1) – (5) are equivalent.*

Additionaly, we have

Theorem 2.1.2 *Let $(X, \mathcal{T}, \mu)$ be a measurable space endowed with a complete σ-finite measure μ defined on $\mathcal{T}$. If*

$$G \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n),$$

then the projection $\mathrm{pr}_X G$ of G into X belongs to $\mathcal{T}$.

For the proof of these theorems see [86].

Later on the following result will be useful.

Theorem 2.1.3 *Let $(X, \mathcal{T}, \mu)$ be a measurable space, where μ is a complete σ-finite measure defined on $\mathcal{T}$, $F : X \longrightarrow \mathbf{R}^n \times \mathbf{R}^m$ a multivalued map with non-empty closed values, and $H : X \times \mathbf{R}^n \longrightarrow \mathbf{R}^m$ the multivalued map defined by the formula*

$$H(x, \xi) = \{\eta \in \mathbf{R}^m : (\xi, \eta) \in Fx\}. \tag{2.1.1}$$

Then the following statements are equivalent:

(i) *F is measurable with respect to $\mathcal{T}$ and $\mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$;*

(ii) $\mathrm{gr}(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$*;*

(iii) *H is measurable with respect to $\mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^m)$;*

(iv) $\mathrm{gr}(H) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m)$.

Proof. Theorem 2.1.1 implies easy that (i) $\Leftrightarrow$ (ii) and (iii) $\Rightarrow$ (iv). Evidently, $\mathrm{gr}(F) = \mathrm{gr}(H)$. Hence, (ii) $\Leftrightarrow$ (iv). So, to complete the proof we have to show that (ii) $\Rightarrow$ (iii). To do this it is sufficient to prove that (ii) implies

$$H^{-1}(C) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$$

for any compact subset $C \subset \mathbf{R}^m$. For such a subset C we have, by (2.1.1),

$$H^{-1}(C) = \{(x,\xi) \in X \times \mathbf{R}^n : \exists \eta \in \mathbf{R}^m, (\xi,\eta) \in Fx \cap (\mathbf{R}^n \times C)\}. \tag{2.1.2}$$

Let

$$B = \{x \in X : Tx \cap (\mathbf{R}^n \times C) \neq \emptyset\}.$$

Theorem 2.1.1 and (ii) imply that $B \in \mathcal{T}$.

Define the multivalued function $\Phi : X \longrightarrow \mathbf{R}^n \times \mathbf{R}^m$ by the formula

$$\Phi x = Fx \cap (\mathbf{R}^n \times C).$$

Evidently, $D(\Phi) = B$ and (2.1.2) becomes

$$H^{-1}(C) = \{(x,\xi) \in X \times \mathbf{R}^n : \exists \eta \in \mathbf{R}^m, (\xi,\eta) \in \Phi x.\} \tag{2.1.3}$$

We have

$$\mathrm{gr}(\Phi) = \mathrm{gr}(F) \cap (X \times \mathbf{R}^n \times C) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^m).$$

Hence, by Theorem 2.1.1, there exists a countable family (φ_k, g_k) of measurable functions from B into $\mathbf{R}^n \times \mathbf{R}^m$ such that

$$\Phi x = \mathrm{cl}\{(\varphi_k(x), g_k(x)) : k \in \mathbf{N}\} \tag{2.1.4}$$

for every $x \in B$.

Consider the set

$$M = \{(x,\xi) \in X \times \mathbf{R}^n : x \in B, \xi \in \mathrm{cl}\{\varphi_k(x) : k \in \mathbf{N}\}\}. \tag{2.1.5}$$

From (2.1.3), (2.1.4) and (2.1.5) we conclude easy that $H^{-1}(C) \subset M$. To prove that $M \subset H^{-1}(C)$, consider any fixed $(x,\xi) \in M$. By (2.1.5), there exists a subsequence $(\varphi_{\sigma(k)})$ such that $\xi = \lim \varphi_{\sigma(k)}(x)$. Since C is a compact set and $g_k(x) \in C$, we may assume (passing to a further subsequence if necessary) that $(g_{\sigma(k)}(x))$ converges to some $\eta \in \mathbf{R}^m$. Then (2.1.4) implies that $(\xi,\eta) \in \Phi(x)$. Hence, $(x,\xi) \in H^{-1}(C)$. Therefore, we have proved that $M = H^{-1}(C)$.

Since we may rewrite (2.1.5) as

$$M = \{(x,\xi) \in X \times \mathbf{R}^n : x \in B, \inf_{k \in \mathbf{N}} |\xi - \varphi_k(x)| = 0\},$$

we have

$$M = H^{-1}(C) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^m)$$

and the theorem is proved. □

Now we recall the well-known Aumann-von Neuman theorem [86] on existence of measurable selections.

Theorem 2.1.4 *Let $(X, \mathcal{T}, \mu)$ be a measurable space endowed with a complete σ-finite measure μ, and $F : X \longrightarrow \mathbf{R}^n$ a multivalued map with non-empty values. If* $\mathrm{gr}(F) \in \mathcal{T} \otimes \mathcal{B}(\mathbf{R}^n)$*, then F has a measurable selection.*

For further applications we sketch briefly some results on so-called decomposable families of measurable functions. Here we restrict ourself to a particular case we shall use later on.

Let $Q \subset \mathbf{R}^n$ be a bounded open set and $\mathcal{L}(Q)$ the σ-field of all Lebesgue measurable subsets of Q. Denote by $\mathcal{F}$ the set of all measurable multivalued maps from Q into $\mathbf{R}^n \times \mathbf{R}^n$ having non-empty closed values. Let $1 < p < \infty$ and $1/p + 1/p' = 1$. We set

$$\mathcal{S}_F^p = \{f \in (\mathrm{L}^p(Q))^n \times (\mathrm{L}^{p'}(Q))^n : f(x) \in Fx \text{ a.e. on } Q\},$$

i.e. $\mathcal{S}_F^p$ is the set of all $(\mathrm{L}^p(Q))^n \times (\mathrm{L}^{p'}(Q))^n$-selections of F.

The following result on the so-called Castaign representation holds true.

Lemma 2.1.1 *Let $F \in \mathcal{F}$. If $\mathcal{S}_F^p \neq \emptyset$, then there exists a countable family $(f_k) \subset \mathcal{S}_F^p$ such that*

$$Fx = \mathrm{cl}\{f_k(x) : k \in \mathbf{N}\}$$

for any $x \in Q$.

Lemma 2.1.2 *Let $F_1, F_2 \in \mathcal{F}$. If $\mathcal{S}_{F_1}^p = \mathcal{S}_{F_2}^p \neq \emptyset$, then $F_1x = F_2x$ a.e. on Q.*

Now let M be a set consisting of single-valued measurable functions $f : Q \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$. The set M is called *descomposable* (with respect to $\mathcal{L}(Q)$) if for any f_1, $f_2 \in M$ and $U \in \mathcal{L}(Q)$ we have

$$1_U f_1 + 1_{Q \setminus U} f_2 \in M,$$

where 1_U and $1_{Q\setminus U}$ stand for the characteristic functions of U and $Q \setminus U$, respectively. It turns out that any closed decomposable set may be characterized as a set of selection for an appropriate multivalued map.

Theorem 2.1.5 *Let M be a non-empty closed subset of $(L^p(Q))^n \times (L^{p'}(Q))^n$. The set M is decomposable if and only if there exists $F \in \mathcal{F}$ such that $M = \mathcal{S}_F^p$.*

For the proofs of the last three statements we refer to [161].

2.1.2 Multivalued Monotone Elliptic Operators

We will study a class of multivalued monotone operators of the form

$$-\operatorname{div} a(x, \nabla u)$$

acting on appropriate Sobolev spaces. Here we denote by ∇u the gradient of u.

To be more precise, let us fix a real number $p \in (1, +\infty)$. By p' we denote the dual exponent, $1/p + 1/p' = 1$. Let Q be a bounded open subset of $\mathbf{R}^n$. We fix two non-negative functions $m_1, m_2 \in \mathrm{L}^1(Q)$ and two constants $c_1 > 0$, $c_2 > 0$. We denote by $|\cdot|$ and by $\cdot$ the Euclidian norm and the inner product in $\mathbf{R}^n$, respectively.

Definition 2.1.1 *Denote by M_Q the set of all multivalued functions*

$$a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

which have closed values and satisfy the folllowing conditions:

(i) *for almost all $x \in Q$, the function $a(x, \cdot) : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is maximal monotone;*

(ii) *a is measurable with respect to $\mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n)$ and $\mathcal{B}(\mathbf{R}^n)$, i.e.*

$$a^{-1}(C) = \{(x, \xi) \in Q \times \mathbf{R}^n : a(x, \xi) \cap C \neq \emptyset\} \in \mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n)$$

for any closed subset $C \subset \mathbf{R}^n$;

(iii) *for almost all $x \in Q$, the inequalities*

$$|\eta|^{p'} \leq m_1(x) + c_1 \eta \cdot \xi, \tag{2.1.6}$$

$$|\xi|^p \leq m_2(x) + c_2 \eta \cdot \xi \tag{2.1.7}$$

hold true for any $\xi \in \mathbf{R}^n$ and $\eta \in a(x, \xi)$.

Evidently, inequalities (2.1.6) and (2.1.7) imply the existence of functions $m_3 \in \mathrm{L}^{p'}(Q)$ and $m_4 \in \mathrm{L}^1(Q)$, and two constants $c_3 > 0$, $c_4 > 0$ such that for almost all $x \in Q$

$$|\eta| \leq m_3(x) + c_3 |\xi|^{p-1}, \tag{2.1.8}$$

$$\eta \cdot \xi \geq m_4(x) + c_4 |\xi|^p \tag{2.1.9}$$

for any $\xi \in \mathbf{R}^n$ and $\eta \in a(x, \xi)$. Conversely, (2.1.8) and (2.1.9) imply inequalities (2.1.6) and (2.1.7), with appropriate m_1, m_2, c_1 and c_2.

Notice that the set $a(x,\xi)$ is closed and convex for almost all $x \in Q$ and for any $\xi \in \mathbf{R}^n$. Moreover, by Theorem 2.1.1,

$$\mathrm{gr}(a) \in \mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^N).$$

Since, by (2.1.8), the maximal monotone operator $a(x,\cdot)$ is locally bounded for almost all $x \in Q$, we have $a(x,\xi) \neq \emptyset$ for almost all $x \in Q$ and for any $\xi \in \mathbf{R}^N$.

Now we describe the functional spaces we will use. Denote by $\overline{V} = W^{1,p}(Q)$ the Sobolev space consisting of all functions $u \in \mathrm{L}^p(Q)$ such that he first order distributional derivatives of u belongs to $\mathrm{L}^p(Q)$. Endowed with the norm

$$\|u\|_{\overline{V}} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{1/p},$$

$\overline{V}$ is a separable reflexive Banach space. By $V = W_0^{1,p}(Q)$ we denote the closure of the space $C_0^\infty(Q)$ in $\overline{V}$. This space is separable and reflexive as well. The space V will be considered with the following equivalent norm

$$\|u\|_V = \|\nabla u\|_p.$$

The dual, V^*, to V is the negative Sobolev space $W^{-1,p'}(Q)$. The duality pairing on $V^* \times V$ will be denoted by $(\cdot,\cdot)$, as well as the pairing on $\mathrm{L}^{p'}(Q)^n \times \mathrm{L}^p(Q)^n$. The same notation is used for members of Cartesian products, but no confusion may occur.

For $a \in M_Q$, we consider the following Dirichlet boundary value problem

$$-\operatorname{div} a(x,\nabla u) \ni f, \quad u \in W_0^{1,p}(Q), \tag{2.1.10}$$

where $f \in W^{-1,p'}(Q)$ is a given function. Also one can consider the inhomogeneous Dirichlet problem

$$-\operatorname{div} a(x,\nabla u) \ni f, \quad u \in W^{1,p}(Q), \quad u - \varphi \in W_0^{1,p}(Q),$$

where $f \in W^{-1,p'}(Q)$ and $\varphi \in W^{1,p}(Q)$ are given functitons. It is useful to introduce an operator framework for these problems.

Definition 2.1.2 *Denote by M_V (resp. $M_{\overline{V}}$) the set of all multivalued operators $A : V \longrightarrow L^{p'}(Q)^n$ (resp. $A : \overline{V} \longrightarrow L^{p'}(Q)^n$) satisfying the following conditions:*

(i) *for any $u_i \in V$ (resp. $\overline{V}$) and $g_i \in Au_i$, $i = 1,2$, we have*

$$(\nabla u_1 - \nabla u_2) \cdot (g_1 - g_2) \geq 0 \quad \textit{a.e. on } Q;$$

(ii) *for any $u \in V$ (resp. $\overline{V}$) and $g \in Au$, the inequalities*

$$|g|^{p'} \leq m_1 + c_1 g \cdot \nabla u, \tag{2.1.11}$$

$$|\nabla u|^p \leq m_2 + c_1 g \cdot \nabla u \tag{2.1.12}$$

hold true a.e. on Q.

Definition 2.1.3 *By $\hat{\mathcal{M}}_V$ (resp. $\hat{\mathcal{M}}_{\overline{V}}$) we denote the set of all multivalued operators $\mathcal{A} : V \longrightarrow V^*$ (resp. $\mathcal{A} : \overline{V} \longrightarrow V^*$) of the form*

$$\mathcal{A}u = \{-\operatorname{div} g : g \in Au\}, \tag{2.1.13}$$

where $A \in M_V$ (resp. $A \in M_{\overline{V}}$). By $\mathcal{M}_V$ we denote the set of all maximal monotone operators from $\hat{\mathcal{M}}_V$.

Any operator of the class $\hat{\mathcal{M}}_V$ is monotone, as it follows from Definition 2.1.2 (i). If $\mathcal{A} \in \mathcal{M}_V$, i.e. $\mathcal{A}$ is a maximal monotone operator, then $D(\mathcal{A}) = V$. This follows from the local boundedness of $\mathcal{A}$ (estimate (2.1.11)).

Definition 2.1.4 *For any $a \in M_Q$ we denote by $A \in M_{\overline{V}}$ and $\mathcal{A} \in \mathcal{M}_{\overline{V}}$ the associated operators defined by*

$$Au = \{g \in L^{p'}(Q)^n : g(x) \in a(x, \nabla u(x)) \text{ a.e. on } Q\},$$

$$\mathcal{A}u = \{-\operatorname{div} g : g \in Au\}.$$

For any $\varphi \in \overline{V}$, we define the operators $A^\varphi \in M_V$ and $\mathcal{A}^\varphi \in \hat{\mathcal{M}}_V$ by

$$A^\varphi u = A(\varphi + u),$$

$$\mathcal{A}^\varphi u = \mathcal{A}(\varphi + u).$$

Now problem (2.1.10) may be rewriten as follows: given $f \in V^*$ find $u \in V$ such that

$$\mathcal{A}^0 u \ni f,$$

or, equivalently, find $u \in V$ and $g \in \mathrm{L}^{p'}(Q)^n$ such that

$$g \in A^0 u, \quad -\operatorname{div} g = f.$$

In the similar way the operators A^φ and $\mathcal{A}^\varphi$ serve the inhomogeneous Dirichlet problem with the boundary data $\varphi \in \overline{V}$.

In the case $\varphi = 0$ we shall sometimes omit the superscript '0' in the notations A^0 and $\mathcal{A}^0$ (of course, if no confusion may occur). Thus, we do not distinguish by notations operators defined on $\overline{V}$ and their restrictions to V.

Let us denote by J the duality operator from $\mathrm{L}^p(Q)$ into $\mathrm{L}^{p'}(Q)$ defined by

$$Ju = |u|^{p-2}u.$$

We can also consider J as an operator acting from V into V^*.

The following theorem contains the main solvability result for problem (2.1.10) and some additional information we need later on. Certainly, under the same assumptions the corresponding inhomogeneous Dirichlet problem is solvable as well.

Theorem 2.1.6 *Let $a \in M_Q$ and $\mathcal{A} \in \hat{\mathcal{M}}_V$ be the associated operator. Then*

(i) *$\mathcal{A}$ is a maximal monotone operator, i.e $\mathcal{A} \in \mathcal{M}_V$;*

(ii) *$R(\mathcal{A} + \lambda J) = V^*$ for every $\lambda \geq 0$.*

Proof. To prove (i) we show that the operator $\mathcal{A}$ satisfies the assumptions of Theorem 1.1.1.

First of all, notice that $\mathcal{A}u \neq \emptyset$ for any $u \in V$. As we have pointed out after Definition 2.1.1, the set $a(x, \nabla u(x))$ is a non-empty closed and convex subset of $\mathbf{R}^n$ for almost all $x \in Q$. Moreover, it is easy to see that $x \mapsto a(x, \nabla u(x))$ is a measurable multivalued map from Q into $\mathbf{R}^n$. By Theorem 2.1.1, there exists a measurable selection $g : Q \longrightarrow \mathbf{R}^n$ of this map, i.e.

$$g(x) \in a(x, \nabla u(x))$$

a.e. on Q. Estimate (2.1.8) implies that $g \in \mathrm{L}^{p'}(Q)^n$ and

$$-\operatorname{div} g \in \mathcal{A}u.$$

Since $a(x, \nabla u(x))$ is a convex subset of $\mathbf{R}^n$ for almost all $x \in Q$, we see that $\mathcal{A}u$ is a convex subset of V^*.

Now we prove that $\mathcal{A}u$ is weakly closed in V^*, for any $u \in V$, and $\mathcal{A}$ is an upper semicontinuous map from V, with the strong topology, into V^*, with the weak topology. To do this, by (2.1.8), it is sufficient to prove the following assertion: if $u_k \to u$ strongly in V, $f_k \to f$ weakly in V^*, and $f_k \in \mathcal{A}u_k$ for any $k \in \mathbf{N}$, then $f \in \mathcal{A}u$. For such u_k, u, f_k and f, the definition of $\mathcal{A}$ and inequality (2.1.8) imply the existence of functions $g_k \in \mathrm{L}^{p'}(Q)^n$ and $g \in \mathrm{L}^p(Q)^n$ such that (up to a subsequence) $g_k \to g$ weakly in $\mathrm{L}^{p'}(Q)^n$,

$$g_k(x) \in a(x, \nabla u_k(x)) \quad \text{a.e. on } Q,$$

$$-\operatorname{div} g_k = f_k,$$

and

$$-\operatorname{div} g = f.$$

Therefore, we have to verify that

$$g(x) \in a(x, \nabla u(x)) \quad \text{a.e. on } Q.$$

In view of monotonicity of a, to do this it is enough to show that the Lebesgue measure, $|Y|$, of the set

$$Y = \{x \in Q : \exists \xi \in \mathbf{R}^n, \exists \eta \in a(x, \xi),\ (g(x) - \eta) \cdot (\nabla u(x) - \xi) < 0\}$$

is equal to zero. To prove this statement we observe that

$$Y = \{x \in Q : Gx \neq \emptyset\},$$

where

$$Gx = \{(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n : \eta \in a(x, \xi),\ (g(x) - \eta) \cdot (\nabla u(x) - \xi) < 0\}.$$

Since

$$\mathrm{gr}(a) \in \mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n),$$

the same is true for $\mathrm{gr}(G)$. Hence, by Theorem 2.1.2, $Y \in \mathcal{L}(Q)$. Now the Aumann-von Neuman Theorem 2.1.4 garantees the existence of a measurable selection $(\xi(x), \eta(x))$ of G defined on Y. Thus, we have $\eta(x) \in a(x, \xi(x))$ and

$$(g(x) - \eta(x)) \cdot (\nabla u(x) - \xi(x)) < 0, \quad x \in Y. \tag{2.1.14}$$

On the other hand, by the monotonicity assumption for a, we have

$$(g_k(x) - \eta(x)) \cdot (\nabla u_k(x) - \xi(x)) \geq 0 \quad \text{a.e. on } Y \tag{2.1.15}$$

for any $k \in \mathbf{N}$. If $|Y| > 0$, then there exists a measurable subset $Y' \subset Y$, with $|Y'| > 0$, such that $(\xi(x), \eta(x))$ is bounded on Y'. Integrating (2.1.15) over Y' and passing to the limit we get

$$\int_{Y'} (g(x) - \eta(x)) \cdot (\nabla u(x) - \xi(x)) dx \geq 0.$$

Since $|Y'| > 0$, this contradicts (2.1.14). Hence, $|Y| = 0$ and we complete the proof of statement (i).

To prove (ii) we observe that both $\mathcal{A}$ and J are maximal monotone operators and $D(\mathcal{A}) = D(J) = V$. By Theorem 1.1.3, the operator $\mathcal{A} + \lambda J$ is maximal monotone for every $\lambda \geq 0$. Inequality (2.1.9) implies its coerciveness. Hence, $R(\mathcal{A} + \lambda J) = V^*$, by Theorem 1.1.4. The proof is complete. □

Remark 2.1.1 Of course, the statement of Theorem 2.1.6 is still valid with $\mathcal{A}$ replaced by $\mathcal{A}^{\varphi}$ for any $\varphi \in \overline{V}$. This implies the solvability of corresponding nonhomogeneous Dirichlet problem with the right hand part $f \in V^*$ and the boundary data $\varphi \in \overline{V}$.

It turns out to be that all maximal monotone operators of the class $\mathcal{M}_V$ have an explicite description. To prove the corresponding result we need the following

Lemma 2.1.3 **(i)** *Let $\psi \in L^{\alpha}_{loc}(\mathbf{R}^n)$, $1 \leq \alpha \leq \infty$, be a function which is 1-periodic in each its variable. Then $\psi(\varepsilon^{-1}x) \to \langle\psi\rangle$ as $\varepsilon \to 0$ weakly in $L^{\alpha}(Q)$ if $1 \leq \alpha < \infty$, and $*$-weakly if $\alpha = \infty$, where*

$$\langle\psi\rangle = \int_K \psi(y) dy,$$

K is a unit cube in $\mathbf{R}^n$.

(ii) *For any* $t \in (0,1)$ *there exists a sequence* (Q_k) *of subsets of* Q *such that* $1_{Q_k} \to t \cdot 1_Q$ *$*$-weakly in* $L^\infty(Q)$.

Proof. Statement (i) is well-known (see, e.g., [164]). To prove (ii), let us consider any subset $S \subset K$, $|S| = t$, and define ψ to be a 1-periodic function which coincides with 1_S on K. Then $\psi(kx)$ restricted to Q is the characteristic function of a subset Q_k. Now (ii) follows directly from (i). □

Theorem 2.1.7 *Any operator of the class* $\mathcal{M}_V$ *is associated to a function* $a \in M_Q$ *according to Definition 2.1.4.*

As we pointed out earlier, the domain of any operator of the class $\mathcal{M}_V$ coincides with the whole space V. Therefore, Theorem 2.1.7 is a consequence of the following

Proposition 2.1.1 *Let* $B \in M_{\overline{V}}$ *and* $D(B) \supset C_0^\infty(Q)$. *Then there exists a unique multivalued function* $a \in M_Q$ *such that* $B \subset A$, *where* $A \in M_{\overline{V}}$ *is the operator associated to* a.

Proof. Let us define a subset E of $\mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$ by

$$E = \{(\nabla u, g) \in \mathrm{L}(Q)^p \times \mathrm{L}^{p'}(Q)^n : u \in \overline{V}, g \in Bu\}.$$

Then $E \neq \emptyset$ and satisfies the following monotonicity condition:

$$if\ (\varphi_i, g_i) \in E,\ i = 1, 2,\ then\ (g_1 - g_2) \cdot (\varphi_1 - \varphi_2) \geq 0\ a.e.\ on\ Q. \tag{2.1.16}$$

Moreover, for any $(\varphi, g) \in E$ we have

$$|g|^{p'} \leq m_1 + c_1 g \cdot \varphi \quad \text{a.e. on } Q, \tag{2.1.17}$$

$$|\varphi|^p \leq m_2 + c_2 g \cdot \varphi \quad \text{a.e. on } Q. \tag{2.1.18}$$

Let us introduce the set $\operatorname{dec}(E)$ being the smallest decomposable set containing E. It is not difficult to see that $(\varphi, g) \in \operatorname{dec}(E)$ if and only if there exists a finite Borel partition (Q_i) of Q and a finite family $\{(\varphi_i, g_i)\} \subset E$ such that $(\varphi, g) = (\varphi_i, g_i)$ a.e. on Q_i. It is easy that $\operatorname{dec}(E) \neq \emptyset$, and conditions (2.1.16), (2.1.17), and (2.1.18), hold true with E replaced by $\operatorname{dec}(E)$.

Now let us define the set

$$\tilde{E} = \mathrm{cl}_{s\times w}(\operatorname{dec}(E)),$$

the closure of $\operatorname{dec}(E)$ in $\mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$, where $\mathrm{L}^p(Q)^n$ is considered with its strong topology, while $\mathrm{L}^{p'}(Q)$ is considered with the weak one. The set $\tilde{E}$ possesses the following properties:

(a) *for any* $(\varphi, g) \in \tilde{E}$ *there exists a sequence* $(\varphi_k, g_k) \in \operatorname{dec}(E)$ *such that* $\varphi_k \to \varphi$ *strongly in* $\mathrm{L}^p(Q)^n$ *and* $g_k \to g$ *weakly in* $\mathrm{L}^{p'}(Q)^n$;

(b) $\tilde{E}$ *is decomposable and* (2.1.16), (2.1.17), (2.1.18) *hold with* E *replaced by* $\tilde{E}$;

(c) $\tilde{E}$ *is a maximal monotone set.*

We postpone the proofs of these properties.

Since $\tilde{E}$ is non-empty, closed, and decomposable, Theorem 2.1.5 implies the existence of a measurable multivalued function $F : Q \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$, having non-empty closed values, such that

$$\tilde{E} = \{(\varphi, g) \in \mathrm{L}^p(Q)^N \times \mathrm{L}^{p'}(Q)^n : (\varphi(x), g(x)) \in Fx \text{ a.e. on } Q\}. \tag{2.1.19}$$

Now we define the multivalued function $a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ by the formula

$$a(x, \xi) = \{\eta \in \mathbf{R}^n : (\xi, \eta) \in Fx\}. \tag{2.1.20}$$

Later on we shall prove that $a \in M_Q$ (see Lemma 2.1.4). The definition of E, (2.1.19), and (2.1.20), imply easy that $B \subset A$, where A stands for the operator associated to a. The uniqueness of a will be proved separately, in Lemma 2.1.5.

We conclude with the proofs of properties (a) – (c) of $\tilde{E}$. Let us start with (a). Suppose $(\varphi_0, g_0) \in \tilde{E}$, and $\mathcal{U}_1$ the unit ball in $\mathrm{L}^p(Q)^n$ centered at φ_0. By (2.1.17) (which is valid for $\operatorname{dec}(E)$), there exists a constant $R > 0$ such that $(\varphi, g) \in \operatorname{dec}(E)$ and $\varphi \in \mathcal{U}_1$ imply that $g \in \mathcal{B}_R$, the ball of radius R in $\mathrm{L}^{p'}(Q)^n$ centered at 0. We may also assume that $g_0 \in \mathcal{B}_R$. Hence,

$$\operatorname{dec}(E) \cap (\mathcal{U} \times (\mathcal{V} \cap \mathcal{B}_R)) = \operatorname{dec}(E) \cap (\mathcal{U} \times \mathcal{V}) \neq \emptyset \tag{2.1.21}$$

for any weak neighborhood $\mathcal{V}$ of g_0 in $\mathrm{L}^{p'}(Q)^n$ and for any strong neighborhood $\mathcal{U} \subset \mathcal{U}_1$ of φ_0.

It is well-known that the weak topology is metrizable on $\mathcal{B}_R$. Therefore, there is a countable base, $(\mathcal{V}_k)$, of neighborhoods of g_0 in $\mathcal{B}_R$ endowed with the weak topology. Without loss of generality we may assume that $\mathcal{V}_{k+1} \subset \mathcal{V}_k$, $k \in \mathbf{N}$. Denote by $\mathcal{U}_k \subset \mathrm{L}^p(Q)^n$ the ball of the radius $1/k$ centered at φ_0. By (2.1.21),

$$\operatorname{dec}(E) \cap (\mathcal{U}_k \times \mathcal{V}_k) \neq \emptyset.$$

Let (φ_k, g_k) be a member of this set. Then it is obvious that $\varphi_k \to \varphi_0$ strongly in $\mathrm{L}^p(Q)^n$ and $g_k \to g_0$ weakly in $\mathrm{L}^{p'}(Q)^n$.

Property (b) follows in a straitforward way from the similar property of $\operatorname{dec}(E)$ and (a).

To prove (c) we consider $\tilde{E}$ as a multivalued operator indetified with its graph and apply Theorem 1.1.1. First of all, we prove that $\tilde{E}(\varphi) \neq \emptyset$ for any $\varphi \in \mathrm{L}^p(Q)^n$. In the case of piecewise constant compactly suppoted φ, the last follows directly from the assumption $D(B) \supset C_0^\infty(Q)$ and the definition of $\operatorname{dec}(E)$. The general case can be covered by means of approximation, with respect to the strong topology, of $\varphi \in \mathrm{L}^p(Q)^n$ by functions φ_k of the previous kind. In fact, such an approximation procedure gives rise to a family of functions $g_k \in \mathrm{L}^{p'}(Q)^n$ such that $g_k \in \tilde{E}(\varphi_k)$. Then estimate (2.1.17) for $\tilde{E}$ (see property (b)) garantees that (g_k) is bounded in $\mathrm{L}^{p'}(Q)^n$. Hence, we may assume (passing to a subsequence if necessary) that $g_k \to g$ weakly in $\mathrm{L}^{p'}(Q)^n$. Now it is easy that $g \in \tilde{E}(\varphi)$ and $\tilde{E}(\varphi) \neq \emptyset$.

Evidently, the set $\tilde{E}(\varphi)$ is decomposable and weakly closed in $\mathrm{L}^{p'}(Q)^n$ for any $\varphi \in \mathrm{L}^p(Q)^n$. To prove that $\tilde{E}(\varphi)$ is convex, let us fix $g_1, g_2 \in \tilde{E}(\varphi)$. By Lemma 2.1.3 (ii), there exists a sequence of subsets (Q_k) such that $1_{Q_k} \to t1_Q$ $*$-weakly in $\mathrm{L}^\infty(Q)$, where $t \in (0,1)$. Then $1_{Q\setminus Q_k} \to (1-t)1_Q$ $*$-weakly. Decomposability of $\tilde{E}$ implies that

$$1_{Q_k}g_1 + 1_{Q\setminus Q_k}g_2 \in \tilde{E}(\varphi).$$

Since $\tilde{E}(\varphi)$ is weakly closed in $\mathrm{L}^{p'}(Q)^n$, we can pass to the limit. Hence,

$$tg_1 + (1-t)g_2 \in \tilde{E}(\varphi)$$

and $\tilde{E}(\varphi)$ is convex.

Finally, we prove that $\tilde{E}$ is an upper semi-continuous multivalied map from $\mathrm{L}^p(Q)^n$, with the strong topology, into $\mathrm{L}^{p'}(Q)^n$, with the weak topology. Given $\varphi \in \mathrm{L}^p(Q)^n$ let $\mathcal{V}$ be an open neighborhood of $\tilde{E}(\varphi)$ in the weak topology. We have to prove that, for any sequence φ_k converging to φ strongly in $\mathrm{L}^p(Q)^n$, there exists $k_0 \in \mathbf{N}$ such that $\tilde{E}(\varphi_k) \subset \mathcal{V}$ for $k \geq k_0$. Assume the contrary. Then there are a subsequence $(\varphi_{\sigma(k)})$ and a sequence (g_k) such that $g_k \in \tilde{E}(\varphi_{\sigma(k)})$ and $g_k \notin \mathcal{V}$ for $k \in \mathbf{N}$. By estimate (2.1.17) for $\tilde{E}$ (see property (b)), the sequence (g_k) is bounded in $\mathrm{L}^{p'}(Q)^n$. Hence, we may assume (g_k) being convergent to g weakly in $\mathrm{L}^{p'}(Q)^n$. Since $\tilde{E}$ is weakly closed, we have $g \in \tilde{E}(\varphi)$, hence, $g \in \mathcal{V}$. Then $g_k \in \mathcal{V}$ for k large enough, which contradicts our assumption. Thus, $\tilde{E}$ is upper semi-continuous and we conclude the proof of (c). □

Now let us prove two statements we have left aside in the proof of Proposition 2.1.1. Thereby, we complete the proof of Proposition 2.1.1 and Theorem 2.1.7.

Lemma 2.1.4 *The function a defined by (2.1.20) belongs to M_Q.*

Proof. The measurability of F and Theorem 2.1.3 imply that a is measurable. Moreover, (2.1.16) for $\tilde{E}$ and the Castaign representation for F (Lemma 2.1.1) imply that Fx, hence $a(x,\cdot)$, is monotone a.e on Q.

Now let us prove maximal monotonicity of a. To do this it is sufficient to show that the set

$$\begin{aligned} Y = \{x \in Q : \exists(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n, \quad & (\xi,\eta) \notin Fx, \\ & (\xi-\xi')\cdot(\eta-\eta') \geq 0 \quad \forall(\xi',\eta') \in Fx\} \end{aligned}$$

is of zero Lebesgue measure. We have

$$Y = \{x \in Q : \Phi x \neq \emptyset\},$$

where

$$\Phi x = \{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n : (\xi,\eta) \notin Fx,\ (\xi - \xi')\cdot(\eta - \eta') \geq 0\ \forall(\xi',\eta') \in Fx\}.$$

Since $F \in \mathcal{F}$ and $\tilde{E} = \mathcal{S}_F^p \neq \emptyset$, Lemma 2.1.1 implies the existence of a sequence

$$(\varphi_k, g_k) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$$

such that

$$\begin{aligned}\Phi x &= \{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n : (\xi,\eta) \notin Fx,\ (\xi - \varphi_k(x))\cdot(\eta - g_k(x)) \geq 0 \\ &\qquad \forall k \in \mathbf{N}\} = \\ &= \bigcap_{k\in\mathbf{N}} \{(\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^n : (\xi,\eta) \notin Fx,\ (\xi - \varphi_k(x))\cdot(\eta - g_k(x)) \geq 0\}.\end{aligned}$$

It follows easily from measurability of F, φ_k, and g_k, that

$$\mathrm{gr}(\Phi) \in \mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n).$$

Therefore, by Theorem 2.1.2, $Y \in \mathcal{L}(Q)$. Now, by the Aumann-von Neimann Theorem 2.1.4, there is a measurable selection (φ_0, g_0) of Φ over Y. Hence,

$$(\varphi_0(x), g_0(x)) \notin Fx, \quad x \in Y, \tag{2.1.22}$$

and

$$(\varphi_0(x) - \xi)\cdot(g_0(x) - \eta) \geq 0 \quad \forall(\xi,\eta) \in Fx,\ x \in Y. \tag{2.1.23}$$

Assume that $|Y| > 0$. Then one can choose a measurable subset $Y' \subset Y$ such that $|Y'| > 0$ and $(\varphi_0(x), g_0(x))$ is bounded on Y'. Now let us fix $(\varphi_*, g_*) \in \tilde{E}$ and consider the functions $\overline{\varphi}$ and $\overline{g}$ defined by

$$\overline{\varphi}(x) = \begin{cases} \varphi_0(x) & \text{if } x \in Y', \\ \varphi_*(x) & \text{if } x \notin Y', \end{cases}$$

and

$$\overline{g}(x) = \begin{cases} g_0(x) & \text{if } x \in Y', \\ g_*(x) & \text{if } x \notin Y'. \end{cases}$$

Obviously, $(\overline{\varphi}, \overline{g}) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$. By (2.1.23) and (2.1.16) for $\tilde{E}$, we have

$$\begin{aligned}\int_Q (\overline{\varphi} - \varphi)\cdot(\overline{g} - g)dx &= \int_{Y'} (\varphi_0 - \varphi)\cdot(g_0 - g)dx + \\ &\quad + \int_{Q\setminus Y'} (\varphi_* - \varphi)\cdot(g_* - g)dx \geq 0\end{aligned}$$

for any $(\varphi, g) \in \tilde{E}$. Since $\tilde{E}$ is maximal monotone, this implies that $(\overline{\varphi}, \overline{g}) \in \tilde{E}$. Hence, $(\overline{\varphi}(x), \overline{g}(x)) \in Fx$ a.e on Q. However, then we have $(\varphi_0(x), g_0(x)) \in Fx$ for almost all $x \in Y'$. Since $|Y'| > 0$, the last contradicts (2.1.22). Therefore, $|Y| = 0$, and $a(x,\cdot)$ is a maximal monotone map a.e. on Q.

Inequalities (2.1.6) and (2.1.7) for the function a follow easily from the Castaign representation (Lemma 2.1.1) and inequalities (2.1.17) and (2.1.18) for $\tilde{E}$. □

Lemma 2.1.5 *Let C be an operator of the class $M_{\overline{V}}$ such that $\psi + C_0^\infty(Q) \subset D(C)$ for some $\psi \in \overline{V}$. Let a and b be two functions of the class M_Q, and $A, B \in M_{\overline{V}}$ the corresponding operators. Assume that $C \subset A$ and $C \subset B$. Then $a(x,\xi) = b(x,\xi)$ a.e. on Q for any $\xi \in \mathbf{R}^n$.*

Proof. Without loss of generality we may assume that $\psi = 0$. Now we define the subset $E \subset \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$ by

$$E = \{(\nabla u, g) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n : u \in \overline{V}, g \in Cu\}$$

(as in the proof of Proposition 2.1.1 with B replaced by C). Let

$$E_a = \{(\varphi, g) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n : g(x) \in a(x, \varphi(x)) \text{ a.e. on } Q\}.$$

Since $C \subset A$, we have $E \subset E_a$. The set E_a is decomposable. Hence, $\mathrm{dec}(E) \subset E_a$. It is not hard to see that E_a is a maximal monotone set. Therefore, it is sequentially closed in the space $\mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$ endowed with the product of strong and weak topologies, respectively. This implies that $\tilde{E} \subset E_a$. By the maximal monotonicity of $\tilde{E}$ (see property (c) of $\tilde{E}$ in the proof of Proposition 2.1.1), $\tilde{E} = E_a$.

In the similar way we may define E_b, with a replaced by b, and show that $\tilde{E} = E_b$. Then Lemma 2.1.2 implies that $a(x,\xi) = b(x,\xi)$ a.e. on Q for any $\xi \in \mathbf{R}^n$. □

2.1.3 Some Classes of Single-Valued Elliptic Operators

Now we discuss briefly some important subclasses of the class of operators we just have studed and then consider non-monotone elliptic operators.

Definition 2.1.5 *Given a non-negative function $m \in L^1(Q)$ and two constants $c > 0$ and α, with*

$$0 < \alpha \leq \min\left[\frac{p}{2}, p-1\right],$$

we denote by $U_Q = U_Q(\alpha, c, m)$ the set of all $a \in M_Q$ such that

$$m(x) + \eta_1 \cdot \xi_1 + \eta_2 \cdot \xi_2 \geq 0 \tag{2.1.24}$$

and

$$|\eta_1 - \eta_2| \leq c\Phi^{(p-1-\alpha)/p}[(\eta_1 - \eta_2)\cdot(\xi_1 - \xi_2)]^{\alpha/p} \tag{2.1.25}$$

a.e. on Q, for any ξ_1, $\xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x,\xi_1)$, $\eta_2 \in a(x,\xi_2)$, where $\Phi = \Phi(x,\xi_1,\xi_2,\eta_1,\eta_2)$ denotes the left hand part of (2.1.24).

Definition 2.1.6 *Given a non-negative function $m \in L^1(Q)$ and two constants $c > 0$ and β, with $\beta \geq \max[p, 2]$, denote by $S_Q = S_Q(\beta, c, m)$ the set of all $a \in M_Q$ satisfying (2.1.24) and*

$$(\eta_1 - \eta_2) \cdot (\xi_1 - \xi_2) \geq c\Phi^{(p-\beta)/p}|\xi_1 - \xi_2|^\beta \tag{2.1.26}$$

a.e. on Q, for any $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x, \xi_1)$, $\eta_2 \in a(x, \xi_2)$.

Using inequalities (2.1.6) and (2.1.7) it is easy to see that there exists a non-negative function $m \in \mathrm{L}^1(Q)$ such that inequality (2.1.24) holds for any $a \in M_Q$. Moreover, inequality (2.1.25) implies that any $a \in U_Q$ is single-valued.

Example 2.1.1 Suppose $b(x)$ is a measurable function on Q such that

$$0 < b_1 \leq b(x) \leq b_2 < +\infty \quad \text{a.e. on } Q,$$

and

$$a(x, \xi) = b(x)|\xi|^{p-2}\xi.$$

Then $a \in U_Q \cap S_Q$, with

$$\alpha = \min\left[\frac{p}{2}, p - 1\right], \qquad \beta = \max[p, 2],$$

and suitable c and m.

Definition 2.1.7 *Given a non-negative function $m \in L^p(Q)$ and two constants $c > 0$ and α, with*

$$0 < \alpha \leq \min[1, p - 1],$$

denote by $U_Q^* = U_Q^*(\alpha, c, m)$ the set of all single-valued function $a \in M_Q$ such that

$$|a(x, \xi_1) - a(x, \xi_2)| \leq c(m(x) + |\xi_1| + |\xi_2|)^{p-1-\alpha}|\xi_1 - \xi_2|^\alpha \tag{2.1.27}$$

a.e. on Q for any ξ_1, $\xi_2 \in \mathbf{R}^n$.

Definition 2.1.8 *Given a non-negative function $m \in L^p(Q)$ and two constants $c > 0$ and β, with $\beta \geq \max[p, 2]$, denote by $S_Q^* = S_Q^*(\beta, c, m)$ the set of all $a \in M_Q$ such that*

$$(\eta_1 - \eta_2) \cdot (\xi_1 - \xi_2) \geq c(m(x) + |\xi_1| + |\xi_2|)^{p-\beta}|\xi_1 - \xi_2|^\beta \tag{2.1.28}$$

a.e. on Q for any $\xi_1, \xi_2 \in \mathbf{R}^n$ and $\eta_1 \in a(x, \xi_1)$, $\eta_2 \in (a, \xi_2)$.

According to Definition 2.1.4, any function $a(x,\xi)$ from the classes we just have introduced generates an operator $\mathcal{A} : V \longrightarrow V^*$. Corresponding classes of such operators will be denoted by $\mathcal{U}_V$, $\mathcal{S}_V$, $\mathcal{U}_V^*$ and $\mathcal{S}_V^*$, respectively. We do not consider the subsets of the class M_V generated by the classes U_Q, S_Q, U_Q^* and S_Q^*. Notice that the classes of differential opęrators we consider are containėd in the corresponding classes of abstract monotone operators discussed in n^0 1.2.1.

Now we collect some basic propoties of the operator classes under consideration. The proofs are quite sinilar to that of Propositions 1.2.1 and 1.2.2.

Proposition 2.1.2 **(i)** *If*

$$0 < \alpha' \leq \alpha \leq \min\left[\frac{p}{2}, p-1\right],$$

then

$$U_Q(\alpha, c, m) \subset U_Q(\alpha', c', m'),$$

with suitable $c' > 0$ and $m'(x) \geq 0$.

(ii) *If*

$$\max[p, 2] \leq \beta \leq \beta',$$

then

$$S_Q(\beta, c, m) \subset S_Q(\beta', c', m'),$$

with suitable $c' > 0$ and $m'(x) \geq 0$.

Proposition 2.1.3 **(i)** *We have*

$$U_Q(\alpha, c, m) \subset U_Q^*(\alpha', c', m'),$$

with $\alpha' = \alpha(p-\alpha)^{-1}$ and appropriate $m'(x) \geq 0$ and $c' > 0$.

(ii) *Given c', c'', m' and m'' we have*

$$U_Q^*(\alpha, c', m') \cap S_Q^*(\beta, c'', m'') \subset U_Q(\alpha', c, m),$$

with $\alpha' = \alpha p/\beta$ and suitable $c > 0$ and $m(x) \geq 0$.

(iii) *We have*

$$S_Q(\beta, c, m) \subset S_Q^*(\beta, c', m') \subset S_Q(\beta, c'', m''),$$

with suitable $c', c'' > 0$ *and* $m'(x) \geq 0$, $m''(x) \geq 0$.

Later on we shall also consider non-monotone elliptic operators, but only single-valued ones. Let

$$a : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

and

$$a_0 : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

be two Carathéodory functions, i.e. $a(\cdot, \xi_0, \xi)$ and $a_0(\cdot, \xi_0', \xi)$ are measurable for any $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$, while $a(x, \cdot, \cdot)$ and $a_0(x, \cdot, \cdot)$ are continuous for almost all $x \in Q$. We assume the following conditions to be valid:

- *for any* $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$

$$|a(x, \xi_0, \xi)|^{p'} + |a_0(x, \xi_0, \xi)|^{p'} \leq c(x) + c_0 \left(|\xi_0|^p + |\xi|^p\right) \tag{2.1.29}$$

a.e. on Q, *where* $c_0 > 0$ *and* $c_1 \in \mathrm{L}^1(Q)$ *is non-negative;*

- *for any* $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$, $\zeta' = (\xi_0, \xi') \in \mathbf{R} \times \mathbf{R}^n$

$$\begin{aligned}[a(x, \xi_0, \xi) - a(x, \xi_0, \xi')] \cdot (\xi - \xi') \quad \geq \quad & \kappa \left(h(x) + |\zeta|^p + |\zeta'|^p\right)^{1-\beta/p} \times \\ & \times |\xi - \xi'|^\beta \end{aligned} \tag{2.1.30}$$

a.e. on Q, *where* $\kappa > 0$ *and* $h \in \mathrm{L}^1(Q)$ *is non-negative.*

Now we consider an operator

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u). \tag{2.1.31}$$

This operator acts continuously from V into V^* and from $\overline{V}$ into V^*.

Definition 2.1.9 *Denote by* $\hat{\mathcal{E}}$ *the set of operators, acting from* V *into* V^*, *which are of the from (2.1.31) and satisfy conditions (2.1.29) and (2.1.30).*

The following statment is well-known (see, e.g., [200], Theorem 2.8 of Chapter 2).

Proposition 2.1.4 . *Let* $\mathcal{A} \in \hat{\mathcal{E}}$. *Then the operator* $\mathcal{A}$ *is pseudomonotone. Moreover,* $\mathcal{A}$ *is surjective provided it is coercive.*

Remark 2.1.2 We note that any operator of the class $\hat{\mathcal{E}}$ is an operator of the calculus of variation. Indeed, $\mathcal{A}u = \mathcal{A}(u, u)$, where $\mathcal{A}(u, v)$ is defined by the formula

$$\mathcal{A}(u, v) = -\operatorname{div} a(x, u, \nabla v) + a_0(x, u, \nabla u).$$

For such the representation of $\mathcal{A}$ one can verify [200] all the conditions 1) – 4) of n^0 1.1.2.

2.2 Strong G-convergence for Multivalued Elliptic Operators

2.2.1 Definition of Strong G-convergence

By many reasons, the general concept of G-convergence we have considered in Chapter 1 is not adequate for study of differential operators. Really, we need to introduce more stronger notion of convergence, taking into account the differential structure of operators we consider. This is the notion of strong G-convergence we want now to define.

As in Section 1, we fix $p > 1$ and keep the notations V and V^* for the spaces $W_0^{1,p}(Q)$ and $W^{-1,p'}(Q)$, respectively.

Denote by σ_1 the weak topology of the space $\mathrm{L}^{p'}(Q)^n$ and by σ_2 the topology on the space $\mathrm{L}^{p'}(Q)^n$ induced by the pseudo-metric

$$d(g_1, g_2) = \|\operatorname{div}(g_1 - g_2)\|_{V^*}.$$

Let us introduce the weakest topology, σ, on $\mathrm{L}^{p'}(Q)^n$ such that σ is stronger that σ_1 and σ_2. One may characterize this topology as follows. The sequence (g_k) converges to g with respect to σ if and only if (g_k) converges to g weakly in $\mathrm{L}^{p'}(Q)^n$ and $(\operatorname{div} g_k)$ converges to $\operatorname{div} g$ strongly in V^*. We shall also denote by w the weak topology on the space $\overline{V} = W^{1,p}(Q)$. The Hahn-Banach Theorem implies that on V the topology w coincides with the weak topology of V.

Lemma 2.2.1 *Let $(u_k) \subset \overline{V}$ and $(g_k) \subset L^{p'}(Q)^n$. Assume that $u_k \to u$ weakly in $\overline{V}$ and $g_k \to g$ with respect to σ. Then*

$$\int_Q g_k \nabla u_k \varphi dx \to \int_Q g \nabla u \varphi dx$$

for any $\varphi \in C_0^\infty(Q)$.

Proof. It follows from the formula

$$\int_Q (g_k \cdot \nabla u_k)\varphi dx = (-\operatorname{div} g_k, u_k\varphi) - \int_Q g_k \cdot u_k \nabla\varphi dx,$$

since, by the Sobolev Embedding Theorem, $u_k\nabla\varphi \to \nabla\varphi$ strongly in $\mathrm{L}^p(Q)^n$. □

As usual, we shall identify any multivalued map with its graph.

Definition 2.2.1 *Let (a_k) be a sequence in M_Q and $a \in M_Q$. We say that (a_k) strongly G-converges to a if*

$$K_s(w \times \sigma)\text{-}\limsup A_k \subset A, \tag{2.2.1}$$

where A_k and A are the operators in M_V assiciated to a_k and a. We also say that $(\mathcal{A}_k)$ strongly G-converges to $\mathcal{A}$, where $\mathcal{A}_k, \mathcal{A} \in \mathcal{M}_V$ are the corresponding operators of the class $\mathcal{M}_V$. In this situation we shall use the notations $a_k \stackrel{G}{\Longrightarrow} a$ and $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$.

Proposition 2.2.1 *Strong G-limit is unique provided it exists.*

We postpone the proof of this statement to n^0 2.2.2.

It is not difficult to verify that strong G-convergence satisfies the following axioms:

(i) axiom of the constant sequence: *if $\mathcal{A}_k = \mathcal{A}$ for any $k \in \mathrm{N}$, then $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$;*

(ii) axiom of the subsequence: *if $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$, then $\mathcal{A}_{\sigma(k)} \stackrel{G}{\Longrightarrow} \mathcal{A}$ for any subsequence $(\sigma(k))$.*

The following statement explains why the convergence defined above is named "strong G-convergence".

Proposition 2.2.2 *Assume that $(\mathcal{A}_k) \subset \mathcal{M}_V$ and $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A} \in \mathcal{M}_V$. Then $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$.*

The proof will be given in n^0 2.2.3.

Now we describe the notion of strong G-convergence more explicitly. Condition (2.2.1) is equivalent to the following one:

(j) *for any increasing sequence $\tau(k)$ of integers, for any $f \in V^*$, for any sequence (f_k) converging to f strongly in V^*, for any sequence $(u_k) \subset V$ of solutions to*

$$-\operatorname{div} a_{\tau(k)}(x, \nabla u_k) \ni f_k \quad \text{on} \quad Q,$$

and for any sequence $(g_k) \in \mathrm{L}^{p'}(Q)^n$ such that

$$g_k \in a_{\tau(k)}(x, \nabla u_k(x)) \quad \text{a.e. in} \quad Q \quad \text{and} \quad -\operatorname{div} g_k = f_k,$$

there exists a subsequence of integers $\sigma(k) \to +\infty$ such that

$$u_{\sigma(k)} \to u \quad \text{weakly in} \quad V$$

and

$$g_{\sigma(k)} \to g \quad \text{weakly in} \quad \mathrm{L}^{p'}(Q)^n,$$

where $u \in V$ is a solution of the problem

$$-\operatorname{div} g = f \quad \text{on} \quad Q, \tag{2.2.2}$$

and

$$g(x) \in a(x, \nabla u) \quad \text{a.e. on} \quad Q. \tag{2.2.3}$$

Indeed, assume (2.2.1) and suppose that $\tau(k)$, f, f_k, u_k, g_k are as in (j). The uniform coerciveness of $\mathcal{A}_k$ implies that the sequence (u_k) is bounded in V. Therefore, (g_k) is bounded in $\mathrm{L}^{p'}(Q)^n$, by growth condition (2.1.6), or (2.1.8). Hence, there exists a sequence $(u_{\sigma(k)},\ g_{\sigma(k)})$ which converges to (u, g) weakly in the space $V \times \mathrm{L}^{p'}(Q)^n$. This implies that $\operatorname{div} g_k$ converges to $\operatorname{div} g$ weakly in V^* and, therefore,

$$f = -\operatorname{div} g.$$

As consequence,

$$(u_{\sigma(k)}, g_{\sigma(k)}) \longrightarrow (u, g)$$

in the topology $w \times \sigma$ and (2.2.1) implies (2.2.3). The converse implication is evident.

The statement $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$ is valid if and only if both the following conditions (a) and (b) are satisfied:

(a) *if $f_k \to f$ strongly in the space V^*, $u_k \to u$ weakly in the space V, and*

$$-\operatorname{div} a_k(x, \nabla u_k) \ni f_k \quad \text{on} \quad Q \tag{2.2.4}$$

for infinitely many $k \in \mathbf{N}$, then

$$-\operatorname{div} a(x, \nabla u) \ni f \quad \text{on} \quad Q; \tag{2.2.5}$$

(b) *if $f \in V^*$ and $u \in V$ is a solution of* (2.2.5), *then there exist (f_k) converging to f strongly in the space V^* and (u_k) converging to u weakly in the space V such that u_k satisfies* (2.2.4) *for any $k \in \mathbf{N}$.*

It is easy that (j) implies (a). Hence, a non-trivial part of Proposition 2.2.2 is that (j) implies (b) as well.

It must be pointed out that G-convergence $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$ does not imply strong G-convergence. The reason is that a differential operator may possess different representations in the divergence form as the following example shows:

Example 2.2.1 Let $Q \subset \mathbf{R}^3$, and $\varphi \in C_0^\infty(Q)$. Define

$$a(x, \xi) = \xi$$

and

$$b(x, \xi) = \xi + \nabla\varphi \times \xi.$$

where $\times$ stands for the usual vector product in $\mathbf{R}^3$. It is easy that a and b belongs to the class M_Q with

$$p = 2, \quad m_1 = m_2 = 0,$$

and

$$c_1 = (1 + \max_Q |\nabla\varphi|^2), \quad c_2 = 1.$$

Since

$$\int_Q (\nabla\varphi \times \nabla u) \cdot \nabla v dx = 0 \quad \forall u, v \in W_0^{1,2}(Q),$$

we have

$$\int_Q a(x, \nabla u) \cdot \nabla u dx = \int_Q b(x, \nabla u) \cdot \nabla u dx \quad \forall u, v \in W_0^{1,2}(Q).$$

Hence, the differential operators generated by a and b according to Definition 2.1.4 coincide.

2.2.2 Strong G-compactness

Now we prove the following compactness theorem which is the main result of the theory.

Theorem 2.2.1 *Let $a_k \in M_Q$. Suppose $\mathcal{A}_k \in \mathcal{M}_V$ is the operator associated to the function a_k. Then there exist a subsequence $\tau(k)$ and a function $a \in M_Q$ such that $\mathcal{A}_{\tau(k)} \overset{G}{\Longrightarrow} \mathcal{A}$, where $\mathcal{A} \in \mathcal{M}_V$ is the operator associated to a.*

To prove this theorem we start with the following

Proposition 2.2.3 *Let $B_k \in M_V$. Then there exist a subsequence $\tau(k)$ and an operator $B \in M_V$ such that*

$$B = K_s(w \times \sigma)\text{-}\lim B_{\tau(k)}.$$

Proof. Recall that, for any separable reflexive Banach space X, there is a metric d on X such that $x_k \to x$ weakly in X if and only if (x_k) is bounded and $d(x_k, x) \to 0$. We denote by τ_1 the topology induced by such the metric on V. By τ_2 we denote the topology on the space $\mathrm{L}^{p'}(Q)^n$ induced by the metric

$$d_2(g_1, g_2) = d(g_1, g_2) + \|\mathrm{div} g_1 - \mathrm{div} g_2\|_{V^*},$$

where d is the above mentioned metric on the space $X = \mathrm{L}^{p'}(Q)^n$.

The topology $\tau_1 \times \tau_2$ has a countable base. Hence, by Kuratowski compactness theorem (see Theorem 1.1.6), there exists a subsequence of (B_k) still denoted by (B_k) which $K_s(\tau_1 \times \tau_2)$-converges to a set $B \subset V \times \mathrm{L}^{p'}(Q)^n$.

Now we prove that really

$$B = K_s(w \times \sigma)\text{-}\lim B_k.$$

To do this it is sufficient to show that

$$K_s(w \times \sigma)\text{-}\limsup B_k \subset B \tag{2.2.6}$$

and

$$B \subset K_s(w \times \sigma)\text{-}\liminf B_k. \tag{2.2.7}$$

To verify (2.2.6), let us take a couple (u, g) such that

$$(u, g) \in K_s(w \times \sigma)\text{-}\limsup B_k.$$

Then there exist a subsequence $\tau(k)$ and elements (u_k, g_k) such that $(u_k, g_k) \in B_{\tau(k)}$ for any $k \in \mathbf{N}$ and $(u_k, g_k) \to (u, g)$ with respect to the topology $w \times \sigma$. By definition of the topologies τ_1 and τ_2, we see immediately that $(u_k, g_k) \to (u, g)$ with respect to the topology $\tau_1 \times \tau_2$. Hence, $(u, g) \in B$.

Let us prove (2.2.7). If $(u, g) \in B$, then there is a sequence $(u_k, g_k) \in B_k$ which converges to (u, g) with respect to $\tau_1 \times \tau_2$. Since $\operatorname{div} g_k$ is bounded in V^*, inequality (2.1.12) implies that u_k is bounded in V. Hence, g_k is bounded in $\mathrm{L}^{p'}(Q)^n$ by (2.1.11). Therefore, $u_k \to u$ weakly in V and $g_k \to g$ weakly in $\mathrm{L}^{p'}(Q)^n$. Since $\operatorname{div} g_k \to g$ strongly in V^*, we see that $(u_k, g_k) \to (u, g)$ with respect to the topology $w \times \sigma$, which implies (2.2.7).

Finally, we prove that $B \in M_V$. Here we verify only condition (i) of Definition 2.1.2. The boundedness and coerciveness conditions (see (2.1.11) and (2.1.12)) may be proved in the similar way. Let $u^i \in V$ and $g^i \in Bu^i, i = 1, 2$. By (2.2.7), there is a sequence (u_k^i, g_k^i) such that $(u_k^i, g_k^i) \in B_k$ and $(u_k^i, g_k^i) \to (u^i, g^i)$ with respect to the topology $w \times \sigma$. Since $B_k \in M_V$, we have

$$\int_Q (\nabla u_k^1 - \nabla u_k^2) \cdot (g_k^1 - g_k^2) \varphi dx \geq 0$$

for any $\varphi \in C_0^\infty(Q), \varphi \geq 0$ on Q. By Lemma 2.2.1, this implies that

$$\int_Q (\nabla u^1 - \nabla u^2) \cdot (g^1 - g^2) \varphi dx \geq 0$$

for any $\varphi \in C_0^\infty(Q), \varphi \geq 0$ on Q. As consequence,

$$(\nabla u^1 - \nabla u^2) \cdot (g^1 - g^2) \geq 0 \quad \text{a.e. on } Q,$$

and B satisfies condition (i) of Definition 2.1.2. The proof is complete. □

Now we prove

Proposition 2.2.4 *Let $B_k \in M_V$ and $\mathcal{B}_k$ the associated operator of the class $\hat{\mathcal{M}}_V$. Assume that*

$$B = K_s(w \times \sigma)\text{-}\lim B_k.$$

Then $\mathcal{B}_k \xrightarrow{G} \mathcal{B}$, where $\mathcal{B}$ is the operator of the class $\hat{\mathcal{M}}_V$ associated to $B \in M_V$.

Proof. The statement $\mathcal{B}_k \xrightarrow{G} \mathcal{B}$ means that

$$\mathcal{B} = K_s(w \times \rho)\text{-}\lim \mathcal{B}_k.$$

Recall that ρ stands for the norm topology of V^*. The inclusion

$$\mathcal{B} \subset K_s(w \times \rho)\text{-}\liminf \mathcal{B}_k$$

is trivial. So, we need to prove that

$$K_s(w \times \rho)\text{-}\limsup \mathcal{B}_k \subset \mathcal{B}.$$

With this aim we fix

$$(u, f) \in K_s(w \times \rho)\text{-}\limsup \mathcal{B}_k.$$

This means that there are a subsequence $\sigma(k)$ and a sequence (u_k, f_k) such that $(u_k, f_k) \in \mathcal{B}_{\sigma(k)}$ for $k \in \mathbf{N}$ and $(u_k, f_k) \to (u, f)$ with respect to the topology $w \times \rho$. By Definition 2.1.3, there exists $g_k \in B_{\sigma(k)}u_k$ such that

$$-\operatorname{div} g_k = f_k.$$

By (2.1.11), we have

$$\int_Q |g_k|^{p'} dx \leq c \left[1 + \int_Q |\nabla u_k|^p dx\right],$$

which implies that g_k is bounded in $\mathrm{L}^{p'}(Q)^n$. Therefore, there exists a subsequence $(g_{\tau(k)})$ converging to a function g weakly in $\mathrm{L}^{p'}(Q)^n$. As consequence,

$$-\operatorname{div} g_{\tau(k)} \to -\operatorname{div} g$$

weakly in V^*. However, by assumption, $f_k \to f$ strongly in V^* and we conclude that

$$f = -\operatorname{div} g.$$

Hence, $(u_{\tau(k)}, g_{\tau(k)})$ converges to (u, g) with respect to the topology $w \times \sigma$ and

$$(u_{\tau(k)}, g_{\tau(k)}) \in B_{\sigma(\tau(k))}.$$

Thus, $(u, g) \in B$. As consequence, $(u, f) \in \mathcal{B}$ and we conclude. $\square$

Proof of Theorem 2.2.1. By Proposition 2.2.3, there are a subsequence $A_{\tau(k)}$ and an operator $B \in M_V$ such that

$$B = K_s(w \times \sigma)\text{-}\lim A_{\tau(k)}.$$

Then, by Proposition 2.2.4, $\mathcal{A}_{\tau(k)} \xrightarrow{G} \mathcal{B}$, where $\mathcal{B} \in \hat{\mathcal{M}}_V$ is the operator associated to B. The operator $J : V \longrightarrow V^*$ defined by the formula

$$Ju = |u|^{p-2}u$$

is single-valued and strictly monotone. The Sobolev Embedding Theorem implies that J is continuous from V with the weak topology into V^* with the strong one. Since $\mathcal{A}_k$ is a maximal monotone operator, Proposition 1.2.6(i) implies that $\mathcal{B}$ is a maximal monotone operator as well. Therefore, $\mathcal{B} \in \hat{\mathcal{M}}_V$. Hence, $\mathcal{B}^{-1}$ is a maximal monotone operator. By (2.1.11), it is coercive, and $R(\mathcal{B}^{-1}) = D(\mathcal{B}) = V$. Thus, $D(B) = V$.

Now, by Proposition 2.1.1, there exists a multivalued function $a \in M_Q$ such that $B \subset A$, where $A \in M_V$ is the operator associated to a. It is obvious that $\mathcal{A}_{\tau(k)} \overset{G}{\Longrightarrow} \mathcal{A}$ and the proof is complete. □

Proof of Proposition 2.2.1. Let

$$C = K_s(w \times \sigma)\text{-}\limsup A_k,$$

and let $a, b \in M_Q$ be such that $C \subset A$ and $C \subset B$. By Proposition 2.2.3, there are a subsequence $\tau(k)$ and an operator $C_1 \in M_V$ such that

$$C_1 = K_s(w \times \sigma)\text{-}\lim A_{\tau(k)}.$$

Obviously, $C_1 \subset C$. As in the proof of Theorem 2.2.1, we see that $D(C_1) = V$. Now the conclusion of Proposition 2.2.1 follows from Lemma 2.1.5. □

2.2.3 Additional Results

First of all we prove the so-called localization property of strong G-convergence. Let Q_1 be an open subset of Q. For the sake of brevity we set $V_1 = W_0^{1,p}(Q_1)$ and $\overline{V}_1 = W^{1,p}(Q_1)$. Besides the topologies w and σ on $\overline{V}$ and $\mathrm{L}^{p'}(Q)^n$ introduced in n^0 2.2.1, we consider the topologies w_1 and σ_1 defined on $\overline{V}_1$ and $\mathrm{L}^{p'}(Q_1)^n$ in the similar manner. For any $a \in M_Q$, we denote by $a^{(1)} \in M_{Q_1}$ the function defined by

$$a^{(1)} = a_{|Q_1 \times \mathbf{R}^n}.$$

Associated to $a^{(1)}$ there are two operators $A^{(1)} \in M_{V_1}$ and $\mathcal{A}^{(1)} \in \mathcal{M}_{V_1}$. The natural extensions of these operators to $\overline{V}_1$ will be still denoted by $A^{(1)}$ and $\mathcal{A}^{(1)}$, respectively.

Theorem 2.2.2 *Let $a_k \in M_Q$. Assume that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Then $\mathcal{A}_k^{(1)} \overset{G}{\Longrightarrow} \mathcal{A}^{(1)}$.*

This theorem follows from the next statement.

Proposition 2.2.5 *Assume that $a_k \in M_Q$ and $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Then*

$$K_s(w_1 \times w_2)\text{-}\limsup A_k^{(1)} \subset A^{(1)} \quad \textit{in } \overline{V}_1 \times L^{p'}(Q_1)^n. \tag{2.2.8}$$

Proof. Evidently, it is sufficient to show that for any subsequence $\sigma(k)$ there is a further subsequence $\sigma(\tau(k))$ such that

$$K_s(w_1 \times \sigma_1)\text{-}\liminf A_{\sigma(\tau(k))}^{(1)} \subset A^{(1)} \quad \text{in } \overline{V}_1 \times \mathrm{L}^{p'}(Q_1)^n. \tag{2.2.9}$$

By the definition of strong G-convergense and by Proposition 2.2.3, for a sequence $\sigma(k)$ there are a subsequence $\sigma(\tau(k))$ and an operator $C \in M_V$ such that

$$C = K_s(w \times \sigma)\text{-}\lim A_{\sigma(\tau(k))} \subset A \quad \text{in } V \times \mathrm{L}^{p'}(Q)^n. \tag{2.2.10}$$

Moreover, as in the proof of Theorem 2.2.1, we see that $D(C) = V$.

Now let

$$C_1 = K_s(w_1 \times \sigma_1)\text{-}\liminf A_{\sigma(\tau(k))}^{(1)}$$

in $\overline{V}_1 \times \mathrm{L}^{p'}(Q_1)^n$. We have

$$C_0^\infty(Q_1) \subset D(C_1). \tag{2.2.11}$$

Indeed, let $u_1 \in C_0^\infty(Q_1)$. Then there is $u \in C_0^\infty(Q)$ such that $u|_{Q_1} = u_1$. Since $D(C) = V$, we can find $g \in \mathrm{L}^{p'}(Q)^n$ such that $(u, g) \in C$. Hence, there exists a sequence

$$(u_k, g_k) \in V \times \mathrm{L}^{p'}(Q)^n$$

converging to (u, g) in the topology $w \times \sigma$ such that

$$(u_k, g_k) \in A_{\sigma(\tau(k))}$$

for any $k \in \mathbf{N}$. Evidently,

$$(u_{k|Q_1}, g_{k|Q_1}) \to (u_{|Q_1}, g_{|Q_1})$$

in the topology $w_1 \times \sigma_1$. Hence, $(u_1, g_{|Q_1}) \in C_1$ and (2.2.11) is proved.

As in the proof of Proposition 2.2.3, we see that $C_1 \in M_{\overline{V}_1}$. By Proposition 2.1.1, there is $b^{(1)} \in M_{Q_1}$ such that

$$C_1 \subset B^{(1)},$$

where $B^{(1)} \in M_{\overline{V}_1}$ is the operator associated to $b^{(1)}$. Now we define

$$C^{(1)} = \{(u_{|Q_1}, g_{|Q_1}) : (u, g) \in C\}.$$

It is obvious that $C^{(1)} \in M_{\overline{V}_1}$. Since $D(C) = V$, we see that

$$D(C^{(1)}) \supset C_0^\infty(Q_1).$$

By (2.2.10), we have

$$C^{(1)} \subset A^{(1)}. \tag{2.2.12}$$

Moreover, it is easy that

$$C^{(1)} \subset C_1 \subset B^{(1)}. \tag{2.2.13}$$

Now Lemma 2.1.5 implies that $a^{(1)} = b^{(1)}$. Thus, $A^{(1)} = B^{(1)}$, and (2.2.13) implies (2.2.9). The proof is complete. □

As a consequence we have the following

Corollary 2.2.1 *Let $a_k, b_k \in M_Q$ and $a_k^{(1)} = b_k^{(1)}$ for any $k \in \mathbf{N}$. Assume that $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$ and $\mathcal{B}_k \xrightarrow{G} \mathcal{B}$. Then $a^{(1)} = b^{(1)}$, where $\mathcal{A}$ and $\mathcal{B}$ are generated by $a \in M_Q$ and $b \in M_Q$.*

Now let us consider a finite family $(Q_i)_{i \in I}$ of open subsets of Q such that

$$|Q \setminus \cup Q_i| = 0.$$

For $a \in M_Q$, we denote by $a^{(i)}$ the restriction of a to $Q_i \times \mathbf{R}^n$. Let $\mathcal{A}$ and $\mathcal{A}^{(i)}$ be the corresponding operators from the classes $\mathcal{M}_V$ and $\mathcal{M}_{V_i}$, respectively, and $V_i = W_0^{1,p}(Q_i)$. Theorem 2.2.1 and Corollary 2.2.1 imply

Corollary 2.2.2 *A sequence $\mathcal{A}_k \in \mathcal{M}_V$ strongly G-converges to $\mathcal{A} \in \mathcal{M}_V$ if and only if $\mathcal{A}_k^{(i)} \xrightarrow{G} \mathcal{A}^{(i)}$ for any $i \in I$.*

The following result shows, in particular, that the Dirichlet boundary value problem play no special role in the theory of G-convergenve.

Theorem 2.2.3 *Let $\varphi \in \overline{V}$, $a_k \in M_Q$, and $a \in M_Q$. Then the following statements for the associated operators are equivalent:*

(i) $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$;

(ii) $K_s(w \times \sigma)\text{-}\limsup A_k \subset A \quad$ *in* $\overline{V} \times L^{p'}(Q)^n$;

(iii) $\mathcal{A}_k^\varphi \xrightarrow{G} \mathcal{A}^\varphi$.

Proof. (i) $\Rightarrow$ (ii): This implication follows from Proposition 2.2.5, with $Q_1 = Q$.

(ii) $\Rightarrow$ (iii): By definition, the statement $\mathcal{A}_k^\varphi \xrightarrow{G} \mathcal{A}^\varphi$ means that

$$B = K_s(w \times \sigma)\text{-}\limsup A_k^\varphi \subset A^\varphi.$$

Let $(u, g) \in B$. Then there are a subsequence $\sigma(k)$ and a sequence (u_k, g_k) converging to (u, g) with respect to the topology $w \times \sigma$ such that $(u_k, g_k) \in A_{\sigma(k)}^\varphi$. By definition of A_k^φ, we have

$$(\varphi + u_k, g_k) \in A_{\sigma(k)}$$

and $(\varphi + u_k, g_k) \to (\varphi + u, g)$ with respect to the topology $w \times \sigma$. Then, by (ii), $(\varphi + u, g) \in A$ and we have $(u, g) \in A^\varphi$, which yields (iii).

(iii) $\Rightarrow$ (i): By Theorem 2.2.1, there exists a subsequence $\mathcal{A}_{\sigma(k)}$ and an operator $\mathcal{B}$ associated to a function $b \in M_Q$ such that $\mathcal{A}_{\sigma(k)} \xrightarrow{G} \mathcal{B}$. Since (i) implies (iii), we get $\mathcal{A}_{\sigma(k)}^\varphi \xrightarrow{G} \mathcal{B}^\varphi$. On the other hand, by assumption, we have $\mathcal{A}_{\sigma(k)}^\varphi \xrightarrow{G} \mathcal{A}^\varphi$. Now Proposition 2.2.1 yields $a(x, \xi) = b(x, \xi)$ a.e. on Q. Hence, the passage to the subsequence $\sigma(k)$ is superfluous and we conclude. □

Remark 2.2.1 It is easy to see that statement (ii) of Theorem 2.2.3 is equivalent to the following one. Let $\sigma(k)$ be a subsequence of integers, $f_k \to f$ strongly in V^*, $u_k \to u$ weakly in V and

$$-\operatorname{div} a_{\sigma(k)}(x, \nabla u_k) \ni f_k.$$

Then u satisfies the equation

$$-\operatorname{div} a(x, \nabla u) \ni f$$

and, for any $g_k \in \mathrm{L}^{p'}(Q)^n$ such that

$$g_k(x) \in a_{\sigma(k)}(x, \nabla u_k(x)) \text{ a.e. on } Q \quad \text{and} \quad -\operatorname{div} g_k = f_k,$$

there is a subsequence $g_{\tau(k)}$ such that $g_{\tau(k)} \to g$ weakly in $\mathrm{L}^p(Q)^n$ and

$$g(x) \in a(x, \nabla u(x)) \text{ a.e. on } Q.$$

Proof of Proposition 2.2.2. Assume that $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$. By the definition of strong G-convergence and by Proposition 2.2.3, for any subsequence $\sigma(k)$ there is a further subsequence $\sigma(\tau(k))$ such that

$$B = K_s(w \times \sigma)\text{-}\lim A_{\sigma(\tau(k))} \subset A.$$

in $V \times \mathrm{L}^{p'}(Q)^n$. By Proposition 2.2.4, we have

$$\mathcal{A}_{\sigma(\tau(k))} \xrightarrow{G} \mathcal{B} \subset \mathcal{A}.$$

Theorem 2.1.6 implies that

$$R(\mathcal{A}_{\sigma(\tau(k))} + \lambda J) = V^*, \ \lambda \geq 0,$$

where $Ju = |u|^{p-2}u$. Hence, by Proposition 1.2.6(i),

$$R(\mathcal{B} + \lambda J) = V^*.$$

As consequence, the operator $\mathcal{B}$ is maximal monotone. Since $\mathcal{A}$ is also a monotone operator and $\mathcal{B} \subset \mathcal{A}$, we have $\mathcal{B} = \mathcal{A}$ and $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$. □

Remark 2.2.2 In the similar way one can prove that

$$K_s(w \times \rho)\text{-}\lim \mathcal{A}_k = \mathcal{A},$$

where $\mathcal{A}_k$ and $\mathcal{A}$ are regarded as operators acting from $\overline{V}$ into V^*.

We prove now that the classes of operators $\mathcal{U}_V$ and $\mathcal{S}_V$ defined in n^0 2.1.3 are stable with respect to strong G-convergence. With this aim we need the following

Lemma 2.2.2 *Let $\gamma \geq 0$, $\delta \geq 0$, and $\gamma + \delta \leq 1$. Let ψ_k, ζ_k, θ_k, ψ, ζ, $\theta \in L^1(Q)$ and $\psi_k \to \psi$, $\zeta_k \to \zeta$, $\theta_k \to \theta$ weakly in the sense of distributions. Assume that $\zeta_k \geq 0$, $\theta_k \geq 0$ and*

$$|\psi_k| \leq \zeta_k^\gamma \theta_h^\delta \quad a.e. \ in\, Q.$$

Then

$$|\psi| \leq \zeta^\gamma \theta^\delta \quad a.e. \ in\, Q.$$

Proof. Let $\varepsilon = 1 - \gamma - \delta$. For any $\varphi \in C_0^\infty(Q)$, $\varphi \geq 0$, we have

$$\int_Q |\psi_k|\varphi dx \leq \left(\int_Q \zeta_k \varphi dx\right)^\gamma \left(\int_Q \theta_k \varphi dx\right)^\delta \left(\int_Q \varphi dx\right)^\varepsilon.$$

Since $\psi_k\varphi \to \psi\varphi$ in the sense of distributions, we obtain immediately

$$\int_Q |\psi|\varphi dx \leq \liminf \int_Q |\psi_k|\varphi dx.$$

Hence,

$$\int_Q |\psi|\varphi dx \leq \left(\int_Q \zeta \varphi dx\right)^\gamma \left(\int_Q \theta \varphi dx\right)^\delta \left(\int_Q \varphi dx\right)^\varepsilon$$

for any $\varphi \in C_0^\infty(Q)$, $\varphi \geq 0$. Then, by the standard approximation argument, the last inequality is valid for any $\varphi \in \mathrm{L}^\infty(Q)$, $\varphi \geq 0$. If we take here φ to be the characteristic function of the ball of the radius r centered at $x \in Q$ and then pass to the limit as $r \to 0$, we obtain the required. □

Theorem 2.2.4 *Let $\mathcal{A}_k \in \mathcal{U}_V$ (resp. $\mathcal{A}_k \in \mathcal{S}_V$). Assume that $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$. Then $\mathcal{A} \in \mathcal{U}_V$ (resp. $\mathcal{A} \in \mathcal{S}_V$).*

Proof. We shall use the notations introduced in n^0 2.1.3. Assume that $\mathcal{A}_k \in \mathcal{U}_V$. We have to prove that the function $a \in M_Q$ associated to $\mathcal{A}$ belongs to U_Q. By assumption, we have

$$K_s(w \times \sigma)\text{-}\limsup A_k \subset A,$$

where A_k, $A \in M_V$ are the operators associated to a_k and a, respectively. By Proposition 2.2.3, there exists a subsequence of $\mathcal{A}_k$ still denoted by $\mathcal{A}_k$ such that

$$B = K_s(w \times \sigma)\text{-}\lim A_k \subset A. \tag{2.2.14}$$

As in the proof of Proposition 2.1.1, we introduce the set E defined by

$$E = \{(\nabla u, g) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n : g \in Bu\}.$$

We denote by dec(E) the smallest decomposable subset of $\mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$ contaning E and consider the set

$$\tilde{E} = \mathrm{cl}_{s\times w}(\mathrm{dec}(E)),$$

the closure of dec(E) in $\mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$, where $\mathrm{L}^p(Q)^n$ is endowed with its strong topology and $\mathrm{L}^{p'}(Q)^n$ with its weak topology. As in the proof of Lemma 2.1.5, we have

$$\tilde{E} = \{(\varphi, g) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n : g(x) \in a(x, \varphi(x)) \text{ a.e. in } Q\}. \tag{2.2.15}$$

Now we are able to prove that $a(x,\xi)$ satisfies inequalities (2.1.24) and (2.1.25). At first, we show that if (u^1, g^1), $(u^2, g^2) \in B$, then

$$|g^1 - g^2| \le c\zeta^{(p-1-\alpha)/p}\left[(g^1 - g^2)\cdot(\nabla u^1 - \nabla u^2)\right]^{\alpha/p} \tag{2.2.16}$$

a.e. on Q, where

$$\zeta = m + g^1 \cdot \nabla u^1 + g^2 \cdot \nabla u^2 \ge 0.$$

Indeed, by (2.2.14), there exists a sequence (u_k^i, g_k^i) such that $(u_k^i, g_k^i) \to (u^i, g^i)$ with respect to the topology $w \times \sigma$ and $(u_k^i, g_k^i) \in A_k$ for any $k \in \mathbf{N}$. Since $a_k \in U_Q$, we have

$$|g_k^1 - g_k^2| \le c\zeta_k^{(p-1-\alpha)/p}\left[(g_k^1 - g_k^2)\cdot(\nabla u_k^1 - \nabla u_k^2)\right]^{\alpha/p},$$

where

$$\zeta_k = m + g_k^1 \cdot \nabla u_k^1 + g_k^2 \cdot \nabla u_k^2 \ge 0.$$

Let us set

$$\psi_k = g_k^1 - g_k^2, \quad \psi = g^1 - g^2,$$

$$\theta_k = (g_k^1 - g_k^2) \cdot (\nabla u_k^1 - \nabla u_k^2), \quad \theta = (g^1 - g^2) \cdot (\nabla u^1 - \nabla u^2).$$

By Lemma 2.2.1, $\zeta_k \to \zeta$ and $\theta_k \to \theta$ weakly in the sense of distributions . Hence, $\zeta \geq 0$ a.e. in Q and Lemma 2.2.2 yields

$$|\psi| \leq \zeta^{(p-1-\alpha)/p} \theta^{\alpha/p} \quad \text{a.e. in } Q.$$

Thus, we have proved (2.2.16).

As a second step, we prove, for (φ^1, g^1), $(\varphi^2, g^2) \in \tilde{E}$, the inequality

$$|g^1 - g^2| \leq c\omega^{(p-1-\alpha)/p} \left[(g^1 - g^2) \cdot (\varphi^1 - \varphi^2)\right]^{\alpha/p} \quad \text{a.e. in } Q, \tag{2.2.17}$$

where

$$\omega = m + g^1 \cdot \varphi^1 + g^2 \cdot \varphi^2 \geq 0.$$

Inequality (2.2.16) implies that (2.2.17) holds for $(\varphi^i, g^i) \in E$. The characterization of dec(E) in terms of Borel partitions mentioned in the proof of Proposition 2.1.1 shows us that (2.2.17) holds for $(\varphi^i, g^i) \in$ dec(E). To prove the same property for $\tilde{E}$, let $(\varphi^i, g^i) \in \tilde{E}$. By property (a) of $\tilde{E}$ (see the proof of Proposition 2.1.1), there exists a sequence $(\varphi_k^i, g_k^i) \in$ dec$E$ such that $\varphi_k^i \to \varphi^i$ strongly in $\mathrm{L}^p(Q)^n$ and $g_k^i \to g^i$ weakly in $\mathrm{L}^{p'}(Q)^n$. Since (2.2.17) holds on dec(E), we have

$$|g_k^1 - g_k^2| \leq c\omega_k^{(p-1-\alpha)/p} \left[(g_k^1 - g_k^2) \cdot (\varphi_k^1 - \varphi_k^2)\right]^{\alpha/p},$$

where

$$\omega_k = m + g_k^1 \cdot \varphi_k^1 + g_k^2 \cdot \varphi_k^2 \geq 0.$$

Now Lemma 2.2.2, with

$$\psi_k = g_k^1 - g_k^2$$

and

$$\theta_k = (g_k^1 - g_k^2) \cdot (\varphi_k^1 - \varphi_k^2),$$

yields (2.2.17). Moreover, $\omega \geq 0$ a.e. on Q.

Finally, let us prove that inequalities (2.1.24) and (2.1.25) hold true. Denote by Y the set of all $x \in Q$ such that (2.1.25) is not satisfied for some ξ^1, ξ^2 and $\eta^1 \in a(x, \xi^1)$, $\eta^2 \in a(x, \xi^2)$. We have to prove that $|Y| = 0$. To do this we observe that

$$Y = \{x \in Q : Gx \neq \emptyset\};$$

where

$$Gx = \{(\xi^1, \xi^2, \eta^1, \eta^2) : |\eta^1 - \eta^2| > c\Phi^{(p-1-\alpha)/p} \left[(\eta^1 - \eta^2) \cdot (\xi^1 - \xi^2)\right]^{\alpha/p}, \\ \eta^i \in a(x, \xi^i),\ i = 1, 2\}$$

and

$$\Phi = m + \eta^1 \cdot \xi^1 + \eta^2 \cdot \xi^2.$$

Since, by Theorem 2.1.1, $\mathrm{gr}(a) \in \mathcal{L}(Q) \otimes \mathcal{B}(\mathbf{R}^n) \otimes \mathcal{B}(\mathbf{R}^n)$, we see that

$$\mathrm{gr}(G) \in \mathcal{L}(Q) \otimes \underbrace{\mathcal{B}(\mathbf{R}^n) \otimes \cdots \otimes \mathcal{B}(\mathbf{R}^n)}_{4}.$$

Theorem 2.1.2 implies that $Y \in \mathcal{L}(Q)$. Hence, by Aumann-von Neumann Theorem 2.1.4, there exists a measurable selection $(\varphi_0^1, \varphi_0^2, g_0^1, g_0^2)$ of G defined on Y. Therefore, for any $x \in Y$ we have

$$|g_0^1 - g_0^2| > c(m + g_0^1 \cdot \varphi_0^1 + g_0^2 \cdot \varphi_0^2)^{(p-1-\alpha)/p} \left[(g_0^1 - g_0^2) \cdot (\varphi_0^1 - \varphi_0^2)\right]^{\alpha/p} \quad (2.2.18)$$

and

$$g_0^i(x) \in a(x, \varphi_0^i(x)), \quad i = 1, 2.$$

If $|Y| > 0$, then there is a measurable subset $Y' \subset Y$, with $|Y'| > 0$, such that $(\varphi_0^1, \varphi_0^2, g_0^1, g_0^2)$ is bounded on Y'. Now, as at the end of the proof of Lemma 2.1.4, we can construct $g_* \in \mathrm{L}^{p'}(Q)^n$ such that $g_*(x) \in a(x, 0)$ a.e. on Q. For $i = 1, 2$, we set

$$\varphi^i(x) = \begin{cases} \varphi_0^i(x) & \text{if } x \in Y', \\ 0 & \text{if } x \in Q \setminus Y', \end{cases}$$

$$g^i(x) = \begin{cases} g_0^i(x) & \text{if } x \in Y', \\ g_*(x) & \text{if } x \in Q \setminus Y'. \end{cases}$$

Then $(\varphi^i, g^i) \in \mathrm{L}^p(Q)^n \times \mathrm{L}^{p'}(Q)^n$ and $g^i(x) \in a(x, \varphi^i(x))$ a.e. on Q. Therefore, by (2.2.15), $(\varphi^i, g^i) \in \tilde{E}$. Hence, (2.2.17) implies that

$$|g_0^1 - g_0^2| \le c(m + g_0^1 \cdot \varphi_0^1 + g_0^2 \cdot \varphi_0^2)^{(p-1-\alpha)/p} \left[(g_0^1 - g_0^2) \cdot (\varphi_0^1 - \varphi_0^2)\right]^{\alpha/p}$$

a.e. on Y'. Since $|Y'| > 0$, this contradicts (2.2.18). Thus, we have porved that $|Y| = 0$ and (2.1.25) is satisfied. The proof of (2.1.24) is similar.

So, we have proved that $\mathcal{U}_V$ is stable with respect to strong G-convergence. The statement for $\mathcal{S}_V$ may be proved along the same lines. □

Remark 2.2.3 Strong G-convergence for operators of the class $\mathcal{U}_V \cap \mathcal{S}_V$ may be characterized more directly (cf. Proposition 1.2.7). Let $\mathcal{A}_k$, $\mathcal{A} \in \mathcal{U}_V \cap \mathcal{S}_V$. For any $f \in V^*$, there exist $u_k \in V$ and $u \in V$ such that $\mathcal{A}_k u_k = f$ and $\mathcal{A} u = f$. The sequence $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if and only if $u_k \to u$ weakly in V and $a_k(x, \nabla u_k) \to a(x, \nabla u)$ weakly in $\mathrm{L}^{p'}(Q)^n$, for any $f \in V^*$.

2.2.4 Variational Problems

Let us discuss briefly the case when operators we consider have variational structure, i.e. they are subdifferentials of integral functionals. Given an open bounded subset $Q \subset \mathbf{R}^n$ let $h_1,\ h_2 \in \mathrm{L}^1(Q)$ be two nonnegative functions, $k_1,\ k_2$ two positive constants. We denote by $\mathcal{F}_Q$ the set of all functions $f : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$ such that $f(x,\xi)$ is measurable in $x \in Q$, convex in $\xi \in \mathbf{R}^n$, and

$$k_1|\xi|^p - h_1(x) \leq f(x,\xi) \leq k_2|\xi|^p + h_2(x)$$

for any $(x,\xi) \in Q \times \mathbf{R}$. Associated to such a function f, there is an integral functional

$$F(u) = \int_Q f(x, \nabla u)dx$$

defined on $V = W_0^{1,p}(Q)$.

It is not difficult to see that for any $f \in \mathcal{F}_Q$ the subdifferential $\partial_\xi f$ belongs to the class M_Q, with suitable $m_1,\ m_2,\ c_1$ and c_2 depending on $h_1,\ h_2,\ k_1$ and k_2. Conversely, given multivalued function $a \in M_Q$ which is cyclically monotone in ξ there is a function f such that $a(x,\xi) = \partial_\xi f(x,\xi)$ for a.e. $x \in Q$, $f \in \mathcal{F}_Q$ with suitable $h_1,\ h_2,\ k_1$ and k_2 depending on $m_1,\ m_2,\ c_1$ and c_2, and $f(x,0) = 0$ for a.e. $x \in Q$. Notice that $a(x,\cdot)$ is odd (resp. positively homogeneous of degree $p-1$) if and only if the associated function $f(x,\cdot)$ is even (resp. positively homogeneous of degree p).

Without any difficulty one can prove the following

Proposition 2.2.6 *Let $f \in \mathcal{F}_Q$. Then $\partial F = \mathcal{A}$, where $\mathcal{A} \in \mathcal{M}_V$ is the operator associated to $\partial_\xi f$.*

Proposition 2.2.7 *Let $f : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$ be a Carathéodory function satisfying the following conditions:*

(i) *for any $R > 0$, there exists $h_R \in \mathrm{L}^1(Q)$ such that $|f(x,\xi)| \leq h_R(x)$ for a.e. $x \in Q$ and for $|\xi| \leq R$;*

(ii) *$f(x,0) = 0$ for a.e. $x \in Q$;*

(iii) *for any $u \in W_0^{1,p}(Q)$,*

$$\int_Q f(x, \nabla u)dx = 0.$$

Then $f(x,\xi) = g(x) \cdot \xi$ for a.e. $x \in Q$ and for any $\xi \in \mathbf{R}^n$, where $g \in \mathrm{L}^1(Q)^n$ and $\operatorname{div} g = 0$.

Proof. First of all, we show that for any ξ_1, $\xi_2 \in \mathbf{R}^n$ and $\lambda \in (0,1)$

$$f(x, \lambda\xi_1 + (1-\lambda)\xi_2) = \lambda f(x, \xi_1) + (1-\lambda) f(x, \xi_2) \tag{2.2.19}$$

a.e. on Q. Set

$$\xi = \lambda\xi_1 + (1-\lambda)\xi_2$$

and

$$\xi_0 = \frac{\xi_2 - \xi_1}{|\xi_2 - \xi_1|}.$$

For any $k \in \mathbf{N}$ and $j \in \mathbf{Z}$, let us define the sets

$$Q^1_{kj} = \left\{ y \in Q : \frac{j-1}{k} < y \cdot \xi_0 < \frac{j-1+\lambda}{k} \right\},$$

$$Q^2_{kj} = \left\{ y \in Q : \frac{j-1+\lambda}{k} < y \cdot \xi_0 < \frac{j}{k} \right\},$$

$$Q^1_k = \bigcup_{j \in \mathbf{Z}} Q^1_{kj}$$

and

$$Q^2_k = \bigcup_{j \in \mathbf{Z}} Q^2_{kj}.$$

It is easy that, in the $*$-weak topology of $\mathrm{L}^\infty(Q)$, the sequence of characteristic functions $1_{Q^1_k}$ of Q^1_k converges to the constant function λ, while the sequence $1_{Q^2_k}$ converges to the constant function $(1-\lambda)$. Now we define the piecewise affine function $u_k(y)$ on Q by the formula

$$u_k(y) = \begin{cases} c^1_{kj} + \xi_1 \cdot y & \text{if } y \in Q^1_{kj}, \\ c^2_{kj} + \xi_2 \cdot y & \text{if } y \in Q^2_{kj}, \end{cases}$$

where

$$c^1_{kj} = \frac{(j-1)(1-\lambda)}{k} |\xi_2 - \xi_1|$$

and

$$c^2_{kj} = -\frac{j\lambda}{k} |\xi_2 - \xi_1|.$$

We set also $u_\xi(y) = \xi \cdot y$.

It is not difficult to verify that

$$|u_k(y) - u_\xi(y)| \leq \frac{\lambda(1-\lambda)}{k}|\xi_2 - \xi_1|$$

for any $y \in Q$. Hence, u_k converges to u_ξ uniformly on Q.

Let Q' be an open set such that $\overline{Q}' \subset Q$. Let $\varphi \in C_0^\infty(Q')$ be a function such that $0 \leq \varphi \leq 1$ on Q', and let $\psi \in C_0^\infty(Q)$ be a function such that $0 \leq \psi \leq 1$ on Q and $\psi = 1$ on Q'. Then the function

$$v_k = \psi u_\xi + \varphi(u_k - u_\xi)$$

belongs to $W_0^{1,p}(Q)$. We have

$$\begin{aligned}
0 &= \int_Q f(x, \nabla v_k)dx = \int_{Q'} f(x, \xi + (u_k - u_\xi)\nabla\varphi + \varphi(\nabla u_k - \xi))dx + \\
&+ \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx = \\
&= \int_{Q'\cap Q_k^1} f(x, \xi + (u_k - u_\xi)\nabla\varphi + \varphi\cdot(\xi_1 - \xi))dx + \\
&+ \int_{Q'\cap Q_k^2} f(x, \xi + (u_k - u_\xi)\nabla\varphi + \varphi\cdot(\xi_2 - \xi))dx + \\
&+ \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx = \\
&= \int_{Q'} f(x, \xi + (u_k - u_\xi)\nabla\varphi + \varphi\cdot(\xi_1 - \xi))1_{Q_k^1}dx + \\
&+ \int_{Q'} f(x, \xi + (u_k - u_\xi)\nabla\varphi + \varphi\cdot(\xi_2 - \xi))1_{Q_k^2}dx + \\
&+ \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx.
\end{aligned}$$

Since $u_\xi = \lim u_k$ uniformly on Q, $\lambda = \lim 1_{Q_k^1}$ and $(1-\lambda) = \lim 1_{Q_k^2}$ $*$-weakly in $L^\infty(Q)$, we get, passing to the limit as $k \to \infty$,

$$\begin{aligned}
0 &= \lambda\int_{Q'} f(x, \xi + \varphi\cdot(\xi_1 - \xi))dx + (1-\lambda)\int_{Q'} f(x, \xi + \varphi\cdot(\xi_2 - \xi))dx + \\
&+ \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx.
\end{aligned}$$

If we take $\varphi = \varphi_k$, where $\varphi_k \to 1_{Q'}$ pointwise and $\varphi_k(x)$ is monotone increasing with respect to k for any $x \in Q'$, we obtain easy

$$\begin{aligned}
0 &= \lambda\int_{Q'} f(x, \xi)dx + (1-\lambda)\int_{Q'} f(x, \xi_2)dx + \\
&+ \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx. \qquad (2.2.20)
\end{aligned}$$

On the other hand, assumption (iii) implies

$$0 = \int_Q f(x, \nabla(\psi u_\xi))dx = \int_{Q'} f(x, \xi)dx + \int_{Q\setminus Q'} f(x, \nabla(\psi u_\xi))dx.$$

Together with (2.2.20), this gives rise to

$$\int_{Q'} f(x,\xi)dx = \lambda \int_{Q'} f(x,\xi_1)dx + (1-\lambda)\int_{Q'} f(x,\xi_2)dx,$$

and we get immediately (2.2.19).

Now we show that $f(x,\cdot)$ is homogeneous of degree 1. Indeed, let $t < 0$. If we set $\lambda = 1/(1-t)$, $\xi_1 = t\xi$ and $\xi_2 = \xi$, we get directly from (2.2.19) that

$$f(x, t\xi) = tf(x,\xi).$$

In particular, $f(x,-\xi) = \varphi(x,\xi)$. If $t > 0$, we have

$$f(x,t\xi) = f(x,(-t)(-\xi)) = tf(x,\xi).$$

Homogeneity of f and (2.2.19) imply that f is linear with respect to ξ, i.e. $f(x,\xi) = g(x)\cdot\xi$. It is obvious that $g \in \mathrm{L}^1(Q)$ and $\operatorname{div} g = 0$. □

Proposition 2.2.8 *Let a_1, $a_2 \in M_Q$ be two functions which are cyclically monotone with respect to ξ a.e on Q. Assume that $a_1(x,\cdot)$ and $a_2(x,\cdot)$ are odd (resp. positively homogeneous of degree $p-1$). Let $\mathcal{A}_1$, $\mathcal{A}_2 \in \mathcal{M}_V$ be the operators associated to a_1 and a_2, respectively. If $\mathcal{A}_1 = \mathcal{A}_2$, then $a_1(x,\xi) = a_2(x,\xi)$ for a.e. $x \in Q$ and for any $\xi \in \mathbf{R}^n$.*

Proof. Our assumptoins imply that there exist two functions f_1, $f_2 \in \mathcal{F}_Q$ such that

$$a_i(x,\xi) = \partial_\xi f_i(x,\xi), \quad i = 1,2,$$

for a.e. $x \in Q$ and for any $\xi \in \mathbf{R}^n$. Moreover, $f_i(x,0) = 0$, $i = 1,2$, a.e. on Q. Since $\mathcal{A}_1 = \partial F_1 = \mathcal{A}_2 = \partial F_2$, we have $F_1(u) = F_2(u)$ for any $u \in V = W_0^{1,p}(Q)$. By Proposition 2.2.7, there exists $g \in \mathrm{L}^1(Q)^n$ such that $\operatorname{div} g = 0$ and

$$f_1(x,\xi) - f_2(x,\xi) = g(x)\cdot\xi$$

for a.e. $x \in Q$, for any $\xi \in \mathbf{R}^n$. On the other hand, $f_i(x,\xi)$ is even (resp. positively homogeneous of degree p). Hence, $g = 0$ a.e. on Q and we conclude that $f_1(x,\xi) = f_2(x,\xi)$ for a.e. $x \in Q$, for any $\xi \in \mathbf{R}^n$. This implies immediately the assertion of the proposition. □

Now we are able to treate strong G-convergence of cyclically monotone (variational) elliptic operators. First of all, the following simple statement holds true.

Proposition 2.2.9 *Let $(\mathcal{A}_k) \subset \mathcal{M}_V$ be a sequence which srongly G-converges to $\mathcal{A} \in \mathcal{M}_V$. If $a_k(x,\cdot)$ is cyclically monotone (resp. odd, or positively homogeneous of degree $p-1$) for a.e. $x \in Q$, for any $k \in \mathbf{N}$, then $a(x,\cdot)$ is cyclically monotone (resp. odd, or positively homogeneous of degree $p-1$) for a.e. $x \in Q$.*

Next, we have the following

Theorem 2.2.5 *Let $\mathcal{A}_k$, $\mathcal{A} \in \mathcal{M}_V$ be operators such that for a.e. $x \in Q$ the associated functions $a_k(x,\cdot)$, $a(x,\cdot)$ are cyclically monotone and odd (resp. positively homogeneous of degree $p-1$). Then the following statements are equivalent:*

(i) *$\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$;*

(ii) *$\mathcal{A}_k$ G-converges to $\mathcal{A}$.*

*Proof.*The implication (i) $\Rightarrow$ (ii) follows immediately from Proposition 2.2.2

To prove the implication (ii) $\Rightarrow$ (i) we may assume that there exists $b \in M_Q$ such that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{B}$, where $\mathcal{B} \in \mathcal{M}_V$ is associated to the function b. Therefore, we have only to prove that $a = b$. Since (i) implies (ii), it follows that $\mathcal{A}_k \xrightarrow{G} \mathcal{B}$ and we have $\mathcal{A} = \mathcal{B}$. By Proposition 2.2.9, $b(x,\cdot)$ is cyclically monotone and odd (resp. positively homogeneous of degree $p-1$). Now Proposition 2.2.8 implies that $a(x,\xi) = b(x,\xi)$ for a.e. $x \in Q$, for any $\xi \in \mathbf{R}^n$, and the proof is complete. $\square$

2.2.5 Other Boundary Conditions

Let us come back to the situation of Theorem 2.2.3. Let $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, $f_k \to f$ strongly in V^*, and $u_k \to u$ weakly in $\overline{V}$. If u_k solves the equation

$$\mathcal{A}_k u_k = f_k,$$

then, as we have seen in Remark 2.2.1, u is a solution of the limit equation

$$\mathcal{A}u = f.$$

Moreover, if we subject u_k to a fixed variational boundary condition, not depending on k, then it is not difficult to show that u satisfies the same boundary condition. Perhaps, it is more interesting that in the definition of strong G-convergence one can use any such boundary condition instead of the Dirichlet condition [137]. We outline here this result.

Suppose V_0 is a closed linear subspase of $\overline{V} = W^{1,p}(Q)$ such that

$$V = W_0^{1,p}(Q) \subset V_0.$$

We consider the following boundary value problem:

$$-\operatorname{div} a(x, \nabla u) \ni f, \quad u \in V_0. \tag{2.2.21}$$

Of course, more general case of boundary condition $u - \varphi \in V_0$, $\varphi \in \overline{V}$, may be treated. We always assume that $a \in M_Q$.

The choice $V_0 = V$ leads to the Dirichlet boundary condition, while $V_0 = \overline{V}$ corresponds to the Neumann one. Suppose Γ_0 is a proper open part of ∂Q. Then one can take the space V_0 consisting of all functions $v \in \overline{V}$ such that $v = 0$ on Γ_0. This choice of V_0 serves mixed Dirichlet-Neumann boundary conditions. At last, let us define V_0 to be the space of all functions $v \in \overline{V}$ such that $v = const$ on ∂Q. Then, assuming for simplicity $a(x, \xi)$ to be singlr-valued, the boundary value problem under consideration reads, formally,

$$-\operatorname{div} a(x, \nabla u) = f, \quad \text{on } Q,$$

$$u = const \quad \text{on } \partial Q,$$

$$\int_{\partial Q} a(x, \nabla u) \cdot \nabla u ds = -\int_Q f dx.$$

This is the so-called electrostatic boundary value problem.

Now let us present an operator setting for boundary value problems we consider. First, we define the operator

$$\Lambda_{V_0} : \mathrm{L}^{p'}(Q)^n \longrightarrow V_0^*$$

by the formula

$$\langle \Lambda_{V_0} g, v \rangle = \int_Q g \cdot \nabla v dx$$

for any $g \in \mathrm{L}^{p'}(Q)^n$, $v \in V_0$, where $\langle \cdot, \cdot \rangle$ stands now for the duality pairing on $V_0^* \times V_0$. Exactly as in Definition 2.1.2, we introduce the set M_{V_0} of operators

$$A : V_0 \longrightarrow \mathrm{L}^{p'}(Q)^n$$

satisfying conditions (i) and (ii) of that Definition. Associated to $A \in M_{V_0}$, there exists the operator $\mathcal{A}^{V_0}$ defined by

$$\mathcal{A}^{V_0} = \Lambda_{V_0} A. \tag{2.2.22}$$

This operator acts from V_0 into V_0^*. We denote by $\hat{\mathcal{M}}_{V_0}$ the set of all such operators, while $\mathcal{M}_{V_0}$ stands for the set of all maximal monotone operators which belongs to $\hat{\mathcal{M}}_{V_0}$. Any function $a(x, \xi) \in M_Q$ generates the operator $A \in M_{V_0}$ defined by

$$Au = \left\{ g \in \mathrm{L}^{p'}(Q)^n : g(x) \in a(x, \nabla u(x)) \text{ a.e. on } Q \right\}, \quad u \in V_0,$$

and, accoding to (2.2.22), the operator $\mathcal{A}^{V_0} \in \mathcal{M}_{V_0}$.Now boundary value problem (2.2.21) becomes

$$\mathcal{A}^{V_0} u \ni f.$$

As usually, we denote by w and σ_1 the weak topologies on $\overline{V}$ and $\mathrm{L}^{p'}(Q)^n$, respectively. Let σ_{2,V_0} be the topology on $\mathrm{L}^{p'}(Q)^n$ induced by the pseudo-metric

$$d(g_1, g_2) = \|\Lambda_{V_0}(g_1 - g_2)\|_{V_0^*}.$$

Then we denote by σ_{V_0} the weakest topology on $\mathrm{L}^{p'}(Q)^n$ which is stronger than σ_1 and σ_{2,V_0}. This topology is stronger than the topology σ considered in n^0 2.2.1. In general, $\sigma_{V_0} \neq \sigma$. To see this, assume that $p = 2$, $V_0 = \overline{V}$, and ∂Q is smooth. Let $\varphi_k \to 0$ weakly but not strongly in $H^{1/2}(\partial Q)$. Consider the solution u_k to the Dirichlet problem

$$-\Delta u_k = 0, \quad u_k = \varphi_k \quad \text{on } \partial Q.$$

Obviously, $u_k \to 0$ weakly in $\overline{V} = H^1(Q)$. Then $g_k = \nabla u_k \to 0$ in the topology σ. However, g_k does not converges to 0 in σ_{V_0}. Indeed, if $g_k \to 0$ in σ_{V_0}, then

$$(\Lambda_{V_0} g_k, u_k) = \int_Q |\nabla u_k|^2 dx \to 0,$$

which implies that $u_k \to 0$ strongly in $H^1(Q)$, hence, $\varphi_k \to 0$ strongly in $H^{1/2}(\partial Q)$.

Theorem 2.2.6 *Let V_0 be a closed linear subspace between V and $\overline{V}$, $a_k \in M_Q$ and $a \in M_Q$. Then the following statements are equivalent:*

(i) *a_k strongly G-converges to a;*

(ii) *$K_s(w \times \sigma_{V_0})$-$\limsup A_k \subset A$,*

where A_k and A are the operators in M_{V_0} associated to a_k and a, respectively.

For the proof we refer to [137].

Of course, as in the case $V_0 = V$, condition (ii) implies that $\mathcal{A}_k^{V_0} \xrightarrow{G} \mathcal{A}^{V_0}$. We have also to point out that, in general, the operator $\mathcal{A}^{V_0}$ associated to $a \in M_Q$ does not satisfy the coerciveness condition. However, the operator $\mathcal{A}^{V_0} + \lambda J$, where $\lambda > 0$ and $Ju = |u|^{p-2}u$, is coercive and strictly monotone. Notice that $\mathcal{A}_k^{V_0} \xrightarrow{G} \mathcal{A}^{V_0}$ if and only if

$$\mathcal{A}_k^{V_0} + \lambda J \xrightarrow{G} \mathcal{A}^{V_0} + \lambda J, \quad \lambda > 0,$$

and the last property may be expressed in terms of convergence of u_k to u, where u_k is a unique solution of the equation

$$\mathcal{A}_k^{V_0} u_k + \lambda J u_k \ni f$$

and u is a unique solution of the corresponding limit problem.

2.3 Strong G-convergence for Single-Valued Elliptic Operators

2.3.1 Main Results

In this section we consider strong G-convergence of single-valued, but, generally, non-monotone, elliptic operators acting from

$$V = W_0^{1,p}(Q)$$

into

$$V^* = W^{-1,p'}(Q).$$

We define this concept in a different way than in the previous section. However, later on we shall show that on a reasonable class of operators these two definitions are equivalent.

Thus, we consider an operator $\mathcal{A} : V \longrightarrow V^*$ of the class $\hat{\mathcal{E}}$ (see Definition 2.1.9), i.e $\mathcal{A}$ is defined by the formula

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u), \tag{2.3.1}$$

where

$$a : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

and

$$a_0 : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

are two Carathéodory functions satisfying inequalities (2.1.29) and (2.1.30). For any $v \in \overline{V}$, we set

$$\mathcal{A}^1(u, v) = -\operatorname{div} a(x, v, \nabla u),$$

$$\mathcal{A}^0(u, v) = a_0(x, v, \nabla u),$$

and

$$\mathcal{A}(u, v) = \mathcal{A}^1(u, v) + \mathcal{A}^0(u, v).$$

So, for any $v \in \overline{V}$ we have defined the operators

$$\mathcal{A}^i(\cdot, v) : V \longrightarrow V^*, \quad i = 0, 1,$$

and

$$\mathcal{A}(\cdot, v) : V \longrightarrow V^*.$$

Obviously,

$$\mathcal{A}u = \mathcal{A}(u, u), \quad u \in V.$$

The operator $\mathcal{A}^1(\cdot, v)$ belongs to a suitable class $\mathcal{U} \cap \mathcal{S}$. Hence, for any $v \in \overline{V}$, this operator is invertible as an operator from V into V^*.

Now let us consider operators $\mathcal{A}_k \in \hat{\mathcal{E}}$, $k \in \mathbf{N}$,

$$\mathcal{A}_k u = -\operatorname{div} a^k(x, u, \nabla u) + a_0^k(x, u, \nabla u),$$

and $\mathcal{A} \in \hat{\mathcal{E}}$ of the form (2.3.1). For $u, v \in V$, we define $u_k \in V$ as a unique solution of the equation

$$\mathcal{A}_k^1(u_k, v) = \mathcal{A}^1(u, v).$$

We set

$$\Gamma^k(u, v) = a^k(x, v, \nabla u_k),$$

$$\Gamma_0^k(u, v) = a_0^k(x, v, \nabla u_k),$$

$$\Gamma(u, v) = a(x, v, \nabla u),$$

and

$$\Gamma_0(u, v) = a_0(x, v, \nabla u).$$

It is easy to see that

$$\Gamma^k, \Gamma : V \times V \longrightarrow \mathrm{L}^{p'}(Q)^n$$

and

$$\Gamma_0^k, \Gamma_0 : V \times V \longrightarrow \mathrm{L}^{p'}(Q)$$

are well-defined continuous operators. We call these operators *momenta* or *generalized gradients* for the system of operators $(\mathcal{A}_k, \mathcal{A})$.

Definition 2.3.1 *We say that the sequence of operators $(\mathcal{A}_k)$ is strongly G-convergent to $\mathcal{A}$ if for any $u, v \in V$*

1. *$u_k \rightarrow u$ weakly in V;*

2. *$\Gamma^k(u, v) \rightarrow \Gamma(u, v)$ weakly in $L^{p'}(Q)^n$ and $\Gamma_0^k(u, v) \rightarrow \Gamma_0(u, v)$ weakly in $L^{p'}(Q)$.*

As above, we shall write $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ if $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$.

Remark 2.3.1 It is evident that, for any $v \in V$, operators $\mathcal{A}_k^1(\cdot, v)$ strongly G-converge to $\mathcal{A}^1(\cdot, v)$ in the sense of Section 2.2 provided $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ in the sense of Definition 2.3.1. Hence, if the operators we consider depend only on ∇u, not on u, and contain no lower order terms, then the notion of strong G-convergence we have just introduced coincides with that we considered in Section 2.2. Also it is easy that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ if and only if $\mathcal{A}_k(\cdot, v) \overset{G}{\Longrightarrow} \mathcal{A}(\cdot, v)$ for any $v \in V$.

Proposition 2.3.1 *Strong G-limit of a sequence of operators $(\mathcal{A}_k) \subset \hat{\mathcal{E}}$ is unique.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ are two strong G-limit for $(\mathcal{A}_k)$. Then, for any $v \in V$, we have

$$\mathcal{A}_k^{(1)}(\cdot, v) \xrightarrow{G} \mathcal{A}^{(1)}(\cdot, v),$$

and

$$\mathcal{A}_k^{(1)}(\cdot, v) \xrightarrow{G} \mathcal{B}^{(1)}(\cdot, v),$$

and $\mathcal{A}^{(1)}(\cdot, v) = \mathcal{B}^{(1)}(\cdot, v)$ as abstract operators from V into V'. Now Definition 2.3.1 implies immediately that $a(x, v, \nabla u) = b(x, v, \nabla u)$ a.e. in Q for any $u, v \in V$ and the similar statement holds for a_0 and b_0. Choosing appropriate test functions u and v we conclude that $a(x, \xi_0, \xi) = b(x, \xi_0, \xi)$ and $a_0(x, \xi_0, \xi) = b_0(x, \xi_0, \xi)$ for all $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$ and almost all $x \in Q$. □

We note the following properties which can be easily verified:

(i) *if* $\mathcal{A}_k = \mathcal{A}$ for any $k \in \mathbf{N}$, *then* $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$;

(ii) *if* $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, *then* $\mathcal{A}_{\sigma(k)} \overset{G}{\Longrightarrow} \mathcal{A}$ *for any subsequence* $\sigma(k)$;

(iii) $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ *if and only if for any subsequence* $\sigma(k)$ *there exists a further subsequence* $\tau(\sigma(k))$ *such that* $\mathcal{A}_{\tau(\sigma(k))} \overset{G}{\Longrightarrow} \mathcal{A}$.

It must be pointed out that the class of operators $\hat{\mathcal{E}}$ is too wide in order to get reasonable results on strong G-convergence. Therefore, we want to define a more restricted class of operators. Recall that in Definition 2.1.9 we have fixed constants $p > 1$, $c_0 > 0$, $\kappa > 0$, and

$$\beta \geq \max(p, 2),$$

and nonnegative functions $h, c \in \mathrm{L}^1(Q)$. Additionally, let us fix constants $\theta > 0$ and

$$s \in (0, \min(p, p')]$$

and a modulus of continuity $\nu(r)$, i.e. a continuous nondecreasing function on $[0, +\infty)$ such that $\nu(0) = 0$, $\nu(r) > 0$ for $r > 0$, and $\nu(r) = 1$ for $r \geq 1$ (the last assumption is imposed for the sake of convenience only). We shall consider operators from the class $\hat{\mathcal{E}}$, i.e. of the form (2.3.1) satisfying inequalities (2.1.29) and (2.1.30), such that the following inequality is fulfilled for almost all $x \in Q$ and for all $\zeta = (\xi_0, \xi)$, $\zeta' = (\xi_0', \xi) \in \mathbf{R}^{n+1}$:

$$\begin{aligned} |a(x,\xi_0,\xi) - a(x,\xi_0',\xi')|^{p'} + |a_0(x,\xi_0,\xi) - a_0(x,\xi_0',\xi)|^{p'} \leq \\ \leq \theta \left[(h(x) + |\zeta|^p + |\zeta'|^p) \cdot \nu(|\xi_0 - \xi_0'|) + \right. \\ \left. + (h(x) + |\zeta|^p + |\zeta'|^p)^{1-s/p} |\xi - \xi'|^s \right]. \end{aligned} \tag{2.3.2}$$

Definition 2.3.2 *Denote by* $\mathcal{E} = \mathcal{E}(c_0, c, \kappa, h, \theta, \nu, s, \beta)$ *the set of operators* $\mathcal{A} \in \hat{\mathcal{E}}$ *satisfying inequality (2.3.2). If we need to indicate the domain* Q *explicitly, we write* $\mathcal{E}_Q$.

Now we state the main result – the compactness theorem.

Theorem 2.3.1 *For any sequence* $(\mathcal{A}_k) \subset \mathcal{E}(c_0, c, \kappa, h, \theta. \nu, s, \beta)$, *there exists a subsequence* $\sigma(k)$ *such that* $\mathcal{A}_{\sigma(k)} \stackrel{G}{\Longrightarrow} \mathcal{A}$, *where* $\mathcal{A} \in \mathcal{E}(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{h}, \overline{\theta}, \overline{\nu}, \overline{s}, \beta)$, *with*

$$\overline{s} = \frac{sp}{\beta p - sp + s},$$

$$\overline{\nu}(r) = \nu^{s/\beta}(r),$$

$$\overline{c}(x) = K\overline{h}(x),$$

$$\overline{h}(x) = c(x) + h(x),$$

and the positive constants $\overline{c}_0$, $\overline{\kappa}$, $\overline{\theta}$ *and* K *depending on* c_0,κ, θ *only.*

In what follows all the overlined parameters are the same as in Theorem 2.3.1.

For any operator $\mathcal{A} \in \mathcal{E}_Q$ and for any subdomain $Q_1 \subset Q$, it is defined the restriction $\mathcal{A}|_{Q_1}$ which belongs, evidently, to the class $\mathcal{E}_{Q_1}$. Strong G-convergence possesses the following localization property.

Theorem 2.3.2 *Let* $(\mathcal{A}_k) \subset \mathcal{E}_Q$ *and* $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$. *Then* $\mathcal{A}_k|_{Q_1} \stackrel{G}{\Longrightarrow} \mathcal{A}|_{Q_1}$ *for any subdomain* $Q_1 \subset Q$.

Corollary 2.3.1 *Let* $(\mathcal{A}_k)$, $(\mathcal{B}_k) \subset \mathcal{E}_Q$. *Assume that* $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$ *and* $\mathcal{B}_k \stackrel{G}{\Longrightarrow} \mathcal{B}$. *If* $\mathcal{A}_k|_{Q_1} = \mathcal{B}_k|_{Q_1}$ *for a subdomain* $Q_1 \subset Q$, *then* $\mathcal{A}|_{Q_1} = \mathcal{B}|_{Q_1}$.

Corollary 2.3.2 *Let $(Q_i)_{i\in I}$ be a finite family of open subsets of Q such that*

$$|Q \setminus \cup Q_i| = 0.$$

A sequence $(\mathcal{A}_k) \subset \mathcal{E}_Q$ strongly G-converges to $\mathcal{A}$ if and only if

$$\mathcal{A}_{k|Q_i} \overset{G}{\Longrightarrow} \mathcal{A}_{|Q_i}$$

for any $i \in I$.

The next result is called often the theorem on convergence of arbitrary solutions. It shows that, as in monotone case, the Dirichlet problem plays no special role in the theory of G-convergence. This result is also very useful in applications to homogenization problems.

Theorem 2.3.3 *Assume that $(\mathcal{A}_k) \subset \mathcal{E}$ and $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Let $(v_k) \in \overline{V}$ be a sequence such that $\mathcal{A}_k v_k = f_k \to f$ strongly in V^* and $v_k \to u$ weakly in $\overline{V}$. Then $\mathcal{A}u = f$,*

$$a^k(x, v_k, \nabla v_k) \to a(x, u, \nabla u)$$

weakly in $L^{p'}(Q)^n$, and

$$a_0^k(x, v_k, \nabla v_k) \to a_0(x, u, \nabla u)$$

weakly in $L^{p'}(Q)$.

The proofs of these theorems will be presented in the next two subsections. Fiist, in n^0 2.3.2, we prove these results in the particular case when the operators we consider depend only on the gradient, ∇u, and do not contain the lower order term, a_0. Such operators are monotone and in this case the statements of Theorems 2.3.1 – 2.3.3 are contained in the results of Section 2. Nevertheless, we prefer to give different proofs based on another, more elementary, techiniques. The general case will be treated later on, in n^0 2.3.3.

Associated to an operator $\mathcal{A} \in \hat{\mathcal{E}}$, there is the energy density defined by the formula

$$E(u)(x) = a(x, u, \nabla u) \cdot \nabla u + a_0(x, u, \nabla u)u. \tag{2.3.3}$$

The following result shows that strong G-convergence of elliptic operators is accompanied by convergence of corresponding energy densities.

Theorem 2.3.4 *Under the assumptions of Theorem 2.3.3, $E_k(v_k) \to E(u)$ weakly in the sense of distributions.*

Proof. For any $\varphi \in C_0^\infty$ we have

$$(E_k(v_k), \varphi) = (\mathcal{A}_k v_k, \varphi v_k) - \int_Q a^k(x, v_k, \nabla v_k) \cdot (v_k \nabla \varphi) dx.$$

By assumption, $\mathcal{A}_k v_k \longrightarrow f$ strongly in V^* and $\varphi v_k \longrightarrow \varphi v$ weakly in V. Moreover, by the Sobolev Embedding Theorem,

$$v_k \nabla \varphi \longrightarrow u \nabla \varphi$$

strongly in $\mathrm{L}^p(Q)^n$. Since, by Theorem 2.3.3,

$$a^k(x, v_k, \nabla v_k) \longrightarrow a(x, u, \nabla u)$$

weakly in $\mathrm{L}^{p'}(Q)^n$, we get our statement. □

2.3.2 Proofs of Main Results: Particular Case

Here we consider the subclass $\mathcal{E}_0 = \mathcal{E}_0(c_0, c, \kappa, h, \theta, s, \beta) \subset \mathcal{E}$ of operators of the form

$$\mathcal{A}u = -\operatorname{div} a(x, \nabla u),$$

where the Carathéodory function $a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$ satisfies inequalities (2.1.29), (2.1.30) and (2.3.2). Of course, the class $\mathcal{E}_0$ does not depend on ν, the modulus of continuity involved in the definition of $\mathcal{E}$. It is not difficult to see that the set of operators $\mathcal{E}_0$ is contained in $\mathcal{K}(c_0, c_1, c_2, \kappa, \theta, \beta, s)$, the class of abstract operators from V into V^* introduced in Definition 1.2.6, where

$$c_1 = c_1(Q) = \int_Q c(x) dx, \tag{2.3.4}$$

and

$$c_2 = c_2(Q) = \int_Q h(x) dx. \tag{2.3.5}$$

Recall that everywhere in the present chapter $V = W_0^{1,p}(Q)$. As usual, we do not distinguish by notations an operator $\mathcal{A} \in \mathcal{E}_0$ and its natural extension acting from $\overline{V} = W^{1,p}(Q)$ into $V^* = W^{-1,p'}(Q)$.

First of all, we need the following technical result.

Lemma 2.3.1 *Let $(\mathcal{A}_k) \subset \mathcal{E}_0$ and let (u_k), (v_k) be bounded sequences in $\overline{V}$ such that $z_k = u_k - v_k \longrightarrow 0$ weakly in $\overline{V}$. Assume that $\mathcal{A}_k u_k \longrightarrow f$ and $\mathcal{A}_k v_k \longrightarrow g$ strongly in V^*. Then $f = g$ and $z_k \longrightarrow 0$ strongly in $W_{loc}^{1,p}(Q)$.*

Proof. Let

$$Z^k = a^k(x, \nabla u_k) - a^k(x, \nabla v_k).$$

The sequence (Z^k) is bounded in $\mathrm{L}^{p'}(Q)^n$. By (2.1.30), for any $\varphi \in C_0^\infty(Q)$, with $0 \leq \varphi \leq 1$, we have

$$\int_Q (Z^k \cdot \nabla z_k)\varphi dx \geq C \cdot \|\varphi \nabla z_k\|_p^\beta. \tag{2.3.6}$$

The left hand side of the last inequality is equal to

$$\int_Q Z^k \cdot \nabla(\varphi z_k)dx - \int_Q Z^k \cdot z_k \nabla \varphi dx = (\mathcal{A}_k u_k - \mathcal{A}_k v_k, \varphi z_k) - \\ - \int_Q Z^k \cdot z_k \nabla \varphi dx.$$

The assumptions of the lemma imply obviously that the first term in the right hand side of the last identity tends to zero. As for the second term, the Sobolev Embedding Theorem shows that $z_k \nabla \varphi \to 0$ strongly in $\mathrm{L}^p(Q)^n$. Since (Z^k) is bounded, this term tends to zero as well. So, we have stated that the left hand side of (2.3.6) converges to zero. Hence, $\varphi \nabla z_k \to 0$ strongly in $\mathrm{L}^p(Q)^n$ and $z_k \to 0$ strongly in $W^{1,p}_{loc}(Q)$.

Now inequality (2.3.2) for a^k implies that $Z^k \to 0$ strongly in $\mathrm{L}^{p'}_{loc}(Q)^n$. Since this sequence is bounded in $\mathrm{L}^{p'}(Q)^n$, it converges to zero weakly in $\mathrm{L}^{p'}(Q)^n$. Hence,

$$\mathcal{A}_k u_k - \mathcal{A}_k v_k = -\operatorname{div} Z^k \to 0$$

weakly in V^* and this implies that $f = g$. □

Associated to an operator $\mathcal{A}$ of the form (2.3.1), there is an operator

$$\overline{\mathcal{A}} : \mathrm{L}^p(Q)^n \longrightarrow V^*$$

defined by

$$\overline{\mathcal{A}}\psi = -\operatorname{div} a(x, \psi).$$

Obviously, $\mathcal{A} = \overline{\mathcal{A}} \circ \nabla$. Now we introduce the family of "shifted" operators

$$\overline{\mathcal{A}}^\psi : \mathrm{L}^p(Q)^n \longrightarrow V^*, \quad \psi \in \mathrm{L}^p(Q)^n,$$

defined by the formula

$$\overline{\mathcal{A}}^\psi \chi = \overline{\mathcal{A}}(\psi + \chi).$$

We set also

$$\mathcal{A}^\psi = \overline{\mathcal{A}}^\psi \circ \nabla.$$

The last operator is considered as a map from V (or $\overline{V}$) into V^*. If $\mathcal{A} \in \mathcal{E}_0$, it is easy to verify the following properties

$$\mathcal{A}^{\psi+\nabla w}(u) = \mathcal{A}^{\psi}(u+w), \tag{2.3.7}$$

$$\|\mathcal{A}^{\psi}u\|_*^{p'} \leq \overline{c}_0 \left(\|u\|^p + \|\psi\|_p^p\right) + c_1, \tag{2.3.8}$$

$$(\mathcal{A}^{\psi}u - \mathcal{A}^{\psi}w, u-w) \geq \overline{\kappa} H(u,w,\psi)^{1-\beta/p} \cdot \|u-w\|^{\beta}, \tag{2.3.9}$$

$$\|\mathcal{A}^{\psi}u - \mathcal{A}^{\psi}w\|_*^{p'} \leq \overline{\theta} H(u,w,\psi)^{1-s/p} \cdot \|u-w\|^{s}, \tag{2.3.10}$$

$$\|\mathcal{A}^{\psi}u - \mathcal{A}^{\psi'}u\|_*^{p'} \leq \overline{\theta} H(u,\psi,\psi')^{1-s/p} \cdot \|\psi-\psi'\|_p^{s}, \tag{2.3.11}$$

for any $u, w \in V$ and $\psi, \psi' \in \mathrm{L}^p(Q)^n$. Here we use the notation

$$H(u,w,\ldots) = c_2 + \|u\|^p + \|w\|^p + \ldots .$$

For any $\psi \in \mathrm{L}^p(Q)^n$, the operator $\mathcal{A}^{\psi} : V \to V^*$ is invertible. Hence, there is a well-defined operator

$$\mathcal{R} : V^* \times \mathrm{L}^p(Q)^n \longrightarrow V$$

acting by the formula

$$\mathcal{R}(f,\psi) = (\mathcal{A}^{\psi})^{-1} f, \quad f \in V^*, \ \psi \in \mathrm{L}^p(Q)^n. \tag{2.3.12}$$

We have

$$\mathcal{R}(f, \psi + \nabla w) = \mathcal{R}(f,\psi) + w, \tag{2.3.13}$$

$$\|\mathcal{R}(f,\psi)\|^p \leq K \cdot \left(\|f\|_*^{p'} + \|\psi\|_p^p + c_1 + c_2\right), \tag{2.3.14}$$

for any $f \in V^*$, $\psi \in \mathrm{L}^p(Q)^n$, and $w \in V$, where $K > 0$ does not depend on c_1 and c_2.

Now we are able to prove the following particular case of Theorem 2.3.1:

Lemma 2.3.2 *Any sequence* $\mathcal{A}_k \in \mathcal{E}_0(c_0, c, \kappa, h, \theta, s, \beta)$ *contains a subsequence which strongly* G*-converges to an operator* $\mathcal{A} \in \mathcal{E}_0(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{h}, \overline{\theta}, \overline{s}, \beta)$.

Proof. We divide the proof into several steps.

Step 1. First of all, we recall that the spase $\mathrm{L}^p(Q)^n$ is separable. Hence, by Theorem 1.2.1, we may assume that the sequence $\mathcal{A}_k^{\psi}$ is G-convergent to an abstract operator $\mathcal{A}^{\psi} : V \longrightarrow V^*$, for any ψ belonging to a dense countable subset of $\mathrm{L}^p(Q)^n$. Since $\mathcal{A}_k^{\psi}$ depends continuously, in the metric defined by (1.2.24), on $\psi \in \mathrm{L}^p(Q)^n$ uniformly with respect to $k \in \mathbf{N}$, Corollary 1.2.1 implies that $\mathcal{A}_k^{\psi} \xrightarrow{G} \mathcal{A}^{\psi}$ for *any*

$\psi \in \mathrm{L}^p(Q)^n$ and $\mathcal{A}^\psi$ depends continuously on ψ. Moreover, by Proposition 1.2.14, we see that $\mathcal{A}^\psi$ satisfies inequality (2.3.9) and the following inequalities

$$\|\mathcal{A}^\psi u\|_*^{p'} \le \bar{c}_0(\|u\|^p + \|\psi\|_p^p) + K(c_1 + c_2), \tag{2.3.15}$$

$$\|\mathcal{A}^\psi u - \mathcal{A}^\psi w\|_*^{p'} \le \bar{\theta} H_1(u, w, \psi)^{1-\bar{s}/p}\|u - v\|^{\bar{s}} \tag{2.3.16}$$

with, possibly, new constants $\bar{c}_0$ and $\bar{\theta}$, where

$$H_1(\cdot) = H(\cdot) + c_1.$$

Now we set $\mathcal{A} = \mathcal{A}^0$ and $\mathcal{R}(f, \psi) = (\mathcal{A}^\psi)^{-1} f$. Then, by definition of G-convergence and Proposition 1.2.7, $\mathcal{R}_k(f, \psi) \to \mathcal{R}(f, \psi)$ weakly in V for any $(f, \psi) \in V^* \times \mathrm{L}^p(Q)^n$, where $\mathcal{R}_k$ is associated to $\mathcal{A}_k$ according to (2.3.12). It is evident that $\mathcal{A}^\psi$ and $\mathcal{R}$ satisfy relations (2.3.7) and (2.3.13), respectively.

Let us define the operator $\overline{\mathcal{A}} : \mathrm{L}^p(Q)^n \longrightarrow V^*$ by the formula

$$\overline{\mathcal{A}}\psi = \mathcal{A}^\psi(0).$$

By (2.3.7), we have

$$\mathcal{A} = \overline{\mathcal{A}} \circ \nabla,$$

while (2.3.15) yields

$$\|\overline{\mathcal{A}}\psi\|_*^{p'} \le \bar{c}_0\|\psi\|_p^p + K(c_1 + c_2), \quad \psi \in \mathrm{L}^p(Q)^n. \tag{2.3.17}$$

Finally, the operator $\mathcal{A}$ may be extended to $\overline{V}$ as the composition $\overline{\mathcal{A}} \circ \nabla$.

Step 2. For any $\psi \in \mathrm{L}^p(Q)^n$, let us define

$$\psi_k = \psi + \nabla\mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi) = \psi + \nabla u_k^1, \tag{2.3.18}$$

where

$$u_k^1 = \mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi) \in V.$$

In the case $\psi = \nabla u$, $u \in \overline{V}$, we have $\psi_k = \nabla u_k$, where $u_k = u + u_k^1$. Obviously, $u_k^1 \to 0$ weakly in V.

Now we introduce the operator

$$\overline{\Gamma}^k : \mathrm{L}^p(Q)^n \longrightarrow \mathrm{L}^{p'}(Q)^n$$

by the formula

$$\overline{\Gamma}^k(\psi) = a^k(x, \psi_k), \quad \psi \in \mathrm{L}^p(Q)^n.$$

Using inequality (2.3.17), inequality (2.3.14) for the operator $\mathcal{R}_k$, and inequality (2.1.29) for a^k, we see immediately that

$$\|\overline{\Gamma}^k(\psi)\|_p^{p'} \le \bar{c}_0\|\psi\|_p^p + K(c_1 + c_2), \quad \psi \in \mathrm{L}^p(Q)^n. \tag{2.3.19}$$

We set also $\Gamma^k(u) = \overline{\Gamma}^k(\nabla u)$ for $u \in \overline{V}$.

Step 3. Let Q_1 be an open subset of Q and let $\varphi, \psi \in \mathrm{L}^p(Q)^n$ be two vector functions such that $\varphi_{|Q_1} = \psi_{|Q_1}$. We claim that

$$(\overline{\mathcal{A}}\varphi)_{|Q_1} = (\overline{\mathcal{A}}\psi)_{|Q_1}.$$

To prove this we consider the functions ψ_k defined by (2.3.18) and the functions φ_k associated to φ in the similar way, i.e.

$$\varphi_k = \varphi + \nabla v_k^1,$$

where

$$v_k^1 = \mathcal{R}(\overline{\mathcal{A}}\varphi, \varphi).$$

Since $u_k^1, v_k^1 \to 0$ weakly in V, we see that

$$(u_k^1 - v_k^1)_{|Q_1} \to 0$$

weakly in $W^{1,p}(Q_1)$. However,

$$\mathcal{A}_k^\psi(u_k^1)_{|Q_1} = \overline{\mathcal{A}}(\psi)_{|Q_1},$$

$$\mathcal{A}_k^\varphi(v_k^1)_{|Q_1} = \overline{\mathcal{A}}(\varphi)_{|Q_1},$$

and the operators $\mathcal{A}_k^\psi$ and $\mathcal{A}_k^\varphi$ coincide on Q_1. Applying Lemma 2.3.1, we get our claim.

Moreover, passing to a further subsequence one can construct, as in Step 1, the strong G-limit operator $\mathcal{A}_{(1)}^\psi$ for the sequence $\mathcal{A}_{k|Q_1}^\psi$, and the associated operators $\overline{\mathcal{A}}_{(1)}$ and $\overline{\Gamma}_{(1)}^k$, with underlying domain Q_1. As above, one can show that

$$\overline{\mathcal{A}}_{(1)}(\psi_{|Q_1}) = \overline{\mathcal{A}}(\psi)_{|Q_1}$$

for any $\psi \in \mathrm{L}^p(Q)^n$. In particular, this implies that the passage to a subsequence at this point is supperfluous. Now let us assume that

$$\overline{\Gamma}^k(\psi) \to \overline{\Gamma}(\psi)$$

weakly in $\mathrm{L}^{p'}(Q)$ for any $\psi \in \mathrm{L}^p(Q)^n$, where $\overline{\Gamma}$ is an operator acting from $\mathrm{L}^p(Q)^n$ into $\mathrm{L}^{p'}(Q)^n$. (This will be proved later on). Using the same argument as above one can show that $\overline{\Gamma}$ is a local operator, i.e

$$\overline{\Gamma}(\psi)_{|Q_1} = \overline{\Gamma}(\varphi)_{|Q_1}$$

provided $\psi_{|Q_1} = \varphi_{|Q_1}$. Moreover, associated to the operators $\mathcal{A}_{k|Q_1}$ and $\mathcal{A}_{(1)}$, there is the sequence of momenta $\overline{\Gamma}_{(1)}^k$ (they do not coincide with the restrictions,

$\overline{\Gamma}^k_{|Q_1}$, of $\overline{\Gamma}^k$ to Q_1). In this situation the sequence of operators $\overline{\Gamma}^k_{(1)}$ converges weakly in $\mathrm{L}^{p'}(Q_1)$ to the restriction $\overline{\Gamma}_{|Q_1}$ of $\overline{\Gamma}$ to Q_1 defined in a natural way.

Step 4. Using inequalities (2.3.8) for $\mathcal{A}_k$, (2.3.14) for $\mathcal{R}_k$, and (2.1.30) for a^k, it is not difficult to see that the operator $\mathcal{R}_k$ is Hölderian on any ball in $V^* \times \mathrm{L}^p(Q)^n$, uniformly with respect to $k \in \mathbf{N}$. Hence, by definition of $\overline{\Gamma}^k$, the family of operators $\overline{\Gamma}^k$ is equicontinuous with respect to $k \in \mathbf{N}$ on any ball in $\mathrm{L}^p(Q)^n$. By (2.3.19), we may assume, passing to a subsequense, that $\overline{\Gamma}^k(\psi)$ is weakly convergent in $\mathrm{L}^{p'}(Q)^n$, for any ψ from a dense countable subset of $\mathrm{L}^p(Q)^n$. The equicontinuity of $\overline{\Gamma}^k$ implies that the last is true for any $\psi \in \mathrm{L}^p(Q)^n$. Thus, there exists an operator

$$\overline{\Gamma} : \mathrm{L}^p(Q)^n \longrightarrow \mathrm{L}^{p'}(Q)^n$$

such that $\overline{\Gamma}^k(\psi) \to \overline{\Gamma}(\psi)$ weakly in $\mathrm{L}^{p'}(Q)^n$, for any $\psi \in \mathrm{L}^p(Q)^n$.

Now we shall derive some estimates for the operator Γ. Passing to the limit in (2.3.19) we have

$$\|\overline{\Gamma}(\psi)\|_{p'}^{p'} \le \overline{c}_0 \|\psi\|_p^p + K(c_1 + c_2), \quad \psi \in \mathrm{L}^p(Q)^n. \tag{2.3.20}$$

Moreover,

$$\|\overline{\Gamma}(\psi) - \overline{\Gamma}(\varphi)\|_{p'}^{p'} \le \overline{\theta} H_1(\psi, \varphi)^{1-\overline{s}/p} \cdot \|\psi - \varphi\|_p^{\overline{s}}, \quad \psi, \varphi \in \mathrm{L}^p(Q)^n. \tag{2.3.21}$$

Indeed, given $\psi, \varphi \in \mathrm{L}^p(Q)^n$ we consider $\psi_k = \psi + \nabla u_k^1$ defined by (2.3.18), and φ_k defined in the similar way, i.e. $\varphi = \varphi + \nabla v_k^1$, where $v_k^1 = \mathcal{R}_k(\overline{\mathcal{A}}\varphi, \varphi)$. Let us introduce the following notations:

$$Z^k = \overline{\Gamma}^k(\psi) - \overline{\Gamma}^k(\varphi),$$

$$z_k^1 = u_k^1 - v_k^1,$$

$$\sigma_k = \psi_k - \varphi_k,$$

and

$$\sigma = \psi - \varphi.$$

We note that, by inequality (2.3.2) for $\mathcal{R}_k$, we have

$$H(\psi_k, \varphi_k) \le C H_1(\psi, \varphi). \tag{2.3.22}$$

This, together with inequality (2.3.2) for a^k, yields

$$\|Z^k\|_{p'}^{p'} \le \overline{\theta} H_1^{1-s/p} \|\sigma_k\|_p^s, \tag{2.3.23}$$

where $H_1 = H_1(\varphi, \psi)$. For $y = \overline{\mathcal{A}}\psi - \overline{\mathcal{A}}\varphi$, we have

$$y = -\mathrm{div}\, Z^k.$$

Hence, using (2.1.30) for a^k, (2.3.14) for $\mathcal{R}_k$, and (2.3.26), we obtain

$$\begin{aligned}(y, z_k^1) &= \int_Q Z^k \sigma_k dx - \int_Q Z^k \sigma dx \geq \\ &\geq \overline{\kappa} H_1^{1-\beta/p} \|\sigma_k\|_p^\beta - \overline{\theta} \left(H_1^{1-s/p} \|\sigma_k\|_p^s\right)^{1/p'} \|\sigma\|_p.\end{aligned}$$

To estimate the second term in the right-hand side we use the Young inequality

$$ab \leq \varepsilon a^r + C_\varepsilon b^{r'},$$

with $r = p'\beta/s$ and

$$a = H_1^{s(p-\beta)/(p'p\beta)} \|\sigma_k\|_p^{s/p'}, \quad b = H_1^{(\beta-s)/(\beta p')} \|\sigma\|_p.$$

This yields

$$\left(H_1^{1-s/p} \|\sigma_k\|_p^s\right)^{1/p'} \|\sigma\|_p \leq \varepsilon H_1^{1-\beta/p} \|\sigma_k\|_p^\beta + C_\varepsilon H_1^{(\beta-s)/(\beta p'-s)} \|\sigma\|_p^{\beta p'/(\beta p'-s)}.$$

With ε being sufficiently small, we get

$$\overline{\kappa} H_1^{1-\beta/p} \|\sigma_k\|_p^\beta \leq (y, z_k^1) + C H_1^{(\beta-s)/(\beta p'-s)} \|\sigma\|_p^{\beta p'/(\beta p'-s)},$$

or

$$\|\sigma_k\|_p^s \leq C H_1^{s(\beta-p)/(\beta p)} \left[(y, z_k^1) + H_1^{(\beta-s)/(\beta p'-s)} \|\sigma\|_p^{\beta p'/(\beta p'-s)}\right]^{s/\beta}$$

If we combine this inequality with (2.3.23) and then pass to the limit using weak convergence of z_k^1 to zero, we get, after simple calculations, inequality (2.3.21).

We note that inequality (2.3.21) implies, in particular, the continuity of $\overline{\Gamma}$. Moreover, as we have shown at the end of Step 3, $\overline{\Gamma}$ is a local operator.

Now we prove the inequality

$$\int_Q [\overline{\Gamma}(\psi) - \overline{\Gamma}(\varphi)] \cdot (\psi - \varphi) dx \geq \overline{\kappa} H_1(\psi, \varphi)^{1-\beta/p} \|\psi - \varphi\|_p^\beta. \tag{2.3.24}$$

Besides of the notations we have introduced after inequality (2.3.21), we set

$$Z = \overline{\Gamma}(\psi) - \overline{\Gamma}(\varphi).$$

Let us fix a function $\varphi \in C_0^\infty$ such that $0 \leq \varphi \leq 1$. As in the proof of Lemma 2.3.1, we have

$$\int_Q Z^k \cdot (\varphi \sigma_k) dx \to \int_Q Z \cdot (\varphi \sigma) dx.$$

Inequalities (2.1.30) for a^k and (2.3.22) imply that the last integral is estimated from below by

$$\overline{\kappa} H_1^{1-\beta/p} \|\varphi \sigma_k\|_p^\beta.$$

Since

$$\liminf \|\varphi\sigma_k\|_p \geq \|\varphi\sigma\|_p,$$

passing to the limit we get

$$\int_Q Z \cdot (\varphi\sigma)dx \geq \overline{\kappa} H_1^{1-\beta/p} \|\varphi\sigma\|_p^\beta.$$

This implies (2.3.24).

By definition of $\overline{\Gamma}^k$, we have the identity

$$\int_Q \overline{\Gamma}^k(\psi)\nabla v dx = (\overline{\mathcal{A}}_k\psi_k, v) = (\overline{\mathcal{A}}\psi, v)$$

for any $\psi \in \mathrm{L}^p(Q)^n$ and $v \in V$. Passing to the limit we obtain the representation

$$\overline{\mathcal{A}}\psi = -\operatorname{div}\overline{\Gamma}(\psi), \quad \psi \in \mathrm{L}^p(Q)^n.$$

In particular,

$$\mathcal{A}u = -\operatorname{div}\Gamma(u), \quad u \in V, \tag{2.3.25}$$

where $\Gamma = \overline{\Gamma} \circ \nabla$.

Step 5. To complete the proof we need only to show, in view of (2.3.25), that the operator $\overline{\Gamma}$ is of the form

$$\overline{\Gamma}(\psi) = a(x, \psi) \tag{2.3.26}$$

with an appropriate function $a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$. However, before to do this, we summarize what we have just proved:

- Modulo the passage to a subsequence, we have constructed the operators $\overline{\mathcal{A}}$ and $\overline{\Gamma}$ such that $\mathcal{A}_k^\psi \xrightarrow{G} \mathcal{A}^\psi$ and $\overline{\Gamma}^k(\psi) \rightarrow \overline{\Gamma}(\psi)$ weakly in $\mathrm{L}^{p'}(Q)$, for any $\psi \in \mathrm{L}^p(Q)^n$.
- Both $\overline{\mathcal{A}}$ and $\overline{\Gamma}$ are local operators.
- For any open subset $Q_1 \subset Q$ we have

  $$(\mathcal{A}_k^\psi)_{|Q_1} \xrightarrow{G} \mathcal{A}_{|Q_1}^\psi$$

  and the associated momenta converges weakly to $\overline{\Gamma}_{|Q_1}$, without further passage to a subsequence.
- The operator $\mathcal{A}$ is represented by (2.3.25).

- The operator $\overline{\Gamma}$ satisfies inequalities (2.3.20), (2.3.21) and (2.3.24). Moreover, for any open subset $Q_1 \subset Q$, the operator $\overline{\Gamma}_{|Q_1}$ satisfies the same inequalities, where the constants $\overline{c}_0, \overline{\theta}, \overline{\kappa}$ and K do not depend on Q_1, and $c_1 = c_1(Q_1), c_2 = c_2(Q_1)$ are given by (2.3.4), (2.3.5), respectively.

Recall, that

$$H_1(\varphi, \psi) = c_1 + c_2 + \|\varphi\|_p^p + \|\psi\|_p^p.$$

Let us define $a(x, \xi)$ by the formula

$$a(x, \xi) = \overline{\Gamma}(\xi),$$

where $\xi \in \mathbf{R}^n$ is regarded as a constant function. Evidently, $a(x, \xi)$ is measurable in $x \in Q$ for any fixed $\xi \in \mathbf{R}^n$. The function $a(x, \xi)$ satisfies the inequalities

$$|a(x, \xi)|^{p'} \leq \overline{c}_0 |\xi|^p + K\overline{h}(x), \tag{2.3.27}$$

$$|(a(x, \xi) - a(x, \xi')|^{p'} \leq \overline{\theta} \left[\overline{h}(x) + |\xi|^p + |\xi'|^{p'}\right]^{1-\overline{s}/p} |\xi - \xi'|^{\overline{s}}, \tag{2.3.28}$$

and

$$[a(x, \xi) - a(x, \xi')] \cdot (\xi - \xi) \geq \overline{\kappa} \left[h_1(x) + |\xi\|^p + |\xi'|^{p'}\right]^{1-\beta/p} |\xi - \xi'|^{\beta}, \tag{2.3.29}$$

a.e. on Q for any $\xi, \xi' \in \mathbf{R}^n$. Here

$$\overline{h}(x) = c(x) + h(x).$$

Indeed, let $x_0 \in Q$ be a common Lebesgue point of the functions $\overline{h}(x)$, $a(x, \xi)$, and $a(x, \xi')$, and Q_ε the ball of the radius ε, centered at the point x_0. Applying inequality (2.3.21), with Q_ε insted of Q and $\psi \equiv \xi$, $\psi' \equiv \xi'$, dividing by $|Q_\varepsilon|$, and passing to the limit as $\varepsilon \to 0$, we obtain (2.3.28). In the similar way (2.3.27) and (2.3.29) follow from (2.3.20) and (2.3.24), respectively.

Now all we need is to prove (2.3.26). Inequality (2.3.28) implies that $a(x, \xi)$ satisfies the Carathéodory condition. Hence, almost all points of Q are common Lebesgue points of the family of functions $\{a(\cdot, \xi), \xi \in \mathbf{R}^n\}$. Then,evidenttly, the same is true for the functions $\{a(\cdot, \xi), \xi \in \mathbf{R}^n\}$, $\overline{h}, \psi$ and $\overline{\Gamma}(\psi)$. Let us fix any such point x_0 and set $\varphi \equiv \xi = \psi(x_0)$. Applying (2.3.21), with Q_ε instead of Q, and making use the passage to the limit, as above, we obtain (2.3.26), which completes the proof. □

As by-product, we have proved the statement of Theorem 2.3.2 for the particular case of operators from the class $\mathcal{E}_0$:

Lemma 2.3.3 *Let $\mathcal{A}_k \in \mathcal{E}_0$ and $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$ in Q. Then $\mathcal{A}_{k|Q_1} \stackrel{G}{\Longrightarrow} \mathcal{A}_{|Q_1}$ for any open subset $Q_1 \subset Q$.*

Now we prove the following particular case of Theorem 2.3.3.

Lemma 2.3.4 *Assume that $\mathcal{A}_k \in \mathcal{E}_0$ and $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Let $(v_k) \subset \overline{V}$ be a sequence such that*

$$\mathcal{A}_k v_k = f_k \to f$$

strongly in V^* and $v_k \to u$ weakly in $\overline{V}$. Then $\mathcal{A}u = f$ and

$$a^k(x, \nabla v_k) \to a(x, \nabla u)$$

weakly in $L^{p'}(Q)^n$.

Proof. Let u_k^1 be defined by (2.3.18), with $\psi = \nabla u$, and $u_k = u + u_k^1$. Then $u_k \to u$ and $u_k - v_k \to 0$ weakly in $\overline{V}$. Moreover,

$$\mathcal{A}_k u_k = \mathcal{A}u$$

and, using the notations of Step 2 (proof of Lemma 2.3.2);

$$a^k(x, \nabla u_k) = \Gamma^k(u) \to \Gamma(u) = a(x, \nabla u)$$

weakly in $\mathrm{L}^{p'}(Q)^n$. By Lemma 2.3.1, $\mathcal{A}u = f$ and $u_k - v_k \to 0$ in $W^{1,p}_{loc}(Q)$. The sequence

$$a^k(x, \nabla u_k) - a^k(x, \nabla v_k)$$

is bounded in $\mathrm{L}^{p'}(Q)^n$, by inequality (2.1.29) for a^k. Inequality (2.3.2) for a^k implies that this sequence converges to zero in $\mathrm{L}^{p'}_{loc}(Q)^n$, hence, weakly in $\mathrm{L}^{p'}_{loc}(Q)^n$. Since $a^k(x, \nabla v_k)$ is bounded in $\mathrm{L}^{p'}(Q)^n$, this gives rise to the last statement of the lemma. □

From the proof of Lemma 2.3.2 we obtain the following

Corollary 2.3.3 *For operators of the class $\mathcal{E}_0$ the following three statements are equivalent:*

(i) $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$;

(ii) $\mathcal{A}_k^\psi \overset{G}{\Longrightarrow} \mathcal{A}^\psi$ *for any* $\psi \in L^p(Q)^n$;

(iii) $\mathcal{A}_k^\xi \overset{G}{\Longrightarrow} \mathcal{A}^\xi$ *for any* $\xi \in \mathbf{R}^n$.

Remark 2.3.2 Given $\psi \in \mathrm{L}^p(Q)^n$ let us consider a unique solution, $u_k \in V$, of the equation

$$-\operatorname{div} a^k(x, \psi(x) + \nabla u_k) = -\operatorname{div} a(x, \psi(x)). \tag{2.3.30}$$

Then statement (ii) of Corollary 2.3.3 means that $u_k \to 0$ weakly in V and

$$a^k(x, \psi + \nabla u_k) \to a(x, \psi)$$

weakly in $\mathrm{L}^{p'}(Q)^n$, for any $\psi \in \mathrm{L}^p(Q)^n$. Statement (iii) is a particular case of (ii), with $\psi \equiv \xi$.

Remark 2.3.3 Lemma 2.3.3 implies the statement of Corollary 2.3.1 for the case of operators of the class $\mathcal{E}_0$.

2.3.3 Proofs of Main Results: General Case

To prove our main results in the full generality we start with the following comparison lemma for strong G-limit operators.

Lemma 2.3.5 *Let* $(\mathcal{A}_k)$, $(\mathcal{B}_k) \subset \mathcal{E}_0$, $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, *and* $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$. *Assume that*

$$|a^k(x,\xi) - b^k(x,\xi)|^{p'} \leq (\gamma_k(x) + |\xi|^p) \cdot \delta_k(x), \tag{2.3.31}$$

where $0 \leq \gamma_k \in L^1(Q)$, $\gamma_k \to \gamma$ *strongly in* $L^1(Q)$, *and* (δ_k) *is a bounded sequence in* $L^\infty(Q)$ *such that* $\delta_k(x) \to \delta(x)$ *a.e. on* Q. *Then*

$$|a(x,\xi) - b(x,\xi)|^{p'} \leq \overline{\theta} \cdot \left(\gamma(x) + \overline{h}(x) + |\xi|^p\right) \cdot \left(\delta^{s/\beta}(x) + \delta(x)\right), \tag{2.3.32}$$

where $\overline{h}(x) = c(x) + h(x)$.

Proof. Without loss of generality we can assume that $\gamma_k = \gamma$ and $\delta_k = \delta$. Indeed, if the statement is valid in this case, one can apply it with γ_k and δ_k replaced by $\sup\{\gamma, \gamma_k : k \geq k_0\}$ and $\sup\{\delta, \delta_k : k \geq k_0\}$, respectively. Then, to get the required we pass to the limit, as $k_0 \to \infty$, in the inequality of the type (2.3.31) just obtained. In the similar way, we can assume δ to be a step-function each step foot of which is an open set up to a set of zero measure. Hence, by localization Lemma 2.3.3, we may restrict ourself to the case $\delta \equiv$ const.

Now let $u_k \in V$ be a unique solution of (2.3.30), with $\psi \equiv \xi$, and $\psi_k = \psi + \nabla u_k$. Define $v_k \in V$ and $\varphi_k = \psi + \nabla v_k$ in the similar way, but with $\mathcal{A}_k$ replaced by $\mathcal{B}_k$. We set also

$$y = \overline{\mathcal{A}}\psi - \overline{\mathcal{B}}\psi = \overline{\mathcal{A}}_k\psi_k - \overline{\mathcal{B}}_k\varphi_k,$$

$$Z^k = a^k(x,\psi_k) - a^k(x,\varphi_k),$$

and

$$z_k = u_k - v_k.$$

Then we have

$$\begin{aligned}(y, z_k) &= \int_Q \left[a^k(x,\psi_k) - b^k(x,\varphi_k)\right] \cdot \nabla z_k dx = \\ &= \int_Q Z^k \cdot \nabla z_k dx - \int_Q \left[a^k(x,\varphi_k) - b^k(x,\varphi_k)\right] \cdot \nabla z_k dx.\end{aligned}$$

Making use of the Young inequality to estimate the last integral and taking into account inequality (2.3.31) we obtain

$$\begin{aligned}(y, z_k) &\geq \overline{\kappa} H_{(k)}^{1-\beta/p} \cdot \|z_k\|^\beta - \varepsilon \|z_k\|^p - C \cdot \delta \cdot L_k \geq \\ &\geq \frac{\overline{\kappa}}{2} H_{(k)}^{1-\beta/p} \cdot \|z_k\|^\beta - C \cdot \delta \cdot L_k,\end{aligned}$$

where $H_{(k)} = H(\psi_k, \varphi_k)$ and

$$L_k = \int_Q (\gamma(x) + |\varphi_k(x)|^p)\, dx.$$

Therefore,

$$\|z_k\|^\beta \leq \overline{\theta} H_{(k)}^{\beta/p-1} \cdot [(y, z_k) + \delta \cdot L_k].$$

Further, inequalities (2.3.2) and (2.3.31) imply easy that

$$\|a^k(x, \psi_k) - b^k(x, \varphi_k)\|_{p'}^{p'\beta/s} \leq \overline{\theta} \cdot \left[\delta^{\beta/s} L_k^{\beta/s} + H_{(k)}^{\beta/s-\beta/p} \cdot \|z_k\|^\beta\right].$$

By (2.3.14),

$$H_{(k)} \leq C \cdot H_1(\psi).$$

The same argument, with $\mathcal{A}_k$ replaced by $\mathcal{B}_k$, proves that

$$\|\varphi_k\|_p^p \leq C \cdot H_1(\psi).$$

Moreover, we have the following trivial inequalities:

$$H_1(\psi) \leq L = \int_Q \left(\gamma(x) + \overline{h}(x) + |\xi|^p\right) dx$$

and

$$L_k \leq L.$$

Therefore,

$$\begin{aligned}\|a^k(x, \psi_k) - b^k(x, \varphi_k)\|_{p'}^{p'\beta/s} &\leq \overline{\theta}\{\delta^{\beta/s} L_k^{\beta/s} + H_1^{\beta/s-1}[(y, z_k) + \delta \cdot L_k]\} \leq \\ &\leq \overline{\theta}\{(\delta^{\beta/s} + \delta) \cdot L^{\beta/s} + L \cdot (y, z_k^1)\}.\end{aligned}$$

By Corollary 2.3.3, $z_k \to 0$ weakly in V,

$$a^k(x, \psi_k) \to a(x, \xi),$$

and

$$b^k(x, \varphi_k) \to b(x, \xi)$$

weakly in $L^{p'}(Q)^n$. Then, passing to the limit in the last inequality we get

$$\|a(x,\xi) - b(x,\xi)\|_{p'}^{p'} \le \overline{\theta}(\delta^{s/\beta} + \delta) \int_Q \left(\gamma(x) + \overline{h}(x) + |\xi|^p\right) dx.$$

By the localization property (see Lemma 2.3.3), the previous inequality still holds true with Q replaced by any open subset of Q. This implies that inequality (2.3.32) is valid at any common Lebesgue point, $x \in Q$, of the functions $a(x,\xi)$ and $b(x,\xi)$, $\xi \in \mathbf{R}^n$. □

Remark 2.3.4 Additionally to the setting of Lemma 2.3.5, let us consider Carathéodory functions

$$a_0^k : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

and

$$b_0^k : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

satisfying inequalities of the type (2.1.29) and (2.3.2). Given $\psi \in L^p(Q)^n$ let ψ_k be the function defined by (2.3.18) and φ_k the function defined in the similar way, with $\mathcal{A}_k$ replaced by $\mathcal{B}_k$. Suppose that

$$a_0^k(x,\psi_k) \to a_0(x,\psi)$$

and

$$b_0^k(x,\varphi_k) \to b_0(x,\psi)$$

weakly in $L^p(Q)$ for any $\psi \in L^p(Q)^n$, with suitable functions a_0 and b_0. As in the proof of Lemma 2.3.2, one can show that a_0 and b_0 must satisfy inequalities like (2.1.29) and (2.3.2) as well. If estimate (2.3.31) is assumed to be valid for the difference $a_0^k(x,\xi) - b_0^k(x,\xi)$, then the difference $a_0(x,\xi) - b_0(x,\xi)$ satisfies inequality (2.3.32).

Proof of Theorem 2.3.1. Let us given a sequence of operators

$$(\mathcal{A}_k) \subset \mathcal{E}(c_0, c, \kappa, h, \theta, \nu, s, \beta).$$

For any $\lambda \in L^p(Q)$ we define the operator

$$\mathcal{A}_k^{(1),\lambda} u = -\operatorname{div} a^k(x,\lambda,\nabla u).$$

It is easy that

$$\mathcal{A}_k^{(1),\lambda} \in \mathcal{E}_0(c_0, c + |\lambda|^p, \kappa, h + |\lambda|^p, \theta, s, \beta).$$

By Lemma 2.3.2, passing to a subsequence still denoted by (k) we have

$$\mathcal{A}_k^{(1),\lambda} \xrightarrow{G} \mathcal{A}^{(1),\lambda}$$

for any $\lambda \in \Lambda$, a countable dense subset of $\mathrm{L}^p(Q)$, where

$$\mathcal{A}^{(1),\lambda} u = -\operatorname{div} a[\lambda](x, \nabla u).$$

Now we notice that, in fact, this statement is still valid for all $\lambda \in \mathrm{L}^p(Q)$. Indeed, by Lemma 2.3.2, for any $\lambda \in \mathrm{L}^p(Q)$ there is a subsequence $\sigma(k)$ such that

$$\mathcal{A}^{(1),\lambda}_{\sigma(k)} \xrightarrow{G} \mathcal{A}^{(1),\lambda}.$$

On the other hand, there is a sequence $(\lambda_j) \subset \Lambda$ such that $\lambda_j \to \lambda$ strongly in $\mathrm{L}^p(Q)$. Moreover, one can assume additionally that $\lambda_j \to \lambda$ a.e. in Q. Applying Lemma 2.3.5, with

$$\gamma_k(x) = h(x) + |\lambda(x)|^p + |\lambda_j(x)|^p$$

and

$$\delta_k(x) = \nu(|\lambda(x) - \lambda_j(x)|),$$

we have

$$\begin{aligned} |a[\lambda_j](x,\xi) - a[\lambda](x,\xi)|^{p'} \quad &\leq \quad \overline{\theta}\,(h_1(x) + |\lambda(x)|^p + |\lambda_j(x)|^p) \times \\ &\quad \times \nu^{s/\beta}(|\lambda(x) - \lambda_j(x)|). \end{aligned}$$

Therefore,

$$a[\lambda_j](x,\xi) \to a[\lambda](x,\xi),$$

for a.e. $x \in Q$. Thus, the passage to the subsequence $\sigma(k)$ is superfluous and we obtain our claim. Obviously

$$\mathcal{A}^{(1),\lambda} \in \mathcal{E}_0(\overline{c}, \overline{c}_\lambda, \overline{h}_\lambda, \overline{\theta}, s, \beta), \tag{2.3.33}$$

where $\overline{h}_\lambda = \overline{h} + |\lambda|^p$ and $\overline{c}_\lambda = K\overline{h}_\lambda$.

Now we define

$$\overline{\Gamma}^k : \mathrm{L}^p(Q)^n \times \mathrm{L}^p(Q) \longrightarrow \mathrm{L}^{p'}(Q)$$

and

$$\overline{\Gamma}^k_0 : \mathrm{L}^p(Q)^n \times \mathrm{L}^p(Q) \longrightarrow \mathrm{L}^{p'}(Q)$$

in the following way. Given $\psi \in \mathrm{L}^p(Q)^n$ and $\lambda \in \mathrm{L}^p(Q)$ let $u_k \in V$ be a unique solution of the equation

$$-\operatorname{div} a^k(x, \lambda, \psi + \nabla u_k) = -\operatorname{div} a[\lambda](x, \psi).$$

We set

$$\overline{\Gamma}^k(\psi, \lambda) = a^k(x, \lambda, \psi + \nabla u_k)$$

and

$$\overline{\Gamma}_0^k(\psi,\lambda) = a_0^k(x,\lambda,\psi+\nabla u_k).$$

As in the proof of Lemma 2.3.2 (see (2.3.19)), we have the estimate

$$\|\overline{\Gamma}^k(\psi,\lambda)\|_{p'}^{p'} + \|\overline{\Gamma}_0^k(\psi,\lambda)\|_{p'}^{p'} \le \overline{c}_0\left(\|\psi\|_p^p + \|\lambda\|_p^p\right) + K(c_1+c_2). \tag{2.3.34}$$

Since $\mathcal{A}_k^{(1),\lambda} \xRightarrow{G} \mathcal{A}^{(1),\lambda}$, there exists an operator

$$\overline{\Gamma} : \mathrm{L}^p(Q)^n \times \mathrm{L}^p(Q) \longrightarrow \mathrm{L}^{p'}(Q)^n$$

such that

$$\overline{\Gamma}^k(\psi,\lambda) \to \overline{\Gamma}(\psi,\lambda)$$

weakly in $\mathrm{L}^{p'}(Q)^n$, for any $\psi \in \mathrm{L}^p(Q)^n$ and $\lambda \in \mathrm{L}^p(Q)$ (see Corollary 2.3.3 and Remark 2.3.3). In fact,

$$\Gamma(\psi,\lambda) = a[\lambda](x,\psi).$$

Further, for any $\lambda \in \mathrm{L}^p(Q)$, the family of operators $\overline{\Gamma}_0^k$ is equicontinuous in the first variable on any ball in $\mathrm{L}^p(Q)^n$. This may be stated exactly as for $\overline{\Gamma}^k$ (see the proof of Lemma 2.3.2, Step 4). Therefore, by (2.3.34), fixed $\lambda \in \mathrm{L}^p(Q)$ (and, then, for a countable dense set of such λ's) passing to a subsequence we may assume that the sequence $\overline{\Gamma}_0^k(\psi,\lambda)$ is weakly convergent in $\mathrm{L}^{p'}(Q)$ for all $\psi \in \mathrm{L}^p(Q)^n$. Using Remark 2.3.4 we see that this is so, really, for all $\lambda \in \mathrm{L}^p(Q)$. Hence, there exists an operator

$$\overline{\Gamma}_0 : \mathrm{L}^p(Q)^n \times \mathrm{L}^p(Q) \longrightarrow \mathrm{L}^{p'}(Q)$$

such that

$$\overline{\Gamma}_0^k(\psi,\lambda) \to \overline{\Gamma}_0(\psi,\lambda)$$

weakly in $\mathrm{L}^{p'}(Q)^n$, for any $\psi \in \mathrm{L}^p(Q)$ and $\lambda \in \mathrm{L}^p(Q)$. Exactly as in the proof of the Lemma 2.3.2, $\overline{\Gamma}$ and $\overline{\Gamma}_0$ are local operators.

Now we define the operator $\mathcal{A}$ by the formula

$$\mathcal{A}u = -\operatorname{div}\overline{\Gamma}(\nabla u, u) + \overline{\Gamma}_0(\nabla u, u) = -\operatorname{div} a[u](x,\nabla u) + \overline{\Gamma}_0(\nabla u, u).$$

We have to prove that $\mathcal{A}$ belongs to a suitable class $\mathcal{E}$. First of all, (2.3.34) implies that

$$\|\overline{\Gamma}(\psi,\lambda)\|_{p'}^{p'} + \|\overline{\Gamma}_0(\psi,\lambda)\|_{p'}^{p'} \le \overline{c}_0\left(\|\psi\|_p^p + \|\lambda\|_p^p\right) + K(c_1+c_2). \tag{2.3.35}$$

Next, the following inequality is valid:

$$\|\overline{\Gamma}(\psi-\varphi,\lambda-\mu)\|_{p'}^{p'} + \|\overline{\Gamma}_0(\psi-\varphi,\lambda-\mu)\|_{p'}^{p'} \le$$

$$\leq \overline{\theta} H_1(\psi, \varphi, \lambda, \mu)^{1-\overline{s}/p} \|\psi - \varphi\|_p^{\overline{s}} + \\ + \overline{\theta} \int_Q \left(\overline{h} + |\psi|^p + |\varphi|^p + |\lambda|^p + |\mu|^p\right) \cdot \nu^{s/\beta}(|\mu - \lambda|) dx. \tag{2.3.36}$$

Indeed, for $\lambda = \mu$ this follows from inequality (2.3.21), with $\mathcal{A}$ replaced by $\mathcal{A}^{(1),\lambda}$ (the estimate for $\overline{\Gamma}_0$ may be derived in a quite similar way). In the case $\varphi = \psi$, (2.3.36) may be obtained, using Lemma 2.3.5 and Remark 2.3.4. Now we set

$$a(x, \xi_0, \xi) = \overline{\Gamma}(\xi, \xi_0)$$

and

$$a_0(x, \xi_0, \xi) = \overline{\Gamma}_0(\xi, \xi_0)$$

for $\xi_0 \in \mathbf{R}$ and $\xi \in \mathbf{R}^n$. Here in the right-hand sides we consider ξ and ξ_0 as constant functions. As in the proof of Lemma 2.3.2,

$$\overline{\Gamma}(\psi, \lambda) = a(x, \lambda, \psi)$$

and

$$\overline{\Gamma}_0(\psi, \lambda) = a_0(x, \lambda, \psi).$$

Hence,

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u).$$

It is not difficult to see that $\mathcal{A} \in \mathcal{E}(\overline{c}_0, \overline{c}, \kappa, \overline{h}, \overline{\theta}, \overline{\nu}, \overline{s}, \beta)$. Moreover, $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. We note that

$$\Gamma^k(u, v) = \overline{\Gamma}^k(\nabla u, v),$$

$$\Gamma_0^k(u, v) = \overline{\Gamma}_0^k(\nabla u, v),$$

$$\Gamma(u, v) = \overline{\Gamma}(\nabla u, v)$$

and

$$\Gamma_0(u, v) = \overline{\Gamma}_0(\nabla u, v),$$

where $\Gamma^k, \Gamma_0^k, \Gamma$ and Γ_0 was introduced in n 2.3.1. □

Proof of Theorem 2.3.2. It follows immediately from the previous considerations and Lemma 2.3.3. □

Proof of Theorem 2.3.3. In view of the localization property (see Theorem 2.3.2) and boundedness of $a^k(x, v_k, \nabla v_k)$ and $a_0^k(x, v_k, \nabla v_k)$, we may assume the boundary, ∂Q, of Q to be smooth. Then, by the Sobolev Embedding Theorem, $v_k \to u$ strongly in $L^p(Q)$. Using the notations introduced in the proof of Theorem 2.3.1 we claim that $\mathcal{A}_k^{(1),v_k} \overset{G}{\Longrightarrow} \mathcal{A}^{(1),u}$. Indeed, passing to a subsequence we may assume that $\mathcal{A}^{(1),v_k} \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$. Now we set

$$\gamma_k(x) = h(x) + |u(x)|^p + |v_k(x)|^p,$$

$$\delta_k(x) = \nu(|u(x) - v_k(x)|),$$

and then apply Lemma 2.3.5 to the operators $\mathcal{A}_k^{(1),v_k}$ and $\mathcal{A}_k^{(1),u}$. We conclude that $\hat{\mathcal{A}} = \mathcal{A}^{(1),u}$ and the passage to a subsequence is superfluous.

Now let $u_k \in \overline{V}$ be a unique solution of the equation

$$-\operatorname{div} a^k(x, v_k, \nabla u_k) = -\operatorname{div} a(x, u, \nabla u),$$

such that $u_k - u \in V$. Since $\mathcal{A}^{(1),v_k} \overset{G}{\Longrightarrow} \mathcal{A}^{(1),u}$, it follows from Lemma 2.3.4 that

$$a^k(x, v_k, \nabla u_k) \to a(x, u, \nabla u) \tag{2.3.37}$$

weakly in $L^{p'}(Q)^n$. Exactly as in the proof of Theorem 2.3.1, we may assume, passing to a subsequence, that there exists a function $b_0(x, \xi)$ such that the functions $a_0^k(x, u, \xi)$, $a_0(x, u, \xi)$,

$$b_0^k(x, \xi) = a_0^k(x, v_k, \xi),$$

and $b_0(x, \xi)$ satisfy the condition of Remark 2.3.4 with respect to the operators $\mathcal{A}_k^{(1),u}$ and $\mathcal{A}_k^{(1),v_k}$. As above, we conclude that

$$b_0(x, \xi) = a_0(x, u, \xi)$$

and, really, we do not need any passage to a subsequence at this point. Hence,

$$a_0^k(x, v_k, \nabla u_k) \to a_0(x, u, \nabla u) \tag{2.3.38}$$

weakly in $L^{p'}(Q)$.

Further, we have

$$\mathcal{A}^{(1),v_k}(v_k) = f_k - a_0^k(x, v_k, \nabla v_k).$$

Since $a_0^k(x, v_k, \nabla v_k)$ is bounded in $L^{p'}(Q)$, we may assume that

$$a_0^k(x, v_k, \nabla v_k) \to g$$

weakly. Therefore, $a_0^k(x, v_k, \nabla v_k) \to g$ strongly in V^*. From Lemma 2.3.4 it follows that

$$\mathcal{A}^{(1),u}(u) = -\operatorname{div} a(x, u, \nabla u) = f - g. \tag{2.3.39}$$

Since $u_k \to u$, $v_k \to u$ weakly in $\overline{V}$ and

$$\mathcal{A}^{(1),v_k}(u_k) = \mathcal{A}^{(1),u}(u),$$

we have, by Lemma 2.3.1, $u_k - v_k \to 0$ in $W^{1,p}_{loc}(Q)$. Using inequality of the type (2.3.2) we see that

$$a^k(x, v_k, \nabla v_k) - a^k(x, v_k, \nabla u_k) \to 0,$$

$$a_0^k(x, v_k, \nabla v_k) - a_0^k(x, v_k, \nabla u_k) \to 0$$

in $L^{p'}_{loc}(Q)$ and, hence, weakly. Together with (2.3.37) and (2.3.38), these relations imply that

$$a^k(x, v_k, \nabla v_k) \to a(x, u, \nabla u),$$

$$a_0^k(x, v_k, \nabla v_k) \to a_0(x, u, \nabla u)$$

weakly in $L^{p'}(Q)^n$ and $L^p(Q)$, respectively. In particular,

$$g = \lim a_0^k(x, v_k, \nabla v_k) = a_0(x, u, \nabla u)$$

and, by (2.3.39),

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u) = f.$$

The proof is complete. □

Remark 2.3.5 For further references we point out the following statement. Let $\mathcal{A}_k \in \mathcal{E}$ and $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Suppose $(u_k) \subset \overline{V}$ is a sequence such that $u_k \to u$ weakly in $\overline{V}$ and $\mathcal{A}_k^{(1)}(u_k, v) \to f$ strongly in V^*. Then $\mathcal{A}^{(1)}(u, v) = f$ and

$$a^k(x, v, \nabla u_k) \to a(x, v, \nabla u),$$

$$a_0^k(x, v, \nabla u_k) \to a_0(x, v, \nabla u)$$

weakly in $L^{p'}(Q)^n$ and $L^{p'}(Q)$, respectively. This may be proved by an argument quite similar to that used in the proof of Theorem 2.3.3.

Remark 2.3.6 For any $\mathcal{A} \in \mathcal{E}$ consider the family of operators

$$\mathcal{A}^{(\lambda,\psi)}u = -\operatorname{div} a(x, \lambda + u, \psi + \nabla u) + a_0(x, \lambda + u, \psi + \nabla u),$$

where $\lambda \in L^p(Q)$ and $\psi \in L^p(Q)^n$. From the previous consideration one can extract, without any difficulty, that the following three statements are equivalent:

(i) $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$;

(ii) $\mathcal{A}^{(\lambda,\psi)} \overset{G}{\Longrightarrow} \mathcal{A}^{(\lambda,\psi)}$ for any $\lambda \in \mathrm{L}^p(Q)$, $\psi \in \mathrm{L}^p(Q)^n$;

(iii) $\mathcal{A}^{(\xi_0,\xi)} \overset{G}{\Longrightarrow} \mathcal{A}^{(\xi_0,\xi)}$ for any $(\xi_0, \xi) \in \mathbf{R}^{n+1}$.

Remark 2.3.7 The statement of Lemma 2.3.5 may be extended immediately to the case of operators from the class $\mathcal{E}$. More precisely, if $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$, where $\mathcal{A}_k$, $\mathcal{B}_k \in \mathcal{E}$ satisfy the inequality

$$|a^k(x,\xi_0,\xi) - b^k(x,\xi_0,\xi)|^{p'} + |a_0^k(x,\xi_0,\xi) - b_0^k(x,\xi_0,\xi)|^{p'} \leq$$
$$\leq (\gamma_k(x) + |\xi_0|^p + |\xi|^p) \cdot \delta_k(x),$$

with γ_k and δ_k as in the statement of Lemma 2.3.5, then

$$|a(x,\xi_0,\xi) - b(x,\xi_0,\xi)|^{p'} + |a_0(x,\xi_0,\xi) - b_0(x,\xi_0,\xi)|^{p'} \leq$$
$$\leq \overline{\theta} \cdot (\gamma(x) + \overline{h}(x) + |\xi_0|^p + |\xi|^p) \cdot (\delta^{s/\beta}(x) + \delta(x)).$$

The definition of strong G-convergence we have used in this section seems to be somewhat technical. Now we give another description of this notion in the spirit of n^0 2.2.1. Let $\mathcal{A}_k$, $\mathcal{A} \in \mathcal{E}$. The sequence $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if and only if the followig condition holds true (cf. page 63, condition (j)):

(j1) *for any increasing sequence $\tau(k)$ of integers, for any $f \in V^*$, for any sequence f_k converging to f strongly in V^*, and for any sequence $u_k \in V$ of solutions to the equation*

$$\mathcal{A}_{\tau(k)} u_k = f_k,$$

there exist a subsequence $\sigma(k)$ of $\tau(k)$ and a solution $u \in V$ of the equation

$$\mathcal{A}u = f$$

such that

$$u_{\sigma(k)} \to u \quad \textit{weakly in } V,$$

$$a^{\tau(\sigma(k))}(x, u_{\sigma(k)}, \nabla u_{\sigma(k)}) \to a(x,u,\nabla u) \quad \textit{weakly in } \mathrm{L}^{p'}(Q)^n,$$

and

$$a_0^{\tau(\sigma(k))}(x, u_{\sigma(k)}) \to a_0(x,u,\nabla u) \quad \textit{weakly in } \mathrm{L}^{p'}(Q).$$

Indeed, if $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, then (j1) follows immediately from Theorem 2.3.3. Conversely, suppose (j1) to be valid. By Theorem 2.3.1, we can assume that $\mathcal{A}_k \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$. Now Theorem 2.3.3 implies that $\hat{\mathcal{A}} = \mathcal{A}$.

Additionally, suppose $f \in V^*$ and $u \in V$ is a solution of the equation

$$\mathcal{A}u = f.$$

If $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, then there exist a sequence f_k converging to f strongly in V^* and a sequence $u_k \in V$ of solutions to the equation

$$\mathcal{A}_k u_k = f_k$$

such that $u_k \to u$ weakly in V (cf. statement (b), page 64). Indeed, define u_k to be a unique solution of the equation

$$-\operatorname{div} a^k(x, u, \nabla u_k) = f - a_0(x, u, \nabla u)$$

and set $f_k = \mathcal{A}_k u_k$. We leave it to the reader to prove that $f_k \to f$ strongly in V^*.

Now we discuss the case of monotone operators. Replacing in the definition of the class $\mathcal{E}$ inequality (2.1.30) by the inequality

$$[a(x,\xi_0,\xi) - a(x,\xi_0',\xi')] \cdot (\xi - \xi') + [a_0(x,\xi_0,\xi) - a_0(x,\xi_0',\xi')] \cdot (\xi_0 - \xi_0') \geq$$

$$\geq \kappa\,(h(x) + |\zeta|^p + |\zeta'|^p)^{1-\beta/p}\,|\xi - \xi'|^\beta \tag{2.3.40}$$

a.e. on Q for any $\zeta = (\xi_0, \xi) \in \mathbf{R}^{n+1}$, $\zeta' = (\xi_0', \xi') \in \mathbf{R}^{n+1}$, we obtain a subclass of $\mathcal{E}$ denoted by $\mathcal{E}_{(m)}$. Any operator of the class $\mathcal{E}_{(m)}$ is strictly monotone and coercive as an operator acting from V into V^*. Hence, any such operator has a single-valued inverse operator. Let $\mathcal{A}_k$, $\mathcal{A} \in \mathcal{E}_{(m)}$. The sequence $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if and only if for any $f \in V^*$

$$u_k = \mathcal{A}_k^{-1} f \to u = \mathcal{A}^{-1} f \quad \text{weakly in } V,$$

$$a^k(x, u_k, \nabla u_k) \to a(x, u, \nabla u) \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n,$$

and

$$a_0^k(x, u_k, \nabla u_k) \to a_0(x, u, \nabla u) \quad \text{weakly in } \mathrm{L}^{p'}(Q).$$

The last statement follows immediately from Theorems 2.3.3 and 2.3.1.

2.4 Further Results on Strong G-convergence

2.4.1 Criteria for Strong G-convergenvce

Let $\mathcal{A}_k$ be a sequence of operators of the class $\mathcal{E}$ such that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Then, as we have pointed out in Remark 2.3.8, $\mathcal{A}_k^{(\eta,\xi)} \overset{G}{\Longrightarrow} \mathcal{A}^{(\eta,\xi)}$ for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$. Hence, if we consider a unique solution, $u_k \in V$, of the equation

$$-\operatorname{div} a^k(x, \eta, \xi + \nabla u_k) = -\operatorname{div} a(x, \eta, \xi)$$

and set $v_k(x) = \xi \cdot x + u_k(x)$, then v_k possesses the following properties:

1. $v_k \to \xi \cdot x$ *weakly in* $\overline{V}$;
2. *the sequences* $a^k(x, \eta, \nabla v_k)$ *and* $a_0^k(x, \eta, \nabla v_k)$ *are weakly convergent in the spaces* $\mathrm{L}^{p'}(Q)^n$ *and* $\mathrm{L}^{p'}(Q)$ *respectively*;
3. *the sequence* $\operatorname{div} a^k(x, \eta, \nabla v_k)$ *is precompact in the space* V^*.

It turns out to be that the converse statement holds true.

Theorem 2.4.1 *Let* $\mathcal{A}_k \in \mathcal{E}$. *Assume that, for any* $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$, *there exists* $v_k \in \overline{V}$ *such that the above mentioned properties 1 – 3 are fulfilled. Then the sequence* $\mathcal{A}_k$ *is strongly G-convergent and for the G-limit operator,* $\mathcal{A}$, *we have*

$$a(x, \eta, \xi) = \lim_{k\to\infty} a^k(x, \eta, \nabla v_k), \tag{2.4.1}$$

$$a_0(x, \eta, \xi) = \lim_{k\to\infty} a_0^k(x, \eta, \nabla v_k) \tag{2.4.2}$$

weakly in $\mathrm{L}^{p'}(Q)^n$ *and* $\mathrm{L}^{p'}(Q)$, *respectively.*

Proof. Let $r(x) = r(x, \eta, \xi)$ and $r_0(x) = r_0(x, \eta, \xi)$ be the weak limits of $a^k(x, \eta, \nabla v_k)$ and $a_0^k(x, \eta, \nabla v_k)$, respectively. By Theorem 2.3.1, there exists a subsequence $\sigma(k)$ and an operator $\mathcal{A}$ of the form

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u),$$

such that $\mathcal{A}_{\sigma(k)} \overset{G}{\Longrightarrow} \mathcal{A}$. To prove the theorem it is enough to show that

$$a(x, \eta, \xi) = r(x) \tag{2.4.3}$$

and

$$a_0(x, \eta, \xi) = r_0(x). \tag{2.4.4}$$

From properties 2 and 3 it follows that

$$\operatorname{div} r(x) = \lim_{k\to\infty} \operatorname{div} a^k(x, \eta, \nabla v_k)$$

strongly in $\overline{V}$. Since the embedding $\mathrm{L}^{p'}(Q)^n \subset V^*$ is compact, we have also

$$r_0(x) = \lim_{k\to\infty} a_0^k(x, \eta, \nabla v_k)$$

strongly in V^*. By Remark 2.3.5, we obtain immediately (2.4.3) and (2.4.4). The proof is complete. □

Corollary 2.4.1 *Let $\mathcal{A}_k \in \mathcal{E}$ be a sequence such that, for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$, the sequence $a^k(x, \eta, \xi)$ converges in measure and the sequence $a_0^k(x, \eta, \xi)$ converges weakly in $\mathrm{L}^1(Q)$. Then $\mathcal{A}_k$ is strongly G-convergent.*

Proof. Inequality (2.1.29) and the dominated convergence theorem imply that $a^k(x, \eta, \xi)$ convegres strongly in $\mathrm{L}^{p'}(Q)^n$. Also, $a_0^k(x, \eta, \xi)$ converges, really, weakly in $\mathrm{L}^{p'}(Q)$. Now the statement follows directly from Theorem 2.4.1, with $v_k \equiv \xi \cdot x$. □

Another consequence of Theorem 2.4.1 is the following

Corollary 2.4.2 *Suppose $\mathcal{A}_k \in \mathcal{E}$ is a sequence such that*

$$\operatorname{div} a^k(x, \eta, \xi) = 0$$

for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$. Then the sequence $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if and only if

$$a^k(x, \eta, \xi) \to a(x, \eta, \xi)$$

and

$$a_0^k(x, \eta, \xi) \to a_0(x, \eta, \xi),$$

for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$, weakly in $\mathrm{L}^1(Q)^n$ and $\mathrm{L}^1(Q)$, respectively.

With the previous results in hand, we consider the following problem. Let us given an operator $\mathcal{A} \in \mathcal{E}$. Fixed $x_0 \in Q$ we define the operator $\mathcal{A}_\rho$, $0 < \rho \le 1$, by the formula

$$\mathcal{A}_\rho u = -\operatorname{div} a(x_0 + \rho x, u, \nabla u) + a_0(x_0 + \rho x, u, \nabla u). \tag{2.4.5}$$

We look for the asymptotic behaviour of $\mathcal{A}_\rho$, as $\rho \to 0$, assuming that ρ runs a subsequence which tends to 0.

Proposition 2.4.1 *For any common Lebesgue point $x_0 \in Q$ of the functions $a(x,\eta,\xi)$ and $a_0(x,\eta,\xi)$, i.e. for almost all $x_0 \in Q$, the sequence $\mathcal{A}_\rho$ strongly G-converges, as $\rho \to 0$, to the operator*

$$\hat{\mathcal{A}}u = -\operatorname{div} a(x_0, u, \nabla u) + a_0(x_0, u, \nabla u). \tag{2.4.6}$$

Proof. By Lebesgue's differentiation theorem $a(x_0+\rho x,\eta,\xi) \to a(x,\eta,\xi)$ and $a_0(x_0+\rho x,\eta,\xi) \to a_0(x_0,\eta,\xi)$, as $\rho \to 0$, strongly in $\mathrm{L}^1(Q)^n$ and $\mathrm{L}^1(Q)$ respectively. Now Corollary 2.4.1 implies the required statement. □

Let us state another result which provides a criterion for strong G-convergence and a representation formula for the G-limit operator. With this aim, given $\mathcal{A} \in \mathcal{E}$ define the functions

$$\Psi(\eta,\xi,Q_1) = \int_{Q_1} a(x,\eta,\xi+\nabla v(x))dx \tag{2.4.7}$$

and

$$\Psi_0(\eta,\xi,Q_1) = \int_{Q_1} a_0(x,\eta,\xi+\nabla v(x))dx, \tag{2.4.8}$$

for any $(\eta,\xi) \in \mathbf{R}\times\mathbf{R}^n$ and for any open subset Q_1 of Q, where the function v is defined as a unique solution $v \in W_0^{1,p}(Q_1)$ of the equation

$$-\operatorname{div} a(x,\eta,\xi+\nabla v) = 0 \quad \text{on } Q_1. \tag{2.4.9}$$

Proposition 2.4.2 *Let $\mathcal{A} \in \mathcal{E}$. Then there exists a measurable subset N of Q, with $|N| = 0$, such that for any $x_0 \in Q\setminus N$ and $(\eta,\xi) \in \mathbf{R}\times\mathbf{R}^n$*

$$a(x_0,\eta,\xi) = \lim_{\rho\to 0} \frac{\Psi(\eta,\xi,U_\rho(x_0))}{|U_\rho(x_0)|} \tag{2.4.10}$$

and

$$a_0(x_0,\eta,\xi) = \lim_{\rho\to 0} \frac{\Psi_0(\eta,\xi,U_\rho(x_0))}{|U_\rho(x_0)|}, \tag{2.4.11}$$

where $U_\rho(x_0) = x_0+\rho U$, with U being an open bounded subset of $\mathbf{R}^n$.

Proof. Let N be the complement of the set of all common Lebesgue points of the family of functions $\{a(x,\eta,\xi), a_0(x,\eta,\xi)\}_{(\eta,\xi)\in\mathbf{R}\times\mathbf{R}^n}$. Given $x_0 \in Q\setminus N$, $\rho > 0$, and $(\eta,\xi) \in \mathbf{R}\times\mathbf{R}^n$, we consider the function $v \in W_0^{1,p}(U_\rho(x_0))$ defined to be a unique solution of (2.4.9), with $Q_1 = U_\rho(x_0)$. By performing the change of variables $y = (x-x_0)/\rho$, equation (2.4.9) becomes

$$-\mathrm{div}_y a(x_0+\rho y,\eta,\xi+\nabla_y u_\rho(y)) = 0, \quad y \in U,$$

where $u_\rho(y) = v(x_0 + \rho y)/\rho$. Since $w = 0$ is a unique solution of the equation

$$-\mathrm{div}_y a(x_0, \eta, \xi + \nabla_y w) = 0 \quad \text{on } U,$$

Proposition 2.4.1 and Remark 2.3.6 imply that $u_\rho \to 0$ weakly in $W_0^{1,p}(U)$,

$$a(x_0 + \rho y, \eta, \xi + \nabla_y u_\rho) \to a(x_0, \eta, \xi),$$

weakly in $\mathrm{L}^{p'}(Q)^n$, and

$$a_0(x_0 + \rho y, \eta, \xi + \nabla_y u_\rho) \to a_0(x_0, \eta, \xi)$$

weakly in $\mathrm{L}^{p'}(Q)$. Then

$$a(x_0, \eta, \xi) = \lim_{\rho\to 0} \frac{1}{|U|} \int_U a(x_0 + \rho y, \eta, \xi + \nabla_y u_\rho(y))dy,$$

which, by the change of variables, proves (2.4.10). In the similar way one can derive (2.4.11) and we conclude. □

Theorem 2.4.2 *Suppose $\mathcal{A}_k$ is a sequence of operators of the class $\mathcal{E}$. Let Ψ^k and Ψ_0^k be the functions associated to $\mathcal{A}_k$ by (2.4.7) and (2.4.8), respectively, U a bounded open subset in $\mathbf{R}^n$, and $U_\rho(x_0) = x_0 + \rho U$. Then the following statements are equivalent:*

(i) *for almost all $x_0 \in Q$, the limits*

$$\lim_{k\to\infty} \Psi^k(\eta, \xi, U_\rho(x_0))$$

and

$$\lim_{k\to\infty} \Psi_0^k(\eta, \xi, U_\rho(x_0))$$

exist for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$ and $\rho > 0$ small enough;

(ii) *the sequence $\mathcal{A}_k$ strongly G-converges to an operator $\mathcal{A}$.*

Moreover, if these statements hold true, then, for almost all $x \in Q$,

$$a(x, \eta, \xi) = \lim_{\rho\to 0} \lim_{k\to\infty} \frac{\Psi^k(\eta, \xi, U_\rho(x))}{|U_\rho(x)|} \tag{2.4.12}$$

and

$$a_0(x, \eta, \xi) = \lim_{\rho\to 0} \lim_{k\to\infty} \frac{\Psi_0^k(\eta, \xi, U_\rho(x))}{|U_\rho(x)|} \tag{2.4.13}$$

for any $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$, where $\mathcal{A}$ is associated to a and a_0.

Proof. Assume (i). By Theorem 2.3.1, we may suppose that a subsequence of $\mathcal{A}_k$ still denoted by $\mathcal{A}_k$ is strongly G-covergent to an operator $\mathcal{A}$. If we prove formulae (2.4.12) and (2.4.13), we conclude that the initial sequence is strongly G-convergent.

Fix $x_0 \in Q$ such that the limits in (i) exist. Given $(\eta, \xi) \in \mathbf{R} \times \mathbf{R}^n$ and $\rho > 0$ we consider a unique solution $v_k \in W_0^{1,p}(U_\rho(x_0))$ of the equation

$$-\operatorname{div} a^k(x, \eta, \xi + \nabla v_k) = 0,$$

and a unique solution $v \in W_0^{1,p}(U_\rho(x_0))$ of the equation

$$-\operatorname{div} a(x, \eta, \xi + \nabla v) = 0.$$

From Theorem 2.3.2 and Remark 2.3.6 it follows that $v_k \rightarrow v$ weakly in $W_0^{1,p}(U_\rho(x_0))$,

$$a^k(x, \eta, \xi + \nabla v_k) \rightarrow a(x, \eta, \xi + \nabla v) \tag{2.4.14}$$

weakly in $\mathrm{L}^{p'}(U_\rho(x_0))^n$, and

$$a_0^k(x, \eta, \xi + \nabla v_k) \rightarrow a_0(x, \eta, \xi + \nabla v) \tag{2.4.15}$$

weakly in $\mathrm{L}^{p'}(U_\rho(x_0))$. Hence,

$$\lim_{k \to \infty} \Psi^k(\eta, \xi, U_\rho(x_0)) = \int_{U_\rho(x_0)} a(x, \eta, \xi + \nabla v) dx = \Psi(\eta, \xi, U_\rho(x_0))$$

and the similar statement holds for Ψ_0. Now, by Proposition 2.4.2, we get (2.4.12) and (2.4.13).

Assume (ii). Then, for v and v_k defined in the first part of the proof, statements (2.4.14) and (2.4.15) hold true and (i) follows immediality. □

2.4.2 Stability and Comparison Results

At first we consider the following problem. Let us given two functions r, $r_0 \in \mathrm{L}^\infty(Q)$ such that $\operatorname{ess\,inf} r(x) > 0$. Assume a sequence of operators $\mathcal{A}_k$ to be strongly G-convergent to an operator $\mathcal{A}$. What can one say about G-convergence of the sequence of operators $\mathcal{B}_k$, where

$$\mathcal{B}_k u = -\operatorname{div}\left[r(x) a^k(x, u, \nabla u)\right] + r_0(x) a_0^k(x, u, \nabla u), \tag{2.4.16}$$

and its limit? The answer is the following.

Proposition 2.4.3 *Let* r, $r_0 \in \mathrm{L}^\infty(Q)$, *with* $\operatorname{ess\,inf} r(x) > 0$. *Assume that* $\mathcal{A}_k \in \mathcal{E}$ *and* $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. *Then for the sequence* $\mathcal{B}_k$ *defined by (2.4.16) we have* $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$, *where*

$$\mathcal{B} u = -\operatorname{div}\left[r(x) a(x, u, \nabla u)\right] + r_0(x) a_0(x, u, \nabla u). \tag{2.4.17}$$

Proof. For simplicity let us assume that $a_0^k \equiv 0$. If $r(x) \equiv$ const, the statement follows trivially from Theorem 2.4.1. The case when $r(x)$ is piecewise constant, with each its level set being open up to a set of zero Lebesgue measure, reduces easily to the previous one, by means of Theorem 2.3.2. To cover the general case we choose a bounded sequence $r_j \in \mathrm{L}^\infty(Q)$ of piecewise constant functions such that $\operatorname{ess\,inf} r_j(x) > 0$ and $r_j \to r$ a.e. in Q. Then our statement holds with r replaced by r_j. Let $\mathcal{B}_{k,j}$ be the corresponding operator. Passing to a subsequence we way assume that $\mathcal{B}_k$ strongly G-converges to an operator $\mathcal{B}$ and all we need is to derive (2.4.17). But we get this immediately, applying Lemma 2.3.5 to the operators $\mathcal{B}_k$ and $\mathcal{B}_{k,j}$ (we have to set $\delta_k(x) = |r_j(x) - r(x)|$ there). □

To prove our next result we need the following theorem of N. Meyers [210]. Let us consider a Carathéodory function $a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$ such that

$$|a(x,\xi)| \le \lambda_0 |\xi|^{p-1} + \lambda_1, \tag{2.4.18}$$

$$a(x,\xi)\cdot\xi \ge \mu_0|\xi|^p - \mu_1 \tag{2.4.19}$$

for any $\xi \in \mathbf{R}^n$ a.e. on Q, with the constants $\lambda_0 > 0$, $\mu_0 > 0$, $\lambda_1 \ge 0$ and $\mu_1 \ge 0$.

Theorem 2.4.3 *Assume that $a(x,\xi)$ satisfies (2.4.18), (2.4.19) and the boundary ∂Q is regular, e.g. piesewise C^1. Let $w \in W_0^{1,p}(Q)$ be a weak solution of the equation*

$$-\operatorname{div} a(x, \nabla w) = 0.$$

Then there exists $\alpha > 0$ such that $w \in W^{1,p+\alpha}(Q)$ and

$$\|\nabla w\|_{p+\alpha} \le C(Q)\|\nabla w\|_p.$$

Here the constant α depends only on λ_0, μ_0, n and p, while $C(Q)$ depends, in addition, on λ_1, μ_1 and Q.

Remark 2.4.1 A rescaling argument shows that one can take $C(Q)$ in such a way that

$$C(tQ) = t^{-\alpha n/(p+\alpha)} C(Q), \quad t > 0.$$

In the rest of this subsection we shall consider operators of the class $\mathcal{E}$, with the parameters c and h being constant:

$$c \equiv const, \quad h \equiv const. \tag{2.4.20}$$

We shall apply Theorem 2.4.3 to the equation

$$-\operatorname{div} a(x, \eta, \xi + \nabla u) = 0,$$

where $a(x,\cdot,\cdot)$ is the "principal part" of an operator from $\mathcal{E}$, i.e. it satisfies inequalities (2.1.29), (2.1.30) and (2.3.2). In this case the constant α in Theorem 2.4.3 depends only on c, h, n and β.

To state the next result we need the following notations. Let $\mathcal{A}_k \in \mathcal{E}$ and $\mathcal{B}_k \in \mathcal{E}$. We set

$$g^k(x,r) = \sup_{|\xi|,|\eta|\le r} |a^k(x,\eta,\xi) - b^k(x,\eta,\xi)|,$$

$$g_0^k(x,r) = \sup_{|\xi|,|\eta|\le r} |a_0^k(x,\eta,\xi) - b_0^k(x,\eta,\xi)|.$$

Assuming $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$, we introduce also the functions

$$g(x,r) = \sup_{|\xi|,|\eta|\le r} |a(x,\eta,\xi) - b(x,\eta,\xi)|$$

and

$$g_0(x,r) = \sup_{|\xi|,|\eta|\le r} |a_0(x,\eta,\xi) - b_0(x,\eta,\xi)|.$$

Given a bounded open subset $U \subset \mathbf{R}^n$, with the regular boundary ∂U, we set

$$U_\rho(x_0) = x_0 + \rho U.$$

Let us define the functions $\overline{g}(x,r)$ and $\overline{g}_0(x,r)$ by

$$\overline{g}(x,r) = \limsup_{\rho\to 0} \limsup_{k\to\infty} \frac{1}{|U_\rho(x)|} \int_{U_\rho(x)} g^k(y,r)dy$$

and

$$\overline{g}_0(x,r) = \limsup_{\rho\to 0} \limsup_{k\to\infty} \frac{1}{|U_\rho(x)|} \int_{U_\rho(x)} g_0^k(y,r)dy.$$

For any $r \ge 0$, these functions are well-defined a.e. on Q and belong to $\mathrm{L}^\infty(Q)$. Also we shall use the notation

$$\varphi(r) = r^{-p} + r^{-\alpha p/(p+\alpha)}, \quad r > 0,$$

with the constant $\alpha > 0$ appeared in Theorem 2.4.3.

Theorem 2.4.4 *Suppose $\mathcal{A}_k$ and $\mathcal{B}_k$ are sequences of operators of the class $\mathcal{E}$, $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$, and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$. Then given $R > 0$ there exists a constant $K = K(R)$ such that*

$$g(x,R) \le \overline{g}(x,r) + K\left[\varphi(r)^{1/p'} + \varphi^\gamma(r) + (1+r)\overline{g}(x,r)^\gamma\right] \tag{2.4.21}$$

and

$$g_0(x,R) \le \overline{g}_0(x,r) + K\left[\varphi(r)^{1/p'} + \varphi^\gamma(r) + (1+r)\overline{g}(x,r)^\gamma\right] \tag{2.4.22}$$

for almost all $x \in Q$ and for all $r > 0$, where

$$\gamma = \frac{s}{(p')^2(\beta-1)}.$$

Proof. We consider the case of inequality (2.4.21) only. Inequality (2.4.22) may be proved in a similar way. For the sake of brevity we shall write U_ρ instead of $U_\rho(x)$. With the same purpose we shall suppress the variable η in the functions a^k and b^k.

Given $R > 0$ we fix $\xi, \eta \in \mathbf{R}$ such that $|\xi| \leq R$, $|\eta| \leq R$. Let $v_k \in W_0^{1,p}(U_\rho)$ be a unique solution of the equation

$$-\operatorname{div} a^k(x, \xi + \nabla v_k) = 0$$

and $w_k \in W_0^{1,p}(U_\rho)$ be a unique solution of the equation

$$-\operatorname{div} b^k(x, \xi + \nabla w_k) = 0.$$

In view of Theorem 2.4.2, we need to estimate the quantity

$$\begin{aligned} J &= \int_{U_\rho} |a^k(x, \xi + \nabla v_k) - b^k(x, \xi + \nabla w_k)| dx \leq \\ &\leq \int_{U_\rho} |a^k(x, \xi + \nabla v_k) - a^k(x, \xi + \nabla w_k)| dx + \\ &\quad + \int_{U_\rho} |a^k(x, \xi + \nabla w_k) - b^k(x, \xi + \nabla w_k)| dx = \\ &= J_1 + J_2. \end{aligned} \tag{2.4.23}$$

In what follows we shall denote by K any constant depending only on R.

First of all, we have

$$\|\xi + \nabla w_k\|_p^p \leq K|U_\rho| \leq c\rho^n. \tag{2.4.24}$$

By Theorem 2.4.3 and Remark 2.4.1,

$$\|\xi + \nabla w_k\|_{p+\alpha} \leq C(U_\rho)\|\xi + \nabla w_k\|_p \leq c\rho^{-\alpha n/(p+\alpha)}|U_\rho|^{1/p}. \tag{2.4.25}$$

Of course, the similar estimates hold true for v_k as well. Now let us define the set

$$A_r = \{x \in U_\rho : |\xi + \nabla w_k(x)| > r\}.$$

Then we have

$$|A_r| r^p \leq \int_{A_r} |\xi + \nabla w_k|^p dx.$$

Hence,

$$|A_r| \leq K|U_\rho| r^{-p}.$$

Using the Hölder inequality and (2.4.25) we obtaine

$$\begin{aligned} \int_{A_r} (1 + |\xi + \nabla w_k|^p) dx &\leq |A_r| + |A_r|^{\alpha/(p+\alpha)} \|\xi + \nabla w_k\|_{p+\alpha}^p \leq \\ &\leq K|U_\rho|(r^{-p} + r^{-\alpha/(p+\alpha)}) = K|U_\rho| \cdot \varphi(r). \end{aligned} \tag{2.4.26}$$

To derive an estimate for J, first we consider the integral J_2. We have, using (2.4.26),

$$\begin{aligned}
J_2 &= \int_{A_r} |a^k(x,\xi+\nabla w_k) - b^k(x,\xi+\nabla w_k)|dx + \\
&\quad + \int_{U_\rho\setminus A_r} |a^k(x,\xi+\nabla w_k) - b^k(x,\xi+\nabla w_k)|dx \le \\
&\le K\int_{A_r}(1+|\xi+\nabla w_k|^{p-1})dx + \int_{\mathcal{U}_\rho\setminus A_r} g^k(x,r)dx \le \\
&\le \int_{U_\rho} g^k(x,r)dx + K|U_\rho|^{1/p}\left[\int_{A_r}(1+|\xi+\nabla w_k|^p)dx\right]^{1/p'} \le \\
&\le \int_{U_\rho} g^k(x,r)dx + K|U_\rho|\cdot\varphi(r)^{1/p'}. \qquad (2.4.27)
\end{aligned}$$

In the similar way,

$$\begin{aligned}
J_3 &= \int_{U_\rho} |a^k(x,\xi+\nabla w_k) - b^k(x,\xi+\nabla w_k)|^{p'}dx \le \\
&\le K(1+r)\int_{U_\rho} g^k(x,r)dx + K|U_\rho|\cdot\varphi(r). \qquad (2.4.28)
\end{aligned}$$

Here to get the factor $(1+r)$ in the first term of the right-hand part, we have used (2.1.29) to estimate the integral

$$\int_{U_\rho} g^k(x,r)^{p'}dx = \int_{U_\rho} g^k(x,r)^{p'/p}g^k(x,r)dx.$$

Before to handle J_1, we need an estimate for $\nabla v_k - \nabla w_k$. Using the Hölder inequality, the definition of $\mathcal{E}$, and (2.4.28), we have

$$\begin{aligned}
0 &= \int_{U_\rho}\left[a^k(x,\xi+\nabla v_k) - b^k(x,\xi+\nabla w_k)\right]\cdot\nabla(v_k-w_k)dx = \\
&= \int_{U_\rho}\left[a^k(x,\xi+\nabla v_k) - a^k(x,\xi+\nabla w_k)\right]\cdot\nabla(v_k-w_k)dx + \\
&\quad + \int_{U_\rho}\left[a^k(x,\xi+\nabla w_k) - b^k(x,\xi+\nabla w_k)\right]\cdot(v_k-w_k)dx \ge \\
&\ge K\int_{U_\rho}(1+|\xi+\nabla v_k|^p+|\xi+\nabla w_k|^p)^{1-\beta/p}\cdot|\nabla(v_k-w_k)|^\beta dx - \\
&\quad - J_3^{1/p'}\cdot\|\nabla(v_k-w_k)\|_p \ge \\
&\ge K\left[\int_{U_\rho}(1+|\xi+\nabla v_k|^p+|\xi+\nabla w_k|^p)\,dx\right]^{1-\beta/p}\cdot\|\nabla(v_k-w_k)\|_p^\beta - \\
&\quad - J_3^{1/p'}\cdot\|\nabla(v_k-w_k)\|_p \ge \\
&\ge K|U_\rho|^{1-\beta/p}\cdot\|\nabla(v_k-w_k)\|_p^\beta - J_3^{1/p'}\cdot\|\nabla(v_k-w_k)\|_p.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|\nabla(v_k - w_k)\|_p^{\beta-1} &\le K|U_\rho|^{\beta/p-1} J_3^{1/p'} \le \qquad (2.4.29)\\
&\le K|U_\rho|^{\beta/p-1}\left[(1+r)\int_{U_\rho} g^k(x,r)dx + |U_\rho|\cdot\varphi(r)\right]^{1/p'}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
J_1 &\le |U_\rho|^{1/p}\left[\int_{U_\rho} |a^k(x,\xi+\nabla v_k) - a^k(x,\xi+\nabla w_k)|^{p'}dx\right]^{1/p'} \le\\
&\le |U_\rho|^{1/p}\left[\int_{U_\rho} (1+|\xi+\nabla v_k|^p + |\xi+\nabla w_k|^p)^{1-s/p}|\nabla(v_k-w_k)|^s dx\right]^{1/p'} \le\\
&\le K|U_\rho|^{1/p'}\left[\int_{U_\rho} (1+|\xi+\nabla v_k|^p + |\xi+\nabla w_k|^p)\,dx\right]^{(p-s)/(pp')} \times\\
&\quad\times \|\nabla(v_k-w_k)\|_p^{s/p'}.
\end{aligned}
$$

Using (2.4.24) and (2.4.29), we obtain after a simple calculation

$$
J_1 \le K|U_\rho|^{1-\gamma}\left[(1+r)\int_{U_\rho} g^k(x,r)dx + |U_\rho|\cdot\varphi(r)\right]^{\gamma}. \qquad (2.4.30)
$$

By Theorem 2.4.2, inequalities (2.4.27) and (2.4.30) imply immediately inequality (2.4.21). □

Now we state some direct consequences of the last result.

Corollary 2.4.3 *Let $\mathcal{A}_k$ and $\mathcal{B}_k$ be two sequences of operators of the class $\mathcal{E}$ such that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$ and $\mathcal{B}_k \overset{G}{\Longrightarrow} \mathcal{B}$. Assume that for any $r \ge 0$*

$$
\begin{aligned}
&\lim_{k\to\infty} \sup_{|\xi|,|\eta|\le r} |a^k(x,\eta,\xi) - b^k(x,\eta,\xi)| =\\
&= \lim_{k\to\infty} \sup_{|\xi|,|\eta|\le r} |a_0^k(x,\eta,\xi) - b_0^k(x,\eta,\xi)| = 0
\end{aligned}
$$

strongly in $\mathrm{L}^1(Q)$. Then $\mathcal{A} = \mathcal{B}$.

We shall say that a sequence $\mathcal{A}_k \in \mathcal{E}$ converges to $\mathcal{A} \in \mathcal{E}$ *component-wise* in L^1 (c.-w. in L^1), if for any $r \ge 0$

$$
\begin{aligned}
&\lim_{k\to\infty} \sup_{|\xi|,|\eta|\le r} |a^k(x,\eta,\xi) - a(x,\eta,\xi)| =\\
&= \lim_{k\to\infty} \sup_{|\xi|,|\eta|\le r} |a_0^k(x,\eta,\xi) - a_0(x,\eta,\xi)| = 0
\end{aligned}
$$

strongly in $\mathrm{L}^1(Q)$.

Corollary 2.4.4 *Let $\mathcal{A}_k^l$ be a double sequence of operators of the class $\mathcal{E}$ such that $\mathcal{A}_k^l \xrightarrow{G} \mathcal{A}^l$ for any $l \in \mathbf{N}$, as $k \to \infty$. Assume that $\mathcal{A}_k^l \to \mathcal{A}_k$ c.-w. in L^1 uniformly with respect to $k \in \mathbf{N}$ and $\mathcal{A}^l \to \mathcal{A}$ c.-w. in L^1, as $l \to \infty$. Then $\mathcal{A}_k \xrightarrow{G} \mathcal{A}$.*

Remark 2.4.2 Recall that we have imposed here assumption (2.4.20). The statement of Corollary 2.4.4 is still valid without this assumption if we replace c.-w. convergence in L^1 by the following condition:

$$\lim_{l\to\infty} \operatorname*{ess\,sup}_{x\in Q} \sup_{(\eta,\xi)\in\mathbf{R}^{n+1}} \frac{|a_l^k(x,\eta,\xi) - a^k(x,\eta,\xi)|^{p'}}{(c(x) + |\eta|^p + |\xi|^p)} = 0,$$

and similarly for the differences $a_{0,l}^k(x,\eta,\xi) - a_0^k(x,\eta,\xi)$, $a_l(x,\eta,\xi) - a(x,\eta,\xi)$, and $a_{0,l}(x,\eta,\xi) - a_0(x,\eta,\xi)$.

2.4.3 One-Dimentional Case

Now we consider the case $n = 1$ and $Q = (x_0, x_1) \subset \mathbf{R}$. Therefore, we look at differential operators of the form

$$\mathcal{A}u = -(a(x,u,u'))' - a_0(x,u,u'), \tag{2.4.31}$$

where the Carathéodory functions

$$a : (x_0,x_1) \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$$

and

$$a_0 : (x_0,x_1) \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$$

satisfy inequalities (2.1.29), (2.1.30), and (2.3.2). In this case strong G-convergence may be characterized in a more explicit way.

Let us consider a sequence $\mathcal{A}_k$ of operators of the form (2.4.31) belonging to the class $\mathcal{E}$. Inequality (2.1.30) implies that, for any $\eta \in \mathbf{R}$ and for almost all $x \in (x_0, x_1)$, the function $a^k(x,\eta,\xi)$ is strictly monotone with respect to $\xi \in \mathbf{R}$. Denote by $r^k(x,\eta,\tau)$ the inverse function to $a^k(x,\eta,\xi)$ with respect to the variable $\xi \in \mathbf{R}$, i.e. $\xi = r^k(x,\eta,\tau)$ is a unique solution of the equation

$$a^k(x,\eta,\xi) = \tau.$$

It is not hard to see that $r^k(\cdot,\eta,\tau)$ is a bounded sequence in $\mathrm{L}^p(x_0,x_1)$, for any $(\eta,\tau) \in \mathbf{R} \times \mathbf{R}$.

Theorem 2.4.5 *Let $\mathcal{A}_k$ be a sequence of operators of the form (2.4.31) belonging to the class $\mathcal{E}$. Then $\mathcal{A}_k$ is strongly G-convergent if and only if for any $(\eta, \tau) \in \mathbf{R}$ there exist the limits*

$$\lim_{k\to\infty} r^k(x,\eta,\xi) = r(x,\eta,\tau) \tag{2.4.32}$$

weakly in $L^p(x_0, x_1)$ *and*

$$\lim_{k\to\infty} a_0^k(x,\eta,r^k(x,\eta,\tau)) = q(x,\eta,\tau) \tag{2.4.33}$$

weakly in $L^{p'}(x_0,x_1)$. *Moreover, for the strong G-limit operator $\mathcal{A}$, $a(x,\eta,\xi)$ is the inverse function to $r(x,\eta,\tau)$ with respect to the variable τ and*

$$a_0(x,\eta,\xi) = q(x,\eta,a(x,\eta,\xi)).$$

Proof. Let us suppose the limits in (2.4.32) and (2.4.33) to exist. Passing to a subsequence we may assume that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. Therefore, we have only to prove the second statement of the theorem. This implies, evidently, that the passage to a subsequence is superfluous. Let us consider the function $u_k \in W^{1,p}(x_0, x_1)$ defined by the formula

$$u_k(x) = \int_{x_0}^{x} r^k(y,\eta,\tau)dy.$$

Then $u_k \to u$ weakly in $W^{1,p}(x_0,x_1)$, where

$$u(x) = \int_{x_0}^{x} r(y,\eta,\tau)dy.$$

Since u_k solves the equation

$$-(a^k(x,\eta,u_k'))' = 0$$

and $a^k(x,\eta,u_k') = \tau$, we have, by Remark 2.3.5,

$$a(x,\eta,r(x,\eta,\tau)) = \tau \tag{2.4.34}$$

and

$$a_0(x,\eta,r(x,\eta,\tau)) = q(x,\eta,\tau). \tag{2.4.35}$$

This implies immediately the required.

Now we assume that $\mathcal{A}_k \overset{G}{\Longrightarrow} \mathcal{A}$. To prove the existence of the limits in (2.4.32) and (2.4.33) we have to show that given $(\eta,\tau) \in \mathbf{R}\times\mathbf{R}$ equations (2.4.34) and (2.4.35) are satisfied for any weak limit points $r(x,\eta,\tau)$ and $q(x,\eta,\tau)$ of the sequences $r^k(x,\eta,\tau)$ and $a_0^k(x,\eta,r^k(x,\eta,\tau))$, respectively. However, this may be done exactly as above and we conclude. □

2.5 Strong Nonlinearity in Lower Order Term

Following [223] we discuss G-convergence for a more wide class of elliptic operators then that we considered in Section 2.3. Let Q be a bounded open subset of $\mathbf{R}^n$. We consider differential operators of the form

$$\mathcal{A}u = -\operatorname{div} a(x,u,\nabla u) + a_0(x,u,\nabla u) \tag{2.5.1}$$

where the functions

$$a : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

and

$$a_0 : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

satisfy the Carathéodory condition and, for almost all $x \in Q$ and all $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$, the inequality

$$|a(x,\xi_0,\xi)|^{p'} + |a_0(x,\xi_0,\xi)|^{p_0'} \le c_0\,(|\xi_0|^{p_0} + |\xi|^p) + c(x), \tag{2.5.2}$$

where $p > 1$, $p_0 > 1$, $c_0 > 0$, and $c \in \mathrm{L}^1(Q)$ is a nonnegative function. Furtheremore, we suppose the following inequalities to be satisfied for almost all $x \in Q$ and for all $\xi_0, \xi_0' \in \mathbf{R}$, $\xi, \xi' \in \mathbf{R}^n$:

$$[a(x,\xi_0,\xi) - a(x,\xi_0',\xi')]\cdot(\xi-\xi') \ge \kappa\,[H_{p_0}(\xi_0,0) + H_p(\xi,\xi')]^{1-\beta/p}\,|\xi-\xi'|^\beta, \tag{2.5.3}$$

$$\begin{aligned} &|a(x,\xi_0,\xi) - a(x,\xi_0',\xi')|^{p'} + |a_0(x,\xi_0,\xi) - a_0(x,\xi_0',\xi')|^{p_0'} \le \\ &\quad \le \theta\,[H_{p_0}(\xi_0,\xi_0') + H_p(\xi,\xi')]\,\nu\,(|\xi_0 - \xi_0'|) + \theta\,[H_{p_0}(\xi_0,\xi_0') + H_p(\xi,\xi')]^{1-s/p} \times \\ &\qquad \times |\xi - \xi'|^s, \end{aligned} \tag{2.5.4}$$

where

$$\beta \ge \max[2;p],$$

$$0 < s \le \min[p;p'],$$

$$H_p(\tau,\tau') = h(x) + |\tau|^p + |\tau'|^p,$$

$\kappa > 0$, $\theta > 0$, $h \in \mathrm{L}^1(Q)$ is a nonnegative function, and $\nu(r)$ is a modulus of continuity. The class of all such operators will be denoted by

$$\mathcal{E}^{p,p_0} = \mathcal{E}^{p,p_0}(c_0, c, \kappa, \theta, \nu, s, \beta).$$

If $p_0 = p$, then $\mathcal{E}^{p,p_0}$ coincides, essentially, with the class $\mathcal{E}$ considered in Section 2.3. Moreover, if $p_0 < p$, the class $\mathcal{E}^{p,p_0}$ is contained in a suitable class $\mathcal{E}^{p,p_0}$. Therefore, from the point of veiw of G-convergence, the case $p_0 \le p$ is covered by the results of Section 2.3.

Now we assume that $p_0 > p$ and

$$\frac{1}{p_0} > \frac{1}{p} - \frac{1}{n}. \tag{2.5.5}$$

In this case $V = W_0^{1,p}(Q) \subset \mathrm{L}^{p_0}(Q)$ and the embedding is compact. Therefore, any operator $\mathcal{A} \in \mathcal{E}^{p,p_0}$ acts continuously from V into V^*. As in Section 2.3, given $\mathcal{A} \in \mathcal{E}^{p,p_0}$ we set

$$\mathcal{A}^1(u,v) = -\operatorname{div} a(x,v,\nabla u).$$

In the case under consideration, we say that $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if, for any $u \in V$ and $v \in V$, we have $u_k \to u$ weakly in V,

$$\Gamma^k(u,v) \to \Gamma(u,v) \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n,$$

$$\Gamma_0^k(u,v) \to \Gamma_0(u,v) \quad \text{weakly in } \mathrm{L}^{p_0'}(Q),$$

where $u_k \in V$ is a unique solution of the equation

$$\mathcal{A}^k(u,v) = a^k(x,v,\nabla u_k),$$

and

$$\Gamma_0^k(u,v) = a_0^k(x,v,\nabla u_k).$$

Making evident modifications in the technique we have used in Section 2.2.3 (and using, in particular, compactness of the embedding) one can obtain corresponding versions of all the results of that Section.

Now we turn to the more delicate case

$$\frac{1}{p_0} \le \frac{1}{p} - \frac{1}{n}. \tag{2.5.6}$$

In this situation we set $V_0 = \mathrm{L}^{p_0}(Q)$, $V_1 = W_0^{1,p}(Q)$, and $V = V_0 \cap V_1$. Then

$$V^* = V_0^* + V_1^* = \mathrm{L}^{p_0'}(Q) + W^{-1,p'}(Q).$$

Notice that in the case of equality in (2.5.6) we have $V = V_1 = W_0^{1,p}(Q)$. However, in contrast to the case of (2.5.5), the embedding $V \subset V_0 = \mathrm{L}^{p_0}(Q)$ is not compact now.

It is evident that any operator of the class $\mathcal{E}^{p,p_0}$ acts continuously from V into V^* (and from $\overline{V} = \overline{V}_1 \cap V_0 = W^{1,p}(Q) \cap \mathrm{L}^{p_0}(Q)$ into V^*). However, to overcome the lack of compactness we restrict ourself to the case of monotone operators. More precisely, denote by

$$\mathcal{E}^{p,p_0}_{(m)} = \mathcal{E}^{p,p_0}_{(m)}(c_0, c, \kappa, \nu, s, \beta, \beta_0)$$

the set of all operators $\mathcal{A} \in \mathcal{E}^{p,p_0}$ satisfying, instetad of (2.5.3), the inequality

$$\begin{aligned}
&[a(x,\xi_0,\xi) - a(x,\xi_0',\xi')] \cdot (\xi - \xi') + [a_0(x,\xi_0,\xi) - a_0(x,\xi_0',\xi')] \cdot (\xi_0 - \xi_0') \geq \\
&\quad \geq \kappa \left[H_{p_0}(\xi_0 - \xi_0') + H_p(\xi,\xi')\right]^{1-\beta/p} |\xi - \xi'|^\beta + \\
&\qquad + \kappa \left[H_{p_0}(\xi_0, \xi_0') + H_p(\xi,\xi')\right]^{1-\beta_0/p_0} |\xi_0 - \xi_0'|^{\beta_0},
\end{aligned} \tag{2.5.7}$$

where $\beta_0 \geq \max[2; p_0]$. Any such operator is invertible as an operator acting from V into V^*.

It turns out to be that all the results of Section 2.3 are still valid for the class of operators $\mathcal{E}^{p,p_0}_{(m)}$. First of all, we have the following version of Lemma 2.3.1 with, essentially, the same proof:

Lemma 2.5.1 *Let $\mathcal{A}_k \in \mathcal{E}^{p,p_0}_{(m)}$. Suppose u_k, v_k are bounded sequences in $\overline{V}$ such that $z_k = u_k - v_k \to 0$ weakly in $\overline{V}$. Assume that $\mathcal{A}_k u_k \to f$ and $\mathcal{A}_k v_k \to g$ strongly in V^*. Then $f = g$ and $z_k \to 0$ strongly in $W^{1,p}_{loc}(Q) \cap \mathrm{L}^{p_0}_{loc}(Q)$.*

Now we state the main result.

Theorem 2.5.1 *Any sequence $\mathcal{A}_k \in E^{p,p_0}_{(m)}(c_0, c, \kappa, h, \nu, s, \beta, \beta_0)$ contains a subsequence which strongly G-converges to an operator $\mathcal{A} \in \mathcal{E}^{p,p_0}_{(m)}(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{h}, \nu, \overline{s}, \beta, \beta_0)$, with suitable values of the overlined parameters.*

Proof. It goes along the same lines as that of Lemma 2.3.2 and we indicate here only the principal changes, refering to the consecutive steps of the proof of Lemma 2.3.2 when it is needed.

We start with some preliminaries. Set $X = \mathrm{L}^{p_0}(Q) \times \mathrm{L}^p(Q)^n$. Associated to $\mathcal{A} \in \mathcal{E}^{p,p_0}_{(m)}$, there is the operator $\overline{\mathcal{A}} : X \longrightarrow V^*$ defined by

$$\overline{\mathcal{A}}\psi = -\operatorname{div} a(x, \psi_0, \psi') + a_0(x, \psi, \psi'), \quad \psi = (\psi_0, \psi') \in X.$$

We have $\mathcal{A} = \overline{\mathcal{A}} \circ \overline{\nabla}$, where $\overline{\nabla} : V \longrightarrow X$ is defined by $\overline{\nabla} u = (u, \nabla u)$. As in n^0 2.3.2, given $\psi \in X$ we define the operator $\overline{\mathcal{A}}^\psi : X \longrightarrow V^*$ by

$$\overline{\mathcal{A}}^\psi \chi = \overline{\mathcal{A}}(\chi + \psi), \quad \chi \in X,$$

and set $\mathcal{A}^\psi = \overline{\mathcal{A}}^\psi \circ \overline{\nabla}$. For any $\psi \in X$, the operator $\mathcal{A}^\psi : V \longrightarrow V^*$ is invertible and we introduce the operator $\mathcal{R} : V^* \times X \longrightarrow V$ defined by the formula

$$\mathcal{R}(f, \psi) = (\mathcal{A}^\psi)^{-1} f, \quad f \in V^*, \ \psi \in X.$$

We devide the rest of the proof into several steps.

Step 1. Let us consider a sequence $\mathcal{A}_k \in \mathcal{E}^{p,p_0}_{(m)}$. One can show that the sequence of operators $\mathcal{R}_k$ is equicontinuous on any ball in $V^* \times X$. Hence, using the diagonal process we see that, as on Step 1, n^0 2.3.2, there exist a subsequence of $\mathcal{A}_k$ still denoted by $\mathcal{A}_k$ and an operator $\mathcal{R} : V^* \times X \longrightarrow V$ with the property:

$$\mathcal{R}_k(f, \psi) \to \mathcal{R}(f, \psi) \quad \text{weakly in } V,$$

for any $(f, \psi) \in V^* \times X$. Moreover, for any $\psi \in X$, the map $f \mapsto \mathcal{R}(f, \psi)$ is invertible, as a map from V^* into V. Therefore, there exists an operator $\mathcal{A}^\psi : V \longrightarrow V^*$, $\psi \in X$, such that

$$\mathcal{R}(f, \psi) = (\mathcal{A}^\psi)^{-1} f.$$

We set also $\overline{\mathcal{A}}\psi = \mathcal{A}^\psi(0)$ and $\mathcal{A} = \overline{\mathcal{A}} \circ \overline{\nabla}$. The last operator is well-defined on the space $\overline{V}$, hence, on V. Additionally, one can obtain an upper bound for $\|\overline{\mathcal{A}}\psi\|_{V^*}$ by means of $\|\psi\|_X$, like (2.3.17), but a little more comlicated.

Step 2. The construction of Step 2, n^0 2.3.2, must be modified in the following way. Given $\psi \in X$ we set (cf. (2.3.18))

$$\psi_k = \psi + \overline{\nabla}\mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi) = \psi + \overline{\nabla} u^1_k, \tag{2.5.8}$$

where $u^1_k = \mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi) \in V$. If $\psi = \overline{\nabla} u$, $u \in \overline{V}$, then $\psi_k = \overline{\nabla} u_k$, where $u_k = u + u^1_k$. Obviously, $u^1_k \to 0$ weakly in V and $\psi_k \to \psi$ weakly in X.

Now we define the operators

$$\overline{\Gamma}^k : X \longrightarrow \mathrm{L}^{p'}(Q)^n, \quad \overline{\Gamma}^k_0 : X \longrightarrow \mathrm{L}^{p_0'}(Q)$$

by the formulae

$$\overline{\Gamma}^k(\psi) = a^k(x, \psi_{0,k}, \psi'_k), \quad \overline{\Gamma}^k_0(\psi) = a_0(x, \psi_{0,k}, \psi'_k),$$

where $\psi = (\psi_0, \psi') \in X$ and $\psi_k = (\psi_{0,k}, \psi'_k)$. We set also $\Gamma^k = \overline{\Gamma}^k \circ \overline{\nabla}$ and $\Gamma^k = \overline{\Gamma}^k_0 \circ \overline{\nabla}$. For these operators there is an upper bound similar to that given by (2.3.19).

Step 3. All the localization properties established on Step 3, n^0 2.3.2, are still valid in the case under consideration.

Step 4. Now we want to study the continuity of $\overline{\mathcal{A}}$ using a version of the arguments we applied on Step 4, n^0 2.3.2. Let $\varphi, \psi \in X$. We consider $\psi_k = \psi + \overline{\nabla} u^1_k$ defined by (2.5.8) and $\varphi_k = \varphi + \overline{\nabla} v^1_k$ defined in the similar way, with ψ replaced by φ. Set

$$Z^k = \overline{\Gamma}^k(\psi) - \overline{\Gamma}(\varphi), \quad Z^k_0 = \overline{\Gamma}^k_0(\psi) - \overline{\Gamma}^k_0(\varphi),$$

$$z^1_k = u^1_K - v^1_k, \quad \sigma_k = \psi_k - \varphi_k, \quad \sigma = \psi - \varphi,$$

and

$$y = \overline{\mathcal{A}}\psi - \overline{\mathcal{A}}\varphi = -\operatorname{div} Z^k + Z_0^k.$$

Also we shall use the notations like $\psi = (\psi_0, \psi')$, $\psi_k = (\psi_{0,k}, \psi'_k)$, etc., where $\psi_0, \psi_{0,k} \in \mathrm{L}^{p_0}(Q)$, $\psi', \psi'_k \in \mathrm{L}^p(Q)^n$.

Let $T \subset Q$ be a measurable subset, $S = Q \setminus T$. For the sake of brevity we introduce the following notations:

$$N_T(f) = \|\nu(|f|)\|_{\infty,T},$$

$$H_{T,k} = H_T(\varphi_k, \psi_k) = \int_T [H_{p_0}(\varphi_{0,k}, \psi_{0,k}) + H_p(\varphi'_k, \psi'_k)]\, dx,$$

$$\overline{H}_T = \overline{H}_T(\varphi, \psi) = \int_T \left[\overline{H}_{p_0}(\varphi_0, \psi_0) + \overline{H}_p(\varphi', \psi')\right] dx,$$

where $\overline{H}_p$ is defined like H_p with $h(x)$ replaced by $\overline{h}(x)$ which depends only on $h(x)$ and $c(x)$.

From (2.5.4) we deduce easily the inequality

$$\|Z^k\|_{p'}^{p'} \le \theta \left[H_{T,k} \cdot N_T(\sigma_{0,k}) + H_{S,k} + H_{Q,k}^{1-s/p} \|\sigma'_k\|_p^s\right]. \tag{2.5.9}$$

In the other hand, we have, using (2.5.3) and (2.5.4),

$$\begin{aligned}(y, z_k^1) &= \int_Q Z^k \sigma'_k dx + \int_Q Z_0^k \sigma_{0,k} dx - \int_Q Z^k \sigma' dx - \int_Q Z_0^k \sigma_0 dx \ge \\ &\ge \kappa H_{Q,k}^{1-\beta/p} \|\sigma'_k\|_p^\beta - \\ &\quad - \theta \left[H_{T,k} \cdot N_T(\sigma_{0,k}) + H_{S,k} + H_{Q,k}^{1-s/p} \|\sigma'_k\|_p^s\right]^{1/p'} \|\sigma'\|_p - \\ &\quad - \theta \left[H_{T,k} \cdot N_T(\sigma_{0,k}) + H_{S,k} + H_{Q,k}^{1-s/p} \|\sigma'_k\|_p^s\right]^{1/p'_0} \|\sigma_0\|_{p_0}.\end{aligned}$$

As in the proof of inequality (2.3.21), we obtain, by means of the Young inequality,

$$\begin{aligned}(y, z_k^1) &\ge \overline{\kappa} H_{Q,k}^{1-\beta/p} \|\sigma'_k\|_p^\beta - \overline{\theta} H_{Q,K}^{1-\beta/s} \left[H_{T,k} \cdot N_T(\sigma_{0,k}) + H_{S,k}\right]^{\beta/s} - \\ &\quad - \theta H_{Q,k}^{(\beta-s)/(\beta p'-s)} \|\sigma'\|^{(\beta p'/(\beta p'-s)} - \overline{\theta} H^{(\beta-s)/(\beta p'_0-s)} \|\sigma_0\|^{\beta p'_0/(\beta p'_0-s)}.\end{aligned}$$

This implies (we suppress somewhat long calculations)

$$\begin{aligned}\|\sigma'_k\|_p^s &\le \overline{\theta} \left[H_{Q,k}^{s/p-s\beta} (y, z_k^1)^{s/\beta} + H_{Q,k}^{s/p-1} (H_{T,k} \cdot N_T(\sigma_{0,k}) + H_{S,k}) + \right. \\ &\quad \left. + H_{Q,k}^{s/p-\overline{s}/p} \|\sigma'\|_p^{\overline{s}} + H_{Q,k}^{s/p-\overline{s}/p_0} \|\sigma_0\|_{p_0}^{\overline{s}_0}\right],\end{aligned}$$

where

$$\overline{s} = \frac{sp}{\beta p - sp + s}, \quad \overline{s}_0 = \frac{sp_0}{\beta p_0 - sp + s}.$$

Substituting into (2.5.9) we get

$$\|Z^k\|_p^{p'} \geq \overline{\theta}\Big[H_{Q,k}^{1-s/\beta}(y, z_k^1)^{s/\beta} + H_{T,k}\cdot N_T(\sigma_{0,k}) + H_{S,k} + \quad (2.5.10)$$

$$+ H_{Q,k}^{1-\overline{s}/p}\|\sigma'\|_p^{\overline{s}} + H_{Q,k}^{1-\overline{s}_0/p}\|\sigma_0\|_{p_0}^{\overline{s}_0}\Big]. \quad (2.5.11)$$

Now, as in n^0 2.3.2, we have

$$H_{T,k} \leq H_{Q,k} \leq C\overline{H}_Q, \quad (2.5.12)$$

with an appropriate function $\overline{h}(x)$ and a constant $C > 0$.

Notice that $z_k^1 \to 0$ weakly in V and, hence, strongly in $\mathrm{L}^p(Q)$. As consequence, $\sigma_{0,k} \to \sigma_0$ strongly in $\mathrm{L}^p(Q)$. Passing to a subsequence, we can assume that $\sigma_{0,k} \to \sigma$ a.e. on Q and, by the Egorov Theorem, almost uniformly. Therefore, for any $\varepsilon > 0$ there exists a measurable subset $T_\varepsilon \subset T$ such that $|T \setminus T_\varepsilon| < \varepsilon$ and $\sigma_{0,k} \to \sigma_0$ in $\mathrm{L}^\infty(T_\varepsilon)$. Hence,

$$\lim N_{T_\varepsilon}(\sigma_{0,k}) = N_{T_\varepsilon}(\sigma_0) \leq N_T(\sigma_0). \quad (2.5.13)$$

Let $S_\varepsilon = Q \setminus T_\varepsilon$. Suppose Q_ε is an open subset of Q such that $\overline{S}_\varepsilon \subset Q_\varepsilon$ and $|Q_\varepsilon \setminus \overline{S}_\varepsilon| < \varepsilon$. We define the function

$$\tilde{\psi}_k \in X(Q_\varepsilon) = \mathrm{L}^{p_0}(Q_\varepsilon) \times \mathrm{L}^p(Q_\varepsilon)^n$$

in the same way as ψ_k, but with Q replaced by Q_ε, i.e.

$$\tilde{\psi}_k = \psi_{|Q_\varepsilon} + \overline{\nabla}\tilde{u}_k^1 = \psi_{|Q_\varepsilon} + \overline{\nabla}\mathcal{R}_k^\varepsilon(\overline{\mathcal{A}}\psi_{|Q_\varepsilon}, \psi_{|Q_\varepsilon}),$$

where the operator $\mathcal{R}_k^\varepsilon$ is defined like $\mathcal{R}_k$, with the underlining domain Q_ε instead of Q. Similarly, we define the function $\tilde{\varphi}_k$. We have

$$\tilde{\psi}_k \to \psi_{|Q_\varepsilon}, \quad \tilde{\varphi}_k \to \varphi_{|Q_\varepsilon}$$

weakly in $X(Q_\varepsilon)$. At the same time, Lemma 2.5.1 implies that

$$\tilde{\psi}_k - \psi_{k|Q_\varepsilon} \to 0, \quad \tilde{\varphi}_k - \varphi_{k|Q_\varepsilon} \to 0$$

strongly in

$$X_{loc}(Q_\varepsilon) = \mathrm{L}_{loc}^{p_0}(Q_\varepsilon) \times \mathrm{L}_{loc}^p(Q_\varepsilon)^n.$$

Similarly to (2.5.12), we have

$$H_{Q_\varepsilon}(\tilde{\varphi}_k, \tilde{\psi}_k) \leq C\overline{H}_{Q_\varepsilon},$$

where $C > 0$ does not depend on ε. Therefore, for any compact subset $K \subset Q_\varepsilon$, we have

$$H_{S_\varepsilon \cap K,k} \leq C\Big[H_{Q_\varepsilon}(\tilde{\varphi}_k, \tilde{\psi}_k) + \|\tilde{\varphi}_{0,k} - \varphi_{0,k}\|_{p_0,K}^{p_0} + \|\tilde{\psi}_{0,k} - \psi_{0,k}\|_{p_0,K}^{p_0} +$$

$$+ \|\tilde{\varphi}_k' - \varphi_k'\|_{p,K}^p + \|\tilde{\psi}_k' - \psi_k'\|_{p,K}^p\Big]$$

and, consequently,

$$\limsup_{k\to\infty} H_{S_\varepsilon\cap K,k} \le C\overline{H}_{Q_\varepsilon}.$$

Since K is an arbitrary compact subset of Q_ε,

$$\limsup_{t\to\infty} H_{S_\varepsilon,k} \le C\overline{H}_{Q_\varepsilon}. \tag{2.5.14}$$

We set

$$\gamma = \gamma(\varphi,\psi) = \limsup \|Z^k\|_{p'}^{p'}.$$

Combining inequalities (2.5.10), with T replaced by T_ε, (2.5.12), (2.5.13), and (2.5.14), we get

$$\gamma \le \overline{\theta}\left[\overline{H}_Q \cdot N_T(\sigma_0) + \overline{H}_Q^{1-\overline{s}/p}\|\sigma'\|_p^{\overline{s}} + \overline{H}_Q^{1-\overline{s}_0/p_0}\|\sigma_0\|_{p_0}^{\overline{s}_0}\right].$$

Since $S = \cap S_\varepsilon = \cap Q_\varepsilon$, we obtain finally

$$\gamma \le \overline{\theta}\left[\overline{H}_Q \cdot N_T(\sigma_0) + \overline{H}_S + \overline{H}_Q^{1-s/p}\|\sigma'\|_p^{\overline{s}} + \overline{H}_Q^{1-\overline{s}_0/p_0}\|\sigma_0\|_{p_0}^{\overline{s}_0}\right] \tag{2.5.15}$$

There is also a similar inequality for

$$\gamma_0 = \gamma_0(\varphi,\psi) = \limsup_{k\to\infty} \|Z_0^k\|_{p_0'}^{p_0'}.$$

Now we are able to prove the continuity of $\overline{\mathcal{A}}$. Let $\psi^{(l)} \to \varphi$ in X. Passing to a subsequence one can assume this convergence to be almost uniform. Hence, we can choose T such that $\psi^{(l)} \to \varphi$ in $\mathrm{L}^\infty(T)$, while $\overline{H}_T(\psi^{(l)},\varphi)$ is arbitrarily small. By inequality (2.5.15), we see that

$$\lim_{l\to\infty} \gamma(\varphi,\psi^{(l)}) = 0.$$

Similarly,

$$\lim_{l\to\infty} \gamma_0(\varphi,\psi^{(l)}) = 0.$$

Since

$$\overline{\mathcal{A}}\psi - \overline{\mathcal{A}}\varphi = -\operatorname{div} Z^k + Z_0^k,$$

we have

$$\|\overline{\mathcal{A}}\psi^{(l)} - \overline{\mathcal{A}}\varphi\|_* \le \gamma(\varphi,\psi^{(l)})^{1/p'} + \gamma_0(\varphi,\psi^{(l)})^{1/p_0'}.$$

Thus, we have proved that the operator $\overline{\mathcal{A}}$ is continuous.

Step 5. Now suppose $\psi^{(l)} \to \psi$ in X. Let ψ_k be defined by (2.5.8), and let $\psi_k^{(l)}$ be defined in the same manner, with ψ replaced by $\psi^{(l)}$. Since the operators $\mathcal{R}_k$ are equicontinuous, $\psi_k^{(l)} \to \psi_k$ in X uniformly with respect to k, as $l \to \infty$. Given $\varepsilon > 0$ there exist a measurable set $T_{\varepsilon,k} \subset Q$ and a constant $C > 0$ such that $|Q \setminus T_{\varepsilon,k}| < \varepsilon$ and

$$h(x) + |\psi_k^{(l)}(x)| + |\psi_k(x)| \leq C, \quad x \in T_{\varepsilon,k}.$$

By (2.5.4), we have, for $x \in T_{\varepsilon,k}$,

$$\left|\left[\overline{\Gamma}^k(\psi^{(l)} - \overline{\Gamma}^k(\psi)\right](x)\right|^{p'} \leq C_1 \left[\nu\left(\left|(\psi_k^{(l)} - \psi_k)_0(x)\right|\right) + \left|\left(\psi^{(l)} - \psi_k\right)'(x)\right|^s\right].$$

Denote by $F_{lk}(x)$ and $\Phi_{lk}(x)$ the left- and right-hand parts of the last inequality. It is evident that $\Phi_{lk} \to 0$ in measure, as $l \to \infty$, uniformly with respect to k. Since

$$\{x \in Q : F_{lk}(x) \geq r\} \subset ((Q \setminus T_{\varepsilon,k}) \cup \{x \in Q \setminus \Phi_{lk}(x) \geq r\},$$

we have

$$|\{x \in Q : F_{lk} \geq k\}| < 2\varepsilon$$

for all $k \in \mathbf{N}$ provided l is large enough. Hence, $\overline{\Gamma}^k(\psi^{(l)})) \to \Gamma^k(\psi)$ in measure, as $l \to \infty$, uniformly with respect to k. Inequality (2.5.2) implies that for the family of functions $\{\overline{\Gamma}^k(\psi^{(l)}), \overline{\Gamma}^k(\psi)\}$ the $\mathrm{L}^{p'}$-norm is absolutely equicontinuous. (Recall that L^r-norm is absolutely equicontinuous for a family $\mathcal{F}$ if $\lim \|f\|_{r,S} = 0$, as $|S| \to 0$, uniformly with respect to $f \in \mathcal{F}$). Hence (see, e.g., [239]),

$$\lim_{l \to \infty} \overline{\Gamma}^k\left(\psi^{(l)}\right) = \overline{\Gamma}^k(\psi) \quad \text{in } \mathrm{L}^{p'}(Q)$$

uniformly with respect to k. Thus, $\overline{\Gamma}^k$ is an equicontinuous sequence of operators. The same is true for $\overline{\Gamma}_0^k$. Therefore, after passing to a subsequence, there exist operators $\overline{\Gamma} : X \longrightarrow \mathrm{L}^{p'}(Q)^n$ and $\overline{\Gamma}_0 : X \longrightarrow \mathrm{L}^{p_0'}(Q)$ such that, for any $\psi \in X$,

$$\lim \overline{\Gamma}^k(\psi) = \overline{\Gamma}(\psi), \quad \lim \overline{\Gamma}_0^k(\psi) = \overline{\Gamma}(\psi)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p_0'}(Q)$, respectively. Moreover, inequality (2.5.15) implies

$$\begin{aligned}\|\overline{\Gamma}(\psi) - \overline{\Gamma}(\varphi)\|_{p'}^{p'} \quad \leq \quad & \overline{\theta}\Big[\overline{H}_Q \cdot N_T(\psi_0 - \varphi_0) + \overline{H}_S + \overline{H}_Q^{1-\overline{s}/p}\|\psi' - \varphi'\|_p^{\overline{s}} + \\ & + \overline{H}_Q^{1-\overline{s}_0/p}\|\psi_0 - \varphi_0\|_{p_0}^{\overline{s}_0}\Big].\end{aligned}$$

There is a similar inequality for $\overline{\Gamma}_0$. In particular, these inequalities imply the continuity of $\overline{\Gamma}$ and $\overline{\Gamma}_0$.

Step 6. To complete the proof it is now sufficient to repeat the arguments used on Step 5, n^0 2.3.2. □

Exactly as in n^0 2.3.2, one can prove the following results.

Theorem 2.5.2 *Let $\mathcal{A}_k \in \mathcal{E}^{p,p_0}_{(m)}$. Suppose $\mathcal{A}_k$ is strongly G-convergent to $\mathcal{A}$. Then $\mathcal{A}_{k|Q_1} \stackrel{G}{\Longrightarrow} \mathcal{A}_{Q_1}$ for any open subset $Q_1 \subset Q$.*

Theorem 2.5.3 *Assume that $\mathcal{A}_k \in \mathcal{E}^{p,p_0}_{(m)}$ and $\mathcal{A}_k \stackrel{G}{\Longrightarrow} \mathcal{A}$. Suppose $v_k \in \overline{V}$ is a sequence such that $\mathcal{A}_k u_k = f_k \to f$ strongly in V^* and $v_k \to u$ weakly in $\overline{V}$. Then $\mathcal{A}u = f$,*

$$a^k(x, v_k, \nabla v_k) \to a(x, u, \nabla u) \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n,$$

and

$$a_0^k(x, v_k, \nabla v_k) \to a_0(x, u, \nabla u) \quad \text{weakly in } \mathrm{L}^{p_0'}(Q).$$

Now we discuss briefly the nonmonotone case. It must be pointed out that we cannot investigate the whole class $\mathcal{E}^{p,p_0}$ provided (2.5.6) is fulfilled. Instead we consider a more restricted class of operators of the form

$$\mathcal{A}u = -\operatorname{div} a(x, u, u, \nabla u) + a_0(x, u, \nabla u) + a_1(x, u, \nabla u).$$

Here

$$a : Q \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n, \quad a_0 : Q \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R},$$

and

$$a_1 : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

are Carathéodory functions satisfying the following inequalities:

$$|a(x,\eta,\xi_0,\xi)|^{p'} + |a_0(x,\eta,\xi_0,\xi)|^{p_0'} + |a_1(x,\xi_0,\xi)|^{p_1'} \le$$

$$\le c_0\left(|\xi|^p + |\xi_0|^{p_0} + |\eta|^{p_1}\right) + c(x),$$

$$[a(x,\eta,\xi_0,\xi) - a(x,\eta,\xi_0',\xi')] \cdot (\xi - \xi') +$$

$$+ [a_0(x,\eta,\xi_0,\xi) - a_0(x,\eta,\xi_0',\xi')] \cdot (\xi_0 - \xi_0') \ge$$

$$\ge \kappa\left[H_p(\xi,\xi') + H_{p_0}(\xi_0,\xi_0') + H_{p_1}(\eta)\right]^{1-\beta/p} |\xi - \xi'|^\beta +$$

$$+ \kappa\left[H_p(\xi,\xi') + H_{p_0}(\xi_0,\xi') + H_{p_0}(\xi_0,\xi') + H_{p_1}(\eta)\right]^{1-\beta_0/p_0} |\xi_0 - \xi_0'|^{\beta_0},$$

$$|a(x,\eta,\xi_0,\xi) - a(x,\eta',\xi_0',\xi')|^{p'} + |a_0(x,\eta,\xi_0,\xi) - a_0(x,\eta',\xi_0',\xi')|^{p_0'} +$$

$$+ |a_1(x,\xi_0,\xi) - a_1(x,\xi_0',\xi')|^{p_1'} \le$$

$$\le \theta\left[H_p(\xi,\xi') + H_{p_0}(\xi_0,\xi_0') + H_{p_1}(\eta,\eta')\right] \cdot \left[\nu(|\xi_0 - \xi_0'|) + \nu(|\eta - \eta'|)\right] +$$

$$+\,\theta\left[H_p(\xi,\xi')+H_{p_0}(\xi_0,\xi_0')+H_{p_1}(\eta,\eta')\right]^{1-s/p}|\xi-\xi'|^s.$$

Here c_0, $c(x)$, $h(x)$, s, β, and β_0 are the same as above, and inequalities (2.5.6) and

$$\frac{1}{p_1}>\frac{1}{p}-\frac{1}{n}$$

are assumed to be valid. The definition of strong G-convergence must be changed in the following way. Set

$$\mathcal{A}^1(u,v)=-\operatorname{div}a(x,v,u,\nabla u)+a_0(x,v,u,\nabla u).$$

For any $v\in V=V_0\cap V_1$, this operators belongs to $\mathcal{E}^{p,p_0}_{(m)}$. We say that a sequence $\mathcal{A}_k$ strongly G-converges to $\mathcal{A}$ if $\mathcal{A}_k^1(\cdot,v)\stackrel{G}{\Longrightarrow}\mathcal{A}^1(\cdot,v)$, for any $v\in V$, and additionally,

$$a_1^k(x,u_k,\nabla u_k)\to a_1(x,u_k,\nabla u_k)\quad\text{weakly in } \mathrm{L}^{p_1'}(Q),$$

where $u_k\in V$ is a unique solution of the equation

$$\mathcal{A}_k^1(u_k,v)=\mathcal{A}^1(u,v).$$

The technique of n^0 2.3.3 may be extended to this case and one can state all the standard properties of strong G-convergence for operators of such kind.

Finally, we indicate a simple example. Let

$$\mathcal{A}u=-\operatorname{div}\left[a(x)|\nabla u|^{p-2}\nabla u\right]+a_0(x,u).$$

where the matrix $a(x)\in\mathrm{L}^\infty(Q)$ and

$$a(x)\xi\cdot\xi\geq\alpha|\xi|^2,\qquad\alpha>0,$$

while $a_0(x,\xi_0)=c(x)|\xi_0|^{p_0-2}\xi_0$ for $|\xi_0|$ being large enough, with p_0 satisfying (2.5.6). It is obvious that $\mathcal{A}\in\mathcal{E}^{p,p_0,p_1}$ with any choice of p_1.

Comments

G-convergence for linear second order elliptic operators is investigated by many authors. We do not discuss it here and refer the reader to [138, 164, 266]. In [266] even more general case of high order operators is considered.

In the nonlinear case the first results on strong G-convergence are due to L. Tartar [260] who studied the class $\mathcal{E}_0$ of operators of the form

$$\mathcal{A}u=-\operatorname{div}a(x,\nabla u),$$

with $p = \beta = s = 2$ ($s = 2$ means that $\mathcal{A}$ satisfies the Lipschitz condition). More general class of monotone elliptic operators was considered by U. Raitum [234]. This class contains operators of the form (2.3.1)

$$\mathcal{A}u = -\operatorname{div} a(x, u, \nabla u) + a_0(x, u, \nabla u)$$

satisfying growth condition (2.1.29), monotonicity condition (2.3.40), and the regularity assumption

$$|a(x, \xi_0, \xi) - a(x, \xi_0', \xi')|^{p'} + |a_0(x, \xi_0, \xi) - a_0(x, \xi_0', \xi')|^{p'} \leq$$
$$\leq \theta\,[h(x) + |\zeta|^p + |\zeta'|^p]^{1-s/p}\,[|\xi_0 - \xi_0'|^s + |\xi - \xi'|^s]$$

instead of (2.3.2). This class of operators contains, evidently, our class $\mathcal{E}_0$, but, in general, is not quite satisfactory. Indeed, even for very simple operators like

$$\mathcal{A}u = -\operatorname{div}(a(x, u)\nabla u) + a_0(x)u,$$

where $a(x, \xi_0)$ satisfies the Lipschitz condition in ξ_0, the case considered in [58, 59, 60], this regularity assumption is not satisfied.

Strong G-convergence for operators of the class $\mathcal{E}$ was studied by the author [221, 224, 226]. Later on similar results was obtained in [143]. We have to point out that in above mentioned author's papers the case of high order operators is also treated.

The case of multivalued monotone elliptic operators was investigated in [98]. These results are presented in Section 2.2, exepting n^0 2.2.4 and n^0 2.2.5 which are based on [136] and [137], respectively. The paper [137] contains also some results on homogenization in perforated domains. Many things described in Section 2.1 are also taken from [98].

In Section 2.3 a simplified version of the results of [221, 224, 226] is presented. In the nonlinear setting Theorem 2.4.1 is new (cf. [164] for the linear case), while Theorem 2.4.2 for the class $\mathcal{E}_0$ is obtained in [74]. The result of Theorem 2.4.4, in its explicite form, appears here for the first time. Implicitly, a particular case of this result is contained in [74].

Strong G-convergence for operators with strong nonlinearity in the lower order term (see Section 2.5) was studied in [223]. For another result in this direction we refer to [55].

We also mention the papers [135, 244] which concern G-convergence of nonlinear degerate elliptic operators of the form

$$\mathcal{A}u = -\operatorname{div} a(x, \nabla u)$$

satisfying an appropriate monotonicity condition.

CHAPTER 3

Homogenization of Elliptic Operators

3.1 Random Homogeneous Fields

3.1.1 Definitions and Main Properties

We start with an analytic description of random homogeneous fields on $\mathbf{R}^n$. In the case $n = 1$ they are called, usually, random stationary processes. Let us consider a probability space Ω, i.e. a set equipped with a σ-algebra $\mathcal{F}$ of measurable subsets and a countably additive non-negative measure μ on $\mathcal{F}$ normalized by $\mu(\Omega) = 1$. We always assume the measure μ to be complete. An *n-dimensional dynamical system* is defined as a family of selfmaps

$$T(x) : \Omega \longrightarrow \Omega, \quad x \in \mathbf{R}^n,$$

with the following properties:

(1) $T(x + y) = T(x)T(y)$, $x, y \in \mathbf{R}^n$, *and* $T(0) = I$;

(2) *the map $T(x)$ is measure preserving, i.e. for any $x \in \mathbf{R}^n$ and for any μ-measurable subset $\mathcal{U} \subset \Omega$, the set $T(x)\mathcal{U}$ is μ-measurable and*

$$\mu(T(x)\mathcal{U}) = \mu(\mathcal{U});$$

(3) *the map*

$$T : \mathbf{R}^n \times \Omega \longrightarrow \Omega, \quad T : (x, \omega) \longmapsto T(x)\omega,$$

is measurable, where $\mathbf{R}^n \times \Omega$ is endowed with the measure $dx \otimes \mu$, dx stands for the Lebesgue measure.

A. Pankov, *G-Convergence and Homogenization of Nonlinear Partial Differential Operators*

Condition (3) may be rewritten in the following equivalent form:

(3′) *for any measurable function f on Ω, the function $f(T(x)\omega)$ defined on $\mathbf{R}^n \times \Omega$ is measurable.*

Associated to $T(x)$, there exists an n-parameter group of operators $U(x)$ in the space $L^p(\Omega)$, $1 \le p \le \infty$, defined by

$$(U(x)f)(\omega) = f(T(x)\omega), \quad f \in L^p(\Omega).$$

It is easy to verify that $U(x)$, $x \in \mathbf{R}^n$, is an isometric operator in the space $L^p(\Omega)$, $1 \le p \le \infty$. Moreover,

$$U^*(x) = U(-x), \quad x \in \mathbf{R}^n. \tag{3.1.1}$$

Here we consider $U(x)$ as an operator acting in $L^{p'}(\Omega)$, hence $U^*(x)$ acts in $L^p(\Omega)$. In particular, $U(x)$ is a group of unitary operators in the space $L^2(\Omega)$.

Proposition 3.1.1 *The group $U(x)$ is strongly continuous in $L^p(\Omega)$, i.e.*

$$\lim_{x\to 0} \|U(x)f - f\|_{p,\Omega} = 0, \quad f \in L^p(\Omega), \tag{3.1.2}$$

provided $p \in [1, \infty)$.

Proof. Since $U(x)$ is uniformly bounded and $L^\infty(\Omega)$ is dense in $L^p(\Omega)$, it is sufficient to verify (3.1.2) for $f \in L^\infty(\Omega)$. By the Fubini Theorem,

$$\begin{aligned} \|U(x)f - f\|_{p,\Omega}^p &= \int_\Omega |f(T(x)\omega) - f(\omega)|^p d\mu(\omega) = \\ &= \frac{1}{|B|} \int_\Omega \int_B |f(T(x+y)\omega) - f(T(y)\omega)|^p dy d\mu(\omega), \end{aligned}$$

where

$$B = \{x \in \mathbf{R}^n : |x| < 1\}$$

is the unit ball in $\mathbf{R}^n$. Now we recall that the translations form a strongly continuous group of operators in $L^p_{loc}(\mathbf{R}^n)$. Therefore, the Lebesgue Dominated Convergence Theorem implies the required. □

To deal with "trajectories" of the dynamical system $T(x)$ it is useful the following

Proposition 3.1.2 *Let Ω_0 be a measurable subset of Ω such that $\mu(\Omega_0) = 1$. Then there exists a measurable subset $\Omega_1 \subset \Omega_0$ such that $\mu(\Omega_1) = 1$ and, for any $\omega \in \Omega_1$, we have $T(x)\omega \in \Omega_0$ for almost all $x \in \mathbf{R}^n$.*

Proof. Let χ be the characteristic function of Ω_0, i.e. χ is equal to 1 on Ω_0 and is equal to 0 outside of Ω_0. Since $\chi(T(x)\omega)$ is a measurable function on $\mathbf{R}^n \times \Omega$, the Fubini Theorem implies

$$|B_t| = \int_{B_t} \left(\int_\Omega \chi(T(x)\omega) d\mu(\omega) \right) dx = \int_\Omega \left(\int_{|x|<t} \chi(T(x)\omega) dx \right) d\mu(\omega),$$

where

$$B_t = \{x \in \mathbf{R}^n : |x| < t\}.$$

Therefore,

$$\frac{1}{|B_t|} \int_{B_t} \chi(T(x)\omega) dx = 1, \quad t > 0,$$

a.e. in Ω. Hence, for any natural number m there exists a measurable subset $\mathcal{U}_m \subset \Omega$ such that $\mu(\mathcal{U}_m) = 1$ and, for any $\omega \in \mathcal{U}_m$, we have

$$\chi(T(x)\omega) = 1$$

for almost all $x \in B_m$. Now the set $\Omega_1 = \cap \mathcal{U}_m$ satisfies all the requirements. $\square$

A measurable function f on Ω is called *random homogenious field*. For any fixed $\omega \in \Omega$, the function $f(T(x)\omega)$ in the variable $x \in \mathbf{R}^n$ is said to be a *realization* of the field f. We say that the realization is *generic* if ω is taken from a measurable subset of full measure, i.e. of measure equals to 1. Usually we shall consider more restrected classes of random fields, like $\mathrm{L}^p(\Omega)$. For such classes of random fields the Fubini Theorem gives rise easily to the following two important properties:

- *if* $f \in \mathrm{L}^p(\Omega)$, $1 \le p < \infty$, *than its generic realization* $f(T(x)\omega)$ *belongs to* $\mathrm{L}^p_{loc}(\mathbf{R}^n)$;
- *if* $f_m \to f$ *in* $\mathrm{L}^p(\Omega)$, $1 \le p < \infty$, *than, for a generic realization, there is a subsequence* m' *depending on* ω *such that* $f_{m'}(T(x)\omega) \to f(T(x)\omega)$ *in* $\mathrm{L}^p_{loc}(\mathbf{R}^n)$.

Now we recall that a measurable function f on Ω is said to be *invariant* if

$$f(T(x)\omega) = f(\omega) \quad \text{a.e. in } \Omega,$$

for any $x \in \mathbf{R}^n$. The dynamical system $T(x)$ is called *ergodic* if any invariant function is constant a.e. in Ω.

We need also a notion of the mean value for functions defined on $\mathbf{R}^n$. Let $f \in \mathrm{L}^1_{loc}(\mathbf{R}^n)$. Suppose that for any bounded Lebesque measurable subset $K \subset \mathbf{R}^n$, $|K| \ne 0$, there exists the limit

$$\lim_{\varepsilon \to 0} \frac{1}{|K|} \int_K f(\varepsilon^{-1} x) dx \tag{3.1.3}$$

which does not depend on K. In this case the limit in (3.1.3) is said to be the *mean value* of f and is denoted by $M\{f\}$. Let

$$K_t = \{x \in \mathbf{R}^n : t^{-1}x \in K\}, \quad t > 0.$$

Then one can rewrite the definition of $M\{f\}$ as follows:

$$M\{f\} = \lim_{t\to\infty} \frac{1}{t^n|K|} \int_{K_t} f(x)dx. \tag{3.1.4}$$

Assume additionally that the family of functions $\{f(\varepsilon^{-1}x)\}$ is bounded in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$, $1 \le p < \infty$. In this case, it is not difficult to verify that f has the mean value $M\{f\}$ if and only if $f(\varepsilon^{-1}x) \to M\{f\}$ weakly in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$.

Theorem 3.1.1 (Birkhoff Ergodic Theorem) *Let*

$$f \in \mathrm{L}^p(\Omega), \quad 1 \le p < \infty.$$

Then a generic realization $f(T(x)\omega)$ prossesses a mean value. The mean value $M\{f(T(x)\omega)\}$ is invariant, as a function of $\omega \in \Omega$, and

$$\langle f\rangle = \int_\Omega f(\omega)d\mu(\omega) = \int_\Omega M\{f(T(x)\omega)\}d\mu(\omega).$$

If the system $T(x)$ is ergodic, then

$$\langle f\rangle = M\{f(T(x)\omega)\} \quad a.e. \text{ on } \Omega.$$

For the proof we refer to [144].

In what follows we shall always assume the dynamical system $T(x)$ to be ergodic.

Now we consider several examples.

Example 3.1.1 (Periodic functions) Let $\Omega = \mathbf{T}^n$ be an n-dimentional torus. We consider $\mathbf{T}^n$ as the factor group $\mathbf{R}^n/\mathbf{Z}^n$. The action of $\mathbf{R}^n$ on Ω is defined by

$$T(x)\omega = \omega + x \mod \mathbf{Z}^n.$$

One can consider Ω as the unit cube in $\mathbf{R}^n$ with all the pairs of antipodal faces being identified. The standard Lebesgue measure on $\mathbf{R}^n$ induces the measure μ which is invariant with respect to the action of $T(x)$ on $\mathbf{T}^n$. Moreover, $T(x)$ is ergodic. In this case any function f on Ω, i.e. random field, may be regarded as a periodic function on $\mathbf{R}^n$ whose period, in each coordinate variable, is equal to 1. All the realizations of f are of the form $f(\omega + x)$. Thus, up to translations, we have only one realization of f.

Example 3.1.2 (Quasiperiodic functions) Let $\Omega = \mathbf{T}^m$, $m > n$, endowed with the Lebesgue measure. To define the dynamical system $T(x)$ we fix an $(m \times n)$-matrix $\Lambda = (\lambda_{ij})$ and set

$$T(x)\omega = \omega + \Lambda x \mod \mathbf{Z}^m.$$

The map $T(x)$ preserves the measure μ. For $T(x)$ to be ergodic, it is necessary and sufficient that $\Lambda k \neq 0$ for any $k \in \mathbf{Z}^n$, $k \neq 0$. Again, any measurable function f on Ω may be identified with a unique measurable 1-periodic function on $\mathbf{R}^m$. However, in this case we have a lot of essentially different realizations $f(\omega + \Lambda x)$. If we assume f to be continuous, all such realizations are quasiperiodic functions.

Example 3.1.3 (Almost periodic functions) Let $\Omega = \mathbf{R}^n_B$ be the Bohr compactification of $\mathbf{R}^n$. Recall that $\mathbf{R}^n_B$ is a compact abelian group and there is the canonical dense embedding $\mathbf{R}^n \subset \mathbf{R}^n_B$ which is a group homomorphism. Let μ be the Haar measure on $\mathbf{R}^n_B$ normalized by $\mu(\mathbf{B}^n_B) = 1$. We define the dynamical system $T(x)$ by

$$T(x)\omega = \omega + x, \quad x \in \mathbf{R}^n,\ \omega \in \mathbf{R}^n_B,$$

where "+" stands for the group operation in $\mathbf{R}^n_B$. It is obvious that $T(x)$ is measure preserving. Moreover, it is known that the dynamical system $T(x)$ is ergodic. In this case realizations are almost periodic functions. More precisely, if f is continuous on $\mathbf{R}^n_B$, then all its realizations are almost periodic in the sense of Bohr. The case $f \in \mathrm{L}^p(\mathbf{R}^n_B)$ gives rise to Besicovich almost periodic functions as realizations. Later on we shall discuss this example in more details.

Now we explain briefly the relation betwen the standard definition of random homogeneous fields and that we introduced here. Let Ξ be a probability space endowed with a probability measure P. Let f be a random vector valued function, i.e. a measurable map

$$f : \Xi \times \mathbf{R}^n \longrightarrow \mathbf{R}^N.$$

Probabilists say that f is a random homogeneous field if all its finite dimentional distributions are translation invariant. The last means that, for any natural number k, any x^1, x^2, ..., $x^k \in \mathbf{R}^n$, and any Borel subsets B_1, B_2, ..., $B_k \subset \mathbf{R}^N$,

$$P\{\xi \in \Xi : f(\xi, x^1 + h) \in B_1, \dots, f(\xi, x^k + h) \in B_k\}$$

does not depend on $h \in \mathbf{R}^n$. Let us construct a new probability space Ω and a dynamical system $T(x)$ acting on Ω. We define Ω to be the set of all measurable functions $\omega : \mathbf{R}^n \longrightarrow \mathbf{R}^N$ and set

$$T(x)\omega(y) = \omega(x + y), \quad x, y \in \mathbf{R}^n.$$

Let $\mathcal{F}$ be the σ-algebra generated by "cylinder" sets, i.e. the sets of the form

$$B = \{\omega : \omega(x^1) \in B_1, \dots, \omega(x^k) \in B_k\},$$

where x^1, x^2, ..., $x^k \in \mathbf{R}^n$ and B_1, B_2, ..., B_k are Borel subsets in $\mathbf{R}^N$. We define the measure μ on "cylinder" sets by

$$\mu(B) = P\{\xi \in \Xi : f(\xi, \cdot) \in B\}$$

and then extend it to $\mathcal{F}$ by σ-additivity. Thus, we have constructed the probability space Ω and the measure preserving dynamical system $T(x)$, $x \in \mathbf{R}^n$, on Ω. Moreover, consider the μ-measurable function

$$\hat{f} : \Omega \longrightarrow \mathbf{R}^N$$

defined by the formula $\hat{f}(\omega) = \omega(0)$. Then we have

$$f(\xi, x) = \hat{f}(T(x)\omega),$$

where $\omega(\cdot) = f(\xi, \cdot)$.

3.1.2 Vector Fields and Compensated Compactness

Now we leave for a moment the random setting and recall some basic facts about vector fields on $\mathbf{R}^n$. Let

$$f = (f_1, \dots, f_n) \in \mathrm{L}^p_{loc}(\mathbf{R}^n)^n, \quad 1 < p < \infty.$$

The field f is said to be *vortex-free* if $\operatorname{curl} f = 0$ in the weak sense, i.e.

$$\int_{\mathbf{R}^n} \left(f_i \frac{\partial \varphi}{\partial x_j} - f_j \frac{\partial \varphi}{\partial x_i} \right) dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n).$$

It is known that f is vortex-free if and only if it is *potential*, i.e. there exists a function $u \in W^{1,p}_{loc}(\mathbf{R}^n)$ such that $f = \nabla u$. A vector field f is called *solenoidal* if

$$\int_{\mathbf{R}^n} f_i \frac{\partial \varphi}{\partial x_i} dx = 0 \quad \forall \varphi \in C_0^\infty(\mathbf{R}^n),$$

in other words, $\operatorname{div} f = 0$.

Later on we shall need the concept of $*$-weak convergence in L^1 and a simple compensated compactness result. Let u_k, $u \in \mathrm{L}^1(Q)$, where Q is a bounded domain in $\mathbf{R}^n$. We say that u_k converges to u $*$-weakly in $\mathrm{L}^1(Q)$ (in symbols, $u = *\text{-}\lim u_k$) if u_k is bounded in $\mathrm{L}^1(Q)$ and

$$\lim_{k \to \infty} \int_Q u_k \varphi dx \quad \forall \varphi \in C_0^\infty.$$

Weak L^1-convergence implies $*$-weak convergence, but the converse statement does not hold, in general. The following result is well-known in the case $p = 2$.

Lemma 3.1.1 *Let $f_k \in \mathrm{L}^p(Q)^n$ and $g_k \in \mathrm{L}^{p'}(Q)^n$ be vector fields such that $f_k \to f$ weakly in $\mathrm{L}^p(Q)^n$ and $g_k \to g$ weakly in $\mathrm{L}^{p'}(Q)^n$. Assume that* $\operatorname{curl} f_k = 0$ *and* $\operatorname{div} g_k$ *converges strongly in $W^{-1,p'}(Q)$. Then $*\text{-}\lim f_k \cdot g_k = f \cdot g$.*

The proof is essentially the same as for the standard case $p = 2$ (see, e.g., [164]).

3.1.3 Random Vector Fields

Now we come back to the probability space Ω endowed with the dynamical system $T(x)$ and random vector fields defined on Ω. A vector field

$$f \in \mathbf{L}^p(\Omega) = \mathrm{L}^p(\Omega)^n, \quad 1 < p < \infty,$$

is said to be *potential* (resp., *solenoidal*) if its generic realization $f(T(x)\omega)$ is a potential (resp., solenoidal) vector field defined on $\mathbf{R}^n$. Recall that in this case the generic realization belongs to the space $\mathrm{L}^p_{loc}(\mathbf{R}^n)^n$. We denote by $\mathbf{L}^p_{pot}(\Omega)$ (resp., $\mathbf{L}^p_{sol}(\Omega)$) the subspace of $\mathbf{L}^p(\Omega)$ formed by potential (resp., solenoidal) vector fields. Since $\mathbf{L}^p(\Omega)$-convergence implies $\mathrm{L}^p_{loc}(\mathbf{R}^n)$-convergence of generic realizations, the spaces $\mathbf{L}^p_{pot}(\Omega)$ and $\mathbf{L}^p_{sol}(\Omega)$ are closed in $\mathbf{L}^p(\Omega)$.

We shall use the following spaces of vector fields with vanishing mean value:

$$\mathbf{V}^p_{pot} = \{f \in \mathbf{L}^p_{pot}(\Omega) : \langle f \rangle = 0\},$$

$$\mathbf{V}^p_{sol} = \{f \in \mathbf{L}^p_{sol}(\Omega) : \langle f \rangle = 0\}.$$

We have obviously

$$\mathbf{L}^p_{pot}(\Omega) = \mathbf{V}^p_{pot} \oplus \mathbf{R}^n, \quad \mathbf{L}^p_{sol}(\Omega) = \mathbf{V}^p_{sol} \oplus \mathbf{R}^n.$$

Proposition 3.1.3 *If $f \in \mathbf{L}^p_{pot}(\Omega)$ and $g \in \mathbf{L}^{p'}_{sol}(\Omega)$, $1 < p < \infty$, then*

$$\langle f \cdot g \rangle = \langle f \rangle \cdot \langle g \rangle. \tag{3.1.5}$$

Proof. Let

$$f(x) = f(T(x)\omega), \quad g(x) = g(T(x)\omega)$$

be generic realizations of the vector fields f and g, respectively. By the Ergodic Theorem, we have

$$w\text{-}\lim_{\varepsilon \to 0} f(\varepsilon^{-1}x) \cdot g(\varepsilon^{-1}x) = \langle f \cdot g \rangle \quad \text{in } \mathrm{L}^1_{loc}(\mathbf{R}^n).$$

On the other hand, Lemma 3.1.1 together with the Ergodic Theorem implies that

$$*\text{-}\lim_{\varepsilon \to 0} f(\varepsilon^{-1}x) \cdot g(\varepsilon^{-1}x) = \langle f \rangle \cdot \langle g \rangle$$

and we conclude. □

As consequence, we have

$$\mathbf{L}^{p'}_{sol}(\Omega) \subset (\mathbf{V}^{p}_{pot})^{\perp}, \quad \mathbf{L}^{p'}_{pot}(\Omega) \subset (\mathbf{V}^{p}_{sol})^{\perp}, \tag{3.1.6}$$

where $\perp$ stands for the orthogonal complement with respect to the duality pairing between $\mathbf{L}^p(\Omega)$ and $\mathbf{L}^{p'}(\Omega)$ given by

$$\langle f, g \rangle = \langle f \cdot g \rangle.$$

Really, we have the following result which is, in the case $p = 2$, a random version of the well-known Weyl orthogonal decomposition.

Theorem 3.1.2 *Let $1 < p < \infty$. Then $\mathbf{L}^{p'}_{sol}(\Omega) = (\mathbf{V}^{p}_{pot})^{\perp}$ and $\mathbf{L}^{p'}_{pot}(\Omega) = (\mathbf{V}^{p}_{sol})^{\perp}$.*

To prove the theorem we need more details on the group $U(x)$. Let us consider the one-parameter group of operators

$$U_i(t) = U(te_i), \quad i = 1, \dots, n,$$

where $\{e_i\}$ is the standard basis in $\mathbf{R}^n$. The group $U_i(t)$ is strongly continuous in $\mathrm{L}^p(\Omega)$, $1 < p < \infty$. The generator, ∂_i, of the group U_i is a closed unbounded linear operator acting in the space $\mathrm{L}^p(\Omega)$. From (3.1.1) one can deduce immediately that ∂_i is a skew-symmetric operator in the following sense:

$$\langle (\partial_i f) \cdot g \rangle = -\langle f \cdot \partial_i g \rangle, \quad \forall f \in D(\partial_i, \mathrm{L}^p(\Omega)),\ \forall g \in D(\partial_i, \mathrm{L}^{p'}(\Omega)),$$

where $D(\partial_i, \mathrm{L}^p(\Omega))$ stands for the domain of ∂_i in the space $\mathrm{L}^p(\Omega)$. In particular, $\langle \partial_i f \rangle = 0$ for any $f \in D(\partial_i, \mathrm{L}^p(\Omega))$, $i = 1, \dots, n$.

The operators ∂_i, $i = 1, \dots, n$, may be viewed as "derivatives" along trajectories of the dynamical system $T(x)$. Indeed, we have

$$(\partial_i f)(T(x)\omega) = \frac{\partial}{\partial x_i} f(T(x)\omega), \quad f \in D(\partial_i, \mathrm{L}^p(\Omega)), \tag{3.1.7}$$

for almost all $\omega \in \Omega$. To prove this formula we recall that, by definition,

$$(\partial_i f)(\omega) = \lim_{t \to 0} \frac{f(T(te_i)\omega) - f(\omega)}{t},$$

the convergence in the sense of $\mathrm{L}^p(\Omega)$. Then we have

$$(\partial_i f)(T(x)\omega) = \lim_{t \to 0} \frac{1}{t}[f(T(te_i)T(x)\omega) - f(T(x)\omega)],$$

the limit in the sense of $\mathrm{L}^p_{loc}(\mathbf{R}^n)$ for almost all $\omega \in \Omega$. Hence, for a generic realization $f(T(x)\omega)$, there exists a weak derivative with respect to x and formula (3.1.7) holds true.

Now let us introduce smoothing operators we need to prove Theorem 3.1.2. Let $K \in C_0^\infty(\mathbf{R}^n)$ be a non-negative even function such that

$$\int_{\mathbf{R}^n} K(x)dx = 1. \tag{3.1.8}$$

We set

$$K_\delta(x) = \delta^{-n} K(x/\delta), \quad \delta > 0,$$

and then define the operator J_δ by the formula

$$(J_\delta f)(\omega) = \int_{\mathbf{R}^n} K_\delta(y) f(T(y)\omega)dy. \tag{3.1.9}$$

The last equality may be written in the form

$$J_\delta f = \int_{\mathbf{R}^n} K_\delta(y) U(y) f dy, \tag{3.1.10}$$

where the integral in the right-hand part is regarded in the sense of Bochner, with values in $\mathrm{L}^p(\Omega)$. It is easy to see that J_δ is a bounded linear operator in the space $\mathrm{L}^p(\Omega)$ and its norm is not greater then 1. At the level of realizations, we have the identity

$$\begin{aligned}(J_\delta f)(T(x)\omega) &= \int_{\mathbf{R}^n} K_\delta(y) f(T(x+y)\omega)dy = \\ &= \int_{\mathbf{R}^n} K_\delta(y-x) f(T(y)\omega)dy \end{aligned} \tag{3.1.11}$$

which shows us that a generic realization of $(J_\delta f)$ belongs to $C^\infty(\mathbf{R}^n)$. Using (3.1.10) we see, in the similar manner, that, for $f \in \mathrm{L}^p(\Omega)$, the function $J_\delta f$ belongs to the domain $D(\partial^\alpha, \mathrm{L}^p(\Omega))$ for any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\partial^\alpha = \partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}$. In addition, we recall that the function K is even. This implies that the operator J_δ is symmetric:

$$\langle (J_\delta f) \cdot g\rangle = \langle f \cdot J_\delta g\rangle \quad \forall f \in \mathrm{L}^p(\Omega),\ \forall g \in \mathrm{L}^{p'}(\Omega). \tag{3.1.12}$$

As consequence,

$$\langle J_\delta f\rangle = \langle f\rangle. \tag{3.1.13}$$

An important property of J_δ is stated in the following

Lemma 3.1.2 *For any $f \in \mathrm{L}^p(\Omega)$*

$$\lim_{\delta\to 0} \|J_\delta f - f\|_{p,\Omega} = 0. \tag{3.1.14}$$

Proof. Using (3.1.8) and 3.1.10) we have

$$\begin{aligned} J_\delta f - f &= \int_{\mathbf{R}^n} K_\delta(y)(U(y)f - f)dy = \\ &= \int_{\mathbf{R}^n} K(y)(U(\delta y)f - f)dy. \end{aligned}$$

Let $Q = \mathrm{supp} K$. Then, again taking (3.1.8) into account we obtain

$$\begin{aligned} \|J_\delta f - f\|_{p,\Omega} &\leq \int_Q K(y)\|U(\delta y)f - f\|_{p,\Omega} dy \leq \\ &\leq \sup_{y\in Q}\|U(\delta y)f - f\|_{p,\Omega}. \end{aligned}$$

Since Q is compact and the group $U(x)$ is strongly continuous, this implies (3.1.14). □

Remark 3.1.1 One can show that

$$\lim_{\delta\to+\infty} \|J_\delta f - \langle f\rangle\|_{p,\Omega} \quad \forall f \in \mathrm{L}^p(\Omega)$$

provided $p \in [1, +\infty)$.

Proof of Theorem 3.1.2. At first, we observe that the spaces $\mathbf{V}^p_{pot}$ and $\mathbf{V}^p_{sol}$ are invariant with respect to the operator J_δ acting on vector fields component-wise. This follows immediately from the first identity in (3.1.11) and (3.1.13). Since J_δ is symmetric (see (3.1.12)), the spaces $(\mathbf{V}^p_{pot})^\perp$ and $(V^p_{sol})^\perp$ are J_δ-invariant as well.

Now let

$$f = (f_1, \ldots, f_n) \in (\mathbf{V}^p_{pot})^\perp \subset \mathbf{L}^{p'}(\Omega).$$

In view of (3.1.6), to prove the first statement of the theorem we need to show that $f \in \mathbf{L}^{p'}_{sol}(\Omega)$. Since $(\mathbf{V}^p_{pot})^\perp$ is J_δ-invariant, we have

$$f_\delta = J_\delta f \in (\mathbf{V}^p_{pot})^\perp.$$

Given $u \in \mathrm{L}^p(\Omega)$ we set

$$v_\gamma = (\partial_1 J_\gamma u, \ldots, \partial_n J_\gamma u).$$

It is easy that $\langle v_\gamma\rangle = 0$. Hence, identity (3.1.7) implies that $v_\gamma \in \mathbf{V}^p_{pot}$. We have

$$0 = \langle f_\delta \cdot v_\gamma\rangle = \sum_i \langle (J_\delta f_i) \cdot \partial_i (J_\gamma u)\rangle = -\langle (J_\gamma u) \cdot \sum_i \partial_i (J_\delta f_i)\rangle.$$

Passing to the limit, as $\gamma \to 0$, and using Lemma 3.1.2 we get

$$\langle u \cdot \sum_i \partial_i (J_\delta f_i)\rangle = 0,$$

for any $u \in \mathrm{L}^p(\Omega)$. Therefore

$$\sum_i \partial_i(J_\delta f_i) = 0 \quad \text{a.e. on } \Omega.$$

Now equation (3.1.7) and Proposition 3.1.2 show us that

$$\operatorname{div} f_\delta(T(x)\omega) = 0$$

for almost all $\omega \in \Omega$, hence, $f_\delta \in \mathbf{L}^{p'}_{sol}(\Omega)$. Since $\mathbf{L}^{p'}_{sol}(\Omega)$ is a closed subspace of $\mathbf{L}^{p'}(\Omega)$, Lemma 3.1.2 implies that $f \in \mathbf{L}^{p'}_{sol}(\Omega)$.

In the similar way one can prove the second statement of the theorem. □

3.2 Homogenization of Random Elliptic Operators

3.2.1 Multivalued Monotone Operators and Auxiliary Problem

First of all we introduce the set of random homogeneous multivalued operators we shall homogenize. As in Section 3.1, let Ω be a probability space and $T(x)$, $x \in \mathbf{R}^n$, be an n-dimentional dynamical system which is assumed to be ergodic. We fix $p \in (1, +\infty)$, two nonnegative constants m_1 and m_2, and two positive constants c_1 and c_2. Denote by M_Ω the set of all multivalued maps

$$a : \Omega \times \mathbf{R}^n \longrightarrow \mathbf{R}^n,$$

with closed values, which satisfy the conditions:

(i) *for almost all $\omega \in \Omega$, the function $a(\omega, \cdot) : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ is maximal monotone;*

(ii) *a is measurable, i.e.*

$$a^{-1}(C) = \{(\omega, \xi) \in \Omega \times \mathbf{R}^n : a(x, \xi) \cap C \neq \emptyset\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^n)$$

for any closed subset $C \subset \mathbf{R}^n$;

(iii) *for almost all $\omega \in \Omega$, the inequalities*

$$|\eta|^{p'} \leq m_1 + c_1 \eta \cdot \xi, \tag{3.2.1}$$

$$|\xi|^p \leq m_2 + c_2 \eta \cdot \xi \tag{3.2.2}$$

hold true for any $\xi \in \mathbf{R}^n$ and $\eta \in a(\omega, \xi)$.

Proposition 3.1.2 implies immediately that given $a \in M_\Omega$ almost all realizations $a(T(x)\omega, \xi)$ are well-defined. Moreover, such realizations belong to the class $M_{\mathbf{R}^n}$ introduced in Definition 2.1.1, with m_1 and m_2 being constant now.

Let Q be an open bounded subset of $\mathbf{R}^n$. Since $a \in M_Q$, then for almost all $\omega \in \Omega$ the operator

$$\mathcal{A}_\varepsilon u = \mathcal{A}_\varepsilon(\omega)u = -\operatorname{div} a(T(\varepsilon^{-1}x)\omega, \nabla u), \quad \varepsilon > 0, \tag{3.2.3}$$

is well-defined. Here ε runs a sequence which tends to 0. Moreover, $\mathcal{A}_\varepsilon(\omega) \in \mathcal{M}_V$ for almost all $\omega \in \Omega$. In particular, $\mathcal{A}_\varepsilon(\omega)$ is a maximal monotone operator acting from $V = W_0^{1,p}(Q)$ into $V^* = W^{-1,p}(Q)$ and this operator is coercive. We are interested to understand an asymptotic behaviour, as $\varepsilon \to 0$, of a solution, $u_\varepsilon \in V$, to the equation $\mathcal{A}_\varepsilon u_\varepsilon = f$. A natural tool for this is the concept of strong G-convergence introduced in Chapter 2. More precisely, we shall prove the existence of a translation invariant operator $\hat{\mathcal{A}}$ of the form

$$\hat{\mathcal{A}}u = -\operatorname{div} \hat{a}(\nabla u) \tag{3.2.4}$$

such that $\mathcal{A}_\varepsilon(\omega) \xrightarrow{G} \hat{\mathcal{A}}$, for almost all $\omega \in \Omega$ and for any open bounded subset $Q \subset \mathbf{R}^n$. Moreover, the function $\hat{a}$ may be calculated by means of an auxiliary problem on the probability space Ω. The operator $\hat{\mathcal{A}}$ is called the *homogenized operator* for the family $\mathcal{A}_\varepsilon$. It must be pointed out that in the random setting the homogenization takes place, in general, only in the statistical sense, i.e. for generic realizations.

To state and prove the exact homogenization theorem we start with the above mentioned auxiliary problem. Let $a \in M_\Omega$. We consider the following problem. Given $\xi \in \mathbf{R}^n$ find a couple of functions

$$(v, h) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$$

such that

$$h(\omega) \in a(\omega, \xi + v(\omega)) \quad \text{a.e. on } \Omega, \tag{3.2.5}$$

or, equivalently,

$$\xi + v(\omega) \in a^{-1}(\omega, h(\omega)) \quad \text{a.e. on } \Omega, \tag{3.2.6}$$

where $a^{-1}(\omega, \cdot)$ is the inverse of the maximal monotone mapping

$$a(\omega, \cdot) : \mathbf{R}^n \longrightarrow \mathbf{R}^n.$$

Maximal monotonicity of a^{-1} is obvious, while its measurability follows from Theorem 2.3.1.

To prove the existence of a solution to problem (3.2.5), or (3.2.6), we shall make, at first, a passage to a corresponding operator setting. For the sake of brevity we set $\mathbf{W} = \mathrm{L}^{p'}_{sol}(\Omega)$. Now we introduce the multivalued operator

$$B = B_\xi : \mathbf{W} \longrightarrow \mathbf{W}^*$$

defined in the following way. Let $h \in \mathbf{W}$. Then a functional $\varphi \in \mathbf{W}^*$ belongs to Bh if and only if there exists $v \in \mathbf{L}^p(\Omega)$ such that

$$\xi + v(\omega) \in a^{-1}(\omega, h(\omega)) \quad \text{a.e. on } \Omega \tag{3.2.7}$$

and

$$(\varphi, w) = \langle v(\omega) \cdot w(\omega) \rangle \quad \forall w \in \mathbf{W}. \tag{3.2.8}$$

Here $(\cdot,\cdot)$ denotes the duality pairing on $\mathbf{W}^* \times \mathbf{W}$. Now Theorem 3.1.2 implies that problem (3.2.6) (hence, (3.2.5)) has a solution if and only if the range $R(B)$ of B contains $0 \in \mathbf{W}^*$.

Proposition 3.2.1 *The operator B is maximal monotone and $R(B) = \mathbf{W}^*$.*

Proof. To prove that B is maximal monotone it is sufficient to verify, in view of Theorem 1.1.1, the following properties of B:

1. *B is a monotone operator*;
2. *for any $h \in \mathbf{W}$, the value Bh is a nonempty convex subset of $\mathbf{W}^*$*;
3. *the values of B are weakly closed and B is upper-semicontinuous from $\mathbf{W}$, with its strong topology, into $\mathbf{W}^*$, with its weak topology.*

As for property 1, let $h_i \in \mathbf{W}$ and $\varphi_i \in Bh_i$, $i = 1, 2$. Then there exists $v_i \in \mathrm{L}^p(\Omega)^n$, $i = 1, 2$, such that

$$\xi + v_i(\omega) \in a^{-1}(\omega, h_i(\omega)) \quad \text{a.e. on } \Omega.$$

We have

$$\begin{aligned}(\varphi_1 - \varphi_2, h_1 - h_2) &= \langle (v_1 - u_2) \cdot (h_1 - h_2) \rangle = \\ &= \langle [(\xi + v_1) - (\xi + v_2)] \cdot (h_1 - h_2) \rangle.\end{aligned}$$

Hence, monotonicity of a^{-1} implies the same property for B.

Now we prove property 2. Since $a^{-1}(\omega, \cdot)$ is measurable and maximal monotone, almost all sets $a^{-1}(\omega, h(\omega))$ are nonempty and convex. Therefore, by Theorem 2.1.1, there exists a measurable selection $w(\omega)$ of $a^{-1}(\omega, h(\omega))$. Inequality (3.2.2) implies that $w \in \mathbf{L}^p(\Omega)$, hence,

$$v = w - \xi \in \mathbf{L}^p(\Omega).$$

Thus, v induces a functional which belongs to Bh. Convexity of Bh follows immediately from convexity of $a^{-1}(\omega, h(\omega))$.

Let us prove 3. Suppose $h \in \mathbf{W}$, and U is a weak open neighbourhood of Bh in $\mathbf{W}^*$. Consider a sequence h_k converging to h strongly in $\mathbf{W}$. Let us show that $Bh_k \subset U$ for k being large enough. If not, then there are a subsequense $(h_{k'})$ of (h_k) and $\varphi_{k'} \in Bh_{k'}$ such that $\varphi_{k'} \notin U$. By definition of B, there exists $v_{k'} \in \mathbf{L}^p(\Omega)$ such that

$$\xi + v_{k'}(\omega) \in a^{-1}(\omega, h_{k'}(\omega)) \quad \text{a.e. on } \Omega;$$

and

$$(\varphi_{k'}, w) = \langle v_{k'}(\omega) \cdot w(\omega) \rangle \quad \forall w \in \mathcal{W}.$$

Inequality (3.2.2) and strong convergence of (h_k) imply that $(v_{k'})$ is bounded in $\mathbf{L}^p(\Omega)$. Therefore, we can assume that $v_{k'} \to v$ weakly in $\mathbf{L}^p(\Omega)$. Now to obtain a contradiction we need only to prove that

$$\xi + v(\omega) \in a^{-1}(\omega, h(\omega)) \quad \text{a.e. on } \Omega. \tag{3.2.9}$$

Indeed, let $\varphi \in \mathbf{W}^*$ be the functional associated to v. Then (3.2.9) implies that $\varphi \in Bh$, hence, $\varphi \in U$. The last ensures that $\varphi_{k'} \in U$ for k' being suffuciently large, the contradiction to our assumption.

Now we prove (3.2.9). To do this, consider the multivalued mapping

$$H : \Omega \longrightarrow \mathbf{R}^n \times \mathbf{R}^n$$

defined by the formula

$$H\omega = \{(\eta, \chi) \in \mathbf{R}^n : \chi \in a^{-1}(\omega, \eta)\}.$$

Since a^{-1} is measurable, Theorem 2.1.3 garantees measurability of H and, hence, the existence of a countable family $(\eta_m(\omega), \chi_m(\omega))$ of measurable selections of H such that the set $\{(\eta_m(\omega), \chi_m(\omega))\}$ is dense in $H\omega$, for any $\omega \in \Omega$. By monotonicity of a^{-1}, we have

$$(h_{k'}(\omega) - \eta_m(\omega)) \cdot (\xi + v_{k'}(\omega) - \chi_m(\omega)) \geq 0 \quad \text{a.e. on } \Omega. \tag{3.2.10}$$

Given $N > 0$ we set

$$\Omega_N = \{\omega \in \Omega : |\eta_m(\omega)| + |\chi_m(\omega)| \leq N\}.$$

Recall that $v_{k'} \to v$ weakly in $\mathbf{L}^{p'}(\Omega)$ and $h_{k'} \to h$ strongly in $\mathbf{L}^p(\Omega)$. Taking any measurable subset $S \subset \Omega_N$, integrating (3.2.10) over S, and then passing to the limit as $k' \to \infty$ we get, in view of arbitrariness of S, the inequality

$$(h(\omega) - \eta_m)) \cdot (\xi + v(\omega) - \chi_m(\omega)) \geq 0 \quad \text{a.e. on } \Omega_N.$$

Since N is arbitrary choosen, the last inequality takes place a.e. on Ω. By density of the set $\{(\eta_m(\omega), \chi_m(\omega))\}$ in $H\omega$, we have

$$(h(\omega) - \eta) \cdot (\xi + v(\omega) - \chi) \geq 0 \quad \text{a.e. on } \Omega$$

for any $\eta \in \mathbf{R}^n$, $\chi \in a^{-1}(\omega, \eta)$. Now the maximal monotonicity of a^{-1} implies (3.2.9) and the proof of property 3 is complete.

In view of Theorem 1.1.4, to end the proof we need only to verify the coerciveness of B. If $\varphi \in Bh$, then, by definition, there exist $v \in \mathbf{L}^p(\Omega)$ such that equations (3.2.7) and (3.2.8) hold true. Now inequality (3.2.1) implies that

$$\lim_{\|h\| \to \infty} \frac{(\varphi, h)}{\|h\|} \geq c \lim_{\|h\| \to \infty} \|h\|^{p'-1} = +\infty, \quad c > 0.$$

Thus, we have proved the coerciveness of B. The proof is complete. □

Corollary 3.2.1 *For any $\xi \in \mathbf{R}^n$, there exists a solution $(v, h) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$ of problem (3.2.5).*

Remark 3.2.1 Let

$$a(x, \xi) = a(T(x)\omega, \xi)$$

be a generic realization of $a(\omega, \xi)$. Then equation (3.2.5) becames

$$h(x) \in a(x, \xi + v(x)) \tag{3.2.11}$$

where

$$h(x) = h(T(x)\omega), \quad v(x) = v(T(x)\omega)$$

are generic realizations of $h(\omega)$ and $v(\omega)$, respectively. Since $v \in \mathrm{L}^p_{loc}(\mathbf{R}^n)^n$ is a potential vector field, there exists a function $N(x) \in W^{1,p}_{loc}(\mathbf{R}^n)$ such that

$$v(x) = \nabla N(x).$$

Taking into account that $h \in \mathrm{L}^{p'}_{loc}(\mathbf{R}^n)^n$ is solenoidal, we may rewrite (3.2.11) in the form of differential equation

$$-\operatorname{div} a(x, \xi + \nabla N) \ni 0. \tag{3.2.12}$$

However, it must be pointed out that, in general, $N(x)$ is *not* a realization of a random homogeneous field defined on Ω, although it is so in the periodic case.

3.2.2 Homogenization Theorem

Now we are able to treat homogenization problem for the family of operators (3.2.3). First of all, we define the operator $\hat{\mathcal{A}}$ which will turn out to be the homogenized operator. More precisely, given $a \in M_\Omega$ we define the new function $\hat{a}(\xi)$ as follows:

$$\hat{a}(\xi) = \{\eta \in \mathbf{R}^n : \exists (v,h) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega) \quad \text{satisfying (3.2.5)} \quad \text{and } \eta = \langle h \rangle\}. \tag{3.2.13}$$

Using Theorem 3.1.1, we see that $\hat{a}(\xi)$ consists of all vectors η which are mean values $M\{h(T(x)\omega)\}$ for generic realizations of solutions to problem (3.2.5). We consider $\hat{a}$ as a map

$$\hat{a} : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

which is constant with respect to the first variable.

Proposition 3.2.2 *Given $a \in M_\Omega$ the function $\hat{a}$ belongs to $M_{\mathbf{R}^n}$.*

Proof. To prove inequality (3.2.1) for $\hat{a}$, consider a vector $\eta \in \hat{a}(\xi)$. Then $\eta = \langle h(\omega) \rangle$, where

$$(v,h) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$$

is a solution of (3.2.5). Inequality (3.2.1) for $a(\omega,\xi)$ implies

$$|\eta|^{p'} \le m_1 + c_1 \langle h(\omega) \cdot \xi \rangle + c_1 \langle h(\omega) \cdot v(\omega) \rangle.$$

Since h is solenoidal and v is potential with zero mean value, Proposition 3.1.4 implies that the third term in the right-hand part vanishes, and we get immediately the required inequality. The verification of (3.2.2) and of the monotonicity of $\hat{a}$ may be done in the similar manner.

Now we prove the maximal monotonicity of $\hat{a}$ using the same criterion, as in the proof of Proposition 3.2.1. Since B is a maximal monotone operator and $0 \in R(B)$, the set $B^{-1}0$ is nonempty, closed, and convex. Then $\hat{a}(\xi)$ is nonempty, closed, and convex, as a set of mean values of members of $B^{-1}0$.

Let us prove the upper semicontinuity of $\hat{a}$. Assume that $(\xi_k, \eta_k) \to (\xi, \eta)$ in $\mathbf{R}^n \times \mathbf{R}^n$ and $\eta_k \in \hat{a}(\xi_k)$. We have to show that $\eta \in \hat{a}(\xi)$. Consider a solution

$$(v_k, h_k) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$$

of (3.2.5), with $\xi = \xi_k$, such that $\eta_k = \langle h_k(\omega) \rangle$. It is not difficult to see that the coerciveness of B is uniform if ξ stays in any bounded subset of $\mathbf{R}^n$. This implies that the sequence h_k is bounded in $\mathbf{L}^{p'}_{sol}(\Omega)$. Therefore, by inequality (3.2.2) for a, the sequence v_k is bounded in $\mathbf{V}^p_{pot}$. Passing to a subsequence, we may assume that

$h_k \to h$ weakly in $\mathbf{L}^{p'}_{sol}(\Omega)$ and $v_k \to v$ weakly in $\mathbf{V}^p_{pot}$. Moreover, $\eta = \langle h(\omega) \rangle$. Thus, we need only to prove that $h(\omega) \in a(\omega, \xi + v(\omega))$.

The last may be carried out exactly as in the proof of Propotion 3.2.1, property 3 of the operator B. □

Now we are able to prove that $\hat{\mathcal{A}}$ defined by (3.2.4) is the homogenized operator for the family $\mathcal{A}_\varepsilon$ defined by (3.2.3).

Theorem 3.2.1 *Assume that $a \in M_\Omega$. Then, for almost all $\omega \in \Omega$, the family $A_\varepsilon(\omega)$ strongly G-converges to $\hat{A}$ for any open bounded domain $Q \subset \mathbf{R}^n$.*

Proof. According to Definition 2.2.1, we need to prove that

$$K_s(w \times \sigma)\text{-}\limsup A_\varepsilon \subset \hat{A},$$

where $A_\varepsilon : V \longrightarrow \mathrm{L}^{p'}(Q)^n$ and $\hat{A} : V \longrightarrow \mathrm{L}^{p'}(Q)^n$ are the operators of the class M_V associated to a generic realization

$$a^\varepsilon(x, \xi) = a(T(\varepsilon^{-1}x)\omega, \xi)$$

and the function $\hat{a}(\xi)$, respectively. As usual, we identify these operators with their graphs. Recall also that w denotes the weak topology of $\overline{V} = W^{1,p}(Q)$, while the topology σ is generated by the weak topology on $\mathrm{L}^{p'}(Q)^n$ and the topology induced by the seminorm $\|\operatorname{div} g\|_{V^*}$ (see Section 2.2).

Let

$$(u, g) \in K(w \times \sigma)\text{-}\limsup A_\varepsilon.$$

Then, by definition, there exist a subsequence of (ε) still denoted by (ε) and $(u_\varepsilon, g_\varepsilon) \in A^\varepsilon$ such that $(u_\varepsilon, g_\varepsilon) \to (u, g)$ in the topology $w \times \sigma$ on $\overline{V} \times \mathrm{L}^{p'}(Q)^n$.

We have to prove that $(u, g) \in \hat{A}$. Let $\xi \in \mathbf{R}^n$ and $\eta \in \hat{a}(\xi)$. Then there exists a solution

$$(v(\omega), h(\omega)) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$$

of problem (3.2.5) such that $\langle h(\omega) \rangle = \eta$. Consider generic realizations

$$v_\varepsilon(x) = v(T(\varepsilon^{-1}x)\omega)$$

and

$$h_\varepsilon(x) = h(T(\varepsilon^{-1}x)\omega).$$

Since $v_\varepsilon(x)$ is potential and, by Theorem 3.1.1, $v_\varepsilon \to 0$ weakly in $\mathrm{L}^p(Q)$, we have

$$v_\varepsilon(x) = \nabla N_\varepsilon(x),$$

where $N_\varepsilon \in \overline{V}$ and $N_\varepsilon \longrightarrow 0$ strongly in $\mathrm{L}^p(Q)$. Let us define the function w_ε by

$$w_\varepsilon(x) = \xi \cdot x + N_\varepsilon(x).$$

Then,

$$w_\varepsilon \longrightarrow \xi \cdot x \quad \text{weakly in } \overline{V},$$

$$\nabla w_\varepsilon(x) \longrightarrow \xi \quad \text{weakly in } \mathrm{L}^p(Q)^n,$$

$$h_\varepsilon(x) \longrightarrow \eta \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n.$$

Moreover,

$$h_\varepsilon(x) \in a^\varepsilon(x, \nabla w_\varepsilon(x)) \quad \text{a.e. on } Q$$

and $\operatorname{div} h_\varepsilon = 0$. Thus,

$$(w_\varepsilon(x), h_\varepsilon(x)) \longrightarrow (\xi \cdot x, \eta)$$

in the topology $w \times \sigma$ on the space $\overline{V} \times \mathrm{L}^{p'}(Q)^n$.
Now the monotonicity of a implies

$$\int_Q (g_\varepsilon(x) - h_\varepsilon(x)) \cdot (\nabla u_\varepsilon(x) - \nabla w_\varepsilon(x))\varphi(x)dx \geq 0$$

for any nonnegative $\varphi \in C_0^\infty(Q)$. By Lemma 3.1.1, we may pass to the limit in the last inequality. Hence,

$$\int_Q (g(x) - \eta) \cdot (\nabla u(x) - \xi)\varphi(x)dx \geq 0 \quad \forall \varphi \in C_0^\infty(Q),\ \varphi \geq 0.$$

This implies that, for any $\xi \in \mathbf{R}^n$ and $\eta \in \hat{a}(\xi)$,

$$(g(x) - \eta) \cdot (\nabla u(x) - \xi) \geq 0 \quad \text{a.e on } Q. \tag{3.2.14}$$

Since the graph of $\hat{a}$ is separable, there exists a subset $Q_0 \subset Q$ such that $|Q \setminus Q_0| = 0$ and, for $x \in Q_0$, inequality (3.2.14) is valid for all

$$(\xi, \eta) \in \mathbf{R}^n \times \mathbf{R}^n, \quad \eta \in \hat{a}(\xi).$$

In view of maximal monotonicity of $\hat{a}$, the last property ensures that

$$g(x) \in \hat{a}(\nabla u(x)) \quad \text{a.e. on } Q.$$

Therefore, $(u, g) \in \hat{A}$ and the proof is complete. □

Now let us consider the periodic case. More precisely, let

$$a : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

be a multivalued map satisfying all the conditions (i) – (iii) above, with Ω repliced by $\mathbf{R}^n$. Suppose, additionally, that $a(y,\xi)$ is 1-periodic with respect to each component of the first variable. We consider the family of operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x, \nabla u). \tag{3.2.15}$$

If we take Ω and $T(x)$ as in Example 3.1.1, we may regard $\mathcal{A}_\varepsilon$ as a realization of a random operator of the form (3.2.3) and apply Theorem 3.2.1. But in this case we have only one realization up to translations. Therefore, we have got the following *individual* homogenization result.

Corollary 3.2.2 *For any periodic family of operators (3.2.15) there exists a homogenized operator $\hat{\mathcal{A}}$ which is translation invariant.*

3.2.3 Properties of Homogenizated Operators

We collect here some additional results on homogenized operators, concerning situations when such an operator turns out to be single-valued or strictly monotone.

Proposition 3.2.3 *Let $a \in M_\Omega$. Assume that $a(\omega,\cdot)$ is strictly monotone for almost all $\omega \in \Omega$. Then $\hat{a}$ defined by (3.2.13) is strictly monotone.*

Proof. Let $\xi_i \in \mathbf{R}^n$, $\nu_i \in \hat{a}(\xi_i)$, $i = 1,2$, and $\xi_1 \neq \xi_2$. By definition of $\hat{a}$, there exists a solution

$$(v_i, h_i) \in \mathbf{V}^p_{pot} \times \mathbf{L}^p_{sol}(\Omega)$$

of problem (3.2.5) such that

$$\nu_i = \langle h_i \rangle, \quad i = 1,2.$$

Let

$$\Omega_0 = \{\omega \in \Omega : v_1(\omega) + \xi_1 \neq v_2(\omega) + \xi_2\}.$$

Then $\mu(\Omega_0) > 0$. Indeed, if not, then

$$v_1(\omega) + \xi_1 = v_2(\omega) + \xi_2 \quad \text{a.e. on } \Omega.$$

Since $v_1 \in \mathbf{V}^p_{pot}$ and, hence, $\langle v_i \rangle = 0$, we have $\xi_1 = \xi_2$, the contradiction. Now the strict monotonicity of a implies

$$(h_1(\omega) - h_2(\omega)) \cdot [(v_1(\omega) + \xi_1) - (v_2(\omega) + \xi_2)] > 0 \quad \text{on } \Omega_0.$$

Integrating, we get

$$(\nu_1 - \nu_2) \cdot (\xi_1 - \xi_2) > 0.$$

Thus, we have proved the strict monotonicity of $\hat{a}$. □

Proposition 3.2.4 *Let $a \in M_\Omega$. Assume that $a^{-1}(\omega, \cdot)$ is strictly monotone for almost all $\omega \in \Omega$. Then the map $\hat{a}$ is single-valued.*

Proof. Assume the contrary, i.e. there exist $\xi \in \mathbf{R}^n$ and ν_1, $\nu_2 \in \hat{a}(\xi)$ such that $\nu_1 \neq \nu_2$. Then there exists a solution

$$(v_i, h_i) \in \mathbf{V}^p_{pot} \times \mathbf{L}^{p'}_{sol}(\Omega)$$

of problem (3.2.5) such that

$$\nu_i = \langle h_i \rangle, \quad i = 1, 2.$$

We have

$$v_i(\omega) + \xi \in a^{-1}(\omega, h_i(\omega)) \quad \text{a.e. on } \Omega.$$

By strict monotonicity of a^{-1},

$$(h_1(\omega) - h_2(\omega)) \cdot (v_1(\omega) - v_2(\omega)) > 0 \tag{3.2.16}$$

on the set

$$\Omega_0 = \{\omega \in \Omega : h_1(\omega) \neq h_2(\omega)\}.$$

Since $\nu_1 \neq \nu_2$, the set Ω_0 is of positive measure. Then, integrating (3.2.16) we obtain

$$\langle (h_1(\omega) - h_2(\omega)) \cdot (v_1(\omega) - v_2(\omega)) \rangle > 0.$$

But $\mathbf{V}^p_{pot}$ is ortogonal to $\mathbf{L}^{p'}_{sol}(\Omega)$ and we get a contradiction. Hence, $\nu_1 = \nu_2$ and $\hat{a}$ is single-valued. □

Corollary 3.2.3 *Let $a \in M_\Omega$. Assume that the map a is single-valued and $a(\omega, \cdot)$ is strictly monotone for almost all $\omega \in \Omega$. Then the map $\hat{a}$ is single-valued and strictly monotone as well.*

Proof. By Proposition 3.2.3, the map $\hat{a}$ is strictly monotone. Hence, in view of Proposition 3.2.4, we need only to show that $a^{-1}(\omega, \cdot)$ is strictly monotone. Since $a \in M_\Omega$ and $a(\omega, \cdot)$ is strictly monotone, $a^{-1}(\omega, \cdot)$ is an everywhere defined single-valued map. Now strict monotonicity of $a^{-1}(\omega, \cdot)$ comes out from the same property of a. □

Remark 3.2.2 Let

$$f : \Omega \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

be a measurable function such that $f(\omega, \xi)$ is convex in the variable ξ. Assume that

$$c_3|\xi|^p - c_4 \leq f(\omega, \xi) \leq c_5(1 + |\xi|^p),$$

where $0 < c_3 \leq c_5$, and $c_4 \geq 0$. If $f(\omega, \xi)$ is differentiable with restect to ξ, for almost all $\omega \in \Omega$, then $a = \partial_\xi f \in M_\Omega$. In this case one can prove, by means of duality argument, that $\hat{a}$ is single-valued (see [99] for the periodic case).

Remark 3.2.3 Let $n = 1$ and let $a \in M_\Omega$ be a single-valued map. Letting

$$f(\omega, \xi) = \int_0^\xi a(\omega, t)dt,$$

we see that the assumptions of Remark 3.2.2 are automatically fulfilled. Therefore, in this case $\hat{a}$ is single-valued.

Now we give an example of a single-valued operator, the homogenized operator of which is multivalued.

Example 3.2.1 Let $n = 2$ and $p = 2$. Consider the function

$$r : \mathbf{R}^2 \times \mathbf{R}^2 \longrightarrow \mathbf{R}^2$$

such that $r(\omega, \xi)$ is 1-periodic in y_i, $i = 1, 2$, and

$$r(\omega, \xi) = \begin{cases} (\xi_2, -\xi_1) & \text{if } \omega_1 \in (0, 1/2), \\ (-\xi_2, \xi_1) & \text{if } \omega_1 \in [1/2, 1], \end{cases}$$

where $\omega = (\omega_1, \omega_2)$ and $\xi = (\xi_1, \xi_2)$. Let

$$f : \mathbf{R}^2 \times \mathbf{R} \longrightarrow \mathbf{R}$$

be the function defined by

$$f(\xi) = \frac{1}{2}[\max(|\xi| - 1; 0)]^2, \quad \xi \in \mathbf{R}^2.$$

We set

$$a(\omega, \xi) = \partial_\xi f(\xi) + r(\omega, \xi) = \max(|\xi| - 1; 0)|\xi|^{-1}\xi + r(\omega, \xi).$$

It is not difficult to see that a is a single-valued map which belongs to M_Ω with $\Omega = \mathbf{T}^2$. Moreover, $a(\omega, \xi)$ is continuous in ξ. Therefore, $a(\omega, \cdot)$ is maximal monotone. However, it is clear that a is not of the form $\partial_\xi f$, since it is not cyclically monotone. We show that the homogenized map $\hat{a}$ is actually multivalued. In fact, $\hat{a}(0)$ contains at least two members. Evidently, $0 \in \hat{a}(0)$. Therefore, it is enough to state the existence of $\nu \in \hat{a}(0)$ such that $\nu \neq 0$. To do this we consider a function $w \in W_0^{1,2}(0, 1)$ (then, w is continuous) such that $w(1/2) \neq 0$ and

$$\int_0^1 w(t)dt = 0.$$

Now we want to define the functions $v(\omega)$ and $h(\omega)$ which are involved in the definition of $\hat{a}$ (see (3.2.13)). We put

$$v(\omega) = (w'(\omega_1), 0), \quad \omega = (\omega_1, \omega_2),$$

$$h(\omega) = r(\omega, v(\omega)),$$

where $w'(t) = dw(t)/dt$. It is easy to verify that (v, h) is a solution of equation (3.2.5), with $\xi = 0$. A direct calculation show us that

$$\nu = \int_\Omega h(\omega) d\omega = (0, -2w(1/2)) \neq 0.$$

Thus, we have proved that $\hat{a}$ is multivalued.

3.2.4 Single-Valued Elliptic Operators

Now we turn to homogenization of elliptic operators which are single-valued, but non-monotone, in general. As usual, let Ω be a probability space endowed with an ergodic n-dimensional dynamical system $T(x)$. We fix constants $p > 1$, $c_0 > 0$, $\kappa > 0$, $\beta \geq \max(p, 2)$, $s \in (0, \min(p, p')]$, and nonnegative constants c, m, and h. Furthermore, let $\nu(r)$ be a modulus of continuity. Denote by

$$E_\Omega = E_\Omega(c_0, c, \kappa, h, \theta, \nu, s, \beta)$$

the set formed by couples (a, a_0) of Carathéodory maps

$$a : \Omega \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n,$$

$$a_0 : \Omega \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

which satisfy the following conditions:

- *for any* $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$,

$$|a(\omega, \xi_0, \xi)|^{p'} + |a_0(\omega, \xi_0, \xi)|^{p'} \leq c_1 + c(|\xi_0|^p + |\xi|^p) \quad a.e.\ on\ \Omega; \tag{3.2.17}$$

- *for any* $\xi_0 \in \mathbf{R}$, $\xi \in \mathbf{R}^n$ *and* $\xi' \in \mathbf{R}^n$

$$[a(\omega, \xi_0, \xi) - a(\omega, \xi_0, \xi')] \cdot (\xi - \xi') \geq$$
$$\geq \kappa \left(h + |\xi_0|^p + |\xi|^p + |\xi'|^p\right)^{1-\beta/p} |\xi - \xi'|^\beta \quad a.e.\ on\ \Omega; \tag{3.2.18}$$

- *for any* $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$ *and* $\zeta' = (\xi_0', \xi') \in \mathbf{R} \times \mathbf{R}^n$,

$$|a(\omega, \xi_0, \xi) - a(\omega, \xi_0', \xi')|^{p'} + |a_0(\omega, \xi_0, \xi) - a_0(\omega, \xi_0', \xi')|^{p'} \leq$$
$$\leq \theta \left[(h + |\zeta|^p + |\zeta'|^p) \cdot \nu(|\xi_0 - \xi_0'|) + \right.$$
$$\left. + (h + |\zeta|^p + |\zeta'|^p)^{1-s/p} |\xi - \xi'|^s\right] \quad a.e.\ on\ \Omega. \tag{3.2.19}$$

It is easily seen that given $(a, a_0) \in E_\Omega$ the realization

$$(a(T(x)\omega, \zeta), a_0(T(x)\omega, \zeta))$$

is well-defined for almost all $\omega \in \Omega$. Moreover, for any such realization and for any open bouned subset $Q \subset \mathbf{R}^n$, one can consider the operator

$$\mathcal{A}_\varepsilon(\omega)u = -\operatorname{div} a(T(\varepsilon^{-1}x)\omega, u, \nabla u) + a_0(T(\varepsilon^{-1}x)\omega, u, \nabla u), \quad \varepsilon > 0, \tag{3.2.20}$$

acting from $V = W_0^{1,p}(Q)$ into $V^* = W^{-1,p'}(Q)$. In addition, $\mathcal{A}_\varepsilon(\omega) \in \mathcal{E}_Q$ for a generic $\omega \in \Omega$. As above, we say that the family of operators $\mathcal{A}_\varepsilon$ admits homogenization if there exist a non-random operator $\hat{\mathcal{A}}$ such that $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$, as $\varepsilon \to 0$ (of course, for a generic $\omega \in \Omega$ and for any bounded open subset $Q \subset \mathbf{R}^n$).

To construct the homogenized operator for the family $\mathcal{A}_\varepsilon$ we need to solve an auxiliary problem. In our case this problem is the following. Given

$$\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$$

find $v \in \mathbf{V}_{pot}^p$ such that

$$a(\omega, \xi_0, \xi + v(\omega)) \in \mathbf{L}_{sol}^{p'}(\Omega). \tag{3.2.21}$$

Under the assumptions we imposed, problem (3.2.21) is simpler then (3.2.5). Indeed, Theorem 3.1.2 implies that the space $[\mathbf{V}_{pot}^p]^*$ may be naturally identified with the space

$$\mathbf{L}^{p'}(\Omega)/\mathbf{L}_{sol}^{p'}(\Omega). \tag{3.2.22}$$

Now we define the operator

$$\mathbf{A}_\zeta : \mathbf{V}_{pot}^p \longrightarrow [\mathbf{V}_{pot}^p]^*$$

by the following rule:

for $v \in \mathbf{V}_{pot}^p$, set *$\mathbf{A}_\zeta v$ to be the image in the space $\mathbf{L}^{p'}(\Omega)/\mathbf{L}_{sol}^{p'}(\Omega)$ of the function $a(\omega, \xi_0, \xi + v(\omega))$ which belongs, evidently, to $\mathbf{L}^{p'}(\Omega)$.*

It is clear that $v \in \mathbf{V}_{pot}^p$ is a solution of (3.2.21) if and only if $\mathbf{A}_\zeta v = 0$.

Since $(a, a_0) \in E_\Omega$, it is not difficult to verify that operator $\mathbf{A}_\zeta$ is bounded, continuous, strictly monotone, and coercive. Therefore,

$$R(\mathbf{A}_\zeta) = [\mathbf{V}_{pot}^p]^*.$$

As consequence, there exists a unique $v = v_\zeta \in \mathbf{V}_{pot}^p$ such that v_ζ is a solution of the equation $\mathbf{A}_\zeta v_\zeta = 0$, hence, of our auxilary problem (3.2.21).

Now let us introduce new functions $\hat{a}$ and $\hat{a}_0$ by the formulas

$$\hat{a}(\xi_0, \xi) = \langle a(\omega, \xi_0, \xi + v_\zeta(\omega)) \rangle, \tag{3.2.23}$$

$$\hat{a}_0(\xi_0, \xi) = \langle a_0(\omega, \xi_0, \xi + v_\zeta))\rangle, \tag{3.2.24}$$

where $\zeta = (\xi_0, \xi)$. We define the homogenized operators of the family $\mathcal{A}_\varepsilon$ by

$$\hat{\mathcal{A}}u = -\operatorname{div} \hat{a}(u, \nabla u) + \hat{a}_0(u, \nabla u). \tag{3.2.25}$$

Certainly, for the time being we have no information on this operator. In particular, we do not know is $\hat{\mathcal{A}}$ of the class $\mathcal{E}$, with suitable values of the parameters? One can obtain this property studying auxiliary problem (3.2.21) directly. But, this way requires a lot of work. Therefore, we prefer to get the relevant information on $\hat{\mathcal{A}}$ indirectly, as a by-product of the proof of the homogenization theorem.

Theorem 3.2.2 *Let $(a, a_0) \in E_\Omega$. Then $(\hat{a}, \hat{a}_0) \in E_{\mathbf{R}^n}$, with suitable values of the parameters of $E_{\mathbf{R}^n}$, and $\hat{\mathcal{A}}$ defined by (3.2.25) is the homogenized operator of $\mathcal{A}_\varepsilon$, i.e., for almost all $\omega \in \Omega$, we have $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$ for any bounded open subset $Q \subset \mathbf{R}^n$.*

Proof. Let us denote by $a^\varepsilon(x, \xi_0, \xi)$ and $a_0^\varepsilon(x, \xi_0, \xi)$ the generic realizations

$$a(T(\varepsilon^{-1}x)\omega, \xi_0, \xi) \quad \text{and} \quad a_0(T(\varepsilon^{-1}x)\omega, \xi), \xi),$$

respectively. Then $(a^\varepsilon, a_0^\varepsilon) \in E_{\mathbf{R}^n}$. Given open bounded subset $Q \subset \mathbf{R}^n$ the operator

$$\mathcal{A}^\varepsilon u = -\operatorname{div} a^\varepsilon(x, u, \nabla u) + a_0^\varepsilon(x, u, \nabla u)$$

belongs to the class $\mathcal{E}_Q$. Therefore, by Theorem 2.3.1, there exist a subsequence $\varepsilon' \to 0$ and an operator

$$\tilde{\mathcal{A}}u = -\operatorname{div} \tilde{a}(x, u, \nabla u) + \tilde{a}_0(x, u, \nabla u)$$

such that $\tilde{\mathcal{A}} \in \mathcal{E}_Q$, with, possibly, new values of the parameters of $\mathcal{E}_Q$, and $\mathcal{A}^{\varepsilon'} \overset{G}{\Longrightarrow} \tilde{\mathcal{A}}$. To prove the theorem it is now sufficient to show that $\tilde{a} = \hat{a}$ and $\tilde{a}_0 = \hat{a}_0$. To simplity the notations we still denote the subsequence ε' by ε.

Now let

$$v_\varepsilon(x) = v_\zeta(T(\varepsilon^{-1}x)\omega),$$

where $v_\zeta(\omega)$ is the solution of auxiliary problem (3.2.21). For almost all $\omega \in \Omega$, the vector field $v_\varepsilon \in \mathrm{L}^p_{loc}(\mathbf{R}^n)^n$ is potential and, by the Ergodic Theorem, we have $v_\varepsilon \to 0$ weakly in $\mathrm{L}^p_{loc}(\mathbf{R}^n)^n$. Therefore, there exists a function $N_\varepsilon \in W^{1,p}_{loc}(\mathbf{R}^n)$ such that

$$v_\varepsilon = \nabla N_\varepsilon,$$

$N_\varepsilon \to 0$ in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$ and weakly in $W^{1,p}_{loc}(\mathbf{R}^n)$. Set

$$w_\varepsilon(x) = \xi \cdot x + N_\varepsilon(x).$$

Restricting to the set Q we have easily

$$w_\varepsilon \to \xi \cdot x \quad \text{weakly in } \overline{V} = W^{1,p}(Q).$$

Moreover, since v_ζ is a solution of the auxiliary problem, we see, using the Ergodic Theorem, that

$$a^\varepsilon(x, \xi_0, \nabla w_\varepsilon) \to \hat{a}(\xi_0, \xi) \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n,$$

$$a_0^\varepsilon(x, \xi_0, \nabla w_\varepsilon) \to \hat{a}_0(\xi_0, \xi) \quad \text{weakly in } \mathrm{L}^{p'}(Q)^n,$$

and

$$\operatorname{div} a^\varepsilon(x, \xi_0, \nabla w_\varepsilon) = 0$$

Now Theorem 2.4.1 implies that $\tilde{a}(x, \xi_0, \xi) = \hat{a}(\xi_0, \xi)$ and $\tilde{a}_0(x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi)$ a.e. on Q. Since Q is an arbitrary bounded open subset of $\mathbf{R}^n$, we conclude. □

Remark 3.2.4 In fact, Theorem 2.3.1 implies that

$$(\hat{a}, \hat{a}_0) \in E(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{\theta}, \overline{\nu}, \overline{s}, \beta),$$

where

$$\overline{s} = \frac{sp}{\beta p - sp + s}, \qquad \overline{\nu} = \nu^{s/p}, \qquad \overline{c} = K \cdot h, \qquad \overline{h} = c + h,$$

and the positive constants $\overline{c}_0$, $\overline{\kappa}$ and K depend on c_0, κ and θ only.

Now let us consider the periodic case. Assume that $(a, a_0) \in E_{\mathbf{R}^n}$, and $a(y, \cdot)$ and $a_0(y, \cdot)$ are 1-periodic in $y \in \mathbf{R}^n$. Then, as in Corollary 3.2.2, the "individual" operator

$$\mathcal{A}^\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x, u, \nabla u) + a_0(\varepsilon^{-1}x, u, \nabla u), \quad \varepsilon > 0,$$

admits homogenization.

3.3 Almost Periodic Homogenization

3.3.1 Almost Periodic Functions

Let $C_b(\mathbf{R}^n)$ be the Banach space of all bounded and continuous (complex valued) functions on $\mathbf{R}^n$, endowed with the standard supremum norm. Denote by $\mathrm{Trig}(\mathbf{R}^n)$ the vector space of all trigonometric polinomials, i.e. all finite sums of the form

$$u(x) = \sum u_k \exp(i\xi_k \cdot x), \quad \xi_k \in \mathbf{R}^n, \ u_k \in \mathbf{C}.$$

The closure of the space Trig($\mathbf{R}^n$) in $C_b(\mathbf{R}^n)$ is called the space of *Bohr almost periodic* (a.p.) *functions* and is denoted by $CAP(\mathbf{R}^n)$. It is well-known that $f \in C_b(\mathbf{R}^n)$ is a.p. if and only if the family of shifts

$$\{f(\cdot + y)\}_{y\in\mathbf{R}^n} \subset C_b(\mathbf{R}^n)$$

is precompact. Moreover, any Bohr a.p. function has a mean value in the sense of n^0 3.1.1. In fact, for any such a function a mean value exists in a more strong sense, but we do not use this fact here.

Now we recall the concept of Bohr compactification of $\mathbf{R}^n$ [225]. There exist a compact abelian group $\mathbf{R}^n_B$ and a continuous group monomorphism

$$i_B : \mathbf{R}^n \longrightarrow \mathbf{R}^n_B$$

with the following property:

$f \in C_b(\mathbf{R}^n)$ is a.p. if and only if there exists a unique function $\tilde{f} \in C(\mathbf{R}^n_B)$ such that $f(x) = \tilde{f}(i_B x)$.

Such a couple $(\mathbf{R}^n_B, i_B)$ is unique up to a natural equivalence and is called the *Bohr compactification.* In the following we identity $\mathbf{R}^n$ with its dense image $i_B(\mathbf{R}^n)$ in $\mathbf{R}^n_B$. Frequently, we do not distinguish an a.p. function f and its extension $\tilde{f}$ to $\mathbf{R}^n_B$. Therefore, $CAP(\mathbf{R}^n_B)$ may be isometrically identified with $C(\mathbf{R}^n_B)$. We define a dynamical system $T(x)$ on $\mathbf{R}^n_B$ by

$$T(x)\omega = \omega + x, \quad \omega \in \mathbf{R}^n_B,\ x \in \mathbf{R}^n$$

Let us denote by μ the Haar measure on $\mathbf{R}^n_B$ normalized by $\mu(\mathbf{R}^n_B) = 1$. It is know that

$$M\{f\} = \int_{\mathbf{R}^n} \tilde{f}(\omega)d\mu(\omega), \tag{3.3.1}$$

where $\tilde{f}$ is the continuous extension of f to $\mathbf{R}^n_B$.

Now we turn to Besicovitch almost periodicity. For a function

$$f \in \mathrm{L}^p_{loc}(\mathbf{R}^n), \quad 1 \le p < \infty,$$

we set

$$\|f\|^p_{B^p} = \limsup_{t\to\infty} \frac{1}{|K_t|}\int_{K_t} |f(x)|^p dx, \tag{3.3.2}$$

where

$$K_t = \{x \in \mathbf{R}^n : |x_i| \le t,\ i = 1, 2, \dots, n\}.$$

A function $f \in \mathrm{L}^p_{loc}(\mathbf{R}^n)$ is said to be *Besicovitch a.p.* with the exponent p if there is a sequence $f_k \in \mathrm{Trig}(\mathbf{R}^n)$ such that

$$\lim_{k\to\infty} \|f - f_k\|_{B^p} = 0.$$

Of course, in this definition one can replace the space $\mathrm{Trig}(\mathbf{R}^n)$ by $CAP(\mathbf{R}^n)$. The space of all such functions is denoted by $B^p(\mathbf{R}^n)$. For any $f \in B^p(\mathbf{R}^n)$ the quantity $\|f\|_{B^p}$ is finite and defines a semi-norm on $B^p(\mathbf{R}^n)$. With respect to this semi-norm the space $B^p(\mathbf{R}^n)$ possesses a kind of completeness: *if* $f_k \in B^p(\mathbf{R}^n)$ *and*

$$\|f_k - f_m\|_{B^p} \to 0, \quad \text{as } k, m \to \infty,$$

then there exists a function $f \in B^p(\mathbf{R}^n)$ *such that*

$$\|f - f_k\|_{B^p} \to 0.$$

However, $B^p(\mathbf{R}^n)$ is not a Banach space, since the kernel of the semi-norm $\|\cdot\|_{B^p}$ is non-trivial. We say that two functions f_1, $f_2 \in B^p(\mathbf{R}^n)$ are *equivalent* if

$$\|f_1 - f_2\|_{B^p} = 0.$$

A vector space formed by equivalence classes of members of $B^p(\mathbf{R}^n)$ will be denoted by $\overline{B}^p(\mathbf{R}^n)$. The semi-norm $\|\cdot\|_{B^p}$ induces a norm on $\overline{B}^p(\mathbf{R}^n)$ and the last space is a Banach space with respect to that norm.

Let us look at the notion of mean value for Besicovitch a.p. functions. Assume that $f \in \mathrm{L}^p_{loc}(\mathbf{R}^n)$ and f has a finite norm (3.3.2). Since the family $f(\varepsilon^{-1}x)$ is bounded in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$, the mean value $M\{f\}$, if exists, may be characterized as the weak limit in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$:

$$M\{f\} = w\text{-}\lim_{\varepsilon\to 0} f(\varepsilon^{-1}x).$$

Using this statement it is very easy to verify that $M\{f\}$ depends continuously on f with respect to $\|\cdot\|_{B^p}$. More precisely, assume that $f, f_k \in \mathrm{L}^p_{loc}(\mathbf{R}^n)$, f_k has a mean value, and

$$\|f - f_k\|_{B^p} \to 0.$$

Then f also has a mean value and

$$M\{f_k\} \to M\{f\}.$$

Since any trigonometrical polynomial has a mean value, we see that for each $f \in B^p(\mathbf{R}^n)$ there exists the mean value $M\{f\}$. Moreover,

$$\|f\|_{B^p} = M\{|f|^p\}^{1/p}, \quad f \in B^p(\mathbf{R}^n). \tag{3.3.3}$$

Now we invoke the Bohr conpactification. Using (3.3.3) and (3.3.1) one can extend, by continuity, the isomorphism $f \longmapsto \tilde{f}$ between $CAP(\mathbf{R}^n)$ and $C(\mathbf{R}^n_B)$ to the map from $B^p(\mathbf{R}^n)$ into $\mathrm{L}^p(\mathbf{R}^n_B)$, the last space being regarded with respect to the measure μ. In fact, the density of $C(\mathbf{R}^n_B)$ in $\mathrm{L}^p(\mathbf{R}^n_B)$ implies that this map is onto. Moreover,

$$\|\tilde{f}\|_{p,\mathbf{R}^n_B} = \|f\|_{B^p}. \tag{3.3.4}$$

Therefore, the map $f \longmapsto \tilde{f}$ induces an isometric isomorphism between $\overline{B}^p(\mathbf{R}^n)$ and $\mathrm{L}^p(\mathbf{R}^n_B)$.

Consider more closely how to approximate Besicovitch a.p. functions by Bohr a.p. functions. Let $\{U_\gamma\}$ be a base of symmetric neighbourhoods of zero in $\mathbf{R}^n_B$, indexed by a directed set Γ in such a way that $U_{\gamma_1} \subset U_{\gamma_2}$ if $\gamma_1 \geq \gamma_2$. By Urysohn's Lemma, for any U_γ there exists an even non-negative function $\tilde{\varphi}_\gamma \in C(\mathbf{R}^n_B)$ such that $\mathrm{supp}\tilde{\varphi}_\gamma \subset U_\gamma$ and

$$\int_{\mathbf{R}^n_B} \tilde{\varphi}_\gamma(\omega)d\mu(\omega) = 1.$$

Let $\varphi_\gamma \in CAP(\mathbf{R}^n)$ be the restriction of $\tilde{\varphi}_\gamma$ to $\mathbf{R}^n \subset \mathbf{R}^n_B$. For any function $\tilde{f} \in \mathrm{L}^1(\mathbf{R}^n_B)$ we set

$$(\tilde{L}_\gamma \tilde{f})(\omega) = \tilde{\varphi}_\gamma *_B \tilde{f} = \int_{\mathbf{R}^n_B} \tilde{\varphi}_\gamma(\omega - \theta)\tilde{f}(\theta)d\mu(\theta), \tag{3.3.5}$$

where $*_B$ stands for the convolution on $\mathbf{R}^n_B$. In the similar way, for $f \in B^1(\mathbf{R}^n)$ we set

$$(L_\gamma f)(x) = M_y\{\varphi_\gamma(x-y)f(y)\}, \tag{3.3.6}$$

where M_y stands for the mean value with respect to the variable y. It is obvious that

$$\widetilde{L_\gamma f} = \tilde{L}_\gamma \tilde{f}. \tag{3.3.7}$$

Moreover, for $\tilde{f} \in \mathrm{L}^p(\mathbf{R}^n_B)$ (resp., $f \in B^p(\mathbf{R}^n)$) we have $\tilde{L}_\gamma \tilde{f} \in C(\mathbf{R}^n_B)$ (resp., $L_\gamma f \in CAP(\mathbf{R}^n)$). The operators L_γ and $\tilde{L}_\gamma$ are uniformly bounded:

$$\|L_\gamma f\|_{B^p} \leq \|f\|_{B^p}, \quad f \in B^p(\mathbf{R}^n), \tag{3.3.8}$$

$$\|\tilde{L}_\gamma \tilde{f}\|_{p,\mathbf{R}^n_B} \leq \|\tilde{f}\|_{p,\mathbf{R}^n_B}, \quad \tilde{f} \in \mathrm{L}^p(\mathbf{R}^n_B). \tag{3.3.9}$$

Directly from the definition of $\tilde{\varphi}_\gamma$ one can deduce that $\tilde{L}_\gamma \tilde{f} \to \tilde{f}$ in $\mathrm{L}^p(\mathbf{R}^n_B)$ for any $\tilde{f} \in \mathrm{L}^p(\mathbf{R}^n_B)$, $1 \leq p < \infty$. Now (3.3.4) gives rise to the following

Proposition 3.3.1 *For any $f \in B^p(\mathbf{R}^n)$, $1 \leq p < \infty$, we have*

$$\lim_\gamma \|f - L_\gamma f\|_{B^p} = 0. \tag{3.3.10}$$

Remark 3.3.1 Let $f \in \mathcal{F} \subset B^p(\mathbf{R}^n)$. If the image $\tilde{\mathcal{F}}$ of $\mathcal{F}$ in $\mathrm{L}^p(\mathbf{R}^n_B)$ is precompact, then the convergence in (3.3.10) is uniform with fespect to $f \in \mathcal{F}$. Moreover, if $\tilde{\mathcal{F}}$ is a separable subset in $\mathrm{L}^p(\mathbf{R}^n_B)$, then the net $\{L_\gamma\}$ may be replaced by a subsequence $\{L_m\}$.

Proposition 3.3.2 *The map $f \longmapsto \tilde{f}$ is order preserving: if $f_1 \leq f_2$, then $\tilde{f}_1 \leq \tilde{f}_2$.*

Proof. On $CAP(\mathbf{R}^n)$ this is obvious. Since $\tilde{\varphi}_\gamma \geq 0$, the operator L_γ is order presirving. Therefore, the general statement follows from the previous one by approximation. □

As a consequence, for any $f \in B^1(\mathbf{R}^n) \cap \mathrm{L}^\infty(\mathbf{R}^n)$ we have $\tilde{f} \in \mathrm{L}^\infty(\mathbf{R}^n_B)$.

3.3.2 Individual Homogenization

Now we consider almost periodic operators of the class $\mathcal{E}$. In this case we shall prove that homogenization take place in the individual sense, not only in the statistical one. More precisely, let us consider a couple of functions

$$(a, a_0) \in E_{\mathbf{R}^n} = E_{\mathbf{R}^n}(c_0, c, \kappa, h, \theta, \nu, s, \beta),$$

where c_0, c, κ, h, θ, ν, s, and β are constants subject to the stantard assumptions (see, for example, the begining of n^0 3.2.4). We deal with the family of operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x, u, \nabla u) + a_0(\varepsilon^{-1}x, u, \nabla u), \quad \varepsilon > 0. \tag{3.3.11}$$

Assume that

$$\left.\begin{array}{l} \textit{for any } \zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n, \textit{ the functions } a(x, \xi_0, \xi) \textit{ and} \\ a_0(x, \xi_0, \xi) \textit{ are } B^1\textit{-a.p. with respect to the variable } x \in \mathbf{R}^n. \end{array}\right\} \tag{3.3.12}$$

Associated to the family $\mathcal{A}_e$, there is a family of random operators $\mathcal{A}_e(\omega)$ defined on the probability space $\Omega = \mathbf{R}^n_B$ in the following way. According to n^0 3.3.1, one can extend the functions $a(x, \xi_0, \xi)$ and $a_0(x, \xi_0, \xi)$ to the functions $\tilde{a}(\omega, \xi_0, \xi)$ and $\tilde{a}_0(\omega, \xi_0, \xi)$, respectively, defined on $\mathbf{R}^n_B$. Proposition 3.3.2 implies that

$$(\tilde{a}, \tilde{a}_0) \in E_{\mathbf{R}^n_B}.$$

To simplify the notations we suppress the tilde here and still denote $\tilde{a}$ and $\tilde{a}_0$ by a and a_0, respectively. Let

$$\mathcal{A}_\varepsilon(\omega) u = -\operatorname{div} a(\omega + \varepsilon^{-1}x, u, \nabla u) + a_0(\omega + \varepsilon^{-1}x, u, \nabla u). \tag{3.3.13}$$

Then, we have formally

$$\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(0).$$

By Theorem 3.2.2, for the family $\mathcal{A}_\varepsilon(\omega)$ there exists a homogenized operator $\hat{\mathcal{A}}$. However, since the conclusion of the theorem is fulfilled in the statistical sence, i.e. for almost all $\omega \in \Omega$ only, we cannot conclude directly that $\hat{\mathcal{A}}$ serves the particular operator $\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon(0)$. Nevertheless, this is true, as it is stated in the following

Theorem 3.3.1 *Assume that $(a, a_0) \in E_{\mathbf{R}^n}$ and condition (3.3.12) is fulfilled. Then for any open bounded subset $Q \subset \mathbf{R}^n$ we have $\mathcal{A}_\varepsilon \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$.*

Proof. First we prove the statement under a more restrictive assumption then (3.3.12). Namely, let us assume that for any

$$\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$$

the functions $a(x, \xi_0, \xi)$ and $a_0(x, \xi_0, \xi)$ are a.p. in the sense of Bohr with respect to the variable $x \in \mathbf{R}^n$. Then, being extended to $\mathbf{R}^n_B$, the functions $a(\omega, \xi_0, \xi)$ and $a_0(\omega, \xi_0, \xi)$ are continuous with respect to $\omega \in \mathbf{R}^n_B$. Moreover, since

$$(a, a_0) \in E_{\mathbf{R}^n_B},$$

it is easy to verify that these functions are equicontinuous in ω if ζ belongs to any bounded subset of $\mathbf{R} \times \mathbf{R}^n$.

By Theorem 3.2.2, there exists a subset $\Omega_0 \subset \mathbf{R}^n_B$, with $\mu(\Omega_0) = 1$, such that $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$ for $\omega \in \Omega_0$. But any subset $\Omega_0 \subset \mathbf{R}^n_B$ of full measure is dense in $\mathbf{R}^n_B$. Using Corollary 2.4.4 we conclude now that $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$ for any $\omega \in \mathbf{R}^n_B$.

Let us return to the general case. Consider new functions $a^m(x, \xi_0, \xi)$ and $a_0^m(x, \xi_0, \xi)$ defined by

$$a^m(x, \xi_0, \xi) = L_m a(x, \xi_0, \xi),$$

$$a_0^m(x, \xi_0, \xi) = L_m a_0(x, \xi_0, \xi),$$

where L_m is the sequence of "smoothing" operators introduced in n^0 3.3.1 (see Remark 3.3.1). Then $a^m(x, \xi_0, \xi)$ and $a_0^m(x, \xi_0, \xi)$ are a.p. in $x \in \mathbf{R}^n$ in the sense of Bohr, and

$$(a^m, a_0^m) \in E_{\mathbf{R}^n},$$

with the same values of the parameters. The last follows from the fact that the kernel function of L_m is non-negative and has the mean value equals to 1. By definition, for any

$$\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$$

we have

$$\lim_{m \to \infty} a^m(\cdot, \xi_0, \xi) = a(\cdot, \xi_0, \xi), \tag{3.3.14}$$

$$\lim_{m \to \infty} a_0^m(\cdot, \xi_0, \xi) = a_0(\cdot, \xi_0, \xi), \tag{3.3.15}$$

in the B^1-norm. Moreover, since a and a_0 are continuous functions of ζ with values in $B^1(\mathbf{R}^n)$, these limits are uniform with respect to ζ whenever ζ belongs to any bounded subset of $\mathbf{R} \times \mathbf{R}^n$. By G-conpactness, we may assume that

$$\mathcal{A}_\varepsilon \overset{G}{\Longrightarrow} \mathcal{B} \in \mathcal{E},$$

where

$$\mathcal{B}u = -\operatorname{div} b(x, u, \nabla u) + b_0(x, u, \nabla u).$$

Since individual homogenization take place for the operator

$$\mathcal{A}_\varepsilon^m u = -\operatorname{div} a^m(\varepsilon^{-1}x, u, \nabla u) + a_0^m(\varepsilon^{-1}x, u, \nabla u),$$

we have $\mathcal{A}_\varepsilon^m \overset{G}{\Longrightarrow} \hat{\mathcal{A}}^m$.

Now we apply Theorem 2.4.4. With this aim, let us consider the functions

$$g(x,r) = \sup_{|\xi_0|,|\xi| \le r} |a^m(x,\xi_0,\xi) - a(x,\xi_0,\xi)|,$$

$$\hat{g}(x,r) = \sup_{|\xi_0|,|\xi| \le r} |\hat{a}^m(\xi_0,\xi) - b(x,\xi_0,\xi)|,$$

and the functions $g_0(x,r)$ and $\hat{g}_0(x,r)$ defined similary in terms of a_0, a_0^m, $\hat{a}_0^m$, and b_0. For simplicity of notations we suppress here the explicit dependence of the functions $g, \hat{g}$, etc. on m. Suppose K is the unit cube in $\mathbf{R}^n$ centered at the origin and

$$K_\rho(x_0) = x_0 + \rho K.$$

We set

$$\overline{g}(x,r) = \limsup_{\rho \to 0} \limsup_{\varepsilon \to 0} \frac{1}{|K_\rho(x)|} \int_{K_\rho(x)} g(\varepsilon^{-1}y, r)dy.$$

Replacing here g by g_0 we define the function $\overline{g}_0$. By Theorem 2.4.4, we have

$$\hat{g}(x,R) \le \overline{g}(x,r) + c(R)\left[\varphi(r)^{1/p'} + (1+r)\overline{g}(x,r)^\gamma\right], \tag{3.3.16}$$

$$\hat{g}_0(x,R) \le \overline{g}_0(x,r) + c(R)\left[\varphi(r)^{1/p'} + (1+r)\overline{g}(x,r)^\gamma\right], \tag{3.3.17}$$

for any $r > 0$, where $\gamma > 0$ and $\varphi(r) \to 0$, as $r \to +\infty$. Taking into account (3.3.3) and (3.1.4) we see that there exists

$$\lim_{\varepsilon \to 0} \frac{1}{|K_\rho(x)|} \int_{K_\rho(x)} g(\varepsilon^{-1}y, r)dy = \|g(\cdot, r)\|_{B^1}$$

which does not depend on x and ρ. By (3.3.14) and (3.3.15), for any $r > 0$, we have

$$\|g(\cdot, r)\|_{B^1} \to 0, \quad \|g_0(\cdot, r)\|_{B^1} \to 0, \text{ as } m \to \infty.$$

Passing in (3.3.16) and (3.3.17) to the limit as $m \to +\infty$ and, then, as $r \to \infty$ we see that

$$b(x, \xi_0, \xi) = \lim_{m\to\infty} \hat{a}^m(\xi_0, \xi),$$

$$b_0(x, \xi_0, \xi) = \lim_{m\to\infty} \hat{a}_0^m(\xi_0, \xi).$$

The same argument works for the operators $\mathcal{A}_\varepsilon(\omega)$ and $\mathcal{A}_\varepsilon^m(\omega)$, with $\omega \in \mathbf{R}^n_B$. But, for a generic $\omega \in \mathbf{R}^n_B$, the homogenized operators for $\mathcal{A}_\varepsilon(\omega)$ and $\mathcal{A}_\varepsilon^m(\omega)$ coincide with $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}^m$, respectively. Therefore,

$$b(x, \xi_0, \xi) = \hat{a}(\xi_0, \xi),$$

$$b_0(x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi),$$

and the proof is complete. □

Remark 3.3.2 By Theorem 2.4.2, we have the following representation formulas for the homogenized operator $\hat{\mathcal{A}}$:

$$\hat{a}(\xi_0, \xi) = \lim_{t\to\infty} \frac{1}{t^n} \int_{K_t} a(x, \xi_0, \xi + \nabla v_t^\zeta)dx,$$

$$\hat{a}_0(\xi_0, \xi) = \lim_{t\to\infty} \frac{1}{t^n} \int_{K_t} a_0(x, \xi_0, \xi + \nabla v_t^\zeta)dx,$$

where K_t is the cube, with the side length t, centered at the origin and

$$v_t^\zeta \in W_0^{1,p}(K_t)$$

is a unique solution of the problem

$$\operatorname{div} a(x, \xi_0, \xi + \nabla v) = 0 \quad \text{on } K_t.$$

Here, as usual, $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$.

3.4 One-Dimentional Problems

We discuss some cases when homogenized operators may be calculated more or less explicitly. For the sake of simplicity we restrict ourself to periodic operators of the form

$$\mathcal{A}_\varepsilon u = -\left(a(\varepsilon^{-1}x, u, u')\right)' \tag{3.4.1}$$

assuming $\mathcal{A}_\varepsilon$ to be of the class $\mathcal{E}$. It is not difficult to treat in the same manner the case when a lower order term is added. Considering (3.4.1) auxiliary problem (3.2.21) becomes: find a 1-periodic function

$$v = v_\zeta \in \mathrm{L}^p_{loc}(\mathbf{R}), \quad \zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n,$$

such that

$$\langle v \rangle = \int_0^1 v(x)dx = 0, \tag{3.4.2}$$

$$a\left(x, \xi_0, \xi + v(x)\right) = \tau, \tag{3.4.3}$$

where $\tau \in \mathbf{R}$ is an undefined constant. Given τ one can solve (3.4.3) uniquely and determine the function $v_\zeta(\tau, x)$. Then equation (3.4.2) permits us to find the unique value $\tau = \tau_0$ such that $v_\zeta(x) = v_\zeta(\tau_0, x)$ is the solution of the auxiliary problem. Now the homogenized operator

$$\hat{\mathcal{A}}u = -\hat{a}(u, u')' \tag{3.4.4}$$

is defined by

$$\hat{a}(\xi_0, \xi) = \langle a\left(\cdot, \xi_0, \xi + v_\zeta(\cdot)\right)\rangle. \tag{3.4.5}$$

The same may be done in the case of multivalued ordinary differential operators of the class $\mathcal{M}$, with only difference that in this case a solution of the auxiliary problem is not unique, in general.

Example 3.4.1 Let us consider the family of operators

$$\mathcal{A}_\varepsilon u = -\left[a(\varepsilon^{-1}x)|u'(x)|^{p-2}u'(x)\right]',$$

where $p \in (1, \infty)$ and $a(x) \in \mathrm{L}^\infty(\mathbf{R})$ is a 1-periodic function such that $a(x) \geq \alpha > 0$. In this case (3.4.3) becomes

$$a(x)|\xi + v|^{p-2}(\xi + v) = \tau.$$

It is not difficult to see that the solution is given by the formula

$$\xi + v = \left[a(x)^{-1}|\tau|\right]^{1/(p-1)} \cdot \frac{\tau}{|\tau|}.$$

Taking into account (3.4.2) we have

$$\tau = \langle a^{1/(1-p)}\rangle^{1-p}|\xi|^{p-2}\xi$$

and

$$\xi + v = a(x)^{1/(1-p)}\langle a^{1/(1-p)}\rangle^{-1}\xi.$$

From (3.4.5) we conclude that the homogenized operator is defined by the formula

$$\hat{\mathcal{A}}u = -\left[\hat{a}|u'|^{p-2}u'\right]',$$

where

$$\hat{a} = \langle a(\cdot)^{1/(1-p)}\rangle^{1-p}.$$

In the classical case $p = 2$ we get the well-known formula for the homogenized coefficient.

Now we consider the case of operators of the form

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x_1, u, \nabla u), \tag{3.4.6}$$

where

$$a : \mathbf{R} \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

is 1-periodic with respect to the first variable $x_1 \in \mathbf{R}$ and satisfies all the estimates which define the class $\mathcal{E}$. Look more closely at the auxiliary problem (3.2.21). We have to find a 1-periodic function

$$v \in \mathrm{L}^p_{loc}(\mathbf{R}^n)^n$$

such that

$$\int_K v(x)dx = 0, \tag{3.4.7}$$

$$\operatorname{div} a(x_1, \xi_0, \xi + v(x)) = 0, \tag{3.4.8}$$

where K is the unit cube in $\mathbf{R}^n$. Let us try to find a solution which does not depend on x_i, $i = 2, \dots, n$. Then, in components, we must have $v_i = 0$ for $i = 2, \dots, n$, $v_1 \in \mathrm{L}^p_{loc}(\mathbf{R})$, and

$$\langle v_1 \rangle = \int_0^1 v_1(x_1)dx_1 = 0. \tag{3.4.9}$$

Moreover, (3.4.8) becomes,

$$a_1(x_1, \xi_0, \xi_1 + v_1, \xi') = const, \tag{3.4.10}$$

where $\xi = (\xi_1, \xi')$ and $a = (a_1, \dots, a_n)$. It is a standard consequence of the monotonicity method that given $\zeta = (\xi_0, \xi)$ problem (3.4.9), (3.4.10) has a unique solution $v_1 = v_{1,\zeta}$. But the original auxiliary problem has only one solution. Therefore, this solution is of the form

$$v = (v_1(x_1), 0, \dots, 0),$$

where v_1 is the solution of (3.4.9), (3.4.10). The last problem is completely similar to (3.4.2), (3.4.3) and can be solved exactly in the same way. Thus, the homogenized operator

$$\hat{\mathcal{A}}u = -\operatorname{div} \hat{a}(u, \nabla u) \tag{3.4.11}$$

may be constructed by the formula

$$\hat{a}(\xi_0, \xi_1, \xi') = \int_0^1 a\left(x_1, \xi_0, \xi_1 + v_{1,\zeta}(x_1), \xi'\right) dx. \tag{3.4.12}$$

Of course, one can add to $\mathcal{A}_\varepsilon$ a lower order term of the form $a_0(\varepsilon^{-1}x, u, \nabla u)$. In this case the homogenized operator $\hat{\mathcal{A}}$ will contain on additional term of the form $\hat{a}_0(u, \nabla u)$, where

$$\hat{a}_0(\xi_0, \xi_1, \xi') = \int_K a_0(x, \xi_0, \xi_1 + v_{1,\zeta}(x_1), \xi') dx.$$

Moreover, the previous reduction may be done in the case of operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x_1, \nabla u)$$

from the class $\mathcal{M}$ provided the function $a(x_1, \xi)$ is *single-valued* and *strictly monotone* for almost all $x_1 \in \mathbf{R}$. Unfortunately, in general multivalued case we know nothing about a possibility of such a reduction.

Example 3.4.2 Consider the family of operators

$$\mathcal{A}_\varepsilon u = -\sum_{i,j=1}^{n} \partial_i \left(a_{ij}(\varepsilon^{-1}x_1)|\partial_j u|^{p-2}\partial_j u \right),$$

where $\partial_i = \partial/\partial x_i$ and the matrix $\{a_{ij}(x_1)\}$, not necessary symmetric, belongs to $\mathrm{L}^\infty(\mathbf{R})$ and is 1-periodic. Moreover, we assume that $\mathcal{A}_\varepsilon$ is of the class $\mathcal{E}$. In general, we cannot solve corresponding auxiliary problem (3.4.9), (3.4.10). By this reason, we impose the following *additional* structural condition:

$$a_{1k}(x_1) = 0, \quad k = 2, \dots, n.$$

In this case equation (3.4.10) becomes

$$a_{11}(x_1)|\xi_1 + v_1|^{p-2}(\xi_1 + v_1) = \tau,$$

the same as in Example 3.4.1. Hence,

$$\xi_1 + v_1 = a_{11}(x_1)^{1-p}\langle a(x_1)^{1/(1-p)}\rangle^{-1}\xi_1.$$

Using (3.4.12) we see that the homogenized operator $\hat{\mathcal{A}}$ is of the form

$$\hat{\mathcal{A}}u = -\sum_{ij} \partial_i \left(\hat{a}_{ij}|\xi_j|^{p-2}\xi_j\right),$$

where

$$\hat{a}_{11} = \langle a_{11}^{1/(1-p)}\rangle^{1-p},$$

$$\hat{a}_{1j} = 0, \quad j = 2, \dots, n,$$

$$\hat{a}_{i1} = \langle a_{11}^{1/(1-p)}\rangle^{1-p}\langle a_{i1}a_{11}^{-1}\rangle, \quad i = 2, \dots, n,$$

$$\hat{a}_{ij} = \langle a_{ij}\rangle, \quad i = 2, \dots, n, \ j = 2, \dots, n.$$

In general, if $a_{1i} \neq 0$, we have no explicit formulas for the homohenized operator, exept $p = 2$, the classical case.

3.5 Additional Results

3.5.1 Operators with Strong Nonlinearity

Here, following [223], we show how to extend the homogenization results presented in n^0 3.2.4 and n^0 3.3.2 to the class $\mathcal{E}^{p,p_0}$ considered in Section 2.5. If p and p_0 satisfy (2.5.5), all may be done exactly as in the case of $\mathcal{E}^p$. Therefore, we assume that

$$\frac{1}{p_0} \leq \frac{1}{p} - \frac{1}{n}.$$

Moreover, for the sake of simplicity we consider only the case of periodic operators of the class $\mathcal{E}^{p,p_0}_{(m)}$. Also we assume $c(x)$ and $h(x)$ to be constant functions.

Thus, we consider the family of operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x, u, \nabla u) + a_0(\varepsilon^{-1}x, u, \nabla u)$$

and assume that the periodic functions a and a_0 satisfy inequalities (2.5.2), (2.5.4), and (2.5.7). Now the auxiliary equation reads

$$-\operatorname{div}_y a(y, \xi_0, \xi + \nabla_y w) = 0. \tag{3.5.1}$$

Without any difficulty one can show that given

$$\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$$

there exists a unique solution of (3.5.1) such that $w \in W^{1,p}_{loc}(\mathbf{R}^n)$, w is 1-periodic in each variable $y_1, \ldots, y_n$, and has the mean value equals 0, $\langle w \rangle = 0$. Moreover, formulae (3.2.23), (3.2.24) for the homogenized operator still hold true and become now

$$\hat{a}(\xi_0, \xi) = \langle a\,(y, \xi_0, \xi + \nabla_y w_\zeta(y))\rangle, \tag{3.5.2}$$

$$\hat{a}_0(\xi_0, \xi) = \langle a_0\,(y, \xi_0, \xi + \nabla_y w_\zeta(y))\rangle. \tag{3.5.3}$$

Theorem 3.5.1 *Assume that the family $\mathcal{A}_\varepsilon$ satisfies the above mentioned conditions. Then $\mathcal{A}_\varepsilon$ admits homogenization. Moreover, the homogenized operator $\hat{\mathcal{A}}$ is of the form*

$$\hat{\mathcal{A}}u = -\operatorname{div} \hat{a}(u, \nabla u) + \hat{a}_0(u, \nabla u),$$

where $\hat{a}$ and $\hat{a}_0$ are defined by (3.5.2) and (3.5.3), respectively.

Proof. Let Q be an open bounded subset of $\mathbf{R}^n$. By Theorem 2.5.1, we can assume that $\mathcal{A}_\varepsilon \overset{G}{\Longrightarrow} \tilde{\mathcal{A}}$, as ε runs a sequence tending to 0. Now all we need is to show that $\tilde{a}(x, \xi_0, \xi) = \hat{a}(\xi_0, \xi)$ and $\tilde{a}_0(x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi)$.

With this aim, we consider the following reqularized auxiliary equation

$$-\operatorname{div}_y a(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) + \varepsilon a(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) = 0, \tag{3.5.4}$$

where $\varepsilon \in (0, 1)$. It is a standard consequence of the monotonicity method that there exists a unique solution

$$w_\varepsilon \in W^{1,p}_{per} \cap \mathrm{L}^{p_0}_{per}$$

of (3.5.4). Here $W^{1,p}_{per}$ (resp. L^p_{per}) denotes the space of all 1-periodic functions which belong to $W^{1,p}_{loc}(\mathbf{R}^n)$ (resp. $\mathrm{L}^p_{loc}(\mathbf{R}^n)$)).

We need to study the dependence of w_ε on ε, as $\varepsilon \to 0$. Multiplying (3.5.4) by w_ε and integrating we get

$$\int_K a(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla w_\varepsilon) \nabla_y w_\varepsilon dy +$$

$$+ \int_K a_0(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) \cdot (\varepsilon w_\varepsilon) dy = 0,$$

where K is the unit cube in $\mathbf{R}^n$. This and (2.5.7) yield the bounds

$$\varepsilon \|w_\varepsilon\|_{p_0, K} \le C, \tag{3.5.5}$$

$$\|\nabla_y w_\varepsilon\|_{p,K} \leq C, \tag{3.5.6}$$

where $C > 0$ does not depend on ε.

Now we consider the solution w of equation (3.5.1). We want to prove that

$$\lim_{\varepsilon \to 0} \|\nabla_y (w_\varepsilon - w)\|_{p,K} = 0. \tag{3.5.7}$$

Indeed, let $\delta > 0$. By density, one can choose $v_\delta \in W^{1,p}_{per} \cap \mathrm{L}^{p_0}_{per}$ such that

$$\|\nabla_y (w - v_\delta)\|_{p,K} \leq \delta.$$

Equations (3.5.1) and (3.5.4) imply

$$\int_K [a(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) - a(y, \xi_0, \xi + \nabla_y w)] \cdot \nabla_y (w_\varepsilon - v_\delta) dy +$$

$$+ \varepsilon \int_K a_0(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) \cdot (w_\varepsilon - v_\delta) dy = 0.$$

Next we have

$$\int_K [a(y, \xi_0 + \varepsilon w_\varepsilon, \xi + \nabla_y w_\varepsilon) - a(y, \xi_0, \xi + \nabla_y v_\delta)] \cdot \nabla_y (w_\varepsilon - v_\delta) dy +$$

$$+ \varepsilon \int_K [a_0(y, \xi_0 + \varepsilon w_\varepsilon, \nabla_y w_\varepsilon) - a_0(y, \xi_0, \xi + \nabla_y v_\delta)] \cdot (w_\varepsilon - v_\delta) dy =$$

$$= J_1 + J_2,$$

where

$$J_1 = \int_K [a(y, \xi_0, \xi + \nabla_y w) - a(y, \xi_0, \xi + \nabla_y v_\delta)] \cdot \nabla_y (w_\varepsilon - v_\delta) dy,$$

$$J_2 = -\varepsilon \int_K a_0(y, \xi_0, \xi + \nabla_y v_\delta) \cdot (w_\varepsilon - v_\delta) dy.$$

Using (2.5.7), (3.5.6), and the trivial bound

$$\|\nabla_y v_\delta\|_{p,K} \leq C \|\nabla_y w\|_{p,K}, \tag{3.5.8}$$

we obtain the inequality

$$\|\nabla_y (v_\delta - w_\varepsilon)\|^\beta_{p,K} \leq C(|J_1| + |J_2|).$$

Hence

$$\|\nabla_y (w - w_\varepsilon)\|_{p,K} \leq \delta + C(|J_1| + |J_2|)^{1/\beta}. \tag{3.5.9}$$

Inequalities (2.5.4), (3.5.6), and (3.5.8) show us that $|J_1| \leq \Delta(\delta)$ for all $\varepsilon \in (0,1)$, where $\Delta(\delta) \to 0$ as $\delta \to 0$. Finally, by (3.5.5), the set $\{\varepsilon w_\varepsilon\}$ is weakly precompact in $L^{p_0}_{per}$. At the same time, by (3.5.6) and the embedding theorem, $\varepsilon w \to 0$ strongly in L^p_{per}. Moreover, $\varepsilon v_\delta \to 0$ strongly in $L^{p_0}_{per}$ for any fixed $\delta > 0$. Therefore, by the definition of J_2, we see that $J_2 \to 0$, as $\varepsilon \to 0$, for any fixed $\delta > 0$. Thus, we have proved (3.5.7).

Let us now consider the function v^ε and v_0^ε defined by

$$v^\varepsilon(x) = \varepsilon w_\varepsilon(\varepsilon^{-1}x),$$

$$v_0^\varepsilon(x) = \varepsilon w(\varepsilon^{-1}x)$$

for any $x \in Q$. Obviously $\lim_{\varepsilon\to 0} v^\varepsilon = 0$ weakly in $\overline{V}$ and strongly in $L^p(Q)$, while $\lim_{\varepsilon\to 0} v_0^\varepsilon = 0$ weakly in $W^{1,p}(Q)$ and strongly in $L^p(Q)$. Moreover, by the definition of w_ε, we have

$$\mathcal{A}_\varepsilon^\zeta v^\varepsilon = 0,$$

where $\mathcal{A}_\varepsilon^\zeta$ is the shifted operator

$$\mathcal{A}_\varepsilon^\zeta u = -\operatorname{div} a(\varepsilon^{-1}x, \xi_0 + u, \xi + \nabla u) + a_0(\varepsilon^{-1}x, \xi_0 + u, \xi + \nabla u).$$

Since $\mathcal{A}_\varepsilon \overset{G}{\Longrightarrow} \tilde{\mathcal{A}}$, then $\mathcal{A}_\varepsilon^\zeta \overset{G}{\Longrightarrow} \tilde{\mathcal{A}}^\zeta$ for any $\zeta \in \mathbf{R} \times \mathbf{R}^n$. Theorem 2.5.3 implies that 0 is a solution of $\tilde{\mathcal{A}}^\zeta v = 0$ and

$$\lim_{\varepsilon\to 0} a(\varepsilon^{-1}x, \xi_0 + v^\varepsilon, \xi + \nabla v^\varepsilon) = \tilde{a}(x, \xi_0, \xi),$$

$$\lim_{\varepsilon\to 0} a_0(\varepsilon^{-1}x, \xi_0 + v^\varepsilon, \xi + \nabla v^\varepsilon) = \tilde{a}_0(x, \xi_0, \xi)$$

weakly in $L^{p'}(Q)$ and $L^{p_0'}(Q)$, respectively. However, $\lim_{\varepsilon\to 0} v^\varepsilon = 0$ strongly in $L^p(Q)$, hence, in measure. Using this statement and inequality (2.5.4) one can show that

$$\lim_{\varepsilon\to 0} \left[a(\varepsilon^{-1}x, \xi_0 + v^\varepsilon, \xi + \nabla v^\varepsilon) - a(\varepsilon^{-1}x, \xi_0, \xi + \nabla v^\varepsilon)\right] = 0,$$

$$\lim_{\varepsilon\to 0} \left[a_0(\varepsilon^{-1}x, \xi_0 + v^\varepsilon, \xi + \nabla v^\varepsilon) - a_0(\varepsilon^{-1}x, \xi_0, \xi + \nabla v^\varepsilon)\right] = 0$$

in measure, hence, weakly in $L^{p'}(Q)$ and $L^{p_0'}(Q)$, respectively. By (3.5.7),

$$\lim_{\varepsilon\to 0} \|\nabla(v^\varepsilon - v_0^\varepsilon)\|_{p,Q} = 0$$

and, using inequality (2.5.4), we see that

$$\lim_{\varepsilon\to 0} \left[a(\varepsilon^{-1}x, \xi_0, \xi + \nabla v^\varepsilon) - a(\varepsilon^{-1}x, \xi_0, \xi + \nabla v_0^\varepsilon)\right] = 0,$$

$$\lim_{\varepsilon\to 0} \left[a_0(\varepsilon^{-1}x, \xi_0, \xi + \nabla v^\varepsilon) - a_0(\varepsilon^{-1}x, \xi_0, \xi + \nabla v_0^\varepsilon)\right] = 0$$

strongly in $L^p(Q)$ and $L^{p'_0}(Q)$, respectively. Therefore,

$$\lim_{\varepsilon\to 0} a(\varepsilon^{-1}x, \xi_0, \xi + \nabla v_0^\varepsilon) = \tilde{a}(x, \xi_0, \xi),$$

$$\lim_{\varepsilon\to 0} a_0(\varepsilon^{-1}x, \xi_0, \xi + \nabla v_0^\varepsilon) = \tilde{a}_0(x, \xi_0, \xi)$$

weakly in the corresponding spaces. But, on the other hand, these limits coincide with $\hat{a}(\xi_0, \xi)$ and $\hat{a}_0(\xi_0, \xi)$, respectively. Thus, we have proved that

$$\tilde{a}(x, \xi_0, \xi) = \hat{a}(\xi_0, \xi),$$

$$\tilde{a}_0(x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi).$$

The proof is complete. □

3.5.2 Correctors

Let us consider a family of operators

$$\mathcal{A}_\varepsilon u = -\operatorname{div} a(\varepsilon^{-1}x, \nabla u), \quad x \in Q,$$

of the class $\mathcal{E}_0$. More precisely, we assume that the Carathéodory function $a(y, \xi)$ is 1-periodic in y and saticfies inequalities (3.2.17), (3.2.18), and (3.2.19), with ω repliced by y and $a_0 = 0$. As we have seen in n^0 3.2.5, there exists the homogenized operator

$$\hat{\mathcal{A}}u = -\operatorname{div} \hat{a}(\nabla u)$$

for the family $\mathcal{A}_\varepsilon$. For any $u \in V$, we have $u_\varepsilon \to u$ weakly in V, where $u_\varepsilon \in V$ is a unique solution of the equation

$$\mathcal{A}_\varepsilon u_\varepsilon = \mathcal{A}u. \tag{3.5.10}$$

Moreover, by the Sobolev Embedding Theorem, $u_\varepsilon \to u$ strongly in $L^p(Q)$. But, in general, the corresponding gradients converge only in the weak sence.

Now, following [116], we want to construct the so-called correctors, an additional terms which improve the convergence of gradients. First of all, we define a family M_ε of linear operators approximating the identity operator on $L^p(Q)^n$. Let K be the unit cube in $\mathbf{R}^n$. For $k \in \mathbf{Z}$ and $\varepsilon > 0$, we consider the set $K_\varepsilon^k = k + \varepsilon K$. Given $\varphi \in L^p(Q)^n$ we define $M_\varepsilon\varphi$ by the formula

$$(M_\varepsilon\varphi)(x) = \sum_{k\in I_\varepsilon} 1_{K_\varepsilon^k}(x)\frac{1}{|K_\varepsilon^k|}\int_{K_\varepsilon^k} \varphi(z)dz,$$

where

$$I_\varepsilon = \{k \in \mathbf{Z}^n : K_\varepsilon^k \subset Q\}$$

and 1_T is the characteristic function of a set $T \subset Q$. It is easy to see that, for any $\varphi \in \mathrm{L}^p(Q)^n$,

$$\|M_\varepsilon \varphi\|_{p,Q} \leq \|\varphi\|_{p,Q}$$

and $M_\varepsilon \varphi \to \varphi$ strongly in $\mathrm{L}^p(Q)^n$.
In the case under consideration auxiliary problem (3.2.21) becomes

$$-\operatorname{div} a(y, \xi + \nabla_y w) = 0.$$

Given $\xi \in \mathbf{R}^n$ there exists a unique solution $w = w_\xi$ of this equation such that $w \in W^{1,p}_{per}$ and $\langle w \rangle = 0$. Let us define the function P_ε by the formula

$$P_\varepsilon(x, \xi) = \xi + \nabla w_\xi(\varepsilon^{-1} x).$$

The following result was obtained in [116].

Theorem 3.5.2 *Let $u \in V$ and let $u_\varepsilon \in V$ be a solution of (3.5.10). Then*

$$\nabla u_\varepsilon = P_\varepsilon(\cdot, M_\varepsilon \nabla u) + r_\varepsilon,$$

where $r_\varepsilon \to 0$ strongly in $\mathrm{L}^p(Q)^n$.

In the case $p = 2$ and the operator $\mathcal{A}_\varepsilon$ is linear, one can eliminate the approximation of identity M_ε. More precisely, in this case P_ε is linear in ξ and one can show that

$$P_\varepsilon(\cdot, M_\varepsilon \varphi) - P_\varepsilon(\cdot, \varphi) \to 0$$

strongly in $\mathrm{L}^1(Q)^n$ for any $\varphi \in \mathrm{L}^2(Q)^n$. Therefore,

$$\nabla u_\varepsilon = P_\varepsilon(\cdot, \nabla u_\varepsilon) + r_\varepsilon, \tag{3.5.11}$$

where $r_\varepsilon \to 0$ strongly in $\mathrm{L}^1(Q)^n$. If, in addition, $a(y, \xi)$ is Hölder continuous with respect to y, then, in (3.5.11), $r_\varepsilon \to 0$ strongly in $\mathrm{L}^2(Q)^n$. However, in general, there is no satisfactory result in this direction.

Comments

The contents of Section 3.1 is more or less standard. Here we follow closely to [164].

The results of n^0 3.2.1 – 3.2.3 was proved in [228]. Earlier, the periodic versions of these results was obtained in [99]. Theorem 3.2.2 of n^0 3.2.4 is taken from [221, 224, 226], where the case of high order elliptic operators was considered.

Almost periodic homogenization for nonlinear elliptic operators was investigated in [221, 222, 224, 226] and [74]. In author's papers, it is considered the case of operators of the class $\mathcal{E}$ under the assumption of uniform almost periodicity. The paper [74] deals with the more restricted class $\mathcal{E}_0$, but under weaker assumption of B^1-almost periodicity. Theorem 3.3.1 covers all those results. We deduce it form the statistical homogenization theorem using the Bohr compactification and some general properties of strong G-convergence. In [74], it is used a completely different approach based on a reduction to the quasi-periodic case by means of an appropriate approximation. Theorem 3.5.1 is obtained in [223]. Another approach to problems of such kind may be found in [55, 56, 100].

For a detailed account of the theory of almost periodic functions, including Borh compactifications, we refer to [225] (see, also, [245]). In the books [12] and [199] the classical approach to this theory is presented.

Theorem 3.5.2 is taken from [116]. Its version for almost periodic homogenization was proved in [69]. The last paper contains also some applications of correctors.

In the periodic case the first homogenization results for nonlinear elliptic operators was obtained in [18, 58, 260]. Slightly more general class of operators depending on u, not only on ∇u, was studied in [153, 154]. This class of operators is similar to that considered in [234] and is not so satisfactory (see Comments to Chapter 2).

Also we mention the paper [134] which is devoted to homogenization of nonlinear degenerate elliptic operators.

Finally, there is a quite different approach to periodic homogenization problems [7, 8]. This approach is based on the so-called two-scale convergence.

CHAPTER 4

Nonlinear Parabolic Operators

4.1 Strong G-convergence

4.1.1 Main Definitions

Let $Q_0 \subset \mathbf{R}^n$ be a bounded open set and $Q = (0,T) \times Q_0$. On Q, we shall consider evolution operators of the form

$$\mathcal{L}u = \partial_t u - \operatorname{div} a(t,x,u,\nabla u) + a_0(t,x,u,\nabla u), \tag{4.1.1}$$

where $\partial_t = \partial/\partial t$. We assume that the functions

$$a : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

and

$$a_0 : Q \times \mathbf{R} \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

satisfy the Catathéodory condition and the following inequalities:

- *for any* $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$

$$|a(t,x,\xi_0,\xi)|^{p'} + |a_0(t,x,\xi_0,\xi)|^{p'} \le c_0|\zeta|^p + c(t,x) \qquad \textit{a.e. on } Q, \tag{4.1.2}$$

where $p > 1$, $c_0 > 0$ *and* $c(t,x) \ge 0$ *belongs to* $\mathrm{L}^1(Q)$;

- *for any* $\zeta = (\xi_0, \xi)$, $\zeta' = (\xi_0, \xi') \in \mathbf{R} \times \mathbf{R}^n$

$$\begin{aligned}[a(t,x,\xi_0,\xi) - a(t,x,\xi_0,\xi')] \cdot (\xi - \xi') \ge \\ \ge \kappa\left[h(t,x) + |\zeta|^p + |\zeta'|^p\right]^{1-\beta/p} \times |\xi - \xi'|^\beta\end{aligned} \tag{4.1.3}$$

a.e. on Q, *where* $\kappa > 0$ *and* $h \in \mathrm{L}^1(Q)$ *is non-negative.*

A. Pankov, *G-Convergence and Homogenization of Nonlinear Partial Differential Operators*

Additionally, we assume that

$$p > \frac{2n}{n+2} \tag{4.1.4}$$

Now we introduce corresponding functional spaces. As before, we set

$$V = W_0^{1,p}(Q_0), \quad \overline{V} = W^{1,p}(Q_0), \quad V^* = W^{-1,p'}(Q_0).$$

Also we denote by H the Hilbert space $L^2(Q_0)$. Inequality (4.1.4) implies that the space V is embedded compactly into the space H. To treat the operators like (4.1.1) we need the following spaces of vector valued functions on (0,T):

$$\mathcal{V} = L^p(0,T;V),$$

$$\mathcal{V}^* = L^{p'}(0,T;V^*),$$

$$\mathcal{W} = \{u \in \mathcal{V} : \partial_t u \in \mathcal{V}^*\},$$

$$\mathcal{W}_0 = \{u \in \mathcal{W} : u(0) = 0\}.$$

Here the time derivative ∂_t is regarded in the sence of vector valued distributions. Recall that, by Proposition 1.3.1, the space $\mathcal{W}$ is embedded continuously into $C([0,T];H)$. Therefore, $\mathcal{W}_0$ is well-defined. We shall use also the spaces

$$\overline{\mathcal{V}} = L^p(0,T;\overline{V})$$

and

$$\overline{\mathcal{W}} = \{u \in \overline{\mathcal{V}} : \partial_t u \in \mathcal{V}^*\}.$$

It is a standard fact (see, e.g. [200], Theorem 5.1 of Ch. 1) that the embedding

$$\mathcal{W}_0 \subset L^p(Q) = L^p(0,T;L^p(Q_0))$$

is compact. Moreover, the embedding

$$\mathcal{W}_0 \subset L^p(0,T;H)$$

is also compact. All the spaces we have just introduced are reflexive Banach spaces. Under the previous conditions expression (4.1.1) generates an operator acting from $\mathcal{W}_0$ (or, even, from $\overline{\mathcal{W}}$) into $\mathcal{V}^*$.

As for solvability of the homogeneous Cauchy problem for the operator $\mathcal{L}$ defined by (4.1.1), we have (see [200])

Proposition 4.1.1 *Additionally to (4.1.2) and (4.1.3), assume that, for any* $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$, *the inequality*

$$a(t,x,\xi_0,\xi)\cdot\xi + a_0(t,x,\xi_0,\xi)\cdot\xi_0 \geq \gamma|\zeta|^p - \gamma_0(t,x) \tag{4.1.5}$$

holds true for almost all $(t,x) \in Q$, *where* $\gamma > 0$ *and* $\gamma_0 \in \mathrm{L}^1(Q)$ *is non-negative (i.e. the operator* $\mathcal{L}$ *is coercive). Then for any* $f \in \mathcal{V}^*$ *there exists a solution* $u \in \mathcal{W}_0$ *of the equation*

$$\mathcal{L}u = f.$$

Moreover, if $\mathcal{L}$ *satisfies the strict monotonicity condition*

$$[a(t,x,\xi_0,\xi) - a(t,x,\xi_0',\xi')]\cdot(\xi-\xi') + [a_0(t,x,\xi_0,\xi) - a_0(t,x,\xi_0',\xi')]\times \\ \times(\xi_0-\xi_0') > 0 \quad a.e.\ on\ Q, \tag{4.1.6}$$

for any $\zeta = (\xi_0,\xi)$, $\zeta' = (\xi_0',\xi')$, $\zeta \neq \zeta'$, *then the solution is unique.*

Remark 4.1.1 In the case $a_0 \equiv 0$ and a does not depend explicitly on ξ_0, both the coerciveness and strict monotonicity follow from (4.1.3).

As in the case of elliptic operators, we impose additionally the following condition:

- *for any* $\zeta = (\xi_0,\xi)$, $\zeta' = (\xi_0',\xi) \in \mathbf{R}\times\mathbf{R}^n$,

$$|a(t,x,\xi_0,\xi) - a(t,x,\xi_0',\xi')|^{p'} + |a_0(t,x,\xi_0,\xi) - a_0(t,x,\xi_0',\xi')|^{p'} \leq \\ \leq \theta\left[(h(t,x)+|\zeta|^p+|\zeta'|^p)\,\nu(|\xi_0-\xi_0'|) + (h(t,x)+|\zeta|^p+|\zeta'|^p)^{1-s/p}\times \right. \\ \left. \times\ |\xi-\xi'|^s\right] \quad \text{a.e. on } Q, \tag{4.1.7}$$

where $\theta > 0$, $0 < s \leq \min(p,p')$ *and* $\nu(r)$ *is a continuity modulus.*

Fixed $c_0, c, \kappa, h, \theta, \nu$ and β as above we denote by

$$\Pi = \Pi(c_0, c, \kappa, h, \theta, s, \nu, \beta)$$

the class of all operators $\mathcal{L}$ satisfying inequalities (4.1.2), (4.1.3), and (4.1.7). By

$$\Pi_0 = \Pi_0(c_0, c, \kappa, h, \theta, s, \beta)$$

we denote the subset of Π consisting of all operators $\mathcal{L}$ such that $a_0 \equiv 0$ and $a(t,x,\xi_0,\xi) = a(t,x,\xi)$ does not depend on ξ_0.

For any $\mathcal{L} \in \Pi$ we set

$$\mathcal{L}^1(u,v) = \partial_t u - \operatorname{div} a(t,x,v,\nabla u). \tag{4.1.8}$$

For $v \in \mathcal{V}$, the operator $\mathcal{L}^1(\cdot, v)$ acts from $\mathcal{W}_0$ into $\mathcal{V}^*$ and, by Remark 4.1.1, satisfies the coerciveness and strict monotonicity conditions. Therefore, any such operator is invertible. Now let $\mathcal{L}$, $\mathcal{L}_k \in \Pi$. We introduce the *momenta* or the *generalized gradients* of the system $\{\mathcal{L}, \mathcal{L}_k\}$ in the following way. Given $u \in \mathcal{W}_0$ and $v \in \mathcal{V}$ we set

$$\Gamma^k(u, v) = a^k(t, x, v, \nabla u_k),$$

$$\Gamma_0^k(u, v) = a_0^k(t, x, v, \nabla u_k),$$

$$\Gamma(u, v) = a(t, x, v, \nabla u)$$

and

$$\Gamma_0(u, v) = a_0(t, x, v, \nabla u),$$

where $u_k \in \mathcal{W}_0$ is a unique solution of the equation

$$\mathcal{L}_k^1(u_k, v) = \mathcal{L}^1(u, v); \tag{4.1.9}$$

it exists, by Remark 4.1.1. It is easy to verify that the operators Γ^k and Γ (resp. Γ_0^k and Γ_0) act continuously from $\mathcal{W}_0 \times \mathcal{V}$ into $\mathrm{L}^{p'}(Q)^n$ (resp. $\mathrm{L}^{p'}(Q)$). Indeed, condition (4.1.3) implies that $u_k \in \mathcal{W}_0$ depends continuously on $(u, v) \in \mathcal{W}_0 \times \mathcal{V}$.

Definition 4.1.1 *A sequence $\mathcal{L}_k \in \Pi$ is said to be strongly G-convergent to $\mathcal{L} \in \Pi$ (in symbols, $\mathcal{L}_k \stackrel{G}{\Longrightarrow} \mathcal{L}$) if for any $v \in \mathcal{V}$ and $u \in \mathcal{W}_0$*

$$w\text{-}\lim u_k = u \quad \text{in } \mathcal{W}_0, \tag{4.1.10}$$

where u_k is defined by (4.1.9), and

$$\left.\begin{aligned} w\text{-}\lim \Gamma^k(u, v) &= \Gamma(u, v), \\ w\text{-}\lim \Gamma_0^k(u, v) &= \Gamma_0(u, v) \end{aligned}\right\} \tag{4.1.11}$$

in $\mathrm{L}^{p'}(Q)$.

Remark 4.1.2 Condition (4.1.10) means that $\mathcal{L}_k^1(\cdot, v) \xrightarrow{G} \mathcal{L}(\cdot, v)$ for any $v \in \mathcal{V}$ in the sense of Section 1.3. In the case $\mathcal{L}_k$, $\mathcal{L} \in \Pi_0$, the operators Γ^k and Γ do not depend on v. Moreover, Γ_0^k and Γ_0 vanish. Therefore, in this case we shall suppress the variable v in our notations and write simply $\Gamma^k(u)$ and $\Gamma(u)$.

4.1.2 Monotone Operators

Here we consider the special case of operators of the form

$$\mathcal{L}u = \partial_t u - \operatorname{div} a(t, x, \nabla u)$$

which belong to the class Π_0. Any such operator may be regarded as an abstract parabolic operator. Therefore, we can apply the results of Section 1.3. To study strong G-convergence we shall use the approach similar to that of Section 2.3. Hence, some points will be discussed briefly. As in n^0 2.3.2, we start with the following technical result which is similar to Lemma 2.3.1.

Lemma 4.1.1 *Suppose $\mathcal{L}_k \in \Pi_0$ and u_k, v_k are bounded sequences in $\overline{\mathcal{W}}$ such that $z_k = u_k - v_k \to 0$ weakly in $\overline{\mathcal{W}}$. Let $f_k = \mathcal{L}_k u_k$ and $g_k = \mathcal{L}_k v_k$. Assume that* $\lim f_k = f$ *and* $\lim g_k = g$ *strongly in $\mathcal{W}_0^*$. Then $f = g$ and $z_k \to 0$ strongly in $\mathcal{V}_{loc} = \mathrm{L}^p_{loc}(0, T; W^{1,p}_{loc}(Q_0))$.*

Proof. Set

$$Z^k = a^k(t, x, \nabla u_k) - a^k(t, x, \nabla v_k);$$

the sequence Z^k is bounded in $\mathrm{L}^{p'}(Q)$. Let $\varphi \in C_0^\infty(Q)$ be a function such that $0 \le \varphi \le 1$. Multiplying the equation

$$\mathcal{L}_k u_k - \mathcal{L}_k v_k = f_k - g_k$$

by $\varphi^2 z_k$ and integrating we get, after a simple calculation,

$$\langle \partial_t(\varphi z_k), \varphi z_k \rangle - \langle (\partial_t \varphi) z_k, \varphi z_k \rangle + \int_Q (Z^k \cdot \nabla z_k)\varphi^2 dtdx +$$
$$+ \int_Q Z^k \cdot (z_k \nabla \varphi^2) dtdx = \langle f_k - g_k, \varphi^2 z_k \rangle.$$

Since φz_k is compactly supported in Q, we have $\langle \partial_t(\varphi z_k), \varphi z_k \rangle = 0$. Therefore, using (4.1.3) we obtain

$$\langle f_k - g_k, \varphi^2 z_k \rangle + \langle (\partial_t \varphi) z_k, \varphi z_k \rangle - \int_Q Z^k \cdot (z_k \nabla \varphi^2) dtds \ge$$
$$\ge c \cdot \|\varphi^2 \nabla z_k\|_p^\beta.$$

Now we need to show that the left-hand part of the last inequality tends to zero. As for the first term, it is obvious that $\varphi^2 z_k \to 0$ weakly in $\mathcal{W}_0$ and, therefore, this term tends to zero. Furthermore, since $\{(\partial_t \varphi) z_k\}$ is bounded in $C([0, T]; H)$ and $\varphi z_k \to 0$ strongly in $\mathrm{L}^p(0, T; H)$ due to compactness of the embedding $\mathcal{W}_0 \subset \mathrm{L}^p(0, T; H)$, the second term also tends zero. By a similar reason, the third term has the limit equals zero as well.

Thus, $z_k \to 0$ strongly in $\mathrm{L}^p_{loc}(0,T;W^{1,p}_{loc}(Q_0))$. This implies that $Z^k \to 0$ strongly in $\mathrm{L}^{p'}_{loc}(0,T;\mathrm{L}^{p'}_{loc}(Q_0))$. Since, $z_k \to 0$ weakly in $\overline{\mathcal{W}}$, we have

$$\langle \mathcal{L}_k u_k - \mathcal{L}_k v_k, \varphi\rangle = -\langle z_k, \partial_t \varphi\rangle + \int_Q Z^k \cdot \nabla\varphi dtdx \to 0$$

for any $\varphi \in C_0^\infty(Q)$. Hence, $f = g$. □

As in the elliptic case, we need to consider a family of shifted operators associated to a parabolic operator. Let us denote by $\mathcal{A}$ the elliptic part

$$\mathcal{A}u = -\operatorname{div} a(t,x,\nabla u)$$

of $\mathcal{L}$. This operator acts continuously from $\mathcal{V}$ into $\mathcal{V}^*$. We introduce the operator

$$\overline{\mathcal{A}} : \mathrm{L}^p(Q)^n \longrightarrow \mathcal{V}^*$$

defined by

$$\overline{\mathcal{A}}\psi = -\operatorname{div} a(t,x,\psi).$$

Obviously, $\mathcal{A} = \overline{\mathcal{A}} \circ \nabla$. The family of operators

$$\overline{\mathcal{A}}^\psi : \mathrm{L}^p(Q)^n \longrightarrow \mathcal{V}^*,$$

parametrized by $\psi \in \mathrm{L}^p(Q)^n$, is defined by the formula

$$\overline{\mathcal{A}}^\psi \chi = \overline{\mathcal{A}}(\psi + \chi),$$

and we set also $\mathcal{A}^\psi = \overline{\mathcal{A}}^\psi \circ \nabla$. The last operator acts from $\mathcal{V}$ into $\mathcal{V}^*$. We list the main properties of this family which are completely similar to those presented in (2.3.7) – (2.3.10).

$$\mathcal{A}^{\psi+\nabla w}(u) = \mathcal{A}^\psi(u+w), \tag{4.1.12}$$

$$\|\mathcal{A}^\psi u\|_*^{p'} \le \overline{c}_0 \left(\|u\|^p + \|\psi\|_p^p\right) + c_1, \tag{4.1.13}$$

$$\langle \mathcal{A}^\psi u - \mathcal{A}^\psi w, u-w\rangle \ge \overline{\kappa} H(u,w,\psi)^{1-\beta/p}\|u-w\|^\beta, \tag{4.1.14}$$

$$\|\mathcal{A}^\psi u - \mathcal{A}^\psi w\|_*^{p'} \le \overline{\theta} H(u,w,\psi)^{1-s/p}\|u-w\|^s, \tag{4.1.15}$$

$$\|\mathcal{A}^\psi u - \mathcal{A}^{\psi'} u\|_*^{p'} \le \overline{\theta} H(u,\psi,\psi')^{1-s/p}\|\psi-\psi'\|_p^s, \tag{4.1.16}$$

for $u, w \in \mathcal{V}$ and $\psi, \psi' \in \mathrm{L}^p(Q)^n$. Here

$$H(u,w,\ldots) = c_2 + \|u\|^p + \|w\|^p + \cdots,$$

$$c_1 = \int_Q c(t,x)dtdx,$$

$$c_2 = \int_Q h(t,x)dtdx.$$

For any $\psi \in \mathrm{L}^p(Q)^n$ we consider the operator

$$\mathcal{L}^\psi u = \partial_t u + \mathcal{A}^\psi u$$

acting from $\mathcal{V}$ into $\mathcal{V}^*$, with the domain $\mathcal{W}_0$. Any such operator is invertible and we define the operator

$$\mathcal{R} : \mathcal{V}^* \times \mathrm{L}^p(Q)^n \longrightarrow \mathcal{W}_0$$

by the formula

$$\mathcal{R}(f,\psi) = \mathcal{L}_\psi^{-1} f. \tag{4.1.17}$$

It is easy to verify that

$$w + \mathcal{R}(f - \partial_t w, \psi + \nabla w) = \mathcal{R}(f,\psi) \tag{4.1.18}$$

and

$$\|\mathcal{R}(f,\psi)\|^p \le K\left(\|f\|_*^{p'} + \|\psi\|_p^p + c_1 + c_2\right), \tag{4.1.19}$$

for any $f \in \mathcal{V}^*$, $\psi \in \mathrm{L}^p(Q)^n$ and $w \in \mathcal{W}_0$, where $K > 0$ does not depend on c_1 and c_2.

As usually, we set

$$\overline{s} = \frac{sp}{\beta p - sp + s}, \quad \overline{h} = c + h, \quad \overline{c} = K\overline{h}.$$

The positive constants $K, \overline{c}_0, \overline{\kappa}$ and θ may depend on c_0, κ and θ only.

Lemma 4.1.2 *For any sequence $\mathcal{L}_k \in \Pi_0(c_0, c, \kappa, h, \theta, s, \beta)$ there exists an operator $\mathcal{L} \in \Pi_0(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{h}, \overline{\theta}, \overline{s}, \beta)$ and a subsequence $\mathcal{L}_{k'}$ such that $\mathcal{L}_{k'}$ strongly G-converges to $\mathcal{L}$.*

Proof is similar to that of Lemma 2.3.2. Therefore, we only sketch it following the main steps given there.

Step 1. Passing to a subsequence we may assume, by Theorem 1.3.2, that $\mathcal{L}_k^\psi \xrightarrow{G} \mathcal{L}^\psi$ for any ψ from a dense countable subset of $\mathrm{L}^p(Q)^n$. Here

$$\mathcal{L}^\psi = \partial_t + \mathcal{A}^\psi$$

is an abstract parabolic operator. By (4.1.16), the operator $\mathcal{A}_k^\psi$ depends continuously on $\psi \in \mathrm{L}^p(Q)^n$ in the metric defined by (1.3.30), uniformly with respect to $k \in \mathbf{N}$. Hence, by Corollary 1.3.1, $\mathcal{L}_k^\psi \xrightarrow{G} \mathcal{L}^\psi$ for any $\psi \in \mathrm{L}^p(Q)^n$ and the operator

$\mathcal{A}^\psi$ depends continuously on ψ. Moreover, this operator satisfies inequality (4.1.14) and

$$\|\mathcal{A}^\psi u\|_*^{p'} \le \bar{c}_0 \left(\|u\|^p + \|\psi\|_p^p\right) + K(c_1 + c_2), \tag{4.1.20}$$

$$\|\mathcal{A}^\psi u - \mathcal{A}^\psi v\|_*^{p'} \le \bar{\theta} H_1(u, w, \psi)^{1-\bar{s}/p} \|u - v\|^{\bar{s}}, \tag{4.1.21}$$

where

$$H_1(\cdot) = H(\cdot) + c_1.$$

Now let

$$\mathcal{R}(f, \psi) = (\mathcal{L}^\psi)^{-1} f$$

and $\mathcal{R}_k(f, \psi)$ be associated to $\mathcal{L}_k$ by (4.1.17). Then

$$\mathcal{R}_k(f, \psi) \to \mathcal{R}(f, \psi)$$

weakly in $\mathcal{W}_0$ for any

$$(f, \psi) \in \mathcal{V}^* \times \mathrm{L}^p(Q)^n.$$

Evidently, $\mathcal{R}$ satisfies (4.1.18). We set $\mathcal{A} = \mathcal{A}^0$ and define

$$\overline{\mathcal{A}} : \mathrm{L}^p(Q)^n \longrightarrow \mathcal{V}^*$$

by the formula

$$\overline{\mathcal{A}}\psi = \mathcal{A}^\psi(0).$$

Using (4.1.18), it is easy to verify that

$$\mathcal{A} = \overline{\mathcal{A}} \circ \nabla.$$

By (4.1.20),

$$\|\overline{\mathcal{A}}\psi\|_*^{p'} \le \bar{c}_0 \|\psi\|_p^p + K(c_1 + c_2), \quad \psi \in \mathrm{L}^p(Q)^n. \tag{4.1.22}$$

Certainly, one can extend $\mathcal{A}$ to $\overline{\mathcal{V}}$ as the composition $\overline{\mathcal{A}} \circ \nabla$.

Step 2. Given $\psi \in \mathrm{L}^p(Q)^n$ we set

$$\psi_k = \psi + \nabla \mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi) = \psi + \nabla u_k^1, \tag{4.1.23}$$

where $u_k^1 = \mathcal{R}_k(\overline{\mathcal{A}}\psi, \psi)$. Since, obviously,

$$\mathcal{R}(\overline{\mathcal{A}}\psi, \psi) = 0$$

and

$$\mathcal{R}_k(f, \psi) \to \mathcal{R}(f, \psi)$$

weakly in $\mathcal{W}_0$, we have $u_k^1 \to 0$ weakly in $\mathcal{W}_0$. Hence, $\psi_k \to \psi$ weakly in $\mathrm{L}^p(Q)^n$. In the case $\psi = \nabla u$, $u \in \mathcal{W}$, we have $\psi_k = \nabla u_k$, where $u_k = u + u_k^1 \in \mathcal{W}$. It is easy to verify that

$$\mathcal{L}_k u_k = \mathcal{L} u.$$

Moreover, if $u \in \mathcal{W}_0$, then $u_k \in \mathcal{W}_0$, i.e. u_k satisfies zero initial condition.

By setting

$$\overline{\Gamma}^k(\psi) = a^k(t, x, \psi_k)$$

we define the operator

$$\overline{\Gamma}^k : \mathrm{L}^p(Q)^n \longrightarrow \mathrm{L}^{p'}(Q)^n.$$

Inequalities (4.1.19) for $\mathcal{R}_k$, (4.1.22), and (4.1.2) for a^k give rise to the inequality

$$\|\overline{\Gamma}^k(\psi)\|_{p'}^{p'} \leq \bar{c}_0 \|\psi\|_p^p + K(c_1 + c_2), \quad \psi \in \mathrm{L}^p(Q)^n. \tag{4.1.24}$$

Also we set $\Gamma^k(u) = \overline{\Gamma}^k(\nabla u)$ for $u \in \overline{\mathcal{V}}$.

Step 3. Exactly as in the proof of Lemma 2.3.2, one can prove the localization property for $\overline{\mathcal{A}}$. Only one change is in order: one need to use Lemma 4.1.1 instead of Lemma 2.3.1.

Step 4. First of all we observe that the operator $\mathcal{R}_k$ is Hölderian on any ball of $\mathcal{V}^* \times \mathrm{L}^p(Q)^n$, uniformly with respect to k. Indeed, let

$$u_i = \mathcal{R}_k(f_i, \psi_i), \quad i = 1, 2.$$

Then $u_i \in \mathcal{W}_0$ is a solution of the equation

$$\mathcal{L}_k^{\psi_1} u_i = g_i,$$

where $g_1 = f_1$ and

$$g_2 = f_2 + \nabla(a^k(t, x, \psi_2 + \nabla u_2) - a^k(t, x, \psi_1 + \nabla u_2)).$$

Using (4.1.24), (4.1.7) for a^k, and Lemma 1.3.2, we get the required.

Now, by definition of $\overline{\Gamma}^k$, the sequence of operators $\overline{\Gamma}^k$ is equicontinuous on any ball in $\mathrm{L}^p(Q)^n$. By (4.1.24), we may assume, passing to a subsequence, that for any ψ from a dense countable subset of $\mathrm{L}^p(Q)^n$ the sequense $\overline{\Gamma}^k(\psi)$ is weakly convergent in $\mathrm{L}^{p'}(Q)^n$. By equicontinuity of $\overline{\Gamma}^k$, the last assertion still holds true for all $\psi \in \mathrm{L}^p(Q)^n$. Hence, there exists an operator

$$\overline{\Gamma} : \mathrm{L}^p(Q) \longrightarrow \mathrm{L}^{p'}(Q)^n$$

such that $\overline{\Gamma}^k(\psi) \longrightarrow \overline{\Gamma}(\psi)$ weakly in $\mathrm{L}^{p'}(Q)^n$ for any $\psi \in \mathrm{L}^p(Q)^n$. The operator $\overline{\Gamma}$ is local and the following inequalities hold true for any $\varphi, \psi \in \mathrm{L}^p(Q)^n$:

$$\|\overline{\Gamma}(\psi)\|_{p'}^{p'} \leq \overline{c}_0\|\psi\|_p^p + K(c_1 + c_2), \tag{4.1.25}$$

$$\|\overline{\Gamma}(\varphi) - \overline{\Gamma}(\psi)\|_{p'}^{p'} \leq \overline{\theta} H_1(\varphi, \psi)^{1-\overline{s}/p}\|\varphi - \psi\|_{p'}^{\overline{s}}, \tag{4.1.26}$$

$$\int_Q [\overline{\Gamma}(\varphi) - \overline{\Gamma}(\psi)] \cdot (\varphi - \psi) dtdx \geq \kappa H_1(\varphi, \psi)^{1-\beta/p}\|\varphi - \psi\|_p^\beta. \tag{4.1.27}$$

Inequality (4.1.25) follows immediatelly from (4.1.24). To prove (4.1.26) we set

$$y = \overline{\mathcal{A}}\varphi - \overline{\mathcal{A}}\psi, \quad Z^k = \overline{\Gamma}^k\varphi - \overline{\Gamma}^k\psi.$$

Let

$$v_k^1 = \mathcal{R}_k(\varphi, \overline{\mathcal{A}}\varphi), \quad u_k^1 = \mathcal{R}_k(\psi, \overline{\mathcal{A}}\psi)$$

and $z_k = v_k^1 - u_k^1$. It is easy to check that $\langle \partial_t z_k, z_k \rangle \geq 0$. Hence,

$$\langle y, z_k \rangle \geq \int_Q Z^k \cdot (\varphi_k - \psi_k) dtdx - \int_Q Z^k \cdot (\varphi - \psi) dtdx,$$

where

$$\varphi_k = \varphi + \nabla v_k^1, \quad \psi_k = \psi + \nabla u_k^1.$$

Now we may proceed exactly as in the proof of (2.3.21).

Proof of (4.1.27) is similar to that of (2.3.24), using $\langle \partial_t z_k, z_k \rangle \geq 0$.

By definition of $\overline{\Gamma}^k$, we have

$$\int_Q \overline{\Gamma}^k(\psi)\nabla v dtdx = \langle \overline{\mathcal{A}}_k\psi_k, v \rangle = \langle \overline{\mathcal{A}}\psi, v \rangle - \langle \partial_t u_k^1, v \rangle$$

for any $\psi \in \mathrm{L}^p(Q)^n$ and $v \in \mathcal{W}_0$, where u_k^1 is as above. Since $\overline{\Gamma}^k(\psi) \to \overline{\Gamma}(\psi)$ weakly in $\mathrm{L}^p(Q)^n$ and $\partial_t u_k^1 \to 0$ weakly in $\mathcal{V}^*$, we get

$$\overline{\mathcal{A}}\psi = -\mathrm{div}\, \overline{\Gamma}(\psi), \quad \psi \in \mathrm{L}^p(Q)^n.$$

In particular,

$$\mathcal{A}u = -\mathrm{div}\, \Gamma(u),$$

where $\Gamma = \overline{\Gamma} \circ \nabla$, and

$$\mathcal{L}u = \partial_t u - \mathrm{div}\, \Gamma(u).$$

Step 5. Now we need only to show that

$$\overline{\Gamma}(\psi) = a(t, x, \psi),$$

where

$$a : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

is a Carathéodory function satisfying inequalities like (4.1.2), (4.1.3), and (4.1.20), with suitably changed parameters. This may be done exactly as in Step 5 of the proof of Lemma 2.3.2, using inequalities (4.1.25) – (4.1.27). □

Exactly as in the elliptic case, we have the following statements the proofs of which we leave to the reader.

Lemma 4.1.3 *Let $\mathcal{L}_k \in \Pi_0$ and $\mathcal{L}_k$ strongly G-convergent to $\mathcal{L}$. Then, for any cylindrical subdomain $Q' \subset Q$, we have $\mathcal{L}_{k|Q'} \overset{G}{\Longrightarrow} \mathcal{L}_{|Q'}$.*

Lemma 4.1.4 *Let $\mathcal{L}_k \in \Pi_0$, $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, and $v_k \in \overline{\mathcal{W}}$. Assume that $\mathcal{L}_k v_k = f_k$, $v_k \to u$ weakly in $\overline{\mathcal{W}}$, and $f_k \to f$ weakly in $\mathcal{W}_0^*$, where $f, f_k \in \mathcal{V}^*$. Then $\mathcal{L}u = f$ and $a^k(t,x,\nabla v_k) \to a(t,x,\nabla u)$ weakly in $\mathrm{L}^{p'}(Q)^n$.*

4.1.3 General Parabolic Operators

Now we extend the results of n^0 4.1.2 to general parabolic operators. As in the elliptic case, to do this we need the following comparison result.

Lemma 4.1.5 . *Let*

$$\mathcal{L}_k u = \partial_t u - \operatorname{div} a^k(t,x,\nabla u),$$

$$\mathcal{P}_k u = \partial_t u - \operatorname{div} b^k(t,x,\nabla u)$$

be operators of the class Π_0, $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, and $\mathcal{P}_k \overset{G}{\Longrightarrow} \mathcal{P}$. Assume that

$$|a^k(t,x,\xi) - b^k(t,x,\xi)|^{p'} \le (\gamma_k(t,x) + |\xi|^p)\,\delta_k(t,x), \tag{4.1.28}$$

where $0 \le \gamma_k \in \mathrm{L}^1(Q)$, $\gamma_k \to \gamma$ in $\mathrm{L}^1(Q)$, δ_k is a bounded sequence in $\mathrm{L}^\infty(Q)$ such that $\delta_k \to \delta$ a.e. on Q. Then

$$|a(t,x,\xi) - b(t,x,\xi)|^{p'} \le \theta \cdot \left(\gamma(t,x) + \overline{h}(t,x) + |\xi|^p\right) \times \\ \times \left(\delta^{s/\beta}(t,x) + \delta(t,x)\right), \tag{4.1.29}$$

where $\overline{h}(t,x) = c(t,x) + h(t,x)$.

Proof. Exactly as in the proof of Lemma 2.3.5, we may assume that $\gamma_k = \gamma$, $\delta_k = \delta$, and δ is a picewise constant function, with cylindrical foots of steps. Moreover, by Lemma 4.1.3, one may assume that δ is a constant function.

Setting $\psi \equiv \xi$ we consider $\psi_k = \psi + \nabla u_k^1$ defined by (4.1.23), and $\varphi_k = \psi + \nabla v_k^1$ defined in the same way, with $\mathcal{L}_k$ and $\mathcal{L}$ replaced by $\mathcal{P}_k$ and $\mathcal{P}$, respectively. We have

$$\mathcal{L}_k^\psi(u_k^1) = \overline{\mathcal{A}}\psi, \quad \mathcal{P}_k^\psi(v_k^1) = \overline{\mathcal{B}}\psi.$$

Set $y = \overline{\mathcal{A}}\psi - \overline{\mathcal{B}}\psi$ and $z_k = u_k^1 - v_k^1$. Then we obtain the following evident identity

$$\int_Q \Big[a^k(t,x,\psi_k) - b^k(t,x,\varphi_k)\Big] \cdot \nabla z_k dtdx + \langle \partial_t z_k, z_k\rangle = \langle y, z_k\rangle.$$

Since $z_k \in \mathcal{W}_0$, we have $\langle \partial_t z_k, z_k\rangle \geq 0$. Hence,

$$\langle y, z_k\rangle \geq \int_Q \Big[a^k(t,x,\psi_k) - a^k(t,x,\varphi_k)\Big] \cdot \nabla z_k dtdx + \\ + \int_Q \Big[a^k(t,x,\varphi_k - b^k(t,x,\varphi_k)\Big] \cdot \nabla z_k dtdx.$$

Now to complete the proof it is sufficient to repeat the arguments used in the proof of Lemma 2.3.5. □

Remark 4.1.3 Under the condition of Lemma 4.1.5 let us consider, additionally, two sequences of Carathéodory functions

$$a_0^k : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

and

$$b_0^k : Q \times \mathbf{R}^n \longrightarrow \mathbf{R}$$

satisfying inequalities like (4.1.2), (4.1.7). Using the notations introduced in the proof of Lemma 4.1.5 suppose that

$$a_0^k(t,x,\psi_k) \to a_0(t,x,\psi)$$

and

$$b_0^k(t,x,\varphi_k) \to b_0(t,x,\psi)$$

weakly in $\mathrm{L}^p(Q)$ for any $\psi \in \mathrm{L}^p(Q)^n$, with appropriate functions a_0 and b_0. If we assume (4.1.2) to be valid for the difference $a_0^k - b_0^k$, then $a_0 - b_0$ satisfies inequality of the type (2.1.25).

Now we are able to state the main results of G-convergence theory for general parabolic operators.

Theorem 4.1.1 *Let $\mathcal{L}_k \in \Pi(c_0, c, \kappa, h, \theta, s, \nu, \beta)$. Then there exist an operator $\mathcal{L} \in \Pi(\overline{c}_0, \overline{c}, \overline{\kappa}, \overline{h}, \overline{\theta}, \overline{s}, \overline{\nu}, \overline{\beta})$ and a subsequence $\mathcal{L}_{k'}$ such that $\mathcal{L}_{k'}$ strongly G-converges to $\mathcal{L}$, where*

$$\overline{s} = \frac{sp}{\beta p - sp + s}, \quad \overline{\nu} = \nu^{s/p}, \quad \overline{c} = K\overline{h}, \quad \overline{h} = c + h,$$

and the positive constants $\overline{c}_0, \overline{\kappa}, \overline{\theta}$ and K depend on c_0, κ, θ only.

The proof is quite similar to that of Theorem 2.3.1, using Lemma 4.1.5 instead of Lemma 2.3.5. Exactly as in the elliptic case, we have the following localization property:

Theorem 4.1.2 *Let $\mathcal{L}_k \in \Pi$ and $\mathcal{L}_k$ strongly G-convergent to $\mathcal{L}$. Then, for any cylindrical open subset $Q' \subset Q$, we have*

$$\mathcal{L}_{k|Q'} \overset{G}{\Longrightarrow} \mathcal{L}_{|Q'}.$$

The next result is the so-called property of convergence of arbitrary solutions:

Theorem 4.1.3 . *Let $\mathcal{L}_k \in \Pi$, $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, $v_k \in \overline{\mathcal{W}}$, and $f, f_k \in \mathcal{V}^*$. Assume that $\mathcal{L}_k v_k = f_k$, $v_k \to v$ weakly in $\overline{\mathcal{W}}$, and $f_k \to f$ strongly in $\mathcal{W}_0^*$. Then $\mathcal{L}u = f$,*

$$a^k(t, x, v_k, \nabla v)k) \to a(t, x, u, \nabla u),$$

and

$$a_0^k(t, x, v_k, \nabla v_k) \to a_0(t, x, u, \nabla u)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively.

Proof. First of all, the localization property permits us to assume that ∂Q_0 is smooth (recall that $Q = Q_0 \times (0, T)$). Then, by the Sobolev Embedding Theorem, $v_k \to u$ strongly in $\mathrm{L}^p(Q)$.

Let us consider the operators

$$\mathcal{L}_k^{(1),v_k} w = \partial_t w - \operatorname{div} a^k(t, x, v_k, \nabla w)$$

and

$$\mathcal{L}^{(1),u} w = \partial_t w - \operatorname{div} a(t, x, u, \nabla w).$$

Exactly as in the proof of Theorem 2.3.3, one can show that

$$\mathcal{L}_k^{(1),v_k} \overset{G}{\Longrightarrow} \mathcal{L}^{(1),u}.$$

Moreover, let $u_k \in \overline{\mathcal{W}}$ be a unique solution of the equation

$$\mathcal{L}_k^{(1),v_k} u_k = \mathcal{L}^{(1),u} u \tag{4.1.30}$$

such that $u_k - u \in \mathcal{W}_0$. Then

$$a^k(t,x,v_k,\nabla u_k) \to a(t,x,u,\nabla u),$$

$$a_0^k(t,x,v_k,\nabla u_k) \to a(t,x,u,\nabla u)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively (cf. (2.3.37) and (2.3.38)).

Now we have

$$\mathcal{L}_k^{(1),v_k} v_k = f_k - g_k, \tag{4.1.31}$$

where $g_k = a_0^k(t,x,v_k,\nabla v_k)$. It is easy that $\{g_k\}$ is bounded in $\mathrm{L}^{p'}(Q)$. Hence, passing to a subsequence one can assume that $g_k \to g$ weakly in $\mathrm{L}^{p'}(Q)$. Since the space $\mathcal{W}_0$ is embedded compactly into $\mathrm{L}^p(Q)$, the emmbedding $\mathrm{L}^{p'}(Q) \subset \mathcal{W}_0^*$ is also compact. Therefore, $g_k \to g$ strongly in $\mathcal{W}_0^*$.

Set

$$Z^k = a^k(t,x,v_k,\nabla v_k) - a^k(t,x,v_k,\nabla u_k),$$

$$z_k = v_k - u_k,$$

and choose a function $\varphi \in C_0^\infty(Q)$ such that $0 \le \varphi \le 1$. A simple calculation (using (4.1.30) and (4.1.31)) implies

$$\langle f_k - g_k, \varphi^2 z_k\rangle - \langle \mathcal{L}^{(1),u} u, \varphi^2 z_k\rangle + \langle (\partial_t\varphi) z_k, \varphi z_k\rangle -$$

$$- \int_Q Z^k \cdot (\nabla\varphi^2) z_k dtdx = \langle \partial_k(\varphi z_k), \varphi z_k\rangle + \int_Q Z^k \cdot (\nabla z_k)\varphi^2 dtdx.$$

Since φz_k is compactly supported, we see that $\langle \partial_t(\varphi z_k), \varphi z_k\rangle = 0$. Therefore, by (4.1.3), we have

$$\langle f_k - g_k\rangle, \varphi^2 z_k\rangle - \langle \tilde{f}, \varphi^2 z_k\rangle + \langle (\partial_t\varphi) z_k, \varphi z_k\rangle - \int_Q Z^k \cdot (\nabla\varphi^2) z_k dtdx \ge$$

$$\ge C\|\varphi^2 \nabla z_k\|_p^\beta,$$

where $\tilde{f} = \mathcal{L}^{(1),u} u$. Now it is not difficult to see that the left-hand side of the last inequality tends to zero. Hence, we claime that $z_k \to 0$ strongly in $\mathcal{V}_{loc} = \mathrm{L}^p_{loc}(0,T;W^{1,p}_{loc}(Q_0))$.

Using the last statement we can complete the proof exactly as in the proof of Theorem 2.3.3. □

Remark 4.1.4 Evidently, the statement of Lemma 4.1.5 may be extended straightforwardly to the case of general parabolic operators of the class Π.

4.1.4 Further Results

Now we will explain the connection between strong G-convergence for elliptic and parabolic operators. However, before to do this, we present a statement which concerns double sequences of parabolic operators and follows directly from Lemma 4.1.5 (cf. Corollary 1.3.1). For any two operators

$$\mathcal{L}u = \partial_t u - \operatorname{div} a(t, x, u, \nabla) + a_0(t, x, u, \nabla u)$$

and

$$\mathcal{P}u = \partial_t u - \operatorname{div} b(t, x, u, \nabla u) + b_0(t, x, u, \nabla u)$$

of the class Π, we introduce the quantity

$$d(\mathcal{L}, \mathcal{P}) = \operatorname*{ess\,sup}_{(t,x,\zeta)\in Q\times\mathbf{R}^{n+1}} \left[\frac{|a(t,x,\zeta) - b(t,x,\zeta)|^{p'}}{c(t,x) + c_0|\zeta|^p} + \right.$$
$$\left. + \frac{|a_0(t,x,\zeta) - b_0(t,x,\zeta)|^{p'}}{c(t,x) + c_0|\zeta|^p} \right].$$

Let us consider parabolic operators

$$\mathcal{L}^l_k u = \partial_t u - \operatorname{div} a^k_l(t, x, u, \nabla u) + a^k_{0,l}(t, x, u, \nabla u),$$

$$\mathcal{L}^l u = \partial_t u - \operatorname{div} a_l(t, x, u, \nabla u) + a_{0,l}(t, x, u, \nabla u),$$

and

$$\mathcal{L}_k u = \partial_t u - \operatorname{div} a^k(t, x, u, \nabla u) + a^k_0(t, x, u, \nabla u).$$

Proposition 4.1.2 *Assume that $d(\mathcal{L}^l_k, \mathcal{L}_k) \to 0$, uniformly with respect to k, and $d(\mathcal{L}^l, \mathcal{L}) \to 0$, as $l \to \infty$. If $\mathcal{L}^l_k \overset{G}{\Longrightarrow} \mathcal{L}^l$, as $k \to \infty$, for any $l \in \mathbf{N}$, then $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$.*

Proof. Passing to a subsequence we can assume that there exists $\hat{\mathcal{L}} \in \Pi$ such that $\mathcal{L}_k \overset{G}{\Longrightarrow} \hat{\mathcal{L}}$. Applying Lemma 4.1.5 and Remark 4.1.4 to $\mathcal{L}_k$ and $\mathcal{L}^l_k$ we see that $\mathcal{L} = \hat{\mathcal{L}}$. □

Now we assume that the functions $c(t,x)$ and $h(t,x)$ do not depend on t.

Theorem 4.1.4 *Let $\mathcal{L}_k \in \Pi$ be a sequence of parabolic operators such that*

$$\operatorname*{ess\,sup}_{(x,\zeta)\in Q_0\times\mathbf{R}^{n+1}} \frac{|a^k(t+\Delta t, x, \zeta) - a^k(t,x,\zeta)|^{p'}}{c(x) + c_0(x)|\zeta|^p} \to 0,$$

$$\operatorname*{ess\,sup}_{(x,\zeta)\in Q_0\times\mathbf{R}^{n+1}} \frac{|a^k_0(t+\Delta t, x, \zeta) - a^k_0(t,x,\zeta)|^{p'}}{c(x) + c_0|\zeta|^p} \to 0,$$

as $\Delta t \to 0$, uniformly with respect to $t \in [0,T]$ and $k \in \mathbf{N}$. Assume that, for the elliptic parts, we have $\mathcal{A}^k(t) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}(t)$ for any $t \in [0,T]$. Then $\mathcal{L}_k \overset{G}{\Longrightarrow} \hat{\mathcal{L}}$, where

$$\hat{\mathcal{L}}u = \partial_t u + \hat{\mathcal{A}}(t)u.$$

Proof. By Theorem 4.1.1, we can assume that $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, where

$$\mathcal{L}u = \partial_t u + \mathcal{A}(t)u = \partial_t u - \operatorname{div} a(t,x,u,\nabla u) + a_0(t,x,u,\nabla u).$$

We need only to show that $\mathcal{A} = \hat{\mathcal{A}}$.

First, we consider the case when $\mathcal{A}^k(t)$ does not depend on t. By definition of strong G-convergence of elliptic operators, we have the following. For any $v \in V = W_0^{1,p}(Q_0)$ and for any $u \in V$, let us consider the unique solution $u_k \in V$ of the equation

$$-\operatorname{div} a^k(x,v,\nabla u_k) = -\operatorname{div} \hat{a}(x,u,\nabla u).$$

Then $u_k \to u$ weakly in V,

$$a^k(x,v,\nabla u_k) \to \hat{a}(x,v,\nabla u), \tag{4.1.32}$$

and

$$a_0^k(x,v,\nabla u_k) \to \hat{a}_0(x,v,\nabla u) \tag{4.1.33}$$

weakly in $\mathrm{L}^{p'}(Q_0)^n$ and $\mathrm{L}^{p'}(Q_0)$, respectively.

Considering u_k and u as constant functions in the variable t, we see that $u_k \to u$ weakly in $\mathcal{W}$. Evidently, we have

$$\mathcal{L}_k^1(u_k, v) = -\operatorname{div} \hat{a}(x,v,\nabla u).$$

Let

$$\mathcal{L}_k(u,v) = \partial_t u - \operatorname{div} a^k(x,v,\nabla u) + a_0^k(x,v,\nabla u).$$

Then

$$\mathcal{L}_k(u_k, v) = g_k, \tag{4.1.34}$$

where

$$g_k = -\operatorname{div} \hat{a}(x,v,\nabla u) + a_0^k(x,v,\nabla u_k).$$

In view of (4.1.33),

$$g_k \to g = -\operatorname{div} \hat{a}(x,v,\nabla u) + \hat{a}_0(x,v,\nabla u)$$

strongly in $\mathcal{W}_0^*$ (recall that the embedding $\mathrm{L}^{p'}(Q) \subset \mathcal{W}_0^*$ is compact). Moreover, since $\mathcal{L}_k \overset{G}{\Longrightarrow} \mathcal{L}$, it is not difficult to see that $\mathcal{L}_k(\cdot, v) \overset{G}{\Longrightarrow} \mathcal{L}(\cdot, v)$ for any $v \in \mathcal{V}$. Now (4.1.34) and Theorem 4.1.3 imply that

$$\mathcal{L}(u,v) = g,$$

$$a^k(x,v,\nabla u_k) \to a(t,x,v,\nabla u),$$

and

$$a_0^k(x, v, \nabla u_k) \rightarrow a_0(t, x, v, \nabla u)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. Using (4.1.32) and (4.1.33) we see that

$$a(t, x, v, \nabla u) = \hat{a}(x, v, \nabla u),$$

$$a_0(t, x, v, \nabla u) = \hat{a}_0(x, v, \nabla u).$$

Thus, we have proved the theorem for the autonomous case.

To prove the required in the full generality we define new operators $\mathcal{L}_k^l$ by setting

$$a_l^k(t, x, \zeta) = a^k(h \cdot s, x, \zeta),$$

$$a_{0,l}^k(t, x, \zeta) = a_0^k(h \cdot s, x, \zeta),$$

for $t \in [h \cdot s, h \cdot (s+1))$, where $h = T/l$. By the localization property and the statement we have just proved, $\mathcal{L}_k^l \overset{G}{\Longrightarrow} \hat{\mathcal{L}}^l$, where $\hat{\mathcal{L}}^l$ is constructed in the similar way as $\mathcal{L}_k^l$. Now to complete the proof we need only to apply Proposition 4.1.2 (cf. the proof of Theorem 1.3.5). □

4.2 Homogenization

4.2.1 Setting of the Problem

Now we want to study the homogenization problem for parabolic operators. More precisely, we consider the family of operators of the form

$$\mathcal{L}_\varepsilon u = \partial_t u - \operatorname{div} a(\varepsilon^{-\alpha} t, \varepsilon^{-\beta} x, u, \nabla u) + a_0(\varepsilon^{-\alpha} t, \varepsilon^{-\beta} x, u, \nabla u), \tag{4.2.1}$$

where $\varepsilon > 0, \alpha \geq 0, \beta \geq 0$, and $\alpha + \beta > 0$. Throughout what follows we assume that $a(\tau, y, \xi_0, \xi)$ and $a_0(\tau, y, \xi_0, \xi)$ are Carathéodory functions satisfying inequalities (4.1.2), (4.1.3), and (4.1.7), with (t, x) replaced by (τ, y). We suppose also inequality (4.1.4) to be satisfied.

As usually, we say that the family $\mathcal{L}_\varepsilon$ admits homogenization if there exists a parabolic operator $\hat{\mathcal{L}}$ such that $\mathcal{L}^\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$, as $\varepsilon \rightarrow 0$, for any cylindrical open set $Q = (0, T) \times Q_0$, where $T > 0$ and $Q_0 \subset \mathbf{R}^n$ is an open bounded set.

Depending on the value of the ratio α/β we shall distinguish the following five cases:

1) $\alpha = 2\beta$ – *self-similar homogenization*

2) $\alpha > 2\beta$ – *non self-similar homogenization*

3) $\alpha < 2\beta$ – *non self-similar homogenization*

4) $\alpha = 0$ – *spatial homogenization*

5) $\beta = 0$ – *time homogenization*

In the cases 1 – 3, we assume the functions c and h to be constant, while $a(\tau, y, \xi_0, \xi)$ and $a_0(\tau, y, \xi_0, \xi)$ to be 1-periodic in each variable τ and y_i, $i = 1, 2, \dots, n$. In the case 4, it is assumed that c and h may depend only on t and a, a_0 are 1-periodic in y_i, $i = 1, 2, \dots, n$; $T > 0$ is fixed. Finally, in the case 5, we assume that c and h do not depend on t and a, a_0 are 1-periodic in the time variable only; the spatial domain Q_0 is fixed. Evidently, one can assume, without loss of generality, that $\beta = 1$ in the cases 1 – 4, while $\alpha = 1$ in the case 5.

In all these cases we shall prove that $\mathcal{L}_\varepsilon$ admits homogenization. However, the construction of $\hat{\mathcal{L}}$ depends essentially on the kind of homogenization we consider. It shoud be pointed out that $\hat{\mathcal{L}}$ is a translation invariant operator in the cases 1 – 3 listed above. In the case 4, $\hat{\mathcal{L}}$ is translation invariant with respect to spatial variables, while in the case 5 the operator $\hat{\mathcal{L}}$ is autonomous.

The key tool is the investigation of various auxiliary equations for a periodic unknown function. One of them reads

$$\mu \partial_\tau w = \operatorname{div}_y a(\tau, y, \xi_0, \xi + \nabla_y w), \tag{4.2.2}$$

where $\mu > 0$ is an additional parameter; its value will be specified in accordance to the kind of homogenization. Equation (4.2.2) and all further auxiliary equations are regarded in the sense of distributions.

To study such auxiliary equations we need to introduce some spaces of periodic functions. Denote by $\langle u \rangle$ the mean value of a periodic function u. We shall write $\langle u \rangle_x$ if the mean value is regarded with respect to a specific variable x.

Denote by H_{per} the space of all functions $v \in \mathrm{L}^2_{loc}(\mathbf{R}^n)$ such that $v(y)$ is $\square$-periodic and $\langle v \rangle_y = 0$; $\square$ stands for the unit cube in $\mathbf{R}^n$ centered at 0 and $\square$-periodic means 1-periodic in each variable y_i, $i = 1, 2, \dots, n$. H_{per} is a separable Hilbert space with respect to the inner product

$$\langle v, w \rangle = \int_\square v(y) w(y) dy.$$

By V_{per} we denote the space which consists of all functions $v \in W^{1,p}_{loc}(\mathbf{R}^n)$ such that $v(y)$ is $\square$-periodic and $\langle v \rangle_y = 0$. Endowed with the norm

$$\|v\|_{V_{per}} = \left(\int_\square |\nabla_y v(y)|^p dy \right)^{1/p},$$

V_{per} is a separable reflexive Banach space. As usually, its dual space will be denoted by V_{per}^*.

Now we introduce the Hilbert space $\mathcal{H}_{per}$ which consists of all functions $v \in \mathrm{L}^2_{loc}(\mathbf{R}^{n+1})$ such that $v(\tau, y)$ is 1-periodic in τ, $\Box$-periodic in y, and $\langle v \rangle_{\tau,y} = 0$. Obviously, $\mathcal{H}_{per}$ is a Hilbert space. For any Banach space E, we denote by $\mathrm{L}^p_{per}(\mathbf{R}; E)$ the subspace of $\mathrm{L}^p_{loc}(\mathbf{R}; E)$ which consists of all 1-periodic functions. This is a Banach space with respect to the norm

$$\|f\| = \|f\|_{\mathrm{L}^p(0,1;E)}.$$

We set

$$\mathcal{V}_{per} = \{v(\tau, y) \in \mathrm{L}^p_{per}(\mathbf{R}; V_{per}) : \langle v(\tau, y) \rangle_{\tau,y} = 0\};$$

$\mathcal{V}_{per}$ is a reflexive Banach space with the dual $\mathcal{V}_{per}^*$. Set also

$$\mathcal{W}_{per} = \{v \in \mathcal{V}_{per} : \partial_\tau v \in \mathcal{V}_{per}^*\}.$$

Endowed with the graph norm, $\mathcal{W}_{per}$ is a reflexive Banach space and its dual is denoted by $\mathcal{W}_{per}^*$. It is not difficult to see that $\mathcal{W}_{per}$ is embedded continuously into the space of all continuous 1-periodic functions on $\mathbf{R}$ with values in H_{per}.

We have the following simple

Lemma 4.2.1 *Given $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$ there exists a unique solution $w = w_\mu \in \mathcal{W}_{per}$ of equation (4.2.2). Moreover,*

$$\|w_\mu\|_{\mathcal{V}_{per}} \leq C,$$

where $C > 0$ does not depend on $\mu > 0$.

Proof. Let us define the operator

$$\mathbf{A} = \mathbf{A}_{\xi_0,\xi} : \mathcal{W}_{per} \longrightarrow \mathcal{W}_{per}^*$$

by the formula

$$\langle \mathbf{A}w, v \rangle = \int_0^1 \int_\Box a(\tau, y, \xi_0, \xi + \nabla_y w) \cdot \nabla v d\tau dy$$

for any $v, w \in \mathcal{W}_{per}$. It is easy to verify that the operator $\mathbf{A}$ is bounded, continuous, strictly monotone, and coercive. Equation (4.2.2) may be rewritten as

$$\mu \partial_\tau w + \mathbf{A}w = 0.$$

Applying standard solvability results (see, e.g. [200]) we get the existence and uniqueness of w_μ. (It should be pointed out that, since any periodic test function is of the form $const + \varphi$, where $\langle \varphi \rangle = 0$, our operator framework serves equation (4.2.2) in the sense of distributions). Since

$$\langle \mu \partial_\tau w, w \rangle = 0,$$

we have

$$\langle \mathbf{A} w_\mu, w_\mu \rangle = 0.$$

This and the coerciveness of $\mathbf{A}$ imply the last statement of the proposition. $\square$

We need also the following straightforward generalization of the well-known property of periodic functions ([164], see, also, Lemma 2.1.3 (i)).

Lemma 4.2.2 *Let $f \in \mathrm{L}^p_{loc}(\mathbf{R}^n)$, $1 < p < \infty$, be a periodic function. Then*

$$f(\varepsilon_1^{-1} x_1, \ldots, \varepsilon_n^{-1} x_n) \to \langle f \rangle$$

weakly in $\mathrm{L}^p_{loc}(\mathbf{R}^n)$ as $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \to 0$ in $\mathbf{R}^n$.

The proof is essentially the same as in the standard situation [164].

4.2.2 Self-Similar Case

We treat here the homogenization problem for the family of operators $\mathcal{L}_\varepsilon$ defined by (4.2.1) in the self-similar case $\alpha = 2$, $\beta = 1$. Let us consider the solution $w(\tau, y) \in \mathcal{W}_{per}$ of equation (4.2.2), with $\mu = 1$; it exists by Lemma 4.2.1. Of course, w depends on $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$. We set

$$\left.\begin{aligned} \hat{a}(\xi_0, \xi) &= \langle a(\tau, y, \xi_0, \xi + \nabla_y w(\tau, y)) \rangle_{\tau, y}, \\ \hat{a}_0(\xi_0, \xi) &= \langle a_0(\tau, y, \xi_0, \xi + \nabla_y w(\tau, y)) \rangle_{\tau, y}, \end{aligned}\right\} \tag{4.2.3}$$

and then define the operator $\hat{\mathcal{L}}$ by

$$\hat{\mathcal{L}} u = \partial_t u - \operatorname{div} \hat{a}(u, \nabla u) + \hat{a}_0(u, \nabla u). \tag{4.2.4}$$

Analyzing the dependence of w on (ξ_0, ξ), one can prove that $\hat{\mathcal{L}}$ belongs to a suitable class Π. However, we do not use this result, since it will be derived as by-product of the proof of the homogenization theorem.

Theorem 4.2.1 *Assume that $\alpha = 2\beta$. Then there exists a parabolic operator $\hat{\mathcal{L}}$ such that for any cylindrical open set $Q = (0, T) \times Q_0$, we have $\hat{\mathcal{L}} \in \Pi$ and $\mathcal{L}_\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$. Moreover, $\hat{\mathcal{L}}$ is defined by (4.2.4), (4.2.4).*

Proof. Fix a cylindrical open set $Q = (0, T) \times Q_0$. By the localization property (Theorem 4.1.2), we may assume that T is an integer and Q_0 is a cube of an integer edge length, centered at the origin. Theorem 4.1.1 implies that there exists a parabolic operator

$$\hat{\mathcal{L}} u = \partial_t u - \operatorname{div} \tilde{a}(t, x, u, \nabla u) + \tilde{a}_0(t, x, u, \nabla u)$$

of the class Π such that $\mathcal{L}_\varepsilon \stackrel{G}{\Longrightarrow} \tilde{\mathcal{L}}$, where ε runs along a subsequence which tends to zero. All we need now is to prove that

$$\tilde{a}(t, x, \xi_0, \xi) = \hat{a}(\xi_0, \xi)$$

and

$$\tilde{a}_0(t, x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi).$$

Let $w(\tau, y) \in \mathcal{W}_{per}$ be a unique solution of equation (4.2.2), with $\mu = 1$. We set

$$w^\varepsilon(t, x) = \varepsilon w(\varepsilon^{-2}t, \varepsilon^{-1}x).$$

Equation (4.2.2) implies

$$\partial_t w^\varepsilon - \operatorname{div} a(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) = 0. \tag{4.2.5}$$

We have

$$\|w^\varepsilon\|^p_{p,Q} = \int_Q |\varepsilon w(\varepsilon^{-2}t, \varepsilon^{-1}x)|^p dt dx = \varepsilon^{n+2} \int_{Q_\varepsilon} |\varepsilon w(\tau, y)|^p d\tau dy,$$

where

$$Q_\varepsilon = \{(\tau, y) : (\varepsilon^2\tau, \varepsilon y) \in Q\}.$$

The set Q_ε is contained in the union

$$\tilde{Q}_\varepsilon = \bigcup_{j=1}^{N_\varepsilon} (z_j + \square'),$$

where $\square' = (0,1) \times \square$, $z_j = \mathbf{Z}^n$ for $j = 1, \dots, N_\varepsilon$, and N_ε is of order $\sim \varepsilon^{-(n+2)}$. Therefore, using the periodicity of w we get

$$\|w^\varepsilon\|^p_{p,Q} \le \varepsilon^{n+2} \sum_{j=1}^{N_\varepsilon} \int_{z_j + \square'} |\varepsilon w(\tau, y)|^p d\tau dy \le C \int_{\square'} |\varepsilon w(\tau, y)|^p d\tau dy.$$

Thus,

$$\|w^\varepsilon\|_{p,Q} \le C \cdot \varepsilon. \tag{4.2.6}$$

In the similar manner, we have

$$\|\nabla w^\varepsilon\|_{p,Q} \le C. \tag{4.2.7}$$

In particular,

$$\|w^\varepsilon\|_{\overline{\mathcal{V}}} \le C.$$

Together with (4.2.5), the last inequality implies

$$\|\partial_t w^\varepsilon\|_{\mathcal{V}^*} \le C.$$

Thus, w^ε is a bounded (hence, weakly precompact) sequence in $\overline{\mathcal{W}}$. Therefore, by (4.2.6), $w^\varepsilon \to 0$ strongly in $\mathrm{L}^p(Q)$ and weakly in $\overline{\mathcal{W}}$.

For $\zeta = (\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$, let

$$\mathcal{L}_\varepsilon^\zeta u = \partial_t u - \operatorname{div} a\left(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla u\right) + a_0\left(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla u\right).$$

Then $\mathcal{L}_\varepsilon^\zeta \xrightarrow{G} \tilde{\mathcal{L}}^\zeta$, where $\tilde{\mathcal{L}}^\zeta$ is defined in the similar way, with $\mathcal{L}_\varepsilon$ replaced by $\tilde{\mathcal{L}}$. Equation (4.2.5) implies easily

$$\mathcal{L}_\varepsilon^\zeta w^\varepsilon = f_\varepsilon,$$

where

$$f_\varepsilon = a_0\left(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w_\varepsilon\right).$$

By (4.2.7), f_ε is bounded in $\mathrm{L}^{p'}(Q)$. Hence, we can assume that $f_\varepsilon \to f$ weakly in $\mathrm{L}^{p'}(Q)$. Since the embedding $\mathcal{W}_0 \subset \mathrm{L}^p(Q)$ is compact, $f_\varepsilon \to f$ strongly in $\mathcal{W}_0^*$. Now Theorem 4.1.3 implies that $u = 0$ is a solution of

$$\tilde{\mathcal{L}}^\zeta u = f,$$

and

$$a(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) \to \tilde{a}(t, x, \xi_0, \xi),$$

$$a_0(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) \to \tilde{a}_0(t, x, \xi_0, \xi)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. On the other hand,

$$a(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) = a(\tau, y, \xi_0, \xi + \nabla_y w)_{|\tau=\varepsilon^{-2}t, y=\varepsilon^{-1}x},$$

$$a_0(\varepsilon^{-2}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) = a_0(\tau, y, \xi_0, \xi + \nabla_y w)_{|\tau=\varepsilon^{-2}t, y=\varepsilon^{-1}x},$$

and, by Lemma 4.2.2, the last quantities tend to the mean values

$$\langle a(\tau, y, \xi_0, \xi + \nabla_y w)\rangle$$

and

$$\langle a_0(\tau, y, \xi_0, \xi + \nabla_y w)\rangle$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. Thus,

$$\tilde{a}(t, x, \xi_0, \xi) = \hat{a}(\xi_0, \xi),$$

$$\tilde{a}_0(t, x, \xi_0, \xi) = \hat{a}_0(\xi_0, \xi),$$

and the proof is complete. □

4.2.3 Non Self-Similar Cases

First, we consider the case $\alpha < 2$, $\beta = 1$. To construct the homogenized operator we need to study the limit case, as $\mu \to 0$, of auxiliary equation (4.2.2):

$$-\operatorname{div} a(\tau, y, \xi_0, \xi + \nabla_y w) = 0. \tag{4.2.8}$$

Let $w_\mu \in \mathcal{W}_{per}$ be a unique solution of (4.2.2). Then we have

Lemma 4.2.3 *Given $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$ there exists a unique solution $w_0 \in \mathcal{V}_{per}$ of (4.2.8). Moreover, $w_\mu \to w_0$ strongly in $\mathcal{V}_{per}$, as $\mu \to 0$.*

Proof. Since equation (4.2.8) may be written as $\mathbf{A}w = 0$, with the operator $\mathbf{A}$ introduced in the proof of Lemma 4.2.2, the existence and uniqueness of w_0 follows immediately from the standard results on monotone operators (see, e.g., n^0 1.1.1, or [200]).

Let $w_{0,k} \in \mathcal{W}_{per}$ be a sequence such that $w_{0,k} \to w_0$ strongly in $\mathcal{V}_{per}$. Such a sequence exists, since $\mathcal{W}_{per}$ is dense in $\mathcal{V}_{per}$. Since w_μ is bounded in $\mathcal{W}^*_{per}$, inequality (4.1.3) implies easily

$$\|w_\mu - w_{0,k}\|^p_{\mathcal{V}_{per}} \le K\langle \mathbf{A}w_\mu - \mathbf{A}w_{0,k}, w_\mu - w_{0,k}\rangle.$$

Taking into account that $\langle \partial_\tau v, v\rangle = 0$ for any $v \in \mathcal{W}_{per}$ and w_μ is a solution of (4.2.2), we have

$$\begin{aligned}
\|w_\mu - w_{0,k}\|^\beta_{\mathcal{V}_{per}} &\le K\langle \mu\partial_\tau(w_\mu - w_{0,k}) + \mathbf{A}w_\mu - \mathbf{A}w_{0,k}, w_\mu - w_{0,k}\rangle = \\
&= K\langle -\mu\partial_\tau w_{0,k} - \mathbf{A}w_{0,k}, w_\mu - w_{0,k}\rangle \le \\
&\le K\left(\mu\|\partial_\tau w_{0,k}\|_{\mathcal{V}^*_{per}} + \|\mathbf{A}w_{0,k}\|_{\mathcal{V}^*_{per}}\right)\|w_\mu - w_{0,k}\|_{\mathcal{V}_{per}}.
\end{aligned}$$

This and the identity $\mathbf{A}w_0 = 0$ imply

$$\|w_\mu - w_{0,k}\|_{\mathcal{V}_{per}} \le K\left(\mu\|\partial_\tau w_{0,k}\|_{\mathcal{V}^*_{per}} + \|\mathbf{A}w_{0,k} - \mathbf{A}w_0\|_{\mathcal{V}^*_{per}}\right)^{1/(\beta-1)}$$

Since $w_{0,k} \to w_0$ strongly in $\mathcal{V}_{per}$ and

$$\mathbf{A} : \mathcal{V}_{per} \longrightarrow \mathcal{V}^*_{per}$$

is continuous, we can choose k being large enough so that $\|w_{0,k} - w_0\|_{\mathcal{V}_{per}}$ and $\|\mathbf{A}w_{0,k} - \mathbf{A}w_0\|_{\mathcal{V}^*_{per}}$ are small. Then, given $\delta > 0$ we have

$$\|w_\mu - w_{0,k}\|_{\mathcal{V}_{per}} < \delta,$$

for all sufficiently small $\mu > 0$. Hence,

$$\|w_\mu - w_0\|_{\mathcal{V}_{per}} \le \|w_\mu - w_{0,k}\|_{\mathcal{V}_{per}} + \|w_{0,k} - w_0\|_{\mathcal{V}_{per}} < 2\delta$$

and we conclude. □.

Now let us define the functions $\hat{a}(\xi_0, \xi)$ and $\hat{a}_0(\xi_0, \xi)$ by

$$\left.\begin{aligned} \hat{a}(\xi_0, \xi) &= \langle a(\tau, y, \xi_0, \xi + \nabla_y w_0(\tau, y))\rangle_{(\tau,y)}, \\ \hat{a}_0(\xi_0, \xi) &= \langle a_0(\tau, y, \xi_0, \xi + \nabla_y w_0(\tau, y))\rangle_{(\tau,y)}, \end{aligned}\right\} \tag{4.2.9}$$

where $w_0(\tau, y)$ is a unique solution of (4.2.8).

Theorem 4.2.2 *Assume that $\alpha < 2\beta$. Then there exists a parabolic operator $\hat{\mathcal{L}}$ such that for any cylindrical open set $Q = (0,T) \times Q_0$ we have $\hat{\mathcal{L}} \in \Pi$ and $\mathcal{L}_\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$. Moreover, the operator $\hat{\mathcal{L}}$ is defined by (4.2.4) and (4.2.9).*

Proof. Starting exactly as in the proof of Theorem 4.2.1, one can assume that Q_0 is a cube of integer size, $T > 0$ is an integer, and $\mathcal{L}_\varepsilon \xRightarrow{G} \tilde{\mathcal{L}} \in \Pi$.

Let us consider the solution $w_\mu(\tau, y)$ of (4.2.2), with $\mu = \varepsilon^{2-\alpha}$. Recall that we assume $\beta = 1$. Setting

$$w^\varepsilon(t, x) = \varepsilon w_\mu(\varepsilon^{-\alpha} t, \varepsilon^{-1} x)$$

we have

$$\partial_t w^\varepsilon - \nabla a(\varepsilon^{-\alpha} t, \varepsilon^{-1} x, \xi_0, \xi + \nabla w^\varepsilon) = 0.$$

As in the proof of Theorem 4.2.1, we see that $w^\varepsilon \to 0$ weakly in $\overline{\mathcal{W}}$ and strongly in $\mathrm{L}^p(Q)$. Moreover,

$$a(\varepsilon^{-\alpha} t, e^{-1} x, \xi_0, \xi + \nabla w^\varepsilon) \to \tilde{a}(t, x, \xi_0, \xi),$$

$$a_0(\varepsilon^{-\alpha} t, e^{-1} x, \xi_0, \xi + \nabla w^\varepsilon) \to \tilde{a}_0(t, x, \xi_0, \xi),$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively.

On the other hand, setting

$$w_0^\varepsilon(t, x) = \varepsilon w_0(\varepsilon^{-\alpha} t, \varepsilon^{-1} x)$$

we have, by Lemma 4.2.2, that the functions

$$a(\varepsilon^{-\alpha} t, \varepsilon^{-1} x, \xi_0, \xi + \nabla w_0^\varepsilon) = a(\tau, y, \xi_0, \xi + \nabla_y w_0)_{|\tau=\varepsilon^{-\alpha} t, y=\varepsilon^{-1} x},$$

$$a_0(\varepsilon^{-\alpha} t, \varepsilon^{-1} x, \xi_0, \xi + \nabla w_0^\varepsilon) = a_0(\tau, y, \xi_0, \xi + \nabla_y w_0)_{|\tau=\varepsilon^{-\alpha} t, y=\varepsilon^{-1} x}$$

converge to the mean values $\langle a(\tau, y, \xi_0, \xi + \nabla_y w_0)\rangle$ and $\langle a_0(\tau, y, \xi_0, \xi + \nabla_y w_0)\rangle$ weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively.

Finally, Lemma 4.2.3 implies that

$$\|w_\mu - w_0\|_{\mathcal{V}_{per}} \to 0, \quad \text{as } \mu \to 0,$$

and we conclude that

$$\|w^\varepsilon - w_0^\varepsilon\|_{\mathcal{V}} \to 0, \quad \text{as } \varepsilon \to 0.$$

By inequality (4.1.7), it follows that

$$a(\varepsilon^{-\alpha}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) - a(\varepsilon^{-\alpha}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w_0^\varepsilon) \to 0,$$

$$a_0(\varepsilon^{-\alpha}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w^\varepsilon) - a_0(\varepsilon^{-\alpha}t, \varepsilon^{-1}x, \xi_0, \xi + \nabla w_0^\varepsilon) \to 0$$

in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. As consequence, we claim that $\tilde{a} = \hat{a}$ and $\tilde{a}_0 = \hat{a}_0$. The proof is complete. □

Now we consider the case $\alpha > 2$, $\beta = 1$. First of all we look at the asymptotic behaviour of the solution w_μ to equation (4.2.2), as $\mu \to \infty$. Consider the equation

$$-\operatorname{div}_y \overline{a}(y, \xi_0, \xi + \nabla_y w) = 0, \tag{4.2.10}$$

where

$$\overline{a}(y, \xi_0, \xi) = \langle a(\tau, y, \xi_0, \xi)\rangle_\tau. \tag{4.2.11}$$

Lemma 4.2.4 *Given $(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$ there exists a unique solution $w_\infty(y) \in V_{per}$ of equation (4.2.10). Moreover, $w_\mu \to w_\infty$ strongly in $\mathcal{V}_{per}$, as $\mu \to \infty$, where w_μ is a unique solution of (4.2.2).*

Proof. The right-hand part of (4.2.10) generates the operator

$$\overline{\mathbf{A}} : V_{per} \longrightarrow V_{per}^*.$$

It is easy to verify that $\overline{\mathbf{A}}$ is bounded, continuous, coercive, and strictly monotone. Thus, the existence and uniqueness of w_∞ follows immediately.

To prove the second assertion we set

$$w_{\mu,\delta}(\tau, y) = w_\infty(y) + \mu^{-1} v_\delta(\tau, y),$$

where the function v_δ will be specified later on. We have the following identity

$$\mu \partial_\tau w_{\mu,\delta} + \mathbf{A} w_{\mu,\delta} = \partial_\tau v_\delta + f_1 + f_{2,\delta}, \tag{4.2.12}$$

where

$$f_1 = \operatorname{div}_y a(\tau, y, \xi_0, \xi + \nabla_y w_\infty),$$

$$f_{2,\delta} = -\operatorname{div}_y [a(\tau, y, \xi_0, \xi + \nabla_y w_{\mu,\delta}) - a(\tau, y, \xi_0, \xi + \nabla_y w_\infty)].$$

Now let us consider ∂_τ as a closed linear operator acting from $\mathrm{L}^p_{per}(\mathbf{R}; V_{per})$ into $\mathrm{L}^{p'}_{per}(\mathbf{R}; V^*_{per})$. Its kernel consists of all functions which does not depend on τ. Since ∂_τ is skew-adjoint, its image is dense in the orthogonal complement

$$(\ker \partial_\tau)^\perp \subset \mathrm{L}^{p'}_{per}(\mathbf{R}; V^*_{per})$$

consisting of all $f \in \mathrm{L}^{p'}_{per}(\mathbf{R}; V^*_{per})$ such that

$$\langle f, \varphi \rangle = 0 \quad \forall \varphi \in V_{per}.$$

Now we have, for $\varphi = \varphi(y) \in V_{per}$,

$$\begin{aligned}\langle f_1, \varphi \rangle &= \int_0^1 \int_\square a(\tau, y, \xi_0, \xi + \nabla_y w_\infty) \nabla \varphi d\tau dy = \\ &= \int_\square \overline{a}(y, \xi_0, \xi + \nabla_y w_\infty) \nabla_y \varphi dy = \langle \mathbf{A} w_\infty, \varphi \rangle = 0,\end{aligned}$$

since w_∞ is a solution of (4.2.10). Hence, there exist $v_\delta \in \mathcal{W}_{per}$, and $h_\delta, g_\delta \in \mathcal{V}^*_{per}$ such that

$$f_1 = h_\delta + g_\delta,$$

$$-\partial_\tau v_\delta = h_\delta,$$

and

$$\|g_\delta\|_{\mathcal{V}^*_{per}} < \delta.$$

Using inequality (4.1.5) and equations (4.2.2), (4.2.12) we get

$$\begin{aligned}c^{\beta-1}_{\mu,\delta} \|w_{\mu,\delta} - w_\mu\|^\beta_{\mathcal{V}_{per}} &\leq \langle \mu \partial_\tau (w_{\mu,\delta} - w_\mu) + \mathbf{A} w_{\mu,\delta} - \mathbf{A} w_\mu, w_{\mu,\delta} - w_\mu \rangle = \\ &= \langle \mu \partial_\tau w_{\mu,\delta} + \mathbf{A} w_{\mu,\delta}, w_{\mu,\delta} - w_\mu \rangle = \\ &= \langle \partial_\tau v_\delta + f_1 + f_{2,\delta}, w_{\mu,\delta} - w_\mu \rangle = \\ &= \langle g_\delta + f_{2,\delta}, w_{\mu,\delta} - w_\mu \rangle \leq \\ &\leq (\|g_\delta\|_{\mathcal{V}^*_{per}} + \|f_{2,\delta}\|_{\mathcal{V}^*_{per}}) \|w_{\mu,\delta} - w_\mu\|_{\mathcal{V}_{per}},\end{aligned}$$

where

$$c^{\beta-1}_{\mu,\delta} = K(1 + \mu^{-p} \|v_\delta\|^p_{\mathcal{V}_{per}})^{1-\beta/p}.$$

This implies

$$c_{\mu,\delta} \|w_{\mu,\delta} - w_\mu\|_{\mathcal{V}_{per}} \leq \left(\|g_\delta\|_{\mathcal{V}^*_{per}} + \|f_{2,\delta}\|_{\mathcal{V}^*_{per}} \right)^{1/(\beta-1)} \leq \left(\delta + \|f_{2,\delta}\|_{\mathcal{V}^*_{per}} \right)^{1/(\beta-1)}$$

Given $\delta > 0$ one can find $\mu_\delta > 0$ such that $c_{\mu,\delta} \geq K_1 > 0$ for $\mu > \mu_\delta$, where K_1 does not depend on δ. Furthermore, using inequality (4.1.7) we see, choosing μ_δ to be larger if one need, that $\|f_{2,\delta}\|_{\mathcal{V}^*_{per}} < \delta$ for $\mu > \mu_\delta$. Thus, for $\mu > \mu_\delta$ we have

$$\begin{aligned}\|w_\mu - w_\infty\|_{\mathcal{V}_{per}} &\leq \|w_\mu - w_{\mu,\delta}\|_{\mathcal{V}_{per}} + \|w_{\mu,\delta} - w_\infty\|_{\mathcal{V}_{per}} = \\ &= \|w_\mu - w_{\mu,\delta}\|_{\mathcal{V}_{per}} + \mu^{-1} \|v_\delta\|_{\mathcal{V}_{per}} \leq \\ &\leq K_1^{-1} (2\delta)^{1/(\beta-1)} + \mu^{-1} \|v_\delta\|_{\mathcal{V}_{per}}.\end{aligned}$$

Therefore, $\|w_\mu - w_\infty\|_{\mathcal{V}_{per}}$ is small for $\mu > 0$ being large enough and we conclude. $\square$

Now let us define the functions $\hat{a}(\xi_0, \xi)$ and $\hat{a}_0(\xi_0, \xi)$ by

$$\left.\begin{aligned} \hat{a}(\xi_0,\xi) &= \langle a(\tau, y, \xi_0, \xi + \nabla_y w_\infty(y))\rangle_{\tau,y}, \\ \hat{a}_0(\xi_0,\xi) &= \langle a_0(\tau, y, \xi_0, \xi + \nabla_y w_\infty(y))\rangle_{\tau,y}, \end{aligned}\right\} \tag{4.2.13}$$

where w_∞ is a unique solution of equation (4.2.10).

Theorem 4.2.3 *Assume that $\alpha > 2\beta$. Then there exists a parabolic operator $\hat{\mathcal{L}}$ such that for any cylindrical open set $Q = (0,T) \times Q_0$ we have $\hat{\mathcal{L}} \in \Pi$ and $\mathcal{L}_\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$. The operator $\hat{\mathcal{L}}$ is defined by (4.2.4) and (4.2.13).*

Proof. The proof of the theorem is almost identical to that of Theorem 4.2.2. Only one change is in order. One need to define the function w^ε by

$$w^\varepsilon(t,x) = \varepsilon w_\mu(\varepsilon^{-\alpha} t, \varepsilon^{-1} x),$$

where w_μ is a unique solution of (4.2.2), with $\mu = \varepsilon^{2-\alpha} \to \infty$, as $\varepsilon \to 0$. $\square$

Remark 4.2.1 Let $n = 1$. In the non self-similar cases just considered, auxiliary equations (4.2.8) and (4.2.13) are one-dimensional and sometimes may be solved explicitly (cf. Section 3.4). At the same time auxiliary equation (4.2.2) for the self-similar case ($\mu = 1$) is two-dimensional and cannot be solved explicitly.

4.2.4 Spatial Homogenization

The case of spatial homogenization reduces to the standard homogenization of the elliptic part of $\mathcal{L}_\varepsilon$. More precisely, consider the equation

$$-\operatorname{div}_y a(t, y, \xi_0, \xi + \nabla w) = 0. \tag{4.2.14}$$

This is the standard auxiliary equation of the elliptic homogenization theory, depending additionally on the parameter $t \in [0,1]$. As an application of the standard monotonicity method, we see that given

$$(\xi_0, \xi) \in \mathbf{R} \times \mathbf{R}^n$$

there exists a unique soluttion $w_0 \in \mathrm{L}^p(0,T; V_{per})$ of (4.2.14). Then we define the operator $\hat{\mathcal{L}}$ by the formula

$$\hat{\mathcal{L}} u = \partial_t u - \operatorname{div} \hat{a}(t, u, \nabla u), \tag{4.2.15}$$

where

$$\left.\begin{aligned} \hat{a}(t,\xi_0,\xi) &= \langle a(t, y, \xi_0, \xi + \nabla_y w_0)\rangle_y, \\ \hat{a}_0(t,\xi_0,\xi) &= \langle a_0(t, y, \xi_0, \xi + \nabla_y w_0)\rangle_y. \end{aligned}\right\} \tag{4.2.16}$$

Theorem 4.2.4 *Assume $\alpha = 0$, $\beta = 1$. Then $\mathcal{L}_\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$, where the operator $\hat{\mathcal{L}}$ is defined by (4.2.15) and (4.2.16).*

Proof. Let us consider the equation

$$\varepsilon^2 \partial_t w - \mathrm{div}_y a(t, y, \xi_0, \xi + \nabla_y w) = 0.$$

It is easily seen that this equation has the unique solution

$$w_\varepsilon = w_\varepsilon(t, y) \in \mathrm{L}^p(0, T; V_{per})$$

such that $\partial_t w_\varepsilon \in \mathrm{L}^p(0, T; V^*_{per})$ and $w_\varepsilon(0, y) = 0$. Exactly as in Lemma 4.2.3, one can prove that $w_\varepsilon \to w_0$, as $\varepsilon \to 0$, strongly in $\mathrm{L}^p(0, T; V_{per})$. Set

$$w^\varepsilon(t, x) = \varepsilon w_\varepsilon(t, \varepsilon^{-1} x)$$

and

$$w_0^\varepsilon(t, x) = \varepsilon w_0(t, \varepsilon^{-1} x).$$

Now to complete the proof it is sufficient to repeat the arguments from the proof of Theorem 4.2.2. □

4.2.5 Time Homogenization

Finally, we consider the case of time homogenization. In this case the homogenized operator may be constructed explicitly, namely,

$$\hat{\mathcal{L}} u = \partial_t - \mathrm{div}\, \hat{a}(x, u, \nabla u) + \hat{a}_0(x, u, \nabla u), \tag{4.2.17}$$

where

$$\left.\begin{aligned} \hat{a}(x, \xi_0, \xi) &= \langle a(t, x, \xi_0, \xi) \rangle_t = \textstyle\int_0^1 a(t, x, \xi_0, \xi) dt, \\ \hat{a}_0(x, \xi_0, \xi) &= \langle a_0(t, x, \xi_0, \xi) \rangle_t = \textstyle\int_0^1 a_0(t, x, \xi_0, \xi) dt. \end{aligned}\right\} \tag{4.2.18}$$

Theorem 4.2.5 *Assume $\alpha = 1$, $\beta = 0$. Then $\mathcal{L}_\varepsilon$ strongly G-converges to $\hat{\mathcal{L}}$, where $\hat{\mathcal{L}}$ is defined by (4.2.17) and (4.2.18).*

Proof. First of all, we change the notations introduced at the beginning of the section. Denote by $\mathcal{V}_{per}$ the space of all functions

$$v(\tau, x) \in \mathrm{L}^p_{loc}(\mathbf{R}; V)$$

such that v is 1-periodic in τ and $\langle v \rangle_\tau = 0$. Set

$$\mathcal{W}_{per} = \{ v \in \mathcal{V}_{per} : \partial_\tau v \in \mathcal{V}^*_{per} \},$$

where $\mathcal{V}^*_{per}$ stands for the dual space to $\mathcal{V}_{per}$. As in the proofs of all previous parabolic homogenization results, we may assume that $\mathcal{L}_\varepsilon$ strongly G-converges to $\tilde{\mathcal{L}}$, where

$$\tilde{\mathcal{L}}u = \partial_t u - \operatorname{div} \tilde{a}(t,x,u,\nabla u) + \tilde{a}_0(t,x,u,\nabla u),$$

and we need only to show that

$$\tilde{a}(t,x,\xi_0,\xi) = \hat{a}(x,\xi_0,\xi)$$

and

$$\tilde{a}_0(t,x,\xi_0,\xi) = \hat{a}(x,\xi_0,\xi).$$

The function

$$f(\tau,x) = -\operatorname{div}_y[a(\tau,x,\xi_0,\xi) - \hat{a}(x,\xi_0,\xi)]$$

has zero mean value with respect to the variable τ. Hence, as in the proof of Lemma 4.2.4, given $\delta > 0$ there exist $w_\delta \in \mathcal{W}_{per}$ and $h_\delta, g_\delta \in \mathcal{V}^*_{per}$ such that

$$f = h_\delta + g_\delta,$$

$$-\partial_t w_\delta = h_\delta,$$

and

$$\|g_\delta\|_{\mathcal{V}^*_{per}} < \delta.$$

Set

$$w_{\delta,\varepsilon}(t,x) = \varepsilon w_\delta(\varepsilon^{-1}t,x).$$

We have $w_{\delta,\varepsilon} \to 0$, as $\varepsilon \to 0$, strongly in $\mathcal{V}$ and weakly in $\mathcal{W}$ for any $\delta > 0$. Indeed,

$$\begin{aligned}\|w_{\delta,\varepsilon}\|^p_{\mathcal{V}} &= \int_0^T \|\varepsilon w_\delta(\varepsilon^{-1}t,\cdot)\|^p_V dt = \varepsilon^{p+1}\int_0^{T/\varepsilon} \|w_\delta(\tau,\cdot)\|^p_V d\tau \le \\ &\le \varepsilon^{p+1}\left([T/\varepsilon]+1\right)\int_0^1 \|w_\delta(\tau,\cdot)\|^p_V d\tau \le K\varepsilon^p \|w_\delta\|_{\mathcal{V}^p_{per}},\end{aligned}$$

where $[x]$ is the integer part of x, and similarly for $\partial_t w_{\delta,\varepsilon}$. Moreover,

$$g_{\delta,\varepsilon}(t,x) = g_\delta(\varepsilon^{-1}t,x) \to 0, \quad \text{as } \delta \to 0,$$

strongly in $\mathcal{V}^*$, uniformly with respect to $\varepsilon > 0$.

Now we can choose two sequences $\varepsilon_k \to 0$ and $\delta_k \to 0$ such that $w_k = w_{\delta_k,\varepsilon_k} \to 0$ weakly in $\mathcal{W}$ and strongly in $\mathcal{V}$. Setting $g_k = g_{\delta_k,\varepsilon_k}$ we have the identity

$$\begin{aligned}\partial_t w_k - \operatorname{div} a(\varepsilon_k^{-1}t,x,\xi_0,\xi+\nabla w_k) + a_0(\varepsilon_k^{-1}t,x,\xi_0,\xi+\nabla w_k) = \\ = g_k - \operatorname{div}\hat{a}(x,\xi_0,\xi) + \varphi_k + \psi_k,\end{aligned}$$

where

$$\varphi_k = -\operatorname{div}[a(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k) - a(\varepsilon_k^{-1}t, x, \xi_0, \xi)],$$

$$\psi_k = a_0(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k).$$

By inequality (4.1.7), $\varphi_k \to 0$ strongly in $\mathcal{V}^*$. Since ψ_k is bounded in $\mathrm{L}^{p'}(Q)$, we may assume, passing to a subsequence, that $\psi_k \to \psi$ weakly in $\mathrm{L}^{p'}(Q)$, hence, strongly in $\mathcal{W}^*$. As we have seen above, $g_k \to 0$ strongly in $\mathcal{V}^*$. Therefore, by the theorem on convergence of arbitrary solutions, we have

$$-\operatorname{div} \tilde{a}(t, x, \xi_0, \xi) + \tilde{a}_0(t, x, \xi_0, \xi) = -\operatorname{div} \hat{a}(x, \xi_0, \xi) + \psi,$$

and

$$a(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k) \to \tilde{a}(t, x, \xi_0, \xi),$$

$$a_0(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k) \to \tilde{a}_0(t, x, \xi_0, \xi),$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^p(Q)$, respectively.

On the other hand, applying a version of Lemma 4.2.2 for vector valued functions we see that

$$a(\varepsilon^{-1}t, x, \xi_0, \xi) \to \hat{a}(x, \xi_0, \xi),$$

$$a_0(\varepsilon^{-1}t, x, \xi_0, \xi) \to \hat{a}_0(x, \xi_0, \xi)$$

weakly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. Finally, since $w_k \to 0$ strongly in $\mathcal{V}$, inequality (4.1.7) implies that

$$a(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k) - a(\varepsilon_k^{-1}t, x, \xi_0, \xi) \to 0,$$

$$a_0(\varepsilon_k^{-1}t, x, \xi_0, \xi + \nabla w_k) - a_0(\varepsilon_k^{-1}t, x, \xi_0, \xi) \to 0$$

strongly in $\mathrm{L}^{p'}(Q)^n$ and $\mathrm{L}^{p'}(Q)$, respectively. Thus, $\tilde{a} = \hat{a}$, $\tilde{a}_0 = \hat{a}_0$ and the proof is complete. $\square$

4.3 An Equation of Nonstationary Filtration

In this section, we want to study the homogenization process for a very particular problem of mathematical physics, not covered by the previous results directly. Let $Q = (0,T) \times Q_0$, where Q_0 is an open bounded subset of $\mathbf{R}^n$. We consider the following equation arising, e.g., from the filtration theory:

$$\partial_t u - \operatorname{div}(a(x)\nabla\varphi(u)) = 0. \tag{4.3.1}$$

We examine the solutions subject to the initial and boindary conditions

$$u(0,x) = u_0(x), \quad x \in Q_0, \tag{4.3.2}$$

$$u(t,x) = 0, \quad x \in \partial Q_0. \tag{4.3.3}$$

Throughout this section we shall always use the following assumptions:

(h1) *Q_0 is a bounded arcwise connected open set and $\partial Q \in C^3$;*

(h2) *the symmetric matrix $a(x)$ belongs to $\mathrm{L}^\infty(Q_0)$, $\|a\|_{\infty,Q_0} \leq \Lambda$, and*

$$a(x)\xi \cdot \xi \geq \lambda|\xi|^2, \quad \lambda > 0, \quad a.e \text{ on } Q;$$

(h3) *the function φ belongs to $C\left([0,\infty)\right) \cap C^2(0,\infty)$ and*

$$\varphi(0) = 0, \quad \varphi'(s) > 0, \quad \varphi''(s) > 0, \quad for\ s > 0;$$

(h4) *the initual data u_0 is a nonnegative function and $u_0 \in \mathrm{L}^\infty(Q_0)$.*

Now we introduce the notion of weak solution of problem (4.3.1) – (4.3.3). Here and later on we use the notations

$$H^1(Q) = W^{1,2}(Q), \quad H_0^1(Q) = W_0^{1,2}(Q).$$

Definition 4.3.1 *A function $u(t,x)$ is called a soluton of problem (4.3.1) – (4.3.3) if*

(1) *$u \in \mathrm{L}^\infty(Q)$ and $u(t,x) \geq 0$ a.e. on Q;*

(2) *the distributional derivatives $\partial_t\varphi(u)$ and $\nabla\varphi(u)$ belong to $\mathrm{L}^2(Q)$;*

(3) *$\varphi(u) = 0$ on $(0,T) \times \partial Q_0$ and $u(0,x) = u_0(x)$;*

(4) *equation (4.3.1) is satisfied in the weak sense, i.e.*

$$\int_Q \{u \cdot \partial_t h + [a(x)\nabla\varphi(u)] \cdot \nabla h\}\, dtdx = 0$$

for any $h \in H_0^1(Q)$.

Conditions (1) and (2) imply that $\varphi(u) \in H^1(Q)$. Therefore, the trace of $\varphi(u)$ on $(0,T)\times\partial Q_0$ is well-defined and the first part of condition (3) makes sense. Moreover, $\varphi(u) \in C([0,T]; \mathrm{L}^2(Q_0))$. Since $u \in \mathrm{L}^\infty(Q)$, this implies that $u \in C([0,T]; \mathrm{L}^2(Q_0))$ as well and the second part of condition (3) also makes sense. Using this fact it is not difficult to verify that, for a weak solution u, we have the identity

$$\int_Q \{u \cdot \partial_t h + [a(x)\nabla\varphi(x)] \cdot \nabla h\}\, dtdx = \int_{Q_0} h(0,x)u_0(x)dx - \\ - \int_{Q_0} h(T,x)u(T,x)dx, \tag{4.3.4}$$

for any $h \in H^1(Q)$ such that $h = 0$ on $(0,T)\times\partial Q_0$. Moreover,

$$\partial_t u \in \mathrm{L}^2\left(0,T; H^{-1}(Q_0)\right),$$

where $H^{-1}(Q_0) = W^{-1,2}(Q_0)$.

First of all, we state the following uniqueness result.

Lemma 4.3.1 *Given $u_0(x)$ problem (4.3.1) – (4.3.3) has a most one weak solution.*

Proof. Suppose $u_1(t,x)$ and $u_2(t,x)$ are two weak solutions of (4.3.1) – (4.3.3). Let

$$h(t,x) = \int_T^t [\varphi(u_1(\tau,x)) - \varphi(u_2(\tau,x))]\, d\tau.$$

It is easily seen that $h \in H^1(Q)$, and $h = 0$ on $(0,T)\times\partial Q_0$ and at $t = T$. Identity (4.3.5) yields

$$\int_Q \{(u_1 - u_2)\cdot\partial_t h - [a(x)\nabla(\varphi(u_1) - \varphi(u_2))]\cdot\nabla h\}\, dtdx = 0.$$

By means of a simple calculation the last identity may be rewritten as

$$-I_1 = I_2,$$

where

$$I_1 = \int_Q (u_1 - u_2)(\varphi(u_1) - \varphi(u_2))\, dtdx,$$

$$I_2 = -\frac{1}{2}\int_Q \partial_t\left[(a(x)\nabla h)\cdot\nabla h\right] dtdx.$$

Making use the Fubini Theorem we see that

$$I_2 = \frac{1}{2}\int_Q \left[a(x)\int_T^0 \nabla(\varphi(u_1) - \varphi(u_2))\, d\tau\right]\cdot\left[\int_T^0 \nabla(\varphi(u_1) - \varphi(u_2))\, d\tau\right] dtdx.$$

Hence, $I_2 \geq 0$ and $I_1 \leq 0$. Now (h3) implies that $u_1 = u_2$ a.e. on Q. □

Now we prove an existence result imposing some additional restrictions which will be removed next.

Lemma 4.3.2 *Suppose that* $a \in C^2(\overline{Q}_0)$, $u_0 \in C(\overline{Q}_0)$ *is a nonnegative function such that* $\operatorname{supp} u_0$ *is compact in* Q_0, *and*

$$v_0 = \varphi(u_0) \in C^1(Q_0).$$

Then there exists a unique solution $u(t,x)$ *of problem (4.3.1) – (4.3.3). Moreover, if*

$$\|u_0\|_{\infty,Q_0} \leq C_0, \quad \|\nabla v_0\|_{2,Q_0} \leq C_0,$$

then

$$0 \leq u(t,x) \leq C, \quad \|\partial_t \varphi(u)\|_{2,Q} \leq C, \quad \|\nabla \varphi(u)\|_{2,Q} \leq C,$$

where the constant $C > 0$ *depends only on* C_0, λ, *and* Λ.

Proof. First, let us introduce some notations. Denote by Φ the inverse function to φ and set

$$v_0(x) = \varphi\left(u_0(x)\right),$$

$$M = \|\varphi(u_0)\|_{\infty,Q_0} + 1.$$

For any integer $r > 0$, there exists a monotone increasing function $\chi_r(s) \in C^2(\mathbf{R})$ such that

$$\chi_r(s) = \varphi'\left(\Phi(s)\right) \quad \text{if } s \in \left[\frac{1}{2r}, M+1\right],$$

$$\chi_r(s) = \alpha_r \quad \text{if } s \leq \frac{1}{2r+1},$$

$$\chi_r(s) = \beta \quad \text{if } s \geq M+2,$$

where

$$0 < \alpha_r < \varphi'\left(\Phi\left(\frac{1}{2r}\right)\right), \quad \beta > \varphi'\left(\Phi(M+1)\right).$$

It is a consequence of well-known existence results for quasilinear parabolic equation (see, e.g., [196]) that there exists a classical solution $v = v_r$ of the following initual boundary value problem:

$$\partial_t v - \chi_r(v) \cdot \operatorname{div}\left[a(x)\nabla v\right] = 0, \tag{4.3.5}$$

$$v(0,x) = v_{0,r}(x) \quad \text{on } Q_0,$$

$$v(t,x) = \frac{1}{r} \quad \text{on } (0,T) \times \partial Q_0,$$

where $v_{0,r}(x) = v_0(x) + 1/r$.

For the function $v_r(t,x)$, we have

$$\frac{1}{r} \leq v_r(t,x) \leq M, \tag{4.3.6}$$

$$v_{r+1}(t,x) \leq v_r(t,x). \tag{4.3.7}$$

Indeed, suppose w_1 and w_2 are two classical solutions of (4.3.5) such that $w_1 \geq w_2$ on $\{0\} \times Q_0$ and on $(0,T) \times \partial Q_0$. Then $w_1 \geq w_2$ everywhere on Q. To prove this, let us consider the function $z = w_1 - w_2$. Then $z \geq 0$ on $(\{0\} \times Q_0) \cup ((0,T) \times \partial Q_0)$ and, by the Theorem of the Mean,

$$\partial_t z - f(t,x) \cdot \operatorname{div}\left[a(x)\nabla z\right] - g(t,x)z \geq 0,$$

where $f = \chi_r(w_2)$ and

$$g = \chi_r'\left(\theta w_1 + (1-\theta)w_2\right) \cdot \operatorname{div}\left[a(x)\nabla w_1\right]$$

for some $\theta = \theta(t,x) \in (0,1)$. It is easily seen that

$$f(t,x) \geq \alpha_r > 0$$

and $g(t,x)$ is bounded above on $\overline{Q}$. The standard maximum principle for parabolic equations (see, e.g., [197, 257]) implies that $z \geq 0$. The statement we have just proved yields (4.3.6) immediately. Now, by (4.3.6), we see that v_{r+1} is also a solution of (4.3.5). Again using the same comparison principle we get (4.3.7). In particular, inequality (4.3.6) implies that

$$\chi_r(v_r) = \varphi'\left(\Phi(v_r)\right).$$

Thus, for $u_r = \Phi(v_r)$, we have

$$\partial_t u_r - \operatorname{div}\left[a(x)\nabla\varphi(u_r)\right] = 0, \tag{4.3.8}$$

$$u_r(0,x) = \Phi(v_{0,r}(x)) \quad \text{on } Q_0, \tag{4.3.9}$$

$$u_r(t,x) = \Phi\left(\frac{1}{r}\right) \quad \text{on } (0,T) \times \partial Q_0, \tag{4.3.10}$$

$$\Phi\left(\frac{1}{r+1}\right) \leq u_{r+1}(t,x) \leq u_r(t,x) \leq \Phi(M). \tag{4.3.11}$$

Now we want to estimate the derivatives of $\varphi(u_r)$. Let

$$F(s) = \int_0^s \varphi(\tau)d\tau.$$

Multiplying (4.3.8) by $\varphi(u_r) - 1/r$ and integrating we get

$$\begin{aligned} 0 &= \int_Q \left\{ \partial_t u_r \cdot \left(\varphi(u_r) - \frac{1}{r} \right) + [a(x)\nabla\varphi(u_r)] \cdot \nabla\varphi(u_r) \right\} dtdx = \\ &= \int_Q \partial_t \left[F(u_r) - \frac{u_r}{r} \right] dtdx + \int_Q [a(x)\nabla\varphi(u_r)] \cdot \nabla\varphi(u_r) dtdx \geq \\ &\geq \int_{Q_0} \left[F(u_r) - \frac{u_r}{r} \right] \Big|_0^T dx + \lambda \int_Q |\nabla\varphi(u_r)|^2 \, dtdx. \end{aligned}$$

This implies immediately the inequality

$$\|\nabla\varphi(u_r)\|_{2,Q} \leq C_1, \tag{4.3.12}$$

where the constant $C_1 > 0$ depends only on λ and M.

Next, multiply (4.3.8) by $\partial_t\varphi(u_r)$ and integrate over Q. After a simple calculation, we obtain

$$\begin{aligned} 0 &= \int_Q \{ \partial_t u_r \cdot \partial_t\varphi(u_r) - \operatorname{div}[a(x)\nabla\varphi(u_r)] \cdot \partial_t\varphi(u_r) \} \, dtdx = \\ &= \int_Q \varphi'(u_r) \cdot (\partial_t u_r)^2 dtdx + \int_Q [a(x)\nabla\varphi(u_r)] \cdot \nabla\partial_t\varphi(u_r) dtdx = \\ &= \int_Q [\partial_t G(u_r)]^2 \, dtdx + \frac{1}{2} \int_Q \partial_t \{ [a(x)\nabla\varphi(u_r)] \cdot \nabla\varphi(u_r) \} \, dtdx, \end{aligned}$$

where

$$G(s) = \int_k^s \varphi'(\tau)^{1/2} d\tau, \quad k = \frac{1}{2}\Phi\left(\frac{1}{r}\right).$$

This implies

$$\begin{aligned} \int_Q [\partial_t G(u_r)]^2 \, dtdx &= -\frac{1}{2} \int_Q \partial_t \{ [a(x)\nabla\varphi(u_r)] \cdot \varphi(u_r) \} \, dtdx \leq \\ &\leq \frac{1}{2} \int_Q [a(x)\nabla\varphi(u_r)] \cdot \nabla\varphi(u_r)|_{t=0} \, dtdx \\ &\leq \frac{\Lambda}{2} \int_Q |\nabla v_0|^2 \, dx. \end{aligned}$$

Since $\varphi'(s)$ is monotone decreasing, we have

$$[\partial_t\varphi(u_r)]^2 \leq \varphi'(\Phi(M)) \cdot [\partial_t G(u_r)]^2.$$

Therefore,

$$\|\partial_t\varphi(u_r)\|_{2,Q} \leq C_2, \tag{4.3.13}$$

where $C_2 > 0$ depends on C_0 and Λ.

Inequality (4.3.11) implies the existence of pointwise limits

$$u(t,x) = \lim_{r\to\infty} u_r(t,x),$$

$$\varphi(u(t,x)) = \lim_{r\to\infty} \varphi(u_r(t,x)).$$

Moreover, due to (4.3.12) and (4.3.13), we have

$$\nabla\varphi(u) = \lim_{r\to\infty} \nabla\varphi(u_r),$$

$$\partial_t\varphi(u) = \lim_{r\to\infty} \partial_t\varphi(u_r)$$

weakly in $L^2(Q)$. Now the standard passage to the limit shows us that u is a solution of the problem. The required bounds for $\nabla\varphi(u)$ and $\partial_t\varphi(u)$ follow immediately from (4.3.12) and (4.3.13). □

Now let us consider a sequence of problems

$$\partial_t u_k - \operatorname{div}\left(a^k(x)\nabla\varphi(u)\right) = 0, \tag{4.3.14}$$

$$u_k(0,x) = u_{0,k}(x), \quad x \in Q_0, \tag{4.3.15}$$

$$u_k(t,x) = 0, \quad x \in \partial Q_0. \tag{4.3.16}$$

We assume that $a^k(x)$ satisfies (h2) uniformly with respect to k.

Lemma 4.3.3 *Let u_k be a solution of problem (4.3.14) – (4.3.16). Assume that*

(i) *$u_{0,k}, u_0 \in L^\infty(Q_0)$ are nonnegative and $u_{0,k} \to u_0$ strongly in $L^2(Q_0)$;*

(ii) *there exists a constant $C > 0$ such that*

$$0 \le u_k(t,x) \le C, \quad \|\partial_t\varphi(u_k)\|_{2,Q} \le C, \quad \|\nabla\varphi(u_k)\|_{2,Q} \le C;$$

(iii) *there exists $a(x)$ such that $a^k(x)$ is strongly G-convergent to $a(x)$.*

Then there exists a unique solution $u(t,x)$ of problem (4.3.1) – (4.3.3) and

$$\lim u_k = u \quad \text{in } L^2(Q),$$

$$\lim \partial_t\varphi(u_k) = \partial_t\varphi(u),$$

$$\lim \nabla\varphi(u_k) = \nabla\varphi(u),$$

$$\lim a^k(x)\nabla\varphi(u_k) = a(x)\nabla\varphi(u)$$

weakly in $L^2(Q)$.

Proof. Let us put $v_k = \varphi(u_k)$. Passing to a subcequence we may assume that $v_k \to v$ weakly in $H^1(Q)$ and

$$a^k(x)\nabla v_k \to g(x)$$

weakly in $L^2(Q)$. Then, by the Sobolev Embedding Theorem,

$$v_k \to v \quad \text{in } C\left([0,T];L^2(Q_0)\right).$$

Since $0 \le v_k(t,x) \le \varphi(C)$, we have

$$u_k = \Phi(v_k) \to u = \Phi(v) \quad \text{in } C\left([0,T];L^2(Q_0)\right).$$

It is obvious that $u(0,x) = u_0(x)$ and $v = 0$ on $(0,T) \times \partial Q_0$. Moreover, passing to the limit in the corresponding integral identity we see that

$$\partial_t u - \operatorname{div} g = 0. \tag{4.3.17}$$

Now all we need is to show that

$$g = a(x) \cdot \nabla v. \tag{4.3.18}$$

Notice that, by Lemma 3.1.1, $u(t,x)$ should be a unique solution of the limit problem and the passage to a subsequence at the begining of the proof is superfluous.

To prove (4.3.18) we fix arbitrary functions $\theta \in C_0^\infty(0,T)$, $w \in C_0^\infty(Q_0)$ and set

$$h(t,x) = \theta(t)w(x).$$

Let $w_k \in H_0^1(Q_0)$ be a unique solution of the equation

$$-\operatorname{div}\left(a^k \cdot \nabla w_k\right) = -\operatorname{div}(a \cdot \nabla w). \tag{4.3.19}$$

Since $a^k \stackrel{G}{\Longrightarrow} a$, we see that $w_k \to w$ weakly in $H_0^1(Q_0)$.

Multiplying (4.3.14) by $h_k = \theta w_k$ and integrating we get the identity

$$\int_Q \theta \cdot \left(a^k \nabla v_k\right) \cdot \nabla w_k dtdx = \int_Q u_k w_k \theta' dtdx.$$

Using the symmetry of the matrix $a(x)$ and equation (4.3.19) we have

$$\begin{aligned}\int_Q \theta \cdot \left(a^k \nabla v_k\right) \cdot \nabla w_k dtdx &= \int_Q \theta \cdot \left(a^k \nabla w_k\right) \cdot \nabla v_k dtdx = \\ &= \int_Q \theta \cdot (a\nabla w) \cdot \nabla v_k dtdx = \int_Q u_k \cdot w_k \cdot \theta' dtdx.\end{aligned}$$

Letting here $k \to \infty$ we obtain

$$\int_Q \theta \cdot (a\nabla w) \cdot \nabla v dtdx = \int_Q \theta \cdot (a\nabla v) \cdot \nabla w dtdx = \int_Q u \cdot w \cdot \theta' dtdx.$$

However, due to (4.3.17),

$$\int_Q uw\theta' dtdx = \int_Q g \cdot \theta \cdot \nabla w dtdx.$$

Therefore,

$$\int_Q (a\nabla v) \cdot \nabla h dtdx = \int_Q g \cdot \nabla h dtdx$$

which implies (4.3.18). □

Our last result permits us to relax considerably the hyposeses of Lemma 4.3.2.

Lemma 4.3.4 *Suppose $u_0 \in \mathrm{L}^\infty(Q_0)$ is a nonnegative function such that*

$$v_0 = \varphi(u_0) \in H_0^1(Q_0).$$

Then there exists a unique solution $u(t,x)$ of problem (4.3.1) – (4.3.3). Moreover, if

$$\|u_0\|_{\infty,Q_0} \le C_0, \quad \|\nabla v_0\|_{2,Q_0} \le C_0,$$

then there exists a constant $C > 0$, depending only on C_0, λ and Λ, such that

$$0 \le u(t,x) \le C, \quad \|\partial_t \varphi(u)\|_{2,Q} \le C, \quad \|\nabla \varphi(u)\|_{2,Q} \le C.$$

Proof. By a standard approximation argument, there exists $v_{0,k} \in C_0^\infty(Q_0)$ such that

$$0 \le v_{0,k} \le C_1$$

and $v_{0,k} \to v_0$ in $H_0^1(Q_0)$. Then it is easily seen that

$$u_{0,k} = \Phi(v_{0,k}) \to u_0 \quad \text{in } \mathrm{L}^2(Q_0).$$

Moreover, the sequence $u_{0,k}$ is bounded in $\mathrm{L}^\infty(Q_0)$.

Next, again by standard approximation arguments, there exists a real symmetric matrix $a^k(x) \in C^2(\overline{Q}_0)$ such that $a^k \to a$ in $\mathrm{L}^2(Q_0)$ and (h2) is satisfied uniformly with respect to k. Corollary 2.4.1 implies that $a^k \stackrel{G}{\Longrightarrow} a$.

By Lemma 4.3.2, all the conditions on Lemma 4.3.3 are fulfilled and we conclude. □

Combining Lemmas 4.3.3 and 4.3.4 we get

Theorem 4.3.1 *Suppose $u_{0,k} \in \mathrm{L}^\infty(Q_0)$ is a bounded sequence of nonnegative functions such that*

$$v_{0,k} = \varphi(u_{0,k}) \in H_0^1(Q_0)$$

is bounded and $u_{0,k} \to u_0$ in $\mathrm{L}^2(Q_0)$. Assume that $a^k(x) \xrightarrow{G} a(x)$. Then for any k there exists a unique solution u_k of problem (4.3.14) – (4.3.16) and

$$\lim u_k = u \quad \textit{in } \mathrm{L}^2(Q),$$

$$\lim \partial_t \varphi(u_k) = \partial_t \varphi(u),$$

$$\lim \nabla \varphi(u_k) = \nabla \varphi(u),$$

$$\lim a^k(x) \nabla \varphi(u_k) = a(x) \nabla \varphi(u)$$

weakly in $\mathrm{L}^2(Q)$, where u is a unique solution of (4.3.1) – (4.3.3).

Now let $a(y) \in \mathrm{L}^\infty(\mathbf{R}^n)$ be a 1-periodic in y real symmetric matrix such that

$$a(y)\xi \cdot \xi \geq \lambda |\xi|^2, \quad \lambda > 0.$$

Then the matrix $a^\varepsilon(x) = a(\varepsilon^{-1}x)$ admits homogenization and the homogenized matrix $\hat{a}$ is defined by

$$\hat{a}\xi = \langle a(y)(\xi + \nabla_y w(y)) \rangle,$$

where $w \in H^1_{loc}(\mathbf{R}^n)$ is a unique 1-periodic solution of the problem

$$-\mathrm{div}_y a(y)(\xi + \nabla_y w) = 0,$$

$$\langle w \rangle = 0.$$

In this case, Theorem 4.3.1 shows us that the corresponding filtration problem admits homogenization. Of course, the same take place in the case of random homogeneous, or almost periodic, matrix a.

Comments

In the case of linear second order parabolic operators, the homogenization problem is well understood now (see [40, 47, 164] for a detailed account). For the case of high order parabolic operators, we refer the reader to [267, 268, 269]. The results presented in Sections 4.1 and 4.2 was obtained in [191, 192]. Some results of such kind was also proved in [209]. As in [191], we restrict ourself to the case of periodic homogenization only. However, it seems to be possible to get nonlinear versions the results of [268, 269] dealing with linear almost periodic and random parabolic operators, respectively.

In Section 4.3, we present an improved version of the results of [206]. Those results concern only the case of periodic homogenization. Moreover, in this paper a stronger notion of solution is used. More precisely, it is assumed that the initial data u_0 satisfies the following assumption:

$$u_0 \in C(\overline{Q}_0), \quad u_0 = 0 \text{ on } \partial Q_0, \quad \varphi(u_0) \in C(\overline{Q}_0).$$

At the same time, using a more delicate approximation of u_0 suggested in [17] it is shown there that, for a solution of problem (4.3.1) – (4.3.3),

$$\lim u(t,x) = 0 \quad \text{as } x \to x_0 \in \partial Q_0$$

along the normal direction. Recall that, in Section 4.3, boundary condition (4.3.3) is regarded as:

$$\text{trace of } \varphi(u) \text{ on } (0,T) \times \partial Q_0 \text{ is equal to } 0.$$

Appendix A

Homogenization of Nonlinear Difference Schemes

Investigation of homogenization problems for difference schemes was started by S. Kozlov [183] who considered the linear case. There are several sourses which lead to problems of such kind: random walks, discrete models of microstructured media, electric networks, etc. In many such problems, unlike to usual situations in the theory of difference schemes, coefficients of corresponding difference operators contain nonvanishing, as the mesh width tends to zero, oscillations. The key idea of S. Kozlov is to revise the standard concept of difference approximation of a differential operator (see, for example, [240]) in the spirit of G-convergence and homogenization theory. Additional important feature of Kozlov's work is the presence of a kind of selfsimilarity in the underlying geometric structure (rescaled integer lattice), due to which the problem under consideration involves a natural small parameter. Later on, M. Vogelius [262] had obtained some results on homogenization of linear electric networks without any selfsimilarity condition. Some degenerate homogenization problems for linear difference operators are consider in [190].

In contrast to the above mentioned papers, we consider here similar problems for *nonlinear* difference operators. We remark that for this case there is no probabilistic interpretation. However, the problems we deal with are still of interest for the theory of microstructured media, e.g., for nonlinear counterparts of Kunin's modeles [193], as well as for nonlinear electric network theory. As in [183], we consider difference operators on the rescaled integer lattice, but we use completely different approach which is similar to that developed in Chapters 2 and 3. For the sake of simplicity we restrict ourself to the case of monotone operators only. However, all the results may be extended to the case of nonmonotone elliptic difference operators in the same way as it was done in n^0 2.3.3 for differential operators.

A.1 Mesh Functions

Let

$$\mathbf{Z}_\varepsilon^n = \varepsilon \mathbf{Z}^n = \{x \in \mathbf{R}^n : \varepsilon^{-1} x \in \mathbf{Z}^n\}.$$

Fix $\lambda > 0$ and set

$$\Lambda = [-\lambda, \lambda]^n \cap \mathbf{Z}^n \setminus \{0\}.$$

In what follows we shall always assume that ε runs a sequence which tends to 0. Let $Q \subset \mathbf{R}^n$ be a bounded domain and $Q_\varepsilon = Q \cap \mathbf{Z}_\varepsilon^n$. The set

$$\dot{Q}_\varepsilon = \{x \in Q_\varepsilon : \max_{1 \le i \le m} |x_i - y_i| \ge \lambda\varepsilon \quad \forall y \in \partial Q\}$$

is said to be the ε-*interior* of Q_ε. Define the ε-*boundary* ∂Q_ε of Q_ε to be $Q_\varepsilon \setminus \dot{Q}_\varepsilon$. For $z \in \Lambda$, we define the operator d_z by the formula

$$(d_z u)(x) = \varepsilon^{-1}\left(u(x + \varepsilon z) - u(x)\right),$$

where $u(x)$ is a mesh function on $\mathbf{Z}_\varepsilon^n$. For $\{e_i\}$ being the standard basis in $\mathbf{R}^n$, we shall write $d_{\pm i} = d_{\pm e_i}$.

For $p \ge 1$ we introduce the following spaces of mesh functions:

$$L^p(Q_\varepsilon) = \{u : Q_\varepsilon \longrightarrow \mathbf{R} : |u|_{p,\varepsilon}^p = \varepsilon^n \sum_{x \in Q_\varepsilon} |u(x)|^p < \infty\},$$

$$W^{1,p}(Q_\varepsilon) = \{u : Q_\varepsilon \longrightarrow \mathbf{R} : [u]_{p,\varepsilon}^p = |u|_{p,\varepsilon}^p + \varepsilon^n \sum_{x \in Q_\varepsilon} \sum_{i=1}^n |d_i u(x)|^p < \infty\},$$

where u is externed by zero outside of Q_ε,

$$W_0^{1,p}(Q_\varepsilon) = \{u \in W^{1,p}(Q_\varepsilon) : u = 0 \text{ on } \partial Q_\varepsilon\}.$$

The last space will be considered with the equivalent, uniformly with respect to $\varepsilon > 0$, norm

$$\|u\|_{p,\varepsilon}^p = \varepsilon^n \sum_{x \in Q_\varepsilon} \sum_{i=1}^n |d_i u(x)|^p.$$

For $p > 1$ we shall consider also the dual space

$$W^{-1,p'}(Q_\varepsilon) = W_0^{1,p}(Q_\varepsilon)^*, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

endowed with the dual norm $\|\cdot\|_{*,p',\varepsilon}$. Note that any element $f \in W^{-1,p'}(Q_\varepsilon)$ may be written in the form

$$f = \sum_{i=1}^{n} d_{-i} f_i,$$

where $f_i \in L^{p'}(Q_\varepsilon)$ and

$$\|f\|^{p'}_{*,p',\varepsilon} = \sum_{i=1}^{n} |f_i|^{p'}_{p',\varepsilon}.$$

To show this, consider the equation

$$\sum_{i=1}^{n} d_{-i}(|d_i u|^{p-2} d_i u) = f,$$

the difference counterpart of the well-known p-Laplacian equation. By means of standard monotonicity method [200] one can prove that this equation has a unique solution $u \in W_0^{1,p}(Q_\varepsilon)$. Then we put

$$f_i = |d_i u|^{p-2} d_i u.$$

Let us discuss connections between functions of discret variables and of continuous ones [195]. At first, we consider mesh approximations for functions of continuous variables. For $f \in L^p(Q)$, we introduce the mesh function f^ε defined by

$$f^\varepsilon(x) = \varepsilon^{-n} \int_{x+K_\varepsilon} f(y)dy, \quad x \in Q_\varepsilon, \tag{A.1}$$

where $K_\varepsilon = \{y \in \mathbf{R}^n : 0 \le y \le \varepsilon,\ i = 1, 2, \ldots, n\}$. If

$$f = \sum_{i=1}^{n} \partial_i f_i \in W^{-1,p'}(Q),$$

where $f_i \in \mathrm{L}^{p'}(Q)$, $i = 1, 2, \ldots, n$, $\partial_i = \partial/\partial x_i$, we set

$$f^\varepsilon = \sum_{i=1}^{n} d_i f_i^\varepsilon$$

with f_i^ε defined by (A.1).

Next we introduce a completion, $\tilde{f}_\varepsilon$, of a mesh function f_ε in the following way. For $f_\varepsilon \in \mathrm{L}^p(Q_\varepsilon)$, we define $\tilde{f}_\varepsilon$ to be the piecewise constant interpolation function, i.e.

$$\tilde{f}_\varepsilon(y) = f_\varepsilon(x), \quad y \in x + K_\varepsilon, \quad x \in \mathbf{Z}^n_\varepsilon.$$

For

$$f_\varepsilon = \sum_{i=1}^{n} d_{-i} f_{i,\varepsilon} \in W^{-1,p'}(Q_\varepsilon),$$

we set

$$\tilde{f}^{\varepsilon} = -\sum_{i=1}^{n} \partial_i \tilde{f}_{i,\varepsilon}.$$

Finally, for $u_\varepsilon \in W^{1,p}(Q_\varepsilon)$, we shall use the piecewise linear interpolation function

$$u'_\varepsilon \in W^{1,p}(Q) \cap C(\overline{Q})$$

defined by

$$\begin{aligned} u'_\varepsilon(y) &= u_\varepsilon(x) + \sum_{i=1}^{n} (d_i u_\varepsilon)(x) \cdot (y_i - x_i) + \dots \\ &+ \sum_{i=2}^{n-1} (d_1 d_2 \cdots d_{i-1} d_{i+1} \cdots d_n u_\varepsilon)(x) \prod_{j=1, j\neq i}^{n} (y_i - x_i) + \\ &+ (d_1 \cdots d_n u_\varepsilon)(x) \prod_{j=1}^{n} (y_i - x_i), \end{aligned}$$

where $y \in x + K_\varepsilon$, $x \in \mathbf{Z}^n_\varepsilon$.

Now let $f_\varepsilon \in \mathrm{L}^p(Q_\varepsilon)$ (resp. $f_\varepsilon \in W^{-1,p'}(Q_\varepsilon)$). We say that f_ε converges to $f \in \mathrm{L}^p(Q)$ (resp. $f \in W^{-1,p'}(Q)$) if the completions $\tilde{f}_\varepsilon$ converges to f in the corresponding space. Similarly, for $u_\varepsilon \in W^{1,p}(Q_\varepsilon)$, we say that u_ε converges to $u \in W^{1,p}(Q)$ if u'_ε converges to u in the space $W^{1,p}(Q)$. The notion of weak convergence of mesh functions may be introduced in the same manner.

One can show that for any bounded family $f_\varepsilon \in \mathrm{L}^p(Q_\varepsilon)$ (resp. $f_\varepsilon \in W^{-1,p'}(Q_\varepsilon)$) of mesh functions the family $\tilde{f}_\varepsilon$ of completions is bounded in $\mathrm{L}^p(Q)$ (resp. in $W^{-1,p'}(Q)$) as well. Therefore, f_ε contains a weakly convergent subsequence provided $p \in (1, +\infty)$. The similar statement holds true for $u_\varepsilon \in W^{1,p}(Q_\varepsilon)$. Moreover, if $u_\varepsilon \in W^{1,p}_0(Q_\varepsilon)$ and $u_\varepsilon \to u$ weakly in $W^{1,p}_0(Q)$, then $u_\varepsilon \to u$ strongly in $\mathrm{L}^p(Q)$. The same is true for $u_\varepsilon \in W^{1,p}(Q_\varepsilon)$ provided ∂Q is smooth enough. For more details we refer to [195].

A.2 G-convergence

First, we want to introduce the class of difference operators we shall consider. Let γ be the map

$$\gamma : \mathbf{R}^\Lambda \longrightarrow \mathbf{R}^n$$

defined by

$$\gamma\eta = \xi,$$

where $\xi_i = \eta_{e_i}$, $i = 1, 2, \dots, n$, $\eta = \{\eta_\lambda, \lambda \in \Lambda\}$. We fix constants $p \geq 2$, $c_0 > 0$, $\kappa > 0$, and a nonnegative function $c \in L^1(Q)$. The assumption $p \geq 2$ is imposed only for simplicity. It is possible to take any $p > 1$ modifying conditions (A.3), (A.7), (A.9), and (A.10) in the spirit of Section 2.3.

Let

$$a_z^\varepsilon : Q_\varepsilon \times \mathbf{R}^\Lambda \longrightarrow \mathbf{R}, \quad z \in \Lambda,$$

be a function which is continuous with respect to the second variable. We assume that

$$|a_z^\varepsilon(x, \eta)|^{p'} \leq c_0 |\eta|^p + c^\varepsilon(x), \quad x \in Q_\varepsilon, \ \eta \in \mathbf{R}^\Lambda, \tag{A.2}$$

and

$$\sum_{z \in \Lambda} \left[a_z^\varepsilon(x, \eta) - a_z^\varepsilon(x, \eta')\right] (\eta_z - \eta_z') \geq \kappa |\gamma(\eta - \eta')|^p, \quad x \in Q_\varepsilon, \ \eta, \eta' \in \mathbf{R}^\Lambda. \tag{A.3}$$

Consider an operator $\mathcal{A}_\varepsilon$ defined on mesh functions by the formula

$$(\mathcal{A}_\varepsilon u)(x) = \sum_{z \in \Lambda} d_{-z} a_z^\varepsilon(x, du), \quad x \in Q_\varepsilon, \tag{A.4}$$

where $d = \{d_z, \ z \in \Lambda\}$. More precisely, the "integral identity"

$$\langle A_\varepsilon u, v \rangle_\varepsilon = \varepsilon^n \sum_{x \in Q_\varepsilon} \sum_{z \in \Lambda} a_z^\varepsilon(x, du(x)) d_z v(x), \quad u, v \in W_0^{1,p}(Q_\varepsilon) \tag{A.5}$$

defines $\mathcal{A}_\varepsilon$ as an operator acting from $W_0^{1,p}(Q_\varepsilon)$ into $W^{-1,p'}(Q_\varepsilon)$. Here $\langle \cdot, \cdot \rangle_\varepsilon$ stands for the duality pairing between $W^{-1,p'}(Q_\varepsilon)$ and $W_0^{1,p}(Q_\varepsilon)$. It is not hard to see that the operator $\mathcal{A}^\varepsilon$ is continuous, coercive, and strictly monotone. Hence, it is invertible.

Side by side with difference operators of the form (A.4), we shall also consider differential operators. For $z = (z_1, z_2, \dots, z_n) \in \Lambda$, we set

$$\partial_z = \sum_{i=1}^n z_i \partial_i,$$

and $\partial = \{\partial_z : z \in \Lambda\}$. Let

$$a_z : Q \times \mathbf{R}^\Lambda \longrightarrow \mathbf{R}, \quad z \in \Lambda,$$

be a Carathéodory function such that

$$|a_z(x, \eta)|^{p'} \leq \bar{c}_0 |\eta|^p + \bar{c}(x), \quad x \in Q, \ \eta \in \mathbf{R}^\Lambda, \tag{A.6}$$

$$\sum_{z \in \Lambda} \left[a_z(x, \eta) - a_z(x, \eta')\right] (\eta_z - \eta_z') \geq \bar{\kappa} |\gamma(\eta - \eta')|^p, \quad x \in Q, \ \eta, \eta' \in \mathbf{R}^\Lambda, \tag{A.7}$$

where $\overline{c}_0 > 0$, $\overline{\kappa} > 0$, and $\overline{c} \in \mathrm{L}^1(Q)$ is a nonnegative function. In the usual way, the formula

$$(\mathcal{A}u)(x) = \sum_{z\in\Lambda} \partial_{-z} a_z(x, \partial u(x)) \tag{A.8}$$

defines an operator

$$\mathcal{A} : W_0^{1,p}(Q) \longrightarrow W^{-1,p'}(Q)$$

which is continuous, coercive, and strictly monotone. Hence, $\mathcal{A}$ is invertible. Evidently, the operator $\mathcal{A}$ may be rewritten in the standard divergence form

$$\mathcal{A}u = -\sum_{i=1}^{n} \partial_i a_i(x, \nabla u),$$

with suitable $a_i(\cdot,\cdot)$.

Now let us introduce the concept of G-convergence of operators $\mathcal{A}_\varepsilon$ to an operator $\mathcal{A}$ of the form (A.8). For any $u \in W_0^{1,p}(Q)$, there exists a unique $u_\varepsilon \in W_0^{1,p}(Q_\varepsilon)$ such that

$$\mathcal{A}_\varepsilon u_\varepsilon = (\mathcal{A}u)^\varepsilon.$$

We say that $\mathcal{A}_\varepsilon$ is *G-convergent* to $\mathcal{A}$ (in symbols, $\mathcal{A}_\varepsilon \xrightarrow{G} \mathcal{A}$) if $u_\varepsilon \to u$ weakly in $W_0^{1,p}(Q_\varepsilon)$, for any $u \in W_0^{1,p}(Q)$. If, in addition,

$$a_z^\varepsilon(x, du_\varepsilon) \to a_z(x, \partial u), \quad z \in \Lambda,$$

weakly in $\mathrm{L}^{p'}(Q_\varepsilon)$, for any $u \in W_0^{1,p}(Q)$, we say that $\mathcal{A}_\varepsilon$ is *strongly G-convergent* to $\mathcal{A}$ (in symbols, $\mathcal{A}^\varepsilon \overset{G}{\Longrightarrow} \mathcal{A}$). These definitions give us "discrete" counterparts of corresponding concepts for differential operators.

To study G-convergence we need to impose an additional assumption. We shall suppose that, for $\eta, \eta' \in \mathbf{R}^\Lambda$,

$$|a_z^\varepsilon(x,\eta) - a_z^\varepsilon(x,\eta')|^{p'} \le \theta \cdot \left(h^\varepsilon(x) + |\eta|^p + |\eta'|^{1-s/p}|\eta - \eta'|^s\right), \quad x \in Q_\varepsilon, \tag{A.9}$$

and

$$|a_z(x,\eta) - a_z(x,\eta')|^{p'} \le \overline{\theta} \cdot \left(h(x) + |\eta|^p + |\eta'|^p)^{1-\overline{s}/p}|\eta - \eta'|^{\overline{s}}\right), \quad x \in Q, \tag{A.10}$$

where $\theta, \overline{\theta} > 0$, $0 < s, \overline{s} \le p'$, and $h, \overline{h} \in \mathrm{L}^1(Q)$ are nonnegative functions.

Theorem A.1 *Assume conditions (A.2), (A.3) and (A.9) to be valid. Then there exist a subsequence $\varepsilon' \to 0$ and an operator $\mathcal{A}$ satisfying (A.6), (A.7) and (A.10) with suitable $\overline{c}_0$, $\overline{c}(x)$, $\overline{h}(x)$ and $\overline{s} \le s$ such that $\mathcal{A}_{\varepsilon'} \overset{G}{\Longrightarrow} \mathcal{A}$.*

The exponent $\overline{s}$ may be estimated as in n^0 2.3.1.

Let $Q' \subset Q$. Then we may consider operators $\mathcal{A}_{\varepsilon|Q'_\varepsilon}$ and $\mathcal{A}_{|Q'}$ defined in the evident way. We have

Theorem A.2 *Under conditions (A.2), (A.3), and (A.9) assume that* $\mathcal{A}_\varepsilon \xRightarrow{G} \mathcal{A}$. *Then* $\mathcal{A}_{\varepsilon|Q'_\varepsilon} \xRightarrow{G} \mathcal{A}_{|Q'}$ *for any subdomain* $Q' \subset Q$.

Furthermore, the following result on convergence of arbitrary solutions takes place.

Theorem A.3 *Under the assumptions of Theorem A.2 suppose that* $v_\varepsilon \in W^{1,p}(Q_\varepsilon)$, $v_\varepsilon \to v$ *weakly in* $W^{1,p}(Q_\varepsilon)$, *and* $\mathcal{A}_\varepsilon v_\varepsilon \to f$ *strongly in* $W^{-1,p}(Q_\varepsilon)$. *Then* $\mathcal{A}u = f$ *and* $a_z^\varepsilon(x, dv_\varepsilon) \to a_z(x, \partial v)$ *weakly in* $\mathrm{L}^{p'}(Q_\varepsilon)$, *for any* $z \in \Lambda$.

The last result shows us that, exactly as in the case of differential operators, the Dirichlet problem does not play any special role in the theory of G-convergence. Moreover, using Theorem A.3 one can see evidently that the definition of G-convergence may be reformulated as follows:

> $\mathcal{A}_\varepsilon \xrightarrow{G} \mathcal{A}$ if and only if for any $u \in W_0^{1,p}(Q)$ there exists $\psi_\varepsilon \in W_0^{1,p}(Q)$ such that $\psi_\varepsilon \to 0$ weakly in $W^{1,p}(Q_\varepsilon)$ and
>
> $$\lim \|(\mathcal{A}u)^\varepsilon - \mathcal{A}_\varepsilon(u^\varepsilon + \psi_\varepsilon)\|_{*,p',\varepsilon} = 0.$$

If, additionaly,

$$a_z^\varepsilon(x, du^\varepsilon + \psi_\varepsilon)) \to a_z(x, \partial u), \quad z \in \Lambda,$$

weakly in $\mathrm{L}^{p'}(Q_\varepsilon)$, then $\mathcal{A}_e \xRightarrow{G} \mathcal{A}$. In the linear case the last description of G-convergence reduces to that used in [183]. We see also that, due to ψ_ε, the notion of G-convergence is weaker that the usual notion of approximation of differential operator by difference schemes [240].

The proofs of Theorems A.1 – A.3 may be carried out along the same lines as in n^0 2.3.2 for differential operators. As a by-product, one obtains the following useful statement. For a vector function $\psi \in \mathrm{L}^p(Q)^\Lambda$, let us define

$$\mathcal{A}_\varepsilon^\psi(u) = \sum_{z\in\Lambda} d_{-z} a_z^\varepsilon(x, \psi^\varepsilon + du),$$

$$\mathcal{A}^\psi(u) = \sum_{z\in\Lambda} \partial_{-z} a_z(x, \psi + \partial u).$$

Theorem A.4 *Under conditions (A.2), (A.3), (A.6), (A.7), (A.9), and (A.10), the following statements are equivalent:*

1. $\mathcal{A}_\varepsilon \xRightarrow{G} \mathcal{A}$;
2. $\mathcal{A}_\varepsilon^\psi \xRightarrow{G} \mathcal{A}^\psi$ for any $\psi \in \mathrm{L}^p(Q)^\Lambda$;
3. $\mathcal{A}_\varepsilon^\eta \xRightarrow{G} \mathcal{A}^\eta$ for any $\eta \in \mathbf{R}^\Lambda \subset \mathrm{L}^p(Q)^\Lambda$.

In the situation we consider convergence of solutions is accompanied by convergence of corresponding energies. Let

$$E_\varepsilon(v_\varepsilon)(x) = \sum_{z\in\Lambda} a_z^\varepsilon(x, dv_\varepsilon(x))d_z v_\varepsilon(x),$$

$$E(v)(x) = \sum_{z\in\Lambda} a_z(x, \partial v(x))\partial_z v(x),$$

Theorem A.5 *Under the same conditions as in Theorem A.3, we have*

$$\langle E_\varepsilon(v_\varepsilon), \varphi^\varepsilon\rangle \to \langle E(v), \varphi\rangle$$

for any $\varphi \in C_0^\infty(Q)$.

A.3 Homogenization

We consider homogenization of nonlinear difference schemes with random homogeneous coefficients in the following setting.

Let $T(x)$, $x \in \mathbf{Z}^n$, be a discrete multi-time dynamical system on a probability space $(\Omega, \mathcal{F}, \mu)$, i.e

$$T(x) : \Omega \longrightarrow \Omega$$

is a measurable map preserving the measure μ,

$$T(x+y) = T(x)T(y), \quad T(0) = I.$$

We shall assume $T(x)$ to be ergodic. Associated to $T(x)$, there is a group $U(x)$ of isometries of $\mathrm{L}^p(\Omega)$ defined by

$$(U(x)f)(\omega) = f(T(x)\omega), \quad \omega \in \Omega, \quad x \in \mathbf{Z}^n.$$

Let

$$a_z : \Omega \times \mathbf{R}^\Lambda \longrightarrow \mathbf{R}, \quad z \in \Lambda,$$

be a Carathéodory function satisfying inequalities of the type (A.6), (A.7) and (A.10), with $\overline{s}$ replased by s and nonnegative *constants* c and h.

Set

$$\mathcal{A}_\varepsilon(\omega)u = \sum_{z\in\Lambda} d_{-z}a_z(T(\varepsilon^{-1}x)\omega, du).$$

The operator $\mathcal{A}_\varepsilon(\omega)$ is well-defined for almost all $\omega \in \Omega$.

Theorem A.6 *Under the above mentioned conditions, there exists a translation invariant operator*

$$\hat{\mathcal{A}}u = \sum_{z\in\Lambda} \partial_{-z}\hat{a}_z(\partial u) \tag{A.11}$$

satysfying (A.6), (A.7) and (A.10) such that $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \hat{\mathcal{A}}$ *for any bounded domain* $Q \subset \mathbf{R}^n$ *and almost all* $\omega \in \Omega$.

To construct the function $\hat{a}_z(\eta)$ we need to solve an auxiliary equation. Let $D_z = U(z) - I$, $z \in \mathbf{Z}^n$, and $D = \{D_z,\ z \in \Lambda\}$. Define the subspace $\mathbf{V} \subset \mathrm{L}^p(\Omega)^\Lambda$ as the closure of the set of all vector valued functions $D\varphi$, where $\varphi \in \mathrm{L}^p(\Omega)$. In the terminology of [182], $\mathbf{V}$ is the space of exact forms. From the isometry property of $U(x)$ it follows easily that the standard norm on $\mathbf{V}$ is equivalent to that defined by

$$\|v\|^p = \sum_{i=1}^n \|v_{e_i}\|^p_{p,\Omega}, \quad v = \{v_z\}_{z\in\Lambda}.$$

This remark and the well-known monotonicity method imply that, for any $\eta \in \mathbf{R}^\Lambda$, there exists a unique solution $v = \{v_z(\omega,\eta)\}_{z\in\Lambda}$ of the problem

$$\sum_{z\in\Lambda} \langle a_z(\omega, \eta + v) \cdot D_z\varphi(\omega)\rangle = 0 \quad \forall\varphi \in \mathrm{L}^p(\Omega), \tag{A.12}$$

where $\langle\cdot\rangle$ stands for the mean value over Ω. Next we define $\hat{a}_z(\eta)$ by the formula

$$\hat{a}_z(\eta) = \langle a_z(\omega, \eta + v(\omega,\eta))\rangle. \tag{A.13}$$

Now we outline the proof of Theorem A.6. The key point is the existence of a primitive function for any exact form. More precisely, let $v = \{v_z\} \in \mathbf{V}$. Then there exists a random field $w(\omega, x)$ satisfying the following properties:

(i) $w(\cdot, x) \in \mathrm{L}^p(\Omega)$ *and* $\langle w(\cdot, x)\rangle = 0$ *for any* $x \in \mathbf{Z}^n$;

(ii) $w(\cdot, z) = v_z(\cdot)$ *for any* $z \in \Lambda$;

(iii) *for any* $x, y \in \mathbf{Z}^n$,

$$w(\omega, x + y) - w(\omega, x) = w(T(x)\omega, y) \quad \textit{a.e. on } \Omega;$$

(iv) $\|w(\cdot, x)\|_{p,\Omega} \le c\|v\| \cdot |x|, \quad x \in \mathbf{Z}^n.$

It is sufficient to prove this statement only for a dense set of such v's. Then, due to (iv), one can pass to the limit. If $v = D\varphi$, where $\varphi \in \mathrm{L}^p(\Omega)$, we put $w(\omega, x) = D_x\varphi(\omega)$. For such w, it is not difficult to verify (i) – (iii) and all we need is to prove (iv). Let $x = (x_1, \ldots, x_n)$,

$$y_k = \sum_{i=1}^k x_i e_i, \quad y_0 = 0.$$

Then we have

$$\begin{aligned} w(\omega, x) &= \sum_{i=1}^{n} (U(y_k) - U(y_{k-1}))\, \varphi(\omega, z) = \\ &= \sum_{i=1}^{k} U(y_{k-1})\, (U(x_k e_k) - I)\, \varphi(\omega, z). \end{aligned}$$

Furthermore,

$$(U(x_k e_k) - I) = \left(\sum_{l=0}^{x_k} U(le_k) \right) D_{e_k}.$$

The last two identities, together with the isometry property of the group $U(x)$, imply (iv) immediately.

Now we proceed as follows. By Theorem A.1, given Q we may assume that, for generic $\omega \in \Omega$, $\mathcal{A}_\varepsilon(\omega) \overset{G}{\Longrightarrow} \mathcal{A}$, where $\mathcal{A}$ is of the form (A.8), and to end the proof it is sufficient to show that

$$a_z(x, \eta) = \hat{a}_z(\eta), \quad z \in \Lambda, \quad \eta \in \mathbf{R}^n.$$

Let $v = v(\omega, \eta)$ be the solution of (A.12), and $w = w(\omega, x, \eta)$ the corresponding primitive function. We define the function $w_\varepsilon(x)$ on Q_ε by

$$w_\varepsilon(x) = \varepsilon w(T(\varepsilon^{-1} x)\omega, \varepsilon^{-1} x, \eta).$$

One can show that $w_\varepsilon \longrightarrow 0$ weakly in $W^{1,p}(Q)$. Moreover,

$$dw_\varepsilon(x) = v\left(T(\varepsilon^{-1} x)\omega\right) + r_\varepsilon(x),$$

where, by (iii) and (iv), the rest $r_\varepsilon \longrightarrow 0$ strongly in $\mathrm{L}^p(Q)^\Lambda$. Since v is a solution of (A.12), we see that $\mathcal{A}_\varepsilon(\eta + w_\varepsilon) \longrightarrow 0$ strongly in $W^{-1,p'}(Q)$. Now, by Theorem A.3,

$$a_\varepsilon\left(T(\varepsilon^{-1} x)\omega, \eta + dw_\varepsilon\right) \longrightarrow a_z(x, \eta)$$

weakly in $\mathrm{L}^{p'}(Q)$. On the other hand, by the Discrete Ergodic Theorem,

$$\begin{aligned} \lim a_z\left(T(\varepsilon^{-1} x)\omega, \eta + dw_\varepsilon\right) &= \lim a_z\left(T(\varepsilon^{-1} x)\omega, \eta + v\left(T(\varepsilon^{-1} x)\omega\right)\right) = \\ &= \langle a_z(\omega), \eta + v(\omega) \rangle. \end{aligned}$$

Thus, $a_z(x, \eta) = \hat{a}_z(\eta)$ and we conclude.

As usual, in the case of almost periodic operators Theorem A.6 permits us to obtain an individual homogenization theorem. With this aim, let us recall the notion of almost periodic function on $\mathbf{Z}^n$. A bounded function $f : \mathbf{Z}^n \longrightarrow \mathbf{R}$ is said to be *almost periodic* if the family of shifts $\{f(\cdot + y)\}_{y \in \mathbf{Z}^n}$ is precompact in the space of all bounded functions on $\mathbf{Z}^n$. Any a.p. functon may be extended to a contunuous function defined on the Bohr compactification $\mathbf{Z}^n_B$ of $\mathbf{Z}^n$ [225].

Now we consider a family of functions

$$a_z : \mathbf{Z}^n \times \mathbf{R}^\Lambda \longrightarrow \mathbf{R}, \quad z \in \Lambda,$$

satisfying (A.6), (A.7) and (A.10), with Q replaced by $\mathbf{Z}^n$ and the constant functions c and h. Assume also that the function

$$\frac{a_z(x,\eta)}{1+|\eta|^{p-1}}, \quad z \in \Lambda,$$

is almost periodic in $x \in \mathbf{Z}^n$, uniformly with respect to $\eta \in \mathbf{R}^\Lambda$.

Theorem A.7 *Under the above mentioned conditions, there exists an operator*

$$\hat{\mathcal{A}}u = -\sum_{z\in\Lambda} \partial_z \hat{a}_z(\partial u)$$

such that, for any open bounded subset Q, the family

$$\mathcal{A}_\varepsilon u = -\sum_{z\in\Lambda} d_z a_z(\varepsilon^{-1}x, du)$$

strongly G-converges to $\hat{\mathcal{A}}$ on Q.

To prove the theorem one need to introduce the family of operators

$$\mathcal{A}_\varepsilon(\omega)u = -\sum_{z\in\Lambda} d_z a_z(\omega + \varepsilon^{-1}x, du),$$

where $\omega \in \Omega = \mathbf{Z}^n_B$ and $a_z(x,\eta)$ is extended to $\Omega \times \mathbf{R}^\Lambda$ by almost periodicity. By Theorem A.6, $\mathcal{A}_\varepsilon(\omega)$ has a homogenized operator $\hat{\mathcal{A}}$ for generic $\omega \in \Omega$. Now, as in Section 3.3, one can deduce from the almost periodicity condition that the same is true for *all* $\omega \in \Omega$.

Appendix B

Open Problems

Here we collect some open problems which seem to be of interest.

1. In [134, 135], it is studied G-convergence and periodic homogenization for a class of single-valued monotone degenerate elliptic operators. Degeneration is characterized by means of suitable weight function which belongs to the so-called Muckenhoupt class A_p. It seems to be possible to extend the results of those papers to the cases of non-monotone single-valued and monotone multivalued operators. However, homogenization for random operators of such kind is, perhaps, a more delicate problem.

2. It would be of interest to find a version of the notion of two-scale convergence for the case of random homogeneous fields (see [7, 8] for the case of periodic functions). Any progress in this direction may clarify our understanding of random homogenization. In particular, this provides one of possible ways to construct correctors in the random setting (see, also, the next problem).

3. In [116], correctors are constructed for periodic homogenization of nonlinear elliptic operators (see, also, n^0 3.5.2 for a brief account). This construction is extended to the case of almost periodic homogenization in [69]. Hence, it would be of interest to construct such correctors in the framework of random operators.

4. Several authors obtained homogenization results for the case of *non-divergent* second order linear elliptic operators. A natural way to handle this case is to subject an operator under consideration to a simple transformation in order to make it divergent (see, e.g., [164] for more details). However, it is completely unclear is it possible to find a reasonable class of non-divergent *nonlinear* operators which admit homogenization.

5. Let $a(x)$ be a 1-periodic function defined by

$$a(x) = \begin{cases} \alpha & \text{if } |x| \le t, \\ 1 & \text{if } x \in \square,\ |x| > t, \end{cases}$$

where $\square = \{x \in \mathbf{R}^n : |x_i| \le 1/2\}$, $\alpha > 0$, and $0 < t \le 1/2$. The homogenized operator, $\hat{\mathcal{A}}$, for

$$\mathcal{A}_\varepsilon u = -\operatorname{div}\left(a(\varepsilon^{-1}x)\nabla u\right)$$

is of the form

$$\hat{\mathcal{A}}u = -\operatorname{div}(\hat{a}\nabla u),$$

with $\hat{a}$ being a scalar constant. There is no explicit expression for $\hat{a}$. However, it is known the so-called Rayleigh-Maxwell asymptotic formula for $\hat{a}$, as $t \to 0$. See [164] for details. It is of interest to get a similar formula in the nonlinear case, e.g., for the operator

$$\mathcal{A}_\varepsilon u = -\operatorname{div}\left(a(\varepsilon^{-1}x)|\nabla u|^{p-2}\nabla u\right).$$

6. Is it possible to remove the assumption

$$p > \frac{2n}{n+2}$$

in the results of Sections 4.1 and 4.2?

7. If seems to be very natural that G-convergence of nonlinear parabolic operators should be accompanied by a kind of convergence for their attractors. A very partial theorem of such type is obtained in [64]. However, no general result seems to be known. There is also a more deep question concerning the limit behaviour of a fine structure of atractors under G-convergence.

8. In Section 4.3, we have presented a very restricted result on homogenization of filtration problems. It would be interesting to extend it to the case when the function φ depends also on k and x, not only on u. More general problem concerns homogenization of doubly nonlinear evolutiton operators of the form

$$\partial_t\beta(x, u) + \mathcal{A}(u),$$

or

$$\beta(x, \partial_t u) + \mathcal{A}(u),$$

where $\mathcal{A}$ is a nonlinear second order elliptic operator. Some existence results for operators of such kind may be found in [109].

9. Another class of operators interesting from the point of view of homogenization is the class of Sobolev type operators, i.e. operators of the form

$$\partial_t \mathcal{B}(u) + \mathcal{A}(u),$$

or

$$\mathcal{B}(\partial_t u) + \mathcal{A}(u),$$

where $\mathcal{A}$ and $\mathcal{B}$ are second order elliptic operators. For the theory of these operators, we refer to [155]. In the linear case, it is not so difficult to homogenize such operators. By means of the Laplace transform, the last problem reduces to the standard elliptic homogenization problem of the form

$$-\mathrm{div}(a^{\varepsilon}(x,p)\nabla u),$$

where p is the dual variable to t. Returning to the original variable t one can see that the homogenized operator will be *non-local* in t. This effect must occur in general situation as well. We have to point out that homogenization of some nonlinear non-local evolution operators was studied in [26].

10. Homogenization of parabolic equations with changing time direction seems to be unstudied. The following example is of particular interest:

$$p(\varepsilon^{-1}x)\partial_t u + \mathcal{A}_{\varepsilon}(u),$$

where $\mathcal{A}_{\varepsilon}$ is an elliptic operator and $p(y)$ is a periodic function which takes both positive and negative values. Of course, one need to impose the forward initial condition at the set $\{(0,x) : p(\varepsilon^{-1}x) > 0\}$ and the backward one at $\{(T,x) : p(\varepsilon^{-1}x) < 0\}$. We refer to [200], Ch. 3, n^0 2.6, and [229] for some existence results concerning equations of such kind.

11. *Boundary homogenization.* Assume that the boundary ∂Q of an open subset Q, or some its part, is devided into two subsets Γ_1^{ε} and Γ_2^{ε} which are "highly intermixed". Let us consider a solution u_{ε} of a *nonlinear* elliptic boundary value problem, with different type of boundary conditions imposed on Γ_1^{ε} and Γ_2^{ε}. What kind of assymptotic behaviour of u_{ε} may occur? For the case of linear boundary value problems see, e.g., [97, 126, 148, 152].

12. *Reinforcement by a thin layer.* Suppose Q is an open bounded subset of $\mathbf{R}^n$, $Q \subset Q_{\varepsilon}$, and $\mathrm{dist}(Q, \partial Q_{\varepsilon}) < \varepsilon$. Additionally, the surrounding strip $Q_{\varepsilon} \setminus Q$ may have an oscillating thickness. Let u_{ε} be a solution of a boundary value problem of the form, for example,

$$-\mathrm{div}\, a(x, \nabla u) = f \quad \text{on } Q,$$

$$-\varepsilon^{-1}\mathrm{div}\, a(x, \nabla u) = f \quad \text{on } Q_\varepsilon \setminus Q,$$

with suitable boundary and transmission conditions on ∂Q_ε and ∂Q, respectively. What may one claim on the asymptotic behaviour of u_ε, as $\varepsilon \to 0$? This problem seems to be studied in the linear case only (see, e.g., [42, 85]).

13. Is it possible to extend the results presented in Appendix A to the case of multivalued monotone difference schems? Perhaps, it is not so difficult to do this for general results of Section A.2 devoted to G-convergence. However, the proof of Theorem A.6 is based essentially on the continuity of $a_z(x, \eta)$ with respect to η.

14. Another interesting problem is to study homogenization of nonlinear difference operators on general networks, not only on rescaled lattices. The particular case of linear operators on planar polygonal networks was considered in [262]. We would like to point out that at least two fundamentally different situation may occur:

    (a) in the limit, the networks considered fill entirely an open set, like in [262];

    (b) the networks approximate a fractal set (cf. [185]).

    In the case (a) the homogenized operator should be a standard differential operator, while in the case (b) a "differential" operator on a fractal set have to appear.

15. In connection with homogenization of difference operators, it is useful to study Γ-convergence of functionals defined on spaces of mesh functions, i.e. functions whose domain is the set of vortices of a network. It appears that many results on Γ-convergence of integral functionals may have their own counterparts in the present setting. Of course, as in the operator setting, different types of limit functionals may occur: usual integral functionals and functionals on fractal sets, depending on the type of limit behaviour of underlying networks. However, to our knowledge, problems of such kind are still unstudied.

16. *Homogenization on the Heisenberg group.* In [54], it is studied the homogenization problem for the family of *degenerate* elliptic operators of the form

$$\mathcal{A}_\varepsilon u = -\sum_{i,j=1}^{2} X_i \left(a_{ij}(\varepsilon^{-1}\theta) X_j u \right)$$

on $\mathbf{R}^3$, where $\theta = (x, y, z)$,

$$X_1 = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z}, \quad X_2 = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial z}.$$

It is assumed that the real symmetric (2×2)-matrix $a(\theta) = \{a_{ij}(\theta)\}$ belongs to $L^\infty(\mathbf{R}^3)$, satisfies the uniform ellipticity condition, and is "2-periodic" with respect to the Heisenberg group srtucture on $\mathbf{R}^3$, i.e.

$$a(\theta + k) = a(\theta), \quad k \in 2 \cdot \mathbf{Z}^3,$$

where

$$\theta + \theta' = (x + x', y + y', z + z' + 2(x'y - xy')), \quad \theta, \theta' \in \mathbf{R}^3.$$

All those results are still valid in the case of $\mathbf{R}^{2n+1}$. In this setting, many things are still unclear, e.g., nonlinear and/or random homogenization.

17. It is studied in [218] the homogenizaton problem for the equation

$$-\mathrm{div}(a^\varepsilon(x)\nabla u) - \mathrm{div}\,(\Phi(u)) = f,$$

where the function Φ is assumed to be continuous only, no growth condition is imposed. By this reason, the so-called renormalized solutions are considered there. Is it possible to obtain a similar result in the case when the second term in the left-hande part is replaced by $-\mathrm{div}\,(\Phi^\varepsilon(x, u))$?

18. There is a lot of papers on nonlinear homogenization in perforated domains and a lot of problems is still open there. We mention here only two such problems. In the first one, we suggest to prove homogenization results for operators of the class $\mathcal{E}$ in domains with holes, both for the cases of Neumann and Dirichlet conditions imposed on the boundaries of holes. For more restricted classes of operators such results was obtained in [101, 137, 174, 178, 179] (the case of Neumann conditions) and in [114, 121, 176, 248, 249, 251, 252] (the case of Dirichlet conditions). Secondly, it would be intetesting to extend the results of [1] to operators of the class $\mathcal{E}$ and to the case of multivalued operators.

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