THE LONG LINE AND LIMIT POINT COMPACTNESS

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1. Introduction

The Long Line is a classic counterexample in point-set topology, which is associated with the idea of limit point compactness. This is a brief summary of how the long line is constructed, limit point compactness, and several proofs relative to both topics.

2. The Long Line

First, the idea is to stick uncountably many copies of [0,1) end to end. The real line is countably many of these. Consider the set [0,1) x $\mathbb{N}$ this is [0,∞). Now we do the same thing uncountable times. take S_Ω to be the minimal, uncountable, well-ordered set. So let S_Ω be the least uncountable ordinal, the smallest uncountable set that is well-ordered. We define the longline to be S_Ω x [0,1) - $\{s_o, 0\}$. We take out the smallest element since $\mathbb{R}$ has no least element, and we want the long line to resemble the reals. So this is the least uncountable ordinal with it's smallest element removed. This is the long line. Basically we are taking this ordinal space [0,Ω) and between each ordinal, call it α and it's successor, $\alpha + 1$, a copy of the open unit interval (0,1). We take all this and then apply the order topology to it. We can define an 'extended' long line, by consider the ordinal space [0,Ω] instead of [0,Ω). This is the long line, we will call it L. We will refer to the extended long line as L_*.

2.1. First countable and second countable.

Basically a space is first countable, if for each point there is a countable collection of neighborhoods around it, each containing such that any neighborhood of x has one of the collection in it. Formally a space is **first-countable** if it has a countable basis at each of its points. For each $x \in X$, X has a **countable basis at x** if there is a collection of neighborhoods of x such that each neighborhood of x contains at least one of the elements of . A space X is **second countable** if it has a countable basis for its topology. Since L is not second countable, it is not homeomorphic to $\mathbb{R}$. Although we will see that L behaves locally just like $\mathbb{R}$ is has some unique properties.

Theorem 1. *The long line is first countable.*

Proof. Consider a point $x \in L$. x can be considered as the least upper bound of points which precede it. These points, $\{X_n\}$ is countable, so let $\{(x_n, x + 1/n)\}$ be the collection of nieghborhoods, it is countable, and so each neighborhood contains one of the points in it, so L is first countable. □

Theorem 2. *The long line is not second countable.*

Proof. Let $\beta = \{\text{w} : \text{w} < \alpha, \alpha \in [0, \Omega)\}$ be a basis for L. β is a basis since for L since for each $x \in L$, $x < \alpha$ for some α in $[0, \Omega)$ so $x \subseteq B$, for some B $\in \beta$, and also consider if $x \subset (B_1 \cap B_2)$. Without loss of generality, let B_2 be $\{\text{w} : \text{w} < \alpha + 1, \alpha + 1 \in [0, \Omega)\}$ and let B_1 be $\{\text{w} : \text{w} < \alpha, \alpha \in [0, \Omega)\}$. Then $(B_1 \cap B_2) = B_1$. So if $x \subset (B_1 \cap B_2)$, then let $B_3 = B_1$, then $x \subset B_3$ for some basis element B_3. This is a lot more simple than it looks. Basically for any two basis element,s their intersection is equal to the 'smaller' basis element, cardinality wise. So, for any x in their intersection, it is contained completely in the smaller element, so just let B_3 be that element.

Now consider β, the basis for L. β is clearly uncountable, since we defined the set to continue until S_Ω, the first uncountable ordinal. So taking each ordinal α one at time until S_Ω is impossible, So β is not countable, so L is not second countable. □

2.2. **Some properties of the long line.**

Theorem 3. *In the long line* every *countable subset is bounded.*

Proof. Let $\{x_i\}$ be a set in the long line such that $P_i = \{y|y \leq x_i\}$. So each P_i is countable. Let P be the union of the P_i's, it is a countable union of countable sets, so P $\neq$ L. Consider $w \in P$. Every $x \in L$ is less than w. So consider L - P. For every $z \in L - P$, $z > w$. So there is some element in L that is larger than every element in P, and so that element is an upper bound for P, so every countable subset in L is bounded. □

2.3. **Compactness of the long line.**

Theorem 4. *The long line is not compact.*

Proof. Consider an open covering {w : w $< \alpha, \alpha \in [0, \Omega)$}. This set has no finite subcover. To show this, assume it did, call it F. Since F is finite, it is countable, so let γ be the least upper bound of F, then γ is countable, and F covers L, but L remember is the least uncountable ordinal, so $F \neq L$. □

Theorem 5. *The extended long line is compact.*

Proof. Consider an open neighborhood U = $(U_0, U_n) \subset [0, \Omega]$. Now consider $U^c = [0, U_0] \cup [U_n, \Omega]$ Both these sets are countable, so since U was arbitrary, each open neighborhood of U has a compact complement, and so the extended long line is compact. □

2.4. **Seperability of the long line.**

Theorem 6. *The long line is not seperable.*

Proof. Let $C \subseteq L$, C countable. Let γ be the least upper bound of C. Now suppose C is dense in L, that is, $\overline{C} = L$. But now consider a point y $\in$ A, where $A =\{y \in L : y > \gamma\}$. This set is nonempty, open, and disjoint from C. So since $A \cap C = \emptyset$ and $A \subseteq L$, then $\overline{C} \neq L$, contradiction, so C is not dense in L. So L has no countable dense set, and so is not seperable. □

2.5. **Connectedness of the long line.** Both the long line and extended long line are connected. The long line L is path connected and locally path connected, but the extended long line is not. Recall a path is a continuous map $f : [a, b] \to X$ of some closed interval of R into X so f(a) = x and f(b)=y. A space is path connected if *every* pair of points can be joined by a path in X.

Theorem 7. *The long line and extended long line are connected.*

Proof. Assume L and L_* are *not* connected. So L = $A \cup B$ where $A, B \subset L$, $A \cap B = \emptyset$. Without loss of generality (?) let every element $b \in B$ be such that for every $a \in A$, $b > a$. So choose the minimal element such that everything to the 'right' of it is in B. Let this be x. So $x = LUB\{a : \exists b \in B, b > a\}$. Now consider if $x \in A$ or $x \in B$. By definition, every neighborhood of x contains elements of A, but if $x \in A$, then since A and B are disjoint, then no points of the neighborhood of x can be in B, but they must have points. Similar is the case for $x \in B$. So B and A are not neighborhoods of x, but x must be somewhere in L by definition, but we have shown it is not, a contradiction, so L is connected. □

Lemma 8. *Let a_0 be the smallest element of S_Ω, for each $a \in S_\Omega$ such that $a \neq a_0$ then $[a_0\ x\ 0, ax\ 0) \in S_\Omega$ has order type of [0,1).*

Proof. Assume it is true for all $\alpha < \beta$ we want to show it is true for β. We proceed by cases, either β has an immediate predeccesor or it does not. Assume β has an immediate predecessor in S_Ω, call it α_1 By assumption, $[\alpha_0$ x $0, \alpha_1$x 0) has order type [0,1). Consider $[\alpha_1$x $0, \beta$x $0) = (\alpha_1$x $[0, 1)) \cup \{\beta$ x0$\}$, this, by considering the seperate order types, has order type [0,2) which is equivalent to order type [0,1). Now assume β has no immediate predecessor in S_Ω. So β is the suprememum of some increasing sequence $\alpha_1, \alpha_2, ...of S_\Omega$. $[\alpha_0$ x $0, \alpha_{i+1}$ x 0), by hypothesis, has order tpye [0,1). Let α_i x 0 = c, c $\in \mathbb{R}$. So $[\alpha_i$ x $0, \alpha_{i+1}$x 0) has order type [c, 1), which is equivalent to [0,1). Note that J = $[\alpha_0$ x $0, \beta$x $0) = [\alpha_0$ x $0, \alpha_1$ x $0) \cup [\alpha_1$ x $0, \alpha_2$ x $0) \cup ...[\alpha_i$ x $0, \alpha_{i+1}$ x $0) \cup ...$ Since J = $[0, 1) \cup [1, 2) \cup ... = [o, \infty)$ which has order type [0,1). □

Theorem 9. *The long line is path connected.*

Proof. Let $x \in L$ such that given $\alpha \in S_\Omega$ x $< \alpha$ x 0. So x $\in (\alpha_0$ x $0, \alpha$x 0), which has order type (0,1). So L is the union of open intervals, each of which are path connected, and since they have a common point, namely, α_0 x 1/2, L is path connected.

□

Theorem 10. *The extended long line is* not *path connected.*

Proof. Consider the points 0, $\Omega \in$ L*. There is no closed interval [a,b] $\in \mathbb{R}$ such that f(a)=0 and f(b)=Ω since Ω is the least uncountable ordinal, $\Omega \notin \mathbb{R}$. So there is not a path for every two point in L*, so the extended long line is not path connected. □

3. Limit Point Compactness

Let's start with a definition.

Definition 11 (Limit point compact)**.** *A space X is said to be* **limit point compact** *if every infinite subset of X has a limit point.*

Limit point compactness is a weaker kind of property than regular old compactness, our first theorem demonstrates this.

Theorem 12. *If X is compact, then X is limit point compact.*

Proof. Suppose X is compact. Let A $\subseteq$ X, A finite and arbitrary. Assume A has no limit point, so A contains all its limit points, and so A is closed. Therefore A^c is open. For each $a \in A$ let U_a be a neighorhood of a such that $U_a \cap A = \{a\}$. So U_a and A^c are an open cover of X, and since X is compact, then the covering is finite. Therefore A is finite, (since U_a contains only one point of A) and we have proven the contrapositive (if A has no limit point, then A is finite). □

Now let's apply this to the long line (sort-of). Consider S_Ω, this space is not compact, and so we will see the converse of the above statement does not hold, since S_Ω is limit point compact.

Theorem 13. *S_Ω is limit point compact.*

Proof. Let A be an infinite subset of S_Ω. Let $B \subseteq A$, B countably infinite. Since B is countable, it has a least upper bound, call it b. So $B \subseteq [a_0, b]$, a_0 the smallest element of S_Ω. We see that $[a_0, b]$ is compact , so it is limit point compact, so let x be the limit point of B in $[a_0, b]$, x is also a limit point of A, so S_Ω is limit point compact.

□

There's yet another version of compactness available to us, a space can slo be sequentially compact. A space X is **sequentially compact** if every sequence of point of X has a convergent subsequence. This is defined the same way here as in analysis.

Theorem 14. *Let X be metrizable, then the following are equivalent:*
1) *X is compact*
2) *X is limit point compact*
3) *X is sequentially compact*

Proof.
$1 \rightarrow 2$) Covered above (Theorem 11).
$2 \rightarrow 3$) Assume X is limit point compact. Let $\{x_n\}$ be a sequence in X. Consider A = $\{x_n | n \in (Z)\}$. Either A is finite or A is infinite. If A is finite, then there is a point x such that x x_n for *infinitely* many n, so x_n has a subsequence which is constant, and so converges. If A is infinite, then let x be a limit point of A. Let $x_{n_1} \in B(x, 1)$. Now consider n_{i-1}, so there is a ball $B(x, 1/i)$ that intersects A in infinitely many points, choosing index $n_i > n_{i-1)}$ such that $x_{n_i} \in B(x, 1/i)$. Then $x_{n_1}, x_{n_2}, ...$ converges to x.
$3 \rightarrow 1$) □