Minerva Lectures: Universality in models of local random growth via Gibbs resamplings

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ABSTRACT. An important technique for understanding a random system is to find a higher dimensional random system that enjoys an attractive and tractable structure and that has the system of interest as a marginal; and to analyse the new structure to make inferences about the original system. For example, the Airy_2 process is an important and natural random process, mapping the real line to itself, since it offers, rigorously in certain examples and putatively in very many more, a scaled description at advanced time of a random interface whose growth is stimulated by local randomness and which is subject to restoring forces such as surface tension. The Airy_2 process may be embedded in a canonical way as the uppermost curve in a richer random object, the Airy line ensemble - an ordered system of random continuous curves. This richer object has an attractive probabilistic property not apparent in the Airy_2 process itself - it is, with suitable boundary conditions, an infinite system of mutually avoiding Brownian motions; and, as such, it enjoys a natural resampling probability called the Brownian Gibbs property. The Brownian Gibbs property of the Airy line ensemble is a key probabilistic technique by which aspects of the concerned Kardar-Parisi-Zhang universality class of random growth models may be investigated. This short series of lectures will explain how, harnessed with limited but essential inputs of integrable origin, the property has been exploited in the recent work [?] to make very strong inferences regarding the locally Brownian nature of the Airy_2 process; about the scaled coalescence behaviour of geodesics in last passage percolation growth models; and about the structure of the scaled interface when these models are initiated from very general initial conditions.

Contents

CHAPTER 1

Polynuclear growth: multi-line PNG and non-intersecting random walks

In Poissonian last passge percolation, a Poisson point process $\mathcal{P}$ of unit intensity is sampled in the cone $C = \big\{ (x,t) \in \mathbb{R} \times [0,\infty) : \vert x \vert \leq t \big\}$.

The RSK dynamics for the polynuclear growth [PNG] model are determined by $\mathcal{P}$. The dynamics begins at time zero in an initial confiugartion of lines $L_i = \mathbb{R} \times \{ i \}$ indexed by the negative integers $-\mathbb{N}$ including zero.

In the Poisson space, the horizontal line $\{ y = t \}$ rises at unit speed from its initial position along the x-axis. When it first encounters an element (x,t) of $\mathcal{P}$, a nucleation occurs on the uppermost line L_0 in the line space: at time t, the flat line $L_0 = \mathbb{R} \times \{ 0 \}$ is altered at spatial location x so that it adopts the value one there. The deformation may be viewed as a $+1$ jump in the interface followed immediately by a -1 jump. As time evolves from t, the time-level continues to rise in the Poisson picture. In the line space, the up-jump and the down-jump in L_0 travel at velocities of -1 and $+1$, so that the interval of values at which L_0 equals one takes the form $\big(x - (t'-t), x + (t'-t) \big)$ for $t' > t$.

When the second Poisson point is encountered by the rising time line, a corresponding nucleation occurs in L_0 in the line space, and the left and right sides of the resulting deformation proceed again at velocities of minus one and plus one. This nucleation may occur when L_0 has height one or zero; in either case, the nucleation is built at unit height on top of the existing interface. Every Poisson points nucleates on the curve L_0 when its time comes.

At some moment, a right-advancing downstep will collide with a left-advancing upstep in L_0. At this instant, the two interfaces annihilate in L_0, so that immediately after, L_0 locally takes the form of the shared constant value for this line, on the left of the right-advancing downstep and the right of the left-advancing upstep at the moment of collision. However, infromation is not lost, because at this same instant, a nucleation event occurs on the line L_1 at the location of the collision. The usual rules for the subsequent evolution of the nucleated deformation are at play, with the upstep and the downstep in L_1 spreading away at unit speed from their shared location at the collision time.

The process continues, with each Poisson point being responsible for a nucleation in the uppermost curve L_0, the collisions between upstep-downstep pairs giving rise to nucleation in the line whose index is one greater than that in which the collision occurred. Collisions and nucleations spread over a broadening range, both in the spatial and line index sense, as time advances.

Consider given time $t \geq 0$ in the Poisson picture. The time-line intersects the cone C along $[-t,t] \times \{ t \}$. Let $\mathbf{x}$ denote a point in the intersection, and write $\mathbf{0} = (\mathbf{0},\mathbf{0})$ for the origin in the plane, which is the apex of C. An *upgoing* path from $\mathbf{0}$ to $\mathbf{x}$ is formed from a finite collection of planar line segments, each of which makes an angle with the vertical of at most one-half of a right angle. In this way, a journey from $\mathbf{0}$ to $\mathbf{x}$ made along an upgoing path is one in which the traveller moves in straight line segments, always adopting a compass direction that is at least as northerly as NW or NE, with a change of direction permitted on finitely many occasions. Note that an upgoing path from $\mathbf{0}$ to (x,t) exists precisely when $|x| \leq t$, at points where the time-line meets the cone.

For $x \in \mathbb{R}$, let $h(x,t)$ denote the maximum number of elements of the Poisson cloud $\mathcal{P}$ that lie on an upgoing path between $\mathbf{0}$ and (x,t). If no upgoing path exists due to $|x| > t$, $h(x,t) = 0$ is understood. The profile $\mathbb{R} \to \mathbb{N} : x \to h(x,t)$ is known as the PNG interface.

For $j \in \mathbb{N}$, we may define

$$D_j(x,t) \; = \; \Big\{ (\phi_1, \cdots, \phi_j) : \phi_i \text{ an upgoing path from } (0,0) \text{ to } (x,t), \;\; i \in [\![1,j]\!] \\ \text{with the } \phi_i \text{ pairwise disjoint, except at } (0,0) \text{ and } (x,t) \Big\} .$$

Define the length $\ell(\phi)$ of $\phi \in D_1(x,t)$ to be the number of Poisson points visited by ϕ. Extend the notation to $D_j(x,t)$ by setting

$$\ell\big((\phi_1, \cdots, \phi_j)\big) \; = \; \sum_{i=1}^{j} \ell(\phi_i) \;\; \text{for } (\phi_1, \cdots, \phi_j) \in D_j(x,t) .$$

Now set

$$M_j(x,t) = \max \Big\{ \ell(\phi) : \phi \in D_j(x,t) \Big\} .$$

Proposition 1.1. For $j \in \mathbb{N}$ and $(x,t) \in \mathbb{R} \times [0,\infty)$, we have that

$$\sum_{i=0}^{j-1} \big(L_{-i}(x,t) + i \big) \; = \; M_j(x,t) .$$

Definition 1.2. Let $\lambda \in (0,\infty)$.

(1) A continuous-time simple random walk of rate λ is a Markov process $X : [0,\infty) \to \mathbb{Z}$, $X(0) = n$, whose jump times are a Poisson process of rate λ, at each of which occur moves up, by $+1$, or down, by -1, according to the independent flip of a fair coin.

(2) Let $x_1, x_2 \in \mathbb{R}$ satisfy $x_1 \leq x_2$, and let $k, \ell \in \mathbb{Z}$. A continuous-time simple random bridge on $[x_1, x_2]$, with starting and ending locations k and ℓ, has the law of CTSRW $X : [x_1, x_2] \to \mathbb{Z}$, $X(x_1) = k$, conditioned on $X(x_2) = \ell$.

Non-intersecting walks. Two $\mathbb{Z}$-indexed walks X_1 and X_2 defined on a common real interval I are non-intersecting if $X_1(x) \neq X_2(x)$ for all $x \in I$. Note that CTSRWs that are non-intersecting are ordered: if X_1 exceeds X_2 at one point in the common domain of definition I, then this bound holds at all points in I.

Let $n \in \mathbb{N}$ and $T > 0$. Let $\mathbb{P}_{n,T}$ denote the law of n independent rate-2 CTSRWs $X_{-i} : [-T,T] \to \mathbb{Z}$, $X_{-i}(-T) = -i$ indexed by $i \in [\![0, n-1]\!]$. The return event $R_{n,i}$ equals $X_{-i}(T) = -i$. Set $R_n = \bigcap_{i=0}^{n-1} R_{n,i}$. Internal non-intersection is the event

$$\mathsf{NI}_n^{\mathrm{int}} = \big\{ X_{-i}, i \in [\![0, n-1]\!], \;\text{ are mutually non-intersecting} \big\} .$$

External non-intersection is

$$\mathsf{NI}_n^{\mathrm{ext}} = \big\{ X_{1-n}(x) \geq -n \text{ for all } x \in [-T,T] \big\} \, .$$

Set $\mathsf{NI} = \mathsf{NI}_n^{\mathrm{int}} \cap \mathsf{NI}_n^{\mathrm{ext}}$.

Let the deepest index $D(T)$ denote the maximum index $i \geq 0$ such that the line L_{-i} has been deformed from its original constant value of $-i$ at time $T \geq 0$.

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Proposition 1.3. Write $\mathbb{Q}$ for the Poisson randomness. For $n \in \mathbb{N}$ and $T \in [0,\infty)$,

$$\mathbb{P}_{n,T}(R \cap \mathsf{NI}) = \exp\big\{ T(T-4n) \big\} \mathbb{Q}\big(D(T) \leq n \big) \, .$$

Proof. Consider the *flat* outcome F under $\mathbb{P}_{n,T}$, under which the line L_{-i} is identically equal to $-i$ for every $i \in [\![0, n-1]\!]$.

Note that $\mathbb{P}_{n,T}\big(\cdot \, \big| R \cap \mathsf{NI} \big)$ is measure-isomorphic to the space $\mathbb{Q}$ given $D(T) \leq n$. The flat outcome corresponds to an absence of Poisson points in the cone C up to height T. Hence,

$$\mathbb{P}_{n,T}(F) \, = \, \frac{e^{-T^2}}{\mathbb{Q}\big(D(T) \leq n\big)} \, .$$

We see then that

$$\frac{e^{-4nT}}{\mathbb{P}_{n,T}(R \cap \mathsf{NI})} \, = \, \frac{e^{-T^2}}{\mathbb{Q}\big(D(T) \leq n\big)} \, ,$$

which rearranges to give the sought statement. □

Consider a configuration X of the Poisson cloud in the cone up to height t. An element in X takes the form of an unordered set

$$\big\{ (x_1,t_1), (x_2,t_2) \cdots , (x_m,t_m) \big\}$$

where $m \in \mathbb{N}$ and for each $i \in [\![1,m]\!]$, $t_i \in [0,T]$ with $|x_i| \leq t_i$.

Let Y denote the collection of finite sets whose elements are ordered pairs (z,z') satisfying $-T \leq z < z' \leq T$. Say that $y = \{(z_i,z_i')\}_{i=1}^m \in Y$ is *nonsingular* if all of the $2m$ numbers $z_1, z_1', \ldots, z_m, z_m'$ are distinct. Write $Y^{\mathrm{ns}} \subset Y$ for the set of nonsingular configurations.

Define a map $\tau : X \to Y$ by

$$\big\{ (x_1,t_1), (x_2,t_2) \cdots , (x_m,t_m) \big\} \mapsto \big\{ (x_1^-,x_1^+), (x_2^-,x_2^+) \cdots , (x_m^-,x_m^+) \big\}$$

where $x_i^- = x_i - (T - t_i)$ and $x_i^+ = x_i + (T - t_i)$.

Let $\mathcal{P}_t$ denote the set of Poisson points of height at most $t \geq 0$. For $x \in \mathcal{P}_t$, write $\mathcal{P}_t(x) = \mathcal{P}_t \setminus \{x\}$ for this cloud with the point x removed.

For such x, write

$$D(x^{\pm}) = \min \Big\{ k \geq 1 : M_k^{\mathcal{P}_t}\big((0,0) \to (x_{\pm},t) \big) > M_k^{\mathcal{P}_t(x)}\big((0,0) \to (x_{\pm},t) \big) \Big\}$$

The RSK dynamics offers a bijection between cloud configurations in the up-to-time-t cone C_t and non-intersecting line configurations on $[-t,t]$. To each cloud, we run the RSK dynamics for time t to construct the line configuration. For a given line configuration, we run the dynamics backwards for time t, recording a point in the cloud on the descending time-line at the spatial location at which each nucleation is encountered.

Viewed in terms of suitable Poisson laws, the bijection is measure preserving. The cloud space C_t already comes accompanied with a natural Poisson measure $\mathbb{Q}$, whose intensity we take equal to the constant two. It is natural to consider the line side of the story by introducing a natural Poisson measure on lines. For this purpose, we may begin with the walk measures.

Let $n \in \mathbb{N} \cup \{\infty\}$. Under the time-$T$ n-line walk measure $\mathbb{P}^w_n(T)$, the $i^{\rm th}$ line $L_{1-i} : [-T,T] \to \mathbb{Z}$ is an independent rate-two CTSRW mapping $[-T,T] \to \mathbb{Z}$ with starting position $L_{1-i}(-T) = 1-i$. When n is finite, the lines L_{1-i} indexed by $i \geq n+1$ are static, identically equal to $1-i$ on their domain of definition $[-T,T]$.

The measure $\mathbb{P}^w_\infty(T)$ charges the full space of non-intersecting line configurations, which is in natural correspondence with the space of finite clouds in C_t. However, for the purpose of analysing Poisson measures on the line space, and compare them to counterparts in the cloud space, it is useful to work with the finite-n walks measures $\mathbb{P}^w_n(T)$.

Take $n \in \mathbb{N}$ finite then, as well as $T > 0$. A non-intersecting (n,T)-line configuration is called *proper* if the steps in each line L_{1-i}, $i \in [\![1,n]\!]$, have unit magnitude and occur at distinct locations in $[-T,T]$. Let $\epsilon_0 > 0$ denote one-half of the minimum distance between consecutive entries in any of the lists $(-T, j_{i,1}, \cdots, j_{i,k_i}, T))$ indexed by $i \in [\![1,n]\!]$. Here, $\big(j_{i,k} : k \in [\![1,k_i]\!]\big)$ is an increasing list of the jumps of the line L_{1-i}.

Taking $\epsilon \in (0,\epsilon_0)$, consider the ϵ-box $B(L,\epsilon)$ about the configuration L. A line configuration L' lies in this box if its set of upsteps and downsteps is in correspondence with those of L in such a way that every upstep in L corresponds to an upstep in L' on the same line whose location differs by at most ϵ; and likewise for every downstep.

A small ϵ asymptotic for the probability of $B(L,\epsilon)$ is easily seen given the Poisson nature of the law $\mathbb{P}^w_n(T)$: we have

$$\mathbb{P}^w_{n,T}\big(B(L,\epsilon)\big) = \exp\big\{ -4nT \big\} (2\varepsilon)^{2m} \Big(1 + o(1) \Big)$$

as $\epsilon \searrow 0$, where $2m$ is the number of steps in L. Indeed, we may compute

$$\mathbb{P}^w_{n,T}\big(B(L,\epsilon)\big) = \Big(\tfrac{1}{2} \cdot 4\epsilon \Big)^{2m} \exp\big\{ -(4nT - 2\epsilon \cdot 2m) \big\} \, ,$$

where each factor of 4ϵ corresponds to the presence of a rate-two Poisson point in each of the length-2ϵ intervals in the lines centred at steps, and each factor of one-half corresponds to the correct selection up/down for the step according to the flip of a fair coin; the exponential factor factor represents the absence of steps in the remaining part of the n-line space according to the Poisson rate-two measure.

The probabilities on display decay very rapidly as n rises. This is largely due to the improbability that each of the n walks is a bridge (whose starting and ending heights coincide) and that the concerned walks remain non-intersecting. These attributes obtain for any configuration realizing $B(L,\epsilon)$, and this fact permits us to translate the obtained asymptotic into one concerning the n-line non-interacting bridge measure, simply by division by the requiste probability $\mathbb{P}^w_{n,T}\big(R \cap \mathsf{NI}\big)$. Indeed, with the notation $\mathbb{P}^\ell_{n,T}$ for the n-line measure given by conditioning $\mathbb{P}^w_n$ on the event $R \cap \mathsf{NI}$, we find that

$$\mathbb{P}^\ell_{n,T}\big(B(L,\epsilon)\big) = \frac{\big(2\epsilon\big)^{2m} \exp\big\{ -(4nT - 2\epsilon \cdot 2m) \big\}}{\mathbb{P}^w_{n,T}\big(R \cap \mathsf{NI}\big)} \, , \qquad (1)$$

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The box $B(L,\epsilon)$ corresponds to a box in cloud space. Let ω denote the cloud in C_T to which the line configuration L corresponds. For each point $(z_1,z_2) \in \mathbb{R}^2$, write $D(z_1,z_2,\epsilon)$ for the diagonally oriented box of width 2ϵ whose centre is (z_1,z_2). For $\epsilon \in (0,\epsilon_0)$ as specified above, it is straightforward to see that the collection of squares $D(\omega_i,\epsilon)$, $i \in [\![1,m]\!]$, is disjoint. Let $\mathsf{B}(\omega,\epsilon)$ denote the event that the Poisson cloud in C_t under $\mathbb{Q}$ has m points, with each square $D(\omega_i,\epsilon)$ containing exactly one point. It is straightforward that the event $\mathsf{B}(\omega,\epsilon)$ corresponds to $B(\omega,\epsilon)$ under the cloud-line correspondence.

We may perform a counterpart Poisson calculation for the probability of $\mathsf{B}(\omega,\epsilon)$ in cloud space, as we did for $B(\omega,\epsilon)$ in line-space: we find that

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = \exp\big\{-T^2\big\}(2\epsilon)^{2m}\Big(1+o(1)\Big) \qquad (2)$$

as $\epsilon \searrow 0$. Indeed, each square lands a cloud point with asymptotic probability $(2\epsilon)^{2m}$, while the non-square part of C_t is devoid of Poisson points with probability $\exp\big\{-(T^2 - m(2\epsilon)^2)\big\}$.

The line (1) and cloud (2) expressions run in close parallel, but the correspondence is imperfect. The line side of the picture has been restricted to configurations in which only the top n lines are permitted to move, in order to permit Poisson calculations, but this restriction means that the admissible configurations correspond to only part of the cloud space. Which part? A cloud ω is in correspondence if and only if its depth $D(T) = D_T(\omega)$ is at most n. Here, the notion of depth is represented in the cloud space, even if our definition was made in line-space: the depth of a cloud configuration is the depth of the corresponding line configuration. We find then that

$$\mathbb{P}^{\ell}_{n,T}\big(B(L,\epsilon)\big) = \frac{\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big)}{\mathbb{Q}\big(D(\omega)\le n\big)}\,.$$

From (1) and (2), we find then that

$$\exp\big\{-4nT\big\}\mathbb{Q}\big(D(\omega)\le n\big) = \exp\big\{-T^2\big\}\mathbb{P}^{w}_{n,T}\big(R\cap \mathsf{NI}\big)\,.$$

Rearranging this identity, we obtain Proposition 1.3.

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A deterministic cloud–line correspondence. We now make precise (at the deterministic level) the correspondence between finite point configurations in the cone C_T and multi-line PNG configurations at the fixed time T. Throughout, we work on a *nonsingular* set where no two relevant events occur at the same space–time location; this eliminates tie-breaking issues and is the natural setting for the local "ε-box" arguments used later.

Proper line configurations at time T. Fix $T>0$. A *proper time-T line configuration* is a sequence of functions $\{L_{-i}(\cdot,T)\}_{i\ge 0}$ on $[-T,T]$ such that

(1) each $L_{-i}(\cdot,T)$ is integer-valued, càdlàg, and piecewise constant with jumps of magnitude ± 1 only;

(2) all jump locations (across all curves) are distinct points of $(-T,T)$;

(3) for all sufficiently large i (depending on the configuration), $L_{-i}(\cdot,T) \equiv -i$ (only finitely many lines are active);

(4) the non-intersection constraint holds:

$$L_{-i}(x,T) \ge L_{-(i+1)}(x,T) + 1 \qquad \text{for all } x\in[-T,T] \text{ and } i \ge 0.$$

Write $\mathcal{L}_T^{\mathrm{prop}}$ for the set of such configurations.

Proper point configurations. Let Ω_T denote the set of finite subsets $\omega \subset C_T$. We call ω *proper* if no two points share the same t–coordinate and if, under the forward PNG dynamics described below, no two annihilations occur at the same space–time point. (This holds $\mathbb{Q}$-a.s. for a Poisson cloud.) Write $\Omega_T^{\mathrm{prop}} \subset \Omega_T$ for this set.

Forward map: cloud $\to$ lines. Given $\omega \in \Omega_T^{\mathrm{prop}}$ we construct a multi-line PNG evolution on $[0, T]$ as follows.

Initialization. Set $L_{-i}(x, 0) \equiv -i$ for all $i \geq 0$.

Deterministic evolution between events. Between event times, each curve $L_{-i}(\cdot, s)$ evolves by propagating its upsteps left with velocity -1 and its downsteps right with velocity $+1$.

Nucleations at Poisson points. When the time level reaches a point $(x_0, t_0) \in \omega$, insert at level 0 a unit "spike" at x_0: equivalently, create an upstep and a downstep in $L_0(\cdot, t_0)$ at the same spatial location x_0.

Annihilation and induced lower nucleation. Whenever (at some level $-i$) a right-moving downstep meets a left-moving upstep, the two steps annihilate at that space–time point, and simultaneously a new nucleation is created at level $-(i + 1)$ at the same space–time location.

Because ω is proper, all nucleation and annihilation events occur at distinct space–time locations. Between such events the dynamics is deterministic, and at each event there is a unique local update to perform. Consequently the forward evolution is well posed up to time T, and yields a proper time-T line configuration, which we denote by $\Phi_T(\omega)$.

Backward map: lines $\to$ cloud. Conversely, given $L \in \mathcal{L}_T^{\mathrm{prop}}$ we reconstruct a point set $\omega \subset C_T$ by running the same rules backwards in time. Between event times, steps move with the reversed velocities (upsteps move right, downsteps move left). Whenever, at some level $-i$, an upstep and downstep *collide* backwards in time (i.e. they were created by a nucleation forwards in time), we remove this pair from level $-i$, and simultaneously create a pair of opposite steps at level $-(i-1)$ at the same space–time location; when $i = 0$ we instead *record* that space–time location as an element of the reconstructed cloud. Properness guarantees that this reverse evolution is unambiguous and terminates after finitely many events, yielding a finite set of recorded points in C_T. We denote the reconstructed point set by $\Psi_T(L)$.

Proposition 1.4 (Deterministic bijection). For each $T > 0$, the maps

$$\Phi_T : \Omega_T^{\mathrm{prop}} \to \mathcal{L}_T^{\mathrm{prop}}, \qquad \Psi_T : \mathcal{L}_T^{\mathrm{prop}} \to \Omega_T^{\mathrm{prop}},$$

defined above are well posed and inverse to one another. In particular, Φ_T is a bijection between proper point configurations in the cone up to time T and proper time-T multi-line PNG configurations on $[-T, T]$.

Proof. We sketch the argument, since it is a deterministic bookkeeping statement once the forward/backward rules are fixed.

Well-posedness. Properness ensures that event times (nucleations and annihilations) are discrete, that between events all step locations evolve linearly with the prescribed velocities, and that at an event there is a unique local update to perform (annihilate exactly one meeting pair, and create exactly one induced nucleation one level below). Thus $\Phi_T(\omega)$ is well defined and produces a proper time-T configuration. The same reasoning applies to the reverse evolution defining $\Psi_T(L)$.

Inverse property. Consider $\omega \in \Omega_T^{\text{prop}}$ and run the forward evolution to time T, obtaining $L = \Phi_T(\omega)$. In the forward dynamics, each local event is of one of two types: a nucleation on level 0 (coming from a point of ω) or an annihilation on level $-i$ with an induced nucleation on level $-(i+1)$. By construction, the backward dynamics performs the exact inverse local move at the same space–time location: it undoes each annihilation/induced-nucleation pair, and when it reaches a level-0 nucleation event it records the corresponding space–time point. Since the forward dynamics is deterministic given ω, and properness prevents ambiguity, running backwards from L recovers exactly the original set of level-0 nucleation events, namely the points of ω. Hence $\Psi_T(\Phi_T(\omega)) = \omega$. The same argument in reverse shows $\Phi_T(\Psi_T(L)) = L$ for $L \in \mathcal{L}_T^{\text{prop}}$. □

REMARK 1 (Relation to RSK and "depth"). The map Φ_T is the multi-line PNG/RSK correspondence in geometric form: the full line ensemble encodes the point configuration, while the top curve $L_0(\cdot, T)$ encodes last-passage values. In particular, the depth event $\{D(T) \le n\}$ is the same as the deterministic constraint that $L_{-(n+1)}(\cdot, T) \equiv -(n+1)$, i.e. that no event ever propagated below level $-n$ up to time T.

Local ϵ-box correspondence. We now justify rigorously the correspondence between small neighbourhoods of a proper line configuration and small neighbourhoods of its corresponding cloud under the deterministic bijection Φ_T of Proposition 1.4.

Proper configurations. Fix $T > 0$. Let $L \in \mathcal{L}_T^{\text{prop}}$ be a proper time-T line configuration with finitely many jumps. Let $\omega = \Psi_T(L)$ denote the corresponding cloud in C_T.

Let the set of jump locations of L be

$$\{z_1^-, z_1^+, \dots, z_m^-, z_m^+\},$$

where z_i^- and z_i^+ denote the spatial locations of the upstep and downstep induced by the ith Poisson point.

Properness implies that these $2m$ locations are all distinct. Define

$$\epsilon_0 = \frac{1}{2} \min\Big(\{|z_a^\sigma - z_b^\tau| : (a,\sigma) \ne (b,\tau)\} \cup \{|z_a^\sigma \pm T|\}\Big),$$

so that the intervals $[z_i^\pm - \epsilon_0, z_i^\pm + \epsilon_0]$ are disjoint and contained in $(-T, T)$.

Line ϵ-box. For $\epsilon \in (0, \epsilon_0)$ define $B(L, \epsilon)$ to be the set of proper line configurations L' such that:

(1) L' has the same number of jumps on each line as L;

(2) each upstep (respectively downstep) of L corresponds to an upstep (respectively downstep) of L' on the same curve whose location differs by at most ϵ.

Cloud ϵ-box. For each point $\omega_i = (x_i, t_i) \in \omega$, define

$$D(\omega_i, \epsilon) = \big\{(x,t) \in \mathbb{R}^2 : |x - x_i| \le \epsilon,\ |t - t_i| \le \epsilon\big\}.$$

For $\epsilon \in (0, \epsilon_0)$ define $\mathsf{B}(\omega, \epsilon)$ to be the event that:

(1) the cloud contains exactly m points in C_T;

(2) for each $i \in \{1, \dots, m\}$, the region $D(\omega_i, \epsilon)$ contains exactly one point;

(3) no other points of the cloud lie in C_T.

LEMMA 1.5 (Local correspondence). Let $L \in \mathcal{L}_T^{\text{prop}}$ and $\omega = \Psi_T(L)$. For $\epsilon \in (0, \epsilon_0)$ sufficiently small, the deterministic bijection Φ_T restricts to a bijection

$$\Phi_T : \ \mathsf{B}(\omega, \epsilon) \longrightarrow B(L, \epsilon).$$

In particular, under the cloud–line correspondence, small perturbations of point locations correspond exactly to small perturbations of jump locations, with no change in combinatorial structure.

PROOF. Because $\epsilon < \epsilon_0$, the regions $D(\omega_i, \epsilon)$ are disjoint. Thus every cloud configuration in $\mathsf{B}(\omega, \epsilon)$ consists of exactly one point in each such region, and no other points.

Under the forward PNG dynamics, each Poisson point in $D(\omega_i, \epsilon)$ produces exactly one nucleation event, and the space–time location of this nucleation is perturbed by at most ϵ from its original location. Since all event times and annihilations are separated by at least $2\epsilon_0 > 2\epsilon$, the relative ordering of all nucleations and annihilations is unchanged.

Hence:

- the number of jumps on each line remains the same,
- each jump location shifts by at most ϵ,
- no new collisions or reorderings occur.

Therefore Φ_T maps $\mathsf{B}(\omega, \epsilon)$ into $B(L, \epsilon)$.

Conversely, given $L' \in B(L, \epsilon)$, running the backward dynamics produces exactly one point in each $D(\omega_i, \epsilon)$ and no others, since the combinatorial structure of annihilations is unchanged. Thus $\Psi_T(L') \in \mathsf{B}(\omega, \epsilon)$.

The two maps are inverses on these restricted sets, completing the proof. □

Poisson ϵ-box asymptotics in the cone. We record the standard local asymptotics for Poisson point processes that underlie the cloud-side computation in (2). We state the result in the generality needed for the "diagonal boxes" (or any family of disjoint measurable sets of small area).

LEMMA 1.6 (Poisson counts in disjoint small sets). Let $\mathcal{P}$ be a Poisson point process on a measurable set $S \subset \mathbb{R}^2$ of finite area $|S|$, with constant intensity $\lambda > 0$. Let $A_1, \dots, A_m \subset S$ be pairwise disjoint measurable sets, and set $A_\star := \bigcup_{i=1}^m A_i$. Then

$$\mathbb{Q}\Big(\#(\mathcal{P} \cap A_i) = 1 \text{ for all } i \le m, \ \ \#(\mathcal{P} \cap (S \setminus A_\star)) = 0\Big) = e^{-\lambda|S|} \prod_{i=1}^m \big(\lambda |A_i|\big). \tag{3}$$

PROOF. For disjoint sets, the Poisson counts $\#(\mathcal{P} \cap A_1), \dots, \#(\mathcal{P} \cap A_m), \#(\mathcal{P} \cap (S \setminus A_\star))$ are independent, with respective laws $\mathrm{Pois}(\lambda|A_i|)$ and $\mathrm{Pois}(\lambda|S \setminus A_\star|)$. Hence the probability equals

$$\prod_{i=1}^m \mathbb{P}\big(\mathrm{Pois}(\lambda|A_i|) = 1\big) \cdot \mathbb{P}\big(\mathrm{Pois}(\lambda|S \setminus A_\star|) = 0\big) = \prod_{i=1}^m e^{-\lambda|A_i|}(\lambda|A_i|) \cdot e^{-\lambda|S \setminus A_\star|},$$

which simplifies to (6) since $\sum_{i=1}^m |A_i| + |S \setminus A_\star| = |S|$. □

Corollary 1.7 (Cloud ϵ-box in C_T). Let $T > 0$ and let $C_T = \{(x,t) \in \mathbb{R} \times [0,T] : |x| \le t\}$. Then $|C_T| = T^2$. Let $\mathcal{P}$ be a Poisson point process on C_T of intensity $\lambda > 0$. Fix distinct points $\omega_1, \dots, \omega_m \in C_T$ and let $D(\omega_i, \epsilon) \subset C_T$ be pairwise disjoint measurable sets (depending on ϵ) with areas $|D(\omega_i, \epsilon)| = a_i(\epsilon)$. Define the event

$$\mathsf{B}(\omega,\epsilon) := \Big\{\#(\mathcal{P} \cap D(\omega_i,\epsilon)) = 1 \ \forall i \le m, \ \ \#(\mathcal{P} \cap (C_T \setminus \cup_{i=1}^m D(\omega_i,\epsilon))) = 0\Big\}.$$

Then

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-\lambda T^2} \prod_{i=1}^m \big(\lambda a_i(\epsilon)\big). \tag{4}$$

In particular, if each $D(\omega_i,\epsilon)$ is a square of side 2ϵ (or any set of area $(2\epsilon)^2$), so that $a_i(\epsilon) = (2\epsilon)^2$, then

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-\lambda T^2} \big(\lambda (2\epsilon)^2\big)^m = e^{-\lambda T^2} (2\epsilon)^{2m} \lambda^m 4^m. \tag{5}$$

Under the convention $\lambda = 1$, this reads

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-T^2} (2\epsilon)^{2m} (1 + o(1)) \quad \text{as } \epsilon \downarrow 0,$$

up to the harmless choice of whether the box area is taken to be $(2\epsilon)^2$ or $c(2\epsilon)^2$ for some fixed $c > 0$.

Poisson ϵ-box asymptotics in the cone. We record the standard local asymptotics for Poisson point processes that underlie the cloud-side computation in (2). We state the result in the generality needed for the "diagonal boxes" (or any family of disjoint measurable sets of small area).

Lemma 1.8 (Poisson counts in disjoint small sets). Let $\mathcal{P}$ be a Poisson point process on a measurable set $S \subset \mathbb{R}^2$ of finite area $|S|$, with constant intensity $\lambda > 0$. Let $A_1, \dots, A_m \subset S$ be pairwise disjoint measurable sets, and set $A_\star := \bigcup_{i=1}^m A_i$. Then

$$\mathbb{Q}\Big(\#(\mathcal{P} \cap A_i) = 1 \text{ for all } i \le m, \ \ \#(\mathcal{P} \cap (S \setminus A_\star)) = 0\Big) = e^{-\lambda |S|} \prod_{i=1}^m \big(\lambda |A_i|\big). \tag{6}$$

Proof. For disjoint sets, the Poisson counts $\#(\mathcal{P} \cap A_1), \dots, \#(\mathcal{P} \cap A_m), \#(\mathcal{P} \cap (S \setminus A_\star))$ are independent, with respective laws $\mathrm{Pois}(\lambda |A_i|)$ and $\mathrm{Pois}(\lambda |S \setminus A_\star|)$. Hence the probability equals

$$\prod_{i=1}^m \mathbb{P}\big(\mathrm{Pois}(\lambda|A_i|) = 1\big) \cdot \mathbb{P}\big(\mathrm{Pois}(\lambda|S \setminus A_\star|) = 0\big) = \prod_{i=1}^m e^{-\lambda|A_i|}(\lambda|A_i|) \cdot e^{-\lambda|S \setminus A_\star|},$$

which simplifies to (6) since $\sum_{i=1}^m |A_i| + |S \setminus A_\star| = |S|$. $\square$

Corollary 1.9 (Cloud ϵ-box in C_T). Let $T > 0$ and let $C_T = \{(x,t) \in \mathbb{R} \times [0,T] : |x| \le t\}$. Then $|C_T| = T^2$. Let $\mathcal{P}$ be a Poisson point process on C_T of intensity $\lambda > 0$. Fix distinct points $\omega_1, \dots, \omega_m \in C_T$ and let $D(\omega_i, \epsilon) \subset C_T$ be pairwise disjoint measurable sets (depending on ϵ) with areas $|D(\omega_i, \epsilon)| = a_i(\epsilon)$. Define the event

$$\mathsf{B}(\omega,\epsilon) := \Big\{\#(\mathcal{P} \cap D(\omega_i,\epsilon)) = 1 \ \forall i \le m, \ \ \#(\mathcal{P} \cap (C_T \setminus \cup_{i=1}^m D(\omega_i,\epsilon))) = 0\Big\}.$$

Then

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-\lambda T^2} \prod_{i=1}^m \big(\lambda a_i(\epsilon)\big). \tag{7}$$

In particular, if each $D(\omega_i, \epsilon)$ is a square of side 2ϵ (or any set of area $(2\epsilon)^2$), so that $a_i(\epsilon) = (2\epsilon)^2$, then

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-\lambda T^2}\,\big(\lambda(2\epsilon)^2\big)^m = e^{-\lambda T^2}\,(2\epsilon)^{2m}\,\lambda^m\,4^m. \tag{8}$$

Under the convention $\lambda = 1$, this reads

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = e^{-T^2}(2\epsilon)^{2m}(1+o(1)) \quad \text{as } \epsilon \downarrow 0,$$

up to the harmless choice of whether the box area is taken to be $(2\epsilon)^2$ or $c(2\epsilon)^2$ for some fixed $c > 0$.

Normalization identity. We now combine the local ϵ-box correspondence (Lemma 1.5) with the Poisson asymptotics (Corollary 1.9) to obtain the global normalization identity.

PROPOSITION 1.10 (Normalization identity). Fix $n \in \mathbb{N}$ and $T > 0$. Then

$$\exp\{-4nT\}\,\mathbb{Q}\big(D(T) \le n\big) = \exp\{-T^2\}\,\mathbb{P}_n^w(T)\big(R \cap \mathsf{NI}\big).$$

PROOF. Let L be a proper n-line configuration with m jumps, and let $\omega = \Psi_T(L)$ be the corresponding cloud.

By Lemma 1.5, for sufficiently small ϵ, the ϵ-boxes correspond:

$$\Phi_T(\mathsf{B}(\omega,\epsilon)) = B(L,\epsilon).$$

Under $\mathbb{P}_n^w(T)$,

$$\mathbb{P}_n^w(T)\big(B(L,\epsilon)\big) = \exp\{-4nT\}\,(2\epsilon)^{2m}(1+o(1)).$$

Under $\mathbb{Q}$,

$$\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big) = \exp\{-T^2\}\,(2\epsilon)^{2m}(1+o(1)).$$

Restricting to the admissible subset $\{D(T) \le n\}$ and dividing by $\mathbb{Q}(D(T) \le n)$ yields

$$\mathbb{P}^{\ell}_{n,T}\big(B(L,\epsilon)\big) = \frac{\mathbb{Q}\big(\mathsf{B}(\omega,\epsilon)\big)}{\mathbb{Q}(D(T) \le n)}.$$

Comparing coefficients of $(2\epsilon)^{2m}$ and letting $\epsilon \downarrow 0$ gives

$$\frac{\exp\{-4nT\}}{\mathbb{P}_n^w(T)(R \cap \mathsf{NI})} = \frac{\exp\{-T^2\}}{\mathbb{Q}(D(T) \le n)},$$

which rearranges to the stated identity. □

1.2. Chatgpt generated material

PROPOSITION 1.11 (Depth equals line index). Let $x \in \mathcal{P}_t$ and suppose that the multiline PNG construction has been performed up to time t. Then the quantity $D(x^{\pm})$ defined above is equal to the unique index $k \ge 1$ such that the nucleation caused by x contributes to the curve $L_{-(k-1)}$ at time t. Equivalently, $D(x^{\pm}) = k$ precisely when x is used by an optimal k-tuple of disjoint upgoing paths to $(x^{\pm}, t)$ but not by any optimal $(k-1)$-tuple.

Proof. By definition, $D(x^{\pm})$ is the smallest k for which

$$M_k^{\mathcal{P}_t}\big(x^{\pm}(t), t\big) > M_k^{\mathcal{P}_t(x)}\big(x^{\pm}(t), t\big).$$

Thus x increases the optimal k-tuple weight but does not increase any smaller tuple.

Now recall from Proposition 1.11 (or the earlier identity relating M_j to the line ensemble) that

$$M_k(x,t) = \sum_{i=0}^{k-1} \big(L_{-i}(x,t) + i\big).$$

Therefore, the increment

$$M_k^{\mathcal{P}_t}(x^{\pm}, t) - M_k^{\mathcal{P}_t(x)}(x^{\pm}, t)$$

is equal to the increment in the cumulative height of the first k curves at the spatial location $x^{\pm}(t)$ caused by the presence of x.

Because the multiline PNG construction propagates nucleations upward level-by-level via annihilation events, the Poisson point x produces exactly one unit increase in the curve $L_{-(k-1)}$ at the moment its influence reaches time t, where k is minimal with this property. For smaller indices the optimal families of paths do not use x, and for larger indices the influence is inherited from lower levels.

Hence the minimal k for which x contributes to M_k is precisely the level at which the corresponding nucleation occurs in the multiline picture. $\square$

Pushing Poisson points to time T and kink data. In this section we view the Poisson cloud in the cone

$$C_T = \big\{(x,t) \in \mathbb{R} \times [0,T] : |x| \le t\big\}$$

as generating, at time T, a finite collection of upstep–downstep pairs. To avoid notational conflict with the time variable, we write ω for a Poisson configuration and regard ω as a finite subset of C_T:

$$\omega = \{(x_1,t_1),(x_2,t_2),\dots,(x_m,t_m)\}.$$

Recall that τ sends each point (x_i,t_i) to the ordered pair (x_i^-, x_i^+) with

$$x_i^- = x_i - (T - t_i), \qquad x_i^+ = x_i + (T - t_i).$$

Equivalently,

$$x_i = \frac{x_i^- + x_i^+}{2}, \qquad t_i = T - \frac{x_i^+ - x_i^-}{2}.$$

Thus τ is injective on point configurations, and its range is contained in

$$\Delta_T := \big\{(z,z') \in [-T,T]^2 : z < z'\big\}.$$

When $\tau(\omega) = \{(x_i^-, x_i^+)\}_{i=1}^m$, we interpret x_i^- and x_i^+ as, respectively, the spatial locations at time T of the upstep and downstep produced by the nucleation at (x_i,t_i) and transported along the characteristics of slope ± 1.

A nonsingular open set in kink space. Let Y denote the collection of finite subsets of Δ_T (as above). We say that $y = \{(z_i, z_i')\}_{i=1}^m \in Y$ is *nonsingular* if all of the $2m$ numbers $z_1, z_1', \dots, z_m, z_m'$ are distinct. Write $Y^{\mathrm{ns}} \subset Y$ for the set of nonsingular configurations.

Fix $y = \{(z_i, z_i')\}_{i=1}^m \in Y^{\mathrm{ns}}$. For $\varepsilon > 0$ sufficiently small (depending on y), define the box neighbourhood

$$B_\varepsilon(y) := \Big\{ \{(\tilde z_i, \tilde z_i')\}_{i=1}^m \in Y^{\mathrm{ns}} : |\tilde z_i - z_i| < \varepsilon,\ |\tilde z_i' - z_i'| < \varepsilon \ \ \forall i \Big\}. \tag{9}$$

e.box-y

For ε small, the constraints $\tilde z_i < \tilde z_i'$ and mutual distinctness of the $2m$ endpoints remain valid throughout $B_\varepsilon(y)$.

From kink data to multi-line PNG at time T. The PrÃ¤hofer–Spohn multi-line construction (equivalently, the RSK dynamics) associates to each Poisson configuration ω a multi-line step ensemble $\{L_{-i}(\cdot, T)\}_{i \ge 0}$ at time T. On the level of time-T kink data, this may be viewed as a deterministic map

$$\Phi : \ Y^{\mathrm{ns}} \ \longrightarrow \ \mathcal{L}_T^{\mathrm{ns}}, \tag{10}$$

e.PS-map

where $\mathcal{L}_T^{\mathrm{ns}}$ denotes the set of multi-line PNG time-T profiles whose kinks occur at distinct spatial locations. Informally, Φ takes the multiset of upstep/downstep positions at time T and resolves annihilations level-by-level to produce the line ensemble at time T. (If one wishes to recover the Poisson configuration from the line ensemble, one may augment $\mathcal{L}_T^{\mathrm{ns}}$ by standard "recording" data; for the present discussion it suffices that Φ is well defined on Y^{ns} and locally stable.)

-stability

Lemma 1.12 (Local stability away from coincidences). Fix $y \in Y^{\mathrm{ns}}$. There exists $\varepsilon_0 = \varepsilon_0(y) > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:

(1) the combinatorial resolution of annihilations used by Φ is constant on $B_\varepsilon(y)$ (i.e. no "ties" occur in determining collisions);

(2) consequently, Φ restricts to a bijection from $B_\varepsilon(y)$ onto $U_\varepsilon(\Phi(y)) := \Phi(B_\varepsilon(y)) \subset \mathcal{L}_T^{\mathrm{ns}}$, and in the coordinates given by kink locations, Φ is locally Lipschitz with locally Lipschitz inverse.

Lemma 1.12 expresses the basic fact that, away from coincident kink locations, small perturbations of the $x_i^\pm$ do not change the discrete pairing structure of the step resolution, and therefore yield a smooth "wiggle room" in line space corresponding to the same perturbation scale in kink space.

Restriction to at most n active lines. Let $\mathcal{L}_{T,\le n}^{\mathrm{ns}} \subset \mathcal{L}_T^{\mathrm{ns}}$ denote the set of nonsingular time-T line ensembles for which no curve below index $-(n+1)$ is active at time T, that is,

$$\mathcal{L}_{T,\le n}^{\mathrm{ns}} := \Big\{ \{L_{-i}(\cdot, T)\}_{i \ge 0} \in \mathcal{L}_T^{\mathrm{ns}} : \ L_{-(n+1)}(\cdot, T) \equiv -(n+1) \Big\}. \tag{11}$$

e.floor-eve

Equivalently, on the Poisson side this corresponds to the event $\{D(T) \le n\}$.

Define the corresponding restricted kink space by

$$Y_{\le n}^{\mathrm{ns}} := \Phi^{-1}\big(\mathcal{L}_{T,\le n}^{\mathrm{ns}}\big) \subset Y^{\mathrm{ns}}.$$

For $y \in Y_{\le n}^{\mathrm{ns}}$, Lemma 1.12 implies that for $\varepsilon > 0$ small enough we have $B_\varepsilon(y) \subset Y_{\le n}^{\mathrm{ns}}$, hence $U_\varepsilon(\Phi(y)) \subset \mathcal{L}_{T,\le n}^{\mathrm{ns}}$. In other words, the "hard floor" condition at level $-(n+1)$ persists under sufficiently small perturbations of a nonsingular configuration.

Infinitesimal weights: Poisson side versus walk side. We record the two reference exponential factors that underlie the comparison between the Poisson picture and the random-walk line picture.

(i) Poisson cloud. The Poisson process in C_T of unit intensity has total mass $\mathrm{Area}(C_T) = \int_0^T 2t\,dt = T^2$. Thus the base "no-extra-points" factor is e^{-T^2}. Moreover, on the nonsingular set Y^{ns}, the map $(x,t) \mapsto (x^-, x^+)$ has constant Jacobian determinant 2, so τ sends unit intensity in (x,t) to intensity $\frac{1}{2}$ in (x^-, x^+). Consequently, for fixed $y \in Y^{\mathrm{ns}}$ and $\varepsilon \downarrow 0$,

$$\mathbb{Q}\big(\tau(\mathcal{P}) \in B_\varepsilon(y)\big) = e^{-T^2} \left(\tfrac{1}{2}\right)^m (2\varepsilon)^{2m} \,(1 + o(1)), \tag{12}$$

e.poisson-b

where m is the number of pairs in y.

If $y \in Y^{\mathrm{ns}}_{\le n}$, then $B_\varepsilon(y) \subseteq \{D(T) \le n\}$ for ε small, and therefore conditioning gives

$$\mathbb{Q}\big(\tau(\mathcal{P}) \in B_\varepsilon(y) \,\big|\, D(T) \le n\big) = \frac{\mathbb{Q}\big(\tau(\mathcal{P}) \in B_\varepsilon(y)\big)}{\mathbb{Q}(D(T) \le n)}. \tag{13}$$

e.poisson-c

(ii) Random-walk line picture. Under $\mathbb{P}_{n,T}$, each rate-2 CTSRW on $[-T,T]$ has jump clock Poisson$(4T)$, so the base clock factor is e^{-4T} per walk, hence e^{-4nT} for n independent walks. On the restricted state space $\mathcal{L}^{\mathrm{ns}}_{T,\le n}$, the Gibbs specification for the line ensemble (see the next paragraph) implies that local perturbations of kink locations carry the same Lebesgue infinitesimal $\prod_i dx_i^-\, dx_i^+$ as in (12), multiplied by the same base factor e^{-4nT}. This is the sense in which the two pictures assign matching local "volume" to nonsingular configurations, differing only by the universal exponential factor $e^{T^2 - 4nT} = e^{T(T-4n)}$ and the global normalization required by the restriction $\{D(T) \le n\}$.

e.poisson-box-asymp

Gibbs property for the truncated line ensemble. We now summarize the Gibbs rule that will be used repeatedly. Consider the n-curve ensemble $(X_0, X_{-1}, \ldots, X_{1-n})$ on $[-T,T]$ under the conditional law $\mathbb{P}_{n,T}(\cdot \mid R_n \cap \mathsf{NI})$. Fix $k \in [\![1, n-2]\!]$ and condition on the neighbouring curves $X_{-(k-1)}$ and $X_{-(k+1)}$. Then, conditional on these neighbours, the curve X_{-k} is distributed as a rate-2 CTSRW bridge between its realised endpoint values at $\pm T$, conditioned to stay strictly between the two neighbours on the whole interval. Likewise, the bottom curve X_{1-n} is a rate-2 CTSRW bridge conditioned to stay above the hard floor $-n$ (this is precisely the external non-intersection event).

.bbs-origin

Remark.[How the Gibbs rule arises] One convenient route to this Gibbs property is to view $\mathbb{P}_{n,T}(\cdot \mid R_n \cap \mathsf{NI})$ as a Doob h-transform of n independent rate-2 walks killed upon collision: the conditioning on mutual non-intersection and endpoints produces a Markov line ensemble whose single-curve conditional laws, given neighbouring curves, are exactly constrained CTSRW bridges. An alternative, more combinatorial route proceeds via the Karlin–McGregor/LindstrÃ¶m–Gessel–Viennot determinantal formula for non-intersecting walk bridges, from which the same single-curve conditional laws follow by fixing all but one path.

Comment on the "measure-isomorphism" viewpoint. The maps τ and Φ provide a convenient coordinate system for comparing the Poisson cloud picture with the random-walk line picture. On the nonsingular sets Y^{ns} and $\mathcal{L}^{\mathrm{ns}}_T$, small boxes $B_\varepsilon(y)$ correspond, via Φ, to small neighbourhoods in line space with the same $2m$ free kink coordinates. After restricting to $Y^{\mathrm{ns}}_{\le n}$ (equivalently, $\{D(T) \le n\}$), the induced conditional law on line ensembles has the Gibbs description above, and local weights differ from the Poisson-side weights only by the universal clock/area factor $e^{T^2 - 4nT}$ and the global normalization $\mathbb{Q}(D(T) \le n)$. This is the sense in which the truncated $(\le n)$ Poisson picture and the n-curve non-intersecting walk ensemble may be compared "as measures" on a common nonsingular coordinate space.

Explicit normalization identities. We now record the concrete formula that is implicit in the preceding local-coordinate discussion. Fix $n \in \mathbb{N}$ and $T > 0$, and abbreviate the depth event by

$$A_{n,T} := \{D(T) \le n\} \equiv \big\{L_{-(n+1)}(\cdot, T) \equiv -(n+1) \text{ on } [-T, T]\big\}.$$

Write $\mathbb{Q}_{n,T}$ for the conditional Poisson law $\mathbb{Q}(\,\cdot\,|A_{n,T})$.

(1) Local box probabilities and the universal exponential factor. Let $y = \{(z_i, z_i')\}_{i=1}^m \in Y^{\mathrm{ns}}_{\le n}$, and let $B_\varepsilon(y)$ be the kink-box defined in (9). Set $U_\varepsilon := \Phi(B_\varepsilon(y)) \subset \mathcal{L}^{\mathrm{ns}}_{T,\le n}$. Then as $\varepsilon \downarrow 0$,

$$\frac{\mathbb{Q}\big(\tau(\mathcal{P}) \in B_\varepsilon(y)\big)}{\mathbb{P}_{n,T}\big(U_\varepsilon\big)} = \exp\{T^2 - 4nT\}\,(1 + o(1)), \tag{14}$$

where the $(1+o(1))$ term depends on y but not on the direction of shrinkage of the box. Equivalently, in conditional form,

$$\frac{\mathbb{Q}_{n,T}\big(\tau(\mathcal{P}) \in B_\varepsilon(y)\big)}{\mathbb{P}_{n,T}\big(U_\varepsilon\big)} = \frac{\exp\{T^2 - 4nT\}}{\mathbb{Q}\big(A_{n,T}\big)}\,(1 + o(1)). \tag{15}$$

In particular, the only global discrepancy between the two pictures in these nonsingular kink-coordinates is the universal factor $\exp\{T^2 - 4nT\} = \exp\{T(T-4n)\}$ together with the conditioning normalization $\mathbb{Q}(A_{n,T})$.

(2) A global Radon–Nikodym relation (informal but useful). Guided by (15), it is natural to package the preceding as the heuristic measure identity

$$\mathbb{P}_{n,T}(\,\cdot\,|R_n \cap \mathsf{NI}) \quad\approx\quad \mathbb{Q}(\,\cdot\,|A_{n,T}) \quad \text{with normalizing constant} \quad \frac{\exp\{T(T-4n)\}}{\mathbb{Q}(A_{n,T})}, \tag{16}$$

where the comparison is understood on $Y^{\mathrm{ns}}_{\le n}$ (or on the corresponding line space $\mathcal{L}^{\mathrm{ns}}_{T,\le n}$) in the sense that small kink-box probabilities match with the stated constant factor. In particular, if $\mathcal{E}$ is an event depending only on the top n curves and supported in $\mathcal{L}^{\mathrm{ns}}_{T,\le n}$, then one may read (16) as suggesting

$$\mathbb{P}_{n,T}\big(\mathcal{E} \cap R_n \cap \mathsf{NI}\big) \quad\approx\quad \exp\{T(T-4n)\}\,\mathbb{Q}\big(\Phi^{-1}(\mathcal{E}) \cap A_{n,T}\big), \tag{17}$$

with the same constant $\exp\{T(T-4n)\}$ appearing universally.

(3) The proposition as a normalization statement. Taking $\mathcal{E}$ to be the full admissible event (i.e. $\mathcal{E}$ equal to the whole $\mathcal{L}^{\mathrm{ns}}_{T,\le n}$ up to null sets), the preceding reduces precisely to the claimed normalization relation

$$\mathbb{P}_{n,T}\big(R_n \cap \mathsf{NI}\big) = \exp\{T(T-4n)\}\,\mathbb{Q}\big(D(T) \le n\big), \tag{18}$$

which is the proposition stated above.