Data Assimilation in the Boussinesq Approximation for Mantle Convection

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Outline

1. Introduction
2. Infinite Prandtl Assimilation
3. Numerical Simulations
4. Other Assimilation Schemes

"All models are wrong, but some are useful."

– George Box

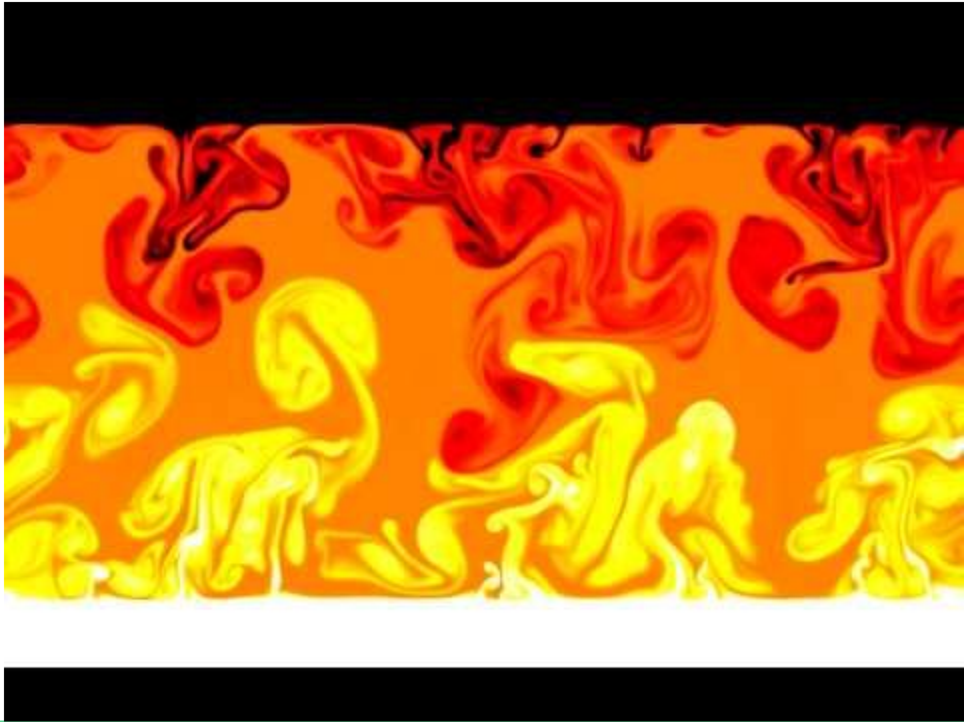
Section 1: Introduction

Rayleigh-Bénard Convection

Problem:

When placed between a hot plate and a cold plate, how does temperature diffuse through a fluid?

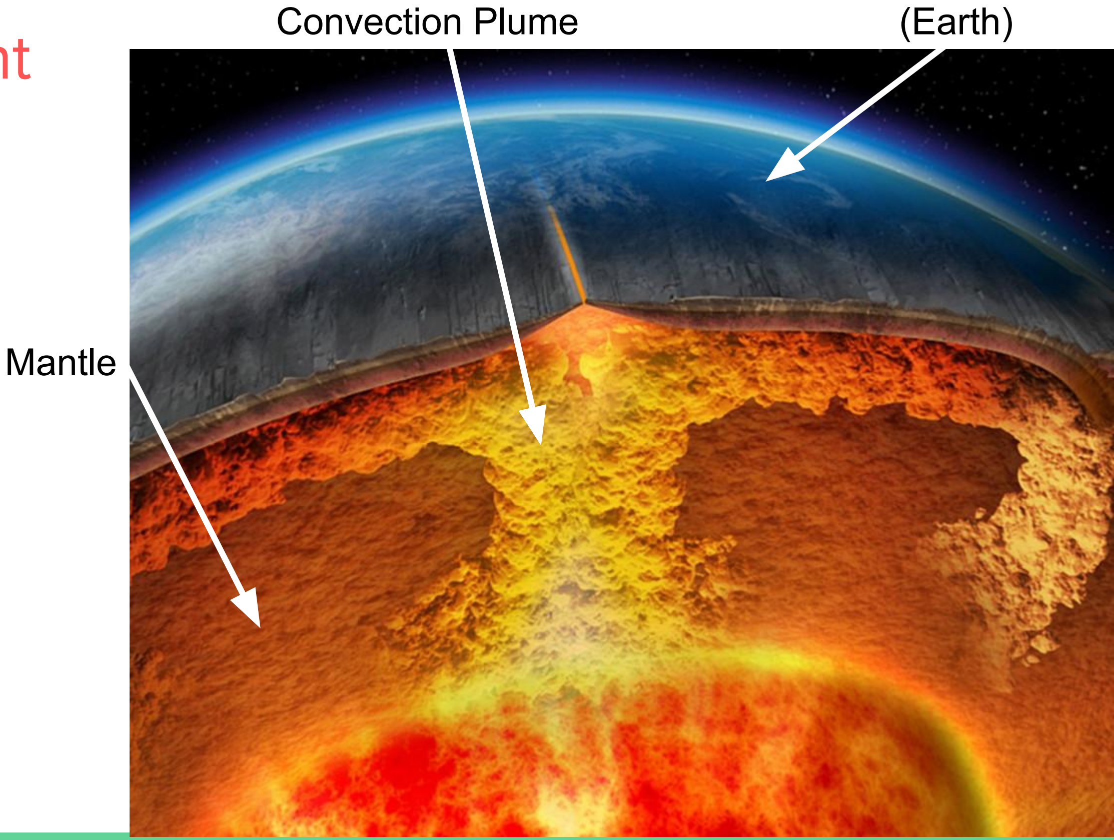
How does the temperature flow affect the movement of mass throughout the system?



Mantle Convection

How is the mantle different from other kinds of convective systems?

- Extremely Viscous
- Geologic Time Scales
- Periodic Domain



Boussinesq Equations

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} = -\nabla p + g \mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \kappa \Delta T$$

Boussinesq Equations

Kinematic Viscosity Pressure Gravity

$$\partial_t \underline{\mathbf{u}} + (\underline{\mathbf{u}} \cdot \nabla)\underline{\mathbf{u}} - \underline{\nu}\Delta \underline{\mathbf{u}} = -\nabla \underline{p} + \underline{g}\mathbf{e}_3 \underline{T}$$

Fluid Velocity

$$\nabla \cdot \underline{\mathbf{u}} = 0$$

Temperature

$$\partial_t \underline{T} + \underline{\mathbf{u}} \cdot \nabla \underline{T} = \underline{\kappa}\Delta \underline{T}$$

Thermal Diffusivity

Boussinesq Equations

$$\mathrm{Ra} = \frac{g\alpha}{\nu\kappa}(\delta T)h^3$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

$$\frac{1}{\mathrm{Pr}}\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \Delta\mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \Delta T$$

$$\mathbf{u}|_{x_3=0} = \mathbf{u}|_{x_3=1} = \mathbf{0}$$

$$T|_{x_3=0} = 1 \qquad T|_{x_3=1} = 0$$

$$\mathbf{u}, T \text{ are periodic in } x_1 \text{ and } x_2$$

$$T(\mathbf{x},0) = T_0(\mathbf{x})$$

Boussinesq Equations

$$\mathrm{Ra} = \frac{g\alpha}{\nu\kappa}(\delta T)h^3$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

$$\xcancel{\frac{1}{\mathrm{Pr}}[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}]} - \Delta \mathbf{u} = -\nabla p + \mathrm{Ra}\,\mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \Delta T$$

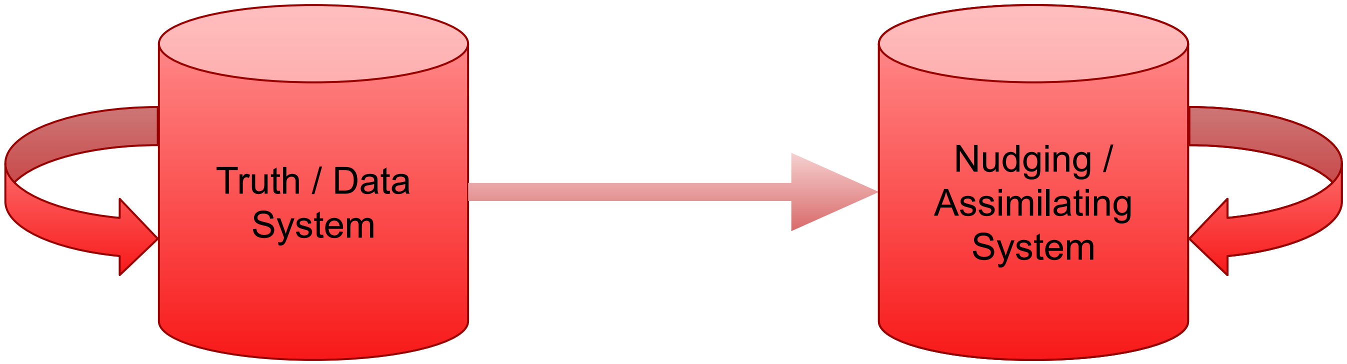
$$\mathbf{u}\big|_{x_3=0} = \mathbf{u}\big|_{x_3=1} = \mathbf{0}$$

$$T\big|_{x_3=0} = 1 \qquad T\big|_{x_3=1} = 0$$

$$\mathbf{u}, T \text{ are periodic in } x_1 \text{ and } x_2$$

$$T(\mathbf{x}, 0) = T_0(\mathbf{x})$$

Data Assimilation



Data Assimilation Problem (Infinite Prandtl)

Data Equations

$$-\Delta \mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

Assimilating Equations

$$-\Delta \tilde{\mathbf{u}} = -\nabla \tilde{p} + \mathrm{Ra}\mathbf{e}_3 \tilde{T}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \Delta \tilde{T} - \mu P_N(\tilde{T} - T)$$

Data Assimilation Problem (Infinite Prandtl)

Data Equations

$$-\Delta \mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

Relaxation Parameter

Assimilating Equations

$$-\Delta \tilde{\mathbf{u}} = -\nabla \tilde{p} + \mathrm{Ra}\mathbf{e}_3 \tilde{T}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \Delta \tilde{T} - \boxed{\underline{\mu} P_N(\tilde{T} - T)}$$

$$\boxed{f = \sum_{n=-N}^{N} f_n e^{int}}$$

Driving Term

Data Assimilation Problem (Infinite Prandtl)

Data Equations

$$-\Delta \mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

Assimilating Equations

$$-\Delta \tilde{\mathbf{u}} = -\nabla \tilde{p} + \mathrm{Ra}\mathbf{e}_3 \tilde{T}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \Delta \tilde{T} - \mu P_N(\tilde{T} - T)$$

$$\lim_{t\to\infty} \|(\tilde{\mathbf{u}} - \mathbf{u})(t)\| = \lim_{t\to\infty} \|(\tilde{T} - T)(t)\| = 0$$

Previous Work: Rayleigh-Bénard Data Assimilation

Comput Geosci (2017) 21:393–410
DOI 10.1007/s10596-017-9619-2

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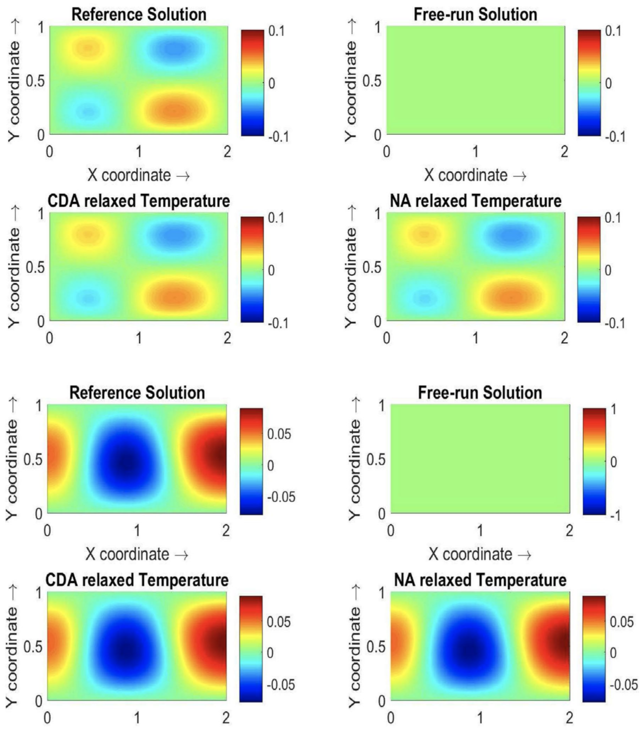
ORIGINAL PAPER

Downscaling the 2D Bénard convection equations using continuous data assimilation

M. U. Altaf[1] · E. S. Titi[2] · T. Gebrael[3] · O. M. Knio[1] · L. Zhao[4] · M. F. McCabe[1] · I. Hoteit[1]



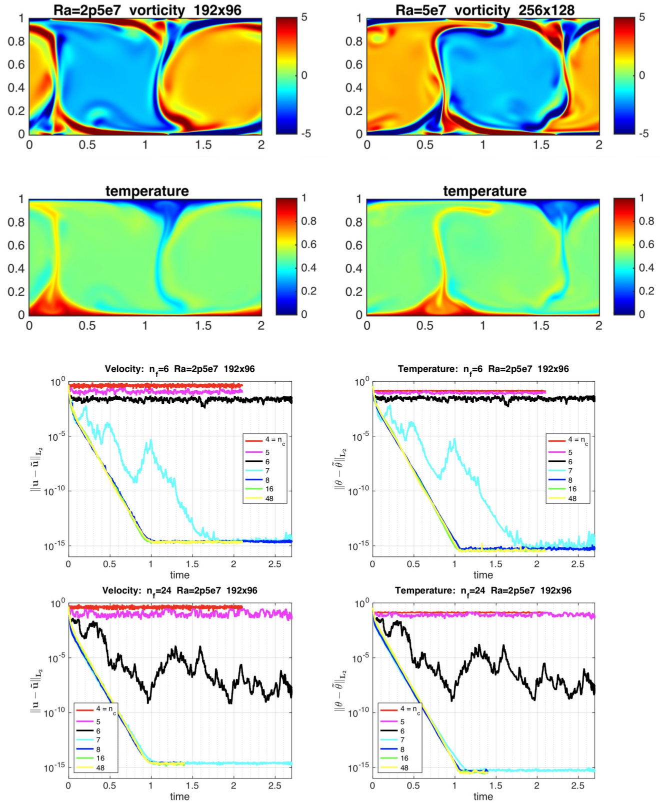
- ➢ Compares two types of projection schemes
- ➢ Limited numerics on a nonturbulent state
- ➢ Nudging equations for temperature **and** velocity

Previous Work: Rayleigh-Bénard Data Assimilation

ASSIMILATION OF NEARLY TURBULENT RAYLEIGH-BÉNARD FLOW THROUGH VORTICITY OR LOCAL CIRCULATION MEASUREMENTS: A COMPUTATIONAL STUDY

ASEEL FARHAT, HANS JOHNSTON, MICHAEL JOLLY, AND EDRISS S. TITI

ABSTRACT. We introduce a continuous (downscaling) data assimilation algorithm for the 2D Bénard convection problem using vorticity or local circulation measurements only. In this algorithm, a nudging term is added to the vorticity equation to constrain the model. Our numerical results indicate that the approximate solution of the algorithm is converging to the unknown reference solution (vorticity and temperature) corresponding to the measurements of the 2D Bénard convection problem when only spatial coarse-grain measurements of vorticity are assimilated. Moreover, this convergence is realized using data which is much more coarse than the resolution needed to satisfy rigorous analytical estimates.



- ➢ Compares number of projection nodes in each direction, always with $\mu = 1 = Pr$
- ➢ Nudging on the vorticity and local circulation
- ➢ Bound $\mu \geq C(Ra^{3/2})$ is not very sharp

Section 2: Infinite Prandtl Assimilation

Infinite Prandtl Assimilation

Data Equations

$$-\Delta \mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \Delta T$$

Assimilating Equations

$$-\Delta \tilde{\mathbf{u}} = -\nabla \tilde{p} + \mathrm{Ra}\mathbf{e}_3 \tilde{T}$$

$$\nabla \cdot \tilde{\mathbf{u}} = 0$$

$$\partial_t \tilde{T} + \tilde{\mathbf{u}} \cdot \nabla \tilde{T} = \Delta \tilde{T} - \mu P_N(\tilde{T} - T)$$

$$\lim_{t \to \infty} \|(\tilde{\mathbf{u}} - \mathbf{u})(t)\| = \lim_{t \to \infty} \|(\tilde{T} - T)(t)\| = 0$$

Infinite Prandtl Assimilation: Theorem

Theorem 2.6 (Infinite Prandtl Synchronization)**.** *Let* $(\mathbf{u}, T)$ *and* $(\tilde{\mathbf{u}}, \tilde{T})$ *satisfy the infinite-Prandtl Boussinesq data assimilation problem, and suppose that* $\|T_0\|_{L^2(\Omega)}, \|\tilde{T}_0\|_{L^2(\Omega)} \leq M$ *for some* $M > 0$*. Then there exists an absolute constant* $C_0 > 0$ *such that if*

$$\frac{1}{2}\lambda_N \geq \mu \quad \textit{and} \quad \mu \geq \frac{C_0}{2} Ra^2,$$

then

$$\|(\tilde{T} - T)(t)\|^2_{L^2(\Omega)} + Ra^{-2}\|(\tilde{\mathbf{u}} - \mathbf{u})(t)\|^2_{H^2(\Omega)} \in O(e^{-\mu t})$$

for all $t \geq 0$*.*

Infinite Prandtl Assimilation: Proof

$$\mathbf{w} = \tilde{\mathbf{u}} - \mathbf{u}$$

$$S = \tilde{T} - T$$

$$q = \tilde{p} - p$$

$$-\Delta \mathbf{w} = -\nabla q + \mathrm{Ra}\mathbf{e}_3 S$$

$$\nabla \cdot \mathbf{w} = \mathbf{0}$$

$$\partial_t S + \tilde{\mathbf{u}} \cdot \nabla S + \mathbf{w} \cdot \nabla T = \Delta S - \mu P_N(S)$$

$$\mathbf{w}|_{x_3=0} = \mathbf{w}|_{x_3=1} = \mathbf{0} \qquad S|_{x_3=0} = S|_{x_3=1} = 0$$

$$\mathbf{w}, S \text{ are periodic in } x_1 \text{ and } x_2$$

$$S(\mathbf{x}, 0) = \tilde{T}_0(\mathbf{x}) - T_0(\mathbf{x})$$

Infinite Prandtl Assimilation: Proof

$$\|\mathbf{w}\|_{H^2(\Omega)} + \|q\|_{H^1(\Omega)} \leq C\|\mathrm{Ra}\mathbf{e}_3 S\|_{L^2(\Omega)}$$

Infinite Prandtl Assimilation: Proof

$$\frac{1}{2}\frac{d}{dt}\|S\|^2_{L^2(\Omega)} + \|\nabla S\|^2_{L^2(\Omega)} + \mu\|P_N(S)\|^2_{L^2(\Omega)} = -\int_\Omega (\mathbf{w}\cdot\nabla T)S\,d\mathbf{x}$$

$$\left|\int_\Omega (\mathbf{w}\cdot\nabla T)S\,d\mathbf{x}\right| \leq \frac{1}{2}\|\nabla S(t)\|^2_{L^2(\Omega)} + C\mathrm{Ra}^2(1+\|\eta(t)\|^2_{L^2(\Omega)})\|S(t)\|^2_{L^2(\Omega)}$$

$$\frac{1}{2}\|\nabla S\|^2_{L^2(\Omega)} + \mu\|P_N(S)\|^2_{L^2(\Omega)} \geq \mu\|S\|^2_{L^2(\Omega)}$$

$$\frac{d}{dt}\|S(t)\|^2_{L^2(\Omega)} + 2\left(\mu - C\mathrm{Ra}^2(1+\|\eta(t)\|^2_{L^2(\Omega)})\right)\|S(t)\|^2_{L^2(\Omega)} \leq 0$$

Infinite Prandtl Assimilation: Proof

$$\begin{aligned}\|S(t)\|^2 &\leq \|S(0)\|^2_{L^2(\Omega)} \exp\left(-2\mu t + C\mathrm{Ra}^2 \int_0^t (1 + \|\eta(s)\|^2_{L^2(\Omega)}) ds\right) \\ &\leq \|S(0)\|^2_{L^2(\Omega)} \exp\left(C_0 \mathrm{Ra}^2 \left(\|T_0\|^2_{L^2(\Omega)} + 1\right)\right) \exp\left(-\mu t\right) \qquad \square\end{aligned}$$

Section 3: Numerical Simulations

Reduction to 2 Dimensions

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_3 \end{bmatrix} = \begin{bmatrix} -\psi_{x_3} \\ \psi_{x_1} \end{bmatrix}$$

$$\nabla \cdot \mathbf{u} = -\psi_{x_3 x_1} + \psi_{x_1 x_3} = 0$$

$$\nabla \times \mathbf{u} = \partial_{x_1} u_3 - \partial_{x_3} u_1 = \psi_{x_1 x_1} + \psi_{x_3 x_3} = \Delta\psi = \zeta$$

$$\nabla \times ((\mathbf{u} \cdot \nabla)\mathbf{u}) = (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{u})$$

Reduction to 2 Dimensions

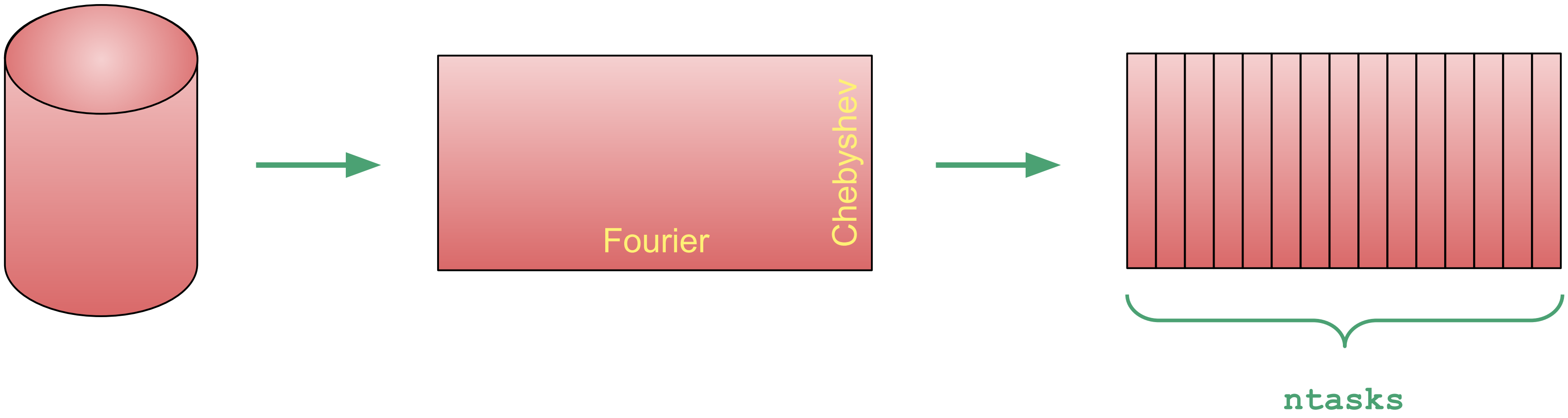
$$\nabla \times (-\Delta \mathbf{u}) = \nabla \times (-\nabla p + \mathrm{Ra}\mathbf{e}_3 T)$$

$$-\Delta(\nabla \times \mathbf{u}) = -\cancel{(\nabla \times \nabla p)} + \mathrm{Ra}(\nabla \times \mathbf{e}_3 T)$$

$$-\Delta(\Delta\psi) = \mathrm{Ra}(T_{x_1} - 0)$$

$$-\Delta\zeta = \mathrm{Ra}T_{x_1}$$

Dedalus: A Flexible Spectral Solver



Dedalus: A Flexible Spectral Solver

```
import numpy as np
from dedalus import public as de

x_basis = de.Fourier('x', 256, interval=(0, 4), dealias=3/2)
z_basis = de.Chebyshev('z', 128, interval=(0, 1), dealias=3/2)
domain = de.Domain([x_basis, z_basis], grid_dtype=np.float64)

problem = de.IVP(domain, variables=['T', 'Tz', 'psi', 'psiz', 'zeta', 'zetaz'])
problem.parameters['Ra'] = 10000
problem.substitutions['u1']  = "-dz(psi)"
problem.substitutions['u3']  =  "dx(psi)"
```

Dedalus: A Flexible Spectral Solver

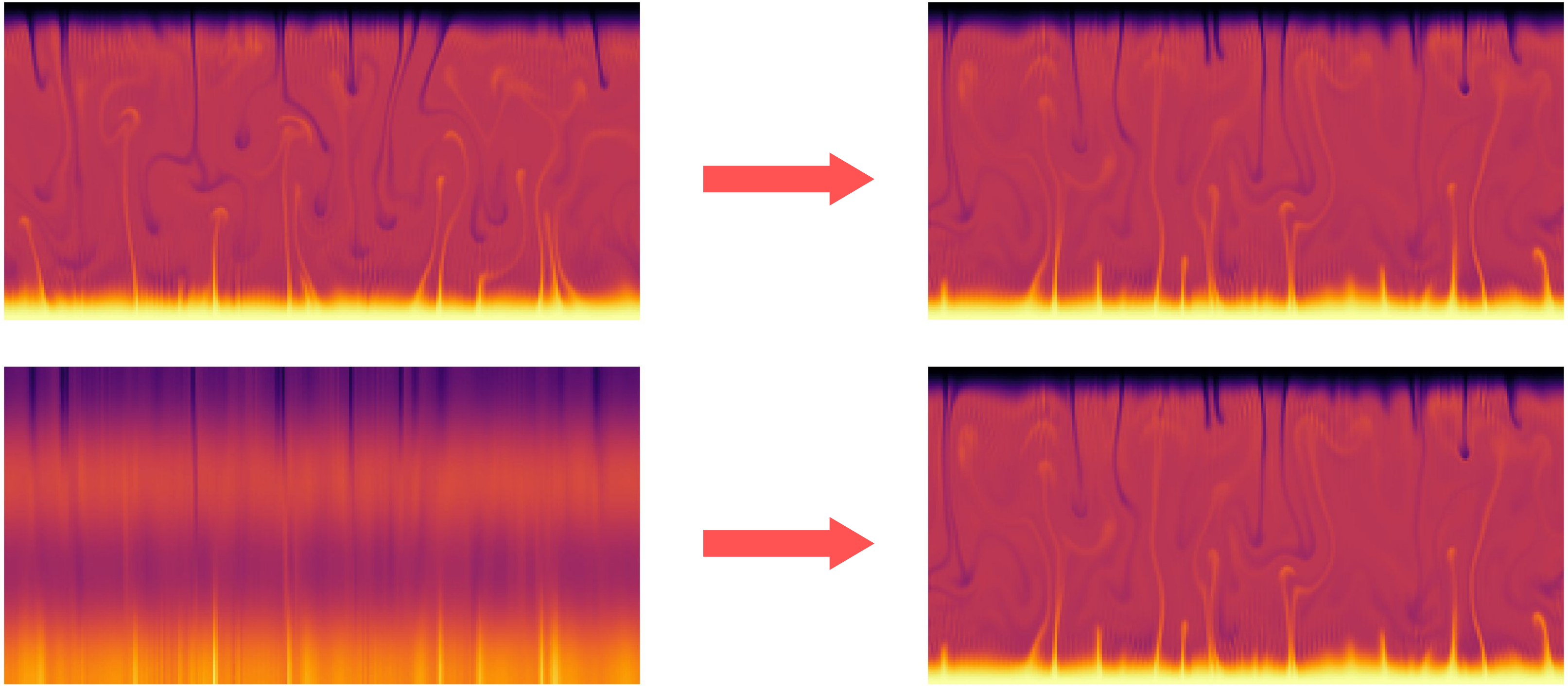
```
problem.add_equation("Tz – dz(T)  = 0")
problem.add_equation("psiz – dz(psi)  = 0")
problem.add_equation("zetaz – dz(zeta)  = 0")
problem.add_equation("zeta – dx(dx(psi)) – dz(psiz) = 0")

problem.add_equation("Ra*dx(T) + dx(dx(zeta)) + dz(zetaz)  = 0")
problem.add_equation("dt(T) – dx(dx(T)) – dz(Tz) = –u1*dx(T) – u3*Tz")
```

Dedalus: A Flexible Spectral Solver

```
problem.add_bc("left(T)  = 1")
problem.add_bc("right(T)  = 0")
problem.add_bc("left(psi)  = 0")
problem.add_bc("right(psi)  = 0")
problem.add_bc("left(psiz)  = 0")
problem.add_bc("right(psiz)  = 0")
```

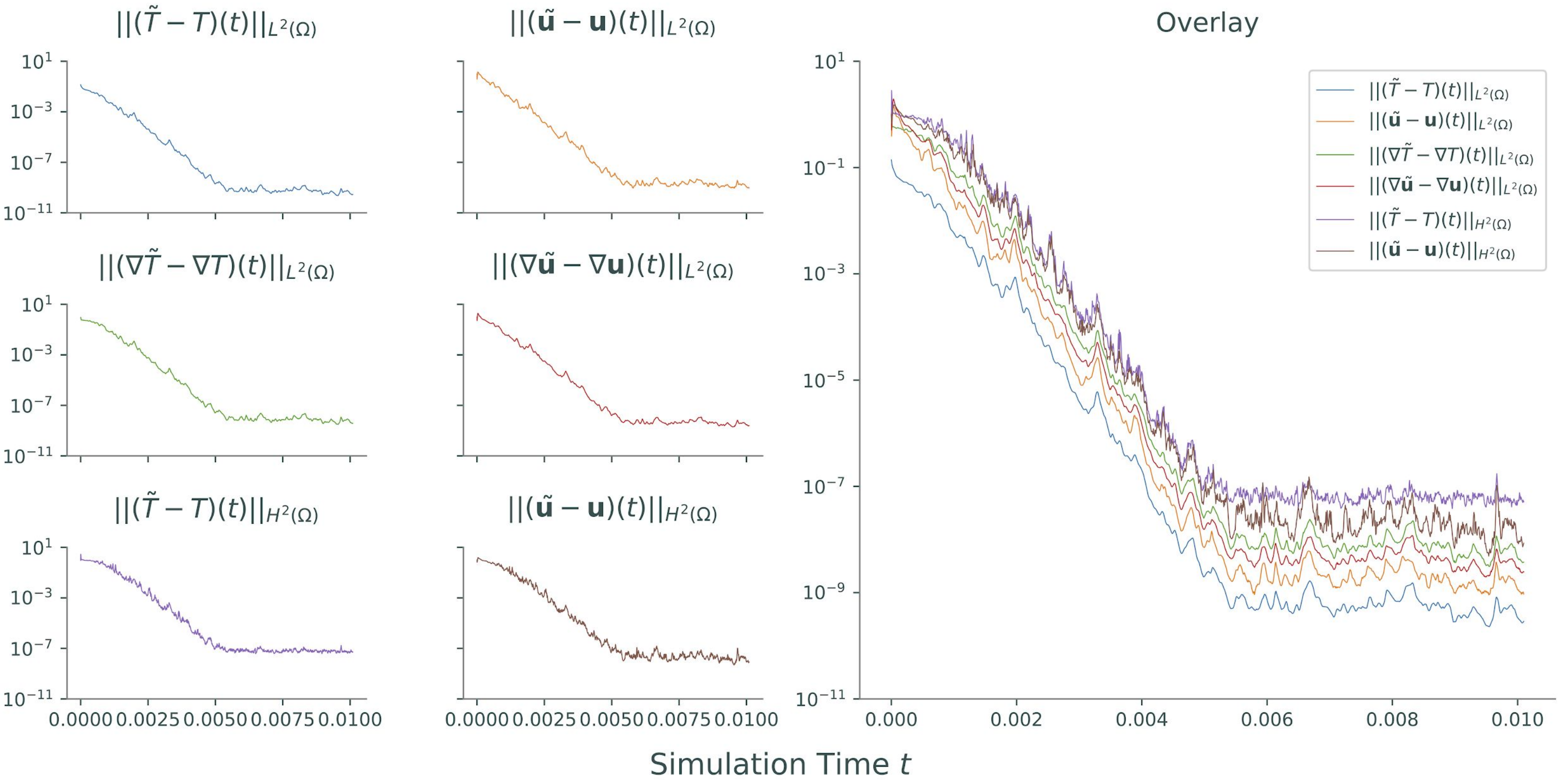
Simulation Results

Ra = 5.22e7, μ = 12700, N = 32



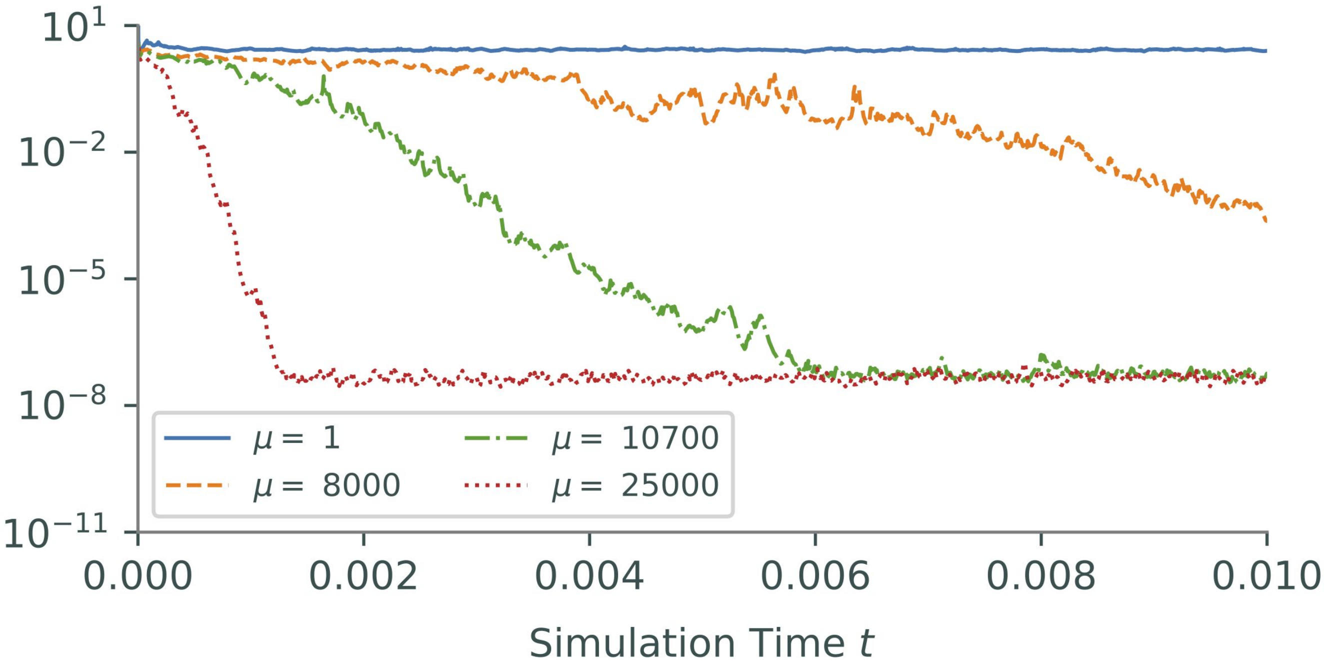
Simulation Results

Ra = 5.22e7, μ = 12700, N = 32



Simulation Results

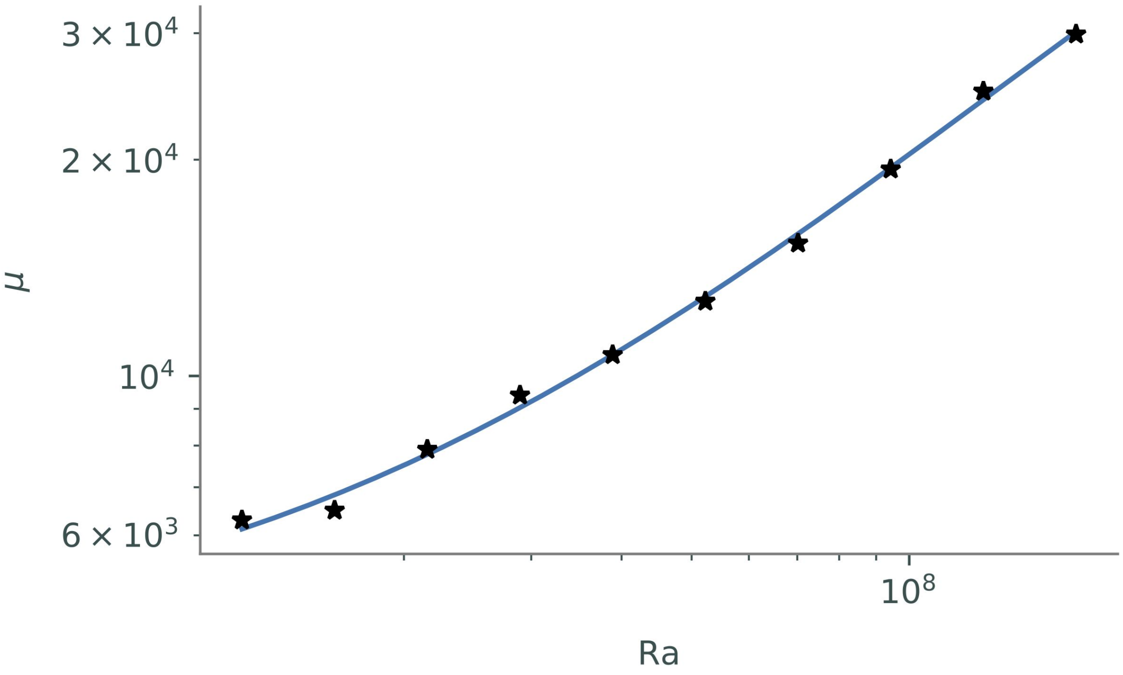
Ra = 3.89e7, N = 32



Simulation Results

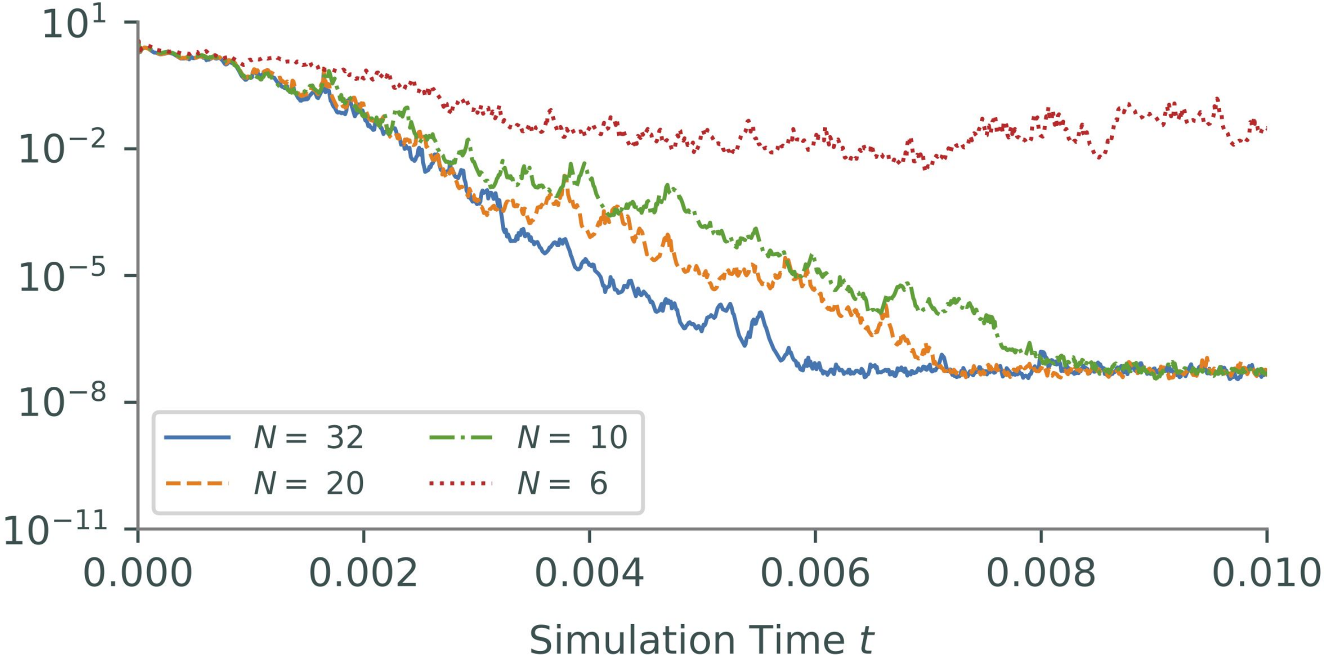
N = 32

Ra	μ
1.1937766×10^7	$6,300$
1.6037187×10^7	$6,500$
2.1544346×10^7	$7,900$
2.8942661×10^7	$9,400$
3.8881551×10^7	$10,700$
5.2233450×10^7	$12,700$
7.0170382×10^7	$15,300$
9.4266845×10^7	$19,400$
1.26638017×10^8	$24,900$
1.70125427×10^8	$29,900$



Simulation Results

Ra = 3.89e7, μ = 10700



Section 4: Other Assimilation Schemes

Finite Prandtl Assimilation

Data Equations

$$\frac{1}{\mathrm{Pr}}\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \Delta\mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \Delta T$$

Assimilating Equations

$$\frac{1}{\mathrm{Pr}}\left[\partial_t \hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\right] - \Delta\hat{\mathbf{u}} = -\nabla \hat{p} + \mathrm{Ra}\mathbf{e}_3 \hat{T}$$

$$\nabla\cdot\hat{\mathbf{u}} = 0$$

$$\partial_t \hat{T} + \hat{\mathbf{u}}\cdot\nabla \hat{T} = \Delta \hat{T} - \mu P_N(\hat{T} - T)$$

$$\lim_{t\to\infty}\|(\hat{\mathbf{u}} - \mathbf{u})(t)\| = \lim_{t\to\infty}\|(\hat{T} - T)(t)\| = 0$$

Finite Prandtl Assimilation: Theorem

Theorem 4.1. *Let* $(\mathbf{u}, T)$ *and* $(\hat{\mathbf{u}}, \hat{T})$ *satisfy the finite-Prandtl Boussinesq data assimilation problem, with* $(\mathbf{u}_0, T_0), (\hat{\mathbf{u}}_0, \hat{T}_0) \in \mathcal{B}(\rho_1, \rho_2)$*, respectively. There exist absolute constants* $C_0, C_1, C_2 > 0$ *such that if*

$$\frac{1}{2}\lambda_N \geq \mu, \quad \mu \geq \left(\frac{1}{2} + Ra^2\right) Pr, \tag{4.3}$$

and

$$Pr \geq \max\{1, 4C_0^{1/2} \max\{C_2^{1/2}, C_1^{1/2}\}(1 + Ra)^5\}, \tag{4.4}$$

then

$$\|(\hat{\mathbf{u}} - \mathbf{u})(t)\|^2_{L^2(\Omega)} + \|(\hat{T} - T)(t)\|^2_{L^2(\Omega)} \leq O(e^{-(Pr/2)t})$$

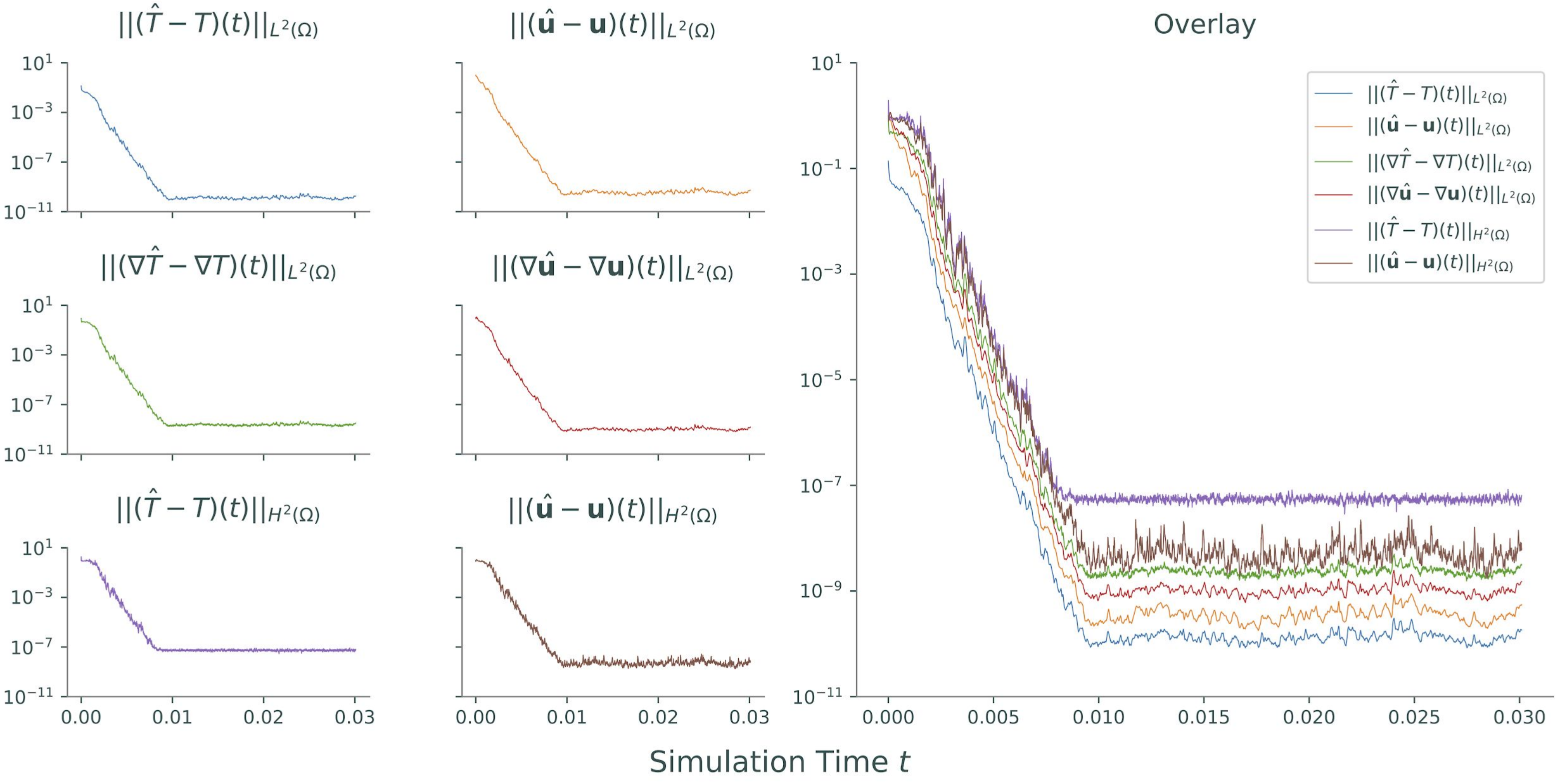
holds for $t \geq 0$.

Reduction to 2 Dimensions

$$
\begin{aligned}
\nabla \times \left(\frac{1}{\mathrm{Pr}} \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] \right) &= \frac{1}{\mathrm{Pr}} \left[(\nabla \times \partial_t \mathbf{u}) + \nabla \times ((\mathbf{u} \cdot \nabla) \mathbf{u}) \right] \\
&= \frac{1}{\mathrm{Pr}} \left[\partial_t (\nabla \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{u}) \right] \\
&= \frac{1}{\mathrm{Pr}} \left[\zeta_t + (\mathbf{u} \cdot \nabla \zeta) \right] \\
&= \frac{1}{\mathrm{Pr}} \left[\zeta_t - \psi_{x_3} \zeta_{x_1} + \psi_{x_1} \zeta_{x_3} \right]
\end{aligned}
$$

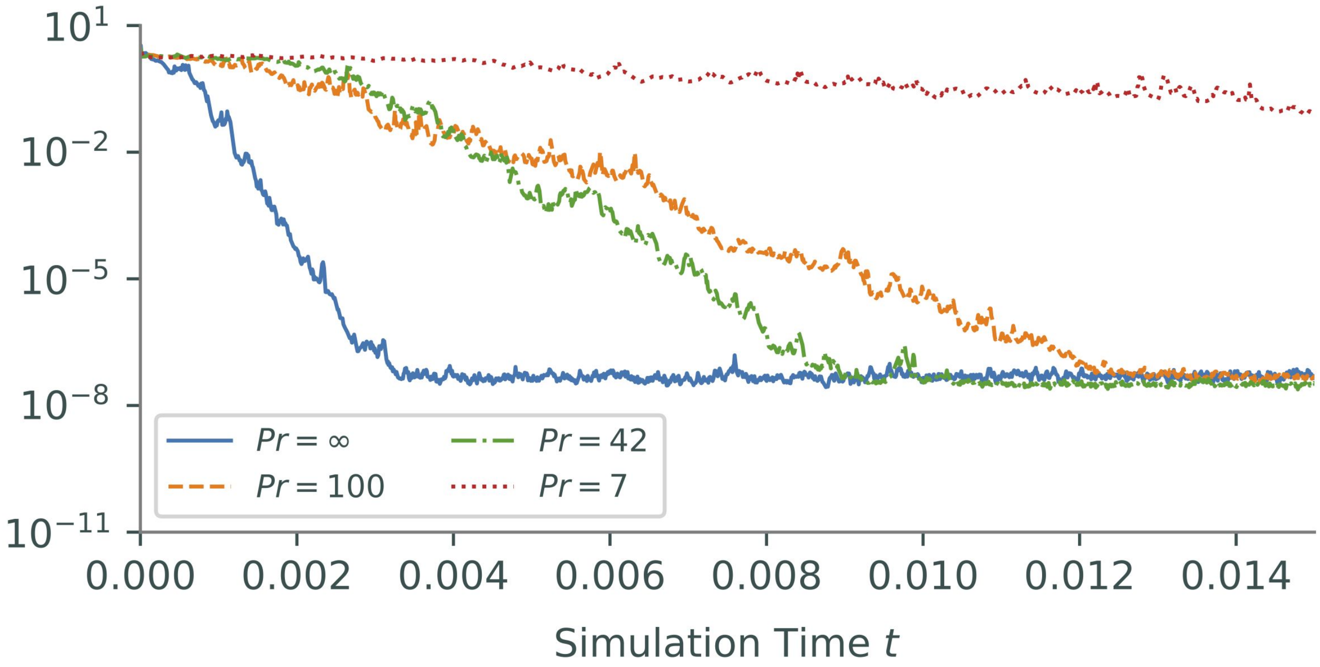
Simulation Results

Ra = 5.22e7, Pr = 100, μ = 18000, N = 32



Simulation Results

Ra = 3.89e7, μ = 14000, N = 32



Simulation Results

Ra	μ	Pr
1.1937766×10^7	$10,000$	75
1.6037187×10^7	$11,000$	56
2.1544346×10^7	$12,000$	56
2.8942661×10^7	$13,000$	100
3.8881551×10^7	$14,000$	42
5.2233450×10^7	$18,000$	42
7.0170382×10^7	$20,000$	42
9.4266845×10^7	$25,000$	31
1.26638017×10^8	$32,000$	31
1.70125427×10^8	$40,000$	31

Hybrid Prandtl Assimilation

Data Equations

$$\frac{1}{\mathrm{Pr}}\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] - \Delta\mathbf{u} = -\nabla p + \mathrm{Ra}\mathbf{e}_3 T$$

$$\nabla\cdot\mathbf{u} = 0$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \Delta T$$

Assimilating Equations

$$\frac{1}{\mathrm{Pr}}\left[\partial_t \hat{\mathbf{u}} + (\hat{\mathbf{u}}\cdot\nabla)\hat{\mathbf{u}}\right] - \Delta\hat{\mathbf{u}} = -\nabla \hat{p} + \mathrm{Ra}\mathbf{e}_3 \hat{T}$$

$$\nabla\cdot\hat{\mathbf{u}} = 0$$

$$\partial_t \hat{T} + \hat{\mathbf{u}}\cdot\nabla \hat{T} = \Delta \hat{T} - \mu P_N(\hat{T} - T)$$

$$\lim_{t\to\infty}\|(\hat{\mathbf{u}} - \mathbf{u})(t)\| = \lim_{t\to\infty}\|(\hat{T} - T)(t)\| = 0$$

Hybrid Prandtl Assimilation: Theorem

Theorem 4.2. *Let* $(\mathbf{u}, T)$, $(\tilde{\mathbf{u}}, \tilde{T})$ *be a regular solution of the hybrid-Prandtl Boussinesq data assimilation problem, with initial conditions* $(\mathbf{u}_0, T_0) \in \mathcal{B}_V(\rho_1) \times \mathcal{B}_{H^1(\Omega)}(\rho_2)$ *and* $\tilde{T}_0 \in \mathcal{B}_{H^1(\Omega)}(\rho_2)$. *There exists an absolute constant* $C_0 > 0$ *such that if*

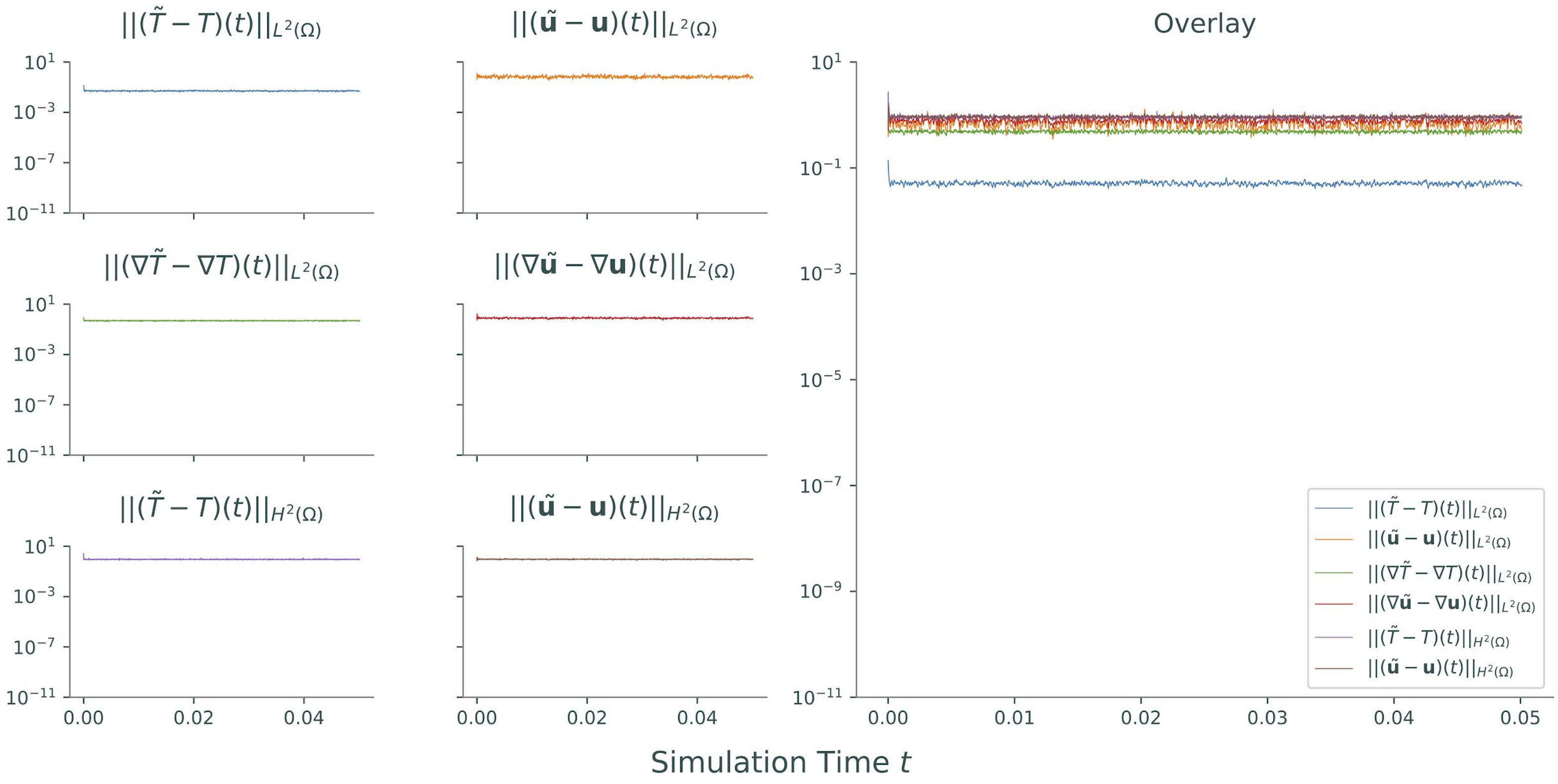
$$\frac{\lambda_N}{2} \geq \mu \quad \textit{and} \quad \mu \geq \frac{4}{3} C_0^2 \left(\rho_2 + 1\right)^2 Ra^2, \tag{4.9}$$

then for any given fixed time interval $[0, t]$,

$$Ra^{-1} \|\tilde{\mathbf{u}}(\tau) - \mathbf{u}(\tau)\|_{H^2(\Omega)} + \|\tilde{T}(\tau) - T(\tau)\|^2_{L^2(\Omega)} \leq O(e^{-\mu t}) + O(Pr^{-1}), \quad 0 \leq \tau \leq t. \tag{4.10}$$

Simulation Results

Ra = 5.22e7, Pr = 100, μ = 18000, N = 32



Thank You

Sources

Videos / Images

- https://www.youtube.com/watch?v=5ApSJe4FaLI
- https://kaiserscience.wordpress.com/earth-science/earths-layered-structure/mantle-convection/

Dedalus

K. J. Burns, G. M. Vasil, J. S. Oishi, D. Lecoanet, B. P. Brown, and E. Quataert. "Dedalus: A Flexible Framework For Spectrally Solving Differential Equations" (in preparation). http://dedalus-project.org/