A Gentle Introduction to Model Order Reduction via Proper Orthogonal Decomposition

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Outline

1. The Model Reduction Problem
2. Solution Strategy
3. Example: Heat Equation

Large Dynamical Systems

Many computational science, engineering, and mathematics applications eventually reduce to the problem of solving a (very) large system of ordinary differential equations:

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{f}(t, \mathbf{x}(t)), \qquad \mathbf{x}(0) = \mathbf{x}_0, \tag{$*$}$$

where

$$\mathbf{x} : \mathbb{R} \to \mathbb{R}^n, \qquad \mathbf{f} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \qquad \mathbf{x}_0 \in \mathbb{R}^n.$$

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Some assumptions:

- We know $\mathbf{f}$ and that it is "nice"
- The dimension n is large ($n \sim 10^8$)
- We can solve $(*)$, but it is computationally expensive to do so

Small Dynamical Systems

The goal of model order reduction (MOR) is to construct a low-dimensional system

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The goal is typically $r < 50$ or so.

Strategy: Projection to a Linear Subspace

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But how tho?

Strategy: Collect Snapshot Data

To approximate the solution set $\{\mathbf{x}(t) \mid t \geq 0\}$, simulate the FOM $(*)$ and record the solution $\mathbf{x}(t)$ at times $0 = t_0 < t_1 < \cdots < t_{k-1}$. Stack the results column-wise to get the *snapshot matrix*,

$$X = \left[\begin{array}{c|c|c|c} \mathbf{x}(t_0) & \mathbf{x}(t_1) & \cdots & \mathbf{x}(t_{k-1}) \end{array}\right] \in \mathbb{R}^{n \times k}.$$

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But we want $V \in \mathbb{R}^{n \times r}$ (probably $r < k < n$)...

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Theorem

Let $A \in \mathbb{R}^{n\times k}$ *have the singular value decomposition*

$$A = \Phi\Sigma\Psi^{\mathsf{T}}.$$

Then the first r *columns of* Φ *are the best rank-*r *approximation of the column space of* A*. In other words,* $\{\phi_j\}_{j=1}^{r}$ *solves the minimization problem*

$$\underset{\tilde{\phi}_1,\dots,\tilde{\phi}_r\in\mathbb{R}^n}{argmin} \sum_{j=1}^{k} \left\| \mathbf{x}_j - \sum_{i=1}^{r} \left\langle \mathbf{x}_j, \tilde{\phi}_i \right\rangle_{\mathbb{R}^n} \tilde{\phi}_i \right\|_{\mathbb{R}^n}^2$$

$$such\ that$$

$$\left\langle \tilde{\phi}_i, \tilde{\phi}_j \right\rangle_{\mathbb{R}^n} = \delta_{ij} \quad for \quad 1 \le i, j \le r.$$

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Define

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- $V_r^\mathsf{T} V_r = I \in \mathbb{R}^{r \times r}$
- $V_r V_r^\mathsf{T}$ is the orthogonal projector onto

$$\text{Range}(V_r) \approx \text{Range}(X) \approx \{\mathbf{x}(t) \mid t \geq 0\}$$

Strategy: Substitute the Approximation

Since V_r is orthonormal,

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3. Reconstruct the solution in the initial space as $\mathbf{x}_{\mathtt{ROM}}(t) = V_r\hat{\mathbf{x}}(t)$.

Example: Heat Equation

Consider the following one-dimensional parabolic PDE.

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\begin{aligned}
x_t(t,\omega) &= \frac{1}{\pi^2} x_{\omega\omega}(t,\omega), && (t,\omega) \in (0,T] \times \Omega, \\
x_\omega(t,0) &= x_\omega(t,1) = 0, && t > 0, \\
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We can reduce this system!

`https://tinyurl.com/y28qhsfl`