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Assignment 4 - Matrix

Denote matrix $M = \sigma A$. Then,

$$
\begin{aligned}
X_1 &= IX_0 + MX_0 = (I+M)X_0 \\
X_2 &= X_1 + M^2X_0 = (I+M)X_0 + M^2X_0 = (I+M+M^2)X_0 \\
&... \\
X_t &= (I+M+M^2+...+M^t)X_0 \\
(M-I)X_t &= (M-I)(I+M+M^2+...+M^t)X_0 \\
(M-I)X_t &= (M-I+M^2-M+M^3-M^2+...+M^{t+1}-M^t)X_0 \\
(M-I)X_t &= (M^{t+1}-I)X_0 \\
X_t &= (M-I)^{-1}(M^{t+1}-I)X_0
\end{aligned}
$$

By decomposing matrix M, we get$M = PDP^{-1}$, where D is a diagonal matrix formed from the eigenvalues of M, and the columns of P are the corresponding eigenvectors of M.

Because $P^{-1}P = I$ then $M^{t+1} = (PDP^{-1})^{t+1} = PD^{t+1}P^{-1}$.

Therefore, $X_t = (M-I)^{-1}(PD^{t+1}P^{-1} - I)X_0$
In this problem,

$$
D = \begin{bmatrix} w_1 & 0 & 0 & 0 & 0 \\ 0 & w_2 & 0 & 0 & 0 \\ 0 & 0 & w_3 & 0 & 0 \\ 0 & 0 & 0 & w_4 & 0 \\ 0 & 0 & 0 & 0 & w_5 \end{bmatrix}
$$

Then,

$$
D^{t+1} = \begin{bmatrix} w_1^{t+1} & 0 & 0 & 0 & 0 \\ 0 & w_2^{t+1} & 0 & 0 & 0 \\ 0 & 0 & w_3^{t+1} & 0 & 0 \\ 0 & 0 & 0 & w_4^{t+1} & 0 \\ 0 & 0 & 0 & 0 & w_5^{t+1} \end{bmatrix}
$$

For w_i, we have:

$$w_i^\infty = \begin{cases} 0, & \text{if } -1 < w_i < 1 \\ 1 \text{ or } -1, & \text{if } w_i = -1 \text{ or } 1 \\ \infty, & \text{if } w_i < -1 \text{ or } w_i > 1 \end{cases}$$

We consider two cases:

- One of $w_i \to \infty$ then X_∞ is **divergent**.
  This happens when $1 > \sigma \geq \textbf{\textit{threshold}} = 0.47$. The threshold is calculated by generating w_i on several values of σ in range (0,1).

- All of $w_i \to 0$ then D^∞ is zero matrix. Then,

$$X_\infty = (M - I)^{-1}(PD^\infty P^{-1} - I)X_0$$
$$X_\infty = -(M - I)^{-1}X_0$$

  This happens when $0 < \sigma < \textit{threshold}$.

The below result is obtained in case $\sigma = 0.33$.

$$X_\infty = \begin{bmatrix} 1.6322411 \\ 0.68862128 \\ 0.53863956 \\ 0.45449004 \\ 0.68862128 \end{bmatrix}$$