Path Planning with Robot Dynamics using Mathematical Programming

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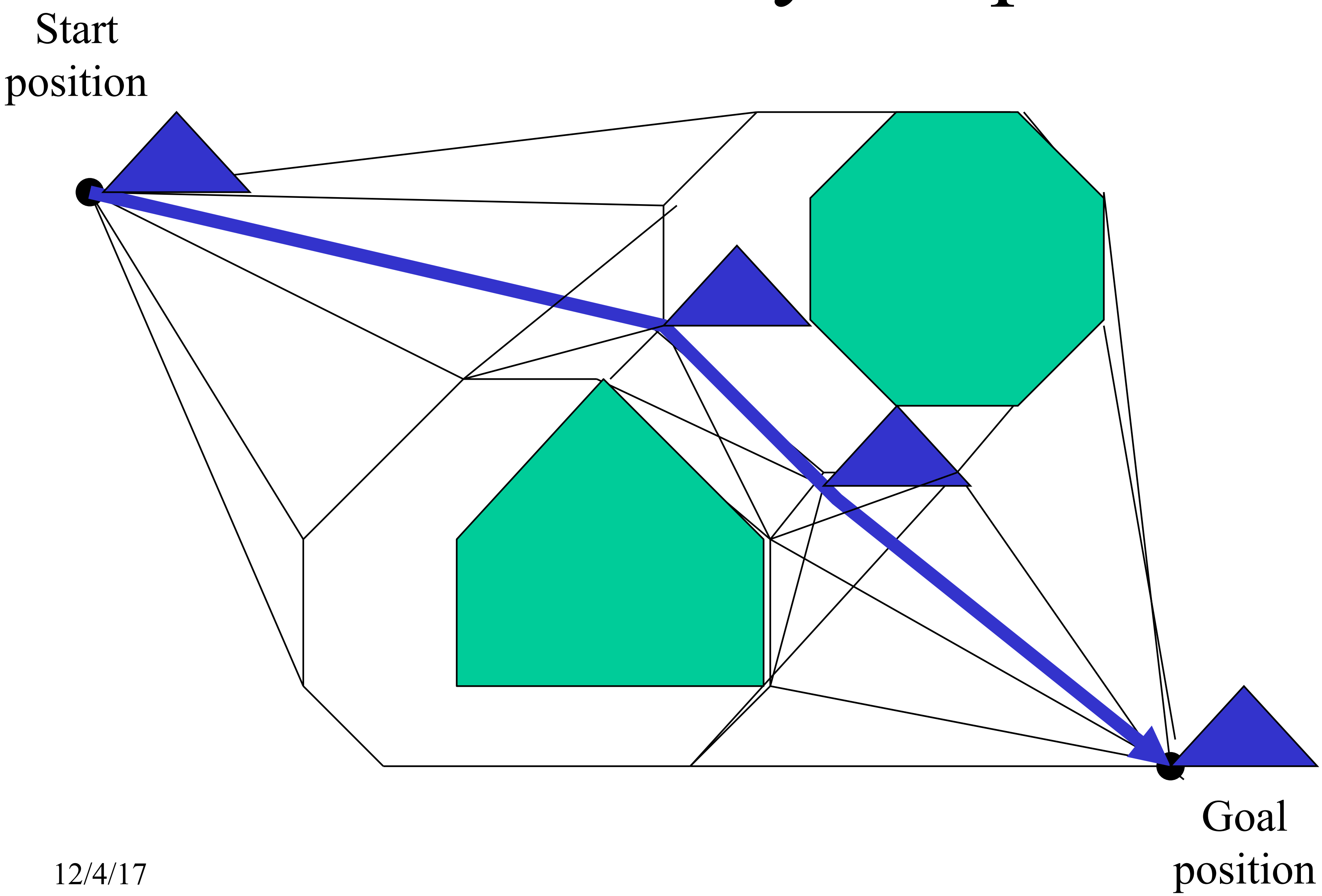
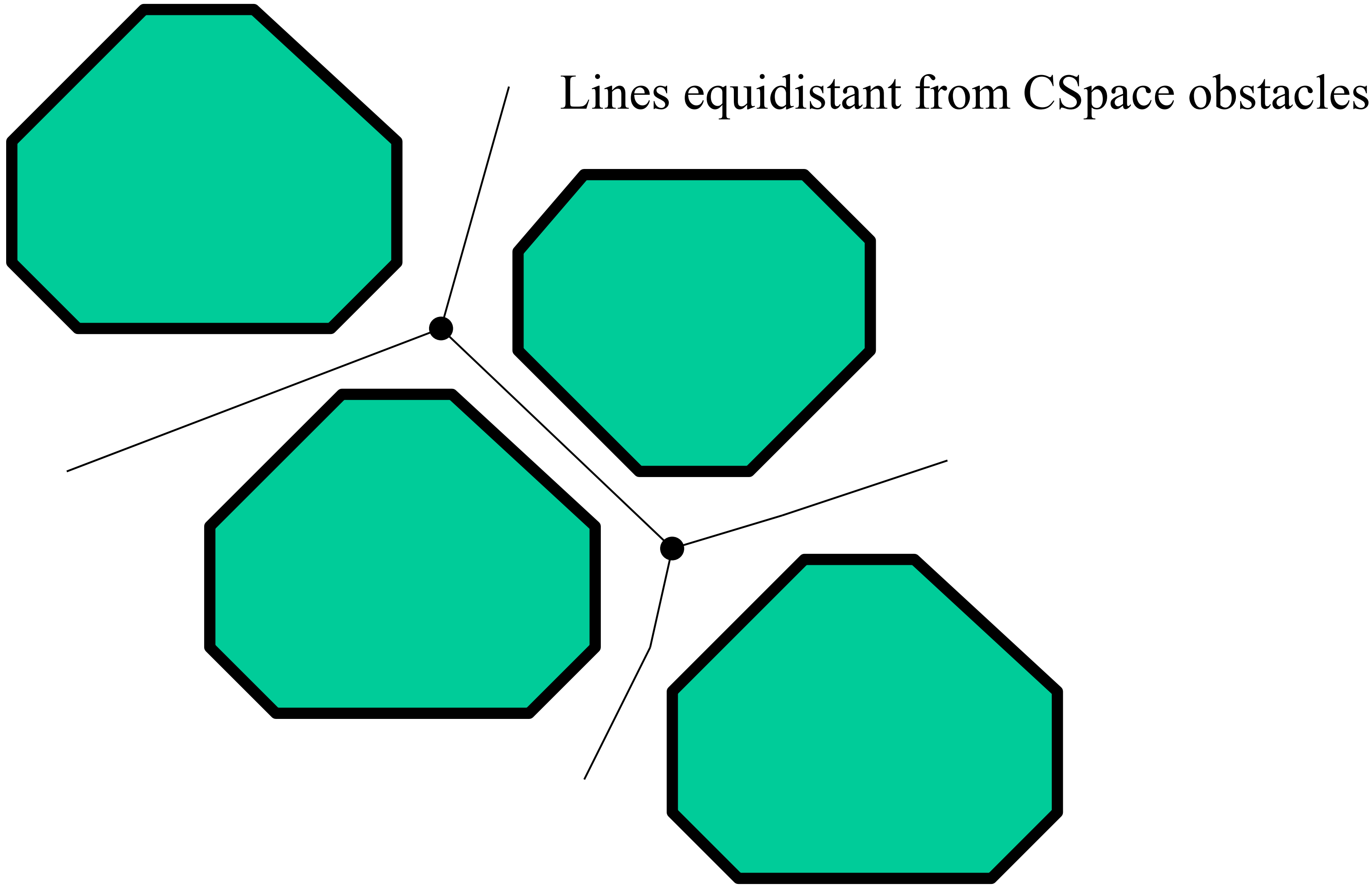
Reference

T. Schouwenaars, B. De Moor, E. Feron, and J. How, "MIXED INTEGER PROGRAMMING FOR MULTI-VEHICLE PATH PLANNING," ECC2001.

Shortest Path Using A Visibility Graph



Robust Paths Using A Voronoi Diagrams



Linear Programming

Can we formulate kino-dynamic path planning as an LP?

In part, we can incorporate:
- robot dynamics,
- time (arrival) constraints,
- vehicle actuation limits, and
- provide a minimum fuel solution.

Obstacle avoidance requires a little more:

Mixed Integer/Linear Programming

Objectives of Path Planning Problem

From a start state (position and velocity) get to a target state.

LP Constraints and Objective function

1. At a fixed time T ? Or by T ?
2. Minimizing fuel burn ?
3. As quickly as possible ?
4. Respecting maximum fuel flow constraints ?
5. Respecting maximum velocity ?
6. Respecting a mission velocity profile ?
7. Avoiding static obstacles ?
8. Avoiding other moving vehicles?
9. Computing paths in real time ?

Robot Control and State

$$\ddot{S} = \frac{F}{M}$$

Linearized as:

$$S_i = (x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i)^T$$

$$U_i = (u_{xi}, u_{yi})^T$$

$$\dot{S} = A_c s + B_c u$$

Spacecraft with Mass M

Consider A Fixed Arrival Time (FAT)

Arrival at T, must reach goal state $\mathbf{s}(G)$. Implement the following constraints:

$$\min J = \int_0^{\infty} f(|\mathbf{u}|)dt$$

$\lvert\mathbf{u}\rvert \le \mathbf{u}_{\max}$	Maximum fuel flow constraint
$\lvert\dot{\mathbf{s}}(t)\rvert \le \dot{\mathbf{s}}_{\max}(t)$	Predefined maximum speed profile
$\mathbf{s}(T) - \mathbf{s}(G) = 0$ and $\dot{\mathbf{s}}(T) = 0$	Termination constraint
$\dot{\mathbf{s}} = \mathbf{A}_c\mathbf{s} + \mathbf{B}_c\mathbf{u}$	Equation of motion constraint

Discretizing A Fixed Arrival Time

Use the linearization $f(|\mathbf{u}|) \approx r^T |\mathbf{u}|$

$$\min J = \min \sum_{i=0}^{N-1} r^T |\boldsymbol{u}_i| \Delta t \qquad N = \frac{T}{\Delta t}$$

To use $|\mathbf{u}_i|$ in linear equations, assume $\frac{\partial J}{\partial u_{ik}} > 0$

$$|u_{ik}| \geq u_{ik}$$

$$|u_{ik}| \geq -u_{ik}$$

(Time step i, vector element k.)

Linear Dynamics with FAT

$$\boldsymbol{s}_{i+1} = A_d \boldsymbol{s}_i + B_d \boldsymbol{u}_i \qquad\qquad \boldsymbol{s}_N = \boldsymbol{s}_g$$

With force proportional to fuel flow:

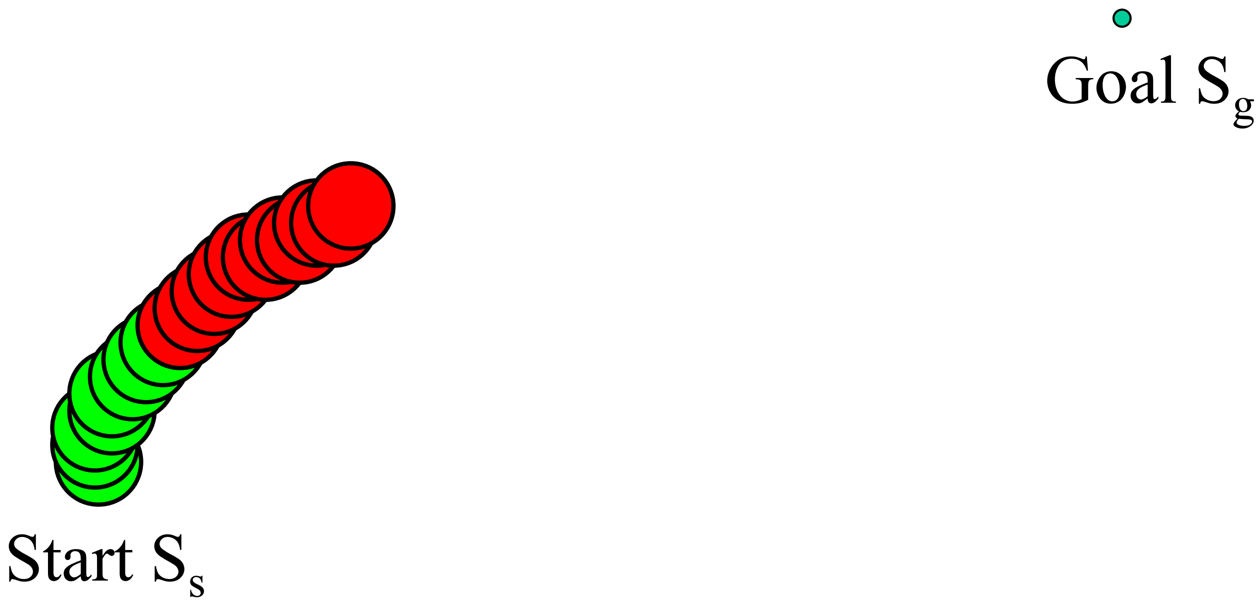
$$\begin{pmatrix} x_{i+1} \\ y_{i+1} \\ \dot{x}_{i+1} \\ \dot{y}_{i+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_i \\ y_i \\ \dot{x}_i \\ \dot{y}_i \end{pmatrix} + \begin{pmatrix} 0.5\Delta t^2 / m & 0 \\ 0 & 0.5\Delta t^2 / m \\ \Delta t / m & 0 \\ 0 & \Delta t / m \end{pmatrix} \begin{pmatrix} F_{i,x} \\ F_{i,y} \end{pmatrix}$$

Receding Horizon

Receding Horizon

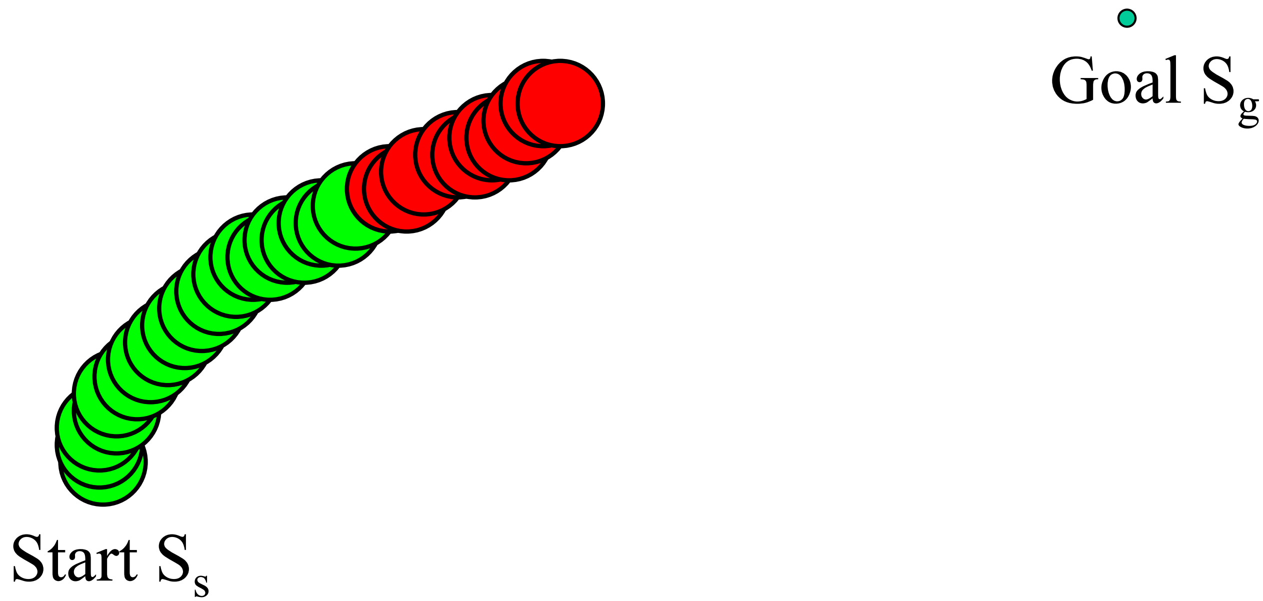
Look ahead T_{hor} and plan with mixed fuel-distance cost function

Receding Horizon



After T_{replan} steps (1<=Treplan<=T_{hor}) plan again to a horizon of T_{hor}.

Receding Horizon



Repeat until goal reached.

Weighted Cost Functions With A Finite Receding Horizon

$$\min J = \min \underbrace{q^T \left| \boldsymbol{s}_N - \boldsymbol{s}_g \right|}_{\text{Estimate of remaining cost}} + \sum_{i=0}^{N-1} r^T |\boldsymbol{u}_i| \, \Delta t$$

$$N = \frac{T_{hor}}{\Delta t}$$

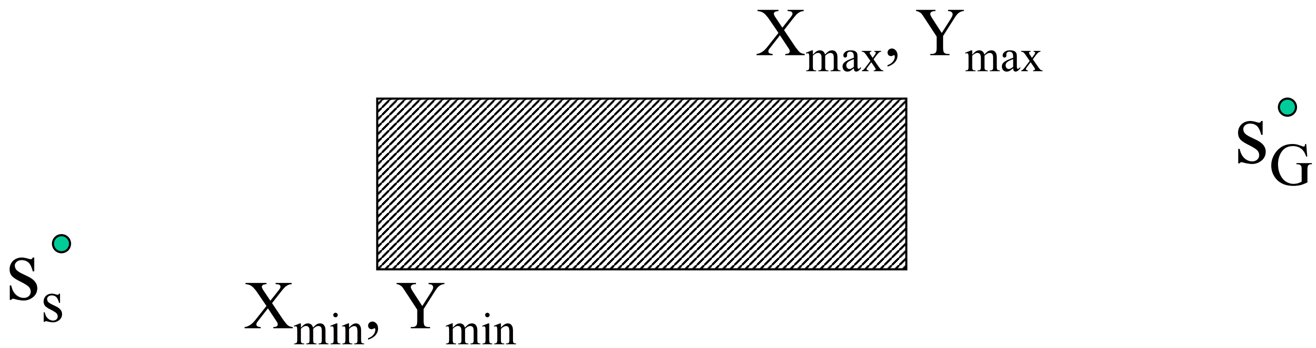
The Multi-Vehicle Case

$$\min J = \min \sum_{p=1}^{v} \left(q_p^T \left| \boldsymbol{s}_{pN} - \boldsymbol{s}_{pg} \right| + \sum_{i=0}^{N-1} r_p^T \left| \boldsymbol{u}_{pi} \right| \Delta t \right)$$

With dynamics

$$\boldsymbol{s}_{p,i+1} = A_{pd} \boldsymbol{s}_{pi} + B_{pd} \boldsymbol{u}_{pi}$$

Avoiding Static Obstacles



$$\forall i \in [1 \ldots N]:$$

$$\text{Not LP !} \quad \left\{ \begin{array}{l} x_i \le x_{\min} \\ \text{or } x_i \ge x_{\max} \\ \text{or } y_i \le y_{\min} \\ \text{or } y_i \ge y_{\max} \end{array} \right.$$

Reformulation as a MILP

$$\forall i \in [1 \ldots N]:$$

$$x_i \le x_{\min} + Mt_{i1}$$

$$\text{and } -x_i \le -x_{\max} + Mt_{i2}$$

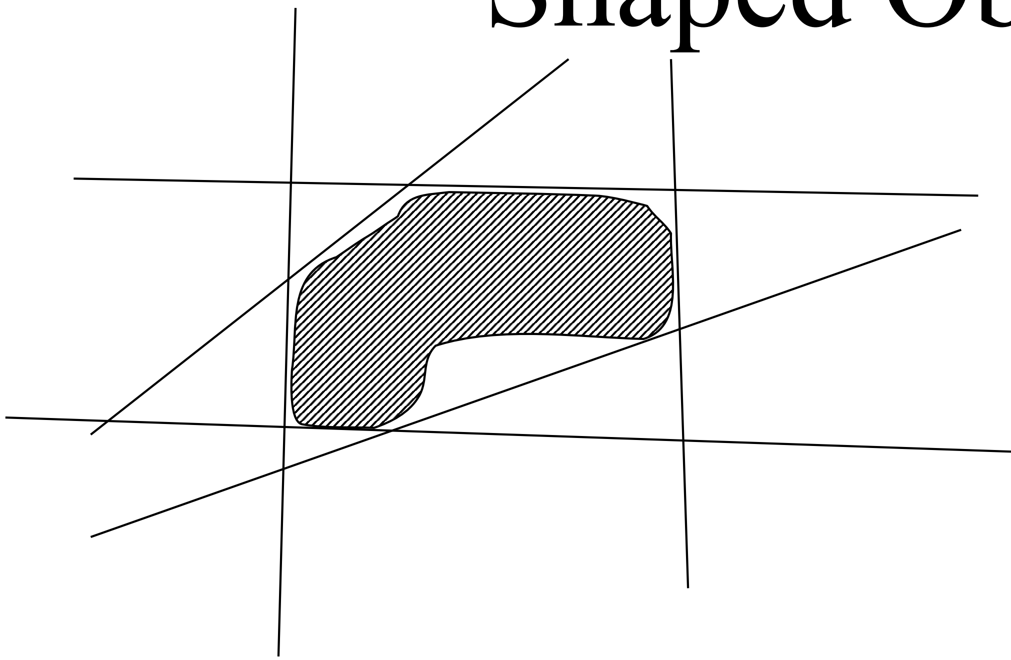
$$\text{and } y_i \le y_{\min} + Mt_{i3}$$

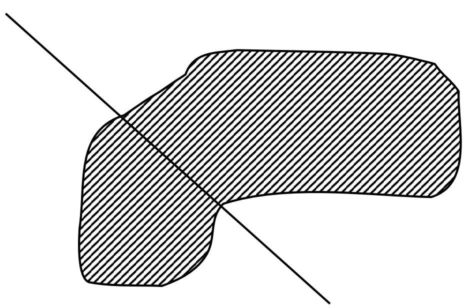
$$\text{and } -y \le -y_{\max} + Mt_{i4}$$

$$\text{and } \sum_{k=1}^{4} t_{ik} \le 3$$

$$\text{with } t_{ik} \in [0,1]$$

Encoding Arbitrarily Shaped Obstacles





- **Describe obstacle as a convex shape using N constraints. Exactly 1 of them must be applied (as before).**
- **Concave shapes can be dealt with by chopping them into multiple convex objects.**
- **Trivially extend to 3D using convex polyhedra instead of convex polygons.**

Moving Obstacles and Multiple Vehicles

- Moving obstacles with predefined motion (fixed).
  - Index all obstacles at each time step.
- Multi non-colliding vehicles.
  - Keep all vehicle pairs a minimum distance apart in x,y, or z

$$
\begin{aligned}
&\forall i \in [1 \ldots N]: \\
&\forall p,q \mid q > p: \\
&|x_{pi} - x_{qi}| \geq d_x \text{ or} \\
&|y_{pi} - y_{qi}| \geq d_y
\end{aligned}
$$

$$
\begin{aligned}
&\forall i \in [1 \ldots N]: \\
&\forall p,q \mid q > p: \\
&x_{pi} - x_{qi} \geq d_x \text{ or} \\
&x_{qi} - x_{pi} \geq d_x \text{ or} \\
&y_{pi} - y_{qi} \geq d_y \text{ or} \\
&y_{qi} - y_{pi} \geq d_y
\end{aligned}
$$

Encoding Multi-Vehicle Collision Avoidance as a MILP

$$\forall i \in [1 \ldots N]:$$

$$\forall p, q \mid q > p:$$

$$x_{pi} - x_{qi} \geq d_x - Mb_{pqi1} \text{ and}$$

$$x_{qi} - x_{pi} \geq d_x - Mb_{pqi2} \text{ and}$$

$$y_{pi} - y_{qi} \geq d_y - Mb_{pqi3} \text{ and}$$

$$y_{qi} - y_{pi} \geq d_y - Mb_{pqi4} \text{ and}$$

$$\sum_{k=1}^{4} b_{pqik} \leq 3$$

Tradeoffs: Receding Horizon vs. Fixed Arrival Time

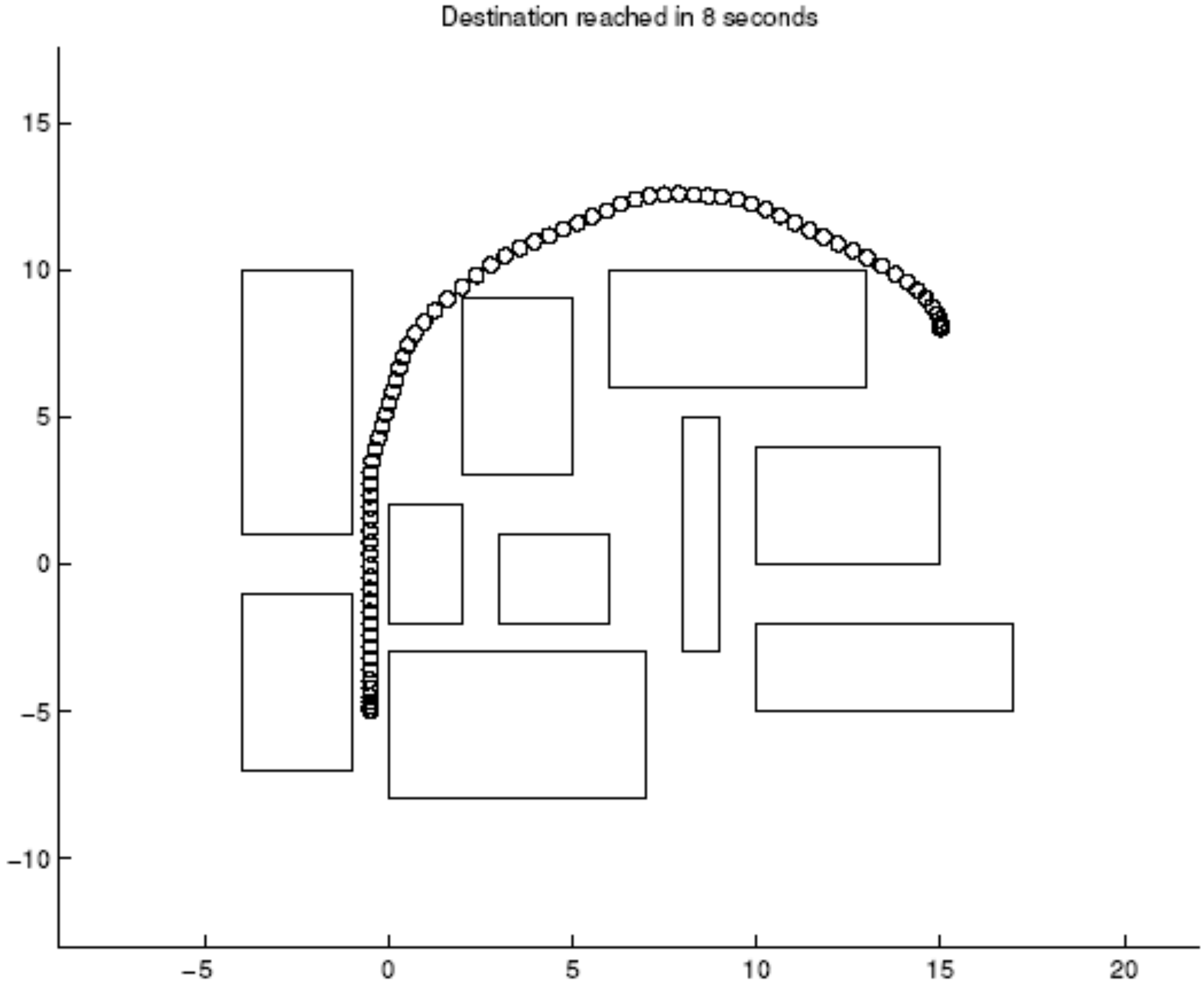
- Advantage of Receding Horizon:
  - Lower computational cost due to significantly less variables and constraints.
- Advantage of Fixed Arrival time:
  - The cost function does not include distance so it it's solutions are purely optimal in fuel

Empirical Performance Data

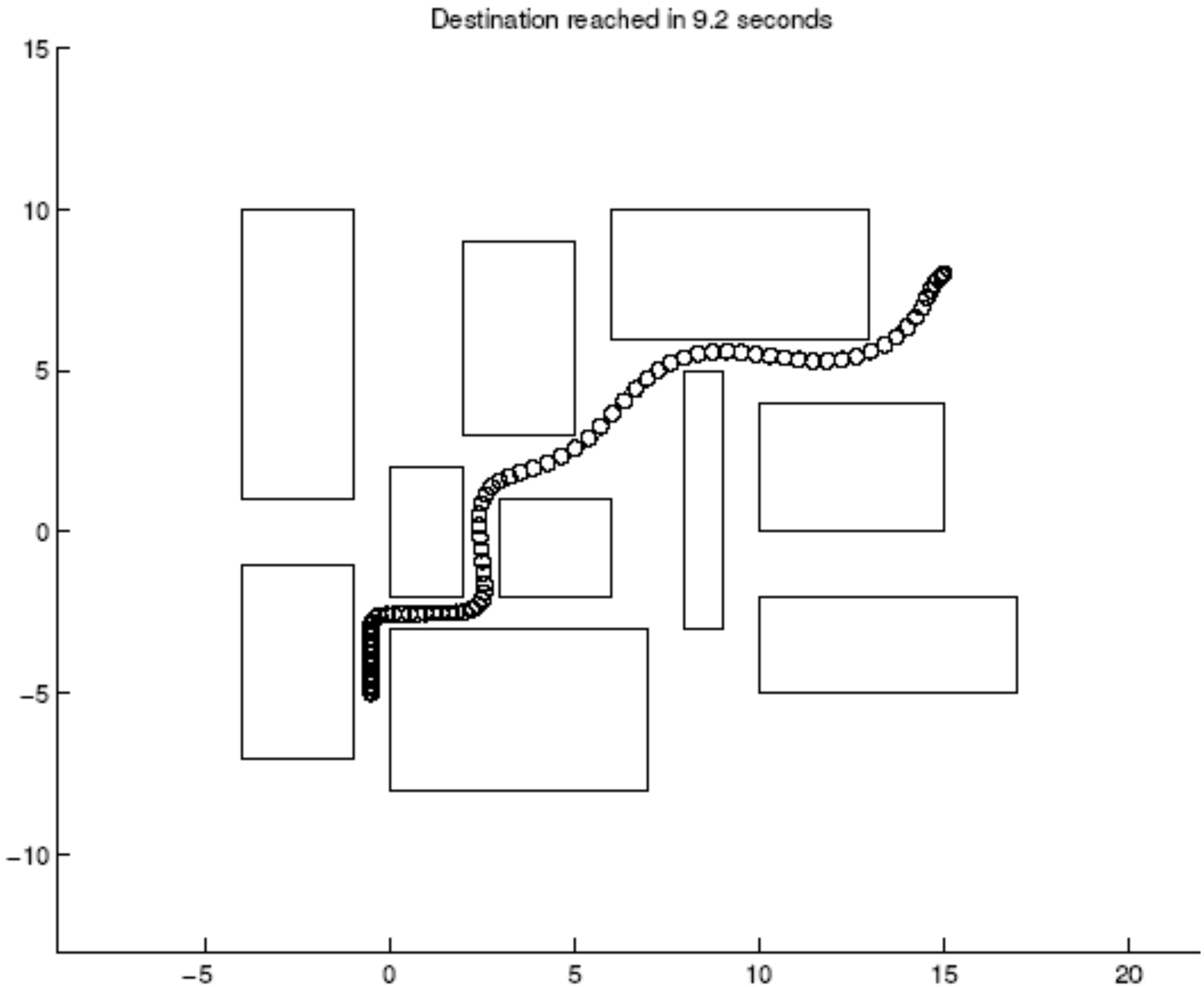
$T_{hor}(s)$	$T_{arr}(s)$	E_{tot}	$T_{comp}(s)$	$T_{it}(s)$
3.0	9.2	35.7	129	1.40
3.5	9.0	34.1	267	2.97
4.0	8.4	31.6	441	5.25
4.5	8.2	30.4	728	8.88
5.0	8.2	30.4	1213	14.79

$T_{arr}(s)$	E_{tot}	$T_{comp}(s)$
9.2	17.4	613
9.0	17.8	442
8.4	19.2	193
8.2	20.0	189
8	21.2	162

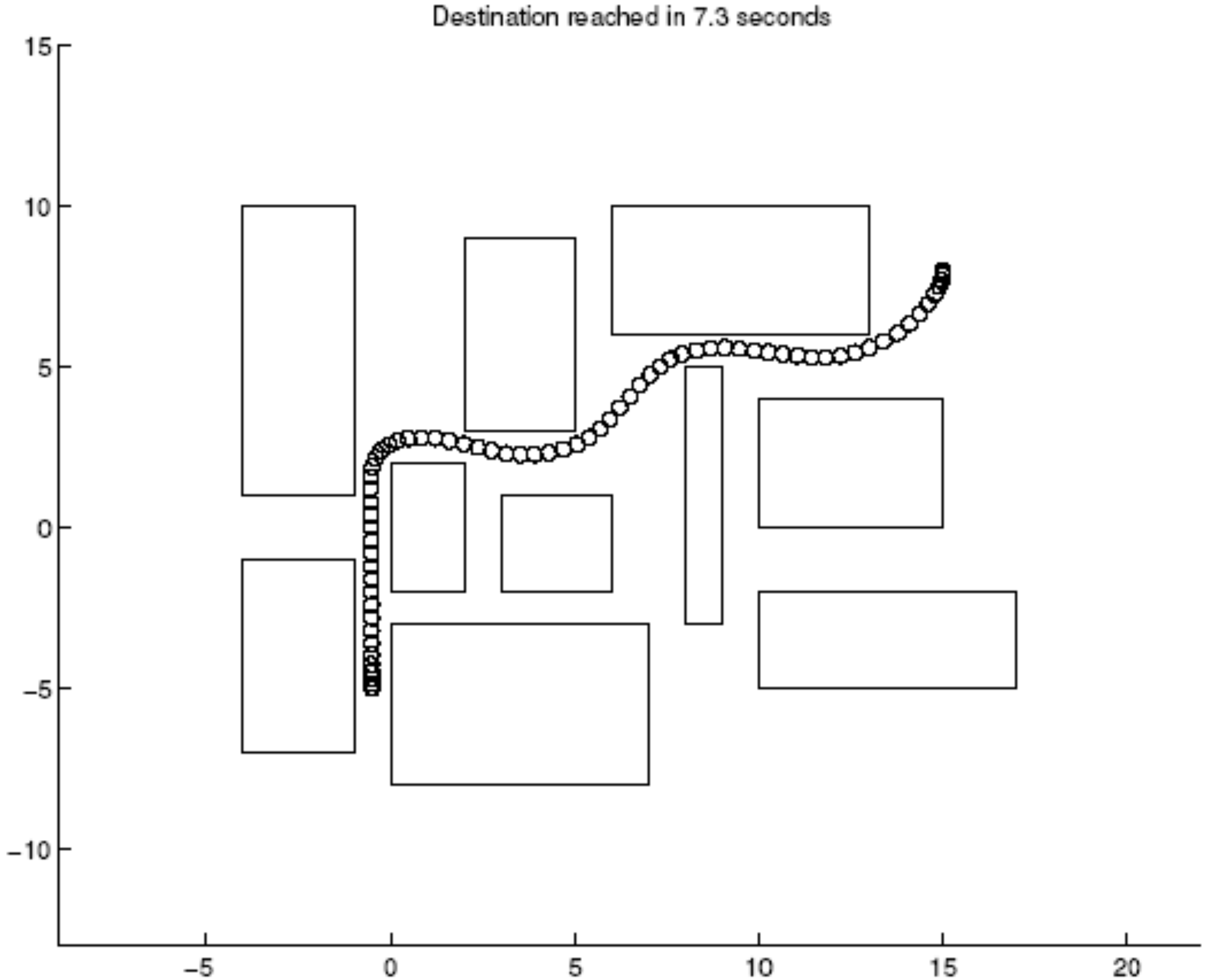
FAT $T_{arr} = 8s$



Receding Horizon T_{hor}=3s



FAT T_{min}=7.3



Spacecraft Dynamics

Given a thrust w we can accelerate a spacecraft at a=w/M.

Starting from a position s(0) and velocity ds/dt(0)

Rendezvous at time T at position s(T). The spacecraft must be stationary at the rendezvous position at time T, hence: ds/dt(T)=0 and s(T)=0 (for notational convenience make the target position be zero).

If thrust intensity is proportional to fuel flow and we want to seek a minimum (fuel) cost path our objective function would be:

$$Z = \int_0^T |u| dt.|$$

Additional Constraints

$|u| \leq u_{max}$ Upper limit on available thrust (Maximum fuel flow).

$|ds/dt(t)| \leq ds_{max(t)}$ Upper limit on speed.

$|s(t)| \geq s_{min}$ Lower limit on distance from goal.

Advanced Activity and Path Planning using Mathematical Programs

Add:

- Time-evolved Goals (QSPs): Sulu.
- Hybrid Actions (PDDL+): Kongming.
- Uncertainty and Risk Constraints: pSulu.