Topic 4: Playing a Guitar

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Introduction and Background

The topic of wave motion appears in almost every field of physics and can be discussed at various levels. In this course alone we have covered simple waves on a string, ripples in a pond (a 2D wave), and electromagnetic waves. In this topic we revisit the wave on a string but now try to make it more realistic.

In most areas of physics, especially mechanics, we tend to take a more simplified approach to problems by assuming ideal conditions. For example, in the case of waves on a string, we often assume that the string is perfectly flexible. That is, the only force on a particular segment of this idealized string would be due to the tension. So, while strings can be designed and manufactured to approach this ideal, it can never actually be achieved. There will always be a stiffness force that directly opposes bending in the string. Also, we often assume that the frictional force on the string is negligible because of how slim the string is. However, to imitate reality as closely as possible we also incorporate this friction parameter in our simulation.

First and foremost, we must describe how a typical guitar works. When talking about waves on a string we often times assume something uniform like a sinusoidal or Gaussian wave form about the center of the string. However, for a guitar we typically pluck the strings at about $1/4^{th}$ the length of the string with a displacement of about 0.5 cm from its resting position, creating two clean edges. In essence a guitar is six strings each of which is fixed at both ends such that each can be adjusted to change the tension of the string. The tension of each string is typically adjusted until the frequency it produces when strummed matches the corresponding standard tuning frequency (Table 1)[4]. We will simplify our guitar by instead tuning each string by varying its respective stiffness parameter.

Table 1: Standard Tuning Frequency			
String	Note	Frequency	Scientific Pitch Notation
1	E	329.63 Hz	E4
2	B	246.94 Hz	B3
3	G	196.00 Hz	G3
4	D	146.83 Hz	D3
5	A	110.00 Hz	A2
6	E	82.41 Hz	E2

Table 1: This is a table of the standard tuning frequency for each string represented by both its string number and note letter.

We have discussed in previous classes how the frequency of a standing wave on a string even under ideal conditions is affected by changing the length of the string. So, it should come as no surprise to find that changing a string's length on the guitar also affects the frequency of said string. This is done often when playing the guitar by pressing down on the space between frets. A fret is a raised element on the neck of the guitar that divides it into fixed segments and helps the player locate common notes on the guitar. For standard playing the musician should place their finger between the frets. So as to be concise, we will instead identify the fret spacings as the fret number which is below it. For example, the "2^{nd} fret spacing" will be called the "2^{nd} fret."

By placing your fingers on different frets, consequently changing the lengths of the respective guitar strings, we change the frequency of the string. By combining several of these strings and strumming them simultaneously, we can create a chord. For the sake of simplicity, we will approximate that all of the frets are equally spaced since we will only be pressing down the first few frets on the guitar anyways. Also, when we press down on a fret, technically we have changed the note, but for consistency we will continue to address that string by its open name.

The goal of this topic is to ultimately simulate a chord on a guitar: six strings played together, all at different frequencies and how the character of the chord changes with time. While trying to explore the result we will inevitably come across additional characteristics of the wave equation that we will further examine. Specifically, we will explore the process of tuning each string on the guitar using the stiffness parameter, how the friction parameter affects the motion of the string and how it affects the frequency of the graph, what happens to the energy of the string over time, and lastly what the frequency distribution of our chord looks like.

Based on what we've seen in previous courses, we can make some preliminary guesses as to what we should expect. As mentioned before, we are not working with a symmetric wave form we are used to seeing so we should see something different from the typical standing wave. Furthermore, we should actually see a superposition of waves in our simulation due to the multiple forces acting on the string acting at once. When examining our frequencies, we should not expect a perfect peak that describes the frequency of the string on our Fourier transform plots. It will more than likely look like a large distribution of peaks with some being taller than others. As mentioned in our textbook**,** "for an ideal string the frequencies of the standing waves are space by $c/2L$…However, for the stiff string this separation increases as we move to higher frequencies" (Giordano, Nakanishi 171).[2] While tuning the instrument, we can expect that strings with higher frequencies will have larger stiffness values as "a stiff string resists bending and this extra resistance makes the string effectively tighter, leading to a higher wave velocity and a higher frequency" (Giordano, Nakanishi 172).[2] Under ideal conditions, the mechanical energy of the wave should always be constant; however, due to our decay term, we can no longer expect this because it would imply our realistic string would oscillate forever. As for the chord, we can expect to see the strings decaying at different times with their frequencies occasionally lining up.

Methodology

Before we get on the topic of solving for our motion, we must first figure out what our initial position (pulse profile) looks like. To do this, we approximate the guitar strings will always be strummed at about 1/4th of the length from the base of the guitar (which we will identify as the origin on the graphs). This will create two straight edges and will form a triangular shape with the height of the triangle being the distance displaced by the pluck.

When dealing with an ideal wave on a string, the typical wave equation given

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

does not take nonconservative factors into account and is often used with ideal conditions in mind. In our search for realistic results we use an equation that takes both the stiffness of the string and the frictional force acting against the string into account

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left(\frac{\partial^2 y}{\partial x^2} - \epsilon L^2 \frac{\partial^4 y}{\partial x^4} \right) - 2b \frac{\partial y}{\partial t}$$

We can use the finite difference form of our wave equation to numerically approximate the solution for the wave at different points in time. We do this by first writing all of its components in finite difference form

$$\frac{\partial^2 y}{\partial t^2} = \frac{y(n+1,i) + y(n-1,i) - 2y(n,i)}{(\Delta t)^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{y(n,i+1) + y(n,i-1) - 2y(n,i)}{(\Delta x)^2}$$

$$\frac{\partial^4 y}{\partial x^4} = \frac{y(n,i+2) - 4y(n,i+1) + 6y(n,i) - 4y(n,i-1) + y(n,i-2)}{(\Delta x)^4}$$

$$\frac{\partial y}{\partial t} = \frac{y(n+1,i) - y(n,i)}{\Delta t}$$

and then plugging them into our actual equation. By solving for the next time step of the evolution of our wave, we can use for loops to compute the wave profile at different times. After a bit of algebra, we get

$$\begin{aligned} y[x,t+1] = \big(&r^2(1+4\epsilon M^2)(y[x+1,t] + y[x-1,t]) \\ &+ y[x,t](2 + 2b\Delta t - 2r^2 - 6\epsilon M^2 r^2)\big) \\ &- 2\epsilon r^2 M^2 (y[x+2,t] + y[x-2,t]) - y[x,t-1]) \ / \ (1 + 2b\Delta t) \end{aligned}$$

where $M = \frac{length}{\Delta x}$ and $r = \frac{c\Delta t}{\Delta x}$. For the remainder of the research paper, we treat our $\Delta t = 0.001$ as our timestep and $\Delta x = 0.01$ as our position step. These values were chosen on the basis of creating efficient code while not sacrificing the accuracy of our results. By taking our phase velocity $c = 1$ to be given, we have only the parameters of stiffness and friction to work with.

As mentioned before, we are using a simplified version of a guitar and as a result will tune the instrument by varying our stiffness parameter ϵ and letting $b = 0$ so as to tune each string in ideal conditions. We will iterate through a range of epsilon values for each string using a for loop that looks at the frequency on the resulting Fourier transform graph, and will return the current epsilon if the frequency is within 0.1 Hz of the accepted tuning frequency.

We will also explore how the friction coefficient b affects the motion. Since it may be hard to see just from the graph of the motion how the b coefficient affects it, we choose to examine the frequency for more accurate and consistent results. It has a chance at affecting the rate of decay of the wave, the sound that we hear, or both, so it is important we test both. To check the former, we will plot the Fourier transforms for two different b values in an attempt to check for any discrepancies between the trends. For the latter, we will also be taking a frequency approach and will plot the difference in frequency vs different b values for each string.

The conservation of energy is always a point of interest when discussing wave motion and it is no different here. We will calculate the energy at every point by using a for loop to iterate through every timestep and produce the sum of the kinetic and potential energies. We will see how energy evolves with time by creating an energy vs time graph with all six strings.

The last thing we will do is create a chord and examine its frequency distribution. Our chord will be created by assigning certain frets to each string and strumming all of the strings simultaneously. We will examine the frequency distribution by plotting the frequency of every string for a given b value and inspecting how certain peaks line up with each other and what the overall trend of the graph looks like.

Analysis

To create something to base our following results on, we graphed the general motion of a wave with the "plucked" wave form with both $\epsilon = 0$ and $b = 0$; that is, the general wave motion in ideal conditions. As can be seen in Figure 1, the initial shape of the wave seemingly reflects across the x-axis and eventually returns to the initial position.

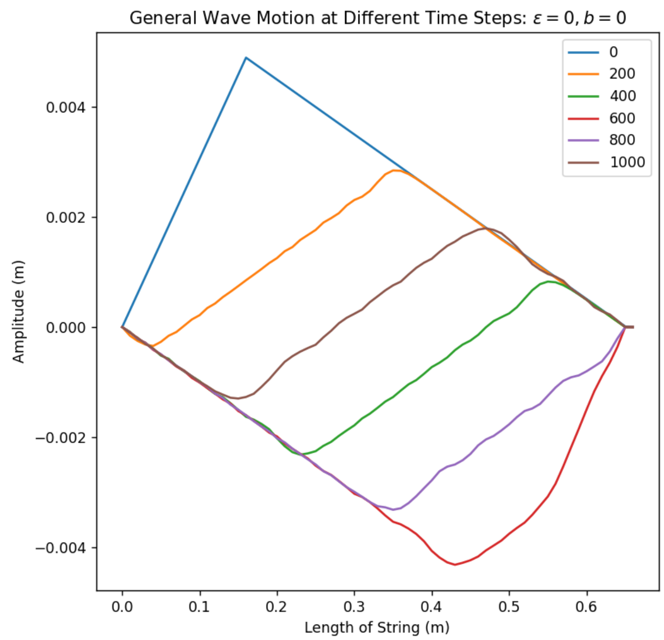
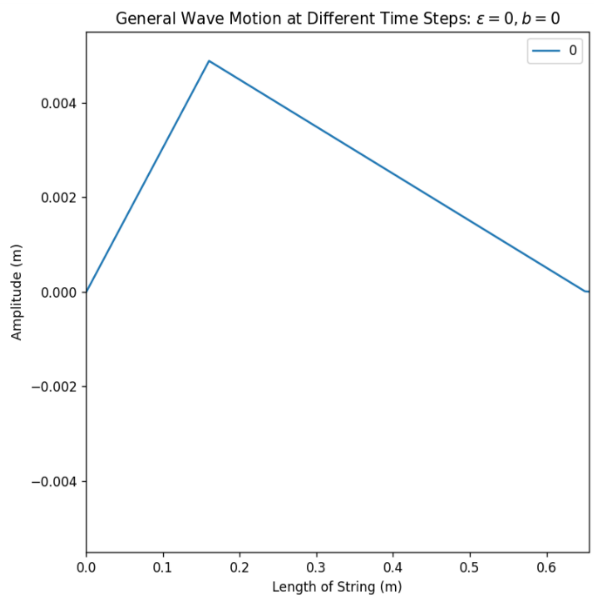


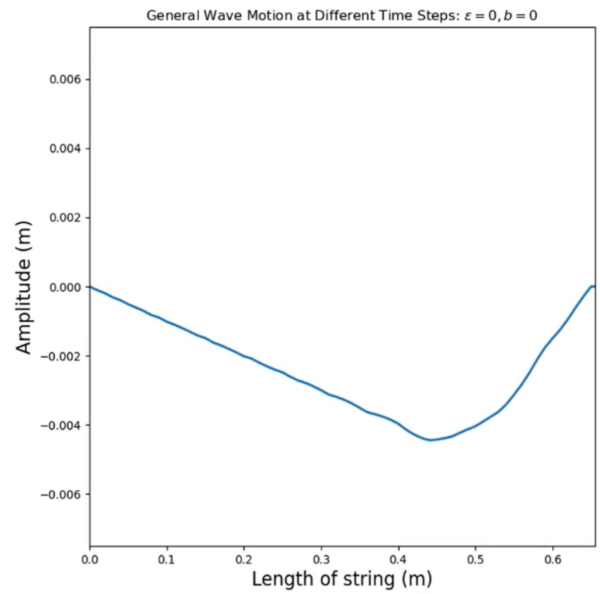
Figure 1. The shape of an ideal wave with a plucked initial wave form at different time steps (indicated by the legend).

Although it is a bit difficult to discern, if we look closely at Figure 2, we can see that there are small standing waves on the large moving wave generated by the pluck. Thus, as expected, there is a superposition of waves. Also notice from these graphs that indeed when our friction and stiffness parameters are not included, the wave returns to its initial position eventually.

By using python's fast Fourier transform and for loops, we successfully cycled through a series of epsilons to find the one that produced a frequency closest to the standard tuning frequency for each string (Table 2).

Table 2: Epsilons and Produced Frequencies				
String	Letter	ϵ	Standard Frequency (Hz)	Frequency Produced (Hz)
1	E	0.004271	329.63	329.53
2	B	0.002812	246.94	246.95
3	G	0.001893	196	195.96
4	D	0.001100	146.83	146.77
5	A	0.000612	110	109.98
6	E	0.000325	82.41	82.38





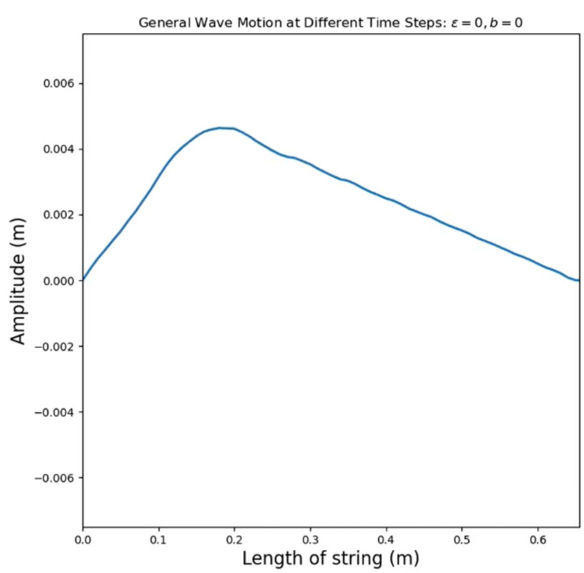


Figure 2: Top left: Initial position of the wave.
Top right: 0.5 seconds into the simulation the small standing waves are present.
Bottom left: 0.75 seconds into the simulation the string returns to its original position with standing waves still there

The fast Fourier transform plots have large distributions of frequencies with the fundamental frequency of the string being at the far right of the distribution. To see if the epsilon produced the correct frequency, we used numpy's "where" function to locate the maximum around the end of the distribution; it is also easier to see exactly the distribution at the end by zooming in on it as we did for each string in Figure 3. As previously mentioned, we wanted to explore if and how the friction parameter has an effect on the large wave produced by the pluck, the standing waves that are visible on the large wave, or both.

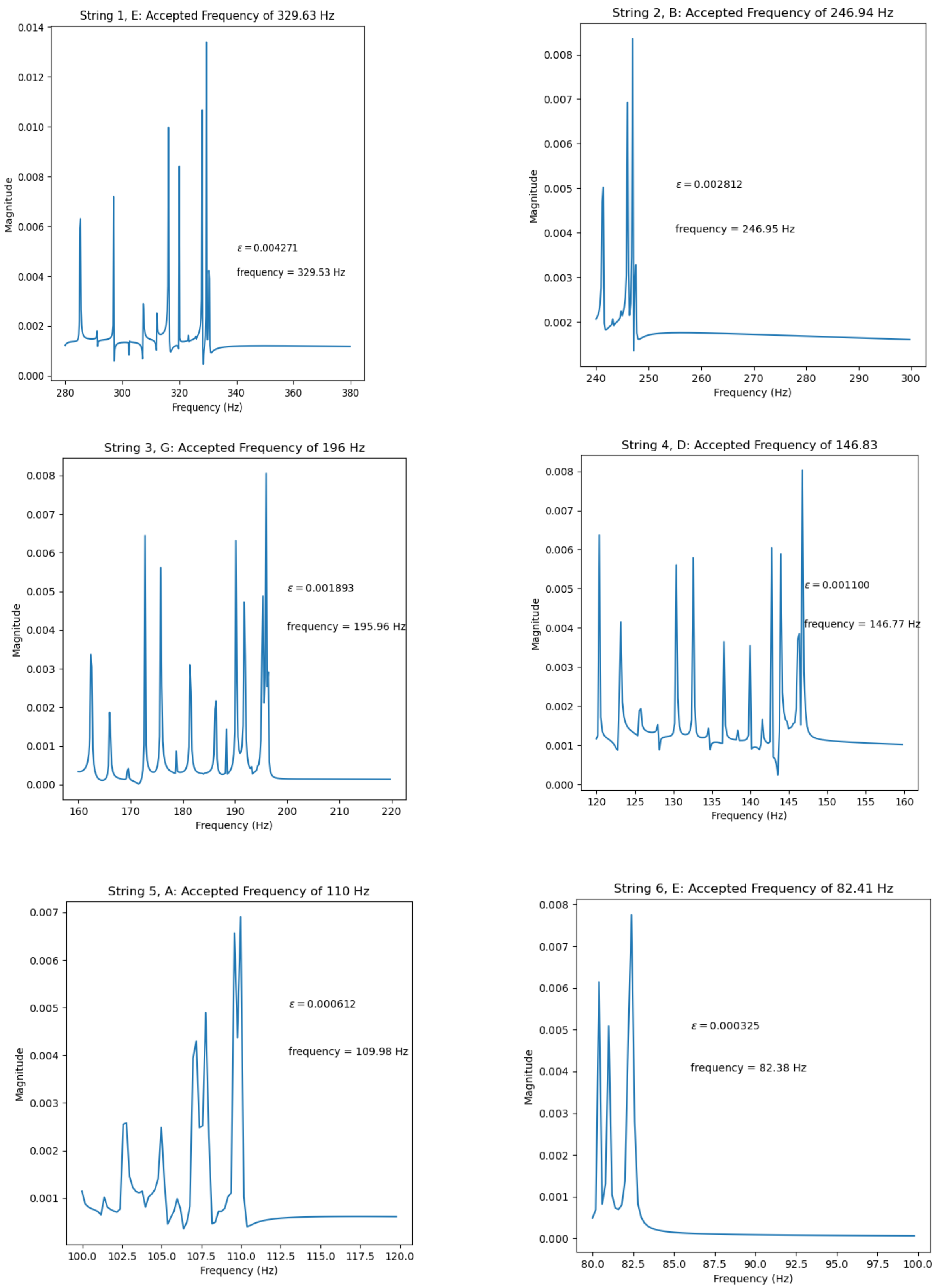
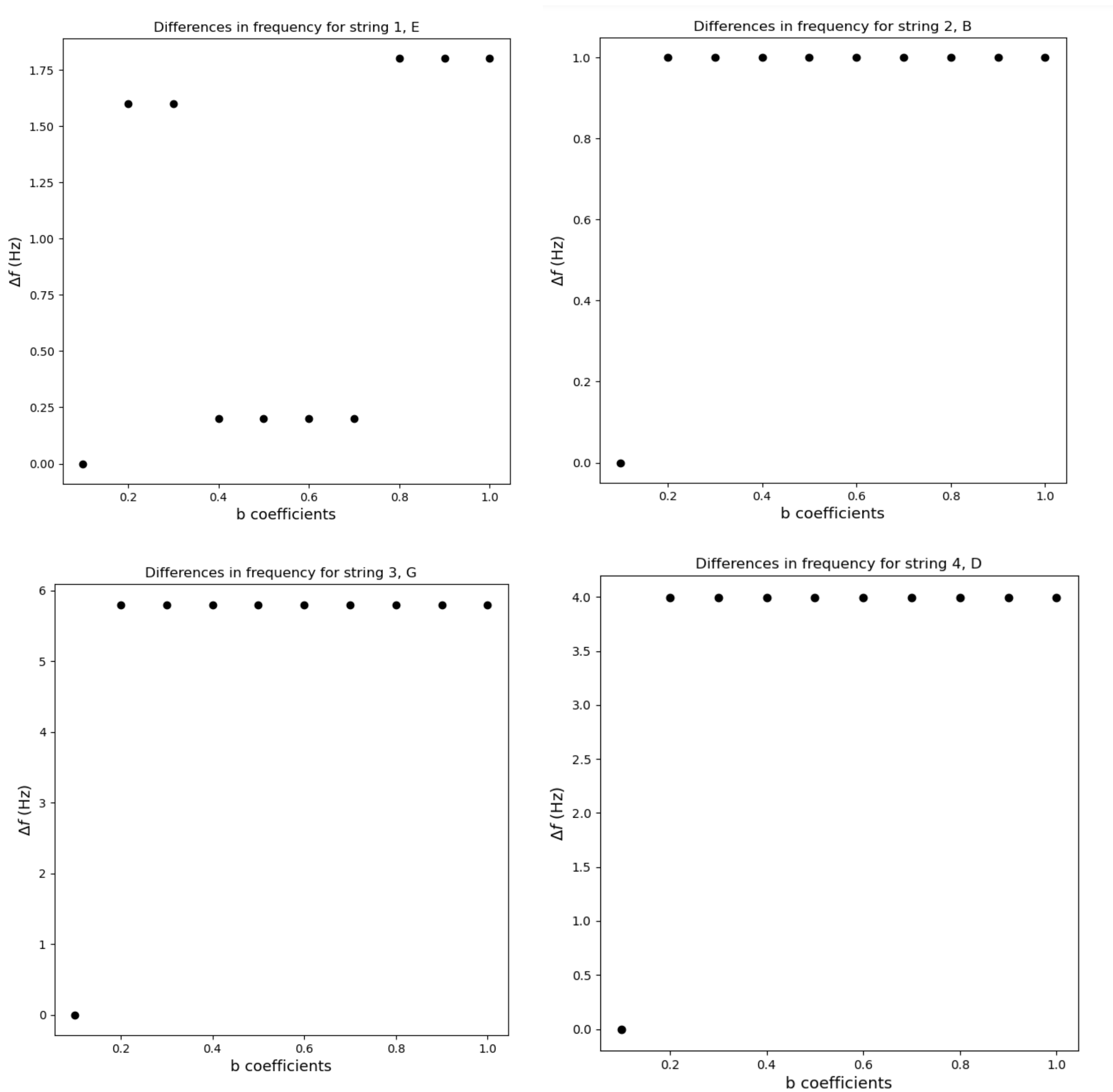


Figure 3. The edges of the Fourier transform graph for each string with its epsilons and corresponding frequencies on the graph. See Table 2 for an organized display.

To see the effects, we took two different approaches: one involved plotting the differences in frequency between a wave with $b = 0$ and a nonzero b value ranging from 0 to 1 versus those different frictional coefficient values, and the other involved plotting the Fourier transforms for two different frictional coefficient values for each string.

As seen in Figure 4, there is a slight difference in the frequencies with the introduction of a friction parameter that is nonzero for all strings. In the case of the less stiff strings (6E, 5A), it seems that regardless of what value is used for the b parameter the difference in frequency does not vary. However, the same cannot be said about the stiffer strings (4D, 3G, 2B, 1E) as different b values have—especially for string 1E—varying effects on the difference in frequency. Although there is a small difference caused by the introduction of the friction coefficient, it is not enough to argue that it has a major effect on what we hear when the string is strummed.



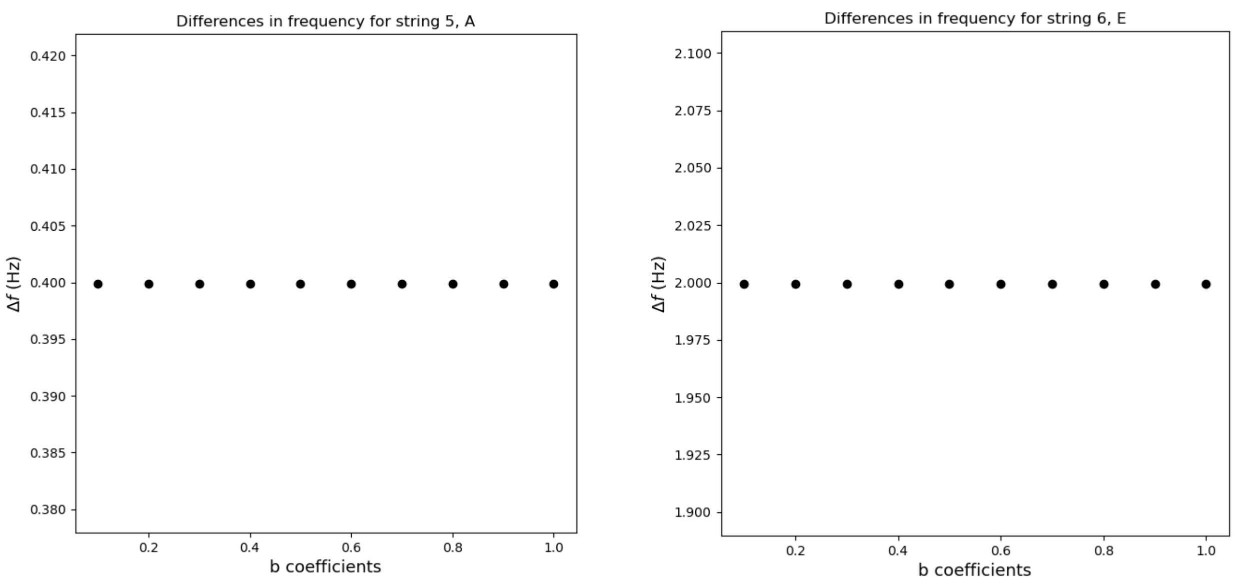
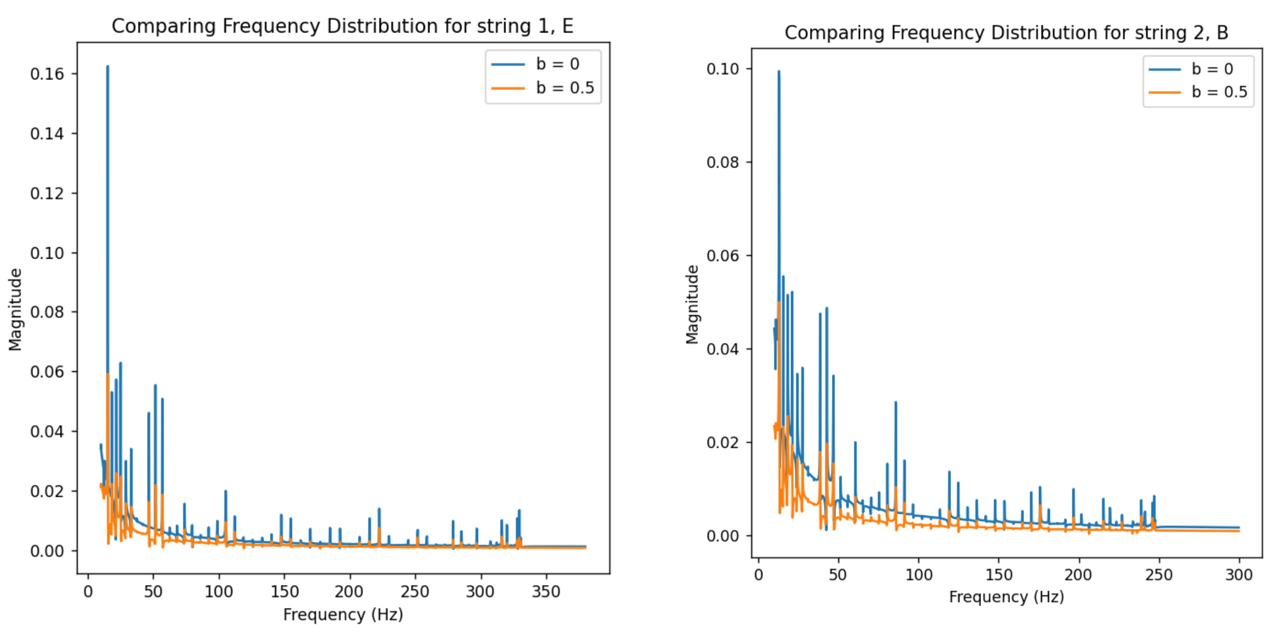


Figure 4: The difference in frequency between the frequency of the graph with $b = 0$ and $b = 0.1, 0.2, \ldots, 1$ for every string. In general, the b value had a minimal effect on the frequency.

Now although the friction parameter has little impact on the fundamental frequency of the string, it does seem to have a vast impact on the overall decay of the magnitude in the frequency distributions. By looking at the Fourier transform graphs in Figure 5 for the 6 different strings at both $b = 0$ and $b = 0.5$, we see just how differently the magnitudes of the frequencies decay. Notice, however, that the locations of the frequency peaks are identical; this further strengthens the argument that the b value has a negligible impact on the fundamental frequencies of our strings. It also introduces a new argument: the friction coefficient has a major impact on the decay of the magnitudes. That is, it affects how the big wave decays over time and returns to its original resting position.



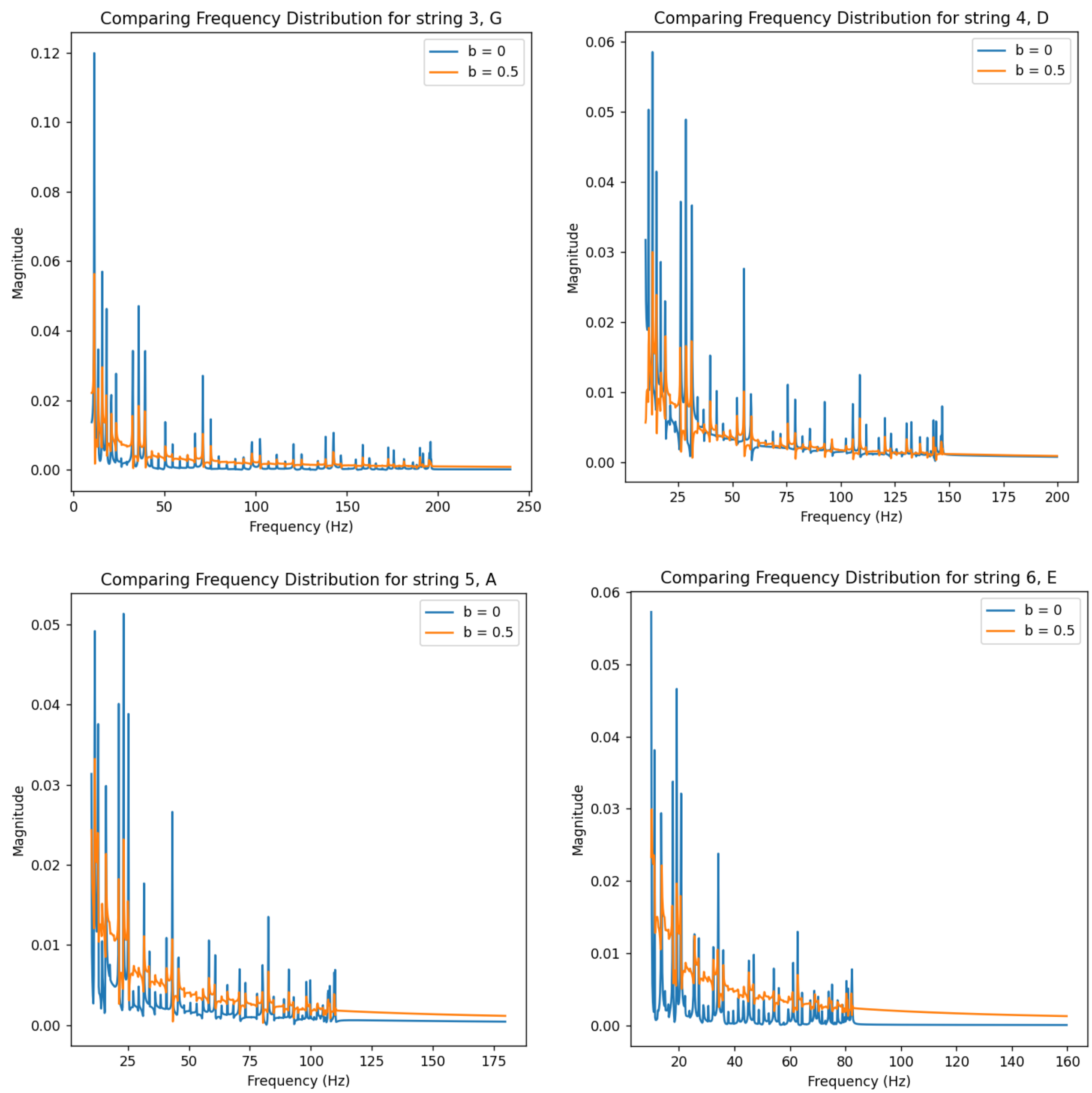


Figure 5: The Fourier transform graphs for $b = 0$ and $b = 0.5$ for all 6 strings. By looking closely, we can see that the peaks of both distributions line up almost perfectly for each string but have very different magnitudes, indicating a change in decay of motion.

Seeing as this was a more realistic case, it was satisfying to find that the mechanical energy of the wave is not constant and does decrease with time. This makes sense considering that we have nonconservative forces such as the frictional component acting against the string as it moves. As our string moves it is losing its energy as it is converted to heat by the medium's molecules rubbing against it,[1] causing it to decay over time as seen in Figure 6.

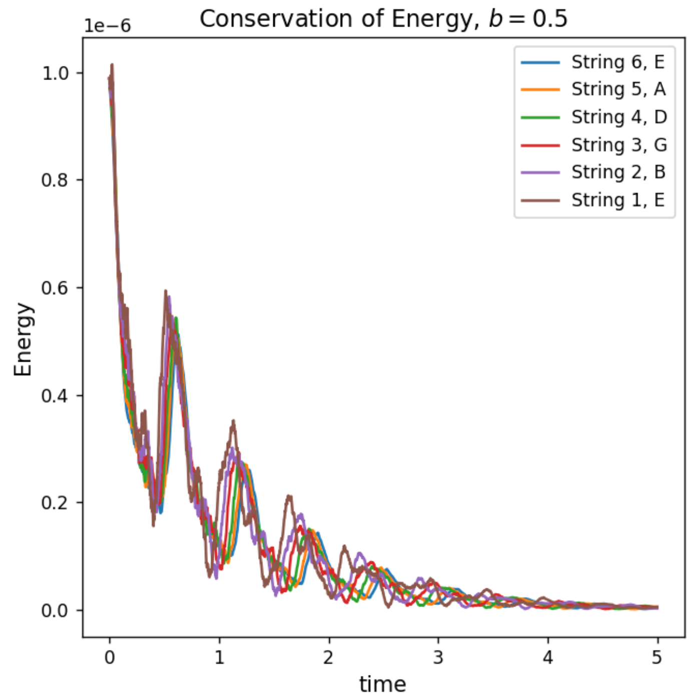


Figure 6: The mechanical energy of each string as time passes. Every trough indicates the bottom of its motion and every peak is the top of its motion. For example, at 0.5 seconds the string reaches the bottom of its motion and the energy jumps as it is slingshot to the top of its motion at 0.75 seconds and then starts decaying again.

We then created the G major chord by pressing down on the correct combination of notes and frets given in the first row of Table 3 and plotted the motion of all six strings (Figure 7) and saw that similar to the single string they all decayed due to the friction and stiffness parameters as seen in Figure 8. They all started at the same displacement of 0.5 cm and were released simultaneously. Interestingly enough, there are instances in the animation where some strings looked entirely like standing waves. As can be seen in the second image of Figure 8, we can almost see what looks like standing waves in some of the strings—most prominently in strings 2, 3, and 4. This is captivating because until now we have only seen and recognized the small standing waves that were riding the large wave. This shows that even in realistic simulations there are some occurrences of idealistic behavior. For the sake of curiosity, we created another chord given in the second row of Table 3 and plotted its motion in Figure 9.

Table 3: Chord Arrangements

	String 1	String 2	String 3	String 4	String 5	String 6
Chord 1	Fret 3	Open	Open	Open	Fret 2	Fret 3
Chord 2	Fret 3	Fret 3	Fret 4	Fret 5	Fret 5	Fret 3

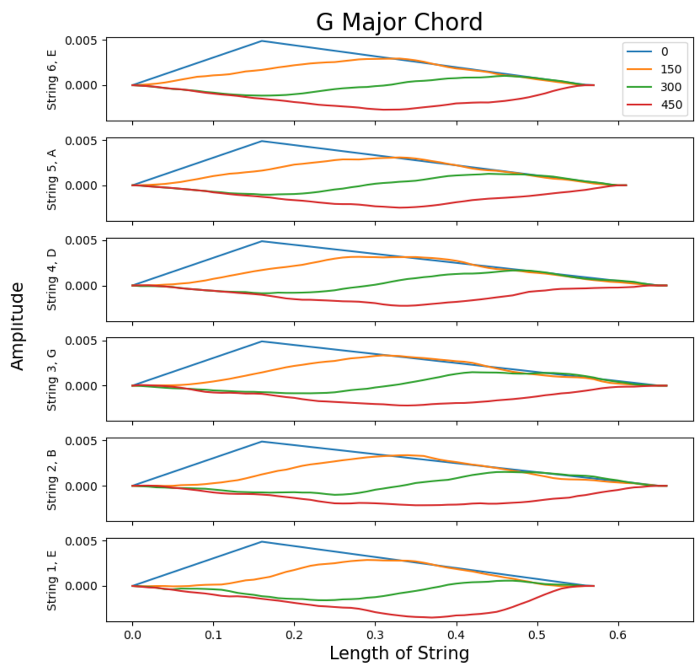
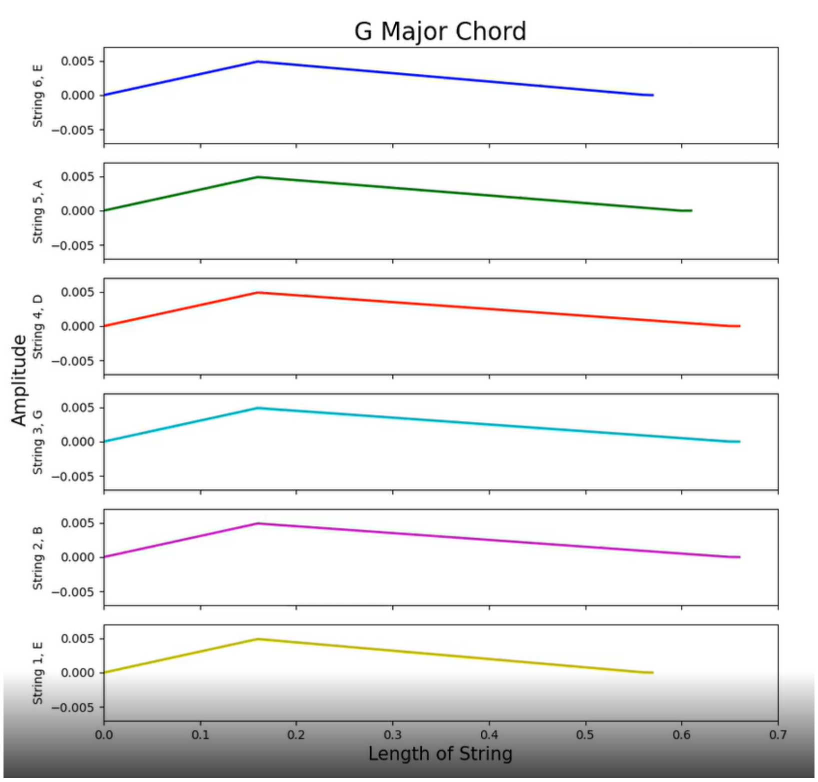


Figure 7: This the motion of the six strings of the first G chord at the 0, 150, 300, and 450 timesteps.

By looking carefully at the graph, we can also see that, as expected, all the strings are at different locations at different timesteps. This is extremely evident at the 450th timestep as string 1 is the shortest and has already reached its minimum point, while the longer strings are lagging behind. Also notice that although the 1st and 6th string are being played with the same fret, string 1 is further in its journey than string 6 is. This is most likely a direct result of the strings' corresponding stiffness parameters. This is also notable in strings 2, 3, and 4.



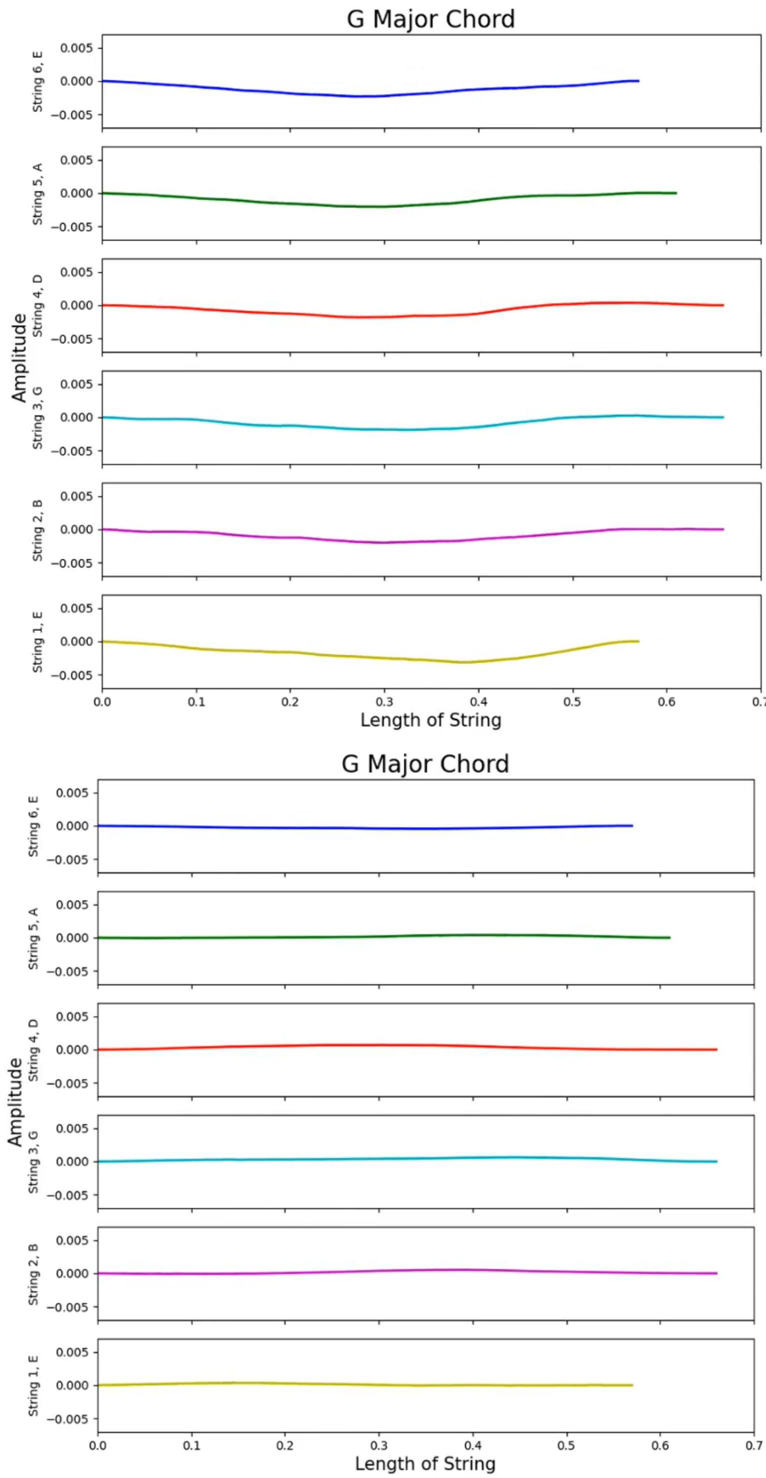


Figure 8: First: This is the initial wave profile for all six strings of the first chord, all displaced 0.5 cm from equilibrium. Second: A few timesteps into the simulation. Third: Towards the end of the simulation

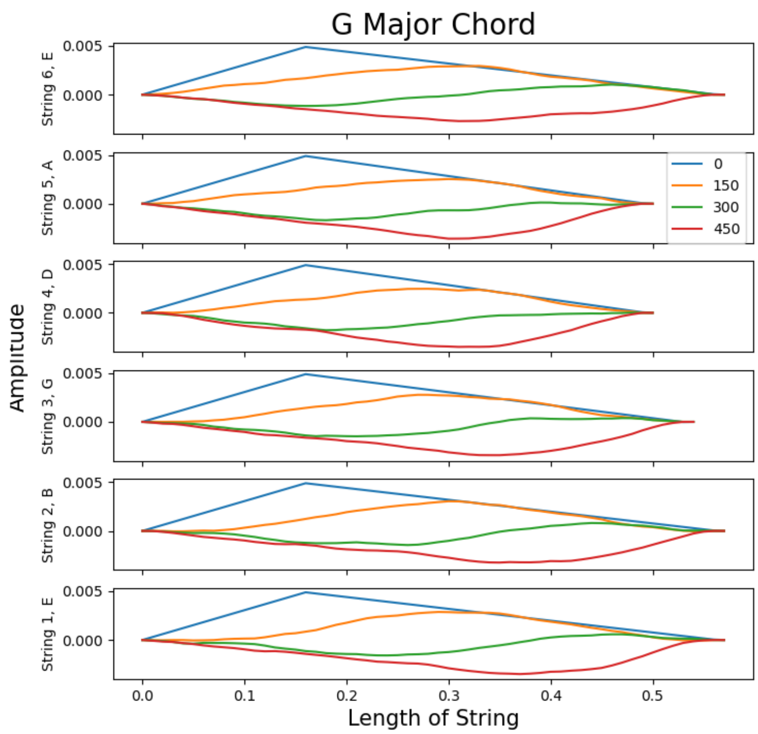
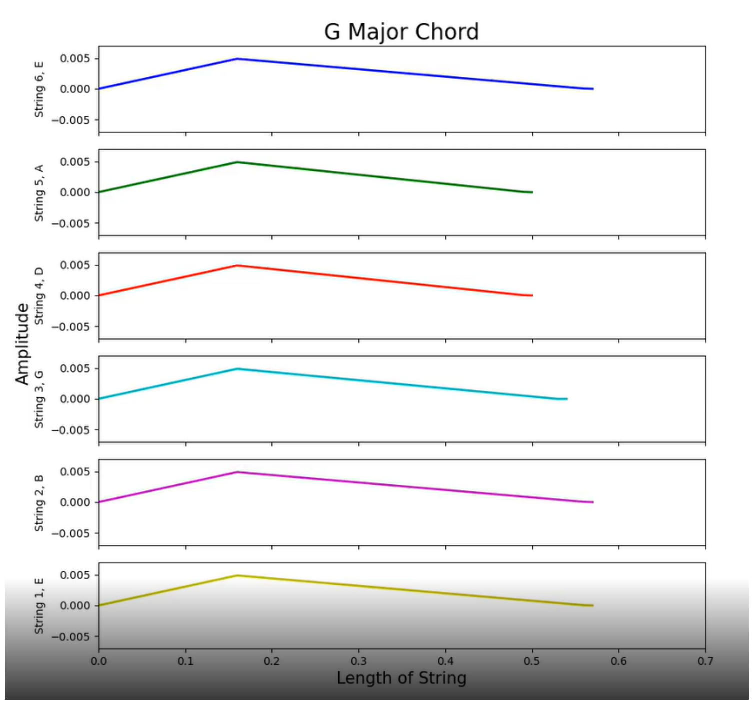
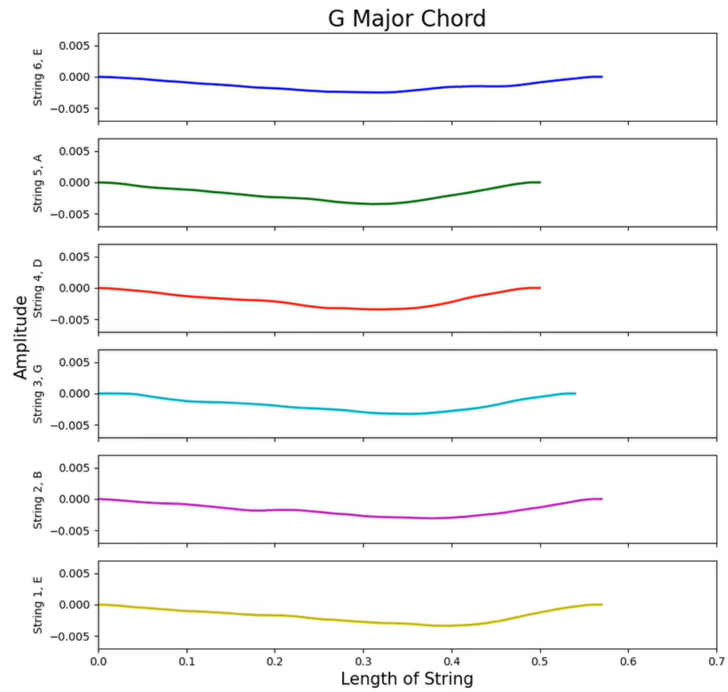


Figure 9: This the motion of the six strings of the second G chord at the 0, 150, 300, and 450 timesteps.

This second chord was plotted for the purpose of testing if what we said about the first chord was correct. Yet again, we see that although some strings have the same length, their shapes differ. This is again almost obvious for string 1 and 6 which once again reinforces our idea that the epsilon values have a major effect on the time evolution.





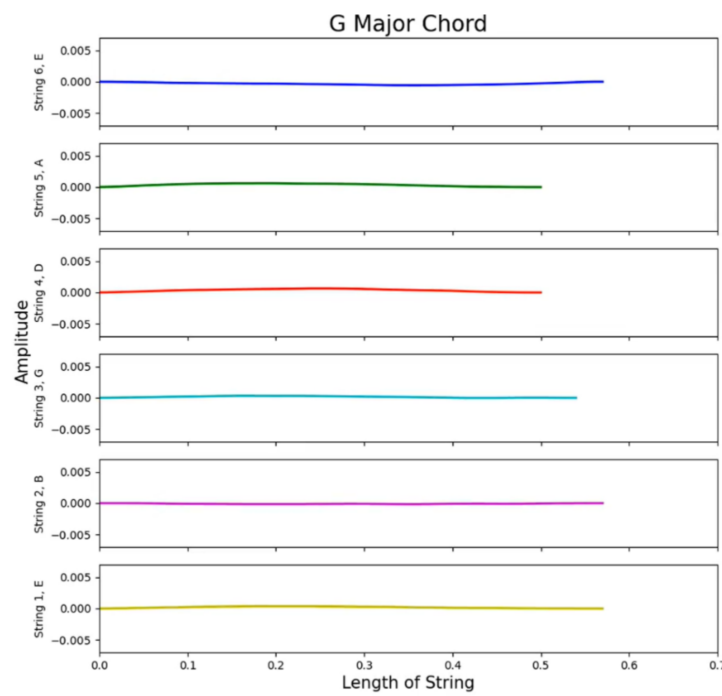
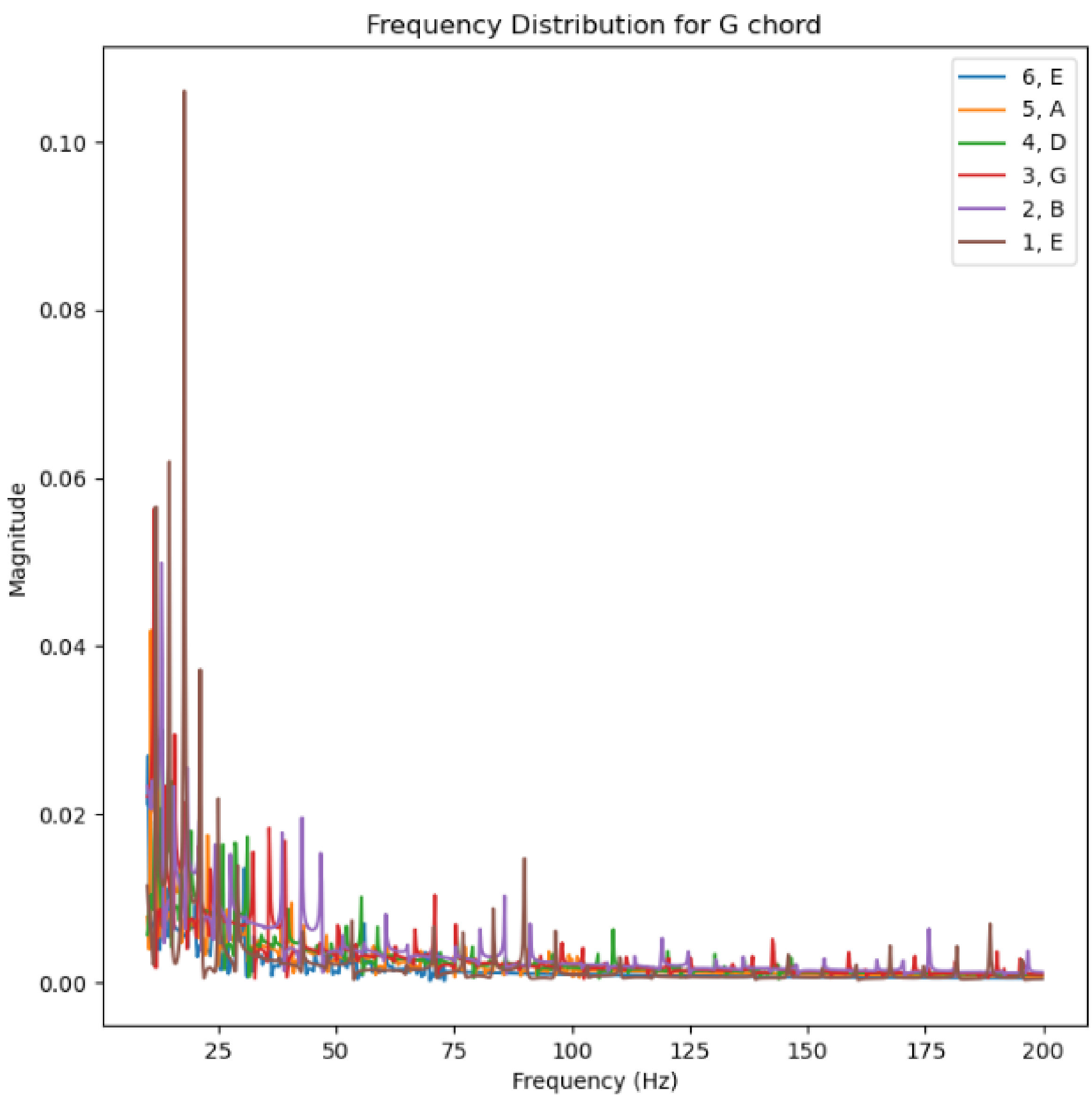


Figure 10: First: This is the initial wave profile for all six strings of the second chord, all displaced 0.5 cm from equilibrium. Second: A few timesteps into the simulation. Third: Towards the end of the simulation

The final item we sought to explore was the frequency distribution of the G chord—in this case, we will only explore the first chord we formed. The first image in Figure 11 is quite cluttered but does a decent job at showing how some strings stop oscillating before others. The second image is a better representation of how certain peaks and troughs of some strings align with those of others. In the ideal case, we would expect to see that these harmonics (peaks) would be harmonically spaced so that one harmonic of a string would be very close in frequency to another harmonic in one of the other strings. But as can be seen in image 2 of Figure 11, although there are instances where some harmonics do line up, for the most part they do not have the same frequencies which affects the way the chord sounds.



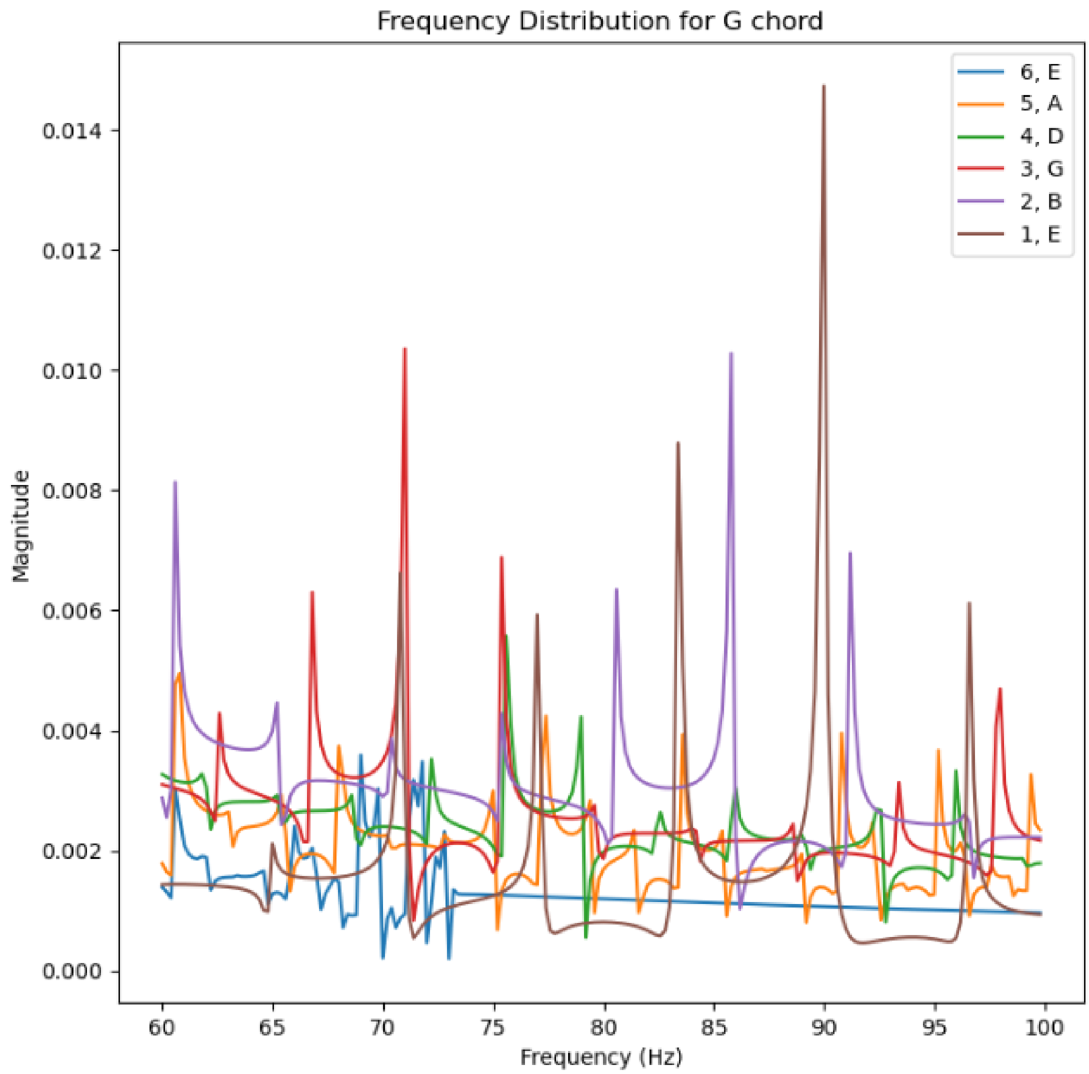


Figure 11: First: A general overview of the Fourier transform graph with all six strings plotted. It is a little difficult to understand exactly what is happening, but it does demonstrate the decay in magnitude of different notes. Second: This more zoomed in version of the first graph shows how certain harmonics

Conclusion

We started by successfully simulating a single string vibrating at a given frequency by tuning the epsilon value of the string until we reached the desired frequency and then extended this method to all six strings to create the G major chord. We looked at how the length, stiffness and frictional parameters affected the frequency and general motion of the strings. For almost every item we were interested in, our approach almost always involved the inspection of the frequency. This was both for efficiency and accuracy since even if we changed a parameter – say the frictional coefficient – and looked at the resulting wave form, there was no guarantee of being able to discern anything from it. However, by looking at the Fourier transforms and finding the fundamental frequencies we were able to closely examine and compare our findings.

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