Chapter 3

Alan R. Vazquez

Excercises

```
import numpy as np
```

Excercises

Problem 1. A function to determine main-effect and interaction contrast vectors

The function `oaf.contrast` takes an array as input and returns two sets of contrast vectors as outputs. The first set includes the main-effect contrast vectors and the second set includes the two-factor interaction contrast vectors. For example,

```
# X1,X2 = oaf.contrast(Array)
```

returns the main-effect contrast vectors of the orthogonal array `Array` in `X1` and the two-factor interaction contrast vectors in `X2`.

The main-effect contrast vectors are Helmert contrast vectors. A factor with s levels results in a set of $s-1$ Helmert contrast vectors. The kth contrast is between level $k+1$ and the levels $l < k$. The `oaf.contrast` function results in normalized main-effect contrast vectors, so that the length of each equals $\sqrt{N}$. The two-factor interaction contrast vectors are calculated by element-wise multiplication of the main-effect contrast vectors. In orthogonal arrays, their length also equals $\sqrt{N}$; see Result 3.3.

- The file `OA18mixed.oa` includes five different $\text{OA}(18;, 3^3 \times 2; 2)$. Take one of these OAs. Check that the main-effect contrast vectors are orthogonal to each other. Also check that the main-effect contrast vectors are not orthogonal to the interaction contrast vectors.
- The file `OA48.oa` includes 19 different $\text{OA}(48; 4 \times 3 \times 2^4; 3)$. Take one of these OAs. Check that the main-effect contrast vectors are orthogonal to each other. Also check that the main-effect contrast vectors are orthogonal to the interaction contrast vectors.

Problem 2. Correlation between contrast vectors

Consider the random variables X and Y. The correlation between these variables, designated ρ_{XY} is defined as

$$\rho_{XY} = \frac{E(X - EX)(Y - EY)}{\sqrt{E(X - EX)^2 E(Y - EY)^2}},$$

where $E(.)$ is the expectation operator.

- Consider the random variables P and Q that take their values from contrast vectors $\mathbf{p}$ and $\mathbf{q}$, each normalized to a length of $\sqrt{N}$, and consider the (P, Q) pairs as bivariate random variables. Show that $\rho_{PQ} = \mathbf{p}^T\mathbf{q}/N$. This correlation is referred to as correlation between contrast vectors $\mathbf{p}$ and $\mathbf{q}$, despite the fact that it is rather the correlation between random variables taking the values of $\{$*elements*$\}$ *of these vectors with equal probability.*
- *What is the value of* ρ_{PQ} *if the vectors are orthogonal? What if they are completely identical? What if* $\mathbf{p} = -\mathbf{q}$*?*
- *Suppose that* $\mathbf{p}$ *is a main-effect contrast vector and* $\mathbf{q}$ *is an interaction contrast vector. Discuss the impact of increasing the absolute correlation between these vectors from 0 to 1 on the interpretation of results from a design with these vectors.*
- *Study the maximum absolute correlations between main-effect contrast vectors and interaction contrast vectors in the five 18-run designs of* `OA18mixed.oa`*. Which design has the smallest maximum correlation?*
- *Study the maximum absolute correlations among interaction contrast vectors in the 19 48-run designs of* `OA48.oa`*. Which design has the smallest maximum correlation?*

If you use a different set of main-effect contrast vectors, the correlations between these vectors and the interaction contrast vectors calculated from them, and those among the interaction contrast vectors, are changed.

- In the five 18-run designs of `OA18mixed.oa`, switch the labels `1' and`2' for the three-level factors. Show that the Helmert contrast vectors constructed from the relabeled factors are equivalent to the linear and quadratic contrast vectors for the original factors. Check the correlations between main-effect and interaction contrast vectors for the original as well as for the relabeled factors.

Problem 3. Sums of squared correlations

This exercise continues with the 18-run designs of `OA18mixed.oa` and the 48-run designs of `OA48.oa`.

- Calculate the sums of squared correlations between main-effect contrast vectors and interaction contrast vectors, SSC_3, for the five 18-run designs. Which design is best?

- Calculate the sums of squared correlations among interaction contrast vectors, SSC_4, for the 19 48-run designs. If you use the $p_2 \times p_2$ matrix of correlations, only use one off the off-diagonal pieces of the matrix. Which design is best?

As explained in Section 3.2.1, $SSC_3/3$ is called the word count of length 3 and indicated with A_3. The divisor of 3 is because each product of three main-effect contrast vectors occurs three times in the matrix $X_1^T X_2$. As explained in Section 3.2.2, the quantity $SSC_4/3$ is called the word count of length 4 and indicated with A_4. The divisor of 3 is because each set of four main-effect contrast vectors can be partitioned in three different ways into two pairs. The corresponding three pairs of interaction contrast vectors have the same correlation. A_3 and A_4 do not depend on the set of normalized orthogonal contrasts used (Xu and Wu 2001). So they can be used as an overall measure expressing the correlations between main-effect contrast vectors and interaction contrast vectors or among interaction contrast vectors.

Xu, Hongquan, and C. F. J. Wu. 2001. "Generalized minimum aberration for asymmetrical fractional factorial designs." *The Annals of Statistics* 29: 1066–77. https://doi.org/10.1214/aos/1013699993.