

PERCOLATION AND ITS VARIATIONS

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An exploration of the current state of percolation models

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May 2020

ABSTRACT

The current state of percolation theory is hard to assess. Research often uses different methods, language and notation, is hard to reproduce, and sometimes even provides conflicting results. We aim to give an overview of the current state of some of the most common and simple percolation models, alongside with open source code that allow the reader to easily reproduce the results presented, as well as present an analysis of a novel model called Stateful Mandelbrot Percolation. We hope to facilitate further research, as well as provide a starting point for anyone interested in the topic.

*We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.*

— **knuth:1974** [knuth:1974]

ACKNOWLEDGEMENTS

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ACRONYMS

DRY Don't Repeat Yourself

Part I

CURRENT STATE

CLASSIC PERCOLATION

1.1 BASIC DEFINITIONS

We begin with a graph G representing a slattice of size L in d dimensions, such that $|G| = L^d$. Each node in the graph is independently set to be "alive"/"white" with probability p and "dead"/"black" with probability $1 - p$. p is often called the occupation probability. Our goal is to study the properties of the lattice after this "coloring" has taken place. This is what such a lattice looks like: (IMAGE)

Definition 1 *A cluster is a set of white connected nodes in the graph.*

Definition 2 *A lattice is said to have percolated if there exists a macroscopic cluster, i.e. a cluster which spans the whole lattice.*

For the case of $d < \infty$, one can (arbitrarily) pick a dimension i from $1, 2, \dots, d$ and use it as the defining dimension for percolation, i.e. a lattice has percolated if there exists a cluster that intersects both boundaries of the lattice. For $d = 2$, for example, we can use the convention that a lattice has percolated if there is a cluster that connects the top boundary and the bottom boundary (left-right would be equally good).

The process is, of course, random. So we define an indicator random variable $H_{p,L}$ which represents whether a particular lattice has percolated:

Definition 3

$$H_{p,L} = \begin{cases} 1 & \text{if lattice has percolated} \\ 0 & \text{otherwise} \end{cases}$$

As we shall see, percolation models are the simplest models that exhibit a phase transition, meaning that there exists a particular occupation probability p_c at which the behavior of the system is expected to change dramatically. This change will be reflected in a number of quantities of interest which will study in the next sections. It turns out that p_c is the probability of seeing a percolating cluster. As we vary p , larger and larger clusters are expected to form. However, in the case that $L < \infty$ and $p < 1$, there is always a possibility that no percolating cluster will be observed (i.e even if $p = 0.999$ there is a non-zero probability all nodes are dead). For this reason, p_c is not well defined for finite L , and we'll define it in the limit $L \rightarrow \infty$.

Definition 4 $\Pi_{p,L} = \mathbb{E}[H_{p,L}]$ is the probability that a particular lattice will percolate.

As we increase L , the behaviour of $\Pi_{p,L}$ as a function of p approach a step function at $p = p_c$. Which means that in the limit of very big L , the lattice percolates with probability 1 for $p > p_c$, and does not percolate with probability 1 for $p < p_c$. (PLOT)

Definition 5 The percolation threshold p_c is the smallest occupation probability p such that a percolating cluster exists with probability 1.

Definition 6 $P_{p,L}$ is the fraction of nodes belonging to a percolating cluster.

Next, we'll look at some quantities of interest and study their behavior as a function of p , in particular for small values of $|p - p_c|$, i.e. for p close to p_c . The reason for studying this particular regime will become apparent later.

Definition 7 $n_s(p)$ is the number of s -sized clusters (clusters with s nodes) per lattice size.

Definition 8 $\chi(p)$ is the average cluster size (excluding the percolating cluster).

Definition 9 $s_\infty(p)$ is the average size of the largest cluster (excluding the percolating cluster).

Definition 10 $\xi(p)$ is the average linear size of the largest cluster. (excluding the percolating cluster).

1.2 2D CASE IN SQUARE LATTICE

1.2.1 Critical exponents

1.2.2 Correlation length

1.2.3 Phase transitions

1.2.4 Real space renormalisation

1.3 BETHE LATTICE

MANDELBROT PERCOLATION

2.1 INTRODUCTION

2.2 DEFINITIONS

2.3 RESULTS

Part II

NOVEL MODEL

CORRELATED MANDELBROT PERCOLATION

3.1 INTRODUCTION

3.2 DEFINITIONS

3.3 RESULTS

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