PERCOLATION AND ITS VARIATIONS

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An exploration of the current state of percolation models

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ABSTRACT

The current state of percolation theory is hard to assess. Research often uses different methods, language and notation, is hard to reproduce, and sometimes even provides conflicting results. We aim to give an overview of the current state of some of the most common and simple percolation models, alongside with open source code that allow the reader to easily reproduce the results presented, as well as present an analysis of a novel model called Stateful Mandelbrot Percolation. We hope to facilitate further research, as well as provide a starting point for anyone interested in the topic.

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— knuth:1974 [knuth:1974]

ACKNOWLEDGEMENTS

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Part I

CURRENT STATE

1 CLASSIC PERCOLATION

1.1 BASIC DEFINITIONS

We begin with a graph G representing a slattice of size L in d dimensions, such that $|G| = L^d$. Each node in the graph is independently set to be "alive"/"white" with probability p and "dead"/"black" with probability $1-p$. p is often called the occupation probability. Our goal is to study the properties of the lattice after this "coloring" has taken place. This is what such a lattice looks like: (IMAGE)

Definition 1 *A cluster is a set of white connected nodes in the graph.*

Definition 2 *A lattice is said to have percolated if there exists a macroscopic cluster, i.e. a cluster which spans the whole lattice.*

For the case of $d < \inf$, one can (arbitrarily) pick a dimension i from $1, 2, ...d$ and use it as the defining dimension for percolation, i.e. a lattice has percolated if there exists a cluster that intersects both boundaries of the lattice. For $d = 2$, for example, we can use the convention that a lattice has percolated if there is a cluster that connects the top boundary and the bottom boundary (left-right would be equally good).

The process is, of course, random. So we define an indicator random variable $H_{p,L}$ which represents whether a particular lattice has percolated:

Definition 3

$$H_{p,L} = \begin{cases} 1 & \textit{if lattice has percolated} \\ 0 & \textit{otherwise} \end{cases}$$

As we shall see, percolation models are the simplest models that exibit a phase transition, meaning that there exists a particular occupation probability p_c at which the behavior of the system is expected to change dramatically. This change will be reflected in a number of quantities of interest which will study in the next sections. It turns out that p_c of the probability of seeing a percolating cluster. As we vary p, larger and larger clusters are expected to form. However, in the case that $L < \inf$ and $p < 1$, there is always a possilibity that no percolating cluster will be observed (i.e even if $p = 0.999$ there is a non-zero probability all nodes are dead). For this reason, p_c is not well defined for finite L, and we'll define it in the limit $L \xrightarrow{\inf}$.

We can think of $E[H_{p,L}]$ as the probability that a particular lattice will percolate. As we increase L, the behaviour of $E[H_{p,L}]$ as a function of p approach a step function at $p = p_c$. Which means that in the limit of very big L, the lattice percolates with probability 1 for $p > p_c$, and does not percolate with probability 1 for $p < p_c$.

Definition 4 *The percolation threshold* p_c *is the smallest occupation probability* p *such that a percolating cluster exists with probability 1.*

1.2 1D CASE: TOY MODEL

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3 CORRELATED MANDELBROT PERCOLATION

3.1 INTRODUCTION

3.2 DEFINITIONS

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