PERCOLATION AND ITS VARIATIONS

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An exploration of the current state of percolation models

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ABSTRACT

The current state of percolation theory is hard to assess. Research often uses different methods, language and notation, is hard to reproduce, and sometimes even provides conflicting results. We aim to give an overview of the current state of some of the most common and simple percolation models, alongside with open source code that allow the reader to easily reproduce the results presented, as well as present an analysis of a novel model called Stateful Mandelbrot Percolation. We hope to facilitate further research, as well as provide a starting point for anyone interested in the topic.

We have seen that computer programming is an art,
because it applies accumulated knowledge to the world,
because it requires skill and ingenuity, and especially
because it produces objects of beauty.

— **knuth:1974 [knuth:1974]**

ACKNOWLEDGEMENTS

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LISTINGS

ACRONYMS

PMF Probability mass function

Part I

CURRENT STATE

CLASSIC PERCOLATION

1.1 BASIC DEFINITIONS

We begin with a graph G representing a slattice of size L in d dimensions, such that $|G| = L^d$. Each node in the graph is independently set to be "alive"/"white" with probability p and "dead"/"black" with probability $1 - p$. p is often called the occupation probability. Our goal is to study the properties of the lattice after this "coloring" has taken place. This is what such a lattice looks like: (IMAGE)

Definition 1 *A cluster is a set of white connected nodes in the graph.*

Definition 2 *A lattice is said to have percolated if there exists a macroscopic cluster, i.e. a cluster which spans the whole lattice.*

For the case of $d < \inf$, one can (arbitrarily) pick a dimension i from $1, 2, ...d$ and use it as the defining dimension for percolation, i.e. a lattice has percolated if there exists a cluster that intersects both boundaries of the lattice. For $d = 2$, for example, we can use the convention that a lattice has percolated if there is a cluster that connects the top boundary and the bottom boundary (left-right would be equally good).

The process is, of course, random. So we define an indicator random variable $H_{p,L}$ which represents whether a particular lattice has percolated:

Definition 3

$$H_{p,L} = \begin{cases} 1 & \text{if lattice has percolated} \\ 0 & \text{otherwise} \end{cases}$$

As we shall see, percolation models are the simplest models that exhibit a phase transition, meaning that there exists a particular occupation probability p_c at which the behavior of the system is expected to change dramatically. This change will be reflected in a number of quantities of interest which will study in the next sections. As we vary p, larger and larger clusters are expected to form. However, in the case that $L < \inf$ and $p < 1$, there is always a possibility that no percolating cluster will be observed (i.e even if $p = 0.999$ there is a non-zero probability all nodes are dead).

Definition 4 $\Pi_{p,L} = \mathbb{E}[H_{p,L}]$ *is the probability that a particular lattice will percolate.*

As we increase L, the behaviour of $\Pi_{p,L}$ as a function of p approach a step function at $p = p_c$. Which means that in the limit of very big L, the lattice percolates with probability 1 for $p > p_c$, and does not percolate with probability 1 for $p < p_c$.

Definition 5 *The percolation threshold* p_c *is the occupation probability* p *such that a percolating cluster exists with probability* 1 *on an infinite lattice (that is, in the limit* $L \to \infty$*).*

At this point, its not entirely obvious why we choose this particular definition. One could, a priori, choose to define p_c as the probability at which there is a $\frac{1}{2}$ probability that the lattice percolates, for example. Or one could choose to define it in terms of a finite lattice. As we shall see, the definition given above is the only one that is well definited when we start working with multiple dimensions and multiple types of lattices.

Definition 6 $P_{p,L}$ *is the fraction of nodes belonging to a percolating cluster.*

Next, we'll look at some quantities of interest and study their behavior as a function of p, in particular for small values of $|p - p_c|$, i.e. for p close to p_c. The reason for studying this particular regime will become apparent later.

Definition 7 $n_s(p)$ *is the cluster size distribution, i.e the number of s-sized clusters (clusters with s nodes) per lattice size.*

Definition 8 $\chi(p)$ *is the average cluster size.*

Definition 9 $s_\xi(p)$ *is the average size of the largest cluster.*

Definition 10 $\xi(p)$ *is the average linear size of the largest cluster.*

1.2   1D CASE: TOY MODEL

Our goal now is to better understand the 1D case. It is one of the few percolation models that can be solved analitically, and therefore can give us some insight into the dynamics of the system. Many of the properties exhibited still extend to higher dimensions, if one knows where to look.

For simplicity, we'll define a new indicator random variable S_i which denotes the state of the i-th node in the lattice. The PMF for S_i is

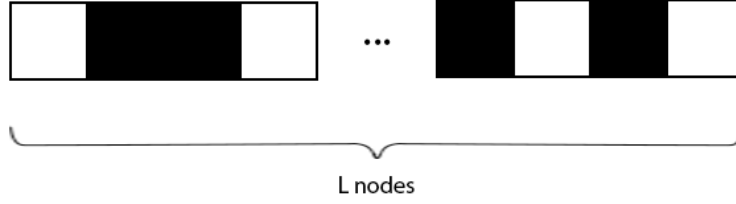$$S(i) = \begin{cases} 1 & \text{with probability p} \\ 0 & \text{with probability 1-p} \end{cases}$$

Figure 1: A 1D lattice of size L, with a few dead nodes

1.2.1 *Percolation threshold* p_c

For a 1D lattice, it is clear that percolation can only happen in a lattice of size L if all nodes in the lattice are dead. Since each coloring is independent, we have that

$$
\begin{aligned}
\Pi_{p,L} &= \Pr[S(1) = 1, \ldots S(L) = 1] \\
&= \Pr[S(1) = 1] \ldots \Pr[S(L) = 1] \\
&= p^L
\end{aligned}
\tag{1}
$$

It is clear that in the limit $L \to \infty$, if $p < 1$, $\Pi_{p,L} = 0$. Therefore, we have that

$$p_c = 1$$

It is interesting to notice here that if we were to define p_c in any simpler way, its value would depend on the size of the lattice. For example, suppose we chose to define p_c as the smallest p such that

$$\Pi_{p,L} \geqslant \alpha$$

For some $0 < \alpha < 1$. In other words, if α was $\frac{1}{2}$, that would be equivalent to defining p_c as the smallest p such that the lattice percolates with probability $\frac{1}{2}$. Then we'd have

$$
\begin{aligned}
\Pi_{p,L} &\geqslant \alpha \\
\Rightarrow p^L &\geqslant \alpha \\
\Rightarrow p &\geqslant \alpha^{\frac{1}{L}}
\end{aligned}
$$

We'd like to make our definitions as universal as possible, in particular we'd like our definition to be indepedent of the size of the lattice. The definition we chose previously accomplishes that. We will see more ways in which any other definition fails in the 2D case as well.

1.2.2 *Cluster size distribution*

The first non-trivial quantity we want to look at is the cluster size distribution, n_s. For a given combination (p, L) of occupation probability and lattice size, n_s describes the number of s-clusters - that is, clusters with s nodes - *per lattice site*.

For the 1D case, we can easily find an analytical expression for n_s. Let's start by considering a very long lattice (or equivalently, $L \gg s$). In this case, an s-cluster happens if and only if we see a sequence of s black cells enclosed by a white cells at the left and right boundaries. The probability for a white cell is $(1-p)$, and the probability of a black cell is p. Therefore, we expect that the probability of seeing an s-cluster somewhere along the lattice is proportional to $(1-p)^2 p^2$. We also expect that by doubling the size of the lattice, we'll see twice as many s-clusters. Therefore, we expect to see

$$\langle n_s \rangle \propto L(1-p)^2 p^s \tag{2}$$

Below, we can see the cluster size distribution obtained from simulations we've run, for various combinations of p and L:
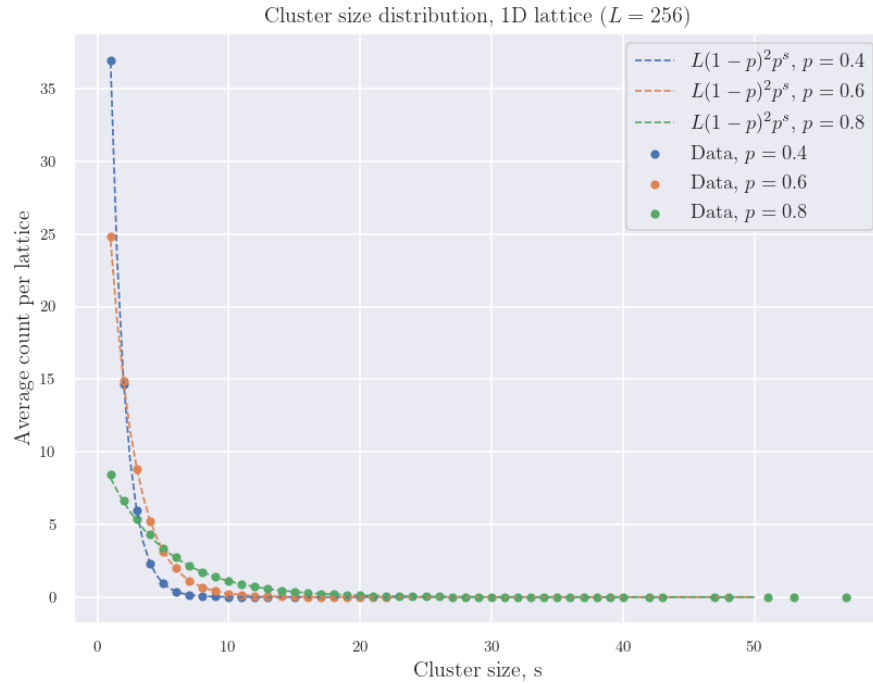

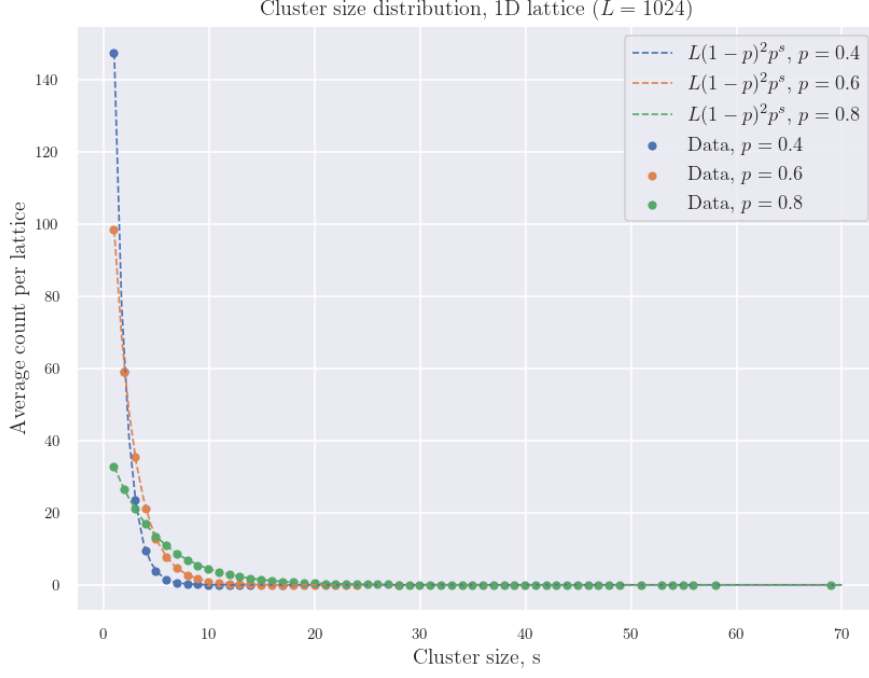Figure 2: Average cluster size distribution in a 1D lattice with L = 256

Figure 3: Average cluster size distribution in a 1D lattice with $L = 1024$

One interesting thing to notice is that as we increase p, the distribution becomes less pronounced towards smaller clusters. Its easy to see why happens by looking at (2): the ratio of 1-cluster and 2-clusters, for examples, decreases as we increase p:

$$\frac{n_1}{n_2} \propto \frac{1}{p}$$

Therefore, we expect a bigger difference between n_1 and n_2 for smaller p.

1.2.3   *Average cluster size*

1.2.4   *Average size of largest cluster*

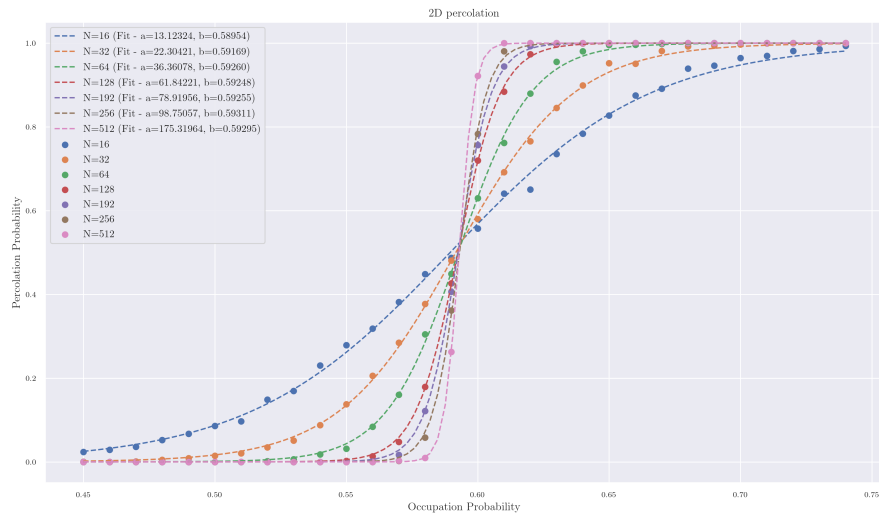### 1.2.5   *Average linear size of the largest cluster*

1.3   2D CASE IN SQUARE LATTICE

1.4   CRITICAL EXPONENTS

The basic quantities we studied in the previous section share an interesting property: all of them behave similarly near p_c. In this section, we study the behavior of these quantities when $|p - p_c|$ is small.

2

MANDELBROT PERCOLATION

2.1 INTRODUCTION

2.2 DEFINITIONS

2.3 RESULTS

Part II

NOVEL MODEL

3

CORRELATED MANDELBROT PERCOLATION

3.1 INTRODUCTION

3.2 DEFINITIONS

3.3 RESULTS

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