Chapter 5

Photon Counting

5.1 Introduction

The purpose of this section is to develop a simple model photon detection. The detector will be a device that responds to the number of photons that it receives with a response function that is nonlinear. This is a model of detectors in which an onset threshold and a saturation level. The photon stream itself will be treated as a Poisson process with an average number q of photons received over an observation interval.

The use of a photon detector to detect and measure changes in the photon stream will be addressed. If one is to measure the brightness of a source, a background level, changes in source brightness, changes in background level and the like, it is necessary to measure changes in the photon stream. This leads to the concepts of detective quantum efficiency (DQE) and noise equivalent power (NEP), which are terms that are used in the detection literature.

5.2 Statistics of Spatial Image Recording

Let us look first at the basic issues that are involved in the recording of two-dimensional image information. Dainty and Shaw have pointed out that the basic principles can be understood without reference to the technology of any specific imaging device[1].

A basic goal for an imaging device is that there should be a one-to-one relationship between the incident quanta at each location in the image plane and some measurable response of the device. We will use a model in which the device is composed of a spatial distribution of photon counters.

Each counter gathers the photons that fall on its aperture. In the simplest model they will be of uniform size and will be distributed to cover the imaging plane. In a more complicated model the collecting area may vary with location, and even be chosen from a random size distribution. The locations need not be on any kind of uniform grid, but can be scattered with some distribution about the image plane. The response function of the counters can also show a variation. It is not difficult to see that the model can be quite complicated in practice. We will use the simplest possible model in which the detectors are close-packed, have a uniform area a, and have identical response functions. This simple model will be adequate to demonstrate the basic principles.

An example of an array of closely packed hexagonal elements with uniform size is shown in Figure 5.1. The array is illuminated by a photon beam with a uniform brightness such that the average number of photons per element is $q = 4$. Note that the numbers actually range from 0 to 7 because the number is a random variable. The proportion of cells in a large array that receive k photons is governed by

$$P[k] = \frac{q^k e^{-q}}{k!} \tag{5.1}$$

Let us assume that each detector has a finite saturation level L beyond which it will not respond. Then each detector will count up to L and no more. The effect of saturation can be modeled by use of the Poisson distribution. In effect, each element has a response function that is linear up to saturation and then constant after that.

Let X be the number of photons that arrive at a detector. Then its response is $Y = h(X)$, where

$$h(x) = \begin{cases} x, & x \leq L-1 \\ L, & x \geq L \end{cases} \tag{5.2}$$

[1]Dainty, J. C. and R. Shaw, *Image Science: principles, analysis and evaluation of photographic-type imaging processes*, Academic Press, 1974. Chapter 1.

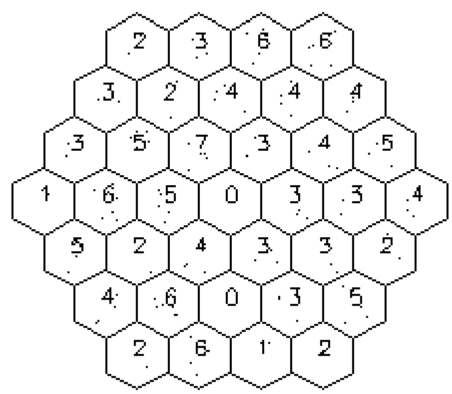

Figure 5.1: An array of detectors with a number of photon counts for each element. The number of photons is drawn from a Poisson distribution with a mean value of 4.

The average response is

$$\mu_Y = E[Y] = \sum_{k=0}^{\infty} P[X = k]h(k) \tag{5.3}$$

Then

$$\mu_Y = \sum_{k=0}^{L-1} \frac{kq^k e^{-q}}{k!} + \sum_{k=L}^{\infty} L \frac{q^k e^{-q}}{k!} \tag{5.4}$$

We would like to evaluate this equation in a closed form. If one writes out a few terms it will be seen that the above equation can be written in terms of

L summations. Each of these can be itself summed.

$$\mu_Y = \sum_{k=1}^{\infty} \frac{q^k e^{-q}}{k!} + \sum_{k=2}^{\infty} \frac{q^k e^{-q}}{k!} + \sum_{k=3}^{\infty} \frac{q^k e^{-q}}{k!} + \cdots + \sum_{k=L}^{\infty} \frac{q^k e^{-q}}{k!} \tag{5.5}$$

Each of these terms is equal to the sum from 0 to ∞ less the sum of the missing terms. The sum from 0 to ∞ in each case is just the sum over the Poisson distribution, so must equal 1.

$$\mu_Y = (1-e^q) + \left(1 - \sum_{k=0}^{1} \frac{q^k e^{-q}}{k!}\right) + \left(1 - \sum_{k=0}^{2} \frac{q^k e^{-q}}{k!}\right) + \cdots + \left(1 - \sum_{k=0}^{L-1} \frac{q^k e^{-q}}{k!}\right) \tag{5.6}$$

or,

$$\mu_Y = L\left(1 - f_1(q, L)e^{-q}\right) \tag{5.7}$$

where

$$f_1(L, q) = \frac{1}{L}\left(1 + \sum_{k=0}^{1} \frac{q^k}{k!} + \sum_{k=0}^{2} \frac{q^k}{k!} + \cdots + \sum_{k=0}^{L-1} \frac{q^k}{k!}\right) \tag{5.8}$$

The response μ_Y vs q is shown in Figure 5.2 for the case $L = 10$. The solid curve is the actual response and the dashed curve is the ideal response as given by equation (5.2). The plot will be similar for other values of L, in which it is nearly linear at the beginning and then bends over as the average photon rate approaches the saturation point.

The saturation threshold may not be the only practical limitation of a real photon detector. It may also be the case that the detector needs to receive some minimum number of photons before it begins to respond. Let us consider an array of elements that have a lower detection threshold of T and a saturation limit of $L+T$ when the active region is of width L as before. The response function of

$$h(x) = \begin{cases} 0, & x < T \\ x - T + 1, & T \leq x < L + T \\ L, & x \geq L + T \end{cases} \tag{5.9}$$

The equation for the average count now becomes

$$\mu_Y = \sum_{k=0}^{\infty} P[X = k]h(k) \tag{5.10}$$

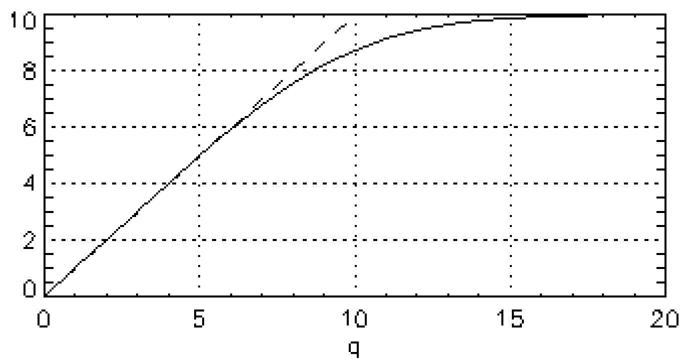

Figure 5.2: The relationship between the average response of the detector elements of the array vs the photon arrival rate for an array with a saturation level of $L = 10$. The dashed line has a slope of unity and illustrates the ideal response.

We now substitute (5.1) and (5.9) into (5.10). Following a procedure that is similar to that above,

$$\mu_Y = \sum_{k=T}^{L+T-2} (k - T + 1)\frac{q^k e^{-q}}{k!} + \sum_{k=L+T-1}^{\infty} L\frac{q^k e^{-q}}{k!} \tag{5.11}$$

This equation can be rewritten in the form of equantion (5.7) if we modify the function f_1 so that

$$f_1(L, q) = \frac{1}{L}\left(\sum_{k=0}^{T-1}\frac{q^k}{k!} + \sum_{k=0}^{T}\frac{q^k}{k!} + \sum_{k=0}^{T+1}\frac{q^k}{k!} + \cdots + \sum_{k=0}^{L+T-1}\frac{q^k}{k!}\right) \tag{5.12}$$

A plot of the average response with this detector is shown in Figure 5.3. Note that the response curve now rolls over at both the threshold and the saturation limit. The response approximates the ideal curve in the middle of the active region.

The examples above show that an array of detectors will have a composite response. This is the response that will be seen by looking at a region of the image in a manner that causes the eye to integrate the response of a large number of cells. The image will begin to look noisy if the eye can resolve down to a small number of cells. In effect, the viewer forms the average, and the response is approximated by the derived curves.

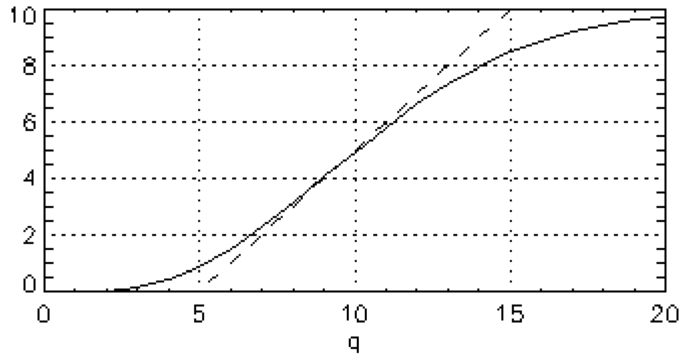

Figure 5.3: The relationship between the average response of the detector elements of the array vs the photon arrival rate for an array with a threshold of $T = 5$ and an active region of width $L = 10$ so that the saturation is at $S = 15$. The dashed line has a slope of unity and illustrates the ideal response.

5.2.1 Contrast

An important factor associated with images is the contrast. Suppose that one looks at the average response in two regions with different exposures to light. We would like to know how much the response changes with a change in exposure. This is determined by the gradient of μ_Y with changes in q. In effect, we are using a linear approximation in which the response

$$\mu_Y(q + \Delta q) = \mu_Y(q) + g(q)\Delta q \tag{5.13}$$

and $g(q)$ is the gradient

$$g(q) = \frac{d\mu_Y}{dq} = Le^{-q}\left(f_1 - \frac{df_1}{dq}\right) \tag{5.14}$$

The derivative of $q^k/k!$ is $q^{k-1}/(k-1)!$. Then differentiation of f_1 is, from equation (5.12),

$$\frac{df_1(L,q)}{dq} = \frac{1}{L}\left(\sum_{k=0}^{T-2}\frac{q^k}{k!} + \sum_{k=0}^{T-1}\frac{q^k}{k!} + \sum_{k=0}^{T}\frac{q^k}{k!} + \cdots + \sum_{k=0}^{L+T-2}\frac{q^k}{k!}\right) \tag{5.15}$$

Effectively, each term is reduced in length by one. You may have to write out an example to see how this works. If we now subtract the derivative from f_1 all that remains is the last term in each sum.

$$f_2(L,q) = \left(f_1 - \frac{df_1}{dq}\right) = \frac{1}{L}\sum_{k=0}^{L-1}\frac{q^k}{k!} \tag{5.16}$$

The response gradient is

$$g(q) = Le^{-q}f_2(L,q) \tag{5.17}$$

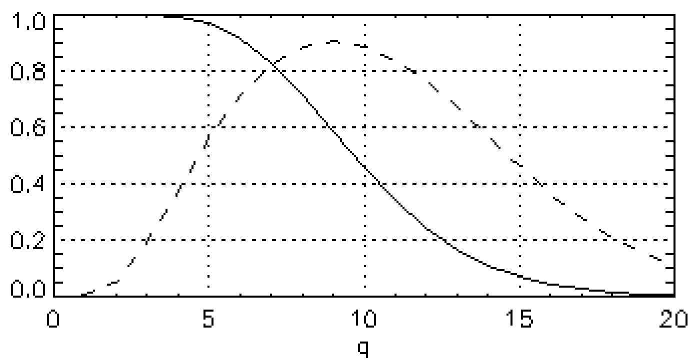

Figure 5.4: The gradient as a function of the average quantum exposure. Solid curve: $L = 10$, $T = 0$; dashed curve: $L = 10$, $T = 5$. These correspond to the response curves in the earlier figures.

The gradient as a function of quantum exposure for two cases is shown in Figure 5.4. The solid curve corresponds to the detector of Figure 5.2. It is unity at low exposure levels and then decreases and falls toward zero as the saturation level is approached and crossed. The dashed curve corresponds to the detector of Figure 5.3. It rises to a maximum near unity in the central region of the exposure range, but is near zero at both low and high exposures. This is typical of a detector that has an onset threshold as well as a saturation threshold. The contrast that will be seen in an image can be related to the gradient curves, with the maximum being 1.0. The range of exposure values that can be accommodated is indicated by the width of the gain curves.

5.2.2 Comparative Noise Level

The cells in a region of the detector that is under uniform quantum exposure will exhibit statistical differences in their responses because different numbers of quanta fall upon them as illistrated in Figure 5.1. We have used Y as the random variable that represents the detector response and have calculated the mean value μ_Y in the analysis above. The mean-squared value can be found by a similar analysis. Let us look first at an array with a simple linear response with saturation as in (5.2). The mean-squared response is

$$
\begin{aligned}
E[Y^2] &= \sum_{k=0}^{\infty} h^2(k) P[X=k] && (5.18)\\
&= \sum_{k=0}^{L-1} k^2 \frac{q^k e^{-q}}{k!} + \sum_{k=L}^{\infty} L^2 \frac{q^k e^{-q}}{k!} && (5.19)
\end{aligned}
$$

This can be expressed in terms of a function $f_3(L, q)$ as

$$E[Y^2] = L^2 \left(1 - f_3 e^{-q}\right) \tag{5.20}$$

where

$$
\begin{aligned}
f_3(L,q) &= \frac{1}{L^2}\left[1 + 3\sum_{k=0}^{1}\frac{q^k}{k!} + 5\sum_{k=0}^{2}\frac{q^k}{k!} + \cdots + (2L-1)\sum_{k=0}^{L-1}\frac{q^k}{k!}\right] && (5.21)\\
&= \frac{1}{L^2}\sum_{n=0}^{L-1}(2n+1)\sum_{k=0}^{n}\frac{q^k}{k!} && (5.22)
\end{aligned}
$$

The variance of the detector responses can now be found by combining (5.7) and (5.20).

$$\sigma_Y^2 = E[Y^2] - \mu_Y^2 = L^2\left[(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2\right] \tag{5.23}$$

The variance of the detector response is shown in Figure 5.5 where the upper curve shows the mean-squared value and mean-value squared and the lower curve shows their difference. It is interesting that the noise seems to be greatest in the midpoint of the operating region of the detector.

The image noise σ_Y^2 is measured in detecor counts (squared). It would be of interest to characterize this noise on a scale that would enable a comparison between detectors. To this end, let us attempt to relate the variance of the

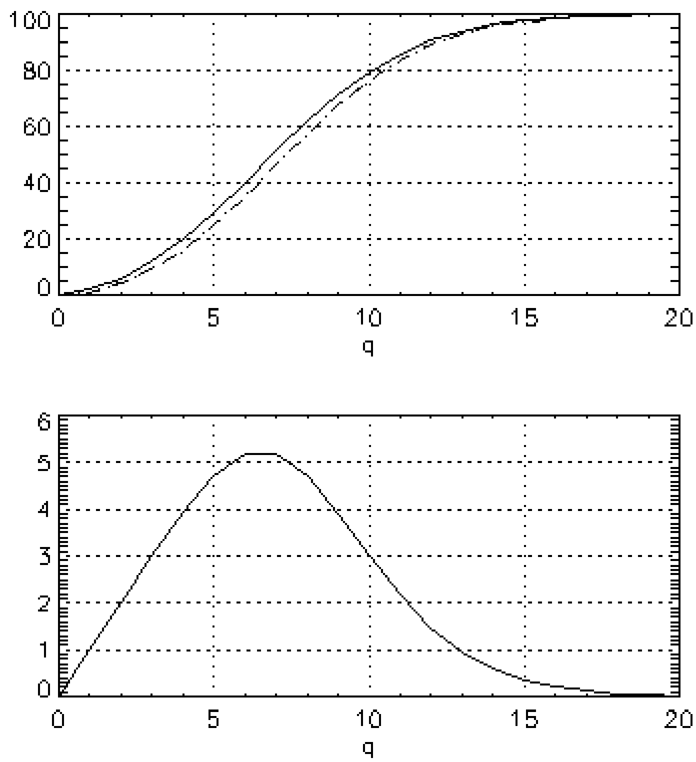

Figure 5.5: The upper figure shows $E[Y^2]$ (solid) and μ_Y^2 (dashed) for a quantum detector with $L = 10$ and $T = 0$. The lower curve shows the variance, which is the difference between the two curves in the upper figure.

response to the variance in the input quantum flux. It is well-known that the variance of a Poisson distribution with mean value q is equal to q. Thus, the variance of the input flux is

$$\sigma_X^2 = q \tag{5.24}$$

At low exposures, where the input and output are related by $Y = X$ we must have $\sigma_Y^2 = \sigma_X^2 = q$, which is reflected in the straight-line relationhip of σ_Y^2 vs q at low exposures in Figure 5.5. The input noise provides a normalizing factor that can be used for comparison purposes.

The output noise can be referred to the system input by dividing by the square of the gain. Let

$$\sigma_{YI}^2 = \frac{\sigma_Y^2}{g^2} \tag{5.25}$$

This is the noise level that would be required at the input of a perfect detector with gain g if it were to produce noise at level σ_Y^2 at the output. The second law of thermodynamics requires that this be at least as large as the noise that is actually present at the input. It is impossible to reduce entropy in this manner. The ratio

$$\varepsilon = \frac{\sigma_X^2}{\sigma_{YI}^2} \tag{5.26}$$

must have an upper limit of unity. If we substitute (5.17), (5.23) and (5.25) into (5.26) we will obtain

$$\varepsilon = \frac{q(f_2)^2}{(1 - f_3 e^{-q}) - (1 - f_1 e^{-q})^2} \tag{5.27}$$

A plot of the comparative noise level vs q can be easily done. By varying the values of the threshold and saturation we can get some insight into the effect of these parameters on the noise characteristics of the detector relative to the ideal. Note that any variation from the ideal includes only effects due to the detector characteristic, and not from any noise introduced separately in the detector. The detector output can seem to have additional noise simply through the way in which the mean and variance of the response are affected by its response characteristic. Any internal detector noise will be in addition to this effect.

We can look at the various relationships in terms of the threshold T and saturation point $S = T + L - 1$. Consider two cases for comparison as shown in the following figure. A set of curves have been drawn for $T = 4$, $L = 5$, $S = 8$ and $T = 2$, $L = 15$, $S = 16$. The first graph shows the average detector response $E[Y]$ as a function of average photon count, q. Each curve has a maximum level L that is reached past the saturation level S. Neither curve is completely linear, although the curve with larger L has a longer response region that is nearly linear. This can be seen well in the second graph, which is a plot of the gradient g as a function of q. We see that the L = 15 case has a region of gradient equal to unity, but that the $L = 5$ case never achieves unity gradient.

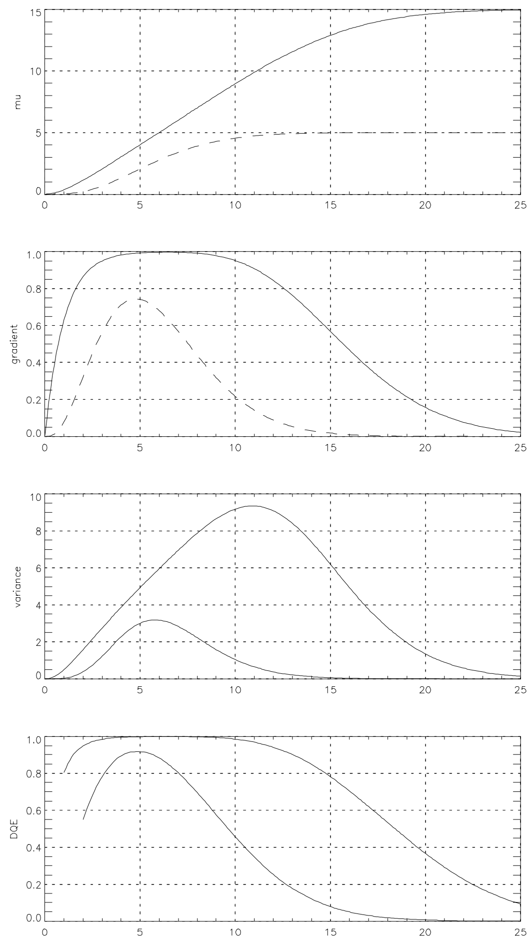

Image characteristics for $L = 2$, $T = 15$ (upper curves) and $L = 5$, $T = 4$ (lower curves). First graph: average count; Second graph: the gradient; Third graph: The mean-squared fluctuation; Fourth graph: comparative noise level.

Plots of the mean-squared fluctuation are shown in the third graph. Here we see that both curves rise to a maximum and then fall to zero. The maximum is greatest at the upper end of the best operating region of the detector. The comparative noise level is shown in the fourth graph. The form of these curves is similar to the gradient curves. For the case $L = 15$ there is a region where the comparative noise level is unity. However, the unity level is never achieved in the $L = 5$ case. Remember that a comparative noise level less than unity means that the system is somehow "adding" noise to the output, which is not desirable.

We see that an effect of introducing a threshold point T as well as a saturation point is to limit the range over which the detector can provide unity gradient and maximum comparative noise performance. However, if the linear region L is sufficiently long then there is a useful operating region in terms of both gain and comparative noise.

5.3 IDL Programs for Detector Response

The programs in this section are implementations of the equations in the notes. They are based upon the corresponding equations in the first chapter of Dainty and Shaw.

Program f1

```
FUNCTION f1,q,T,S
;+
; f=f1(q,T,S) is the average quantum count in a detector cell
; when the detector has a threshold T and saturation level S.
; The width of the active region is L=S-T+1.
;
; This program implements equation (2) of Chapter 1, page 9
; of Dainty and Shaw, Image Science.
;
; The quantum flux has an average value of q quante per pixel.
;
; Quantum count: 0 1 2 ... T-1 T T+1 ... T+L-2 T+L-1 T+L ....
; Counter value: 0 0 0 ...  0  1  2       L-1    L    L  ....
;
; This function is written so that q can be a vector of values
; that covers a range of interest.
;
; INPUT
; q A vector of values for the photon flux.
; T A scalar integer value for the detector threshold. T>0.
; S A scalar integer value >T for the detector saturation level.
;
; USAGE EXAMPLE
; q=FINDGEN(301)/10 A vector of values for q
```

```
; T=5 & S=16        Threshold and saturation values
; ybar=L*(1-f1(q,T,S)*exp(-q))
; plot,q,ybar
;
; HISTORY
; March, 1999 First version by H. Rhody
;-

ON_ERROR,2
msg1='The input must have three parameters: f1(q,T,S)'
IF N_PARAMS() LT 3 THEN MESSAGE,msg1
IF T LT 1 THEN MESSAGE,'The threshold T must be 1 or greater.'
nq=N_ELEMENTS(q)
qf=float(q)
u=FLTARR(nq)+1.0 ;The inner term of the sums
v=u     ;The value of the sum
L=S-T+1    ;The operating range

;Form the first sum
FOR k=1,T-1 DO BEGIN
 u=u*qf/k
 v=v+u
ENDFOR
;Form the sum of sums
w=v     ;The sum of sums
FOR k=1,L-1 DO BEGIN
 u=u*qf/float(T+k-1)
 v=v+u
 w=w+v
ENDFOR

RETURN,float(w)/L ; This is the vector of values of f1(q,T,S)
END
```

5.3.1 Program f2

```
FUNCTION f2,q,T,S
;+
```

```
; f=f2(q,T,S) is used in computing the gradient
; at the output of a detector with a Poisson flux input.
;
; This program implements equation (21) of Chapter 1, page 10
; of Dainty and Shaw, Image Science.
;
; The quantum flux has an average value of q quante per pixel.
;
; Quantum count: 0 1 2 ... T-1 T T+1 ... T+L-2 T+L-1 T+L ....
; Counter value: 0 0 0 ...  0  1  2       L-1    L    L  ....
;
; This function is written so that q can be a vector of values
; that covers a range of interest.
;
; INPUT
; q A vector of values for the photon flux.
; T A scalar integer value for the detector threshold. T>0.
; S A scalar integer value >T for the detector saturation level.
;
; USAGE EXAMPLE
; q=FINDGEN(301)/10 A vector of values for q
; T=5 & S=16        Threshold and saturation values
; grad=L*f2(q,T,S)*exp(-q)
; plot,q,grad
;;
; HISTORY
; March, 1998 First version by H. Rhody
;-

ON_ERROR,2
msg1='The input must have three parameters: f2(q,T,S)'
IF N_PARAMS() LT 3 THEN MESSAGE,msg1
IF T LT 1 THEN MESSAGE,'The threshold T must be 1 or greater.'
nq=N_ELEMENTS(q)
qf=float(q)
u=FLTARR(nq)+1.0 ;The inner term of the sum
u=qf^(T-1)/FACTORIAL(T-1)
v=u
```

```
L=S-T+1    ;The operating range

FOR k=T,S-1 DO BEGIN
 u=u*qf/k
 v=v+u
ENDFOR

RETURN,v/L ; This is the vector of values of f2(q,T,S)
END
```

5.3.2 Program f3

```
FUNCTION f3,q,T,S
;+
; f=f3(q,T,S) is used in computing the mean squared fluctuation
; at the output of a detector with a Poisson flux input.
;
; This program implements equation (22) of Chapter 1, page 10
; of Dainty and Shaw, Image Science.
;
; The quantum flux has an average value of q quante per pixel.
;
; Quantum count: 0 1 2 ... T-1 T T+1 ... T+L-2 T+L-1 T+L ....
; Counter value: 0 0 0 ...  0  1  2       L-1    L    L  ....
;
; This function is written so that q can be a vector of values
; that covers a range of interest.
;
; INPUT
; q A vector of values for the photon flux.
; T A scalar integer value for the detector threshold. T>0.
; S A scalar integer value >T for the detector saturation level.
;
; USAGE EXAMPLE
; q=FINDGEN(301)/10 A vector of values for q
; T=5 & S=16        Threshold and saturation values
; msf=L^2*((1-f3(q,T,S)*exp(-q))-(1-f1(q,T,S)^2*exp(-q)))
; plot,q,msf
```

```
;
; where f1(q,T,S) is a companion function
;
; HISTORY
; March, 1998 First version by H. Rhody
;-

ON_ERROR,2
msg1='The input must have three parameters: f1(q,T,S)'
IF N_PARAMS() LT 3 THEN MESSAGE,msg1
IF T LT 1 THEN MESSAGE,'The threshold T must be 1 or greater.'
nq=N_ELEMENTS(q)
qf=float(q)
u=FLTARR(nq)+1.0 ;The inner term of the sums
v=u     ;The value of the sum
L=S-T+1    ;The operating range

FOR k=1,T-1 DO BEGIN
 u=u*qf/k
 v=v+u
ENDFOR

;Form the sum of sums
w=v     ;The sum of sums
FOR k=0,S-T-1 DO BEGIN
 u=u*qf/(T+k)
 v=v+u
 w=w+(2*k+3)*v
ENDFOR

RETURN,w/L^2 ; This is the vector of values of f3(q,T,S)
END
```

5.4 Exercises

1. A certain kind of film is coated with an emulsion of photo-sensitive grains. The grains are of uniform size with a diameter of 1 micron. Determine the number of grains that would be included in a spot that covers a 0.2 degree angle at a viewing distance of 36 cm.

2. A film is exposed with a quantum flux of q photons per gain. Each grain can absorb only one photon. If it is struck by additional photons then nothing more happens to it.

   (a) Compute the fraction of the grains that would be exposed as a function of q. Plot this fraction as a function of q.

   (b) Compute the rate of change of the average exposure as a function of q and plot the result.

   (c) Let X be the number of photons that hit a grain. Let Y be the response of the grain. Then $Y = 0$ if $X = 0$ and $Y = 1$ if $X > 0$. Find the variance of Y. Plot var(Y) as a function of q.

   (d) Find an expression for the comparative noise level (detective quantum efficiency) of the film. Plot this expression as a function of q.

3. An optical fiber has the property that a section of length 1 km transmits a fraction η of the photons in a beam. Find the detective quantum efficiency of the fiber as a function of q. Describe how the DQE varies as a function of fiber length.

4. Some cells in a detector array are receiving photons at a rate of r per second and other cells are receiving photons at a rate of $1.1r$ per second. When a photon strikes a cell it has a probability q of being absorbed. The cells count all absorbed photons. Estimate the amount of exposure time T_e as a function of r and η that would be needed to be able to reliably differentiate the cells in the low-exposure set from those in the high-exposure set.