Next, consider the average continuous power spectral component $\bar{S}_c{}^*(f_n)$ over all possible K users at f_n. Under the assumption that the channel symbol sequences $(b_k{}^{(i,j)})$ $(1 \leqslant k \leqslant K)$ for any i and j are independent of one another, $\bar{S}_c{}^*(f_n)$ is obtained by the summation of the averages of the individual users' spectra $s_{c,k}{}^{*(i)}(f_n)$ $(1 \leqslant k \leqslant K)$ over all possible m-sequences. Furthermore, the averages of the individual users' spectra can be regarded as identical from (26).

Therefore, we have

$$\bar{S}_c{}^*(f_n) = \frac{K}{N_c} \sum_{i=0}^{N_c-1} s_{c,k}{}^{*(i)}(f_n) \tag{35}$$

for $1 \leqslant k \leqslant K$.

The simulated results in Figs. 3 and 5(a) are the average continuous power spectral components computed by using (35) for the above case. These results are in good agreement with the calculated values.

IV. DISCUSSION

We have obtained the details of the power spectrum of the transmitted signal in synchronous M-ary DS-SSMA communication systems in baseband. These spectra depend on the statistics of the channel symbol ensemble by means of the spectral shaping function. Furthermore, they can be applied to the system employing a linear modulation.

When m-sequences are used as the spreading code sequences assigned to all potential users and each user's information symbol takes on two possible values with equal probability in the system under consideration, we have illustrated the calculated and the simulated results of the continuous power spectral component for two cases, where the channel symbol sequences are binary, and where they are bipolar. In the illustrations for the case of binary sequences, the peak value in the power spectral component of the user's transmitted signal (corresponding to a DS signal) is about three times as large as compared with the corresponding value in the $(\sin x/x)^2$ spectrum. Also, the average power spectral component forms a $(\sin x/x)^2$ distribution, except in the neighborhood of zero frequency. From these results, it is clear that the spectrum of a DS signal is influenced considerably by the spreading code sequence.

Furthermore, in the illustrations for the case of bipolar, the user's power spectral component is influenced by the channel symbol and the spreading code sequences and has almost the same peak value as the one in the case of binary code. Also, the average power spectral component forms a distribution which is modified by the spectral shaping function and which differs from the average power spectral component in the case of binary.

These power spectral characteristics have not been well known. Hence, the spectral analysis presented in this paper will be useful for the investigation of the spread spectrum in M-ary DS-SSMA communication systems.

REFERENCES

[1] R. C. Dixon, *Spread Spectrum Systems*. New York: Wiley, 1976.
[2] R. A. Scholtz, "The spread spectrum concept," *IEEE Trans. Commun.*, vol. COM-25, pp. 748–755, Aug. 1977.
[3] J. K. Holmes, *Coherent Spread Spectrum Systems*. New York: Wiley, 1982.
[4] W. R. Bennett, "Statistics of regenerative digital transmission," *Bell Syst. Tech. J.*, vol. 37, pp. 1501–1542, 1958.
[5] S. O. Rice, "Mathematical analysis of random noise," *Bell Syst. Tech. J.*, vol. 23, pp. 282–332, 1944; vol. 24, pp. 46–156, 1945.
[6] S. W. Golomb, *Shift Register Sequences*. San Francisco, CA: Holden-Day, 1967.
[7] M. B. Pursley, "Performance evaluation for phase-coded spread-spectrum multiple-access communication—Part I: System analysis," *IEEE Trans. Commun.*, vol. COM-25, pp. 795–799, Aug. 1977.
[8] L. E. Franks, *Signal Theory*. Englewood Cliffs, NJ: Prentice-Hall, 1969.

Capacity Limit of the Noiseless, Energy-Efficient Optical PPM Channel

JAMES R. LESH, SENIOR MEMBER, IEEE

***Abstract*—The energy-efficient capacity of the noiseless optical PPM channel is examined. It is shown that, although the capacity per photon can be made to increase without bound, the capacity per channel use (for best energy efficiency) is always less than 2 nats and approaches 2 nats/symbol as the bandwidth expansion factor goes to infinity.**

I. INTRODUCTION

In [1], a method was described for maximizing the energy efficiency of a noiseless PPM optical channel subject to simultaneous constraints on the throughput capacity and bandwidth. Specifically, a Q-ary erasure channel was used to model a Q-ary PPM optical communication system. This channel is known to have a capacity C per channel use given by

$$C = (1 - e^{-\lambda_S \Delta T}) \ln Q \quad \text{nats/channel use} \tag{1}$$

where λ_S is the intensity (measured in photons/s) of the optical source as seen by the receiver during the PPM pulse and ΔT is the PPM slot width (in seconds). Since $Q\Delta T$ seconds are required for each channel use and each PPM symbol contains, on the average, $\lambda_S \Delta T$ photons, the channel capacity can alternatively be expressed as

$$C_T = \frac{C}{Q\Delta T} = \frac{1 - e^{-\lambda_S \Delta T}}{\Delta T} \frac{\ln Q}{Q} \quad \text{nats/s} \tag{2}$$

or

$$C_{\text{Ph}} = \frac{C}{\lambda_S \Delta T} = \frac{1 - e^{-\lambda_S \Delta T}}{\lambda_S \Delta T} \ln Q \quad \text{nats/photon.} \tag{3}$$

In [1], the throughput capacity C_T and the slot width (or equivalently the system bandwidth) ΔT were both fixed. Then,

Paper approved by the Editor for Communication Theory of the IEEE Communications Society for publication without oral presentation. Manuscript received February 26, 1982; revised September 8, 1982.

The author is with the Jet Propulsion Laboratory, Pasadena, CA 91109.

0090-6778/83/0400-0546$01.00 © 1983 IEEE

TABLE I
OPTIMIZED VALUES OF Q AND $C_T \Delta TQ$

α	Q^*	$C_T \Delta TQ^*$
$10^{-.7}$	8	1.40
10^{-1}	16	1.60
10^{-2}	176	1.76
10^{-3}	1.83×10^3	1.83
10^{-4}	1.87×10^4	1.87
10^{-5}	1.89×10^5	1.89
10^{-6}	1.91×10^6	1.91
10^{-7}	1.92×10^7	1.92
10^{-8}	1.93×10^8	1.93
10^{-9}	1.94×10^9	1.94
10^{-10}	1.94×10^{10}	1.94
10^{-11}	1.95×10^{11}	1.95
10^{-12}	1.95×10^{12}	1.95
10^{-14}	1.96×10^{14}	1.96
10^{-20}	1.97×10^{20}	1.97
10^{-30}	1.98×10^{30}	1.98

C_{Ph} was plotted as a function of Q. From such plots one could determine Q^*, the value of Q for which the capacity per photon C_{Ph} was maximized. In [1], it was observed that the product $C_T \Delta TQ^*$ was approximately constant. In this paper, we will examine this phenomenon in more detail. We show that, provided one adheres to the constraint of optimum power efficiency, the capacity of the channel, measured in nats per channel use, is upper bounded by 2 and asymptotically approaches this limit as the bandwidth expansion factor increases.

II. ANALYSIS

We first determine the requirements on the optimizing value of Q. A necessary condition on Q^* can be obtained by setting the derivative of C_{Ph} with respect to Q equal to zero. However, before this can be done, note that if C_T and ΔT are fixed, any variation of Q must be offset by compensating changes in λ_S [see (2)]. Thus, we must explicitly show the dependence of λ_S on Q. From (2) we have that

$$\lambda_S = -\frac{1}{\Delta T} \ln \left(1 - \frac{C_T \Delta TQ}{\ln Q}\right). \tag{4}$$

Now, substituting (4) into (3) and differentiating we obtain the necessary condition

$$\frac{Z(Q^*)}{1 - Z(Q^*)} \left[1 - \frac{1}{\ln Q^*}\right] + \ln [1 - Z(Q^*)] = 0 \tag{5}$$

where

$$Z(Q) = \frac{C_T \Delta TQ}{\ln Q}. \tag{6}$$

Thus, once we specify the product $\alpha \triangleq C_T \Delta T$ (which is equivalently the capacity per PPM slot), Q^* can be determined by solving (5) numerically. When the solution of (5) gives a noninteger value for Q^*, one must, of course, check which of the adjacent pair of integers is the actual Q^*. This distinction is important only when Q^* is small.

Table I shows the results of such calculations for a wide range of α's. Both Q^* and $C_T \Delta TQ^*$ are shown for each α. Clearly, as α decreases there is a compensating increase in Q^* such that for most values of α, $C_T \Delta TQ^*$ is slightly less than 2.

We will now interpret these results. From (1) and (2), we recognize that $C_T \Delta TQ$ is simply C, the channel capacity in nats/channel use. Thus, $C_T \Delta TQ^*$ *is nothing more than* C *evaluated at the most energy-efficient operating point.* We shall denote this quantity by C^*. Additionally, we note that, if α decreases, then either ΔT decreases (bandwidth increases) for fixed C_T, or the required throughput capacity decreases for fixed bandwidth. Both statements are equivalent to saying that the ratio of available bandwidth to information bandwidth is increasing. Thus, *the reciprocal of* α *can be interpreted as a bandwidth expansion factor* (doubling α^{-1} doubles available bandwidth). In Fig. 1, C^* is plotted as a function of the logarithm of this bandwidth expansion factor.

We can now state our main result.

For the noiseless optical PPM channel, the most energy-efficient use of the channel results in a capacity, C^*, *of less than 2 nats per channel use. Furthermore,* C^* *increases with the bandwidth expansion factor and approaches 2 nats per channel use as that factor approaches* ∞.

To prove these assertions, note from (5) that the optimizing Q for any α satisfies the relation

$$\frac{Z^*}{1 - Z^*} \left[1 - \frac{1}{\ln Q^*}\right] + \ln (1 - Z^*) = 0 \tag{7}$$

where we have defined

$$Z^* = \frac{C_T \Delta TQ^*}{\ln Q^*} = \frac{C^*}{\ln Q^*}. \tag{8}$$

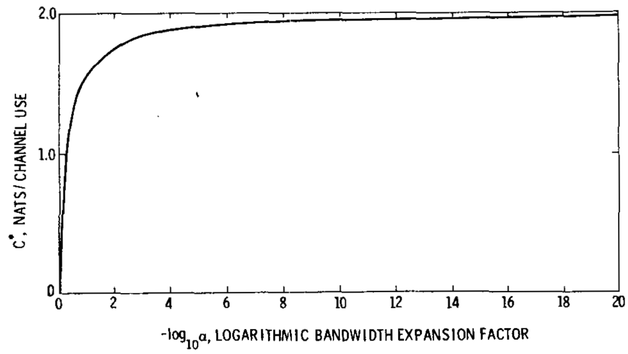

Fig. 1. Variation of energy-efficient channel capacity/channel use with bandwidth expansion.

Using this last equality, we can rewrite the optimality condition (7) as

$$C^* = f(Z^*)$$

where

$$f(Z) = \frac{Z^2}{Z + (1 - Z) \ln (1 - Z)}.$$

From (2) we see that

$$Z = 1 - e^{-\lambda_S \Delta T}$$

so that $0 < Z < 1$; in fact, Z is the probability that the PPM pulse is not erased. ($Z = 1$ corresponds to infinite pulse energy, whereas $Z = 0$ corresponds to zero pulse energy.) We now wish to show that $f(Z) < 2$ for all $Z \in (0, 1)$. Defining

$$g(Z) = Z + (1 - Z) \ln (1 - Z)$$

so that

$$f(Z) = \frac{Z^2}{g(Z)}$$

we find that

$$f'(Z) = \frac{2Z^2 + Z(2 - Z) \ln (1 - Z)}{g^2(Z)}. \tag{10}$$

By expanding the logarithm in a Taylor series we have that

$$\ln (1 - Z) < -Z - \frac{Z^2}{2} - \frac{Z^3}{3} - \frac{Z^4}{4}$$

for $0 < Z < 1$ so that

$$2Z^2 + Z(2 - Z) \ln (1 - Z) < -\frac{Z^4}{6} (1 + Z - \tfrac{3}{2} Z^2) < 0 \tag{11}$$

for all $0 < Z < 1$. Thus, (10) and (11) together show that $f'(Z) < 0$ and, hence, $f(Z)$ is monotonically decreasing with Z $(0 < Z < 1)$. Furthermore, by applying l'Hospital's rule twice we see that $f(0+)$ equals 2. Thus, $f(Z) < 2$ for all $0 < Z < 1$ and approaches 2 as $Z \to 0$.

Finally, we must show that increasing the bandwidth expansion factor (and, hence, decreasing α) corresponds to decreasing Z^*, thus increasing $C^* = f(Z^*)$. From (8), we see that decreasing Z^* must be compensated for by an increase in Q^* (due to the monotonicity of $f(Z)$ and, hence, C^*). However, from (8) and the definition of α, we know that Z^* can decrease in the presence of an increasing Q^* (assuming $Q^* \geqslant 3$) if and only if α decreases, which completes the proof.

III. DISCUSSION

It is well known that the capacity of the noiseless PPM channel *measured in nats/photon* can be made infinite by allowing the word size (and, hence, the bandwidth expansion) to go to infinity. It is therefore surprising that the capacity *in nats/channel use* is at most 2. This apparent paradox can be resolved by noting that as α decreases, Q^* increases, Z^* goes to zero, and the average energy per pulse (i.e., per channel use) also goes to zero as seen by (4). Thus,

$$C_{\mathrm{Ph}}{}^* = \frac{C^*}{\lambda_S{}^* \Delta T}$$

goes to infinity not because C^* goes to infinity but because $\lambda_S{}^* \Delta T$ goes to zero.

The limiting behavior of C^* bears a resemblance to the Shannon limit for the additive white Gaussian noise channel. For the AWGN channel, Shannon showed that the energy E, required to transmit reliably (i.e., zero error probability) one nat of information, normalized by the one-sided noise power spectral density N_0, is lower bounded by

$$\frac{E}{N_0} > 1$$

and that E/N_0 can be made to approach this bound as the bandwidth expansion of the signal approaches ∞. For the noiseless optical PPM channel, N_0 vanishes (quantum noise does not) so the energy required per nat can be made arbitrarily small. However, the capacity per energy-efficient use of the channel is still limited and approaches 2 nats/channel use as the bandwidth expands. As in the Gaussian channel case, the optical channel limiting behavior is obtained by a sequence of increasingly complex orthogonal (PPM in this case) modulation schemes. This similarity should be tempered, however, by the fact that the Shannon limit is a hard limit for the AWGN channel, requiring a substantial loss in data reliability if the limit is violated. For the optical channel, the optimization of C_{Ph} over Q is quite broad and, as a result, one need only relax the power efficiency constraint a small amount to substantially exceed C^*. It is only when we demand that photons be used with maximum efficiency that the 2 nat per channel use upper bound applies.

ACKNOWLEDGMENT

The author would like to express his thanks to the anonymous reviewer who suggested the simplified derivation of the main result presented herein.

REFERENCES

[1] S. A. Butman, J. Katz, and J. R. Lesh, "Bandwidth limitations on noiseless optical channel capacity," *IEEE Trans. Commun.*, vol. COM-30, pp. 1262–1264, May 1982.