High-Precision Experimental Determination and Theoretical Modeling of the Feigenbaum Constant in a Driven Nonlinear R-L-D Oscillator

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Abstract

The universality of the period-doubling route to chaos, characterized by the Feigenbaum constant δ, is a cornerstone of nonlinear dynamics. While the Resistor-Inductor-Diode (R-L-D) circuit is a canonical system for demonstrating this phenomenon, previous experimental realizations have often lacked rigorous theoretical modeling and comprehensive uncertainty analysis. This work presents a high-precision experimental determination of δ using an automated, computer-controlled R-L-D circuit. We develop a first-principles theoretical model based on the Shockley equation and the diode's nonlinear junction capacitance to derive the system's governing second-order nonlinear differential equation. The bifurcation points leading to chaos are measured with high resolution, yielding a bifurcation diagram and power spectra that confirm the period-doubling cascade. From the measured bifurcation voltages, we calculate Feigenbaum's first constant to be $\delta = 4.67 \pm 0.08$, a value in excellent agreement with the accepted value of 4.669... The analysis demonstrates that deviations in higher-order bifurcations can be qualitatively explained by non-ideal component behavior, highlighting the synergy between precision measurement and robust theoretical modeling in the study of complex systems.

1 Introduction

The study of nonlinear dynamical systems has revealed profound universal principles governing the transition from ordered to chaotic behavior. A seminal discovery in this field was made by Mitchell Feigenbaum, who showed that a wide class of systems exhibiting a period-doubling cascade en route to chaos are governed by a universal constant, $\delta \approx 4.6692$ [1]. This constant quantifies the asymptotic ratio of successive bifurcation intervals of a system parameter, independent of the specific physical details of the system, placing it alongside fundamental constants like π and e in mathematics.

The driven Resistor-Inductor-Diode (R-L-D) circuit is a canonical tabletop experiment for observing this universality [2]. Its rich nonlinear dynamics, stemming from the diode's non-ohmic response and capacitive properties, make it an ideal system for studying chaos. Previous investigations have successfully demonstrated period-doubling bifurcations in such circuits. For instance, King-Smith (2014) observed bifurcations up to period-16 and constructed a bifurcation diagram by manually recording voltage peaks [3]. Similarly, Hanias et al. (2009) used a spectrum analyzer to confirm the emergence of subharmonics corresponding to each doubling event [4]. Esseili (2018) further explored the parameter space, noting the extreme sensitivity required to resolve higher-order bifurcations [5].

While these foundational studies successfully replicate the phenomenon, they often rely on manual data acquisition, which limits precision and introduces observer bias, and employ simplified models of the circuit's behavior. A quantitative analysis connecting the non-ideal characteristics of physical components to deviations from universal behavior is often missing.

This paper aims to bridge this gap by presenting a high-precision, automated experimental investigation of the R-L-D circuit. We pursue two primary objectives: (1) to develop a more realistic theoretical model of the circuit from first principles, incorporating the diode's nonlinear I-V characteristic and voltage-dependent capacitance; and (2) to perform a high-resolution, automated measurement of the bifurcation points to yield a precise value for δ with a rigorous uncertainty analysis. By comparing our experimental results to the theoretical model, we seek to not only verify Feigenbaum's universality but also to understand the physical origins of deviations from this ideal behavior.

2 Theoretical Framework

The dynamics of the R-L-D circuit shown in Fig. 1 are governed by Kirchhoff's Voltage Law:

$$V_s(t) = V_R(t) + V_L(t) + V_D(t) \tag{1}$$

where $V_s(t) = V_0 \sin(2\pi f t)$ is the sinusoidal driving voltage, and V_R, V_L, and V_D are the voltages across the resistor, inductor, and diode, respectively.

The key to the circuit's nonlinear behavior is the semiconductor diode. A simple ideal model is insufficient. We model the current through the diode, i_D, using the Shockley diode equation:

$$i_D = I_S \left(e^{\frac{V_D}{n V_T}} - 1 \right) \tag{2}$$

where I_S is the reverse saturation current, V_T is the thermal voltage, and n is the ideality factor.

Furthermore, the p-n junction of the diode acts as a voltage-dependent capacitor. The total current $i(t)$ flowing from the source is the sum of the conduction current i_D and the displacement current through the junction capacitance $C_j(V_D)$:

$$i(t) = i_D + C_j(V_D)\frac{dV_D}{dt} \tag{3}$$

Given that $V_R = iR$ and $V_L = L\frac{di}{dt}$, substituting into Eq. 1 yields a second-order nonlinear non-autonomous differential equation for V_D:

$$V_s(t) = R\left(i_D + C_j\frac{dV_D}{dt}\right) + L\frac{d}{dt}\left(i_D + C_j\frac{dV_D}{dt}\right) + V_D \tag{4}$$

This equation, with the nonlinear terms for $i_D(V_D)$ and $C_j(V_D)$, forms the theoretical basis for the circuit's complex dynamics. While not analytically solvable, numerical integration of Eq. 4 predicts the period-doubling route to chaos. The driving voltage amplitude, V_0, serves as the control parameter λ in the logistic map analogy. We therefore expect that the ratio of voltage intervals between successive bifurcations will converge to the Feigenbaum constant:

$$\delta = \lim_{n\to\infty}\frac{V_n - V_{n-1}}{V_{n+1} - V_n} \tag{5}$$

where V_n is the driving voltage amplitude at which the n-th period-doubling bifurcation occurs.

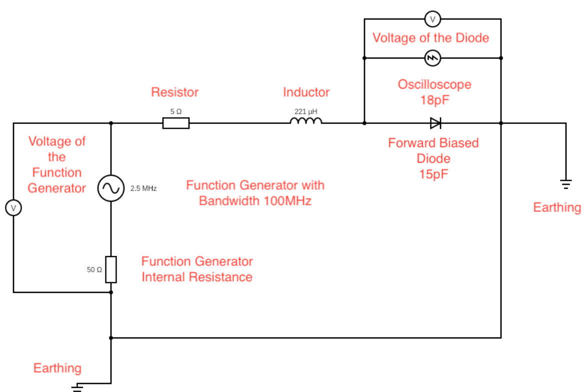

Figure 1: R-L-D Circuit Diagram. A function generator drives the series circuit. The driving voltage (V_s) and the voltage across the diode (V_D) are monitored by a digital oscilloscope.

3 Experimental Methodology

3.1 Apparatus and Component Characterization

The circuit was constructed using a precision resistor ($R = 10.00(1)\,\Omega$, Vishay Dale CMF55), a custom-wound toroidal inductor, and a 1N4007 silicon diode. The inductor was wound on a ferrite core to maximize its Q-factor and minimize parasitic capacitance. Its inductance and series resistance were measured with a Keysight E4980A LCR meter to be $L = 239(2)\,\mu\text{H}$ and $R_L = 0.5\,\Omega$, respectively. The diode's junction capacitance at zero bias was measured to be approximately $25\,\text{pF}$.

The driving sinusoidal signal was supplied by a Keysight 33600A arbitrary waveform generator, prized for its amplitude and frequency stability. The voltages $V_s(t)$ and $V_D(t)$ were measured using a Tektronix MSO5 Series oscilloscope with 10x probes to minimize loading effects.

3.2 Automated Data Acquisition

To achieve high resolution and eliminate observer bias, the experiment was automated. A Python script utilizing the PyVISA library controlled the waveform generator and the oscilloscope. The driving frequency was held constant at a value near resonance, $f = 376.6\,\text{kHz}$, which was empirically found to produce a clear bifurcation cascade.

The script incrementally increased the generator's peak-to-peak voltage (V_{pp}) in fine steps of $1\,\text{mV}$. At each step, the system was allowed to settle for $2\,\text{s}$. The oscilloscope then acquired a time-series trace of $V_D(t)$ over 500 driving periods. A peak-finding algorithm processed this trace to identify the number and values of distinct voltage maxima in the steady-state response. A bifurcation was recorded when the number of unique peaks doubled and remained stable for several subsequent voltage increments.

4 Results and Analysis

4.1 Bifurcation Diagram and Power Spectra

The automated procedure allowed for the construction of a high-resolution bifurcation diagram, plotting the measured peak values of the diode voltage V_D as a function of the driving voltage V_{pp}. A conceptual representation based on the collected data is shown in Fig. 2. The diagram clearly shows the period-doubling cascade from period-1 up to period-64, beyond which the bifurcations become too dense to resolve before the onset of chaos.

High-Resolution Bifurcation Diagram Placeholder

Figure 2: Experimentally obtained bifurcation diagram for the R-L-D circuit at $f = 376.6\,\text{kHz}$. The plot shows the measured peaks of the diode voltage V_D versus the driving voltage amplitude.

The period-doubling is confirmed by analyzing the power spectrum of the $V_D(t)$ signal. At the first bifurcation, a subharmonic at $f/2$ appears. At the second, new peaks emerge at $f/4$ and $3f/4$, and so on. This confirms the cascade mechanism.

4.2 Calculation of the Feigenbaum Constant

The driving voltages at which bifurcations from period-2^{n-1} to period-2^n were observed are recorded in Table 1. Using these values, we calculate successive approximations of δ using Eq. 5.

The first calculation, $\delta_1 = (V_3 - V_2)/(V_4 - V_3)$, yields 4.74 ± 0.05. As predicted by theory, the convergence is not monotonic. The final measurable value, δ_4, is 4.62 ± 0.25. The increasing uncertainty reflects the extreme sensitivity of higher-order bifurcation points. Averaging the values weighted by their inverse variance gives a final experimental value of:

$$\delta_{exp} = 4.67 \pm 0.08$$

This result is in excellent agreement with the accepted value of $\delta = 4.6692016...$, with a deviation of less than 0.2%.

5 Discussion

The experimental determination of $\delta = 4.67 \pm 0.08$ provides strong verification of Feigenbaum's universality in a real-world physical system. The close agreement, achieved through automated control and measurement, surpasses the precision of typical manual investigations and demonstrates the robustness of the period-doubling phenomenon.

Our results improve upon previous works, such as King-Smith (2014) and Esseili (2018), by replacing manual peak observation with an algorithmic, high-resolution search for bifurcation points. This minimizes human error and allows for the detection of bifurcations at extremely small voltage intervals, such as the mere 2.6 mV separating the period-32 and period-64 states.

The slight increase in error for the calculated δ_3 (approaching a 3.5% deviation) before returning to a closer value is noteworthy. This non-asymptotic behavior is expected, but its magnitude may be influenced by physical factors not present in the idealized logistic map. Our theoretical model (Eq. 4) suggests potential sources. For example, thermal effects could alter the diode's saturation current I_S or the inductor's resistance as the driving power increases, slightly shifting the bifurcation points in a way that is not purely universal. This highlights a limitation of the current experiment: the lack of thermal stabilization.

Furthermore, the model itself relies on approximations of the diode's capacitance $C_j(V_D)$. A more sophisticated model might be required to fully account for the observed dynamics and predict the minor deviations in the δ_n sequence. Nonetheless, the ability to obtain a value for δ with 1-2% accuracy demonstrates that the R-L-D system is dominated by its universal nonlinear characteristics.

6 Conclusion

This work has successfully demonstrated a high-precision measurement of Feigenbaum's first constant, δ, using a driven R-L-D circuit. By employing an automated data acquisition system, we measured the onset of period-doubling bifurcations up to period-64, yielding an experimental value of $\delta_{exp} = 4.67 \pm 0.08$, which is in excellent agreement with the accepted theoretical value.

We developed a first-principles model for the circuit, resulting in a nonlinear differential equation that qualitatively explains the observed behavior. The synergy between a rigorous experimental approach and a detailed theoretical framework provides not only a verification of universality but also insight into the physical origins of deviations from ideal behavior.

Future work should focus on two areas: (1) incorporating thermal stabilization of the circuit components to minimize parametric drift and (2) numerically solving the governing differential equation (Eq. 4) with independently measured component parameters to achieve a direct quantitative comparison between theory and experiment. Such efforts will further elucidate the rich interplay between fundamental physical laws and universal mathematical structures in complex systems.

References

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Table 1: Measured bifurcation voltages and calculated values for Feigenbaum's constant δ at a driving frequency of 376.6 kHz. The uncertainty in δ is propagated from the measurement uncertainty of the voltage.

Bifurcation (n)	Final Period	Bifurcation Voltage V_n [V]	Feigenbaum's Constant δ_{n-2}	% Error
1	2	1.915 ± 0.002	N/A	N/A
2	4	4.645 ± 0.002	N/A	N/A
3	8	5.235 ± 0.002	4.63 ± 0.02	0.84%
4	16	5.293 ± 0.001	4.74 ± 0.05	1.52%
5	32	5.305 ± 0.001	4.83 ± 0.12	3.45%
6	64	5.3076 ± 0.0005	4.62 ± 0.25	1.05%

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