Millersville University

Generalizations of the Brachistochrone Problem

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By John A. Gemmer

Millersville Pennsylvania
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This Senior Thesis was completed in the Departments of Mathematics and Physics, defended before and approved by the following members of the thesis committee

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Michael Nolan, Ph.D (Thesis Advisor)

Professor of Physics

Millersville University

Ronald Umble, Ph.D (Thesis Advisor)

Professor of Mathematics

Millersville University

Scott Walck, Ph.D

Associate Professor of Physics

Lebanon Valley College

Abstract

Consider a frictionless surface S in a gravitational field that need not be uniform. Given two points A and B on S, what curve is traced out by a particle that starts at A and reaches B in the shortest time? This paper discusses this problem for simple surfaces such as surfaces of revolution. We first solve this more general problem using the Euler-Lagrange equation and conservation of mechanical energy. We then use geometrical optics to give an alternative method for solving the problem. Finally, we also consider particles that are falling with relativistic velocities.

Generalizations of the Brachistochrone Problem

Contents

1 Introduction

The original Brachistochrone problem, posed in 1696, was stated as follows: Find the shape of the curve down which a bead sliding from rest and accelerated by gravity will fall from one point to another in the least time. In the original problem it was assumed that the particle is falling on a vertical plane lying in a uniform gravitational field. Newton showed that the solution is a cycloid, the curve traced out by a point on the rim of a rolling circle.

This paper extends this idea and considers the following problem: *Consider a frictionless surface S in gravitational field that need not be uniform. Given two points A and B on S, what curve is traced out by a particle that starts at A and reaches B in the shortest time.* To solve this problem we use conservation of mechanical energy and the Euler-Lagrange equation. Unfortunately, for many surfaces this differential equation is nonlinear. To simplify our problem, we only consider surfaces and gravitational fields that reduce the problem to a separable differential equation.

The second section of this paper reviews a solution of the classical Brachistochrone problem found in most introductory Calculus of Variations texts. In section 3 we consider the more general problem in terms of generalized coordinates on a surface. We first develop the notions of orthogonal coordinate patches and show how they can be used to find the metric coefficients on a surface. We then show how these metric coefficients can be used to develop a general solution to the Brachistochrone problem for a large class of surfaces and gravitational fields. Next, we illustrate some applications of this general solution to surfaces of revolution and inverse-square fields. We conclude with a detailed explanation of some interesting properties that arise in inverse fields.

In section 4 we explore the relationship between geometrical optics and Brachistochrone solutions. Using the Eikonal equation, we give an alternative method for finding solution curves. We also use the Eikonal equation to find solution curves for particles falling with relativistic velocities. Finally, using the light ray curvature equation we discuss the curvature and torsion of Brachistochrone solutions and find that torsion always vanishes for inverse-square fields.

2 The Classical Brachistochrone Problem

In 1696 Johann Bernoulli challenged the European mathematical world to solve the Brachistochrone problem: *Given two points A and B in a vertical plane, find the curve connecting A and B along which a point acted on only by gravity starts at A and reaches B in the shortest time.* In May 1697, Newton, Leibniz, Johann Bernoulli, and Jakob Bernoulli presented four solutions in Acta Eruditorum, (see [1]). In this section we will review a solution similar to Newton's, but using some modern techniques from Calculus of Variations, (see [5]).

First, we need to relate how far and how long the particle falls. To do this, let V be the potential energy of the particle and apply classical conservation of mechanical energy:

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 + V = V(A),$$

where m is the mass of the particle and ds is the element of arclength of the curve. Now, near the surface of the earth the potential energy of the particle is given by $V(y) = mgy$, where g is the acceleration due to gravity. Also, since this problem is confined to the xy-plane, $ds = \sqrt{1+(x')^2}dy$, where $'$ denotes differentiation with respect to y. Let a and b be the respective y coordinates of A and B and obtain the differential equation:

$$\frac{1}{2}m\left(1+(x')^2\right)\left(\frac{dy}{dt}\right)^2 + mgy = mga.$$

Solving this equation for T, the time for the particle to fall from A to B is

$$\frac{1}{2}\left(1+(x')^2\right)\left(\frac{dy}{dt}\right)^2 = g(a-y)$$

$$\frac{2g(a-y)}{1+(x')^2} = \left(\frac{dy}{dt}\right)^2$$

$$\pm\sqrt{\frac{1+(x')^2}{2g(a-y)}}dy = dt$$

$$T = \pm\frac{1}{\sqrt{2g}}\int_a^b \frac{\sqrt{1+(x')^2}}{\sqrt{a-y}}dy$$

$$T = \pm\frac{1}{\sqrt{2g}}\int_a^b \mathcal{F}[y,x']dy$$

where $\mathcal{F} = \sqrt{\frac{1+(x')^2}{a-y}}$. To minimize T, we must solve the Euler-Lagrange equation

$$\frac{\partial \mathcal{F}}{\partial x} - \frac{d}{dy}\frac{\partial \mathcal{F}}{\partial x'} = 0. \tag{1}$$

Assuming independence of x, x' and y, we have $\frac{\partial \mathcal{F}}{\partial x} = 0$. The fact that one can make this assumption follows by a standard argument (see [2]). Therefore, equation (1) simplifies to $\frac{\partial \mathcal{F}}{\partial x'} = c$, where c is a constant of integration chosen so that the solution curve passes through B. Differentiating $\mathcal{F}$ with respect to x', we obtain the separable differential equation

$$\pm\frac{1}{\sqrt{A-y}}\frac{x'}{\sqrt{1+x'^2}} = c,$$

whose solution is

$$\begin{aligned}
(x')^2 &= (a-y)(1+(x')^2)c^2 \\
(x')^2 &= (a-y)c^2 + (a-y)(x')^2c^2 \\
(x')^2(1-(a-y)c^2) &= (a-y)c^2 \\
\frac{dx}{dy} &= \pm\sqrt{\frac{(a-y)c^2}{1-(a-y)c^2}} \\
x &= \pm\int \frac{\sqrt{(a-y)c^2}}{\sqrt{1-(a-y)c^2}}\,dy. \qquad (2)
\end{aligned}$$

To simplify this integral set $c' = \frac{1}{c}$ and make the substitution $u = a - y$. Then

$$x = \pm\int \frac{\sqrt{u}}{\sqrt{(c')^2 - u}}\,du,$$

which we evaluate by making the substitution $w = \pm\sqrt{(c')^2 - u}$ to obtain

$$x = -2\left(\pm\int \sqrt{(c')^2 - w^2}\,dw\right).$$

Then, with $w = c'\sin\theta$ we have

$$x = -2(\pm\int \sqrt{(c')^2 - w^2}\,dw) = -2(c')^2\int \cos^2\theta\,d\theta,$$

and consequently,

$$x = -\frac{(c')^2}{2}(2\theta + \sin 2\theta) + r,$$

where r is another constant. The substitutions above imply that $a - y = u = (c')^2 \cos^2\theta$ or equivalently

$$y = a - (c')^2 \cos^2\theta = a - \frac{(c')^2}{2}(1 + \cos\frac{\theta}{2}).$$

We can now express the solution curves to the classical Brachistochrone problem in the following parametric form:

$$\begin{aligned} x &= -\frac{(c')r}{2}(2\theta + \sin 2\theta) + r \\ y &= a - \frac{(c')^2}{2}\left(1 + \cos\frac{\theta}{2}\right). \end{aligned}$$

Remarkably, this is the parametrization of a cycloid, the curve traced out by the rim of a rolling circle. Figure 1 displays a particle with coordinates (x, y) falling along the cycloid.

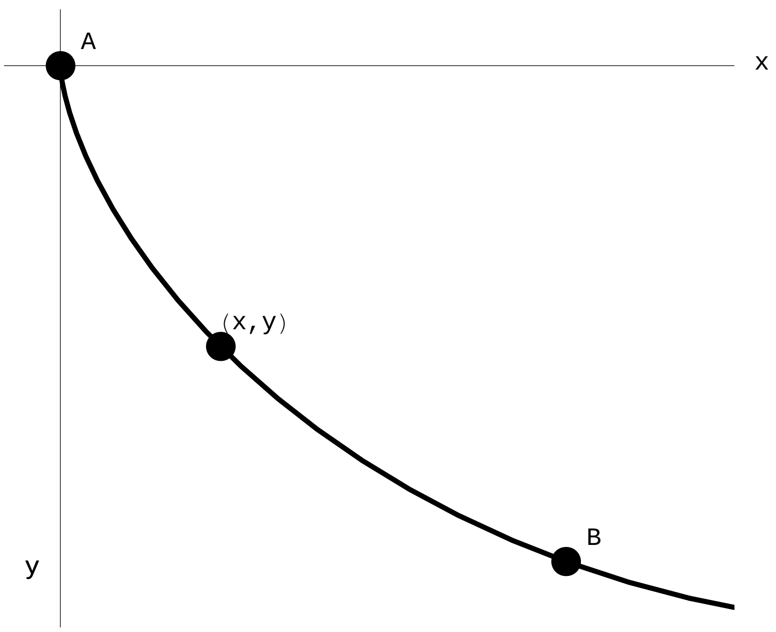

Figure 1: Particle falling on a solution curve

3 General non-relativistic solutions

3.1 Generalized coordinates

To find Brachistochrone solutions on surfaces, we need to describe the coordinates of a particle on the surface. Since surfaces are 2-dimensional objects, it is advantageous to describe them using 2 coordinates. To do this, first let U be some open set in $\mathbb{R}^2$ and recall the following definitions from the classical theory of surfaces, (see [5]).

Definition 1. *A parametrization is a differentiable mapping* $\mathbf{x} : U \to \mathbb{R}^3$.

Let $\mathbf{x} : U \to S$ be a parametrization of some surface S. Then S is the image of the function

$$\mathbf{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v)).$$

The function $\mathbf{x}$ is commonly called a *coordinate chart* on S.

Definition 2. *A parametrization* $\mathbf{x} : U \to S$ *is regular if* $\mathbf{x}_u \times \mathbf{x}_v \neq 0$.

Definition 3. *A coordinate patch is an injective regular mapping* $\mathbf{x} : U \to S$.

Now, if we fix $u = u_0$ and let v vary, then the image of $\mathbf{x}(u_0, v)$ is a curve on S. Likewise, if we fix $v = v_0$ and allow u to vary, the image of $\mathbf{x}(u, v_0)$ is another curve on S. These curves are called u and v *parameter curves* or *coordinate curves* (see Figure 2).

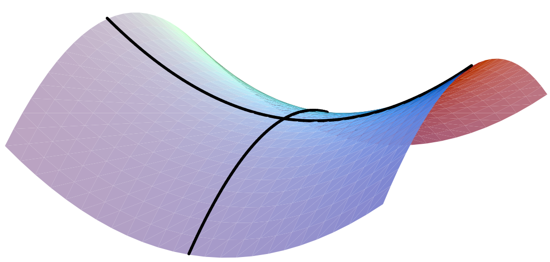

Figure 2: Parameter curves on the saddle surface

The partial derivatives

$$\begin{aligned}\mathbf{x}_u &= \frac{\partial \mathbf{x}}{\partial u}(u_0, v_0),\\ \mathbf{x}_v &= \frac{\partial \mathbf{x}}{\partial v}(u_0, v_0)\end{aligned}$$

are tangent vectors at the point of intersection. Therefore, coordinates on S must satisfy two properties:

1. $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent.

2. $\mathbf{x}$ is injective ($\mathbf{x}(U)$ has no self-intersections).

It follows that a coordinate patch allows us to impose local coordinates on a surface, for a more detailed discussion see [5].

Unfortunately, knowing the coordinates of a particle on S is not enough to solve the Brachistochrone problem on S. We also need to compute the arclength of curves on S. Let $\alpha : (-\epsilon, \epsilon) \to S$ be a curve on S and let $a, t \in (-\epsilon, \epsilon)$. Then, the length of the curve between $\alpha(a)$ and $\alpha(t)$ is given by

$$s(t) = \int_a^t \parallel \alpha'(r) \parallel dr = \int_a^t \sqrt{\alpha'(r) \bullet \alpha'(r)} dr.$$

Let $\mathbf{x} : U \to S$ be a coordinate chart whose image contains the curve α. Then $\alpha(t) = \mathbf{x}\left(u(t), v(t)\right)$, and its arclength $s(t)$ can be expressed in local coordinates as

$$s(t) = \int_a^t \sqrt{\left(\mathbf{x}_u \frac{du}{dr} + \mathbf{x}_v \frac{dv}{dr}\right) \bullet \left(\mathbf{x}_u \frac{du}{dr} + \mathbf{x}_v \frac{dv}{dr}\right)}\, dr. \tag{3}$$

Let

$$E(u,v) = \mathbf{x}_u \bullet \mathbf{x}_u, \;\; F(u,v) = \mathbf{x}_u \bullet \mathbf{x}_v, \;\; G(u,v) = \mathbf{x}_v \bullet \mathbf{x}_v.$$

Expanding the inner product in (3) we have

$$s(t) = \int_a^t \sqrt{E(u(r), v(r)) \left(\frac{du}{dr}\right)^2 + 2F(u(r), v(r)) \frac{du}{dr}\frac{dv}{dr} + G(u(r), v(r)) \left(\frac{dv}{dr}\right)^2}\, dr,$$

which when in expressed in differential form is

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where ds is the element of arclength on S. The coefficients E, F, and G are called the *component functions of the metric on* S or simply the *metric coefficients.*

For example, the xz-plane can be parametrized by the coordinate chart $\mathbf{x}(u,v) = (u,0,v)$. Therefore, it is trivial to show that $E = G = 1$ and $F = 0$. Thus, the element of arclength is $ds^2 = du^2 + dv^2$, which is the element of arc length used to solve the classical Brachistochrone problem.

A more interesting surface is a surface of revolution about the z-axis. Such surfaces can be parametrized by the coordinate patch

$$\mathbf{x}(u,v) = (h(u)\cos v, h(u)\sin v, g(u)),$$

$0 < v < 2\pi$, $u \in \mathbb{R}$, in which case $E = h'(u)^2 + g'(u)^2$, $F = 0$ and $G = h^2$, and the element of arc length is

$$ds^2 = \left(h'(u)^2 + g'(u)^2\right) du^2 + h^2 dv^2. \tag{4}$$

Finally, on a surface S, the angle θ between the tangent vectors of the coordinate curves is given by

$$\theta = \arccos\left(\frac{\mathbf{x}_u \bullet \mathbf{x}_v}{\sqrt{(\mathbf{x}_u \bullet \mathbf{x}_u)(\mathbf{x}_v \bullet \mathbf{x}_v)}}\right) = \arccos\frac{F}{\sqrt{EG}}.$$

Therefore $\mathbf{x}_u$ and $\mathbf{x}_v$ are orthogonal if and only if $F = 0$. This leads to the following useful definition.

Definition 4. *A coordinate patch $\mathbf{x} : U \to S$ is orthogonal if and only if F is identically zero on U.*

Note that the coordinate patches given above for the vertical plane and the surface of revolution are orthogonal. Furthermore, we have the nice result that if a coordinate chart $\mathbf{x} : U \to S$ is orthogonal, then $ds^2 = Edu^2 + Gdv^2$.

3.2 General non-relativistic theorem

In this section we develop a solution to the problem for a large class of surfaces and gravitational fields. Throughout this section, S denotes a surface on which some particle falls.

Furthermore, m denotes the particle's mass and V the potential energy of the particle at a point.

Choose two distinct points A and B such that $V(A) > V(B)$ and assume that the velocity of the particle is much less than the speed of light. Then classical conservation of mechanical energy

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 + V = V(A)$$

holds, where ds is the arclength on S. Solving this equation for $\left(\frac{ds}{dt}\right)^2$ gives:

$$\frac{2(V(A)-V)}{m} = \left(\frac{ds}{dt}\right)^2.$$

Separating variables yields:

$$\pm\sqrt{\frac{m}{2(V(A)-V)}}\,ds = dt.$$

Therefore, the total time T is given by:

$$T = \pm\sqrt{\frac{m}{2}}\int_A^B \sqrt{\frac{1}{V(A)-V}}\,ds. \tag{5}$$

Now, let $\mathbf{x} : U \to S$ be an orthogonal coordinate patch. Then, the element of arc length on S is given by $ds^2 = Edu^2 + Gdv^2$ and we may rewrite equation (5) in the form

$$T = \pm\sqrt{\frac{m}{2}}\int_A^B \sqrt{\frac{1}{V(A)-V}}\,\sqrt{Edu^2 + Gdv^2}. \tag{6}$$

Furthermore we may rewrite this integral as

$$T = \pm\sqrt{\frac{m}{2}}\int_A^B \sqrt{\frac{E\left(\frac{du}{dv}\right)^2 + G}{V(A)-V}}\,dv = \pm\sqrt{\frac{m}{2}}\int_A^B \mathcal{F}(u,v,u')dv, \tag{7}$$

$$T = \pm\sqrt{\frac{m}{2}}\int_A^B \sqrt{\frac{E + G\left(\frac{dv}{du}\right)^2}{V(A)-V}}\,du = \pm\sqrt{\frac{m}{2}}\int_A^B \mathcal{F}(u,v,v')du, \tag{8}$$

where $'$ denotes differentiation with respect to the other variable. So, to minimize T in (7), we again need to solve the Euler-Lagrange equation

$$\frac{d}{dv}\frac{\partial\mathcal{F}}{\partial u'} - \frac{\partial\mathcal{F}}{\partial u} = 0. \tag{9}$$

Now, if we suppose that $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$, then $\frac{\partial \mathcal{F}}{\partial u} = 0$ and equation (9) simplifies to

$$\frac{\partial \mathcal{F}}{\partial u'} = C,$$

where C is a constant that depends upon B. Differentiating $\mathcal{F}$ with respect to u', we obtain the separable differential equation

$$\pm \frac{Eu'}{\sqrt{(V(A) - V)(E(u')^2 + G)}} = C.$$

Squaring both sides and simplifying gives

$$\begin{aligned}
E^2(u')^2 &= C^2(V(A) - V)(E(u')^2 + G) \\
E^2(u')^2 &= C^2(V(A) - V)E(u')^2 + C^2(V(A) - V)G \\
(u')^2 E(E - C^2(V(A) - V)) &= C^2(V(A) - V)G \\
\frac{du}{dv} &= \pm\sqrt{\frac{C^2(V(A) - V)G}{E(E - C^2(V(A) - V))}}.
\end{aligned}$$

Consequently, we can express u as a function of v:

$$u = \pm \int_A^v \sqrt{\frac{C^2G(w)(V(A) - V(w))}{E(w)[E(w) - C^2(V(A) - (V(w)]}}\, dw. \tag{10}$$

Similarly, if $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$ we can use equations (8) and a modified form of (9) to express v as a function of u by

$$v = \pm \int_A^u \sqrt{\frac{C^2E(w)(V(A) - V(w))}{G(w)[G(w) - C^2(V(A) - (V(w)]}}\, dw. \tag{11}$$

We summarize these results in the following theorem:

Theorem 1. *Let $\mathbf{x} : U \to S$ be an orthogonal coordinate patch on a frictionless surface S.*

1. *If $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$, then the solution to the Brachistochrone problem on S is given by the curve $\mathbf{x}(u(v), v)$, where*

$$u(v) = \pm \int_A^v \sqrt{\frac{C^2G(w)(V(A) - V(w))}{E(w)[E(w) - C^2\,(V(A) - V(w))]}}\, dw.$$

2. *If $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$, then the solution to the Brachistochrone problem on S is given by the curve $\mathbf{x}(u, v(u))$, where*

$$v(u) = \pm \int_A^u \sqrt{\frac{C^2E(w)(V(A) - V(w))}{G(w)[G(w) - C^2\,(V(A) - V(w))]}}\, dw.$$

3.3 Applications of Theorem 1

3.3.1 Classical Brachistochrone Problem

Let us apply Theorem 1 to the classical Brachistochrone Problem. In this case, the particle is falling on the vertical plane given by the coordinate patch $\mathbf{x}(u, v) = (u, 0, v)$ with metric coefficients $E = G = 1$ and $F = 0$. Applying Theorem 1 with $V(v) = v$ we get a solution curve of the form,

$$u(v) = \pm \int_a^v \sqrt{\frac{C^2(a-v)}{1 - C^2(a-v)}}\, dw,$$

where a is the particle's initial v-coordinate. Comparing this solution with equation (2), we see that we recover the classical solution.

Figure 3 illustrates several solution curves for a particle starting at the origin. Each of the curves is uniquely determined by C and the choice of the sign in front of the radical. Physically, the $\pm$ sign determines if the particle moves to left or right and the value of C is chosen so that the curve passes through some final point. These curves in some sense represent a family of solution curves since one can terminate anywhere along the curve.

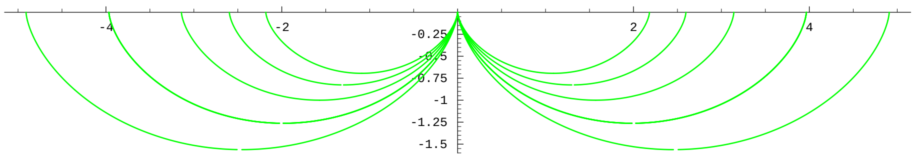

Figure 3: Cycloid solutions in a uniform field

3.3.2 Surfaces of Revolution

Another application of Theorem 1 is to surfaces of revolution in uniform gravitational fields in which the axis of revolution is parallel to the direction of the field. If we assume that the axis of revolution is the z-axis, the metric coefficients and the potential energy are given by equation (4) and $V = g(u)$ respectively. Therefore, applying the theorem we have the following corollary:

Corollary 1. *Let S be a surface of revolution about the z-axis defined by the parameterization $\mathbf{x}(u,v) = (h(u)\cos v, h(u)\sin v, g(u))$. If S is in a uniform gravitational field parallel to the z-axis then the solution to the Brachistochrone Problem on S is given by the curve $\mathbf{x}(u, v(u))$ where,*

$$v(u) = \pm \int_{u_0}^{u} \sqrt{\frac{C^2(h'(w)^2 + g'(w)^2)(g(u_0) - g(w))}{h(w)^2((h(w)^2 - C^2(g(u_0) - g(w)))}}\, dw$$

and $A = \mathbf{x}(u_0, v_0)$ is the initial position of the particle.

For example, consider the right circular cone parameterized by $\mathbf{x}(u,v) = (u\cos v, u\sin v, u)$, $u \geq 0$, $0 < v < 2\pi$. By Corollary 1, the solution curves on the cone are given are given by

$$\mathbf{x}\left(u, \pm \int_{u_0}^{u} \sqrt{\frac{2C^2(u_0 - u)}{u^2(u^2 - C^2(u_0 - u))}}\, dw\right).$$

Figure 4 illustrates several solution curves for a particle starting at the point $A = \mathbf{x}(1,0)$. In this case, the $\pm$ sign determines on which side of the cone the particle falls.

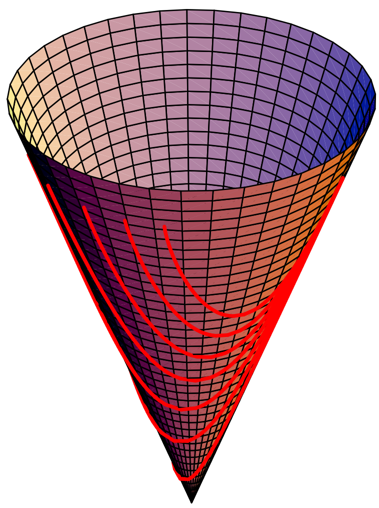

Figure 4: Solution curves on the cone

Another interesting surface of revolution is the hyperboloid of one sheet. Using the coordinate chart $\mathbf{x}(u,v) = (\cosh u \cos v, \cosh u \sin v, \sinh u), u \in \mathbb{R}, 0 < v < 2\pi$, the solution curves on the surface given by Corollary 1 are

$$\mathbf{x}\left(u, \pm \int_A^u \sqrt{\frac{C^2(\sinh u_0 - \sinh u)((\cosh u)^2 + (\sinh u)^2))}{(\cosh u)^2((\cosh u)^2 - C^2(\sinh u_0 - \sinh u))}}\, dw\right).$$

Unlike solutions on the cone, some of these solutions do not obtain a minimum value but continue downward, spiraling around the hyperboloid. These spiraling curves intersect other solution curves on the surface. When two spiraling solution curves intersect, we need to explicitly carry out the integration in (6) to determine which curve actually minimizes time.

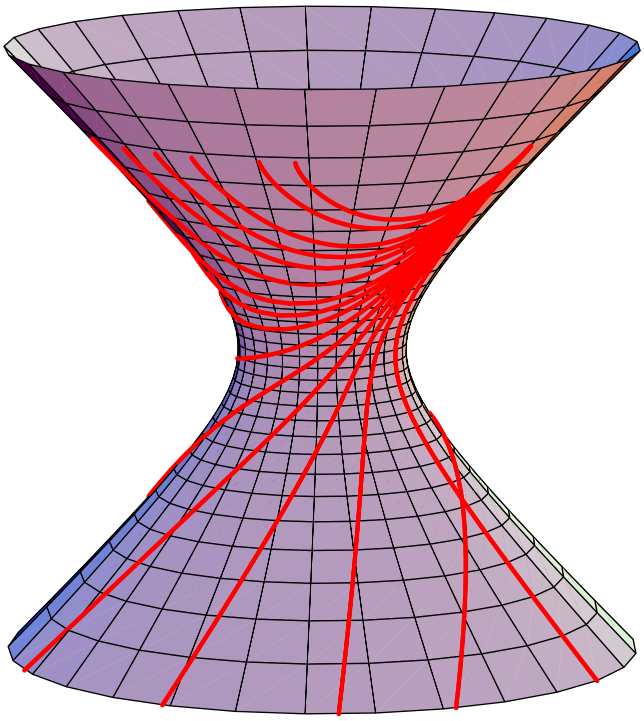

Figure 5: Solution curves on the hyperboloid of one-sheet

3.3.3 Inverse-Square Fields

Consider a particle confined to a plane and falling in an inverse-square field. Using polar coordinates, we can parametrize $\mathbb{R}^2$ by $\mathbf{x}(u, v) = (u \cos v, u \sin v)$. Consequently, the metric coefficients are simply $E = 1$, $F = 0$ and $G = u^2$. Now, letting $V = -\frac{1}{u}, \frac{1}{C^2} = D$ and applying Theorem 1 we find that the solution curves are given by $\mathbf{x}(u, v(u))$ where,

$$\begin{aligned} v(u) &= \pm \int_{u_0}^{u} \sqrt{\frac{C^2(-\frac{1}{u_0}+\frac{1}{w})}{w^2[w^2 - C^2(-\frac{1}{u_0}+\frac{1}{w})]}} dw \\ &= \pm \int_{u_0}^{u} \sqrt{\frac{(u_0 - w)}{w^2(w^3 u_0 D - (u_0 - w))}} dw \end{aligned} \tag{12}$$

and $A = (u_0, 0)$ is the initial position of the particle.

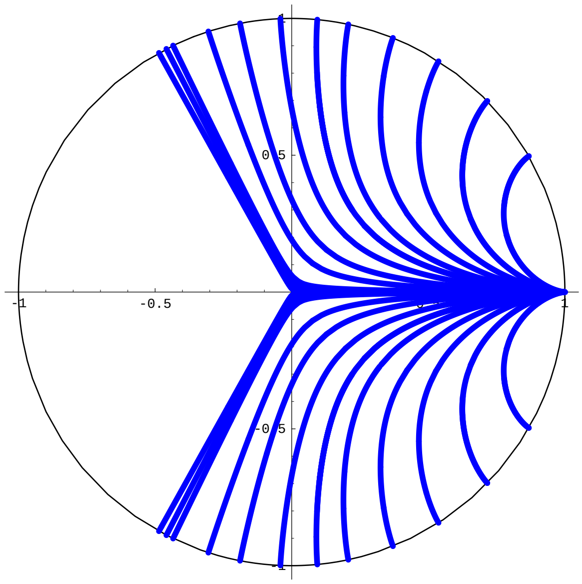

Figure 6: Solution curves for the inverse square field

Interestingly, when numerically plotting solution curves in the region $-\pi < v < 0$ we notice that as $C \to 0$ the curves asymptotically approach a line at an angle of $-\frac{2\pi}{3}$ from the horizontal. Therefore, since solution curves in the region $0 < v < \pi$ are mirror images of the solution curves in the region $-\pi < v < 0$, we have numerical evidence that no solution curves exist in the sector $\frac{2\pi}{3} < \theta < \frac{5\pi}{3}$.

More specifically, note that the solution curves approach two limit curves that follow the x-axis to the origin then continue along the rays at 120° or 240°. Thus a time minimizing curve from (1,0) to a point P within the "forbidden" 120° sector necessarily passes through the singularity at the origin. Since every path from (1,0) to P misses the origin, any such path can be perturbed to one that passes "closer" to the origin and decreases time.

To better understand this phenomenon, let us take a careful look at how these curves are plotted. First, note that when

$$\frac{du}{dv} = 0 \tag{13}$$

the particle will begin to move away from the origin and the sign in (12) is negative. To show that there exists only one positive real valued point where (13) holds we will investigate the roots of

$$r(w) = w^3 D + w - 1. \tag{14}$$

Since $r(0) = -1$ and $r(1) = D$, by the Intermediate Value Theorem there exists some u_f such that $r(u_f) = 0$. Furthermore, we have that

$$r'(w) = 3w^2 + 1.$$

Since $r'(w) > 0$ for all $0 < w < D$ we conclude that $r(w)$ is monotonically increasing and must contain only one real root.

Since solution curves are symmetric about a line passing through the origin and the point $\mathbf{x}(u_f, \theta)$, where

$$\theta = \int_{u_0}^{u_f} \sqrt{\frac{(u_0 - w)}{w^2(w^3 u_0 D - (u_0 - w))}}\, dw, \tag{15}$$

we need to find the value of θ as $u_f \to 0$ to determine the maximum angle. Using equation (14) we can express D in terms of u_f by

$$D = u_f^{-3} + u_f^{-2}.$$

Now, without loss of generality assume that $A = \mathbf{x}(1, 0)$ then (15) can be rewritten as

$$\theta = \int_{1}^{u_f} \sqrt{\frac{(u_0 - w)}{w^2(w^3 u_0 (u_f^{-3} + u_f^{-2}) - (u_0 - w))}}\, dw. \tag{16}$$

Factoring out the term $(w - u_f)$ in the denominator we have that

$$\begin{aligned} \theta &= (u_0)^{\frac{3}{2}} \int_{1}^{u_f} \frac{1}{w} \sqrt{\frac{(1 - w)}{(w - u_f)(w^2 - u_f w(w-1) - u_f^2(w-1))}}\, dw \\ &= (u_0)^{\frac{3}{2}} \int_{1}^{u_f} \frac{1}{w} \sqrt{\frac{(1 - w)}{(w - u_f)(w^2(1 - u_f) + w u_f (1 - u_f) + u_f^2)}}\, dw. \end{aligned} \tag{17}$$

Let $x = (\frac{u_f}{w})^{\frac{2}{3}}$; then (17) reduces to

$$\theta = \frac{2}{3}\int_1^{(u_f)^{\frac{2}{3}}} \sqrt{\frac{(1 - u_f x^{-\frac{2}{3}})}{1 - u_f - x^2 + u_f x^{\frac{4}{3}}}}\, dx.$$

Finally, taking the limit as $u_f \to 0$ we have that the maximum angle is given by

$$\theta = \frac{2}{3}\int_1^0 \frac{1}{\sqrt{1 - x^2}}\, dx = \frac{2}{3}\left(\arcsin(0) - \arcsin(1)\right) = -\frac{\pi}{3}.$$

But, since the limiting angle obtained by the particle is twice this value, we have proved that the limiting angle of a particle falling on a solution curve is $\frac{2\pi}{3}$.

4 Geometrical Optics and the Brachistochrone Problem

In this section we develop an alternative way of looking at the Brachistochrone problem. Instead of thinking about a particle falling in a gravitational field, we investigate the related problem of finding the path of a light ray in a medium of nonuniform index of refraction $n(u, v)$, where u and v are the generalized coordinates on a surface S.

According to Fermat's Principle, light propagates in a medium in such a way as to minimize the total time of travel between two points A and B. Therefore, the time taken by the light to travel along a curve connecting A to B is

$$T = \frac{1}{c}\int_A^B n(u, v)\, ds, \tag{18}$$

where c is the speed of light in a vacuum, (see [6]). Now, if $n(u, v) = \sqrt{\frac{1}{V(A) - V(u,v)}}$ and $c = 1\, m/s$, then equation (18) is equal to equation (5). Therefore, finding the path of a light ray in this medium simultaneously produces a Brachistochrone solution.

4.1 The Eikonal Equation

One can describe light rays in terms of their corresponding wavefronts which are level surfaces of a function $L : \mathbb{R}^3 \to \mathbb{R}$. Given a light ray α with unit tangent vector $\mathbf{T}$ passing through a medium with index of refraction n, L is defined by the Eikonal equation:

$$\nabla L = n\mathbf{T}, \tag{19}$$

(see [6]). In general, L is completely determined by a light source and n. For example, if the light source is an infinite plane and n is constant then in Cartesian Coordinates $L = ax + by + cz$, where a, b, and c are chosen so that the propagating wavefronts are parallel to the source.

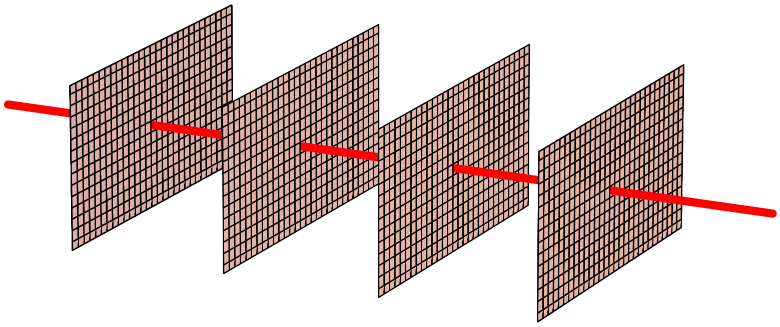

Figure 7: Wavefronts in a uniform medium

Using the Eikonal equation we can solve the Brachistochrone Problem without appealing to the Euler-Lagrange equation. Let $ds^2 = Edu^2 + Gdv^2$ be the metric for some surface S and $n(u, v)$ be the index of refraction. Suppose that $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$, then if

$$L = Cv + f(u) \tag{20}$$

equation (19) yields the differential equation

$$\nabla L \bullet \nabla L = G^{-1}C^2 + E^{-1}\left(\frac{df(u)}{du}\right)^2 = n^2.$$

Solving this separable equation for $f(u)$ gives:

$$\begin{aligned} f(u) &= \pm \int \sqrt{E\left(n^2 - \frac{C^2}{G}\right)}\, du \\ &= \pm \int \sqrt{\frac{E\left(G - \frac{C^2}{n^2}\right)}{\frac{G}{n^2}}}\, du. \end{aligned} \tag{21}$$

Using equations (19) and (20), and letting $\mathbf{e}_u = \frac{x_u}{\|x_u\|}$ and $\mathbf{e}_v = \frac{x_v}{\|x_v\|}$ we have that

$$n\mathbf{T} = E^{-\frac{1}{2}}\sqrt{\frac{E(G-\frac{C^2}{n^2})}{\frac{G}{n^2}}}\mathbf{e}_u + CG^{-\frac{1}{2}}\mathbf{e}_v.$$

Now, expressing the tangent vector in differential form gives

$$n\left(E^{\frac{1}{2}}\frac{du}{ds}\mathbf{e}_u + G^{\frac{1}{2}}\frac{dv}{ds}\mathbf{e}_v\right) = E^{\frac{1}{2}}\sqrt{\frac{E(G-\frac{C^2}{n^2})}{\frac{G}{n^2}}}\mathbf{e}_u + CG^{-\frac{1}{2}}\mathbf{e}_v.$$

Therefore,

$$\frac{dv}{du}\frac{G^{\frac{1}{2}}}{E^{\frac{1}{2}}} = \sqrt{\frac{\frac{C^2}{n^2}}{G-\frac{C^2}{n^2}}}.$$

Solving this equation gives

$$\frac{dv}{du} = \sqrt{\frac{E\frac{C^2}{n^2}}{G(G-\frac{C^2}{n^2})}}. \tag{22}$$

Similarly, if $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$ we have that

$$\frac{du}{dv} = \sqrt{\frac{G\frac{C^2}{n^2}}{E(E-\frac{C^2}{n^2})}}. \tag{23}$$

We can then find the light rays confined to a surface S with index of refraction n by the following theorem:

Theorem 2. *Let $\mathbf{x} : U \to S$ be an orthogonal coordinate patch of a frictionless surface S.*

1. *If $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$, the light rays on S are given by $\mathbf{x}(u(v), v)$, where*

$$u(v) = \pm\int_{v_0}^{v}\sqrt{\frac{G\frac{C^2}{n^2}}{E(E-\frac{C^2}{n^2})}}\,dw$$

2. *If $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$, the light rays on S are given by $\mathbf{x}(u, v(u))$, where*

$$v(u) = \pm\int_{u_0}^{u}\sqrt{\frac{E\frac{C^2}{n^2}}{G(G-\frac{C^2}{n^2})}}\,dw$$

Corollary 2. *Let $\mathbf{x} : U \to S$ be an orthogonal coordinate patch of a frictionless surface S with index of refraction $n = \sqrt{\frac{1}{V_0-V}}$.*

1. If $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$*, we obtain result 1 in Theorem 1.*

2. If $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$*, we obtain result 2 in Theorem 1.*

We can also use equations (20) and (21), and similar equations for the case where $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$ to obtain the wavefronts themselves. This is summarized in the following theorem.

Theorem 3. *Let* $\mathbf{x} : U \to S$ *be an orthogonal coordinate patch of a frictionless surface* S*.*

1. If $\frac{\partial E}{\partial u} = \frac{\partial G}{\partial u} = \frac{\partial V}{\partial u} = 0$*, the wavefronts on* S *are given by* $\mathbf{x}(u(v), v)$*, where*

$$u(v) = \frac{D}{C} \pm \frac{1}{C} \int_{v_0}^{v} \sqrt{\frac{G(E - \frac{C^2}{n^2})}{\frac{E}{n^2}}}\, dw$$

and D *is an arbitrary constant.*

2. If $\frac{\partial E}{\partial v} = \frac{\partial G}{\partial v} = \frac{\partial V}{\partial v} = 0$*, the wavefronts on* S *are given by* $\mathbf{x}(u, v(u))$*, where*

$$v(u) = \frac{D}{C} \pm \frac{1}{C} \int_{u_0}^{u} \sqrt{\frac{E(G - \frac{C^2}{n^2})}{\frac{G}{n^2}}}\, du$$

and D *is an arbitrary constant.*

For example, again consider a particle confined to a plane and falling in an inverse square-field. Applying Theorem 3 we obtain the family of wavefronts $\mathbf{x}(u, v(u))$ where,

$$v(u) = \frac{D}{C} \pm \frac{1}{C} \int_{1}^{u} \sqrt{\frac{u(w)^3 - C^2(1 - u(w))}{u(w)^2(1 - u(w))}}\, dw.$$

Figure (8) illustrates two plots of a Brachistochrone solution with its wavefronts corresponding to the values $D = 1, 2, \ldots, 10$. The two plots differ by the choice of sign in the expression above; the plot on the left is given by the positive sign while the plot on the right is given by the negative sign. From equation (19) it follows that at intersection points, wavefronts are perpendicular to the Brachistochrone solution when the signs in front of the radicals in Theorem 2 and Theorem 3 are identical. Also, the wavefronts do not penetrate a circle of radius u_f, where u_f is the critical radius discussed in Section 3.3.3. Physically, when the

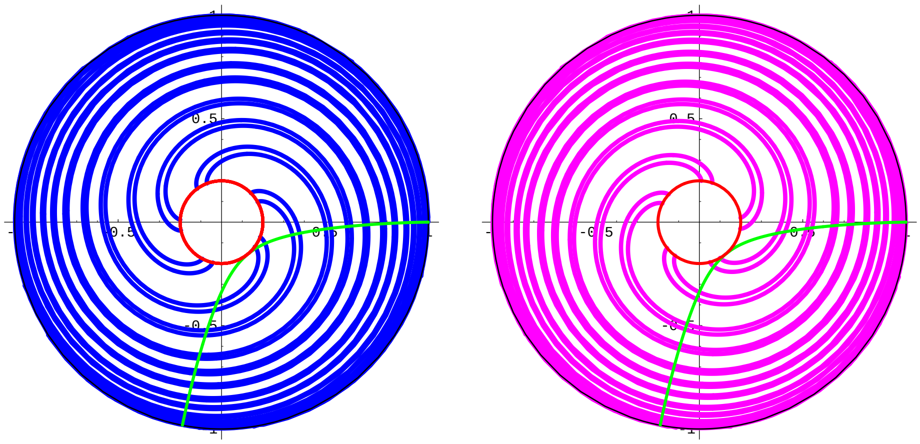

Figure 8: Brachistochrone solution and wave fronts in an inverse square field.

wavefronts in the left plot propagate, they determine a light ray travelling from the initial point to the critical radius u_f, while the wavefronts in the right plot determine the second half of the light ray which recedes away from u_f and terminates at the initial radius.

Another interesting application of Theorem 2 and 3 is to surfaces of revolution in uniform gravitational fields. For example, consider the unit sphere parametrized by

$$\mathbf{x}(u,v) = \left(\sqrt{1-u^2}\cos v, \sqrt{1-u^2}\sin v, u\right),$$

where $-1 < u < 1$, $0 < v < 2\pi$. If we let $n = \frac{1}{\sqrt{u_0 - u}}$, the Brachistochrone solutions given by Theorem 2 are $\mathbf{x}(u, v_1(u))$, where

$$v_1(u) = \pm \int_{u_0}^{u} \sqrt{\frac{C^2(w^2(1-w^2)^{-1}+1)(u_0-w)}{(1-w^2)((1-w^2)-C^2(u_0-w))}}\, dw.$$

Furthermore, the wave fronts given by Theorem 3 are $\mathbf{x}(u, v_2(u,v))$ where,

$$v_2(u) = \frac{D}{C} \pm \frac{1}{C}\int \sqrt{\frac{(u^2(1-u^2)^{-1}+1)((1-u^2)-C^2(u_0-u))}{(1-u^2)(u_0-u)}}\, du.$$

Figure 9 illustrates a solution curve on the sphere with several of its corresponding wave fronts. Again, notice that the wave fronts are perpendicular to the solution curve when the signs in Theorem 2 and 3 agree.

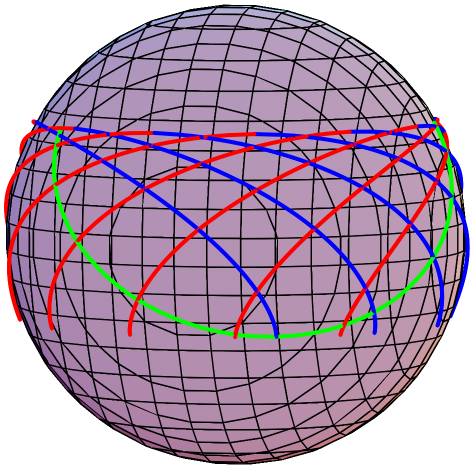

Figure 9: Brachistochrone solution and wave fronts on the unit sphere.

4.2 Special Relativistic Solutions

The strength of Theorem 2 lies in its ability to give Brachistochrone solutions for situations in which classical conservation of mechanical energy does not hold. For example, if the particle's velocity is near the speed of light, the Newtonian mechanical energy equation is replaced with its special relativistic counterpart

$$\gamma c^2 - c^2 + V = V_0, \tag{24}$$

where c is the speed of light in a vacuum, V is the gravitational potential, and $\gamma = (1 - (\frac{ds}{dt})^2 c^{-2})^{-\frac{1}{2}}$, (see [3]). Solving (24) for γ and then solving for $(\frac{ds}{dt})^2$ gives:

$$\gamma = \frac{c^2}{V_0 - V + c^2},$$

$$1 - (\frac{ds}{dt})^2 \frac{1}{c^2} = \frac{c^4}{(V_0 - V + c^2)^2},$$

$$\begin{aligned} \left(\frac{ds}{dt}\right)^2 &= \frac{c^2(V_0 - V + c^2)^2 - c^6}{(V_0 - V + c^2)^2} \\ &= \frac{c^2(V_0 - V)(V_0 - V + 2c^2)}{(V_0 - V + c^2)^2}. \end{aligned}$$

Now, from equation (18) it follows that

$$\left(\frac{1}{n}\right)^2 = \frac{(V_0 - V)(V_0 - V + 2c^2)}{(V_0 - V + c^2)^2}. \tag{25}$$

Therefore, depending upon the geometry of the surface on which the particle is falling and V, we can apply Theorem 2 to find relativistic Brachistochrone solutions.

For example, let us again consider a particle falling in a uniform gravitational field but confined to the vertical plane $\mathbf{x}(u,v) = (u, 0, v)$. But, we will now allow the particle to fall at relativistic speeds. In this case,

$$\left(\frac{1}{n}\right)^2 = \frac{(v_0 - v)(v_0 - v + 2c^2)}{(v_0 - v + c^2)^2},$$

where v_0 is the initial height of the particle. Letting $v_0 = 0$ and applying Theorem 2 gives the solution curves $\mathbf{x}(u(v), v)$, where

$$\begin{aligned} u(v) &= \pm \int_0^v \sqrt{\frac{k^2 \frac{1}{n^2}}{1 - k^2 \frac{1}{n^2}}}\, dw \\ &= \pm \int_0^v \sqrt{-\frac{k^2(2c^2 - w)w}{c^4 + 2c^2(k^2-1)w - (k^2-1)w^2}}\, dw, \end{aligned}$$

and to reduce confusion I have replaced the constant of integration C with k. Figure 10 illustrates several relativistic solution curves and classical Brachistochrone solutions with $c = 10\, m/s$. The relativistic solutions are plotted in solid blue while the classical solutions are plotted with dashed green lines. We can see that the classical solution closely approximates the relativistic solutions in this region. Figure 11 illustrates where the classical solutions give poor approximations to the relativistic solutions.

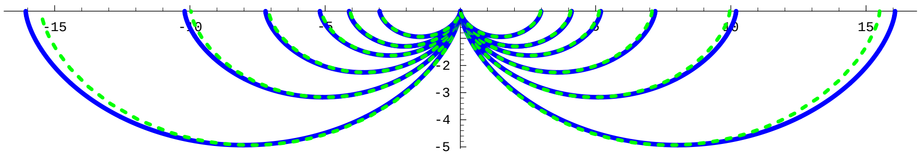

Figure 10: Relativistic and classical Brachistochrone solutions.

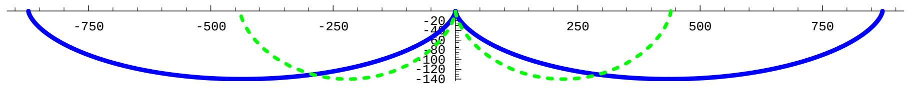

Figure 11: Relativistic and classical Brachistochrone solutions.

4.3 Curvature of Light Rays

Now, consider the curvature of light rays in a medium with index of refraction n. Recall the following definitions and theorem from the classical theory of curves, (see [4]).

Definition 5. *Let $\alpha(s)$ be a light ray with arc length parameter s. Let $\mathbf{T}(s)$ be the unit tangent vector field along $\alpha(s)$. Define $\kappa(s) = \|\mathbf{T}'(s)\|$ to be the curvature of $\alpha(s)$, where $'$ denotes differentiation with respect to s. When $\kappa(s) \neq 0$, let $\mathbf{N}(s) = \mathbf{T}'(s)\kappa(s)^{-1}$ denote the unit normal vector field along α. Finally, let $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$ denote the unit binormal vector field along α.*

Theorem 4. ***(The Frenet-Serret Theorem).*** *If $\alpha(s)$ is a curve with nonzero curvature $\kappa(s)$, then*

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(\mathbf{s}),\ \mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s),\ \textit{and}\ \mathbf{B}'(s) = -\tau(s)\mathbf{N}(s).$$

Definition 6. *The function $\tau(s)$ is called the torsion of $\alpha(s)$ at s.*

It follows that the set $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ forms an orthonormal basis called a *moving frame*. Note that a curve is planar if and only if $\tau(s) \equiv 0$. Now, light rays are distinguished curves with the following property:

$$\nabla n = n\kappa\mathbf{N} + (\mathbf{T} \bullet \nabla n)\mathbf{T}, \tag{26}$$

(see [6]). It follows from this equation that ∇n lies in the *osculatingplane*, the plane spanned by $\mathbf{T}$ and $\mathbf{N}$. Figure (12) is a plot of a cycloid in a uniform gravitational field along with its unit tangent and normal vector fields and ∇n.

In fact, when falling in an inverse-square field, the torsion vanishes. To see this, differ-

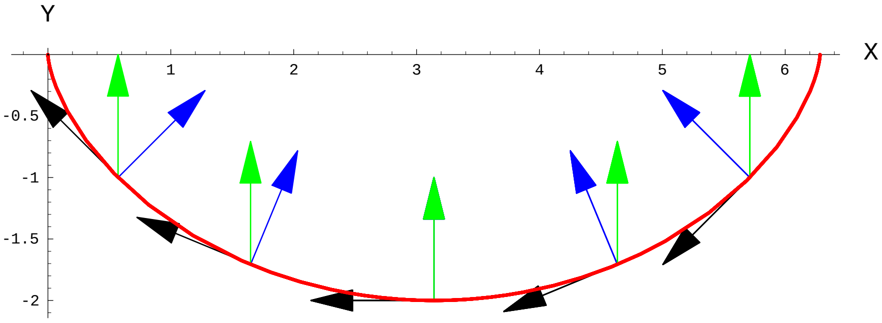

Figure 12: $\mathbf{T}$, $\mathbf{N}$, and ∇n for the cycloid in a uniform gravitational field.

entiate (26) to get

$$\begin{aligned}
\mathbf{N}' &= \left(\frac{1}{\kappa n}\right)' \kappa n \mathbf{N} + \frac{1}{\kappa n}\left[(\nabla n)' - \kappa(\mathbf{T} \bullet \nabla n)\mathbf{N} - (\mathbf{T} \bullet \nabla n)'\mathbf{T}\right] \\
&= \left[\left(\frac{1}{\kappa n}\right)' \kappa n - \frac{1}{n}(\mathbf{T} \bullet \nabla n)\right]\mathbf{N} - \frac{1}{\kappa n}(\mathbf{T} \bullet \nabla n)'\mathbf{T} + \frac{1}{\kappa n}(\nabla n)'.
\end{aligned}$$

By Theorem 4 it follows that the torsion of a light ray is determined by the component of $(\nabla n)'$ lying in the direction of $\mathbf{B}$. But, for a particle falling in an inverse square-field, $n(r) = \sqrt{\frac{r_0 r}{r_0 - r}}$, where r is the radial distance from the origin and r_0 is the particle's initial radial distance from the origin. Furthermore, it follows that $\nabla n = \frac{dn}{dr}\hat{\mathbf{e}}_\mathbf{r}$ and $(\nabla n)' = \left(\frac{dn}{dr}\right)' \hat{\mathbf{e}}_\mathbf{r}$, where $\hat{\mathbf{e}}_\mathbf{r}$ is the unit radial vector. Therefore, $(\nabla n)'$ and ∇n are parallel. Consequently, since ∇n only has components lying in the plane of $\mathbf{T}$ and $\mathbf{N}$ it follows that $\tau = 0$. Therefore, Brachistochrone solutions in inverse-square fields are planar. More specifically, since ∇n is a vector pointing in the direction of the origin it follows that the osculating plane is a plane containing the origin and the initial and final points of the curve. But, this is exactly the problem we investigated when we confined the particle to a plane. Therefore, a solution curve with critical radius u_f and starting point A generates a surface of revolution whose meridians are solution curves. Figures 13, 14, and 15 illustrate several of these surfaces of revolution with the starting point $A = (1, 0, 0)$.

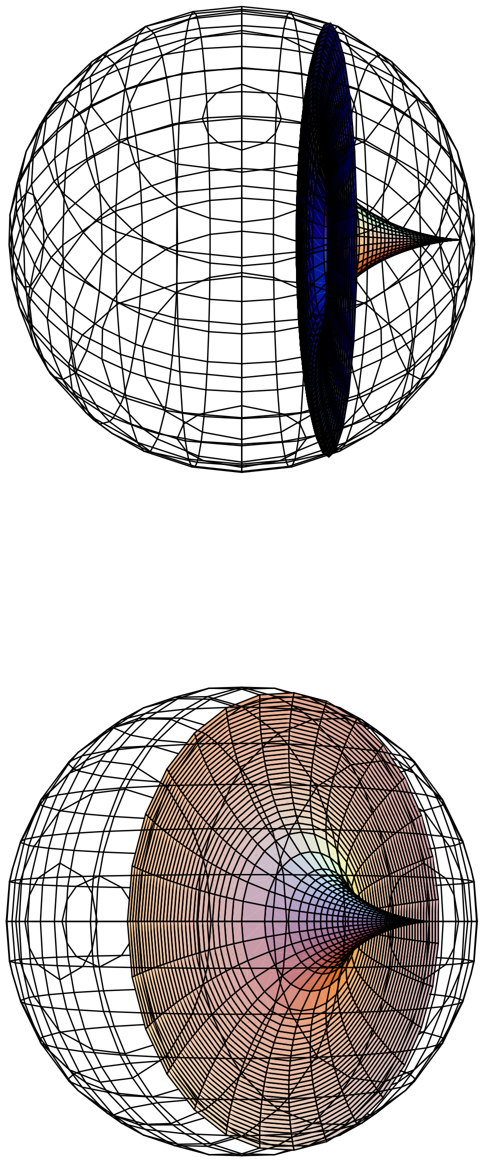

Figure 13: Two points of view for a surface of revolution generated by a solution curve.

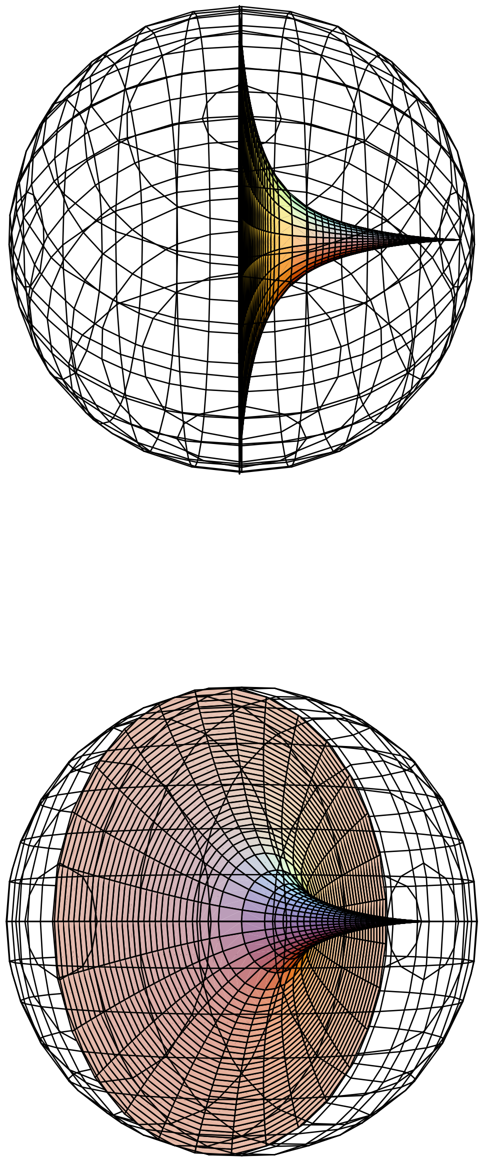

Figure 14: Two points of view for a surface of revolution generated by a solution curve.

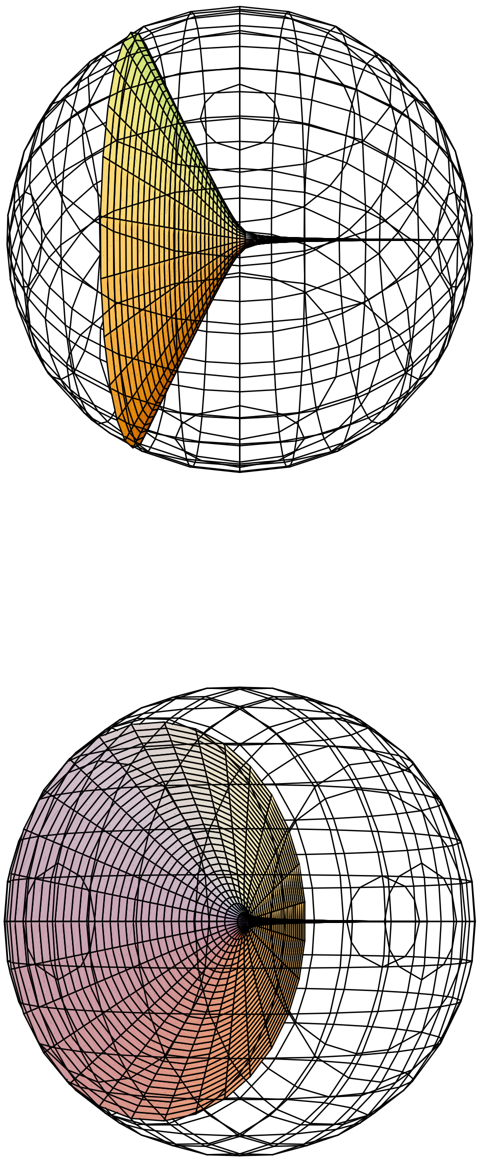

Figure 15: Two points of view for a surface of revolution generated by a solution curve.

5 Conclusion

In this paper we found Brachistochrone solution curves for a large class of surfaces in uniform and inverse-square gravitational fields as well as solutions that incorporate special relativistic effects. Nevertheless, the problem still remains open for particles falling on surfaces in gravitational fields that do not satisfy the conditions in Theorem 1. A simple example of such a surface is the hyperbolic paraboloid parametrized by $\mathbf{x}(u, v) = (u, v, uv)$. It may be possible to use the Eikonal equation to solve these more general problems.

We would also like to solve the problem for particles falling in gravitational fields not confined to a surface. We have already done this for an inverse-square field. One could also consider the problem in a dipole field with gravitational potential given in spherical coordinates by $V = k\frac{\cos\theta}{r^2}$.

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