Comprehensive Internal Guide: Arolla Glacier FEM Project

Simulation Development Team

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Contents

Introduction

This guide provides a deep-dive technical reference for the Arolla Glacier project. It is structured to facilitate clear explanations of the mathematical derivations, proofs, and implementation details during team meetings.

1 Part I: Problem Analysis

Question 1: Order of the Equation

The Equation:

$$2\partial_x(\eta\partial_x u) + \frac{1}{2}\partial_z(\eta\partial_z u) = \rho g \partial_x h$$

Derivation: To determine the order, we look at the highest derivative of the dependent variable u.

1. Expand the outer differential operator using the product rule (assuming η is constant or smooth):

$$\partial_x(\eta\partial_x u) = \eta \underbrace{\partial_{xx}u}_{\text{2nd derivative}} + (\partial_x\eta)(\partial_x u)$$

2. The term $\partial_{xx}u$ represents a second-order derivative with respect to x.

3. Similarly, the z-term contains $\partial_{zz}u$.

Conclusion: The highest derivative is of order 2. Thus, it is a **Second-Order PDE**.

Question 2: Classification of the PDE

General Form: A linear second-order PDE in two variables (x, z) is defined as:

$$Au_{xx} + 2Bu_{xz} + Cu_{zz} + Du_x + Eu_z + Fu = G$$

Step-by-Step Classification:

1. **Identify Coefficients:** From our linearized equation ($2\eta u_{xx} + 0.5\eta u_{zz} = \dots$), we map the terms:

$$A = 2\eta, \quad B = 0, \quad C = \frac{1}{2}\eta$$

2. **Calculate Discriminant:**

$$\Delta = B^2 - AC = 0^2 - (2\eta)\left(\frac{1}{2}\eta\right) = -\eta^2$$

3. **Analyze Sign:** Since viscosity η is a physical property, $\eta > 0$. Therefore, $-\eta^2 < 0$.

4. **Classify:** A negative discriminant ($\Delta < 0$) defines an **Elliptic** equation.

Physical Interpretation: Elliptic equations characterize equilibrium states. Unlike hyperbolic equations (waves) where information travels at a finite speed c, in elliptic problems, a disturbance at the boundary is felt *instantaneously* throughout the entire domain. This justifies the "Stokes flow" assumption of negligible inertia.

2 Part II: Finite Element Formulation

Question 3: Weak Form Derivation

This is the core of our FEM solver. We convert the coordinate-based differential equation into an integral form.

Start: Strong Form

$$\nabla \cdot \boldsymbol{\sigma} = f$$

where $\boldsymbol{\sigma}$ contains our stress terms $(2\eta u_x, 0.5\eta u_z)$ and $f = \rho g h_x$.

Step 1: Multiply and Integrate Multiply by a test function v from the space $V_0 = \{v \in H^1(\Omega) \mid v|_{\Gamma_b} = 0\}$ and integrate over volume Ω:

$$\int_\Omega (\nabla \cdot \boldsymbol{\sigma}) v \, d\Omega = \int_\Omega f v \, d\Omega$$

Step 2: Integration by Parts (Green's Identity) We transfer one derivative from $\boldsymbol{\sigma}$ to v. rule: $\int(\nabla \cdot A)v = -\int A \cdot \nabla v + \oint v(A \cdot n)$.

$$-\int_\Omega \boldsymbol{\sigma} \cdot \nabla v \, d\Omega + \underbrace{\oint_{\partial\Omega} v(\boldsymbol{\sigma} \cdot \mathbf{n}) \, ds}_{\text{Boundary Terms}} = \int_\Omega f v \, d\Omega$$

Step 3: Analyze Boundary Terms The boundary $\partial\Omega$ has two parts:

1. **Bedrock (Γ_b):** Here $u = 0$ (Dirichlet). Our test space V_0 requires $v = 0$ on Γ_b. Thus, the integral is **zero**.
2. **Surface (Γ_s):** The physical condition is "traction-free" or "stress-free", meaning $\boldsymbol{\sigma} \cdot \mathbf{n} = 0$. Thus, the integral is **zero**.

Step 4: Final Form Rearranging to standard $a(u, v) = L(v)$ form:

$$\int_\Omega \left(2\eta \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}\eta \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} \right) d\Omega = -\int_\Omega \rho g \frac{\partial h}{\partial x} v \, d\Omega$$

3 Part III: Theoretical Guarantees

Question 5: Galerkin Orthogonality

This property proves geometrically that the FEM error is "perpendicular" to the approximation space.

1. **Exact Problem:** $a(u, v) = L(v)$ for all $v \in V_0$.
2. **Discrete Problem:** $a(u_h, v_h) = L(v_h)$ for all $v_h \in V_h$.
3. **Subset Property:** Since $V_h \subset V_0$, we can choose $v = v_h$ in the exact problem.
4. **Subtraction:**
$$a(u, v_h) - a(u_h, v_h) = L(v_h) - L(v_h) = 0$$
$$a(u - u_h, v_h) = 0$$

Question 6: Best Approximation Property

We prove that u_h is the best possible approximation in the energy norm $\|w\|_E = \sqrt{a(w, w)}$. **Proof steps:**

1. Take any function $v_h \in V_h$. We want to measure the distance $\|u - v_h\|_E^2$.
2. Write $u - v_h$ as $(u - u_h) + (u_h - v_h)$.
3. Expand the norm:
$$\|u - v_h\|_E^2 = a(u - u_h + w_h, u - u_h + w_h)$$
where $w_h = u_h - v_h$.
4. Distribute terms:
$$= \underbrace{a(u - u_h, u - u_h)}_{\|u-u_h\|^2} + 2 \underbrace{a(u - u_h, w_h)}_{0 \text{ by Orthogonality}} + \underbrace{a(w_h, w_h)}_{\geq 0}$$
5. Conclusion:
$$\|u - v_h\|_E^2 \geq \|u - u_h\|_E^2$$
The error of our solution u_h is smaller than or equal to the error of any other function v_h.

Question 7: A Priori Error Estimate

How big is the error?

1. From Q6, we know $\|u - u_h\|_E \leq \|u - \pi_h u\|_E$ (where $\pi_h u$ is the interpolation).
2. Standard interpolation theory tells us that for mesh size h, the interpolation error gradients scale with h:
$$\|\nabla(u - \pi_h u)\| \leq Ch\|u\|_{H^2}$$
3. Combining these:
$$\|u - u_h\|_E \leq C\sqrt{\eta}h\|u\|_{H^2}$$
4. **Meaning:** If we halve the mesh size h, the error is cut in half (Linear Convergence).

4 Part IV: Implementation & Results

Question 4: Linear Implementation Code

The FEniCSx implementation directly mirrors the Weak Form derived in Q3.

```
# Define Function Space (P1 Elements)
V = fem.functionspace(msh, basix.ufl.element("Lagrange", "triangle", 1))

# Define Variational Problem
a = (2 * eta * ufl.Dx(u, 0) * ufl.Dx(v, 0) +
     0.5 * eta * ufl.Dx(u, 1) * ufl.Dx(v, 1)) * ufl.dx
L = - rho * g * ufl.Dx(h, 0) * v * ufl.dx

# Solve
problem = LinearProblem(a, L, bcs=bcs, petsc_options={...})
uh = problem.solve()
```

Question 8: Non-Linear Algorithm (Picard)

Since η depends on u, we cannot solve in one step. We use **Picard Iteration**:

```
Initialize u = 0
Loop k = 1 to max_iter:
    1. Calculate Strain Rate E from u_{k-1}
    2. Update Viscosity: eta = 0.5 * A^(-1/n) * (E + epsilon)^((1-n)/2n)
    3. Define linear forms a(u, v) using new eta
    4. Solve linear system -> u_k
    5. Check convergence: norm(u_k - u_{k-1}) < tolerance
```

Question 9: Impact of Regularization (ϵ)

The parameter ϵ prevents division by zero when the glacier is not moving (strain rate ≈ 0).

- **Small ϵ (10^{-10}):** The physics governs. Viscosity becomes very high at stagnation points. This "stiff" behavior is hard to solve (10 iterations).
- **Large ϵ (10^{-1}):** The regularization dominates. Viscosity is capped and smooth. The problem looks linear to the solver (3 iterations), but the result is physically inaccurate.