1. The proof for this is almost identical to the proof of Markov's inequality:

$$\begin{aligned}\mu(\{x \in X : |h(x)| > c\}) &= \frac{1}{c^p}\int_{\{|h(x)|>c\}} c^p d\mu,\\ &\leq \frac{1}{c^p}\int_{\{|h(x)|>c\}} |h(x)|^p d\mu,\\ &\leq \frac{1}{c^p}\|h^p\|_1.\end{aligned}$$

2. We apply the previous problem,

$$\begin{aligned}\mu\left(\left\{x \in X : \left|h(x) - \int h d\mu\right| \geq c\right\}\right) &\leq \frac{1}{c^2}\int \left(h - \int h d\mu\right)^2 d\mu,\\ &= \frac{1}{c^2}\int\left(h^2 - 2h\int h d\mu + \left(\int h d\mu\right)^2\right) d\mu,\\ &= \frac{1}{c^2}\left(\int h^2 d\mu - \left(\int h d\mu\right)^2\right).\end{aligned}$$

4. By way of contradiction, suppose that the constant in the Vitali covering lemma can be replaced by 3ε for $0 < \varepsilon < 3$. Then, let $I = (a, b)$ such that $b - \frac{3}{2}\varepsilon b - \varepsilon > 0$ and $a < b - \varepsilon$. Now, consider $I' = (b - \varepsilon, b - \varepsilon + \ell(I))$. Note that I and I' are not disjoint. So, we proceed to show that the Vitali covering lemma no longer holds with 3ε. It follows that

$$3\varepsilon * (a, b) = \left(a - \frac{3\varepsilon}{2}\ell(I), b + \frac{3\varepsilon}{2}\ell(I)\right).$$

Then,

$$b + \frac{3\varepsilon}{2}\ell(I) = b\left(1 + \frac{3}{2}\varepsilon\right) - a,$$

Since $b - \frac{3}{2}\varepsilon b - \varepsilon > 0$, it follows that $2 - \frac{\varepsilon}{b} > 1 - \frac{3}{2}\varepsilon$. Then,

$$< b\left(2 - \frac{\varepsilon}{b}\right) - a.$$

The last inequality implies that $3\varepsilon * (a, b) \not\supset I'$. By symmetry, $3\varepsilon * I' \not\supset I$.

6. For any $x \in (0, 1)$, there exists an open interval, I, such that $\chi_{[0,1]}(I) = 1$. Thus, $h^*((0, 1)) = 1$. On the other hand, if $x \leq 0$, then $\int_{b-t}^{b+t} |h| > 0$ only if $b + t > 0$. It also follows that if $b + t > 1$, then $\int_{b-t}^{b+t} |h|$ decreases monotonically. Thus, $h^*(b) = \frac{1}{2(1-b)}$ for $b < 0$. The same holds if $b > 1$. So, applying the same logic, $h^*(b) = \frac{1}{2b}$ for $b > 1$.

9. Let $A = \{b \in \mathbf{R} : h^*(b) > c\}$. Then, it suffices to show that for every $b \in A$, there exists some $\varepsilon > 0$ such that $(b - \varepsilon, b + \varepsilon) \subset A$. We first consider the case of extending $[b, b]$ to the interval $[b, b + \varepsilon)$ and show that there exists an $\varepsilon > 0$ such that $[b, b + \varepsilon) \subset A$, then by symmetry, $(b - \varepsilon, b + \varepsilon) \subset A$. So,

$$\begin{aligned}\sup_{t>0}\frac{1}{2t}\int_{b-t}^{b+t}|h| &= \sup_{t>0}\frac{1}{2t}\left(\int_{b-t}^{b+t}|h| - \int_{b-t}^{b+\varepsilon-t}|h| + \int_{b+t}^{b+\varepsilon+t}|h|\right),\\ &= \sup_{t>0}\frac{1}{2t}\left(\int_{b-t}^{b+t}|h| - \underbrace{\varepsilon\cdot\sup_{(b-t,b+\varepsilon-t)}|h|}_{\text{(a)}} + \varepsilon\cdot\sup_{(b+t,b+\varepsilon+t)}|h|\right).\end{aligned}$$

It remains to show that there exists $\varepsilon > 0$ where $|h|$ is bounded on $(b - t, b + \varepsilon + t)$. Then, the ε-terms on the right-hand side can be made arbitrarily small so, it follows that $[b, b + \varepsilon) \subset A$. By way of contradiction, suppose for all $\varepsilon > 0$, there does not exist some bounded Lebesgue measurable set L such that $(b - t, b + \varepsilon - t) \subset f^{-1}(L)$. Then, there does not exist a bounded Lebesgue measurable set L such that $\cap_{\varepsilon > 0}(b - t, b + \varepsilon - t) \subset f^{-1}(L)$. This implies that $f(b - t) = \infty$ which is a contradiction since $f : \mathbf{R} \to \mathbf{R}$.

11.

13.