3. Since $f \in \mathcal{L}^1(\mathbf{R})$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\int_B |f| < \varepsilon,$$

for any B such that $\mu(B) < \delta$. Thus, for given $\varepsilon > 0$, choose $\delta > 0$ such that

$$\int_B |f| < \frac{\varepsilon}{2}.$$

Then, it follows that for any $x, x' \in X$ if $|x - x'| < \delta$, then (assuming without the loss of generality that $x > x'$)

$$\begin{aligned}|g(x) - g(x')| &= \left| \int_{(-\infty,x)} f\,d\mu - \int_{(-\infty,x')} f\,d\mu \right|, \\ &= \left| \int_{(x',x)} f\,d\mu \right|,\end{aligned}$$

note that by construction $\mu((x', x')) < \delta$. So,

$$\begin{aligned}&= \left| \int_{(x',x)} f^+\,d\mu - \int_{(x',x)} f^-\,d\mu \right|, \\ &\leq \int_{(x',x)} f^+\,d\mu + \int_{(x',x)} f^-\,d\mu, \\ &< \varepsilon.\end{aligned}$$

The last inequality follows from the fact that $|f| \geq f^+, f^-$.

4. (a) Consider any $\mathcal{S}$-partition $A_1, \ldots, A_m$, then

$$\sum_{i=1}^m \mu(A_i) \sup_{A_i} f \geq \sum_{i=1}^m \mu(A_i) \inf_{A_i} f.$$

Thus,

$$\int f\,d\mu \leq \inf \left\{ \sum_{i=1}^m \mu(A_i) \sup_{A_i} f : \text{for any } \mathcal{S}\text{-partition } A_1, \ldots, A_n \right\}.$$

Now, it remains to show the inequality in the other direction. See the following claim,

Claim 1. For any $\varepsilon > 0$, there exists a $\mathcal{S}$-partition $A_1, \ldots, A_n$ such that

$$\mathcal{L}(f; A_1, \ldots, A_n) + \varepsilon \geq \mathcal{U}(f; A_1, \ldots, A_n),$$

where $\mathcal{L}(\cdot;\cdot)$ denotes the lower Lebesgue sum and $\mathcal{U}(\cdot;\cdot)$ denotes the upper Lebesgue sum.

Proof of claim: Since f is $\mathcal{S}$-measurable and bounded, there exists a sequence of monotontically increasing functions $f_1, f_2, \ldots$ that converge uniformly to f. Thus, there exists $N \in \mathbf{Z}^+$ such that for any $n > N$,

$$f(x) - f_n(x) < \frac{\varepsilon}{2\mu(X)},$$

for all $x \in X$. Since f_n is simple, it follows that

$$f_n = \sum_{i=1}^m c_i \chi_{A_i},$$

where $A_1, \ldots, A_n$ is an $\mathcal{S}$-partition of X. It also follows that

$$\int f - f_n d\mu < \frac{\varepsilon}{2\mu} \cdot \mu(X) < \frac{\varepsilon}{2}.$$

Since $\chi_{A_i} f \geq \chi_{A_i} \inf f \geq \chi_{A_i} f_n$, for any A_i, it must be that $\mathcal{L}(f; A_1, \ldots, A_n) - \int f_n d\mu < \frac{\varepsilon}{2}$. Now, we want to show that the upper Lebesgue sum is also "close" to the simple approximation. By uniform convergence, $\sup f - f_n \leq \varepsilon$ and since f_n is constant on any A_i, it follows that

$$\sum_{i=1}^{m} \sup_{A_i} f \cdot \mu(A_i) - \sum_{i=1}^{m} c_i \chi_{A_i} \leq \frac{\varepsilon}{2\mu(X)} \cdot \mu(X) = \frac{\varepsilon}{2}.$$

Thus, it follows by triangle inequality that

$$\mathcal{U}(f; A_1, \ldots, A_m) - \mathcal{L}(f; A_1, \ldots, A_m) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

□

Using the claim, it follows that for any $\varepsilon > 0$,

$$\begin{aligned}\sup_{\{A_i\}_{i=1}^n} \mathcal{L}(f; A_1, \ldots, A_n) + \varepsilon &\geq \mathcal{U}(f; A_1, \ldots, A_n), \\ &\geq \inf_{\{A_i\}_{i=1}^n} \mathcal{U}(f; A_1, \ldots, A_n).\end{aligned}$$

Since the last inequality holds for arbitrary $\varepsilon > 0$, it follows that $\int f d\mu = \inf_{\{A_i\}_{i=1}^n} \mathcal{U}(f; A_1, \ldots, A_n)$.

(b) Consider the function $f = \frac{1}{\sqrt{x}}$, over the measure space $((0,1), \mathcal{B}((0,1)), \lambda)$. Then, it follows that $\int f d\mu = 2$ (by the dominated convergence theorem and problem 5). However, for any partition $A_1, \ldots, A_n$, there exists A_i such that $\mu(A_i) > 0$ and $\sup A_i = \infty$. Thus, the lower and upper Lebesgue sums do not coincide.

(c) Let $f(x) = \frac{1}{x^2}$ and consider the measure space $((0,1), \mathcal{B}(0,1), \lambda)$. Then, it follows that for any partition $A_1, \ldots, A_n$ there exists a A_i such that $\mu(A_i) > \infty$. Thus, $\inf \mathcal{U}(f; A_1, \ldots, A_n) = \infty$.

5. It is given that $f \in \mathcal{L}^1(\mathbf{R})$ and by definition,

$$\lim_{k \to \infty} \int_{[-k,k]} f d\lambda = \lim_{k \to \infty} \int \chi_{[-k,k]} f d\lambda.$$

Since $f \in \mathcal{L}^1(\mathbf{R})$ and for any $k \in \mathbf{R}$, $\chi_{[-k,k]} f \leq f$ the hypotheses for the dominated convergence theorem holds. Thus, because $\lim_{k\to\infty} \chi_{[-k,k]} f = f$,

$$\lim_{k \to \infty} \int_{[-k,k]} f d\lambda = \int f d\lambda.$$

7. Note that we need to violate the hypotheses of the dominated convergence theorem. That is, we need to find a function f such that there does not exists a function $g \in \mathcal{L}^1(\mathbf{R})$ such that $|f_k| \leq g$ for all $k \in \mathbf{Z}^+$. Consider the function

$$f(x) = -\frac{1}{x} - \frac{1}{x-1},$$

its domain restricted to the interval $(0,1)$. Then, for any $n \in \mathbf{R}$, $\int_{(\frac{1}{n},1)} f d\lambda = \infty$. Thus,

$$\lim_{n \to \infty} \int_{(\frac{1}{n},1)} f d\lambda = \infty.$$

But, $\int_{(0,1)} f d\mu$ does not exist.

8. First, assume that f is not continuous at x. Then, there exists $\varepsilon > 0$, for any interval I_x which contains x, there exists $y \in I_x$ such that $|f(x) - f(y)| \geq \varepsilon$. Therefore, it follows that $\sup_{I_x} f \neq \inf_{I_x} f$. Consider a sequence of closed intervals $I_1, \ldots, I_{2^n}$ which cover $[a, b]$. If there exists I_i which contains I_x, then naturally $g_n(x) \neq h_n(x)$. On the other hand, if x lies on the boundary of some interval I_i, then by definition g_n and h_n take on the infinimum and supremum of f on the union of the two intervals that contain that point. So, x must lie on the interior of this union. Then, there exists an open interval around x contained in the union of these two closed intervals. As a result, $g_n(x) \neq h_n(x)$. Since $h_n(x) - g_n(x) \geq \varepsilon$ for any $n \in \mathbf{Z}^+$, $f^U(x) \neq f^L(x)$ and the proof in this direction is completed.

   Now, assume that $f^U(x) \neq f^L(x)$. This implies that there exists some $\varepsilon > 0$ and $N \in \mathbf{Z}^+$ such that for any $n > N$, $h_n(x) - g_n(x) > \varepsilon$. By the definition of h_n, g_n (see above), it follows for any n there exists some open interval, I, of length $(b-a)/2^n$ containing x such that $\sup_I f - \inf_I f > \varepsilon$. Thus, there must exist some $y \in I$ such that $|f(x) - f(y)| \geq \varepsilon$. Therefore, f is not continuous at x.

9. Let $f = \sum_{i=1}^n a_i \chi_{A_i}$. Then,

$$\begin{aligned}\int |f| d\mu &= \int f^+ d\mu + \int f^- d\mu, \\ &= \sum_{\{i: a_i \geq 0\}} a_i \mu(A_i) + \sum_{\{i: a_i < 0\}} -a_i \mu(A_i), \\ &= \sum_{i=1}^n |a_i| \mu(A_i).\end{aligned}$$

   Let us first show the "if"-direction. Suppose that $E_k \in \mathcal{S}$ and $\mu(E_k) < \infty$ for all $k \in \{1, \ldots, n\}$. Then, clearly, f is $\mathcal{S}$-measurable. And it also follows thta $\int |f| d\mu < \infty$ (from the expression we derived above). Thus, by the definition of $\mathcal{L}^1$, $f \in \mathcal{L}^1(\mu)$. Now, consider the "only-if"-direction. If $f \in \mathcal{L}^1$, then f must be $\mathcal{S}$-measurable. If $E_k \notin \mathcal{S}$, then $f^{-1}(\{a_k\}) \notin \mathcal{S}$ (by disjointness). So, it must be that each $E_k \in \mathcal{S}$. Also, $\int |f| d\mu < \infty$ implies that $\sum_{i=1}^n |a_i| \mu(A_i) < \infty$. Since each $a_i \in (-\infty, \infty)$, it must be that $\mu(E_k)$ for every $k \in \{1, \ldots, n\}$.