5. From the equivalent definitions of the Lebesgue measurable sets, there must exist a sequence of closed sets $F_1, F_2, \ldots$ where $F_k \subset A$ and
$$\left| A \setminus \bigcup_{k=1}^{\infty} F_k \right| = 0.$$
Since any finite union of closed sets is closed, then it follows that $F_1, F_1 \cup F_2, F_1 \cup F_2 \cup F_3, \ldots$, is a sequence of increasing closed sets also contained in A. And since $\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{k} F_\ell$, then the previous equation holds.

6. Let us first assume that A is a Lebesgue measurable set. Then, first consider the following claim:

Claim 1. If $|A| < \infty$, then for every $\varepsilon > 0$, there exists a closed, bounded set $F \subset A$ such that $|A \setminus F| < \varepsilon$.

Proof of claim: By problem 9 in section 2A, for any set $A \subset \mathbf{R}$, it follows that
$$\lim_{t \to \infty} |(-t, t) \cap A| = |A|.$$
From the definition of the limit, for any $\varepsilon > 0$, there exists some $t \in \mathbf{R}$ such that $|A \setminus ((-t, t) \cap A)| = |A| - |(-t, t) \cap A| < \varepsilon$. For any $t \in \mathbf{R}$, $(-t, t) \cap A$ is bounded. Moreover, $(-t, t) \cap A$ must be a Lebesgue measurable set (since every Borel set is also a Lebesgue measurable set). Since $(-t, t) \cap A$ is a Lebesgue measurable set, there must exist a closed set $F \subset (-t, t) \cap A$ such that $|(-t, t) \cap A \setminus F| < \varepsilon$. Then, by countable additivity, it must be that $|A \setminus F| = |A \setminus (-t, t) \cap A| + |(-t, t) \cap A \setminus F| < 2\varepsilon$. □

Now, we show the forward claim:

Claim 2. If A is Lebesgue measurable $|A| < \infty$, then for every $\varepsilon > 0$, there exists G, a union of finite, disjoint, bounded open intervals such that $|G \Delta A| < \varepsilon$.

Proof of claim:

By claim 1, it follows that there exists a closed, bounded set $F \subset A$ such that $|A \setminus F| < \varepsilon$. Since F is a Borel set, there must be a open cover $G \supset F$ such that $|G \setminus F| < \varepsilon$. G is the union of finite disjoint open intervals $G_1, G_2, \ldots$, by the Heine-Borel theorem, there exists a finite subcover of $G' = \bigcup \{G_k\}_{k=1}^{n}$ of these open intervals. Note that because F is bounded and $|G \setminus F| < \varepsilon$ each of these open intervals must also be bounded. Since $F \subset G' \subset G$, it must be that $|G' \setminus F| < \varepsilon$. So,
$$\begin{aligned} |G' \Delta A| &= |G' \setminus A| + |A \setminus G'|, \\ &\leq |G' \setminus F| + |A \setminus F|, \\ &< 2\varepsilon. \end{aligned}$$
□

Now, for the converse:

Claim 3. By the definition of outer measure, there exists a sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $|\bigcup_{k=1}^{\infty} I_k| < |A \setminus G| + \varepsilon$. Thus, it follows that $G \cup \bigcup_{k=1}^{\infty} I_k$ is open and $|G \cup \bigcup_{k=1}^{\infty} I_k| < |A| + 2\varepsilon$.

12. The forward is trivial and follows directly from the definition of the Lebesgue measure since $A, (b, c) \setminus A$ are disjoint. Now, we show the converse directly,

Claim 4. If $A \subset (b, c)$ and $|A| + |(b, c) \setminus A| = c - b$, then A is Lebesgue measurable.

Proof of claim: By the definition of outer measure, there exists a sequence of open intervals $\{I_k\}_{k=1}^{\infty}$ such that $|\bigcup_{k=1}^{\infty} I_k| < |(b, c) \setminus A| + \varepsilon$. Now consider $B = [b + \varepsilon, c - \varepsilon] \setminus \bigcup_{k=1}^{\infty} I_k$. such that $\varepsilon < \min\{b, c\}$. It follows that B is a closed subset of A. It remains to show that $A \setminus B$ can be made arbitrarily small. By order-preservation of outer measure $|B| \leq |A|$ and since B is closed,
$$\begin{aligned} |A \setminus B| &= |A| - |B|, \\ |B| &> (c - b + 2\varepsilon) - (|(b, c) \setminus A| + \varepsilon), \\ &= (c - b + 2\varepsilon) - ((c - b) - |A| + \varepsilon), \\ &= |A| + \varepsilon. \end{aligned}$$

Then, it follows that $|A| - |B| < \varepsilon$. So, A is Lebesgue measurable. $\square$

13. It must be that $(-n, n) \cap A \subset (-n, n)$. So, let $B = (-n, n) \cap A$. Then, $(-n, n) \setminus A = (-n, n) \setminus B$. Since

$$\begin{aligned} x \in (-n, n) \setminus A &\iff x \in (-n, n) \text{ and } x \notin \setminus A, \\ &\iff x \in (-n, n) \text{ and } x \notin A \cap (-n, n), \\ &\iff x \in (-n, n) \setminus B. \end{aligned}$$

So, by problem 12 in this section, this equality only holds if and only if $(-n, n) \cap A$ is Lebesgue measurable for all $n \in \mathbf{Z}^+$. Thus, in the limit as $n \to \infty$, the equality only holds as A is Lebesgue measurable.

24. (a) Follows directly from claim 1. Since this holds for every $\varepsilon > 0$, and the limit in claim 1 is monotonically increasing, then it must exist and so $\sup_{t \in \mathbf{R}} |F| = |A|$.

    (b) Consider the Vitali set.