5. Suppose that $X = \mathbf{R}$, then $\mu(X) \not< \infty$. It remains to show that we can construct a function along with a sequence of functions that does not converge uniformly on a large part of the domain. Consider problem 30 from chapter 2B, where

$$f_n(x) = \cos(n\pi x)^{2n},$$

we've already show that this converges pointwise to the function,

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{o/w.} \end{cases}$$

However, the sequence does not converge uniformly on any irrational point. Thus, there does not exist an arbitrarily large subset of $\mathbf{R}$ where the sequence converges uniformly.

7. The converse has already been proved in the text, so it remains to show the forward direction. For every $\varepsilon > 0$, it suffices to show that there exists some $N \in \mathbf{Z}^+$ such that for all $n > N$, $g(x) - g_n(x) < \varepsilon$ for all $x \in F$. By pointwise convergence and continuity of $g$, for every $x \in F$, there exists some $N_x$, $\delta_x$ such that $(g - g_n)(x') < \varepsilon$ for all $x' \in (x - \delta_x, x + \delta_x)$. Now, consider the open cover of $F$, $\mathcal{U} = \{(x - \delta_x, x + \delta_x) : x \in F\}$. Since $F$ is compact, there exists a finite subcover of $\mathcal{U}$ which we denote as $\mathcal{U}'$. Thus, let $N = \max_{(x-\delta_x, x+\delta_x) \in \mathcal{U}'} N_x$. Then, by the monotonicity of $g_n$s, it follows that for all $n > N$, $g(x) - g_n(x) < \varepsilon$ for all $x \in F$.

8. For any $\varepsilon > 0$, define $n \in \mathbf{Z}^+$ such that

$$\sum_{i=1}^{n} \frac{1}{2^i} > 1 - \varepsilon.$$

Also, let $E = \{1, 2, \ldots, n\}$. So, it follows by the definition of $\mu$ that $\mu(\mathbf{Z}^+ \setminus E) < \varepsilon$. Now, consider any sequence of functions $f_1, f_2, \ldots$ which converges pointwise to $f$. By the definition of pointwise convergence, it must be that for fixed $\varepsilon' > 0$,

$$\bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \{x \in E : |f(x) - f_k(x)| < \varepsilon'\} = E.$$

Since $E$ is finite, there must exist some $N \in \mathbf{Z}^+$ such that

$$\bigcup_{m=1}^{N} \bigcap_{k=m}^{\infty} \{x \in E : |f(x) - f_k(x)| < \varepsilon'\} = E.$$

Thus, it follows that if $k > N$, then $|f(x) - f_k(x)| < \varepsilon'$ for all $x \in E$. Therefore, $f_1, f_2, \ldots$ converges uniformly on $E$.

9. It suffices to show that for any fixed $1 \leq k \leq n$, if $x \in F_k$, then there exists some $\delta > 0$, such that $\Delta = (x - \delta_x, x + \delta_x)$ and for every $x' \in \Delta$, either $x' \in F_k$ or $x' \notin F_1 \cup \ldots F_{k-1} \cup F_{k+1} \cup \ldots \cup F_n$. If this is true, then it follows that for every $x \in \operatorname{dom}(g)$, by the continuity of $g|_{F_k}$, there exists some $\delta > 0$ such that $|g(x) - g(x')| < \varepsilon$ for all $x' \in \Delta$, where $\Delta = (x - \min(\delta, \delta_x), x + \min(\delta, \delta_x))$.

   Fix some $1 \leq k \leq n$, and by way of contradiction suppose that for every $x \in F_k$ and every $\delta_1, \delta_2, \ldots$ that converges to 0, there exists some $x_i' \in (x - \delta_i, x + \delta_i)$ such that $x' \in \bigcup_{i \neq k} F_i$. Then, clearly $x_1', x_2', \ldots$ convergs to $x$, but since each $x_i' \in \bigcup_{i \neq k} F_i$ and $\bigcup_{i \neq k} F_i$ is a closed set, it must be that $x \in \bigcup_{i \neq k} F_i$, which is a contradiction since each $F_k$ is disjoint.

10. By way of contradiction, suppose that $F$ is not closed, then $F$ does not contain all of its limit points. Thus, there exists a sequence $x_1, x_2, \ldots$ such that each $x_k \in F$, but $\lim_{k \to \infty} x_k \notin F$. Let this limit point be denoted as $a$. Then, consider

$$g(x) = \frac{1}{x - a}.$$

It follows that $g(x)$ is continuous on $F$ since $a \notin F$, but $g(x)$ cannot be extended to a continuous function on $\mathbf{R}$.

11. By way of contraposition, suppose that $F$ is not closed. Then, I show that not every continuous function can be extended to a continuous function on $\mathbf{R}$. If $F$ is not closed, then it does not contain all of its limit points. Thus, there exists a sequence $x_1, x_2, \ldots$ such that each point is in $F$, but $\lim_{k \to \infty} x_k = a \notin F$. Now, consider the function
$$g(x) = \sin\left(\frac{1}{x-a}\right).$$
$g$ is continuinous every except at $a$, and $g$ is also bounded. However, $g$ cannot be extended to $a$, since its limit at $a$ does not exist.

14. Let us only consider the case where $x \notin \{b_1, b_2, \ldots\}$ since $\{b_1, b_2, \ldots\}$ is a measure-zero set. From the definition of $f$, if $2^k|x - b_k| > 1$ for all $k \in \mathbf{Z}^+$, then $f(x) < 1$. So, consider the following set,
$$A = \left\{x \in \mathbf{R} : 2^k|x - b_k| > 1, \forall k \in \mathbf{Z}^+\right\}.$$
It suffices to show that $|A| = \infty$ since $A \subset \{x \in \mathbf{R} : f(x) < 1\}$. So, if $x \in A$, then
$$x \notin \bigcup_{k=1}^{\infty} \left[b_k - \frac{1}{2^k}, b_k + \frac{1}{2^k}\right].$$
Now,
$$\begin{aligned} |A| &\geq \left|\mathbf{R} \setminus \bigcup_{k=1}^{\infty} \left[b_k - \frac{1}{2^k}, b_k + \frac{1}{2^k}\right]\right| \\ &\geq |\mathbf{R}| - \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}, \\ &= \infty. \end{aligned}$$
And since $A \subset \{x \in \mathbf{R} : f(x) < 1\}$ and outer measure preserves order, it must be that $|\{x \in \mathbf{R} : f(x) < 1\}| = \infty$.