1. Since outer measure preserves order and $A \cup B \supset A$, then $|A \cup B| \geq |A|$. Now, consider the sequence $A_1, A_2, \ldots$ such that $A_1 = A$, $A_2 = B$, and $A_k = \emptyset$ for $k > 2$. It follows from the countable subadditivity of outer measure that

$$\begin{aligned} \left| \bigcup_{k=1}^{\infty} A_k \right| &= |A \cup B| \\ &\leq \sum_{k=1}^{\infty} |A_k| \\ &= |A| + |B| = |A|. && \text{(by given)} \end{aligned}$$

Thus, $|A \cup B| = |A|$.

2. We first show that for any $A \subset \mathbb{R}$ that $\ell(tA) = |t| \cdot \ell(A)$. If $t = 0$, then since $tA = \emptyset$, $\ell(tA) = 0$. Then, if $t > 0$, it follows that

$$tA = \begin{cases} (ta, tb) & \text{if } A = (a, b), \\ (ta, \infty) & \text{if } A = (a, \infty), \\ (-\infty, ta) & \text{if } A = (-\infty, a), \\ \emptyset & \text{if } A = \emptyset. \end{cases}$$

On the other hand, if $t < 0$, then

$$tA = \begin{cases} (tb, ta) & \text{if } A = (a, b), \\ (-\infty, ta) & \text{if } A = (a, \infty), \\ (ta, \infty) & \text{if } A = (-\infty, a), \\ \emptyset & \text{if } A = \emptyset. \end{cases}$$

Thus, it follows that $\ell(tA) = |t| \cdot \ell(A)$, by the definition of $\ell(\cdot)$. Note that if $t = 0$, then trivially, $|tA| = |\emptyset| = 0$. So, we only consider the case where $t \neq 0$. Consider a sequence of open intervals $I_1, I_2, \ldots$ whose union contains A. It follows that $tI_1, tI_2, \ldots$ is a sequence of open intervals whose union contains tA. Then,

$$|tA| \leq \sum_{k=1}^{\infty} |tI_k| = |t| \sum_{k=1}^{\infty} |I_k|,$$

by taking the *infimum* over all such sequences, we have that $|tA| \leq |t| \cdot |A|$. Now, observe that $|\frac{1}{t}(tA)| = |A|$, so

$$|t| \left| \frac{1}{t}(tA) \right| \leq |t| \sum_{k=1}^{\infty} \ell(\frac{1}{t}(tA)) \leq \frac{|t|}{|t|} \sum_{k=1}^{\infty} \ell(tI_k) = \sum_{k=1}^{\infty} |tI_k|.$$

Again, by taking the *infimum* over all such sequences we have that $|t| \cdot |A| \leq |tA|$. Thus, $|tA| = |t| \cdot |A|$.

3. It suffices to show that $|B \setminus A| + |A| \geq |B|$. By the subadditivity of outer measure, $|B \setminus A| + |A| \geq |(B \setminus A) \cup A|$. Since $(B \setminus A) \cup A \supset B$ and outer measure preserves order then $|(B \setminus A) \cup A| \geq |B|$. Thus, $|B \setminus A| \geq |B| - |A|$.

9. Suppose that A is bounded, then there exists some $x, r \in \mathbb{R}$ such that $A \subset (x - r, x + r)$. Then, for any $t > \max\{|x - r|, |x + r|\}$, it follows that $A \cap (-t, t) = A$. Thus, the limit expression holds when A is bounded. Now, consider the case where A is unbounded. First, the limit must exist since for any sequence of $t_k \to \infty$, it follows that $|A \cap (-t_k, t_k)|$ is monotone. Moreover, it must be that $|A| \geq \lim_{t \to \infty} |A \cap (-t, t)|$ since for any $t > 0$, $|A| \geq |A \cap (-t, t)|$. So, it remains to show that $|A| \leq \lim_{t \to \infty} |A \cap (-t, t)|$.

11. By the definition of outer measure, since

12. (a) The arbitrary union of open sets is open. Thus, $\bigcup_{k=1}^{\infty} (r_k - 1/2^k, r_k + 1/2^k)$ is open. And, by definition, the complement of an open set is closed. So, F is closed.

(b) By way of contradiction, suppose that there exists an interval $I \subset F$ such that I contains more than one element. That is, $I = (a, b)$ for $a < b$. Then, it follows from the denseness of $\mathbb{Q}$ in $\mathbb{R}$ that there exists a rational number in I. Thus, there is an centered around this rational number that is not contained in F. Thus, I is not connected and therefore, I is not an interval. So, any intervals in F contain only one element.

(c) It follows from problem 3 that

$$\begin{aligned}
|F| &\geq |\mathbb{R}| - \left|\bigcup_{k=1}^{\infty}\left(r_k - \frac{1}{2^k}, r_k + \frac{1}{2^k}\right)\right|, \\
&\geq |\mathbb{R}| - \sum_{k=0}^{\infty}\frac{1}{2^k}, \\
&\geq |\mathbb{R}| - 2, \\
&\geq \infty.
\end{aligned}$$