5. First note that for any $x \in \mathbf{R}$, there can only be infinitely many digits after the decimal point. So, it suffices to only consider the decimal digits after the decimal point that are 5. Thus, define

$$f_k = \begin{cases} 1 & \text{if the } k^{\text{th}} \text{ decimal digit of } x = 5, \\ 0 & \text{o/w.} \end{cases}$$

Claim 1. f_k is measurable.

Proof of claim: It suffices to show that $f_k^{-1}(\{0\})$ is an open set. Suppose that $x \in f_k^{-1}(\{0\})$. Then, it follows that the k^{th} decimal digit of x is not 5. So, there exists some $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset f_k^{-1}(\{0\})$. Then, consider the k decimal digit of x which I denote by x_k. If $x_k \neq 4$ and $x_k \neq 6$. Then, $(x - \frac{1}{10^k}, x + \frac{1}{10^k}) \in f_k^{-1}(\{0\})$. Otherwise, if $x_k = 4$ or $x_k = 6$, then it must be that $(x - \frac{1}{10^{k+1}}, x + \frac{1}{10^{k+1}}) \in f_k^{-1}(\{0\})$. Thus, it follows that $f_k^{-1}(\{0\})$ is a Borel set, so f must be measurable. □

Let $g_k = \sum_{i=1}^k f_i$. Then, g_k is also measurable, since it is a sum of measurable functions. Now, it follows that $h(x) = \sup\{g_k(x) : k \in \mathbf{Z}^+\}$ must also be measurable. Thus, $h^{-1}(\{\infty\})$ is exactly the set where there is an infinite number of decimal points and this set by definition must also be a Borel set.

14. (a) If

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcap_{k=j}^{\infty} (f_j - f_k)^{-1}((-1/n, 1/n)),$$

then it follows that for all $n \geq 1$, there exists $j \geq 1$ such that for all $k \geq j$, it must be that $|f_j(x) - f_k(x)| < 1/n$. This implies that for any $n, m > j$, it must be that $|f_n(x) - f_m(x)| \leq |f_n(x) - f_j(x)| + |f_j(x) - f_m(x)| < 2/n$. Thus, $f_1(x), f_2(x), \ldots$ is a Cauchy sequence, so it follows that $f_1(x), f_2(x), \ldots$ converges in $\mathbf{R}$.

(b) We use the equivalence in part (a). Recall that a function is $\mathcal{S}$-measurable if and only if for every Borel set, $\mathcal{B}$, $f^{-1}(\mathcal{B}) \in \mathcal{S}$. Since $(-1/n, 1/n)$ is an open set, then by definition it is a Borel set. Thus, since each f_j is $\mathcal{S}$-measurable, it follows that $f_j - f_k$ is also $\mathcal{S}$-measurable. So, $(f_j - f_k)^{-1}((-1/n, 1/n)) \in \mathcal{S}$. Because $\mathcal{S}$ is a σ-algebra, it must be closed under countable unions and intersections. Therefore, the set must be $\mathcal{S}$-measurable.

15. (a) It must be that $X \in \mathcal{S}$ (this is trivial).

Claim 2. $\mathcal{S}$ is closed under complements.

Proof of claim: Suppose that $A \in \mathcal{S}$. Then, it follows that there exists some $\mathcal{K} \subset \mathbf{Z}^+$ such that $A = \bigcup_{k \in \mathcal{K}} E_k$. Then, it follows that

$$\begin{aligned} (X \setminus A) &= X \setminus \bigcup_{k \in \mathcal{K}} E_k, \\ &= \bigcap_{k \in \mathcal{K}} (X \setminus E_k), \\ &= \bigcap_{k \in \mathcal{K}} \bigcup_{j \in \mathbf{Z}^+ \setminus \{k\}} E_j, \\ & \qquad \text{(since } E_1, E_2, \ldots \text{ are disjoint)} \\ &= \bigcup_{k \in \mathbf{Z}^+ \setminus \mathcal{K}} E_k. \end{aligned}$$

Thus, since $\mathbf{Z}^+ \setminus \mathcal{K} \subset \mathbf{Z}^+$, then it follows that $X \setminus A \in \mathcal{S}$. □

Claim 3. $\mathcal{S}$ is closed under countable unions.

Proof of claim: Suppose a sequence of sets $A_1, A_2, \ldots$ such that any $A_k \in \mathcal{S}$. Then, by the definition of $\mathcal{S}$, it follows that $A_k = \bigcup_{n \in N_k} E_n$. So,

$$\begin{aligned} \bigcup_{k=1}^{\infty} A_k &= \bigcup_{k=1}^{\infty} \bigcup_{n \in N_k} E_n, \\ &= \bigcup_{n \in N} E_n, \end{aligned}$$

for $N = \bigcup_{k=1}^{\infty} N_k$. And since $N \subset \mathbf{Z}^+$, as each $N_k \subset \mathbf{Z}^+$. Thus, $\mathcal{S}$ is closed under countable unions. □

(b)

Claim 4. If f is constant on every E_k, then f is an $\mathcal{S}$-measurable function.

Proof of claim: Formally, if f is constant on every E_k, then

$$f(x) = \sum_{k=1}^{\infty} c_k \chi_{E_k},$$

for $c_k \in \mathbf{R}$. Thus, for any Borel set $\mathcal{B}$,

$$f^{-1}(\mathcal{B}) = \bigcup \{E_k : f(x) = c_k, c_k \in (\mathcal{B} \cap \{c_1, c_2, \ldots\})\}$$

So, it follows by the definition of $\mathcal{S}$ that $f^{-1}(\mathcal{B}) \in \mathcal{S}$. Thus, f is $\mathcal{S}$-measurable. □

Claim 5. If f is $\mathcal{S}$-measurable, then f is constant on every E_k.

Proof of claim: By way of contradiction, assume that f is not constant on every E_k. That is, there exists some $k \in \mathbf{Z}^+$ such that there exists some $x, y \in E_k$ where $f(x) \neq f(y)$. Then, consider the Borel set $\{f(y)\}$. By assumption, $x \notin f^{-1}(\{f(y)\})$, but $y \in f^{-1}(\{f(y)\})$. It remains to show that there does not exist a set $A \in \mathcal{S}$ such that $x \notin A$ and $y \in A$. Since $A = \bigcup_{k \in \mathcal{K}} E_k$ this implies that for all $k \in \mathcal{K}$, $x \notin E_k$, but there exists a set such that $y \in E_k$. This is a contradiction, since by assumption $x, y \in E_j$ for some E_j, but all $E_1, E_2, \ldots$ are disjoint. Thus, if $y \in E_k$ then $x \in E_k$. So, it must be that f is constant on every E_k. □

21. We want to show that $\mathcal{T} = \{A \subset \mathbf{R} : f^{-1}(A) \subset \mathcal{S}\}$ is a σ-algebra. Since for any open interval

$$(a, b) = (a, \infty] \cap \left(\bigcup_{n=1}^{\infty} [-\infty, b - \frac{1}{n}] \right).$$

Since any open set is the countable union of disjoint open intervals, by the construction above $\mathcal{T}$ contains all open sets. Thus, since $\mathcal{T}$ is a σ-algebra, it must also contain all Borel subsets. It remains to show that $\mathcal{T}$ is a σ-algebra. Firstly, $\emptyset \in \mathcal{T}$ since $f^{-1}(\emptyset) = \emptyset$.

Claim 6. $\mathcal{T}$ is closed under complements.

Proof of claim: If $A \in \mathcal{T}$. Then, $f^{-1}(A) \in \mathcal{S}$. So, $f^{-1}(\mathbf{R} \setminus A) = X \setminus f^{-1}(A)$ which is in $\mathcal{S}$ since $\mathcal{S}$ is a σ-algebra. Thus, $\mathbf{R} \setminus A \in \mathcal{T}$. □

Claim 7. $\mathcal{T}$ is closed under countable unions.

Proof of claim: Suppose a sequence of sets $A_1, A_2, \ldots$ such that $A_k \in \mathcal{T}$. Then, it follows that

$$f^{-1}\left(\bigcup_{k=1}^{\infty} A_k \right) = \bigcup_{k=1}^{\infty} f^{-1}(A_k).$$

This last expression is in $\mathcal{S}$ since each $f^{-1}(A_k) \in \mathcal{S}$ and $\mathcal{S}$ is closed under countable unions. So, $\mathcal{T}$ is also closed under countable unions. $\square$

25. We want to construct a sequence of functions such that for any f_k, if $x < y$, then $f_k(x) < f_k(y)$. Let $f_k(x) = f(x) + \frac{x}{k}$. Since by assumption $f(x)$ is an increasing function, for any $x < y$, it must be that $f(x) \leq f(y)$. Moreover, $\frac{x}{k} < \frac{y}{k}$ since $k > 0$. Thus, for any $x < y$, it must be that $f_k(x) < f_k(y)$. Furthermore,

$$\lim_{k\to\infty} f_k(x) = \lim_{k\to\infty} f(x) + \frac{x}{k} = f(x) + \lim_{k\to\infty} \frac{x}{k} = f(x).$$

27. *Remark.* As mentioned in the text, for a measure space $(X, \mathcal{S}$, if $\mathcal{S}$ is the set of all subsets of X, then every function is $\mathcal{S}$-measurable. However, if $\mathcal{S}$ does not contain all subsets of X, say there exists $E \subset X$ such that $E \notin \mathcal{S}$, then the function $\chi_E(x)$ is not $\mathcal{S}$-measurable.

    Thus, the set of measurable functions depends only on $\mathcal{S}$. Also, note that in the proof of Theorem 2.39, since f^{-1} has "σ-algebra" properties, the sets (a, ∞) for all $a \in \mathbf{R}$ can be replaced by any collection of sets as long as they can be used to generate all open intervals in $\mathbf{R}$.

    Since the set of all open interval in $[-\infty, \infty]$ is the same as the set of all open intervals in $\mathbf{R}$, it follows from the proof of Theorem 2.39, that all open intervals can be constructued from (a, ∞). As a result, f must also be measurable.

30. First consider the inside iterated limit as $k \to \infty$. We first show that if $-1 < \cos(j!\pi x) < 1$, then the inner limit goes to 0. See the following claim.

    Claim 8. If $x \in [0, 1]$, then

$$\lim_{a\to\infty} x^a = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

    Proof of claim: If $x = 1$, then $x^a = 1$ for all $a \in \mathbf{R}$. So, $\lim_{a\to\infty} x^a = 1$. On the other hand, consider the case where $x \in [0, 1)$. Let $a = \tilde{a} + b$ such that $\tilde{a} \in \mathbf{Z}$ and $0 \leq b < 1$. Then, it follows that $x^a = x^{\tilde{a}+b} = x^b \cdot x^{\tilde{a}}$. Since $x \in [0, 1)$, $x \leq x^b \leq 1$. Thus, it follows that $x^a \leq x^{\tilde{a}}$, where $\tilde{a} \in \mathbf{Z}^+$. Since $\lim_{a\to\infty} x^{\tilde{a}} = 0$ and $0 \leq x^a$ for all $a \in \mathbf{R}$, it must be that $\lim_{a\to\infty} x^a = 0$. $\square$
    Thus, it follows that

$$\lim_{k\to\infty} \cos(j!\pi x)^{2k} = \begin{cases} 1 & \text{if } \cos(j!\pi x)^2 = 1, \\ 0 & \text{otherwise.} \end{cases}$$

    So, it suffices to show that $\cos(j!\pi x)^2 = 1$ only if $x \in \mathbf{Q}$, as $j \to \infty$. If $x \in \mathbf{Q}$, then by definition, there exists some $a, b \in \mathbf{Z}$ such that $x = \frac{a}{b}$. Thus, by the definition of the limit, for every integer sequence which converges to ∞, $j_1, j_2, \ldots$, there must exist some N such that for all $n > N$, $j_n > |b|$. For all such j_n, it must be that

$$\cos\left(j_n!\pi\frac{a}{b}\right) = \cos(\underbrace{j_n(j_n-1)\cdot\ldots\cdot(|b|+1)(|b|-1)}_{\text{(a)}}\ldots\cdot\pi a).$$

    Since (a) is an integer and $a \in \mathbf{Z}$, it must be that the cosine term is in $\{-1, 1\}$. Thus, the limit converges to 1. On the other hand, if $x \in \mathbf{R}\setminus\mathbf{Q}$, then for every integer sequence $j_1, j_2, \ldots$, there does not exist some j_n such that $j_n!x$ is an integer, otherwise, x would be rational.