Chapter 1

Random Variables and Their Distributions

Problem 1.

Solution. $300 \times 10^6 \times 0.5 \times \dfrac{2}{365^3}$

Problem 2.

Solution. Suppose n identical balls are thrown into m boxes. Let Ω be the set of m-tuples where each entry is a non-negative integer and the sum of the entries is n. Then, by a stars and bars argument, we know that $|\Omega| = \dbinom{n+m-1}{m-1}$. Therefore,

$$\mathbb{P}\left[\text{first box is empty}\right] = \frac{\binom{n+m-2}{m-2}}{\binom{n+m-1}{m-1}} = \frac{m-1}{n+m-1}.$$

Problem 3.

Solution. Let Ω be the set of tuples where the first entry ist he number of defective items and the second is the number of good items. Then, $\mathbb{P}\left[(0,10)\right] = \dfrac{\binom{90}{10}}{\binom{100}{100}} = \dfrac{90 \cdot 89 \cdot \ldots \cdot 81}{100 \cdot 99 \cdot \ldots \cdot 91}$.

Problem 4.

Solution. By way of contradiction, suppose that $\mathbb{P}\left[|\xi| > C\right] > 0$ for $C > 0$. Then, it follows that there exists some $A' \geq 1$ such that $\mathbb{P}\left[|\xi| \geq A'C\right] > 0$. So,

$$\begin{aligned}
\mathbb{E}\left[|\xi|^m\right] &= \sum_{\omega \in \Omega} |\xi(\omega)|^m, \\
&\geq \sum_{\omega \in \{|\xi| \geq A'C\}} |\xi(\omega)|^m, \\
&\geq A'^m C^m \mathbb{P}\left[|\xi| \geq A'C\right].
\end{aligned}$$

Since $\mathbb{P}\left[|\xi| \geq A'C\right]$ is constant, clearly there does not exist $A > 0$ such that $A \geq A'^m$ for all m, thus a contradiction.

Problem 5.

Solution. This is the famous hat-check problem. We find the probability that no one gets the right letter. Let $\Omega = S_n$, the permutation group of n elements, also let C_i be the set of all permutations where the i^{th} element is in the i^{th} position. Formally, $\sigma \in C_i$ if and only if $\sigma(i) = i$. Then, by the inclusion-exclusion principle, we have that

$$\begin{aligned}\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= \mathbb{P}\left[\bigcup_{i=1}^{n} C_i\right],\\ &= \sum_{i=1}^{n} \mathbb{P}\left[C_i\right] - \sum_{i<j} \mathbb{P}\left[C_i \cap C_j\right] + \sum_{i<j<k} \mathbb{P}\left[C_i \cap C_j \cap C_k\right] - \dots.\end{aligned}$$

Now, we derive the expression for each term in the previous sum. For any $\sigma \in C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}$, $\sigma(i_1) = i_1$, $\sigma(i_2) = i_2, \ldots, \sigma(i_k) = i_k$. Thus, $\mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \dfrac{(n-k)!}{n!}$. Therefore, $\displaystyle\sum_{i_1 < i_2 < \ldots < i_k} \mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \binom{n}{k} \frac{(n-k)!}{n!} = \frac{1}{k!}$. So,

$$\begin{aligned}\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + \frac{(-1)^{n+1}}{n!},\\ \lim_{n\to\infty} \mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= 1 - \frac{1}{e}.\end{aligned}$$

Problem 6. Not too sure how to take the limit here.

Solution. The first part of the question is a direct application of a stars and bars argument. Then, there are $\binom{n+r-1}{r-1}$ solutions to the equation $x_1 + \ldots + x_r = n$. By the same argument, we find that

$$\mathbb{P}\left[x_1 = a\right] = \binom{n-a+r-2}{r-2} \Big/ \binom{n+r-1}{r-1} = \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!}.$$

Now, we take the limit as $r, n \to \infty$ and $n/r \to \rho > 0$.

Problem 7.

Solution. Recall, that the Poisson distribution is the measure on $\mathbb{Z}^+$ such that for any elementary outcome k, $\mathbb{P}\left[k\right] = \dfrac{\lambda^k}{e^\lambda k!}$. Let $\xi = \mathrm{Id}_{\mathbb{Z}^+}$. Then,

$$\begin{aligned}\mathbb{E}\left[\xi\right] &= \sum_{n=0}^{\infty} \xi(n)\mathbb{P}[n],\\ &= \sum_{n=1}^{\infty} \frac{\lambda^n}{e^\lambda (n-1)!},\end{aligned}$$

expanding the sum to see the Taylor expansion more clearly, we see that

$$\begin{aligned}&= \frac{\lambda}{e^\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right),\\ &= \frac{\lambda}{e^\lambda} e^\lambda, && \text{(by Taylor expansion)}\\ &= \lambda.\end{aligned}$$

Now, we find the variance using the formula $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}[\xi])^2$. So,

$$
\begin{aligned}
\mathbb{E}\left[\xi^2\right] &= \sum_{n=0}^{\infty} \xi(n)^2 \mathbb{P}[n], \\
&= e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n (n+1)}{n!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} + \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \right), \\
&= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda e^{\lambda} \right), \\
&= \lambda^2 + \lambda.
\end{aligned}
$$

Therefore, $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}[\xi])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$ and the variance of the Poisson distribution is the same as its expected value.

Problem 9.

Solution. Recall that if F is a distribution function of a random variable ξ then, $F_\xi(x) = \mathbb{P}\left[\{\xi(\omega) < x : \omega \in \Omega\}\right]$. We then evaluate the expression on the right-hand side first

$$
\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \lim_{\delta \downarrow 0} \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x - \delta\}\right] = \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\}\right],
$$

where $\delta_1, \delta_2, \ldots$ is any sequence where $\delta_i \ge 0$ and converges to 0. We first show that

$$
\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\} = \{\omega \in \Omega : \xi(\omega) < x\},
$$

That the left-hand side is a subset of the right-hand side is trivial. Now, suppose an element ω in the set on the right-hand side. There must exist some $\varepsilon > 0$, such that $\xi(\omega) = x - \varepsilon$. By convergence of $\delta_1, \delta_2, \ldots$, there must exist some δ_i where $\xi(\omega) = x - \varepsilon \le x - \delta_i$. Thus, the two sets are equivalent. Therefore,

$$
\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) < x\}\right], \\
&= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) = x\}\right], \\
&= \mathbb{P}\left[\xi = x\right].
\end{aligned}
$$

The second equality follows σ-additivity of $\mathbb{P}$.

Problem 10.

Solution. Let F be the distribution of ξ and let $\eta = a\xi + b$ where $a \ne 0$. We first find the distribution of η:

$$
\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \eta(\omega) \le x\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : a\xi(\omega) + b \le x\}\right], \\
&= \mathbb{P}\left[\left\{\omega \in \Omega : \xi(\omega) \le \frac{x-b}{a}\right\}\right], \\
&= F\left(\frac{x-b}{a}\right).
\end{aligned}
$$

Thus, taking the derivative to yield the density function, we have that $p_\eta(x) = \frac{1}{a} p_\xi \left(\frac{x-b}{a} \right)$ where p_ξ is the density of ξ.

Problem 11.

Solution. To show this we need a few simple measure-theoretic facts:

1. By definition, the density function of a random variable uniformly distributed on some set $A \subset \mathbb{R}$ where $\lambda(A) < \infty$ is $\frac{1}{\lambda(A)}$. Moreover, for any measurable-subset $B \subset A$, $\mathbb{P}_A[B] = \frac{\lambda(B)}{\lambda(A)}$. For brevity, We call $\mathbb{P}_A$ the Lebesgue probability measure on A.
2. Let $C \subset D \subset G \subset \mathbb{R}$ (be measurable-subsets of $\mathbb{R}$) and $\mathbb{P}_D, \mathbb{P}_G$ be the Lebesgue probability measures on D, G, respectively. Then, by definition, $\mathbb{P}_D[C] = \frac{\lambda(C)}{\lambda(D)}$ and $\mathbb{P}_G[D] = \frac{\lambda(D)}{\lambda(G)}$. It follows that $\mathbb{P}_G[C] = \frac{\lambda(C)}{\lambda(G)} = \mathbb{P}_D[C]\mathbb{P}_G[D]$.

Let $(\Omega_1, \mathcal{G}_1, \mathbb{P}_{\Omega_1})$ and $(\Omega_2, \mathcal{G}_2, \mathbb{P}_{\Omega_2})$ be the probability subspaces of $[0, 2\pi]$ with the Lebesgue probability measure, where $\Omega_1 = [0, \pi/2] \cup [3\pi/2, 2\pi]$ and $\Omega_2 = [\pi/2, 3\pi/2]$. Also, let $\mathbb{P}_{\Omega_1}, \mathbb{P}_{\Omega_2}$ be the Lebesgue probability measures on Ω_1, Ω_2, respectively. Suppose $\xi = \mathrm{Id}_{[0,2\pi]}$, $\xi_1 = \mathrm{Id}_{[0,\pi/2]\cup[3\pi/2,2\pi]}$, $\xi_2 = \mathrm{Id}_{[\pi/2,3\pi/2]}$, and $\eta = \sin(\xi)$. Then, we start by finding the distribution of η:

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in [0, 2\pi] : \eta(\omega) \le x\}\right] &= \mathbb{P}\left[\{\omega \in [0, \pi/2] \cup [3\pi/2, 2\pi] : \eta(\xi(\omega)) \le x\}\right] \\
&\qquad + \mathbb{P}\left[\{\omega \in [\pi/2, 3\pi/2] : \eta(\xi(\omega)) \le x\}\right], \\
&= \frac{1}{2}\mathbb{P}_{\Omega_1}\left[\{\omega \in \Omega_1 : \eta(\xi_1(\omega)) \le x\}\right] + \frac{1}{2}\mathbb{P}_{\Omega_2}\left[\{\omega \in \Omega_2 : \eta(\xi_2(\omega)) \le x\}\right], \\
&= \frac{1}{2}F_1(\arcsin(x)) + \frac{1}{2}F_2(\arcsin(x)),
\end{aligned}$$

where F_1, F_2 are the distribution functions for ξ_1, ξ_2, respectively. Differentiating with respect to x, we yield the density function:

$$\begin{aligned}
\frac{1}{2}\left(F_1(\arcsin(x)) + F_2(\arcsin(x))\right)' &= \frac{1}{2\sqrt{1-x^2}}\left(F_1'(\arcsin x) + F_2'(\arcsin x)\right), \\
&= \frac{1}{2\sqrt{1-x^2}}\left(\frac{1}{\pi} + \frac{1}{\pi}\right), \\
&= \frac{1}{\pi\sqrt{1-x^2}}.
\end{aligned}$$

Problem 12.

Solution. Let $\mathcal{B}(X)$ be the Borel sets of X, the σ-algebra of all of the open sets in X. Also let $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n)$ be the product σ-algebra of $X_1, \ldots, X_n$. That $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n) \subset \mathcal{B}(X)$ is trivial since the product of open sets in $X_1, \ldots, X_n$ must be an open set in X. It remains to show inclusion in the other direction. Since X is a separable metric space, it is second-countable. Moroever, we know that $B = \{x_1 \times \ldots \times x_n : x_1 \in X_1, \ldots, x_n \in X_n\}$ forms a basis of the product topology. Thus, any open set in X can be constructed through the countable unions of sets in B. And so, since σ-algebras are closed under countable unions the inclusion holds in the other direction.

Problem 13.

Solution. For any $n > 5$, $n^4 \ge 1000$. Therefore, we are mostly concerned with the set of all perfect squares and cubes less than or equal to 1000. Thus, we find these first using the inclusion-exclusion principle. Then, brute-force count the remaining numbers that do not fall into these sets.

$$\begin{aligned}
\sharp\left\{1 \leq n \leq 1000 : n^2 \leq 1000 \text{ or } n^3 \leq 1000\right\} &= \sharp\left\{1 \leq n \leq 1000 : n^2 \leq 1000\right\} + \sharp\left\{1 \leq n \leq 1000 : n^3 \leq 1000\right\} \\
&\quad - \sharp\left\{1 \leq n \leq 1000 : n^6 \leq 1000\right\}, \\
&= \lfloor\sqrt{1000}\rfloor + \lfloor\sqrt[3]{1000}\rfloor - 2, \\
&= 31 + 10 - 3, \\
&= 39.
\end{aligned}$$

It remains to account for numbers where the exponential by $5, 7$ are within 1000. This only holds for $2^5, 2^7, 3^5$. Therefore, there are 41 numbers less than or equal to 1000 that are integer powers (greater than 1) of another integer. Thus, supposing a uniform distribution, the probability of choosing such an integer is $\dfrac{41}{1000}$.

Problem 14.

Solution. This is a weaker version of the Borel-Cantelli lemma. Consider the Lebesgue probability measure on $[0, 1]$. And then consider the events $\left[0, \dfrac{1}{n}\right]$ for $n = 1, 2, \ldots$. It follows that $\displaystyle\sum_{k=1}^{\infty} \mathbb{P}\left[\left[0, \frac{1}{n}\right]\right] \not< \infty$, but we know that the limit superior of the probability of this sequence of events is 0. Such a sequence of events cannot be described by the stronger version of the Borel-Cantelli lemma, however, the preconditions in this problem allow for this "counter example."

First, we show the following lemma:

Lemma 1.1. For any indexed collection of sets $C_1, C_2, \ldots, C_n$,

$$\bigcup_{k=1}^{n} C_k = C_1 \cup \bigcup_{k=1}^{n-1} C_{k+1} \setminus C_k.$$

Proof. We proceed by way of induction. Suppose that $n = 2$, then $C_1 \cup (C_2 \setminus C_1) = C_1 \cup (C_1^c \cap C_2) = (C_1 \cup C_1^c) \cap (C_2 \cup C_1) = C_1 \cup C_2$. Assuming the claim true for all $n < N$, we now prove it for $n = N$:

$$\begin{aligned}
C_1 \cup \bigcup_{k=1}^{n-1} (C_{k+1} \setminus C_k) &= \left(C_1 \cup \bigcup_{k=1}^{n-2} (C_{k+1} \setminus C_k)\right) \cup (C_n \setminus C_{n-1}), \\
&= \left(\bigcup_{k=1}^{n-1} C_k\right) \cup (C_{n-1}^c \cap C_n), && \text{(by ind. hyp.)} \\
&= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup \left(C_{n-1} \cup (C_{n-1}^c \cap C_n)\right), \\
&= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup (C_n \cup C_{n-1}), \\
&= \bigcup_{k=1}^{n} C_k.
\end{aligned}$$

□

With this, we show the desired result:

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} C_k\right] &= \lim_{n\to\infty}\mathbb{P}\left[\bigcup_{k=n}^{\infty} C_k\right], \\
&= \lim_{n\to\infty}\mathbb{P}\left[C_n \cup \bigcup_{k=n}^{\infty} C_{k+1}\setminus C_k\right], \\
&\leq \lim_{n\to\infty}\mathbb{P}[C_n] + \lim_{n\to\infty}\sum_{k=n}^{\infty}\mathbb{P}[C_{k+1}\setminus C_k], \\
&= 0.
\end{aligned}$$

The second equality follows from Lemma 1.1 and the last equality follows from the given.

Problem 15.

Solution. If F is a continuous distribution of a random variable ξ on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$, then it must be that F is injective on $\operatorname{img}\xi$. Consider, now a random variable $\eta = F(\xi)$. We proceed to find the distribution of this η:

$$\begin{aligned}
\mathbb{P}[\{\omega \in \Omega : \eta(\xi(\omega)) \leq x\}] &= \mathbb{P}[\big\{\omega \in \Omega : \xi(\omega) \leq F^{-1}(x)\big\}], \\
&= F(F^{-1}(x)).
\end{aligned}$$

Note that unless F is bijective, $F \circ F^{-1} \neq \mathrm{Id}$. Recall, that there exists a right-inverse if and only if F is surjective.

Chapter 2

Sequences of Independent Trials

Some brief notes before the problems:

- Koralov and Sinai use $\times$ instead of $\otimes$ for the product σ-algebra. Moving forward, the reader needs to be cautious about what the operands are. For if they are σ-algebras, we understand the result of the expression to be the aforementioned product.

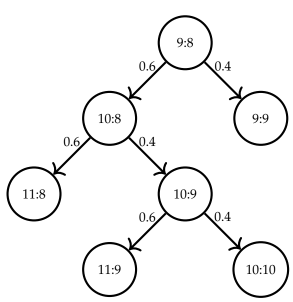
Problem 1.

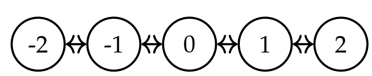
Solution. Directly apply the definition of the binomial distribution. Let $X = \{0, 1\}$. Define the random variable $\chi_i(\omega)$ for $1 \leq i \leq 5$ to be 1 if $\omega = 1$ and zero otherwise. Then, let $\nu_5 = \sum_{i=1}^{5} \chi_i$. ν_5 is a binomial random variable. Thus,

$$\mathbb{P}[\nu_5 = 3] = \binom{5}{3} \frac{1}{2^5} = \frac{5}{16}.$$

Problem 2.

Solution. We present two alternative ways to solve this problem one involving chains and the other geometric series. We first present the solution involving chains. We draw the state space below, where each node denotes a distinct score and the edges between states represent the transition probabilities





It remains to find the probability that Andrew wins from either 9:9 or 10:10 which is the same since both states require Andrew to win two consecutive points. This, then induces a new chain (see above). Denote by a, b, c the probability that Andrew wins from being one point down, drawn, and one point up. Then,

$$\begin{aligned} a &= 0.6b, \\ b &= 0.4a + 0.6c, \\ c &= 0.4b + 0.6b. \end{aligned}$$

So, $c = 0.36/0.52$. Thus, the probability that Andrew wins is then the aggregate of all states in the chain multiplied by their transition probabilities:

$$\mathbb{P}[\text{Andrew wins}] = 0.4\left(\frac{0.36}{0.52}\right) + 0.6\left(0.6 + 0.4\left(0.4\left(\frac{0.36}{0.52}\right) + 0.6\right)\right) = 0.847.$$

Problem 3.

Solution. We directly apply the de Moivre-Laplace Theorem. Let $X_i(\omega_i)$ be a random variable that is 1 if ω_i = heads and 0 otherwise. Then, let $\nu_n = \sum_{i=1}^{n} X_i$. Then, for $n = 1000$, $\mathbb{E}[\nu_n] = 500$ and $\operatorname{Var}\nu_n = 250$. $k = 600$ as the actual number of heads that we saw during all of the coin flips. Therefore, $z = (600 - 500)/\sqrt{250} = 20/\sqrt{(10)}$. Therefore, by the de Moivre-Laplace theorem:

$$\begin{aligned} \mathbb{P}\left[\frac{\nu_n - \mathbb{E}[\nu_n]}{\sqrt{\operatorname{Var}(\nu_n)}} \geq z\right] &\geq \int_{20/\sqrt{10}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)\, dz, \\ &\leq 10^{-5}. \end{aligned}$$

So, it is highly unlikely that the coin is fair.

Problem 4.

Solution.

Problem 5.

Solution. Let ν^n be the number of sixes that appear from n tosses of a die. Then, it follows that $\mathbb{E}\left[\nu^{12000}\right] = 2000$ and $\operatorname{Var}\left[\nu^{12000}\right] = 12000 \cdot \frac{5}{36} \approx 1666$. Using the de Moivre-Laplace Theorem, we approximate $\frac{\nu^n - \mathbb{E}[\nu^n]}{\sqrt{\operatorname{Var}[\nu^n]}}$ as a standard normal distribution. Therefore,

$$\mathbb{P}[1900 \leq \nu^n \leq 2100] = \Phi(2.45) - \Phi(-2.45),$$

where $\Phi(\cdot)$ is the distribution function of the normal distribution.

Problem 6.

Solution.

Problem 7.

Solution.

Problem 8.

Solution.

Chapter 3

Lebesgue Integral and Mathematical Expectation

Problem 1.
Solution.