

Need to Knows

Theory of Probability and Random Processes: Koralov and Sinai

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Here are a list of things, both theorems and problems, that I think are important to know how to do by heart. Therefore, they warrant frequent review.

Key

❄: Things that I need to know off the top of my head.

📖: Recall the result off the top of my head and if you gave me 15 minutes, I could give you a formal proof or explanation.

📌: Elegant results that serve as great examples of ❄ or 📖 concepts. These are also results that if you gave me 30 minutes to an hour, I should be able to derive.

👉: I can recall this theorem without searching it up or referencing any text.

1 Random Variables and Their Distributions

- ❄ Define a probability space.
- ❄ Define a random variable
- ❄ Define a probability distribution of a random variable (or the probability distribution induced by a random variable).
- ❄ Define the expectation of a random variable.
- ❄ Define the variance of a random variable. *Prove that* $\text{Var } \xi = \mathbb{E} [\xi^2] - \mathbb{E} [\xi]^2$.
- 📖 *Prove* Markov's inequality.
- 📖 *Prove* Chebyshev's inequality.
- ❄ Define the correlation coefficient of two random variables.
- 📖 *Prove* that the correlation coefficient is less than or equal to 1. And if the correlation coefficient of ξ_1, ξ_2 is 1, then almost surely $\mathbb{E} [\xi_1] = a\mathbb{E} [\xi_2] + b$ for some $a, b \in \mathbb{R}$.
- 📖 *Define* a distribution function.
- 📖 *Prove* that any distribution function is right-continuous.
- ❄ *Prove* the Borel-Cantelli lemma.
- 📖 *Prove* that both the mean and variance of the Poisson distribution, $\text{Pois}(\lambda)$, is λ .
- 📌 Let x_1, x_2 be two integers randomly and independently chosen from the set $\{1, 2, \dots, n\}$ according to the uniform distribution. Denote this probability measure as \mathbb{P}^n . What is the space of elementary outcomes? Let A^n be the probability that the two numbers are coprime. What is $\lim_{n \rightarrow \infty} \mathbb{P}^n[A^n]$.
- 📌 Suppose there are n letters addressed to n different people, and n envelopes with addresses. The letters are mixed and then randomly placed into the envelopes. Find the probability that at least one letter is in the correct envelope. Find the limit of this probability as $n \rightarrow \infty$.

2 Sequences of Independent Trials

- ❄ Define a homogeneous sequence of independent random variables.
- ❄ *Prove* the law of large numbers.
- 👉 *Prove* the MacMillan Theorem.
- 📌 *Prove* using Bernstein polynomials, Weierstrass' Theorem.
- 📖 *Prove* the Glivenko-Cantelli Theorem.
- 📌 *Prove* the Poisson Limit Theorem.

3 Lebesgue Integral and Mathematical Expectation

- ❄ Define the Riemann-Stieltjes integral.

- ❄ Define almost surely convergence.
- ❄ Define convergence in measure.
- ❄ Define L^p convergence.

4 Conditional Probabilities and Independence

- ❄ Define conditional probability.
- ❄ Prove Bayes' Theorem.
- ❄ Define independence of two or more events, σ -algebras, and random variables.
- ☞ Prove if two random variables are independent, then the expectation of their product is equal to the product of their expectations.
- ❄ Define a π -system and Dynkin system.
- ☞ Let ξ_1, \dots, ξ_n be independent random variables, $m_1 + \dots + m_k = n$ and f_1, \dots, f_k be measurable functions of m_1, \dots, m_k variables respectively. Then, prove the random variables $f_1(\xi_1, \dots, \xi_{m_1}), \dots, f_k(\xi_{m_1+\dots+m_{k-1}+1}, \dots, \xi_n)$ are independent.

5 Markov Chains with a Finite Number of States

- ❄ Define a stochastic matrix.
- ❄ Prove that a product of stochastic matrices is a stochastic matrix.
- ❄ Define a Markov chain.
- ❄ Define a homogeneous Markov chain.
- ❄ Define an ergodic stochastic matrix. Define a homogeneous, ergodic, Markov chain.
- ❄ Define a stationary distribution for a matrix of transition probabilities.

6 Random Walks on the Lattice \mathbb{Z}^d

- ❄ Define a random walk on \mathbb{Z}^d .
- ❄ Define a spatially homogeneous random walk.
- 👉 What is Polya's Theorem?
- ☞ Prove the reflection principle.