Chapter 1

Random Variables and Their Distributions

Problem 1.

Solution. $300 \times 10^6 \times 0.5 \times \dfrac{2}{365^3}$

Problem 2.

Solution. Suppose n identical balls are thrown into m boxes. Let Ω be the set of m-tuples where each entry is a non-negative integer and the sum of the entries is n. Then, by a stars and bars argument, we know that $|\Omega| = \dbinom{n+m-1}{m-1}$. Therefore,

$$\mathbb{P}\left[\text{first box is empty}\right] = \frac{\binom{n+m-2}{m-2}}{\binom{n+m-1}{m-1}} = \frac{m-1}{n+m-1}.$$

Problem 3.

Solution. Let Ω be the set of tuples where the first entry ist he number of defective items and the second is the number of good items. Then, $\mathbb{P}\left[(0,10)\right] = \dfrac{\binom{90}{10}}{\binom{100}{100}} = \dfrac{90 \cdot 89 \cdot \ldots \cdot 81}{100 \cdot 99 \cdot \ldots \cdot 91}$.

Problem 4.

Solution. By way of contradiction, suppose that $\mathbb{P}\left[|\xi| > C\right] > 0$ for $C > 0$. Then, it follows that there exists some $A' \geq 1$ such that $\mathbb{P}\left[|\xi| \geq A'C\right] > 0$. So,

$$\begin{aligned}
\mathbb{E}\left[|\xi|^m\right] &= \sum_{\omega \in \Omega} |\xi(\omega)|^m, \\
&\geq \sum_{\omega \in \{|\xi| \geq A'C\}} |\xi(\omega)|^m, \\
&\geq A'^m C^m \mathbb{P}\left[|\xi| \geq A'C\right].
\end{aligned}$$

Since $\mathbb{P}\left[|\xi| \geq A'C\right]$ is constant, clearly there does not exist $A > 0$ such that $A \geq A'^m$ for all m, thus a contradiction.

Problem 5.

Solution. This is the famous hat-check problem. We find the probability that no one gets the right letter. Let $\Omega = S_n$, the permutation group of n elements, also let C_i be the set of all permutations where the i^{th} element is in the i^{th} position. Formally, $\sigma \in C_i$ if and only if $\sigma(i) = i$. Then, by the inclusion-exclusion principle, we have that

$$\begin{aligned}\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= \mathbb{P}\left[\bigcup_{i=1}^{n} C_i\right], \\ &= \sum_{i=1}^{n} \mathbb{P}\left[C_i\right] - \sum_{i<j} \mathbb{P}\left[C_i \cap C_j\right] + \sum_{i<j<k} \mathbb{P}\left[C_i \cap C_j \cap C_k\right] - \dots.\end{aligned}$$

Now, we derive the expression for each term in the previous sum. For any $\sigma \in C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}$, $\sigma(i_1) = i_1$, $\sigma(i_2) = i_2, \ldots, \sigma(i_k) = i_k$. Thus, $\mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \dfrac{(n-k)!}{n!}$. Therefore, $\displaystyle\sum_{i_1 < i_2 < \ldots < i_k} \mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$. So,

$$\begin{aligned}\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + \frac{(-1)^{n+1}}{n!}, \\ \lim_{n\to\infty} \mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] &= 1 - \frac{1}{e}.\end{aligned}$$

Problem 6. Not too sure how to take the limit here.

Solution. The first part of the question is a direct application of a stars and bars argument. Then, there are $\binom{n+r-1}{r-1}$ solutions to the equation $x_1 + \ldots + x_r = n$. By the same argument, we find that

$$\mathbb{P}\left[x_1 = a\right] = \binom{n-a+r-2}{r-2} \Big/ \binom{n+r-1}{r-1} = \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!}.$$

Now, we take the limit as $r, n \to \infty$ and $n/r \to \rho > 0$.

Problem 7. Missing variance of the Poisson distribution.

Solution. Recall, that the Poisson distribution is the measure on $\mathbb{Z}^+$ such that for any elementary outcome k, $\mathbb{P}\left[k\right] = \dfrac{\lambda^k}{e^\lambda k!}$. Let $\xi = \mathrm{Id}_{\mathbb{Z}^+}$. Then,

$$\begin{aligned}\mathbb{E}\left[\xi\right] &= \sum_{n=0}^{\infty} \xi(n)\mathbb{P}[n], \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n}{e^\lambda (n-1)!},\end{aligned}$$

expanding the sum to see the Taylor expansion more clearly, we see that

$$\begin{aligned}&= \frac{\lambda}{e^\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \ldots\right), \\ &= \frac{\lambda}{e^\lambda} e^\lambda, && \text{(by Taylor expansion)} \\ &= \lambda.\end{aligned}$$

Now, we find the variance using the formula $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}\left[\xi\right])^2$.

Problem 9.

Solution. Recall that if F is a distribution function of a random variable ξ then, $F_\xi(x) = \mathbb{P}\left[\{\xi(\omega) < x : \omega \in \Omega\}\right]$. We then evaluate the expression on the right-hand side first

$$\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \leq x\}\right] - \lim_{\delta \downarrow 0} \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \leq x - \delta\}\right] = \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \leq x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\}\right],$$

where $\delta_1, \delta_2, \ldots$ is any sequence where $\delta_i \geq 0$ and converges to 0. We first show that

$$\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\} = \{\omega \in \Omega : \xi(\omega) < x\},$$

That the left-hand side is a subset of the right-hand side is trivial. Now, suppose an element ω in the set on the right-hand side. There must exist some $\varepsilon > 0$, such that $\xi(\omega) = x - \varepsilon$. By convergence of $\delta_1, \delta_2, \ldots$, there must exist some δ_i where $\xi(\omega) = x - \varepsilon \leq x - \delta_i$. Thus, the two sets are equivalent. Therefore,

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \leq x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \leq x\}\right] - \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) < x\}\right], \\
&= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) = x\}\right], \\
&= \mathbb{P}\left[\xi = x\right].
\end{aligned}$$

The second equality follows σ-additivity of $\mathbb{P}$.

Problem 10.

Solution. Let F be the distribution of ξ and let $\eta = a\xi + b$ where $a \neq 0$. We first find the distribution of η:

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \eta(\omega) \leq x\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : a\xi(\omega) + b \leq x\}\right], \\
&= \mathbb{P}\left[\left\{\omega \in \Omega : \xi(\omega) \leq \frac{x-b}{a}\right\}\right], \\
&= F\left(\frac{x-b}{a}\right).
\end{aligned}$$

Thus, taking the derivative to yield the density function, we have that $p_\eta(x) = \frac{1}{a} p_\xi\left(\frac{x-b}{a}\right)$ where p_ξ is the density of ξ.

Problem 11.

Solution. To show this we need a few simple measure-theoretic facts:

1. By definition, the density function of a random variable uniformly distributed on some set $A \subset \mathbb{R}$ where $\lambda(A) < \infty$ is $\frac{1}{\lambda(A)}$. Moreover, for any measurable-subset $B \subset A$, $\mathbb{P}_A[B] = \frac{\lambda(B)}{\lambda(A)}$. For brevity, We call $\mathbb{P}_A$ the Lebesgue probability measure on A.
2. Let $C \subset D \subset G \subset \mathbb{R}$ (be measurable-subsets of $\mathbb{R}$) and $\mathbb{P}_D, \mathbb{P}_G$ be the Lebesgue probability measures on D, G, respectively. Then, by definition, $\mathbb{P}_D[C] = \frac{\lambda(C)}{\lambda(D)}$ and $\mathbb{P}_G[D] = \frac{\lambda(D)}{\lambda(G)}$. It follows that $\mathbb{P}_G[C] = \frac{\lambda(C)}{\lambda(G)} = \mathbb{P}_D[C]\mathbb{P}_G[D]$.

Let $(\Omega_1, \mathcal{G}_1, \mathbb{P}_{\Omega_1})$ and $(\Omega_2, \mathcal{G}_2, \mathbb{P}_{\Omega_2})$ be the probability subspaces of $[0, 2\pi]$ with the Lebesgue probability measure, where $\Omega_1 = [0, \pi/2] \cup [3\pi/2, 2\pi]$ and $\Omega_2 = [\pi/2, 3\pi/2]$. Also, let $\mathbb{P}_{\Omega_1}, \mathbb{P}_{\Omega_2}$ be the Lebesgue probability measures on Ω_1, Ω_2, respectively. Suppose $\xi = \mathrm{Id}_{[0,2\pi]}$, $\xi_1 = \mathrm{Id}_{[0,\pi/2]\cup[3\pi/2,2\pi]}$, $\xi_2 = \mathrm{Id}_{[\pi/2,3\pi/2]}$, and

$\eta = \sin(\xi)$. Then, we start by finding the distribution of η:

$$\begin{aligned}\mathbb{P}\left[\{\omega \in [0,2\pi] : \eta(\omega) \le x\}\right] &= \mathbb{P}\left[\{\omega \in [0,\pi/2] \cup [3\pi/2, 2\pi] : \eta(\xi(\omega)) \le x\}\right] \\ &\qquad + \mathbb{P}\left[\{\omega \in [\pi/2, 3\pi/2] : \eta(\xi(\omega)) \le x\}\right], \\ &= \frac{1}{2}\mathbb{P}_{\Omega_1}\left[\{\omega \in \Omega_1 : \eta(\xi_1(\omega)) \le x\}\right] + \frac{1}{2}\mathbb{P}_{\Omega_2}\left[\{\omega \in \Omega_2 : \eta(\xi_2(\omega)) \le x\}\right], \\ &= \frac{1}{2}F_1(\arcsin(x)) + \frac{1}{2}F_2(\arcsin(x)),\end{aligned}$$

where F_1, F_2 are the distribution functions for ξ_1, ξ_2, respectively. Differentiating with respect to x, we yield the density function:

$$\begin{aligned}\frac{1}{2}\left(F_1(\arcsin(x)) + F_2(\arcsin(x))\right)' &= \frac{1}{2\sqrt{1-x^2}}\left(F_1'(\arcsin x) + F_2'(\arcsin x)\right), \\ &= \frac{1}{2\sqrt{1-x^2}}\left(\frac{1}{\pi} + \frac{1}{\pi}\right), \\ &= \frac{1}{\pi\sqrt{1-x^2}}.\end{aligned}$$

Problem 12.

Solution.

Problem 13.

Solution.

Problem 14.

Solution. This is essentially the Borel-Cantelli lemma.

Problem 15.

Solution. If F is a continuous distribution of a random variable ξ on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$, then it must be that F is injective on $\operatorname{img} \xi$. Consider, now a random variable $\eta = F(\xi)$. We proceed to find the distribution of this η:

$$\begin{aligned}\mathbb{P}[\{\omega \in \Omega : \eta(\xi(\omega)) \le x\}] &= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \le F^{-1}(x)\}], \\ &= F(F^{-1}(x)).\end{aligned}$$

Note that unless F is bijective, $F \circ F^{-1} \neq \mathrm{Id}$. Recall, that there exists a right-inverse if and only if F is surjective.

Chapter 2

Sequences of Independent Trials