

Chapter 1

Random Variables and Their Distributions

Problem 1.

Solution. $300 \times 10^6 \times 0.5 \times \frac{2}{365^3}$

Problem 2.

Solution. Suppose n identical balls are thrown into m boxes. Let Ω be the set of m -tuples where each entry is a non-negative integer and the sum of the entries is n . Then, by a stars and bars argument, we know that

$|\Omega| = \binom{n+m-1}{m-1}$. Therefore,

$$\mathbb{P}[\text{first box is empty}] = \frac{\binom{n+m-2}{m-2}}{\binom{n+m-1}{m-1}} = \frac{m-1}{n+m-1}.$$

Problem 3.

Solution. Let Ω be the set of tuples where the first entry is the number of defective items and the second is the number of good items. Then, $\mathbb{P}[(0, 10)] = \frac{\binom{90}{10}}{\binom{100}{100}} = \frac{90 \cdot 89 \cdot \dots \cdot 81}{100 \cdot 99 \cdot \dots \cdot 91}$.

Problem 4.

Solution. By way of contradiction, suppose that $\mathbb{P}[|\xi| > C] > 0$ for $C > 0$. Then, it follows that there exists some $A' \geq 1$ such that $\mathbb{P}[|\xi| \geq A'C] > 0$. So,

$$\begin{aligned} \mathbb{E}[|\xi|^m] &= \sum_{\omega \in \Omega} |\xi(\omega)|^m, \\ &\geq \sum_{\omega \in \{|\xi| \geq A'C\}} |\xi(\omega)|^m, \\ &\geq A'^m C^m \mathbb{P}[|\xi| \geq A'C]. \end{aligned}$$

Since $\mathbb{P}[|\xi| \geq A'C]$ is constant, clearly there does not exist $A > 0$ such that $A \geq A'^m$ for all m , thus a contradiction.

Problem 5.

Solution. This is the famous hat-check problem. We find the probability that no one gets the right letter. Let $\Omega = S_n$, the permutation group of n elements, also let C_i be the set of all permutations where the i^{th} element is in the i^{th} position. Formally, $\sigma \in C_i$ if and only if $\sigma(i) = i$. Then, by the inclusion-exclusion principle, we have that

$$\begin{aligned} \mathbb{P}[\{\sigma : \exists i, \sigma(i) = i\}] &= \mathbb{P}\left[\bigcup_{i=1}^n C_i\right], \\ &= \sum_{i=1}^n \mathbb{P}[C_i] - \sum_{i < j} \mathbb{P}[C_i \cap C_j] + \sum_{i < j < k} \mathbb{P}[C_i \cap C_j \cap C_k] - \dots \end{aligned}$$

Now, we derive the expression for each term in the previous sum. For any $\sigma \in C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}$, $\sigma(i_1) = i_1$, $\sigma(i_2) = i_2, \dots, \sigma(i_k) = i_k$. Thus, $\mathbb{P}[C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}] = \frac{(n-k)!}{n!}$. Therefore, $\sum_{i_1 < i_2 < \dots < i_k} \mathbb{P}[C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}] = \binom{n}{k} \frac{(n-k)!}{n!} = \frac{1}{k!}$. So,

$$\begin{aligned} \mathbb{P}[\{\sigma : \exists i, \sigma(i) = i\}] &= 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!}, \\ \lim_{n \rightarrow \infty} \mathbb{P}[\{\sigma : \exists i, \sigma(i) = i\}] &= 1 - \frac{1}{e}. \end{aligned}$$

Problem 6.

Solution. The first part of the question is a direct application of a stars and bars argument. Then, there are $\binom{n+r-1}{r-1}$ solutions to the equation $x_1 + \dots + x_r = n$. By the same argument, we find that

$$\mathbb{P}[x_1 = a] = \frac{\binom{n-a+r-2}{r-2}}{\binom{n+r-1}{r-1}} = \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!}.$$

Now, we take the limit as $r, n \rightarrow \infty$ and $n/r \rightarrow \rho > 0$.

Problem 7.

Solution. Recall, that the Poisson distribution is the measure on \mathbb{Z}^+ such that for any elementary outcome k , $\mathbb{P}[k] = \frac{\lambda^k}{e^\lambda k!}$. Let $\xi = \text{Id}_{\mathbb{Z}^+}$. Then,

$$\begin{aligned} \mathbb{E}[\xi] &= \sum_{n=0}^{\infty} \xi(n) \mathbb{P}[n], \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n}{e^\lambda (n-1)!}, \end{aligned}$$

expanding the sum to see the Taylor expansion more clearly, we see that

$$\begin{aligned} &= \frac{\lambda}{e^\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \dots \right), \\ &= \frac{\lambda}{e^\lambda} e^\lambda, \quad (\text{by Taylor expansion}) \\ &= \lambda. \end{aligned}$$

Problem 9.

Solution. Recall that if F is a distribution function of a random variable ξ then, $F_\xi(x) = \mathbb{P}[\{\xi(\omega) < x : \omega \in \Omega\}]$. We then evaluate the expression on the right-hand side first

$$\mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \lim_{\delta \downarrow 0} \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x - \delta\}] = \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\}\right],$$

where $\delta_1, \delta_2, \dots$ is any sequence where $\delta_i \geq 0$ and converges to 0. We first show that

$$\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\} = \{\omega \in \Omega : \xi(\omega) < x\},$$

That the left-hand side is a subset of the right-hand side is trivial. Now, suppose an element ω in the set on the right-hand side. There must exist some $\varepsilon > 0$, such that $\xi(\omega) = x - \varepsilon$. By convergence of $\delta_1, \delta_2, \dots$, there must exist some δ_i where $\xi(\omega) = x - \varepsilon \leq x - \delta_i$. Thus, the two sets are equivalent. Therefore,

$$\begin{aligned} \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\}\right] &= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \mathbb{P}[\{\omega \in \Omega : \xi(\omega) < x\}], \\ &= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) = x\}], \\ &= \mathbb{P}[\xi = x]. \end{aligned}$$

The second equality follows σ -additivity of \mathbb{P} .

Problem 10.

Solution. Let F be the distribution of ξ and let $\eta = a\xi + b$ where $a \neq 0$. We first find the distribution of η :

$$\begin{aligned} \mathbb{P}[\{\omega \in \Omega : \eta(\omega) \leq x\}] &= \mathbb{P}[\{\omega \in \Omega : a\xi(\omega) + b \leq x\}], \\ &= \mathbb{P}\left[\left\{\omega \in \Omega : \xi(\omega) \leq \frac{x-b}{a}\right\}\right], \\ &= F\left(\frac{x-b}{a}\right). \end{aligned}$$

Thus, taking the derivative to yield the density function, we have that $p_\eta(x) = \frac{1}{a}p_\xi\left(\frac{x-b}{a}\right)$ where p_ξ is the density of ξ .

Problem 11.

Solution. We proceed in the same way as the previous problem, by finding the distribution function of $\eta = \sin \xi$, where $\xi = \text{Id}_{[0, 2\pi]}$. Define $\xi' = \frac{1}{2}(\xi - \pi)$.

Problem 12.

Solution.

Problem 13.

Solution.

Problem 14.

Solution.

Problem 15.

Solution.

Chapter 2

Sequences of Independent Trials