PROBLEMS AND SOLUTIONS:
Theory of Probability and Random Processes, 2nd Edition

Leonid B. Koralov and Yakov G. Sinai

Alan Sun

Chapter 1

Random Variables and Their Distributions

Problem 1. A man's birthday is on March 1st. His father's birthday is on March 2nd. One of his grandfather's birthday is on March 3rd. How would you estimate the number of such people in the USA?

Solution. $300 \times 10^6 \times 0.5 \times \dfrac{2}{365^3}$

Problem 2. Suppose that n identical balls are distributed randomly among m boxes. Construct the corresponding space of elementary outcomes. Assuming that each ball is placed in a random box with equal probability, find the probability that the first box is empty.

Solution. Suppose n identical balls are thrown into m boxes. Let Ω be the set of m-tuples where each entry is a non-negative integer and the sum of the entries is n. Then, by a stars and bars argument, we know that $|\Omega| = \dbinom{n+m-1}{m-1}$. Therefore,

$$\mathbb{P}\left[\text{first box is empty}\right] = \frac{\binom{n+m-2}{m-2}}{\binom{n+m-1}{m-1}} = \frac{m-1}{n+m-1}.$$

Problem 3. A box contains 90 good items and 10 defective items. Find the probability that a sample of 10 items has no defective items.

Solution. Let Ω be the set of tuples where the first entry ist he number of defective items and the second is the number of good items. Then, $\mathbb{P}\left[(0,10)\right] = \dfrac{\binom{90}{10}}{\binom{100}{100}} = \dfrac{90 \cdot 89 \cdot \ldots \cdot 81}{100 \cdot 99 \cdot \ldots \cdot 91}$.

Problem 4. Let ξ be a random variable such that $\mathbb{E}\left[|\xi|^m\right] \leq AC^m$ for some positive constants A and C, and all integers $m \geq 0$. Prove that $\mathbb{P}[|\xi| > C] = 0$.

Solution. By way of contradiction, suppose that $\mathbb{P}\left[|\xi| > C\right] > 0$ for $C > 0$. Then, it follows that there exists some $A' \geq 1$ such that $\mathbb{P}\left[|\xi| \geq A'C\right] > 0$. So,

$$\begin{aligned}
\mathbb{E}\left[|\xi|^m\right] &= \sum_{\omega \in \Omega} |\xi(\omega)|^m, \\
&\geq \sum_{\omega \in \{|\xi| \geq A'C\}} |\xi(\omega)|^m,
\end{aligned}$$

$$\geq A'^m C^m \mathbb{P}\left[|\xi| \geq A'C\right].$$

Since $\mathbb{P}\left[|\xi| \geq A'C\right]$ is constant, clearly there does not exist $A > 0$ such that $A \geq A'^m$ for all m, thus a contradiction.

Problem 5. Suppose there are n letters addressed to n different people, and n envelopes with addresses. The letters are mixed and then randomly placed into the envelopes. Find the probability that at least one letter is in the correct envelope. Find the limit of this probability as $n \to \infty$.

Solution. This is the famous hat-check problem. We find the probability that no one gets the right letter. Let $\Omega = S_n$, the permutation group of n elements, also let C_i be the set of all permutations where the i^{th} element is in the i^{th} position. Formally, $\sigma \in C_i$ if and only if $\sigma(i) = i$. Then, by the inclusion-exclusion principle, we have that

$$\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = \mathbb{P}\left[\bigcup_{i=1}^{n} C_i\right],$$
$$= \sum_{i=1}^{n} \mathbb{P}\left[C_i\right] - \sum_{i<j} \mathbb{P}\left[C_i \cap C_j\right] + \sum_{i<j<k} \mathbb{P}\left[C_i \cap C_j \cap C_k\right] - \ldots.$$

Now, we derive the expression for each term in the previous sum. For any $\sigma \in C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}$, $\sigma(i_1) = i_1$, $\sigma(i_2) = i_2, \ldots, \sigma(i_k) = i_k$. Thus, $\mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \dfrac{(n-k)!}{n!}$. Therefore, $\displaystyle\sum_{i_1<i_2<\ldots<i_k} \mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \ldots \cap C_{i_k}\right] = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$. So,

$$\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = 1 - \frac{1}{2!} + \frac{1}{3!} - \ldots + \frac{(-1)^{n+1}}{n!},$$
$$\lim_{n\to\infty} \mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = \frac{1}{e}.$$

Problem 6. For integers n and r, find the number of solutions of the equation

$$x_1 + \ldots + x_r = n$$

where $x_i \geq 0$ are integers. Assuming the uniform distribution on the space of the solutions, find $\mathbb{P}[x_1 = a]$ and its limit as $r \to \infty$, $n \to \infty$, and $n/r \to \rho > 0$.

Solution. The first part of the question is a direct application of a stars and bars argument. Then, there are $\binom{n+r-1}{r-1}$ solutions to the equation $x_1 + \ldots + x_r = n$. By the same argument, we find that

$$\mathbb{P}\left[x_1 = a\right] = \binom{n-a+r-2}{r-2} \Big/ \binom{n+r-1}{r-1} = \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!}.$$

Now, we take the limit as $r, n \to \infty$ and $n/r \to \rho > 0$.

$$\lim_{\substack{n,r\to\infty \\ n/r\to\rho}} \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!} = \lim_{\substack{n,r\to\infty \\ n/r\to\rho}} \frac{(r-1)n(n-1)\cdots(n-a+1)}{(n+r-1)(n+r-2)\cdots(n-a+r-1)},$$
$$\approx \lim_{\substack{n,r\to\infty \\ n/r\to\rho}} \frac{rn^a}{(n+r)^{a+1}},$$
$$= \frac{r(\rho r)^a}{(\rho r + r)^{a+1}},$$

$$= \frac{\rho^a}{(\rho+1)^a}.$$

Problem 7. Find the mathematical expectation and the variance of a random variable with Poisson distribution with parameter λ.

Solution. Recall, that the Poisson distribution is the measure on $\mathbb{Z}^+$ such that for any elementary outcome k, $\mathbb{P}[k] = \frac{\lambda^k}{e^\lambda k!}$. Let $\xi = \mathrm{Id}_{\mathbb{Z}^+}$. Then,

$$\begin{aligned}
\mathbb{E}[\xi] &= \sum_{n=0}^{\infty} \xi(n)\mathbb{P}[n], \\
&= \sum_{n=1}^{\infty} \frac{\lambda^n}{e^\lambda (n-1)!},
\end{aligned}$$

expanding the sum to see the Taylor expansion more clearly, we see that

$$\begin{aligned}
&= \frac{\lambda}{e^\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \dots\right), \\
&= \frac{\lambda}{e^\lambda} e^\lambda, && \text{(by Taylor expansion)} \\
&= \lambda.
\end{aligned}$$

Now, we find the variance using the formula $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}[\xi])^2$. So,

$$\begin{aligned}
\mathbb{E}\left[\xi^2\right] &= \sum_{n=0}^{\infty} \xi(n)^2 \mathbb{P}[n], \\
&= e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n (n+1)}{n!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} + \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left(e^\lambda + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\right), \\
&= \lambda e^{-\lambda}\left(e^\lambda + \lambda e^\lambda\right), \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Therefore, $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}[\xi])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$ and the variance of the Poisson distribution is the same as its expected value.

Problem 9. Prove that if F is the distribution function of the random variable ξ, then $\mathbb{P}[\xi = x] = F(x) - \lim_{\delta \downarrow 0} F(x - \delta)$.

Solution. Recall that if F is a distribution function of a random variable ξ then, $F_\xi(x) = \mathbb{P}[\{\xi(\omega) < x : \omega \in \Omega\}]$. We then evaluate the expression on the right-hand side first

$$\mathbb{P}[\{\omega \in \Omega : \xi(\omega) \le x\}] - \lim_{\delta \downarrow 0} \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \le x - \delta\}] = \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \le x\}] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\}\right],$$

where $\delta_1, \delta_2, \ldots$ is any sequence where $\delta_i \geq 0$ and converges to 0. We first show that

$$\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\} = \{\omega \in \Omega : \xi(\omega) < x\},$$

That the left-hand side is a subset of the right-hand side is trivial. Now, suppose an element ω in the set on the right-hand side. There must exist some $\varepsilon > 0$, such that $\xi(\omega) = x - \varepsilon$. By convergence of $\delta_1, \delta_2, \ldots$, there must exist some δ_i where $\xi(\omega) = x - \varepsilon \leq x - \delta_i$. Thus, the two sets are equivalent. Therefore,

$$\begin{aligned}
\mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \leq x - \delta_i\}\right] &= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq x\}] - \mathbb{P}[\{\omega \in \Omega : \xi(\omega) < x\}], \\
&= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) = x\}], \\
&= \mathbb{P}[\xi = x].
\end{aligned}$$

The second equality follows σ-additivity of $\mathbb{P}$.

Problem 10. A random variable ξ has density p. Find the density of $\eta = a\xi + b$ for $a, b \in \mathbb{R}$, and $a \neq 0$.

Solution. Let F be the distribution of ξ and let $\eta = a\xi + b$ where $a \neq 0$. We first find the distribution of η:

$$\begin{aligned}
\mathbb{P}[\{\omega \in \Omega : \eta(\omega) \leq x\}] &= \mathbb{P}[\{\omega \in \Omega : a\xi(\omega) + b \leq x\}], \\
&= \mathbb{P}\left[\left\{\omega \in \Omega : \xi(\omega) \leq \frac{x-b}{a}\right\}\right], \\
&= F\left(\frac{x-b}{a}\right).
\end{aligned}$$

Thus, taking the derivative to yield the density function, we have that $p_\eta(x) = \frac{1}{a} p_\xi\left(\frac{x-b}{a}\right)$ where p_ξ is the density of ξ.

Problem 11. A random variable ξ has uniform distribution on $[0, 2\pi]$. Find the density of the distribution of $\eta = \sin \xi$.

Solution. To show this we need a few simple measure-theoretic facts:

1. By definition, the density function of a random variable uniformly distributed on some set $A \subset \mathbb{R}$ where $\lambda(A) < \infty$ is $\frac{1}{\lambda(A)}$. Moreover, for any measurable-subset $B \subset A$, $\mathbb{P}_A[B] = \frac{\lambda(B)}{\lambda(A)}$. For brevity, We call $\mathbb{P}_A$ the Lebesgue probability measure on A.
2. Let $C \subset D \subset G \subset \mathbb{R}$ (be measurable-subsets of $\mathbb{R}$) and $\mathbb{P}_D, \mathbb{P}_G$ be the Lebesgue probability measures on D, G, respectively. Then, by definition, $\mathbb{P}_D[C] = \frac{\lambda(C)}{\lambda(D)}$ and $\mathbb{P}_G[D] = \frac{\lambda(D)}{\lambda(G)}$. It follows that $\mathbb{P}_G[C] = \frac{\lambda(C)}{\lambda(G)} = \mathbb{P}_D[C]\mathbb{P}_G[D]$.

Let $(\Omega_1, \mathcal{G}_1, \mathbb{P}_{\Omega_1})$ and $(\Omega_2, \mathcal{G}_2, \mathbb{P}_{\Omega_2})$ be the probability subspaces of $[0, 2\pi]$ with the Lebesgue probability measure, where $\Omega_1 = [0, \pi/2] \cup [3\pi/2, 2\pi]$ and $\Omega_2 = [\pi/2, 3\pi/2]$. Also, let $\mathbb{P}_{\Omega_1}, \mathbb{P}_{\Omega_2}$ be the Lebesgue probability measures on Ω_1, Ω_2, respectively. Suppose $\xi = \mathrm{Id}_{[0,2\pi]}$, $\xi_1 = \mathrm{Id}_{[0,\pi/2]\cup[3\pi/2,2\pi]}$, $\xi_2 = \mathrm{Id}_{[\pi/2,3\pi/2]}$, and $\eta = \sin(\xi)$. Then, we start by finding the distribution of η:

$$\begin{aligned}
\mathbb{P}[\{\omega \in [0, 2\pi] : \eta(\omega) \leq x\}] &= \mathbb{P}[\{\omega \in [0, \pi/2] \cup [3\pi/2, 2\pi] : \eta(\xi(\omega)) \leq x\}] \\
&\quad + \mathbb{P}[\{\omega \in [\pi/2, 3\pi/2] : \eta(\xi(\omega)) \leq x\}], \\
&= \frac{1}{2}\mathbb{P}_{\Omega_1}[\{\omega \in \Omega_1 : \eta(\xi_1(\omega)) \leq x\}] + \frac{1}{2}\mathbb{P}_{\Omega_2}[\{\omega \in \Omega_2 : \eta(\xi_2(\omega)) \leq x\}],
\end{aligned}$$

$$= \frac{1}{2}F_1(\arcsin(x)) + \frac{1}{2}F_2(\arcsin(x)),$$

where F_1, F_2 are the distribution functions for ξ_1, ξ_2, respectively. Differentiating with respect to x, we yield the density function:

$$\begin{aligned}
\frac{1}{2}\left(F_1(\arcsin(x)) + F_2(\arcsin(x))\right)' &= \frac{1}{2\sqrt{1-x^2}}\left(F_1'(\arcsin x) + F_2'(\arcsin x)\right), \\
&= \frac{1}{2\sqrt{1-x^2}}\left(\frac{1}{\pi} + \frac{1}{\pi}\right), \\
&= \frac{1}{\pi\sqrt{1-x^2}}.
\end{aligned}$$

Problem 12. Let $(X_1, d_1), \ldots, (X_n, d_n)$ be separable metric spaces, and define $X = X_1 \times \ldots \times X_n$ to be the product space with the metric

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{d_1^2(x_1, y_1) + \ldots + d_n^2(x_n, y_n)}.$$

Prove that $\mathcal{B}(X) = \mathcal{B}(X_1) \times \mathcal{B}(X_n)$.

Solution. Let $\mathcal{B}(X)$ be the Borel sets of X, the σ-algebra of all of the open sets in X. Also let $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n)$ be the product σ-algebra of $X_1, \ldots, X_n$. That $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n) \subset \mathcal{B}(X)$ is trivial since the product of open sets in $X_1, \ldots, X_n$ must be an open set in X. It remains to show inclusion in the other direction. Since X is a separable metric space, it is second-countable. Moroever, we know that $B = \{x_1 \times \ldots \times x_n : x_1 \in X_1, \ldots, x_n \in X_n\}$ forms a basis of the product topology. Thus, any open set in X can be constructed through the countable unions of sets in B. And so, since σ-algebras are closed under countable unions the inclusion holds in the other direction.

Problem 13. An integer from 1 to 1000 is chosen at random (with uniform distribution). What is the probability that it is an integer power (higher than the first) of an integer.

Solution. For any $n > 5$, $n^4 \geq 1000$. Therefore, we are mostly concerned with the set of all perfect squares and cubes less than or equal to 1000. Thus, we find these first using the inclusion-exclusion principle. Then, brute-force count the remaining numbers that do not fall into these sets.

$$\begin{aligned}
\sharp\left\{1 \leq n \leq 1000 : n^2 \leq 1000 \text{ or } n^3 \leq 1000\right\} &= \sharp\left\{1 \leq n \leq 1000 : n^2 \leq 1000\right\} + \sharp\left\{1 \leq n \leq 1000 : n^3 \leq 1000\right\} \\
&\quad - \sharp\left\{1 \leq n \leq 1000 : n^6 \leq 1000\right\}, \\
&= \lfloor\sqrt{1000}\rfloor + \lfloor\sqrt[3]{1000}\rfloor - 2, \\
&= 31 + 10 - 3, \\
&= 39.
\end{aligned}$$

It remains to account for numbers where the exponential by $5, 7$ are within 1000. This only holds for $2^5, 2^7, 3^5$. Therefore, there are 41 numbers less than or equal to 1000 that are integer powers (greater than 1) of another integer. Thus, supposing a uniform distribution, the probability of choosing such an integer is $\frac{41}{1000}$.

Problem 14. Let $C_1, C_2, \ldots$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\lim_{n\to\infty} \mathbb{P}[C_n] = 0$ and $\sum_{n=1}^{\infty} \mathbb{P}[C_{n+1} \setminus C_n] < \infty$. Prove that

$$\mathbb{P}\left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} C_k\right] = 0.$$

Solution. This is a weaker version of the Borel-Cantelli lemma. Consider the Lebesgue probability measure on $[0,1]$. And then consider the events $\left[0, \dfrac{1}{n}\right]$ for $n = 1, 2, \ldots$. It follows that $\displaystyle\sum_{k=1}^{\infty} \mathbb{P}\left[\left[0, \frac{1}{n}\right]\right] \not< \infty$, but we know that the limit superior of the probability of this sequence of events is 0. Such a sequence of events cannot be described by the stronger version of the Borel-Cantelli lemma, however, the preconditions in this problem allow for this "counter example."

First, we show the following lemma:

Lemma 1.1. For any indexed collection of sets $C_1, C_2, \ldots, C_n$,

$$\bigcup_{k=1}^{n} C_k = C_1 \cup \bigcup_{k=1}^{n-1} C_{k+1} \setminus C_k.$$

Proof. We proceed by way of induction. Suppose that $n = 2$, then $C_1 \cup (C_2 \setminus C_1) = C_1 \cup (C_1^c \cap C_2) = (C_1 \cup C_1^c) \cap (C_2 \cup C_1) = C_1 \cup C_2$. Assuming the claim true for all $n < N$, we now prove it for $n = N$:

$$\begin{aligned}
C_1 \cup \bigcup_{k=1}^{n-1} (C_{k+1} \setminus C_k) &= \left(C_1 \cup \bigcup_{k=1}^{n-2} (C_{k+1} \setminus C_k)\right) \cup (C_n \setminus C_{n-1}), \\
&= \left(\bigcup_{k=1}^{n-1} C_k\right) \cup (C_{n-1}^c \cap C_n), && \text{(by ind. hyp.)} \\
&= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup \left(C_{n-1} \cup (C_{n-1}^c \cap C_n)\right), \\
&= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup (C_n \cup C_{n-1}), \\
&= \bigcup_{k=1}^{n} C_k.
\end{aligned}$$

$\square$

With this, we show the desired result:

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} C_k\right] &= \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{k=n}^{\infty} C_k\right], \\
&= \lim_{n\to\infty} \mathbb{P}\left[C_n \cup \bigcup_{k=n}^{\infty} C_{k+1} \setminus C_k\right], \\
&\leq \lim_{n\to\infty} \mathbb{P}[C_n] + \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}[C_{k+1} \setminus C_k], \\
&= 0.
\end{aligned}$$

The second equality follows from Lemma **??** and the last equality follows from the given.

Problem 15. Let ξ be a random variable with continuous distribution function F. Find the distribution function of the random variable $F(\xi)$.

Solution. If F is a continuous distribution of a random variable ξ on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$, then it must be that F is injective on $\operatorname{img} \xi$. Consider, now a random variable $\eta = F(\xi)$. We proceed to find the distribution of this η:

$$\begin{aligned}\mathbb{P}[\{\omega \in \Omega : \eta(\xi(\omega)) \leq x\}] &= \mathbb{P}[\{\omega \in \Omega : \xi(\omega) \leq F^{-1}(x)\}], \\ &= F(F^{-1}(x)).\end{aligned}$$

Note that unless F is bijective, $F \circ F^{-1} \neq \mathrm{Id}$. Recall, that there exists a right-inverse if and only if F is surjective.

Chapter 2

Sequences of Independent Trials

Some brief notes before the problems:

- Koralov and Sinai use $\times$ instead of $\otimes$ for the product σ-algebra. Moving forward, the reader needs to be cautious about what the operands are. For if they are σ-algebras, we understand the result of the expression to be the aforementioned product.

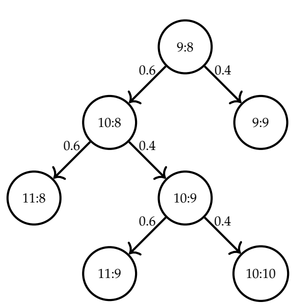
Problem 1. Find the probability that there are exactly three heads after five tosses of a symmetric coin.

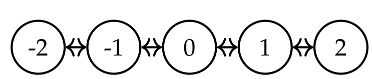
Solution. Directly apply the definition of the binomial distribution. Let $X = \{0, 1\}$. Define the random variable $\chi_i(\omega)$ for $1 \leq i \leq 5$ to be 1 if $\omega = 1$ and zero otherwise. Then, let $\nu_5 = \sum_{i=1}^{5} \chi_i$. ν_5 is a binomial random variable. Thus,

$$\mathbb{P}[\nu_5 = 3] = \binom{5}{3} \frac{1}{2^5} = \frac{5}{16}.$$

Problem 2. Andrew and Bob are playing a game of table tennis. The game ends when the first player reaches 11 points if the other player has 9 points or less. However, if at any time the score is 10:10, then the game continues until one of the players is 2 points ahead. The probability that Andrew wins any given points is 60 percent. What is the probability that Andrew will go on to win the game if he is currently ahead 9:8.

Solution. We present two alternative ways to solve this problem one involving chains and the other geometric series. We first present the solution involving chains. We draw the state space below, where each node denotes a distinct score and the edges between states represent the transition probabilities





It remains to find the probability that Andrew wins from either 9:9 or 10:10 which is the same since both states require Andrew to win two consecutive points. This, then induces a new chain (see above). Denote by a, b, c the probability that Andrew wins from being one point down, drawn, and one point up. Then,

$$\begin{aligned}
a &= 0.6b, \\
b &= 0.4a + 0.6c, \\
c &= 0.4b + 0.6b.
\end{aligned}$$

So, $c = 0.36/0.52$. Thus, the probability that Andrew wins is then the aggregate of all states in the chain multiplied by their transition probabilities:

$$\mathbb{P}[\text{Andrew wins}] = 0.4\left(\frac{0.36}{0.52}\right) + 0.6\left(0.6 + 0.4\left(0.4\left(\frac{0.36}{0.52}\right) + 0.6\right)\right) = 0.847.$$

Problem 3. Will you consider a coin asymmetric if after 1000 coin tosses the number of heads is equal to 600?

Solution. We directly apply the de Moivre-Laplace Theorem. Let $X_i(\omega_i)$ be a random variable that is 1 if ω_i = heads and 0 otherwise. Then, let $\nu_n = \sum_{i=1}^{n} X_i$. Then, for $n = 1000$, $\mathbb{E}[\nu_n] = 500$ and $\operatorname{Var}\nu_n = 250$. $k = 600$ as the actual number of heads that we saw during all of the coin flips. Therefore, $z = (600 - 500)/\sqrt{250} = 20/\sqrt{(10)}$. Therefore, by the de Moivre-Laplace theorem:

$$\begin{aligned}
\mathbb{P}\left[\frac{\nu_n - \mathbb{E}[\nu_n]}{\sqrt{\operatorname{Var}(\nu_n)}} \geq z\right] &\geq \int_{20/\sqrt{10}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right)\, dz, \\
&\leq 10^{-5}.
\end{aligned}$$

So, it is highly unlikely that the coin is fair.

Problem 4. Let ε_n be a numeric sequence such that $\varepsilon_n\sqrt{n} \to \infty$ as $n \to \infty$. Show that for a sequence of Bernoulli trials we have

$$\mathbb{P}\left[\left|\frac{\nu^n}{n} - p\right| < \varepsilon_n\right] \to 1$$

as $n \to \infty$.

Solution. We show that $\mathbb{P}\left[\left|\frac{\nu^n}{n} - p\right| > \varepsilon_n\right] \to 0$ as $n \to \infty$:

$$\begin{aligned}
\mathbb{P}\left[\left|\frac{\nu^n}{n} - p\right| > \varepsilon_n\right] &= \mathbb{P}[|\nu^n - np| > n\varepsilon_n], \\
&\leq \frac{\operatorname{Var}\nu^n}{n^2\varepsilon_n^2}, \\
&= \frac{np(1-p)}{n(n\varepsilon_n)^2}, \\
&= \frac{p(1-p)}{(\sqrt{n}\varepsilon_n)^2} \to 0,
\end{aligned}$$

as $n \to \infty$ since $\sqrt{n}\varepsilon_n \to \infty$ as $n \to \infty$.

Remark: Even though this problem is fairly simple, it demonstrates something profound about the law of large numbers — its convergence rate. Specifically, the probability that ν^n lies in a neighborhood of $O(\sqrt{n})$ tends to 1, as $n \to \infty$. Moreover, the probability that $\nu^n = k$ decays with rate $O(1/\sqrt{n})$ if $k \in (np - O(\sqrt{n}), np + O(\sqrt{n}))$. This is easy to see since we can approximate the distribution within this neighborhood with a crude uniform distribution. Note that this is the exact setup of the De Moivre-Laplace Theorem.

Problem 5. Using the de Moivre-Laplace Theorem, estimate the probability that during 12000 tosses of a die the number 6 appeared between 1900 and 2100 times.

Solution. Let ν^n be the number of sixes that appear from n tosses of a die. Then, it follows that $\mathbb{E}\left[\nu^{12000}\right] = 2000$ and $\mathrm{Var}\left[\nu^{12000}\right] = 12000 \cdot \dfrac{5}{36} \approx 1666$. Using the de Moivre-Laplace Theorem, we approximate $\dfrac{\nu^n - \mathbb{E}\left[\nu^n\right]}{\sqrt{\mathrm{Var}[\nu^n]}}$ as a standard normal distribution. Therefore,

$$\mathbb{P}[1900 \leq \nu^n \leq 2100] = \Phi(2.45) - \Phi(-2.45),$$

where $\Phi(\cdot)$ is the distribution function of the normal distribution.

Problem 6. Let Ω be the space of sequences $\omega = (\omega_1, \ldots, \omega_n)$, where $\omega_i \in [0,1]$. Let $\mathbb{P}_n$ be the probability distribution corresponding to the homogeneous sequence of independent trials, each ω_i having uniform distribution on $[0,1]$. Let $\eta_n = \min\limits_{1\leq i\leq n} \omega_i$. Find $\mathbb{P}_n(\eta_n \leq t)$ and $lim_{n\to\infty}\mathbb{P}_n(n\eta_n \leq t)$.

Solution.

$$\begin{aligned}
\mathbb{P}_n[\eta_n \leq t] &= 1 - \mathbb{P}_n[\eta_n > t], \\
&= 1 - \mathbb{P}_n[\omega_i \geq t]^n, , && \text{(by i.i.d.)} \\
&= 1 - (1-t)^n.
\end{aligned}$$

Next, we evaluate the limit as $n \to \infty$ of $n\eta_n$:

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}_n[n\eta_n \leq t] &= \lim_{n\to\infty} 1 - \mathbb{P}_n[n\eta_n > t], \\
&= \lim_{n\to\infty} 1 - \mathbb{P}_n\left[\eta_n > \frac{t}{n}\right], \\
&= \lim_{n\to\infty} 1 - \left(1 - \frac{t}{n}\right)^n, \\
&= 1 - e^{-t}.
\end{aligned}$$

Problem 7.

Solution.

Problem 8.

Solution.

Problem 9. Suppose that during a day the price of a certain stock either goes up by 3 percent with probability $1/2$ or goes down by 3 percent with probability $1/2$, and that outcomes on different days are independent. Approximate the probability that after 250 days the price of the stock will be at least as high as the current price.

Solution. Let Ω be the space of all sequences $(\omega_1, \ldots, \omega_n)$ of length n such that $\omega_i \in \{1.03, 0.97\}$ chosen uniformly. Suppose that the stock price is A, then the stock price after a sequence of percent changes $(\omega_1, \ldots, \omega_n)$ is $A \cdot \prod_{i=1}^{n} \omega_i$. In order to apply de Moivre-Laplace, we need a sum of Bernoulli random variables. Thus, we naturally proceed to apply $\exp(\log(\cdot))$:

$$\exp\left(\log\left(A\prod_{i=1}^{n}\omega_i\right)\right) = \exp\left(\log A + \sum_{i=1}^{n}\log(\omega_i)\right),$$

$$= A \exp\left(\sum_{i=1}^{n} \log(\omega_i)\right),$$

where in the last equality the term in the exponent is normally-distributed by the de Moivre-Laplace Theorem. We now derive this distribution. Let $X \sim N(\mu, \sigma^2)$ and $Z \sim \exp X$:

$$\begin{aligned}
\mathbb{P}[Z \leq z] &= \mathbb{P}[\exp X \leq z], \\
&= \mathbb{P}[X \leq \log(z)], \\
&= \Phi_{\mu,\sigma^2}(\log(z)),
\end{aligned}$$

where $\Phi_{\mu,\sigma^2}(\cdot)$ is the distribution function of a normally-distributed random variable with mean μ and variance σ^2. Then, it follows that the density function is $\frac{1}{z}p(\mu, \sigma^2; \log(z))$, where $p(\mu, \sigma; \cdot)$ is the density of a normally-distributed random variable.

It remains to derive the distribution of X using the de Moivre-Laplace Theorem. First, we redefine the sample space. Let Ω' be the space of all sequences $(\omega_1, \ldots, \omega_n)$ of length n such that $\omega_i \in \{\log(1.03), \log(0.97)\}$ chosen uniformly. Then,

$$\mathbb{E}\left[\sum_{i=1}^{n} \omega_i\right] = \sum_{i=1}^{n} \mathbb{E}\left[\omega_i\right] = \frac{n}{2}(\log(0.9991)).$$

And

$$\begin{aligned}
\operatorname{Var}\left[\sum_{i=1}^{n} \omega_i\right] &= \mathbb{E}\left[\left(\sum_{i=1}^{n} \omega_i\right)^2\right] - \mathbb{E}\left[\sum_{i=1}^{n} \omega_i\right]^2, \\
&= \sum_{i=1}^{n} \mathbb{E}\left[\omega_i^2\right] + \sum_{1 \leq i \neq j \leq n} \mathbb{E}\left[\omega_i\right]\mathbb{E}\left[\omega_j\right] - \mathbb{E}\left[\sum_{i=1}^{n} \omega_i\right]^2, \\
&= \frac{n}{2}\left(\log^2(1.03) + \log^2(0.97)\right) + \frac{n^2 - n}{4}\log^2(0.9991) - \frac{n^2}{4}\log^2(0.9991).
\end{aligned}$$

Evaluating the mean and variance for $n = 250$, we have that $X \sim N(-0.112551, 0.225135)$. Applying the distribution function of the log-normal distribution with these parameters, we have $\mathbb{P}[Z \geq 1] = 0.4111$.

Chapter 3

Lebesgue Integral and Mathematical Expectation

Problem 1.

Solution. It suffices to show that

$$\{\omega \in \Omega : \lim_{n\to\infty} f_n(\omega) = f(\omega)\} = \bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}\left\{\omega \in \Omega : |f_k(\omega) - f(\omega)| < \frac{1}{i}\right\}. \tag{3.1}$$

This is because

$$\left\{\omega \in \Omega : |f_k(\omega) - f(\omega)| < \frac{1}{i}\right\} = (f_k - f)^{-1}\left(-\frac{1}{i}, \frac{1}{i}\right).$$

Since both f_k and f are $\mathcal{F}$-measurable functions, then $f_k - f$ is also a measurable function. Moreover, since $(-1/i, 1/i)$ is clearly a Borel set, then $(f_k - f)^{-1}(-1/i, 1/i) \in \mathcal{F}$. It follows from the axioms of a σ-algebra that the right hand-side of Eq. **??** is also a set in $\mathcal{F}$.

Now, we show Eq. **??**:

$$\begin{aligned} x \in \bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}\left\{\omega \in \Omega : |f_k(\omega) - f(\omega)| < \frac{1}{i}\right\} &\Rightarrow x \in \bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}\left\{\omega \in \Omega : |f_k(\omega) - f(\omega)| < \frac{1}{i}\right\}, && (\forall i \in \mathbb{N}) \\ &\Rightarrow \exists j, \forall k > j, x \in \left\{\omega \in \Omega : |f_k(\omega) - f(\omega)| < \frac{1}{i}\right\},, && (\forall i \in \mathbb{N}) \\ &\Rightarrow \forall i \in \mathbb{N}, \exists j, \forall k > j, |f_k(x) - f(x)| < \frac{1}{i}. \end{aligned}$$

It follows that $f_1, f_2, \ldots$ to f at x. The other direction of the inclusion is trivial and its proof is omited as it results from the same logic above.

Problem 2.

Solution.

$$\begin{aligned} \mathbb{E}[\xi] &\geq \sum_{k=1}^{\infty} k \cdot \mathbb{P}[k \leq \xi < k+1], \\ &= \mathbb{P}[1 \leq \xi < 2] + 2\mathbb{P}[2 \leq \xi < 3] + 3\mathbb{P}[3 \leq \xi < 4] + \ldots, \\ &= \left(\sum_{k=1}^{\infty} \mathbb{P}[k \leq \xi < k+1]\right) + \mathbb{P}[2 \leq \xi < 3] + 2\mathbb{P}[3 \leq \xi < 4] + \ldots, \end{aligned}$$

$$
\begin{aligned}
&= 1 + \left(\sum_{k=2}^{\infty} \mathbb{P}[k \leq \xi < k+1] \right) + \ldots, \\
&= 1 + \frac{1}{2} + \frac{1}{3} + \ldots, \\
&\geq \infty.
\end{aligned}
$$

Note that we just proved that if ξ is a random variable that only takes on positive integers, then $\mathbb{E}[\xi] = \sum_{i=1}^{\infty} \mathbb{P}[\xi \geq i]$.

Problem 3.

Solution. Let ξ_n be a noramlly-distributed random variable parameterized by mean n and variance 1. Then, it is clear that ξ_n converges pointwise to the random variable $\xi(\omega) = 0$, however,

$$\lim_{n \to \infty} \mathbb{E}[\xi_n] = \lim_{n \to \infty} n = \infty.$$

Problem 4.

Solution. Let ξ be a random variable such that $\mathbb{P}[\xi = A] = \mathbb{P}[\xi = B] = \frac{1}{2}$. Then it follows that

$$
\begin{aligned}
\operatorname{Var} \xi &= \mathbb{E}\left[\xi^2\right] - \mathbb{E}[\xi]^2, \\
&= \frac{1}{2}A^2 + \frac{1}{2}B^2 - \left(\frac{1}{2}A + \frac{1}{2}B \right)^2, \\
&= \frac{1}{2}A^2 + \frac{1}{2}B^2 - \frac{1}{4}(A+B)^2, \\
&= \frac{1}{4}A^2 - \frac{1}{2}AB + \frac{1}{4}B^2, \\
&= \left(\frac{B-A}{2} \right)^2.
\end{aligned}
$$

Interestingly, this is the maximum variance a bounded random variable can achieve (see Popoviciu's Inequality on Variances).

Problem 5.

Solution. Let x be an arbitrary irrational number and also let $\varepsilon > 0$. Then, choose $N \in \mathbb{N}$ such that $\log_2(1/\varepsilon) + 1 < N$. Then, let

$$\delta = \min \{|x - x_k|\}_{k=1}^{N}.$$

So, for any x' such that $|x - x'| < \delta$, it must be that

$$
\begin{aligned}
|F_\xi(x) - F_\xi(x')| &= \left| \sum_{x_j \text{ s.t. } x_j < x} \frac{1}{2^j} - \sum_{x_j \text{ s.t. } x_j < x'} \frac{1}{2^j} \right|, \\
&= \sum_{x_j \text{ s.t.} x' \leq x_j < x} \frac{1}{2^j}.
\end{aligned}
$$

This equality comes from assuming, without the loss of generality, $x' < x$. So, then

$$\leq \sum_{j=N}^{\infty} \frac{1}{2^j},$$

$$= \frac{1}{2^{N-1}},$$
$$< \varepsilon. \qquad \text{(by def.)}$$

Thus, $F_x i(\omega)$ is continuous for irrational ω.

Problem 6.

Solution. We first find the density of η. Consider first the distribution of η:

$$\begin{aligned}\mathbb{P}[\eta \le x] &= \mathbb{P}\left[\frac{1}{\xi} \le x\right], \\ &= 1 - \mathbb{P}\left[\xi < \frac{1}{x}\right], \\ &= 1 - F_\xi\left(\frac{1}{x}\right).\end{aligned}$$

Therefore, the density function of η, $p_\eta = \left(1 - F_\xi\left(\frac{1}{x}\right)\right)' = \frac{1}{x^2} p_\xi\left(\frac{1}{x}\right)$. Now, we apply the definition of the expected value of ξ:

$$\mathbb{E}[\eta] = \int_{-\infty}^{\infty} \frac{1}{x} p_\xi\left(\frac{1}{x}\right)\, dx,$$

make the substitution for $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}$ and also apply the definition of the improper integeral. So, let $L, R \gg 0$, then

$$= \lim_{L\to\infty} \int_{1/L}^{\infty} -\frac{1}{u} p_\xi(u)\, du + \lim_{R\to\infty} \int_0^{-1/R} \frac{1}{u} p_\xi(u)\, du.$$

Since $p_\xi(0) > 0$ and p_ξ is continuous, there exists $a > 0$ such that $p_\xi([0,a)) \subset \mathbb{R}^+$ and $p_\xi((-a,0]) \subset \mathbb{R}^+$. Thus, it follows from the definition of the Lebesgue integral that both

$$\begin{aligned}\lim_{L\to\infty} \int_{1/L}^{\infty} -\frac{1}{u} p_\xi(u)\, du &\le \lim_{L\to\infty} -\frac{1}{1/L} \mathbb{P}[[0,a)], \\ \lim_{R\to\infty} \int_0^{-1/R} \frac{1}{u} p_\xi(u)\, du &\ge \lim_{R\to\infty} \frac{1}{1/R} \mathbb{P}[(-a,0]],\end{aligned}$$

Since the integrals go to $-\infty$ and ∞, respectively, the expected value is then undefined.

Problem 7.

Solution.

Problem 8.

Solution. We first show that convergence almost surely implies convergence in measure.

Problem 9.

Solution. Apply integration by-parts:

$$\int_{-\infty}^{\infty} F(x+10) - F(x)\, dx = x(F(x+10) - F(x))\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} x(p(x+10) - p(x))\, dx,$$

where $p(x)$ is the density function corresponding to $F(\cdot)$. First, we evaluate the term on the left, since $lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$, it must be that $x(F(x+10) - F(x))\Big|_{-\infty}^{\infty} = 0$. Therefore, it suffices to evaluate the term on the right,

$$-\int_{-\infty}^{\infty} x(p(x+10) - p(x))\, dx = -\int_{-\infty}^{\infty} xp(x+10) - xp(x)\, dx,$$

$$\begin{aligned}
&= -\int_{-\infty}^{\infty} (x+10)p(x+10) - xp(x)\;dx - \int_{-\infty}^{\infty} 10p(x+10)\;dx,\\
&= -\int_{-\infty}^{\infty} up(u)\;du - \int_{-\infty}^{\infty} xp(x) + 10,\\
&= 10.
\end{aligned}$$

In general, it follows that for any $u \in \mathbb{R}$, $\int F(x+u) - F(x)\;dx = u$ through the same calculation that we just performed.

Problem 10.

Solution.

Problem 11.

Solution.

Problem 12.

Solution. We state Young's inequality here without proof. This is crucial to proving Hölder's ineqaulity.

Theorem 3.1 (Young's inequality)**.** Suppose $1 < p < \infty$. Then,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where q is the dual exponent of p, for all $a, b \geq 0$.

Now, we prove Hölder's inequality: Suppose first that $p = 1, q = \infty$.

Problem 13.

Solution.

Chapter 4

Conditional Probabilities and Independence

Problem 1.

Solution. We directly apply the definition of conditional probability:

$$\begin{aligned}\mathbb{P}[\omega_1 = 1 | \omega_1 + \ldots + \omega_n = m] &= \frac{\mathbb{P}[\omega_1 = 1 \cap \omega_1 + \ldots + \omega_n = m]}{\mathbb{P}[\omega_1 + \ldots + \omega_n = m]}, \\ &= \frac{\binom{n-1}{m-1} p^m (1-p)^{n-m}}{\binom{n}{m} p^m (1-p)^{n-m}}, \\ &= \binom{n-1}{m-1} / \binom{n}{m}, \\ &= \frac{m}{n}.\end{aligned}$$

Problem 2.

Solution. We are looking for a distribution that is *memoryless*, both the exponential distribution and the Poisson distribution satisfy this property. Let ξ be a exponentially-distributed random variable, parameterized by λ. Then, $\mathbb{P}[\xi > x] = e^{-\lambda x}$ and

$$\begin{aligned}\mathbb{P}[\xi > x + y | \xi > x] &= \frac{\mathbb{P}[\xi > x + y]}{\mathbb{P}[\xi > x]}, \\ &= \frac{e^{-\lambda(x+y)}}{e^{-\lambda x}}, \\ &= e^{-\lambda y}.\end{aligned}$$

Problem 3.

Solution. Let H^1, H^2 denote the event that heads appears on the first and second toss, respectively. Then,

$$\begin{aligned}\mathbb{P}[H^2 | H^1] &= \frac{\mathbb{P}[H^1 \cap H^2]}{\mathbb{P}[H^1]}, \\ &= \frac{1/2(1/2^2) + 1/2(0.6^2)}{1/2(1/2) + 1/2(0.6)},\end{aligned}$$

$$= 0.55.$$

Problem 4.

Solution. Suppose that ξ, η are two random variables such that both of their images is $\{a, b\}$. We show then that if $\mathbb{E}[\xi\eta] = \mathbb{E}[\xi]\mathbb{E}[\eta]$, then η and ξ are independent. Let p, q be the probability that $\xi = a, \eta = a$, respectively. Then,

$$\begin{aligned}
\mathbb{E}[\xi]\mathbb{E}[\eta] &= (pa + (1-p)b)(qa + (1-q)b), \\
&= a^2 pq + ab(p)(1-q) + ba(q)(1-p) + b^2(1-p)(1-q), \\
\mathbb{E}[\xi\eta] &= a\mathbb{P}[\eta = a, \xi = a] + ab\mathbb{P}[\eta = a, \xi = b] + ba\mathbb{P}[\eta = b, \xi = a] + b^2\mathbb{P}[\xi = b, \eta = b],
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbb{P}[\xi = a, \eta = a] &= pq = \mathbb{P}[\xi = a]\mathbb{P}[\eta = a], \\
\mathbb{P}[\xi = b, \eta = b] &= (1-q)(1-p) = \mathbb{P}[\xi = b]\mathbb{P}[\eta = b].
\end{aligned}$$

By Lemma 4.3, then events $\xi = a, \eta = b$ and $\eta = a, \xi = b$ are also independent.

Problem 5.

Solution.

Problem 6.

Solution.

Problem 7.

Solution.

Chapter 5

Markov Chains with a Finite Number of States

Problem 1.

Solution.