Chapter 1

Random Variables and Their Distributions

Problem 1.

Solution. $300 \times 10^6 \times 0.5 \times \dfrac{2}{365^3}$

Problem 2.

Solution. Suppose n identical balls are thrown into m boxes. Let Ω be the set of m-tuples where each entry is a non-negative integer and the sum of the entries is n. Then, by a stars and bars argument, we know that $|\Omega| = \dbinom{n+m-1}{m-1}$. Therefore,

$$\mathbb{P}\left[\text{first box is empty}\right] = \frac{\binom{n+m-2}{m-2}}{\binom{n+m-1}{m-1}} = \frac{m-1}{n+m-1}.$$

Problem 3.

Solution. Let Ω be the set of tuples where the first entry ist he number of defective items and the second is the number of good items. Then, $\mathbb{P}\left[(0,10)\right] = \dfrac{\binom{90}{10}}{\binom{100}{100}} = \dfrac{90 \cdot 89 \cdot \ldots \cdot 81}{100 \cdot 99 \cdot \ldots \cdot 91}$.

Problem 4.

Solution. By way of contradiction, suppose that $\mathbb{P}\left[|\xi| > C\right] > 0$ for $C > 0$. Then, it follows that there exists some $A' \geq 1$ such that $\mathbb{P}\left[|\xi| \geq A'C\right] > 0$. So,

$$\begin{aligned}
\mathbb{E}\left[|\xi|^m\right] &= \sum_{\omega \in \Omega} |\xi(\omega)|^m, \\
&\geq \sum_{\omega \in \{|\xi| \geq A'C\}} |\xi(\omega)|^m, \\
&\geq A'^m C^m \mathbb{P}\left[|\xi| \geq A'C\right].
\end{aligned}$$

Since $\mathbb{P}\left[|\xi| \geq A'C\right]$ is constant, clearly there does not exist $A > 0$ such that $A \geq A'^m$ for all m, thus a contradiction.

Problem 5.

Solution. This is the famous hat-check problem. We find the probability that no one gets the right letter. Let $\Omega = S_n$, the permutation group of n elements, also let C_i be the set of all permutations where the i^{th} element is in the i^{th} position. Formally, $\sigma \in C_i$ if and only if $\sigma(i) = i$. Then, by the inclusion-exclusion principle, we have that

$$\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = \mathbb{P}\left[\bigcup_{i=1}^{n} C_i\right],$$
$$= \sum_{i=1}^{n} \mathbb{P}\left[C_i\right] - \sum_{i<j} \mathbb{P}\left[C_i \cap C_j\right] + \sum_{i<j<k} \mathbb{P}\left[C_i \cap C_j \cap C_k\right] - \dots.$$

Now, we derive the expression for each term in the previous sum. For any $\sigma \in C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}$, $\sigma(i_1) = i_1$, $\sigma(i_2) = i_2, \dots, \sigma(i_k) = i_k$. Thus, $\mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}\right] = \dfrac{(n-k)!}{n!}$. Therefore, $\displaystyle\sum_{i_1<i_2<\dots<i_k} \mathbb{P}\left[C_{i_1} \cap C_{i_2} \cap \dots \cap C_{i_k}\right] = \binom{n}{k}\frac{(n-k)!}{n!} = \frac{1}{k!}$. So,

$$\mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^{n+1}}{n!},$$
$$\lim_{n\to\infty} \mathbb{P}\left[\{\sigma : \exists i, \sigma(i) = i\}\right] = 1 - \frac{1}{e}.$$

Problem 6. Not too sure how to take the limit here.

Solution. The first part of the question is a direct application of a stars and bars argument. Then, there are $\binom{n+r-1}{r-1}$ solutions to the equation $x_1 + \dots + x_r = n$. By the same argument, we find that

$$\mathbb{P}\left[x_1 = a\right] = \binom{n-a+r-2}{r-2} \Big/ \binom{n+r-1}{r-1} = \frac{(r-1)n!(n-a+r-2)!}{(n+r-1)!(n-a)!}.$$

Now, we take the limit as $r, n \to \infty$ and $n/r \to \rho > 0$.

Problem 7.

Solution. Recall, that the Poisson distribution is the measure on $\mathbb{Z}^+$ such that for any elementary outcome k, $\mathbb{P}\left[k\right] = \dfrac{\lambda^k}{e^\lambda k!}$. Let $\xi = \mathrm{Id}_{\mathbb{Z}^+}$. Then,

$$\mathbb{E}\left[\xi\right] = \sum_{n=0}^{\infty} \xi(n)\mathbb{P}[n],$$
$$= \sum_{n=1}^{\infty} \frac{\lambda^n}{e^\lambda (n-1)!},$$

expanding the sum to see the Taylor expansion more clearly, we see that

$$= \frac{\lambda}{e^\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \dots\right),$$
$$= \frac{\lambda}{e^\lambda} e^\lambda, \qquad \text{(by Taylor expansion)}$$
$$= \lambda.$$

Now, we find the variance using the formula $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}\left[\xi\right])^2$. So,

$$\begin{aligned}
\mathbb{E}\left[\xi^2\right] &= \sum_{n=0}^{\infty} \xi(n)^2 \mathbb{P}[n], \\
&= e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n (n+1)}{n!}, \\
&= \lambda e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} + \sum_{n=1}^{\infty} \frac{\lambda^n}{(n-1)!}, \\
&= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \right), \\
&= \lambda e^{-\lambda} \left(e^{\lambda} + \lambda e^{\lambda} \right), \\
&= \lambda^2 + \lambda.
\end{aligned}$$

Therefore, $\mathrm{Var}[\xi] = \mathbb{E}\left[\xi^2\right] - (\mathbb{E}\left[\xi\right])^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$ and the variance of the Poisson distribution is the same as its expected value.

Problem 9.

Solution. Recall that if F is a distribution function of a random variable ξ then, $F_\xi(x) = \mathbb{P}\left[\{\xi(\omega) < x : \omega \in \Omega\}\right]$. We then evaluate the expression on the right-hand side first

$$\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \lim_{\delta \downarrow 0} \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x - \delta\}\right] = \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\}\right],$$

where $\delta_1, \delta_2, \ldots$ is any sequence where $\delta_i \ge 0$ and converges to 0. We first show that

$$\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\} = \{\omega \in \Omega : \xi(\omega) < x\},$$

That the left-hand side is a subset of the right-hand side is trivial. Now, suppose an element ω in the set on the right-hand side. There must exist some $\varepsilon > 0$, such that $\xi(\omega) = x - \varepsilon$. By convergence of $\delta_1, \delta_2, \ldots$, there must exist some δ_i where $\xi(\omega) = x - \varepsilon \le x - \delta_i$. Thus, the two sets are equivalent. Therefore,

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\bigcup_{i=1}^{\infty} \{\omega \in \Omega : \xi(\omega) \le x - \delta_i\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) \le x\}\right] - \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) < x\}\right], \\
&= \mathbb{P}\left[\{\omega \in \Omega : \xi(\omega) = x\}\right], \\
&= \mathbb{P}\left[\xi = x\right].
\end{aligned}$$

The second equality follows σ-additivity of $\mathbb{P}$.

Problem 10.

Solution. Let F be the distribution of ξ and let $\eta = a\xi + b$ where $a \neq 0$. We first find the distribution of η:

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in \Omega : \eta(\omega) \le x\}\right] &= \mathbb{P}\left[\{\omega \in \Omega : a\xi(\omega) + b \le x\}\right], \\
&= \mathbb{P}\left[\left\{\omega \in \Omega : \xi(\omega) \le \frac{x-b}{a}\right\}\right], \\
&= F\left(\frac{x-b}{a}\right).
\end{aligned}$$

Thus, taking the derivative to yield the density function, we have that $p_\eta(x) = \frac{1}{a} p_\xi \left(\frac{x-b}{a} \right)$ where p_ξ is the density of ξ.

Problem 11.

Solution. To show this we need a few simple measure-theoretic facts:

1. By definition, the density function of a random variable uniformly distributed on some set $A \subset \mathbb{R}$ where $\lambda(A) < \infty$ is $\frac{1}{\lambda(A)}$. Moreover, for any measurable-subset $B \subset A$, $\mathbb{P}_A[B] = \frac{\lambda(B)}{\lambda(A)}$. For brevity, We call $\mathbb{P}_A$ the Lebesgue probability measure on A.
2. Let $C \subset D \subset G \subset \mathbb{R}$ (be measurable-subsets of $\mathbb{R}$) and $\mathbb{P}_D, \mathbb{P}_G$ be the Lebesgue probability measures on D, G, respectively. Then, by definition, $\mathbb{P}_D[C] = \frac{\lambda(C)}{\lambda(D)}$ and $\mathbb{P}_G[D] = \frac{\lambda(D)}{\lambda(G)}$. It follows that $\mathbb{P}_G[C] = \frac{\lambda(C)}{\lambda(G)} = \mathbb{P}_D[C]\mathbb{P}_G[D]$.

Let $(\Omega_1, \mathcal{G}_1, \mathbb{P}_{\Omega_1})$ and $(\Omega_2, \mathcal{G}_2, \mathbb{P}_{\Omega_2})$ be the probability subspaces of $[0, 2\pi]$ with the Lebesgue probability measure, where $\Omega_1 = [0, \pi/2] \cup [3\pi/2, 2\pi]$ and $\Omega_2 = [\pi/2, 3\pi/2]$. Also, let $\mathbb{P}_{\Omega_1}, \mathbb{P}_{\Omega_2}$ be the Lebesgue probability measures on Ω_1, Ω_2, respectively. Suppose $\xi = \mathrm{Id}_{[0,2\pi]}$, $\xi_1 = \mathrm{Id}_{[0,\pi/2]\cup[3\pi/2,2\pi]}$, $\xi_2 = \mathrm{Id}_{[\pi/2,3\pi/2]}$, and $\eta = \sin(\xi)$. Then, we start by finding the distribution of η:

$$\begin{aligned}
\mathbb{P}\left[\{\omega \in [0, 2\pi] : \eta(\omega) \le x\}\right] &= \mathbb{P}\left[\{\omega \in [0, \pi/2] \cup [3\pi/2, 2\pi] : \eta(\xi(\omega)) \le x\}\right] \\
&\quad + \mathbb{P}\left[\{\omega \in [\pi/2, 3\pi/2] : \eta(\xi(\omega)) \le x\}\right], \\
&= \frac{1}{2}\mathbb{P}_{\Omega_1}\left[\{\omega \in \Omega_1 : \eta(\xi_1(\omega)) \le x\}\right] + \frac{1}{2}\mathbb{P}_{\Omega_2}\left[\{\omega \in \Omega_2 : \eta(\xi_2(\omega)) \le x\}\right], \\
&= \frac{1}{2}F_1(\arcsin(x)) + \frac{1}{2}F_2(\arcsin(x)),
\end{aligned}$$

where F_1, F_2 are the distribution functions for ξ_1, ξ_2, respectively. Differentiating with respect to x, we yield the density function:

$$\begin{aligned}
\frac{1}{2}\left(F_1(\arcsin(x)) + F_2(\arcsin(x))\right)' &= \frac{1}{2\sqrt{1-x^2}}\left(F_1'(\arcsin x) + F_2'(\arcsin x)\right), \\
&= \frac{1}{2\sqrt{1-x^2}}\left(\frac{1}{\pi} + \frac{1}{\pi}\right), \\
&= \frac{1}{\pi\sqrt{1-x^2}}.
\end{aligned}$$

Problem 12.

Solution. Let $\mathcal{B}(X)$ be the Borel sets of X, the σ-algebra of all of the open sets in X. Also let $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n)$ be the product σ-algebra of $X_1, \ldots, X_n$. That $\mathcal{B}(X_1) \times \ldots \times \mathcal{B}(X_n) \subset \mathcal{B}(X)$ is trivial since the product of open sets in $X_1, \ldots, X_n$ must be an open set in X. It remains to show inclusion in the other direction. Since X is a separable metric space, it is second-countable. Moroever, we know that $B = \{x_1 \times \ldots \times x_n : x_1 \in X_1, \ldots, x_n \in X_n\}$ forms a basis of the product topology. Thus, any open set in X can be constructed through the countable unions of sets in B. And so, since σ-algebras are closed under countable unions the inclusion holds in the other direction.

Problem 13.

Solution. For any $n > 5$, $n^4 \ge 1000$. Therefore, we are mostly concerned with the set of all perfect squares and cubes less than or equal to 1000. Thus, we find these first using the inclusion-exclusion principle. Then, brute-force count the remaining numbers that do not fall into these sets.

$$\begin{aligned}\sharp\left\{1 \le n \le 1000 : n^2 \le 1000 \text{ or } n^3 \le 1000\right\} &= \sharp\left\{1 \le n \le 1000 : n^2 \le 1000\right\} + \sharp\left\{1 \le n \le 1000 : n^3 \le 1000\right\} \\ &\quad - \sharp\left\{1 \le n \le 1000 : n^6 \le 1000\right\}, \\ &= \lfloor\sqrt{1000}\rfloor + \lfloor\sqrt[3]{1000}\rfloor - 2, \\ &= 31 + 10 - 3, \\ &= 39.\end{aligned}$$

It remains to account for numbers where the exponential by $5, 7$ are within 1000. This only holds for $2^5, 2^7, 3^5$. Therefore, there are 41 numbers less than or equal to 1000 that are integer powers (greater than 1) of another integer. Thus, supposing a uniform distribution, the probability of choosing such an integer is $\frac{41}{1000}$.

Problem 14.

Solution. This is a weaker version of the Borel-Cantelli lemma. Consider the Lebesgue probability measure on $[0, 1]$. And then consider the events $\left[0, \frac{1}{n}\right]$ for $n = 1, 2, \ldots$. It follows that $\sum_{k=1}^{\infty} \mathbb{P}\left[\left[0, \frac{1}{n}\right]\right] \not< \infty$, but we know that the limit superior of the probability of this sequence of events is 0. Such a sequence of events cannot be described by the stronger version of the Borel-Cantelli lemma, however, the preconditions in this problem allow for this "counter example."

First, we show the following lemma:

Lemma 1.1. For any indexed collection of sets $C_1, C_2, \ldots, C_n$,

$$\bigcup_{k=1}^{n} C_k = C_1 \cup \bigcup_{k=1}^{n-1} C_{k+1} \setminus C_k.$$

Proof. We proceed by way of induction. Suppose that $n = 2$, then $C_1 \cup (C_2 \setminus C_1) = C_1 \cup (C_1^c \cap C_2) = (C_1 \cup C_1^c) \cap (C_2 \cup C_1) = C_1 \cup C_2$. Assuming the claim true for all $n < N$, we now prove it for $n = N$:

$$\begin{aligned}C_1 \cup \bigcup_{k=1}^{n-1} (C_{k+1} \setminus C_k) &= \left(C_1 \cup \bigcup_{k=1}^{n-2} (C_{k+1} \setminus C_k)\right) \cup (C_n \setminus C_{n-1}), \\ &= \left(\bigcup_{k=1}^{n-1} C_k\right) \cup (C_{n-1}^c \cap C_n), && \text{(by ind. hyp.)} \\ &= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup \left(C_{n-1} \cup (C_{n-1}^c \cap C_n)\right), \\ &= \left(\bigcup_{k=1}^{n-2} C_k\right) \cup (C_n \cup C_{n-1}), \\ &= \bigcup_{k=1}^{n} C_k.\end{aligned}$$

$\square$

With this, we show the desired result:

$$\begin{aligned}
\mathbb{P}\left[\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} C_k\right] &= \lim_{n\to\infty} \mathbb{P}\left[\bigcup_{k=n}^{\infty} C_k\right], \\
&= \lim_{n\to\infty} \mathbb{P}\left[C_n \cup \bigcup_{k=n}^{\infty} C_{k+1} \setminus C_k\right], \\
&\leq \lim_{n\to\infty} \mathbb{P}[C_n] + \lim_{n\to\infty} \sum_{k=n}^{\infty} \mathbb{P}[C_{k+1} \setminus C_k], \\
&= 0.
\end{aligned}$$

The second equality follows from Lemma 1.1 and the last equality follows from the given.

Problem 15.

Solution. If F is a continuous distribution of a random variable ξ on the probability space $(\Omega, \mathcal{G}, \mathcal{P})$, then it must be that F is injective on $\operatorname{img} \xi$. Consider, now a random variable $\eta = F(\xi)$. We proceed to find the distribution of this η:

$$\begin{aligned}
\mathbb{P}[\{\omega \in \Omega : \eta(\xi(\omega)) \leq x\}] &= \mathbb{P}\big[\big\{\omega \in \Omega : \xi(\omega) \leq F^{-1}(x)\big\}\big], \\
&= F(F^{-1}(x)).
\end{aligned}$$

Note that unless F is bijective, $F \circ F^{-1} \neq \mathrm{Id}$. Recall, that there exists a right-inverse if and only if F is surjective.

Chapter 2

Sequences of Independent Trials

Some brief notes before the problems:

- Koralov and Sinai use $\times$ instead of $\otimes$ for the product σ-algebra. Moving forward, the reader needs to be cautious about what the operands are. For if they are σ-algebras, we understand the result of the expression to be the aforementioned product.

Problem 1.

Solution. Directly apply the definition of the binomial distribution. Let $X = \{0, 1\}$. Define the random variable $\chi_i(\omega)$ for $1 \leq i \leq 5$ to be 1 if $\omega = 1$ and zero otherwise. Then, let $\nu_5 = \sum_{i=1}^{5} \chi_i$. ν_5 is a binomial random variable. Thus,

$$\mathbb{P}[\nu_5 = 3] = \binom{5}{3} \frac{1}{2^5} = \frac{5}{16}.$$

Problem 2.

Solution.

Problem 3.

Solution. We directly apply the de Moivre-Laplace Theorem. Let $X_i(\omega_i)$ be a random variable that is 1 if ω_i = heads and 0 otherwise. Then, let $\nu_n = \sum_{i=1}^{n} X_i$. Then, for $n = 1000$, $\mathbb{E}[\nu_n] = 500$ and $\operatorname{Var} \nu_n = 250$. $k = 600$ as the actual number of heads that we saw during all of the coin flips. Therefore, $z = (600 - 500)/\sqrt{250} = 20/\sqrt{(10)}$. Therefore, by the de Moivre-Laplace theorem:

$$\begin{aligned}\mathbb{P}\left[\frac{\nu_n - \mathbb{E}[\nu_n]}{\sqrt{\operatorname{Var}(\nu_n)}} \geq z\right] &\geq \int_{20/\sqrt{10}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}z^2\right) dz, \\ &\leq 10^{-5}.\end{aligned}$$

So, it is highly unlikely that the coin is fair.

Problem 4.

Solution.

Problem 5.

Solution.

Problem 6.

Solution.

Problem 7.

Solution.

Problem 8.

Solution.