

# Rare Disasters and Risk Sharing with Heterogeneous Beliefs

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Risks of rare economic disasters can have a large impact on asset prices. At the same time, difficulties in inference regarding both the likelihood and severity of disasters, as well as agency problems, can lead to significant disagreements among investors about disaster risk. We show that such disagreements generate strong risk-sharing motives, such that just a small number of optimists in the economy will significantly reduce the disaster risk premium. Our model highlights the “latent” nature of disaster risk. The disaster risk premium will likely be low and smooth during normal times but increases dramatically when the risk-sharing capacity of the optimists is reduced, e.g., following a disaster. The model also helps reconcile the difference in the amount of disaster risk implied by financial markets and international macroeconomic data, and provides caution to the approach of extracting disaster probabilities from asset prices, which will disproportionately reflect the beliefs of a small group of optimists. Finally, our model predicts an inverse U-shaped relation between the equity premium and the size of the disaster insurance market. (*JEL* E44, G12)

Recent research by [Barro \(2006\)](#), [Gabaix \(2012\)](#), and others has shown that a model of rare disasters calibrated to international macroeconomic data can explain the equity premium and a wide range of other macro and asset pricing puzzles.<sup>1</sup> At the same time, almost by definition, it is difficult to accurately

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<sup>1</sup> Earlier contributions on disaster risk include [Rietz \(1988\)](#), [Longstaff and Piazzesi \(2004\)](#), and [Liu, Pan, and Wang \(2005\)](#). Among the more recent work are [Weitzman \(2007\)](#), [Barro \(2009\)](#), [Wachter \(2012\)](#), [Farhi and Gabaix \(2011\)](#), [Gourio \(2012\)](#), and others.

estimate the likelihood of disasters or their impact, which naturally leads to disagreements among investors about disaster risk. In this article, we show that the relation between the disaster risk premium and the amount of disagreements about disaster risk is highly nonlinear. In particular, just a small amount of optimistic investors will greatly attenuate the impact of disaster risk on asset prices. Our article highlights the “latent” nature of disaster risk in financial markets. It helps reconcile the difference in the amount of disaster risk implied by financial markets and international macroeconomic data and predicts an inverse U-shaped relation between the equity premium and the size of the disaster insurance market.

We study an endowment economy with two types of agents who disagree about disaster risk. A technical contribution of our model is that it captures very general forms of disagreements in a tractable way. For example, the agents can disagree about the intensity of disasters as well as the distribution of disaster size, and both the perceived disaster intensities and the amount of disagreements are allowed to fluctuate over time. We assume markets are complete, so that the agents can trade contingent claims and achieve optimal risk sharing.

Heterogeneous beliefs about disaster risk arise naturally because of the difficulty in estimating the frequency and size of disasters with limited data. For example, a frequentist will not be able to reject the hypothesis of a disaster intensity of 3% per year at the 5% significance level, even after observing a hundred-year sample without a single disaster. Another source of heterogeneous beliefs is agency problems for fund managers and large financial institutions. Limited liability, lack of transparency, compensation contracts that reward short-term performance, and government guarantees can all motivate excessive tail-risk taking, often referred to as “picking up nickels in front of a steamroller.”<sup>2</sup> These agents will effectively act as optimists in our model.

We show that having a new group of agents with different beliefs about disasters can cause the equity premium to drop substantially, even when the new agents have only a small amount of wealth. This result holds whether the disagreement is about the intensity or impact of disasters. We analytically characterize the sensitivity of risk premiums to the wealth distribution and derive its limit as the amount of disagreement increases. When we calibrate the beliefs of one agent using international macroeconomic data (from Barro 2006) and the other using consumption data from the United States (where disasters have been relatively mild), raising the fraction of total wealth for the second agent from 0% to 10% lowers the equity premium from 4.4% to 2.0%. The decline in the equity premium becomes faster when the disagreement is larger or when the new agents also have lower risk aversion.

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<sup>2</sup> It is well documented that shorting out-of-the-money S&P put options can generate superior investment performances in short samples. See, e.g., Lo (2001). Malliaris and Yan (2011) show that reputation concerns can cause fund managers to favor strategies with negative skewness. Makarov and Plantin (2011) show that convex compensation contracts can lead to risk shifting in the form of selling deep out-of-the-money puts.

Why is the disaster risk premium so sensitive to heterogeneous beliefs? First, the equity premium grows exponentially in the size of individual consumption losses during a disaster. Thus, removing just the “tail of the tail” from consumption losses (i.e., the most extreme losses) can dramatically bring down the premium. For example, in a representative agent economy (with relative risk aversion  $\gamma = 4$ ), if the consumption loss in a disaster is reduced from 40% to 35%, the equity premium will fall by 40%. This nonlinearity is an intrinsic property of disaster risk models, which generate high premium from rare events by making marginal utility in the disaster states rise substantially with the size of the consumption losses.

Second, the equity premium in our economy derives primarily from disaster risk, and the compensation for bearing disaster risk must be high. For example, if the equity premium due to disaster risk is 4% per year and the market falls by 40% in a disaster, then a disaster insurance contract that pays one dollar when a disaster strikes within a year must cost at least ten cents, regardless of the actual chance of payoff. Such a high premium provides strong incentive for investors with optimistic beliefs about disasters to provide the insurance. In a benchmark example of our model, the pessimists are willing to pay up to thirteen cents per dollar of disaster insurance, even though the payoff probability is only 1.7% under their own beliefs. The optimists, who believe the payoff probability is just 0.1%, underwrite insurance contracts with notional value up to 40% of their total wealth, despite the risk of losing 70% of their consumption if a disaster strikes.

Our model provides new insights into how disaster risk affects the dynamics of asset prices. The disaster risk premium crucially depends on the wealth distribution among investors with different beliefs. During normal times (when the wealth distribution among heterogeneous investors is relatively disperse), the disaster risk premium will remain low and smooth despite the fluctuations in the average belief of disaster risk in the market. This makes disaster risk “latent” and hard to detect in financial markets. When the wealth share of the pessimists rises (e.g., following a disaster), the disaster risk premium will increase dramatically and become more sensitive to fluctuations in disaster risk going forward. Such changes in the wealth distribution can also occur for other reasons. For example, the optimists’ beliefs about big disasters can converge to those of the pessimists after observing a relatively small market crash. Fund managers and financial institutions that are acting as optimists can also lose their risk-sharing capacity when they face tighter capital constraints.

The model also helps reconcile the tension between the amount of disaster risk indicated by macroeconomic data and asset prices. For example, [Backus, Chernov and Martin \(2011\)](#) and [Collin-Dufresne, Goldstein, and Yang \(2010\)](#) find that the prices of index options and credit derivatives imply significantly smaller probabilities of extreme outcomes than those estimated from macroeconomic data. See also [Mehra and Prescott \(1988\)](#). We show that, in the presence of heterogeneous beliefs about disasters, asset prices tend to

disproportionately reflect the beliefs of those optimistic agents in the economy, which could make asset prices appear little affected by the disaster risks in the macroeconomy.

The above results also provide caution for extracting disaster probabilities from asset prices. The link between the risk-neutral and actual probabilities of disasters is simple and stable in a model with homogeneous agents, which makes it straightforward to estimate the actual disaster probabilities from the prices of financial assets, such as options. However, our model shows that if we ignore the potential effects of risk sharing and directly extract disaster probabilities from financial data, we could substantially underestimate disaster probabilities. Moreover, changes in the wealth distribution among heterogeneous investors can lead to substantial changes in the risk-neutral probabilities of disasters in the absence of any variation in the actual disaster probabilities, which could cause us to overestimate the variations in the actual disaster probabilities over time.

Finally, our model predicts a novel relation between the equity premium and the size of the disaster insurance market. There are two distinct scenarios under which there will be little trading of the disaster insurance contracts: (1) when the market perceived disaster risk is low; or (2) when investors all agree that disaster risk is high and no one is willing to provide the insurance. The disaster risk premium will be low in the first case but high in the second case. A large amount of trading in disaster insurance markets not only indicates strong demand for disaster insurance but also indicates significant heterogeneity across investors, which will keep the disaster risk premium at low levels. It is when the risk-sharing capacity in the economy dries up (when the optimists have little wealth) that the disaster risk premium becomes the highest.

Our article builds on the literature of heterogeneous beliefs and preferences.<sup>3</sup> The two articles closest to ours are [Bates \(2008\)](#) and [Dieckmann \(2011\)](#). [Bates \(2008\)](#) studies investors with heterogeneous attitudes toward crash risk, which is isomorphic to heterogeneous beliefs of disaster risk. He focuses on small but frequent crashes and does not model intermediate consumption, and he shows that investor heterogeneity helps explain various option pricing anomalies. [Dieckmann](#) considers only log utility. In that setting, risk sharing has limited effects on the equity premium, and many of the asset pricing puzzles that disaster risks are able to solve remain. Our model considers power utility and captures more general disagreements about disasters, time-varying disaster intensities, and time-varying disagreement.

The rest of the article is organized as follows. Section 1 presents the model. Section 2 analyzes the effect of risk sharing in a setting with disagreement about disaster intensity. Section 3 compares our results to other forms of

<sup>3</sup> See [Basak \(2005\)](#) for a survey on heterogeneous beliefs and asset pricing. Recent developments include [Kogan et al. \(2006\)](#), [Buraschi and Jiltsov \(2006\)](#), [Yan \(2008\)](#), [David \(2008\)](#), [Dumas, Kurshev, and Uppal \(2009\)](#), [Xiong and Yan \(2009\)](#), and [Dieckmann and Gallmeyer \(2009\)](#) among others. Among the works on heterogeneous preferences are [Dumas \(1989\)](#), [Wang \(1996\)](#), [Chan and Kogan \(2002\)](#), [Dieckmann and Gallmeyer \(2005\)](#), and more recently [Longstaff and Wang \(2008\)](#).

heterogeneity. Section 4 discusses the robustness of the model, and Section 5 concludes.

## 1. Model Setup

We consider a continuous-time endowment economy. There are two agents (A, B), each being the representative of her own class. Agent A believes that the aggregate endowment is  $C_t = e^{c_t^c + c_t^d}$ , where  $c_t^c$  is the diffusion component of log aggregate endowment, which follows the process below:

$$dc_t^c = \bar{g}_A dt + \sigma_c dW_t^c, \quad c_0^c = 0, \quad (1)$$

where  $\bar{g}_A$  and  $\sigma_c$  are the expected growth rate and volatility of consumption without jumps, and  $W_t^c$  is a standard Brownian motion under agent A's beliefs. The term  $c_t^d$  (with  $c_0^d = 0$ ) is a pure jump process, whose jumps arrive with stochastic intensity  $\lambda_t$  under A's beliefs,

$$d\lambda_t = \kappa(\bar{\lambda}^A - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dW_t^\lambda, \quad (2)$$

where  $\bar{\lambda}^A$  is the long-run average jump intensity under A's beliefs, and  $W_t^\lambda$  is a standard Brownian motion independent of  $W_t^c$ . The jumps  $\Delta c_t^d$  have time-invariant distribution  $\nu^A$ . We summarize agent A's beliefs with the probability measure  $\mathbb{P}_A$ .

Agent B believes that the probability measure is  $\mathbb{P}_B$ , which we shall suppose is equivalent to  $\mathbb{P}_A$ .<sup>4</sup> Intuitively, the probability measures are equivalent when the two agents agree on the set of events that cannot occur; this rules out, e.g., the scenario in which one agent believes that there is a small probability of a disaster, whereas the other agent believes such disasters will never occur. Agent B may disagree about the growth rate of consumption without jumps, the likelihood of disasters, or the severity of disasters (when they occur). We assume that the two agents are aware of each others' beliefs but "agree to disagree."<sup>5</sup>

Chen, Joslin, and Tran (2010) show that the differences in beliefs can be characterized by the Radon-Nikodym derivative (or likelihood ratio)  $\eta_t \equiv (d\mathbb{P}_B/d\mathbb{P}_A)_t$ . To develop some intuition for  $\eta_t$ , let's consider the case in which disasters have a constant size, and the only disagreement between the two agents is on the (constant) disaster intensity:  $\bar{\lambda}^A$  versus  $\bar{\lambda}^B$ . Since the number of disasters is Poisson distributed, the relative likelihood of exactly  $n$  disasters occurring between time 0 and  $t$  for the two agents is

$$\frac{f^B(N_t = n)}{f^A(N_t = n)} = \frac{e^{-\bar{\lambda}^B t} (\bar{\lambda}^B)^n}{e^{-\bar{\lambda}^A t} (\bar{\lambda}^A)^n}. \quad (3)$$

<sup>4</sup> More precisely,  $\mathbb{P}^A$  and  $\mathbb{P}^B$  are equivalent when restricted to any  $\sigma$ -field  $\mathcal{F}_T = \sigma(\{c_t^c, c_t^d, \lambda_t\}_{0 \leq t \leq T})$ .

<sup>5</sup> We do not explicitly model learning about disasters. Given the nature of disasters, Bayesian updating of beliefs about disaster risk using realized consumption growth will likely be very slow, and the disagreements in the priors will persist for a long time. See also Section 4.

Thus, whenever a disaster strikes,  $N_t$  will increase by 1. When this happens, the likelihood ratio will jump by a factor of  $\bar{\lambda}^B/\bar{\lambda}^A$ . If  $\bar{\lambda}^B < \bar{\lambda}^A$ , i.e., agent B feels disasters are less likely, then the likelihood ratio jumps down. In contrast, when time goes by and disasters do not occur (i.e.,  $t$  increases but  $N_t$  does not jump up), the likelihood ratio drifts up at the rate  $\bar{\lambda}^A - \bar{\lambda}^B$ . This is because a situation in which no disasters occur over a period of time is more consistent with agent B's beliefs.

For the general case of disagreement about growth rates, stochastic disaster probabilities, and disaster size distributions, the Radon-Nikodym derivative  $\eta_t$  is given by

$$\eta_t = \exp \left( a_t - \int_0^t \lambda_s \left( \frac{\bar{\lambda}^B}{\bar{\lambda}^A} - 1 \right) ds + b c_t^c - \left( b \bar{g}_A + \frac{1}{2} b^2 \sigma_c^2 \right) t \right) \quad (4)$$

for some constant  $b$  and  $\bar{\lambda}^B > 0$ , and  $a_t$  is a pure jump process (with  $a_0 = 0$ ), whose jumps are coincident with the jumps in  $c_t^d$  and have the size

$$\Delta a_t = \log \left( \frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{dv^B}{dv^A} \right), \quad (5)$$

where  $\frac{dv^B}{dv^A}$  is the relative likelihood of the agents' beliefs for a disaster of a particular size, conditional on a disaster having occurred. It will be large (small) for the type of disasters that agent B thinks are relatively more (less) likely than agent A.

The interpretation for the term  $e^{a_t - \int_0^t \lambda_s \left( \frac{\bar{\lambda}^B}{\bar{\lambda}^A} - 1 \right) ds}$  in  $\eta_t$  is similar to the likelihood ratio in Equation (3), except that now jumps in  $a_t$  reflect not only disagreement about the disaster intensity ( $\bar{\lambda}^B/\bar{\lambda}^A$ ) but also disagreement about the distribution of disaster size ( $\frac{dv^B}{dv^A}$ ). The above specification implies that under B's beliefs, a disaster occurs with intensity  $\lambda_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A}$  (with the long-run average intensity  $\bar{\lambda}^B$ ), and the disaster size distribution is  $v^B$  (which is equivalent to  $v^A$ ).

The term  $e^{b c_t^c - \left( b \bar{g}_A + \frac{1}{2} b^2 \sigma_c^2 \right) t}$  captures agent B's potential disagreement about the growth rate of consumption. It implies that agent B believes that the expected growth rate of consumption without jumps is  $\bar{g}_B \equiv \bar{g}_A + b \sigma_c^2$ . When  $b > 0$ , agent B is more optimistic about the growth rate of consumption than is A. Then, large realizations of  $c_t^c$  (when  $c_t^c$  exceeds the average of the two agents' beliefs,  $\frac{1}{2}(\bar{g}_A + \bar{g}_B)t$ ) will be more consistent with B's belief, and in such cases the likelihood ratio will be larger than 1.

We assume that the agents are infinitely lived and have constant relative-risk aversion (CRRA) utility over lifetime consumption:

$$U^i(C^i) = E_0^i \left[ \int_0^\infty e^{-\rho_i t} \frac{(C_t^i)^{1-\gamma_i}}{1-\gamma_i} dt \right], \quad i = A, B, \quad (6)$$

where  $E^i$  denotes the expectation under agent  $i$ 's beliefs  $\mathbb{P}_i$ . We also assume that markets are complete and agents are endowed with some fixed share of aggregate consumption ( $\theta_A, \theta_B = 1 - \theta_A$ ).

The equilibrium allocations can be characterized as the solution of the following planner's problem, specified under the probability measure  $\mathbb{P}_A$ ,

$$\max_{C_t^A, C_t^B} E_0^A \left[ \int_0^\infty e^{-\rho_A t} \frac{(C_t^A)^{1-\gamma_A}}{1-\gamma_A} + \tilde{\zeta}_t e^{-\rho_B t} \frac{(C_t^B)^{1-\gamma_B}}{1-\gamma_B} dt \right], \quad (7)$$

subject to the resource constraint  $C_t^A + C_t^B = C_t$ . Here,  $\tilde{\zeta}_t \equiv \zeta \eta_t$  is the belief-adjusted Pareto weight for agent B. From the first-order condition and the resource constraint, we obtain the equilibrium consumption allocations  $C_t^A = f^A(\hat{\zeta}_t) C_t$  and  $C_t^B = (1 - f^A(\hat{\zeta}_t)) C_t$ , where  $\hat{\zeta}_t = e^{(\rho_A - \rho_B)t} C_t^{\gamma_A - \gamma_B} \tilde{\zeta}_t$ , and  $f^A$  is in general an implicit function.

The stochastic discount factor under A's beliefs,  $M_t^A$ , is given by

$$M_t^A = e^{-\rho_A t} (C_t^A)^{-\gamma_A} = e^{-\rho_A t} f^A(\hat{\zeta}_t)^{-\gamma_A} C_t^{-\gamma_A}. \quad (8)$$

Finally, we solve for the Pareto weight  $\zeta$  through the lifetime budget constraint for one of the agents (Cox and Huang 1989), which is linked to the initial allocation of endowment.

Since our emphasis is on heterogeneous beliefs about disasters, for the remainder of this section we focus on the case in which there is no disagreement about the distribution of Brownian shocks, and the two agents have the same preferences. In this case,  $b=0$ ,  $\gamma_A = \gamma_B = \gamma$ , and  $\rho_A = \rho_B = \rho$ . The equilibrium consumption share then simplifies to

$$f^A(\tilde{\zeta}_t) = \frac{1}{1 + \tilde{\zeta}_t^{\frac{1}{\gamma}}}. \quad (9)$$

When a disaster of size  $d$  occurs,  $\tilde{\zeta}_t$  is multiplied by the likelihood ratio  $\frac{\tilde{\lambda}^B}{\tilde{\lambda}^A} \frac{d\nu^B}{d\nu^A}(d)$  (see Equation (5)). Thus, if agent B is more pessimistic about a particular type of disaster, she will have a higher weight in the planner's problem when such a disaster occurs so that her consumption share increases.

The equilibrium allocations can be implemented through competitive trading in a sequential-trade economy. Extending the analysis of Bates (2008), we can consider three types of traded securities: (1) a risk-free money market account, (2) a claim to aggregate consumption, and (3) a series (or continuum) of disaster insurance contracts with one-year maturity, which pay one dollar on the maturity date if a disaster of size  $d$  occurs within a year.

The instantaneous risk-free rate can be derived from the stochastic discount factor,

$$r_t = -\frac{\mathcal{D}^A M_t^A}{M_t^A} = \rho + \gamma \bar{g}_A - \frac{1}{2} \gamma^2 \sigma_c^2 - \lambda_t \left( \frac{E_t^{D,A} [(C_t^A)^{-\gamma}]}{(C_t^A)^{-\gamma}} - 1 \right), \quad (10)$$

where  $\mathcal{D}^A$  denotes the infinitesimal generator under agent A's beliefs of the state variables  $X_t = (c_t^c, c_t^d, \lambda_t, \eta_t)$ , and we use the shorthand notation  $E_t^{D,i}$  to

denote agent  $i$ 's expectation conditional on a disaster occurring. That is, for any function  $f(X_t)$ ,

$$E_t^{D,i}[f(X_t)] \equiv \int f\left(c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{dv^B}{dv^A}(d)\right) dv^i(d).$$

The price of the aggregate endowment claim is

$$P_t = \int_0^\infty E_t^A \left[ \frac{M_{t+\tau}^A}{M_t^A} C_{t+\tau} \right] d\tau = C_t h(\lambda_t, \tilde{\zeta}_t), \quad (11)$$

where the price/consumption ratio only depends on the disaster intensity  $\lambda_t$  and the stochastic weight  $\tilde{\zeta}_t$ . In the case in which  $\lambda_t$  is constant, the price of the consumption claim is obtained in closed form. Similarly, we can compute the wealth of the individual agents as well as the prices of disaster insurance contracts using the stochastic discount factor.

In order for prices of the aggregate endowment claim to be finite in the heterogeneous-agent economy, it is necessary and sufficient that prices are finite under each agent's beliefs in a single-agent economy (see the online appendix for a proof). As we show in the appendix, finite prices require that the following two inequalities hold:

$$0 < \kappa^2 - 2\sigma_\lambda^2(\phi^i(1-\gamma) - 1), \quad (12a)$$

$$0 > \kappa \bar{\lambda}^i \frac{\kappa - \sqrt{\kappa^2 + 2\sigma_\lambda^2(1 - \phi^i(1-\gamma))}}{\sigma_\lambda^2} - \rho + (1-\gamma)\bar{g}_A + \frac{1}{2}(1-\gamma)^2\sigma_c^2, \quad (12b)$$

where  $\phi^i$  is the moment-generating function for the distribution of jumps in endowment  $v^i$  under measure  $\mathbb{P}_i$ . The first inequality reflects the fact that the volatility of the disaster intensity cannot be too large relative to the rate of mean reversion. It prevents the convexity effect induced by the potentially large intensity from dominating the discounting. The second inequality reflects the need for enough discounting to counteract the growth.

Additionally, the stochastic discount factor characterizes the unique risk-neutral probability measure  $\mathbb{Q}$  (see, e.g., [Duffie 2001](#)), which facilitates the computation and interpretation of excess returns. The risk-neutral disaster intensity  $\lambda_t^\mathbb{Q} \equiv E_t^{D,i}[M_t^i]/M_t^i \lambda_t^i$  is determined by the expected jump size of the stochastic discount factor at the time of a disaster. When the risk-free rate and disaster intensity are close to zero, the risk-neutral disaster intensity  $\lambda_t^\mathbb{Q}$  has the nice interpretation of (approximately) the value of a one-year disaster insurance contract that pays one dollar at  $t+1$  when a disaster occurs between  $t$  and  $t+1$ . The risk-neutral distribution of the disaster size is given by  $\frac{dv^\mathbb{Q}}{dv^i}(d) = M_t^{D,i}(d)/E_t^{D,i}[M_t^i]$ , where  $M_t^{D,i}(d)$  denotes the pricing kernel when the state is  $(c_t^c, c_t^d + d, \lambda_t, \eta_t \times \frac{\bar{\lambda}^B}{\bar{\lambda}^A} \frac{dv^B}{dv^A}(d))$ . These risk adjustments are quite intuitive. The more the stochastic discount factor for agent  $i$  jumps up during



a disaster, the larger is  $\lambda_t^{\mathbb{Q}}$  relative to  $\lambda_t^i$ , i.e., disasters occur more frequently under the risk-neutral measure. Thus, the ratio  $\lambda_t^{\mathbb{Q}}/\lambda_t^i$  is often referred to as the jump-risk premium. Moreover, the risk-adjusted distribution of jump size conditional on a disaster slants the probabilities toward the types of disasters that lead to a bigger jump in the stochastic discount factor, which generally makes severe disasters more likely under  $\mathbb{Q}$ .

Finally, the risk premium for any security under agent  $i$ 's beliefs is the difference between the expected return under  $\mathbb{P}_i$  and under the risk-neutral measure  $\mathbb{Q}$ . In the case of the aggregate endowment claim, the conditional equity premium, under agent  $i$ 's beliefs, which we denote by  $E_t^i[R^e]$ , is

$$E_t^i[R^e] = \gamma \sigma_c^2 + \lambda_t^i E_t^{D,i}[R] - \lambda_t^{\mathbb{Q}} E_t^{D,\mathbb{Q}}[R], \quad i = A, B, \quad (13)$$

where  $E_t^{D,m}[R] \equiv E_t^{D,m}[P_t]/P_t - 1$  is the expected return of the endowment claim under measure  $m$  ( $m$  could be  $\mathbb{P}_i$  or  $\mathbb{Q}$ ) conditional on a disaster.<sup>6</sup> The difference between the last two terms in Equation (13) is the premium for bearing disaster risk. This premium is large if the jump-risk premium is large and/or the expected loss in return in a disaster is large (especially under the risk-neutral measure).

It follows that the difference in equity premium under the two agents' beliefs is

$$E_t^A[R^e] - E_t^B[R^e] = \lambda_t^A E_t^{D,A}[R] - \lambda_t^B E_t^{D,B}[R]. \quad (14)$$

This difference will be small relative to the size of the equity premium when the disaster intensity and expected loss under the risk-neutral measure are large relative to their values under actual beliefs. In the remainder of the article, unless stated otherwise, we will report the equity premium relative to agent A's beliefs,  $\mathbb{P}_A$ . One interpretation for picking  $\mathbb{P}_A$  as the reference measure is that A has the correct beliefs, and we are studying the impact of the incorrect beliefs of agent B on asset prices.

## 2. Heterogeneous Beliefs and Risk Sharing

We start with a special case of the model in which agents only disagree about the frequency of disasters. First, we analyze the impact of heterogeneous beliefs on asset prices and their implications for survival when the risk of disasters is constant, i.e.,  $\lambda_t = \bar{\lambda}^A$  (denoted as  $\lambda^A$  for simplicity). We then extend the analysis to the case of time-varying disaster risk.

<sup>6</sup> To be concrete, we define the risk premium under measure  $i$  for any price process  $P(X_t, t)$  which pays dividends  $D(X_t, t)$  to be  $\mathcal{D}^i P_t / P_t + D_t / P_t - r_t$ .

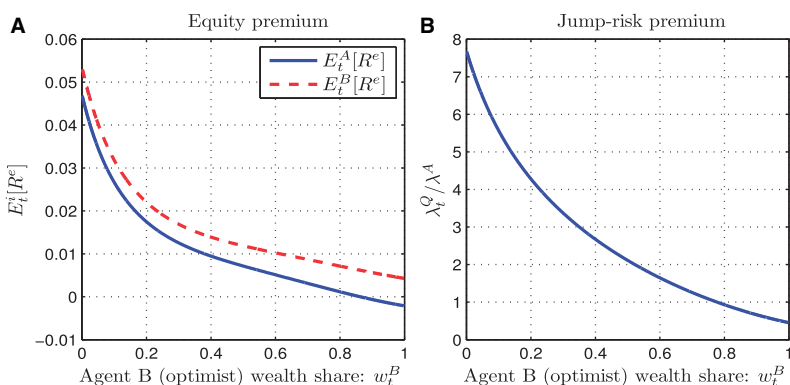
## 2.1 Disagreement about the frequency of disasters

In the benchmark case of our model, the disaster size is deterministic,  $\Delta c_t^d = \bar{d}$ , and the two agents only disagree about the frequency of disasters ( $\lambda$ ). We set  $\bar{d} = -0.51$  so that the moment-generating function (MGF)  $\phi^A(-\gamma)$  in this model matches the calibration of Barro (2006) for  $\gamma = 4$ . It implies that aggregate consumption falls by 40% when a disaster occurs. Agent A (pessimist) believes that disasters occur with intensity  $\lambda^A = 1.7\%$  (once every sixty years), which is also taken from Barro (2006). Agent B (optimist) believes that disasters are much less likely,  $\lambda^B = 0.1\%$  (once every 1,000 years), but she agrees with A on the size of disasters, as well as the Brownian risk in consumption. She also has the same preferences as does agent A. The remaining parameters are the expected consumption growth  $\bar{g} = 2.5\%$ , diffusive consumption volatility  $\sigma_c = 2\%$ , and the subjective discount rate  $\rho = 3\%$ .

Figure 1, Panel A, shows the conditional equity premium under the beliefs of both the pessimist and the optimist. From Equation (14), we obtain the difference in equity premium under the two agents' beliefs in the case of constant disaster risk:

$$E_t^A[R^e] - E_t^B[R^e] = (\lambda^A - \lambda^B) E_t^D[R],$$

where we have suppressed the index for agent type in the expected return conditional on a disaster occurring,  $E_t^D[R]$ , because there is a single type of disaster. Intuitively, disasters and the resulting losses of value in the stock are less likely under the optimist's beliefs; hence, the optimist's perceived equity premium will be higher than that of the pessimist. Compared to Equation (13), we see that the difference in equity premium under the two agents' beliefs will be small relative to the size of the equity premium when the disaster intensity



**Figure 1**

### Disagreement about the frequency of disasters

Panel A plots the equity premium under both agents' beliefs as a function of the wealth share of the optimist. Panel B plots the jump-risk premium  $\lambda_t^Q / \lambda^A$  for the pessimist.

is significantly higher under the risk-neutral measure than under the agents' beliefs, i.e., when the disaster risk premium is large. For this reason, we obtain similar results for the equity premium under either belief.

If all the wealth is owned by the pessimist, the equity premium under her belief is 4.7% (or 5.3% under the optimist's beliefs), and the risk-free rate is also at a reasonable value (1.3%). If the optimist has all the wealth, the equity premium is only -0.21% under the pessimist's beliefs<sup>7</sup> (or 0.43% under the optimist's beliefs), which reflects the low compensation the optimist requires for bearing disaster risk. Thus, it is not surprising to see the premium falling when the optimist owns more wealth. However, the speed at which the premium declines in Panel A is impressive. When the optimistic agent owns 10% of the total wealth, the equity premium under the pessimist's beliefs falls from 4.7% to 2.7%. When the wealth of the optimist reaches 20%, the equity premium falls to just 1.7%.

We can derive the conditional equity premium as a special case of Equation (13) using the assumption of constant disaster size,

$$E_t^A[R^e] = \gamma\sigma_c^2 - \lambda^A \left( \frac{\lambda_t^Q}{\lambda^A} - 1 \right) \left( \frac{h(\tilde{\zeta}_t \frac{\lambda^B}{\lambda^A})e^{\tilde{d}}}{h(\tilde{\zeta}_t)} - 1 \right), \quad (15)$$

where  $h$  is the price-consumption ratio from Equation (11), with  $\lambda_t$  being constant. The first term,  $\gamma\sigma_c^2$ , is the standard compensation for bearing Brownian risk. Heterogeneity has no effect on this term since the two agents agree about the Brownian risk. Given the value of risk aversion and consumption volatility, this term has negligible effect on the premium. The second term reflects the compensation for disaster risk. It can be further decomposed into three factors: (1) the disaster intensity  $\lambda^A$ , (2) the jump-risk premium  $\lambda_t^Q/\lambda^A$ , and (3) the return of the consumption claim in a disaster.

How does wealth distribution affect the jump-risk premium? From the definition of the stochastic discount factor  $M_t^A$  and the risk-neutral intensity  $\lambda_t^Q$ , it is easy to show

$$\lambda_t^Q/\lambda^A = e^{-\gamma\Delta c_t^A}, \quad (16)$$

where  $\Delta c_t^A$  is the jump size of the equilibrium log consumption for agent A in a disaster. Without trading, the individual loss of consumption in a disaster will be equal to that of the endowment,  $\Delta c_t^A = \tilde{d}$ , which under our parameterization generates a jump-risk premium of  $\lambda_t^Q/\lambda^A = 7.7$ . Since  $\lambda_t^Q$  is approximately the premium of a one-year disaster insurance, before any trading the pessimist will be willing to pay an annual premium of about thirteen cents for one dollar of protection against a disaster event that occurs with probability 1.7%.

<sup>7</sup> This negative premium is due to the pessimist acquiring a large amount of insurance against disasters. We discuss this feature in detail later in this section.

The optimist views disasters as very unlikely events and is willing to trade away her claims in the future disaster states in exchange for higher consumption in normal times. Such trades help reduce the pessimist's consumption loss in a disaster  $\Delta c_t^A$ , which in turn lowers the jump-risk premium. However, the optimist's capacity for underwriting disaster insurance is limited by her wealth, as she needs to ensure that her wealth is positive in all future states, including when a disaster occurs (no matter how unlikely such an event is). Thus, the more wealth the optimist has, the more disaster insurance she is able to sell.

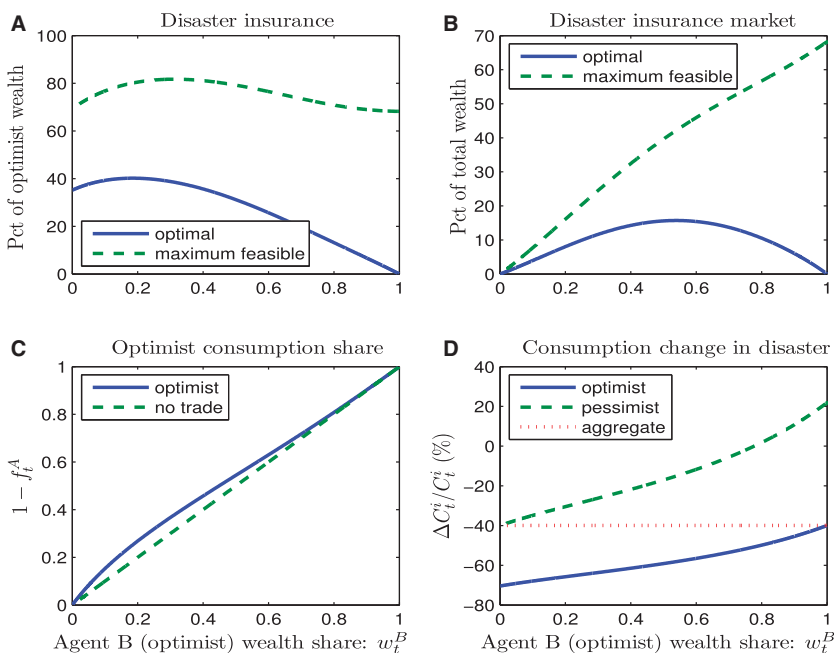
The above mechanism can substantially reduce the disaster risk exposure of the pessimist in equilibrium. Panel B of Figure 1 shows that when the optimist owns 20% of total wealth, the jump-risk premium drops from 7.7 to 4.2. According to Equation (15), such a drop in the jump-risk premium alone will cause the equity premium to fall by about half to 2.2%, which accounts for the majority of the change in the premium (from 4.7% to 1.7%).

Besides the jump-risk premium, the equity premium also depends on the return of the consumption claim in a disaster, which in turn is determined by the consumption loss and changes in the price-consumption ratio. Following a disaster, the risk-free rate drops as the wealth share of the pessimist rises. With CRRA utility, the lower interest rate effect can dominate the effect of the rise in the risk premium, leading to a higher price-consumption ratio.<sup>8</sup> Since a higher price-consumption ratio partially offsets the drop in aggregate consumption, it makes the return less sensitive to disasters, which will contribute to the drop in equity premium. However, our decomposition above shows that the reduction of the jump-risk premium (due to reduced disaster risk exposure) is the main reason behind the fall in premium.

Can we "counteract" the effect of the optimistic agent and restore the high equity premium by making the pessimist even more pessimistic about disasters? We also examine the case in which agent A believes that the disaster intensity is 2.5% ( $\lambda^A = 2.5\%$ ) and everything else remains the same. Whereas the equity premium under the pessimist's beliefs becomes significantly higher (6.8%) when she owns all the wealth, it falls to 4.1% with just 2% of total wealth allocated to the optimist and is below 1% when the optimist's wealth share exceeds 8.5%. Again, the decline in the jump-risk premium is the main reason behind the decline in equity premium. Thus, as the pessimist becomes more pessimistic, she seeks risk sharing more aggressively, which can quickly reverse the effect of her heightened fear of disasters on the equity premium.

To illustrate the risk-sharing mechanism, we compute the agents' portfolio positions in the aggregate consumption claim, disaster insurance, and the money market account. Calculating these portfolio positions amounts to finding a replicating portfolio that matches the exposure to Brownian shocks and jumps in the individual agents' wealth processes. The online appendix provides

<sup>8</sup> Wachter (2012) also finds a positive relation between the price-consumption ratio and the equity premium in a representative agent rare disaster model with time-varying disaster probabilities and CRRA utility.

**Figure 2****Risk sharing**

Panels A and B plot the total notional value of disaster insurance relative to the wealth of the optimist and total wealth in the economy. Panel C plots the consumption share for the optimist in equilibrium. Panel D compares the two agents' consumption drops in a disaster with that of the aggregate endowment. These results are for the case  $\lambda^A = 1.7\%$ .

the details. Notice that each agent will hold a constant proportion of the consumption claim. This is because they agree on the Brownian risk and share it proportionally. Disagreement over disaster risk is resolved through trading in the disaster insurance market, which is financed by the money market account.<sup>9</sup>

We first plot the notional value of the disaster insurance sold by the optimist as a fraction of her total wealth in Panel A of Figure 2. The dashed line is the maximum amount of disaster insurance the optimist can sell (as a fraction of her wealth) subject to her budget constraint. When the optimist has very little wealth, the notional value of the disaster insurance she sells is about 35% of her wealth. This value is initially high and then falls as the optimist gains more wealth. This is because when the optimist has little wealth, the pessimist has great demand for risk sharing and is willing to pay a higher premium, which induces the optimist to sell more insurance relative to her wealth. As the

<sup>9</sup> The implementation of the equilibrium is not unique. For example, instead of disaster insurance, we can use another contract that has exposure to both Brownian and jump risks, in which case the agents will also trade the consumption claim.

optimist gets more wealth, the premium on the disaster insurance falls and so does the relative amount of insurance sold.

We can judge how extreme the risk sharing in equilibrium is by comparing the actual amount of trading to the maximum amount imposed by the budget constraint. At its peak, the amount of disaster insurance sold by the optimist is about half of the maximum amount that she can underwrite, which might appear reasonable. The caveat is that, in reality, underwriters of disaster insurance will likely be required to collateralize their promises to pay in the disaster states, which raises the costs of risk sharing. We will further investigate the feasibility of risk sharing and discuss an alternative implementation that does not require disaster insurance in Section 4.

Panel B plots the size of the disaster insurance market (the total notional value normalized by total wealth). Naturally, the size of this market is zero when either agent has all the wealth, and the market is bigger when wealth is more evenly distributed. Notice that the model generates a nonmonotonic relation between the size of the disaster insurance market and the equity premium. The premium is high when there is a lot of demand for disaster insurance but little supply and is low when the opposite is true. In either case, the size of the disaster insurance market will be small.

Panel C plots the equilibrium consumption share for the optimist. The 45° line corresponds to the case of no trading. The optimist's consumption share is above the 45° line, more so when her wealth share is low. This is because the optimist is giving up consumption in future disaster states in exchange for higher consumption now.<sup>10</sup> Panel D shows that indeed the optimist does bear much greater losses in the event of a disaster. As for the pessimist, the less wealth she possesses, the more disaster insurance she is able to buy relative to her wealth, which lowers her disaster risk exposure and can eventually turn the disaster insurance into a speculative position—her consumption can jump up in a disaster.

## 2.2 The limiting case for risk sharing

In the previous section, we have numerically demonstrated the effects of risk sharing on asset prices. To highlight the key ingredients of the risk-sharing mechanism, we now analytically characterize the equilibrium when a small fraction of wealth is controlled by an optimist who believes disasters are extremely unlikely.<sup>11</sup>

The intuition is as follows. Suppose the pessimist (agent A) consumes fraction  $f_{t-}^A$  of the aggregate endowment  $C_{t-}$  before a disaster at time  $t$ . Since the optimist (agent B) feels disasters are quite unlikely, she is willing to sell her

<sup>10</sup> This result is also due to the low elasticity of intertemporal substitution implied by the CRRA utility, which makes the optimists consume now instead of saving the insurance premium for the future.

<sup>11</sup> We thank Xavier Gabaix for suggesting this analysis.

entire share of the endowment in the disaster state to the pessimist. Thus, when the disaster strikes, aggregate endowment drops to  $C_t = e^{\bar{d}} C_{t-}$ , but agent A now essentially consumes all the endowment ( $f_t^A \approx 1$ ). This argument implies that the jump in the marginal utility of agent A following a disaster, which is also the jump-risk premium she demands, is equal to

$$\frac{\lambda_t^Q}{\lambda^A} \approx \frac{(1 \times e^{\bar{d}} C_{t-})^{-\gamma}}{(f_t^A C_{t-})^{-\gamma}} = (f_t^A)^{\gamma} e^{-\gamma \bar{d}}. \quad (17)$$

For example, when the optimist has just 1% of the endowment before a disaster, the jump-risk premium will be  $(.99)^{\gamma} e^{-\gamma \bar{d}}$ , which, for  $\gamma=4$ , implies approximately a 4% drop in the jump-risk premium from the case in which the pessimist has all the wealth.

Formally, we show in the online appendix that the speed at which the jump-risk premium changes with the optimist's consumption share is given by

$$\lim_{\lambda^B \rightarrow 0^+} \frac{\partial}{\partial f_t^B} \frac{\lambda_t^Q}{\lambda^A} \bigg|_{f_t^B=0} = -\gamma e^{-\gamma \bar{d}}. \quad (18)$$

We see that the effect of risk sharing (in terms of consumption share) becomes stronger with bigger disasters ( $|\bar{d}|$ ) and higher risk aversion ( $\gamma$ ).<sup>12</sup>

The above result only partially reflects the steep slope in the risk premium near  $w_t^B=0$ , as we see in Figure 1. If the optimist consumes a fraction  $f_t^B$  of the endowment at time  $t$ , his fraction of the aggregate wealth,  $w_t^B$ , will be *less* than  $f_t^B$ . This is because the optimist has sold his share of endowment in the disaster state in exchange and consumes more in normal times (see Figure 2, Panel C). This effect implies that the risk premium will decline even faster as a function of the wealth share of the optimist than the consumption share.

To summarize, the limiting differential effect of optimist on the jump-risk premium is given by the following multiplier:

$$\lim_{\lambda^B \rightarrow 0^+} \frac{\partial}{\partial w_t^B} \frac{\lambda_t^Q}{\lambda^A} \bigg|_{f_t^B=0} = \frac{\partial}{\partial f_t^B} \frac{\lambda_t^Q}{\lambda^A} \bigg|_{f_t^B=0} \times \frac{\partial f_t^B}{\partial w_t^B} \bigg|_{f_t^B=0}. \quad (19)$$

The second term reflects the relative wealth-consumption ratios of the two agents, which is determined by their endogenous investment-consumption decisions. In the online appendix, we derive the expression for  $\frac{\partial f_t^B}{\partial w_t^B} \big|_{f_t^B=0}$ . There we show that under very general conditions, a large equity premium due to disasters implies that this ratio will be large, since the claim to consumption is very valuable after disasters occur. In the calibrated example, the multiplier (with  $\lambda^B=0$ ) equals  $-0.581$ . Hence, due to the decline in the jump-risk premium

<sup>12</sup> We take limits since, with  $\lambda^B=0$ , the beliefs are not equivalent and there is no complete markets equilibrium.

alone, allocating only 1% of the endowment to the extreme optimist results in a 58.1-basis-point decline in the equity premium. In comparison, the benchmark case with  $\lambda^B=0.1\%$  generates a multiplier of  $-0.19$ . When  $\lambda^A=2.5\%$  and  $\lambda^B=0$ , the multiplier is  $-2.94$ , which translates into a 2.94% drop in the equity premium when we introduce only 1% of extreme optimist into the economy.

Figure 3 compares the jump-risk premium for several cases. First, the dotted line denotes the benchmark case from Section 2.1. We also plot the jump-risk premium with the same parameters but for the limiting case, where  $\lambda^B$  approaches zero. Additionally, we plot the case in which we decrease the disaster size and increase the risk aversion to maintain the same jump-risk premium for the single agent economy ( $\gamma=6$  and  $\bar{d}=-0.34$ ). The graph shows that the marginal effect of a small amount of optimist with  $\lambda^B=0.1\%$  on the jump-risk premium is visibly smaller than in the limiting case of extreme optimism. Moreover, when we decrease the disaster size but increase risk aversion, the effects become more severe. This is because the larger risk-sharing effect on the jump-risk premium in Equation (18) dominates the smaller consumption-wealth share effect.

2.3 Survival

In models with heterogeneous agents, one type of agents often dominates in the long run (a notable exception is in Chan and Kogan 2002; see also

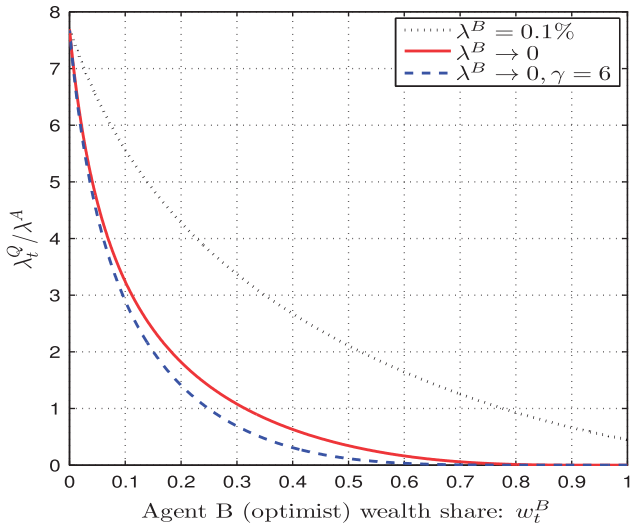


Figure 3  
Limiting jump-risk premia

This figure plots the jump-risk premium  $\lambda^Q/\lambda^A$  for the pessimist, where  $\lambda^A=1.7\%$ . In the benchmark case (dotted line),  $\gamma=4$ , and  $\lambda^B=0.1\%$ . The solid line is for the limiting case where agent B becomes extremely optimistic about disasters ( $\lambda^B \rightarrow 0\%$ ), and the dashed line shows the additional effect of higher risk aversion.



Borovička 2012). Our model also has the property that the agent with correct beliefs will dominate in the long run. For example, let's assume that agent A has the correct beliefs. The strong law of large numbers implies that  $\log \tilde{\zeta}_t \rightarrow -\infty$ , almost surely. Since wealth is monotonic in the relative planner weight,  $\tilde{\zeta}_t$ , this implies that agent A will take over the economy with probability one. We now show that although agents with incorrect beliefs about disasters may not have permanent effects on asset prices, their effects may be long-lived in the sense that these agents can retain, and even build, wealth over long horizons.

With disaster intensity,  $\lambda_t$ , being constant, we need only consider the distribution of the stochastic Pareto weight,  $\tilde{\zeta}_t$ , to analyze the wealth distribution over time. From Equation (4), we see that  $\tilde{\zeta}_t$  has a stochastic component, whereby the Pareto weight (and thus wealth) of the pessimistic agent will jump up when a disaster occurs. This is because the pessimist receives insurance payments from the optimist in a disaster. However, regardless of the occurrence of disasters, there is also a deterministic component in  $\tilde{\zeta}_t$ , whereby the optimist has a deterministic weight increase (and thus her relative wealth increases), which comes from collecting the disaster insurance premium. Thus, even when the pessimist has *correct* beliefs, her relative wealth will *decrease* outside of disasters. Since disasters are rare, it will be common to have extended periods without disasters, during which time an optimistic agent will gain relative wealth.

Table 1 presents a summary of the conditional distribution of wealth after fifty years for various initial wealth distributions. We report the results under the assumption that either the pessimist or the optimist has correct beliefs. If the number of disasters is either 0 or 1, the wealth of the agents remains relatively close to the original distribution. We see that the optimist is likely to retain wealth for long periods of time and will only be wiped out with the occurrence of several disasters, which is unlikely regardless of whose beliefs are correct.

**Table 1**  
**Survival of agents who disagree about the frequency of disasters**

Initial Wealth of B (%)	Final Wealth of B after $N_d$ Disasters (%)			
	$N_d=0$	$N_d=1$	$N_d=2$	$N_d=3$
1.0	1.2	0.6	0.3	0.1
5.0	6.1	3.0	1.5	0.7
10.0	12.2	6.0	2.9	1.4
50.0	55.7	35.5	19.3	9.6
99.0	99.2	98.3	96.7	93.5
Probability under $\mathbb{P}^A$ (%)	42.7	36.3	15.4	4.4
Probability under $\mathbb{P}^B$ (%)	95.1	4.8	0.1	0.0

This table shows the redistribution of wealth over a fifty-year horizon in the model of Section 2.1. Future relative wealth only depends on the initial wealth, the time horizon, and the number of disasters that occur. The top panel provides the possible wealth redistributions throughout time. The bottom panel provides the probabilities of various numbers of disasters (under each agent's beliefs).

The evolution of the wealth distribution over time also has important implications for the equity premium and other dynamic properties of asset prices. For example, when the initial wealth of agent  $B$  is 5% (10%), the equity premium will drop from 3.5% (2.7%) to 3.3% (2.4%) over fifty years if no disasters occurs. If after 120 years there are still no disasters, the equity premium would further drop to 2.9% (2.0%).

There are interesting differences in the survival results between the case of disagreement over disaster risk and the case of disagreement over Brownian risk in consumption growth. As shown by Yan (2008), an agent who has wrong beliefs about the growth rate of aggregate consumption can survive for long periods of time. However, in this case, those agents with wrong beliefs very rarely gain wealth over long horizons. For example, when consumption volatility is 2% per year, the probability that an agent who believes the consumption growth is 1% higher (or lower) than its true value will have a higher wealth share after fifty years is only  $4 \times 10^{-36}$ . In contrast, in the case of disagreement about disaster risk, even if the optimist has incorrect beliefs, there is a 42.7% chance that his wealth share increases relative to the agent with correct beliefs after fifty years.

To understand why the wealth dynamics are so different for the two forms of disagreements, consider first the case of disagreement about the growth rate of consumption. As we discussed in Equation (4), if agent  $B$  believes in a higher growth rate of consumption, he will gain wealth after  $t$  years provided the likelihood ratio is above 1, which occurs when the realized log consumption growth exceeds the average of the two agents' beliefs,  $\frac{1}{2}(\bar{g}_A + \bar{g}_B)t$ . The probability of this event is

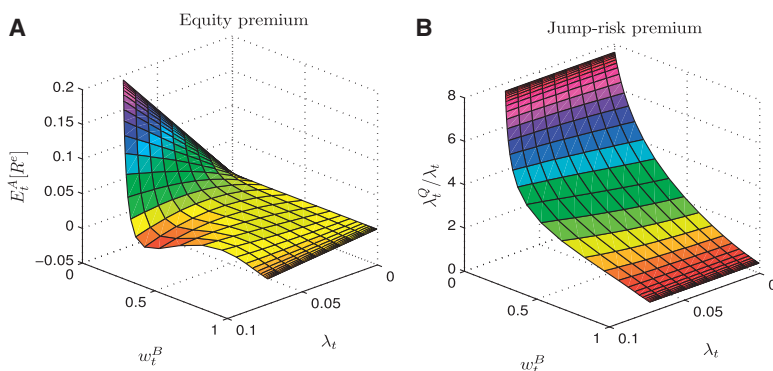
$$\mathbb{P}^A \left( \bar{g}_A t + \sigma_c W_t^c > \frac{\bar{g}_A + \bar{g}_B}{2} t \right) = \mathbb{P}^A \left( \sigma_c W_t^c > \frac{\bar{g}_B - \bar{g}_A}{2} t \right),$$

which drops very rapidly (super-exponentially) to zero as  $t$  increases. In the case of disagreements about disasters, when agent  $B$  believes disasters are less likely ( $\lambda^B < \lambda^A$ ), he will gain wealth as long as disasters do not occur. Since disasters are rare (even under  $A$ 's beliefs), the probability that no disasters occur can be small even for a relatively long period of time.<sup>13</sup>

## 2.4 Time-varying disaster risk

Having analyzed in depth the case of heterogeneous beliefs when disaster intensity is constant, now we extend the analysis to allow the risk of disasters to vary over time, which not only makes the model more realistic but also has important implications for the dynamics of asset prices. As in Gabaix (2012) and Wachter (2012), time-varying disaster intensity serves to drive both asset prices and expected excess returns. We now demonstrate that within

<sup>13</sup> More precisely, agent  $B$  will gain wealth whenever the number of disasters is less than  $(\lambda^B - \lambda^A)t / \log(\frac{\lambda^B}{\lambda^A})$ .



**Figure 4**  
**Time-varying disaster risk**

Panel A plots the equity premium under agent A's beliefs as a function of agent B's wealth share ( $w_t^B$ ) and the disaster intensity under A's beliefs ( $\lambda_t$ ). Panel B plots the jump-risk premium  $\lambda_t^Q / \lambda_t$  for agent A.

our framework, the conditional risk premium could be either very sensitive or insensitive to time variation in disaster risk depending on the wealth distribution among heterogeneous agents. Moreover, when estimating disaster probabilities from asset prices, failing to take into account the effects of risk sharing can lead to significant downward biases in our estimates.

Our calibration of the intensity process  $\lambda_t$  in Equation (2) is as follows. First, the long-run mean intensity of disasters under the two agents' beliefs are  $\bar{\lambda}^A = 1.7\%$  and  $\bar{\lambda}^B = 0.1\%$ . Next, we set the speed of mean reversion  $\kappa = 0.142$  (with a half life of 4.9 years), which is consistent with the value in Gabaix (2012), who calibrates this parameter to the speed of mean reversion of historical price-dividend ratio. The volatility parameter is  $\sigma_\lambda = 0.05$ , so that the Feller condition is satisfied.<sup>14</sup> For simplicity, we assume that the size of disasters is constant,  $\bar{d} = -0.51$ , as in Section 2.1. The remaining preference parameters are also the same as in the constant disaster risk case.

Figure 4 plots the conditional equity premium and the jump-risk premium under agent A's beliefs as functions of agent B's wealth share  $w_t^B$  and the disaster intensity  $\lambda_t$ . First, in Panel A, holding  $\lambda_t$  fixed, the equity premium drops quickly as the wealth share of the optimistic agent rises from zero, which is consistent with the results from the case with constant disaster risk. Moreover, this decline is particularly fast when  $\lambda_t$  is large, suggesting that the agents engage in more risk sharing when disaster risk is high. Indeed, the jump-risk premium in Panel B also declines faster when  $\lambda_t$  is large, which is the result of agent A more aggressively insuring herself against consumption loss in a disaster at times of high disaster risk.

<sup>14</sup> The Feller condition,  $2\kappa\bar{\lambda}^A > \sigma_\lambda^2$ , ensures that  $\lambda_t$  will remain strictly positive under agent A's beliefs.

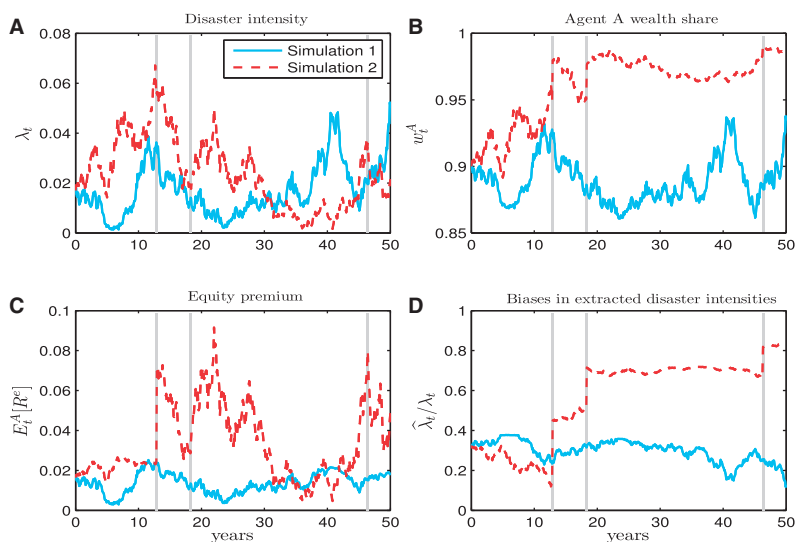
Next, we see that the sensitivity of the equity premium to disaster intensity can be very different depending on the wealth distribution. The sensitivity is largest when the pessimist has all the wealth, but it becomes smaller as the wealth of the optimist increases. When the optimist's wealth share becomes sufficiently high, the equity premium becomes essentially flat as  $\lambda_t$  varies. This result has important implications for the time-series properties of the equity premium. It suggests that when  $\lambda_t$  fluctuates over time, the equity premium can either be volatile or smooth, depending on the wealth distribution.

We can understand the above results through the equity premium formula (13). Variations in the wealth distribution drive  $\lambda_t^Q/\lambda_t$  and  $E_t^D[R]$ . Due to increased risk sharing, the jump-risk premium declines with greater fraction of wealth controlled by the optimistic agent. As a result, the premium becomes less sensitive to variations in  $\lambda_t$ . Moreover, we see in Panel B of Figure 4 that the effect of wealth on the jump-risk premium depends on the disaster intensity. When the disaster intensity is high, the risk-sharing motives are very strong, resulting in a faster decline of the jump-risk premium when the optimistic agent controls just a small amount of wealth. Finally, the returns in disasters also vary somewhat with the wealth distribution as the price-consumption ratio changes after a disaster.

As Figure 4 indicates, a given risk-neutral probability of disasters could be associated with a wide range of beliefs depending on the wealth distribution. This result can help reconcile the differences in disaster risk estimated from macro and financial data. For example, Backus, Chernov and Martin (2011) find that option prices imply smaller probabilities of disasters than those estimated from international macroeconomic data. Collin-Dufresne, Goldstein, and Yang (2010) extract risk-neutral probabilities of extreme events from the prices of CDX tranches. They find that the risk-neutral probabilities of large losses are less than 1% per year. According to our model, these empirical findings might not necessarily imply that the true probability of disasters is low. Rather, they can be explained by our result that a small group of agents with optimistic beliefs about disasters can dramatically reduce the impact of disaster risk on asset prices. At the same time, these results also suggest that when extracting investors' perception of the likelihood of disaster from asset prices, we need to take into account the effects of heterogeneous beliefs and risk sharing.

To further investigate the time-series properties of the model, we simulate the disaster intensity  $\lambda_t$  and the jump component of aggregate endowment  $c_t^d$  under agent A's beliefs, which jointly determine the evolution of the stochastic Pareto weight  $\tilde{\xi}_t$ . Then, along the simulated paths, we compute the equilibrium wealth fraction of agent A,  $w_t^A$ , and the conditional equity premium under A's beliefs,  $E_t^A[R^e]$ . In each simulation, we start with  $\lambda_0 = 1.7\%$  and set the initial wealth share of agent A to  $w_0^A = 90\%$ . The results from two of the simulations are reported in Figure 5.

Panel A plots the paths of  $\lambda_t$  from the simulations. The disaster intensities from both simulations are fairly persistent and show a similar amount of

**Figure 5****Simulation with time-varying disaster risk**

The results are from two simulations of the model with time-varying disaster risk under agent A's beliefs. Panel A plots the simulated paths of disaster intensity. Panels B and C plot the corresponding wealth share of agent A and the conditional equity premium she demands. Panel D plots the time series of disaster intensity extracted from asset prices as a fraction of the true intensities. The shaded areas denote the timing of disasters in Simulation 2. There are no realized disasters in Simulation 1.

variation over time. In Simulation 1, there are no disasters. In Simulation 2, disasters occur three times within the first fifty years, around year 13, 18, and 46, indicated by gray bars in the figure.

What determines the evolution of wealth distribution? When there are no disasters, holding  $\lambda_t$  fixed, agent A is losing her wealth share to B as she pays B the premium for disaster insurance. This effect is captured by the negative drift in the Radon-Nikodym derivative  $\eta_t$  (see Equation (4)) and is stronger when  $\lambda_t^A$  is larger. In addition, as  $\lambda_t$  falls (rises), the value of the disaster insurance that agent A owns falls (rises), causing her wealth to fall (rise) relative to agent B, who is short the disaster insurance. As Panel B shows, the second effect appears to be the main force driving the wealth distribution in Simulation 1.

When a disaster strikes, the wealth distribution can change dramatically. In Simulation 2, the wealth share of agent A jumps up each time a disaster strikes. This is because the disaster insurance that A (pessimist) purchases from B (optimist) pays off at such times, causing the wealth of A to increase relative to B. The size of the jump in  $w_t^A$  is bigger in the first two disasters, which is mainly because agent B has relatively more wealth going into the first two disasters so that he is able to provide more disaster insurance. As a result, he also loses more wealth in these two disasters.

Panel C shows the joint effect of the disaster intensity and wealth distribution on the equity premium. In Simulation 1 (no disasters), despite the fact that the

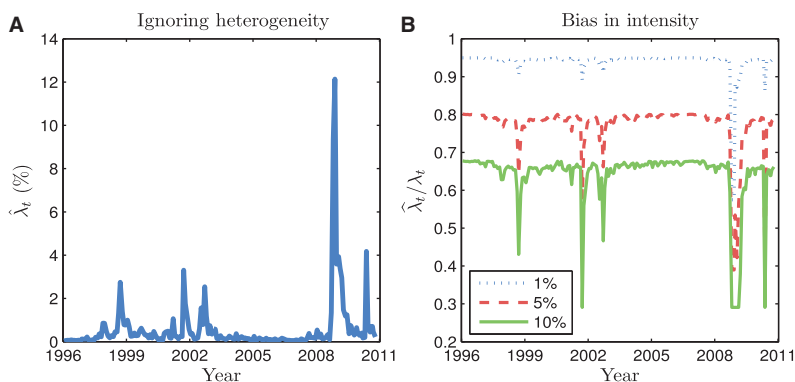
optimistic agent never owns more than 15% of total wealth and that disaster intensity  $\lambda_t$  shows considerable variation over the period, the equity premium is below 2% nearly 90% of the time. This result confirms our finding in Figure 4 that risk sharing between the agents keeps the premium low and smooth when the wealth share of agent B is not too small.

In contrast, the equity premium in Simulation 2 shows large variation, ranging from 0.5% to 9.2%. Following the first disaster in year 13, the premium jumps from 2.4% to 7.0% and becomes significantly more sensitive to fluctuations in  $\lambda_t$  and the wealth distribution afterward. Since the wealth share of agent B drops in a disaster, her risk-sharing capacity is reduced, which drives up both the level and volatility of the equity premium. As shown in Figure 4, this effect is stronger when  $\lambda_t$  is high, which is why the jump in premium is the most visible after the first disaster.

Finally, Panel D of Figure 5 highlights the potentially large biases when extracting investors' beliefs about disaster risk from asset prices. Without considering heterogeneous beliefs, our estimates of disaster probabilities from asset prices can be substantially lower than those of agent A and also substantially lower than the wealth-weighted average belief of the two agents. Consider the procedure in which one takes the disaster size and relative risk aversion to be known ( $\bar{d} = -0.51$  and  $\gamma = 4$  here) and then infers the likelihood of disasters based on asset prices, assuming (incorrectly) that all agents believe the likelihood of disasters is  $\hat{\lambda}_t$ . Under Simulation 1, the extracted disaster intensities are only 20%–40% of the true intensity  $\lambda_t$ . As Simulation 2 shows, even when the wealth distribution becomes highly concentrated, the downward bias in the price-based estimates of disaster risk is still quite sizable. The downward biases are due to the fact that asset prices disproportionately reflect the beliefs of a small group of optimists in the economy. Moreover, there can also be “excessive” variation in these extracted beliefs caused by redistribution of wealth (e.g., following a disaster) rather than actual changes in disaster risk.

In practice, one asset that has often been used to extract information about tail risk is deep out-of-the-money (OTM) index put options. As an example, we compute the disaster probabilities implied by thirty-day S&P 500 index put options (from OptionMetrics) conditional on the optimist owning 1%, 5%, and 10% of total wealth and compare these extracted disaster probabilities to the case in which we ignore belief heterogeneity. Breeden and Litzenberger (1978) show that the derivative of OTM put prices with respect to strike price gives the (discounted) risk-neutral probability of a loss in equity price exceeding a certain level. This method allows us to construct a time series of OTM digital put options on the S&P 500 index, which is robust to the specification of the distribution of disaster size.<sup>15</sup>

<sup>15</sup> Full details of our methodology are in the online appendix.

**Figure 6****Disaster intensities implied by option prices and biases due to ignoring belief heterogeneity**

Panel A plots the implied disaster intensity of the single-agent stochastic intensity model, where the intensity ( $\hat{\lambda}_t$ ) is extracted to match the risk-neutral cumulative probability of a 30% drop in the S&P 500 index in one month. Panel B plots the ratio of the intensities in the single-agent economy to the corresponding intensities ( $\lambda_t$ ) when 1%, 5%, or 10% of the wealth is controlled by optimists.

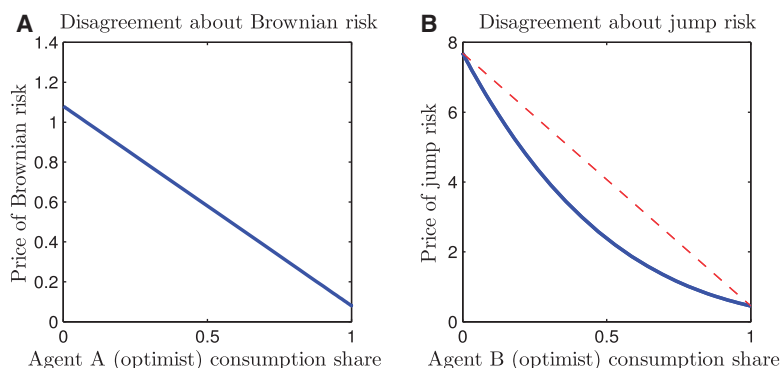
Figure 6, Panel A, shows the inferred probabilities of disasters from the prices of 30% OTM digital index puts when all agents believe in the same stochastic disaster intensity. The implied disaster intensities are low during calm periods (especially during the five years before the financial crisis) but spike up during the Russian Default in 1998, in 2002, and especially in the recent financial crisis, when it reached 12.14% in November 2008.

Panel B shows the biases in the disaster probability estimates of Panel A relative to the cases in which belief heterogeneity is taken into account. Depending on the amount of total wealth owned by the optimist (we consider  $w_t^B = 1\%$ , 5%, or 10%), ignoring belief heterogeneity can lead one to understate the true intensity by 5% to 70%, with the bias becoming more significant when the optimists own more wealth and particularly when the true disaster intensity is high (e.g., during the financial crisis in 2008). These results confirm the results of Panel D of Figure 5 that indeed large biases can exist when one infers disaster probabilities from asset prices while ignoring the presence of belief heterogeneity. An important difference between Panel D of Figure 5 and Figure 6 is that in the former case the wealth fraction of optimist evolves endogenously, whereas in the second case, we fix the wealth of the optimists.

To avoid such biases in our estimates of disaster probabilities, we need to explicitly account for the impact of investor heterogeneity and risk sharing on asset prices. One can potentially measure the amount of heterogeneity using information on the amount of trading on disaster risk in various disaster insurance markets (cf. Figure 2, Panel B).

### 3. Comparison with Other Forms of Heterogeneity

Many studies on heterogeneous beliefs focus on the disagreement about Brownian risks as opposed to jump risks. In this section, we compare these



**Figure 7**

**Disagreement about Brownian risk versus jump risk**

Panel A plots the price of Brownian risk (market Sharpe ratio) under the beliefs of agent A (with perceived consumption growth 2.5%) as a function of her consumption share when agent B believes consumption growth is only 0.5%. Panel B plots the price of jump risk ( $\lambda_t^Q / \lambda^A$ ) for agent A ( $\lambda^A = 1.7\%$ ) as a function of the consumption share of B ( $\lambda^B = 0.1\%$ ).

two forms of disagreements to highlight their different impacts on asset prices, in particular, the prices of Brownian and jump risk. In addition, we also compare our results to a model of heterogeneous risk aversion.

### 3.1 Disagreement about mean growth rate versus jump risk

As a special case of the model presented in Section 1, we can remove the jump component in endowment,  $c_t^d$ , and assume that agents A and B only disagree about the growth rate of endowment. We assume that agent A thinks the growth rate of endowment is  $\bar{g}_A = 2.5\%$ , whereas agent B thinks the growth rate is  $\bar{g} + b\sigma_c^2 = 0.5\%$ . From the stochastic discount factor  $M_t^A$ , one can show (see the online appendix for details) that the price of Brownian risk (which is also the Sharpe ratio of the market portfolio) under A's beliefs is a linear function of her consumption share:

$$SR_t^A = \gamma\sigma_c - (1 - f_t^A)b\sigma_c. \quad (20)$$

Thus, if A has all the wealth in the economy, the price of Brownian risk will be  $\gamma\sigma_c$ , which is small for moderate risk aversion  $\gamma$  and low consumption volatility  $\sigma_c$ . As we allocate more wealth and hence higher consumption share to a pessimistic agent B, the price of equity will fall and the expected return under agent A's beliefs will rise, which leads to a higher Sharpe ratio under the correct beliefs.

In the case of disagreement about jump risk, the price of jump risk under agent A's beliefs can also be expressed explicitly as a function of her consumption share,

$$\frac{\lambda_t^Q}{\lambda^A} = \frac{1}{\lambda^A} \left( f_t^A (\lambda^A)^{\frac{1}{\gamma}} + (1 - f_t^A) (\lambda^B)^{\frac{1}{\gamma}} \right)^\gamma e^{-\gamma \bar{d}}, \quad (21)$$



which converges to  $e^{-\gamma \bar{d}}$  when A's consumption share goes to one. However, unlike the price of the Brownian risk, the price of jump risk changes nonlinearly with the consumption share. This difference is clearly illustrated in Figure 7, where the price of jump risk initially declines quickly when agent B consumes a small share of aggregate endowment, but the decline slows later on.

Another difference between disagreement about growth rates and disagreement about disaster risks is the nonlinearity with respect to the amount of disagreement. In the case of growth rate disagreement, the average belief (weighted by consumption share) determines the price of Brownian risk. This is shown in Equation (20), where the average optimism (assuming agent A is exactly correct so their optimism is zero) is  $(1 - f_A^A)b$ , which is exactly reflected in the Sharpe ratio. In contrast, Equation (21) shows that the jump-risk premium is not a function of the consumption weighted average of the beliefs about the disaster intensity. Instead, in determining the jump-risk premium, more weights are given to the beliefs of the optimist due to risk aversion. One implication of the above difference is that fixing the average belief and increasing the amount of disagreement will have little effect on the risk premium in the case of growth rate disagreement but will tend to lower the equity premium in the case of disagreement about disaster risks.

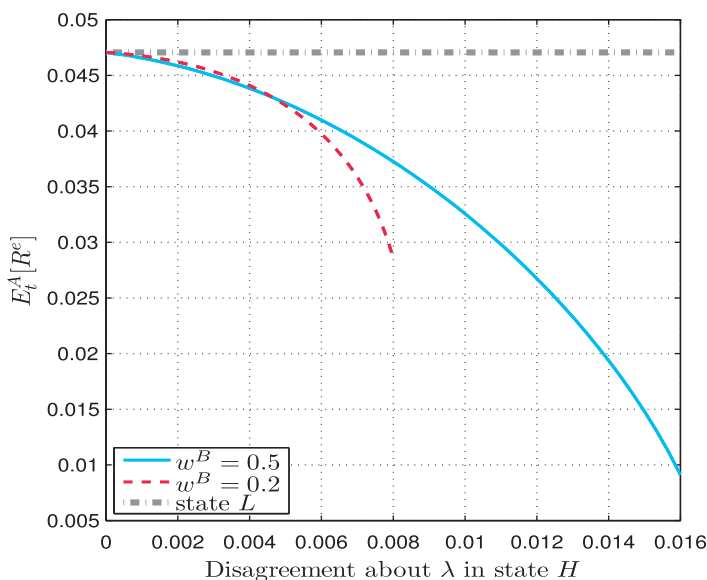
The fact that more disagreement (fixing the consumption-weighted average belief) tends to lower the average belief also holds in a dynamic setting. To this end, consider the following simple extension of our basic model. Suppose that there are two states,  $L$  and  $H$ , and each agent has fixed beliefs about the probability of disasters in a given state. Under the simplifying assumption that transition probabilities between the two states are constant, we show in 1 that our main solution method can be extended to such a model. This regime-switching model then allows us to study the case in which the amount of disagreement is time-varying.

As an example, consider the case in which in state  $L$ , the two agents agree about the frequency of disasters,  $\lambda_L^A = \lambda_L^B = 1.7\%$ . There is disagreement in state  $H$ . In order to isolate the effect of disagreement, we consider different combinations of beliefs in state  $H$  ( $\lambda_H^A > \lambda_H^B$ ) such that the wealth-weighted average belief for a given wealth distribution is the same as in state  $L$ , i.e.,  $(1 - w^B)\lambda_H^A + w^B\lambda_H^B = 1.7\%$ , where  $w^B$  is the wealth share of agent B. We measure the amount of disagreement using the wealth-weighted standard deviation in beliefs,

$$\text{Disagreement measure} = \sqrt{(1 - w^B)(\lambda_H^A - 1.7\%)^2 + w^B(\lambda_H^B - 1.7\%)^2}.$$

Finally, we set the transition probabilities of the Markov chain to be  $\delta_L = 0.1$  and  $\delta_H = 0.5$ .

As Figure 8 shows, holding the average belief constant, the premium can fall substantially as the amount of disagreement increases. As a benchmark, the dashed dotted line gives the equity premium (under agent A's beliefs) in state  $L$ . Since the agents have the same beliefs in that state, the premium remains



**Figure 8**

**Time-varying disagreement**

Panel A plots the equity premium in the case in which beliefs converge in the state with higher disaster risk. Panel B plots the premium as a function of the amount of disagreement for given wealth distribution.

at 4.7% as the amount of disagreement increases in state  $H$ . The solid line plots the equity premium in state  $H$  when the two agents have equal share of total wealth. The premium falls from 4.7% to 0.9% when  $\lambda_H^B$  drops from 1.7% to 0.1% (where the disagreement measure is 1.6%). When agent B has just 20% of total wealth, the premium falls by a smaller amount to 2.9% (when the disagreement measure reaches 0.8%). An interesting implication of this graph is that the premium can be actually *decreasing* while the average belief of disaster risk *increases*, provided that there is enough increase in the amount of disagreement at the same time.

### 3.2 Heterogeneous risk aversion

Intuitively, besides heterogeneous beliefs, heterogeneity in risk aversion should also be able to induce risk sharing among agents and reduce the equity premium in equilibrium. Recall that the jump-risk premium is  $\lambda_t^Q / \lambda_t^i = e^{-\gamma_i \Delta c_t^i}$ , which is not only sensitive to changes in individual consumption loss  $\Delta c_t^i$  but also to the relative risk aversion  $\gamma_i$ . Thus, we expect that heterogeneous risk aversion can have similar effects on the equity premium as do heterogeneous beliefs about disasters.

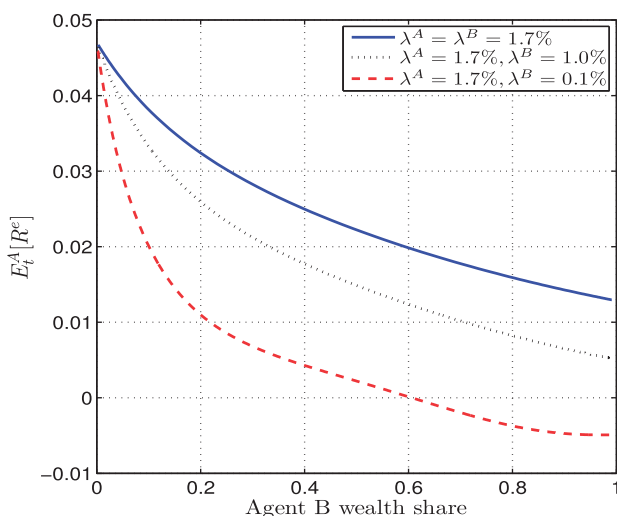
To check this intuition, we consider the following special case of the model. Agent A is the same as in the example of Section 2.1:  $\lambda^A = 1.7\%$ ,  $\gamma_A = 4$ . Agent B

has identical beliefs about disasters but is less risk averse:  $\lambda^B = 1.7\%$ ,  $\gamma_B < \gamma_A$ . We then solve the model using the technique in [Chen and Joslin \(2012\)](#). Figure 9 plots the equity premium as a function of agent B's wealth share for  $\gamma_B = 2$ . The equity premium does decline as agent B's wealth share rises. However, the decline is slow and closer to being linear. In order for the equity premium to fall below 2%, the wealth share of the less risk-averse agent needs to rise to 60%. The decline in the equity premium becomes faster as we further reduce the risk aversion of agent B (not reported here), but the nonlinearity is still less pronounced than in the cases with heterogeneous beliefs.

Combining heterogeneous beliefs about disasters and different risk aversion can amplify risk sharing and accelerate the decline in the equity premium. As shown in Figure 9, if agent B believes disasters are less likely than does agent A, and she happens to be less risk averse, the equity premium falls faster. Consider the case in which agent B believes disasters only occur once every hundred years ( $\lambda^B = 1.0\%$ ). With 20% of total wealth, she drives the equity premium down by almost a half to 2.5%. If  $\lambda^B = 0.1\%$ , the decline in the equity premium will be even more dramatic.

#### 4. Robustness

We have made a number of simplifying assumptions in this article, including complete markets and dogmatic beliefs. In this section, we discuss the potential impact of relaxing these assumptions for our model.



**Figure 9**

##### The effects of heterogeneous risk aversion

This graph plots the equity premium when the two agents have different risk aversion,  $\gamma_A = 4$  and  $\gamma_B = 2$ . Their beliefs about disasters are specified in the legend. Disaster size is constant.

#### 4.1 The assumption of complete markets

In our main analysis, we consider completing the markets with a disaster insurance contract that pays off with certainty exactly when a disaster occurs. The assumption of complete markets greatly simplifies our analysis. However, it also raises some important concerns.

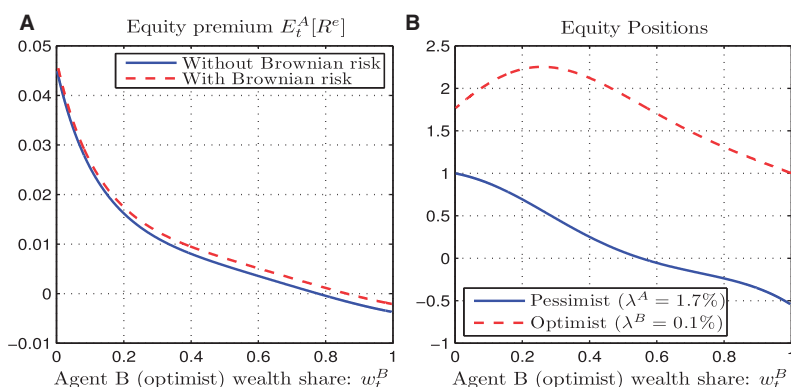
One concern is that a disaster insurance contract might be difficult to implement due to counterparty risk. Within the model, because the marginal utility of the optimist is unbounded as consumption drops to 0, she will never “over-promise” on the amount of disaster insurance she can provide. In fact, we can impose the requirement that disaster insurance be fully collateralized (by the stock), in which case the optimist will have enough wealth to post collateral, and the equilibrium outcome will not change.

Still, there could be other practical reasons for why disaster insurances might be difficult to implement. Our model suggests that any two securities with differential exposure to the Brownian and jump risks would complete the market. For example, high-grade corporate bonds, senior CDX tranches, and put options on the market index can all be used to trade disaster risk. Even if none of these contracts exist, investors will still be able to effectively share disaster risks by trading the stock. This is because in our model, the risk of holding the stock is primarily the exposure to disaster risk (which is bundled with a small amount of Brownian risk that has little effect on the premium). Following this intuition, we consider a variation of the benchmark model by turning off Brownian risk. Then, markets will be dynamically complete via the trading of the aggregate stock and riskless bonds.

Figure 10 plots the equity premium and portfolio positions for both agents. In Panel A, the equity premium in the model with only disaster risk is nearly identical to the benchmark case with Brownian risk. The difference between the two equity premiums is tiny (roughly equal to  $\gamma\sigma_c^2 = 16$  basis points). Panel B shows that the agents now trade disaster risk using the stock market. The pessimist sells part of the stock she owns to the optimist and invests the proceeds in riskless bonds. From the perspective of the optimist, the stock offers a high premium due to disaster risk, which he believes rarely occurs. His capacity to share risk with the pessimist is limited by his wealth, which serves as collateral for taking levered positions in the stock. Because of the budget constraint and the Inada condition, his leverage is in fact fairly modest.

It would be interesting to see whether the intuition we get from the above example holds in an incomplete markets setting with both Brownian and disaster risks but with only one risky asset (the stock). Provided that disaster risk is the main force behind the equity premium relative to the diffusive risk, we conjecture that the optimist would moderately lever up in equity, in a similar way as in Figure 10 (bearing the cost of taking on additional diffusive risk), and the equity premium will be close to the complete markets case.<sup>16</sup>

<sup>16</sup> In the case of log utility, Dieckmann (2011) finds that introducing incomplete markets actually raises the risk premium.

**Figure 10****The relative impact of disaster and Brownian risks**

Panel A plots the equity premium under the pessimist's beliefs as a function of the wealth share of the optimist assuming that the conditional volatility of consumption is  $\sigma_c = 2\%$ . Panel B plots the fraction of wealth of the two agents who invest in the equity claim when there is only disaster risk.

Another important concern is that a big part of total wealth is human capital, which may not be tradable. In that case, the amount of insurance that the optimist can provide will be reduced, and so will the effect of heterogeneous beliefs on the disaster risk premium. For example, in Panel D of Figure 2, the optimist loses up to 70% of his consumption in a disaster when his wealth share is low. Such an allocation might no longer be feasible if a big part of his wealth is nontradable and only tradable wealth can serve as collateral against disaster insurance contracts. In practice, those investors who are selling out-of-the-money index put options or buying senior CDX tranches tend to be institutional investors or high wealth individuals, whose wealth is mostly tradable. Still, it is important to study how much the effects of risk sharing can be weakened by nontradable wealth. We leave this question to future research.

**4.2 Sources of optimistic beliefs**

In the simple version of our model (Section 2.1), the optimist believes that the disaster intensity is only 0.1% per year. How reasonable is this belief? Based on a century of U.S. data, aggregate consumption has never fallen more than 15% in a given year. The maximum cumulative consumption drop over any consecutive number of years is 23%, which occurred during the Great Depression. Thus, it is possible that some agents might (suboptimally) form their beliefs based on only the U.S. experience, even though arguably it is more reasonable to estimate disaster risk using international data.

In Section 2, we calibrate the beliefs of the optimist to the U.S. aggregate consumption data in the last 120 years and of the pessimist based on international macroeconomic data in Barro (2006). The U.S. data suggest that smaller jumps in aggregate consumption are relatively more likely, but these jumps have rather limited effect on the equity premium. Under this calibration, we find very similar effects of risk sharing on the equity premium as in the

benchmark case. For example, raising the fraction of total wealth for the second agent from 0% to 10% lowers the equity premium from 4.4% to 2.0%.

Another source of optimistic beliefs is individual experience. [Malmendier and Nagel \(2011\)](#) argue that individual experiences of macroeconomic outcomes can have long-term effects on their preferences and beliefs. For example, an investor born in the United States who did not experience the Great Depression could assign close to zero probability to a 40% drop of aggregate consumption.

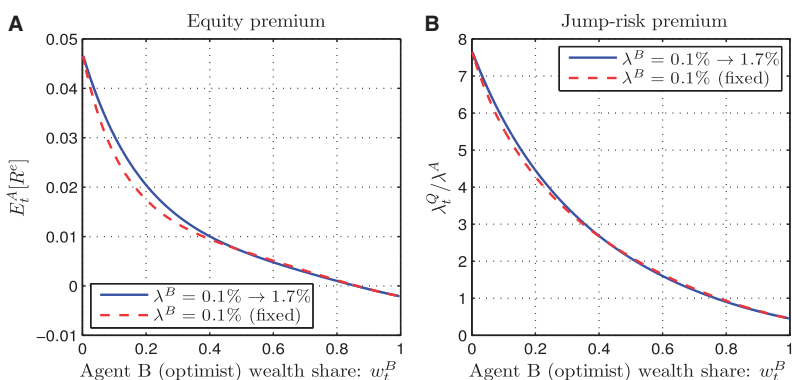
Finally, agency problems could be also an important source of optimistic beliefs in our model. Reputation concerns (see [Malliaris and Yan 2011](#)), convex compensation contracts (see [Makarov and Plantin 2011](#)), and government guarantees can all motivate fund managers and large financial institutions to underwrite insurance against economic disasters. For example, writing deep out-of-the-money index options has long been a popular strategy among hedge funds to manufacture seemingly superior returns in short samples. The recent financial crisis also provides examples of “too-big-to-fail” financial institutions aggressively underwriting so-called “super senior” credit default swaps, which are essentially disaster insurances. Thus, our model provides a link between shocks to the capital supply of these “institutional optimists” and the disaster risk premium.

### 4.3 Effects of learning

In this article, we assume investors have dogmatic beliefs about disaster risk. In reality, investors will update their beliefs about disasters over time, and the beliefs of those who are overly optimistic or pessimistic about disasters might eventually converge to the correct one in the long run. However, due to the nature of disaster risk, learning about either the intensity or size of disasters using realized macro data will be very slow. As we show in this section, the key driver of the conditional equity premium prior to a disaster is risk sharing for the first disaster to come. Even if we assume the belief of the optimist converges fully to that of the pessimist following the first disaster, the risk premium prior to the first disaster will change very little. Thus, learning based on macro data is unlikely to change our results significantly.

To capture the main effects of learning, we consider the following extension of our model. Suppose that agent A correctly believes that the likelihood of a disaster is  $\lambda^A = 1.7\%$  and never changes her belief, whereas agent B is more optimistic. Rather than fully specifying agent B’s prior belief distribution and modeling the Bayesian updating process, we assume that her belief remains constant at  $\lambda^B = 0.1\%$  until the first disaster arrives, at which point she will fully update her belief to the correct one.<sup>17</sup> Thus, the belief of agent B about

<sup>17</sup> Such belief dynamics ignore the fact that the optimist’s belief will be reinforced by each year passed without a disaster, which could further reduce the equity premium.



**Figure 11**  
**Learning through disasters**

Panel A plots the equity premium under the pessimist's beliefs as a function of the wealth share of the optimist assuming that either the optimist holds his beliefs fixed or the optimist updates his beliefs to agree with the pessimist after a disaster occurs. Panel B plots the jump-risk premium  $\lambda_t^Q / \lambda^A$  for the pessimist.

the disaster intensity follows

$$\lambda_t^B = \lambda^B 1_{\{N_t=0\}} + \lambda^A 1_{\{N_t \geq 1\}}.$$

We assume that both agents fully anticipate this updating of beliefs for agent B.

Figure 11 plots the conditional equity premium and jump-risk premium before the first disaster arrives. Both the equity premium and jump-risk premium are slightly higher in the case where beliefs converge after the first disaster, which is consistent with the intuition that learning can reduce risk sharing in the long run. However, the quantitative effect of learning on pricing is very small. As these results show, the majority of the effect of heterogeneous beliefs on asset pricing is due to risk sharing for the first disaster. Thus, any updating of beliefs following the first disaster will only have second-order effects on asset prices.

## 5. Concluding Remarks

We demonstrate the equilibrium effects of heterogeneous beliefs about disasters on risk premia and trading activities. When agents disagree about disaster risk, they will insure each other against the types of disasters they fear most. Because of the highly nonlinear effect of disaster size on risk premia, the risk sharing provided by a small amount of agents with heterogeneous beliefs can significantly attenuate the effect of disasters on the equity premium. The model has important implications for how disaster risks affect the dynamics of asset prices, the potential bias of estimating disaster probabilities from prices, and the link between the size of disaster insurance market and equity premium.

Our results also suggest a few directions for future research on disaster risk. The effectiveness of the risk-sharing mechanism has significant impact

on how disaster risk affects asset prices in the equilibrium. It would be useful to study what happens to asset prices when we limit the risk sharing among investors with heterogeneous beliefs about disasters, perhaps by imposing transaction costs, borrowing constraints, and short-sale constraints as in [Heaton and Lucas \(1996\)](#). Another interesting consideration is ambiguity aversion. As [Hansen \(2007\)](#) and [Hansen and Sargent \(2010\)](#) show, if investors are ambiguity averse, they deal with model/parameter uncertainty by slanting their beliefs pessimistically. In the case with disaster risk, ambiguity-averse investors will behave as if they believe the disaster probabilities are high, even though their actual priors might suggest otherwise. This mechanism could also limit the effects of risk sharing. We leave these questions to future research.

## Appendix

### 1. Time-varying Disagreement

Section 3.1 presents a model with time-varying disagreement through Markov switching. The model solution is generally analogous to the case without Markov regime-switching, so we sketch out the major differences between the models.

The Radon-Nikodym derivative  $\eta_t$  now reflects the change of state  $s_t$ ,

$$\eta_t = e^{\sum_{i \in \{L, H\}} (\Delta a_i N_t^i - \lambda_i^A T_t^i (e^{a_i} - 1))}, \quad (\text{A1})$$

where

$$\Delta a_i = \log \left( \frac{\lambda_i^B}{\lambda_i^A} \right), \quad (\text{A2})$$

$$T_t^i = \int_0^t 1_{\{s_\tau = i\}} d\tau, \quad (\text{A3})$$

and  $N_t^i$  is the number of disasters that have occurred up to time  $t$ , whereas the state is  $s_t = i$ .

The key expectations to compute are of the form

$$E_0^A [e^{aL N_t^L + bH N_t^H + cT_t^L + dT_t^H}], \quad (\text{A4})$$

where  $N_t^i$  is the number of disasters that occur in state  $i$  and  $T_t^i$  is the occupation time in state  $i$  defined in Equation (A3). These expectations can be computed by first conditioning on the path of the Markov state and using the conditional independence of the Poisson process in each state:

$$E_0^A [e^{aL N_t^L + bH N_t^H + cT_t^L + dT_t^H}] = E_0^A \left[ E_0^A [e^{aL N_t^L + bH N_t^H + cT_t^L + dT_t^H} | \{S_\tau\}_{\tau=0}^t] \right] \quad (\text{A5})$$

$$= E_0^A \left[ e^{(\lambda_L^A (e^a - 1) + c)T^L + (\lambda_H^A (e^b - 1) + d)T^H} \right]. \quad (\text{A6})$$

This reduces the problem to computing the joint moment-generating function of the occupation times  $(T_t^L, T_t^H)$ . [Darroch and Morris \(1968\)](#) show that this reduces to

$$E_0^A [e^{aT_t^L + \beta T_t^H}] = \pi_0' \exp(At) \bar{1}, \quad \text{where } A = \Lambda + \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad (\text{A7})$$

and  $\pi_0$  is either  $(1, 0)'$  or  $(0, 1)'$ , as the initial state is  $L$  or  $H$ .

The price of consumption claims involve sums of integrals of such expectations. These integrals can be computed in closed form by diagonalizing  $A$ .



## 2. Calibrating Disagreement: Is the United States Special?

In this section, we calibrate the beliefs of the two types of agents to the data. We assume that agent A believes the United States is no different from the rest of the world in its disaster risk exposure. Hence, her beliefs are calibrated using cross-country consumption data. Agent B, on the other hand, believes that the United States is special. She forms her beliefs on disaster risk using only the U.S. consumption data.

Using maximum likelihood (MLE), we estimate a truncated Gamma distribution for the log disaster size from the Barro (2006) data of major consumption declines across thirty-five countries in the twentieth century. Our estimation is based on the assumption that all the disasters in the sample were independent, and that the consumption declines occurred instantly.<sup>18</sup> We also bound the jump size between  $-5\%$  and  $-75\%$ . The disaster intensity under A's beliefs is still  $\lambda^A = 1.7\%$ . The remaining parameters are the mean growth rate and volatility of consumption without a disaster,  $\bar{g}_A = 2.5\%$  and  $\sigma_c = 2\%$ , which are consistent with the U.S. consumption data post-WWII.

As for agent B, we assume that she agrees with the values of  $\bar{g}_A$  and  $\sigma_c$ , but we estimate the truncated Gamma distribution of disaster size using annual per-capita consumption data in the United States for 1890–2008.<sup>19</sup> Over the sample of 119 years, there are three years where consumption falls by over 5%. Thus, we set  $\lambda^B = 3/119 = 2.5\%$ . Alternatively, we can also jointly estimate  $\lambda^B$  and the jump size distribution.

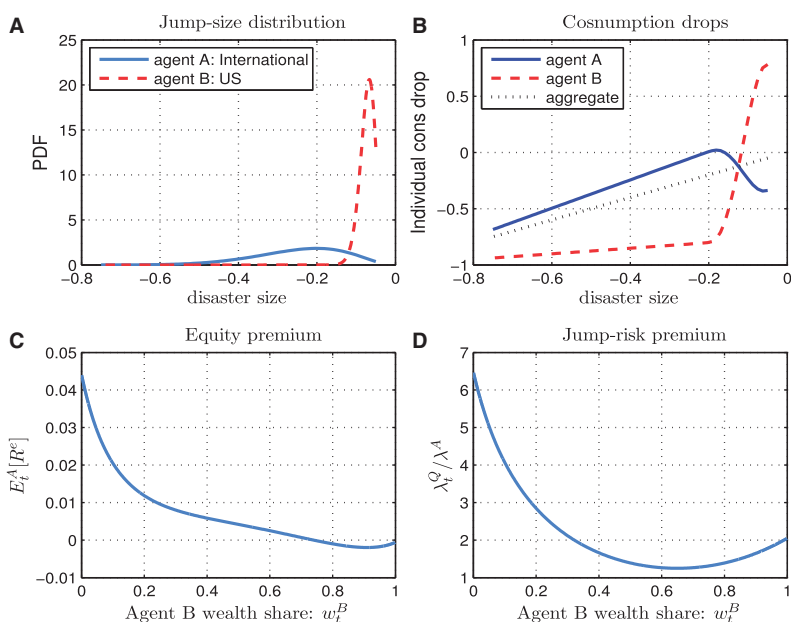
Panel A of A1 plots the probability density functions of the log jump size distributions for the two agents, which are very different from each other. The solid line is the distribution fitted to the international data on disasters. The average log drop is 0.36, which is equivalent to a 30% drop in the level of consumption. In the U.S. data, the average drop in log consumption is only 0.075, or 7.3% in level. In addition, agent A's distribution has a much fatter left tail than B. Thus, whereas A assigns significantly higher probabilities than B to large disasters, agent B assigns more probabilities to small disasters, especially those ranging from 5 to 12%. Agent B's beliefs are close to the calibration by Longstaff and Piazzesi (2004), who assume the jump in aggregate consumption during a disaster is 10%.

The differences in beliefs lead the two agents to insure each other against the types of disasters they fear more, and the trading can be implemented using a continuum of disaster insurance contracts with coverage specific to the various disaster sizes. Panel B plots drops in the equilibrium consumption (level) for the two agents when disasters of different sizes occur, assuming that agent B owns 10% of total wealth. The graph shows that through disaster insurances, agent A is able to reduce her consumption loss in large disasters (comparing the solid line to the dotted line). For example, her own consumption will only fall by 24% in a disaster where aggregate consumption falls by 40%, a sizable reduction especially considering the small amount of wealth that agent B has. At the same time, she also provides insurances to B on smaller disasters, which increases her consumption losses when such disasters strike. Agent B's consumption changes are close to a mirror image of agent A's. However, the changes are magnified both for large and small disasters due to her small wealth share.

Panel C shows the by-now-familiar exponential drop in the equity premium as the wealth share of agent B increases. The equity premium is 4.4% when all the wealth is owned by the agents who form their beliefs about disasters based on international data but drops to 2.0% when just 10% of total wealth is allocated to the agents who form their beliefs using only the U.S. data. The main reason for the lower equity premium is again due to the decrease of the jump-risk premium (Panel D), which falls from 6.5 to 4.0 when agent B's wealth share rises to 10%. This effect alone drives the equity premium down to 2.4%. Notice that the jump-risk premium is no longer monotonic in

<sup>18</sup> See Barro and Ursúa (2008), Donaldson and Mehra (2008), and Constantinides (2008) for more discussion on the measurement of historical disasters.

<sup>19</sup> The data is taken from Robert Shiller's website <http://www.econ.yale.edu/~shiller/data.htm>.



**Figure A1**

**Calibrated disagreements: International versus U.S. experiences**

Panel A plots the truncated Gamma distribution of disaster size for the two agents. Panel B plots the equilibrium consumption drops for the two agents for given disaster size when agent B has 10% of total wealth. Panels C and D plot the equity premium and jump-risk premium under A's beliefs.

the wealth share of agent B. This is because when agent A has little wealth, she would be betting against small disasters so aggressively that the big losses for her during small disasters can cause the jump-risk premium to rise again.

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