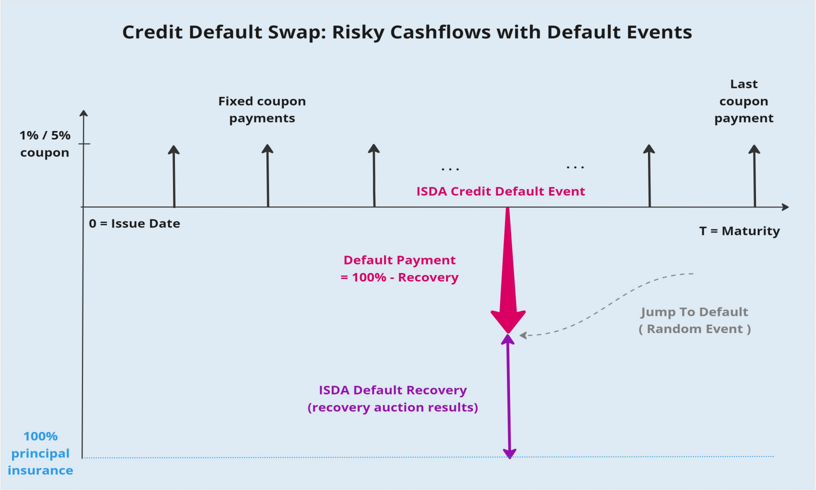
Credit Markets

Session 3

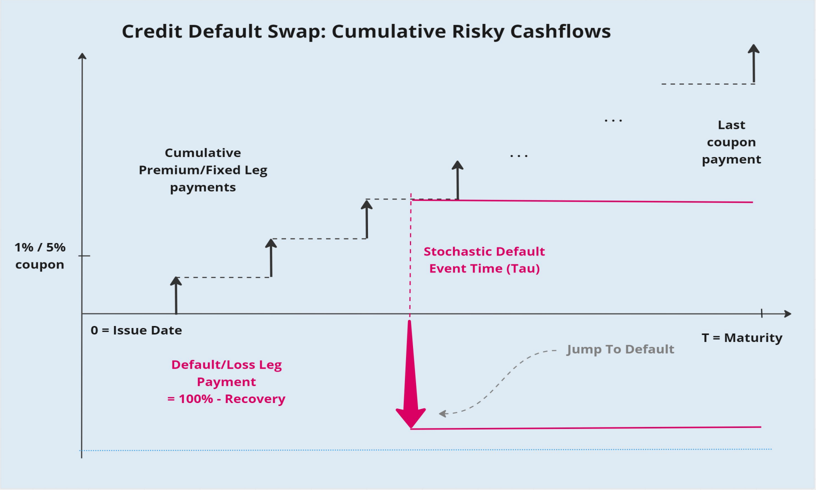
The Hazard Rate Model

1. CDS pricing in the Hazard Rate Model
   - CDS cash-flows
   - CDS pricing
   - Hazard Rates
   - ISDA CDS Hazard Rate model
   - Simple CDS pricing formulas
2. Bond pricing in the hazard rate model
   - Generic bond pricing
   - Simple bond pricing formulas
   - Yield/spread model vs hazard rate model
   - Model Calibration and Fair Values
3. Risks & sensitivities
   - Risk and sensitivities for credit default instruments
   - CDS risks & sensitivities (flat hazard rate)
   - Fixed rate bond risks (flat hazard rate)
4. Q&A

Reminder: CDS cash-flows diagram



CDS cumulative cash-flows (Premium and Default Legs)



Recap: risk neutral valuation framework

- Each instrument uniquely defined by its cumulative future cash-flows
$$CF(t), \quad t \geq 0 \tag{1}$$
- Risk neutral valuation: use a "market implied" probability measure $\mathbb{P}$ for determining prices of securities
- Bank savings account
$$B(t) = e^{\int_0^t r_s ds}, t \geq 0$$
used as numeraire for discounting future cash-flows
- Present value obtained as
$$PV(t) = B(t) \cdot \mathbb{E}\left[\int_t^{\infty} B(s)^{-1} \cdot dCF(s) | \mathcal{F}_t\right] \tag{2}$$

Simple case: deterministic interest rates

- We consider the simple case of deterministic interest rates
- The "time value of money" at time t for time s is captured in the calibrated discount factor curve:

$$DF(t,s) = B(t) \cdot \mathbb{E}\left[B(s)^{-1} | \mathcal{F}_t\right] = e^{-\int_t^s r_u du}, \; 0 \leq t \leq s \tag{3}$$

- The risk free valuation formula simplifies to

$$PV(t) = \mathbb{E}\left[\int_t^{\infty} DF(t,s) \cdot dCF(s) | \mathcal{F}_t\right] \tag{4}$$

- Formula [4] also holds when security cash-flows $CF(t)$ are independent from interest rates.

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CDS Premium/Fixed Leg cash-flows

- Quarterly coupon payments until maturity $T = T_n$:

$$\{c_i, T_i\}_{i=1..n}, \quad 0 \leq t < T_1 < ... < T_n \tag{5}$$

- We denote the stochastic/unknown issuer default time by $\tau \in [0, \infty)$.
- Cumulative Premium Leg cash-flows have stochastic dependence on τ:

$$PL(s) := \sum_{i=1}^{n} c_i \cdot \mathbb{I}_{\{s \geq T_i\}} \cdot \mathbb{I}_{\{\tau > T_i\}} \tag{6}$$

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Valuation framework for credit-risky cash-flows

- We assume $DF(t,s)$ is the deterministic risk-free discount factor (at time t for time s).
- The $DF(t,.)$ curve can be calibrated from market prices via bootstrapping.
- Reminder: $\tau \in [0,\infty)$ is the stochastic/unknown issuer default time
- Market implied issuer survival & default probabilities:

$$SP\left(t,s\right) := \mathbb{P}\left(\tau > s \middle| \tau > t\right), \quad 0 \le t \le s, \tag{7}$$

$$DP\left(t,s\right) := \mathbb{P}\left(\tau \le s \middle| \tau > t\right) = 1 - SP\left(t,s\right). \tag{8}$$

Valuation of CDS Premium/Fixed Leg

$$PV_{CDS\_PL}(t) = \mathbb{E}\left[\int_t^T DF(t,s) \cdot dPL(s) | \tau > t\right] \tag{9}$$

$$= \mathbb{E}\left[\sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot \mathbb{I}_{\{\tau > T_i\}} | \tau > t\right] \tag{10}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot \mathbb{E}\left[\mathbb{I}_{\{\tau > T_i\}} | \tau > t\right] \tag{11}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot \mathbb{P}\left(\tau > T_i | \tau > t\right) \tag{12}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot SP(t, T_i). \tag{13}$$

Intuitive interpretation: premium leg

- Present value of risky "scheduled" cash-flows

$$PV_{CDS\_PL}(t) = \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot SP(t, T_i) \tag{14}$$

- is obtained by summing up the cash-flows amounts c_i
- ... multiplied with the discount factors $DF(t, T_i)$ at cash-flow T_i (time value of money)
- ... and multiplied with probability $SP(t, T_i)$ of issuer being "alive" at cash-flow time T_i (credit risk "adjuster")

CDS Default/Loss Leg cash-flows

- Recovery Rate given default denoted by R
- For simplicity ("basic model") we assume R is constant
- Think of R as the expected average recovery rate for a given issuer and seniority rank
- Default Leg cash-flows until time s given by:

$$DL(s) := (1 - R) \cdot \mathbb{I}_{\{\tau \leq s\}} \tag{15}$$

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Valuation of CDS Default/Loss Leg (using Fubini)

$$PV_{CDS\_DL}(t) = \mathbb{E}\left[\int_t^T DF(t,s)\cdot dDL(s)|\tau > t\right] \tag{16}$$

$$= \mathbb{E}\left[\int_t^T (1-R)\cdot DF(t,s)\cdot d\mathbb{I}_{\{\tau\le s\}}|\tau > t\right] \tag{17}$$

$$= \int_t^T (1-R)\cdot DF(t,s)\cdot d\mathbb{E}\left[\mathbb{I}_{\{\tau\le s\}}|\tau > t\right] \tag{18}$$

$$= (1-R)\cdot\int_t^T DF(t,s)\cdot d\mathbb{P}\left(\tau\le s|\tau > t\right) \tag{19}$$

$$= (1-R)\cdot\int_t^T DF(t,s)\cdot dDP(t,s). \tag{20}$$

Intuitive interpretation: default leg

- Present value of loss payment at (random) issuer default time

$$PV_{CDS\_DL}(t) = \int_t^T (1 - R) \cdot DF(t,s) \cdot dDP(t,s) \qquad (21)$$

- Obtained by splitting the time until maturity $[t, T]$ in infinitesimal (one day) intervals and
- summing up the loss amount on each interval: $1 - R$
- ... multiplied with the discount factor for that interval: $DF(t,s)$
- ... and multiplied with the incremental probability of default on that interval: $dDP(t,s)$.

Quick recap on Forward Interest Rates

- Forward interest rates $f(t,s), s \geq t$ represent information about interest rates at future time s, as seen from time t.
- Information is contained in today's term structure of spot interest rates (discount curves).
- Forward rate curves $f(t,.)$ are defined implicitly from Discount Factor curves:

$$DF(t,s) = \exp\left[-\int_t^s f(t,u)\,du\right], \quad 0 \leq t \leq s, \tag{22}$$

- ... or defined explicitly as

$$f(t,s) = -\frac{\partial}{\partial s}\ln\left[DF(t,s)\right] = -\frac{1}{DF(t,s)} \cdot \frac{\partial DF}{\partial s}(t,s). \tag{23}$$

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Hazard Rates (a.k.a. Default Intensities)

- Hazard rates $h(t, s)$ represent infinitesimal probabilities of default at future time s, using market information as of time t.
- Similar concept to forward interest rates.
- Hazard rate curves $h(t, .)$ defined implicitly from Survival Probability curves:

$$SP(t, s) = \exp\left[-\int_t^s h(t, u)\, du\right], \quad 0 \leq t \leq s, \qquad (24)$$

- ... or defined explicitly as

$$h(t, s) = -\frac{\partial}{\partial s} \ln\left[SP(t, s)\right] = -\frac{1}{SP(t, s)} \cdot \frac{\partial SP}{\partial s}(t, s). \quad (25)$$

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Properties of Hazard Rates

- Hazard rates $h(t,s)$ quantify conditional probabilities of default "around" specific times s in the future

$$\begin{aligned}
&\mathbb{P}\left(s < \tau < s+\epsilon \middle| \tau > s, \tau > t\right) \qquad (26)\\
&= \frac{\mathbb{P}\left(s < \tau < s+\epsilon \middle| \tau > t\right)}{\mathbb{P}\left(\tau > s \middle| \tau > t\right)}\\
&= \frac{SP(t,s) - SP(t,s+\epsilon)}{SP(t,s)}\\
&= 1 - \exp\left[-\int_s^{s+\epsilon} h(t,u)\, du\right]\\
&\simeq h(t,s)\cdot\epsilon, \quad 0 \le t \le s.
\end{aligned}$$

Valuation of CDS Default/Loss Leg using Hazard Rates

$$PV_{CDS\_DL}(t) = (1-R) \cdot \int_t^T DF(t,s) \cdot dDP(t,s) \tag{27}$$

$$= (1-R) \cdot \int_t^T DF(t,s) \cdot d\left[1 - SP(t,s)\right] \tag{28}$$

$$= -(1-R) \cdot \int_t^T DF(t,s) \cdot dSP(t,s) \tag{29}$$

$$= (1-R) \cdot \int_t^T h(t,s) \cdot DF(t,s) \cdot SP(t,s) \cdot ds. \tag{30}$$

Numerical quadrature of CDS Default/Loss Leg

$$PV_{CDS\_DL}(t) = (1-R)\cdot\int_t^T h(t,s)\cdot DF(t,s)\cdot SP(t,s)\cdot ds$$

$$= \sum_{i=1}^{n}(1-R)\cdot\int_{T_{i-1}}^{T_i} h(t,s)\cdot DF(t,s)\cdot SP(t,s)\cdot ds \qquad (31)$$

$$\simeq \sum_{i=1}^{n}(1-R)\cdot h(t,T_i)\cdot\Delta T_i\cdot DF(t,T_i)\cdot SP(t,T_i). \qquad (32)$$

CDS valuation (receiving premium c, paying default/loss leg)

$$PV_{CDS}(t) = PV_{CDS\_PL}(t) - PV_{CDS\_DL}(t) \tag{33}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot SP(t, T_i)$$

$$- (1 - R) \cdot \int_t^T h(t, s) \cdot DF(t, s) \cdot SP(t, s) \cdot ds$$

$$\simeq \sum_{i=1}^{n} [c - (1 - R) \cdot h(t, T_i)] \cdot \Delta T_i \cdot DF(t, T_i) \cdot SP(t, T_i). \tag{34}$$

CDS Par Spread and Unit Premium Leg

- CDS contracts have fixed (contractual) coupons of 1% or 5% of face notional.
- We introduce CDS_PL1 as the "Unit" 1bp Premium Leg PV

$$PV_{CDS\_PL}(t) = c \cdot CDS\_PL1(t), \quad c \in \{1\%, 5\%\}.$$

- Par Spread defined as the coupon that would make the CDS trade "At Par":

$$ParSpread(t) := \frac{PV_{CDS\_DL}(t)}{CDS\_PL1(t)} \tag{35}$$

Relationship between CDS Par Spreads and CDS prices

- CDS Par Spreads = market quotation standard for CDS contracts
- "One-to-one" correspondence to CDS prices via CDS_PL1:

$$\begin{aligned} PV_{CDS}(t) &= PV_{CDS\_PL}(t) - PV_{CDS\_DL}(t) \\ &= [c - ParSpread(t)] \cdot CDS\_PL1(t) \end{aligned} \tag{36}$$

- CDS trades "At Par" when the Par Spread matches the coupon:

$$PV_{CDS}(t) = 0 \Leftrightarrow ParSpread(t) = c \tag{37}$$

CDS Par Spreads, Prices vs Upfronts

- CDS Par Spreads measure the "market implied" issuer default risk for the given maturity.
- ... and can be "bootstrapped" from CDS prices via survival probabilities, using formula (35).
- CDS PV = price from the point of view of a CDS seller:
- CDS Upfront = price from the point of view of a CDS buyer:
- CDS_Upfront = - CDS_PV

ISDA CDS standard pricing & quoting model

- Deterministic discount curve: ISDA CDS SOFR (LIBOR)
- Constant recovery, constant hazard rate.
- Default Leg computed via numerical integration

$$PV_{CDS}(t,c,h,R) = PV_{CDS\_PL}(t,c,h,R) - PV_{CDS\_DL}(t,h,R) \tag{38}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t,T_i) \cdot e^{(t-T_i)\cdot h} - (1-R)\cdot h \cdot \int_t^T DF(t,s)\cdot e^{(t-s)\cdot h}\cdot ds \tag{39}$$

$$\simeq \sum_{i=1}^{n} [c-(1-R)\cdot h]\cdot \Delta T_i \cdot DF(t,T_i)\cdot e^{(t-T_i)\cdot h}. \tag{40}$$

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$$\simeq \sum_{i=1}^{n} \left[c-(1-R)\cdot h\right]\cdot \Delta T_i \cdot DF(t,T_i)\cdot e^{(t-T_i)\cdot h}. \tag{40}$$

Simple CDS valuation with constant parameters

$$PV_{CDS}\left(c, r, h, R, T\right) = \tag{41}$$

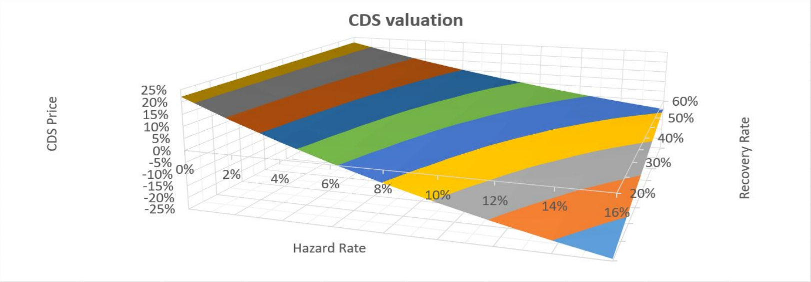
$$= \sum_{k=1}^{4T} \frac{c}{4} \cdot e^{-k \cdot (r+h)/4} - \frac{(1-R) \cdot h}{r+h} \cdot \left[1 - e^{-T \cdot (r+h)}\right] \tag{42}$$

$$= \left[\frac{c/4}{e^{(r+h)/4} - 1} - \frac{(1-R) \cdot h}{r+h}\right] \cdot \left[1 - e^{-T \cdot (r+h)}\right] \tag{43}$$

$$\simeq \left[\frac{c - (1-R) \cdot h}{r+h}\right] \cdot \left[1 - e^{-T \cdot (r+h)}\right]. \tag{44}$$

CDS valuation surface (5% flat interest rates, 5% coupon)

	Hazard Rate	Credit Default Swap with 5% coupon and 5Y maturity							
Recovery Rate	0%	2%	4%	6%	8%	10%	12%	14%	16%
20%	22.1%	14.3%	7.2%	0.8%	-5.1%	-10.6%	-15.5%	-20.0%	-24.1%
25%	22.1%	14.8%	8.1%	1.9%	-3.7%	-8.8%	-13.5%	-17.8%	-21.7%
30%	22.1%	15.2%	8.9%	3.1%	-2.2%	-7.0%	-11.5%	-15.5%	-19.2%
35%	22.1%	15.6%	9.7%	4.2%	-0.7%	-5.3%	-9.4%	-13.2%	-16.7%
40%	22.1%	16.0%	10.5%	5.4%	0.7%	-3.5%	-7.4%	-11.0%	-14.2%
45%	22.1%	16.5%	11.3%	6.5%	2.2%	-1.8%	-5.4%	-8.7%	-11.8%
50%	22.1%	16.9%	12.1%	7.7%	3.7%	0.0%	-3.4%	-6.5%	-9.3%
55%	22.1%	17.3%	12.9%	8.8%	5.1%	1.8%	-1.3%	-4.2%	-6.8%
60%	22.1%	17.7%	13.7%	10.0%	6.6%	3.5%	0.7%	-1.9%	-4.3%



ISDA CDS "curve shape" model

- Uses reference ISDA SNAC discount curve, flat recovery and piece-wise constant hazard rate
- Calibrate piece-wise flat hazard rate curve using sequential "Bootstrapping" over CDS maturities
- Standard CDS maturities:
  - 6M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y
- ISDA SNAC swap/discount curve definition:
  - Curve calibrated to 3M LIBOR until Oct 2022
  - Curve calibrated to 3M SOFR after Oct 2022

Bond pricing in the hazard rate model

- For a given issuer, we first calibrate the implied survival probability curve (hazard rates) to the CDS market
- Risky bond cash-flows: semi-annual coupons + principal payment at maturity $T = T_n$
- The premium/fixed leg valuation formula is similar to (9):

$$PV_{Bond\_PL}(t) = \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot SP(t, T_i) + DF(t, T_n) \cdot SP(t, T_n) \quad (45)$$

- The default recovery leg valuation formula is similar to (16):

$$PV_{Bond\_DL}(t) = R \cdot \int_t^T DF(t, s) \cdot dDP(t, s) \quad (46)$$

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$$PV_{Bond\_DL}(t) = R \cdot \int_t^T DF(t, s) \cdot dDP(t, s) \tag{46}$$

Implied (intrinsic) bond valuation

$$PV_{Bond}\left(t\right) = PV_{Bond\_PL}\left(t\right) + PV_{Bond\_DL}\left(t\right) \tag{47}$$

$$= \sum_{i=1}^{n} c_i \cdot DF(t, T_i) \cdot SP(t, T_i) + DF(t, T_n) \cdot SP(t, T_n) \tag{48}$$

$$+R \int_t^T h\left(t, s\right) \cdot DF(t, s) \cdot SP(t, s) \cdot ds \tag{49}$$

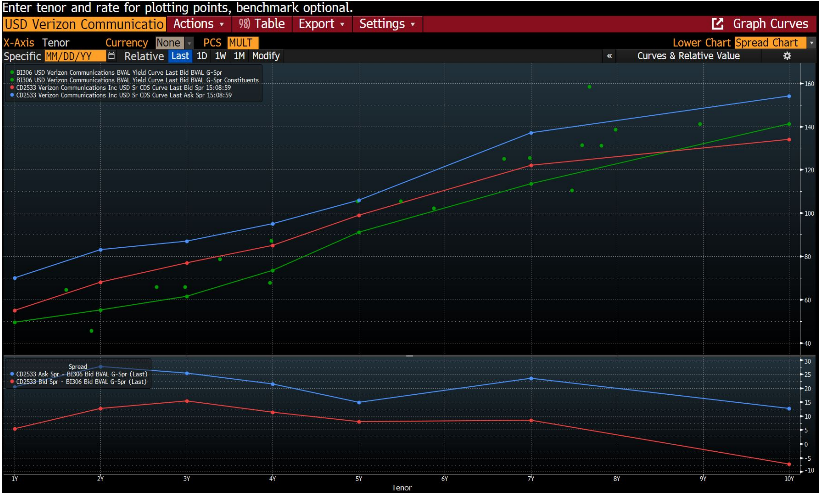
$$\simeq \sum_{i=1}^{n} \left[c + R \cdot h\left(t, T_i\right)\right] \cdot \Delta T_i \cdot DF(t, T_i) \cdot SP(t, T_i) \tag{50}$$

$$+DF(t, T_n) \cdot SP(t, T_n)$$

Bond vs CDS basis arbitrage

- Cash bonds and synthetic CDS for an issuer are linked to same default event!
- Bond CDS implied (intrinsic) prices and market prices "should be close"
- Option Adjusted Spread/OAS = "market - intrinsic" price
- OAS "dislocated" from zero $\Longrightarrow$ opportunity for bond vs CDS basis arbitrage trade!

Bond vs CDS Basis example: VZ Curve



Simple hazard rate bond pricing formula (flat parameters)

$$PV_{Bond}\left(c, r, h, R\right) = \tag{51}$$

$$= \sum_{k=1}^{2T} \frac{c}{2} \cdot e^{-k\cdot(r+h)/2} + e^{-T\cdot(r+h)} + \frac{R\cdot h}{r+h} \cdot \left[1 - e^{-T\cdot(r+h)}\right] \tag{52}$$

$$= 1 + \left[\frac{\frac{c}{2} - \left(e^{\frac{r+h}{2}} - 1\right)}{e^{\frac{r+h}{2}} - 1} + \frac{R\cdot h}{r+h}\right] \cdot \left[1 - e^{-T\cdot(r+h)}\right] \tag{53}$$

$$\simeq 1 + \left[\frac{c - r - (1-R)\cdot h}{r+h}\right] \cdot \left[1 - e^{-T\cdot(r+h)}\right]. \tag{54}$$

Bond valuation surface using hazard rate model

	Hazard Rate	Fixed rate bond with 15Y maturity, 7% coupon and 5% interest rates							
Recovery Rate	0%	2%	4%	6%	8%	10%	12%	14%	16%
20%	121	104	90	79	71	64	59	54	51
25%	121	105	92	82	74	67	62	58	54
30%	121	106	93	84	76	70	65	61	58
35%	121	107	95	86	79	73	69	65	62
40%	121	107	97	88	82	76	72	68	65
45%	121	108	98	90	84	79	75	72	69
50%	121	109	100	93	87	82	78	75	73
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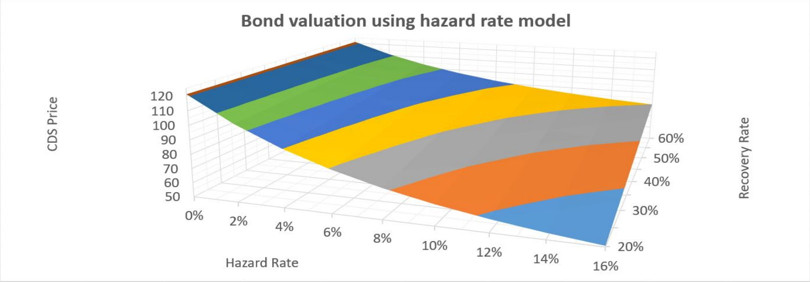

Hazard rate vs yield/spread models (flat parameters)

- Bond valuation in flat "hazard rate" model:

$$PV_{Bond}\left(c, r, h, R\right) \simeq 1 + \left[\frac{c - r - (1 - R) \cdot h}{r + h}\right] \cdot \left[1 - e^{-T \cdot (r+h)}\right]. \tag{55}$$

- Bond valuation in flat "yield/spread" model:

$$PV_{Bond}\left(c, r, s\right) \simeq 1 + \left[\frac{c - r - s}{r + s}\right] \cdot \left[1 - e^{-T \cdot (r+s)}\right]. \tag{56}$$

- Yield/spread model is "simple" case of hazard rate model for

$$R = 0, \quad s = h. \tag{57}$$

Bond pricing in yield model vs hazard rate model

- Bond trading at 100% "par" price in hazard rate model:

$$PV = 1 \iff c = r + (1 - R) \cdot h. \tag{58}$$

- Bond trading at 100% "par" price in yield/spread model:

$$PV = 1 \iff c = y = r + s \tag{59}$$

- Model translations:

$$y = c = r + (1 - R) \cdot h, \tag{60}$$

$$s = (1 - R) \cdot h. \tag{61}$$

Bond pricing in yield model vs hazard rate model

- Bond trading at 100% "par" price in hazard rate model:

$$PV = 1 \iff c = r + (1 - R) \cdot h. \tag{58}$$

- Bond trading at 100% "par" price in yield/spread model:

$$PV = 1 \iff c = y = r + s \tag{59}$$

- Model translations:

$$y = c = r + (1 - R) \cdot h, \tag{60}$$

$$s = (1 - R) \cdot h. \tag{61}$$

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Summary of Hazard Rate Model

- By construction, recovery rates R are increasing with seniority rank in the capital structure
  - Preferred Equity ~0%
  - Subordinated ~20%
  - Senior Unsecured ~40%
  - Senior Secured ~ 65%
  - Senior Loan (First Lien) ~80%
- When using the hazard rate model, bonds of all seniorities are priced simultaneously
  - using same survival probability/hazard rate curve (default time is common to all instruments)
  - ... but different recovery rates.
- Consistent framework for identifying dislocations across maturities and seniority ranks!

Calibration of Hazard Rate Credit Models

- Decide on a functional form for the Hazard Rate curve:
  - piece-wise constant (or linear) curve on pivot maturities grid (e.g. 1Y, 2Y, 3Y, 5Y, 7Y, 10Y, 20 and 30Y)
  - or use smooth parametric curve shapes (e.g. Nelson-Siegel)
- Agree on issuer recovery rates by seniority ranks
  - historical averages: $\sim$40% recovery for CDS and senior unsecured bonds
  - and $\sim$80% recovery for first lean bonds
- Select weights for instrument used as calibration inputs
  - usually weights are proportional to liquidity
  - and inversely proportional to duration/DV01
- Run calibration process: sequential bootstrapping vs numerical minimization of residual errors

Fair Value Prices and "Edges"

- Using the calibrated Hazard Rates model for an issuer, one can compute model prices for arbitrary credit instruments
  - For instruments used as calibration inputs
  - ... as well as instruments outside the calibration set, e.g. less liquid, off-the-run, new issues bonds, etc
- Once can think of the calibrated model price as the "Fair Value" price for an instrument
- The difference between market and model/fair value prices (distance to fair value) is called an "edge"

Risk and sensitivities for credit default instruments

- Interest rates risk
  - dependent on contract maturity (duration)
  - impacts both coupon and face notional cash-flows
- Credit spread risk
  - quantifies risk due to "gradual" changes in issuer credit quality
  - impacts both coupon and face notional cash-flows
- Jump-To-Default / Default Recovery risk
  - quantifies risk due to "sudden" issuer credit default event occurring before contract maturity
  - dependent on recovery value of face notional given default

Interest rate sensitivity (IR01 for -1bp parallel shift in rates)

$$CDS\_IR01 := -\frac{\partial}{\partial r} PV_{CDS}\left(t, c, h, R\right) \tag{62}$$

$$\simeq -\frac{\partial}{\partial r}\left\{\sum_{i=1}^{n}\left[c-(1-R)\cdot h\right]\cdot \Delta T_i \cdot DF(t, T_i)\cdot e^{(t-T_i)\cdot h}\right\}$$

$$= -\sum_{i=1}^{n}\left[c-(1-R)\cdot h\right]\cdot \Delta T_i \cdot (t-T_i)\cdot DF(t, T_i)\cdot e^{(t-T_i)\cdot h}$$

- In particular, if $c = (1-R)\cdot h$, the CDS trades "at par" and

$$CDS\_IR01 = 0$$

Hazard rate sensitivities (HR01 for -1bp shift in hazard rate)

$$CDS\_HR01 := -\frac{\partial}{\partial h} PV_{CDS}\left(t, c, h, R\right) \tag{63}$$

$$\simeq -\frac{\partial}{\partial h}\left\{\sum_{i=1}^{n}\left[c-(1-R)\cdot h\right]\cdot \Delta T_i \cdot DF\left(t, T_i\right)\cdot e^{\left(t-T_i\right)\cdot h}\right\}$$

$$= CDS\_IR01 + (1-R)\cdot CDS\_PL1$$

- In particular, if CDS trades "at par"

$$CDS\_HR01 = (1-R)\cdot CDS\_PL1$$

CDS Par Spread sensitivity (CS01 for -1bp shift in par spread)

- 1bp move in Hazard Rate corresponds to $(1 - R)$ bp move in CDS Par Spread, hence

$$CDS\_CS01 = CDS\_HR01/\left(1 - R\right) \tag{64}$$

$$= CDS\_IR01/\left(1 - R\right) + CDS\_PL1$$

- In particular, if CDS trades "at par", the credit spread sensitivity matches the value of the "Unit" Premium Leg:

$$CDS\_CS01 = CDS\_PL1$$

Recovery rate sensitivity (REC01 for +1% change in R)

$$CDS\_REC01 := \frac{\partial}{\partial R} PV_{CDS}\left(t, h, R\right) \tag{65}$$

$$\simeq -\frac{\partial}{\partial R}\left\{\sum_{i=1}^{n}\left[c - (1 - R) \cdot h\right] \cdot \Delta T_i \cdot DF(t, T_i) \cdot e^{(t - T_i) \cdot h}\right\}$$

$$= \sum_{i=1}^{n} h \cdot \Delta T_i \cdot DF(t, T_i) \cdot e^{(t - T_i) \cdot h}$$

$$= h \cdot CDS\_PL1$$

Risk summary for CDS contracts trading "at par"

$$CDS\_IR01 = 0 \tag{66}$$

$$CDS\_CS01 = CDS\_PL1 \tag{67}$$

$$CDS\_HR01 = (1 - R) \cdot CDS\_PL1 \tag{68}$$

$$CDS\_REC01 = h \cdot CDS\_PL1 \tag{69}$$

$$CDS\_JTD = -(1 - R) \tag{70}$$

- For this reason, CDS_PL1 also called "CDS duration".

Bond interest rate sensitivity (IR01 for -1bp parallel shift in rates)

$$Bond\_IR01 := -\frac{\partial}{\partial r} PV_{Bond}\left(t, c, h, R\right) \qquad (71)$$

$$\simeq -\frac{\partial}{\partial r}\left\{\sum_{i=1}^{n}\left(c + R\cdot h\right)\cdot \Delta T_i \cdot DF\left(t, T_i\right)\cdot e^{(t-T_i)\cdot h} + DF\left(t, T_n\right)\cdot SI\right.$$

$$= \sum_{i=1}^{n}\left(c + R\cdot h\right)\cdot \Delta T_i \cdot \left(T_i - t\right)\cdot DF\left(t, T_i\right)\cdot e^{(t-T_i)\cdot h}$$

$$+ \left(T_n - t\right)\cdot DF\left(t, T_n\right)\cdot e^{(t-T_n)\cdot h}$$

$$= \left(\sum_{i=1}^{n}\left(T_i - t\right)\cdot w_i\right)\cdot PV_{Bond}\left(t, c, h, R\right)$$

$$= Duration_{Bond}\left(t, c, h, R\right)\cdot PV_{Bond}\left(t, c, h, R\right)$$

Bond hazard rate sensitivity (HR01 for -1bp shift in hazard rate)

$$Bond\_HR01 := -\frac{\partial}{\partial h} PV_{Bond}\left(t, c, h, R\right) \tag{72}$$

$$\simeq -\frac{\partial}{\partial h}\left\{\sum_{i=1}^{n} (c + R \cdot h) \cdot \Delta T_i \cdot DF(t, T_i) \cdot e^{(t-T_i)\cdot h} + (t - T_n) \cdot DF(t, T$$

$$= Bond\_IR01 - \sum_{i=1}^{n} R \cdot \Delta T_i \cdot DF(t, T_i) \cdot e^{(t-T_i)\cdot h}$$

$$= Bond\_IR01 - R \cdot CDS\_PL1$$

Bond recovery rate sensitivity (REC01 for $+1\%$ shift in R)

$$Bond\_REC01 := \frac{\partial}{\partial R} PV_{Bond}\left(t, h, R\right) \tag{73}$$

$$= \sum_{i=1}^{n} h \cdot \Delta T_i \cdot DF\left(t, T_i\right) \cdot e^{\left(t - T_i\right) \cdot h}$$

$$= h \cdot CDS\_PL1$$

Summary of factor sensitivities (for fixed rate risky bonds)

$$Bond\_IR01 = Duration_{Bond}\left(t, c, h, R\right) \cdot PV_{Bond}\left(t, c, h, R\right) \quad (74)$$

$$Bond\_HR01 = Bond\_IR01 - R \cdot CDS\_PL1 \quad (75)$$

$$Bond\_REC01 = h \cdot CDS\_PL1 \quad (76)$$

$$Bond\_JTD = R \quad (77)$$

Q&A

- Hazard rate model formulas
- CDS pricing
- Bond pricing
- Calibration
- Yield vs. Hazard Rate pricing models
- Risks and sensitivities