FINM 36702 1 Portfolio Credit Risk: Modeling and Estimation
TA Session 2

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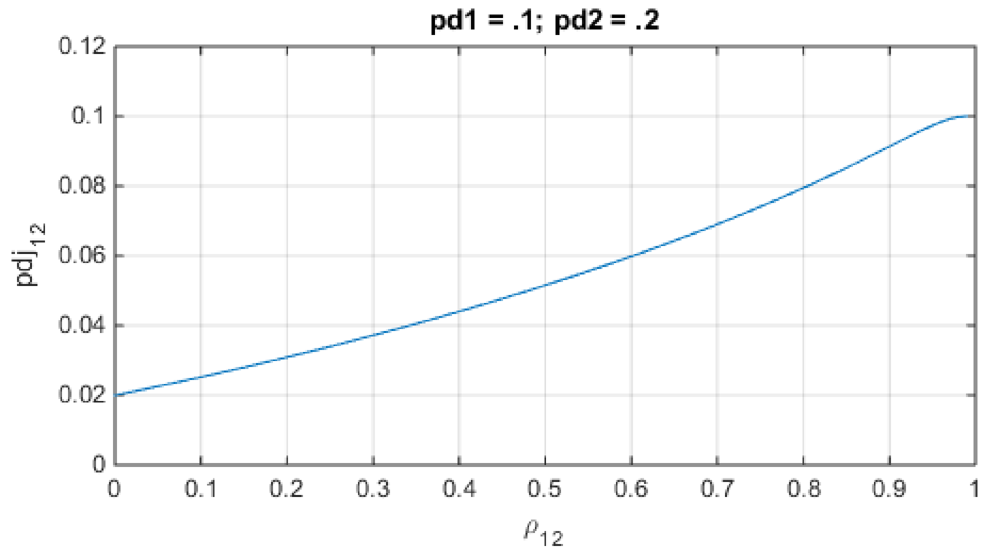
Part I. Homework 1 Review

Q1. Implicitly Inverting a Function

- Given PDs and PDJ, solve ρ_{ij} from $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i \rightarrow \boldsymbol{\rho_{ij}}$
  - FIRST OF ALL: Is this function invertible?
  - Explicitly invertible (rearrange the formula) vs. implicitly invertible ($\underset{\rho_{ij}\in[0,1]}{\arg\min} f := \left\|PDJ_{ij}(\rho_{ij}) - 0.06\right\|$)?



- $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i, \rightarrow \rho_{12}$
- A “special case: when $PD_2 = 0.2, PD_3 = 0.3,$ $and\ PDJ_{23} = PD_2 * PD_3 = 0.06, \rightarrow \rho_{23} = 0$
- $DCorr_{[D_i,D_j]} = \frac{PDJ_{ij} - PD_i \cdot PD_j}{\sqrt{PD_i(1-PD_i)PD_j(1-PD_j)}}$

Q1 Technical notes: Numerically invert the double integral to solve for ρ_{ij} from a given value of PDJ_{ij}

- Step 1, implement the double integral
  - Option 1: Do it the "hard way".
    - Part 1. Code the joint normal formula as a function into a software tool, such as Python, Matlab, R, or Mathematica, etc.

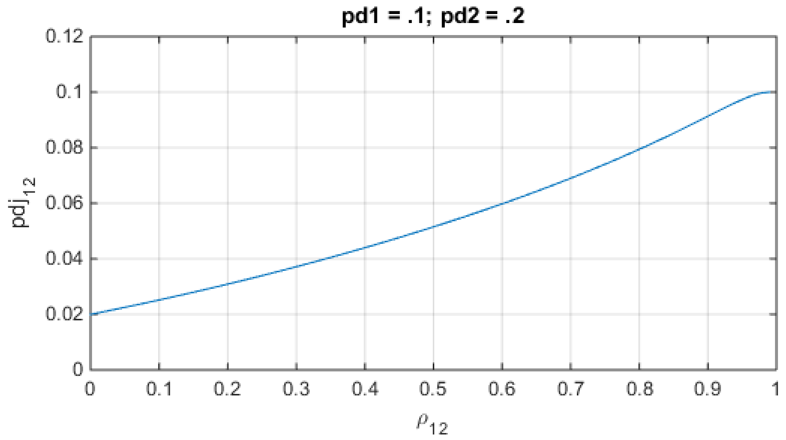
$$f(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} e^{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{z_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{z_1-\mu_1}{\sigma_1}\right)\left(\frac{z_2-\mu_2}{\sigma_2}\right) + \left(\frac{z_2-\mu_2}{\sigma_2}\right)^2\right]}$$

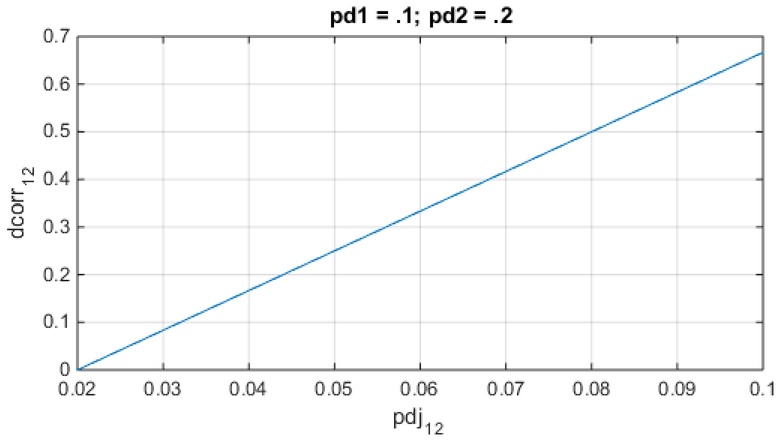
$$= \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} e^{-\frac{1}{2(1-\rho_{12}^2)}[z_1^2 - 2\rho_{12}z_1z_2 + z_2^2]}$$, because $Z_1, Z_2 \sim N[0, 1]$

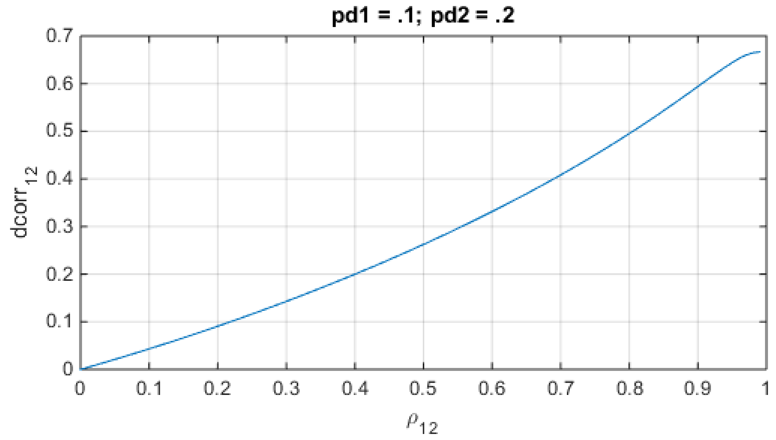
    - Part 2. Integrate f using a built-in numerical integration function in your software tool.
  - Option 2: Or do it the "easier way". Define the double integral using the built-in multivariate normal CDF in your software tool, e.g., MULTIVARIATE_NORMAL.CDF in Python.
- Step 2, numerically invert the double integral to solve for ρ, e.g., what value of ρ_{12} gives you PDJ_{12} = 0.06?
  - Option 1: Engineer your own numerical solver, e.g., loop through different ρ_{12} values in its domain [0, 1] but stop at the value when reaching the "tolerance".
  - Option 2: Utilize the built-in root finding routines in your software tool, e.g., FSOLVE in Python. Note when a software tool offers one-sided numerical algorithm, e.g., only arg max but no arg min, you can use the fact that arg min = - arg max.

Q1. Equivalent Measures of Correlation

- Knowing one quantity of ρ, *PDJ* and *DCorr* means knowing the other two







Firm i & Firm j	Firm1 & Firm2	Firm1 & Firm3	Firm2 & Firm3
$\rho_{i,j}$	0.60	0.43	0
DCorr[D_i, D_j]	0.33	0.22	0

Q2. Joint Probabilities of Default

- Given that each PD = 0.10 and ρ_{ij}'s =

$$\begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \rightarrow$$

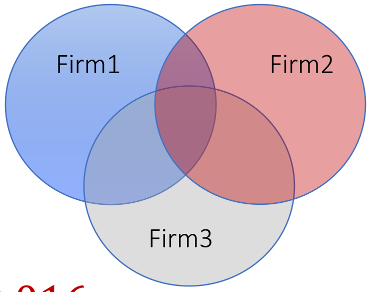
PD_1	PD_2	PD_3	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
0.1	0.1	0.1	0.4	0.5	0.6

1. To find the three values of *PDJ*, plug *PD*s and ρ_{ij} into:

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i = \begin{bmatrix} PDJ_{12} = 0.027 \\ PDJ_{13} = 0.032 \\ PDJ_{23} = 0.039 \end{bmatrix}$$

2. State the range of possible values for the probability that all three firms default.

$PDJ_{123} \in$ [0, min{PDJ_{12} = 0.027, PDJ_{13} = 0.032, PDJ_{23} = 0.039} = PDJ_{12} = 0.027]



3. State the probability that all three default under the Gauss copula.

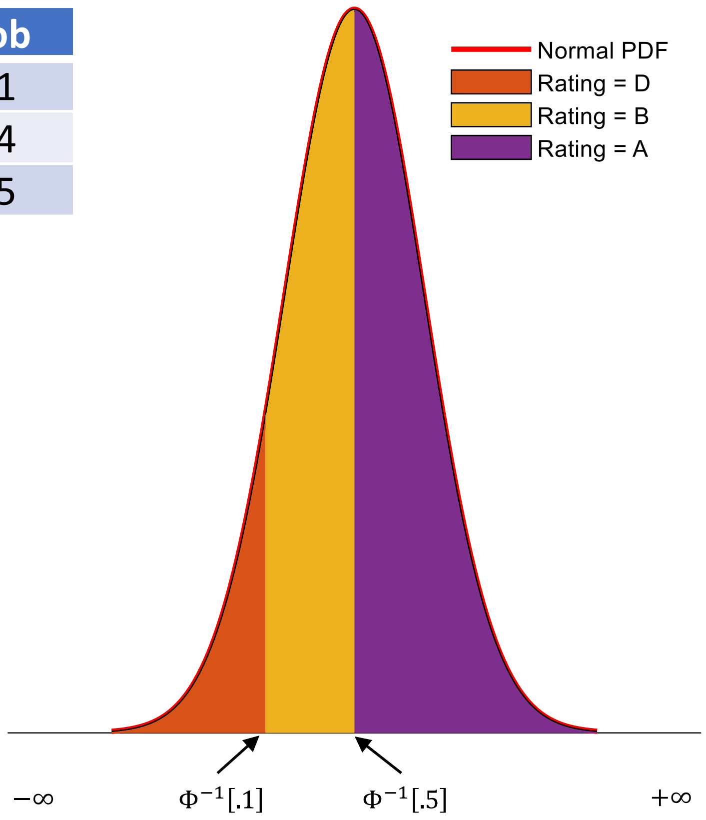
$$PDJ_{123} = \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \phi_3 \left[\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \right] dZ_1\, dZ_2 dZ_3 = 0.016$$

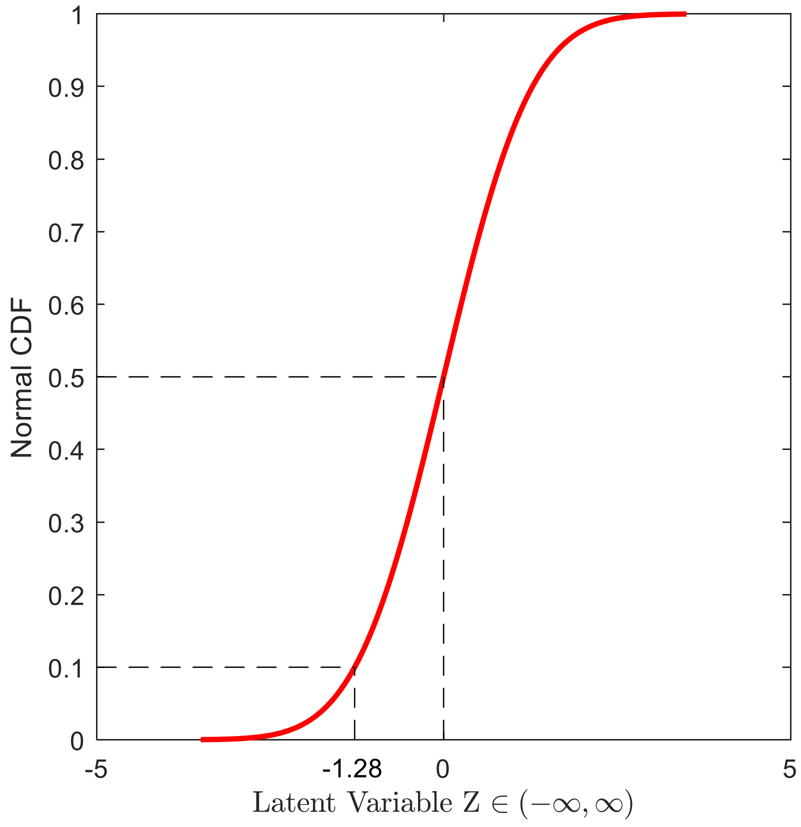
Q3. The Rating Transition Grid

- Rating, a firm's credit worthiness, is modeled as a function of the latent variable Z ~ $N[0, 1]$. For example,
  - $Transition\ Pr[rating = D] = P[D_1 = 1] = \int_{-\infty}^{\Phi^{-1}[0.1]} \phi[z_1]dz_1 = 10\%$
  - $Transition\ Pr[rating = B] = \int_{\Phi^{-1}[0.1]}^{\Phi^{-1}[0.5]} \phi[z_1]dz_1 = 40\%$, and $Transition\ Pr[rating = A] = \int_{\Phi^{-1}[0.5]}^{+\infty} \phi[z_1]dz_1 = 50\%$

Rating	Prob
D	0.1
B	0.4
A	0.5

$Z_1 \sim N[0, 1]$





Q3. Applying the Intuitions

Given Correlation and Transition probabilities:			
ρ = .40	A	B	D
A	0.5	0.4	0.1
B	0.3	0.5	0.2

$$PDJ_{AB}[D_1 = 1, D_2 = 1] = \int_{-\infty}^{\Phi^{-1}[.2]} \int_{-\infty}^{\Phi^{-1}[.1]} \phi[Z_A, Z_B, 0.4]\, dZ_A dZ_B$$

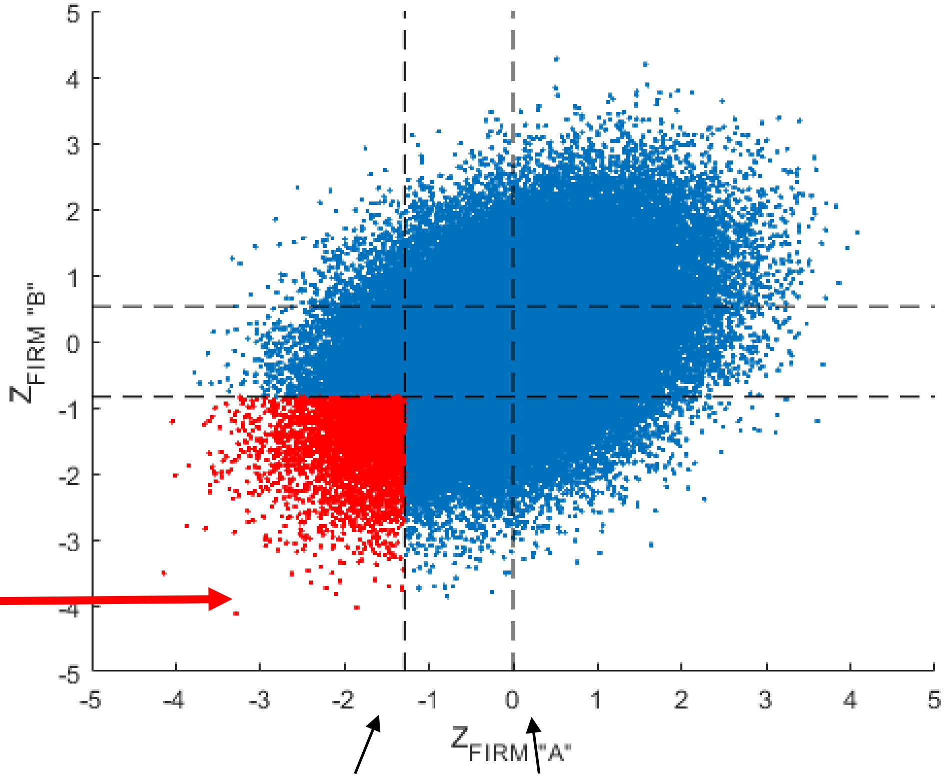
- {AA, AB, AD, BA, BB, BD, DA, DB, DD}
- Probabilities of the joint events: $\sum_1^9(\cdot) = 1$

Firm "B"

A	0.01	0.08	0.21	$+\infty$
				$\Phi^{-1}[.7]$
B	0.05	0.21	0.24	
				$\Phi^{-1}[.2]$
D	$\boxed{0.04}$	0.10	0.06	$-\infty$
	D	**B**	**A**	

$-\infty$ $\Phi^{-1}[.1]$ $\Phi^{-1}[.5]$ $+\infty$

Firm "A"



$+\infty$

$\Phi^{-1}[.7]$

$\Phi^{-1}[.2]$

$-\infty$

$-\infty$ $\Phi^{-1}[.1]$ $\Phi^{-1}[.5]$ $+\infty$

Q4. Determining the Copula Assumption

- Part 1: Assume Gauss copula and find the correlation matrix

r	Firm1	Firm2	Firm3	Firm4
Firm1	1	0.31	0.24	0.18
Firm2	0.31	1	0.10	0.044
Firm3	0.24	0.10	1	-0.036
Firm4	0.18	0.044	-0.036	1

eig(r)
0.61
0.87
1.04
1.48

- Part 2: Are the defaults of the four firms connected by a Gauss copula
  - Yes, the connection between the defaults of the four firms is consistent with a Gauss copula as the PDJs are all invertible to solve for ρ's and the resulting correlation matrix is legitimate- being positive definite.

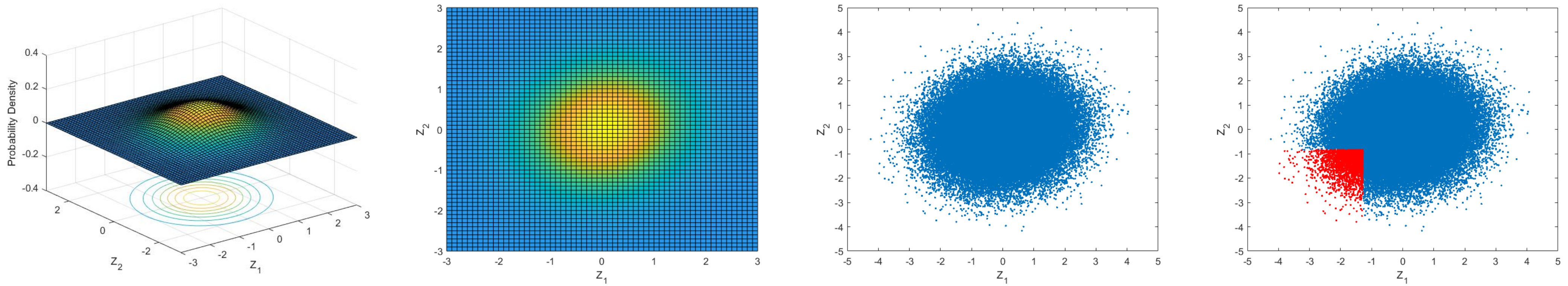
  ALSO CORRECT:
  - Yes, a Gauss copula. However, it is also possible that the defaults are somehow connected by a non-Gauss copula.

Part II. Perspectives and Hints for Homework 2

Simulating Defaults: 2-Firm Case

- From the procedure described on L2.S5, “The standard portfolio simulation”:
  - The meaning of integral: $PDJ_{12} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi[Z_1, Z_2, \rho_{12}]\, dZ_2\, dZ_1$, ρ_{12} = 0.1:

$PDJ_{12} = \int_{-\infty}^{\Phi^{-1}[PD_1]} \int_{-\infty}^{\Phi^{-1}[PD_2]} \phi[Z_1, Z_2, \rho_{12}]\, dZ_2\, dZ_1$



- Technical hints on simulating two correlated random variables:
  - Option 1 (the hard way): Simulate using the given correlation matrix in Cholesky deposition, e.g., cholesky(A) produces upper triangular R, s.t. A = R^T*R. Let Z be the matrix of five IID standard normal variables, then Z_ = Z*R will have correlation A
    - Question: what condition must the correlation matrix A satisfy?
  - Option 2 (the easy way): Simulated with the built-in functions of your software tool, e.g., the MULTIVARIATE_NORMAL method in Python

Asymptotic Answers to L2.S7

Simulation with 4-firm portfolio

Firm	PD_i	Correlation Matrix $\rho_{i,j}$				Simulated Z_i	$\Phi^{-1}[PD_i]$	D_i
1	0.1	1	0.1	0.2	0.3	-1.3559	-1.2816	1
2	0.2	0.1	1	0.4	0.5	-0.6171	-0.8416	0
3	0.3	0.2	0.4	1	0.6	-0.4817	-0.5244	0
4	0.4	0.3	0.5	0.6	1	-0.0562	-0.2533	0
		Number of defaults in this simulation run =						1

- The expected number of defaults (1,000,000 simulations): 1.0009
- The standard deviation of the number of defaults: 1.0774
- The standard deviation assuming all pairwise correlations = zero: 0.8379

Q1. Asymptotic Approach

firm	1	2	3	4	5
pd	0.5	0.4	0.3	0.2	0.1

Pairwise Correlation Matrix, **part a**

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	0.05	0.10	0.15	0.20
Firm2	0.05	1	0.25	0.30	0.35
Firm3	0.10	0.25	1	0.40	0.45
Firm4	0.15	0.30	0.40	1	0.50
Firm5	0.20	0.35	0.45	0.50	1

Pairwise Correlation Matrix, **part b**

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	0.00	0.00	0.00	0.00
Firm2	0.00	1	0.00	0.00	0.00
Firm3	0.00	0.00	1	0.00	0.00
Firm4	0.00	0.00	0.00	1	0.00
Firm5	0.00	0.00	0.00	0.00	1

- Simulate 10,000 times
- Part a: What is the standard deviation of the number of defaults?
- Part b: What would be the standard deviation of the number of defaults if all off-diagonal correlations were set equal to zero instead of the values shown?

Q2. Build Intuitions by Plotting

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	ρ	ρ	ρ	ρ
Firm2	ρ	1	ρ	ρ	ρ
Firm3	ρ	ρ	1	ρ	ρ
Firm4	ρ	ρ	ρ	1	ρ
Firm5	ρ	ρ	ρ	ρ	1

- Plot the standard deviation of the number of defaults in the portfolio as a function of ρ
- For each value of ρ, simulate 1,000 times
- Plotting is very useful in building models

Q3. Simulate or Compute Portfolio Losses

Portfolio	Loan	Firm	PD	ELGD	Exposure
	Loan 1	Firm 1	0.1	0.1	\$700
	Loan 2	Firm 2	0.2	0.2	\$600
	Loan 3	Firm 3	0.3	0.3	\$500
	Loan 4	Firm 4	0.4	0.4	\$400
	Loan 5	Firm 5	0.5	0.5	\$300
	Loan 6	Firm 4	0.4	0.6	\$200
	Loan 7	Firm 5	0.5	0.7	\$100
	Total				\$2800

ρ_{ij}	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Firm 1	1	0.15	0.2	0.25	0.3
Firm 2	0.15	1	0.25	0.3	0.35
Firm 3	0.2	0.25	1	0.35	0.4
Firm 4	0.25	0.3	0.35	1	0.45
Firm 5	0.3	0.35	0.4	0.45	1

- State the four quantities below:
  - Prob[$D_4 = 1$ and $D_5 = 1$]? (What is PDJ for these two firms?)
  - Prob[$D_4 = 1$ and $D_5 = 1 | D_3 = 1$]? (That is, what is the probability that both Firm 4 and Firm 5 default, given that Firm 3 defaults?)
  - What is the portfolio expected loss rate as a fraction of the \$2800 exposure?
  - What is the correlation between D_3 and D_4?

- Hint: Each quantity can be either simulated or computed analytically

Q4. Suppose that PD_X=0.1, PD_Y=0.2, and the latent variables responsible for default obey the 36702 distribution: $f_{X,Y}[\, x, y \,] = (\, 1 + 3\,x - y\,) / 2$. What are the values of PDJ, DCorr, and ρ?

- To solve for ρ, follow the similar strategy in HW1.Q1
  - HW1.Q1: Numerically invert the function $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i \rightarrow \boldsymbol{\rho_{ij}}$
  - HW2.Q4.Step1: Compute PDJ_{xy}; Step2: $PDJ_{xy} = \int_{-\infty}^{\Phi^{-1}[PD_y]} \int_{-\infty}^{\Phi^{-1}[PD_x]} \phi[Z_x, Z_y, \rho_{xy}]\, dZ_x\, dZ_x \rightarrow \boldsymbol{\rho_{xy}}$
- It is acceptable to use a combination of different computing languages / software tools in the homework
  - For example, some questions in Python and some in Mathematica
  - Below is a simulation example in Mathematica for Q1:

```
In[1]:= (*Define the correlation matrix A*)
        A = {{1, 0.5, 0.3, 0.2, 0.1}, {0.5, 1, 0.4, 0.3, 0.2}, {0.3, 0.4, 1, 0.5, 0.3}, {0.2, 0.3, 0.5, 1, 0.4}, {0.1, 0.2, 0.3, 0.4, 1}};

        (*Create a MultinormalDistribution object with the mean vector mu={0,0,0,0,0}*)
        dist = MultinormalDistribution[{0, 0, 0, 0, 0}, A];

        (*Generate 10,000 random samples from the distribution*)
        samples = RandomVariate[dist, 1]

Out[3]= {{0.484656, 0.831133, -1.17957, -0.709066, -1.84841}}
```