

# FINM 36702 1 Portfolio Credit Risk: Modeling and Estimation TA Session 2

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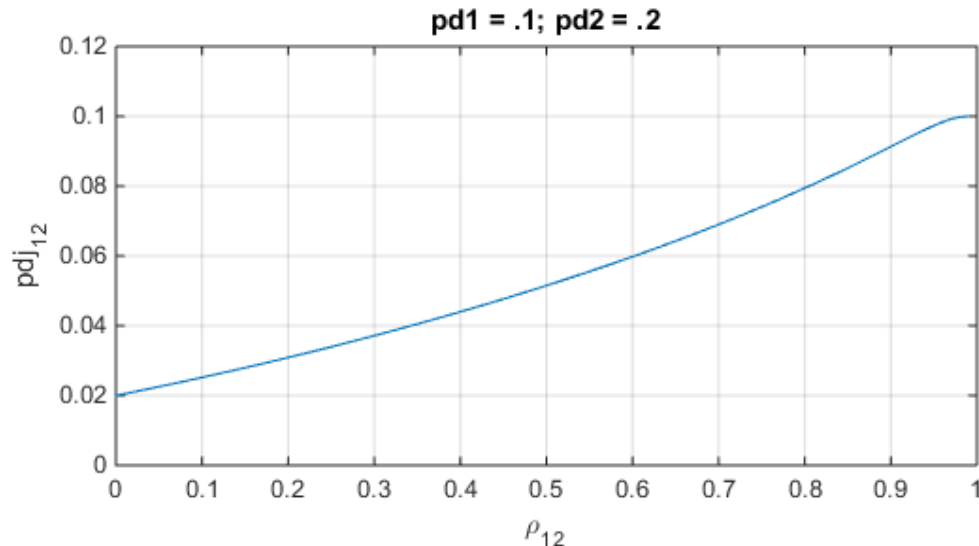
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# Part I. Homework 1 Review

# Q1. Implicitly Inverting a Function

- Given PDs and PDJ, solve  $\rho_{ij}$  from  $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}] dZ_j dZ_i \rightarrow \rho_{ij}$ 
  - FIRST OF ALL: Is this function invertible?
  - Explicitly invertible (rearrange the formula) vs. implicitly invertible ( $\arg \min_{\rho_{ij} \in [0,1]} f := \|PDJ_{ij}(\rho_{ij}) - 0.06\|$ )?



- $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}] dZ_j dZ_i \rightarrow \rho_{12}$
- A “special case: when  $PD_2 = 0.2, PD_3 = 0.3$ , and  $PDJ_{23} = PD_2 * PD_3 = 0.06$ ,  $\rightarrow \rho_{23} = 0$
- $DCorr_{[D_i, D_j]} = \frac{PDJ_{ij} - PD_i \cdot PD_j}{\sqrt{PD_i(1-PD_i)PD_j(1-PD_j)}}$

# Q1 Technical notes: Numerically invert the double integral to solve for $\rho_{ij}$ from a given value of $PDJ_{ij}$

- Step 1, implement the double integral

- Option 1: Do it the “**hard way**”.

- Part 1. Code the joint normal formula as a function into a software tool, such as Python, Matlab, R, or Mathematica, etc.

$$f(z_1, z_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} e^{-\frac{1}{2(1-\rho_{12}^2)}\left[\left(\frac{z_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{z_1-\mu_1}{\sigma_1}\right)\left(\frac{z_2-\mu_2}{\sigma_2}\right) + \left(\frac{z_2-\mu_2}{\sigma_2}\right)^2\right]}$$
$$= \frac{1}{2\pi\sqrt{1-\rho_{12}^2}} e^{-\frac{1}{2(1-\rho_{12}^2)}[z_1^2 - 2\rho_{12}z_1z_2 + z_2^2]}, \text{ because } Z_1, Z_2 \sim N[0, 1]$$

- Part 2. Integrate  $f$  using a built-in numerical integration function in your software tool.

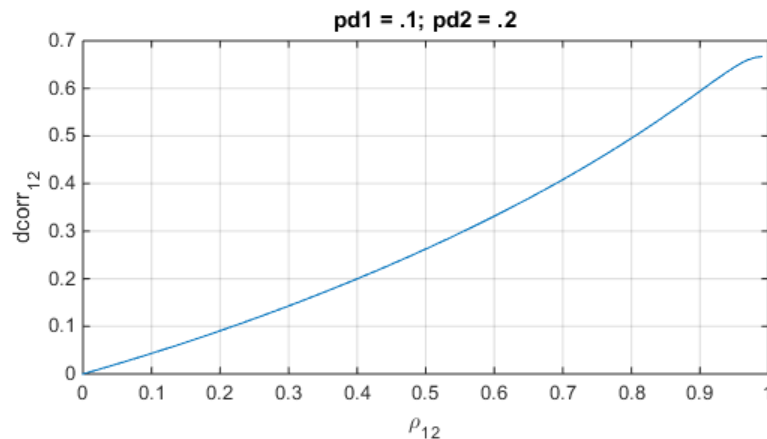
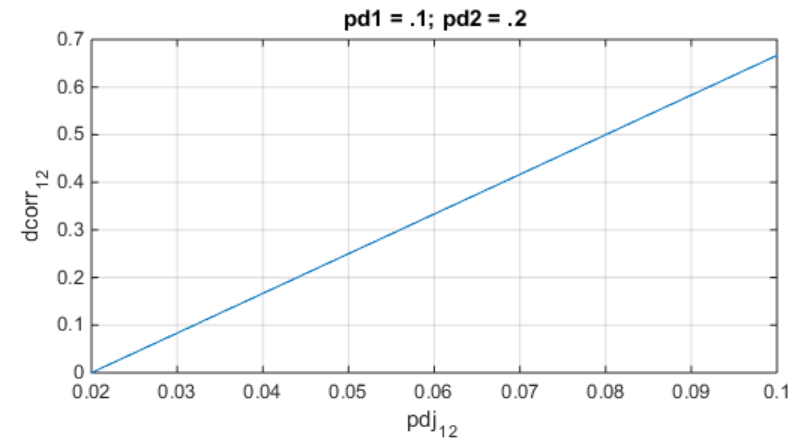
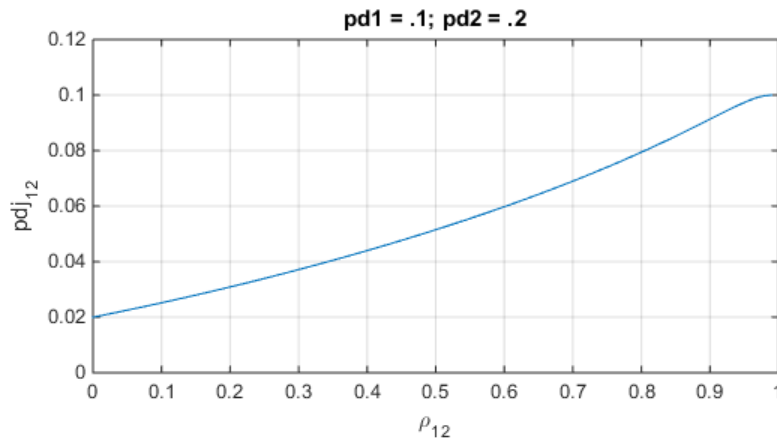
- Option 2: Or do it the “**easier way**”. Define the double integral using the built-in multivariate normal CDF in your software tool, e.g., MULTIVARIATE\_NORMAL.CDF in Python.

- Step 2, numerically invert the double integral to solve for  $\rho$ , e.g., what value of  $\rho_{12}$  gives you  $PDJ_{12} = 0.06$ ?

- Option 1: Engineer your own numerical solver, e.g., loop through different  $\rho_{12}$  values in its domain  $[0, 1]$  but stop at the value when reaching the “tolerance”.
  - Option 2: Utilize the built-in root finding routines in your software tool, e.g., FSOLVE in Python. Note when a software tool offers one-sided numerical algorithm, e.g., only arg max but no arg min, you can use the fact that arg min = - arg max.

# Q1. Equivalent Measures of Correlation

- Knowing one quantity of  $\rho$ ,  $PDJ$  and  $DCorr$  means knowing the other two



Firm i & Firm j	Firm1 & Firm2	Firm1 & Firm3	Firm2 & Firm3
$\rho_{i,j}$	0.60	0.43	0
$DCorr[D_i, D_j]$	0.33	0.22	0

## Q2. Joint Probabilities of Default

- Given that each PD = 0.10 and  $\rho_{ij}$ 's =

$$\begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \rightarrow$$

PD <sub>1</sub>	PD <sub>2</sub>	PD <sub>3</sub>	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
0.1	0.1	0.1	0.4	0.5	0.6

- To find the three values of  $PDJ$ , plug  $PD$ s and  $\rho_{ij}$  into:

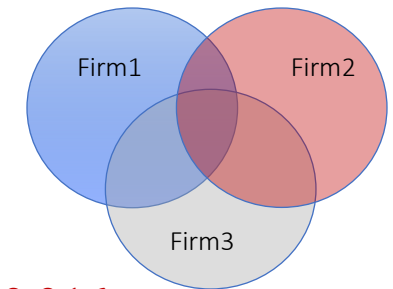
$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}] dZ_j dZ_i = \begin{bmatrix} PDJ_{12} = 0.027 \\ PDJ_{13} = 0.032 \\ PDJ_{23} = 0.039 \end{bmatrix}$$

- State the range of possible values for the probability that all three firms default.

$$PDJ_{123} \in [0, \min\{PDJ_{12} = 0.027, PDJ_{13} = 0.032, PDJ_{23} = 0.039\}] = PDJ_{12} = 0.027$$

- State the probability that all three default under the Gauss copula.

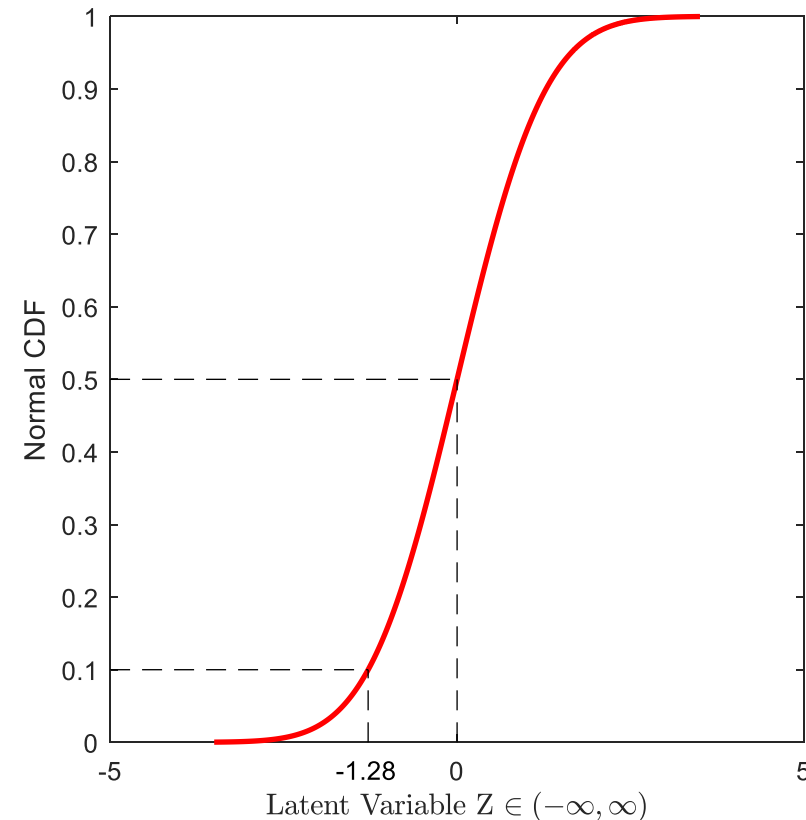
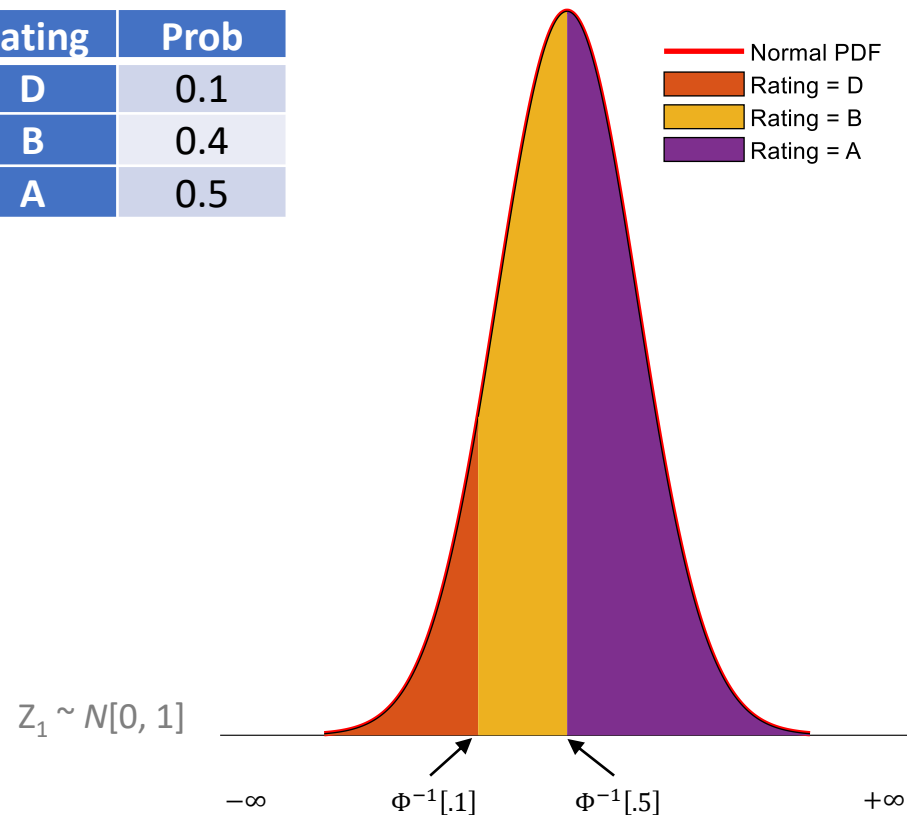
$$PDJ_{123} = \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \phi_3 \left[ \begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \right] dZ_1 dZ_2 dZ_3 = 0.016$$



# Q3. The Rating Transition Grid

- Rating, a firm's credit worthiness, is modeled as a function of the latent variable  $Z \sim N[0, 1]$ . For example,
  - Transition  $Pr[\text{rating} = D] = P[D_1 = 1] = \int_{-\infty}^{\Phi^{-1}[0.1]} \phi[z_1] dz_1 = 10\%$
  - Transition  $Pr[\text{rating} = B] = \int_{\Phi^{-1}[0.1]}^{\Phi^{-1}[0.5]} \phi[z_1] dz_1 = 40\%$ , and Transition  $Pr[\text{rating} = A] = \int_{\Phi^{-1}[0.5]}^{+\infty} \phi[z_1] dz_1 = 50\%$

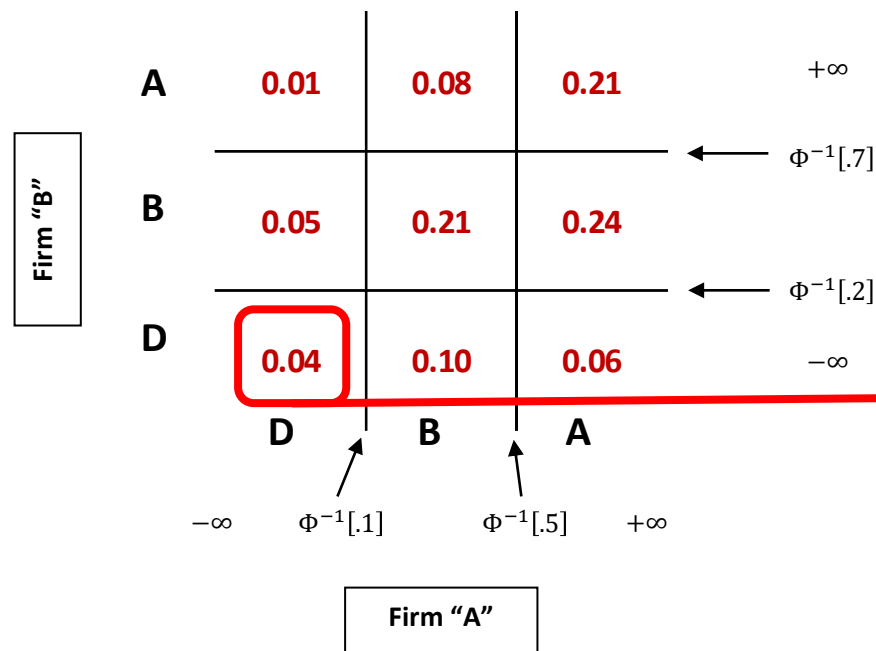
Rating	Prob
D	0.1
B	0.4
A	0.5



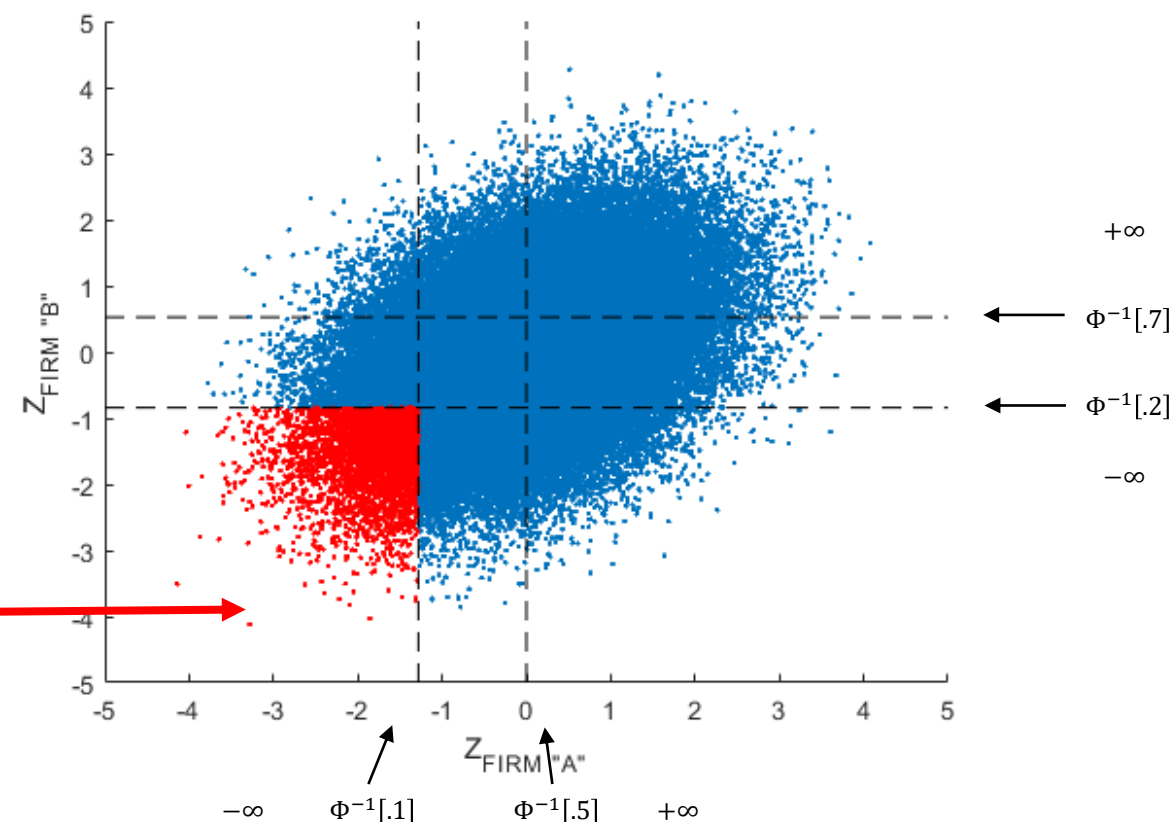
# Q3. Applying the Intuitions

Given Correlation and Transition probabilities:			
$\rho = .40$	A	B	D
A	0.5	0.4	0.1
B	0.3	0.5	0.2

- {AA, AB, AD, BA, BB, BD, DA, DB, DD}
- Probabilities of the joint events:  $\sum_1^9(\cdot) = 1$



$$PDJ_{AB}[D_1 = 1, D_2 = 1] = \int_{-\infty}^{\Phi^{-1}[\cdot 2]} \int_{-\infty}^{\Phi^{-1}[\cdot 1]} \phi[Z_A, Z_B, 0.4] dZ_A dZ_B$$





# Q4. Determining the Copula Assumption

- Part 1: Assume Gauss copula and find the correlation matrix

r	Firm1	Firm2	Firm3	Firm4	eig(r)
Firm1	1	0.31	0.24	0.18	0.61
Firm2	0.31	1	0.10	0.044	0.87
Firm3	0.24	0.10	1	-0.036	1.04
Firm4	0.18	0.044	-0.036	1	1.48

- Part 2: Are the defaults of the four firms connected by a Gauss copula
  - Yes, the connection between the defaults of the four firms is consistent with a Gauss copula as the PDJs are all invertible to solve for  $\rho$ 's and the resulting correlation matrix is legitimate- being positive definite.

ALSO CORRECT:

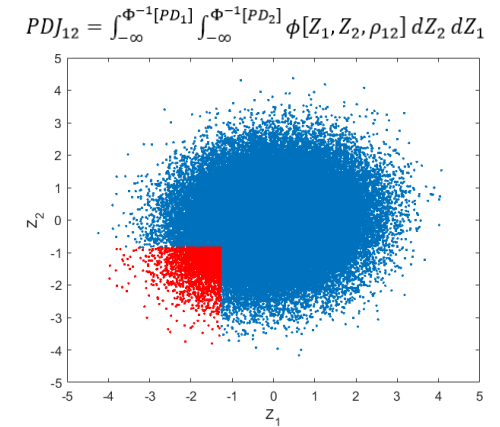
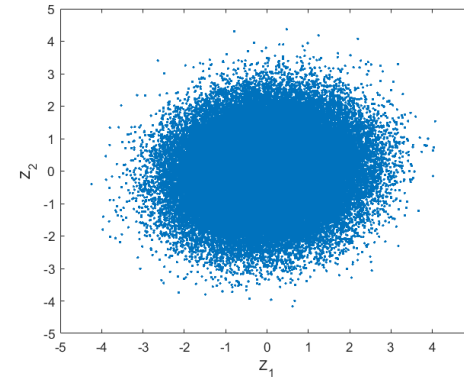
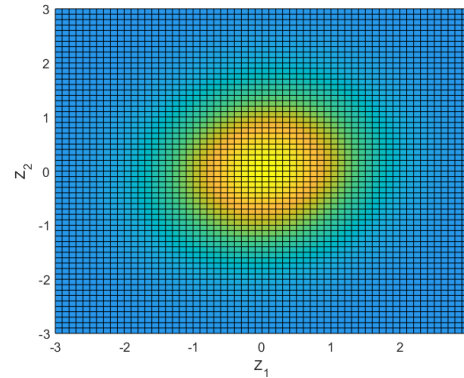
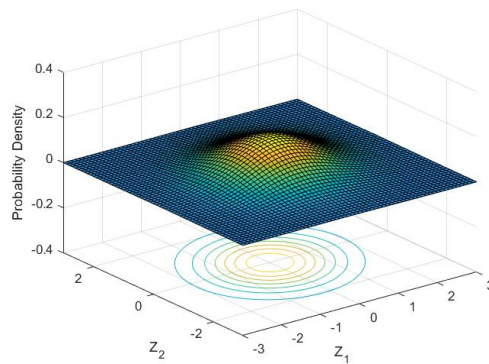
- Yes, a Gauss copula. However, it is also possible that the defaults are somehow connected by a non-Gauss copula.

# Part II. Perspectives and Hints for Homework 2

# Simulating Defaults: 2-Firm Case

- From the procedure described on L2.S5, “The standard portfolio simulation”:

- The meaning of integral:  $PDJ_{12} = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi[Z_1, Z_2, \rho_{12}] dZ_2 dZ_1, \rho_{12} = 0.1$ :



- Technical hints on simulating two correlated random variables:
  - Option 1 (**the hard way**): Simulate using the given correlation matrix in Cholesky deposition, e.g., `cholesky(A)` produces upper triangular  $R$ , s.t.  $A = R^T * R$ . Let  $Z$  be the matrix of five IID standard normal variables, then  $Z\_ = Z * R$  will have correlation  $A$ 
    - Question: what condition must the correlation matrix  $A$  satisfy?
  - Option 2 (**the easy way**): Simulated with the built-in functions of your software tool, e.g., the `MULTIVARIATE_NORMAL` method in Python

# Asymptotic Answers to L2.S7

## Simulation with 4-firm portfolio

Firm	PD <sub>i</sub>	Correlation Matrix $\rho_{i,j}$				Simulated Z <sub>i</sub>	$\Phi^{-1}[\text{PD}_i]$	D <sub>i</sub>
1	0.1	1	0.1	0.2	0.3	-1.3559	-1.2816	1
2	0.2	0.1	1	0.4	0.5	-0.6171	-0.8416	0
3	0.3	0.2	0.4	1	0.6	-0.4817	-0.5244	0
4	0.4	0.3	0.5	0.6	1	-0.0562	-0.2533	0
Number of defaults in this simulation run =								1

- The expected number of defaults (1,000,000 simulations): 1.0009
- The standard deviation of the number of defaults: 1.0774
- The standard deviation assuming all pairwise correlations = zero: 0.8379

# Q1. Asymptotic Approach

firm	1	2	3	4	5
pd	0.5	0.4	0.3	0.2	0.1

Pairwise Correlation Matrix, **part a**

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	0.05	0.10	0.15	0.20
Firm2	0.05	1	0.25	0.30	0.35
Firm3	0.10	0.25	1	0.40	0.45
Firm4	0.15	0.30	0.40	1	0.50
Firm5	0.20	0.35	0.45	0.50	1

Pairwise Correlation Matrix, **part b**

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	0.00	0.00	0.00	0.00
Firm2	0.00	1	0.00	0.00	0.00
Firm3	0.00	0.00	1	0.00	0.00
Firm4	0.00	0.00	0.00	1	0.00
Firm5	0.00	0.00	0.00	0.00	1

- Simulate 10,000 times
- Part a: What is the standard deviation of the number of defaults?
- Part b: What would be the standard deviation of the number of defaults if all off-diagonal correlations were set equal to zero instead of the values shown?

## Q2. Build Intuitions by Plotting

$\rho_{i,j}$	Firm1	Firm2	Firm3	Firm4	Firm5
Firm1	1	$\rho$	$\rho$	$\rho$	$\rho$
Firm2	$\rho$	1	$\rho$	$\rho$	$\rho$
Firm3	$\rho$	$\rho$	1	$\rho$	$\rho$
Firm4	$\rho$	$\rho$	$\rho$	1	$\rho$
Firm5	$\rho$	$\rho$	$\rho$	$\rho$	1

- Plot the standard deviation of the number of defaults in the portfolio as a function of  $\rho$
- For each value of  $\rho$ , simulate 1,000 times
- Plotting is very useful in building models

# Q3. Simulate or Compute Portfolio Losses

Portfolio	<u>Loan</u>	<u>Firm</u>	<u>PD</u>	<u>ELGD</u>	<u>Exposure</u>
	Loan 1	Firm 1	0.1	0.1	\$700
	Loan 2	Firm 2	0.2	0.2	\$600
	Loan 3	Firm 3	0.3	0.3	\$500
	Loan 4	Firm 4	0.4	0.4	\$400
	Loan 5	Firm 5	0.5	0.5	\$300
	Loan 6	Firm 4	0.4	0.6	\$200
	<u>Loan 7</u>	<u>Firm 5</u>	<u>0.5</u>	<u>0.7</u>	<u>\$100</u>
	Total				\$2800

$\rho_{ij}$	Firm 1	Firm 2	Firm 3	Firm 4	Firm 5
Firm 1	1	0.15	0.2	0.25	0.3
Firm 2	0.15	1	0.25	0.3	0.35
Firm 3	0.2	0.25	1	0.35	0.4
Firm 4	0.25	0.3	0.35	1	0.45
Firm 5	0.3	0.35	0.4	0.45	1

- State the four quantities below:
  - Prob[  $D_4 = 1$  and  $D_5 = 1$  ]? (What is PDJ for these two firms?)
  - Prob[  $D_4 = 1$  and  $D_5 = 1 \mid D_3 = 1$  ]? (That is, what is the probability that both Firm 4 and Firm 5 default, given that Firm 3 defaults?)
  - What is the portfolio expected loss rate as a fraction of the \$2800 exposure?
  - What is the correlation between  $D_3$  and  $D_4$ ?
- Hint: Each quantity can be **either simulated or computed analytically**

**Q4.** Suppose that  $PD_X=0.1$ ,  $PD_Y=0.2$ , and the latent variables responsible for default obey the 36702 distribution:  $f_{X,Y}[x, y] = (1 + 3x - y) / 2$ . What are the values of PDJ, DCorr, and  $\rho$ ?

- Follow the similar strategy HW1.Q1
  - $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}] dZ_j dZ_i \rightarrow \rho_{ij}$
  - $PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} f_{X,Y}[x, y, \rho_{xy}] dx dy \rightarrow \rho_{xy}$
- It is acceptable to use a combination of different computing languages / software tools in the homework
  - For example, some questions in Python and some in Mathematica
  - Below is a simulation example in Mathematica for Q1:

```
In[1]:= (*Define the correlation matrix A*)
A = {{1, 0.5, 0.3, 0.2, 0.1}, {0.5, 1, 0.4, 0.3, 0.2}, {0.3, 0.4, 1, 0.5, 0.3}, {0.2, 0.3, 0.5, 1, 0.4}, {0.1, 0.2, 0.3, 0.4, 1}};

(*Create a MultinormalDistribution object with the mean vector mu={0,0,0,0,0}*)
dist = MultinormalDistribution[{0, 0, 0, 0, 0}, A];

(*Generate 10,000 random samples from the distribution*)
samples = RandomVariate[dist, 1]

Out[3]= {{0.484656, 0.831133, -1.17957, -0.709066, -1.84841}}
```