

Portfolio Credit Risk

University of Chicago Masters in Financial Mathematics 36702

<https://uchicago.instructure.com/courses/48373>

Lecture 2

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Simulation and non-simulation models of portfolio default

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Lec 1: The model is complete

The credit loss model says that

- Firm i has a probability of default in the next year, PD_i .
 - PD_i changes because of economic conditions, competitive circumstances, lucky and unlucky management decisions, etc.
- Given PD_i , Firm i defaults at random.
 - If some factor makes the default of Firm i either more likely or less likely than PD_i , that factor should be included in the value of PD_i .
 - We model this with a random variable, $Z_i \sim N[0, 1]$.
 - Then, $D_i = I[Z_i < \Phi^{-1}[PD_i]]$.
- The collection of defaults is connected by a Gauss copula.
 - This is the only assumption in the model!
Some other copula might fit the facts better, but so far no one has found it.
 - We can then calculate probabilities of events involving multiple firms.

Questions? Comments?

Any questions about Lecture 1?

Any comments? How's everything?

Tonight's list of topics

The standard portfolio simulation

Modeling without simulation

The single risk factor (SRF) model

The Vasicek distribution

Basel minimum capital requirement

Multistate simulation models

The standard portfolio simulation

The standard portfolio simulation

The simulation run has inputs:

- There are k firms in the portfolio.
- Each firm in the portfolio has a probability of default, PD_i .
- Each firm has a latent variable, Z_i .
- The latent variables have a joint standard normal distribution.
- Each pair of latent variables has a correlation, $\rho_{i,j}$.

In each simulation run,

- Draw values of the correlated latent variables $\{Z_i\}_{i=1}^k$.
- Calculate the set of default indicators $D_i = I[Z_i < \Phi^{-1}[PD_i]]$.
- Keep track of the defaults.
 - Often the main item of interest is the default rate, $DR = \frac{\sum_{i=1}^k D_i}{k}$.

Last week's optional reading

The standard portfolio simulation was first articulated for a wide audience in the CreditMetrics manual, 1998.

It required a lender to know whether a loan had defaulted.

- At the time, the event of default was not defined.**

It also required a lender to know the probability with which a loan would default within a year.

- The definition of default became more codified.**
- Vendors began selling estimated probabilities to banks.**

And it required estimates of the $\{\rho_{i,j}\}$.

- Banks used equity correlations, and most still do.**

One run with a 4-firm portfolio

Firm	PD_i	Correlation Matrix $\rho_{i,j}$				Simulated Z_i	$\Phi^{-1}[PD_i]$	D_i
1	0.1	1	0.1	0.2	0.3	-1.3559	-1.2816	1
2	0.2	0.1	1	0.4	0.5	-0.6171	-0.8416	0
3	0.3	0.2	0.4	1	0.6	-0.4817	-0.5244	0
4	0.4	0.3	0.5	0.6	1	-0.0562	-0.2533	0
		Number of defaults in this simulation run =						1

Keeping track over many runs, you could estimate the distribution of the default rate, its expectation, and so forth.

You could judge the sensitivity of the results to the inputs.

Time and risk

This is a one-period model.

- It estimates the distribution of the default rate in the next year.
- It says nothing about the world after that.

But real life goes on, usually like it is today:

- If the current quarter is weak, next quarter is likely to be weak.
- If the current quarter is not weak, next quarter is likely to be not weak.

The standard simulation does not model sluggish reality.

- The serial dependence of credit loss data is a separate channel of inquiry that we touch on in Week 5.

Portfolio loss simulation

Simulation can handle portfolio loss as well as default.

More inputs are needed:

- The dollar exposure of each loan to each firm
- The distribution of LGD for each loan

If a loan defaults in a simulation run,

- Draw a value of LGD from its distribution.
- Multiply LGD by exposure to find dollar loss.

The loss portfolio rate: $Loss = \sum_i \$Loss_i / \sum_i \$Exposure_i$.

- We simplify when we assume all the exposures are equal.
- We model conditional LGD in Week 4.

Questions? Comments?

Something like this simulation is run by most banks.

Modeling without simulation

To simulate or not to simulate?

Simulation keeps track of all the relevant details.

- **That is the main reason that simulation is performed:**
 - **The distributions are complicated and difficult to express.**

But it is easy to get lost in the details.

Suppose you want to know the kind of loan that would produce least risk when added to your portfolio.

- **The simulation answer depends on all the PDs, ρ s, etc.**
- **A clear picture might not appear.**

But if you can represent risk symbolically, the answers are easier to see and to comprehend.

- **Distributions must be simpler than simulations allow.**
- **The results appear close enough for some purposes.**

Single risk factor (SRF) model

A single random variable affects every latent variable.

- **This allows for correlation between latent variables.**
 - If each of two firms is highly correlated with the risk factor, then the two firms are highly correlated with each other.
- **The SRF is the only source of correlation between firms.**
 - All other influences on a latent variable are independent.
 - The single risk factor is the single systematic risk factor.

More systematic risk factors?

Different firms are sensitive to different risks:

- 2001: Defaults in the dot-com bust.**
- 2008: Defaults from housing and mortgage speculation.**
- 2012: Defaults at the end of the “fracking” cycle.**
- 2020: Defaults stemming from Covid 19.**

One could develop a risk model with more factors.

- One could be, say, a global warming factor.**
 - Firms are sensitive to warming to differing degrees.**
 - Firms highly sensitive to global warming would default at times that might not align with the rest of the economy.**

Moody’s credit loss model has hundreds of factors.

Our single factor model might not always be enough.

- And Moody’s model might be overfit to the sparse data.**

The systematic risk factor is *bad*.

I define the systematic risk factor, Z , as *bad*.

- When it is positive, it pushes every Z_i downwards.
 - This makes every firm more likely to default.
 - (We ignore firms that prosper in downturns. Bankruptcy lawyers?)

The latent variable Z_i remains standard normal.

- However, it now depends on Z ...

$$Z_i = -\sqrt{\rho_i} Z + \sqrt{1 - \rho_i} X_i$$

Z and X_i are assumed independent standard normal.

- Therefore, Z and $\{X_i\}$ are jointly normal.
- Therefore, $\{Z_i\}$ are jointly normal and have the Gauss copula.

Z affects every Z_i , $i \in \{1, 2, \dots, k\}$.

$\sqrt{\rho_i}$ is Firm i 's “loading” on the systematic risk factor.

- Ignore the negative sign. All firms have it.

X_i is idiosyncratic to Firm i .

- It affects Z_i and no other latent variables.

$$Z_i = -\sqrt{\rho_i} Z + \sqrt{1 - \rho_i} X_i$$

When Z is elevated, each Z_i tends to be depressed.

- Therefore, each firm is more likely to default than otherwise.

A firm with a greater value of ρ_i is more affected by Z .

- Such a firm is said to be a “cyclical” firm, such as an airline.
- A firm with low ρ_i is less affected, like Proctor and Gamble.

Most defaults are idiosyncratic.

- If $\rho_i = 0.1$, then 10% of the variance of Z_i comes from Z .
 - The other 90% comes from X_i .

More about the SRF model

$$E[Z_i] = -\sqrt{\rho_i} E[Z] + \sqrt{1 - \rho_i} E[X_i] = 0$$

$$Var[Z_i] = \rho_i Var[Z] + (1 - \rho_i) Var[X_i] = 1$$

$$\rho_{i,j} = Corr[Z_i, Z_j] = Cov[Z_i, Z_j]$$

$$= E \left[(-\sqrt{\rho_i} Z + \sqrt{1 - \rho_i} X_i) \left(-\sqrt{\rho_j} Z + \sqrt{1 - \rho_j} X_j \right) \right] = \sqrt{\rho_i \rho_j}$$

N values $\{\rho_i\}$ imply the $N(N - 1)/2$ values of $\rho_{i,j}$.

- When ρ has a single subscript, it is the square of the loading on the systematic risk factor.
- When ρ has two subscripts, it is the correlation between two latent variables.

Questions? Comments?

The single risk factor model is the main topic for today.

Conditional expectations

PD is the probability of default in the next year.

- **It can be viewed as an average of what is expected if future conditions are favorable, unfavorable, or neutral to the firm.**
 - **In favorable conditions, the probability of default would be low.**
 - **In unfavorable conditions, the probability of default would high.**

In the single factor model,

- **It is easy to quantify the conditions: it is just the value of Z .**
- **It is also easy to quantify the probability of default given Z .**
 - **This is the “conditionally expected default rate” or “conditional PD”.**
- **Finally, it is easy to derive the PDF of conditional PD.**

Conditionally expected PD: cPD

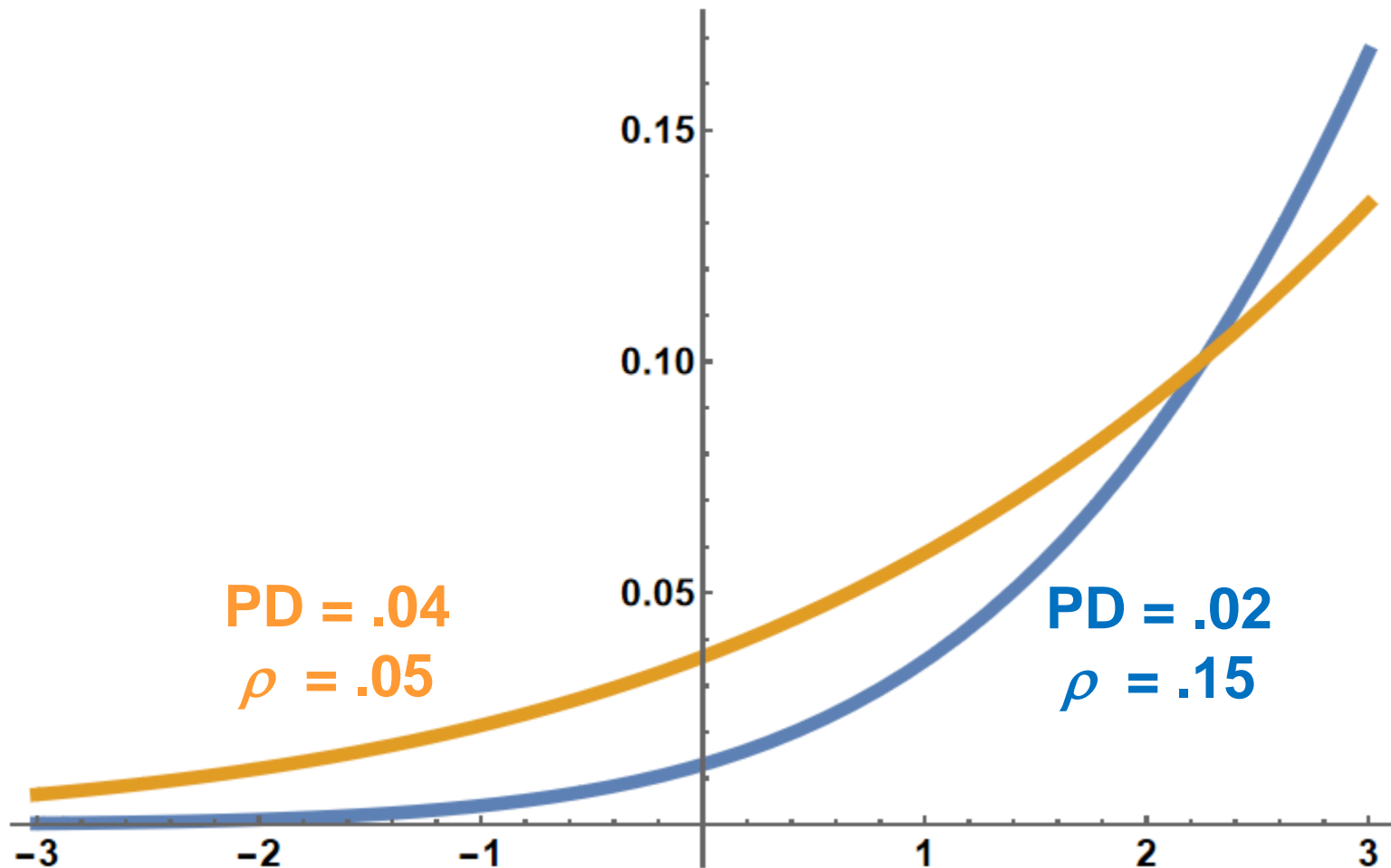
$$\begin{aligned} cPD_i &= Pr [Z_i < \Phi^{-1}[PD_i] \mid Z = z] \\ &= Pr [-\sqrt{\rho_i} z + \sqrt{1 - \rho_i} X_i < \Phi^{-1} [PD_i]] \\ &= Pr \left[X_i < \frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} z}{\sqrt{1 - \rho_i}} \right] = \Phi \left[\frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} z}{\sqrt{1 - \rho_i}} \right] \end{aligned}$$

This applies to each firm in a single risk factor model.

- If every firm had the same PD and the same value of ρ , this would be the conditionally expected PD of the portfolio.
- If in addition the number of firms in the portfolio were very large, then the law of large numbers assures that the default rate equals the conditionally expected rate.

This expression is called the Vasicek formula.

cPD is monotonic in Z



Conditional independence

$Z = z$ fully specifies conditions in a SRF model.

- Then default depends only on X_i , which is independent.
 - The defaults of firms are conditionally independent.

This provides a second way to perform simulation.

- Instead of drawing k correlated values $\{Z_i\}_{i=1}^k$, draw k independent values $\{X_i\}_{i=1}^k$ and an independent value of Z .
- Then calculate each $Z_i : Z_i = -\sqrt{\rho_i} Z + \sqrt{1 - \rho_i} X_i$.
- Proceed with the simulation as before.

The distribution of $\sum_{i=1}^k D_i$

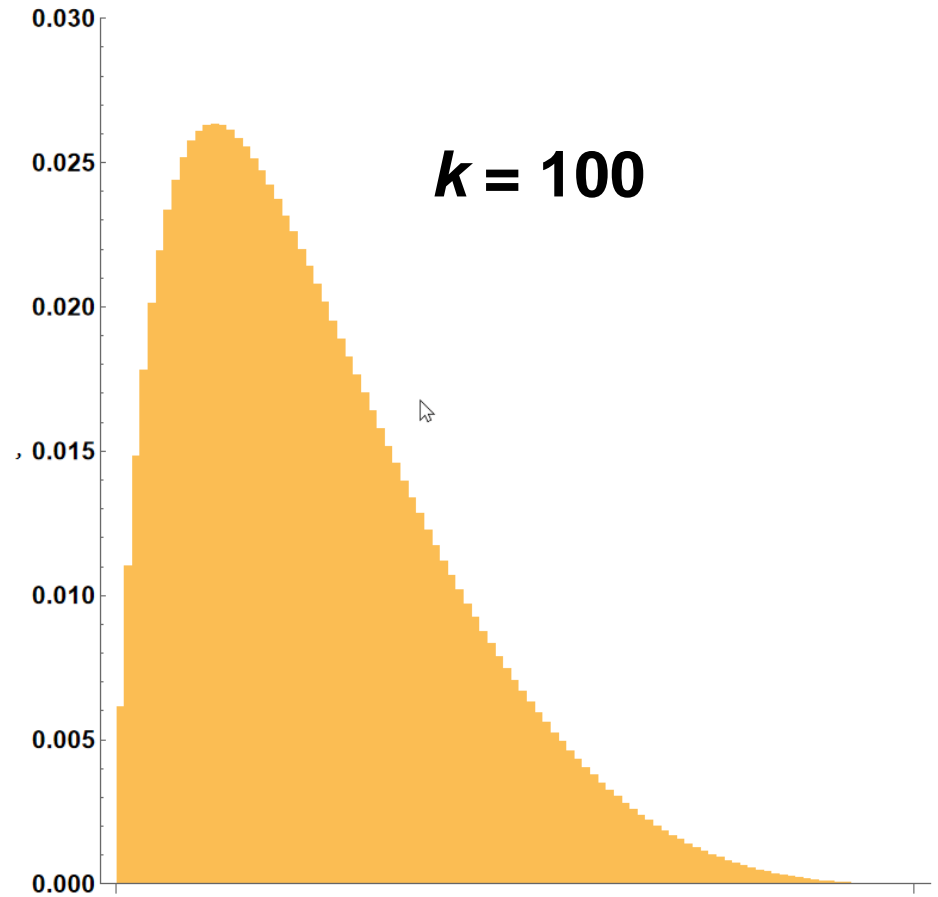
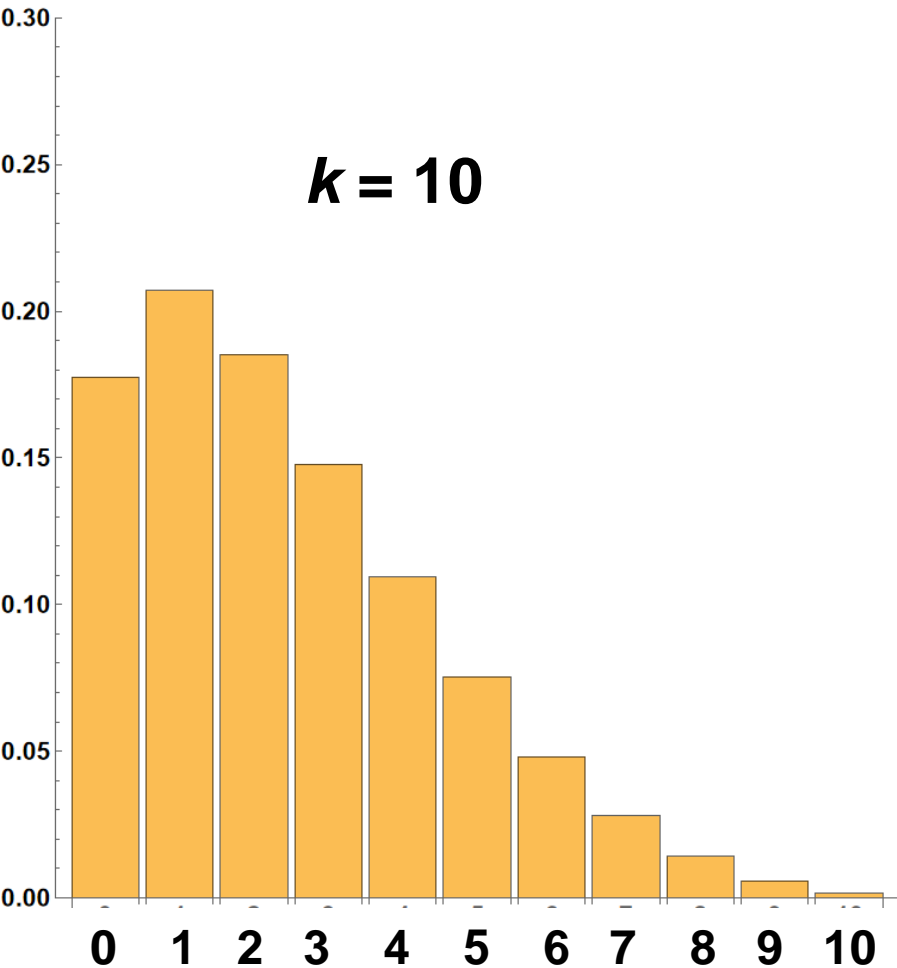
Suppose that there are k firms in the portfolio,
that there is a single factor model,
and that all firms have the same PD and same ρ .

In conditions $Z = z$, the k firms default independently at rate $cPD[z]$. Conditioned on $Z = z$, the number of defaults is distributed Binomial $[k, cPD[z]]$:

$$\left(\sum_{i=1}^k D_i \mid Z = z\right) \sim \text{Bin} [k, cPD[z]].$$

$$\begin{aligned} PMF[\Sigma D] &= \int_{-\infty}^{\infty} PMF[\Sigma D, z] dz = \int_{-\infty}^{\infty} PMF[\Sigma D \mid z] PDF[z] dz \\ &= \int_{-\infty}^{\infty} \binom{N}{\Sigma D} \left(\Phi \left[\frac{\Phi^{-1}[PD] + \sqrt{\rho} z}{\sqrt{1 - \rho}} \right] \right)^{\Sigma D} \left(1 - \Phi \left[\frac{\Phi^{-1}[PD] + \sqrt{\rho} z}{\sqrt{1 - \rho}} \right] \right)^{N - \Sigma D} \phi[z] dz \end{aligned}$$

Two PMFs: $PD = 0.25$, $\rho = 0.25$



Questions? Comments?

**Study the derivations of
the Vasicek formula for cPD and
the PMF of the uniform finite portfolio.**

Letting $k \rightarrow \infty$?

As the number of firms in the portfolio increases, the PMF approaches a continuous distribution.

- We want to discover that distribution, but instead of letting k grow we find the distribution of the cPD of a single firm.
 - The default rate of a large portfolio of identical firms would have the same distribution; therefore, this is the easy way to do it.

To derive the PDF of cPD:

- From the function cPD[Z], find the inverse CDF of cPD.
- Invert the inverse CDF to find the CDF.
- Differentiate the CDF to find the PDF.

The resulting distribution is the Vasicek Distribution.

- Same guy did the interest rate model.
 - It was the Vasicek Distribution stuff that made him rich.

From cPD[Z] to the inverse CDF

$$cPD_i[z] = \Phi \left[\frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} z}{\sqrt{1 - \rho_i}} \right]$$

Make this a function of the quantile of Z: $q = \Phi[z]$:

$$cPD_i[q] = \Phi \left[\frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} \Phi^{-1}[q]}{\sqrt{1 - \rho_i}} \right]$$

**q is the quantile of Z; therefore, q is the quantile of cPD.
For any value of q, this formula produces the value of cPD at that quantile.**

- The formula is the inverse CDF of random variable cPD.

From inverse CDF to CDF

Start with the inverse CDF from the previous slide:

$$cPD_i[q] = \Phi \left[\frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} \Phi^{-1}[q]}{\sqrt{1 - \rho_i}} \right]$$

Solve for q :

$$q = \Phi \left[\frac{\sqrt{1 - \rho_i} \Phi^{-1}[cPD_i] - \Phi^{-1}[PD_i]}{\sqrt{\rho_i}} \right] = CDF_i[cPD_i]$$

Done!

From CDF to PDF

Start with the CDF and differentiate:

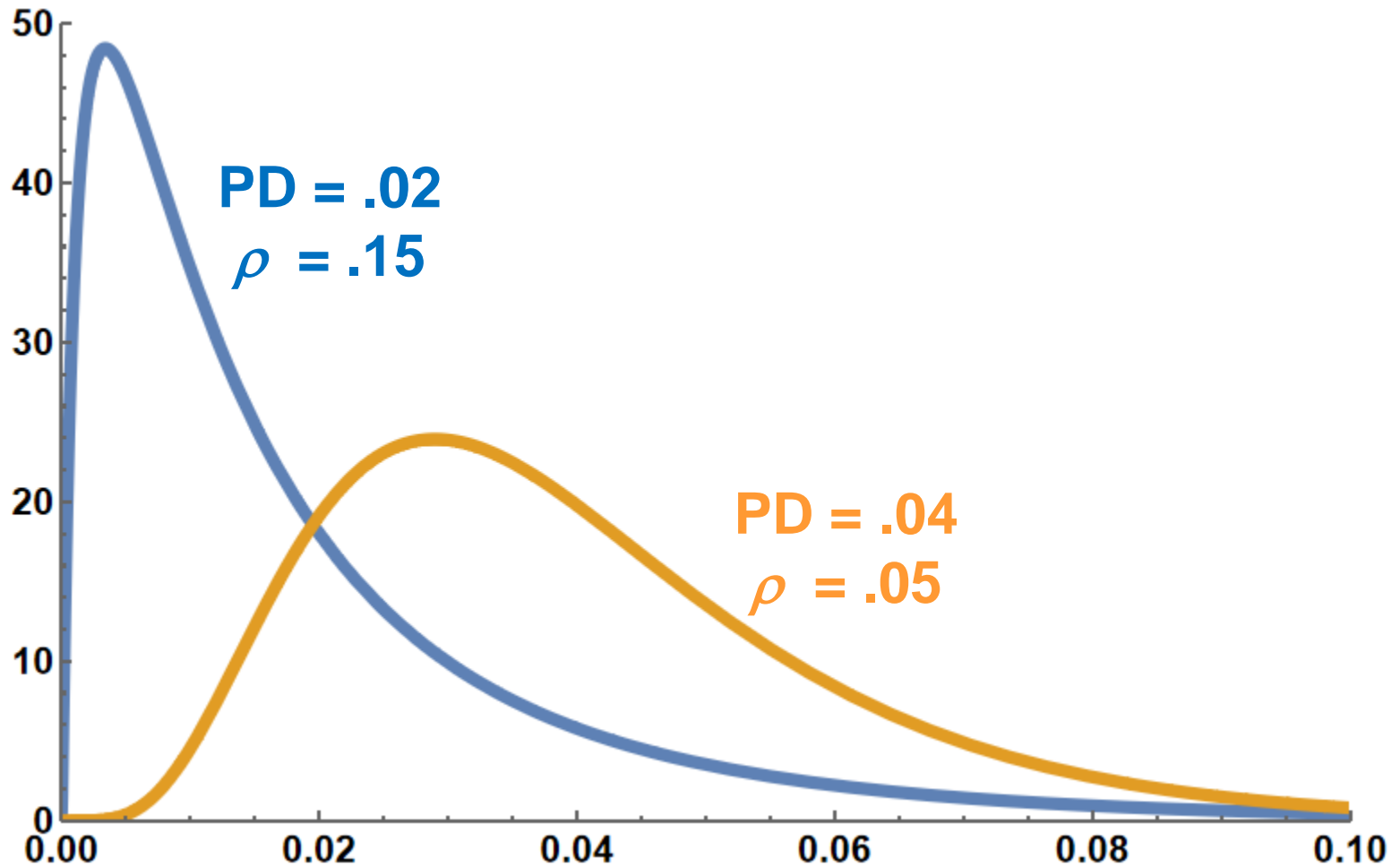
$$CDF_i[cPD_i] = q = \Phi \left[\frac{\sqrt{1 - \rho_i} \Phi^{-1}[cPD_i] - \Phi^{-1}[PD_i]}{\sqrt{\rho_i}} \right]$$

$$PDF_i[cPD_i] = \frac{\sqrt{1 - \rho_i}}{\sqrt{\rho_i} \phi[\Phi^{-1}[cPD_i]]} \phi \left[\frac{\sqrt{1 - \rho_i} \Phi^{-1}[cPD_i] - \Phi^{-1}[PD_i]}{\sqrt{\rho_i}} \right]$$

where ϕ symbolizes the PDF of the standard normal

Done!

Vasicek PDFs



Vasicek Distributions

$$\text{Inverse CDF: } CDF^{-1}[q] = \Phi \left[\frac{\Phi^{-1}[PD] + \sqrt{\rho} \Phi^{-1}[q]}{\sqrt{1-\rho}} \right]$$

$$\text{CDF: } CDF[cPD] = \Phi \left[\frac{\sqrt{1-\rho} \Phi^{-1}[cPD] - \Phi^{-1}[PD]}{\sqrt{\rho}} \right]$$

$$\text{PDF: } PDF[cPD] = \frac{\sqrt{1-\rho}}{\sqrt{\rho} \phi[\Phi^{-1}[cPD]]} \phi \left[\frac{\sqrt{1-\rho} \Phi^{-1}[cPD] - \Phi^{-1}[PD]}{\sqrt{\rho}} \right]$$

Questions? Comments?

**You should understand how to derive these formulas.
But I don't test for it.**

Shortcut to the PDF

$$cPD_i[z] = \Phi \left[\frac{\Phi^{-1}[PD_i] + \sqrt{\rho_i} z}{\sqrt{1 - \rho_i}} \right]$$

cPD is a monotonic function of normally distributed Z.

- This meets the conditions for the change-of-variable technique.
 - This technique is a compact redo of the steps we just completed.

Change-of-variable technique

Suppose the distribution of Z is known, and we want the distribution of $R = g[Z]$, where g is monotonic.

$$CDF_R[r] = \Pr[R < r] = \Pr[g[Z] < r] = \Pr[Z < g^{-1}[r]] = CDF_Z[g^{-1}[r]]$$

$$PDF_R[r] = \frac{\partial CDF_Z[g^{-1}[r]]}{\partial r} = \left| \frac{\partial g^{-1}[r]}{\partial r} \right| PDF_Z[g^{-1}[r]]$$

So, "change-of-variable" is also called "the chain rule".

$$PDF_R[r] = \left| \frac{\partial g^{-1}[r]}{\partial r} \right| PDF_Z[g^{-1}[r]]$$

Applied to cPD and the standard normal distribution of Z , it produces the PDF of the Vasicek distribution.

Nice properties of Vasicek dist.

It has support on $[0,1]$.

- The other common distribution with support on $[0,1]$ is Beta.

The first parameter is the expected value.

- The mean conditionally expected default rate is PD.
- Later we use the Vasicek Distribution to model cLoss.
 - Then the mean parameter is EL, expected loss.

The second parameter takes a limited range of values.

- Most estimated values of ρ seem to be between 5% and 15%.
 - The range reflects the difference between cyclical and non-cyclical firms.

The PDF, CDF, and $\text{CDF}^{-1}[\cdot]$ can all be explicit, as you saw.

Other properties of Vasicek dist.

- It is the distribution of a variable that is unobservable,**
- **unless you have a portfolio with an infinite number of firms.**
 - **However, the default rate is observable, its expectation is cPD, and its distribution resembles the Binomial.**

Its second parameter confuses the average person.

- **But not graduates of this course!**
 - **For them, it is job security.**

Like the CLT, the Vasicek dist. is a plank in a shipwreck.

- **Any bit of help is helpful, since we are otherwise clueless.**
 - **Searching a data set for strong correlations–ad hoc specification search–produces notably poor forecasts, as we'll discuss in Week 5.**

Vasicek summary

The simplest Gauss copula is the independence copula.

- **The next-simplest is the single risk factor model.**
 - **Simpler models have fewer things that can go wrong. That's good!**

If a firm responds to a single risk factor, then its cPD is a monotonic function of the risk factor.

- **And the conditional PD has the Vasicek distribution.**

If a portfolio contains statistically identical firms (uniform values of PD and ρ), each has the same cPD.

Questions? Comments?

**You must know the change of variable formula,
and you must be able to use it.**

You should be able to derive it if you forget it.

The Basel capital requirement

The Basel Committee

The Bank for International Settlements is in Basel, CH.

- There is a Basel Committee on Bank Supervision, BCBS.

The BCBS drafted legislation requiring banks to have minimum capital. “Basel II”, “Basel III”, etc.

- A similar law was adopted by each developed country.
 - The US has other requirements that tend to be more binding than Basel.

Capital is like net worth: assets less liabilities.

- But it is an accounting concept, not fully marked-to-market.

The capital requirement is like a margin requirement.

- To make a given loan, a bank must have minimum capital.
 - Capital lets the bank survive if it loses some money.
 - This protects bank depositors and the public.

One rule to ring them all

The BCBS wanted a function that would set minimum required capital for any loan.

- The characteristics of the loan would imply minimum capital.
- Minimum capital would be the same for every bank,
 - irrespective of which bank makes the loan.
- Minimum capital would be additive.
 - There would be no risk offset through diversification.
 - Otherwise, different banks would have different requirements.

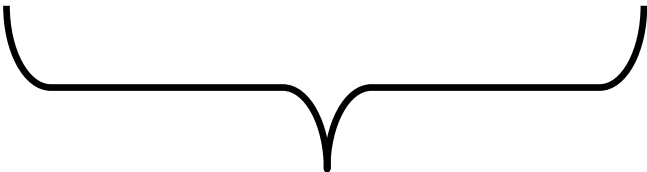
The single risk factor model fills the bill.

- The characteristics of the loan are its PD, ρ , and ELGD.
 - The bank estimates PD and ELGD; BCBS specifies the value of ρ .
- Minimum capital would be loss at quantile 0.999.
- Since there is a single risk factor, there is no diversification.
 - Portfolio required capital is the sum of required capital for each loan.

Basel formula and cPD

Per dollar of a "wholesale" loan, Basel requires a bank to have capital (K) equal to this fraction of the loan amount:

- where “R” is Basel notation for correlation (ρ)
- “N” is the standard normal CDF (Φ), and
- “M” is the maturity of the loan in years ranging from 1 to 5.

$$K = \left[LGD \times N \left(\frac{N^{-1}(PD) + \sqrt{R} \times N^{-1}(0.999)}{\sqrt{1-R}} \right) - (LGD \times PD) \right] \times \left(\frac{1 + (M - 2.5) \times b}{1 - 1.5 \times b} \right)$$


This is cPD at $q = 0.999$.

Three main differences

1. Capital is required for loss, not just for default.

- The formula multiplies by LGD to take care of this.

2. Capital is required only for unexpected loss.

- Reserves should handle expected loss.
- Expected loss, LGD x PD, is subtracted from the risk.

3. Loans might deteriorate but not default.

- Basel adjusts by a maturity adjustment factor.
 - Loans with longer maturity require perhaps 3 times more capital.
 - Don't try to make money trading off this idea!

$$K = \left[\underset{\boxed{1}}{LGD} \times N \left(\frac{N^{-1}(PD) + \sqrt{R} \times N^{-1}(0.999)}{\sqrt{1-R}} \right) - \underset{\boxed{2}}{(LGD \times PD)} \right] \times \underset{\boxed{3}}{\left(\frac{1 + (M - 2.5) \times b}{1 - 1.5 \times b} \right)}$$

More Basel calibration

Basel specifies these parameters in the formula:

- **$R \text{ (correlation)} = 0.12 + .12 * \text{Exp}[-50 \text{ PD}]$**
 - Note: $R [\text{PD} = 0] = .24$, $R [\text{PD} = 1] = .12$, monotonic decreasing.
 - (There is no evidence that correlation and PD are related this way.)
- **$b \text{ (in the maturity adjustment)} = (0.11852 - 0.05478 \text{ Log}[\text{PD}])^2$**
 - Don't ask. Apparently, no one else has.

A bank might estimate Basel parameters like this:

- **A loan's PD equals the average annual default rate**
 - within the rating grade that the bank assigns to the loan.
- **LGD is the average LGD in historical "downturn" conditions,**
 - taken among loans with similar seniority and security.
- **M is maturity in years, bounded between 1 and 5.**
- **All estimates are subject to supervisory oversight.**

Basel formula summary

Basel requires banks to have minimum capital.

- Capital is expensive to banks, so there are games galore surrounding the input estimates.**
- In addition to the credit capital that we've discussed, capital is required for other things like operational risk.**

Minimum capital is a high percentile of the cPD formula.

Minimum capital for the portfolio is the sum of minimum capital for each loan because there's only one risk factor.

The formula depends on estimates of PD and LGD.

- These must be estimated by the bank.**
- The estimation process is overseen by bank supervisors.**

Questions? Comments?

Multistate simulation models

Simulating rating transitions

So far, we have simulated a two-state model.

- A firm either defaults or it doesn't.**

It is possible to model not just the transition to default but also transitions to other states.

- Usually, the other states are internal rating grades.**

This requires the probability that a firm with a given rating experiences transition to a new rating...

A rating transition matrix

High
grade

Rat'g	Rating at year end (%)							
	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	87.74	10.93	0.45	0.63	0.12	0.10	0.02	0.02
AA	0.84	88.23	7.47	2.16	1.11	0.13	0.05	0.02
A	0.27	1.59	89.05	7.40	1.48	0.13	0.06	0.03
BBB	1.84	1.89	5.00	84.21	6.51	0.32	0.16	0.07
BB	0.08	2.91	3.29	5.53	74.68	8.05	4.14	1.32
B	0.21	0.36	9.25	8.29	2.31	63.89	10.13	5.58
CCC	0.06	0.25	1.85	2.06	12.34	24.86	39.97	18.60
D	0	0	0	0	0	0	0	100

The numbers are outdated, but this gives an idea.
The most likely thing is no change of rating.

High
yield

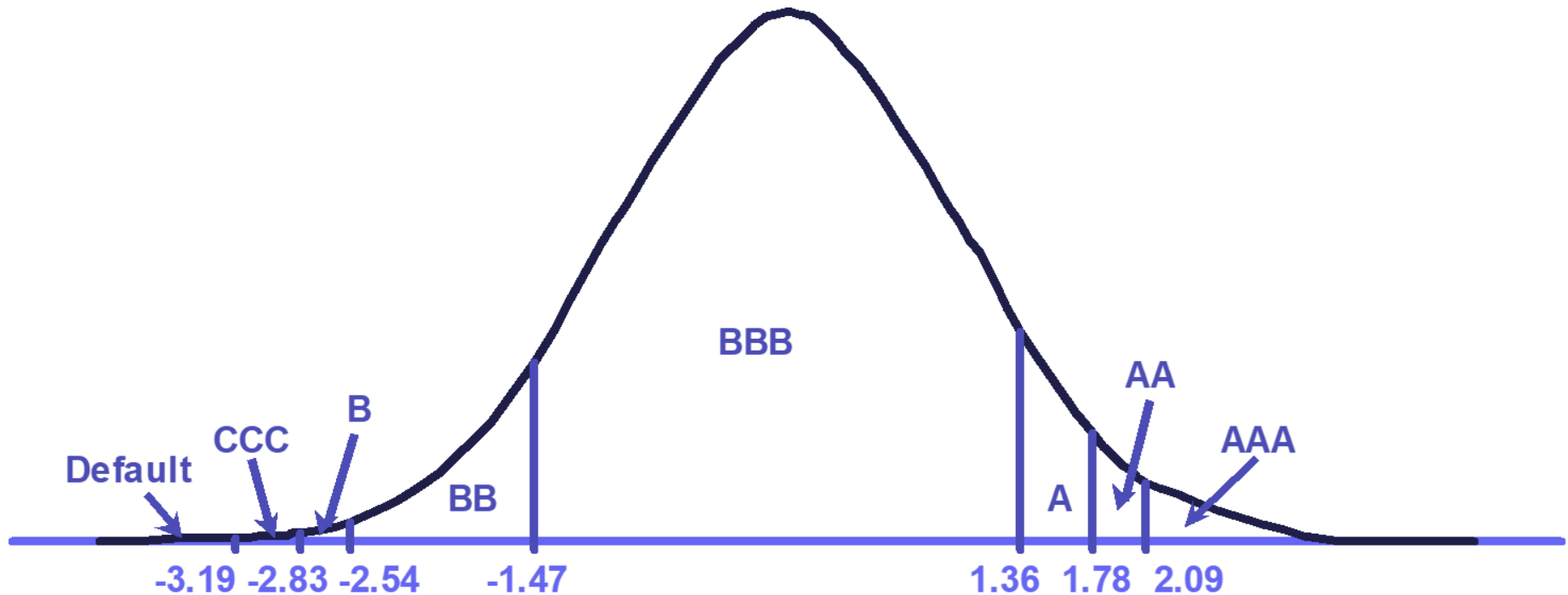
Consider Firm i , rated BBB

Let Z_i control the transition to any other grade.

- According to the previous slide, a firm rated BBB defaults if Z_i is in the worst 0.07% of its range.
- The firm transitions from BBB to CCC if Z_i is very low but above the 0.07 percentile.
 - Specifically, the BBB firm is downgraded to CCC if its value of Z is between the 0.07% quantile and the 0.16% quantile.
- And so forth, right up through upgrades to AAA.

Partition the range of Z_i according to the transition probabilities...

Transitions for a firm rated BBB



The latent variable Z_i controls all the transitions.

Transition matrix reflections

A model of rating transitions requires a cost matrix.

- **In a default-only model, the cost is LGD.**
- **Here you need the cost of transition from every initial state to every other state.**
 - **In practice, the states would be the bank's internal ratings.**
 - **The costs might be fixed amounts or distributions.**

The set-up is too rich for a non-simulation approach.

The rest of the course studies the default-only model.

- **You can always simulate the multi-state model if needed.**

Questions? Comments?

Don't forget

Homework Set 2 is due by 6PM Wednesday April 13.

Lisheng's TA session will be 6PM Sunday April 10.