FINM 36702 1 Portfolio Credit Risk: Modeling and Estimation TA Session 1

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PART I. HOMEWORK 1 HINTS

Q1. Know Thy Correlations

- Given PD's and PDJ's

PD_1	PD_2	PD_3	$PDJ_{1,2}$	$PDJ_{1,3}$	$PDJ_{2,3}$
0.1	0.2	0.3	0.06	0.06	0.06

1. Find the three values of <u>correlation</u>: $\rho_{1,2}, \rho_{1,3}, \rho_{2,3}$

2. Find the three values of <u>default correlation</u>: DCorr[D_1, D_2], DCorr[D_1, D_3], and DCorr[D_2, D_3]

Q1 Hints: ρ_{ij} versus DCorr[D_i, D_j]

- Note the difference between ρ_{ij} and DCorr[D_i, D_j]
  - Theory check: What are the variables underlying each of the two correlation measures? (See L1.S46 – 50)
  - Concept check: How are the two quantities related, i.e., ρ_{ij} versus DCorr[D_i, D_j]?
  - Concept check: Lecture 1 describes "three common ways to state the degree of connection between firms". Given one, is it possible to infer the other two? (Hint: L1.S50)
  - We will build more intuitions in later homework questions

Q1. Hints for Solving ρ_{ij}

- Given the PDs and PDJ, solve ρ_{ij} from

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi\left[Z_i, Z_j, \rho_{ij}\right] dZ_j \, dZ_i \rightarrow \boldsymbol{\rho_{ij}}$$

  – The process calls for
    - numerically implementing the double integral and
    - then inverting the function to solve for ρ_{ij}

  – Theory check: What is the condition for a function to be invertible?

Q1. Python Implementation

- Sample Python:
  - from scipy.stats import mutlivariate_normal
  - from scipy.stats import optimize
- Copula:
  - mutlivariate_normal(mean, cov).cdf(..)
- Numerical solver
  - optimize.fsolve(..)
  - Note the difference between asymptotic solution versus numerical solution

Q2. Joint Probabilities of Default

- Given that each PD = 0.10 and ρ_{ij}'s =

$$\begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \rightarrow$$

PD_1	PD_2	PD_3	$\rho_{1,2}$	$\rho_{1,3}$	$\rho_{2,3}$
0.1	0.1	0.1	0.4	0.5	0.6

1. Find the three values of PDJ:

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi\left[Z_i, Z_j, \rho_{ij}\right] dZ_j \, dZ_i = ?$$

2. State the range of possible values for the probability that all three firms default.

   Hint: try to stylize your solution using the Venn diagrams as examples (L1.S55)

3. State the probability that all three default under the Gauss copula.

$$PDJ_{123} = \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \int_{-\infty}^{\Phi^{-1}[0.1]} \phi_3 \left[\begin{pmatrix} Z_1 \\ Z_2 \\ Z_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix} \right] dZ_1 \, dZ_2 dZ_3 = ?$$

Q2. Hints on Implementation

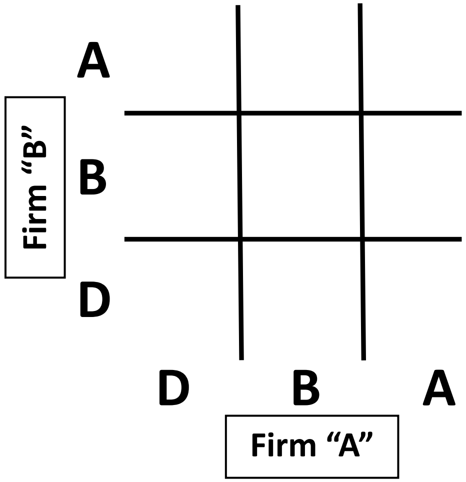
- Implement a numerical function to do a triple integral
  - If you start from scratch: This might become a programming challenge. Try to code one piece of the equation at a time and make sure that the parentheses are balanced.
  - You might want to use a simple case to perform sanity check on the implementation, e.g., what should triple integral produce if all three firms are independent?

Q3. Credit Worthiness and Dynamics

Q3. Suppose a firm rated A has correlation 0.4 with a firm rated B. They will obey the transition matrix in the next period.

Transition probabilities			
	A	B	D
A	0.5	0.4	0.1
B	0.3	0.5	0.2

Create a three-by-three grid and fill in the cells with probabilities that sum to 1.00. Two digits of accuracy is sufficient, e.g., 0.66. Assume that all transitions obey a Gauss copula.



Q3 Hints: Rating Transition Matrix

- A firm's credit worthiness is measured by credit ratings.
  - Firm A is rated A and Firm B is rated B today.
  - Rating A is better than rating B, which is better than rating D.
  - D = default.

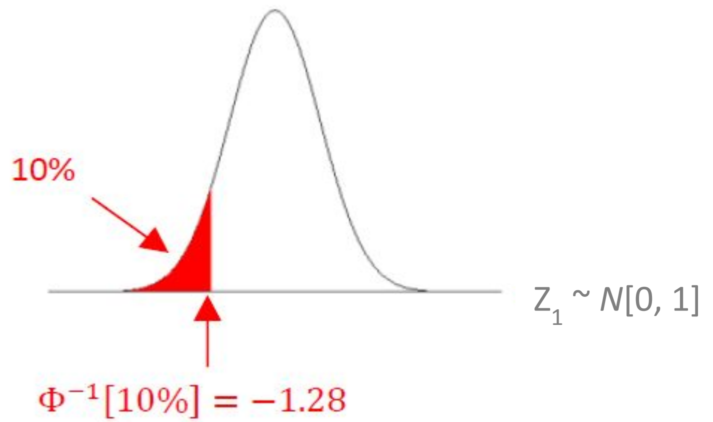
- In the next period (e.g., in 12 months), "things" can change. So would a firm's credit worthiness. The probabilities of a firm's new rating are given in the transition matrix. For example,
  - The probability of Firm A remaining at the rating of A is 50%.
  - The default probabilities are given, PD_1 = 10% and PD_2 = 20%.

Transition probabilities			
Firm/New Rating	**A**	**B**	**D**
Firm A	0.5	0.4	0.1
Firm B	0.3	0.5	0.2

Q3 Hints: The Underlying Dynamics

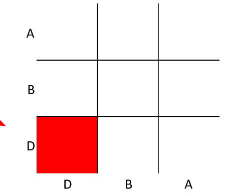
- **Assume Gaussian copula** and build from the previous homework questions.
  - Let Z_1 denote the latent variable that drives the ratings of Firm A, and Z_2 for Firm B.
  - The innovations of the latent variable cause a firm's rating to transition in the next period. For example,

$$Z_1 \sim N[0, 1] \text{ or } P[D_1 = 1] = \int_{-\infty}^{\Phi^{-1}[PD_1]} \phi[z_1] dz_1 = 10\%$$



  - The underlying dynamics also drive the joint default behavior of two firms:

$$PDJ_{AB} = \int_{-\infty}^{\Phi^{-1}[.1]} \int_{-\infty}^{\Phi^{-1}[.2]} \phi[Z_1, Z_2, 0.4]\, dZ_2 dZ_1$$



Q4. Beyond Gaussian Copula

- Suppose that four firms have PDs equal to 1%, 2%, 3%, and 4%, and the probability that any given pair defaults equals 0.1%.
  - Part 1. What is the matrix of correlations?
  - Part 2. Explain why the defaults of the four firms can or cannot be connected by a Gauss copula.

Q4: Validity of a Correlation Matrix Matters

- One way to interpret the question: Assume PDs and PDJs can be observed from data,
  - Assuming that the underlying copula is Gaussian
- Hint: check that if a correlation matrix valid

Applying Math in Modeling

- Focus on capturing the behaviors of drivers and dynamics, e.g., what causes default?
  - Using latent variable, Z, as proxy driver
  - Layered dynamics: how Z translates to default event; correlated Z's to represent herd behaviors
- Set up a collection of machinery
  - Analytical approach and asymptotic approach
  - Always helpful to build intuitions and do sanity checks with simulations and plotting
- Always be curious and always follow disciplines

Appendix. Notations and Greek Letters

- Unless otherwise specified in this course, we define the Gauss copula as following:

$$PDJ_{ij} = \int_{-\infty}^{\Phi^{-1}[PD_i]} \int_{-\infty}^{\Phi^{-1}[PD_j]} \phi[Z_i, Z_j, \rho_{ij}]\, dZ_j\, dZ_i$$

  - Denote a random variable having the standard normal distribution as Z ~ *N*[0,1]
  - Denote normal distribution PDF (probability density function): ϕ, pronounced as /fee/
  - Denote normal distribution CDF (cumulative distribution function): Φ, also pronounced /fee/

- All Greek letters

Greek letters														
Name	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**	**Name**	**TeX**	**HTML**
Alpha	$\mathrm{A}\ \alpha$	Α α	Digamma	$\mathit{Ϝ}\ \digamma$	Ϝ ϝ	Kappa	$\mathrm{K}\ \kappa\ \varkappa$	Κ κ ϰ	Omicron	$\mathrm{O}\ \mathrm{o}$	Ο ο	Upsilon	$\Upsilon\ \upsilon$	Υ υ
Beta	$\mathrm{B}\ \beta$	Β β	Zeta	$\mathrm{Z}\ \zeta$	Ζ ζ	Lambda	$\Lambda\ \lambda$	Λ λ	Pi	$\Pi\ \pi\ \varpi$	Π π ϖ	Phi	$\Phi\ \phi\ \varphi$	Φ ϕ φ
Gamma	$\Gamma\ \gamma$	Γ γ	Eta	$\mathrm{H}\ \eta$	Η η	Mu	$\mathrm{M}\ \mu$	Μ μ	Rho	$\mathrm{P}\ \rho\ \varrho$	Ρ ρ ϱ	Chi	$\mathrm{X}\ \chi$	Χ χ
Delta	$\Delta\ \delta$	Δ δ	Theta	$\Theta\ \theta\ \vartheta$	Θ θ ϑ	Nu	$\mathrm{N}\ \nu$	Ν ν	Sigma	$\Sigma\ \sigma\ \varsigma$	Σ σ ς	Psi	$\Psi\ \psi$	Ψ ψ
Epsilon	$\mathrm{E}\ \epsilon\ \varepsilon$	Ε ϵ ε	Iota	$\mathrm{I}\ \iota$	Ι ι	Xi	$\Xi\ \xi$	Ξ ξ	Tau	$\mathrm{T}\ \tau$	Τ τ	Omega	$\Omega\ \omega$	Ω ω

The contents below merely reflect the author's views and understanding for pedagogical purpose and conceptual discussions with the Financial Mathematics students at the University of Chicago. They do not in any way represent the views of the Federal Reserve Bank of Chicago or the Federal Reserve System.

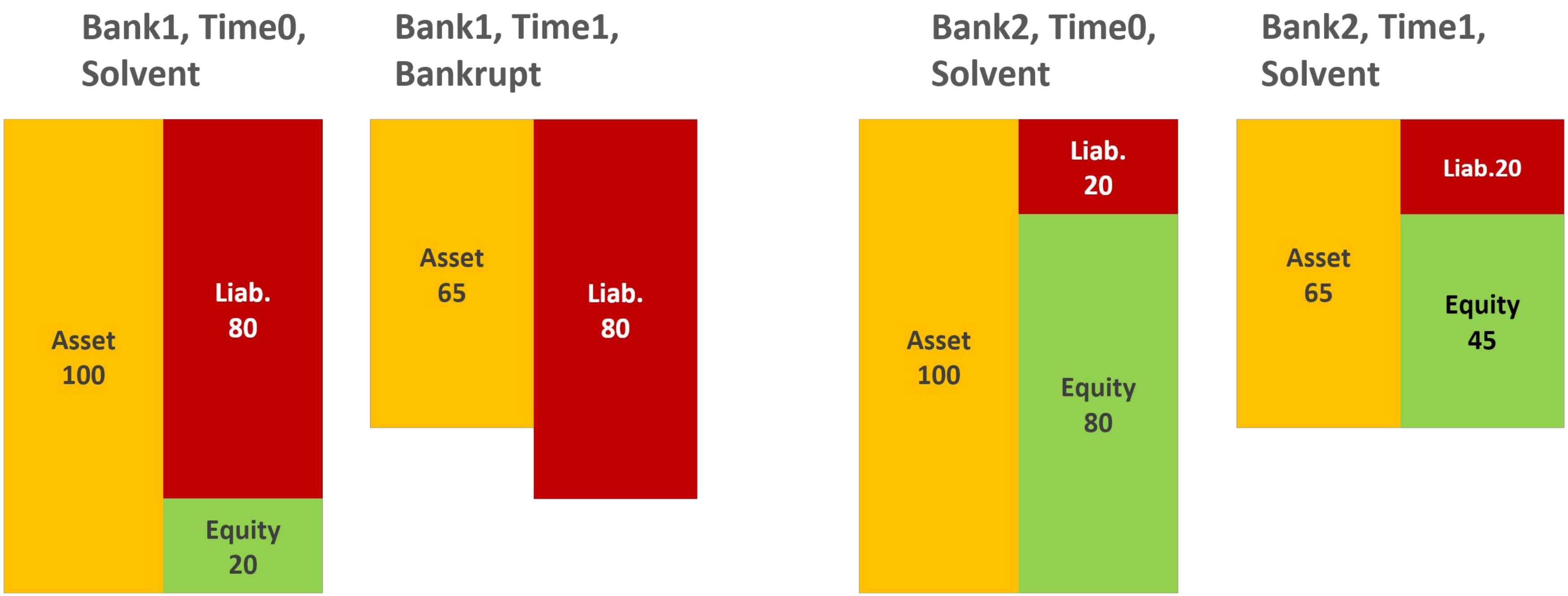
PART II. BANK DEFAULT & CAPITAL

Basics of Banking

- Some examples of banks
  - Bank holding companies = Σ(Retail banks, commercial banks, investment banks, …)
  - Central banks

- Financial intermediary
  - Credit creation: Lending and borrowing
  - What can go wrong: Credit risk

What Makes a Bank Stay Solvent



- Equity = Asset – Liability
- For simplicity, let Insolvency = Bankruptcy = Equity ≤ 0
- Capital is the synonym of equity

What Measuring Risk Means

- Risk = default likelihood
  - Two pedagogical examples:

$$K = Capital\ Ratio = \frac{Capital}{Asset}$$

$$\mathrm{K}_{firm_1,T_0} = \frac{20}{100} = 20\%$$

$$\mathrm{K}_{firm_2,T_0} = \frac{80}{100} = 80\%$$

$$L = Leverage\ ratio = \frac{Asset}{Equity}$$

$$L_{firm_1,T_0} = \frac{100}{20} = 5$$

$$L_{firm_2,T_0} = \frac{100}{80} = 1.25$$

- What risk means
  - Assume \$100 of asset earns \$5 at both firms

$$R_{asset,firm_1} = R_{asset,firm_2} = ROA = \frac{5}{100} = 5\%$$

  - The return of equity (ROE) can be higher than return on assets at both firms!!

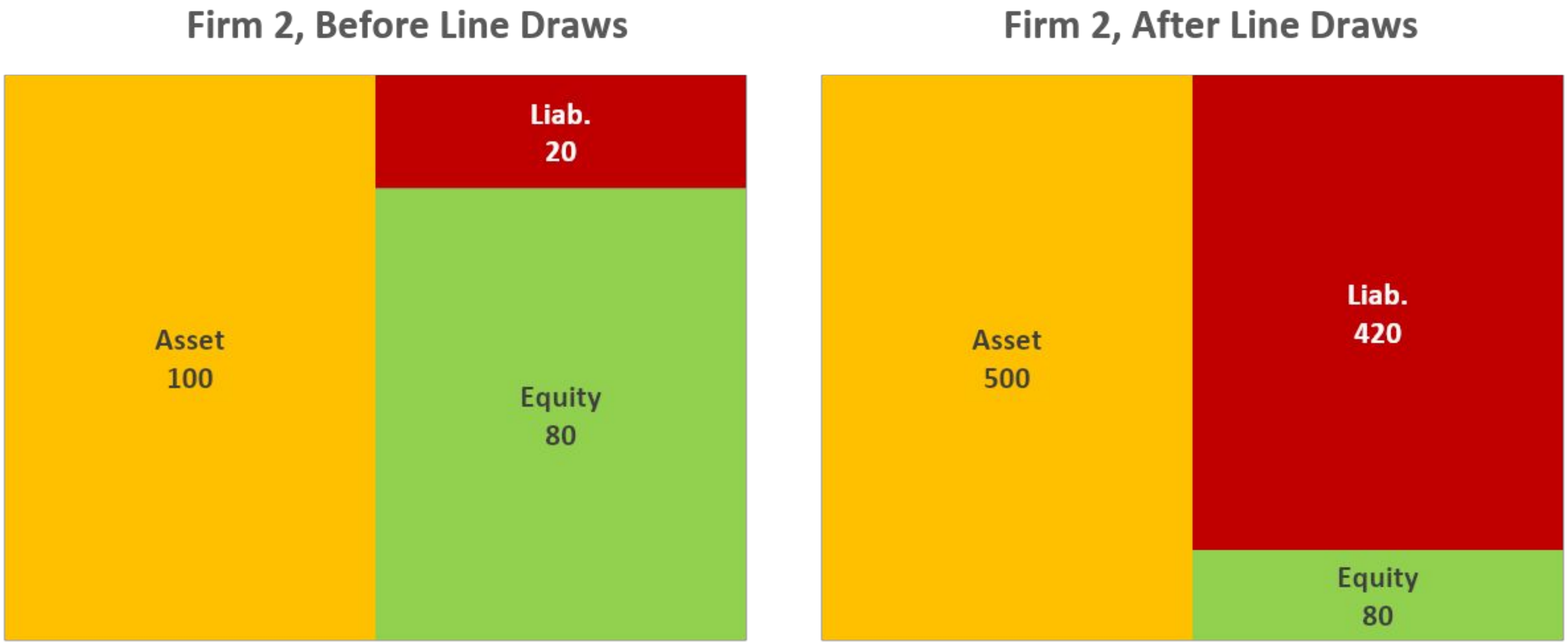
$$ROE_{firm_1} = \frac{5}{20} = 25\%$$

$$ROE_{firm_2} = \frac{5}{80} = 6.25\%$$

  - What does it mean to have a higher return, free lunch?

Revolvers and the Line Draw Scenario

- Revolvers are loans that allow the borrowers to draw on a line of credit up to a limit amount.
- Draw on credit line of $400 in loans => Total asset $500.



Basel: Rules to Require Sufficient Capital

- "An Explanatory Note on the Basel II IRB Risk Weight Functions"
  - "Capital is needed to cover the risks of such peak losses… a loss-absorbing function."

Figure 1

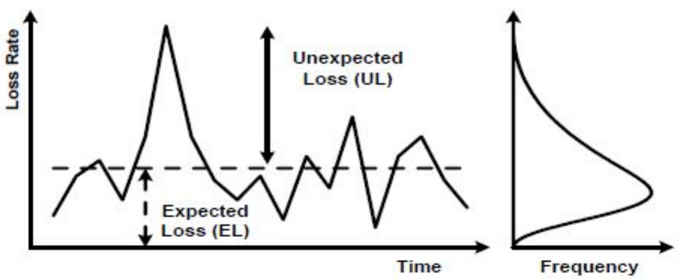
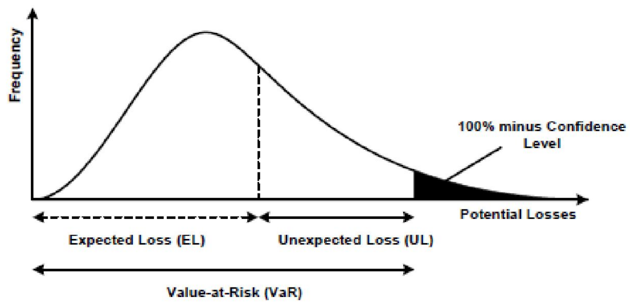


Figure 2



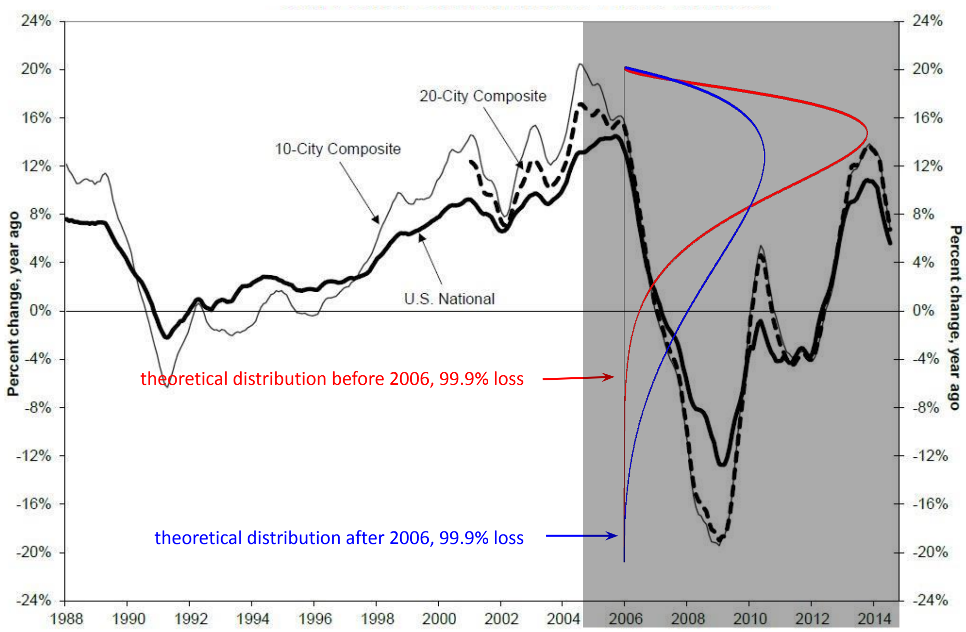
Standard normal distribution (N) applied to threshold and conservative value of systematic factor

Inverse of the standard normal distribution (G) applied to PD to derive default threshold

Inverse of the standard normal distribution (G) applied to confidence level to derive conservative value of systematic factor

Capital requirement (K) = [LGD * N [(1 - R)^-0.5 * G (PD) + (R / (1 - R))^0.5 * G (0.999)] - PD * LGD] * (1 - 1.5 x b(PD))^ -1 × (1 + (M - 2.5) * b (PD))

Stress Testing the Basel Theory



- Basel bank capital requirement can be far exceeded by the actual capital needs. Why?
  - i. The Basel Capital Accord requires the banks to set capital to 99.9% loss based on the *historical experience*
  - ii. But which history?!
- U.S. is moving to more reliance on stress testing (such as CCAR/DFAST) for capital adequacy rules

The Law of Small Numbers

- "Hasty or forced generalization", the tendency of drawing broad conclusions based on small data with coincidental mathematical relations.
  Imposed mathematical relations: model search

- "People have erroneous intuitions about the laws of chance. In particular, they regard a sample randomly drawn from a population as highly representative, i.e., similar to the population in all essential characteristics."
  Tversky, A., & Kahneman, D. (1971). Belief in the law of small numbers. Psychological Bulletin, 76(2), 105–110.

- Conservatism in risk management