Portfolio Credit Risk:
Modeling and Estimation

University of Chicago Masters in Financial Mathematics 36702

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Lecture 1

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Default and the connections between defaults

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Portfolio credit risk

A lender, such as a bank, has a portfolio of loans.

- **The best case for the lender is that each loan pays on time.**
  - **However, the lender is *short* the borrower's option to default.**
  - **The lender experiences loss if the borrower fails to pay on time.**

We study the distribution of portfolio credit loss.

- **The bank has many loans. We model the total loss.**

This approach is different from most other classes.

- **Option pricing finds the *average* outcome for an *instrument*.**
- **We find the loss *distribution* for a credit-risky *portfolio*.**

Why this interests banks

A bank's greatest asset is usually its portfolio of loans (or of other credit exposures).

- The bank finances the loans with equity and liabilities.
  - Equity: This is money the bank gets from selling its stock.
    - Stockholders expect a return, but nothing is guaranteed.
  - Liabilities: Bank liabilities are mostly deposits and bonds.
    - If a bank can't pay these debts, something bad happens.

If the value of a bank's assets becomes less than the value of its liabilities, the bank is said to be ***insolvent***.

Example

A bank has $100 in loans financed by $90 debt, $10 equity.

- **Suppose that the loans become less likely to repay.**
  - **Suppose that value of the loan portfolio declines 20% to $80.**
  - **The value of bank assets is less than the value of bank liabilities.**

There are many things that cannot happen:

- **The value of equity cannot decline to -$10. (Limited liability!)**
- **The bank cannot increase equity by selling more stock.**
  - **The bank has negative value. Would you like to buy part of it?**
- **The bank cannot issue more debt.**
  - **The bank cannot repay its current debts. Would you like to lend it more?**

There are some (bad) things that <u>*can*</u> happen:

- **The depositors can withdraw their deposits.**
  - **If this becomes common and material, it is called a bank “run.”**
- **Bank supervisors can close the bank.**

Risk control at banks

After a bank becomes insolvent, it can't sell stock or issue long-term debt.

- **Investors prefer banks with *positive* value, yes?**
  - **Note: opinions differ as to the values of things.**

To survive, bank equity must be great enough *now* to survive a *future* period of high default.

Banks compare the current *amount* of equity to the *distribution* of future credit loss.

- **Current equity can cover losses up to a certain point.**
  - **If credit loss is greater than that, the bank will fail.**
  - **The probability of bank failure is a *percentile* of the loss distribution. The distribution of credit loss is our subject.**

Why this might interest you

Your job might involve modeling credit risk.

- **Banks want to measure and control portfolio credit risk.**
- **Banks are also required to run portfolio loss models.**

You might trade credit-risky instruments.

- **Your risk will likely be measured by the portfolio model.**

You might create a model using a *small* data set.

- **For example, trading in a new or developing market.**
- **Portfolio credit loss models *always* have a small data set.**
  - **There are only a few thousand big borrowers in the world.**
  - **Big firms have been observed for only a few decades and in only a few stressful periods.**

Questions? Comments?

Please ask away.

After I solicit questions, I often jump to a different topic.

Week-by-week topics

Week 1: Defaults and the copula

Loan defaults involve many complications.

The variables in the credit loss model are stylized.

Defaults of different firms are *not* independent.

There is a *connection* between defaults.

The strength of the connection between the defaults of two firms can be expressed three ways.

With three or more firms, connections between defaults are expressed in a copula. The implications differ from what you would infer from pairwise measurements.

Week 2: The standard model

There is a standard portfolio credit loss model, sometimes called the “PD-and-correlation” model.

Banks perform simulations of the model.

A simplified version produces compact results.

- **The number of firms is infinite, every firm has the same loan amount, and firms have identical risk properties.**

This version is good for many purposes:

- **Bank capital required by “Basel” regulations**
- **Comparing the risks of different kinds of lending.**

***Note*: It is best to read the Week 2 lecture beforehand.**

- **It introduces many key ideas, and it has some math.**

Week 3: Statistical methods

A preview of the methods used in weeks 4 and 5:

- Maximum likelihood estimation, hypothesis testing, statistical confidence, p-values, and the search for a preferred regression specification.

There are weaknesses in vended parameter estimates:

- Probabilities of default are estimated with option theory.
  - The estimates can be wrong by a factor of 500,000.
  - This is a lot, considering that probabilities are between 0 and 1.

    A common-sense adjustment tries to make this less wrong on average.
- Credit correlation is assumed equal to asset correlation.
  - It isn't. But asset correlation or even equity correlation is often used.

We introduce the idea of expected loss given default.

- LGD must be understood before modeling it.

Week 4: Modeling LGD

Previous modelers used three ways to model LGD:

- **Ignore it,**
- **Pretend to not ignore it, then ignore it, and**
- **Model it in isolation from default and loss.**

We model in harmony with default and loss.

- **The two rates go up and down together.**
- **This implies(!) LGD is a function *only* of the default rate.**
  - **The effects of other variables occur via their effects on default.**

We derive the simplest LGD function.

We try to reject it in a statistical test, but we can't.

- **The simplest LGD function is good enough to explain data.**

It helps to look at Week 4 in advance.

Week 5: Regression, instead

The first four weeks develop a model that:

- **forecasts a distribution of credit loss, and**
- **does not depend on observable economic variables.**

In Week 5, we regress credit loss on macro variables.

- **The resulting forecasts are *reliably* poor.**
  - **It isn't just that an odd set of data can produce odd forecasts.**
  - **It is that, on average, forecast errors are greater than those of a different model.**
- **If you want good forecasts, you must analyze forecasts.**
  - **Don't stop after you analyze the data and some models.**

Questions? Comments?

Your questions are likely to help others.

Class personnel

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Tonight's topics

The richness of a default event

The quantities that are modeled

A portfolio containing one loan

A portfolio containing two loans

A portfolio containing many loans

The richness of a default event

Loans, covenants, and default

Every loan places requirements on the borrower:

- **It *must* pay interest and principal on schedule.**
  - **If it doesn't, this is a "money default".**

    Default on a corporate loan is recognized when payments are 90 days late.
- **It *may* be required to obey specified "covenants":**
  - **Maintain earnings above a certain threshold.**
  - **Have cash flow greater than X times the interest on the loan.**
  - **Maintain $Y of equity finance for every $1 of bank loan.**
  - **Do not file for bankruptcy.**
  - **If the firm fails to obey any covenants, this is a "covenant default."**

There are several consequences of default.

Consequences of default

If a firm defaults:

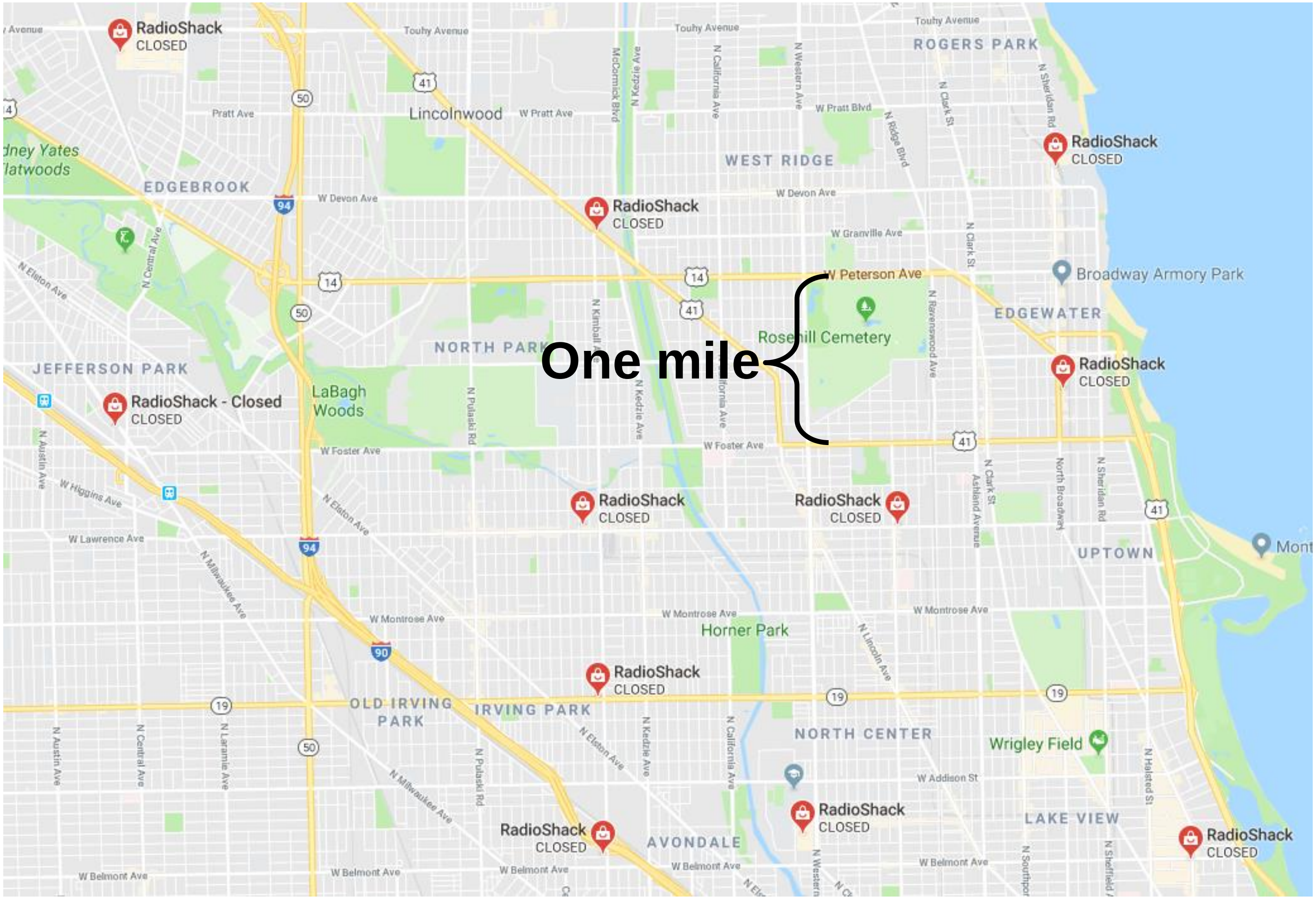
- The lender gains ownership of assets identified as “collateral”.
  - Commonly, collateral assets are real estate, inventories, or financial assets.
  - (Some loans are “unsecured” and specify no collateral.)
- There are likely to be fees.
- The default might be noted publicly.
  - The firm becomes less able to borrow from other lenders.
- A loan agreement usually has a “cross default” provision.
  - If a bank has made more than one loan to a firm, a default on any loan triggers default on every loan.

Usually, it is cheaper to pay on time than to default.

- The borrower is long the option to default.
- The option is usually out of the money.

Consider the convoluted 2015 default by RadioShack.

RadioShack retail stores, 2016



RadioShack: Orientation

RadioShack was an electronics retailer.

- In the early 20th century, an amateur radio operator's workplace was known as his "shack," a very cool thing.
  - By the 21st century, neither radio nor shacks were considered cool.
  - Quarter after quarter, RadioShack lost money.

The company filed for bankruptcy in February 2015.

- Anticipating this, I stored the Wikipedia from January 2015.
- A filing for bankruptcy is covenant default.

The company filed for bankruptcy again in 2017.

- Stuff like this happens.
- We focus on the 2015 bankruptcy.

RadioShack's Wikipedia Jan 2015

- On March 4, 2014
  – RadioShack announced that it planned to close as many as 1,100 lower-performing stores.
- On May 9, 2014
  – the company reported that a conflict with its creditors prevented it from carrying out those closures.
  – Later Standard & Poor's downgraded to "CCC", warning that the company would have "very small amounts of liquidity early next year, which could lead to a liquidity crisis, default, or the company's decision to seek a financial restructuring."
- On September 11, 2014
  – RadioShack admitted it may have to file for bankruptcy.

Translating to English—1

RadioShack "planned to close stores" / "conflict":

- **The land and buildings of some unprofitable stores had been pledged as collateral on certain loans or bonds.**
  - **The lenders prevented RadioShack from closing the stores.**
  - **They believed that it was in their interest that the stores stayed open.**

Later, Standard & Poor's "downgraded" RadioShack:

- **S&P was one of the three major "Nationally Recognized Statistical Rating Organizations (NRSROs)."**
  - **The others were Moody's and Fitch. S&P and Moody's dominated.**
  - **Now there are ten!**
- **Firms can pay an NRSRO to *rate* their bonds or loans.**
  - **A rating gives investors information; the bonds trade with more liquidity.**
  - **In this case, S&P changed its rating lower; this was a *downgrade*.**

Translating to English—2

The new S&P rating was <u>CCC.</u>

- **From best to worst, S&P ratings are:**
  - **AAA, AA, A, and BBB; these are called “Investment grade” ratings.**
  - **BB, B, CCC, CC, and C: these are called “non-investment grade”, “high yield”, or “junk” ratings.**
  - **SD and D: the firm selectively defaults or defaults on all obligations.**
- **Moody’s and Fitch have their own scales.**
  - **The scales have the same number of steps.**
  - **Yet each agency says that their ratings do something different.**

S&P said that the company would have "very small amounts of <u>liquidity</u> early next year.”

- **That is, RadioShack might not have the <u>cash</u> to repay loans.**

Translating to English—3

“This could lead to <u>default</u>…”

- **That is, maybe RadioShack can’t pay the loans it has.**

…or the company might “<u>seek a financial restructuring</u>.”

- **That is, maybe RadioShack could arrange for a new set of loans that it has a better chance of repaying.**
  - **There would be a new schedule of repayment and new covenants.**
  - **The new loans would require less cash to be paid in the short term.**
  - **The proceeds of the new loans would pay off the old loans.**

Then RadioShack admitted it might file for <u>bankruptcy</u>.

- **In bankruptcy a court *<u>enforces</u>* restructuring or liquidation.**
  - **Liquidation is when all the assets are sold, and the corporation dies.**

Normally the end would come soon, but…

Then, this happened

In 10/2014, RSH got a $120 million loan from a hedge fund.

- **At the time, RadioShack was losing about $1 million per day.**
  - **The rate of loss was known to the public.**

About 120 days later, RadioShack filed for bankruptcy.

How it turned out for the lenders

Bankruptcy court (nearly) followed "Strict Seniority".

- The company had enough assets to pay 100% of the amount of the revolving loans.
  - Revolvers have higher seniority than term loans.
- That left enough to pay about 25% of the Term Loan.
- After that, there was approximately zero left to pay the bonds.
  - All lenders know, and agree to, their position on the seniority "ladder."

Debt	Rank	Collateral	Outstanding	Above	Below	Recovery
Revolving Loan	1	All assets	250	0	575	100
Term Loan	2	Non-current assets	250	250	325	25.5
Senior Unsecured Bonds	3	None	325	500	0	0.24

The court put the hedge fund lender into group 3.

- The $120 million loan was pure loss to the hedge fund.
  - Not unexpected!

Questions? Comments?

Please ask away!

I will probably never mention RadioShack again.

You can ask about it later, but the best time is now.

Stylization

Stylizing loss, default and LGD

If the borrower pays on time and obeys all covenants, then there is no default and no loss to the lender.

Otherwise, there is a *<u>default</u>*. The usual reasons are

- **filing for bankruptcy,**
- **violating other loan covenants, or**
- **90 days late on payments.**

The lender often does not lose 100% of the loan amount.

- **For example, the lender might get partial or full recovery by selling the collateral if the loan is secured.**
- **The *<u>fraction</u>* of the loan amount that is lost by the lender is called Loss Given Default (LGD).**
  - **The exact amount is usually not known at the time of the default.**

The definition of default

In the 20th century, most banks did not define default.

- **They had no *<u>reason</u>* to.**

Banks defined "default" when they discovered a model (the one we study) that could help them manage *<u>risk</u>*.

- **The model doesn't care which definition is being used.**
  - **Different kinds of loans and bonds use different definitions.**
- **But the model needs to know whether a *<u>default</u>* has happened.**

Default-side symbols

D The (random) default indicator

= 1 if the firm defaults; = 0 otherwise

If a firm defaults on one loan, it is in default on all.

PD = E [D], the probability that the firm defaults within a period. Usually the period of one year.

PD changes over time depending on events.

We take PD's as given. We focus on the portfolio ***model***.

N The number of ***firms*** having loans in the portfolio.

DR $=\sum_{i=1}^{N} D_i / N$, the default rate in the portfolio.

N is the number of firms; $\sum D_i$ is the number that default.

Historical average default rates

A loan is a private agreement between a bank and a borrower; the loan default rate is unknown.

For perspective, we have US *bond* default rates:

- **Investment grade (BBB or Baa or better): about 0.20%**
  - **The probability of default of Apple is probably less than 0.01%.**
- **Non-investment (junk) (BB or Ba or worse): about 3.6%.**
  - **In December 2014, the PD of RadioShack was greater than 50%.**

Recovery and loss given default

After a default, the bank attempts to *recover* as much as it can of the amount that it is owed.

- **The bank can gain ownership of identified collateral and sell it.**
- **Some loans are guaranteed by other firms, and the bank can make claims on those guarantors.**
- **Sometimes the bank takes over the firm and runs it for a while.**
  - **The loan agreement states whether the lender can do this.**

Expressed in dollars, \$LGD = \$Exposure – \$Recovery.

- **We work in *ratios*: LGD = 1 – \$Recovery / \$Exposure.**

Two ways to measure LGD

If the defaulted loan trades in a market,

- **the lender can measure LGD at the time of default as one minus (post-default market price / par).**
  - **Trouble is, very few loans trade in markets.**

If the defaulted loan does not trade,

- **the lender is usually forced to measure LGD much later, when the bank’s exposure is fully resolved.**
  - **This is whenever the lender stops trying to recover more from the firm.**
- **Recovery equals the net of all <u>*discounted*</u> <u>*cash*</u> <u>*flows*</u>, discounting back to the time of default.**
- **Lender decisions after default (accept a bid on the collateral or continue to own it?) affect this "loss given default."**
  - **It can be difficult to keep track, as when there are two loans.**
  - **It can be difficult to discount properly because there is no other instrument that has risk characteristics like a defaulted loan.**

LGD-side symbols

LGD The (random) fraction of exposure that is lost.
LGD is usually between 0 and 1.
If there is no default, LGD is undefined (not zero).

ELGD The mathematical expectation of a loan's LGD.

Loss Loss is usually measured as a fraction of exposure.

– 36702 usually assumes the exposure of every loan is $1.

For a single loan, $\boldsymbol{Loss = D * LGD}$.

For the k defaulted loans among N loans, $\boldsymbol{Loss = \frac{\sum_{i=1}^{k} D_i\, LGD_i}{N}}$.

Historical average LGD rates

The average US junk bond LGD rate is about 60%.

Loan LGDs tend to be less than bond LGDs.

- **Loans are more frequently secured by collateral.**
- **Bank loans are almost always *senior* to bonds.**
  - **In bankruptcy, senior debt *must* be paid first.**
  - **Less senior debt holders get whatever is left over.**
- **Loan agreements give banks powers they use to reduce loss.**

We model LGD mostly in Week 4. The models of the first three weeks mostly address the portfolio default rate.

Questions? Comments?

If you have a question, it is likely that others have the same question. Please ask away!

Modeling the default rate in a portfolio with only one firm

Modeling a portfolio with 1 firm

Draw a random variable having a known distribution.

If the quantile of the RV is less than PD, simulate default.

- Example: Draw Q ~ U [0, 1]; D = 1 if q < PD, otherwise D = 0.
- Example: Draw Z ~ N [0, 1]; D = 1 if z < Φ^{-1}[PD], otherwise D = 0.
  - Upper-case "phi," Φ, symbolizes the standard normal CDF.
  - $\Phi^{-1}[\,\cdot\,]$ is the inverse cumulative distribution function.
  - Lower-case letter, $\phi[\,\cdot\,]$, symbolizes the standard normal PDF.
  - Q and Z are latent variables responsible for default. They are not observed.

You would not need to believe that firms have random variables to make use of this model.

- The output of a model, not the model itself, should mimic reality.

Questions? Comments?

We could probably spend a long time on that slide because it is our first brush with the idea of a *model*.

Connecting a second firm

The portfolio has a loan to Firm 1 and a loan to Firm 2.

- **The probabilities of default are PD_1 and PD_2.**

Let PDJ symbolize the probability that *both* default.

- **If needed, it can be fully articulated as $PDJ_{1,2}$.**

If the two defaults are independent, $PDJ = PD_1 PD_2$.

- **This is one definition of statistical independence.**

But in general, defaults are *dependent*…

Connection makes sense

Suppose that Apple defaults on its debts. Several other kinds of companies might also default:

- **One that sells stuff to Apple.**
- **One that sells stuff to Apple customers.**
- **One that sells stuff on the Apple Ap Store.**
- **One that has no direct connection to Apple but is indirectly connected through the interaction of the reasons for Apple default and the incomes and preferences of other people and firms at the time.**
- **…**

If defaults were *<u>independent</u>*, then the std. dev. of the default rate data would be *<u>much</u>* *<u>less</u>* than it is.

SD[DR] given independence

Suppose that you have a portfolio of 1000 firms.

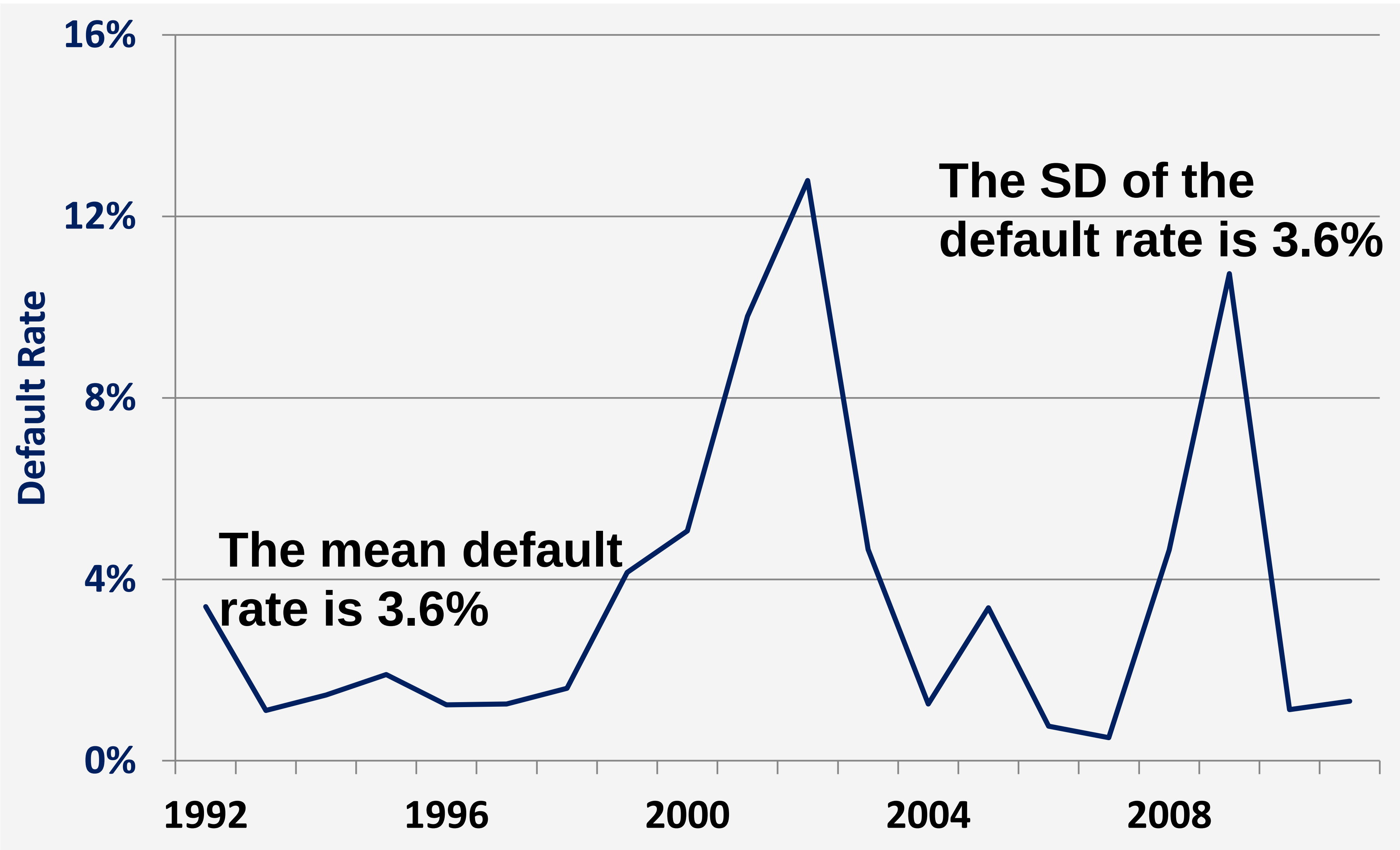
- **Suppose that the PD of each firm is 3.6%.**
- **Suppose that the defaults of firms are independent.**
- **Suppose you have many years of the default rate on this.**

Then the Std. Dev. (SD) of the default rate would be $\sqrt{0.036\,(1-0.036)/1000} = 0.59\%$.

There is a bond portfolio that has a long-term average default rate equal to 3.6%.

- **This is the "junk" bond default rate maintained for decades by Robert Altman at NYU.**
  - **It is *not* known that every PD was 3.6% every year.**
  - **But the SD of Altman's default rate data is so much greater than 0.59% that it appears that firms have positive dependence…**

Altman junk bond default rates



Defaults are not independent

If defaults were independent, then the SD of the default rate would be about 0.59%.

In the real data, the SD is about six times larger.

- A statistical test would reject independence.

Firms are much more likely to default when other firms are defaulting.

- That is, $PDJ > PD_1 PD_2$. Defaults are ***not*** independent.

There are three equivalent measures of the strength of the connection between Firm 1 and Firm 2:

- PDJ itself
- the "default correlation" between the events D_1 and D_2
- the "correlation" between latent variables Z_1 and Z_2.

Default correlation

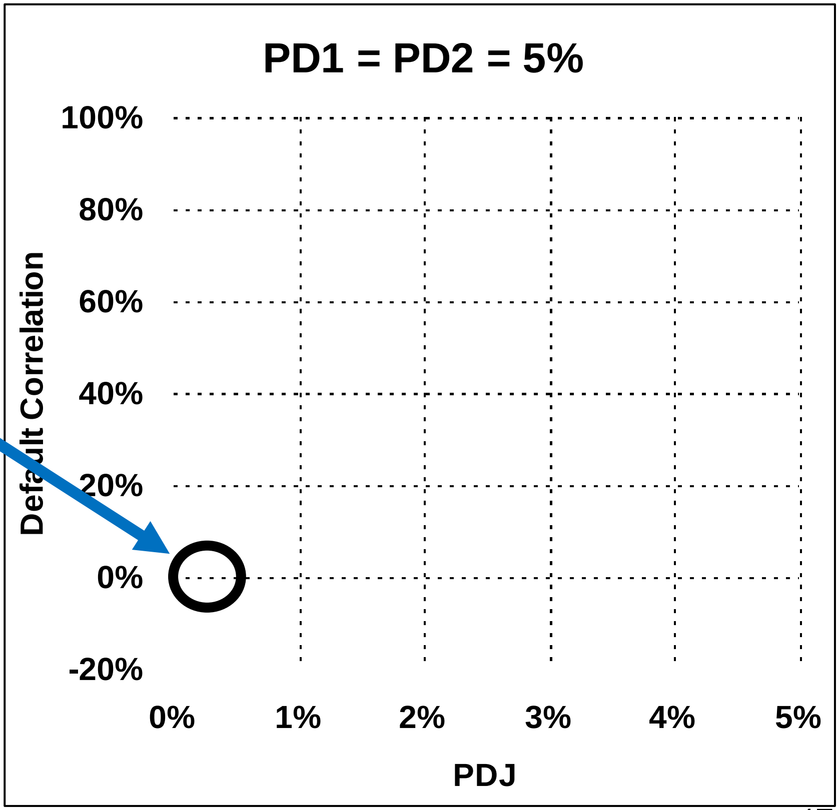
Default correlation is correlation between the events D_1 and D_2. D_1 and D_2 have Bernoulli distributions.

$$Dcorr[D_1, D_2] = \frac{Cov[D_1, D_2]}{\sqrt{Var[D_1]Var[D_2]}}$$

$$= \frac{PDJ - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)\, PD_2(1-PD_2)}}$$

If PDJ = PD_1 * PD_2, then Dcorr = 0.

Holding fixed PD_1 and PD_2,
greater default correlation ⇔
greater PDJ ⇔ greater risk.



Latent variables Z_1 and Z_2

One can simulate a firm's default with a latent variable:

- **Draw Z ~ N [0, 1]; D = 1 if z < Φ^{-1}[PD], otherwise D = 0.**
  - **You can't work backwards and imply Z given D.**

If the defaults of two firms are modeled with two latent normals, then the latent normals are correlated:

- **Draw $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$ given that $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim N\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right]$.**
- **$D_1 = 1$ iff $z_1 < \Phi^{-1}[PD_1]$; $D_2 = 1$ iff $z_2 < \Phi^{-1}[PD_2]$.**

ρ, correlation, implies or is implied by the value of PDJ:

$$PDJ = \int_{-\infty}^{\Phi^{-1}[PD_2]} \int_{-\infty}^{\Phi^{-1}[PD_1]} \phi_2[z_1, z_2, \rho]\, dz_1 dz_2$$

Variables Z_1 and Z_2 are said to be jointly standard normal.

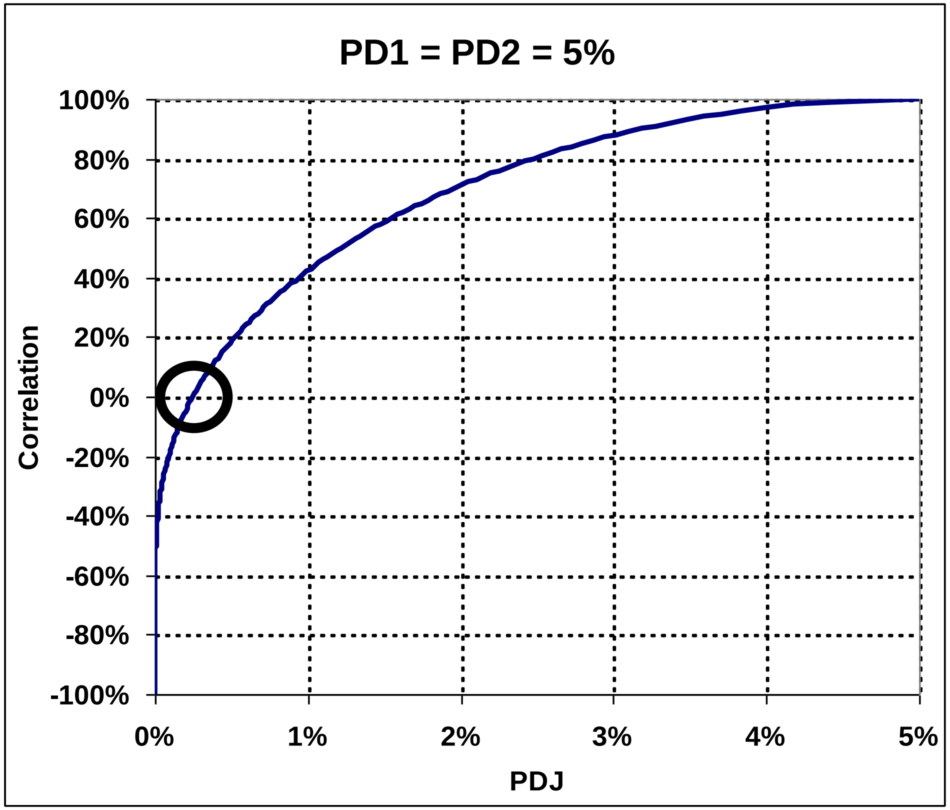
Latent variable correlation

Values of ρ between -1 and 1 produce values of PDJ between 0 and Min [PD_1, PD_2].

For $PD_1 = PD_2 = 5\%$,
PDJ =

$$\int_{-\infty}^{\Phi^{-1}[.05]} \int_{-\infty}^{\Phi^{-1}[.05]} \phi[z_1, z_2, \rho] dz_1 dz_2$$

Greater ρ ⇔ greater PDJ ⇔ greater risk



Three equivalent expressions

PDJ, Dcorr, and ρ express dependence between 2 defaults.

- Given PD_1 and PD_2, each expression implies the other two.

Of the three, risk models make the most use of ρ.

- If a firm's PD changes markedly, then
  - its PDJ with every other firm changes markedly.
  - its Dcorr with every other firm changes markedly.
  - its ρ with another firm does not change markedly and is assumed fixed.
- Second reason: most pairs of firms have ρ in a narrow range.
  - $5\% < \rho < 15\%$ captures the bulk of the pairs of firms.

We refer to ρ as "correlation" or "credit correlation."

- This is easier to say than "the correlation between the latent variables responsible for default."
- Dcorr is named "default correlation." ρ is named "Correlation".

Questions? Comments?

A portfolio with many loans

This might be a trick question

Suppose there is a portfolio containing three firms.

Let each PD equal 10% and let each PDJ equal 1%.

- $PDJ = PD^2$ **in each case.**
- **The matrix of latent variable correlations =** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$**.**

What is the probability that all three firms default?

Can’t say!

There isn’t enough information to answer.

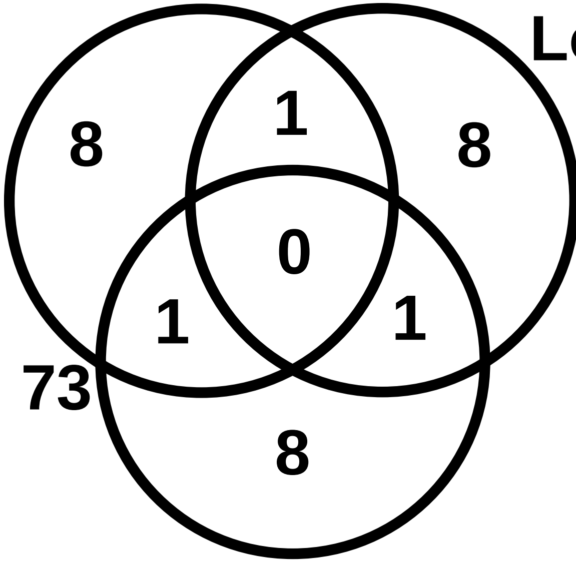
The PDs and PDJs do ***not*** fully describe the distribution of default of the three firms.

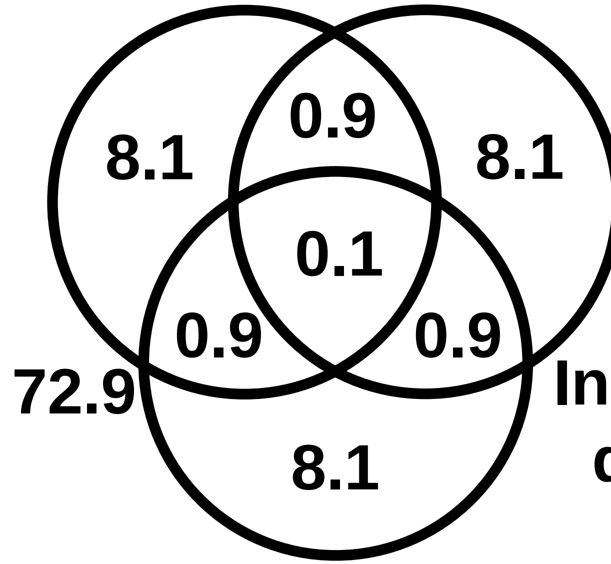
- The distribution of three defaults has more to it than a set of three Bernoulli distributions.
  - The Bernoullis must be ***connected*** to each other in some way.
  - The set of connections between marginal variables is the copula.

The next slide shows ***three*** possible distributions:

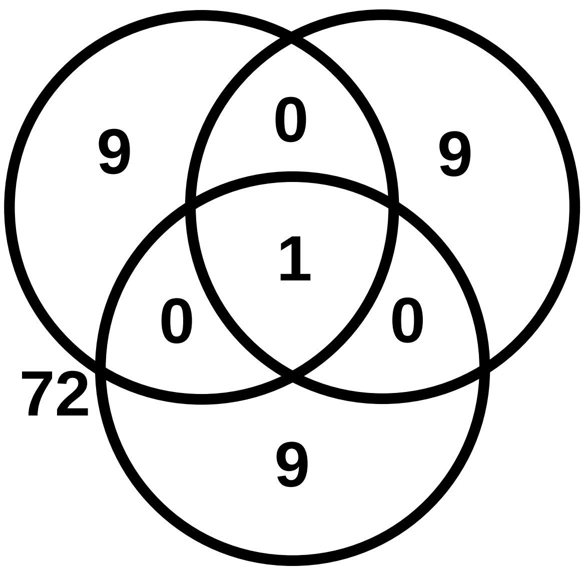
- In each distribution, each firm has PD = 10%, and
- each pair of firms has PDJ = 1%, but
- each distribution gives a ***different*** probability that all three firms default. Distributions of probability ***differ***.
  - It could be as low as 0%, as high as 1%, or anywhere in between.

Low-risk distribution





Independence distribution



High-risk distribution

Independence is more like the low-risk distribution than like the high-risk distribution.

All numbers are in percent

The missing link

Each diagram requires probabilities of 8 events.

- 7 are inside the Venn circles and 1 is outside.
  - In the low-risk copula, Prob[none default] = 0.73.

There are only 7 facts to work with:

- We know 3 PDs, we know 3 PDJs, and we know the sum of eight probabilities equals 1.0.

We need one more fact or assumption.

- Low-risk assumes Prob[$D_1 \cap D_2 \cap D_3$] = 0.
- High-risk assumes Prob[$D_1 \cap D_2 \cap D_3$] = 0.01.
- Independence assumes it is 0.001.
  - If Prob[$D_1 \cap D_2 \cap D_3$] were equal to any number between 0 and 1%, you could find probabilities that are consistent.
- There's no right answer.

Take-aways so far

Each variable has the same marginal distribution.

- **The default of any firm is Bernoulli [PD = 10%].**

The difference between the three joint distributions is in the *<u>connection</u>* between the marginals.

- **The connection between the marginals is called the copula.**

Pairwise behavior does *<u>not</u>* imply joint behavior.

- **Pairwise independence does *<u>not</u>* imply joint independence.**
  - **An exception to the rule is the connection called the Gauss copula.**
  - **The independence copula is a special case of the Gauss copula.**
  - **In a Gauss copula, pairwise correlations *<u>do</u>* imply joint behavior.**
  - **That’s why the Gauss copula is so useful when there is no other guide to the joint behavior! The Gauss copula makes up the answers.**

Questions? Comments?

The copula can be separated

Every continuous distribution has two components:

- **The set of marginal distributions obeyed by each variable.**
  - **For example, each default indicator is distributed Bernoulli[PD].**
- **The copula that connects the marginal distributions.**
  - **Some examples have been Low-risk, High-risk, and Independence.**
- **The copula is defined as a multivariate distribution where every marginal distribution is the uniform distribution.**
  - **Inverse CDFs convert the uniform variables to the desired marginals.**

This section will illustrate with examples:

- **Normal marginals connected with a Gauss copula**
- **Normal marginals connected with a non-Gauss copula**
- **Non-normal marginals connected with a Gauss copula**
- **Non-normal marginals connected with a non-Gauss copula**

The Gauss copula is defined by the vector CLT

Scalar central limit theorem: Let X_i ~ IID [μ, σ^2], i = 1, …, n

- **Define $\bar{X} = \frac{1}{n} \sum X_i$**
- **Law of large numbers: As n $\rightarrow \infty$, $\bar{X} \rightarrow \mu$**
- **Central limit theorem: As n $\rightarrow \infty$, $\sqrt{n}\,(\bar{X} - \mu) \overset{d}{\rightarrow}$ something.**
  - **The proof makes clear that if z = $\sqrt{n}\,(\bar{x} - \mu)/\sigma$, then $f_z[z] = Exp\left[-\frac{z^2}{2}\right]/\sqrt{2\,\pi}$.**
  - **This *__defines__* the standard normal distribution.**

Vector central limit theorem: Let X_{k_i} ~ IID [μ_k, Σ_k], i = 1,..,n

- **LLN: As n $\rightarrow \infty$, $\overline{X_k} \rightarrow \mu_k$**
- **CLT: As n $\rightarrow \infty$, $\sqrt{n}\,(\overline{X_k} - \mu_k) \overset{d}{\rightarrow}$ N [0_k, Σ_k]**
- **This *__defines__* the multinormal distribution. It *__implicitly__* defines the Gauss copula as the *__connection__* between the k distributions.**
  - **Note: $\{X_{k_i}\}$ can be connected by *__any__* permitted copula.**

Separating the copula

A copula is a multivariate distribution where each marginal variable has the distribution Uniform[0,1].

It is easy to separate the Gauss copula from the jointly normal distribution just defined.

- The variables in the standard multinormal distribution each have a standard normal distribution.
- To each variable Z_i, apply the normal CDF: $U_i = \Phi[Z_i]$.
- The joint distribution of $\{U_i\}$ is the Gauss copula.

It is also easy to recreate the joint normal distribution.

- To each of the U_i provided by the Gauss copula, apply the inverse standard normal CDF: $Z_i = \Phi^{-1}[U_i]$.
  - We're back to where we started, which is the joint normal distribution.

Copula variables

In general, the marginal distributions of a copula have uniform distributions.

- These uniform variables are called the “copula variables.”
  - They are not observed.
- A set of transformations produces variables like the ones the modeler works with.

The Gauss copula has a matrix of “copula correlations.”

- They are correlations between $\{Z_i\}$.
- The matrix must be positive definite.

A definition and 3 handy facts

Jointly **normal variables are normal variables that are connected by a Gauss copula.**

1. Independent normal variables are jointly normal.

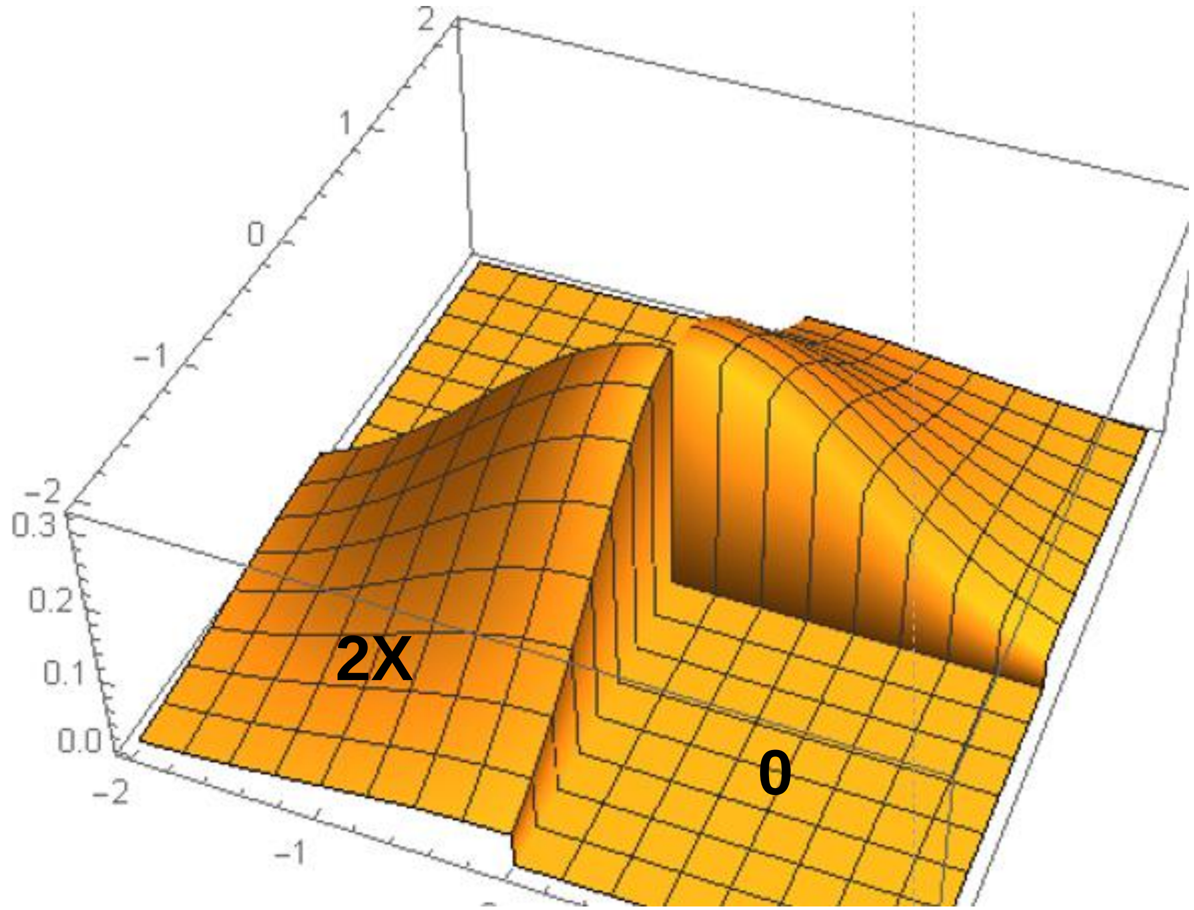
2. Variables that are linear functions of jointly normal variables are jointly normal.

3. The Gauss copula produces a lot from a little.

- **In a portfolio of 300 loans, there are $\mathbf{2^{300}}$ unique probabilities.**
  - **This would require about $\mathbf{2^{300} = 2.04 \times 10^{90}}$ assumptions.**
- **But there are only $\mathbf{299 \times \frac{300}{2} = 4.4 \times 10^{3}}$ unique correlations.**
  - **The Gauss copula is about $\mathbf{10^{87}}$ easier to implement.**

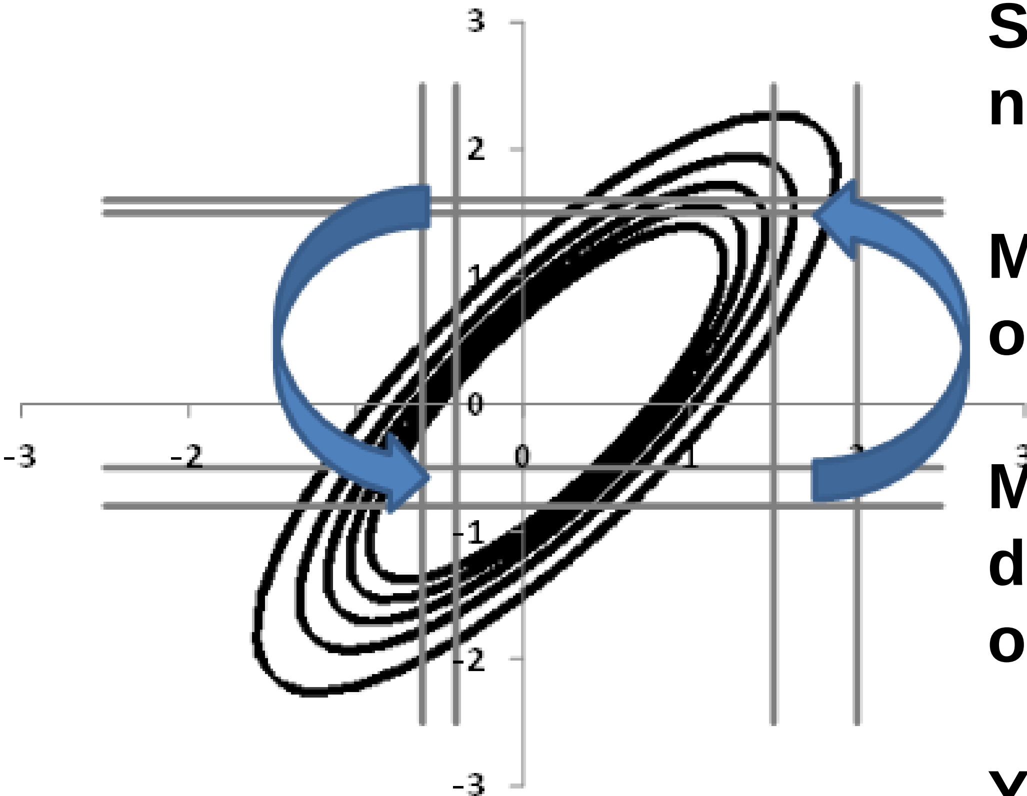
Questions? Comments?

Normal variables, non-Gauss copula



$$pdf_X[x] = \int_{-\infty}^{\infty} pdf_{X,Y}[x,y]\, dy$$

Degaussing a Gauss copula



Start with correlated joint normals.

Move a little density from one box to the other.

Move the same amount of density the other way on the other side.

You have two standard normals connected by a non-Gauss copula.

Two other non-Gauss copulas

Gumbel Clayton

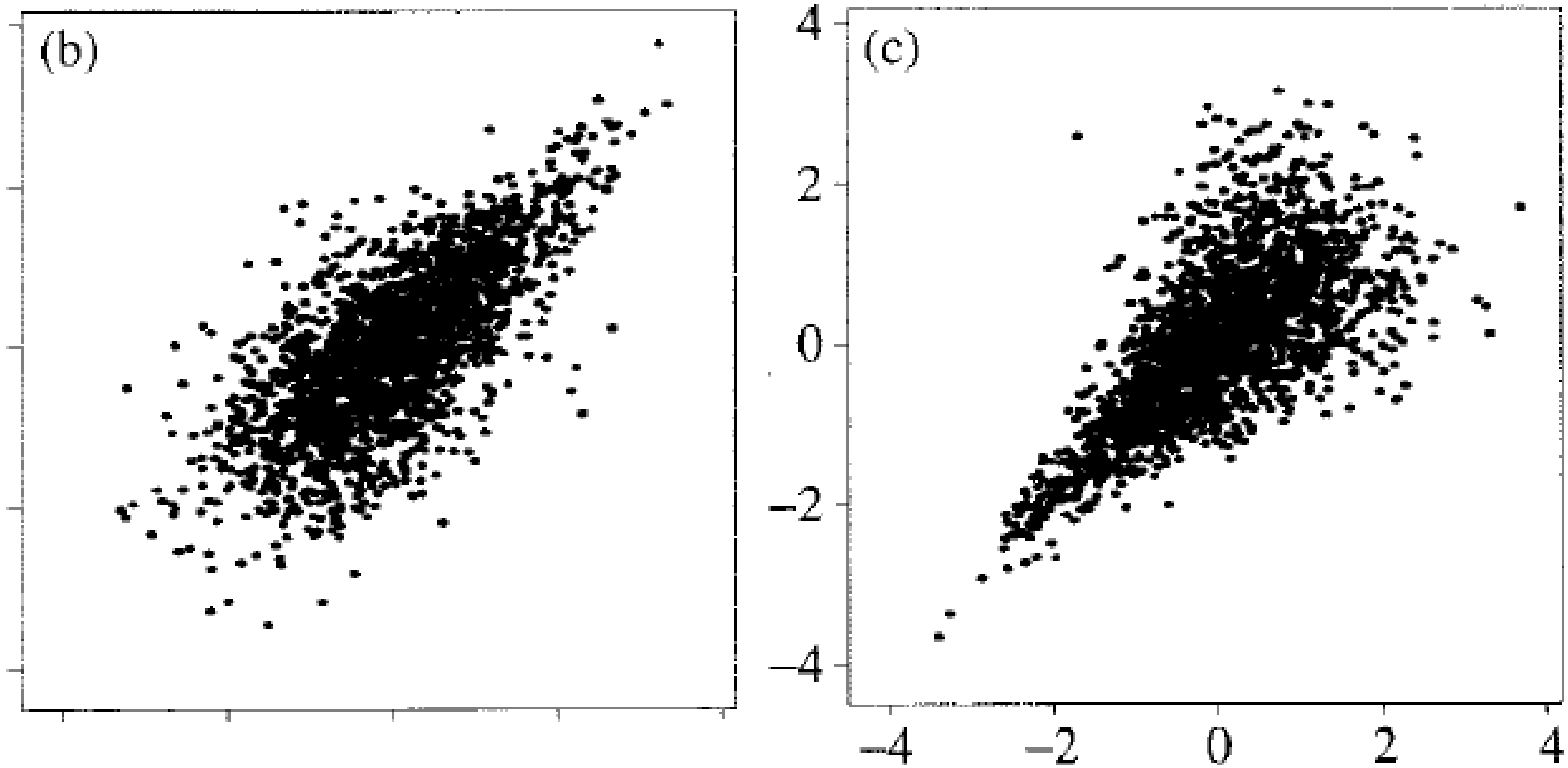


Figure 5.4. Two thousand simulated points from four distributions with standard normal margins, constructed using the copula data from Figure 5.3 ((a) Gaussian, (b) Gumbel, (c) Clayton and (d) *t*). The Gaussian picture shows points from a standard bivariate normal with correlation 70%; other pictures show distributions with non-Gauss copulas constructed to have a linear correlation of roughly 70%. See Example 5.11 for parameter choices and

Non-normal variables, Gauss copula

We already had an example:

- **Each default indicator has a Bernoulli distribution.**
- **Each default indicator is calculated $D_i = I[Z_i < \Phi^{-1}[PD_i]]$.**
- **$\{Z_i\}$ is connected by a Gauss copula.**
- **$\{D_i\}$ is a set of non-normal variables that is connected by a Gauss copula.**

If you have a set of distributions that you want to connect in some way, the Gauss copula is an easy way to do it.

Non-normal RVs, non-Gauss copula

First example: The Winner-Take-All copula.

- **It is a specific case of the “low-risk copula.”**

Second example: The multivariate *t*-copula.

- **It uses a matrix of correlations, but it implies different probabilities than the Gauss copula.**

The winner-take-all copula

Three firms compete in the development of the next gizmo. The one that develops it first will not default on its debt. The other two firms will fail and default.

- **Prob[$D_1 \cap D_2 \cap D_3$] = 0. Very much like the low-risk copula.**
- **Suppose that at present the firms are equal contenders.**

Then, all the PDs are 2/3 and all the PDJs are 1/3.

- **A firm defaults if either of its rivals wins the competition: 2/3.**
- **A pair of firms defaults if the third firm wins the competition: 1/3.**
  - **The PDJs are less than the product of the PD's; values of ρ are negative.**
  - **In fact, each correlation equals -1.**
  - **The matrix correlations of the latent variables is therefore…**

Correlation matrix

The matrix of latent variable correlations is

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{1} & \mathbf{-1} \\ \mathbf{-1} & \mathbf{-1} & \mathbf{1} \end{bmatrix}$$

The Eigenvalues of the matrix are {2, 2, -1}; det = -4.

The matrix is *<u>not</u>* positive definite or semi-definite.

It *<u>cannot</u>* describe jointly normal variables.

Still, this is a realistic copula and easy to simulate:

- **Draw U, a variable that is uniformly distributed on [0,1].**
- **If $U < \frac{1}{3}$, Firms 1 and 2 default. If $\frac{1}{3} < U < \frac{2}{3}$, Firms 1 and 3 default.**
- **If $\frac{2}{3} < U$, Firms 2 and 3 default.**

Questions? Comments?

The (non-Gauss) *t*-copula

The *t*-copula resembles the Gauss copula in some ways.

- **It requires a positive definite correlation matrix.**
  - **From that, it infers the complete connection between variables.**
- **It is the natural connection of a set *t*-distributed variables, just as the Gauss is natural for normals.**
  - **As the number of degrees of freedom rises, the *t*-copula approaches the Gauss copula.**

But *t* copula is different in a most-important way.

- **Just as a *t* distribution has fatter tails than the normal, the *t* copula has greater "tail dependence" than the Gauss.**
  - **If one variable is in its tail, the other is more likely to be in its tail than would be implied by the Gauss copula.**
  - **Therefore, it implies different probabilities than Gauss. Fatter tails.**
  - **Equity returns seem to have elevated tail-dependence.**

Gauss t ($\nu = 4$)

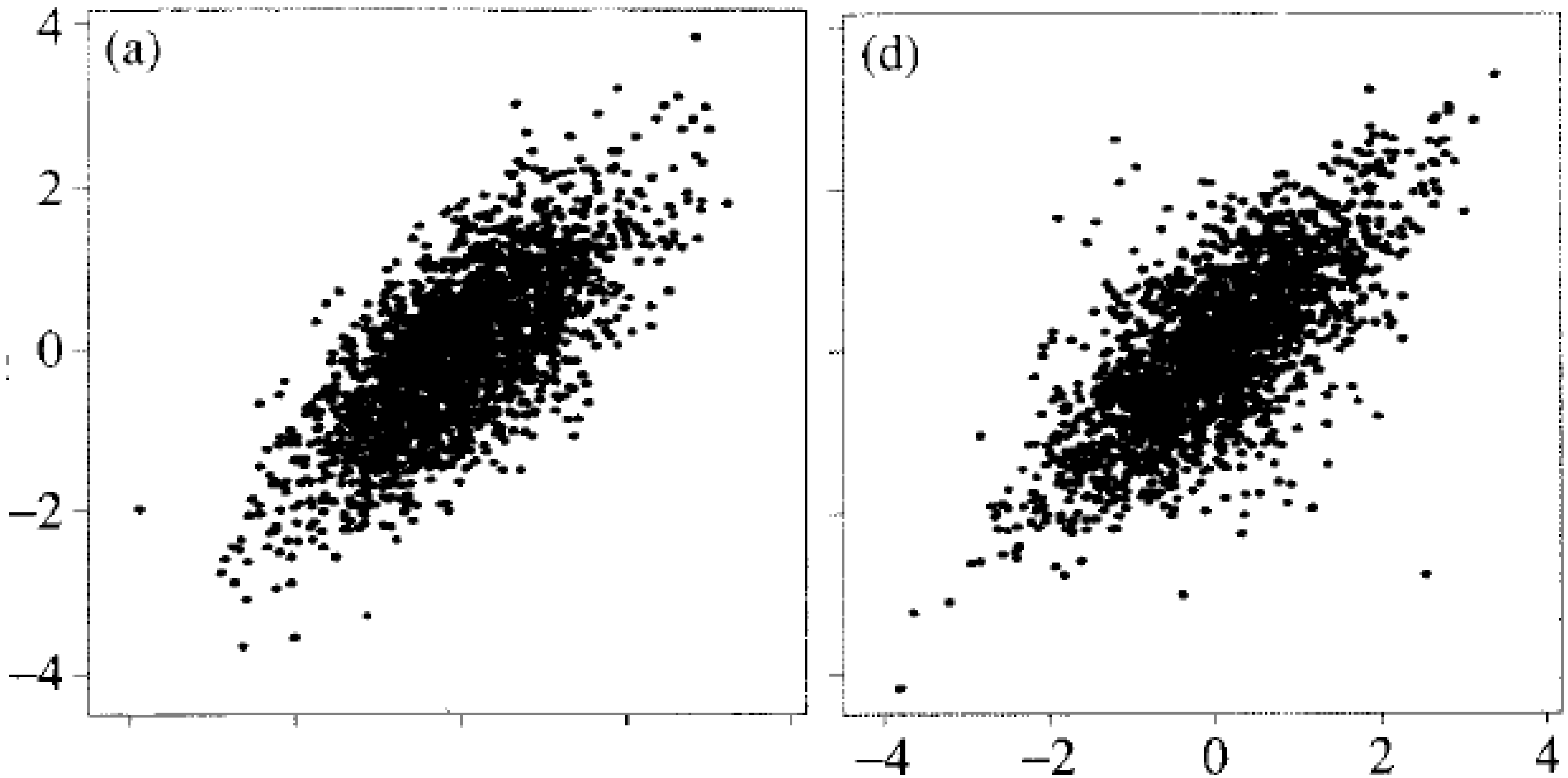


Figure 5.4. Two thousand simulated points from four distributions with standard normal margins, constructed using the copula data from Figure 5.3 ((a) Gaussian, (b) Gumbel, (c) Clayton and (d) t). The Gaussian picture shows points from a standard bivariate normal with correlation 70%; other pictures show distributions with non-Gauss copulas constructed to have a linear correlation of roughly 70%. See Example 5.11 for parameter choices and

The *t*-distribution is a mixture

Define multivariate *normal* *variance* *mixture* distributions:

$$\boldsymbol{X_k = \sqrt{W}\, Z_k;\ Z_k \sim N_k[0_k, \Sigma_k];}$$

$\boldsymbol{W}$ **is a positive scalar;** $\boldsymbol{X_k}$**,** $\boldsymbol{Z_k}$ **and** $\boldsymbol{0_k}$ **are vectors.**

If $\boldsymbol{W}$ **is fixed, then** $\boldsymbol{X_k \sim N_k[0_k, W\,\Sigma_k]}$**.**

If $\boldsymbol{W}$ **is random, independent and nice, then**

$$\boldsymbol{E[X_k] = 0_k;\ \ Var[X_k] = E[W]Var[Z_k];\ \ Corr[X_k] = Corr[Z_k]}$$

$\boldsymbol{X}$ **has the same correlations as** $\boldsymbol{Z}$**, but the** $\boldsymbol{k}$ **variables are *not* connected by a Gauss copula…**

$$\boldsymbol{X = \sqrt{W}\,\mathrm{Z};\quad \mathrm{Z} \sim N_k[0, \mathrm{I}]}$$

Choose two X variables and take their absolute values.

- **If these are values are independent, then the expectation of their product equals the product of their expectations. But no!**

$$\boldsymbol{E[\,|X_1|\,|X_2|\,] = E[\,W\,|Z_1|\,|Z_2|\,] = E[W]\;E[\,|Z_1|\,]\;E[\,|Z_2|\,]}$$

$$>$$

$$\boldsymbol{(E[\sqrt{W}])^2\,E[\,|Z_1|\,]\;E[\,|Z_2|\,] = E[\,|X_1|\,]\;E[\,|X_2|\,]}$$ **; QED.**

Logic: For any random variable, ω:

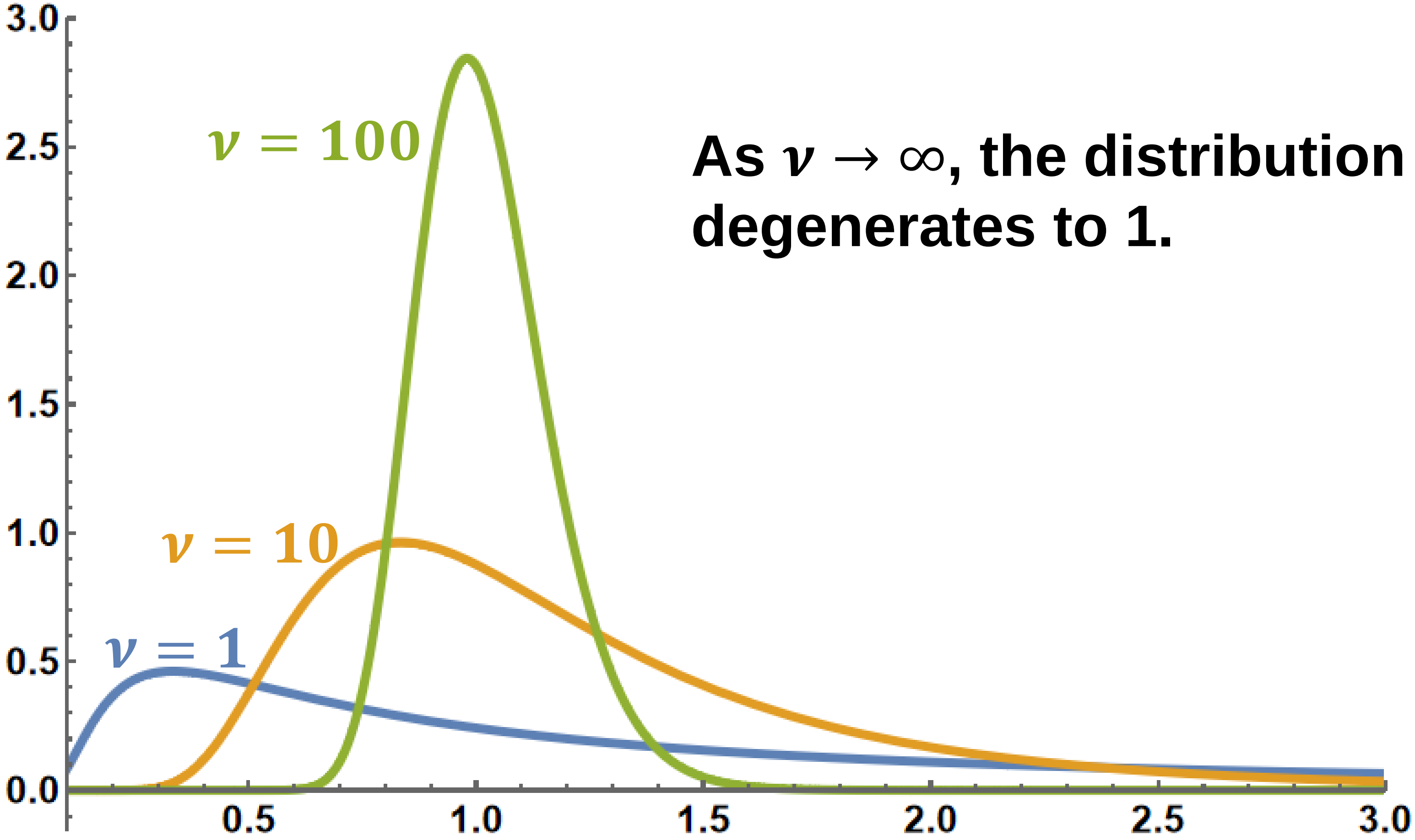
$$\boldsymbol{Var[\omega] = E\left[(\omega - E[\omega])^2\right] = E\left[\omega^2\right] - 2(E[\omega])^2 + (E[\omega])^2}$$

$$\boldsymbol{= E\left[\omega^2\right] - (E[\omega])^2 > 0.}$$

$\boldsymbol{E\left[\omega^2\right] > (E[\omega])^2;\quad \therefore\ |X_1|}$ **and** $|X_2|$ **are *not* independent.**

The mixing distribution for *t*

Let $W \sim Inverse\ Gamma\ Distribution[\frac{\nu}{2}, \frac{\nu}{2}]$, ν an integer.



As $\nu \to \infty$, the distribution degenerates to 1.

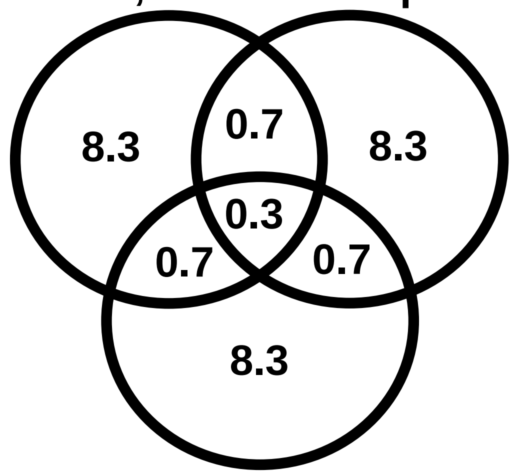
Student's *t*

The marginal distributions that result are Student's *t*.

- **The number of degrees of freedom equals ν.**

The connection between the variables is the t-copula.

- **The copula is different for each value of ν.**
  - **As ν grows, the Inverse Gamma degenerates to 1.0, and the t-copula becomes the same as the Gauss copula.**



With $\boldsymbol{\Sigma = I}$ and $\boldsymbol{\nu}$ = 4,

Prob[$D_1 \cap D_2 \cap D_3$] = 0.00343

The worst case is more than 3 times as likely as with the Gauss copula.

t-copula summary

The variance mixture distribution is attractive.

- **Over the cycle, underlying factors Z might have a normal distribution, but the W factor changes things sometimes.**

The *t*-copula is attractive for portfolio credit loss models.

- **You can dial in more tail dependence by reducing the value of ν.**
  - **That seems more satisfactory than saying, "Correlation goes up in a crisis," every time you have another crisis.**

We have other things to do

Standard practice uses the Gauss copula.

This is easier than learning the *t*-copula.

- **We have other things to get confused about.**

It is beyond this course to distinguish the Gauss copula from *t*-copula in a set of default data.

We use the Gauss copula.

- **If you don't like it, you know what you can do.**
  - **Namely, perform the simulations and analysis using the *t*-copula instead.**

Questions? Comments?

Course mechanics

Here is a quick summary
of what you would find in the Syllabus.

Get in touch

For homeworks, please Email Davide and *copy* Lisheng.

For most course content, please start with Lisheng.

I usually answer Emails early the next morning.

- **Sometimes I answer by sending a clarifying Announcement.**
- **I can meet by appointment before class in MS Room 302.**

After this quarter, contact me any time about anything.

https://uchicago.instructure.com/courses/48373

Canvas has a module for each week. Each contains:

- **The slide deck for the lecture**
  - **A “draft” lecture is usually available well before the “final” one.**
- **Optional readings**
  - **This week's reading is the CreditMetrics manual of 1997. This was the first comprehensive articulation of the model that you study this week.**
- **A homework associated to the lecture.**
  - **This becomes available after the lecture. It is due before the next lecture.**

On-line session notes

- **Lisheng’s on-line sessions will be Sundays at 6:00 pm.**

Announcements

- **You are responsible to act on the announcements on Canvas.**

Homeworks

The homeworks prepare you to perform well on the exam.

- **This is worth more than the points you get.**
  - **Homework questions help you think through certain issues in advance.**
  - **Lisheng's online sessions make the connection more direct.**

Each homework set is due at 6pm the next Thursday.

- **Late submissions are graded like others, but the score is penalized 50% the first 24 hours and 100% afterwards.**
  - **Many strange things happen when people have time pressure.**
  - **Submit early to be safe!**

Five online sessions

Lisheng will discuss the homeworks in online sessions.

- **These take place Sundays at 6:00 pm Chicago time.**
  - **The dates are March 26, April 2, 9, 16, and 23.**
- **Have you received an invitation to each Zoom session?**

Online session topics:

- **Questions and guidance regarding course content**
  - **Please send your questions to Lisheng *in advance* if you can.**
- **In-depth discussion of previous homework questions**
- **Tactics and hints for the current homework**

A paper-and-pencil final exam

6:00 to 7:30 Thursday April 27

Most of your grade is based on the final exam.

Each student takes the exam in person.

- **If you require other arrangements, Email me ASAP.**

You must bring your U of C photo ID to the exam.

- **You cannot use books or written notes of any kind.**
- **You cannot use electronic gear.**
  - **This includes everything: No laptops, tablets, cell phones, or calculators**
- **Any violations will be treated as serious.**

How to prepare for the exam

The best preparation is to think about homeworks.

- Some exam questions go a bit deeper than homeworks.
  - Challenge your own understanding!
  - If you haven’t thought about the homework questions, you are unlikely to do well on the exam.

The range of scores tends to be very large.

- From the mathematical expectation up to almost perfect.

The duration of the exam is 75 minutes.

- Most students feel time pressure.
- Plan to manage time wisely.
  - Recheck your answers on questions you find easier.
  - Do not waste time on questions you find harder.

Grading

Subject to discretion and (usually upward) adjustment,

- **About 84% of your grade stems from the exam.**
- **About 16% of your grade stems from homeworks.**

If you want to take this course Pass / Fail:

- **Contact Meredith Hajinazarian meredith.hajinazarian@uchicago.edu.**
  - **Or use her student Services Page on Canvas. *There is a deadline*.**
  - **She will check that you have unused options remaining.**
- **Then, if you pass the exam, your grade will be "P".**

If you want a grade like A, B, or C (or D or F), do nothing.

Weekly events

Wednesday class sessions

- Before class starts:
  - Homeworks are due.
- After class ends:
  - I post the next homework question set.
  - Michael Jelik posts the recorded session on Canvas.

Sunday TA sessions

- Before the session: Please Email questions to Lisheng.
- After the session is over:
  - Lisheng posts a recording of the session to Canvas with solutions to the previous homework and hints about the current one.

Any questions or comments?

Action items

It is best to read Lecture 2 *in advance*.

- **It contains some novelty and some math.**

Lisheng's first Zoom session is 6PM Sunday March 26.

Homework Set 1 is due by 6PM Thursday March 30.