- We find the loss distribution for a credit-risky portfolio.
- Bank finances the loans with
  - equity → selling stocks.
  - liabilities. → deposits and bonds.

$D \to$ random default indicator. → 1 if defaults. → 0 otherwise.

$PD = \mathbb{E}[D]$ → prob. that firm defaults within a period.

$DR = \sum_{i=1}^{N} \frac{D_i}{N}$ → the default rate in the portfolio.

$\$LGD = \$Exposure - \$Recovery$

→ Ratios: $LGD = 1 - \$Recovery / \$Exposure$

$Loss = D * LGD.$ → for 1 loan.

$Loss = \frac{\sum_{i=1}^{K} D_i LGD_i}{N}$

PDJ → probability that both firms default. $PDJ_{1,2}$. if independent: $PDJ = PD_1 PD_2$

In general, defaults are dependent

$Cov(D_1, D_2) = E[(X-EX)(Y-EY)] = E[XY] - E[X]E[Y]$

Bernoulli trials!!!

$= E[D_1 D_2] - E(D_1)E(D_2)$

$= PDJ - PD_1 PD_2$

- $Dcorr[D_1, D_2] = \frac{Cov(D_1, D_2)}{\sqrt{Var(D_1) Var(D_2)}}$

$= \frac{PDJ - PD_1 PD_2}{\sqrt{PD_1(1-PD_1) PD_2(1-PD_2)}}$, if $PDJ = PD_1 * PD_2 \Rightarrow Dcorr = 0$

Simulation of a firm's default

$Z \sim N(0,1)$ → $D=1$ if $z < \Phi^{-1}(PD)$ → $D=0$ e.o.c.

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \sim N\left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right] \Rightarrow \begin{array}{l} D_1 = 1 \text{ iff } z_1 < \Phi^{-1}[PD_1] \\ D_2 = 1 \text{ iff } z_2 < \Phi^{-1}[PD_2] \end{array}$$

$$PDJ = \int_{-\infty}^{\Phi^{-1}[PD_2]} \int_{-\infty}^{\Phi^{-1}[PD_1]} \phi_2[z_1, z_2, \rho] \, dz_1 \, dz_2$$

greater $\rho \iff$ greater PDJ $\iff$ greater risk

Jointly normal variables are normal variables that are connected by a Gauss copula.

1. Independent normal variables are jointly normal.
2. Variables that are linear functions of jointly normal variables are jointly normal.
3. Gauss copula produces a lot from a little.

PD_1	PD_2	PD_3	$PDJ_{1,2}$	$PDJ_{1,3}$	$PDJ_{2,3}$
0.1	0.2	0.3	0.06	0.06	0.06

$$Dcorr(D_1, D_2) = \frac{PDJ_{1,2} - PD_1 PD_2}{\sqrt{PD_1(1-PD_1)PD_2(1-PD_2)}} = \frac{0.06-(0.1)(0.2)}{\sqrt{0.1(1-0.1)0.2(1-0.2)}} = \frac{0.06-0.02}{\sqrt{.09(.16)}} = \frac{0.04}{\sqrt{.0144}} = \frac{.04}{.12} = \frac{1}{3}$$

$$Dcorr(D_1, D_3) = \frac{PDJ_{1,3} - PD_1 PD_3}{\sqrt{PD_1(1-PD_1)PD_3(1-PD_3)}} = \frac{0.06-(.1)(.3)}{\sqrt{.1(1-.1).3(1-.3)}} = \frac{0.06-.03}{\sqrt{.09(.21)}} = \frac{0.03}{\sqrt{.0189}} = .22$$

$$PDJ = \int_{-\infty}^{\phi^{-1}[PD_2]} \int_{-\infty}^{\phi^{-1}(PD_1)} \phi_2[z_1, z_2, c]\, dz_1\, dz_2$$

Exercise 2.

$PD = 0.10$ and correlations are $\begin{pmatrix} 1 & .4 & .5 \\ .4 & 1 & .6 \\ .5 & .6 & 1 \end{pmatrix}$ 3 values of PDJ.

$$PDJ_{1,2} = \int_{-\infty}^{\phi^{-1}(PD_2)}\int_{-\infty}^{\phi^{-1}(PD_1)} \phi_2(z_1, z_2, e)\, dz_1\, dz_2 = \int_{-\infty}^{\phi^{-1}(.1)}\int_{-\infty}^{\phi^{-1}(.1)} \phi_2[z_1, z_1, .4)\, dz_1\, dz_2$$

inv_cnd(PD$_1$), inv_cnd(PD$_2$)

$PDJ_{1,2} = 0.027$, $PDJ_{1,3} = 0.032$, $PDJ_{2,3} = 0.039$

range: [0, min(PDJ's)]

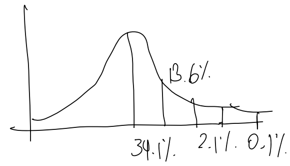
Exercise 3

$e_{A,B} = 0.9$

Initial Rating	Final Rating = A	Final Rating = B	Final Rating = D
A	0.5	0.4	0.1
B	0.3	0.5	0.2

$cPD_i = \phi\left[\dfrac{\phi^{-1}[PD_i] + \sqrt{\rho_i}\, z}{\sqrt{1-\rho_i}}\right]$

std: $\sigma = \sqrt{\dfrac{\Sigma (x_i - \mu)^2}{N}}$



Homework 2.

a) $P(D_4 = 1 \text{ and } D_5 = 1)$ What is PDJ for these 2 firms?

$$\int_{-\infty}^{\phi(.4)} \int_{-\infty}^{\phi(.5)} \phi_2(z_1, z_2, .45)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\mathbb{P}(B|A)\,\mathbb{P}(A)}{\mathbb{P}(B)}$$

b) $P(D_4 = 1 \text{ and } D_5 = 1 \mid D_3 = 1)$

$$P(D_4 = 1 \cap D_5 = 1 \mid D_3 = 1) = \frac{P(D_4 = 1 \cap D_5 = 1 \cap D_3 = 1)}{\mathbb{P}(D_3 = 1)}$$

c) Portfolio expected loss rate as a fraction of the $21,800 exposure?

$$\text{Expected loss} = \underbrace{PD}_{\text{Probability of Default}} \cdot \underbrace{ELGD}_{\text{expected loss given default}} \cdot \underbrace{EAD}_{\text{exposure at default}}$$

d) $$Dcorr(D_3, D_4) = \frac{PDJ_{3,4} - PD_3 \cdot PD_4}{\sqrt{PD_3(1-PD_3) \cdot PD_4(1-PD_4)}}$$

4. 36702 distribution, $f_{x,y}(x,y) = \dfrac{(1+3x-y)}{2}$ cPDJ, Dcorr & ρ?

① Get $f_X(x)$ and $f_Y(y)$ → $f_X(x) = \int_0^1 f_{x,y}(x,y)\,dy$

$f_Y(y) = \int_0^1 f_{x,y}(x,y)\,dx$

② Get $F_X(x)$ and $F_Y(y)$ → $F_X(x) = \int f_X(x)\,dx$

$F_Y(y) = \int f_Y(y)\,dy$

③ Get $F_X^{-1}(x)$ and $F_Y^{-1}(y)$

④ Calculate PD → example $PD_x = 0.1$ $PD_x = F_X^{-1}(.1)$

$PD_y = 0.2$ $PD_y = F_Y^{-1}(.2)$

(5) Get PDJ for that:

(5.1) Get $F_{X,Y}(x,y) = \iint f_{x,y}(x,y)\,dx\,dy$

(5.2) $F_{X,Y}(F_X^{-1}(.1), F_Y^{-1}(.2))$

Homework 4

loss = dr * lgd

Ex. 1.

$pdf_{dr}(dr) = 2 - 2dr$

$lgd(dr) = dr^{1/2} = \sqrt{dr}$

¿ $Pdf_{lgd}(lgd)$

(1) Inverse of lgd(dr)

$f(x) = x^{1/2}$

$y = x^{1/2}$

$x = y^{1/2}$

$y = x^2$

$dr = lgd^2$ *

let
$f(x) = lgd(dr)$
$y = lgd(dr)$
$x = dr$

(2) differentiate.

$$\frac{d(dr)}{d\,lgd} = \frac{d(lgd^2)}{d\,lgd} = 2\,lgd$$

(3) Evaluate the function.

$$Pdf_{lgd}(lgd) = (2 - 2\,lgd^2)^{*} \frac{d(dr)}{d\,lgd}$$

$$= (2 - 2\,lgd^2)\,2\,lgd$$

$$= 4\,lgd - 4\,lgd^3 \checkmark$$

Exercise 2

¿ $Pdf_{loss}(loss)$?

→ loss = dr * lgd

$loss = dr * dr^{1/2}$

$loss = dr^{3/2}$

$loss^{2/3} = dr$

$$\frac{d(dr)}{d\,loss} = \frac{d(loss^{2/3})}{d\,loss}$$

$$= \frac{2}{3} loss^{-1/3}$$

$$pdf_{loss}(loss) = \left[2 - 2(loss^{2/3})\right] \cdot \frac{2}{3} loss^{-1/3}$$

$$= \frac{4}{3} loss^{-1/3} - \frac{4}{3} loss^{1/3} \checkmark$$

Question: $\begin{cases} \bullet\ EL \\ \bullet\ ELGD \\ \bullet\ cLGD. \end{cases}$

$\bullet\ EL = E[c\,loss] = \int_0^1 loss \cdot pdf_{loss}\, d(loss)$

$= \int_0^1 loss \cdot \left(\frac{4}{3} loss^{-1/3} - \frac{4}{3} loss^{1/3}\right) d(loss)$

$= \left(\frac{4}{5} loss^{5/3} - \frac{4}{7} loss^{7/3}\right)\Big|_0^1$

$= 0.23$

$\bullet\ ELGD = \frac{EL}{PD}$, so we need to calulate PD

$PD = E[cPD] = E(dr) = \int_0^1 dr \cdot PDF_{dr} \cdot d(dr)$

$= \int_0^1 dr \cdot (2-2dr)\, d(dr)$

$= 1 - \frac{2}{3} = \frac{1}{3} \approx .33$

$ELGD = \frac{0.23}{.33} \approx 0.69$

"Time-weighted" LGD $= \mathbb{E}[c\,lgd] = \int_0^1 lgd \cdot pdf_{lgd}\, d(lgd) = \int_0^1 4\,lgd^2 - \int_0^1 4\,lgd^4\, d(lgd)$

$= \left[\frac{4}{3} lgd^3 - \frac{4}{5} lgd^5\right]_0^1 = \frac{4}{3} - \frac{4}{5} = \frac{20-12}{15} = \frac{08}{15} = .53$

Homework 4

Questions:
a) EL
b) ELGD
c) cLGD.

①

state	A	B	C	D
prob. of state	.4	.3	.2	.1
cPD	.02	.04	.06	.08
cLGD	.1	.3	.5	.7

a) $EL = \mathbb{E}[\text{closs}]$

*closs = cPD * cLGD.

state	A	B	C	D
prob. of state	.4	.3	.2	.1
cPD	.02	.04	.06	.08
cLGD	.1	.3	.5	.7
* closs	.002	.012	.030	.056

$\mathbb{E}[\text{closs}] = \text{closs} \cdot \text{prob. state}$

$= .002(.4) + .012(.3) + .030(.2) + .056(.1)$

$= .0008 + .0036 + .0060 + .0056 = .0160$

b) $ELGD = \dfrac{EL}{PD}$, so we need PD.

$PD = \mathbb{E}[cPD] = cPD * \text{prob. state}$

$= .008 + .012 + .012 + .008$

$= .040$

$\Rightarrow ELGD = \dfrac{.016}{.04} = .4$

d) $cLGD = \mathbb{E}(clgd) = clgd * \text{prob. state}$

$= .04 + .09 + .10 + .07 = .30$